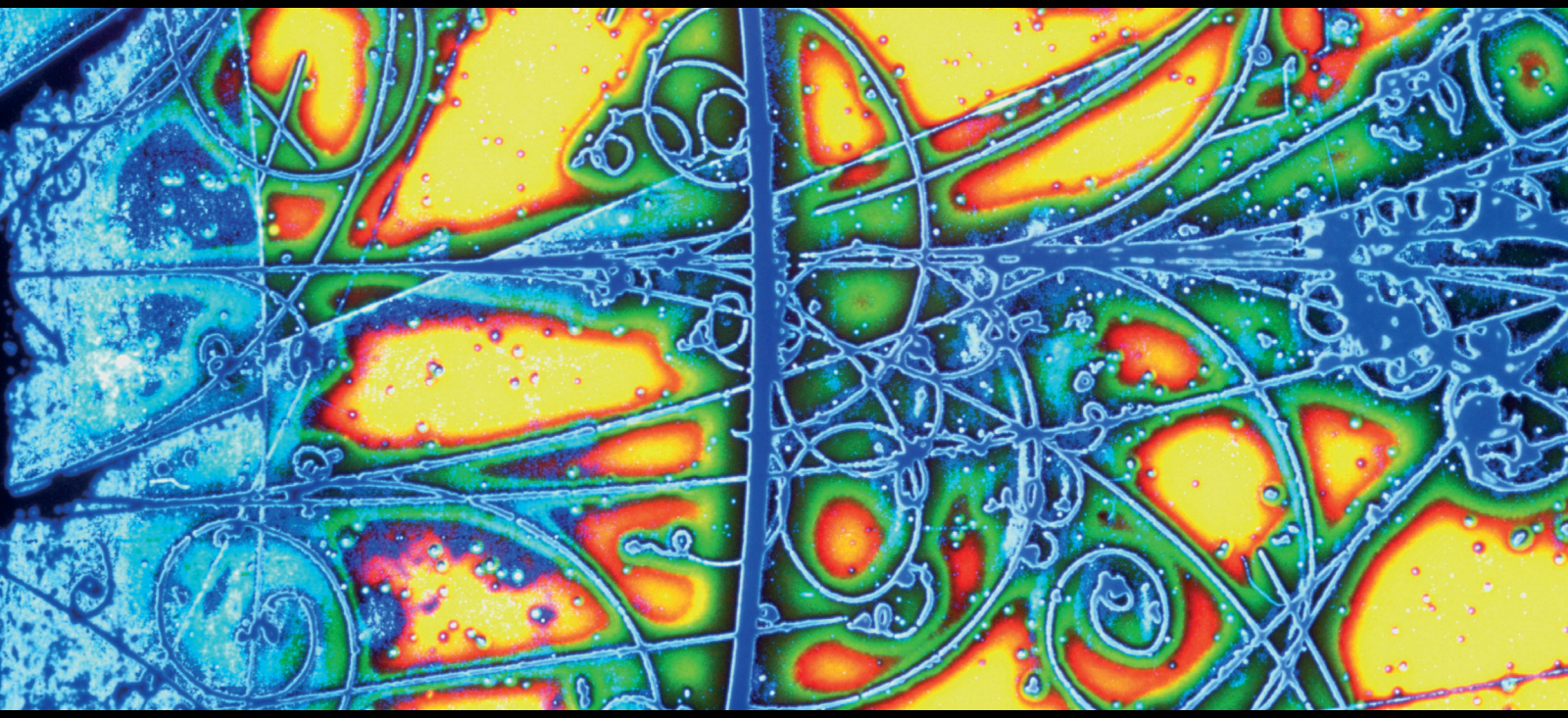
Advances in High Energy Physics

# Black Holes Physics

Guest Editors: Xiaoxiong Zeng, Christian Corda, and Deyou Chen



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# Editorial Board

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*Editorial*

# Black Holes Physics

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Black hole (BH) is perhaps the most fascinating object in the research fields of astrophysics and gravitational physics. This mysterious object is predicted by the general theory of relativity (GTR), but such a classical theory cannot explain all its properties. It is believed that only a more general, definitive theory of quantum gravity, which should unify GTR with quantum mechanics, should clarify all the mysteries of BH physics, starting from the unsolved problem of the singularity in the BH's core to arrive to the BH information puzzle and to the last stages of the BH evaporation, where very high energies are involved. In fact, it is general conviction that black holes result in highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in quantum gravity. On this issue we recall that an exciting consequence of TeV-scale quantum gravity could be the potential production of mini-BHs in high-energy experiments, like the LHC and beyond.

This special issue on BH physics consists of 17 interesting and well written papers.

The paper "*A little quantum help for cosmic censorship and a step beyond all that*", by N. Pappas, discusses the weak and strong versions of the cosmic censorship conjecture and also deals with the well-known problem of naked singularities.

The paper "*On the critical phenomena and Thermodynamics of the Reissner-Nordstrom-de Sitter black hole,*" by R. Zhao et al., deals with the effective thermodynamic quantities in Reissner-Nordstrom-de Sitter BH by also discussing its thermodynamic stability.

The paper "*Intermediate mass black holes: their motion and associated energetics,*" by C. Sivaram and A. Kenath, is devoted to exploring the astrophysical signatures and evidences for the intermediate mass BHs. Especially authors describe the specific features of their motion and energetics, related with the Bondi accretion.

The paper "*Energy Loss of a heavy particle near 3d rotating hairy black hole,*" by J. Naji and H. Saadat, considers rotating BH in 3 dimensions with a scalar charge and discusses energy loss of heavy particle moving near the BH horizon.

The paper "*Holographic screens in ultraviolet self-complete quantum gravity,*" by P. Nicolini and E. Spallucci, investigates the idea of a short distance fundamental scale below which it is not possible to probe and analyzes the geometry and thermodynamics of a holographic screen in the framework of the ultraviolet self-complete quantum gravity.

The paper "*Black holes and quantum mechanics,*" by B. G. Sidharth, reconsiders BHs in the context of general relativity critically reviewing all problems relating these phenomena.

The paper "*Magnetic string with a nonlinear U(1) source,*" by Seyed H. Hendi, deals with the study of magnetic string solutions in Einstein gravity in the presence of nonlinear electrodynamics. Also the effects of these nonlinear fields as well as other properties of the solutions are investigated, precisely.

The paper "*Researching on Hawking effect in a Kerr space time via open quantum system approach,*" by X.-M. Liu and W.-B. Liu, investigates the Hawking effect in a Kerr space time

in the framework of open quantum systems, showing that Hawking effect of the Kerr space time can also be understood as the manifestation of thermalization phenomena via open quantum system approach.

The paper "*Holograghic Brownian motion in three dimensional Gödel black hole,*" by J. Sadeghi et al., uses the AdS/CFT correspondence and Gödel BH background to study the dynamics of heavy quark under a rotating plasma.

The paper "*Entropy spectrum of a KS black hole in IR modified Hovrava-Lifshitz gravity,*" by S. Zhou et al., discusses the entropy spectrum and area spectrum of a KS BH based on the proposal of adiabatic invariant quantity. It is found that that the entropy spectrum is discrete and equidistant spaced and the area spectrum is not equidistant spaced, which depends on the parameter of gravity theory.

The paper "*Particle collisions in the lower dimensional rotating black hole space-time with the cosmological constant,*" by J. Yang et al., deals with the effect of ultrahigh energy collisions of two particles with different energies near the horizon of a 2 + 1 dimensional BTZ BH (BSW effect), finding that the particle with the critical angular momentum could exist inside the outer horizon of the BTZ BH regardless of the particle energy.

The paper "*The geometry of black hole singularities,*" by O. C. Stoica, is a review about singularity problem in general and BH singularity in particular. The importance of dimensional reduction effects is also stressed.

The paper "*State-space geometry, statistical fluctuations and black holes in string theory,*" by S. Bellucci and B. N. Tiwari, considers statistical properties of the charged and anticharged BH configurations by using the notion of the thermodynamic geometry. The authors highlight the utility of thermodynamic geometry in understanding the state space correlations and fluctuations in BHs in string theory while treating them thermodynamically.

The paper "*Quantum tunnelling for Hawking radiation from both static and dynamic black hole,*" by S. Chakraborty and S. Saha, deals with the well-known Hawking radiation and quantum corrections in order to further improve the theory from the semiclassical approach. The authors specially study the Hawking radiation from both static and nonstatic spherically symmetric BHs.

The paper "*Electrostatics in the surroundings of a topologically charged black hole in the brane,*" by A. Larrañaga et al., studies the EM properties of 4D BH due to brane contributions, determining the electrostatic potential generated by a static point-like charge in the brane-world space-time of a BH with topological (or tidal) charge.

The paper "*Analyzing black hole super-radiance emission of particles/energy from a black hole as a Gedankenexperiment to get bounds on the mass of a graviton,*" by A. Beckwith, discusses the process of particles emission and adopted a standard approach proposed by Padmanabhan. In the author's point of view "super-radiance allows massive gravity to be consistent with BH physics and general relativity."

The paper "*Hawking radiation-quasi-normal modes correspondence and effective states for non-extremal Reissner-Nordstrom black holes,*" by C. Corda et al., investigates the correspondence between Hawking radiation and BH quasi-normal modes and defines the concept of "effective state" for the non-extremal Reissner-Nordstrom black holes. The work is the extension of earlier works of the same research group and contributes to the understanding of the quantum properties of BHs.

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*Research Article*

# State-Space Geometry, Statistical Fluctuations, and Black Holes in String Theory

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We study the state-space geometry of various extremal and nonextremal black holes in string theory. From the notion of the intrinsic geometry, we offer a state-space perspective to the black hole vacuum fluctuations. For a given black hole entropy, we explicate the intrinsic geometric meaning of the statistical fluctuations, local and global stability conditions, and long range statistical correlations. We provide a set of physical motivations pertaining to the extremal and nonextremal black holes, namely, the meaning of the chemical geometry and physics of correlation. We illustrate the state-space configurations for general charge extremal black holes. In sequel, we extend our analysis for various possible charge and anticharge nonextremal black holes. From the perspective of statistical fluctuation theory, we offer general remarks, future directions, and open issues towards the intrinsic geometric understanding of the vacuum fluctuations and black holes in string theory.

## 1. Introduction

In this paper, we study statistical properties of the charged and anticharged black hole configurations in string theory. Specifically, we illustrate that the components of the vacuum fluctuations define a set of local pair correlations against the parameters, for example, charges, anticharges, mass, and angular momenta. Our consideration follows from the notion of the thermodynamic geometry, mainly introduced by Weinhold [1, 2] and Ruppeiner [3–9]. Importantly, this framework provides a simple platform to geometrically understand the statistical nature of local pair correlations and underlying structures pertaining to the vacuum phase transitions. In diverse contexts, the state-space geometric perspective offers an understanding of the phase structures of mixtures of gases, black hole configurations [10–26], generalized uncertainty principle [27], strong interactions, for example, hot QCD [28], quarkonium configurations [29], and some other systems, as well.

The main purpose of the present paper is to consider the state-space properties of various possible extremal and nonextremal black holes in string theory, in general. String theory [30], as the most promising framework to understand all possible fundamental interactions, celebrates the physics of black holes, in both the zero and the nonzero temperature domains. Our consideration hereby plays a crucial role in understanding the possible phases and stability of the string theory vacua. A further motivation follows from the consideration of the string theory black holes; namely, $\mathcal{N} = 2$ supergravity arises as a low energy limit of the Type II string theory solution, admitting extremal black holes with the zero Hawking temperature and a nonzero macroscopic attractor entropy.

A priori, the entropy depends on a large number of scalar moduli arising from the compactification of the 10-dimensional theory down to the 4-dimensional physical spacetime. This involves a 6-dimensional compactifying manifold. Interesting string theory compactifications involve $T^6$, $K_3 \times T^2$, and Calabi-Yau manifolds. The macroscopic entropy exhibits a fixed point behavior under the radial flow of the scalar fields. In such cases, the near horizon geometry of an extremal black hole turns out to be an

$AdS_2 \times S^2$ manifold which describes the Bertotti-Robinson vacuum associated with the black hole. The area of the black hole horizon is $A$ and thus the macroscopic entropy [31–42] is given as $S_{\text{macro}} = \pi|Z_\infty|^2$. This is known as the Ferrara-Kallosh-Strominger attractor mechanism, which, as the macroscopic consideration, requires a validity from the microscopic or statistical basis of the entropy. In this concern, there have been various investigations on the physics of black holes, for example, horizon properties [43, 44], counting of black hole microstates [45–47], spectrum of half-BPS states in $\mathcal{N} = 4$ supersymmetric string theory [48], and fractionation of branes [49]. From the perspective of the fluctuation theory, our analysis is intended to provide the nature of the statistical structures of the extremal and nonextremal black hole configurations. The attractor configurations exist for the extremal black holes, in general. However, the corresponding nonextremal configurations exist in the throat approximation. In this direction, it is worth mentioning that there exists an extension of Sen entropy function formalism for $D_1D_5$ and $D_2D_6NS_5$ nonextremal configurations [50–52]. In the throat approximation, these solutions, respectively, correspond to Schwarzschild black holes in $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^2 \times S^1 \times T^4$. In relation with the intrinsic state-space geometry, we will explore the statistical understanding of the attractor mechanism and the moduli space geometry and explain the vacuum fluctuations of the black brane configurations.

In this paper, we consider the state-space geometry of the spherical horizon topology black holes in four spacetime dimensions. These configurations carry a set of electric magnetic charges $(q_i, p_i)$. Due to the consideration of Strominger and Vafa [53], these charges are associated with an ensemble of weakly interacting $D$-branes. Following [53–59], it turns out that the charges $(q_i, p_i)$ are proportional to the number of electric and magnetic branes, which constitute the underlying ensemble of the chosen black hole. In the large charge limit, namely, when the number of such branes becomes large, we have treated the logarithm of the degeneracy of states of the statistical configuration as the Bekenstein-Hawking entropy of the associated string theory black holes. For the extremal black holes, the entropy is described in terms of the number of the constituent $D$-branes. For example, the two charge extremal configurations can be examined in terms of the winding modes and the momentum modes of an excited string carrying $n_1$ winding modes and $n_p$ momentum modes. Correspondingly, the state-space geometry of the nonextremal black holes is described by adding energy to the extremal $D$-branes configurations. This renders as the contribution of the clockwise and anticlockwise momenta in the Kaluza-Klein scenarios and that of the antibrane charges in general to the black hole entropy.

From the perspective of black hole thermodynamics, we describe the structure of the state-space geometry of four-dimensional extremal and nonextremal black holes in a given duality frame. Thus, when we take arbitrary variations over the charges $(q_i, p_i)$ on the electric and magnetic branes, the underlying statistical fluctuations are described by only the numbers of the constituent electric and magnetic branes. From the perspective of the intrinsic state-space geometry, if one pretends that the notion of statistical fluctuations applies to intermediate regimes of the moduli space, then the attractor horizon configurations require an embedding to the higher dimensional intrinsic Riemanian manifold. Physically, such a higher dimensional manifold can be viewed as a possible blow-up of the attractor fixed point phase-space to a nontrivial moduli space. From the perspective of thermodynamic Ruppenier geometry, we have offered future directions and open issues in the conclusion. We leave the explicit consideration of these matters open for further research.

In Section 2, we define the general notion of vacuum fluctuations. This offers the physical meaning of the state-space geometry. In Section 3, we provide a brief review of statistical fluctuations. In particular, for a given black hole entropy, we firstly explicate the statistical meaning of state-space surface and then offer the general meaning of the local and global stability conditions and long range statistical correlations. In Section 4, we provide a set of physical motivations pertaining to the extremal and nonextremal black holes, the meaning of Wienhold chemical geometry, and the physics of correlation. In Section 5, we consider state-space configurations pertaining to the extremal black holes and explicate our analysis for the two and three charge configurations. In Section 6, we extend the above analysis for the four, six, and eight charge-anticharge nonextremal black holes. Finally, Section 7 provides general remarks, conclusion and outlook, and future directions and open issues towards the application of string theory.

## 2. Definition of State-Space Geometry

Considering the fact that the black hole configurations in string theory introduce the notion of vacuum, it turns out for any thermodynamic system, that there exist equilibrium thermodynamic states given by the maxima of the entropy. These states may be represented by points on the state-space. Along with the laws of the equilibrium thermodynamics, the theory of fluctuations leads to the intrinsic Riemannian geometric structure on the space of equilibrium states [8, 9]. The invariant distance between two arbitrary equilibrium states is inversely proportional to the fluctuations connecting the two states. In particular, a less probable fluctuation means that the states are far apart. For a given set of states $\{X_i\}$, the state-space metric tensor is defined by

$$g_{ij}(X) = -\partial_i \partial_j S(X_1, X_2, \ldots, X_n). \tag{1}$$

A physical motivation of (1) can be given as follows. Up to the second order approximation, the Taylor expansion of the entropy $S(X_1, X_2, \ldots, X_n)$ yields

$$S - S_0 = -\frac{1}{2}\sum_{i=1}^{n} g_{ij}\Delta X^i \Delta X^j, \tag{2}$$

where

$$g_{ij} := -\frac{\partial^2 S(X_1, X_2, \ldots, X_n)}{\partial X^i \partial X^j} = g_{ji} \tag{3}$$

is called the state-space metric tensor. In the present investigation, we consider the state-space variables $\{X_1, X_2, \ldots, X_n\}$ as the parameters of the ensemble of the microstates of the underlying microscopic configuration (e.g., conformal field theory [60], black hole conformal field theory [61], and hidden conformal field theory [62, 63]), which defines the corresponding macroscopic thermodynamic configuration. Physically, the state-space geometry can be understood as the intrinsic Riemannian geometry involving the parameters of the underlying microscopic statistical theory. In practice, we will consider the variables $\{X_1, X_2, \ldots, X_n\}$ as the parameters, namely, charges, anticharges, and others if any, of the corresponding low energy limit of the string theory, for example, $\mathcal{N} = 2$ supergravity. In the limit, when all the variables, namely, $\{X_1, X_2, \ldots, X_n\}$, are thermodynamic, the state-space metric tensor equation (1) reduces to the corresponding Ruppenier metric tensor. In the discrete limit, the relative coordinates $\Delta X^i$ are defined as $\Delta X^i := X^i - X_0^i$, for given $\{X_0^i\} \in M_n$. In the Gaussian approximation, the probability distribution has the following form:

$$P(X_1, X_2, \ldots, X_n) = A \exp\left(-\frac{1}{2} g_{ij} \Delta X^i \Delta X^j\right). \quad (4)$$

With the normalization

$$\int \prod_i dX_i P(X_1, X_2, \ldots, X_n) = 1, \quad (5)$$

we have the following probability distribution:

$$P(X_1, X_2, \ldots, X_n) = \frac{\sqrt{g(X)}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} g_{ij} dX^i \otimes dX^j\right), \quad (6)$$

where $g_{ij}$ now, in a strict mathematical sense, is properly defined as the inner product $g(\partial/\partial X^i, \partial/\partial X^j)$ on the corresponding tangent space $T(M_n) \times T(M_n)$. In this connotation, the determinant of the state-space metric tensor,

$$g(X) := \|g_{ij}\|, \quad (7)$$

can be understood as the determinant of the corresponding matrix $[g_{ij}]_{n\times n}$. For a given state-space manifold $(M_n, g)$, we will think of $\{dX^i\}$ as the basis of the cotangent space $T^\star(M_n)$. In the subsequent analysis, by taking an account of the fact that the physical vacuum is neutral, we will choose $X_0^i = 0$.

## 3. Statistical Fluctuations

*3.1. Black Hole Entropy.* As a first exercise, we have illustrated thermodynamic state-space geometry for the two charge extremal black holes with electric charge $q$ and magnetic charge $p$. The next step has thence been to examine the thermodynamic geometry at an attractor fixed point(s) for the extremal black holes as the maxima of their macroscopic entropy $S(q, p)$. Later on, the state-space geometry of nonextremal counterparts has as well been analyzed. In this investigation, we demonstrate that the state-space correlations of nonextremal black holes modulate relatively more swiftly to an equilibrium statistical basis than those of the corresponding extremal solutions.

*3.2. State-Space Surface.* The Ruppenier metric on the state-space $(M_2, g)$ of two charge black holes is defined by

$$g_{qq} = -\frac{\partial^2 S(q,p)}{\partial q^2}, \qquad g_{qp} = -\frac{\partial^2 S(q,p)}{\partial q \partial p}, \qquad g_{pp} = -\frac{\partial^2 S(q,p)}{\partial p^2}. \quad (8)$$

Subsequently, the components of the state-space metric tensor are associated with the respective statistical pair correlation functions. It is worth mentioning that the coordinates on the state-space manifold are the parameters of the microscopic boundary conformal field theory which is dual the black hole space-time solution. This is because the underlying state-space metric tensor comprises of the Gaussian fluctuations of the entropy which is the function of the number of the branes and antibranes. For the chosen black hole configuration, the local stability of the underlying statistical system requires both principle minors to be positive. In this setup, the diagonal components of the state-space metric tensor, namely, $\{g_{x_i x_i} \mid x_i = (n, m)\}$, signify the heat capacities of the system. This requires that the diagonal components of the state-space metric tensor

$$g_{x_i x_i} > 0, \quad i = n, m, \quad (9)$$

be positive definite. In this investigation, we discuss the significance of the above observation for the eight parameter nonextremal black brane configurations in string theory. From the notion of the relative scaling property, we will demonstrate the nature of the brane-brane pair correlations; namely, from the perspective of the intrinsic Riemannian geometry, the stability properties of the eight parameter black branes are examined from the positivity of the principle minors of the space-state metric tensor. For the Gaussian fluctuations of the two charge equilibrium statistical configurations, the existence of a positive definite volume form on the state-space manifold $(M_2(R), g)$ imposes such a global stability condition. In particular, the above configuration leads to a stable statistical basis if the determinant of the state-space metric tensor,

$$\|g\| = S_{nn} S_{mm} - S_{nm}^2, \quad (10)$$

remains positive. Indeed, for the two charge black brane configurations, the geometric quantities corresponding to the underlying state-space manifold elucidate typical features of the Gaussian fluctuations about an ensemble of equilibrium brane microstates. In this case, we see that the Christoffel connections on the $(M_2, g)$ are defined by

$$\Gamma_{ijk} = g_{ij,k} + g_{ik,j} - g_{jk,i}. \quad (11)$$

The only nonzero Riemann curvature tensor is

$$R_{qpqp} = \frac{N}{D}, \quad (12)$$

where

$$N := S_{pp}S_{qqq}S_{qpp} + S_{qp}S_{qqp}S_{qpp} + S_{qq}S_{qqp}S_{ppp} - S_{qp}S_{qqq}S_{ppp} - S_{qq}S_{qpp}^2 - S_{pp}S_{qqp}^2, \quad (13)$$

$$D := \left(S_{qq}S_{pp} - S_{qp}^2\right)^2. \quad (14)$$

The scalar curvature and the corresponding $R_{ijkl}$ of an arbitrary two-dimensional intrinsic state-space manifold $(M_2(R), g)$ may be given as

$$R(q,p) = \frac{2}{\|g\|} R_{qpqp}(q,p). \quad (15)$$

*3.3. Stability Conditions.* For a given set of variables $\{X^1, X^2, \ldots, X^n\}$, the local stability of the underlying state-space configuration demands the positivity of the heat capacities:

$$\left\{g_{ii}\left(X^i\right) > 0; \forall i = 1, 2, \ldots, n\right\}. \quad (16)$$

Physically, the principle components of the state-space metric tensor $\{g_{ii}(X^i) \mid i = 1, 2, \ldots, n\}$ signify a set of definite heat capacities (or the related compressibilities), whose positivity apprises that the black hole solution complies an underlying, locally in equilibrium, statistical configuration. Notice further that the positivity of principle components is not sufficient to insure the global stability of the chosen configuration and thus one may only achieve a locally stable equilibriumstatistical configuration. In fact, the global stability condition constraint over the allowed domain of the parameters of black hole configurations requires that all the principle components and all the principle minors of the metric tensor must be strictly positive definite [6]. The above stability conditions require that the following set of equations must be simultaneously satisfied:

$$\begin{aligned} p_0 &:= 1, \\ p_1 &:= g_{11} > 0, \\ p_2 &:= \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} > 0, \\ p_3 &:= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} > 0, \\ &\vdots \\ p_n &:= \|g\| > 0. \end{aligned} \quad (17)$$

*3.4. Long Range Correlations.* The thermodynamic scalar curvature of the state-space manifold is proportional to the correlation volume [6]. Physically, the scalar curvature signifies the interaction(s) of the underlying statistical system. Ruppenier has in particular noticed for the black holes in general relativity that the scalar curvature

$$R(X) \sim \xi^d, \quad (18)$$

where $d$ is the spatial dimension of the statistical system and the $\xi$ fixes the physical scale [6]. The limit $R(X) \to \infty$ indicates the existence of certain critical points or phase transitions in the underlying statistical system. The fact that "all the statistical degrees of freedom of a black hole live on the black hole event horizon" signifies that the state-space scalar curvature, as the intrinsic geometric invariant, indicates an average number of correlated Plank areas on the event horizon of the black hole [8]. In this concern, [9] offers interesting physical properties of the thermodynamic scalar curvature and phase transitions in Kerr-Newman black holes. Ruppeiner [6] has further conjectured that the global correlations can be expressed by the following arguments: (a) the zero state-space scalar curvature indicates certain bits of information on the event horizon, fluctuating independently of each other; (b) the diverging scalar curvature signals a phase transition indicating highly correlated pixels of the information.

## 4. Some Physical Motivations

*4.1. Extremal Black Holes.* The state-space of the extremal black hole configuration is a reduced space comprising of the states which respect the extremality (BPS) condition. The state-spaces of the extremal black holes show an intrinsic geometric description. Our intrinsic geometric analysis offers a possible zero temperature characterization of the limiting extremal black brane attractors. From the gauge/gravity correspondence, the existence of state-space geometry could be relevant to the boundary gauge theories, which have finitely many countable sets of conformal field theory states.

*4.2. Nonextremal Black Holes.* We will analyze the state-space geometry of nonextremal black holes by the addition of antibrane charge(s) to the entropy of the corresponding extremal black holes. To interrogate the stability of a chosen black hole system, we will investigate the question that the underlying metric $g_{ij}(X_i) = -\partial_i\partial_j S(X_1, X_2, \ldots, X_n)$ should provide a nondegenerate state-space manifold. The exact dependence varies case to case. In the next section, we will proceed in our analysis with an increasing number of the brane charges and antibrane charges.

*4.3. Chemical Geometry.* The thermodynamic configurations of nonextremal black holes in string theory with small statistical fluctuations in a "canonical" ensemble are stable if the following inequality holds:

$$\left\|\partial_i\partial_j S\left(X_1, X_2, \ldots, X_n\right)\right\| < 0. \quad (19)$$

The thermal fluctuations of nonextremal black holes, when considered in the canonical ensemble, give a closer approximation to the microcanonical entropy:

$$S = S_0 - \frac{1}{2}\ln\left(CT^2\right) + \cdots. \quad (20)$$

In (20), the $S_0$ is the entropy in the "canonical" ensemble and $C$ is the specific heat of the black hole statistical configuration. At low temperature, the quantum effects dominate and the above expansion does not hold anymore. The stability condition of the canonical ensemble is just $C > 0$. In other words, the Hessian function of the internal energy with respect to the chemical variables, namely, $\{x_1, x_2, \ldots, x_n\}$, remains positive definite. Hence, the energy as the function of the $\{x_1, x_2, \ldots, x_n\}$ satisfies the following condition:

$$\left\|\partial_i \partial_j E\left(x_1, x_2, \ldots, x_n\right)\right\| > 0. \tag{21}$$

The state-space coordinates $\{X^i\}$ and intensive chemical variables $\{x_i\}$ are conjugate to each other. In particular, the $\{X^i\}$ are defined as the Legendre transform of $\{x_i\}$, and thus we have

$$X^i := \frac{\partial S(x)}{\partial x_i}. \tag{22}$$

*4.4. Physics of Correlation.* Geometrically, the positivity of the heat capacity $C > 0$ turns out to be the positivity condition of $g_{ij} > 0$, for a given $i$. In many cases, the state-space stability restriction on the parameters of the black hole corresponds to the situation away from the extremality condition; namely, $r_+ = r_-$. Far from the extremality condition, even at the zero antibrane charge or angular momentum, we find that there is a finite value of the thermodynamic scalar curvature, unlike the nonrotating or only brane-charged configurations. It turns out that the state-space geometry of the two charge extremal configurations is flat. Thus, the Einstein-Hilbert contributions lead to a noninteracting statistical system. At the tree level, some black hole configurations turn out to be ill-defined, as well. However, we anticipate that the corresponding state-space configuration would become well-defined when a sufficient number of higher derivative corrections [64–67] are taken into account with respect to the $\alpha'$-corrections and the string loop $l_s$ corrections. For the BTZ black holes [13], we notice that the large entropy limit turns out to be the stability bound, beyond which the underlying quantum effects dominate.

For the black hole in string theory, the Ricci scalar of the state-space geometry is anticipated to be positive definite with finitely many higher order corrections. For nonextremal black brane configurations, which are far from the extremality condition, such effects have been seen from the nature of the state-space scalar curvature $R(S(X_1, X_2, \ldots, X_n))$. Indeed, [12, 14] indicate that the limiting state-space scalar curvature $R(S(X_1, X_2, \ldots, X_n))|_{\text{no anticharge}} \neq 0$ gives a set of stability bounds on the statistical parameters. Thus, our consideration yields a classification of the domain of the parameters and global correlation of a nonextremal black hole.

*4.5. String Theory Perspective.* In this subsection, we recall a brief notion of entropy of a general string theory black brane configuration from the viewpoint of the counting of the black hole microstates [53, 53–59, 68]. Given a string theory configuration, the choice of compactification [30] chosen is the factorization of the type $\mathscr{M}_{(3,1)} \times M_6$, where $M_6$ is a compact internal manifold. From the perspective of statistical ensemble theory, we will express the entropy of a nonextremal black hole as the function of the numbers of branes and antibranes. Namely, for the charged black holes, the electric and magnetic charges $(q_i, p_i)$ form a coordinate chart on the state-space manifold. In this case, for a given ensemble of $D$-branes, the coordinate $q_i$ is defined as the number of the electric branes and $p_i$ as the number of the magnetic branes. Towards the end of this paper, we will offer further motivation for the consideration of the state-space geometry of large charged nonspherical horizon black holes in spacetime dimensions $D \geq 5$. In this concern, [68] plays a central role towards the formation of the lower dimensional black hole configuration. Namely, for the torus compatifications, the exotic branes play an important role concerning the physical properties of supertubes, the $D_0$-$F_1$ system and associated counting of the black hole microstates.

In what follows, we consider the four-dimensional string theory black holes in a given duality basis of the charges $(q_i, p_i)$. From the perspective of string theory, the exotic branes and nongeometric configurations offer interesting fronts for the black holes in three spacetime dimensions. In general, such configurations could carry a dipole or a higher pole charge, and they leave the four-dimension black hole configuration asymptotically flat. In fact, for the spacetime dimensions $D \geq 4$, [68] shows that a charge particle corresponds to an underlying gauge field, modulo $U$-duality transformations. From the perspective of nonextremal black holes, by taking appropriate boundary condition, namely, the unit asymptotic limit of the harmonic function which defines the spacetime metric, one can choose the spacetime regions such that the supertube effects arising from nonexotic branes can effectively be put off in an asymptotically flat space [68]. This allows one to compute the Arnowitt-Deser-Misner (ADM) mass of the asymptotic black hole. From the viewpoint of the statistical investigation, the dependence of the mass to the entropy of a nonextremal black hole comes from the contribution of the antibranes to the counting degeneracy of the states.

## 5. Extremal Black Holes in String Theory

*5.1. Two Charge Configurations.* The state-space geometry of the two charge extremal configurations is analyzed in terms of the winding modes and the momentum modes of an excited string carrying $n_1$ winding modes and $n_p$ momentum modes. In the large charge limit, the microscopic entropy obtained by the degeneracy of the underlying conformal field theory states reduces to the following expression:

$$S_{\text{micro}} = 2\sqrt{2n_1 n_p}. \tag{23}$$

The microscopic counting can be accomplished by considering an ensemble of weakly interacting $D$-branes [54]. The counting entropy and the macroscopic attractor entropy of the two charge black holes in string theory which have a $n_4$

number of $D_4$ branes and a $n_0$ number of $D_0$ branes match and thus we have

$$S_{\text{micro}} = 2\pi\sqrt{n_0 n_4} = S_{\text{macro}}. \tag{24}$$

In this case, the components of underlying state-space metric tensor are

$$g_{n_0 n_0} = \frac{\pi}{2n_0}\sqrt{\frac{n_4}{n_0}}, \qquad g_{n_0 n_4} = -\frac{\pi}{2}\frac{1}{\sqrt{n_0 n_4}}, \qquad g_{n_4 n_4} = \frac{\pi}{2n_4}\sqrt{\frac{n_0}{n_4}}. \tag{25}$$

The diagonal pair correlation functions remain positive definite:

$$g_{n_i n_i} > 0 \quad \forall i \in \{0,4\} \mid n_i > 0, \quad g_{n_4 n_4} > 0, \quad \forall (n_0, n_4). \tag{26}$$

For distinct $i, j \in \{0,4\}$, the state-space pair correlation functions admit

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{n_i}{n_j}. \tag{27}$$

The global properties of fluctuating two charge $D_0$-$D_4$ extremal configurations are determined by possible principle minors. The first minor constraint $p_1 > 0$ directly follows from the positivity of the first component of metric tensor:

$$p_1 = \frac{\pi}{2n_0}\sqrt{\frac{n_4}{n_0}}. \tag{28}$$

The determinant of the metric tensor $p_2 := g(n_0, n_4)$ vanishes identically for all allowed values of the parameters. Thus, the leading order large charge extremal black branes having (i) a $n_0$ number of $D_0$-branes and a $n_4$ number of $D_4$ or (ii) excited strings with a $n_1$ number of windings and a $n_p$ number of momenta, where either set of charges forms local coordinates on the state-space manifold, find degenerate intrinsic state-space configurations. For a given configuration entropy $S_0 := 2\pi c$, the constant entropy curve can be depicted as the rectangular hyperbola

$$n_0 n_4 = c^2. \tag{29}$$

The intrinsic state-space configuration depends on the attractor values of the scalar fields which arise from the chosen string compactification. Thus, the possible state-space Ruppenier geometry may become well-defined against further higher derivative $\alpha'$-corrections. In particular, the determinant of the state-space metric tensor may take positive/negative definite values over the domain of brane charges. We will illustrate this point in a bit more detail in the subsequent consideration with a higher number of charges and anticharges.

*5.2. Three Charge Configurations.* From the consideration of the two derivative Einstein-Hilbert action, [53] shows that the leading order entropy of the three charge $D_1$-$D_5$-$P$ extremal black holes is

$$S_{\text{micro}} = 2\pi\sqrt{n_1 n_5 n_p} = S_{\text{macro}}. \tag{30}$$

The concerned components of state-space metric tensor are given in Appendix A. Hereby, it follows further that the local state-space metric constraints are satisfied as

$$g_{n_i n_i} > 0 \quad \forall i \in \{1, 5, p\} \mid n_i > 0. \tag{31}$$

For distinct $i, j \in \{1,5\}$ and $p$, the list of relative correlation functions is depicted in Appendix A. Further, we see that the local stabilities pertaining to the lines and two-dimensional surfaces of the state-space manifold are measured as

$$p_1 = \frac{\pi}{2n_1}\sqrt{\frac{n_5 n_p}{n_1}}, \qquad p_2 = -\frac{\pi^2}{4n_1 n_5^2 n_p}\left(n_p^2 n_1 + n_5^3\right). \tag{32}$$

The stability of the entire equilibrium phase-space configurations of the $D_1$-$D_5$-$P$ extremal black holes is determined by the $p_3 := g$ determinant of the state-space metric tensor:

$$\|g\| = -\frac{1}{2}\pi^3\left(n_1 n_5 n_p\right)^{-1/2}. \tag{33}$$

The universal nature of statistical interactions and the other properties concerning Maldacena, Strominger, and Witten (MSW) rotating black branes [55] are elucidated by the state-space scalar curvature:

$$R\left(n_1, n_5, n_p\right) = \frac{3}{4\pi\sqrt{n_1 n_5 n_p}}. \tag{34}$$

The constant entropy (or scalar curvature) curve defining the state-space manifold is the higher dimensional hyperbola:

$$n_1 n_5 n_p = c^2, \tag{35}$$

where $c$ takes respective values of $(c_S, c_R) = (S_0/2\pi, 3/4\pi R_0)$. In [12, 14, 17, 18], we have shown that similar results hold for the state-space configuration of the four charge extremal black holes.

## 6. Nonextremal Black Holes in String Theory

*6.1. Four Charge Configurations.* The state-space configuration of the nonextremal $D_1$–$D_5$ black holes is considered with nonzero momenta along the clockwise and anticlockwise directions of the Kaluza-Klein compactification circle $S^1$. Following [56], the microscopic entropy and the macroscopic entropy match for given total mass and brane charges :

$$S_{\text{micro}} = 2\pi\sqrt{n_1 n_5}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right) = S_{\text{macro}}. \tag{36}$$

The state-space covariant metric tensor is defined as a negative Hessian matrix of the entropy with respect to the number of $D_1$, $D_5$ branes $\{n_i \mid i = 1, 5\}$ and clockwise-anticlockwise

Kaluza-Klein momentum charges $\{n_p, \overline{n_p}\}$. Herewith, we find that the components of the metric tensor take elegant forms. The corresponding expressions are given in Appendix B. As in the case of the extremal configurations, the state-space metric satisfies the following constraints:

$$g_{n_i n_i} > 0, \quad \forall i = 1, 5; \qquad g_{n_a n_a} > 0, \quad \forall a = p, \overline{p}. \tag{37}$$

Furthermore, the scaling relations for distinct $i, j \in \{1, 5\}$ and $p$, concerning the list of relative correlation functions, are offered in Appendix B. In this case, we find that the stability criteria of the possible surfaces and hypersurfaces of the underlying state-space configuration are determined by the positivity of the following principle minors:

$$\begin{aligned} p_0 &= 1, \qquad p_1 = \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right), \\ p_2 &= 0, \qquad p_3 = -\frac{1}{2n_p}\frac{\pi^3}{\sqrt{n_1 n_5}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right). \end{aligned} \tag{38}$$

The complete local stability of the full nonextremal $D_1$–$D_5$ black brane state-space configuration is ascertained by the positivity of the determinant of the metric tensor:

$$g\left(n_1, n_5, n_p, \overline{n_p}\right) = -\frac{1}{4}\frac{\pi^4}{\left(n_p \overline{n_p}\right)^{3/2}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right)^2. \tag{39}$$

The global state-space properties concerning the four charge nonextremal $D_1$–$D_5$ black holes are determined by the regularity of the invariant scalar curvature:

$$R\left(n_1, n_5, n_p, \overline{n_p}\right) = \frac{9}{4\pi\sqrt{n_1 n_5}}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right)^{-6} f\left(n_p, \overline{n_p}\right), \tag{40}$$

where the function $f(n_p, \overline{n_p})$ of two momenta $(n_p, \overline{n_p})$ running in opposite directions of the Kaluza-Klein circle $S^1$ has been defined as

$$\begin{aligned} f\left(n_p, \overline{n_p}\right) := {} & n_p^{5/2} + 10 n_p^{3/2}\overline{n_p} + 5 n_p^{1/2}\overline{n_p}^2 \\ & + 5 n_p^2 \overline{n_p}^{1/2} + 10 n_p \overline{n_p}^{3/2} + \overline{n_p}^{5/2}. \end{aligned} \tag{41}$$

By noticing the Pascal coefficient structure in (41), we see that the above function $f(n_p, \overline{n_p})$ can be factorized as

$$f\left(n_p, \overline{n_p}\right) = \left(n_p + \overline{n_p}\right)^5. \tag{42}$$

Thus, (40) leads to the following state-space scalar curvature:

$$R\left(n_1, n_5, n_p, \overline{n_p}\right) = \frac{9}{4\pi\sqrt{n_1 n_5}} \times \left(\frac{1}{\sqrt{n_p} + \sqrt{\overline{n_p}}}\right). \tag{43}$$

In the large charge limit, the nonextremal $D_1$–$D_5$ black branes have a nonvanishing small scalar curvature function on the state-space manifold $(M_4, g)$. This implies an almost everywhere weakly interacting statistical basis. In this case, the constant entropy hypersurface is defined by the curve

$$\frac{c^2}{n_1 n_5} = \left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right)^2. \tag{44}$$

As in the case of two charge $D_0$-$D_4$ extremal black holes and $D_1$-$D_5$-$P$ extremal black holes, the constant $c$ takes the same value of $c := S_0^2/4\pi^2$. For a given state-space scalar curvature $k$, the constant state-space curvature curves take the following form:

$$f\left(n_p, \overline{n_p}\right) = k\sqrt{n_1 n_5}\left(\sqrt{n_p} + \sqrt{\overline{n_p}}\right)^6. \tag{45}$$

*6.2. Six Charge Configurations.* We now extrapolate the state-space geometry of four charge nonextremal $D_1$–$D_5$ solutions for nonlarge charges, where we are no longer close to an ensemble of supersymmetric states. In [57], the computation of the entropy of all such special extremal and near-extremal black hole configurations has been considered. The leading order entropy as a function of charges $\{n_i\}$ and anticharges $\{m_i\}$ is

$$\begin{aligned} & S\left(n_1, m_1, n_2, m_2, n_3, m_3\right) \\ & \quad := 2\pi\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right). \end{aligned} \tag{46}$$

For given charges $i, j \in A_1 := \{n_1, m_1\}$; $k, l \in A_2 := \{n_2, m_2\}$; and $m, n \in A_3 := \{n_3, m_3\}$, the intrinsic state-space pair correlations are in precise accordance with the underlying macroscopic attractor configurations which are being disclosed in the special leading order limit of the nonextremal $D_1$–$D_5$ solutions. The components of the covariant state-space metric tensor over generic nonlarge charge domains are not difficult to compute, and, indeed, we have offered their corresponding expressions in Appendix C.

For all finite $(n_i, m_i)$, $i = 1, 2, 3$, the components involving brane-brane state-space correlations $g_{n_i n_i}$ and antibrane-antibrane state-space correlations $g_{m_i m_i}$ satisfy the following positivity conditions:

$$g_{n_i n_i} > 0, \qquad g_{m_i m_i} > 0. \tag{47}$$

The distinct $\{n_i, m_i \mid i \in \{1, 2, 3\}\}$ describing six charge string theory black holes have three types of relative pair correlation functions. The corresponding expressions of the relative statistical correlation functions are given in Appendix C.

Notice hereby that the scaling relations remain similar to those obtained in the previous case, except that (i) the number of relative correlation functions has been increased, and (ii) the set of cross ratios, namely, $\{g_{ij}/g_{kl}, g_{kl}/g_{mn}, g_{ij}/g_{mn}\}$ being zero in the previous case, becomes ill-defined for the six charge state-space configurations. Inspecting the specific pair of distinct charge sets $A_i$ and $A_j$, there are now 24 types of nontrivial relative correlation functions. The set of principle

components denominator ratios computed from the above state-space metric tensor reduces to

$$\frac{g_{ij}}{g_{kk}} = 0, \quad \forall i, j, k \in \{n_1, m_1, n_2, m_2, n_3, m_3\}. \tag{48}$$

For given $i, j \in A_1 := \{n_1, m_1\}$; $k, l \in A_2 := \{n_2, m_2\}$; $m, n \in A_3 := \{n_3, m_3\}$, and $g_{n_i m_i} = 0$, there are the total 15 types of trivial relative correlation functions. There are five such trivial ratios in each family $\{A_i \mid i = 1, 2, 3\}$. The local stability of the higher charged string theory nonextremal black holes is given by

$$\begin{aligned} p_1 &= \frac{\pi}{2n_1^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right), \\ p_2 &= \frac{1}{4}\frac{\pi^2}{(n_1m_1)^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)^2\left(\sqrt{n_3}+\sqrt{m_3}\right)^2, \\ p_3 &= \frac{1}{8}\frac{\pi^3}{(n_1m_1n_2)^{3/2}}\sqrt{m_2}\left(\sqrt{n_3}+\sqrt{m_3}\right)^3 \\ &\quad \times\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_1}+\sqrt{m_1}\right), \\ p_4 &= 0. \end{aligned} \tag{49}$$

The principle minor $p_5$ remains nonvanishing for all values of charges on the constituent brane and antibranes. In general, by an explicit calculation, we find that the hyper-surface minor $p_5$ takes the following nontrivial value:

$$\begin{aligned} p_5 = &-\frac{1}{8}\frac{\pi^5}{(n_1m_1n_2m_2)^{3/2}n_3}\left(\sqrt{n_1}+\sqrt{m_1}\right)^3 \\ &\times\left(\sqrt{n_2}+\sqrt{m_2}\right)^3\left(\sqrt{n_3}+\sqrt{m_3}\right)^3. \end{aligned} \tag{50}$$

Specifically, for an identical value of the brane and antibrane charges, the minor $p_5$ reduces to

$$p_5(k) = -64\frac{\pi^5}{k^{5/2}}. \tag{51}$$

The global stability on the full state-space configuration is carried forward by computing the determinant of the metric tensor:

$$\begin{aligned} \|g\| = &-\frac{1}{16}\frac{\pi^6}{(n_1m_1n_2m_2n_3m_3)^{3/2}}\left(\sqrt{n_1}+\sqrt{m_1}\right)^4 \\ &\times\left(\sqrt{n_2}+\sqrt{m_2}\right)^4\left(\sqrt{n_3}+\sqrt{m_3}\right)^4. \end{aligned} \tag{52}$$

The underlying state-space configuration remains nondegenerate for the domain of given nonzero brane antibrane charges, except for extreme values of the brane and antibrane charges $\{n_i, m_i\}$, when they belong to the set

$$\begin{aligned} B := \{&(n_1, n_2, n_3, m_1, m_2, m_3) \mid \\ &(n_i, m_i) = (0,0), (\infty,\infty), \text{some } i\}, \end{aligned} \tag{53}$$

among the given brane-antibrane pairs $\{(n_1, m_1), (n_2, m_2), (n_3, m_3)\}$. The component $R_{n_1n_2m_3m_4}$ diverges at the roots of the two variables polynomials defined as the functions of brane and antibrane charges:

$$\begin{aligned} f_1(n_2, m_2) &= n_2^4m_2^3 + 2(n_2m_2)^{7/2} + n_2^3m_2^4, \\ f_2(n_3, m_3) &= m_3^{9/2}n_3^4 + n_3^4m_3^{9/2}. \end{aligned} \tag{54}$$

However, the component $R_{n_3,m_3,n_3,m_3}$ with an equal number of brane and antibrane charges diverges at a root of a single higher degree polynomial:

$$\begin{aligned} &f(n_1, m_1, n_2, m_2, n_3, m_3) \\ &\quad := n_2^4m_2^3n_3^{9/2}m_3^4 + n_2^4m_2^3n_3^4m_3^{9/2} \\ &\qquad + 2n_2^{7/2}m_2^{7/2}n_3^{9/2}m_3^4 + 2n_2^{7/2}m_2^{7/2}n_3^4m_3^{9/2} \\ &\qquad + n_2^3m_2^4n_3^{9/2}m_3^4 + n_2^3m_2^4n_3^4m_3^{9/2}. \end{aligned} \tag{55}$$

Herewith, from the perspective of state-space global invariants, we focus on the limiting nature of the underlying ensemble. Thus, we may choose the equal charge and anticharge limit by defining $m_i := k$ and $n_i := k$ for the calculation of the Ricci scalar. In this case, we find the following small negative curvature scalar:

$$R(k) = -\frac{15}{16}\frac{1}{\pi k^{3/2}}. \tag{56}$$

Further, the physical meaning of taking an equal value of the charges and anticharges lies in the ensemble theory, namely, in the thermodynamic limit, all the statistical fluctuations of the charges and anticharges approach to a limiting Gaussian fluctuations. In this sense, we can take the average over the concerned individual Gaussian fluctuations. This shows that the limiting statistical ensemble of nonextremal nonlarge charge $D_1$–$D_5$ solutions yields an attractive state-space configuration. Finally, such a limiting procedure is indeed defined by considering the standard deviations of the equal integer charges and anticharges, and thus our interest in calculating the limiting Ricci scalar in order to know the nature of the long range interactions underlying in the system.

For a given entropy $S_0$, the constant entropy hypersurface is again some nonstandard curve:

$$\left(\sqrt{n_1}+\sqrt{m_1}\right)\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right) = c, \tag{57}$$

where the real constant $c$ takes the precise value of $S_0/2\pi$.

*6.3. Eight Charge Configurations.* From the perspective of the higher charged and anticharged black hole configurations in string theory, let us systematically analyze the underlying statistical structures. In this case, the state-space configuration of the nonextremal black hole involves finitely many nontrivially circularly fibered Kaluza-Klein monopoles. In this process, we enlist the complete set of nontrivial relative state-space correlation functions of the eight charged anticharged

configurations, with respect to the lower parameter configurations, as considered in [12, 14]. There have been calculations of the entropy of the extremal, near-extremal, and general nonextremal solutions in string theory; see, for instance, [58, 59]. Inductively, the most general charge anticharge nonextremal black hole has the following entropy:

$$S\left(n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4\right) = 2\pi \prod_{i=1}^{4} \left(\sqrt{n_i} + \sqrt{m_i}\right). \tag{58}$$

For the distinct $i, j, k \in \{1, 2, 3, 4\}$, we find that the components of the metric tensor are

$$\begin{aligned}
g_{n_i n_i} &= \frac{\pi}{2 n_i^{3/2}} \prod_{j \neq i} \left(\sqrt{n_j} + \sqrt{m_j}\right), \\
g_{n_i n_j} &= -\frac{\pi}{2\left(n_i n_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right), \\
g_{n_i m_i} &= 0, \\
g_{n_i m_j} &= -\frac{\pi}{2\left(n_i m_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right), \\
g_{m_i m_i} &= \frac{\pi}{2 m_i^{3/2}} \prod_{j \neq i} \left(\sqrt{n_j} + \sqrt{m_j}\right), \\
g_{m_i m_j} &= -\frac{\pi}{2\left(m_i m_j\right)^{1/2}} \prod_{i \neq k \neq j} \left(\sqrt{n_k} + \sqrt{m_k}\right).
\end{aligned} \tag{59}$$

From the above depiction, it is evident that the principle components of the state-space metric tensor $\{g_{n_i n_i}, g_{m_i m_i} \mid i = 1, 2, 3, 4\}$ essentially signify a set of definite heat capacities (or the related compressibilities) whose positivity in turn apprises that the black brane solutions comply with an underlying equilibrium statistical configuration. For an arbitrary number of the branes $\{n_i\}$ and antibranes $\{m_i\}$, we find that the associated state-space metric constraints as the diagonal pair correlation functions remain positive definite. In particular, $\forall i \in \{1, 2, 3, 4\}$; it is clear that we have the following positivity conditions:

$$g_{n_i n_i} > 0 \mid n_i, m_i > 0, \qquad g_{m_i m_i} > 0 \mid n_i, m_i > 0. \tag{60}$$

As observed in [12, 14], we find that the ratios of diagonal components vary inversely with a multiple of a well-defined factor in the underlying parameters, namely, the charges and anticharges, which changes under the Gaussian fluctuations, whereas the ratios involving off diagonal components in effect uniquely inversely vary in the parameters of the chosen set $A_i$ of equilibrium black brane configurations. This suggests that the diagonal components weaken in a relatively controlled fashion into an equilibrium, in contrast with the off diagonal components, which vary over the domain of associated parameters defining the $D_1$-$D_5$-$P$-$KK$ nonextremal nonlarge charge configurations. In short, we can easily substantiate, for the distinct $x_i := (n_i, m_i) \mid i \in \{1, 2, 3, 4\}$ describing eight (anti)charge string theory black holes, that the relative pair correlation functions have distinct types of relative correlation functions. Apart from the zeros, infinities, and similar factorizations, we see that the nontrivial relative correlation functions satisfy the following scaling relations:

$$\begin{aligned}
\frac{g_{x_i x_i}}{g_{x_j x_j}} &= \left(\frac{x_j}{x_i}\right)^{3/2} \frac{\sqrt{n_j} + \sqrt{m_j}}{\sqrt{n_i} + \sqrt{m_i}}, \\
\frac{g_{x_i x_j}}{g_{x_k x_l}} &= \left(\frac{x_i x_j}{x_k x_l}\right)^{-1/2} \frac{\prod_{i \neq p \neq j} \left(\sqrt{n_p} + \sqrt{m_p}\right)}{\prod_{k \neq q \neq l} \left(\sqrt{n_q} + \sqrt{m_q}\right)}, \\
\frac{g_{x_i x_i}}{g_{x_i x_k}} &= -\sqrt{\left(\frac{x_k}{x_i^2}\right)} \frac{\prod_{p \neq i} \left(\sqrt{n_p} + \sqrt{m_p}\right)}{\prod_{i \neq q \neq k} \left(\sqrt{n_q} + \sqrt{m_q}\right)}.
\end{aligned} \tag{61}$$

As noticed in [12, 14], it is not difficult to analyze the statistical stability properties of the eight charged anticharged nonextremal black holes; namely, we can compute the principle minors associated with the state-space metric tensor and thereby argue that all the principle minors must be positive definite, in order to have a globally stable configuration. In the present case, it turns out that the above black hole is stable only when some of the charges and/or anticharges are held fixed or take specific values such that $p_i > 0$ for all the dimensions of the state-space manifold. From the definition of the Hessian matrix of the associated entropy concerning the most general nonextremal nonlarge charged black holes, we observe that some of the principle minors $p_i$ are indeed nonpositive. In fact, we discover a uniform local stability criteria on the three-dimensional hypersurfaces, two-dimensional surface, and the one-dimensional line of the underlying state-space manifold. In order to simplify the factors of the higher principle, we may hereby collect the powers of each factor $(\sqrt{n_i} + \sqrt{m_i})$ appearing in the expression of the entropy. With this notation, Appendix D provides the corresponding principle minors for the most general nonextremal nonlarge charged anticharged black hole in string theory involving finitely many nontrivially circularly fibered Kaluza-Klein monopoles.

Notice that the heat capacities, as the diagonal components $g_{ii}$, surface minor $p_2$, hypersurface minors $p_3$, $p_5$, $p_6$, and $p_7$, and the determinant of the state-space metric tensor, as the highest principle minor $p_8$ are examined as the functions of the number of branes $n$ and antibranes $m$. Thus, they describe the nature of the statistical fluctuations in the vacuum configuration. The corresponding scalar curvature is offered for an equal number of branes and antibranes ($n = m$), which describes the nature of the long range statistical fluctuations. As per the above evaluation, we have obtained the exact expressions for the components of the metric tensor, principle minors, determinant of the metric tensor, and the underlying scalar curvature of the fluctuating statistical configuration of the eight parameter black holes in string theory. Qualitatively, the local and the global correlation properties of the limiting vacuum configuration can be realized under the statistical fluctuations. The first seven principle minors describe the local stability properties, and the last minor describes the global ensemble stability.

The scalar curvature describes the corresponding phase space stability of the eight parameter black hole configuration. In general, there exists an akin higher degree polynomial equation on which the Ricci scalar curvature becomes null, and exactly on these points the state-space configuration of the underlying nonlarge charge nonextremal eight charge black hole system corresponds to a noninteracting statistical system. In this case, the corresponding state-space manifold $(M_8, g)$ becomes free from the statistical interaction with a vanishing state-space scalar curvature. As in case of the six charge configuration, we find interestingly that there exists an attractive configuration for the equal number of branes $n := k$ and antibranes $m := k$. In the limit of a large $k$, the corresponding system possesses a small negative value of the state-space scalar curvature:

$$R(k) = -\frac{21}{32}\frac{1}{\pi k^2}. \tag{62}$$

Interestingly, it turns out that the system becomes noninteracting in the limit of $k \to \infty$. For the case of the $n = k = m$, we observe that the corresponding principle minors reduce to the following constant values:

$$\begin{aligned}\{p_i\}_{i=1}^{8} = \{&4\pi, 16\pi^2, 32\pi^3, 0, -2048\pi^5, -16384\pi^6,\\ &-163840\pi^7, -1048576\pi^8\}.\end{aligned} \tag{63}$$

In this case, we find that the limiting underlying statistical system remains stable when at most three of the parameters, namely, $\{n_i = k = m_i\}$, are allowed to fluctuate. Herewith, we find for the case of $n := k$ and $m := k$ that the state-space manifold of the eight parameter brane and antibrane configuration is free from critical phenomena, except for the roots of the determinant. Thus, the regular state-space scalar curvature is comprehensively universal for the nonlarge charge nonextremal black brane configurations in string theory. In fact, the above perception turns out to be justified from the typical state-space geometry, namely, the definition of the metric tensor as the negative Hessian matrix of the duality invariant expression of the black brane entropy. In this case, we may nevertheless easily observe, for a given entropy $S_0$, that the constant entropy hypersurface is given by the following curve:

$$\left(\sqrt{n_1} + \sqrt{m_1}\right)\left(\sqrt{n_2} + \sqrt{m_2}\right)\left(\sqrt{n_3} + \sqrt{m_3}\right)\left(\sqrt{n_4} + \sqrt{m_4}\right) = c, \tag{64}$$

where $c$ is a real constant taking the precise value of $S_0/2\pi$. Under the vacuum fluctuations, the present analysis indicates that the entropy of the eight parameter black brane solution defines a nondegenerate embedding in the viewpoints of intrinsic state-space geometry. The above state-space computations determine an intricate set of statistical properties, namely, pair correlation functions and correlation volume, which reveal the possible nature of the associated parameters prescribing an ensemble of microstates of the dual conformal field theory living on the boundary of the black brane solution. For any black brane configuration, the above computation hereby shows that we can exhibit the state-space geometric acquisitions with an appropriate comprehension of the required parameters, for example, the charges and anticharges $\{n_i, m_i\}$, which define the coordinate charts. From the consideration of the state-space geometry, we have analyzed state-space pair correlation functions and the notion of stability of the most general nonextremal black hole in string theory. From the perspective of the intrinsic Riemannian geometry, we find that the stability of these black branes has been divulged from the positivity of principle minors of the space-state metric tensor.

Herewith, we have explicitly extended the state-space analysis for the four charge and four anticharge nonextremal black branes in string theory. The present consideration of the eight parameter black brane configurations, where the underlying leading order statistical entropy is written as a function of the charges $\{n_i\}$ and anticharges $\{m_i\}$, describes the stability properties under the Gaussian fluctuations. The present consideration includes all the special cases of the extremal and near-extremal configurations with a fewer number of charges and anticharges. In this case, we obtain the standard pattern of the underlying state-space geometry and constant entropy curve as that of the lower parameter nonextremal black holes. The local coordinate of the state-space manifold involves four charges and four anticharges of the underlying nonextremal black holes. In fact, the conclusion to be drawn remains the same, as the underlying state-space geometry remains well-defined as an intrinsic Riemannian manifold $N := M_8 \setminus \widetilde{B}$, where $\widetilde{B}$ is the set of roots of the determinant of the metric tensor. In particular, the state-space configuration of eight parameter black brane solutions remains nondegenerate for various domains of nonzero brane antibrane charges, except for the values, when the brane charges $\{n_i\}$ and antibrane charges $\{m_i\}$ belong to the set

$$\begin{aligned}\widetilde{B} := \{&(n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4) \mid\\ &(n_i, m_i) = (0,0), (\infty,\infty)\},\end{aligned} \tag{65}$$

for a given brane-antibrane pair, $i \in \{1, 2, 3, 4\}$. Our analysis indicates that the leading order statistical behavior of the black brane configurations in string theory remains intact under the inclusion of the Kaluza-Klein monopoles. In short, we have considered the eight charged anticharged string theory black brane configuration and analyzed the state-space pair correlation functions, relative scaling relations, stability conditions, and the corresponding global properties. Given a general nonextremal black brane configuration, we have exposed (i) for what conditions the considered black hole configuration is stable, (ii) how its state-space correlations scale in terms of the numbers of branes and antibranes.

## 7. Conclusion and Outlook

The Ruppenier geometry of two charge leading order extremal black holes remains flat or ill-defined. Thus, the statistical systems are, respectively, noninteracting or require higher derivative corrections. However, an addition of the third brane charge and other brane and antibrane charges

indicates an interacting statistical system. The statistical fluctuations in the canonical ensemble lead to an interacting statistical system, as the scalar curvature of the state-space takes a nonzero value. We have explored the state-space geometric description of the charged extremal and associated charged anticharged nonextremal black holes in string theory.

Our analysis illustrates that the stability properties of the specific state-space hypersurface may exactly be exploited in general. The definite behavior of the state-space properties, as accounted in the specific cases, suggests that the underlying hypersurfaces of the state-space configuration include the intriguing mathematical feature. Namely, we find well-defined stability properties for the generic extremal and nonextremal black brane configurations, except for some specific values of the charges and anticharges. With and without the large charge limit, we have provided explicit forms of the higher principle minors of the state-space metric tensor for various charged, anticharged, extremal and nonextremal black holes in string theory. In this concern, the state-space configurations of the string theory black holes are generically well-defined and indicate an interacting statistical basis. Interestingly, we discover the state-space geometric nature of all possible general black brane configurations. From the very definition of the intrinsic metric tensor, the present analysis offers a definite stability character of string theory vacua.

Significantly, we notice that the related principle minors and the invariant state-space scalar curvature classify the underlying statistical fluctuations. The scalar curvature of a class of extremal black holes and the corresponding nonextremal black branes is everywhere regular with and without the stringy $\alpha'$-corrections. A nonzero value of the state-space scalar curvature indicates an interacting underlying statistical system. We find that the antibrane corrections modify the state-space curvature, but do not induce phase transitions. In the limit of an extremal black hole, we construct the intrinsic geometric realization of a possible thermodynamic description at the zero temperature.

Importantly, the notion of the state-space of the considered black hole follows from the corresponding Wald and Cardy entropies. The microscopic and macroscopic entropies match in the large charge limit. From the perspective of statistical fluctuations, we anticipate the intrinsic geometric realization of two point local correlation functions and the corresponding global correlation length of the underlying conformal field theory configurations. In relation to the gauge-gravity correspondence and extremal black holes, our analysis describes state-space geometric properties of the corresponding boundary gauge theory.

*General Remarks*. For distinct $\{i, j\}$, the state-space pair correlations of an extremal configurations scale as

$$\frac{g_{ii}}{g_{jj}} = \left(\frac{X_j}{X_i}\right)^2, \qquad \frac{g_{ij}}{g_{ii}} = -\frac{X_i}{X_j}. \tag{66}$$

In general, the black brane configurations in string theory can be categorized as per their state-space invariants. The underlying subconfigurations turn out to be well-defined over possible domains, whenever there exists a respective set of nonzero state-space principle minors. The underlying full configuration turns out to be everywhere well-defined, whenever there exists a nonzero state-space determinant. The underlying configuration corresponds to an interacting statistical system, whenever there exists a nonzero state-space scalar curvature. The intrinsic state-space manifold of extremal/nonextremal and supersymmetric/nonsupersymmetric string theory black holes may intrinsically be described by an embedding:

$$\left(M_{(n)}, g\right) \hookrightarrow \left(M_{(n+1)}, \widetilde{g}\right). \tag{67}$$

The extremal state-space configuration may be examined as a restriction to the full counting entropy with an intrinsic state-space metric tensor $g \mapsto \widetilde{g}|_{r_+=r_-}$. Furthermore, the state-space configurations of the supersymmetric black holes may be examined as the BPS restriction of the full space of the counting entropy with an understanding that the intrinsic state-space metric tensor is defined as $g := \widetilde{g}|_{M=M_0}$. From the perspective of string theory, the restrictions $r_+ = r_-$ and $M = M_0(P_i, Q_i)$ should be understood as the fact that it has been applied to an assigned entropy of the nonextremal/nonsupersymmetric (or nearly extremal/nearly supersymmetric) black brane configuration. This allows one to compute the fluctuations in ADM mass of the black hole. In the viewpoint of the present research on the state-space geometry, it is worth mentioning that the dependence of the mass to the entropy of a nonextremal black hole comes from the contribution of the antibranes, see, for instance, Section 4.5, and so we may examine the corresponding Weinhold chemical geometry, as mentioned in Section 4.3.

*Future Directions and Open Issues*. The state-space instabilities and their relation to the dual microscopic conformal field theories could open up a number of new realizations. The state-space perspective includes the following issues.

(i) Multicenter Gibbons-Hawking solutions [69, 70] with generalized base space manifolds having a mixing of positive and negative residues, see [71, 72] for a perspective development of state-space geometry by invoking the role of foaming of black holes and plumbing the Abyss for the microstates counting of black rings.

(ii) Dual conformal field theories and string duality symmetries, see [61] for a quantum mechanical perspective of superconformal black holes and [73, 74] for the origin of gravitational thermodynamics and the role of giant gravitons in conformal field theory.

(iii) Stabilization against local and/or global perturbations, see [75–80] for black brane dynamics, stability, and critical phenomena. Thus, the consideration of state-space geometry is well-suited for examining the domain of instability. This includes Gregory-Laflamme (GL) modes, chemical potential

fluctuations, electric-magnetic charges and dipole charges, rotational fluctuations, and the thermodynamic temperature fluctuations for the near-extremal and nonextremal black brane solutions. We leave this perspective of the state-space geometry open for a future research.

In general, various $D$ dimensional black brane configurations, see, for instance, [75–80] for black rings in $D > 5$ spacetime dimensions with $S^1 \times S^{D-3}$ horizon topology, and the higher horizon topologies, for example, $S^1 \times S^1 \times S^2$, $S^3 \times S^3$, and so forth offer a platform to extend the consideration of the state-space geometry.

On the other hand, the bubbling black brane solutions, namely, Lin, Lunin, and Maldacena (LLM) geometries [81], are interesting from the perspective of Mathur's Fuzzball conjecture(s). From the perspective of the generalized hyper-Kähler manifolds, Mathur's conjecture [82–85] reduces to classifying and counting asymptotically flat four-dimensional hyper Kähler manifolds [71] which have moduli regions of uniform signature $(+,+,+,+)$ and $(-,-,-,-)$.

Finally, the new physics at the length of the Planck scale anticipates an analysis of the state-space configurations. In particular, it materializes that the state-space geometry may be explored with the parameters of the foam geometries [71], and the corresponding empty space virtual black holes, see [81] for the notion of bubbling AdS space and 1/2 BPS geometries. In such cases, the local and global statistical correlations, among the parameters of the microstates of the black hole conformal field theory [60, 61], would involve the foams of two spheres. From the perspective of the string theory, the present exploration thus opens up an avenue for learning new insights into the promising structures of the black brane space-time configurations at very small scales.

## Appendices

In these appendices, we provide explicit forms of the state-space correlation arising from the metric tensor of the charged (non)extremal (non)large black holes in string theory. In fact, our analysis illustrates that the stability properties of the specific state-space hypersurface may exactly be exploited in general. The definite behavior of state-space properties, as accounted in the concerned main sections, suggests that the various intriguing hypersurfaces of the state-space configuration include the nice feature that they do have definite stability properties, except for some specific values of the charges and anticharges.

As mentioned in the main sections, these configurations are generically well-defined and indicate an interacting statistical basis. Herewith, we discover that the state-space geometry of the general black brane configurations in string theory indicates the possible nature of the underlying statistical fluctuations. Significantly, we notice from the very definition of the intrinsic metric tensor that the related statistical pair correlation functions and relative statistical correlation functions take the following exact expressions.

### A. Correlations for Three Charge Configurations

Following the notion of the fluctuations, we see from the Hessian of the entropy equation (30) that the components of state-space metric tensor are

$$\begin{aligned}
g_{n_1n_1} &= \frac{\pi}{2n_1}\sqrt{\frac{n_5n_p}{n_1}}, & g_{n_1n_5} &= -\frac{\pi}{2}\sqrt{\frac{n_p}{n_1n_5}},\\
g_{n_1n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1n_p}}, & g_{n_5n_5} &= \frac{\pi}{2n_5}\sqrt{\frac{n_1n_p}{n_5}},\\
g_{n_5n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5n_p}}, & g_{n_pn_p} &= \frac{\pi}{2n_p}\sqrt{\frac{n_1n_5}{n_p}}.
\end{aligned} \tag{A.1}$$

For distinct $i, j \in \{1, 5\}$ and $p$, the list of relative correlation functions follows the scaling relations:

$$\begin{aligned}
&\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \quad \frac{g_{ii}}{g_{pp}} = \left(\frac{n_p}{n_i}\right)^2, \quad \frac{g_{ii}}{g_{ij}} = -\left(\frac{n_j}{n_i}\right),\\
&\frac{g_{ii}}{g_{ip}} = -\left(\frac{n_p}{n_i}\right), \quad \frac{g_{ip}}{g_{jp}} = \left(\frac{n_j}{n_i}\right), \quad \frac{g_{ii}}{g_{jp}} = -\left(\frac{n_jn_p}{n_i^2}\right),\\
&\frac{g_{ip}}{g_{pp}} = -\left(\frac{n_p}{n_i}\right), \quad \frac{g_{ij}}{g_{ip}} = \left(\frac{n_p}{n_j}\right), \quad \frac{g_{ij}}{g_{pp}} = -\left(\frac{n_p^2}{n_in_j}\right).
\end{aligned} \tag{A.2}$$

### B. Correlations for Four Charge Configurations

For the given entropy as in (36), we find that the components of the metric tensor are

$$\begin{aligned}
g_{n_1n_1} &= \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
g_{n_1n_5} &= -\frac{\pi}{2\sqrt{n_1n_5}}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
g_{n_1n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1n_p}}, \qquad g_{n_1\overline{n_p}} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1\overline{n_p}}},\\
g_{n_5n_5} &= \frac{\pi}{2}\sqrt{\frac{n_1}{n_5^3}}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right), \qquad g_{n_5n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5n_p}},\\
g_{n_5\overline{n_p}} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5\overline{n_p}}}, \qquad g_{n_pn_p} = \frac{\pi}{2}\sqrt{\frac{n_1n_5}{n_p^3}},\\
g_{n_p\overline{n_p}} &= 0, \qquad g_{\overline{n_p}\overline{n_p}} = \frac{\pi}{2}\sqrt{\frac{n_1n_5}{\overline{n_p}^3}}.
\end{aligned} \tag{B.1}$$

For distinct $i, j \in \{1,5\}$, and $k, l \in \{p, \overline{p}\}$ describing four charge nonextremal $D_1$–$D_5$-$P$-$\overline{P}$ black holes, the statistical pair correlations consist of the following scaling relations:

$$\begin{aligned}
&\frac{g_{ii}}{g_{jj}} = \left(\frac{n_j}{n_i}\right)^2, \qquad \frac{g_{ii}}{g_{kk}} = \frac{n_k}{n_i^2}\sqrt{n_k}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
&\frac{g_{ii}}{g_{ij}} = -\frac{n_j}{n_i},\\
&\frac{g_{ii}}{g_{ik}} = -\frac{\sqrt{n_k}}{n_i}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
&\frac{g_{ik}}{g_{jk}} = \frac{n_j}{n_i}, \qquad \frac{g_{ii}}{g_{jk}} = -\frac{n_j}{n_i^2}\sqrt{n_k}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
&\frac{g_{ik}}{g_{kk}} = -\frac{n_k}{n_i}, \qquad \frac{g_{ij}}{g_{ik}} = \frac{\sqrt{n_k}}{n_j}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right),\\
&\frac{g_{ij}}{g_{kk}} = -\frac{n_k}{n_i n_j}\sqrt{n_k}\left(\sqrt{n_p}+\sqrt{\overline{n_p}}\right).
\end{aligned} \tag{B.2}$$

Notice that the list of other mixed relative correlation functions concerning the nonextremal $D_1$–$D_5$-$P$-$\overline{P}$ black holes read as

$$\begin{aligned}
&\frac{g_{ik}}{g_{il}} = \sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{ik}}{g_{jl}} = \frac{n_j}{n_i}\sqrt{\frac{n_l}{n_k}}, \qquad \frac{g_{kl}}{g_{ij}} = 0,\\
&\frac{g_{kl}}{g_{ii}} = 0, \qquad \frac{g_{kk}}{g_{ll}} = \left(\frac{n_l}{n_k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}} = 0.
\end{aligned} \tag{B.3}$$

## C. Correlations for Six Charge Configurations

Over generic nonlarge charge domains, we find from the entropy equation (46) that the components of the covariant state-space metric tensor are given by the following expressions:

$$\begin{aligned}
&g_{n_1n_1} = \frac{\pi}{2n_1^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right), \qquad g_{n_1m_1} = 0,\\
&g_{n_1n_2} = -\frac{\pi}{2\sqrt{n_1n_2}}\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{n_1m_2} = -\frac{\pi}{2\sqrt{n_1m_2}}\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{n_1n_3} = -\frac{\pi}{2\sqrt{n_1n_3}}\left(\sqrt{n_2}+\sqrt{m_2}\right),\\
&g_{n_1m_3} = -\frac{\pi}{2\sqrt{n_1m_3}}\left(\sqrt{n_2}+\sqrt{m_2}\right),\\
&g_{m_1m_1} = \frac{\pi}{2m_1^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{m_1n_2} = -\frac{\pi}{2\sqrt{m_1n_2}}\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{m_1m_2} = -\frac{\pi}{2\sqrt{m_1m_2}}\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{m_1n_3} = -\frac{\pi}{2\sqrt{m_1n_3}}\left(\sqrt{n_2}+\sqrt{m_2}\right),\\
&g_{m_1m_3} = -\frac{\pi}{2\sqrt{m_1m_3}}\left(\sqrt{n_2}+\sqrt{m_2}\right),\\
&g_{n_2n_2} = \frac{\pi}{2n_2^{3/2}}\left(\sqrt{n_1}+\sqrt{m_1}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{n_2m_2} = 0, \qquad g_{n_2n_3} = -\frac{\pi}{2\sqrt{n_2n_3}}\left(\sqrt{n_1}+\sqrt{m_1}\right)\\
&g_{n_2m_3} = -\frac{\pi}{2\sqrt{n_2m_3}}\left(\sqrt{n_1}+\sqrt{m_1}\right),\\
&g_{m_2m_2} = \frac{\pi}{2m_2^{3/2}}\left(\sqrt{n_1}+\sqrt{m_1}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right),\\
&g_{m_2n_3} = -\frac{\pi}{2\sqrt{m_2n_3}}\left(\sqrt{n_1}+\sqrt{m_1}\right),\\
&g_{m_2m_3} = -\frac{\pi}{2\sqrt{m_2m_3}}\left(\sqrt{n_1}+\sqrt{m_1}\right),\\
&g_{n_3n_3} = \frac{\pi}{2n_3^{3/2}}\left(\sqrt{n_1}+\sqrt{m_1}\right)\left(\sqrt{n_2}+\sqrt{m_2}\right), \qquad g_{n_3m_3} = 0,\\
&g_{m_3m_3} = \frac{\pi}{2m_3^{3/2}}\left(\sqrt{n_1}+\sqrt{m_1}\right)\left(\sqrt{n_2}+\sqrt{m_2}\right).
\end{aligned} \tag{C.1}$$

In this case, from the definition of the relative statistical correlation functions, for $i, j \in \{n_1, m_1\}$, and $k, l \in \{n_2, m_2\}$, the relative correlation functions satisfy the following scaling relations:

$$\begin{aligned}
&\frac{g_{ii}}{g_{jj}} = \left(\frac{j}{i}\right)^{3/2}, \qquad \frac{g_{ii}}{g_{kk}} = \left(\frac{k}{i}\right)^{3/2}\left(\frac{\sqrt{n_2}+\sqrt{m_2}}{\sqrt{n_3}+\sqrt{m_3}}\right),\\
&\frac{g_{ij}}{g_{ii}} = 0,\\
&\frac{g_{ii}}{g_{ik}} = -\frac{\sqrt{k}}{i}\left(\sqrt{n_2}+\sqrt{m_2}\right), \qquad \frac{g_{ik}}{g_{jk}} = \sqrt{\frac{j}{i}},\\
&\frac{g_{ii}}{g_{jk}} = -\frac{\sqrt{jk}}{i^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right),\\
&\frac{g_{kk}}{g_{ik}} = -\frac{\sqrt{i}}{k}\left(\sqrt{n_2}+\sqrt{m_2}\right), \qquad \frac{g_{ij}}{g_{ik}} = 0, \qquad \frac{g_{ij}}{g_{kk}} = 0.
\end{aligned} \tag{C.2}$$

The other concerned relative correlation functions are

$$\frac{g_{ik}}{g_{il}} = \sqrt{\frac{l}{k}}, \qquad \frac{g_{ik}}{g_{jl}} = \sqrt{\frac{jl}{ik}}, \qquad \frac{g_{ij}}{g_{kl}} = \text{n.d.},$$
$$\frac{g_{kl}}{g_{ii}} = 0, \qquad \frac{g_{kk}}{g_{ll}} = \left(\frac{l}{k}\right)^{3/2}, \qquad \frac{g_{kl}}{g_{kk}} = 0. \tag{C.3}$$

For $k, l \in \{n_2, m_2\}$, and $m, n \in \{n_3, m_3\}$, we have

$$\frac{g_{kk}}{g_{mm}} = \left(\frac{m}{k}\right)^{3/2}\left(\frac{\sqrt{n_3}+\sqrt{m_3}}{\sqrt{n_2}+\sqrt{m_2}}\right),$$
$$\frac{g_{kl}}{g_{kk}} = 0, \qquad \frac{g_{kk}}{g_{km}} = -\frac{\sqrt{m}}{k}\left(\sqrt{n_3}+\sqrt{m_3}\right),$$
$$\frac{g_{km}}{g_{lm}} = \sqrt{\frac{l}{k}}, \qquad \frac{g_{kk}}{g_{lm}} = -\frac{\sqrt{lm}}{k^{3/2}}\left(\sqrt{n_3}+\sqrt{m_3}\right), \tag{C.4}$$
$$\frac{g_{mm}}{g_{km}} = -\frac{\sqrt{k}}{m}\left(\sqrt{n_2}+\sqrt{m_2}\right),$$
$$\frac{g_{kl}}{g_{km}} = 0, \qquad \frac{g_{kl}}{g_{mm}} = 0.$$

The other concerned relative correlation functions are

$$\frac{g_{km}}{g_{kn}} = \sqrt{\frac{n}{m}}, \qquad \frac{g_{km}}{g_{ln}} = \sqrt{\frac{ln}{km}}, \qquad \frac{g_{kl}}{g_{mn}} = \text{n.d.},$$
$$\frac{g_{mn}}{g_{kk}} = 0, \qquad \frac{g_{mm}}{g_{nn}} = \left(\frac{n}{m}\right)^{3/2}, \qquad \frac{g_{mn}}{g_{mm}} = 0. \tag{C.5}$$

However, for $i, j \in \{n_1, m_1\}$, and $m, n \in \{n_3, m_3\}$, we have

$$\frac{g_{ii}}{g_{mm}} = \left(\frac{m}{i}\right)^{3/2}\left(\frac{\sqrt{n_3}+\sqrt{m_3}}{\sqrt{n_1}+\sqrt{m_1}}\right), \qquad \frac{g_{ij}}{g_{ii}} = 0,$$
$$\frac{g_{ii}}{g_{im}} = -\frac{\sqrt{m}}{i}\left(\sqrt{n_3}+\sqrt{m_3}\right),$$
$$\frac{g_{im}}{g_{jm}} = \sqrt{\frac{j}{i}}, \qquad \frac{g_{ii}}{g_{jm}} = -\frac{\sqrt{jm}}{i^{3/2}}\left(\sqrt{n_3}+\sqrt{m_3}\right),$$
$$\frac{g_{mm}}{g_{im}} = -\frac{\sqrt{i}}{m}\left(\sqrt{n_1}+\sqrt{m_1}\right), \tag{C.6}$$
$$\frac{g_{ij}}{g_{im}} = 0, \qquad \frac{g_{ij}}{g_{mm}} = 0, \qquad \frac{g_{im}}{g_{in}} = \sqrt{\frac{n}{m}},$$
$$\frac{g_{im}}{g_{jn}} = \sqrt{\frac{jn}{im}}, \qquad \frac{g_{ij}}{g_{mn}} = \text{n.d.},$$
$$\frac{g_{mn}}{g_{ii}} = 0, \qquad \frac{g_{mn}}{g_{mm}} = 0.$$

## D. Principle Minors for Eight Charge Configurations

For the entropy equation (58) of the most general nonextremal nonlarge charged anticharged black hole in string involving finitely many nontrivially circularly fibered Kaluza-Klein monopoles, the principle minors take the following expressions:

$$p_1 = \frac{\pi}{2n_1^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)\left(\sqrt{n_3}+\sqrt{m_3}\right)\left(\sqrt{n_4}+\sqrt{m_4}\right),$$
$$p_2 = \frac{\pi^2}{4(n_1m_1)^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)^2\left(\sqrt{n_3}+\sqrt{m_3}\right)^2 \times\left(\sqrt{n_4}+\sqrt{m_4}\right)^2,$$
$$p_3 = \frac{\pi^3}{8(n_1m_1n_2)^{3/2}}\left(\sqrt{n_3}+\sqrt{m_3}\right)^3\left(\sqrt{n_4}+\sqrt{m_4}\right)^3 \times\left(\sqrt{n_2}+\sqrt{m_2}\right)\sqrt{m_2}\left(\sqrt{n_1}+\sqrt{m_1}\right),$$
$$p_4 = 0,$$
$$p_5 = -\frac{\pi^5}{8(n_1n_2m_2m_1)^{3/2}n_3}\left(\sqrt{n_2}+\sqrt{m_2}\right)^3 \times\left(\sqrt{n_3}+\sqrt{m_3}\right)^3\left(\sqrt{n_4}+\sqrt{m_4}\right)^5\left(\sqrt{n_1}+\sqrt{m_1}\right)^3,$$
$$p_6 = -\frac{\pi^6}{16(n_1n_2m_1m_2n_3m_3)^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)^4 \times\left(\sqrt{n_3}+\sqrt{m_3}\right)^4\left(\sqrt{n_4}+\sqrt{m_4}\right)^6\left(\sqrt{n_1}+\sqrt{m_1}\right)^4,$$
$$p_7 = -\frac{\pi^7}{32(n_1m_1n_2m_2n_3m_3n_4)^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)^5 \times\left(\sqrt{n_3}+\sqrt{m_3}\right)^5\left(\sqrt{n_4}+\sqrt{m_4}\right)^5\left(4\sqrt{n_4}+\sqrt{m_4}\right) \times\left(\sqrt{n_1}+\sqrt{m_1}\right)^5,$$
$$p_8 = -\frac{\pi^8}{16\left(\prod_{i=1}^4 n_im_i\right)^{3/2}}\left(\sqrt{n_2}+\sqrt{m_2}\right)^6 \times\left(\sqrt{n_3}+\sqrt{m_3}\right)^6\left(\sqrt{n_4}+\sqrt{m_4}\right)^6\left(\sqrt{n_1}+\sqrt{m_1}\right)^6. \tag{D.1}$$

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Quantum Tunnelling for Hawking Radiation from Both Static and Dynamic Black Holes

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The paper deals with Hawking radiation from both a general static black hole and a nonstatic spherically symmetric black hole. In case of static black hole, tunnelling of nonzero mass particles is considered and due to complicated calculations, quantum corrections are calculated only up to the first order. The results are compared with those for massless particles near the horizon. On the other hand, for dynamical black hole, quantum corrections are incorporated using the Hamilton-Jacobi method beyond semiclassical approximation. It is found that different order correction terms satisfy identical differential equation and are solved by a typical technique. Finally, using the law of black hole mechanics, a general modified form of the black hole entropy is obtained considering modified Hawking temperature.

## 1. Introduction

Hawking radiation is one of the most important effects in black hole (BH) physics. Classically, nothing can escape from the BH across its event horizon. But in 1974, there was a dramatic change in view when Hawking and Hartle [1, 2] showed that BHs are not totally black; they radiate analogous to thermal black body radiation. Since then, there has been lots of attraction to this issue and various approaches have been developed to derive Hawking radiation and its corresponding temperature [3–7]. However, in the last decade, two distinct semiclassical methods have been developed which enhanced the study of Hawking radiation to a great extent. The first approach developed by Parikh and Wilczek [8, 9] is based on the heuristic pictures of visualisation of the source of radiation as tunnelling and is known as radial null geodesic method. The essence of this method is to calculate the imaginary part of the action for the *s-wave* emission (across the horizon) using the radial null geodesic equation and is then related to the Boltzmann factor to obtain Hawking radiation by the relation:

$$\Gamma \propto \exp\left\{-\frac{2}{h}\left(\operatorname{Im} S^{\text{out}} - \operatorname{Im} S^{\text{in}}\right)\right\} = \exp\left\{-\frac{E}{T_H}\right\}, \qquad (1)$$

where $E$ is the energy associated with the tunnelling particle and $T_H$ is the usual Hawking temperature.

The alternative way of looking into this aspect is known as complex paths method developed by Srinivasan et al. [10, 11]. In this approach, the differential equation of the action $S(r,t)$ of a classical scalar particle can be obtained by plugging the scalar field wave function $\phi(r,t) = \exp\{-(i/\hbar)S(r,t)\}$ into the Klein-Gordon (KG) equation in a gravitational background. Then, the Hamilton-Jacobi (HJ) method is employed to solve the differential equation for $S$. Finally, Hawking temperature is obtained using the "principle of detailed balance" [10–12] (time-reversal invariant). It should be noted that the first method is limited to massless particles only. Also, this method is applicable to such coordinate system only in which there is no singularity across the horizon. On the other hand, in complex paths method, the emitted particles are considered without self-gravitation and the action is assumed to satisfy the relativistic HJ equation. Here tunnelling of both massless and massive particles is possible and it is applicable to any coordinate system to describe the BH.

Most of the studies [13–18] dealing with the Hawking radiation are connected to semiclassical analysis. Recently, Banerjee and Majhi [19] and Corda et al. [20, 21] initiated

the calculation of Hawking temperature beyond the semiclassical limit. Mostly, both groups have considered tunnelling of massless particle and evaluated the modified Hawking temperature with quantum corrections.

In the present work, at first we consider a general nonstatic metric for dynamical BH. HJ method is extended beyond semiclassical approximation to consider all the terms in the expansion of the one particle action. It is found that the higher order terms (quantum corrections) satisfy identical differential equations as the semiclassical action and the complicated terms are eliminated considering BH horizon as one way barrier. We derive the modified Hawking temperature using both the above approaches which are found to be identical at the semiclassical level. Also, modified form of the BH entropy with quantum correction has been evaluated.

Subsequently, in the next section, we consider tunnelling of particles having nonzero mass beyond semiclassical approximation. Due to nonzero mass, the imaginary part of the action cannot be evaluated using first approach; only HJ method will be applicable. Further, the complicated form of the equations involved restricted us to only first order quantum correction.

## 2. Method of Radial Null Geodesic: A Survey of Earlier Works

This section deals with a brief survey of the method of radial null geodesics method [8] considering the picture of Hawking radiation as quantum tunnelling. In a word, the method correlates the imaginary part of the action for the classically forbidden process of *s-wave* emission across the horizon with the Boltzmann factor for the black body radiation at the Hawking temperature. We start with a general class of nonstatic spherically symmetric BH metric of the form

$$ds^2 = -A(r,t)\,dt^2 + \frac{dr^2}{B(r,t)} + r^2 d\Omega_2^2, \tag{2}$$

where the horizon $r_h$ is located at $A(r_h,t) = 0 = B(r_h,t)$ and the metric has a coordinate singularity at the horizon. To remove the coordinate singularity, we make the following Painleve-type transformation of coordinates:

$$dt \longrightarrow dt - \sqrt{\frac{1-B}{AB}}\,dr \tag{3}$$

and as a result metric (2) transforms to

$$ds^2 = -Adt^2 + 2\sqrt{A\left(\frac{1}{B}-1\right)}dtdr + dr^2 + r^2 d{\Omega_2}^2. \tag{4}$$

This metric (i.e., the choice of coordinates) has some distinct features over the former one, as follows.

(i) The metric is singularity free across the horizon.

(ii) At any fixed time, we have a flat spatial geometry.

(iii) Both the metric will have the same boundary geometry at any fixed radius.

The radial null geodesic (characterized by $ds^2 = 0 = d{\Omega_2}^2$) has the differential equation (using (3)):

$$\frac{dr}{dt} = \sqrt{\frac{A}{B}}\left[\pm 1 - \sqrt{1-B(r,t)}\right], \tag{5}$$

where outgoing or ingoing geodesic is identified by the + or − sign within the square bracket in (4). In the present case, we deal with the absorption of particles through the horizon (i.e., + sign only) and according to Parikh and Wilczek [8], the imaginary part of the action is obtained as

$$\begin{aligned} \operatorname{Im} S &= \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} p_r\,dr = \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\int_0^{p_r} dp_r'\,dr \\ &= \operatorname{Im}\int_{r_{\text{in}}}^{r_{\text{out}}}\left\{\int_0^H \frac{dH'}{dr/dt}\right\} dr. \end{aligned} \tag{6}$$

Note that in the last step of the above derivation we have used the Hamilton's equation $\dot{r} = (dH/dp_r)|_r$, where $(r,p_r)$ are canonical pair. Further, it is to be mentioned that in quantum mechanics, the action of a tunnelled particle in a potential barrier having energy larger than the energy of the particle will be imaginary as $p_r = \sqrt{2m(E-V)}$. For the present nonstatic BH, the mass of the BH is not constant and hence the $dH'$ integration extends over all the values of energy of outgoing particle, from zero to $E(t)$ [22] (say). As we are dealing with tunnelling across the BH horizon, so using Taylor series expansion about the horizon $r_h$ we write

$$\begin{aligned} A(r,t)|_t &= \left.\frac{\partial A(r,t)}{\partial r}\right|_t (r-r_h) + O(r-r_h)^2\big|_t, \\ B(r,t)|_t &= \left.\frac{\partial B(r,t)}{\partial r}\right|_t (r-r_h) + O(r-r_h)^2\big|_t. \end{aligned} \tag{7}$$

So, in the neighbourhood of the horizon, the geodesic equation (4) can be approximated as

$$\frac{dr}{dt} \approx \frac{1}{2}\sqrt{A'(r_h,t)\,B'(r_h,t)}\,(r-r_h). \tag{8}$$

Substituting this value of $dr/dt$ in the last step of (5) we have

$$\operatorname{Im} S = \frac{2\pi E(t)}{\sqrt{A'(r_h,t)\,B'(r_h,t)}}, \tag{9}$$

where the choice of contour for $r$-integration is on the upper half complex plane to avoid the coordinate singularity at $r_h$. Thus, the tunnelling probability is given by

$$\Gamma \sim \exp\left\{-\frac{2}{\hbar}\operatorname{Im} S\right\} = \exp\left\{-\frac{4\pi E(t)}{\hbar\sqrt{A'B'}}\right\}, \tag{10}$$

which in turn equates with the Boltzmann factor $\exp\{E(t)/T\}$; the expression for the Hawking temperature is

$$T_H = \frac{\hbar\sqrt{A'(r_h,t)\,B'(r_h,t)}}{4\pi}. \tag{11}$$

From the above expression, it is to be noted that $T_H$ is time dependent.

Recently, a drawback of the above approach has been noted [23–25]. It has been shown that $\Gamma \sim \exp\{-(2/\hbar)\,\mathrm{Im}\,S\} = \exp\{-(2/\hbar)\,\mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r dr\}$ is not canonically invariant and hence is not a proper observable; it should be modified as $\exp\{-\,\mathrm{Im}\oint p_r dr/\hbar\}$. The closed path goes across the horizon and back. For tunnelling across the ordinary barrier, it is immaterial whether the particle goes from the left to the right or the reverse path. So in that case

$$\oint p_r\, dr = 2\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r\, dr \tag{12}$$

and there is no problem of canonical invariance. But difficulty arises for BH horizon which behaves as a barrier for particles going from inside of the BH to outside but it does not act as a barrier for particles going from outside to the inside. So relation (12) is no longer valid. Also, using tunnelling the probability is $\Gamma \sim \exp\{-\,\mathrm{Im}\oint p_r dr/\hbar\}$, so there will be a problem of factor two in Hawking temperature [24, 26, 27].

Further, the above analysis of tunnelling approach remains incomplete unless effects of self-gravitation and back reaction are taken into account. But unfortunately, no general approaches to account for the above effects are there in the literature; only few results are available for some known BH solutions [26–32].

Finally, it is worth mentioning that so far the above tunnelling approach is purely semiclassical in nature and quantum corrections are not included. Also, this method is applicable for Painleve-type coordinates only; one cannot use the original metric coordinates to avoid horizon singularity. Lastly, the tunnelling approach is not applicable for massive particles [19].

## 3. Hamilton-Jacobi Method: Quantum Corrections

We will now follow the alternative approach as mentioned in the introduction, that is, the HJ method to evaluate the imaginary part of the action and hence the Hawking temperature. We will analyze the beyond semiclassical approximation by incorporating possible quantum corrections. As this method is not affected by the coordinate singularity at the horizon so we will use the general BH metric (2) for convenience.

In the background of the gravitational field described by the metric (2), massless scalar particles obey the Klein-Gordon equation

$$\frac{\hbar^2}{\sqrt{-g}}\partial\left[g^{\mu\nu}\sqrt{-g}\partial_\nu\right]\psi = 0. \tag{13}$$

For spherically symmetric BH, as we are only considering radial trajectories, so we will consider $(t, r)$-sector in the spacetime given by (2); that is, we concentrate on two-dimensional BH problems. Using (2), the above Klein-Gordon equation becomes

$$\frac{\partial^2\psi}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial\psi}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} - AB\frac{\partial^2\psi}{\partial r^2} = 0. \tag{14}$$

Using the standard ansatz for the semiclassical wave function, namely,

$$\psi(r,t) = \exp\left\{-\frac{i}{\hbar}S(r,t)\right\}, \tag{15}$$

the differential equation for the action $S$ is

$$\left(\frac{\partial S}{\partial t}\right)^2 - AB\left(\frac{\partial S}{\partial r}\right)^2 + i\hbar\left[\frac{\partial^2 S}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S}{\partial r} - AB\frac{\partial^2 S}{\partial r^2}\right]. \tag{16}$$

To solve this partial differential equation we expand the action $S$ in powers of Planck's constant $\hbar$ as

$$S(r,t) = S_0(r,t) + \Sigma\hbar^k S_k(r,t), \tag{17}$$

with $k$ being a positive integer. Note that, in the above expansion, terms of the order of Planck's constant and its higher powers are considered as quantum corrections over the semiclassical action $S_0$. Now substituting ansatz (17) for $S$ into (16) and equating different powers of $\hbar$ on both sides, we obtain the following set of partial differential equations:

$$\hbar^0 : \left(\frac{\partial S}{\partial t}\right)^2 - AB\left(\frac{\partial S}{\partial r}\right)^2 = 0, \tag{18}$$

$$\hbar^1 : \frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - AB\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} + \frac{i}{2}\left[\frac{\partial^2 S_0}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S_0}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S_0}{\partial r} - AB\frac{\partial^2 S_0}{\partial r^2}\right] = 0, \tag{19}$$

$$\hbar^2 : \left(\frac{\partial S_1}{\partial t}\right)^2 + 2\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - AB\left(\frac{\partial S_1}{\partial r}\right)^2 - 2AB\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} + i\left[\frac{\partial^2 S_1}{\partial t^2} - \frac{1}{2AB}\frac{\partial(AB)}{\partial t}\frac{\partial S_1}{\partial t} - \frac{1}{2}\frac{\partial(AB)}{\partial r}\frac{\partial S_1}{\partial r} - AB\frac{\partial^2 S_1}{\partial r^2}\right] = 0, \tag{20}$$

and so on.

Apparently, different order partial differential equations are very complicated but fortunately there will be lot of simplifications if, in the partial differential equation corresponding to $\hbar^k$, all previous partial differential equations are used and finally we obtain identical partial differential equation, namely,

$$\hbar^k : \frac{\partial S_k}{\partial t} = \pm\sqrt{A(r,t)\,B(r,t)}\frac{\partial S_k}{\partial r}, \tag{21}$$

for $k = 0, 1, 2, \ldots$.

Thus, quantum corrections satisfy the same differential equation as the semiclassical action $S_0$. Hence, the solutions will be very similar. To solve $S_0$, it is to be noted that due to nonstatic BHs the metric coefficients are functions of $r$ and $t$ and hence standard HJ method cannot be applied; some generalization is needed. We start with a general metric [22]

$$S_0(r,t) = \int_0^t \omega_0(t')\, dt + D_0(r,t). \tag{22}$$

Here $\omega_0(t)$ behaves as the energy of the emitted particle and the justification of the choice of the integral is that the outgoing particle should have time-dependent continuum energy.

Now substituting the above ansatz for $S_0(r,t)$ into (18) and using the radial null geodesic in the usual metric from (2), namely,

$$\frac{dr}{dt} = \pm\sqrt{AB}, \tag{23}$$

we have

$$\frac{\partial D_0}{\partial r} + \frac{\partial D_0}{\partial t}\frac{dt}{dr} = \mp\omega_0(t)\frac{dt}{dr}. \tag{24}$$

that is,

$$\frac{dD_0}{dr} = \mp\frac{\omega_0(t)}{\sqrt{AB}}, \tag{25}$$

which gives

$$D_0 = \mp\omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}. \tag{26}$$

Hence, the complete semiclassical action takes the form

$$S_0(r,t) = \int_0^t \omega_0(t')\, dt' \mp \omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}. \tag{27}$$

Here the $-$ (or $+$) sign corresponds to absorption (or emission) particle. As solution (27) contains an arbitrary time-dependent function $\omega_0(t)$, so a general solution for $S_k$ can be written as

$$S_k(r,t) = \int_0^t \omega_k(t')\, dt' \mp \omega_0(t)\int_0^r \frac{dr}{\sqrt{AB}}, \quad k = 1,2,3,\ldots. \tag{28}$$

Thus, from (15), using solutions (27) and (28) into (17), the wave functions for absorption and emission of scalar particle can be expressed as

$$\begin{aligned}\psi_{\text{emm.}}(r,t) = \exp\Bigg\{-\frac{i}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' &+ \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &- \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\}, \\ \psi_{\text{abs.}}(r,t) = \exp\Bigg\{-\frac{i}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' &+ \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &+ \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\},\end{aligned} \tag{29}$$

respectively. Due to tunnelling across the horizon, there will be a change of sign of the metric coefficients in the $(r,t)$-part of the metric and as a result, function of $t$ coordinate has an imaginary part which will contribute to the probabilities. So we write

$$\begin{aligned}P_{\text{abs.}} &= \left|\psi_{\text{abs.}}(r,t)\right|^2 \\ &= \exp\Bigg\{\frac{2\,\text{Im}}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &\qquad + \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\},\end{aligned} \tag{30}$$

$$\begin{aligned}P_{\text{emm.}} &= \left|\psi_{\text{emm.}}(r,t)\right|^2 \\ &= \exp\Bigg\{\frac{2\,\text{Im}}{\hbar}\Bigg[\Bigg(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\Bigg) \\ &\qquad - \left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\Bigg]\Bigg\}.\end{aligned} \tag{31}$$

To have some simplification, we will now use the physical fact that all incoming particles certainly cross the horizon; that is, $P_{\text{abs.}} = 1$. So from (30),

$$\begin{aligned}&\text{Im}\left(\int_0^t \omega_0(t')\, dt' + \Sigma_k\hbar^k\int_0^t \omega_k(t')\, dt'\right) \\ &\quad = -\,\text{Im}\left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\int_0^r \frac{dr}{\sqrt{AB}}\end{aligned} \tag{32}$$

and hence $P_{\text{emm.}}$ simplifies to

$$P_{\text{emm.}} = \exp\left\{-\frac{4}{\hbar}\left(\omega_0(t) + \Sigma_k\hbar^k\omega_k(t)\right)\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\right\}. \tag{33}$$

Then from the principle of "detailed balance" [10–12] (which states that transitions between any two states take place with equal frequency in either direction at equilibrium), we write

$$P_{\text{emm.}} = \exp\left\{-\frac{\omega_0(t)}{T_h}\right\} P_{\text{in}} = \exp\left\{-\frac{\omega_0(t)}{T_h}\right\}. \quad (34)$$

So, comparing (33) and (34), the temperature of the BH is given by

$$T_h = \frac{\hbar}{4}\left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right]^{-1} \left[\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1}, \quad (35)$$

where

$$T_h = \frac{\hbar}{4}\left[\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1} \quad (36)$$

is the usual Hawking temperature of the BH. Thus, due to quantum corrections, the temperature of the BH is modified from the Hawking temperature and both temperatures are functions of $t$ and $r$. Note that (36) is the standard expression for semiclassical Hawking temperature and it is valid for nonspherical metric also. However, for spherical metric, one can use the Taylor series expansions (7) near the horizon and obtain $T_H$ as given in (11) by performing the contour integration. The ambiguity of factor of two (as mentioned earlier) in the Hawking temperature does not arise here.

Further, one may note that solutions (27) or (28) are the unique solutions to (18) or (21) except for a premultiplication factor. This arbitrary multiplicative factor does not appear in the expression for Hawking temperature; only the particle energy ($\omega_0$) or $\omega_k$ is rescaled. As quantum correction term contains $\omega_k/\omega_0$, so it does not involve the arbitrary multiplicative factor and hence it is unique.

To have some interpretation about the arbitrary functions $\omega_k(t)$ appearing in the quantum correction terms, we make use of dimensional analysis. As $S_0$ has the dimension $\hbar$, so the arbitrary function $\omega_k(t)$ has the dimension $\hbar^{-k}$. In standard choice of units, namely, $G = c = K_B = 1$, $\hbar \sim M_p^2$ and so $\omega_k \sim M^{-2k}$, where $M$ is the mass of the BH.

Similar to the Hawking temperature, the surface gravity of the BH is modified due to quantum corrections. If $\kappa_c$ is the semiclassical surface gravity corresponding to Hawking temperature, that is, $\kappa_c = 2\pi T_H$, then the quantum corrected surface gravity $\kappa = 2\pi T_H$ is related to the semiclassical value by the relation:

$$\kappa = \kappa_c\left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right]^{-1}. \quad (37)$$

Moreover, based on the dimensional analysis, if we choose, for simplicity,

$$\omega_k(t) = \frac{a^k \omega_0(t)}{M^{2k}}, \quad \text{"}a\text{" is a dimensionless parameter,} \quad (38)$$

then expression (37) is simplified to

$$\kappa = \kappa_0\left(1 - \frac{\hbar a}{M^2}\right)^{-1}. \quad (39)$$

This is related to the one loop back reaction effects in the spacetime [6, 33] with the parameter $a$ corresponding to trace anomaly. Higher order loop corrections to the surface gravity can be obtained similarly by suitable choice of the functions $\omega_k(t)$. For static BHs, Banerjee and Majhi [19] have studied these corrections in detail. Lastly, it is worth mentioning that identical result for BH temperature may be obtained if we use the Painleve coordinate system as in the previous section.

## 4. Entropy Function and Quantum Correction

We will now examine how the semiclassical Bekenstein-Hawking area law, namely, $S_{\text{BH}} = (A/4\hbar)$ ($A$ is the area of the horizon) is modified due to quantum corrections described in the previous section. The first law of the BH mechanics, which is essentially the energy conservation relation, related the change of BH mass ($M$) to the change of its entropy ($S_{\text{BH}}$), electric charge ($Q$), and angular momentum ($J$) as

$$dM = T_h dS_{\text{BH}} + \Phi dQ + \Omega dJ. \quad (40)$$

Here, $\Omega$ is the angular velocity and $\Phi$ is the electrostatic potential. So, for nonrotating uncharged BHs, the entropy has the simple form

$$S_{\text{BH}} = \int \frac{dM}{T_h}, \quad (41)$$

or using (35) for $T_h$, we get

$$S_{\text{BH}} = \int \left[1 + \Sigma_k \hbar^k \frac{\omega_k(t)}{\omega_0(t)}\right] \frac{dM}{T_H}. \quad (42)$$

For choice (38) corresponding to one loop back reaction effects, we have from (42) the quantum corrected BH entropy as

$$S_{\text{BH}} = \int \left[1 + \frac{a\hbar}{M} + \frac{a^2\hbar^2}{M^2} + \cdots\right] \frac{dM}{T_H}. \quad (43)$$

The first term is the usual semiclassical Bekenstein-Hawking entropy and the subsequent terms are the quantum corrections of different order. For static BHs, Banerjee and Majhi [19] have shown the correction terms of which the leading one gives the standard logarithmic correction. On the other hand, for nonstatic BHs, as the proportionality factors are time-dependent and arbitrary (see (42)) so the leading order correction term may not be logarithmic. For future work, we will attempt to determine physical interpretation of the arbitrary time-dependent proportionality factors so that quantum corrections may be evaluated.

## 5. Hamilton-Jacobi Method for Massive Particles: Quantum Corrections

The KG equation for a scalar field $\psi$ describing a scalar particle of mass $m_0$ has the form [10]

$$\left(\Box + \frac{m_0^2}{\hbar^2}\right)\psi = 0, \quad (44)$$

where the box operator "□" is evaluated in the background of a general static BH metric of the form

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_2{}^2. \tag{45}$$

The explicit form of the KG equation for the metric (45) is

$$\begin{aligned} &-\frac{1}{A}\frac{\partial^2\psi}{\partial t^2} + B\frac{\partial^2\psi}{\partial r^2} + \frac{1}{2A}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} + \frac{2B}{r}\frac{\partial\psi}{\partial r} \\ &+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \\ &+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{m_0^2}{\hbar^2}\psi(t,r,\theta,\phi). \end{aligned} \tag{46}$$

Due to spherical symmetry, we can decompose $\phi$ in the form

$$\psi(t,r,\theta,\phi) = \Phi(t,r)\,Y_l^m(\theta,\phi), \tag{47}$$

where $\phi$ satisfies [10]

$$\begin{aligned} &\frac{1}{A}\frac{\partial^2\psi}{\partial t^2} - B\frac{\partial^2\psi}{\partial r^2} - \frac{1}{2A}\frac{\partial(AB)}{\partial r}\frac{\partial\psi}{\partial r} - \frac{2B}{r}\frac{\partial\psi}{\partial r} \\ &+ \left\{\frac{l(l+1)}{r^2} + \frac{m_0^2}{\hbar^2}\right\}\Phi(t,r) = 0. \end{aligned} \tag{48}$$

If we substitute the standard ansatz for the semiclassical wave function, namely,

$$\phi(t,r) = \exp\left\{-\frac{i}{\hbar}S(r,t)\right\}, \tag{49}$$

then the action $S$ will satisfy the following differential equation:

$$\begin{aligned} &\left[\frac{1}{A}\left(\frac{\partial S}{\partial t}\right)^2 - B\left(\frac{\partial S}{\partial r}\right)^2 - E_0^2(r)\right] \\ &- \frac{\hbar}{i}\left[\frac{1}{A}\frac{\partial^2 S}{\partial t^2} - B^2\frac{\partial^2 S}{\partial r^2} - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S}{\partial r}\right] = 0, \end{aligned} \tag{50}$$

where $E_0^2 = m_0^2 + (L^2/r^2)$ and $L^2 = l(l+1)\hbar^2$ is the angular momentum. To incorporate quantum corrections over the semiclassical action, we expand the actions in powers of Planck constant $\hbar$ as

$$S(r,t) = S_0(r,t) + \Sigma_k \hbar^k S_k(r,t), \tag{51}$$

where $S_0$ is the semiclassical action and $k$ is a positive integer. Now substituting this ansatz for $S$ in the differential equation (50) and equating different powers of $\hbar$ on both sides, we obtain the following set of partial differential equations:

$$\hbar^0 : \frac{1}{A}\left(\frac{\partial S}{\partial t}\right)^2 - B\left(\frac{\partial S}{\partial r}\right)^2 - E_0^2(r) = 0, \tag{52}$$

$$\begin{aligned} \hbar^1 : &\frac{2}{A}\frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - 2B\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} \\ &- \frac{1}{i}\left[\frac{1}{A}\frac{\partial^2 S_0}{\partial t^2} - B^2\frac{\partial^2 S_0}{\partial r^2}\right. \\ &\left. - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S_0}{\partial r}\right] = 0, \end{aligned} \tag{53}$$

$$\begin{aligned} \hbar^2 : &\frac{1}{A}\left(\frac{\partial S_1}{\partial t}\right)^2 + \frac{2}{A}\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - B\left(\frac{\partial S_1}{\partial r}\right)^2 - 2B\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} \\ &- \frac{1}{i}\left[\frac{1}{A}\frac{\partial^2 S_1}{\partial t^2} - B^2\frac{\partial^2 S_1}{\partial r^2}\right. \\ &\left. - \left\{\frac{1}{2A}\frac{\partial(AB)}{\partial r} + \frac{2B}{r}\right\}\frac{\partial S_1}{\partial r}\right] = 0, \end{aligned} \tag{54}$$

and so on.

To solve the semiclassical action $S_0$, we start with the standard separable choice [10]

$$S_0(r,t) = \omega_0 t + D_0(r). \tag{55}$$

Substituting this choice in (52), we obtain

$$D_0 = \pm\int_0^r \sqrt{\frac{\omega_0^2 - AE_0^2}{AB}}\,dr = \pm I_0 \quad (\text{say}), \tag{56}$$

where + or − sign corresponds to absorption or emission of scalar particle. Now substituting this choice for $S_0$ in (53), we have the differential equation for first order corrections $S_1$ as

$$\begin{aligned} &\frac{\partial S_1}{\partial t} \mp \sqrt{AB}\sqrt{1 - \frac{AE_0^2}{\omega_0^2}}\frac{\partial S_1}{\partial r} \\ &\mp \frac{\sqrt{AB}}{i}\left[-\frac{1}{r}\sqrt{1 - \frac{AE_0^2}{\omega^2}}\right. \\ &\left. + \frac{(\partial A/\partial r)\left(E_0^2/\omega^2\right) - \left(2AL^2/\omega_0^2 r^3\right)}{4\sqrt{1 - \left(AE_0^2/\omega^2\right)}}\right] = 0. \end{aligned} \tag{57}$$

As before, $S_1$ can be written in separable form as

$$S_1 = \omega_1 t + D_1(r), \tag{58}$$

where

$$D_1 = \int_0^r \frac{dr}{\sqrt{AB}\sqrt{1-\left(AE_0^2/\omega_0^2\right)}} \times \left[ \pm\omega_1 - \frac{\sqrt{AB}}{i} \times \left\{ -\frac{1}{r}\sqrt{1-\frac{AE_0^2}{\omega^2}} + \frac{(\partial A/\partial r)\left(E_0^2/\omega^2\right) - \left(2AL^2/\omega_0^2 r^3\right)}{4\sqrt{1-\left(AE_0^2/\omega^2\right)}} \right\} \right] = \pm I_1 - I_2. \tag{59}$$

Now due to complicated form, if we retain terms up to first order quantum corrections, that is,

$$S = S_0 + \hbar S_1 = \left(\omega_0 + \hbar\omega_1\right)t + \left\{D_0 + \hbar D_1(r)\right\}, \tag{60}$$

then the wave function denoting absorption and emission solutions of the KG equation (48) using (49) are of the form

$$\begin{aligned} \phi_{\text{abs.}} &= \exp\left\{-\frac{i}{\hbar}\left(\overline{\omega_0+\hbar\omega_1}t + \overline{I_0+\hbar I_1-\hbar I_2}\right)\right\}, \\ \phi_{\text{emm.}} &= \exp\left\{-\frac{i}{\hbar}\left(\overline{\omega_0+\hbar\omega_1}t - \overline{I_0+\hbar I_1-\hbar I_2}\right)\right\}. \end{aligned} \tag{61}$$

It is to be noted that in course of tunnelling across the horizon, the coordinate nature changes; that is, more precisely the signs of the metric coefficients in the $(r,t)$-hyperplane are altered. Thus, we can interpret this as that the time coordinate has an imaginary part in crossing the horizon and accordingly the temporal part has contribution to the probabilities [19, 33]. Thus, absorption and emission probabilities are given by

$$P_{\text{abs.}} = \left|\phi_{\text{in}}\right|^2 = \exp\left\{\frac{2}{\hbar}\left(\text{Im}\ \overline{\omega_0+\hbar\omega_1}t\right) + \text{Im}\ \overline{I_0+\hbar I_1} - \text{Im}\ \hbar I_2\right\}, \tag{62}$$

$$P_{\text{emm.}} = \left|\phi_{\text{out}}\right|^2 = \exp\left\{-\frac{i}{\hbar}\left(\text{Im}\ \overline{\omega_0+\hbar\omega_1}t\right) - \text{Im}\ \overline{I_0+\hbar I_1} - \text{Im}\ \hbar I_2\right\}. \tag{63}$$

In the classical limit $\hbar \to 0$, there is no reflection, so all ingoing particles should be absorbed and hence [33]

$$\lim_{\hbar\to 0} P_{\text{abs.}} = 1. \tag{64}$$

So, from (62), we must have

$$\text{Im}\ \omega_0 t = \text{Im}\ I_0, \qquad \text{Im}\left(\omega_1 t - I_2\right) = \text{Im}\ I_1 \tag{65}$$

and as a result $P_{\text{emm.}}$ simplifies to

$$P_{\text{emm.}} = \exp\left[-\frac{4\omega_0}{\hbar} \times \text{Im}\left\{\int_0^r \frac{dr}{\sqrt{AB}}\left(\sqrt{1-\frac{AE_0^2}{\omega_0^2}} + \frac{\hbar\left(\omega_1/\omega_0\right)}{\sqrt{1-\left(AE_0^2/\omega_0^2\right)}}\right)\right\}\right]. \tag{66}$$

Using the principle of "detailed balance" [10, 11, 20, 21], namely,

$$P_{\text{emm.}} = \exp\left\{-\frac{E}{T_h}\right\} P_{\text{in}} = \exp\left\{-\frac{E}{T_h}\right\}, \tag{67}$$

the temperature of the BH is given by

$$T_h = \frac{\hbar E}{4\omega_0}\left[\text{Im}\left\{\int_0^r \frac{dr}{\sqrt{AB}} \times \left(\sqrt{1-\frac{AE_0^2}{\omega_0^2}} + \frac{\hbar(\omega_1/\omega_0)}{\sqrt{1-\left(AE_0^2/\omega_0^2\right)}}\right)\right\}\right]^{-1}, \tag{68}$$

where the semiclassical Hawking temperature of the BH has the expression

$$T_H = \frac{\hbar E}{4\omega_0}\left[\text{Im}\int_0^r \frac{dr}{\sqrt{AB}}\sqrt{1-\frac{AE_0^2}{\omega_0^2}}\right]^{-1}. \tag{69}$$

Now, to obtain the modified form of the surface gravity of the BH, we start with the usual relation between surface gravity and Hawking temperature, namely,

$$\kappa_H = 2\pi T_H, \tag{70}$$

where $T_H$ is given by (69).

So the quantum corrected surface gravity is given by

$$\kappa_{\text{QC}} = 2\pi T_h. \tag{71}$$

Further, for the present nonrotating, uncharged, static BHs, using the law of BH thermodynamics $dM = T_h dS$, we have the expression for the entropy of the BH as

$$S_{\text{BH}} = \int \frac{4\omega_0}{\hbar E}\left(1+\frac{\hbar\omega_1}{\omega_0}\right) dM \int_0^r \frac{dr}{\sqrt{AB}}. \tag{72}$$

Finally, it is easy to see from (68) that near the horizon the presence of $E_0^2$ term can be neglected as it is multiplied

by the metric coefficient $A$. Therefore, the quantum corrected (up to first order) temperature of the BH (in (68)) reduces to

$$T_h = \frac{\hbar E}{4\omega_0}\left(1 + \frac{\hbar\omega_1}{\omega_0}\right)^{-1}\left[\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1} \tag{73}$$

and the Hawking temperature (given in (69)) becomes

$$T_H = \frac{\hbar E}{4\omega_0}\left[\int_0^r \frac{dr}{\sqrt{AB}}\right]^{-1}. \tag{74}$$

So we have

$$T_h = \left(1 + \frac{\hbar\omega_1}{\omega_0}\right)^{-1} T_H. \tag{75}$$

We see that if the energy of the tunnelling particle is chosen as $\omega_0$ (i.e., $E = \omega_0$) and $\omega_1 = \beta_1/M$ (for notations see Banerjee and Majhi [19]) then the Hawking temperature given by (74) is the usual one derived for massless particles and the quantum corrected temperature $T_h$ given in (75) agrees with that of Banerjee and Majhi [19] for massless particle. Therefore, Hawking temperature near the horizon remains the same for both massless and nonzero mass tunnelling particles and it agrees with the claim of Srinivasan and Padmanabhan [10] and Banerjee and Majhi [19]. For future work, it will be interesting to calculate the temperature of the BH for tunnelling nonzero mass particle with full quantum corrections and examine whether the result agrees with that of Banerjee and Majhi [19] near the horizon. Finally, it will be nice to determine quantum corrected entropy of the BH in a convenient form.

## 6. Summary of the Work

This work is an attempt to study quantum corrections to Hawking radiation of massless particle from a dynamical BH as well as for massive particle from a static BH. At first, radial null geodesic tunnelling approach has been used with Painleve-type choice of coordinate system to derive semiclassical Hawking temperature. Then full quantum mechanical calculations have performed writing action in a power series of the Planck constant $\hbar$ to evaluate the quantum corrections to the Hawking temperature. Subsequently, quantum corrected surface gravity has been calculated and it is found that one loop back reaction effects in the spacetime can be obtained by suitable choice of the arbitrary functions and parameters. Finally, an expression for the quantum corrected entropy of the BH has been evaluated. It is found that, due to the presence of the arbitrary functions in the expression for entropy, the leading order quantum correction may not be logarithmic in nature. On the other hand, in the case of Hawking radiation of massive particle from static BH, it is found that Hawking temperature near the horizon does not depend on the mass term as predicted by Srinivasan and Padmanabhan [10] and Banerjee et al. [16–18]. For future work, we will try to find a solution for the partial differential equation (18) in a more simple form so that more physical interpretations can be done from the BH parameters. Also, it will be interesting to calculate temperature of the BH for tunnelling nonzero mass particle with full quantum correction and examine whether the result agrees with that of Banerjee and Majhi [19] near the horizon. Finally, it will be nice to determine quantum corrected entropy of the BH in a convenient form.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Analyzing Black Hole Super-Radiance Emission of Particles/Energy from a Black Hole as a Gedankenexperiment to Get Bounds on the Mass of a Graviton

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Use of super-radiance in BH physics, so $dE/dt < 0$ specifies conditions for a mass of a graviton being less than or equal to $10^{65}$ grams, allows for determing what role additional dimensions may play in removing the datum that massive gravitons lead to 3/4th the bending of light past the planet Mercury. The present document makes a given differentiation between super-radiance in the case of conventional BHs and Braneworld BH super-radiance, which may delineate whether Braneworlds contribute to an admissible massive graviton in terms of removing the usual problem of the 3/4th the bending of light past the planet Mercury which is normally associated with massive gravitons. This leads to a fork in the road between two alternatives with the possibility of needing a multiverse containment of BH structure or embracing what Hawkings wrote up recently, namely, a redo of the event horizon hypothesis as we know it.

## 1. Introduction: Massive Gravity and How to Get It to Commensurate with Black Hole Physics

We are now attempting to come up with criteria for either massless or massive gravitons. Our preferred way to do it distinguishing between the two forms of super-radiance. One built about Kerr black holes [1], and the other involving brane theory [2]; with the brane theory version of super-radiance, perhaps correcting a problem as to when a massive graviton would, without brane theory, lead to 3/4th the angular bending of light and be seen experimentally. We briefly allude to both of these cases in the introduction below, before giving more details to this phenomenon in Sections 2 and 3.

In general, relativity of the metric $g_{ab}(x,t)$ is a set of numbers associated with each point which gives the distance to neighboring points. That is, general relativity is a classical theory. As it is designated by GR traditionalists [3], the graviton is usually stated to be massless, with two spin states and with two polarizations. Adding a mass to the graviton results in 5 polarizations plus other problems [4, 5]; that is, in [4], there is a description of how a massive graviton leads to 3/4th the calculated bending of light pass the mass of Mercury, as seen in the 1919 experiment. Reference [5] has details on the five polarization states, which are another problem. One cannot go from a massive graviton and eliminate mass from the graviton and then neatly recover the easier spin dynamics (2 polarization states) and vastly simpler situation where one has recovered the Schwartzshield metric. As [5] discusses, in its page 92, that this easy recovery of the Schwartzshield metric, if a graviton mass goes to zero, is impossible. Also note that note [4] has a discussion on how the bending of light is not commensurate with GR for massive graviton, which is equivalent to a discussion on a phenomenological ghost state for the trace of $h$, which is given by [6] and occurs regardless of whether the mass for graviton nearly goes to zero. In [7], Csáki et al., have given a temporary fix to restore the bending of light for massive gravitons and to remove the 3/4th angle deflection from the 1919 GR test value, and this is by the use of brane theory. What this document will do will be to try to establish massive

gravitons as super-radiant emission candidates from black holes [8] and, in doing so, provide another framework for their analysis which would embed them in GR. In doing so, one should keep in mind that this is a thought experiment and that the author is fully aware of how hard it would be to perform experimental measurements. In coming up with criteria as to graviton mass, we are also, by extension, considering the Myers-Perry higher dimensional model of black holes [9] and commenting upon its applications, some of which are in [10] and all of which start with the implications of $dE/dt < 0$, leading to "leakage" from a black hole. That is, energy of the black hole "decreases" in time. That is, there are ghost states, where $h$ is the trace of $h(i, j)$ which is a GW perturbation of the flat Euclidian metric, a possibility that brane theory and higher dimensions may remove the 3/4th angle of bent light calculated for massive gravitons, and a suitable thought experiment as given below may allow for $dE/dt$ allowing us to determine whether higher dimensional models are justifiable. This is the reason why the super-radiance phenomenology is being investigated, that is, of bending of angle of light divergence from GR models using massive Gravitons. Does $dE/dt < 0$ imply that there are brane theory states which may remove the 3/4th bending of light divergence from GR by massive gravitons? And can a super-radiance model for when $dE/dt < 0$ imply conditions for which brane worlds have to be considered [2], as opposed to the simpler model proposed by Padmanabhan [1].

The paper will differentiate between Kerr BH [1] versions of super-radiance and brane theory BH super-radiance [2] and, secondly, afterwards, inquire about whether a BH in brane theory configuration is satisfied, if the simper Kerr BH super-radiance criteria is not satisfied. After these two versions are distinguished, we will then discuss experimental criteria which may result in determining whther Kerr BH super-radiance occurs [1] or brane theory super-radiance occurs [2]; if only Kerr BH super-radiance occurs, the likelihood of massive Gravitons is remote. If brane theory BH super-radiance occurs, then there may [2] be conditions for which the 3/4th error in light bending is removed, permitting massive Gravitons.

## 2. What is Super-Radiance in Black Hole Physics? First: The Padmanabhan Treatment for Kerr BHs

We, first of all, consider a simplified version of super-radiance. In simple language, super-radiance involves having incoming radiation scattered off the horizon of a BH and radiated outward, so the net flow of energy is $dE/dt < 0$ radiation energy with a frequency bounded by $0 < \omega < m \cdot \Omega_H$ [1]. In this case $m$ is a quantum number, the frequency $\omega$ is for radiation infalling to the event horizon of the BH, and the term $\Omega_H$ is the angular velocity of a KERR black hole [2]. This paper, first of all, examines Padmanabhan's derivation of super-radiance [2] stating its application to the graviton, with mass, and making then a referral to the likelihood of measurement which ties in with the metric $g_{\mu\nu}$ being perturbed from flat space values by $h_{00}$, $h_{0i}$, and $h_{ij}$ [7], thereby making the case, due to the mass dependence of the black hole, that super-radiance would almost certainly not be observable but would firmly embed massive gravitons in GR in spite of the view point offered in [3]. Doing so would mean that [1] has the following formulation; with respect to when $dE/dt < 0$, which occurs for super radiance; in (1) below; we set c1 as a constant, radiation frequency omega as the frequency of radiation approaching a black hole, the number $m$ as a quantum number, and a definite given value for the angular velocity of a black hole. Here, after the Padmanabhan derivation of what $dE/dt < 0$ means, there will be a separate, brane theory derivation of BH super-radiance [2] which has provisionally $0 < \omega < \sum_{J=1}^{N/2} m_J \cdot (\Omega_H)_J$ [2], where $N$ is the number of dimensions. The 2nd frequency dependence for when $N$ can go up to at least 10 or so will be remarked after we finish the Padmanabhan frequency dependence for super-radiance, as given below for a "classical" Kerr BH. To initiate our analysis of the physics happening in due to [1], we formulate super-radiance by (1) given below

$$\frac{dE}{dt} = c_1 \cdot \omega \cdot [\omega - m \cdot \Omega_H]. \tag{1}$$

In this case, according to Padmanabhan, $c_1$ is a constant, which is defined via writing (1) via [1]. Consider

$$\begin{aligned} \frac{dE}{dt} &= \frac{M \cdot r_H}{2\pi} \cdot \left[\int \left(S^2(\theta) \cdot \sin^2\theta \cdot d\theta\right) \cdot d\phi\right] \\ &\quad \cdot \omega \cdot [\omega - m \cdot \Omega_H] \\ &= \text{flux-of-energy-through-horizon} \\ &\Longleftrightarrow c_1 \equiv \frac{M \cdot r_H}{2\pi} \cdot \left[\int \left(S^2(\theta) \cdot \sin^2\theta \cdot d\theta\right) \cdot d\phi\right] = \text{const.}, \end{aligned} \tag{2}$$

where, for a massless scalar field, one has the function $S$ for a "surface area" function, defined as follows [1]:

$$\begin{aligned} \exists S^2(\theta) &\Longleftrightarrow (-g)^{-1/2} \partial_b \left[(-g)^{-1/2} \cdot g^{ab} \cdot \partial_a \Phi\right] = 0 \\ &\Longleftrightarrow \Phi \equiv e^{-i\omega t} e^{im\cdot\phi} \cdot R(r) \cdot S(\theta). \end{aligned} \tag{3}$$

In this case, mass $M$ is for the source, that is, later for the mass $M$ of a GW generator, in this case a BH. Also, here, $r_H$ is the horizon radius, as specified. And this will have its application to the issue of gravitons of a small mass spirialing into a BH, with the BH subsequently releasing radiation via $dE/dt < 0$, with the given versions of a BH set of parameters [1] for a KERR BH. The following (4) comes as far as angular velocity of the BH, as well as the following sets of parameters. Here, the phenomenon of super-radiance is impossible, if (6) below is zero. More on this point about when super-radiance is impossible will be discussed later in the text. Consider

$$\Omega_H = \frac{a}{2M_{\text{BH}} \cdot r_H}, \tag{4}$$

$$r_H = M_{\text{BH}} - \sqrt{{M_{\text{BH}}}^2 - a^2}, \tag{5}$$

$$a = \sqrt{x^2 + y^2}. \tag{6}$$

Then,

$$0 < \frac{\omega}{m} < \Omega_H. \tag{7}$$

Note that there are conditions, based upon (4) above, which go to zero, due to the numerator, in a manner which means when there is no angular velocity for the black hole; that the frequency of the incoming radiation is set equal to zero and that there is, effectively no super-radiance. This will obviously lead to the classical description of BH physics. A problem, though, is that, recently, Hawkings has stated that not all is well in BH event horizons and that scrambled information could possibly leave a BH, in opposition.

*2.1. Examining Super-Radiance When There Is More than 4 Dimensions as to BH Physics.* As said before [2],

$$0 < \omega < \sum_{J=1}^{N/2} m_J \cdot (\Omega_H)_J. \tag{8}$$

In doing so, $N$ as given above is a measure of dimensions as to the BH, and the difference in this from (7) in part is also due to

$$\begin{aligned} \Omega_H &= \left.\frac{a}{2M_{\text{BH}} \cdot r_H}\right|_{\text{Kerr-BH}} \\ &\xrightarrow[\text{Kerr-BH} \to \text{Myers-Perry-BH}]{} \left.\frac{a}{a^2 + r_H^2}\right|_{\text{Myers-Perry-BH}}. \end{aligned} \tag{9}$$

The numerator of the above is still defined by the square root of $x^2 + y^2$ and could go to zero for certain quantum numbers, $m_J$, and we would then paraphrase the right hand side of (9) as functionally being

$$\Omega_j = \frac{a_J}{a_J^2 + r_H^2} \tag{10}$$

as frequency of BH arises due to the jth component of BH angular Momentum $J_j$.

So, then, one has a rewrite of (8) as given, with a slightly different angular frequency for BHs as by [2],

$$0 < \omega < \sum_{j=1}^{N/2} m_j \left.\frac{a_j}{a_j^2 + r_H^2}\right|_{\text{Myers-Perry-BH}}. \tag{11}$$

This is to be compared with the Padmanabhan version of super-radiance as given by [1]:

$$0 < \omega < m \cdot \left.\frac{a}{2M_{\text{BH}} \cdot r_H}\right|_{\text{Kerr-BH}}. \tag{12}$$

We will be commenting upon what the experimental signatures of both (11) and (12) could be and why in the next section.

## 3. Could Super-Radiance Be Observed Experimentally, and What Good Is This Thought Experiment?

Super-radiance is really about the same as particle production from a BH. From Padmanabhan [1] is a vital result which is given in the following quote.

> *If we think of super-radiance as stimulated emission of radiation by the black hole in certain modes, owing the presnce [sic] of the incoming wave, it seems natural to expect spontaneous emission of radiation in various modes by the black hole in quantum field theory. The black hole evaporation (then) can be thought of as spontaneous emission of particles that survives even in the limit of zero angular momentum of the black hole.*

Furthermore, on the same page, page 623 of [1] states the following.

> *It seems natural to assume that this source of energy radiated to infinity is the mass of the collapsing structure.*

Leading to Formula 14.143 of [1] that the "entropy" of a BH is given by, where $M$ is the mass of the BH, $L_P$ is the Plamck length, and $A_{\text{hor}}$ is the area of the Event horizon of a black hole. This area of a BH event horizon is relevant since it directly connects, as we will mention later, to [2] version of super-radiance. Reference [1] version of entropy would also hold for [2] as well, and we state the entropy as

$$S = 4\pi M^2 = \frac{1}{4} \cdot \left(\frac{A_{\text{hor}}}{L_P^2}\right). \tag{13}$$

Here, in [2] we have (27), that its main result is about the differential of the area of an event horizon which is given as follows, if there is a brane theory connection to the formation of BHs:

$$dA_{\text{hor}} = \frac{8\pi r_H}{B} dM_{\text{BH}} \cdot \left(1 - \frac{1}{\omega}\sum_{j=1}^{N/2} m_j \cdot \Omega_j\right). \tag{14}$$

The positive definite nature of this expression for the differential of the area of an event horizon would then be [2] since $dM < 0$, then by [2], so then by (15) below, we recover (11), by [2]; if $dA > 0$, then

$$\begin{aligned} &dM_{\text{BH}} \cdot \left(1 - \frac{1}{\omega}\sum_{j=1}^{N/2} m_j \cdot \Omega_j\right) > 0 \\ &\iff \left(1 - \frac{1}{\omega}\sum_{j=1}^{N/2} m_j \cdot \Omega_j\right) < 0. \end{aligned} \tag{15}$$

We make the following 3 claims as for the analogy to BH physics.

*Claim 1.* Entropy in both Kerr and Myers-Perry BHs has $dA > 0$, where $A$ is the event horizon, and

(i) for Myers-Perry BHs, the following are true (dimensions up to 10, say, i.e., $N = 10$):

$$S = 4\pi M^2 = \frac{1}{4}\cdot\left(\frac{A_{\text{hor}}}{L_P^2}\right)$$
$$dA_{\text{hor}} = \frac{8\pi r_H}{B} dM_{\text{BH}}\cdot\left(1-\frac{1}{\omega}\sum_{j=1}^{N/2} m_j\cdot\Omega_j\right) \tag{16}$$
$$\left(1-\frac{1}{\omega}\sum_{j=1}^{N/2} m_j\cdot\Omega_j\right) < 0;$$

(ii) for Kerr BH, one could arguably have much the same thing; that is,

$$S = 4\pi M^2 = \frac{1}{4}\cdot\left(\frac{A_{\text{hor}}}{L_P^2}\right)$$
$$dA_{\text{hor}} = \frac{8\pi r_H}{B} dM_{\text{BH}}\cdot\left(1-\frac{1}{\omega} m\cdot\Omega_H\right) \tag{17}$$
$$\left(1-\frac{1}{\omega} m\cdot\Omega_H\right) < 0.$$

*Proof.* By (15), (9), (10), and (11), we next consider the following. □

*Claim 2.* If $a$ is zero, then super-radiance as made possible in Claim 1 part (ii) is impossible for Kerr Black holes.

*Proof.* $a$ goes to zero and mean numerator of (9) goes to zero. Hence, $(1-(1/\omega)m\cdot\Omega_H) < 0$ does not happen. Hence, for nonzero frequency of incoming radiation, $0 < \omega < m\cdot\Omega_H$ does not hold. Hence, there is no BH super-radiance. □

*Claim 3.* One could have the following: Claim 1 part (ii) may be false, but Claim 1 part (i) may be true.

*Proof.* For $N \geq 4$ or so, the following decomposition may be true:

$$\left(1-\frac{1}{\omega}\sum_{j=1}^{N/2} m_j\cdot\Omega_j\right) = \left[1-\frac{1}{\omega} m\cdot\Omega_H\right] - \frac{1}{\omega}\sum_{j=2}^{N/2} m_j\cdot\Omega_j < 0. \tag{18}$$

If the first term in [ ] in the RHS of the above formula is equal to 0, Claim 1 part (ii) is false, but one could still have Claim 1 part (i) as true. That is, one could write the following.

Consider $0 < \omega < \sum_{J=2}^{N/2} m_J\cdot(\Omega_H)_J$. Then, Claim 1 part (i) will be true, that is, super-radiance for brane theory BHs. □

The significance of the three claims is as follows. As given by [4], there is a problem, if a massive graviton exists, the bending of light, say about Mercury, the Eddinton 1919 experiment is calculated to be 3/4th the value seen in the 1919 experiment which proved classical GR. By [7], there can be a situation for which if there exists higher than 4 dimensional brane theory, one may correct the 3/4th deficiency. But if Claim 1 part (i) is not true, then the solution allowing for [7] is likely not to be true.

Note that the super-radiance phenomenon as referenced in Claim 1 part (i) and part (ii) has its roots in entropy. Note that entropy of a black hole with its surface area is stated to be a precondition for initial conditions for super-radiance. And, more than that, one needs a spinning black hole. No black hole spin, with a commensurate treatment, could lead to just black hole evaporation, as noted above, but BH evaporation is not the same as the super-radiance phenomenon.

*3.1. Minimum Experimental Bounds Which Can Affect the Results of our Inquiry, Provided That Claim 1 Is True (as well as That Claim 3 Holds). That Is, Myers-Perry as a Higher Representation of Black Holes. Presumably Allowing Massive Gravitons.* IMO, as stated above, the Meyers-Perry condition for BHs is, as a gateway, a probable candidate to experimental observations for BHs. As mentioned earlier, for higher dimensional BHs which may allow for massive gravitons, here are the perturbations due to GW due to higher dimensional black holes. We state these as follows.

The subsequent values by $h_{00}$, $h_{0i}$, and $h_{ij}$ make the case, due to mass dependence of the black holes in the Myers-Perry black holes, has an explicit mass dependence on the mass of the black hole included. [4] has

$$h_{00} \approx \frac{16\pi G}{(d-2)\cdot\Omega_{d-2}}\cdot\frac{M_{\text{BH}}}{r^{d-3}},$$
$$h_{ij} \approx \frac{16\pi G}{(d-2)\cdot(d-3)\cdot\Omega_{d-2}}\cdot\frac{M_{\text{BH}}}{r^{d-3}}\cdot\delta_{ij}, \tag{19}$$
$$h_{oi} \approx -\frac{8\pi G}{\Omega_{d-2}}\cdot\frac{x^k}{r^{d-1}}\cdot J^{ki}.$$

The coefficient $d$ is for dimensions, 4 or above, and in this situation, with angular momentum $J^{ki}$. Here, the term put in, namely (20) is for angular area, and it has no relationship with the formula for angular velocity of BHs; namely, (20) has no relations with (4) and (9) above.

$$\Omega_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}, \tag{20}$$

$$J^{ki} = 2\cdot\int x^k\cdot T^{i0}\cdot d^{d-1}x. \tag{21}$$

The $T^{i0}$ above is a stress energy tensor as part of a $d$ dimensional Einstein equation given in [5] as

$$R_{jl} - \frac{1}{2}\cdot g_{jl}\cdot R = 8\pi G T^{jl}. \tag{22}$$

Also, the mass of the black hole is, in this situation scaled as follows: if $\mu$ is a rescaled mass term [5],

$$M_{\text{BH}} = \mu\cdot\Omega_{d-2}\cdot\frac{(d-2)}{16\pi G}. \tag{23}$$

More generally, the mass of the black hole is written as

$$M_{\mathrm{BH}} \equiv \int T_{00} d^{d-1}x. \tag{24}$$

We will next go to the minimum size of a black hole which would survive as up to 13.6 billion years and then say something about the relative magnitude of the terms in (22) and then their survival today. The variance of black hole masses, from super massive BHs to those smaller than $10^{15}$ grams will be discussed, in the context of (19), and stress strength, with commentary as to what we referred to earlier, namely, strain for detecting GW, is given by $h(t)$ given below, with $D^{ij}$ as the detector tensor, that is, a constant term, so that, by [4, page 336], we write

$$h(t) = D^{ij} h_{ij}. \tag{25}$$

Equation (25) means that the magnitude of strain, $h$, is effected by (19), (20), and (21) and its magnitude, seen next. Note that the magnitude of the strain, $h$, as being brought up, may be affected by the mass of a graviton, due to $T$, which is a feed into (19) above. Namely, consider that the mass assumed for the graviton is of the order of $10^{-65}$ grams, which is given by [5]; if $h$ does not equal zero, then the stress energy tensor of the massive graviton is for nonzero $T_{uv}$ which corresponds to a nonzero concentration in interstellar space, with [5]

$$\begin{aligned} m_g^2 &= -\frac{\kappa}{6h} T \\ T &= \text{trace } T_{uv}. \end{aligned} \tag{26}$$

We will get explicit upper bounds to (26) and use them as commentary in the conclusion of this paper. That will affect the infalling frequency $\omega$ which will be part of the super-radiance discussion.

*3.2. Values of the Meyers-Perry $h_{00}$, $h_{0i}$, and $h_{ij}$ in Magnitude Lead to Nominal $h$ Values.* If $D$ below is redshift corrected distance, in a rough sense, it leads to an approximation of $h$ as roughly proportional to $h_{00}$ with the roughly scaled results of

$$h \sim \frac{GM}{c^2 D}. \tag{27}$$

Note that the tensor $D^{ij}$ is approximately unity, with the results as given by

$$\begin{aligned} M_{\mathrm{BH}}\big|_{\text{min-life.time}} \propto 10^{15} \text{ grams} \Longleftrightarrow h_{00}, h_{ii} \propto 10^{-40} \\ \text{for BHs; } Z \text{ (redshift)} \sim 10. \end{aligned} \tag{28}$$

whereas super massive black holes of about 100 times the mass of our sun, at Z(redshift) of about 10 lead to significantly larger values of $h_{ii}$ as seen below,

$$M_{\mathrm{BH}}\big|_{\text{100-solar-mass}} \Longleftrightarrow h_{00}, h_{ii} \propto 10^{-20}. \tag{29}$$

It is easy from inspection to infer from this that the most early formed black holes would not be accessible and that only the giant ones would do. With that, we next then explore the frequency ranges which could lead to certain Graviton masses, as could be linked to super-radiance. That is, it would mean that a very large SMBH, of about 100 solar masses of a redshift of the order of $Z \sim 10$ at or less than a billion years after the creation of the universe, would lead to the values of (29) above, which could be conceivably detected, which then leads us to the question of what frequencies of the graviton, if presumably massive, would be involved. This would then allow us to make inquiry as to what the Meyers-Perry values for super-radiance and absorption/subsequent reflection of GW radiation which could conceivably be detected for strain values of the order given by (29) above.

*3.3. Frequency and Wavelengths for Ultra Low "Massive Graviton" Masses.* To get the appropriate estimates, we turn to [11], by Goldhaber and Nieta, which can be used to give a set of frequency and mass equivalences for the "massive" graviton; on the order of having the following equivalent values as paired together, namely, starting off, with graviton mass, graviton wavelength, and resulting graviton frequency, we observe the dual pairing of the following, if one also looks at Valev's estimates [12],

$$\begin{aligned} m_g &\sim 2 \times 10^{-65} \text{ grams} \lambda_g \\ &\sim 2 \times 10^{22} \text{ meters} \\ &\sim 10^{-4} \cdot \text{radius-of-universe} \omega_g \\ &\sim \left(\frac{3}{2}\right) \times 10^{-14}/\text{second}. \end{aligned} \tag{30}$$

Obviously, with regard to this, if such an extremely low value for resultant frequency is obtained, and then one is obtaining the value that is inevitable, just in terms of frequency, to have for any spinning Kerr BH,

$$\omega_g \sim \left(\frac{3}{2}\right) \times 10^{-14}/\text{second} < \Omega_H. \tag{31}$$

If we evaluate further for any reasonable value of $a$, we will find that, for a SMBH of about 100 solar masses, one will still have, realistically, $\omega_g \sim (3/2) \times 10^{-14}/\text{second} \ll \Omega_H$. The author has found that for supermassive black holes for masses up to a million times the mass of the sun, the freauency the becomes, $\omega_g \sim (3/2) \times 10^{-14}/\text{second} \leq \Omega_H$ which leads to Claim 4.

*Claim 4.* For super-radiance (Kerr style), $\omega_g \sim (3/2) \times 10^{-14}/\text{second} \ll \Omega_H$ (easy super-radiance), for SMBH 100 times solar mass and $\omega_g \sim (3/2) \times 10^{-14}/\text{second} \leq \Omega_H$ (problematic super-radiance), for SMBH $10^6$ times solar mass.

The proof is in the definition of $\Omega_H = (a/(2M_{\mathrm{BH}} \cdot r_H))|_{\text{Kerr-BH}}$, with a very small numerator.

Claim 4 means that, before the formation of massive spiral galaxies, the super-radiance is doable. However, the author fails to understand how it is possible on another

theoretical ground; that is, what does super-radiance mean for BHs for which

$$\begin{aligned}\lambda_g &\sim 2\times 10^{22}\ \text{meters}\\ &\sim 10^{-4}\cdot \text{radius-of-universe}.\end{aligned} \tag{32}$$

On the face of it, this is absurd. That is, how could wavelengths 1/10,000 the size of the universe interact with a Kerr Black hole?

We claim that the embedding of black holes in five or higher dimensional space time is a way to make a connection with a multiverse, as given in the following supposition [13], and that this may be the only way to reconcile what seems to be an absurd proposition. That is, graviton wavelengths 1/10,000 the size of the standard 4 dimensional universe is interacting with spinning black holes in 4 dimensional space-time, whereas that moderate 100 times the mass of the sun BHs easily satisfy $\omega_g \sim (3/2)\times 10^{-14}/\text{second} \ll \Omega_H$.

## 4. Conclusion

In one way, it is ridiculously easy to obtain super-radiance for massive gravitons, and, in another sense, it is an absurd proposition. Could a multiverse embedding of BHs be a way out of what otherwise seems an impossible Dichotomy? Or will we have to embrace Hawkings' suggestion that the event Horizon has foundationally crippling flaws?

To address this problem, the author looks at two suggestions. Either that the BH is really embedded in a multiverse and has a different geometry in higher dimensions than is supposed, or one goes to the recent Hawkings' hypothesis which changes entirely the supposition of the event horizon. Namely

We will first of all give a brief introduction to the Penrose CCC hypothesis generalized to a multiverse.

*4.1. Extending Penrose's Suggestion of Cyclic Universes, Black Hole Evaporation, and the Embedding Structure our Universe Is Contained within, That Is, Using the Implications of* (32) *for a Multiverse. This Multiverse Embeds BHs and May Resolve What Appears to Be an Impossible Dichotomy.* There are no fewer than $N$ universes undergoing Penrose "infinite expansion" (Penrose, 2006) [13] contained in a mega universe structure. Furthermore, each of the $N$ universes has black hole evaporation, with the Hawkings radiation from decaying black holes. If each of the $N$ universes is defined by a partition function, called $\{\Xi_i\}_{i\equiv N}^{i\equiv 1}$, then there exists information ensemble of mixed minimum information correlated as about $10^7$-$10^8$ bits of information per partition function in the set $\{\Xi_i\}_{i\equiv N}^{i\equiv 1}\big|_{\text{before}}$, so minimum information is conserved between a set of partition functions per universe. Consider

$$\{\Xi_i\}_{i\equiv N}^{i\equiv 1}\Big|_{\text{before}} \equiv \{\Xi_i\}_{i\equiv N}^{i\equiv 1}\Big|_{\text{after}}. \tag{33}$$

However, there is nonuniqueness of information put into each partition function $\{\Xi_i\}_{i\equiv N}^{i\equiv 1}$. Furthermore, Hawkings' radiation from the black holes is collated via a strange attractor collection in the mega universe structure to form a new big bang for each of the $N$ universes represented by $\{\Xi_i\}_{i\equiv N}^{i\equiv 1}$. Verification of this mega structure compression and expansion of information with nonuniqueness of information placed in each of the $N$ universes favors ergodic mixing treatments of initial values for each of $N$ universes expanding from a singularity beginning. The $n_f$ value, will be using (Ng, 2008) $S_{\text{entropy}} \sim n_f$. [14]. How to tie in this energy expression, as in (33), will be to look at the formation of a nontrivial gravitational measure as a new big bang for each of the $N$ universes as by $n(E_i)$. The density of states at a given energy $E_i$ for a partition function (Poplawski, 2011) [15]. Consider

$$\{\Xi_i\}_{i\equiv 1}^{i\equiv N} \propto \left\{\int_0^\infty dE_i\cdot n(E_i)\cdot e^{-E_i}\right\}_{i\equiv 1}^{i\equiv N}. \tag{34}$$

Each of $E_i$, identified with (34) above, is with the iteration for $N$ universes (Ng, 2008) [14]. Then the following claim holds.

*Claim 5.* Consider

$$\begin{aligned}&\frac{1}{N}\cdot\sum_{j=1}^{N}\Xi_j\Big|_{j\text{-before-nucleation-regime}}\\ &\xrightarrow[\text{vacuum-nucleation-tranfer}]{}\ \Xi_i\Big|_{i\text{-fixed-after-nucleation-regime}}.\end{aligned} \tag{35}$$

For $N$ number of universes, with each $\Xi_j\big|_{j\text{-before-nucleation-regime}}$ for $j = 1$ to $N$ being the partition function of each universe just before the blend into the RHS of (35) above for our present universe. Also, each of the independent universes given by $\Xi_j\big|_{j\text{-before-nucleation-regime}}$ are constructed by the absorption of one to ten million black holes taking in energy. That is,(Ng, 2008) [14]. Furthermore, the main point is similar to what was done in [16] in terms of general ergodic mixing.

*Claim 6.* Consider

$$\Xi_j\Big|_{j\text{-before-nucleation-regime}} \approx \sum_{k=1}^{\text{Max}}\widetilde{\Xi}_k\Big|_{\text{black-holes-}j\text{th-universe}}. \tag{36}$$

What is done in Claims 5 and 6 is to come up with a protocol as to how a multidimensional representation of black hole physics enables continual mixing of space and time [17] largely as a way to avoid the Anthropic principle, as to a preferred set of initial conditions. With investigations, this complex multiverse may allow bridging what seems to be an unworkable dichotomy between ultralow graviton frequency, corresponding roughly to $10^{-65}$ grams in rest mass, easily satisfied by Kerr black holes with rotational frequencies, as given in our text as many times greater, combined with the absurdity of what (32) is. How can a graviton with a wavelength $10^{-4}$ the size of the universe interact with a Kere black hole, spatially? Embedding the BH in a multiverse setting may be the only way out.

Claim 5 is particularly important. The idea here is to use what is known as CCC cosmology, which can be thought of as follows.

First. Have a big bang (initial expansion) for the universe. After redshift $z = 10$, a billion years ago, SMBH formation starts. Matter-energy is vacuumed up by the SMBHs, which at a much later date than today (present era) gather up all the matter-energy of the universe and recycle it in a cyclic conformal translation as follows:

$$E = 8\pi \cdot T + \Lambda \cdot g \tag{37}$$

$E$ = source for gravitational field

$T$ = mass energy density

$g$ = gravitational metric

$\Lambda$ = vacuum energy, rescaled as follows

$$\Lambda = c_1 \cdot [\text{Temp}]^{\beta}, \tag{38}$$

where $c_1$ is a constant. Then

The main methodology in the Penrose proposal has been in (38) evaluating a change in the metric $g_{ab}$ by a conformal mapping $\widehat{\Omega}$ to

$$\widehat{g}_{ab} = \widehat{\Omega}^2 g_{ab}. \tag{39}$$

Penrose's suggestion has been to utilize the following [18]

$$\widehat{\Omega} \xrightarrow[\text{ccc}]{} \widehat{\Omega}^{-1}. \tag{40}$$

Infall into cosmic black hopes has been the main mechanism which the author asserts would be useful for the recycling apparent in (40) above with the caveat that $\hbar$ is kept constant from cycle to cycle as represented by

$$\hbar_{\text{old-cosmology-cycle}} = \hbar_{\text{present-cosmology-cycle}}. \tag{41}$$

Equation (40) is to be generalized, as given by a weighing averaging as given by (35) where the averaging is collated over perhaps thousands of universes, call that number $N$, with an ergodic mixing of all these universes, with the ergodic mixing represented by (35) to generalize (40) from cycle to cycle.

*4.2. Conclusion, Future Prospects.* If this does not work, and the multiverse suggestion is unworkable, there, then, has to be a consideration of the zero option, namely, Hawkings throwing out the event horizon as we know it in BH physics. See this reference, namely, [16].

We are in for interesting times. I see turbulence and interesting results ahead.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Magnetic String with a Nonlinear $U(1)$ Source

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Considering the Einstein gravity in the presence of Born-Infeld type electromagnetic fields, we introduce a class of 4-dimensional static horizonless solutions which produce longitudinal magnetic fields. Although these solutions do not have any curvature singularity and horizon, there exists a conic singularity. We investigate the effects of nonlinear electromagnetic fields on the properties of the solutions and find that the asymptotic behavior of the solutions is adS. Next, we generalize the static metric to the case of rotating solutions and find that the value of the electric charge depends on the rotation parameter. Furthermore, conserved quantities will be calculated through the use of the counterterm method. Finally, we extend four-dimensional magnetic solutions to higher dimensional solutions. We present higher dimensional rotating magnetic branes with maximum rotation parameters and obtain their conserved quantities.

## 1. Introduction

One of the interesting topological defects is cosmic string which may be originated during the early universe phase transitions [1] (see Kibble mechanism for more details [2]). Furthermore, considering the inflationary models [3, 4], it has been proposed that cosmic strings can form at the end of inflation. Moreover, one of the predictions of supersymmetric hybrid inflation [5] (and also grand unified models of inflation [6]) is the cosmic string. Interesting properties and interaction of the superconducting cosmic string with astrophysical magnetic fields have been found in [7–9]. Besides, magnetic strings have been studied in Brans-Dicke theory as well as dilaton gravity [10–13]. From cosmological point of view, one can find the properties of the magnetic (cosmic) string in various literatures [14–16].

In addition to cosmic strings, other kinds of strings may be considered in QCD and also gravity. Properties of the QCD static strings have been investigated extensively in [17–20] and it has been shown that QCD magnetic string can contribute to hadron dynamics [21]. Applications of magnetic string in quantum theories have been presented in [22–24]. Magnetic strings in antiferromagnetic crystals have been investigated in [25]. Application of the (chromo)magnetic string model to some experimental data on the inclusive pion asymmetries has been studied in [26, 27]. Some arguments about the magnetic strings in the Yang-Mills plasma have been found in [28].

On gravitational aspect, the horizonless solutions and spacetime with conical singularity have been investigated in gravitating electromagnetic field background (see [7, 8, 29–49] and references therein). Interesting properties of the magnetic string in branes, M-theory, and string theory have been investigated [50, 51]. Calculations of the vacuum energy of two different fields in the background of a magnetic string have been analyzed in [52, 53].

One of the generalizations of the Einstein-Maxwell field equations is gravitating nonlinear electrodynamics (NLED), whose most popular theory is Born-Infeld [54–59]. In addition to Lorentz and $U(1)$ gauge invariances, we know that the Lagrangian of the Maxwell electrodynamics contains only quadratic forms of gauge potential and its first derivative. One can consider both invariances and leave out the third condition to obtain NLED [60]. From historical point of view, NLED were introduced to eliminate infinite quantities in theoretical analysis of charged point-like particles [54–59]. Recently, we have more motivations for considering NLED theories, for example, various limitations of the linear

electrodynamics [61, 62], clarification of the self-interaction of virtual electron-positron pairs [63–65], explanation of electrodynamics on D-branes [66–68], and description of radiation propagation inside specific materials [69–72]. In addition, from astrophysical viewpoint, we know that the effects of NLED become indeed quite important in super-strongly magnetized compact objects, such as pulsars, and particular neutron stars (some examples include the so-called magnetars and strange quark magnetars) [73–75]. Moreover, NLED modifies in a fundamental basis the concept of gravitational redshift and its dependency on any background magnetic field as compared to the well-established method introduced by standard general relativity. Furthermore, it has been recently shown that NLED objects can remove both of the big bang and black hole singularities [76–78].

Amongst the nonlinear generalization of Maxwell electrodynamics, the so-called BI type NLED, whose first nonlinear correction is quadratic function of Maxwell invariant, is completely special. It has been shown that BI type NLED may be arisen as a low energy limit of heterotic string theory [66, 68, 79–83], which led to an increased interest for BI type NLED theories. In addition, BI type theories have some interesting properties; for example, these theories enjoy the birefringence phenomena, free of the shock waves [84, 85] and electric-magnetic duality [86]. Furthermore, considering the relation between AdS/CFT correspondence and superconductivity phenomenon, it was shown that the BI type theories make a crucial effect on the condensation, the critical temperature, and energy gap of the superconductors [87].

In this paper, we investigate the horizonless magnetic strings in the presence of two kinds of the BI type NLED [88, 89]. One of the elemental motivations for analyzing the horizonless string solutions is that they may be interpreted as cosmic strings.

## 2. Basic Field Equations

Our goal in this work is to construct a class of four-dimensional solutions to the Einstein equations with negative cosmological constant in the presence of nonlinear electromagnetic source, $L(\mathcal{F})$, which describes a magnetic string. The Euler-Lagrange equations of motion for the metric $g_{\mu\nu}$ and the gauge potential $A_\mu$ may be written as [90]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = T_{\mu\nu}, \tag{1}$$

$$\partial_\mu \left( \sqrt{-g} L_{\mathcal{F}} F^{\mu\nu} \right) = 0, \tag{2}$$

where $\Lambda = -3/l^2$, $L_{\mathcal{F}} = dL(\mathcal{F})/d\mathcal{F}$, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ denotes the Maxwell invariant, and the energy-momentum tensor is given by

$$T_{\mu\nu} = \left( \frac{1}{2} g_{\mu\nu} L(\mathcal{F}) - 2L_{\mathcal{F}} F_{\mu\lambda} F^{\lambda}_{\nu} \right). \tag{3}$$

It is notable that these field equations can be obtained from variation of the following action:

$$\begin{aligned} I_G = &-\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ R - 2\Lambda + L(\mathcal{F}) \right] \\ &- \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma} \Theta(\gamma), \end{aligned} \tag{4}$$

where the bulk action (first term) is supplemented with a Gibbons-Hawking surface term (second term) whose variation will cancel the extra normal derivative term in deriving the equation of motion. The quantities $\Theta$ and $\gamma$ denote the trace of the extrinsic curvature and the induced metric for the boundary $\partial\mathcal{M}$, respectively.

In this work, we take into account the recently proposed BI type models of NLED [88, 89]. They have been nominated the *Exponential form of Nonlinear Electromagnetic Field* (ENEF) and the *Logarithmic form of Nonlinear Electromagnetic Field* (LNEF), in which their Lagrangians are

$$L(\mathcal{F}) = \begin{cases} \beta^2 \left( \exp\left( -\dfrac{\mathcal{F}}{\beta^2} \right) - 1 \right), & \text{ENEF}, \\ -8\beta^2 \ln\left( 1 + \dfrac{\mathcal{F}}{8\beta^2} \right), & \text{LNEF}. \end{cases} \tag{5}$$

Here, we want to obtain magnetic solutions. It is well known that the electric field comes from the time component of the vector potential ($A_t$), while the magnetic field is associated with the angular component ($A_\phi$). Hence one expects that a magnetic solution may be written in a metric gauge in which the components $g_{tt}$ and $g_{\phi\phi}$ interchange their roles relatively to that present in the Schwarzschild gauge used to describe electric solution. Therefore, we start with a class of the four-dimensional metrics which produces longitudinal magnetic fields along the $z$ direction [48]:

$$ds^2 = -\frac{\rho^2}{l^2} dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\varphi^2 + \frac{\rho^2}{l^2} dz^2, \tag{6}$$

where $f(\rho)$ is an arbitrary function of coordinate $\rho$. It is notable that this metric may be obtained from the horizon flat Schwarzschild-like metric:

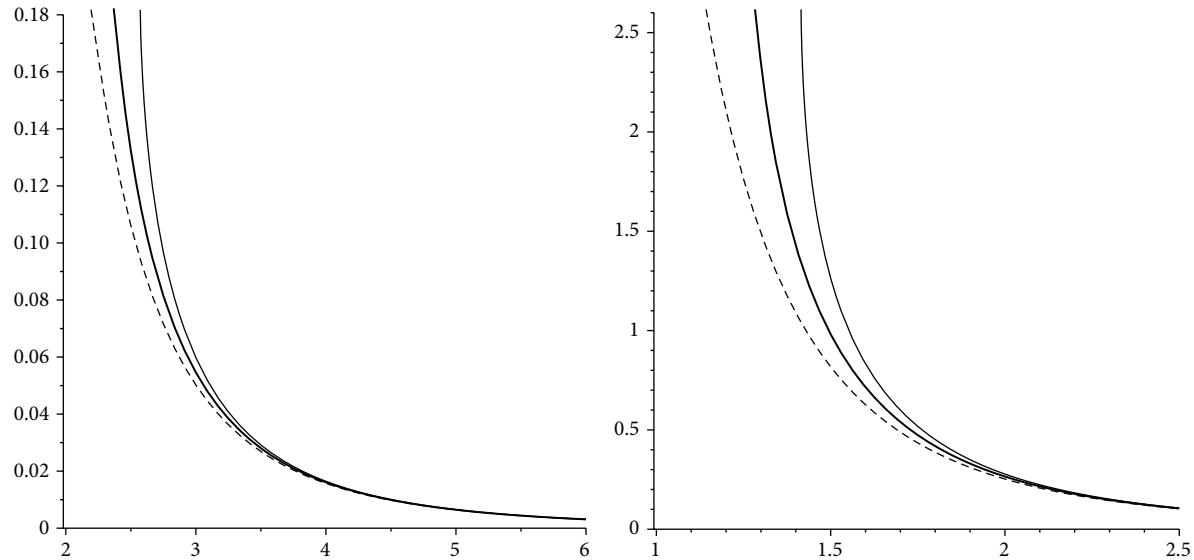
$$ds^2 = -f(\rho) dt^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 d\varphi^2 + \frac{\rho^2}{l^2} dz^2, \tag{7}$$

with the following *local* transformation:

$$t \longrightarrow il\varphi, \quad \varphi \longrightarrow \frac{it}{l}. \tag{8}$$

Since the mentioned transformation is not a global mapping and metric (7) can be locally mapped to metric (6), one can find that both (6) and (7) do not describe a unique spacetime. Using the nonlinear Maxwell equation (2) with the metric (6), one can obtain

$$\begin{aligned} &\left[ 1 - \left( \frac{2F_{\phi\rho}}{\beta l} \right)^2 \right] F'_{\phi\rho} + \frac{2F_{\phi\rho}}{\rho} = 0, \quad \text{ENEF}, \\ &\left[ 1 - \left( \frac{F_{\phi\rho}}{2\beta l} \right)^2 \right] F'_{\phi\rho} + \left[ 4 + \left( \frac{F_{\phi\rho}}{\beta l} \right)^2 \right] \frac{F_{\phi\rho}}{2\rho} = 0, \quad \text{LNEF} \end{aligned} \tag{9}$$




(a) ENEF (b) LNEF

Figure 1: $T_{\widehat{tt}}$ versus $\rho$ for $l = 1$, $q = 1$, and $\beta = 1$ (solid line), $\beta = 1.3$ (bold line), and $\beta = 3$ (dashed line). "ENEF branch (a) and LNEF branch (b)."

with the following solutions:

$$F_{\phi\rho} = \begin{cases} \dfrac{2ql^2}{\rho^2}\exp\left(-\dfrac{1}{2}L_W\right), & \text{ENEF}, \\ \dfrac{\beta^2\rho^2}{q}\left(1-\Gamma\right), & \text{LNEF}, \end{cases} \tag{10}$$

where the prime denotes differentiation with respect to $\rho$, the parameter $q$ is an integration constant, $L_W = \text{Lambert}\,W(-16l^2q^2/\beta^2\rho^4)$ which satisfies $\text{Lambert}\,W(x)\exp[\text{Lambert}\,W(x)] = x$ [91, 92], and $\Gamma = \sqrt{1-(2lq/\beta\rho^2)^2}$. It is worthwhile to note that in order to have a real electromagnetic field, we should consider $\rho > \rho_0$, where

$$\rho_0 = \sqrt{\frac{2lq}{\beta}} \times \begin{cases} \sqrt{2}\exp\left(\dfrac{1}{4}\right), & \text{ENEF}, \\ 1, & \text{LNEF}. \end{cases} \tag{11}$$

Here, we use the orthonormal contravariant (hatted) basis vectors to study the effect of nonlinearity on the energy density. Considering the mentioned diagonal metric in this basis, one should apply $\mathbf{e}_{\hat{t}} = (l/\rho)(\partial/\partial t)$ and therefore the $\widehat{tt}$ component of the stress-energy tensor is

$$T_{\widehat{tt}} = \begin{cases} \dfrac{\beta^2}{2}\left[1-\exp\left(-\dfrac{2F_{\phi\rho}^2}{l^2\beta^2}\right)\right], & \text{ENEF}, \\ \beta^2\ln\left(1+\dfrac{F_{\phi\rho}^2}{4l^2\beta^2}\right), & \text{LNEF}. \end{cases} \tag{12}$$

We plot $T_{\widehat{tt}}$ versus $\rho > \rho_0$ in Figure 1 and find that, for a fixed value of $\rho$, as nonlinearity parameter increases, the energy density of the spacetime decreases and therefore, in order to reduce the concentration volume of the energy density, we should increase the nonlinearity parameter.

Now, we should obtain the metric function $f(\rho)$. One can take into account (6) and (10) in the gravitational field equation (1) to obtain its nonzero components as

$$\begin{aligned} &\left(\Psi(\rho)+\rho\beta^2\right)\exp\left(-\frac{1}{2}L_W\right)-\rho\beta^2\left[1+H^2(\rho)\right] = 0, \quad \text{ENEF}, \\ &\Psi(\rho)+8\rho\beta^2\ln\left[1+\left(\frac{\beta\rho^2(1-\Gamma)}{2ql}\right)^2\right]-J(\rho) = 0, \quad \text{LNEF}, \end{aligned} \tag{13}$$

where $\Psi(\rho) = 2f'(\rho) + g(\rho) - (6\rho/l^2)$ and

$$g(\rho) = \begin{cases} \rho f''(\rho), & tt\,(zz)\text{ component}, \\ \dfrac{2f(\rho)}{\rho}, & \rho\rho\,(\varphi\varphi)\text{ component}, \end{cases}$$

$$H(\rho) = \begin{cases} 0, & tt\,(zz)\text{ component}, \\ \dfrac{4ql}{\beta\rho^2}\exp\left(-\dfrac{1}{2}L_W\right), & \rho\rho\,(\varphi\varphi)\text{ component}, \end{cases}$$

$$J(\rho) = \begin{cases} 0, & tt\,(zz)\text{ component} \\ \dfrac{4\rho^5\beta^4(1-\Gamma)^2}{l^2q^2\left[1+(\beta\rho^2(1-\Gamma)/2ql)^2\right]}, & \rho\rho\,(\varphi\varphi)\text{ component}. \end{cases} \tag{14}$$

After some calculations one can show that these equations have the following solutions:

$$f(\rho) = \frac{\rho^2}{l^2} + \frac{2ml^3}{\rho} - \frac{\chi\beta^2\rho^2}{6} - \frac{2\beta ql}{\rho}\Upsilon(\rho), \tag{15}$$

where $m$ is the integration constant which is related to mass parameter, $\chi$ is equal to 1 and $-16$ for ENEF and LNEF branches, respectively, and

$$\Upsilon(\rho) = \begin{cases} \int (-L_W)^{-1/2} [1 - L_W]\, d\rho, & \text{ENEF}, \\ \dfrac{2\beta}{ql} \int \left[ \rho^2 \ln\left( \dfrac{\beta^2 (1-\Gamma) \rho^4}{2l^2 q^2} \right) + \dfrac{4q^2 l^2}{\beta^2 \rho^2 (1-\Gamma)} \right] d\rho, & \text{LNEF}, \end{cases} \tag{16}$$

where one may calculate these integrations. We should note that the obtained solutions are the same as asymptotically anti-de Sitter magnetic solution of Einstein-Maxwell gravity [49], asymptotically (large values of radial coordinate). In addition, one may expect to recover the solution of [49] for $\beta \to \infty$.

Taking into account the metric (6), it is clearly desirable to have an examination on the geometric structure of the solutions. The first step is investigation of the spacetime curvature. It is easy to show that the Kretschmann scalar is

$$R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} = \left( \frac{d^2 f(\rho)}{d\rho^2} \right)^2 + 4\left( \frac{1}{\rho} \frac{df(\rho)}{d\rho} \right)^2 + 4\left( \frac{f(\rho)}{\rho^2} \right)^2. \tag{17}$$

Numerical calculations show that the Kretschmann is finite for nonzero $\rho$. Furthermore, we can show that

$$\lim_{\rho \to 0} R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} = \frac{48 M^2 l^6}{\rho^6} + \frac{A(M, q, \beta, l)}{\rho^5} + O\left( \frac{1}{\rho^4} \right), \tag{18}$$

$$\lim_{\rho \to \infty} R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} = \frac{8\Lambda^2}{3} + \frac{48 M^2 l^6}{\rho^6} - \frac{384 M l^5 q^2}{\rho^7} + \frac{B(M, q, \beta, l)}{\rho^8} + O\left( \frac{1}{\rho^{11}} \right), \tag{19}$$

where $A$ and $B$ are different functions of metric parameters $M$, $q$, $l$, and $\beta$. Equation (19) confirms that the asymptotic behavior of the solutions is adS. In addition, one may take into account (18) to think about the existence of a curvature singularity located at $\rho = 0$ and therefore conclude that there are magnetically charged black hole solutions. Since $\rho > \rho_0$, one concludes that, for charged solutions with finite $\beta$, the spacetime never achieves $\rho = 0$. In addition, we should obtain the zeroes of the function $f(\rho) = (g_{\rho\rho})^{-1}$. Considering the largest positive real root of $f(\rho) = 0$ by $r_0$ (suppose $r_0 > \rho_0$; for $r_0 < \rho_0$, the metric function $f(\rho)$ is positive definite which we are not interested in), one can find that the function $f(\rho)$ is negative for $\rho < r_0$. We should note that $g_{\rho\rho}$ and $g_{\phi\phi}$ are related by $f(\rho) = g_{\rho\rho}^{-1} = l^{-2} g_{\phi\phi}$, and therefore negativity of $g_{\rho\rho}$ (which occurs for $\rho < r_0$) leads to negativity of $g_{\phi\phi}$ and hence the signature of the metric changes from +2 to −2. This indicates that we could not extend the spacetime from $\rho > r_0$ to $\rho < r_0$. In order to get rid of this incorrect extension, one may introduce a new radial coordinate $r$ in the following form:

$$r^2 = \rho^2 - r_0^2, \qquad d\rho^2 = \frac{r^2}{r^2 + r_0^2} dr^2. \tag{20}$$

Considering this suitable coordinate transformation, the electromagnetic field can be written as

$$F_{\phi r} = \begin{cases} \dfrac{2ql^2}{r^2 + r_0^2} \exp\left( -\dfrac{1}{2} L'_W \right), & \text{ENEF}, \\ \dfrac{\beta^2 (r^2 + r_0^2)}{q} (1 - \Gamma'), & \text{LNEF}, \end{cases} \tag{21}$$

where $L'_W = \text{Lambert}\, W(-16 l^2 q^2 / \beta^2 (r^2 + r_0^2)^2)$ and $\Gamma' = \sqrt{1 - (2lq/\beta(r^2 + r_0^2))^2}$. Moreover, the metric (6) in the new coordinate is

$$ds^2 = -\frac{r^2 + r_0^2}{l^2} dt^2 + l^2 f(r)\, d\phi^2 + \frac{r^2}{(r^2 + r_0^2)} \frac{dr^2}{f(r)} + \frac{r^2 + r_0^2}{l^2} dz^2, \tag{22}$$

with $0 \le r < \infty$, and $f(r)$ is now given as

$$f(r) = \frac{r^2 + r_0^2}{l^2} + \frac{2ml^3}{\sqrt{r^2 + r_0^2}} - \frac{\chi \beta^2 (r^2 + r_0^2)}{6} - \frac{2\beta q l}{\sqrt{r^2 + r_0^2}} \Upsilon(r), \tag{23}$$

where

$$\Upsilon(r) = \begin{cases} \displaystyle\int \frac{[1 - L'_W]\, r}{\sqrt{-(r^2 + r_0^2) L'_W}} dr, & \text{ENEF}, \\ \displaystyle\frac{2\beta}{ql} \int \left( \frac{\ln\left( \left( \beta^2 (1 - \Gamma')(r^2 + r_0^2)^2 \right) / 2l^2 q^2 \right)}{(r^2 + r_0^2)^{-1/2}} + \frac{4q^2 l^2}{\beta^2 (1 - \Gamma')(r^2 + r_0^2)^{3/2}} \right) r\, dr & \text{LNEF}. \end{cases} \tag{24}$$

Numerical calculations show that not only Kretschmann scalar but also other curvature invariants are finite in the range $0 \le r < \infty$ ($r_0 \le \rho < \infty$) and therefore the mentioned spacetime has no curvature singularity and no horizon. It is notable that the above-mentioned magnetic solutions differ from the electric solutions and the properties

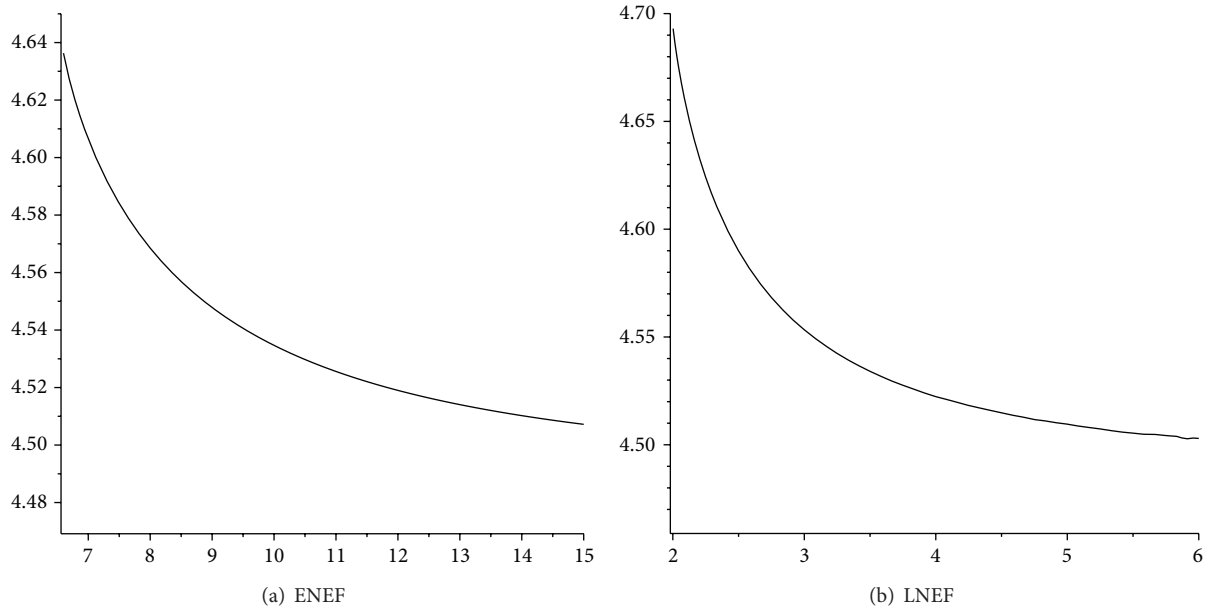



Figure 2: The deficit angle versus $\beta$ for $r_0 = 1$, $l = 1$, and $q = 1$. "ENEF branch (a) and LNEF branch (b)."

of electric and magnetic solutions are distinct. For example, the electric solutions lead to black objects interpretation, while the magnetic solutions do not.

In spite of the fact that the obtained magnetic solutions have no essential singularity, one can show that $\lim_{r\to 0}(1/r)\sqrt{g_{\phi\phi}/g_{rr}} \neq 1$ and so, when $r$ goes to zero, the limit of the ratio "*circumference/radius*" is not $2\pi$. This indicates that there is a conic singularity located at $r = 0$. In order to remove the conic singularity, one can identify the angular coordinate $\phi$ with the period

$$\text{Period}_\phi = 2\pi\left(\lim_{r\to 0}\frac{1}{r}\sqrt{\frac{g_{\phi\phi}}{g_{rr}}}\right)^{-1} = 2\pi\left(1-4\mu\right), \tag{25}$$

where the conical singularity has a deficit angle $\delta\phi = 8\pi\mu$. Expanding the metric function for $r \to 0$, one can show that

$$\begin{aligned} f(r)\big|_{r=0} &= \left.\frac{df(r)}{dr}\right|_{r=0} = 0, \\ f''(0) &= \left.\frac{d^2 f(r)}{dr^2}\right|_{r=0} \neq 0, \end{aligned} \tag{26}$$

and therefore $\mu$ is given by

$$\mu = \frac{1}{4}\left(1 - \frac{2}{l r_0 f''(0)}\right). \tag{27}$$

It is easy to show that the near origin metric can be written as

$$\begin{aligned} ds^2 = &-\frac{r_0^2}{l^2}dt^2 + \frac{r^2 l^2}{2}\left.\frac{d^2 f(r)}{dr^2}\right|_{r=0} d\phi^2 \\ &+ \frac{2}{r_0^2}\frac{dr^2}{\left(d^2 f(r)/dr^2\right)\big|_{r=0}} + \frac{r_0^2}{l^2}dz^2. \end{aligned} \tag{28}$$

Following the Vilenkin procedure [93], one can identify the near origin metric (28) with a cosmic string and interpret $\mu$ as the mass per unit length of the string [93].

Here, we are in a position to investigate the effect of nonlinearity parameter on the deficit angle $\delta\phi$. At first, we should note that $\delta\phi$ is a smooth real function for $> \beta_{\text{ext}}$, where

$$\beta_{\text{ext}} = \frac{2lq}{r_0^2} \times \begin{cases} 2\exp\left(\frac{1}{2}\right), & \text{ENEF}, \\ 1, & \text{LNEF}. \end{cases} \tag{29}$$

Second, it is interesting to note that the minimum and maximum values of the deficit angle are

$$\begin{aligned} \delta\phi\big|_{\text{Min}} &= \lim_{\beta\to\infty}\delta\phi, \\ \delta\phi\big|_{\text{Max}} &= \lim_{\beta\to\beta_{\text{ext}}^+}\delta\phi, \end{aligned} \tag{30}$$

which means that increasing the nonlinearity parameter $\beta$ leads to decreasing the deficit angle (see Figure 2 for more clarifications).

## 3. Spinning Magnetic String

In this section, we apply a local rotation boost to the static metric (22) to obtain rotating spacetime solutions. In 4-dimensional spacetime the rotation group is $SO(3)$, and so one can find that there is only one independent rotation parameter. In order to apply rotation, one may use the following local transformation in the $t-\phi$ plane:

$$t \longmapsto \sqrt{1+\frac{a^2}{l^2}}\,t - a\phi, \quad \phi \longmapsto \sqrt{1+\frac{a^2}{l^2}}\,\phi - \frac{a}{l^2}t, \tag{31}$$

where $a$ is a rotation parameter. Taking into account the static metric (20) and applying (31), we can obtain

$$ds^2 = -\frac{r^2+r_0^2}{l^2}\left(\sqrt{1+\frac{a^2}{l^2}}dt - ad\phi\right)^2 + \frac{r^2dr^2}{(r^2+r_0^2)f(r)} + l^2 f(r)\left(\frac{a}{l^2}dt - \sqrt{1+\frac{a^2}{l^2}}d\phi\right)^2 + \frac{r^2+r_0^2}{l^2}dz^2, \tag{32}$$

where the metric function $f(r)$ is the same as that in (23). According to the mentioned transformation, one can find that, in spite of the static case, $F_{rt}$ does not vanish for rotating solutions. Straightforward calculations show that the nonvanishing components of the electromagnetic fields are

$$F_{rt} = -\frac{a}{l^2\sqrt{1+a^2/l^2}}F_{r\phi} = \frac{a}{\sqrt{1+a^2/l^2}} \times \begin{cases} \dfrac{2q}{r^2+r_0^2}\exp\left(-\dfrac{1}{2}L'_W\right), & \text{ENEF}, \\ \dfrac{\beta^2\left(r^2+r_0^2\right)}{ql^2}\left(1-\Gamma'\right), & \text{LNEF}. \end{cases} \tag{33}$$

Considering (22), (31), and (32), one may think that there is a one-to-one correspondence between static and rotating spacetimes and so they are the same. But this statement is not correct. It is worthwhile to mention that the coordinate $\phi$ is periodic and therefore (31) is not a proper coordinate transformation on the entire manifold. In other words, the metrics (22) and (32) can be locally mapped into each other but not globally, and so (31) generates a new metric (for some details about this local transformation see, e.g., [94]).

In order to finalize this section, we should discuss the conserved quantities of the magnetic string. Using the counterterm method [95–98] and following the procedure of magnetic solutions papers [7, 8, 29–49], one can find that the mass and angular momentum per unit length of the string can be written as

$$M = \frac{\pi}{2}\left(1+\frac{3a^2}{l^2}\right)m, \tag{34}$$

$$J = \frac{3\pi ma}{2}\sqrt{1+\frac{a^2}{l^2}}. \tag{35}$$

Equation (35) shows that considering $a = 0$ leads to vanishing angular momentum and it confirms that $a$ is the rotational parameter of the spacetime. In addition, it is interesting to calculate the electric charge of the solutions. Using Gauss's law and calculating the flux of the electric field at infinity, we find that the electric charge per unit length $Q$ can be given by

$$Q = \pi qa. \tag{36}$$

We should note that the electric charge may be originated from the electric field. Since, for rotating solutions, besides the magnetic field along the $\phi$ coordinate, there is also a radial electric field $F_{tr}$ (see (33)), one may expect to obtain an electric charge which is related to the rotating parameter.

## 4. Magnetic Brane Solutions

Here, we start with a class of the $(n+1)$-dimensional metrics to obtain magnetic brane solutions with the following ansatz:

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho)\,d\varphi^2 + \frac{\rho^2}{l^2}dX_1^2, \tag{37}$$

where $dX_1^2 = \sum_{i=1}^{n-2}(dx^i)^2$ is the Euclidean metric on the $(n-2)$-dimensional submanifold. Using the nonlinear Maxwell equation (2) with the metric (37), we find that the nonzero components of Maxwell field are

$$F_{\phi\rho} = -F_{\rho\phi} = \begin{cases} \dfrac{2ql^{n-1}}{\rho^{n-1}}\exp\left(-\dfrac{1}{2}L_W\right), & \text{ENEF}, \\ \dfrac{\beta^2\rho^{n-1}}{q}(1-\Gamma), & \text{LNEF}, \end{cases} \tag{38}$$

where $L_W = \text{Lambert}\,W(-16l^{2(n-2)}q^2/\beta^2\rho^{2(n-1)})$ and $\Gamma = \sqrt{1-(2lq/\beta\rho^{n-1})^2}$. It is notable that considering the real electromagnetic field leads to $\rho > \rho_0$, where

$$\rho_0^{2n-2} = \left(\frac{2lq}{\beta}\right)^2 \times \begin{cases} 4l^{2(n-3)}\exp(1), & \text{ENEF}, \\ 1, & \text{LNEF}. \end{cases} \tag{39}$$

Now, we are in a position to obtain the metric function $f(\rho)$. Considering (37) with (38), we find that the solution of the gravitational field equation (1) is

$$f(\rho) = \frac{\rho^2}{l^2} + \frac{2ml^3}{\rho^{n-2}} - \frac{\chi\beta^2\rho^2}{n(n-1)} - \frac{4\beta ql^{n-2}}{(n-1)\rho^{n-2}}\Upsilon(\rho), \tag{40}$$

where

$$\Upsilon(\rho) = \begin{cases} \displaystyle\int(-L_W)^{-1/2}[1-L_W]\,d\rho, & \text{ENEF}, \\ \displaystyle\frac{2\beta}{ql^{n-2}}\int\left[\rho^{n-1}\ln\left(\frac{\beta^2(1-\Gamma)\rho^{2(n-1)}}{2l^2q^2}\right) + \frac{4q^2l^2}{\beta^2\rho^{n-1}(1-\Gamma)}d\rho\right] & \text{LNEF}. \end{cases} \tag{41}$$

It is easy to show that the Kretschmann scalar diverges when $\rho \to 0$ and is finite for $\rho \neq 0$. Following the same method, we find that one could not extend the spacetime from $\rho > r_0$ to $\rho < r_0$ in which $r_0$ is the largest positive real root of $f(\rho) = 0$. Therefore, we can use radial coordinate transformation (20) to obtain a real well-defined spacetime for $0 \le r < \infty$. Here, we leave details for reasons of economy.

Final step is generalization of static magnetic branes to spinning ones. We know that the rotation group in $n+1$ dimensions is $SO(n)$ and hence the maximum number of independent rotation parameters is integer part of $n/2$.

Generalization of static solutions to the spinning case with $k \leq [n/2]$ rotation parameters leads to the following metric:

$$\begin{aligned} ds^2 = &-\frac{r^2+r_0^2}{l^2}\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi^i\right)^2 \\ &+ f(r)\left(\sqrt{\Xi^2-1}dt - \frac{\Xi}{\sqrt{\Xi^2-1}}\sum_{i=1}^{k} a_i d\phi^i\right)^2 \\ &+ \frac{r^2 dr^2}{(r^2+r_0^2) f(r)} + \frac{r^2+r_0^2}{l^2(\Xi^2-1)}\sum_{i<j}^{k}\left(a_i d\phi_j - a_j d\phi_i\right)^2 \\ &+ \frac{r^2+r_0^2}{l^2} dX_2^2, \end{aligned} \tag{42}$$

where $\Xi = \sqrt{1+\sum_i^k a_i^2/l^2}$; $dX_2^2$ is the Euclidean metric on the $(n-k-1)$-dimensional submanifold with volume $V_{n-k-1}$. The nonvanishing components of electromagnetic field tensor and the metric function are, respectively,

$$\begin{aligned} F_{rt} = -\frac{(\Xi^2-1)}{\Xi a_i} F_{r\phi^i} = \frac{(\Xi^2-1)}{\Xi a_i} \\ \times \begin{cases} \dfrac{2ql^{n-1}}{(r^2+r_0^2)^{(n-1)/2}} \exp\left(-\dfrac{1}{2}L'_W\right), & \text{ENEF}, \\ \dfrac{\beta^2(r^2+r_0^2)^{(n-1)/2}}{q}\left(1-\Gamma'\right), & \text{LNEF}, \end{cases} \end{aligned} \tag{43}$$

$$\begin{aligned} f(r) = &\frac{r^2+r_0^2}{l^2} + \frac{2ml^3}{(r^2+r_0^2)^{(n-2)/2}} - \frac{\chi\beta^2(r^2+r_0^2)}{n(n-1)} \\ &- \frac{4\beta q l^{n-2}\Upsilon(r)}{(n-1)(r^2+r_0^2)^{(n-2)/2}}, \end{aligned} \tag{44}$$

where $L'_W = \text{Lambert}W(-16l^{2(n-2)}q^2/\beta^2(r^2+r_0^2)^{n-1})$, $\Gamma' = \sqrt{1-4l^2q^2/\beta^2(r^2+r_0^2)^{n-1}}$, and the function $\Upsilon(r)$ is

$$\begin{aligned} &\Upsilon(r) \\ &= \begin{cases} \displaystyle\int \frac{\left[1-L'_W\right]r}{\sqrt{-(r^2+r_0^2)L'_W}}dr, & \text{ENEF}, \\ \displaystyle\frac{2\beta}{ql^{n-2}}\int\left(\frac{\ln\left(\beta^2(1-\Gamma')(r^2+r_0^2)^{n-1}/2l^2q^2\right)}{(r^2+r_0^2)^{1-n/2}}\right. \\ \qquad\left. + \dfrac{4q^2l^2}{\beta^2(1-\Gamma')(r^2+r_0^2)^{n/2}}\right) r\,dr & \text{LNEF}. \end{cases} \end{aligned} \tag{45}$$

Following the known counterterm procedure and Gauss's law, it is easy to calculate the conserved quantities of the magnetic brane solutions. Straightforward calculations show that the mass, angular momentum, and electric charge per unit volume of the magnetic branes may be written as

$$\begin{aligned} M &= \frac{(2\pi)^k}{4}\left[n\left(\Xi^2-1\right)+1\right]m, \\ J_i &= \frac{(2\pi)^k}{4} n\Xi m a_i, \\ Q &= \frac{(2\pi)^k ql}{2}\sqrt{\Xi^2-1}. \end{aligned} \tag{46}$$

We should note that the electric charge is proportional to the rotation parameter and is zero for the case of static magnetic branes. This is due the fact that radial electric field $F_{tr}$ vanishes for the static solutions.

## 5. Conclusions

At the first step, we introduced a class of static magnetic string solutions in Einstein gravity in the presence of negative cosmological constant with two types of NLED. In order to have real solutions, we obtained a lower limit for the radial coordinate, $\rho$. Furthermore, we nominated the largest real root of the metric function as $r_0$ and, in order to get rid of signature changing, we introduced a new radial coordinate $r$.

Calculations of geometric quantities showed that although these solutions do not have curvature singularity, there is a conical singularity at $r = 0$ with a deficit angle $\delta\phi = 8\pi\mu$, where one can interpret $\mu$ as the mass per unit length of the string. Moreover, we found that, unlike the power Maxwell invariant solutions [45–47], the nonlinearity does not have any effect on the asymptotic behavior of the solutions and, in other words, obtained solutions are asymptotically adS.

In addition, we investigated the effects of nonlinearity parameter on the energy density and deficit angle, separately, and found that when one increases the nonlinearity parameter, the concentration volume of the energy density and the deficit angle reduce.

Using a suitable local transformation, we added an angular momentum to the spacetime and found that for rotating solutions there is an electric field in addition to the magnetic one.

Next, we used the counterterm method and Gauss's law to obtain conserved quantities and electric charge, respectively. It is interesting to note that these quantities depend on the rotation parameter and the static string has no net electric charge.

At the final step, we studied magnetic solutions in higher dimensions. We generalized static magnetic branes to spinning ones and obtained consistent electromagnetic field as well as metric function. Moreover, we obtained conserved quantities of the magnetic branes and found that the electric charge vanishes for the static magnetic branes. In addition, we found that, for $n = 3$, the conserved quantities of the magnetic branes reduce to those of magnetic string, as we expected.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Holographic Brownian Motion in Three-Dimensional Gödel Black Hole

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By using the AdS/CFT correspondence and Gödel black hole background, we study the dynamics of heavy quark under a rotating plasma. In that case we follow Atmaja (2013) about Brownian motion in BTZ black hole. In this paper we receive some new results for the case of $\alpha^2 l^2 \neq 1$. In this case, we must redefine the angular velocity of string fluctuation. We obtain the time evolution of displacement square and angular velocity and show that it behaves as a Brownian particle in non relativistic limit. In this plasma, it seems that relating the Brownian motion to physical observables is rather a difficult work. But our results match with Atmaja work in the limit $\alpha^2 l^2 \rightarrow 1$.

## 1. Introduction

In the last several years, the holographic AdS/CFT [1–4] has been exploited to study strongly coupled systems, in particular quark gluon plasmas [5–7]. The quark gluon plasma (QGP) is produced, when two heavy ions collide with each other at very high temperature. A relatively heavy particle, for example, a heavy quark, immerses in a soup of quarks and gluons with small fluctuations due to its interaction with constituent of QGP. The random motion of this particle is well known as Brownian motion [8–10]. The Brownian motion is a universal phenomenon in finite temperature systems and any particle immersed in a fluid at finite temperature undergoes Brownian motion. The Brownian motion opens a wide view from microscopic nature. It offers a better understanding of the microscopic origin of thermodynamics of black holes. Therefore, it is a natural step to study Brownian motion using the AdS/CFT correspondence. Particularly, the AdS/CFT correspondence can be utilized to investigate the Brownian motion for a quark in the quark gluon plasma.

In the field theory or boundary side of AdS/CFT story, a mathematical description of Brownian motion is given by the Langevin equation which phenomenologically describes the force acting on Brownian particles [10–12] which is given by

$$\dot{p}(t) = -\gamma_0 p(t) + R(t), \tag{1}$$

where $p$ is momentum of Brownian particle and $\gamma_0$ is the friction coefficient. These forces originate from losing energy to medium due to friction term (first term) and getting a random kick from the thermal bath (second term). One can learn about the microscopic interaction between the Brownian particle and the fluid constituents, if these forces be clear. By assuming $\langle m\dot{x}^2 \rangle = T$, the time evolution of displacement square is given as follows [10]:

$$\left\langle s(t)^2 \right\rangle = \left\langle [x(t) - x(0)]^2 \right\rangle \approx \begin{cases} \dfrac{T}{m} t^2, & \left(t \ll \dfrac{1}{\gamma_0}\right), \\ 2Dt, & \left(t \gg \dfrac{1}{\gamma_0}\right), \end{cases} \tag{2}$$

where $D = T/\gamma_0 m$ is diffusion constant, $T$ is the temperature, and $m$ is the mass of Brownian particle. At early time, $t \ll 1/\gamma_0$ (ballistic regime), the Brownian particle moves with constant velocity $\dot{x} \sim \sqrt{T/m}$, while at the late time, $t \gg 1/\gamma_0$ (diffusive regime), the particle undergoes a random walk.

In the gravity or bulk side of AdS/CFT version for Brownian motion, we need a gravitational analog of a quark immersed in QGP. This is achieved by introducing a bulk fundamental string stretching between the boundary at infinity and event horizon of an asymptotically AdS black hole background [13–17]. The dual statement of a quark in QGP on the boundary corresponds to the black hole environment that excites the modes of string. In the context of this duality, the end of string at the boundary corresponds to the quark which shows Brownian motion and its dynamics is formulated by Langevin equation. In the formulation of AdS/CFT correspondence, fields of gravitational theory would be related to the corresponding boundary theory operators [3, 4]. In this way, instead of using the boundary field theory to obtain the correlation function of quantum operators, we can determine these correlators by the thermal physics of black holes and use them to compute the correlation functions. In [14–17], the Brownian motion has been studied in holographic setting and the time evolution of displacement square. If we consider different gravity theories, we know that strings live in a black hole background and excitation of the modes is done by Hawking radiation of black hole. So, different theories of gravity can be associated with various plasma in the boundary. In this paper we follow different works to investigate Brownian motion of a particle in rotating plasmas. We try to consider the motion of a particle in two-dimensional rotating plasma whose gravity dual is described by three-dimensional Gödel metric background. In this case, we will see that, for the $\alpha$ parameter in the Gödel metric, new conditions for Brownian motion will be provided. As we know, the Brownian motion of a particle in two-dimensional rotating plasma with the corresponding gravity of BTZ black hole has been studied in [16]. The Gödel metric background in the special case receives to the BTZ black hole [18], so the comparison of our results with [16] gives us motivation to understanding the Brownian motion in rotating plasmas in general form of background as Gödel black hole.

This paper is arranged as follows. In Section 2, we give some review of three-dimensional Gödel black hole and derive string action from this metric background. Section 3 is devoted to investigate a holographic realization of Brownian motion and obtain the solution for equation of motion of string in Gödel black hole geometry. We study the Hawking radiation of the transverse modes near the outer horizon of Gödel black hole to describe the random motion of the external quark in Section 4. In Section 5, we make some summery about our results.

## 2. Background and String Action

*2.1. Gödel Black Hole.* Three-dimensional Gödel spacetime is an exact solution of Einstein-Maxwell theory with a negative cosmological constant and a Chern-Simons term [19]. When the electromagnetic field acquires a topological mass $\alpha$ Maxwell equation will be modified by an additional term. In that case, we receive to Einstein-Maxwell-Chern-Simons system, and the geometry is the Gödel space time [20]. This theory can be viewed as a lower dimensional toy model for the bosonic part of five-dimensional supergravity theory, so it can be an advantage in development of string theory. Three-dimensional Gödel black holes are like their higher dimensional counterparts in special properties. The action of Einstein-Maxwell-Chern-Simons theory in three dimensions is given by [21]

$$I = \frac{1}{16\pi G}\int d^3x\left[\sqrt{-g}\left(R + \frac{2}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \frac{\alpha}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right]. \tag{3}$$

A general spherically symmetric static solution to the above action in various cases for the $\alpha$ parameter can be written by [22]

$$ds^2 = \frac{dr^2}{h^2 - pq} + pdt^2 + 2hdtd\phi + qd\phi^2, \tag{4}$$

where $p$, $q$, and $h$ are functions of $r$ as

$$\begin{aligned} p(r) &= 8G\mu, \\ q(r) &= \frac{-4GJ}{\alpha} + 2r - 2\frac{\gamma^2}{l^2}r^2, \\ h(r) &= -2\alpha r, \end{aligned} \tag{5}$$

with

$$\gamma = \sqrt{\frac{1-\alpha^2 l^2}{8G\mu}}. \tag{6}$$

The gauge potential is given by

$$A = A_t(r)\,dt + A_\phi(r)\,d\phi, \tag{7}$$

with

$$A_t(r) = \frac{\alpha^2 l^2 - 1}{\gamma\alpha l} + \varepsilon, \qquad A_\varphi(r) = \frac{-4GQ}{\alpha} + 2\frac{\gamma}{l}r. \tag{8}$$

The parameters $\mu$ and $J$ are mass and angular momentum. The arbitrary constant $\varepsilon$ is a pure gauge. We can rewrite metric (4) in the ADM form as follows:

$$ds^2 = -\frac{\Delta}{q}dt^2 + \frac{dr^2}{\Delta} + q\left(d\phi + \frac{h}{q}dt\right)^2, \tag{9}$$

where

$$\Delta = h^2 - pq = \lambda(r - r_+)(r - r_-), \qquad \lambda = \frac{2\left(1+\alpha^2 l^2\right)}{l^2},$$
$$r_\pm = \frac{8G}{\lambda}\left[\mu \pm \sqrt{\mu^2 - \frac{\mu J\lambda}{2\alpha}}\right]. \tag{10}$$

The Hawking temperature that gives the temperature of the plasma is [23]

$$T_H = \frac{\lambda(r_+ - r_-)}{8\pi\alpha r_+}. \tag{11}$$

Here $r_-$ is the inner horizon and $r_+$ is the outer horizon. In the sector $\alpha^2 l^2 > 1$, we have real solution only for $\mu$ negative. In this regime, there are Gödel particles and theory supports time-like constants fields. When $\alpha^2 l^2 < 1$, $\mu$ has positive values. In this case black hole will be will be constructed and theory supports space-like constants fields. For $\alpha^2 l^2 = 1$, metric (4) reduces to BTZ metric as can be explicitly seen by transforming to the standard frame that is nonrotating at infinity with respect to anti-de Sitter space:

$$\phi \longrightarrow \phi + \alpha t, \qquad r \longrightarrow \frac{r^2}{2} + \frac{2GJ}{\alpha}. \tag{12}$$

In the standard frame, energy and angular momentum become $M = \mu - \alpha J$ and $J$, instead of $\mu$ and $J$ in rotating frame.

*2.2. String Action.* In the general case for a $d+2$-dimensional black hole metric background is

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu + G_{IJ}(x)\, dx^I dx^J. \tag{13}$$

Here $x^\mu = r, t$ stands for the string worldsheet coordinates and $X^I = X^I(x)$ $(I, J = 1, \ldots, d-2)$ for the spacetime coordinates. If we stretch a string along the $r$ direction and consider small fluctuation in the transverse direction $X^I$, the dynamics of this string follows from the Nambu-Goto action [14]:

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\acute{\alpha}} \int dx^2 \sqrt{\left(\dot{X}X'\right)^2 - \dot{X}^2 X'^2}. \tag{14}$$

If the scalars $X^I$ do not fluctuate too far from their equilibrium values ($X^I = 0$), we can expand the above action up to quadratic order in $X^I$:

$$S_{\mathrm{NG}} \approx -\frac{1}{4\pi\acute{\alpha}} \int dx^2 \sqrt{-g(x)}\, g^{\mu\nu} G_{IJ} \frac{\partial X^I}{\partial x^\mu} \frac{\partial X^J}{\partial x^\nu}. \tag{15}$$

In fact, this quadratic fluctuation Lagrangian can be interpreted as taking the nonrelativistic limit, so we must use the dual Langevin dynamics on boundary in the nonrelativistic case.

## 3. Strings in Gödel Black Hole

As we said in the Introduction, an external quark is dual to an open string that extends from the boundary to the horizon of the black hole [24]. We can obtain the dynamics of this string in a threedimensional Gödel black hole with the metric background (9) by the Nambu-Goto action (14) in the following form:

$$\begin{aligned} S_{\mathrm{NG}} = &-\frac{1}{2\pi\acute{\alpha}} \\ &\times \int dx^2 \left( \frac{r^2}{q(r^2)} + \Delta(r^2)\, \phi'^2 - \frac{q(r^2)\, r^2}{\Delta(r^2)} \right. \\ &\left. \times \left( \dot{\phi} + \alpha + \frac{h(r^2)}{q(r^2)} \right)^2 \right)^{1/2}. \end{aligned} \tag{16}$$

We have obtained the above relation in the standard frame. The equation of motion for $\phi$ derived from (16) is

$$-\frac{\partial}{\partial t}\left[ \frac{r^2 q}{\Delta\sqrt{-g}} \left( \dot{\phi} + \alpha + \frac{h}{q} \right) \right] + \frac{\partial}{\partial r}\left[ \frac{\Delta\phi'}{\sqrt{-g}} \right] = 0. \tag{17}$$

*3.1. Trivial Solution.* The Nambu-Goto action up to quadratic terms after subsisting the small fluctuation $\phi \to C + \phi$ under Gödel metric background in the standard frame is given by

$$\begin{aligned} S_{\mathrm{NG}}^{(2)} = &-\frac{1}{4\pi\acute{\alpha}} \int dt\, dr \frac{\Delta^{3/2}\phi'^2}{r^3\left[\Delta/q - q(h/q+\alpha)^2\right]^{1/2}} \\ &- \frac{\Delta^{1/2}\dot{\phi}^2}{r\left[\Delta/q - q(h/q+\alpha)^2\right]^{3/2}}. \end{aligned} \tag{18}$$

By changing coordinate to $s = x - x_+$ (where $x = r^2/2 + 2GJ/\alpha$) and defining $\phi(t,s) = e^{-i\omega t} f_\omega(s)$, one can write the equation of motion as follows:

$$\begin{aligned} &W(s)\, Z(s)\, \partial_s^2 f_\omega + \frac{1}{2}\left[3\partial Z(s)\, W(s) - \partial W(s)\, Z(s)\right] \partial_s f_\omega \\ &\quad + \omega^2 f_\omega = 0, \end{aligned} \tag{19}$$

where

$$W(s) = \frac{\Delta(s)}{q(s)} - q(s)\left(\frac{h(s)}{q(s)} + \alpha\right)^2, \qquad Z(s) = \Delta(s). \tag{20}$$

The solution for this equation of motion is the trivial solution for the relation (17). In general, solution of this equation is very complicated. However, for the extremal case $\mu = J(1 + \alpha^2 l^2)/\alpha l^2$, we can find an analytical solution as

$$\begin{aligned} &f_\omega^{\pm} = s^{-1\pm D} Y(s), \\ &\text{where } D = \sqrt{1 + \frac{\omega^2 l^2}{16GM}} \text{ with } M = \frac{\mu}{1+\alpha^2 l^2}, \end{aligned} \tag{21}$$

where $Y(s)$ is obtained from the following relation:

$$\begin{aligned} & s\left(cs^2+4z\left(s-\frac{1}{4}\frac{z}{\alpha^2}\right)\right)\frac{d^2}{ds^2}Y(s) \\ & \quad +\left(2\chi\left(cs^2+4z\left(s-\frac{1}{4}\frac{z}{\alpha^2}\right)\right)\right. \\ & \qquad \left.+2cs^2+10sz-3\frac{z^2}{\alpha^2}\right)\frac{d}{ds}Y(s) \\ & \quad +\chi\left((\chi-1)(cs+4z)+2cs+10z\right)Y(s)=0, \end{aligned} \tag{22}$$

where $\chi = -1 \pm D$ and $z = 2x_+/l^2$. The complete solution of $Y(s)$ is derived in the Appendix. With similar argument as in [16], this solution is not acceptable, because it does not have oscillatory modes in radial coordinates and also, for this trivial constant solution, one can investigate that the square root determinant of the worldsheet metric is not real everywhere; therefore, it is not a physical solution. Due to nonphysical motivation about the mentioned solution, we have to consider another approach, which is linear solution.

*3.2. Linear Solution.* We can take the linear ansatz for the small fluctuation in the transverse direction $\phi$ to achieve a nontrivial solution, so we expand it as

$$\phi(t,r) = wt + \eta(r), \tag{23}$$

where $w$ is a constant angular velocity. By replacing this relation into (17), the solution for $\eta$ is obtained as follows:

$$\eta'(r) = -\frac{\pi_\phi}{\Delta}\sqrt{\frac{\left(r^2/q\right)\left(\Delta - q^2\left(h/q+\alpha+w\right)^2\right)}{\Delta-\pi_\phi^2}}, \tag{24}$$

where $\pi_\phi$ is a constant which has a concept as the total force to keep string moving with linear angular velocity $w$ and also is related to momentum conjugate of $\phi$ in $r$ direction. At $r = r_{\mathrm{NH}}$, the numerator becomes zero, so the denominator should also vanish there, because the string solution (23) must be real everywhere along the worldsheet. For $w = 0$ and $\alpha^2 l^2 \neq 1$, $r_{\mathrm{NH}}$ is given by

$$r_{\mathrm{NH}}^2 = -\frac{l^2}{\gamma^2}\left[1 \pm \sqrt{1+\frac{8G\gamma^2}{l^2\alpha^2}(2\mu - J\alpha)}\right] - \frac{4GJ}{\alpha}. \tag{25}$$

When $w \neq 0$ and $\alpha^2 l^2 = 1$, we receive the excepted relation for BTZ black hole [16]. For $w \neq 0$ and $\alpha^2 l^2 \neq 1$, we obtain

$$\begin{aligned} r_{\mathrm{NH}}^2 &= \frac{l^2}{\gamma^2(\alpha+w)} \\ &\quad \times\left[(w-\alpha) \pm \sqrt{(w-\alpha)^2+\frac{16G\gamma^2}{l^2} > \left(\mu - \frac{J(\alpha+w)^2}{2\alpha}\right)}\right] \\ &\quad - \frac{4GJ}{\alpha}. \end{aligned} \tag{26}$$

According to [16] we set dominator to zero, so we have

$$\pi_\phi^2 = \Delta = \left(\frac{h\alpha+p}{\alpha}\right)^2, \qquad h = h\left(r_{\mathrm{NH}}^2|_{w=0}\right). \tag{27}$$

The external force $F_{\mathrm{ext}}$ can be obtained by considering the rotation and the topological mass of black hole which is given by

$$F_{\mathrm{ext}} = \frac{\pi_\phi}{2\pi\acute{\alpha}} = \frac{h\alpha+p}{2\pi\acute{\alpha}\alpha}. \tag{28}$$

After extracting this external force, we can derive the friction coefficient $\gamma_0$ for nonzero $w$, by considering the relation $p_\phi = m_0 w$, as

$$\gamma_0 m_0 = \frac{q_{\mathrm{NH}}}{2\pi\acute{\alpha}} = \frac{r_{\mathrm{NH}}^2 - \left(2\gamma^2/l^2\right)\left(r_{\mathrm{NH}}^2/2 + 2GJ/\alpha\right)^2}{2\pi\acute{\alpha}}. \tag{29}$$

With $\alpha^2 l^2 = 1$ this coefficient reduces to the excepted value $\gamma_0 = r_{\mathrm{NH}}^2/2\pi\acute{\alpha}m_0$ for BTZ black hole [16].

The Nambu-Goto action, with the small fluctuation, $\phi \to wt + \eta(r) + \phi$, under the Gödel background becomes

$$\begin{aligned} S_{\mathrm{NG}}^{(2)} &= -\frac{1}{4\pi\acute{\alpha}}\int dtdr\frac{\Delta^{3/2}\phi'^2}{r^3\left[\Delta/q - q\left(h/q+\alpha+w\right)^2\right]^{1/2}} \\ &\quad - \frac{\Delta^{1/2}\dot{\phi}^2}{r\left[\Delta/q - q\left(h/q+\alpha+w\right)^2\right]^{3/2}}. \end{aligned} \tag{30}$$

The equation of motion from the above Nambu-Goto action is given by

$$\begin{aligned} & \frac{-r\Delta^{1/2}\ddot{\phi}}{\left[\Delta/q - q\left(h/q+\alpha+w\right)^2\right]^{3/2}} \\ & \quad + \frac{\partial}{\partial r}\frac{\Delta^{3/2}\phi'^2}{r\left[\Delta/q - q\left(h/q+\alpha+w\right)^2\right]^{1/2}} = 0. \end{aligned} \tag{31}$$

Solving this equation is quite complicated for more values of $w$. However, one can find that there are some values like

$$\begin{aligned} w &= \frac{x_- + \left(x_+ - x_-\right)\left(\left(1-\alpha^2 l^2\right)/2\right)}{\alpha x_+ l^2} \\ &= \frac{r_- + \left(r_+ - r_-\right)\left(\left(1-\alpha^2 l^2\right)/2\right)}{\alpha r_+ l^2}, \end{aligned} \tag{32}$$

where this makes it possible to solve the equation of motion. In derivation of the right-hand side of the above relation we use $r_- r_+ = 4GJ/\alpha$. The special radius $r_{\mathrm{NH}}$ approaches the outer horizon of the Gödel black hole for this value of angular velocity $w$, where $\Delta(r_+) = 0$ (and $\pi_\phi = 0$); then the steady state solution is the case that $r_{\mathrm{NH}} = r_+$. From relation (32) for the angular velocity, it is evident that we can receive to $w = r_-/r_+$ for BTZ black hole ($\alpha^2 l^2 = 1$). Furthermore, for

($\alpha^2 l^2 > 1$), we can check that $w^2 < \alpha^2$, but there must be some condition on $\mu$ and $J$ to have $w^2 < \alpha^2$ for ($\alpha^2 l^2 < 1$). We can write the equation of motion for this terminal angular velocity with changing coordinate to $s = x - x_+$ as

$$W(s) Z(s) \phi_s'' + \frac{1}{2}\left[3\partial Z(s) W(s) - \partial W(s) Z(s)\right] \phi_s' - \ddot{\phi}_s = 0, \tag{33}$$

where

$$W(s) = \frac{p}{(2\alpha x_+)^2}\left[s(cs + \lambda\zeta)\right], \qquad Z(s) = \lambda s(s + \zeta),$$
$$c = \lambda - 4\alpha^2, \qquad \zeta = x_+ - x_-. \tag{34}$$

As before, we take $\phi(t, s) = e^{-i\omega t} f_\omega(s)$, so (33) reduce to

$$W(s) Z(s) f_\omega'' + \frac{1}{2}\left[3Z'(s) W(s) - W'(s) Z(s)\right] f_\omega' + \omega^2 f_\omega = 0. \tag{35}$$

Consequently, the independent linear solutions to the above equation are obtained as below:

$$f_\omega^\pm(s) = (\lambda\zeta s)^{\pm i\vartheta}\, {}_2F_1\left(\pm i\vartheta, \frac{3}{2} \pm i\vartheta; 1 \pm 2i\vartheta, \frac{(-\lambda + c)s}{cs + \lambda\zeta}\right) \times (cs + \lambda\zeta)^{\mp i\vartheta}, \tag{36}$$

where

$$\vartheta = \frac{2\alpha\omega x_+}{\lambda\zeta\sqrt{p}}, \tag{37}$$

or with $\xi = 2\zeta = r_+^2 - r_-^2$ and $x_+/\sqrt{p} = 4GJ/(\alpha r_- \sqrt{\lambda})$, we have $\vartheta = 16GJ\omega/\lambda^{3/2}\xi r_-$. By considering the following relation for hypergeometric functions,

$$ {}_2F_1\left(\kappa, \kappa + \frac{3}{2}, 2\kappa + 1, z\right) = \frac{2^{2\kappa}}{\kappa + 1/2}\left[1 + (1 - z)^{1/2}\right]^{-2\kappa}\left[\frac{1}{2} + \kappa(1 - z)^{-1/2}\right], \tag{38}$$

(36) reduces as

$$f_w^\pm(s) = \frac{(4\lambda\zeta)^{\pm i\vartheta}}{1 \pm 2i\vartheta}\frac{\left[1 \pm 2i\vartheta/\sqrt{1 + (\lambda - c)s/(cs + \lambda\zeta)}\right]}{\left[1 + \sqrt{1 + (\lambda - c)s/(cs + \lambda\zeta)}\right]^{\pm 2i\vartheta}} \times (cs + \lambda\zeta)^{\mp i\vartheta} s^{\pm i\vartheta}, \tag{39}$$

which gives oscillation modes. We have the following asymptotic behavior from the solutions near the outer horizon ($s \to 0$) and the boundary ($s \to \infty$):

$$f_w^\pm(s) \sim \begin{cases} e^{\pm i\omega s_\star} & (s \longrightarrow 0) \\ \dfrac{(4\lambda\zeta)^{\pm i\vartheta}\left(1 \pm 2i\vartheta/\sqrt{\lambda/c}\right)}{(1 \pm 2i\vartheta)\left(1 + \sqrt{\lambda/c}\right)^{\pm 2i\vartheta}} & (s \longrightarrow \infty), \end{cases} \tag{40}$$

with $s_\star = (\vartheta/\omega)\ln(s) = (16GJ/\lambda^{3/2}\xi r_-)\ln(s)$.

## 4. Displacement Square

So far, we have succeeded to drive oscillation modes for a string moving in the Gödel black hole background. In the following, we follow the same procedure as in [14, 16] to compute the displacement square for Brownian motion. In order to achieve this, we write the solutions for bulk equation of motion as a linear combination of $f_\omega^\pm$:

$$f_\omega(s) = A\left[f_\omega^+(s) + Bf_\omega^-(s)\right] e^{-i\omega t}, \tag{41}$$

where $A$ and $B$ are constants. By exerting the Neumann boundary condition near the boundary, $\partial_s f_s(\omega) = 0$ with $s = s_c \gg 0$, to put the UV-cutoff, we obtain

$$B = \frac{(4\lambda\zeta)^{2i\vartheta}(cs_c + \lambda\zeta)^{-2i\vartheta} s_c^{2i\vartheta}(1 - 2i\vartheta)}{\left[1 + \sqrt{1 + (\lambda - c)s_c/(cs_c + \lambda\zeta)}\right]^{4i\vartheta}(1 + 2i\vartheta)} \times \frac{\left[1 + 2i\vartheta\sqrt{1 + (\lambda - c)s_c/(cs_c + \lambda\zeta)}\right]}{\left[1 - 2i\vartheta\sqrt{1 + (\lambda - c)s_c/(cs_c + \lambda\zeta)}\right]} \equiv e^{i\theta_\omega}. \tag{42}$$

Note that the constant $B$ is a pure phase, so by using (40) in the near horizon we can write

$$\Phi(t, s) = f_\omega(s) e^{i\omega t} \sim e^{-i\omega(t - s_\star)} + e^{i\theta_\omega} e^{-i\omega(t + s_\star)}. \tag{43}$$

To regulate the theory, we implement another cutoff near the outer horizon at $s_h = \epsilon$, $\epsilon \ll 1$, which is called IR-cutoff; we obtain

$$B \approx \epsilon^{2i\vartheta} = e^{-2i\vartheta\ln(1/\epsilon)}. \tag{44}$$

If we take $B$ in the terms of $\omega$ by relation (41) only, then $B$ has continuous values, since the $\omega$ can have any value. Using relation (43) for $B$ will satisfy our requirements to have discrete values in $\epsilon \ll 1$. In this case, the discreteness is [14, 16]

$$\Delta\vartheta = \frac{\pi}{\ln(1/\epsilon)}, \tag{45}$$

where, in terms of $\omega$, it is given by

$$\Delta\omega = \frac{\lambda^{3/2}\pi r_- \xi}{16GJ\ln(1/\epsilon)}. \tag{46}$$

Following the above processes and using IR-cutoff to discrete the continuous spectrum makes it easy to find normalized bases of modes and to quantize $\phi(t, r)$ by extending in these modes.

*4.1. Brownian Particle Location.* In this section we are going to use quantized modes of the string near the outer horizon of Gödel black hole to describe the Brownian motion of an external quark. Therefore we consider the Nambu-Goto action for certain amount of terminal angular velocity, near the outer horizon ($s \to 0$):

$$S_{\mathrm{NG}}^2 \sim \frac{1}{2}\int dt ds_\star\left(\dot{\Phi}^2 - \Phi'^2\right), \tag{47}$$

where $\Phi \equiv (8GJ/(r_-\sqrt{2\pi\lambda\acute{\alpha}}))\phi$. Thus, according to the same procedure for standard scalar fields, we introduce the following mode expansions:

$$\Phi(t,s) = \sum_{\omega>0}\left[a_\omega u_\omega(t,s) + a_\omega^\dagger u_\omega(t,s)^*\right], \tag{48}$$

with

$$u_\omega(t,s) = \sqrt{\frac{\lambda^{3/2}\xi r_-}{32GJ\omega\ln(1/\epsilon)}}\left[f_\omega^+(s) + Bf_\omega^-(s)\right]e^{-i\omega t}, \qquad \left[a_\omega, a_{\acute{\omega}}^\dagger\right] = \delta_{\omega\acute{\omega}}. \tag{49}$$

Now, by considering the above quantum modes on the probe string in the bulk, we want to work out the dynamics of the endpoint which corresponds to an external quark. We investigate the wave-functions of the world-sheet fields in the two interesting regions: (i) near the black hole horizon and (ii) close to the boundary. From (40), near the horizon ($S \sim 0$), expansion (48) becomes

$$\phi(t, S\longrightarrow 0) = \frac{r_-^{3/2}\sqrt{2\pi\acute{\alpha}^{\lambda^{5/2}}\xi}}{(16GJ)^{3/2}\sqrt{\ln(1/\varepsilon)}} \times \sum_{\omega=-\infty}^{\infty}\frac{1}{\sqrt{\omega}}\left(e^{-i\omega(t-S_\star)} + e^{i\theta_\omega}e^{-i\omega(t+S_\star)}\right)a_\omega. \tag{50}$$

We used $S = 2s = r^2 - r_+^2$. On the other hand, expansion (48) at $S = R$ (the location of the regulated boundary) is given by

$$\begin{aligned}\phi(t, S=R) &= \frac{r_-^{3/2}\sqrt{2\pi\acute{\alpha}^{\lambda^{5/2}}\xi}}{(16GJ)^{3/2}\sqrt{\ln(1/\epsilon)}} \times \sum_{\omega>0}\frac{1}{\sqrt{\omega}} \\ &\times\left[\left(2^{1+2i\vartheta}(\lambda\xi R)^{i\vartheta}(cR+\lambda\xi)^{-i\vartheta}(1-2i\vartheta)\right.\right. \\ &\times\left(\left[1+\sqrt{1+\frac{(\lambda-c)R}{cR+\lambda\xi}}\right]^{2i\vartheta}\right. \\ &\left.\left.\left.\times\left[1-2i\vartheta\sqrt{1+\frac{(\lambda-c)R}{cR+\lambda\xi}}\right]\right)^{-1}\right) \\ &\times e^{-i\omega t}a_\omega + h.c.\right].\end{aligned} \tag{51}$$

One can see that there are two modes in the solutions. The outgoing modes ($\omega > 0$) that are excited because of Hawking radiation [25, 26] and incoming modes ($\omega < 0$) which fall into black hole. The outgoing mode correlators are determined by the thermal density matrix:

$$\rho_0 = \frac{e^{-\beta H}}{Tr\left(e^{-\beta H}\right)}, \qquad H = \sum_{\omega>0}\omega a_\omega^\dagger a_\omega, \tag{52}$$

and the expectation value of occupation number is given by the Bose-Einstein distribution:

$$\left\langle a_\omega^\dagger a_{\acute{\omega}}\right\rangle = \frac{\delta\omega\acute{\omega}}{e^{\beta\omega}-1}, \tag{53}$$

with $\beta = 1/T$. Using the knowledge of relation (52) about outgoing modes correlators in the bulk, we can investigate the motion of the endpoint of the string at $S = R \gg 1$. We can also determine the behavior of the Brownian motion, by computing displacement square, as came in (2). So we can predict the nature of Brownian motion of external particle on the boundary. For this purpose, we compute the modes correlators at $S = R \gg 1$ as

$$\begin{aligned}\left\langle\phi_R(t)\phi_R(0)\right\rangle &= \sum_{\omega>0}\frac{r_-^3\pi\acute{\alpha}\lambda^{5/2}\xi}{(16GJ)^3\omega\ln(1/\epsilon)} \\ &\times\frac{\left[1+4\vartheta(\omega)^2\right]}{\left[1+4\vartheta(\omega)^2\left(\lambda(R+\xi)/(cR+\lambda\zeta)\right)\right]} \\ &\times\left[\frac{2\cos\omega t}{e^{\beta\omega}-1} + e^{-i\omega t}\right].\end{aligned} \tag{54}$$

By utilizing (46), we can write the above relation in the integral form. We see that the integral is diverging. So we regularize it by normally ordering the $a, a^\dagger$ oscillators: $a_\omega a_\omega^\dagger \equiv a_\omega^\dagger a_\omega$; then we have

$$\begin{aligned}\left\langle:\phi_R(t)\phi_R(0):\right\rangle &= \frac{8\lambda r_-^2\acute{\alpha}}{(16GJ)^2} \\ &\times\int_0^\infty\frac{d\omega}{\omega}\left(1+4\frac{(16GJ)^2\omega^2}{\lambda^3\xi^2r_-^2}\right) \\ &\times\left(1+4\frac{(16GJ)^2\omega^2}{\lambda^3\xi^2r_-^2}\left(\frac{\lambda(R+\xi)}{cR+\lambda\xi}\right)\right)^{-1} \\ &\times\left[\frac{2\cos(\omega t)}{e^{\beta\omega}-1}\right],\end{aligned} \tag{55}$$

and the displacement square becomes

$$\begin{aligned}S_{\text{reg}}(t)^2 &\equiv \left\langle:\left[\phi_R(t)-\phi_R(0)\right]^2:\right\rangle \\ &= \frac{16\lambda r_-^2\acute{\alpha}}{(16GJ)^2}\left[\frac{(\lambda-c)R}{\lambda(R+\xi)}I_1 + \frac{cR+\lambda\xi}{\lambda(R+\xi)}I_2\right],\end{aligned} \tag{56}$$

with

$$I_1 = 4\int_0^\infty \frac{dy}{y\left(1+a^2y^2\right)}\frac{\sin^2\left(ky/2\right)}{e^y-1},$$
$$I_2 = 4\int_0^\infty \frac{dy}{y}\frac{\sin^2\left(ky/2\right)}{e^y-1}, \tag{57}$$

and we have defined

$$y = \beta\omega, \qquad k = \frac{t}{\beta}, \qquad a^2 = 4\left(\frac{16GJ}{\lambda\xi r_-\beta}\right)^2\left(\frac{R+\xi}{cR+\lambda\xi}\right). \tag{58}$$

The evaluation of these integrals and their behavior for $R \gg 1$ and $a \gg 1$ can be found in Appendix B of [14]. From relation (58), we can see that when $R \gg 1$, we have $a \propto 1/c$. Thus in general case for $a$, we use the following relations for integrals (57):

$$\begin{aligned} I_1 = {} & \frac{1}{2}\left[e^{k/a}Ei\left(-\frac{k}{a}\right)+e^{-k/a}Ei\left(\frac{k}{a}\right)\right] \\ & + \frac{1}{2}\left[\psi\left(1+\frac{1}{2\pi a}\right)+\psi\left(1-\frac{1}{2\pi a}\right)\right] \\ & - \frac{\pi}{2}\left(1-e^{|k|/a}\right)\cot\frac{1}{2a}+\log\left(\frac{2a\sinh\pi k}{k}\right)+\frac{e^{-2\pi|k|}}{2} \\ & \times\left[\frac{{}_2F_1\left(1,1+1/2\pi a,2+1/2\pi a;e^{-2\pi|k|}\right)}{1+1/2\pi a}\right. \\ & \left. + \frac{{}_2F_1\left(1,1-1/2\pi a,2-1/2\pi a;e^{-2\pi|k|}\right)}{1-1/2\pi a}\right], \\ I_2 = {} & \log\left(\frac{\sinh\pi k}{\pi k}\right). \end{aligned} \tag{59}$$

However, for $R \gg 1$ and $\ll 1(\alpha^2l^2 \to 1)$, then $a \gg 1$, one can utilize the following relation for $I_1$ and $I_2$:

$$I_1 = \begin{cases} \dfrac{\pi k^2}{2a}+O\left(a^{-2}\right) \\ \pi k + O\left(\log k\right), \end{cases}$$
$$I_2 = \begin{cases} O\left(a^0\right), & (k \ll a) \\ \pi k + O\left(\log k\right), & (k \gg a). \end{cases} \tag{60}$$

Therefore, $S_{\text{reg}}(t)^2$ has the following form:

$$\left\langle S_{\text{reg}}(t)^2\right\rangle = \begin{cases} \dfrac{16r_-^3\lambda\pi\acute{\alpha}}{(16GJ)^3\beta}\left[\dfrac{\xi(\lambda-c)(cR+\lambda\xi)^{1/2}}{4(R+\xi)^{1/2}}\right]t^2 \\ \qquad +O\left(\dfrac{cR+\lambda\xi}{R+\xi}\right), \qquad (t \ll \beta), \\ \dfrac{16r_-^2\lambda\pi\acute{\alpha}}{(16GJ)^2\beta}t+O\left(\log\dfrac{t}{\beta}\right), \quad (t \gg \beta). \end{cases} \tag{61}$$

One can check that the displacement square (61) is consistent with BTZ black hole in [16] by setting $c = 0$ or $\alpha^2l^2 = 1$. In that case, the $w$ vanishes for $J = 0$ (or $r_- = 0$), but when $\alpha^2l^2 \neq 1$, the $w$ will have zero value only for $J/\alpha l^2 = (\alpha^2l^2-1)\mu$ (see relation (32)). Then our static solution is achieved by this condition. The diffusion constant from (61) is given by

$$D = \frac{\lambda\pi\acute{\alpha}\alpha^2}{2r_+^2}T. \tag{62}$$

So, the relaxation time of Brownian particle is as follows:

$$t_c = \frac{1}{\gamma_0} = \frac{\lambda m_0\pi\acute{\alpha}\alpha^2}{2r_+^2}. \tag{63}$$

The mass of external particle, $m_0$, can be computed by using the total energy and momentum of string [27] under the metric background (4):

$$E = \frac{1}{2\pi\acute{\alpha}}\int dr\pi_t^0, \qquad p_\phi = \frac{1}{2\pi\acute{\alpha}}\int dr\pi_\phi^0, \tag{64}$$

with

$$\pi_t^0 = \frac{\phi'^2}{\sqrt{-g}}\left(g_{t\phi}^2-g_{tt}g_{\phi\phi}\right)-\frac{g_{rr}}{\sqrt{-g}}\left(g_{tt}+g_{t\phi}\dot{\phi}\right),$$
$$\pi_\phi^0 = \frac{g_{rr}}{\sqrt{-g}}\left(g_{t\phi}+g_{\phi\phi}\dot{\phi}\right). \tag{65}$$

Then we have

$$E = \frac{\alpha}{2\pi\acute{\alpha}}\int ds\frac{c\left(s+x_+\right)+\lambda x_+}{\sqrt{\lambda p\left(s+\zeta\right)\left(cs+\lambda\zeta\right)}},$$
$$p_\phi = \frac{1}{2\pi\acute{\alpha}}\int ds\frac{-c\left(s+x_+\right)+\lambda x_-}{\sqrt{\lambda p\left(s+\zeta\right)\left(cs+\lambda\zeta\right)}}. \tag{66}$$

One can check that, after putting $c = 0$ in the above relation, the result of integral is as excepted for BTZ black hole. However, for $c \neq 0$ we obtain

$$\begin{aligned} E = {} & \alpha\frac{\sqrt{(cR+\lambda\xi)(R+\xi)}-\sqrt{\lambda\xi^2}}{\sqrt{\lambda p}} \\ & + 2\alpha\frac{\lambda+c}{\lambda}\sqrt{\frac{p}{\lambda c}}\ln\left[\frac{\sqrt{cR+\lambda\xi}+\sqrt{c(R+\xi)}}{\sqrt{c\xi}+\sqrt{\lambda\xi}}\right], \\ p_\phi = {} & -\frac{\sqrt{(cR+\lambda\xi)(R+\xi)}-\sqrt{\lambda\xi^2}}{\sqrt{\lambda p}} \\ & + 2\frac{\lambda-c}{\lambda}\sqrt{\frac{p}{\lambda c}}\ln\left[\frac{\sqrt{cR+\lambda\xi}+\sqrt{c(R+\xi)}}{\sqrt{c\xi}+\sqrt{\lambda\xi}}\right], \end{aligned} \tag{67}$$

where $\sqrt{p/\lambda} = (r_+ + r_-)/2$. Then the mass is defined as

$$m_0^2 = E^2 - p_\phi^2. \tag{68}$$

From the above relations, we see that relating the physical mass to displacement square is difficult.

## 5. Summary

In this paper, by using AdS/CFT correspondence, we studied the Brownian motion of an external quark in plasma. It is corresponded to a string stretched from horizon of AdS to boundary. By using the Nambu-Goto action, we obtained the equation of motion for this string in the Gödel background. For an acceptable solution with oscillatory modes, we had to redefine the terminal angular velocity. We found that turning on a finite density for a conserved $U(1)$ charge (reflected by a CS term in the bulk) and the rotation of black hole influence oscillatory modes. For realization of the Brownian motion, we derived the time evolution of the displacement square from the modes correlators. We showed that in general case ($\alpha^2 l^2 \neq 1$ Gödel black hole), our results for displacement square are different in comparison with [16]. However, in $\alpha^2 l^2 = 1$ limit (BTZ black hole), we confirmed that our results agree with the work of Atmaja [16]. We derived the physical mass, but we found that relating the displacement square to physical observables is a difficult work. This is the problem that we would like to consider in future work. Also we would like to investigate the Brownian motion of external quarks in different environments, in particular plasmas which correspond to metric backgrounds as Lifshitz geometry [28] and metric backgrounds with hyperscaling violation [29, 30].

## Appendix

The solution to the following differential equation

$$\begin{aligned} & s\left(cs^2+4z\left(s-\frac{1}{4}\frac{z}{\alpha^2}\right)\right)\frac{d^2}{ds^2}Y(s) \\ & \quad +\left(2\chi\left(cs^2+4z\left(s-\frac{1}{4}\frac{z}{\alpha^2}\right)\right)\right. \\ & \qquad \left.+2cs^2+10sz-3\frac{z^2}{\alpha^2}\right)\frac{d}{ds}Y(s) \\ & \quad +\chi\left((\chi-1)(cs+4z)+2cs+10z\right)Y(s) \\ & = 0 \end{aligned} \tag{A.1}$$

can be obtained analytically as

$$\begin{aligned} Y(s) &= C_1 s^{-3/2-\chi}\sqrt[4]{s(cs+4z)\alpha^2-z^2} \\ &\quad \times\left(\frac{s}{-c\alpha s-2z\alpha+z\sqrt{\lambda}}\right)^{(1/2)(1-p(\alpha)/r(\alpha)q(\alpha))} \\ &\quad \times\left(\frac{-\alpha\sqrt{\lambda}s+2s\alpha^2-z}{-z\sqrt{\lambda}+\alpha(cs+2z)}\right)^{(1/2)(1+3\sqrt{-c^2}/4r(\alpha)q(\alpha))} \\ &\quad \times\left(-c\alpha s-2z\alpha+z\sqrt{\lambda}\right) \\ &\quad \times {}_2F_1\left(E(\alpha)(T(\alpha)+U(\alpha)),\right. \\ &\qquad E(\alpha)(T(\alpha)+U(\alpha)), \\ &\qquad \left(1-\frac{p(\alpha)}{q(\alpha)r(\alpha)}\right); \\ &\qquad \left.\frac{c\alpha\sqrt{\lambda s}}{\left(-z\sqrt{\lambda}+\alpha(cs+2z)\right)r(\alpha)}\right) \\ &\quad + C_2 s^{-3/2-\chi}\sqrt[4]{s(cs+4z)\alpha^2-z^2} \\ &\quad \times\left(\frac{s}{-c\alpha s-2z\alpha+z\sqrt{\lambda}}\right)^{(1/2)(1+p(\alpha)/r(\alpha)q(\alpha))} \\ &\quad \times\left(\frac{-\alpha\sqrt{\lambda}s+2s\alpha^2-z}{-z\sqrt{\lambda}+\alpha(cs+2z)}\right)^{(1/2)(1+3\sqrt{-c^2}/4r(\alpha)q(\alpha))} \\ &\quad \times\left(-c\alpha s-2z\alpha+z\sqrt{\lambda}\right) \\ &\quad \times {}_2F_1\left(E(\alpha)(T(\alpha)+R(\alpha)),\right. \\ &\qquad E(\alpha)(T(\alpha)+R(\alpha)), \\ &\qquad \left(1+\frac{p(\alpha)}{q(\alpha)r(\alpha)}\right); \\ &\qquad \left.\frac{c\alpha\sqrt{\lambda s}}{\left(-z\sqrt{\lambda}+\alpha(cs+2z)\right)r(\alpha)}\right), \end{aligned} \tag{A.2}$$

where $E$, $R$, $T$, $U$, $r$, $q$, and $p(\alpha)$ are given by relations (A.2)–(A.6):

$$\begin{aligned} E(\alpha) &= \frac{1}{2\sqrt{-c-8\alpha^2+4\alpha\sqrt{\lambda}}\left(\alpha+1/2\sqrt{\lambda}\right)\left(2\alpha\sqrt{\lambda}+\lambda\right)} \\ &= \frac{1}{4\sqrt{\lambda}r^2(\alpha)q(\alpha)}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} r(\alpha) &= \sqrt{-c-8\alpha^2+4\alpha\sqrt{\lambda}}, \\ q(\alpha) &= \left(\alpha+\frac{1}{2}\sqrt{\lambda}\right), \\ p(\alpha) &= \sqrt{-(\chi+1)^2c^2}, \end{aligned} \tag{A.4}$$

$$T(\alpha) = \left(\left(2\alpha^2+\frac{1}{4}c\right)\sqrt{\lambda}+\alpha\lambda\right)\sqrt{-c-8\alpha^2+4\alpha\sqrt{\lambda}}, \tag{A.5}$$

$$U(\alpha) = \left(\frac{3}{2}\alpha + \frac{3}{4}\sqrt{\lambda}\right)\sqrt{-c^2\lambda} - \sqrt{-(\chi+1)^2 c^2}\left(2\alpha\sqrt{\lambda} + \lambda\right), \tag{A.6}$$

$$R(\alpha) = \left(\frac{3}{2}\alpha + \frac{3}{4}\sqrt{\lambda}\right)\sqrt{-c^2\lambda} + \sqrt{-(\chi+1)^2 c^2}\left(2\alpha\sqrt{\lambda} + \lambda\right). \tag{A.7}$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Holographic Screens in Ultraviolet Self-Complete Quantum Gravity

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This paper studies the geometry and the thermodynamics of a *holographic screen* in the framework of the ultraviolet self-complete quantum gravity. To achieve this goal we construct a new static, neutral, nonrotating black hole metric, whose outer (event) horizon coincides with the surface of the screen. The spacetime admits an extremal configuration corresponding to the minimal holographic screen and having both mass and radius equalling the Planck units. We identify this object as the spacetime fundamental building block, whose interior is physically unaccessible and cannot be probed even during the Hawking evaporation terminal phase. In agreement with the holographic principle, relevant processes take place on the screen surface. The area quantization leads to a discrete mass spectrum. An analysis of the entropy shows that the minimal holographic screen can store only one byte of information, while in the thermodynamic limit the area law is corrected by a logarithmic term.

## 1. Introduction

"Quantum gravity" is the common tag for any attempt to reconcile gravity and quantum mechanics. Since the early proposals by Wheeler [1, 2] and DeWitt [3], up to the recent ultraviolet (UV) self-complete scenario [4], the diverse formulations of a would-be quantum theory of gravity have shown a common feature, that is, a fundamental length/energy scale where the smooth manifold model of spacetime breaks down. Let us refer to this scale as the "Planck scale" irrespectively whether it is $10^{19}$ GeV or $10$–$10^2$ TeV. The very concept of distance becomes physically meaningless at the Planck scale and spacetime "evaporates" into something different, a sort of "foamy" structure, a spin network, a fractal dust, and so forth, according to the chosen model [5]. As a matter of fact, one of the most powerful frameworks for describing the Planckian phase of gravity is definitely (Super) String Theory. The price to pay to have a perturbatively finite, anomaly-free quantum theory is to give up the very idea of point-like building blocks of matter and replace them with one-dimensional vibrating strings. As there does not exist any physical object smaller than a string; there are no physical ways to probe distances smaller than the length of the string itself. In this regard two properties of fundamental strings are worth mentioning:

(i) string excitations correspond to different mass and spin "particle" states;

(ii) highly excited strings share various physical properties with black holes.

Thus, we infer that string theory provides a bridge between particle-like objects and black holes (see for instance [6]). However, it is important to remark that while the Compton wavelength of a particle-type excitation decreases by increasing the mass, the Schwarzschild radius of a black hole increases with its mass. Thus, the first tenet of high energy

particle physics, which is "the higher the energy the shorter the distance," breaks down when gravity comes into play and turns a "particle" into a black hole. The above remark is the foundation of the UV self-complete quantum gravity scenario, where the Planckian and sub-Planckian length scales are permanently shielded from observation due to the production of black hole excitations at Planck energy scattering [7]. Accordingly the Planck scale assumes the additional meaning of scale at which matter undergoes a transition between its two admissible "phases," that is, the particle phase and the black hole phase [8–10]. From this perspective, trans-Planckian physics is dominated by larger and larger black hole configurations. It follows that only black holes larger than, or at most equal to, Planck size objects can self-consistently fit into this scheme. However, classical black hole solutions do not fulfill this requirement, that is, the existence of a lower bound for their mass and size (see Figure 1).

A first attempt to overcome this limitation is offered by the noncommutative geometry inspired solutions of the Einstein equations [11]. The latter are a family of regular black holes which span all possible combinations of parameters, such as mass [12], charge [13], and angular momentum [14, 15]. In addition such regular geometries admit a variety of complementary gravitational configurations such as traversable wormholes [16], dirty black holes [17], dilaton gravity black holes [18], and collapsing matter shells [19]. Recently this family of black holes has been recognized as viable solutions of nonlocal gravity [20, 21], that is, a set of theories exhibiting an infinite number of derivative terms of the curvature scalar [22–24] in place of the mere Ricci scalar as in the standard Einstein-Hilbert action. More importantly extensions of noncommutative geometry inspired metrics to the higher dimensional scenario [25, 26] are currently under scrutiny at the LHC for their unconventional phenomenology [27]: specifically the terascale black holes described by such regular metrics tend to have a slower evaporation rate [28] and emit only soft particles mainly on the four-dimensional brane [29]. A characteristic feature of this type of solutions is that the minimum size configuration is given by the extremal black hole configuration which exists even in the neutral nonspinning case [30–32]. This fact automatically implies a minimum energy for black hole production in particle collisions [33] without any further need of correcting formulas of cross sections with *ad hoc* threshold functions. Extremal configurations play a crucial role in the physics of the decaying de Sitter universe via the nucleation of microscopic black holes. It has been shown that Planck size noncommutative inspired black holes might have been copiously produced during inflationary epochs [34]. This fact has further phenomenological repercussions: being stable, noninteracting objects, extremal black holes turn out to be a reliable candidate for dark matter component. On the theoretical side, extremal configurations in the presence of a negative cosmological term can provide a short scale completion of the Hawking-Page diagram which switches to a more realistic Van der Waals phase diagram [35].

Extremal configurations can be either descending from the introduction of a fundamental length in the line element and can alternatively be interpreted as a phenomenological input from quantum gravity: in the latter case it has been shown that such extremal black holes fit pretty well in the UV self-complete scenario providing a stable, minimum size probe at the transition point between particles and black holes [36].

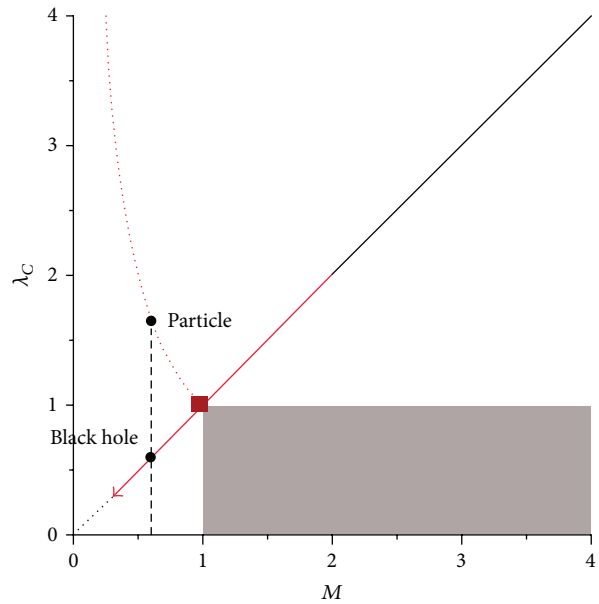


Figure 1: The dotted and the solid curves represent the particle Compton wavelength $\lambda_C$ and the Schwarzschild radius as a function of the energy $M$ in Planck units (quantities are rescaled). The squared bullet is the Planck scale. The grey area of the diagram is actually excluded, meaning that a particle cannot be compressed at distances smaller than the Planck length: at trans-Planckian energy only black hole form. The arrow shows the inadequacy of the Schwarzschild metric: black holes have no lower mass bounds, can have size smaller than the Planck length, and can expose the curvature singularity by decaying through the Hawking process.

In this paper we want to take a step further in the realization of this program by avoiding the introduction of an additional principle to justify the presence of a minimal length, rather we demand the radius of a Planck size extremal black hole to provide the natural UV cutoff of a quantum spacetime. In this framework gravity is expected to be self-regular in the sense that the actual regulator cutting off sub-Planckian length scales is given in terms of the gravitational coupling constant; that is, $\sqrt{G} = L_P$. The paper is organized as follows. In Section 2 we derive a black hole metric, consistent with the above discussion and the concept of holographic screen. The latter coincides with the outer horizon of the black hole whose mass spectrum is bounded from below by the mass of the extremal configuration equalling the Planck mass. Once trans-Planckian length scales are cut-ff, the "interior" of the black hole loses its physical meaning in the sense that all the relevant degrees of freedom are necessarily located on the horizon itself. In Section 3 we discuss the thermodynamics of the screen. We find that the area law is modified by logarithmic corrections and that there exists a minimal holographic screen with zero thermodynamic

entropy. Finally we propose a "holographic quantization" scheme where the area of the extremal configuration provides the quantum of surface. In Section 4 we offer the reader a brief summary of the main results of this work.

## 2. Self-Regular Holographic Screen

A simple but intriguing model of singularity-free black hole has been "guessed" in [37], in the sense that the metric was assigned as an input for the Einstein equations. Sometimes this inverted procedure is called "engineering" because the actual source term of field equations is not known a priori. The distinctive feature of the solution is the presence in the line element of a free parameter with dimension of a length, acting as a short distance regulator for the spacetime curvature, allowing a safe investigation of back-reaction effects of the Hawking radiation. In [38] a higher dimensional extension of this model has been proposed; it was also shown that, by a numerical rescaling of the short-distance regulator, it is possible to identify this fundamental length scale with the radius of the extremal configuration. With hindsight, we are going to take a step forward to improve this inverse procedure. Specifically, we want to follow the "*direct way*" by building up a consistent source for Einstein equations: we introduce a physically motivated energy momentum tensor which allows for transitions between particle-like objects and black holes as consistently required by UV self-complete quantum gravity.

We start from the energy density for a point-particle in spherical coordinates as

$$\rho_p(r) = \frac{M}{4\pi r^2}\delta(r), \tag{1}$$

where $\delta(r)$ is the Dirac delta. The energy distribution (1) implies a black hole for any value of mass $M$ even for sub-Planckian values where one expects just particles. Before proceeding, we would like to recall that a Dirac delta function can be represented as the derivative of a Heaviside step-function $\Theta$:

$$\delta(r) = \frac{d}{dr}\Theta(r). \tag{2}$$

Against this background, we want to accommodate both particles and black holes by a suitable modification of the energy distribution in order to overcome the ambiguities of the Schwarzschild metric in the sub-/trans-Planckian regimes (see also Figure 1). This can be done by considering a "smooth" function $h(r)$ in place of the Heaviside step:

$$\Theta(r) \longrightarrow h(r). \tag{3}$$

The new profile $\rho(r)$ of the energy density is defined through $h(r)$ by the relation

$$\rho(r) = \frac{M}{4\pi r^2}\frac{d}{dr}h(r) \equiv T_0^0. \tag{4}$$

By means of the conservation equation $\nabla_\mu T^{\mu\nu} = 0$ one can determine the remaining components of the stress tensor, which turns to be out of the form

$$T_\mu^\nu = \mathrm{diag}\left(-\rho, p_r, p_\perp, p_\perp\right). \tag{5}$$

The condition for the metric coefficients $g_{00} = -g_{11}^{-1}$ determines the equation of state, namely, the relation between the energy density and the radial pressure, $p_r = -\rho$. The angular pressure is specified by the conservation of the stress tensor and reads $p_\perp = p_r + (r/2)\partial_r p_r$.

By plugging the tensor (5) in Einstein equations, one finds that the metric reads ($G = 1$)

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2m(r)}{r}\right)dt^2 \\ &+ \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned} \tag{6}$$

with

$$m(r) = 4\pi \int dr' \left(r'\right)^2 \rho\left(r'\right). \tag{7}$$

At large distances $r \gg L_P$, the above energy density has to quickly vanish; that is, $\rho(r) \to 0$ in order to match the "vacuum" Schwarzschild metric. Conversely, at shorter scales $r \gtrsim L_P$, the density $\rho(r)$ (and accordingly $h(r)$) has to depart from the point-particle profile in order to fulfill the following requirements:

(i) no curvature singularity in the origin;

(ii) *self-implementation* of a characteristic scale $l_0$ in the spacetime geometry by means of the radius of the extremal configuration $r_0$; that is, $r_0 = l_0$.

The latter condition is crucial. For instance noncommutative geometry inspired black holes [11] are derived by the direct way; they enjoy (i) but fail to fulfill the condition (ii). This means that the characteristic length scale of the system $l_0$ and the extremal configuration radius $r_0$ are independent quantities. Indeed noncommutative geometry is the underlying theory which provides the scale $l_0$ in terms of an "external" parameter, namely, the noncommutative parameter $\theta$. In other words one needs to invoke a principle, like a modification of commutators in quantum mechanics, or the emergence of a quantum gravity induced fundamental length to achieve the regularity of the geometry at short scales. Against this background, we want just to use $r_0$ as fundamental scale, getting rid of any $l_0$ as emerging from any theory or principle not included in Einstein field equations. This is a step forward since it opens the possibility for Einstein gravity to be self-protected in the ultraviolet regime. To emphasize this point, we introduced the word "*self*-implementation" in (ii). Since there exists actually only one additional scale beyond $r_0$, that is, the Planck length $L_P = \sqrt{G}$, or the Planck mass $M_P = 1/\sqrt{G}$, we can implement the condition (ii) in the most natural way by setting $r_0 = L_P$ and accordingly $M_0 = M_P$, where $M_0 \equiv M(r_0)$ is the extremal black hole mass.

Despite the virtues of the above line of reasoning, we feel that the set of conditions (i) and (ii) can be relaxed and a further simplification is possible. Having in mind that for extremal black hole configurations the Hawking emission stops we just need to find a metric for which only the condition (ii) holds. This would be enough for completing

the program of the UV self-complete quantum gravity by protecting the short distance behavior of gravity during the final stages of the evaporation process. In this regard, the resulting extremal black hole is just the smallest object one can use to probe short-distance physics. In other words, in the framework of UV self-complete quantum gravity, it is not physically meaningful to ask about curvature singularity inside the horizon as the very concept of spacetime is no longer defined below this length scale.

According with such a line of reasoning, we can determine the function $h(r)$ by dropping the condition (i) and keeping just the condition (ii). Inside the class of all admissible profiles for $h(r)$, the most natural and algebraically compact choice is given by

$$h(r) = 1 - \frac{L_P^2}{r^2 + L_P^2}. \tag{8}$$

A similar procedure has been already used in [33] and accounts for the fact that in the presence of $L_P$ the step cannot be any longer sharp. Thus, the smeared energy density $\rho(r)$ turns out to be

$$\rho(r) = \frac{M}{2\pi r} \frac{L_P^2}{\left(r^2 + L_P^2\right)^2}. \tag{9}$$

As a result we find the following metric which is derived from a stress tensor modeling a particle-black hole system (5):

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2ML_P^2 r}{r^2 + L_P^2}\right) dt^2 \\ & + \left(1 - \frac{2ML_P^2 r}{r^2 + L_P^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned} \tag{10}$$

where the arbitrary constant $M$ is defined as follows:

$$M \equiv \frac{1}{2L_P^2 r_h}\left(r_h^2 + L_P^2\right). \tag{11}$$

We give $M$ the physical meaning of mass for a spherical, *holographic screen* with radius $r_h$. The basic idea is that gravitational phenomena taking place in three-dimensional space can be projected on a two-dimensional "viewing screen" with no loss of information [39]. The idea of holographic screen has been proposed in [40] and it has mathematically been formulated in [41]: the holographic screen plays the role of "basic constituent of space where the Newton potential is constant." Along this line of reasoning, the idea of holographic screen has been used also in the context of noncommutative inspired metric to derive compelling deviations to Newton's law [42]. For what concerns the current discussion, however, we just need to recall that a special case of holographic screen is given by an event horizon where the entropy is maximized.

Several remarks are in order.

(i) It is easy to show that $M \geq M_P$ and equals the Planck mass only for $r_h = L_P$.

(ii) The line element (10) admits a pair of horizons provided $M \geq M_P$. The radii $r_\pm$ of the horizons are given by

$$r_\pm = L_P^2\left(M \pm \sqrt{M^2 - M_P^2}\right). \tag{12}$$

For $M = M_P$ the two horizons merge into a single (degenerate) null surface at $r_\pm = r_0 = L_P$. For $M \gg M_P$ the outer horizon approaches the conventional value of the Schwarzschild geometry; that is, $r_+ \simeq 2ML_P^2$.

(iii) By inserting (11) into (12) one finds $r_+ = r_h$, $r_- = L_P^2/r_h$. We see that the holographic screen surface coincides with the (outer) black hole horizon $r_+$, while the inner Cauchy horizon has a radius which is always smaller or equal to the Planck length. This fact lets us circumvent the issue of potential blue shift instabilities [43, 44] (see, i.e., recent analyses for noncommutative inspired [45, 46] and other quantum gravity corrected metrics [47, 48]) because $r_-$ simply loses its physical meaning being not accessible to any sort of measurement process. In what follows we can identify the holographic screen with the black hole outer horizon without distinguishing between the two surfaces any longer.

(iv) "Light" objects, with $M < M_P$, are "particles" rather than holographic screens. By particles we mean localized lumps of energy of linear size given by the Compton wavelength $\lambda_C = 1/M$ that can never collapse into a black hole. Rather they give rise to horizonless metrics (see Figure 2) and cannot probe distances smaller than $\lambda_C$. The "transition" particle $\rightarrow$ black holes is discussed below in terms of critical *surface density*.

As a further analysis of this result, it is interesting to consider the *surface energy density* of the holographic screen which is defined as

$$\sigma_h \equiv \frac{M}{4\pi r_h^2} = \frac{1}{8\pi L_P^2} \frac{r_+^2 + L_P^2}{r_+^3}. \tag{13}$$

From the above relation we see that $\sigma_h$ is a monotonically decreasing function of the screen radius. We notice that there exists a minimal screen encoding the physically maximum attainable energy density, that is, the Planck (surface) density:

$$\sigma_h\left(r_+ = L_P\right) = \frac{1}{4\pi L_P^3} = \frac{M_P}{4\pi L_P^2}. \tag{14}$$

We stress that there is no physically meaningful "interior" for the minimal screen; that is, the "volume" of such an object is not even defined, in the sense that it can never be probed. Thus, we can only consider energy per unit area, rather than per unit volume. If we, formally, define a surface energy for a particle as

$$\sigma_p \equiv \frac{M}{4\pi\lambda_C^2} = \frac{1}{4\pi\lambda_C^3} \tag{15}$$

we see that the two curves (13) and (15) cross at $\lambda_C = L_P = r_+$. This result offers an additional interpretation for the Planck length which consistently turns to be the minimal size for a particle as well for a black hole (see Figure 2). Accordingly, the Planck density (14) is the *critical density* for a particle to collapse into a black hole. This argument is usually formulated in terms of volume energy density having in mind the picture of macroscopic body gravitationally collapsing under their own weight. From our holographic vantage point, where "surfaces" are the basic dynamical objects, it is natural to reformulate this reasoning in terms of areal densities [39]. In addition holography offers a way to circumvent potential conflicts between the mechanism of spontaneous dimensional reduction [49, 50] and the UV self-complete paradigm. If we perform the limit for $r \to 0$ the metric (10) would apparently reduce into an effective two-dimensional spacetime:

$$ds^2 \longrightarrow -(1-2Mr)\,dt^2 + (1-2Mr)^{-1}dr^2 + \mathcal{O}\left(\frac{r^2}{L_P^2}\right). \tag{16}$$

As explained in [51], this mechanism would lead the formation of lower dimensional black holes for length scales below the Planck length, in contrast with the predicted semiclassical regime of trans-Planckian black holes in four dimensions. However, contrary to the Schwarzschild metric that eventually reduces into dilaton gravity black holes when $r \simeq L_P$ (for reviews of the mechanism see [52, 53]), the presence of the holographic screen forbids the access to length scales $r < L_P$ and safely protects the arguments at the basis of the UV self-complete quantum gravity.

## 3. Thermodynamics, Area Quantization, and Mass Spectrum

In this section we would like to investigate the thermodynamics of the black hole described by (10) and determine the relation between entropy and area of the event horizon. It is customary to consider the area law for granted in any case, but this assumption leads to an inconsistency with the third law of thermodynamics: extremal black holes have zero temperature but nonvanishing area. Here, we stick to the textbook definition of thermodynamical entropy and not to more exotic quantity like Rényi, or entanglement entropy. To cure this flaw, we will *derive* the relation between entropy and area from the first law, rather than assuming it. The Hawking temperature associated to the metric (10) can be calculated by evaluating the surface gravity $\kappa$:

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\left(\frac{dg_{00}}{dr}\right)_{r=r_+} = \frac{1}{4\pi r_+}\left(1 - \frac{2L_P^2}{r_+^2 + L_P^2}\right), \tag{17}$$

while the heat capacity $C \equiv \partial U/\partial T_H$ is

$$C \equiv \frac{\partial M}{\partial T_H} = -2\pi r_+ \left(\frac{r_+^2 - L_P^2}{L_P^2}\right)\frac{\left(r_+^2 + L_P^2\right)^2}{r_+^4 - 4L_P^2 r_+^2 - L_P^4}. \tag{18}$$

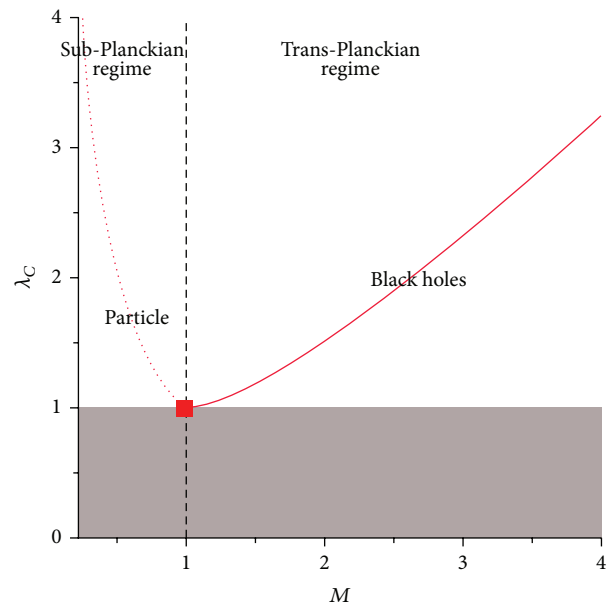


Figure 2: The plot shows a length/energy relation consistent with the self-complete quantum gravity arguments in Planck units. Particles (dotted line) and black holes (solid line) cannot probe length shorter than the Planck length. The grey area is permanently inaccessible and accordingly represents the minimal spacetime time region or fundamental constituent, that is, the "atom" the spacetime is supposed to be made of.

One can check that for large distances, that is, $r_+ \gg L_P$, both (17) and (18) coincide with the conventional results of the Schwarzschild metric; that is, $T_H \approx 1/4\pi r_+$ and $C \approx -2\pi r_+^2$ (see Figures 3 and 4). On the other hand at Planckian scales, contrary to the standard result for which a Planckian black hole has a temperature $T_H = M_P/8\pi$, we have that $T_H \to 0$ as $r_+ \to r_0 = L_P$ as expected for any extremal configurations. This discrepancy with the classical picture is consistent with the genuine *quantum gravitational* character of the black hole and is reminiscent of the modified thermodynamics of noncommutative inspired black holes [54, 55].

The Hawking emission is a semiclassical decay where gravity is considered just in terms of a classical spacetime background. Such a semiclassical approximation conventionally breaks down as the Planck scale is approached. On the other hand for our metric, at $r_+ = r_M = \sqrt{2+\sqrt{5}}L_P \simeq 2.058L_P$ the temperature admits a maximum corresponding to a pole in the heat capacity. In the final stage of the evaporation, that is, $L_P < r_+ < r_M$, the heat capacity is positive; the Hawking emission slows down and switches off at $r_+ = L_P$. From a numerical estimate of the maximum temperature one finds $T_H(r_M) = 0.0239M_P$. This implies that the ratio temperature/mass is $T_H/M < T_H(r_M)/M_0 \simeq 0.0239$. As a consequence, no relevant back reaction occurs during all the evaporation processes and the metric can consistently describe the system "black hole + radiation" for all $r_+ \geq L_P$.

We can summarize the process with the following scheme:

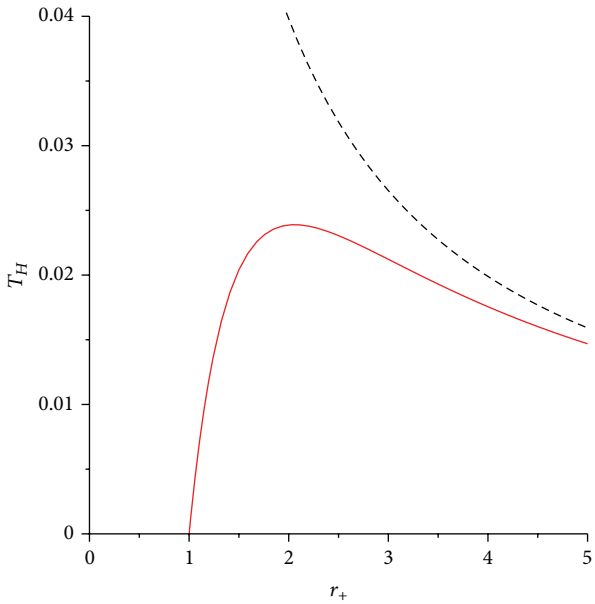


FIGURE 3: The solid curve represents the Hawking temperature $T_H$ and as a function of the horizon radius $r_+$ in Planck units. The dotted curve represents the corresponding classical result in terms of the Schwarzschild metric.

(i) "*large*", far-from-extremality, black holes are semiclassical objects which radiate thermally;

(ii) "*small*", quasi-extremal, black holes are quantum objects;

(iii) $r = r_M$ is "*critical point*" where the heat capacity diverges (see Figure 4). Since $C > 0$ for $r_0 < r_+ < r_M$ and $C < 0$ for $r_M < r_+$, we conclude that a phase transition takes place from large thermodynamically unstable black holes to small stable black holes.

As a matter of fact, the black hole emission preceding the evaporation switching off (often called "SCRAM phase" [11]) might not be thermal. It has been argued that such a quantum regime might be characterized by discrete jumps towards the ground state [7, 56]. To clarify the nature of this mechanism we proceed by studying the black hole entropy profile and the related area quantization. By integrating the first law, taking into account that no black hole can have a radius smaller than $r_0 = L_P$, that is,

$$S(r_+) = \int_{r_0}^{r_+} \frac{dM}{T_H} = \frac{\pi}{L_P^2}\left(r_+^2 - L_P^2\right) + 2\pi \ln\left(\frac{r_+}{L_P}\right), \quad (19)$$

we can cast the entropy in terms of the area of the event horizon $\mathscr{A}_+ \equiv 4\pi r_+^2$ as

$$S(\mathscr{A}_+) = \frac{\pi}{\mathscr{A}_0}\left(\mathscr{A}_+ - \mathscr{A}_0\right) + \pi \ln\left(\frac{\mathscr{A}_+}{\mathscr{A}_0}\right), \quad (20)$$

where $\mathscr{A}_0 = 4\pi L_P^2$ is the area of the extremal event horizon. We remark that the modifications to the Schwarzschild metric, encoded in our model, are in agreement with all the major approaches to quantum gravity, which universally foresee a logarithmic term as a correction to the classical area law. For brevity we recall that this is the case for string theory [57, 58], loop quantum gravity [59–61], and other results based on generic arguments [62, 63], on Cardy's formula [64], conformal properties of spacetimes [65], and other mechanisms for counting microstates [66–68]. We can check that this is the case for the metric (10) by performing the limit $r_+ \gg L_P$ for (20) to obtain

$$S(\mathscr{A}_+) \approx \frac{\mathscr{A}_+}{4L_P^2} + \pi \ln\left(\frac{\mathscr{A}_+}{4\pi L_P^2}\right). \quad (21)$$

Conversely for $r_+ \to L_P$ the entropy vanishes; that is,

$$S(\mathscr{A}_+) \approx \frac{4\pi}{L_P}\left(r_+ - L_P\right) + O\left(\left(r_+ - L_P\right)^2\right). \quad (22)$$

This result is consistent both with the third law of thermodynamics and the entropy statistical meaning. The Planck size, zero temperature, black hole configuration is the unique ground state for holographic screens. Thus, it is a zero entropy state as there is only one way to realize this configuration. To see this we promote the extremal configuration area to the fundamental quantum of area:

$$\mathscr{A}_+ \equiv \mathscr{A}_{n-1} = n\mathscr{A}_0 = 4\pi n L_P^2, \quad (23)$$

where $L_P^2$ represents the basic information pixel and $n = 1, 2, 3\ldots$ is the number of bytes (we borrow here the names of some units of digital information. In the present context, each byte consists of $4\pi$ bits. Each bit, represented by $L_P^2$ is the basic capacity of information of the holographic screen. In the analogy with the theory of information for which a byte represents the minimum amount of bits for encoding a single character of text, here the byte represents the minimum number of basic pixel $L_P^2$ for encoding the smallest holographic screen). From the above condition one obtains

$$\begin{aligned} r_{n-1} &\equiv n^{1/2} L_P, \\ M_{n-1} &\equiv \frac{1}{2}\left(n^{1/2} + n^{-1/2}\right) M_P. \end{aligned} \quad (24)$$

Consistently the ground state of the system is $r_0 = L_P$ and $M_0 = M_P$, while for $n \gg 1$ one finds a continuous spectrum of values. This can be checked through the following relation:

$$\Delta M_n \equiv M_n - M_{n-1} \sim \frac{1}{4} n^{-1/2} M_P. \quad (25)$$

We notice that for $n \leq 4$ we are in the regime of positive heat capacity $C > 0$ and discrete mass spectrum, while for $n > 4$ we approach the semiclassical limit characterized by negative heat capacity $C < 0$ and continuous mass spectrum; that is, $\Delta M_n / M_n \leq 1/12$. This confirms that at $r_+ = r_M$, the system undergoes a phase transition from a semiclassical regime to a genuine quantum gravity regime. As a conclusion we have that large black holes decay thermally, while small objects decay quantum mechanically, by emitting quanta of

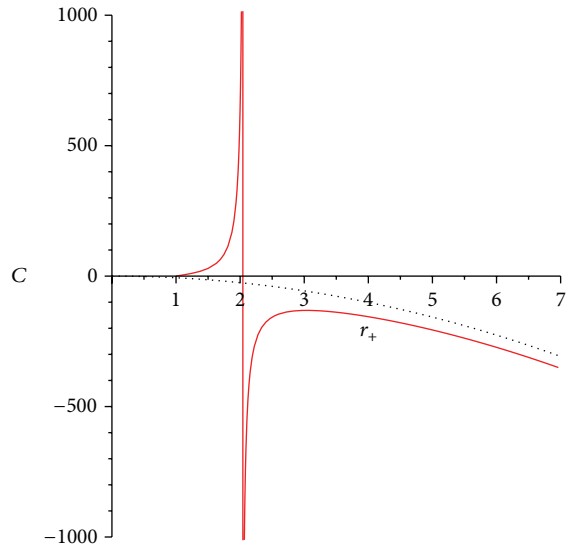


Figure 4: The solid curve represents the black hole heat capacity $C$ as a function of the horizon radius $r_+$ in Planck units. The dotted curve represents the corresponding classical results in terms of the Schwarzschild metric.

energy (for a recent phenomenological analysis of such kind of decay see [69]). The end point of the decay is a Planck mass, holographic screen.

The quantization of the area of the holographic screen lets us disclose further features of the informational content of the holographic screen. We have that the surface density can be written as

$$\sigma_h(n) = \frac{1}{2}\left(\frac{1}{n^{1/2}} + \frac{1}{n^{3/2}}\right)\frac{M_P}{4\pi L_P^2}, \tag{26}$$

while the entropy reads $S(n) = \pi(n + \ln(n) - 1)$. From this relation we learn that, while the entropy increases with the number $n$ of bytes, the surface density decreases. This confirms that the extremal configuration is nothing but a single byte, zero entropy, Planckian density holographic screen.

## 4. Discussion and Conclusions

In this paper we have presented a neutral nonspinning black hole geometry admitting an extremal configuration whose mass and radius coincide with the Planck units. We have reached this goal by suitably modelling a stress tensor able to accommodate both the particle and black hole configurations, undergoing a transition at the Planck scale. We showed that the horizon of the degenerate black hole represents the minimal holographic screen, within which we cannot access any information about the matter-energy content of spacetime.

We showed that a generic holographic screen is described in terms of the outer horizon of the metric (10), while the inner horizon lies within the prohibited region, that is, inside the minimal holographic screen. The whole scheme fits into the gravity self-completeness scenario. For sub-Planckian energy scales one has just a quantum particle able to probe at the most distances of the order of its Compton wavelength. By increasing the degree of compression of the particle, one traverses the Planck scale where a collapse into a black hole occurs, before probing a semiclassical regime at trans-Planckian energies. The virtual curvature singularity of the geometry in $r = 0$ is therefore wiped out since in such a context sub-Planckian lengths have no physical meaning. From this vantage point spacetime stops to exist beyond the Planck scale as there is no physical way to access this regime. Thus, the curvature singularity problem is ultimately resolved by giving up the very concept of spacetime at sub-Planckian length scales.

The study of the associated thermodynamic quantities confirmed that at trans-Planckian energies black holes radiate thermally before undergoing a phase transition to smaller, quantum black holes. The latter decay by emitting a discrete spectrum of quanta of energy and reach the ground state of the evaporation corresponding to the minimal holographic screen. We came to this conclusion by quantizing the black hole horizon area in terms of the minimal holographic screen which actually plays the role of a basic information byte. We showed that in the thermodynamic limit, the area law for the black hole entropy acquires a logarithmic correction in agreement with all the major quantum gravity formulations.

In conclusion, we stress that the line element (10) not only captures the basic features of more "sophisticated" models of quantum gravity improved black holes (e.g., noncommutative geometry inspired black holes [11], loop quantum gravity black holes [70, 71], asymptotically safe gravity black holes [72, 73], and other studies about collapses in quantum gravity [74, 75]) but overcomes some of their current weak points: specifically there is no longer any concern for potential Cauchy instabilities or for conflicts between the gravity self-completeness and the Planck scale spontaneous dimensional reduction mechanism, as well as the scenario of the terminal phase of the evaporation for static, nonrotating, neutral black holes. In addition, for its compact form the new metric allows straightforward analytic calculations and opens the route to testable predictions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Black Holes and Quantum Mechanics

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We look at black holes from different, novel perspectives.

## 1. Introduction

We will first show that black holes, generally, thought to be a general relativistic phenomena could also be understood without invoking general relativity at all. (Indeed, Laplace had anticipated these objects.)

We start by defining a black hole as an object at the surface of which the escape velocity equals the maximum possible velocity in the universe, namely, the velocity of light. We next use the well-known equation of Keplerian orbits [1]

$$\frac{1}{r} = \frac{GM}{L^2}\left(1 + e\cos\theta\right), \tag{1}$$

where $L$, the so-called impact parameter, is given by $Rc$, where $R$ is the point of closest approach, in our case a point on the surface of the object, and $c$ is the velocity of approach, in our case the velocity of light.

Choosing $\theta = 0$ and $e \approx 1$, we can deduce from (1)

$$R = \frac{2GM}{c^2}. \tag{2}$$

Equation (2) gives the Schwarzschild radius for a black hole and can be deduced from the full general relativity theory as well.

We will now use (2) to exhibit black holes at three different scales, the micro-, the macro-, and the cosmic scales.

## 2. Black Holes

Our starting point is the observation that a Planck mass, $10^{-5}$ gms at the Planck length $10^{-33}$ cms, satisfies (2) and as such a Schwarzschild black hole is. Rosen has used nonrelativistic quantum theory to show that such a particle is a mini universe [2].

We next come to stellar scales. It is well known that for an electron gas in a highly dense mass we have [3, 4]

$$K\left(\frac{\overline{M}^{4/3}}{\overline{R}^4} - \frac{\overline{M}^{2/3}}{\overline{R}^2}\right) = K'\frac{\overline{M}^2}{\overline{R}^4}, \tag{3}$$

where

$$\left(\frac{K}{K'}\right) = \left(\frac{27\pi}{64\alpha}\right)\left(\frac{\hbar c}{\gamma m_P^2}\right) \approx 10^{40}, \tag{4}$$

$$\overline{M} = \frac{9\pi}{8}\frac{M}{m_P}, \qquad \overline{R} = \frac{R}{\left(\hbar/m_e c\right)}, \tag{5}$$

$M$ is the mass, $R$ the radius of the body, $m_P$ and $m_e$ are the proton and electron masses, and $\hbar$ is the reduced Planck constant. From (3) and (4), it is easy to see that for $\overline{M} < 10^{60}$, there are highly condensed planet sized stars. (In fact these considerations lead to the Chandrasekhar limit in stellar theory.) We can also verify that for $\overline{M}$ approaching $10^{60}$ corresponding to a mass ~$10^{36}$ gms, or roughly a hundred to a thousand times the solar mass, the radius $R$ gets smaller and

smaller and would be ~$10^8$ cms, so as to satisfy (2) and give a black hole in broad agreement with theory and observation.

Finally for the universe as a whole, using only the theory of Newtonian gravitation, we had deduced [5]

$$R \sim \frac{2GM}{c^2}; \tag{6}$$

that is, (2) where this time $R \sim 10^{28}$ cms is the radius of the universe and $M \sim 10^{55}$ gms is the mass of the universe. (6) can be deduced alternatively from general relativistic considerations also as noted.

Equation (6) is the same as (2) and suggests that the universe itself is a black hole. (This will still be true if there is dark matter.)

It is remarkable that if we consider the universe to be a Schwarzschild black hole as suggested by (6), the time taken by a ray of light to traverse the universe, that is, from the horizon to the singularity, namely, $10^{-5}(M/M_0)$, equals the age of the universe ~$10^{17}$ secs as shown elsewhere [5]. $M_0$ is the mass of the sum. We will deduce this result alternatively a little later.

## 3. Micro Black Holes

Attempts have been made to express elementary particles as tiny black holes by several authors, notably, Markov and Recami [6, 7]. These black holes do not reproduce charge or spin which are so essential.

Let us, instead, observe that if we treat an electron as a Kerr-Newman black hole, then we get the correct quantum mechanical $g = 2$ factor, but the horizon of the black hole becomes complex [4, 8]. Consider

$$r_+ = \frac{GM}{c^2} + \imath b, \qquad b \equiv \left(\frac{G^2M^2}{c^4} - \frac{GQ^2}{c^4} - a^2\right)^{1/2} \tag{7}$$

with $G$ being the gravitational constant, $M$ being the mass, and $a \equiv L/Mc$, $L$ being the angular momentum. While (7) exhibits a naked singularity and as such has no physical meaning, we note that from the realm of quantum mechanics the position coordinate for a Dirac particle is given by

$$x = \left(c^2 p_1 H^{-1} t\right) + \frac{\imath}{2} c\hbar \left(\alpha_1 - c p_1 H^{-1}\right) H^{-1} \tag{8}$$

an expression that is very similar to (7). In the above, the various symbols have their usual meaning. In fact as was argued in detail [4], the imaginary parts of both (7) and (8) are the same, being of the order of the Compton wavelength.

It is at this stage that a proper physical interpretation begins to emerge. Dirac himself observed that to interpret (8) meaningfully it must be remembered that quantum mechanical measurements (unlike classical ones) are really averaged over the Compton scale. Within the scale there are the unphysical Zitterbewegung effects: for a point electron the velocity equals that of light.

Once such a minimum spacetime scale is invoked, then we have a noncommutative geometry as shown by Snyder [9, 10]

$$\begin{gathered}[x, y] = \left(\frac{\imath a^2}{\hbar}\right) L_z, \qquad [t, x] = \left(\frac{\imath a^2}{\hbar c}\right) M_x, \text{etc}, \\ [x, p_x] = \imath\hbar \left[1 + \left(\frac{a}{\hbar}\right)^2 p_x^2\right].\end{gathered} \tag{9}$$

The relations (9) are compatible with special relativity. Indeed, such minimum spacetime models were studied for several decades, precisely to overcome the divergences encountered in quantum field theory [4, 10–13].

All this is symptomatic of the fact that we cannot measure arbitrary small intervals of spacetime in quantum theory, as indeed argued by Dirac himself [14]. Indeed subsequently Salecker and Wigner argued that time within the Compton scale has no physical meaning [15] (and for a detailed discussion cf. [16]). Indeed this quantum mechanical feature explains what Misner et al. termed the greatest crisis of physics [8], namely, the singularity of the black hole. All this has been the matter of detailed study (cf. [16]).

## 4. Black Hole Thermodynamics

The author has approached this problem from the point of view of oscillations at the Planck scale [16]. Briefly, if there are $N$ such oscillators with an amplitude $\Delta x$, then we have

$$R = \sqrt{N\Delta x^2}. \tag{10}$$

This leads to

$$R = \sqrt{N} l_P, \qquad M = \frac{m_P}{\sqrt{N}}, \tag{11}$$

where $M$ is the arbitrary mass, $R$ the extent, and $l_P$ and $m_P$ are the Planck length and Planck mass, respectively. We now use the fact that $l_P$ is the Schwarzschild radius of the Planck mass as was shown by Rosen [2]. Substitution in the above gives us the Schwarzschild radius; that is (4)

$$R = \frac{2GM}{c^2}. \tag{12}$$

It can be immediately seen from (11) that

$$RM = l_P m_P. \tag{13}$$

It must be mentioned that the above is completely consistent with the mass and radius of an arbitrary black hole, including the universe itself.

From the theory of black hole thermodynamics we have as it is well known [17]

$$T = \frac{\hbar c^3}{8\pi k m G}, \tag{14}$$

namely, the Beckenstein temperature. Interestingly, (14) can be deduced alternatively from our above theory of oscillations

at the Planck scale. For this we use the following relations for a Schwarzschild black hole [17]

$$dM = TdS, \qquad S = \frac{kc}{4\hbar G}A, \tag{15}$$

where $T$ is the Bekenstein temperature, $S$ the entropy, and $A$ is the area of the black hole. In our case, the mass $M = \sqrt{N}m_P$ and $A = Nl_P^2$, where $N$ is arbitrary for an arbitrary black hole. This follows from (11). Whence,

$$T = \frac{dM}{dS} = \frac{4\hbar G}{kl_P^2 c}\frac{dM}{dN}. \tag{16}$$

If we use the fact that $l_P$ is the Schwarzschild radius for the Planck mass $m_P$ and use the expression for $M$, the above reduces to (14), the Bekenstein formula.

Equation (14) gives also the thermodynamic temperature of a Planck mass black hole. Further, in this theory as it is known [17],

$$\frac{dM}{dt} = -\frac{\beta}{M^2}, \tag{17}$$

with $M$ being the mass. Before proceeding, we observe that we have deduced a string of $N$ Planck oscillators, $N$ arbitrary, form a Schwarzschild black hole of mass $\sqrt{N}m_P = M$. We can now deduce that

$$\begin{aligned} \frac{dM}{dt} &= \frac{m_P}{t_P}, \\ M &= \left(\frac{m_P}{t_P}\right)\cdot t, \end{aligned} \tag{18}$$

where $t$ is the "Hawking-Bekenstein decay time." For the Planck mass, $M = m_P$, the decay time is the Planck time $t = t_P$. For the universe, the above gives the life time $t$ as $\sim 10^{17}$ sec, the age of the universe again.

Further, we have also seen the emergence of the quantum of area [18] as it is evident from the $N$ elementary Planck areas $l_P^2$ for the black hole (cf. also [18]).

It has also been argued that not only does the universe mimic a black hole but also the black hole is a two dimensional object [16, 19]. Indeed, the interior of a black hole is in any case inaccessible and the two dimensions follow from the area of the black hole which plays a central role in black hole thermodynamics. We have already seen that the area of the black hole is given by

$$A = Nl_P^2. \tag{19}$$

For these quantum gravity considerations, we have to deal with the quantum of area [16, 18]. In other words, we have to consider the black hole to be made up of $N$ quanta of area. It is remarkable that we can get an opportunity to test these quantum gravity features in two-dimensional surfaces such as graphene.

That is, we could model a black hole as a "graphene" ball. Indeed, in the case of graphene as it is well known, and as the author deduced in 1995 [20, 21], this behaviour in two dimensions is given by

$$\nu_F\ \vec{\sigma}\cdot\vec{\nabla}\ \psi(r) = E\psi(r), \tag{20}$$

where $\nu_F \sim 10^6$ m/s is the Fermi velocity replacing $c$, the velocity of light, and $\psi(r)$ is a two-component wave function, $\vec{\sigma}$ and $E$, denoting the Pauli matrices and energy.

Though this resembles the neutrino equation, $\nu_F$ is some three hundred times less than the velocity of light. However the author has argued that for a sufficiently large sheet of graphene, this would approximate the neutrino equation itself, that is, the usual Minkowski spacetime. From this point of view, a black hole can be simulated by a "graphene ball."

It may be mentioned that very recently Hawking has proposed rather shockingly that black holes may not have event horizons [22].

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Particle Collisions in the Lower Dimensional Rotating Black Hole Space-Time with the Cosmological Constant

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We study the effect of ultrahigh energy collisions of two particles with different energies near the horizon of a 2 + 1 dimensional BTZ black hole (BSW effect). We find that the particle with the critical angular momentum could exist inside the outer horizon of the BTZ black hole regardless of the particle energy. Therefore, for the nonextremal BTZ black hole, the BSW process is possible on the inner horizon with the fine tuning of parameters which are characterized by the motion of particle, while, for the extremal BTZ black hole, the particle with the critical angular momentum could only exist on the degenerated horizon, and the BSW process could also happen there.

## 1. Introduction

In the recent paper [1], Banãdos, Silk, and West proposed a mechanism (BSW process) that two particles may collide on the horizon of an extremal Kerr black hole with ultrahigh center-of-mass (CM) energy, although it was pointed out in [2, 3] that the collision in fact takes an infinite proper time. Moreover, there are astrophysical limitations preventing a Kerr black hole from being an extreme one, and the gravitational radiation and backreaction effects should also be included in this process. Due to the potential interest in exploring ultrahigh energy physics, the BSW process has been studied extensively in other kinds of black holes or naked singularities [4–29]. To achieve ultrahigh CM energy under the astrophysical limitation of maximal spin, the multiple scattering was taken into account in the nonextremal Kerr black hole [7, 15]. Another more direct application is to consider different extreme rotating black holes, such as the Kerr-Newman black holes and the Sen black hole [8, 11]. On the other hand, a general explanation of this BSW process was tried to give for a rotating black hole [19] and for other black holes [20, 21]. Some efforts had also been made to draw some implications concerning the effects of gravity generated by colliding particles in [23].

However, all of the works mentioned above have been focused on the black holes embedded in the asymptotically flat space-time without cosmological constant. In our previous work [30], we had considered the BSW process in the background of the Kerr- (anti-) de Sitter black hole with nonzero cosmological constant and had found that the cosmological constant has an important effect on the result. For the case of the general Kerr-de Sitter black hole (with positive cosmological constant), the collision of two particles can take place on the outer horizon of the nonextremal black hole and the CM energy of collision can blow up arbitrarily if one of the colliding particles has the critical angular momentum. In the present paper, we extend the investigation of the BSW process to the background of a 2 + 1 dimensional BTZ black hole [31], and our motivation is to examine whether the BSW effect remains valid in the lower dimensional case. Actually, in [5, 6], Lake had pointed out the divergence of the CM energy of particle collision on the inner horizon of the BTZ black hole, but the process was not discussed in detail. In this paper, we study this process in the BTZ black hole with circumstances.

This paper is organized as follows. In Section 2, we give a brief review of the BTZ black hole. In Section 3, we study the

CM energy of the particle collision on the horizon and derive the critical angular momentum to blow up the CM energy. In Section 4, we investigate the radial motion of colliding particles with the critical angular momentum in detail. The extremal and nonextremal cases are examined, respectively. The conclusion is given in the last section.

## 2. The 2 + 1 Dimensional BTZ Black Hole

In this section we would like to study the horizon structure of the 2 + 1 dimensional BTZ black hole. The metric of the BTZ black hole is usually written as [31] (with units $c = G = 1$)

$$ds^2 = -N_r^2 dt^2 + N_r^{-2} dr^2 + r^2 \left(N_\phi dt + d\phi\right)^2 \tag{1}$$

with

$$\begin{aligned} N_r^2(r) &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \\ N_\phi(r) &= -\frac{J}{2r^2}, \end{aligned} \tag{2}$$

where $M$ and $J$ are the mass and spin angular momentum of the black hole, respectively, and $l^2$ is related to the cosmological constant $\Lambda$ by $l^{-2} = -\Lambda$.

The horizons can be solved from $N_r|_{r=r_h} = 0$, and they are given by

$$r_\pm = \sqrt{\frac{l}{2}\left(lM \pm \sqrt{l^2M^2 - J^2}\right)}. \tag{3}$$

Here, $r_+$ is the outer horizon and $r_-$ is the inner horizon. The existence of the horizon requires

$$|J| \le Ml. \tag{4}$$

The horizon of the extremal black hole (corresponding to $|J| = Ml$) is read as

$$r_e = \sqrt{\frac{M}{2}}\, l. \tag{5}$$

## 3. The Center-of-Mass Energy for the On-Horizon Collision

To investigate the CM energy of the collision on the horizon of the BTZ black hole, we have to derive the 2 + 1 dimensional "4-velocity" component of the colliding particle in the background of the 2 + 1 dimensional BTZ black hole.

The generalized momentum $P_\mu$ is

$$P_\mu = g_{\mu\nu}\dot{x}^\nu, \tag{6}$$

where the dot denotes the derivative with respect to the affine parameter $\lambda$ and $\mu, \nu = t, r, \phi$. Thus, the components $P_t$ and $P_\phi$ of the momentum are turned out to be

$$\begin{aligned} P_t &= g_{tt}\dot{t} + g_{t\phi}\dot{\phi}, \\ P_\phi &= g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t}. \end{aligned} \tag{7}$$

$P_t$ and $P_\phi$ are constants of motion. In fact, they correspond to the test particle's energy per unit mass $E$ and the angular momentum parallel to the symmetry axis per unit mass $L$, respectively. And in the following discussion we will just regard these two constants of motion as $-E \equiv P_t$ and $L \equiv P_\phi$ [24].

The affine parameter $\lambda$ can be related to the proper time by $\tau = \mu\lambda$, where $\tau$ is given by the normalization condition $-\mu^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ with $\mu^2 = 1$ for time-like geodesics and $\mu^2 = 0$ for null geodesics. For a time-like geodesic, the affine parameter can be identified with the proper time, and thus, from (7), we can solve the 2 + 1 dimensional "4-velocity" components $\dot{t}$ and $\dot{\phi}$ (where the dot denotes a derivative with respect to the proper time now) as

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{2E - \left(JL/r^2\right)}{2N_r^2}, \\ \frac{d\phi}{d\tau} &= \frac{J\left(-JL + 2Er^2\right) + 4Lr^2N_r^2}{4r^4N_r^2}. \end{aligned} \tag{8}$$

For the remaining component $\dot{r} = dr/d\tau$ of the radial motion, we can obtain it from the Hamilton-Jacobi equation of the time-like geodesic:

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \tag{9}$$

with the ansatz

$$S = \frac{1}{2}\tau - Et + L\phi + S_r(r), \tag{10}$$

where $S_r(r)$ is a function of $r$. Inserting the ansatz into (9), with the help of the metric (1), we get

$$\left(\frac{dS_r(r)}{dr}\right)^2 = \frac{J^2L^2 - 4EJLr^2 - 4r^2\left[L^2N_r^2 + \left(-E^2 + N_r^2\right)r^2\right]}{4N_r^4r^4}. \tag{11}$$

On the other hand, we have

$$\frac{dS_r(r)}{dr} = P_r = g_{rr}\dot{r} = \frac{\dot{r}}{N_r^2}. \tag{12}$$

Thus we get the square of the 4-velocity radial component:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{K^2 - 4r^2N_r^2\left(L^2 + r^2\right)}{4r^4}, \tag{13}$$

where

$$K = JL - 2Er^2. \tag{14}$$

Here we have obtained all nonzero 2 + 1 dimensional "4-velocity" components for the geodesic equation. Next we would like to study the CM energy of the two-particle collision in the background of the BTZ black hole. Here we consider a more general case that the two colliding particles have different energies $E_1$ and $E_2$ and different

angular momenta per unit mass $L_1$ and $L_2$. For simplicity, the particles under consideration have the same rest mass $m_0$. We can compute the CM energy $E_{\mathrm{CM}}$ of this two-particle collision by using

$$E_{\mathrm{CM}} = \sqrt{2}m_0\sqrt{1 - g_{\mu\nu}u_1^\mu u_2^\nu}, \tag{15}$$

where $u_1^\mu$ and $u_2^\nu$ are the "4-velocity" vectors of the two particles ($u = (\dot{t}, \dot{r}, \dot{\phi})$). With the help of (8) and (13), we obtain the CM energy:

$$\begin{aligned}\frac{E_{\mathrm{CM}}^2}{2m_0^2} = \frac{1}{4r^4N_r^2}\left[\left(JL_1 - 2E_1r^2\right)\left(JL_2 - 2E_2r^2\right)\right.\\ \left. + 4r^2N_r^2\left(-L_1L_2 + r^2\right) - H_1H_2\right],\end{aligned} \tag{16}$$

where

$$\begin{aligned}H_i = \sqrt{\left(JL_i - 2E_ir^2\right)^2 - 4r^2N_r^2\left(L_i^2 + r^2\right)}\\ (i = 1, 2).\end{aligned} \tag{17}$$

For simplicity, we can rescale the CM energy as $\overline{E}_{\mathrm{CM}}^2 \equiv (1/2m_0^2)E_{\mathrm{CM}}^2$. We would like to study $\overline{E}_{\mathrm{CM}}^2$ for the case that the particles collide on the black hole's horizon, which means $N_r = 0$. The denominator of the expression on the right hand of (16) is zero, and the numerator of it is

$$\begin{aligned}K_1K_2 - \sqrt{K_1^2}\sqrt{K_2^2},\\ K_i = K|_{E=E_i, L=L_i}, \quad i = 1, 2.\end{aligned} \tag{18}$$

When $K_1K_2 \geq 0$, the numerator will be zero and the value of $\overline{E}_{\mathrm{CM}}^2$ on the horizon will be undetermined, but when $K_1K_2 < 0$, the numerator will be a negative finite value and $\overline{E}_{\mathrm{CM}}^2$ on the horizon will be negative infinity. So it should have $K_1K_2 \geq 0$, and, for the CM energy on the horizon, we have to compute the limiting value of (16) as $r \to r_h$, where $r_h$ is the horizon of the black hole.

After some calculations, we get the limiting value of (16):

$$\begin{aligned}&\overline{E}_{\mathrm{CM}}^2(r \longrightarrow r_h)\\ &= 2 + \frac{(L_1 - L_2)^2 - l^2(E_1 - E_2)^2 - 2(L_1 - L_2)(L_{C1} - L_{C2})}{2(L_1 - L_{C1})(L_2 - L_{C2})}\\ &\quad + \frac{l\left[(E_2L_1 - E_1L_2)^2 + Ml^2(E_1 - E_2)^2\right]\left(lM + \sqrt{l^2M^2 - J^2}\right)}{J^2(L_1 - L_{C1})(L_2 - L_{C2})},\end{aligned} \tag{19}$$

which can also be rewritten as

$$\overline{E}_{\mathrm{CM}}^2(r \longrightarrow r_h) = 2 + \frac{A}{2K_1K_2}, \tag{20}$$

where

$$\begin{aligned}A = J^2\left[(L_1 - L_2)^2 - (E_1 - E_2)^2l^2\right.\\ \left. - 2(L_1 - L_2)(L_{C1} - L_{C2})\right]\\ + 2l\left[(E_2L_1 - E_1L_2)^2 + (E_1 - E_2)^2l^2M\right]\\ \times\left(lM + \sqrt{l^2M^2 - J^2}\right).\end{aligned} \tag{21}$$

So it can be seen that when $K_i = 0$, the CM energy on the horizon will blow up. Solving $K_i = 0$, we get the critical angular momentum:

$$L_{Ci} = \frac{2r_h^2E_i}{J} = \frac{E_il\left(lM + \sqrt{l^2M^2 - J^2}\right)}{J}, \quad i = 1, 2. \tag{22}$$

It is easy to prove that when $K_1 = 0$ and $K_2 = 0$, the CM energy is finite. So in order to obtain an arbitrarily high CM energy, one and only one of the colliding particles should have the critical angular momentum. For the extremal BTZ black hole $J = lM$, the $\overline{E}_{\mathrm{CM}}^2$ on the extremal horizon is

$$\begin{aligned}&\overline{E}_{\mathrm{CM}}^2(r \longrightarrow r_e)\\ &= 2 + \frac{M\left[(L_1 - E_1l) - (L_2 - E_2l)\right]^2 + 2(E_2L_1 - E_1L_2)^2}{2M(L_1 - E_1l)(L_2 - E_2l)}.\end{aligned} \tag{23}$$

Obviously, when one particle has the critical angular momentum $L_{C1} = E_1l$ (or $L_{C2} = E_2l$) and the other does not, the CM energy on the extremal horizon could be infinite.

From the above derivation, it seems that the CM energy could blow up on the horizon. However, in order to get arbitrarily high CM energy on the horizon of the BTZ black hole, the colliding particle with the critical angular momentum must be able to reach the horizon of the black hole. We will investigate this part in the next section.

## 4. The Radial Motion of the Particle with the Critical Angular Momentum near the Horizon

In this section, we will study the radial motion of the particle with the critical angular momentum and find the region where it can exist. In order for a particle to reach the horizon of the black hole, the square of the radial component of the "4-velocity" $(dr/d\tau)^2$ in (13) has to be positive in the neighborhood of the black hole's horizon. Obviously, $R(r)|_{L=L_{Ci}} = 0$ on the horizon of the BTZ black hole. For a particle with arbitrary energy $E$ and angular momentum $L$, the explicit form of $(dr/d\tau)^2$, which is denoted by $R(r)$, reads

$$\begin{aligned}R(r) \equiv \left(\frac{dr}{d\tau}\right)^2 = E^2 - \frac{L^2}{l^2} + M\\ + \frac{1}{r^2}\left(L^2M - EJL - \frac{J^2}{4}\right) - \frac{r^2}{l^2}.\end{aligned} \tag{24}$$

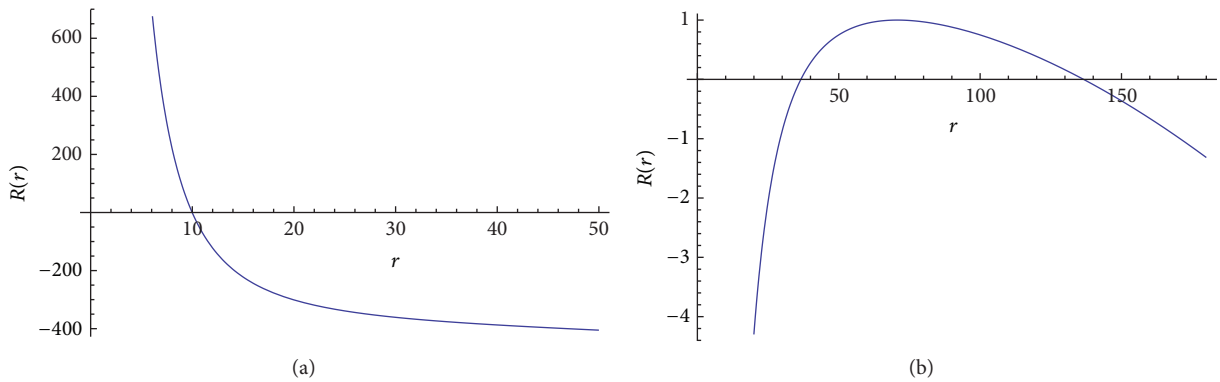



Figure 1: Behaviors of $R(r)$ for (a) $L^2M - EJL - (J^2/4) > 0$ and (b) $L^2M - EJL - (J^2/4) < 0$.

We draw $R(r)$ in Figure 1. It can be seen that, when $L^2M - EJL - J^2/4 > 0$, $R(r \to 0) \to +\infty$ and $R(r \to +\infty) \to -\infty$, so there is only one positive root for $R(r) = 0$ and the particle can exist in the region inside of the root. When $L^2M - EJL - J^2/4 < 0$, $R(r \to 0) \to -\infty$ and $R(r \to +\infty) \to -\infty$, and there are two positive roots and the particle can exist in the region between the two roots. The bigger root of $R(r) = 0$ is

$$r_2 = \left(l^2\left(E^2 + M\right) - L^2 + \sqrt{\left[l^2M - Jl + (L - El)^2\right]\left[l^2M + Jl + (L + El)^2\right]}\right)^{1/2} \times \left(\sqrt{2}\right)^{-1}. \tag{25}$$

We find that it increases with $E$ and $L$, which means that the particle can move arbitrarily far from black hole's horizon with its energy and angular momentum's increase.

Next, we will study the radial motion of the particle with the critical angular momentum:

$$R(r)|_{L=L_c} = \frac{W}{r^2} - \frac{r^2}{l^2} + 2E^2 + M - \frac{2E^2l^2M^2 + 2E^2lM\sqrt{l^2M^2 - J^2}}{J^2}, \tag{26}$$

where

$$W = \frac{E^2l\left[2lM\left(l^2M^2 - J^2\right) + \left(2l^2M^2 - J^2\right)\sqrt{l^2M^2 - J^2}\right]}{J^2} - \frac{J^2}{4}. \tag{27}$$

By solving $W = 0$ we get the critical energy $E_0$:

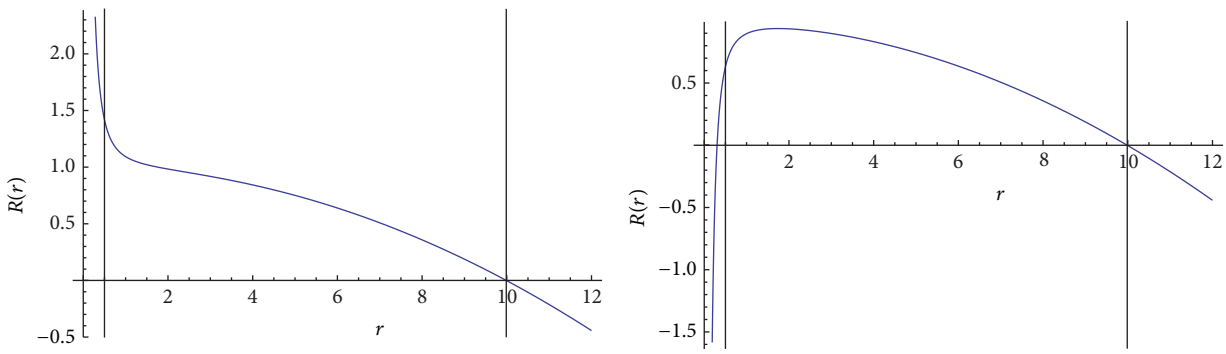
$$E_0 = \frac{J^2}{2\sqrt{2l^2M\left(l^2M^2 - J^2\right) + \left(2l^3M^2 - J^2l\right)\sqrt{l^2M^2 - J^2}}}. \tag{28}$$

When $E > E_0$, $R(r) = 0$ has one root

$$r_0 = \frac{1}{\sqrt{2}J}\left\{l^2\left[J^2M + 2E^2 \times \left(J^2 - lM\left(lM + \sqrt{l^2M^2 - J^2}\right)\right)\right] + l\left[\left(l^2M^2 - J^2\right) \times \left[J^4 + 8E^4l^3M\left(lM + \sqrt{l^2M^2 - J^2}\right) + 4E^2J^2l\left(lM - E^2l + \sqrt{l^2M^2 - J^2}\right)\right]\right]^{1/2}\right\}^{1/2} \tag{29}$$

and the particle with the critical angular momentum can exist inside of it. When $E < E_0$, $R(r) = 0$ has two roots

$$r_{0+} = \frac{1}{\sqrt{2}J}\left\{l^2\left[J^2M + 2E^2 \times \left(J^2 - lM\left(lM + \sqrt{+l^2M^2 - J^2}\right)\right)\right] + l\left[\left(l^2M^2 - J^2\right) \times \left(J^4 + 8E^4l^3M\left(lM + \sqrt{l^2M^2 - J^2}\right) + 4E^2J^2l\left(lM - E^2l + \sqrt{l^2M^2 - J^2}\right)\right)\right]^{1/2}\right\}^{1/2}, \tag{30}$$




(a) (b)

Figure 2: The variation of $R(r)$ versus radius $r$ for the case of the nonextremal BTZ black hole ($l = 10$, $M = 1$, and $J = 1$), $E > E_0$ ((a) $E = 0.003$) and $E < E_0$ ((b) $E = 0.002$). The vertical lines denote the locations of the outer and inner horizons.

$$r_{0-} = \frac{1}{\sqrt{2}J}\left\{l^2\left[J^2M + 2E^2 \times\left(J^2 - lM\left(lM + \sqrt{+l^2M^2 - J^2}\right)\right)\right] - l\left[\left(l^2M^2 - J^2\right) \times\left(J^4 + 8E^4l^3M\left(lM + \sqrt{l^2M^2 - J^2}\right) + 4E^2J^2l\left(lM - E^2l + \sqrt{l^2M^2 - J^2}\right)\right)\right]^{1/2}\right\}^{1/2}, \tag{31}$$

and the particle with the critical angular momentum can exist between them. The above discussion only concerns the square of the "4-velocity" radial component. To find whether the particle with the critical angular momentum can reach the horizon of the BTZ black hole, we should investigate the roots of $R(r) = 0$ and the horizons of the black hole. The nonextremal and extremal cases will be considered in the following.

*4.1. Nonextremal BTZ Black Hole.* For the nonextremal BTZ black hole case, we can prove that the solution (for $E > E_0$ case) or the bigger solution (for $E < E_0$ case) of $R(r) = 0$ is just the outer horizon of black hole:

$$r_0 = r_+ = \sqrt{\frac{l}{2}\left(lM + \sqrt{l^2M^2 - J^2}\right)}. \tag{32}$$

That means that the particle with the critical angular momentum can exist inside the outer horizon of the nonextremal BTZ black hole.

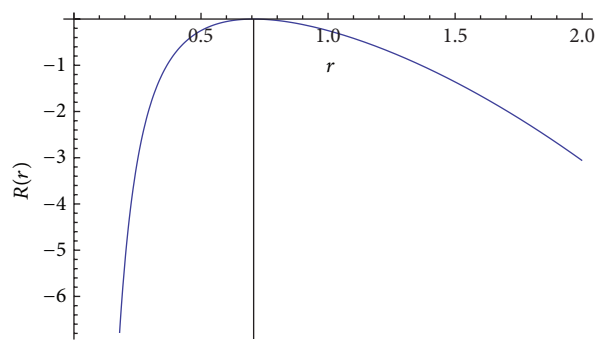



Figure 3: The variation of $R(r)$ versus radius $r$ for the case of the extremal BTZ black hole ($l = 1$, $M = 1$, and $J = 1$). The vertical line denotes the locations of the degenerated horizon.

*4.2. Extremal BTZ Black Hole.* For the extremal BTZ black hole case, $R(r)$ for particle with the critical angular momentum becomes very simple:

$$R(r) = M - \frac{r^2}{l^2} - \frac{J^2}{4r^2}. \tag{33}$$

We solve $R(r) = 0$ and get

$$r_0 = \sqrt{\frac{M}{2}}l. \tag{34}$$

It is just the degenerated horizon of the extremal black hole.

The behaviors of $R(r)$ for the particle with the critical angular momentum are plotted in Figure 2 for the nonextremal black hole and Figure 3 for the extremal black hole. For the nonextremal black hole, we find that the particle with the critical angular momentum can exist inside the outer horizon. So particle collision on the inner horizon could produce unlimited CM energy. For the extremal black hole, the particle with the critical angular momentum could only exist on the degenerated horizon, so if such particle exists, then unlimited CM energy will be approached.

## 5. Conclusion

In this work, we have analyzed the possibility that the 2 + 1 dimensional BTZ black holes can serve as particle accelerator. We first calculate the CM energy for the two-particle collision. In order to obtain unlimited CM energy, one of the particles should have the critical angular momentum. Next, we study the radial motion for the particle with the critical angular momentum. For the extremal BTZ black hole, particles with critical angular momentum can only exist on the outer horizon of the BTZ black hole. So if such particle exists, then unlimited CM energy will be approached. For the nonextremal BTZ black hole, particles can collide on the inner horizon with arbitrarily high CM energy.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Entropy Spectrum of a KS Black Hole in IR Modified Hořava-Lifshitz Gravity

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As a renormalizable theory of gravity, Hořava-Lifshitz gravity, might be an ultraviolet completion of general relativity and reduces to Einstein gravity with a nonvanishing cosmological constant in infrared. Kehagias and Sfetsos obtained a static spherically symmetric black hole solution called KS black hole in the IR modified Hořava-Lifshitz theory. In this paper, the entropy spectrum and area spectrum of a KS black hole are investigated based on the proposal of adiabatic invariant quantity. By calculating the action of producing a pair of particles near the horizon, it is obtained that the action of the system is exactly equivalent to the change of black hole entropy, which is an adiabatic invariant quantity. With the help of Bohr-Sommerfeld quantization rule, it is concluded that the entropy spectrum is discrete and equidistant spaced and the area spectrum is not equidistant spaced, which depends on the parameter of gravity theory. Some summary and discussion will be given in the last.

## 1. Introduction

In the 1970's, with the discovery of Hawking radiation and Bekenstein's proposal of black hole entropy, black hole thermodynamics has been built up successfully, which has opened a new field to study quantum theory and gravity theory [1–4]. Nowadays there has been some trouble on the statistic origin and quantization of black hole entropy for physicists on black hole thermodynamics. Since Bekenstein proposed that the horizon area of a nonextremal black hole is an adiabatic invariant classically and the horizon area of black hole is quantized in units of $l_p^2$ [5–8], there has been much attention paid to the quantization of black hole entropy spectrum and area spectrum. Hod proposed that if one employs Bohr's corresponding principle, the real part of the quasinormal mode frequency is responsible for the area spectrum of black hole [9, 10]. Combining the proposal by Bekenstein for the adiabaticity of black hole horizon area and Hod's proposal, Kunstatter proved that entropy spectrum of a $d$-dimensional black hole is quantized and the result is in agreement with that of Hod and Bekenstein [11]. Later, Maggiore gave a new interpretation of black hole quasinormal modes in connection to the quantization of black hole horizon area. An important statement in Maggiore's work is that the periodicity of a black hole in Euclidean time may be the origin of area quantization [12]. It is well known that for any background spacetime with a horizon in Kruskal coordinates, the period with respect to Euclidean time takes the form of $T = 2\pi/\kappa$, where $\kappa$ is the surface gravity of the horizon. Vagenas exclusively used the fact that the black hole horizon area is an adiabatic invariant quantum and derived an equally spaced entropy spectrum of a black hole with its value to be equal to that of Bekenstein [13]. Zeng et al. considered that the action $I$, action variable $I_v$, and Hamiltonian $H$ of any single periodic system satisfy the relation $I = I_v - \int H dt$. They proposed that the action variable can be quantized with the equally spaced form $I_v = 2\pi n\hbar$. Once the action and Hamiltonian are given, the quantization action variable can be obtained. With the help of Bohr-Sommerfeld quantization rule, they proved that the quantized action variable is nothing but the entropy of black hole; thus the entropy and the horizon area of a black hole

can be quantized. They emphasized that the action variable should be adiabatic invariant [14–16]. Recently, Jiang and Han argued that the adiabatic invariant quantity $\int p_i dq_i$ is not canonically invariant, and the adiabatic invariant quantum should be of the covariance form $\oint p_i dq_i = nh$. They also obtained the equally spaced entropy spectrum with the form of $\Delta S = 2\pi$ [17]. Some more works on the quantization of entropy spectrum of a black hole can be seen in the paper [18–23].

Hořava gravity is a nonrelativistic renormalizable theory of gravitation [24], which is inspired by the anisotropic scaling between time and space in condensed matter systems in particular in the theory of quantum critical phenomena, where the degree of anisotropy between space and time is characterized by the "dynamical critical exponent" $z$. It is well known that relativistic systems automatically satisfy $z = 1$ as a consequence of Lorentz invariance. In Hořava-Lifshitz theory, systems' scaling at a short distance exhibits a strong anisotropy between space and time with $z > 1$. This will improve the short-distance behavior of the theory. The anisotropy at short distance can be lost for long distance while the Lorentz symmetry will appear as an emergent symmetry. The black hole solution of original Hořava-Lifshitz gravity does not recover the usual Schwarzschild-anti-de Sitter black hole with the detailed-balance condition. A relevant operator proportional to the $3D$ geometry Ricci scalar of the original Hořava-Lifshitz theory action was introduced [25] and it deviated from detailed balance. This does not modify the ultraviolet properties of the theory. However, it modifies the infrared Hořava-Lifshitz gravity theory. So a Schwarzschild-anti-de Sitter solution can be realized in infrared modified Hořava-Lifshitz gravity theory and the Minkowski vacuum is also allowed. On the limit of vanishing $\Lambda_\omega$, a spherically symmetric black hole solution has been obtained by Kehagias and Sfetsos [25], which is the analogy of Schwarzschild black hole in general relativity and is exactly asymptotically flat. After Hořava-Lifshitz gravity was proposed, much attention has been paid to it [26–31].

In this paper, based on Vagenas' proposal of adiabatic invariant quantity, we investigate entropy spectrum of a KS black hole in IR modified Hořava-Lifshitz gravity. Vagenas pointed that the general coordinates $q_i$ contain $q_0 = \tau$ and $q_1 = r$, and the contribution of the two parts to the adiabatic invariant quantity are equivalent with each other by defining $\dot{r} = dr/d\tau$. Thus the integration result of adiabatic invariant quantity was directly given by the $\tau$-integration, where the period $T$ of gravity system satisfies $T = 2\pi/\kappa$. We find that the periodicity of gravity system can conveniently be used to calculate the entropy spectrum. However, the physical picture of periodicity is not clear. We give our explanation about it by considering a process that a pair of particles create outside the horizon. One period of the system corresponds to the process with the outgoing positive energy particle crossing outwards the horizon while the negative energy particle tunnels into the black hole. The movement of the two particles can be described as a tunneling process proposed by Parikh and Wilczek's Hawking radiation [32]. After calculating, we find that the action of the system is exactly the black hole entropy which is the adiabatic invariant quantity. With the help of Bohr-Sommerfeld quantization rule, we obtain the quantized entropy and area spectrum. Some summary and discussion will be given in the latter.

## 2. Review of a KS Black Hole in IR Modified Hořava-Lifshitz Gravity

Using the ADM decomposition of the metric

$$ds^2 = -N^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \quad (1)$$

where $N$ and $N^i$ are the "lapse" and "shift" variables, respectively, and $g_{ij}$ is the spatial metric. On the limit of $\Lambda_\omega \to 0$, the action of IR modified Hořava-Lifshitz gravity theory can be given by

$$\begin{aligned} S = &\int dt d^3x \sqrt{g} N \\ &\times \left[\frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda K^2\right)\right. \\ &- \frac{\kappa^2}{2\omega^4} C_{ij}C^{ij} + \frac{\kappa^2\mu}{2\omega^2}\epsilon^{ijk} R^{(3)}_{il}\nabla_j R^{(3)l}_k \\ &- \frac{\kappa^2\mu^2}{8} R^{(3)}_{ij} R^{(3)ij} \\ &\left. + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{1-4\lambda}{4}\left(R^{(3)}\right)^2 + \mu^4 R^{(3)}\right], \end{aligned} \quad (2)$$

which is obtained by introducing a term proportional to the Ricci scalar of the three-geometry $\mu^4 R^{(3)}$ to the original action of Hořava-Lifshitz gravity [26, 27]. Here $K_{ij}$ is the extrinsic curvature, defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \quad (3)$$

where the dot denotes a derivative with respect to $t$, $C^{ij}$ is the Cotton tensor defined by

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^{(3)j}_l - \frac{1}{4}R^{(3)}\delta^j_l\right), \quad (4)$$

and $R^{(3)}$ is the 3-dimensional curvature scalar for $g_{ij}$; $\kappa$, $\lambda$, $\omega$, $\mu$ are all coupling constant parameters.

Comparing the action for the case of $\lambda = 1$ with the standard Einstein-Hilbert action, we find that the Lagrangian will become the usual Einstein-Hilbert Lagrangian when the speed of light $c$, Newton's constant $G$, and the cosmological constant $\Lambda$ are given by

$$c^2 = \frac{\kappa^2\mu^4}{2}, \qquad G = \frac{\kappa^2}{32\pi c}, \qquad \Lambda = \frac{3}{2}\Lambda_W. \quad (5)$$

The spherically symmetric asymptotically flat black hole solution has been obtained by Cai et al. [27], which is the

analogy of Schwarzschild black hole in general relativity. The metric can be written as

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (6)$$

with

$$f(r) = 1 + \omega r^2 - \sqrt{r\left(\omega^2 r^3 + 4\omega M\right)}, \quad (7)$$

where $M$ is an integration constant corresponding to the mass of black hole, and $\omega$ is a coupling constant parameter.

The condition $f(r_\pm) = 0$ gives the outer and inner horizons at

$$r_\pm = M\left(1 \pm \sqrt{1 - \frac{1}{2\omega M^2}}\right). \quad (8)$$

To avoid naked singularity, we should have $\omega M^2 \geq 1/2$. In the regime of traditional general relativity, we have $\omega M^2 \gg 1$, so the outer horizon approaches the usual Schwarzschild horizon $r_+ \simeq 2M$, whereas the inner one approaches the singularity $r_- \simeq 0$.

## 3. Entropy Quantization via Adiabatic Invariant Action

We consider a process that a pair of particles create near the horizon. While the outgoing positive energy particle crossing outwards the horizon, the negative energy particle ingoing towards the black hole along the radial direction. We describe the movement of the two particles as a tunneling process proposed by Parikh and Wilczek's proposal [32].

The action of the system is

$$\begin{aligned} I &= \int p_r dr = \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr + \int_{r_{\text{out}}}^{r_{\text{in}}} p_r dr \\ &= \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp_r' dr + \int_{r_{\text{out}}}^{r_{\text{in}}} \int_0^{p_r} dp_r' dr. \end{aligned} \quad (9)$$

The first term is corresponding to the particles with positive energy, and the second term is the negative energy one. When energy conservation is considered, the black hole mass will decrease with the outgoing particle emitting. The Hamilton $H$, ADM energy $M$, and the particle's energy $m'$ satisfy the relation $H = M - m'$; that is, $dH = -dm'$. Then by use of Hamilton's equation $\dot{r} = dH/dp_r$, the first term of (9) becomes

$$\begin{aligned} I_1 &\equiv \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp_r' dr = \int_{r_{\text{in}}}^{r_{\text{out}}} \int_M^{M-m} \frac{dH}{\dot{r}} dr \\ &= \int_0^m \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{\dot{r}}\left(-dm'\right), \end{aligned} \quad (10)$$

where $r_{\text{in}} = r_h(M) - \epsilon$, $r_{\text{out}} = r_h'(M - m) + \epsilon$, for the reason of that the black hole horizon will decrease with the particle emitting out.

When considering $\dot{r} = dr/d\tau = f(r)$, we get

$$I_1 = \int_0^m \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 + \omega r^2 - r\left[\omega^2 r^3 + 4\omega\left(M - m'\right)\right]} dm'. \quad (11)$$

It is easily found that there is a pole at $r_h' = (M - m') + \sqrt{(M - m')^2 - 1/2\omega}$. We do the integration as follows:

$$\begin{aligned} I_1 &= \int_0^m \int_{r_{\text{in}}}^{r_{\text{out}}} \left(dr\left(\left(r - r_h'\right) \times \left[2\omega r - \frac{2\omega^2 r^3 + 2\omega\left(M - m'\right)}{\sqrt{r\left(\omega^2 r^3 + 4\omega\left(M - m'\right)\right)}}\right]\right)^{-1}\right) dm' \\ &= 2\pi \int_0^m \left(1\left(2\omega r_h' - \frac{2\omega^2 r_h'^3 + 2\omega\left(M - m'\right)}{\sqrt{r_h'\left[\omega^2 r_h'^3 + 4\omega\left(M - m'\right)\right]}}\right)^{-1}\right) dm' \\ &= 2\pi \int_0^m \left(\left(\frac{1}{2} + 2\omega\left(M - m'\right)^2 + 2\omega\left(M - m'\right)\sqrt{\left(M - m'\right)^2 - \frac{1}{2\omega}}\right) \times \left(2\omega\sqrt{\left(M - m'\right)^2 - \frac{1}{2\omega}}\right)^{-1}\right) dm' \\ &= 2\pi \left[\int_0^m \frac{1}{4\omega\sqrt{\left(M - m'\right)^2 - 1/2\omega}} dm' + \int_0^m \frac{\left(M - m'\right)^2}{\sqrt{\left(M - m'\right)^2 - 1/2\omega}} dm' + \int_0^m \left(M - m'\right) dm'\right] \end{aligned}$$

$$= 2\pi \left[ \frac{M}{2}\sqrt{M^2 - \frac{1}{2\omega}} - \frac{M-m'}{2}\sqrt{(M-m')^2 - \frac{1}{2\omega}} + \frac{1}{2\omega}\left[\ln M + \sqrt{M^2 - \frac{1}{2\omega}} - \ln(M-m) + \sqrt{(M-m)^2 - \frac{1}{2\omega}}\right] + Mm - \frac{m^2}{2}\right]. \tag{12}$$

The second term of (9) corresponds to ingoing particles with negative energy. After similar calculation as the first term, we find that the contribution is equivalent to that of $I_1$. That is,

$$I = 2I_1. \tag{13}$$

On the other hand, the Hawking temperature of the outer event horizon has been obtained as [31]

$$T_H = \frac{2\omega r_+^2 - 1}{8\pi(\omega r_+^3 + r_+)} = \frac{\sqrt{M^2 - 1/2\omega}}{2\pi\left(2M^2 + 2M\sqrt{M^2 - 1/2\omega} + 1/2\omega\right)}, \tag{14}$$

which is proportional to the surface gravity $\kappa = (1/2)(\partial f(r)/\partial r)|_{r=r_+}$ on the event horizon of black hole. Considering the first law of thermodynamics and substituting the expression of temperature of black hole, we can obtain the entropy

$$S_{\mathrm{BH}} = \frac{A}{4} + \frac{\pi}{\omega}\ln\frac{A}{4}, \tag{15}$$

where $A = \pi r_+^2$ is the area of event horizon.

After calculation, we find that the change of black hole entropy $\Delta S$, when a particle with energy of $m$ emits out of the black hole, is exactly equivalent to the action $I$; that is,

$$\Delta S = \pi\left[r_+^2(M) - r_+^2(M-m)\right] = I. \tag{16}$$

Now, using the periodicity of the black hole, we calculate the adiabatic invariant quantity. According to the dimensional reduction technique, the two-dimensional spacetime of a KS black hole can be given by

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2. \tag{17}$$

When defining the tortoise coordinate as

$$r_* = r + \frac{1}{2\kappa_+}\ln\frac{r-r_+}{r_+} + \frac{1}{2\kappa_-}\ln\frac{r-r_-}{r_-}, \tag{18}$$

in which $\kappa_\pm = (f'(r)/2)|_{r=r_\pm} = (2\omega r_\pm^2 - 1)/(4r_\pm(\omega r_\pm^2 + 1))$ is the surface gravity on the outer (inner) horizon. Using the null coordinates $u = t - r_*$, $v = t + r_*$, we can get the coordinates $U = -e^{-\kappa_+ u}$ and $V = e^{\kappa_+ v}$ [33, 34]. Then define

$$T = \frac{1}{2}(V+U) = e^{\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)^{1/2}\left(\frac{r-r_-}{r_-}\right)^{\kappa_+/2\kappa_-}\sinh\kappa_+ t,$$
$$R = \frac{1}{2}(V-U) = e^{\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)^{1/2}\left(\frac{r-r_-}{r_-}\right)^{\kappa_+/2\kappa_-}\cosh\kappa_+ t, \tag{19}$$

where $T$, $R$ are the Kruskal-like coordinates.

Different from (18), that is,

$$dr_* = f(r)^{-1}dr; \tag{20}$$

the two-dimensional KS metric becomes

$$ds^2 = -dt^2 + dr_*^2 = \kappa_+^{-2}e^{-2\kappa_+ r}\left(\frac{r-r_-}{r_+}\right)\left(\frac{r_-}{r-r_-}\right)^{\kappa_+/\kappa_-} \times\left[-dT^2 + dR^2\right]. \tag{21}$$

Transforming the time coordinate as $t \to -i\tau$, we have

$$iT = e^{\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)^{1/2}\left(\frac{r-r_-}{r_-}\right)^{\kappa_+/2\kappa_-}\sin\kappa_+\tau,$$
$$R = e^{\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)^{1/2}\left(\frac{r-r_-}{r_-}\right)^{\kappa_+/2\kappa_-}\cos\kappa_+\tau. \tag{22}$$

It is easily found that both $T$, $R$ are periodic functions with respect to the Euclidean time $\tau$ with the period of $2\pi/\kappa_+$. Since we only consider the case of outer horizon, for simplicity we write $\kappa_+$ for $\kappa$ from now on.

When utilizing Vagenas's adiabatic invariant quantity with the following form:

$$I = \int p_i dq_i = \int\int_0^{p_i} dp_i' dq_i = \int\int_0^H \frac{dH'}{\dot{q}_i}dq_i = \int\int_0^H dH' d\tau + \int\int_0^H \frac{dH'}{\dot{r}}dr, \tag{23}$$

where $p_i$ are the conjugate momentum of the general coordinate $q_i$ with $i = 0, 1$ for $q_0 = \tau$ and $q_1 = r$. Considering $\dot{r} = dr/d\tau$, we have

$$I = 2\int\int_0^H dH' d\tau. \tag{24}$$

Because of the periodicity of $\tau$ with $T = 2\pi/\hbar\kappa$, the adiabatic invariant quantity can be calculated as

$$I = 2\pi\int_0^H \frac{dH'}{\kappa} = \hbar\int_0^H \frac{dH'}{T_{\mathrm{BH}}} = \hbar S_{\mathrm{BH}}. \tag{25}$$

Implementing the Bohr-Sommerfeld quantization condition

$$\oint pdq = 2\pi n\hbar, \tag{26}$$

the black hole entropy spectrum can be given as

$$S_{\mathrm{BH}} = 2\pi n, \quad n = 1, 2, 3, \ldots, \tag{27}$$

and the entropy spectrum is discrete and equidistant spaced with

$$\Delta S_{\mathrm{BH}} = 2\pi. \tag{28}$$

To get the area spectrum, differentiate (15),

$$\Delta S_{\mathrm{BH}} = \frac{1}{4}\Delta A + \frac{\pi}{\omega}\frac{4}{A}\Delta A, \tag{29}$$

we have

$$\Delta A = \frac{\Delta S_{\mathrm{BH}}}{1/4 + 4\pi/A\omega} \simeq 8\pi\left(1 - \frac{4}{\omega r_+^2}\right). \tag{30}$$

We find that area spectrum is not equidistant spaced.

## 4. Summary and Conclusion

In this paper, based on the idea of adiabatic invariant quantity, we have investigated entropy spectrum of a KS black hole in IR modified Hořava-Lifshitz gravity. As a modified gravity theory, the entropy of a KS black hole does not satisfy Bekenstein's entropy-area relation. It consists of two terms: one is the Bekenstein-Hawking entropy, the other is a logarithmic term. The discrepancy between the entropy and the Bekenstein-Hawking entropy is the reflection of differences between this modified gravity theory and general relativity. After calculating, we find that the black hole entropy is an adiabatic invariant quantity. With the help of Bohr-Sommerfeld quantization rule, we obtain the quantized entropy and area spectrum. It is concluded that the entropy spectrum can be given as $S_{\mathrm{BH}} = 2\pi n$ with $n = 1, 2, 3, \ldots$, which is discrete and equidistant spaced with $\Delta S_{\mathrm{BH}} = 2\pi$; and the area spectrum is not equidistant spaced, which depends on the parameter of gravity theory. In addition, by calculating the action of a production of a pair of particles near the horizon, we find that the action of the system is exactly equivalent to the change to black hole entropy, which is an adiabatic invariant quantity. The procession of the particle producing, with positive energy outgoing towards the horizon while the one with negative energy is ingoing the horizon, can give a clear explanation to the periodicity of gravity system.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# The Critical Phenomena and Thermodynamics of the Reissner-Nordstrom-de Sitter Black Hole

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It is wellknown that there are two horizons for the Reissner-Nordstrom-de Sitter spacetime, namely, the black hole horizon and the cosmological one. Both horizons can usually seem to be two independent thermodynamic systems; however, the thermodynamic quantities on both horizons satisfy the laws of black hole thermodynamics and are not independent. In this paper by considering the relations between the two horizons we give the effective thermodynamic quantities in Reissner-Nordstrom-de Sitter spacetime. The thermodynamic properties of these effective quantities are analyzed; moreover, the critical temperature, critical pressure, and critical volume are obtained. We also discussed the thermodynamic stability of Reissner-Nordstrom-de Sitter spacetime.

## 1. Introduction

Black hole physics, especially the black hole thermodynamics, refer directly to the theories of gravity, statistical physics, particle physics, field theory, and so forth. This makes the field concerned by many physicists [1–6]. Although the complete statistical description of black hole thermodynamics is still unclear, the research on the properties of black hole thermodynamics is prevalent, such as Hawking-Page phase transition [7], and critical phenomena. More interestingly, the research on the charged and nonrotating RN-AdS black hole shows that there exists a similar phase transition to the van der Waals-Maxwell vapor-liquid phase transition [8, 9].

Motivated by the AdS/CFT correspondence [10], where the transitions have been related with the holographic superconductivity [11, 12], the subject of the phase transitions of black holes in asymptotically anti-de Sitter (AdS) spacetime has received considerable attention [13–17]. The underlying microscopic statistical interaction of the black holes is also expected to be understood via the study of the gauge theory living on the boundary in the gauge/gravity duality.

Recently, by considering the cosmological constant correspond to pressure in general thermodynamic system, namely,

$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{l^2}, \tag{1}$$

the thermodynamic volumes in AdS and dS spacetime are obtained [18–24]. The studies on phase transition of black holes have aroused great interest [25–31]. Connecting the thermodynamic quantities of AdS black holes to ($P \sim V$) in the ordinary thermodynamic system, the critical behaviors of black holes can be analyzed and the phase diagram like van der Waals vapor-liquid system can be obtained. This helps to further understand the black hole entropy, temperature, heat capacities, and so forth. It also has a very important significance in completing the geometric theory of black hole thermodynamics.

As is well known, there are black hole horizon and cosmological horizon in the appropriate range of parameters for de Sitter spacetime. Both horizons have thermal radiation, but with different temperatures. The thermodynamic quantities on both horizons satisfy the first law of thermodynamics,

and the corresponding entropy fulfills the area formula [23, 32, 33]. In recent years, the research on the thermodynamic properties of de Sitter spacetime has drawn a lot of attention [23, 32–36]. In the inflation epoch of early universe, the universe is a quasi-de Sitter spacetime. The cosmological constant introduced in de Sitter space may come from the vacuum energy, which is also a kind of energy. If the cosmological constant is the dark energy, the universe will evolve to a new de Sitter phase. To depict the whole history of evolution of the universe, we should have some knowledge on the classical and quantum properties of de Sitter space [23, 33, 37, 38].

Firstly, we expect the thermodynamic entropy to satisfy the Nernst theorem [34, 35, 39]. At present a satisfactory explanation to the problem in which the thermodynamic entropy of the horizon of the extreme de Sitter spacetime does not fulfill the Nernst theorem is still lacking. Secondly, when considering the correlation between the black hole horizon and the cosmological horizon whether the thermodynamic quantities in de Sitter spacetime still have the phase transition and critical behavior like in AdS black holes. Thus it is worthy of our deep investigation and reflection to establish a consistent thermodynamics in de Sitter spacetime.

Because the thermodynamic quantities on the black hole horizon and the cosmological one in de Sitter spacetime are the functions of mass $M$, electric charge $Q$, and cosmological constant $\Lambda$. The quantities are not independent of each other. Considering the relation between the thermodynamic quantities on the two horizons is very important for studying the thermodynamic properties of de Sitter spacetime. Based on the relation we give the effective temperature and pressure of Reissner-Nordstrom-de Sitter(R-NdS) spacetime and analyze the critical behavior of the equivalent thermodynamic quantities. It is shown that when considering the relation between the two horizons in RN-dS spacetime there is the similar phase transition like the ones in van der Waals liquid-gas system and charged AdS black holes.

The paper is arranged as follows. In Section 2 we introduce the Reissner-Nordstrom-de Sitter(R-NdS) spacetime and give the two horizons and corresponding thermodynamic quantities. In Section 3 by considering the relations between the two horizons we obtain the effective temperature and the equivalent pressure. In Section 4 the critical phenomena of effective thermodynamic quantities are discussed. Finally we discuss and summarize our results in Section 5 (we use the units $G_{n+1} = \hbar = k_B = c = 1$).

## 2. RN-dS Spacetime

The line element of the R-N SdS black holes is given by [32]

$$ds^2 = -f(r)\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2, \tag{2}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \tag{3}$$

The above geometry possesses three horizons: the black hole Cauchy horizon located at $r = r_-$, the black hole event horizon (BEH) located at $r = r_+$, and the cosmological event horizon (CEH) located at $r = r_c$, where $r_c > r_+ > r_-$, the only real, positive zeroes of $f(r) = 0$.

The equations $f(r_+) = 0$ and $f(r_c) = 0$ are rearranged to

$$\begin{aligned} Q^2 &= r_+ r_c \left(1 - \frac{r_c^2 + r_c r_+ + r_+^2}{3}\Lambda\right), \\ 2M &= (r_c + r_+)\left(1 - \frac{r_c^2 + r_+^2}{3}\Lambda\right). \end{aligned} \tag{4}$$

The surface gravity on the black hole horizon and the cosmological horizon is, respectively,

$$\begin{aligned} \kappa_+ &= \frac{1}{2}\frac{df(r)}{dr}\bigg|_{r=r_+} \\ &= \frac{r_+^2 - Q^2 - r_+^4\Lambda}{2r_+^3} \\ &= \frac{(r_+ - r_c)}{2r_+^2}\left(1 - \frac{\left(r_c^2 + 2r_+r_c + 3r_+^2\right)\left(r_c r_+ - Q^2\right)}{r_c r_+\left(r_c^2 + r_+ r_c + r_+^2\right)}\right), \\ \kappa_c &= \frac{1}{2}\frac{df(r)}{dr}\bigg|_{r=r_c} \\ &= \frac{r_c^2 - Q^2 - r_c^4\Lambda}{2r_c^3} \\ &= \frac{(r_c - r_+)}{2r_c^2}\left(1 - \frac{\left(r_+^2 + 2r_+r_c + 3r_c^2\right)\left(r_c r_+ - Q^2\right)}{r_c r_+\left(r_c^2 + r_+ r_c + r_+^2\right)}\right). \end{aligned} \tag{5}$$

The thermodynamic quantities on the two horizons satisfy the first law of thermodynamics [23, 33, 40, 41]:

$$\begin{aligned} \delta M &= \frac{\kappa_+}{2\pi}\delta S_+ + \Phi_+\delta Q + V_+\delta P, \\ \delta M &= \frac{\kappa_c}{2\pi}\delta S_c + \Phi_c\delta Q + V_c\delta P, \end{aligned} \tag{6}$$

where $S_+ = \pi r_+^2$, $S_c = \pi r_c^2$, $\Phi_+ = Q/r_+$, $\Phi_c = -(Q/r_c)$, $V_+ = (4\pi/3)r_+^3$, $V_c = (4\pi/3)r_c^3$, $P = -(\Lambda/8\pi)$.

## 3. Thermodynamic Quantity of RN-dS Spacetime

In Section 2, we have obtained thermodynamic quantities without considering the relationship between the black hole horizon and the cosmological horizon. Because there are three variables $M$, $Q$, and $\Lambda$ in the spacetime, the thermodynamic quantities corresponding to the black hole horizon and the cosmological horizon are functional with respect to $M$, $Q$, and $\Lambda$. The thermodynamic quantities corresponding to the black hole horizon are related to the ones corresponding to the cosmological horizon. When the thermodynamic property of charged de Sitter spacetime is studied, we must consider the relationship with the two horizons. Recently, by studying Hawking radiation of de Sitter

spacetime, [42, 43] obtained that the outgoing rate of the charged de Sitter spacetime which radiates particles with energy $\omega$ is

$$\Gamma = e^{\Delta S_+ + \Delta S_c}, \tag{7}$$

where $\Delta S_+$ and $\Delta S_c$ are Bekenstein-Hawking entropy difference corresponding to the black hole horizon and the cosmological horizon after the charged de Sitter spacetime radiates particles with energy $\omega$. Therefore, the thermodynamic entropy of the charged de Sitter spacetime is the sum of the black hole horizon entropy and the cosmological horizon entropy:

$$S = S_+ + S_c. \tag{8}$$

Substituting (6) into (8), one can obtain

$$\begin{aligned} dS &= 2\pi\left(\frac{1}{\kappa_+} + \frac{1}{\kappa_c}\right)dM \\ &\quad - 2\pi\left(\frac{\varphi_+}{\kappa_+} + \frac{\varphi_c}{\kappa_c}\right)dQ + \frac{\pi}{3}\left(\frac{r_+^3}{\kappa_+} + \frac{r_c^3}{\kappa_c}\right)d\Lambda. \end{aligned} \tag{9}$$

For simplicity, we consider the case with constant. In this case the equation above turns into

$$dS = 2\pi\left(\frac{1}{\kappa_+} + \frac{1}{\kappa_c}\right)dM + \frac{\pi}{3}\left(\frac{r_+^3}{\kappa_+} + \frac{r_c^3}{\kappa_c}\right)d\Lambda. \tag{10}$$

From (4) and (5), we derive

$$dr_c = \frac{dM}{r_c\kappa_c} + \frac{\left(r_c^3/3\right)}{2r_c\kappa_c}d\Lambda, \qquad dr_+ = \frac{dM}{r_+\kappa_+} + \frac{\left(r_+^3/3\right)}{2r_+\kappa_+}d\Lambda. \tag{11}$$

Recently, the thermodynamic volume of RN-dS spacetime [23, 34] is given as

$$V = \frac{4\pi}{3}\left(r_c^3 - r_+^3\right). \tag{12}$$

Substituting (11) into (12), one gets

$$dV = 4\pi\left(\frac{r_c}{\kappa_c} - \frac{r_+}{\kappa_+}\right)dM + \frac{2\pi}{3}\left(\frac{r_c^4}{\kappa_c} - \frac{r_+^4}{\kappa_+}\right)d\Lambda. \tag{13}$$

Substituting (13) into (10), one can derive the thermodynamic equation of thermodynamic quantities in de Sitter spacetime [34]:

$$dM = T_{\text{eff}}dS - P_{\text{eff}}dV, \tag{14}$$

where the effective temperature is

$$\begin{aligned} T_{\text{eff}} &= \frac{\left(x^4 + x^3 - 2x^2 + x + 1\right)}{4\pi r_c x\,(x+1)\left(x^2 + x + 1\right)} \\ &\quad - \frac{Q^2}{4\pi r_c^3 x^3\,(x+1)\left(x^2 + x + 1\right)} \\ &\quad \times \left(1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6\right). \end{aligned} \tag{15}$$

The effective pressure is

$$\begin{aligned} P_{\text{eff}} &= \frac{(1-x)\left(1 + 3x + 3x^2 + 3x^3 + x^4\right)}{8\pi r_c^2 x\,(1+x)\left(1 + x + x^2\right)^2} \\ &\quad - \frac{Q^2\,(1-x)\left(1 + 2x + 3x^2 - 3x^5 - 2x^6 - x^7\right)}{8\pi r_c^4 x^3\,(1+x)\left(1 - x^3\right)\left(1 + x + x^2\right)}, \end{aligned} \tag{16}$$

where $x := r_+/r_c$ and $0 < x < 1$.

From (15) and (16), when $Q = 0$, the effective temperature and pressure are both greater than zero. This fulfills the stable condition of thermodynamic equilibrium. If considering the black hole horizon and the cosmological one as independent of each other, because of the different radiant temperatures on the two horizons, the spacetime is instable.

Another problem of considering the black hole horizon and the cosmological one as independent of each other is that when the two horizons coincide, namely,

$$r_{+/c}^2 = \frac{1 \pm \sqrt{1 - 4Q^2\Lambda}}{2\Lambda} = r_0^2, \tag{17}$$

from (4), the surface gravity $\kappa_{+/c} = 0$; thus the temperature on the black hole horizon and the temperature on the cosmological horizon are both zero. However, both horizons have nonzero area, which means that the entropy for the two horizons should not be zero. This conclusion is inconsistent with Nernst theorem. In the extreme case $r_{+/c}^2 = r_0^2$, the effective temperature from (15) is

$$T_{\text{eff}} = \frac{1}{12\pi r_0}\left(1 - \frac{5Q^2}{2r_0^2}\right). \tag{18}$$

The effective pressure is

$$P_{\text{eff}} = 0. \tag{19}$$

In this case the volume-thermodynamic system becomes area-thermodynamic one. According to (19) the pressure of thermodynamic membrane is zero. However from (18), the temperature of thermodynamic membrane is nonzero. This can partly solve the problem that extreme de Sitter black holes do not satisfy the Nernst theorem, when $Q = 0$, (15) and (16) return to the known result [34].

## 4. Critical Behaviour

To compare with the van der Waals equation, we set $P_{\text{eff}} \to P$, $v \to v$ and discuss the phase transition and the critical phenomena when $r_c$ is invariant. The van der Waals equation is

$$\left(P + \frac{a}{v^2}\right)\left(v - \tilde{b}\right) = kT. \tag{20}$$

Here, $v = V/N$ is the specific volume of the fluid, $P$ is its pressure, $T$ is its temperature, and $k$ is the Boltzmann constant.

Substituting (15) into (16), we obtain

$$P_{\text{eff}} = T_{\text{eff}}\frac{B_4}{2r_cB_2} + \frac{B_2B_3 - B_1B_4}{8\pi r_c^2 x\,(1+x)\,B_2}, \tag{21}$$

where

$$B_1 = \frac{1 + x - 2x^2 + x^3 + x^4}{1 + x + x^2},$$
$$B_2 = \frac{1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6}{1 + x + x^2},$$
$$B_3 = \frac{1 + 2x - 2x^4 - x^5}{\left(1 + x + x^2\right)^2}, \tag{22}$$
$$B_4 = \frac{1 + 2x + 3x^2 - 3x^5 - 2x^6 - x^7}{\left(1 + x + x^2\right)^2}.$$

According to (21), combing dimensional analysis [22] with (12), we conclude that we should identify the specific volume $v$ with

$$v = r_c (1 - x). \tag{23}$$

From the two equations

$$\frac{\partial P_{\text{eff}}}{\partial v} = 0, \qquad \frac{\partial^2 P_{\text{eff}}}{\partial v^2} = 0, \tag{24}$$

we first calculate the position of critical points. Then, one can derive

$$\begin{aligned}
&\left(\frac{\partial P_{\text{eff}}}{\partial v}\right)_{T_{\text{eff}}} \\
&= -1 \times \Big(8\pi r c^5 (1 + x)\left(1 + x + x^2\right)^3 \\
&\qquad \times \left(1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6\right)\Big)^{-1} \\
&\quad \times \Big[r c^2 x^2 \left(-6 - 21x - 38x^2 - 45x^3 - 30x^4 - 10x^5\right. \\
&\qquad \left. + 6x^6 + 7x^7 + 4x^8 + x^9\right) \\
&\qquad \times Q^2 x \left(15 + 54x + 98x^2 + 104x^3 + 59x^4 + 18x^5\right. \\
&\qquad \left. - 9x^6 - 12x^7 - 10x^8 - 4x^9 - x^{10}\right)\Big] = 0
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\partial^2 P_{\text{eff}}}{\partial v^2}\right)_{T_{\text{eff}}} \\
&= 1 \times \Big(4\pi r c^6 (1 + x)\left(1 + x + x^2\right)^4 \\
&\qquad \times \left(1 + x + x^2 - 2x^3 + x^4 + x^5 + x^6\right)^2\Big)^{-1} \\
&\quad \times \Big[-r c^2 x \left(6 + 36x + 115x^2 + 286x^3 + 486x^4\right. \\
&\qquad + 530x^5 + 203x^6 - 357x^7 - 783x^8 \\
&\qquad - 854x^9 - 666x^{10} - 375x^{11} - 127x^{12} \\
&\qquad \left. + 3x^{13} + 32x^{14} + 18x^{15} + 6x^{16} + x^{17}\right) \\
&\qquad + Q^2 \left(30 + 165x + 456x^2 + 837x^3 + 1050x^4\right. \\
&\qquad + 903x^5 + 255x^6 - 720x^7 - 1707x^8 \\
&\qquad - 2113x^9 - 1743x^{10} - 969x^{11} - 314x^{12} \\
&\qquad - 9x^{13} + 63x^{14} + 44x^{15} + 21x^{16} + 6x^{17} \\
&\qquad \left. + x^{18}\right)\Big] = 0.
\end{aligned} \tag{25}$$

When $r_c = 1$, from (15), (16), and (25) one can obtain the position of critical point

$$x^c = 0.871992. \tag{26}$$

The critical electric charge, specific volume, temperature, and the critical pressure are, respectively,

$$\begin{aligned}
Q^c &= 0.60395, \qquad v^c = 0.128008, \\
T^c &= 0.00436559, \qquad p^c = 0.0000145468.
\end{aligned} \tag{27}$$

In Figure 1 we give the figure of $P_{\text{eff}}$ with the change of $v$ at the constant effective temperature near the critical point.

To reflect the influence of $r_c$ on the spacetime, we set $r_c = 5$, from which we derived the position of the critical point

$$x^c = 0.871992. \tag{28}$$

The critical electric charge, specific volume, temperature, and the critical pressure are, respectively,

$$\begin{aligned}
Q^c &= 3.01975, \qquad v^c = 0.640038, \\
T^c &= 0.000873119, \qquad p^c = 5.8187 \times 10^{-7}.
\end{aligned} \tag{29}$$

The diagram of the effective $P_{\text{eff}}$ with the change of $v$ is depicted in Figure 2.

For the van der Waals fluid and the RN-AdS black hole, the relation $P_c v_c / T_c$ is a universal number and is independent of the charge $Q$. For the RN-dS black hole, according to the effective thermodynamic quantities, numerical calculation shows that $P_c v_c / T_c$ still has a $Q$-independent universal value. Certainly, the universal value is no more than 3/8, but ~0.0004265.

## 5. Discussion and Conclusions

From above we can find out that the value of $r_c$ does not influence the critical point $x$; namely, for the RN-dS spacetime, the position of critical point is irrelevant to the value of the cosmological horizon. This indicates that $x = r_c / r_+$ is fixed; however the critical effective temperature, critical volume, and critical pressure are dependent on the value of $r_c$. The critical effective temperature and pressure for the RN-dS system will decrease as the values of $r_c$ increase, while the critical electric charge and the critical volume will increase as the values of $r_c$ increase.

By Figures 1 and 2, the $P_{\text{eff}} - v$ curve at constant temperature of RN-dS spacetime is different from the ones of van der Waals equation and charged AdS black holes.

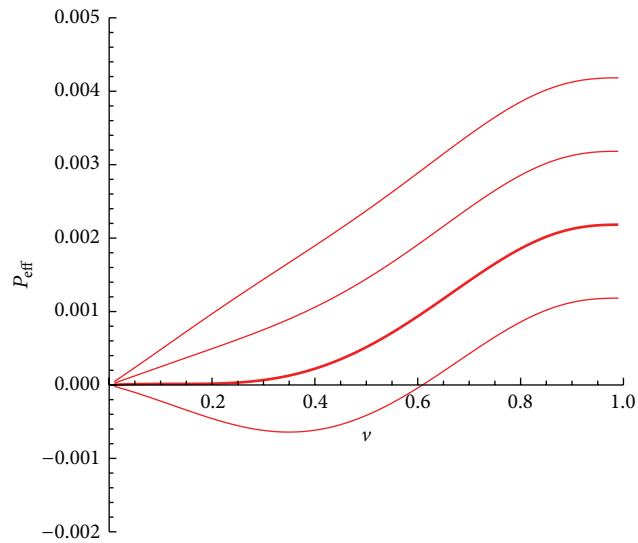


FIGURE 1: $P_{\text{eff}} - v$ diagram of RN-dS black holes. The temperature of isotherms decreases from top to bottom and corresponds to $T^c + 0.004$, $T^c + 0.002$, $T^c$, $T^c - 0.002$.

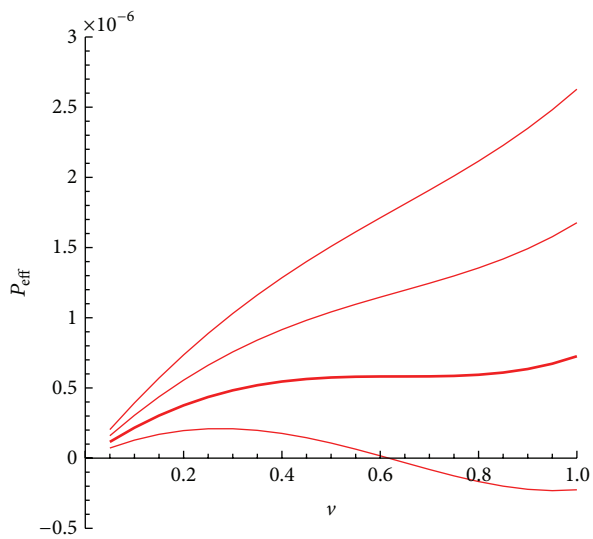


FIGURE 2: $P_{\text{eff}} - v$ diagram of RN-dS black holes. The temperature of isotherms decreases from top to bottom and corresponds to $T^c + 0.00004$, $T^c + 0.00002$, $T^c$, $T^c - 0.00002$.

(1) The first difference lies at the critical pressure $P_{\text{eff}}$ of RN-dS spacetime which increases as the volume increases at the constant temperature when $T_{\text{eff}} > T^c$, and the maximal value turns up at $x \to 0$, namely, de Sitter spacetime.

(2) The first difference is that the critical pressure $P_{\text{eff}}$ of RN-dS spacetime may be negative at the constant temperature when $T_{\text{eff}} < T^c$, which means the system is not stable. From (15) and (16) we can obtain the parameters $x$ and $Q$ when the system is in the unstable state.

(3) On the $P_{\text{eff}} - v$ curves with $T_{\text{eff}} > T^c$ the system lies in a phase; on the $P_{\text{eff}} - v$ curves with $T_{\text{eff}} < T^c$ the same pressure will correspond to two different values of $x$, namely, two-phase coexistence region. Using the equal area criterion we can find out the proportion of the two phases for the RN-dS system.

From the above discussion, in RN-dS spacetime when considering the relation between the two horizons there is the similar phase transition like the one in van der Waals equation and charged AdS black holes. The reason is still unclear. This deserves further study. If the cosmological constant is just the dark energy, the universe will evolve into a new de Sitter phase. According to Figures 1 and 2 the RN-dS system can evolve into de Sitter phase along isothermal curve. However, along the different isothermal curves, which processes will be the evolution of the universe to a new de Sitter phase needs further consideration according to the observations.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Researching on Hawking Effect in a Kerr Space Time via Open Quantum System Approach

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It has been proposed that Hawking radiation from a Schwarzschild or a de Sitter spacetime can be understood as the manifestation of thermalization phenomena in the framework of an open quantum system. Through examining the time evolution of a detector interacting with vacuum massless scalar fields, it is found that the detector would spontaneously excite with a probability the same as the thermal radiation at Hawking temperature. Following the proposals, the Hawking effect in a Kerr space time is investigated in the framework of an open quantum systems. It is shown that Hawking effect of the Kerr space time can also be understood as the the manifestation of thermalization phenomena via open quantum system approach. Furthermore, it is found that near horizon local conformal symmetry plays the key role in the quantum effect of the Kerr space time.

## 1. Introduction

Hawking radiation arising from the quantization of matter field in a curved background space-time with the event horizon is a prominent quantum effect. The research to understand Hawking radiation, which is related to general relativity, quantum theory, and thermodynamics, has attracted widespread interest in the physics community. Since Hawking's original derivation of black hole thermal radiation [1–3], several alternative methods have been proposed, such as Damour-Ruffini method [4, 5], the tunneling method [6–11], and gravitational anomaly method [12, 13].

However, from a physical viewpoint, the black hole thermodynamics system should be more like a nonequilibrium system rather than an equilibrium system. Hawking effect should be investigated in the frame of nonequilibrium statistics physics. In quantum mechanics and nonequilibrium statistics physics, the open quantum theory system has gotten a lot of successful development [14]. The quantum dynamics of an open quantum system characterized by the effects of decoherence and dissipation cannot be represented in terms of a unitary time evolution. It has been applied to quantum information science, modern quantum optics, atomic and many-body systems, soft condensed matter physics, and biophysics. Recently, in the paradigm of open quantum system, based on [15], Yu and Zhang proposed a new insight to understand Hawking radiation in a Schwarzschild space time [16]. Through examining the time evolution of a detector interacting with vacuum massless scalar field, they got a conclusion that the detector in both Unruh and Hartle-Hawking vacua would spontaneously excite with a nonvanishing probability the same as Hawking thermal radiation from the black hole. This new approach has been extended to understand the Gibbons-Hawking effect of de Sitter spacetime [17]. However, there remain some challenges to study Hawking radiation from a generic Kerr space time under the manifestation of thermalization phenomena in an open quantum system.

Our motivation comes from the fact that the near-horizon geometry plays the key role to the character of a black hole space time [18–29]. In 1998, Strominger [18] discussed the near-horizon asymptotic symmetry in a Kerr black hole and found that there was a holographic duality between extremal and near-extremal Kerr black hole and a 2-dimensional

conformal field theory. In [23, 24], it was shown that around the horizon of a Kerr space time, the scalar field theory can be reduced to a 2-dimensional effective field theory. Now, the thermal radiation of scalar particles from a Kerr black hole can be derived on the basis of a conformal symmetry arising from the near-horizon geometry [29].

Using the near-horizon geometry and open quantum system approach, Hawking effect in a Kerr space time will be investigated. We will examine the time evolution of a static detector (modeled by a two-level atom) outside a Kerr space time immersed in a vacuum massless scalar field. The dynamics of the detector can be obtained from the complete time evolution describing the total system (detector plus external field) by integrating over the field degrees of freedom. Our results show that the detector would spontaneously excite in the Unruh vacuum state with a probability the same as the thermal radiation at Hawking temperature, indicating that Hawking radiation from a Kerr space time can be understood as the manifestation of thermalization phenomena in the framework of open quantum systems. The conformal invariance of the wave equation near the horizon is at the key point of Hawking quantum effect in the Kerr space time.

The organization of our paper is as follows. In Section 2, we will review the basic formulae, including the master equation describing the system of the detector plus external vacuum scalar field in the weak-coupling limit and the reduced dynamical equation for the finite time evolution of the detector. In Section 3, the dimensional reduction technique is used to investigate the massless scalar field in a Kerr space time, and the Wightman function is obtained. In Section 4, applying the method and results of the preceding sections to calculate the probability of a spontaneous transition of the detector from the ground state to the excited state outside a Kerr space time. Finally, some discussions and conclusions will be given in Section 5.

## 2. Review of the Open Quantum System Approach

In this section, we will review the open quantum system approach to get the master equation describing the combined system $B + S$, where a static detector (two-level atom) as an open system $S$ which is coupled to another quantum system $B$ of a vacuum massless scalar field in a Kerr space time. Our derivation mostly follows the works in [15–17]. Here, we will consider the evolution of the static detector in the proper time and assume the combined system $(B + S)$ to be initially prepared in a factorized state, with the detector keeping static in the exterior region of the Kerr black hole and the field keeping in vacuum state. The static detector is a two-level simplest quantum system whose Hilbert space is spanned over just two states, an excited state $|+\rangle$, and a ground state $|-\rangle$. The Hilbert space of such a system is equivalent to that of a spin-(1/2) system. So the states of the detector can be represented by a $2 \times 2$ density matrix, which is Hermitian $\rho^{\dagger} = \rho$, and normalized $\mathrm{Tr}(\rho) = 1$ with $\det(\rho) \geqslant 0$. For simplicity, the Hamiltonian of the detector $H_S$ may be taken as

$$H_S = \frac{\omega_0}{2}\sigma_3, \tag{1}$$

where $\sigma_3$ is the Pauli matrix and $\omega_0$ is the energy level spacing. The standard Hamiltonian of massless, free scalar field in a Kerr space time can be denoted as $H_B$, which will be discussed in detail in Section 3. The interaction Hamiltonian of the detector with the scalar field can be denoted as

$$H_I' = \sigma_3 \phi(x). \tag{2}$$

Therefore, the Hilbert space of the total system $S + B$ is given by the tensor product space $\mathscr{H} = \mathscr{H}_S \otimes \mathscr{H}_B$. The total Hamiltonian $H(t)$ can be taken as

$$H = H_S \otimes I_B + I_S \otimes H_B + \lambda H_I', \tag{3}$$

where $\lambda$ is the coupling constant and, $I_S$ and $I_B$ denote the identity operators in $\mathscr{H}_S$ and $\mathscr{H}_B$, respectively.

Now in order to get the reduced dynamics of the subsystem $S$, we assume that the interaction between the detector and the scalar field is weak as $\lambda$ is small and the finite time evolution describing the dynamics of the detector takes the form of a one-parameter semigroup of completely positive map.

Initially, the complete system is described by the total density matrix $\rho_{\mathrm{tot}} = \rho(0) \otimes |0\rangle\langle 0|$, where $\rho(0)$ is the initial reduced density matrix of the detector and $|0\rangle$ is the Kerr space-time vacuum state of field $\phi(x)$. In the frame of the atom, the evolution in the proper time $\tau$ of the total density $\rho_{\mathrm{tot}}$ of the complete system satisfies

$$\frac{\partial \rho_{\mathrm{tot}}(\tau)}{\partial \tau} = -iL_H\left[\rho_{\mathrm{tot}}(\tau)\right], \tag{4}$$

which is often referred to the von Neumann or Liouville-von Neumann equation, where $L_H$ represents the Liouville operator associated with $H$ as follows:

$$L_H[S] \equiv [H, S]. \tag{5}$$

The dynamics of the detector can be obtained by summing over the degrees of freedom of the field $\phi$; that is, by applying to $\rho_{\mathrm{tot}}(\tau)$ with the trace projection operator $P$ as follows:

$$\rho(\tau) = P\left[\rho_{\mathrm{tot}}(\tau)\right] \equiv \mathrm{Tr}_{\phi}\left[\rho_{\mathrm{tot}}(\tau)\right]. \tag{6}$$

In the limit of weak coupling, we can find that the reduced density obeys an equation in the Kossakowski-Lindblad form [30–32] as follows:

$$\frac{\partial \rho(\tau)}{\partial \tau} = -i\left[H_{\mathrm{eff}}, \rho(\tau)\right] + \mathscr{L}\left[\rho(\tau)\right], \tag{7}$$

where

$$\mathscr{L}[\rho] = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}\left[2\sigma_j\rho\sigma_i - \sigma_i\sigma_j\rho - \rho\sigma_i\sigma_j\right]. \tag{8}$$

The matrix $a_{ij}$ and the effective Hamiltonian $H_{\text{eff}}$ are determined by the Fourier transform $\mathcal{G}(\lambda)$ and Hilbert transform $\mathcal{K}(\lambda)$ of the vacuum field correlation function (the Wightman function) as follows:

$$G^{+}(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle, \tag{9}$$

and they are defined as

$$\begin{aligned} \mathcal{G}(\lambda) &= \int d\tau\, e^{i\lambda\tau} G^{+}(x(\tau)), \\ \mathcal{K}(\lambda) &= \frac{P}{\pi i} \int d\omega \frac{\mathcal{G}(\omega)}{\omega - \lambda}. \end{aligned} \tag{10}$$

The coefficients of the Kossakowski matrix $a_{ij}$ can be written as

$$a_{ij} = A\delta_{ij} - iB\epsilon_{ijk}\delta_{k3} + C\delta_{i3}\delta_{j3}, \tag{11}$$

with

$$\begin{aligned} A &= \frac{1}{2}\left[\mathcal{G}(\omega_0) + \mathcal{G}(-\omega_0)\right], \\ B &= \frac{1}{2}\left[\mathcal{G}(\omega_0) - \mathcal{G}(-\omega_0)\right], \\ C &= \mathcal{G}(0) - A. \end{aligned} \tag{12}$$

The effective Hamiltonian $H_{\text{eff}}$ contains a correction term, the so-called Lamb shift, and one can find that it can be obtained by replacing $\omega_0$ in $H_s$ with a renormalized energy level spacing $\Omega$ as follows:

$$H_{\text{eff}} = \frac{\Omega}{2}\sigma_3 = \omega_0 + i\left[\mathcal{K}(-\omega_0) - \mathcal{K}(\omega_0)\right]\sigma_3, \tag{13}$$

where a suitable subtraction is assumed in the definition of $\mathcal{K}(-\omega_0) - \mathcal{K}(\omega_0)$ to remove the logarithmic divergence which would otherwise be presented.

To facilitate the discussion of the properties of solutions for (7) and (8), let us express the density matrix in terms of the Pauli matrices as follows:

$$\rho(\tau) = \frac{1}{2}\left(1 + \sum_{i=1}^{3} \rho_i(\tau)\sigma_i\right). \tag{14}$$

Substituting (14) into (8), the Bloch vector $|\rho(\tau)\rangle$ of components $\rho_1(\tau), \rho_2(\tau), \rho_3(\tau)$ satisfies

$$\frac{\partial}{\partial\tau}|\rho(\tau)\rangle = -2\mathcal{H}|\rho(\tau)\rangle + |\eta\rangle, \tag{15}$$

where $|\eta\rangle$ denotes a constant vector $\{0, 0, -4B\}$. The exact form of the matrix $\mathcal{H}$ reads

$$\mathcal{H} = \begin{pmatrix} 2A + C & \frac{\Omega}{2} & 0 \\ -\frac{2}{\Omega} & 2A + C & 0 \\ 0 & 0 & A \end{pmatrix}. \tag{16}$$

Equation (15) can be solved exactly and its solution is

$$|\rho(\tau)\rangle = e^{-2\mathcal{H}\tau}|\rho(0)\rangle + \left(1 - e^{-2\mathcal{H}\tau}\right)|\rho_\infty\rangle, \tag{17}$$

where

$$|\rho_\infty\rangle = \frac{1}{2}\mathcal{H}^{-1}|\eta\rangle = -\frac{B}{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{18}$$

the matrix $e^{-2\mathcal{H}\tau}$ is defined by series expansion as usual. However, $\mathcal{H}$ obeys a cubic eigenvalue equation, so powers of $\mathcal{H}$ higher than 2 can always be written in terms of combinations of $\mathcal{H}^2$, $\mathcal{H}$, and $I$. Actually, three eigenvalues of $\mathcal{H}$ are $\lambda_1 = 2A$, $\lambda_\pm = (2A + C) \pm i\Omega/2$. We can write

$$\begin{aligned} e^{-2\mathcal{H}\tau} = \frac{4}{\Omega^2 + 4C^2} \Big\{ & e^{-2A\tau}\Lambda_1 + 2e^{-2(2A+C)\tau} \\ & \times \left[\Lambda_2 \cos(\Omega\tau) + \Lambda_3 \frac{\sin(\Omega\tau)}{\Omega}\right]\Big\}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Lambda_1 &= \left[(2A + C)^2 + \frac{(\Omega)^2}{4}\right] I - 2(2A + C)\mathcal{H} + \mathcal{H}^2, \\ \Lambda_2 &= -2A(A + C) I + (2A + C)\mathcal{H} - \frac{1}{2}\mathcal{H}^2, \\ \Lambda_3 &= 2A\left[\frac{\Omega^2}{4} - C(2A + C)\right] I \\ &\quad + \left[C(4A + C) - \frac{\Omega^2}{4}\right]\mathcal{H} - C\mathcal{H}^2. \end{aligned} \tag{20}$$

Equation (19) reveals that a freely falling atom in a Kerr space time is subjected to the effects of decoherence and dissipation by the exponentially decaying factors including the real parts of the eigenvalues of $\mathcal{H}$ and oscillating terms associated with the imaginary part. These nonunitary effects can be analyzed by examining the evolution behavior in time of suitable atom observable. For any observable of the atom represented by a Hermitian operator $\mathcal{O}$, the behavior of its mean value is determined by

$$\langle \mathcal{O} \rangle = \text{Tr}\left[\mathcal{O}\rho(\tau)\right]. \tag{21}$$

Let the observable $\mathcal{O}$ be an admissible atom state $\rho_f$, the probability $\mathcal{P}_{i\to f}$, that the atom evolves to the expected state represented by density matrix $\rho_f(\tau)$ from an initial one $\rho_i \equiv \rho(0)$, should be

$$\mathcal{P}_{i\to f}(\tau) = \text{Tr}\left[\rho_f \rho(\tau)\right]. \tag{22}$$

If initially the atom is in the ground state, its Bloch vector $|\rho(0)\rangle$ is $\{0, 0, -1\}$, and the final state $\rho_f$ is the excited state given by the Bloch vector $|\rho_f\rangle = \{0, 0, 1\}$, according to (17)–(22), we have

$$\mathcal{P}_{i\to f} = \frac{1}{2}\left(1 - e^{-4A\tau}\right)\left(1 - \frac{B}{A}\right). \tag{23}$$

The probability per unit time of the transition from the ground state to the excited state, in the limit of infinitely slow switching on and off the atom-field interaction, that is, the spontaneous excitation rate, can be calculated by taking the time derivative of $\mathscr{P}_{i\to f}(\tau)$ at $\tau = 0$ as

$$\Gamma_{i\to f} = \frac{\partial}{\partial\tau}\mathscr{P}_{i\to f}(\tau)\Big|_{\tau=0} = 2A - 2B = 2\mathscr{G}(-\omega_0). \quad (24)$$

## 3. Scalar Wave Equation Near the Event Horizon in a Kerr Space-Time

*3.1. Dimensional Reduction Near the Horizon.* In order to find out how the reduced density evolves with proper time from (7), we will investigate the scalar wave equation of the Kerr space time. In Boyer-Lindquist coordinates, the stationary Kerr space time can be written as

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\varphi\right)^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - a\,dt\right]^2 \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \end{aligned} \quad (25)$$

where $\Delta = (r - r_+)(r - r_-)$, $\rho^2 = r^2 + a^2\cos^2\theta$ and $r_\pm = M \pm (M^2 - a^2)^{1/2}$. The parameters $M$ and $a$ represent the mass and the angular momentum per unit mass of the black hole, respectively. The event horizon of the Kerr black hole is located at $r = r_+$. The line element in (25) is stationary and axisymmetric, with $\partial_t^\mu$ and $\partial_\varphi^\mu$ as the corresponding Killing vector fields.

And then, we will show that the scalar field theory in the background (25) can be reduced to a 2-dimensional field theory in the near-horizon region with the dimensional reduction technique. This technique firstly has been employed for the Kerr black hole by Murata and Soda [24] and developed with a more general technique by Iso et al. [23].

The action for the scalar field in a Kerr space time is

$$S = \frac{1}{2}\int dx^4\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + S_{\text{int}}, \quad (26)$$

where the first term is the kinetic term and the second term $S_{\text{int}}$ represents the mass, potential, and interaction terms.

By substituting (25) into (26), we obtain

$$\begin{aligned} S = -\frac{1}{2}\int dr\,dt\,d\theta\,d\varphi\sin\theta\phi\Bigg[&-\left(\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right)\partial_t^2 \\ &- \frac{2a\left(r^2+a^2-\Delta\right)}{\Delta}\partial_t\partial_\varphi \\ &+ \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\partial_\varphi^2 + \partial_r\Delta\partial_r \\ &+ \frac{1}{\sin\theta}\partial_\theta\sin\theta\,\partial_\theta\Bigg]\phi + S_{\text{int}}. \end{aligned} \quad (27)$$

Now, we transform the radial coordinate $r$ into the tortoise coordinate $r_*$ defined by

$$\frac{dr_*}{dr} = \frac{1}{F(r)} \equiv \frac{r^2+a^2}{\Delta}. \quad (28)$$

After the transformation, the action (27) can be written as

$$\begin{aligned} S = &-\frac{1}{2}\int dr_*\,dt\,d\theta\,d\varphi\sin\theta\phi \\ &\times\Bigg[-\left(\left(r^2+a^2\right) - F(r)\,a^2\sin^2\theta\right)\partial_t^2 \\ &- 2a\left(1 - F(r)\right)\partial_t\partial_\varphi + \left(\frac{F(r)}{\sin^2\theta} - \frac{a^2}{r^2+a^2}\right)\partial_\varphi^2 \\ &+ \partial_{r_*}\left(r^2+a^2\right)\partial_{r_*} + \frac{F(r)}{\sin\theta}\partial_\theta\sin\theta\,\partial_\theta\Bigg]\phi + S_{\text{int}}. \end{aligned} \quad (29)$$

Now we consider this action in the region near the horizon. Since $F(r_+) = 0$ at $r \to r_+$, we only retain dominant terms in (29). We have

$$\begin{aligned} S = &-\frac{1}{2}\int dr_*\,dt\,d\theta\,d\varphi\sin\theta\phi \\ &\times\Bigg[-\left(r^2+a^2\right)\partial_t^2 - 2a\partial_t\partial_\varphi \\ &- \frac{a^2}{r^2+a^2}\partial_\varphi^2 + \partial_{r_*}\left(r^2+a^2\right)\partial_{r_*}\Bigg]\phi, \end{aligned} \quad (30)$$

where we have ignored $S_{\text{int}}$ by using $F(r_+) = 0$ at $r \to r_+$. Because the theory becomes high-energy case near the horizon and the kinetic term dominates, we can ignore all the terms in $S_{\text{int}}$. After this analysis, we return to the expression written in terms of $r$. So, we have

$$\begin{aligned} S = &-\frac{1}{2}\int dt\,dr\,d\theta\,d\varphi\sin\theta\left(r^2+a^2\right)\phi \\ &\times\Bigg[-\frac{1}{F(r)}\partial_t^2 - \frac{2a}{\Delta}\partial_t\partial_\varphi - \frac{a^2}{\Delta\left(r^2+a^2\right)}\partial_\varphi^2 \\ &+ \partial_r F(r)\,\partial_r\Bigg]\phi. \end{aligned} \quad (31)$$

Following Murata and Soda's method [24], we transform the coordinates to the corotating coordinate system. They

employed a locally corotating coordinate system, and we will use a globally corotating coordinate system as

$$\psi = \varphi - \frac{a}{r^2 + a^2} t, \qquad \xi = t. \tag{32}$$

Under these new coordinates, we can rewrite the action (31) as

$$S[\phi] = -\frac{1}{2} \int d\xi\, dr\, d\psi\, d\theta \left(r^2 + a^2\right) \times \sin\theta\, \phi \left(-\frac{1}{F(r)} \partial_\xi^2 + \partial_r F(r) \partial_r\right) \phi, \tag{33}$$

so the angular terms disappear completely. Using the spherical harmonics expansion $\phi = \Sigma_{l,m} \phi_{lm}(\xi, r) Y_{lm}(\theta, \psi)$, we obtain the effective 2-dimensional action

$$S[\phi] = \sum_{l,m} \frac{1}{2} \int \left(r^2 + a^2\right) d\xi\, dr \phi_{lm} \times \left(-\frac{1}{F(r)} \partial_\xi^2 + \partial_r F(r) \partial_r\right) \phi_{lm}, \tag{34}$$

where we have used the orthonormal condition for the spherical harmonics as follows:

$$\int d\psi\, d\theta \sin\theta\, Y^*_{l'm'} Y_{lm} = \delta_{l',l} \delta_{m',m}. \tag{35}$$

From the action (34), it is obvious to find that $\phi$ can be considered as a (1 + 1)-dimensional massless scalar field in the backgrounds of the dilaton $\Phi$. The effective 2-dimensional metric and the dilaton $\Phi$ can be written as

$$ds^2 = -F(r)\, d\xi^2 + \frac{1}{F(r)} dr^2, \tag{36}$$

$$\Phi = r^2 + a^2. \tag{37}$$

So far, we have reduced the 4-dimensional field theory to a 2-dimensional case. This is consistent with [23]. This 2-dimensional metric tells us that, near the horizon, the geometry of a Kerr space time can be regarded as a Rindler space time when $r_+ > r_-$. In the extremal case $r_+ = r_-$, the near horizon geometry reduces to $AdS_2$ which is consistent with [19, 23]. The same as the Schwarzschild space time [33], we will define two vacuum states by using the two natural notions of time translation of this effective 2-dimensional metric, namely, the Killing time and the proper time as measured by a congruence of freely falling observers.

*3.2. The Boulware Vacuum.* Using the tortoise coordinate in (28), the effective 2-dimensional metric (36) can be changed into

$$ds_I^2 = -F(r) \left(d\xi^2 + dr_*^2\right). \tag{38}$$

We can see that the $(\xi, r_*)$ part of the metric has the form of Minkowski metric. Now in this 2-dimensional space time, the wave equation of $\phi(\xi, r_*)$ can be written as

$$\left[\partial_\xi^2 - \partial_{r_*}^2\right] \phi(\xi, r_*) \equiv \partial_u \partial_v \phi(u, v) = 0. \tag{39}$$

Its standard ingoing and outgoing orthonormal mode solutions are

$$\phi(\xi, r_*) \sim \left(e^{-i\sigma(\xi + r_*)}, e^{-i\sigma(\xi - r_*)}\right) \sim \left(e^{-i\sigma v}, e^{-i\sigma u}\right), \tag{40}$$

where $v = \xi + r_*$, $u = \xi - r_*$ are null coordinates. These modes are positive frequency modes with respect to the killing vector field $\partial/\partial\xi$ for $\sigma > 0$, and they satisfy

$$L_{\partial/\partial\xi} \phi = -i\sigma\phi. \tag{41}$$

It is obvious that the wave equation (39) is manifestly invariant under the infinite-dimensional group of conformal transformation in two dimensions $u \to u'(u)$, $v \to v'(v)$. In the following, we will show how this conformal symmetry does play the key role in the quantum effect of a Kerr space time.

Near the event horizon, we only consider the outgoing modes

$$\phi^{\text{out}}(\xi, r_*) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\sigma(\xi - r_*)} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\sigma u}, \tag{42}$$

along the rays $u$ = constant. Quantizing the field $\phi^{\text{out}}$ in the exterior of the black hole, we can expand it as follows

$$\hat{\phi}^I = \sum_\sigma \left[a_\sigma^I \phi_\sigma^{\text{out}}(\xi, r_*) + a_\sigma^{I\dagger} \phi_\sigma^{\text{out}\dagger}(\xi, r_*)\right], \tag{43}$$

where $a_\sigma^I$ and $a_\sigma^{I\dagger}$ are the annihilation and creation operators acting on the $I$ vacuum state, which corresponds to the Boulware vacuum. The Fock vacuum state can be defined as $a_\sigma^I |0\rangle = 0$. So, with the proper $i\epsilon$ prescription, the Wightman function of $I$ state can be written as

$$G_{\text{Kerr}}^{I+}(x, x') = -\frac{1}{4\pi^2} \frac{1}{\left(x^0 - i\epsilon\right)^2 - \left(x^1\right)^2} = -\frac{1}{4\pi^2} \frac{1}{(\Delta\xi - i\epsilon)^2}, \tag{44}$$

where $x^0 = \xi$, $x^1 = r_*$.

*3.3. The Unruh Vacuum.* In order to define the Unruh vacuum state, one can write the Kerr line element in the near-horizon region in terms of Kruskal-like coordinates defined as

$$U = T - R = -\kappa^{-1} e^{-\kappa u}, \qquad V = T + R = \kappa^{-1} e^{\kappa v}, \tag{45}$$

where $T = \kappa^{-1} e^{\kappa r_*} \sinh \kappa\xi$, $R = \kappa^{-1} e^{\kappa r_*} \cosh \kappa\xi$ and $\kappa = (r_+ - r_-)/2(r_+^2 + a^2)$ is the surface gravity of the event horizon. The effective 2-dimensional space time (36) becomes

$$ds_{II}^2 = C(r) \left[-dT^2 + dR^2\right], \tag{46}$$

where $C(r) = e^{-2\kappa r_*} F(r)$, which is a finite constant near the event horizon $r = r_+$. The $(T, R)$ part of the metric also has the form of Minkowski metric. The interval of time $\Delta T$ then corresponds to the interval of proper time of a radial freely

falling observer crossing the horizon. The 2-dimensional space time in the $(T, R)$ coordinates is well-behaved near the event horizon. One can obtain the wave equation of $\phi(T, R)$ as

$$\left[\partial_T^2 - \partial_R^2\right]\phi(T,R) \equiv \partial_U \partial_V \phi(U,V) = 0. \tag{47}$$

Similar to previous proceeding, we can obtain the outgoing wave solution as $\phi^{\text{out}}(T,R) \sim e^{-i\omega(T-R)} = e^{-i\omega U}$. These modes are positive frequency modes with respect to the freely falling observer for $\omega > 0$, satisfying

$$L_{\partial/\partial_T}\phi = -i\omega\phi. \tag{48}$$

As pointed in previous subsection, there are conformal transformations from $\phi(U,V)$ to $\phi(u,v)$, as (45).

Subsequently, quantizing the field $\phi^{\text{out}}(T,R)$ in the exterior of the black hole, we can expand it as follows:

$$\widehat{\phi}^{II} = \sum_{\omega}\left[a_\omega^{II}\phi_\omega^{\text{out}}(T,R) + a_\omega^{II\dagger}\phi_\omega^{\text{out}\dagger}(T,R)\right], \tag{49}$$

where $a_\omega^{II}$ and $a_\omega^{II\dagger}$ are the annihilation and creation operators acting on the $II$ vacuum state, which can be defined as $a_\omega^{II}|0\rangle = 0$. This vacuum state is just the so-called Unruh vacuum defined in the maximally extended geometry. So, with the proper $i\epsilon$ prescription, the Wightman function of the $II$ state can be written as

$$\begin{aligned} G_{\text{Kerr}}^{II+}\left(x,x'\right) &= -\frac{1}{4\pi^2}\frac{1}{\left(x^0 - i\epsilon\right)^2 - \left(x^1\right)^2} \\ &= -\frac{1}{16\pi^2\kappa^{-2}\sinh^2\left[\left(\xi - \xi'\right)\kappa/2 - i\epsilon\right]}, \end{aligned} \tag{50}$$

where $x^0 = T$, $x^1 = R$.

## 4. Probability of Spontaneous Transition of the Detector in a Kerr Space-Time

In what follows, we will calculate the spontaneous excitation rate in the two vacuum states with the open quantum system approach.

*4.1. The I State-Boulware Vacuum.* Firstly, let us turn to the $I$ state-Boulware vacuum case. Thinking of the relation between the proper time and the coordinate time,

$$d\tau = \sqrt{F(r)}d\xi, \tag{51}$$

the Fourier transform of the Wightman function (44) with respect to the proper time can be expressed as

$$\begin{aligned} \mathcal{G}_{\text{Kerr}}(\lambda) &= \int_{-\infty}^{+\infty} d\tau\, e^{i\lambda\tau} G_{\text{Kerr}}^{I+}\left(x,x'\right) \\ &= -\int_{-\infty}^{+\infty} d\xi\sqrt{F(r)}e^{i\lambda\sqrt{F(r)}\xi}\left[\frac{1}{4\pi^2}\frac{1}{(\Delta\xi - i\epsilon)^2}\right] = 0. \end{aligned} \tag{52}$$

According to (12), we have

$$A = B = 0. \tag{53}$$

So the spontaneous excitation rate can be obtained as

$$\Gamma_{i\to f} = 2(A - B) = 0. \tag{54}$$

Therefore, no spontaneous excitation would ever occur in the $I$ state-Boulware vacuum. In fact, the Boulware vacuum corresponds to our familiar notion of a vacuum state. This result is consistent with the conclusion in [16].

*4.2. The II State-Unruh Vacuum.* Using the Wightman function (50) and the relation between the proper time and coordinate time (51), the Fourier transform can be given as

$$\mathcal{G}_{\text{Kerr}}(\lambda) = \int_{-\infty}^{+\infty} d\tau e^{i\lambda\tau} G_{\text{Kerr}}^{II+}\left(x,x'\right) = \frac{\lambda}{2\pi}\frac{e^{2\pi\kappa_r^{-1}\lambda}}{e^{2\pi\kappa_r^{-1}\lambda} - 1}, \tag{55}$$

where $\kappa_r = \kappa/\sqrt{F(r)} = \kappa\sqrt{(r^2 + a^2)/\Delta}$.

According to (12), we have

$$\begin{aligned} A &= \frac{1}{2}\left[\mathcal{G}_{\text{Kerr}}(\omega_0) + \mathcal{G}_{\text{Kerr}}(-\omega_0)\right] = \frac{\omega_0\coth(\pi\omega_0/\kappa_r)}{4\pi}, \\ B &= \frac{1}{2}\left[\mathcal{G}_{\text{Kerr}}(\omega_0) - \mathcal{G}_{\text{Kerr}}(-\omega_0)\right] = \frac{\omega_0}{4\pi}, \\ C &= \mathcal{G}_{\text{Kerr}}(0) - \frac{\omega_0\coth(\pi\omega_0/\kappa_r)}{4\pi}. \end{aligned} \tag{56}$$

Using (23) and (24), we have

$$\begin{aligned} \mathcal{P}_{i\to f} &= \frac{1}{2}\left(1 - e^{-(\omega_0\coth(\pi\omega_0/\kappa_r)/\pi)t}\right) \\ &\quad\times\left(1 - \frac{1}{\coth(\pi\omega_0/\kappa_r)}\right), \end{aligned} \tag{57}$$

and the spontaneous excitation rate is

$$\Gamma_{i\to f} = \frac{\partial}{\partial t}\mathcal{P}_{i\to f}(t)\Big|_{t=0} = \frac{\omega_0}{\pi\left(e^{2\pi\omega_0/\kappa_r} - 1\right)}, \tag{58}$$

which reveals that, the ground state detector in the $II$ vacuum would spontaneously excite with an excitation rate that one would expect in the case of a flux of thermal radiation at the temperature

$$T = \frac{\kappa_r}{2\pi}. \tag{59}$$

It is obvious that there is a near horizon conformal symmetry from $\phi(U,V)$ to $\phi(u,v)$ as (45), which is just the reason of the nonvanishing spontaneously excitation. This proposal is similar to [29]. In fact, the effective temperature $T$ in (59) approaches to Hawking temperature $T = \kappa/2\pi = (r_+ - r_-)/4\pi(r_+^2 + a^2)$ as $r \to \infty$. This suggests that the thermal radiation emanating from the horizon of a Kerr black hole is just Hawking radiation, which is in agreement with [16, 17].

## 5. Conclusions and Discussions

In brief, we have investigated the Hawking radiation from a Kerr space time through examining the evolution of a detector (modeled by a two-level atom) interacting with the vacuum massless scalar field in the framework of open quantum systems.

First of all, using the dimensional reduction technique, the 4-dimensional spherically nonsymmetric Kerr metric can be regarded as a 2-dimensional effective spherically symmetric metric near the event horizon. So we can construct two conformal vacuum states in this 2-dimensional effective space time: one is the $I$ state-Boulware vacuum, the other is the $II$ state-Unruh vacuum. Then we give the Wightman functions of the two vacuum states, respectively.

On the basis of these, we have calculated the time evolution of the detector in the two vacuum states. It is found that the detector in the $II$ state-Unruh vacuum would spontaneously excite with a nonvanishing probability the same as thermal radiation at Hawking temperature from a Kerr black hole. Hawking-Unruh effect of a Kerr space time can be understood as a manifestation of thermalization phenomena in an open quantum system. Meanwhile, it is also found that the probability of spontaneous transition of the detector would be vanishing in the $I$ state-Boulware vacuum. It suggests that near horizon conformal symmetry plays the key role in the full quantum phenomena in the Kerr space time.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Review Article*

# The Geometry of Black Hole Singularities

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Recent results show that important singularities in General Relativity can be naturally described in terms of finite and invariant canonical geometric objects. Consequently, one can write field equations which are equivalent to Einstein's at nonsingular points but, in addition remain well-defined and smooth at singularities. The black hole singularities appear to be less undesirable than it was thought, especially after we remove the part of the singularity due to the coordinate system. Black hole singularities are then compatible with global hyperbolicity and do not make the evolution equations break down, when these are expressed in terms of the appropriate variables. The charged black holes turn out to have smooth potential and electromagnetic fields in the new atlas. Classical charged particles can be modeled, in General Relativity, as charged black hole solutions. Since black hole singularities are accompanied by dimensional reduction, this should affect Feynman's path integrals. Therefore, it is expected that singularities induce dimensional reduction effects in Quantum Gravity. These dimensional reduction effects are very similar to those postulated in some approaches to make Quantum Gravity perturbatively renormalizable. This may provide a way to test indirectly the effects of singularities, otherwise inaccessible.

## 1. Introduction

For millennia, space was considered the fixed background where physical phenomena took place. Special Relativity changed this, by proposing spacetime as the new arena. Then, while trying to extend the success of Special Relativity to noninertial frames and gravity, Einstein realized that one should let go the idea of an immutable background, and General Relativity (GR) was born. There is a very deep interdependence between matter and the geometry of spacetime, encoded in Einstein's equation. Its predictions were tested with high accuracy and confirmed.

However, the task of decoding the way our universe works from something as abstract as Einstein's equation is not easy, and we are far from grasping all of its consequences. For instance, even from the beginning, when Schwarzschild proposed his model for the exterior of a spherically symmetric object, Einstein's equations led to infinities [1, 2]. The Schwarzschild metric tensor becomes infinite at $r = 0$ and on the event horizon, where $r = 2m$. The big bang also exhibited a singularity [3–10].

The first reaction to the singularities was to somehow minimize their importance, on the grounds that they are exceptions due to the perfect symmetry of the solutions. This hope was ruined by the theorems of Penrose [11, 12] and Hawking [13–16], showing that the singularities are predicted to occur in GR under very general conditions and are not caused by the perfect symmetry.

Singularities, hidden by the event horizon or naked, are very well researched in the literature (e.g., [12, 17–25] and references therein).

Interesting results concerning singularities were obtained in some modified gravity theories, for example, $f(R)$ gravity ([26–30] and references therein). Another way to avoid singularities was proposed in nonlinear electrodynamics [31].

In addition to the singularities, infinities occur in GR when we try to quantize gravity, because gravity is perturbatively nonrenormalizable [32, 33]. It is expected by many that a solution to the problem of quantization will also remove the singularities. For example, *Loop quantum cosmology* obtained significant positive results in showing that quantum effects may prevent the occurrence of singularities [34–37].

There is another possibility: the problem of singularities may be in fact not due to GR but to our limited understanding of GR. Therefore, it would be useful to better understand singularities, even in the eventuality that a better theory will replace GR. In the following we review some recent results showing that by confronting singularities, we realize that they are not that undesirable [38]. Moreover, new possibilities open also for the Quantum Gravity problem.

## 2. The Problem of Singularities in General Relativity

*2.1. Two Types of Singularities.* Not all singularities are born equal. We can roughly classify the singularities in two types:

(1) *Malign singularities*: some of the components of the metric are divergent: $g_{ab} \to \infty$.

(2) *Benign singularities*: $g_{ab}$ are smooth and finite but $\det g \to 0$.

Benign singularities turn out to be, in many cases, manageable [39–41]. The infinities simply disappear, if we use different geometric objects to write the equations and describe the phenomena. At points where the metric is nondegenerate, the proposed description is equivalent to the standard one. But, in addition, it works also at the points where the metric becomes degenerate.

Malign singularities appear in the black hole solutions. They appear to be malign because the coordinates in which they are represented are singular. In nosingular coordinates, they become benign [42–44]. This is somewhat similar to the case of the apparent singularity on the event horizon, which turned out to be a coordinate singularity and not a genuine one [45, 46].

*2.2. What Is Wrong with Singularities?* The geometry of spacetime is encoded in the metric tensor. To write down field equations, we have to use partial derivatives. In curved spaces, partial derivatives are replaced by *covariant derivatives*. They are defined with the help of the *Levi-Civita connection*, which takes into account the parallel translations, to compare fields at infinitesimally closed points. The covariant derivative is written using the Christoffel symbol of the second kind, obtained from the metric tensor by

$$\Gamma^c_{ab} = \frac{1}{2} g^{cs} \left( \partial_a g_{bs} + \partial_b g_{sa} - \partial_s g_{ab} \right). \tag{1}$$

It can be used to define the Riemann curvature tensor:

$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^d_{bs}\Gamma^s_{ac} - \Gamma^d_{cs}\Gamma^s_{ab}. \tag{2}$$

It plays a major part in the Einstein equation:

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{3}$$

since

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{4}$$

Table 1: Singular objects and their nonsingular equivalents.

| Singular | Nonsingular | When $g$ is |
|---|---|---|
| $\Gamma^c_{ab}$ (2nd) | $\Gamma_{abc}$ (1st) | Smooth |
| $R^d_{abc}$ | $R_{abcd}$ | Semiregular |
| $R_{ab}$ | $R_{ab}\sqrt{\lvert\det g\rvert}^W, W \le 2$ | Semiregular |
| $R$ | $R\sqrt{\lvert\det g\rvert}^W, W \le 2$ | Semiregular |
| Ric | $\mathrm{Ric} \circ g$ | Quasi-regular |
| $R$ | $Rg \circ g$ | Quasi-regular |

where $R_{ab} = R^s_{asb}$ is the Ricci tensor and $R = R^s_s$ is the scalar curvature.

In the case of malign singularities, since some of metric's components are singular, the geometric objects like the Levi-Civita connection and the Riemann curvature tensor are singular too. Therefore, it seems that the situation of malign singularities is hopeless.

Even in the case of benign singularities, when the metric is smooth, but its determinant $\det g \to 0$, the usual Riemannian objects are singular. For example, the covariant derivative cannot be defined, because the inverse of the metric, $g^{ab}$, becomes singular ($g^{ab} \to \infty$ when $\det g \to 0$). This makes Christoffel's symbols of the second kind (1) and the Riemann curvature (2) singular.

It is therefore understandable why singularities were considered unsolvable problems for so many years.

*2.3. From Singular to Nonsingular: A Dictionary.* The main variables which appear in the equations are indeed singular. But we can replace them with new variables, which are equivalent to the original ones on the domain where both are defined. Sometimes, we can choose the new variables so that the equations remain valid at points where the original ones were singular.

The geometric objects of interest that become singular when the metric is degenerate are the Levi-Civita connection (1), the Riemann curvature (2), and the Ricci and the scalar curvatures. If the metric is nondegenerate, the Christoffel symbols of the first kind are equivalent to those of the second kind, in the sense that by knowing one of them, we can obtain the other one. Similarly, the Riemann curvature $R^a_{bcd}$ is equivalent to $R_{abcd}$, and the Ricci and scalar curvatures are equivalent to their densitized versions and to their Kulkarni-Nomizu products (see (30)) with the metric. In some important cases, these equivalent objects remain nonsingular even when the metric is degenerate [39, 41]. We summarize these cases in Table 1.

## 3. The Mathematical Methods: *Singular Semi-Riemannian Geometry*

*3.1. Singular Semi-Riemannian Geometry.* We review the main mathematical tool on which the results presented here are based, named *Singular Semi-Riemannian Geometry* [39, 40]. Singular Semi-Riemannian Geometry is mainly

concerned with the study of singular semi-Riemannian manifolds.

*Definition 1* (see [39, 47]). A *singular semi-Riemannian manifold* $(M, g)$ consists in a differentiable manifold $M$ and a symmetric bilinear form $g$ on $M$, named *metric tensor* or *metric*.

If $g$ is nondegenerate, then $(M, g)$ is just a *semi-Riemannian manifold*. If in addition $g$ is positive definite, $(M, g)$ is named *Riemannian manifold*. In General Relativity semi-Riemannian manifolds are normally used, but when we are dealing with singularities, it is natural to use the Singular Semi-Riemannian Geometry, which is more general.

*3.2. Properties of the Degenerate Inner Product.* Let $(V, g)$ be an inner product vector space. Let $\flat : V \to V^*$ be the morphism defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, -)$. We define the *radical* of $V$ as the set of isotropic vectors in $V$: $V_\circ := \ker \flat = V^\perp$. We define the *radical annihilator* space of $V$ as the image of $\flat$, $V^\bullet := \operatorname{im} \flat \subset V^*$. The inner product $g$ induces on $V^\bullet$ an inner product, defined by $g_\bullet(u_1^\flat, u_1^\flat) := g(u_1, u_2)$. This one is the inverse of $g$ if and only if $\det g \neq 0$. The *coannihilator* is the quotient space $V_\bullet := V/V_\circ$, given by the equivalence classes of the form $u+V_\circ$. On the *coannihilator* $V_\bullet$, the metric $g$ induces an inner product $g^\bullet(u_1+V_\circ, u_2+V_\circ) := g(u_1, u_2)$.

Let $p \in M$. In the following, we will denote by $T_{\circ p}M \leq T_pM$ the radical of the tangent space at $p$, by $T_p^\bullet M \leq T_p^* M$ the radical annihilator and by $T_{\bullet p}M$ the coannihilator.

We have seen that one important problem which appears when the metric becomes degenerate is that it does not admit an inverse $g^{ab}$, and fundamental tensor operations like raising indices and contractions between covariant indices are no longer defined. But we can use the reciprocal metric $g_\bullet$ to define metric contraction between covariant indices, for tensors that live in tensor products between $T_pM$ and the subspace $T_p^\bullet M$. This turned out to be enough for some important singularities in General Relativity.

*3.3. Covariant Derivative.* Because at points where the metric is degenerate there is no inverse metric, the Levi-Civita connection is not defined. Then, how can we derivate? We will see that in some cases, which turn out to be enough for our purposes, we still can derivate.

*3.3.1. The Koszul Object.* Let $X, Y, Z$ be vector fields on $M$. We define the *Koszul object* as

$$\mathscr{K}(X,Y,Z) := \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X[Y,Z]\rangle + \langle Y[Z,X]\rangle + \langle Z[X,Y]\rangle\}. \tag{5}$$

Its components in local coordinates are just Christoffel's symbols of the first kind:

$$\mathscr{K}_{abc} = \mathscr{K}(\partial_a, \partial_b, \partial_c) = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) = \Gamma_{abc}. \tag{6}$$

If the metric is nondegenerate, one defines the Levi-Civita connection uniquely, by raising an index of the Koszul object:

$$\nabla_X Y = \mathscr{K}(X, Y, \_)^\sharp. \tag{7}$$

But if the metric is degenerate, one cannot raise the index, and we will have to avoid the usage of the Levi-Civita connection. Luckily, we can do what we do with the Levi-Civita connection and more, just by using the Koszul object instead.

*3.3.2. The Covariant Derivatives.* We define the *lower covariant derivative* of a vector field $Y$ in the direction of a vector field $X$ by

$$\left(\nabla_X^\flat Y\right)(Z) := \mathscr{K}(X,Y,Z). \tag{8}$$

This is not quite a true covariant derivative, because it does not map vector fields to vector fields but to 1-forms. However, we can use it to replace the covariant derivative of vector fields, and it is equivalent to it if the metric is nondegenerate.

If the Koszul object satisfies the condition that $\mathscr{K}(X,Y,W) = 0$ for any $W \in \Gamma(T_\circ M)$, then the singular semi-Riemannian manifold $(M, g)$ is named *radical stationary*. In this case, it makes sense to contract in the third slot of the Koszul object and define by this covariant derivatives of differential forms. The covariant derivative of differential forms is defined by

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - g_\bullet\left(\nabla_X^\flat Y, \omega\right), \tag{9}$$

if $\omega \in \mathscr{A}^\bullet(M) := \Gamma(T^\bullet M)$. More general,

$$\nabla_X(\omega_1 \otimes \cdots \otimes \omega_s) := \nabla_X(\omega_1) \otimes \cdots \otimes \omega_s + \cdots + \omega_1 \otimes \cdots \otimes \nabla_X(\omega_s). \tag{10}$$

The covariant derivative of a tensor $T \in \Gamma(\otimes_M^k T^\bullet M)$ is defined as

$$(\nabla_X T)(Y_1, \ldots, Y_k) = X(T(Y_1, \ldots, Y_k)) - \sum_{i=1}^k \mathscr{K}(X, Y_i, \__\bullet)\, T(Y_1, \ldots, \__\bullet, \ldots, Y_k). \tag{11}$$

*3.4. Riemann Curvature Tensor: Semi-Regular Manifolds.* Let $(M, g)$ be a radical stationary manifold. Then, the *Riemann curvature tensor* is defined as

$$R(X,Y,Z,T) = \left(\nabla_X \nabla_Y^\flat Z\right)(T) - \left(\nabla_Y \nabla_X^\flat Z\right)(T) - \left(\nabla_{[X,Y]}^\flat Z\right)(T). \tag{12}$$

The components of the Riemann curvature tensor in local coordinates are

$$R_{abcd} = \partial_a \mathcal{K}_{bcd} - \partial_b \mathcal{K}_{acd} + \left(\mathcal{K}_{ac\bullet}\mathcal{K}_{bd\bullet} - \mathcal{K}_{bc\bullet}\mathcal{K}_{ad\bullet}\right). \quad (13)$$

The Riemann curvature tensor has the same symmetry properties as in Riemannian geometry and is radical annihilator in each of its slots.

A singular semi-Riemannian manifold is called *semiregular* [39] if

$$\nabla_X \nabla_Y^{\flat} Z \in \mathcal{A}^{\bullet}(M). \quad (14)$$

An equivalent condition is

$$\mathcal{K}(X, Y, \bullet)\, \mathcal{K}(Z, T, \bullet) \in \mathcal{F}(M). \quad (15)$$

It is easy to see that the Riemann curvature of semiregular manifolds is smooth.

*3.5. Examples of Semiregular Semi-Riemannian Manifolds.* We present some examples of semi-Riemannian manifolds [39, 40].

*3.5.1. Isotropic Singularities.* Isotropic singularities have the form

$$g = \Omega^2 \tilde{g}, \quad (16)$$

where $\tilde{g}$ is a nondegenerate bilinear form on $M$.

Such singularities were studied in connection to some cosmological models [48–56].

*3.5.2. Degenerate Warped Products.* Warped products are products of two semi-Riemannian manifolds $(B, g_B)$ and $(F, g_F)$, so that the metric on the manifold $F$ is scaled by a scalar function $f$ defined on the manifold $B$ [57]. The warped product has the form

$$\mathrm{d}s^2 = \mathrm{d}s_B^2 + f^2(p)\, \mathrm{d}s_F^2. \quad (17)$$

Normally, the warping function $f$ is taken to be strictly positive at all points of $B$. However, it may happen to vanish at some points, and in this case the result is a singular semi-Riemannian manifold. The resulting manifold is semiregular [40]. Moreover, if the manifolds $B$ and $F$ are radical stationary and if $\mathrm{d}f \in \mathcal{A}^{\bullet}(M)$, their warped product is radical stationary. If $B$ and $F$ are semiregular, $\mathrm{d}f \in \mathcal{A}^{\bullet}(M)$, and $\nabla_X \mathrm{d}f \in \mathcal{A}^{\bullet}(M)$ for any vector field $X$, and then $B \times_f F$ is semiregular [40].

## 4. Einstein Equations at Singularities

We discuss now two equations which are equivalent to Einstein's when the metric is nondegenerate but remains smooth and finite also at some singularities. The first equation remains smooth at semiregular singularities, while the second at quasi-regular singularities.

*4.1. Einstein's Equation on Semi-regular Spacetimes*

*4.1.1. The Densitized Einstein Equation.* Consider the following densitized version of the Einstein equation:

$$G \det g + \Lambda g \det g = \kappa T \det g, \quad (18)$$

or, in coordinates or local frames:

$$G_{ab} \det g + \Lambda g_{ab} \det g = \kappa T_{ab} \det g. \quad (19)$$

If the metric is nondegenerate, this equation is equivalent to the Einstein equation, the only difference is the factor $\det g \neq 0$. But what happens if the metric becomes degenerate? In this case, it is not allowed to divide by $\det g$, because this is 0.

On four-dimensional semi-regular spacetimes Einstein tensor density $G \det g$ is smooth [39]. Hence, the proposed densitized Einstein equation (18) is smooth, and nonsingular. If the metric is regular, this equation is equivalent to the Einstein equation.

*4.1.2. FLRW Spacetimes.* To better understand black hole singularities, which will be discussed later, we start by taking a look at the Friedmann-Lemaître-Robertson-Walker (FLRW) singularities, which are benign. Black hole singularities are malign but can be made benign by removing the coordinate singularity (see Sections 5, 6, and 7).

FLRW spacetimes are examples of degenerate warped products, with the metric defined by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\, \mathrm{d}\Sigma^2, \quad (20)$$

where

$$\mathrm{d}\Sigma^2 = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \quad (21)$$

where $k = 1$ for $S^3$, $k = 0$ for $\mathbb{R}^3$, and $k = -1$ for $H^3$. It follows that they are semiregular.

Since the FLRW singularities are warped products, they are semiregular. Therefore, we can expect that the densitized Einstein equation holds. In fact, in [58] more is shown than that, as we will see now.

The FLRW stress-energy tensor is

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab}, \quad (22)$$

where $u^a$ is the time-like vector field $\partial_t$, normalized. The scalar $\rho$ represents the mass density and $p$ the pressure density. From the stress-energy tensor (22), in the case of a homogeneous and isotopic universe, follow the *Friedmann equation:*

$$\rho = \frac{3}{\kappa} \frac{\dot{a}^2 + k}{a^2}, \quad (23)$$

and the *acceleration equation:*

$$\rho + 3p = -\frac{6}{\kappa} \frac{\ddot{a}}{a}. \quad (24)$$

Equations (23) and (24) show that the scalars $\rho$ and $p$ are singular for $a = 0$. But $\rho$ and $p$ represent the mass and pressure densities the orthonormal frame obtained by normalizing the comoving frame $(\partial_t, \partial_x, \partial_y, \partial_z)$, where $(x, y, z)$ are coordinates on the space manifold $S$. The mass and pressure density can be identified with the scalars $\rho$ and $p$ only in an orthogonal frame. But at the singularity $a = 0$ there is no orthonormal frame, so we should not normalize the comoving frame. In general, nonnormalized case, the actual densities contain in fact the factor $\sqrt{-g}(= a^3\sqrt{g_\Sigma})$:

$$\begin{aligned} \tilde{\rho} &= \rho\sqrt{-g} = \rho a^3 \sqrt{g_\Sigma}, \\ \tilde{p} &= p\sqrt{-g} = p a^3 \sqrt{g_\Sigma}. \end{aligned} \tag{25}$$

The Friedmann and the acceleration equations become

$$\begin{aligned} \tilde{\rho} &= \frac{3}{\kappa} a \left(\dot{a}^2 + k\right) \sqrt{g_\Sigma}, \\ \tilde{\rho} + 3\tilde{p} &= -\frac{6}{\kappa} a^2 \ddot{a} \sqrt{g_\Sigma}. \end{aligned} \tag{26}$$

We see that $\tilde{\rho}$ and $\tilde{p}$ are smooth and so is the densitized stress-energy tensor:

$$T_{ab}\sqrt{-g} = \left(\tilde{\rho} + \tilde{p}\right) u_a u_b + \tilde{\rho} g_{ab}. \tag{27}$$

We obtain a densitized Einstein equation, from which (18) follows by multiplying with $\sqrt{-g}$.

Hence, the FLRW solution is described by smooth densities even at the big bang singularity. Moreover, the solution extends beyond the singularity.

### 4.2. Einstein's Equation on Quasi-Regular Spacetimes

*4.2.1. The Ricci Decomposition.* Let $(M, g)$ be an $n$-dimensional semi-Riemannian manifold. The Riemann curvature decomposes algebraically [59–61] as

$$R_{abcd} = S_{abcd} + E_{abcd} + C_{abcd}, \tag{28}$$

where

$$\begin{aligned} S_{abcd} &= \frac{1}{n(n-1)} R(g \circ g)_{abcd}, \\ E_{abcd} &= \frac{1}{n-2} (S \circ g)_{abcd}, \\ S_{ab} &:= R_{ab} - \frac{1}{n} R g_{ab}, \end{aligned} \tag{29}$$

where $\circ$ denotes the Kulkarni-Nomizu product:

$$(h \circ k)_{abcd} := h_{ac}k_{bd} - h_{ad}k_{bc} + h_{bd}k_{ac} - h_{bc}k_{ad}. \tag{30}$$

If the Riemann curvature tensor on a semiregular manifold $(M, g)$ admits such a decomposition so that all of its terms are smooth, $(M, g)$ is said to be *quasi-regular.*

*4.2.2. The Expanded Einstein Equation.* For dimension $n = 4$, in [41] we introduced the *expanded Einstein equation:*

$$(G \circ g)_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd} \tag{31}$$

or, equivalently,

$$2E_{abcd} - 6S_{abcd} + \Lambda(g \circ g)_{abcd} = \kappa(T \circ g)_{abcd}. \tag{32}$$

It is equivalent to Einstein's equation if the metric is nondegenerate but in addition extends smoothly at quasi-regular singularities.

*4.2.3. Examples of Quasi-Regular Singularities.* As shown in [41], the following are examples of quasi-regular singularities:

(i) isotropic singularities,

(ii) degenerate warped products $B \times_f F$ with $\dim B = 1$ and $\dim F = 3$,

(iii) FLRW singularities, as a particular case of degenerate warped products [62],

(iv) Schwarzschild singularities (after removing the coordinates singularity, see Section 5). The question whether the Reissner-Nordström and Kerr-Newman singularities are quasi-regular, or at least semi-regular, is still open.

*4.2.4. The Weyl Curvature Hypothesis and Quasi-Regular Singularities.* To explain the low entropy at the big bang and the high homogeneity of the universe, Penrose emitted the Weyl curvature hypothesis, stating that the Weyl curvature tensor vanishes at the big bang singularity [18].

From (28), the *Weyl curvature tensor* is

$$C_{abcd} = R_{abcd} - S_{abcd} - E_{abcd}. \tag{33}$$

In [63] it was shown that when approaching a quasi-regular singularity, $C_{abcd} \to 0$ smoothly. Because of this, any quasi-regular big bang satisfies the Weyl curvature hypothesis. In [63] it has also been shown that a very large class of big bang singularities, which are not homogeneous or isotropic, are quasi-regular.

*4.3. Taming a Malign Singularity.* We have seen that when the singularity is benign; that is, the singularity is due to the degeneracy of the metric tensor, which is smooth; there are important cases when we can obtain a complete description of the fields and their evolution, in terms of finite quantities.

But what can we do if the singularities are malign? This case is important, since all black hole singularities are malign. In [42–44] we show that although the black hole singularities appear to be malign, we can make them benign, by a proper choice of coordinates. This is somewhat analog to the method used in [45, 46] to show that the event horizon singularity is not a true singularity, being due to coordinates. In the following sections, we will review these results.

## 5. Schwarzschild Singularity Is Semi-Regular

The Schwarzschild metric is given in Schwarzschild coordinates by

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \quad (34)$$

where

$$\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \quad (35)$$

Let us change the coordinates to

$$r = \tau^2, \qquad t = \xi\tau^4. \quad (36)$$

The four-metric becomes

$$\mathrm{d}s^2 = -\frac{4\tau^4}{2m-\tau^2}\mathrm{d}\tau^2 + \left(2m-\tau^2\right)\tau^4(4\xi\mathrm{d}\tau + \tau\mathrm{d}\xi)^2 + \tau^4\mathrm{d}\sigma^2, \quad (37)$$

which is analytic and semiregular at $r = 0$ [42].

The problems were fixed by a coordinate change. Does not this mean that the singularity depends on the coordinates? Well, this deserves an explanation. Changing the coordinates does not make a singularity appear or disappear, if the coordinate transformation is a local diffeomorphism. But a regular tensor can become singular or a singular tensor can become regular, if the coordinate transformation itself is singular. This situation is very similar to that of the event horizon singularity $r = 2m$ of the Schwarzschild metric, in Schwarzschild coordinates (34). This singularity vanishes when we go to the Eddington-Finkelstein coordinates. This proves that the Eddington-Finkelstein coordinates are from the correct atlas, while the original Schwarzschild coordinates were in fact singular at $r = 2m$. In our case, the coordinate transformation (36) allows us to move to an atlas in which the metric is analytic and semiregular, showing that the Schwarzschild coordinates were in fact singular at $r = 0$.

## 6. Charged and Nonrotating Black Holes

Charged nonrotating black holes are described by the Reissner-Nordström metric:

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\mathrm{d}t^2 \\ &+\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2.\end{aligned} \quad (38)$$

To make the singularity benign, we choose the new coordinates $\rho$ and $\tau$ [43]; so that

$$t = \tau\rho^T, \qquad r = \rho^S. \quad (39)$$

In the new coordinates, the metric has the following form:

$$\mathrm{d}s^2 = -\Delta\rho^{2T-2S-2}(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho)^2 + \frac{S^2}{\Delta}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2, \quad (40)$$

where

$$\Delta := \rho^{2S} - 2m\rho^S + q^2. \quad (41)$$

To remove the infinity of the metric at $r = 0$ and ensure analiticity, we have to choose

$$S \geq 1, \qquad T \geq S + 1. \quad (42)$$

In the Reissner-Nordström coordinates $(t, r, \phi, \theta)$, the electromagnetic potential is singular at $r = 0$,

$$A = -\frac{q}{r}\mathrm{d}t. \quad (43)$$

But in the new coordinates $(\tau, \rho, \phi, \theta)$, the electromagnetic potential is

$$A = -q\rho^{T-S-1}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right), \quad (44)$$

and the electromagnetic field is

$$F = q\left(2T - S\right)\rho^{T-S-1}\mathrm{d}\tau\wedge\mathrm{d}\rho, \quad (45)$$

and they are analytic everywhere, including at the singularity $\rho = 0$ [43].

The proposed coordinates define a space + time foliation only if $T \geq 3S$ [43].

## 7. Rotating Black Holes

Electrically neutral rotating black holes are represented by the Kerr solution. If they are also charged, they are described by the very similar Kerr-Newman solution.

Consider the space $\mathbb{R}\times\mathbb{R}^3$, where $\mathbb{R}$ represents the time coordinate and $\mathbb{R}^3$ the space, parameterized by the spherical coordinates $(r, \phi, \theta)$. The rotation is characterized by the parameter $a \geq 0$, $m \geq 0$ is the mass, and $q \in \mathbb{R}$ the charge. The following notations are useful:

$$\begin{aligned}\Sigma\left(r,\theta\right) &:= r^2 + a^2\cos^2\theta, \\ \Delta\left(r\right) &:= r^2 - 2mr + a^2 + q^2.\end{aligned} \quad (46)$$

The nonvanishing components of the Kerr-Newman metric are [64]

$$\begin{aligned}g_{tt} &= -\frac{\Delta\left(r\right) - a^2\sin^2\theta}{\Sigma\left(r,\theta\right)}, \\ g_{rr} &= \frac{\Sigma\left(r,\theta\right)}{\Delta\left(r\right)}, \\ g_{\theta\theta} &= \Sigma\left(r,\theta\right), \\ g_{\phi\phi} &= \frac{\left(r^2+a^2\right)^2 - \Delta\left(r\right)a^2\sin^2\theta}{\Sigma\left(r,\theta\right)}\sin^2\theta, \\ g_{t\phi} = g_{\phi t} &= -\frac{2a\sin^2\theta\left(r^2+a^2-\Delta\left(r\right)\right)}{\Sigma\left(r,\theta\right)}.\end{aligned} \quad (47)$$

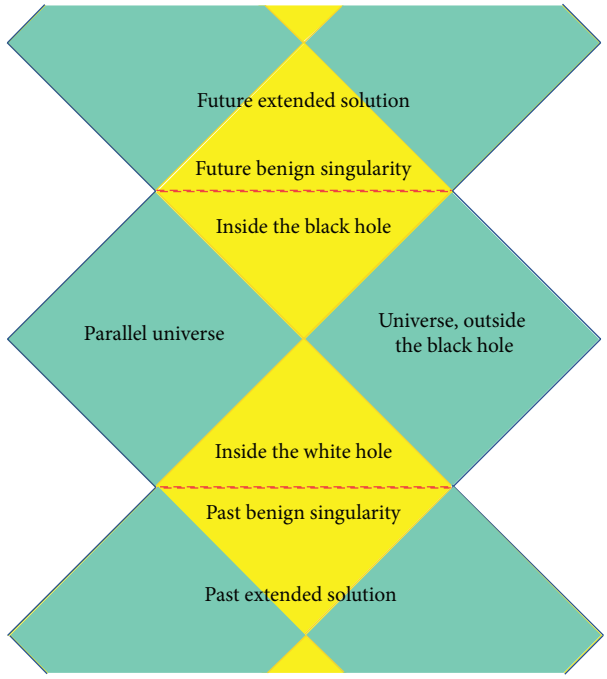


Figure 1: Schwarzschild solution, analytically extended beyond the $r = 0$ singularity.

In [44] it was shown that in the coordinates $\tau$, $\rho$, and $\mu$, are defined by

$$t = \tau\rho^{\mathbb{T}}, \qquad r = \rho^{\mathbb{S}},$$
$$\phi = \mu\rho^{\mathbb{M}}, \qquad \theta = \theta, \tag{48}$$

where $\mathbb{S}, \mathbb{T}, \mathbb{M} \in \mathbb{N}$ are positive integers so that

$$\mathbb{S} \geq 1,$$
$$\mathbb{T} \geq \mathbb{S} + 1, \tag{49}$$
$$\mathbb{M} \geq \mathbb{S} + 1,$$

and the metric is analytic.

Not only the metric becomes analytic in the proposed coordinates, but also the electromagnetic potential and electromagnetic field. The electromagnetic potential of the Kerr-Newman solution is, in the standard coordinates, the 1-form:

$$A = -\frac{qr}{\Sigma(r,\theta)}\left(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi\right). \tag{50}$$

In the proposed coordinates

$$A = -\frac{q\rho^{\mathbb{S}}}{\Sigma(r,\theta)}\left(\rho^{\mathbb{T}}\mathrm{d}\tau + \mathbb{T}\tau\rho^{\mathbb{T}-1}\mathrm{d}\rho - a\sin^2\theta\rho^{\mathbb{M}}\mathrm{d}\mu\right). \tag{51}$$

which is smooth [44]. The electromagnetic field $F = \mathrm{d}A$ is smooth too.

## 8. Global Hyperbolicity and Information Loss

*8.1. Foliations with Cauchy Hypersurfaces.* While Einstein's equation describes the relation between geometry and matter in a block-world view of the universe, there are equivalent formulations which express this relation from the perspective of the time evolution. Einstein's equation can be expressed in terms of a Cauchy problem [65–70].

The standard black hole solutions pose two main problems to the Cauchy problem. First, the solutions have malign singularities. Second, they have in general Cauchy horizons. Luckily, there is more than one way to skin a black hole.

The evolution equations make sense at least locally, if the singularities are benign. The black hole singularities appear to be malign in the coordinates used so far, but by removing the coordinate's contribution to the singularity, they become benign. Even so, to formulate initial value problems globally, spacetime has to admit space + time foliations. The space-like hypersurfaces have to be Cauchy surfaces; in other words, the global hyperbolicity condition has to be true. The topology of the space-like hypersurfaces must remain independent on the time $t$, although the metric is allowed to become degenerate. This seems to be prevented in the case of Reissner-Nordström and Kerr-Newman black holes, by the existence of Cauchy horizons. As shown in [71], the stationary black hole singularities admit such foliations and are therefore compatible with the condition of global hyperbolicity.

*8.2. Schwarzschild Black Holes.* In the proposed coordinates for the Schwarzschild black hole, the metric extends analytically beyond the $r = 0$ singularity (Figure 1).

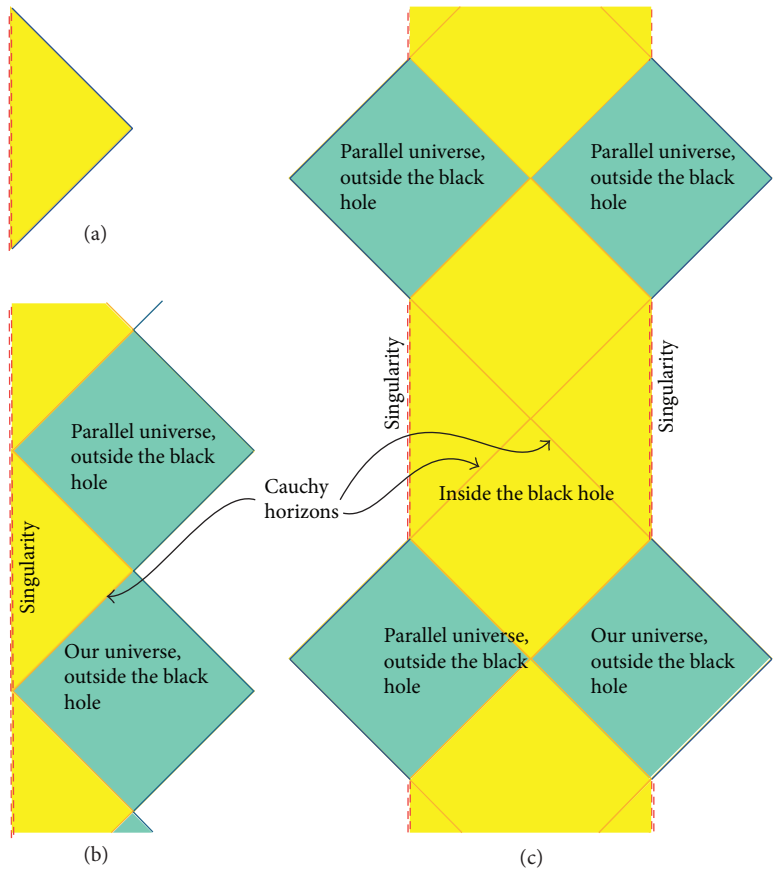


Figure 2: Reissner-Nordström black holes. (a) Naked solutions ($q^2 > m^2$). (b) Extremal solution ($q^2 = m^2$). (c) Solutions with $q^2 < m^2$.

This solution can be foliated in space + time and therefore is globally hyperbolic.

*8.3. Space-Like Foliation of the Reissner-Nordström Solution.* Figure 2 shows the standard Penrose diagrams for the Reissner-Nordström spacetimes [72].

The Penrose diagram 3 shows how our extensions beyond the singularities allow the Reissner-Nordström solutions to be foliated in Cauchy hypersurfaces. In Figures 3(b) and 3(c), in addition to extending the solution beyond the singularity, we cut out the spacetime along the Cauchy horizons. This is justified if the black holes form by collapse at a finite time and then evaporate after a finite lifetime [43, 71].

For the Kerr-Newman black holes, the foliations are similar to those for the Reissner-Nordström solutions [71], especially because the extension proposed in [44] can be chosen so that the closed time-like curves disappear.

*8.4. Black Hole Information Paradox.* Bekenstein and Hawking discovered that black holes obey laws similar to those of thermodynamics and proposed that these laws are in fact thermodynamics (see [73–75], also [76, 77], and references therein). Hawking realized that black holes evaporate, and the radiation is thermal. This led him to the idea that after evaporation, the information is lost [78–80]. Many solutions were proposed, such as [81–94]. It was proposed that Quantum Gravity would naturally cure this problem, but it has been suggested that in fact it would make the problem exist even in the absence of black holes [95].

Since the extended Schwarzschild solution can be foliated in space + time (Sections 5 and 8.2), it can be used to represent evaporating electrically neutral nonrotating black holes. The solution can be analytically extended beyond $r = 0$, and hence the affirmation that the information is lost at the singularity is no longer supported. In Figure 4, it can be seen that our solution extends through the singularity and allows the existence of globally hyperbolic spacetimes containing evaporating black holes.

## 9. Possible Experimental Consequences and Quantum Gravity

*9.1. Can We Do Experiments with Singularities?* We reviewed the foundations of Singular General Relativity (SGR) and its applications to black hole singularities. SGR is a natural extension of GR, but, nevertheless, it would be great to be able to submit it to experimental tests. We have seen that the solutions are the same as those predicted by Einstein's equation, as long as the metric is nondegenerate. The only differences appear where the metric is degenerate, at singularities. But how can we go to the singularities, or how can we generate singularities, and test the results at the singularities?

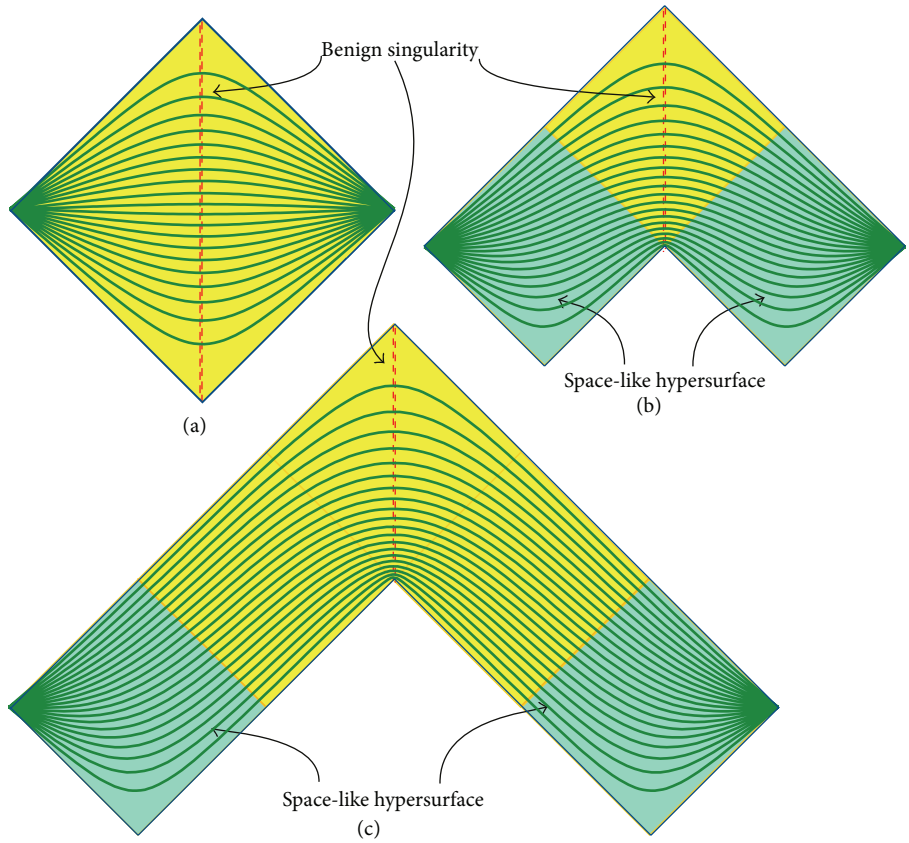


Figure 3: Reissner-Nordström black hole solutions, extended beyond the singularities and restricted to globally hyperbolic regions. (a) Naked solutions ($q^2 > m^2$). (b) Extremal solution ($q^2 = m^2$). (c) Solutions with $q^2 < m^2$.

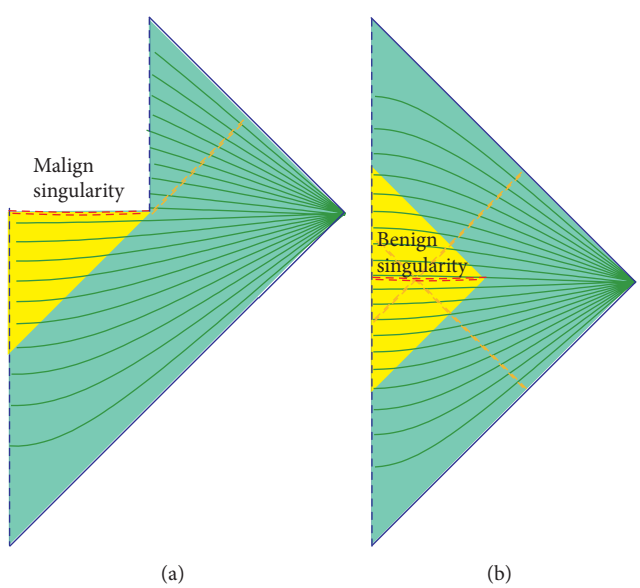


Figure 4: (a) Standard evaporating black hole, whose singularity destroys the information. (b) Evaporating black hole extended through the singularity preserves information and admits a space + time foliation.

How could we design an experimental apparatus which is not destroyed by the singularity? It seems that a direct experiment to test the predictions of SGR is not possible.

What about indirect tests? For example, if information is preserved, this would be evidence in favor of SGR. But how can we test this? Can we monitor a black hole, from the time when it is formed to the time when it evaporates completely, and check that the information is preserved during this entire process? The current knowledge predicts that this information will be anyway extremely scrambled. Even if we would be able to do this someday, the conservation of information is predicted by a long list of other approaches to Hawking's information loss paradox (see Section 8.4).

In General Relativity, classical elementary particles can be considered small black holes. If they are pointlike and have definite trajectories, then they are singularities, like the Schwarzschild, Reissner-Nordström, and Kerr-Newman singularities. To go from classical to quantum, one applies path integrals over the classical trajectories. In this way, possible effects of the singularities may also be present at the points where the metric is nonsingular.

In [96] we suggested that the geometric and topological properties we identified at singularities have implications to Quantum Gravity (QG), as we shall see in the following. This suggests that it might be possible to test our approach by QG effects. One feature that seems to be required by most, if not all approaches to QG, is dimensional reduction. Singular General Relativity shows that singularities are accompanied in a natural way by dimensional reduction.

*9.2. Dimensional Reduction in QFT And QG.* Various results obtained in Quantum Field Theory (QFT) and in QG suggest that at small scales a dimensional reduction should take place. The definition and the cause of this reduction differ from one approach to another. Here is just a small part of the literature using one form of dimensional reduction or another to obtain regularization in QFT and QG:

(i) fractal universe [97, 98], based on a Lebesgue-Stieltjes measure or a fractional measure [99], fractional calculus, and fractional action principles [100–109];

(ii) topological dimensional reduction [110–114];

(iii) vanishing dimensions at LHC [115];

(iv) dimensional reduction in QG [116–118];

(v) asymptotic safety [119];

(vi) Hořava-Lifschitz gravity [120];

(vii) other approaches to Quantum Gravity based on dimensional reduction including [121–126].

Some of these types of dimensional reduction are very similar to those predicted by SGR to occur at benign singularities.

*9.3. Is Dimensional Reduction due to the Benign Singularities?* Quantum Gravity is perturbatively nonrenormalizable, but it can be made renormalizable by assuming one kind or another of dimensional reduction. The above mentioned approaches did this, by modifying General Relativity. In this section we point that several types of dimensional reduction, which were postulated by various authors, occur naturally at our semiregular and quasi-regular singularities [96].

*9.3.1. Geometric Dimensional Reduction.* First, at each point where the metric becomes degenerate, a geometric or *metric reduction* takes place, because the rank of the metric is reduced:

$$\dim T_{p\bullet}M = \dim T_p^{\bullet}M = \operatorname{rank} g_p. \tag{52}$$

*9.3.2. Topological Dimensional Reduction.* From the Kupeli theorem [47] follows that for constant signature, the manifold is locally a *product* $M = P \times_0 N$ between a manifold of lower dimension $P$ and another manifold $N$ with metric 0. In other words, from the viewpoint of geometry, a region where the metric is degenerate and has constant signature can be identified with a lower dimensional space. This suggests a connection with the *topological dimensional reduction* explored by Shirkov and Fiziev [110–114].

*9.3.3. Vanishing of Gravitons.* If the singularity is quasi regular, the Weyl tensor $C_{abcd} \to 0$ as approaching a quasi-regular singularity. This implies that the *local degrees of freedom,* that is, the gravitational waves for GR and the gravitons for QG, vanish, allowing by this the needed renormalizability [116].

*9.3.4. Anisotropy between Space and Time.* In [43] we obtained new coordinates, which make the Reissner-Nordström metric analytic at the singularity. In these coordinates, the metric is given by (40). A *charged particle* with spin 0 can be viewed, at least classically, as a Reissner-Nordström black hole. The above metric reduces its dimension to dim = 2.

To admit space + time foliation in these coordinates, we should take $T \geq 3S$. An open research problem is whether this anisotropy is connected to the similar anisotropy from *Hořava-Lifschitz* gravity, introduced in [120].

*9.3.5. Measure Dimensional Reduction.* In the *fractal universe approach* [97, 98, 127], one expresses the measure in the integral

$$S = \int_{\mathscr{M}} \mathrm{d}\varrho(x)\, \mathscr{L}, \tag{53}$$

in terms of some functions $f_{(\mu)}(x)$, some of them vanishing at low scales:

$$\mathrm{d}\varrho(x) = \prod_{\mu=0}^{D-1} f_{(\mu)}(x)\, \mathrm{d}x^{\mu}. \tag{54}$$

In *Singular General Relativity*,

$$\mathrm{d}\varrho(x) = \sqrt{-\det g}\, \mathrm{d}x^{D}. \tag{55}$$

If the metric is diagonal in the coordinates $(x^{\mu})$, then we can take

$$f_{(\mu)}(x) = \sqrt{\left|g_{\mu\mu}(x)\right|}. \tag{56}$$

This suggests that the results obtained by Calcagni by considering the universe to be fractal follow naturally from the benign metrics.

*9.4. Dimensional Reduction and Quantum Gravity.* The Singular General Relativity approach leads, as a side effect, to various types of dimensional reduction, which are similar to those proposed in the literature to make Quantum Gravity perturbatively renormalizable. By investigating the nonrenormalizability problems appearing when quantizing gravity, many researchers were led to the conclusion that the problem would vanish if one kind of dimensional reduction or another is postulated (sometimes ad hoc). By contrary, our approach led to this as a natural consequence of understanding the singularities.

Of course, in SGR the dimensional reduction appears at the singularity, while QG is expected to be perturbatively renormalizable everywhere. But if classical particles are singularities, quantum particles behave like sums over histories of classical particles. Thus, at any point there will be virtual singularities to contribute to the Feynman integrals. This means that the effects will be present everywhere. They are expected as a reduction of the determinant of the metric, and of the Weyl curvature tensor, which allows the desired regularization. Moreover, as the energy increases, the order of the Feynman diagrams in the same region increases, and we expect that the dimensional reduction effects induced by singularities become more significant too. It is an open question at this time whether this dimensional reduction is enough to regularize gravity, but this research is just at the beginning.

## 10. Conclusions

We reviewed some of our results of Singular General Relativity [38], concerning the black hole singularities. Some singularities allow the canonical and invariant construction of geometric objects which remain smooth and nonsingular. By using these objects, one can write equations which are equivalent to Einstein's equations outside singularities but in addition extend smoothly at singularities. The FLRW big bang singularities turn out to be of this type. The black hole singularities can be made so by removing the coordinate singularity for the charged black hole singularities, the electromagnetic potential and field become smooth. The singularities of the black hole having a finite life span are compatible with global hyperbolicity and conservation of information. Such singularities are accompanied by dimensional reduction, a feature which is desired by many approaches to Quantum Gravity. While in these approaches dimensional reduction is obtained by modifying General Relativity, these singularities lead naturally to it, within the framework of GR.

There is a rich literature concerning gravity, black holes, and singularities in lower or higher dimensions (see e.g., [76, 128–130] and references therein). While the geometric apparatus of Singular Semi-Riemannian Geometry reviewed in Section 3 works for other dimensions too, in this review we focused only on four-dimensional spacetimes, and some of the results do not work in more dimensions.

## Conflict of Interests

The author declares that there is no conflict of interests section regarding the publication of this paper.

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*Research Article*

# Intermediate Mass Black Holes: Their Motion and Associated Energetics

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There is a lot of current astrophysical evidence and interest in intermediate mass black holes (IMBH), ranging from a few hundred to several thousand solar masses. The active galaxy M82 and the globular cluster G1 in M31, for example, are known to host such objects. Here, we discuss several aspects of IMBH such as their expected luminosity, spectral nature of radiation, and associated jets. We also discuss possible scenarios for their formation including the effects of dynamical friction, and gravitational radiation. We also consider their formation in the early universe and also discuss the possibility of supermassive black holes forming from mergers of several IMBH and compare the relevant time scales involved with other scenarios.

## 1. Introduction

Intermediate mass black holes (IMBH) are those black holes having mass between that of stellar black holes and supermassive black holes, that is, in the range of 500 to $10^4$ $M_\Theta$. Recent observations indicate an intermediate mass black hole in the elliptical galaxy NGC 4472, with an X-ray luminosity of $4 \times 10^{32}$ J/s [1].

The Schwarzschild radius corresponding to these masses is of the order of

$$R_S = \frac{2GM}{c^2} \approx 1.5 \times 10^3 - 3 \times 10^4 \text{ km}. \tag{1}$$

An accretion disc is formed by material falling into a gravitational source. Conservation of angular momentum requires that, as a large cloud of material collapses inward, any small rotation it may have will increase. Centrifugal force causes the rotating cloud to collapse into a disc, and tidal effects will tend to align this disc's rotation with the rotation of the gravitational source in the middle.

The Bondi accretion rate for a black hole (of mass M moving with a velocity $v$ in a medium of density $\rho = nm_P$), is given by [2]:

$$\dot{m} = 4\pi R^2 n m_P V, \tag{2}$$

where $4\pi R^2$ is the cross section and it is given by $4\pi R^2 = 4\pi (GM/V^2)^2$.

The velocity is given by

$$V^2 = c_S^2 + v^2, \tag{3}$$

where $c_S$ is the velocity of sound in the medium of density $\rho$ and it is given by $c_S = \sqrt{\gamma T R}$, and $\gamma = 5/3$ is the ratio of specific heats and $R$ is the universal gas constant.

The equation describing the velocity of isothermal winds has many solutions depending on the initial conditions at the base of the wind. There is only one critical solution for which the velocity increases from subsonic at the base to supersonic far out. This velocity passes through the critical point and implies one particular value of the initial velocity at the lower boundary of the isothermal region.

If the density at this point is fixed, the mass loss rate is fixed by (2) as $\dot{m} = 4\pi R_0^2 n m_P V$. The total energy increases from negative at the base of the wind to positive in the supersonic region, so the flow requires the input of energy into the wind. This energy input is needed to keep the flow isothermal and it is this energy that is transferred into kinetic energy of the wind through the gas pressure.

As we will see in the next section, for an intermediate mass black hole, the temperature is typically of the order of $10^4$ K. In this case, the velocity of sound $c_S = \sqrt{\gamma T R}$ is of the order of $10^4$ m/s.

Also we have $R = 2GM/V^2$; hence, the accretion rate is given by

$$\dot{m} = \frac{4\pi n m_P (GM)^2}{V^3}. \tag{4}$$

For a typical number density of $n \approx 10^{20}/m^3$, and the IMBH mass of $M = 10^4\,\mathrm{M}_\Theta$, the accretion rate is: $\dot{m}$=$10^{17}$ kg/s.

The above expression implies that the higher the density of the medium through which the black hole is travelling, the more the accretion rate. Also, the accretion rate is inversely related to the velocity with which the black hole is travelling in the medium. Although there is not much conclusive evidence for the IMBH, there is indirect evidence.

For a given mass of luminous object, there is a maximum value for its luminosity (Eddington luminosity), as the radiation pressure would tend to push the matter apart exceeding the gravitational force supporting it. This is the luminosity a body would have to have so that the force generated by radiation pressure exceeds the gravitational force. Thus, observed luminosity can set a lower limit on the mass of an accreting black hole.

Chandra and XMM-Newton observations in the nearby spiral galaxy have detected X-ray sources of luminosities of the order of $10^{33}$ W, with the source away from the centre [3]. In Section 4, we will discuss the dynamics of the IMBH.

The X-ray luminosity corresponding to the accretion rate of $\dot{m} = 10^{17}$ kg/s is given by $L_X = \varepsilon \dot{m} c^2 \approx 5 \times 10^{31}$ W. This matches with recent observed results [4].

The Eddington luminosity (for an IMBH of mass $10^4\,\mathrm{M}_\Theta$) is given by:

$$L_{\mathrm{Edd}} = \frac{4\pi c G M m_P}{\sigma_T} \approx 10^{35}\ \mathrm{W}, \tag{5}$$

where, the Thomson cross section is given by:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \approx 10^{-28}\ \mathrm{m}^2. \tag{6}$$

A possible candidate for the IMBH is the globular cluster G1 in M31, which is the most massive stellar cluster in the local group, with a mass of the order of $10^7\,\mathrm{M}_\Theta$.

Due to a large number of field stars contained within the accretion radius the Bondi accretion by an IMBH is complicated [4]. The accretion radius is given by

$$R_{\mathrm{acc}} = \frac{2GM_{\mathrm{BH}}}{v^2} \approx 0.4\,\mathrm{pc}, \quad \text{for } M_{\mathrm{BH}} = 10^4\,\mathrm{M}_\Theta; \quad v \approx 15\,\mathrm{km/s}. \tag{7}$$

In the case of globular cluster G1 in M31, there are more than $10^5$ stars within 0.4 pc of the centre [5]. The dynamical effects of these stars should also be considered.

Many more such dense stellar clusters are known to exist. For instance, a recently recognised super star cluster in our own galaxy is the Westerlund I. Most of its estimated half a million stars are crowded in a region hardly 3 parsecs wide. Several dozen of these stars are among the most massive and luminous superhot Wolf-Rayet stars, LBV, red supergiants, yellow hypergiants, and so forth. Also, near the Milky Way centre, we have the well-known Arches and Quintuplet clusters.

In the centre of M15 there are approximately five million stars per cubic parsec, having hundred million times more stellar density than the solar neighbourhood. Also galaxies like M31, M33, and the Milky Way itself have comparable central stellar densities. However M32 (satellite galaxy of Andromeda) has thirty million stars in a cubic parsec at its core. Even HST cannot resolve individual stars in this region.

More and more massive binary stars are being found, like, for instance, WR20a (>80 $\mathrm{M}_\Theta$ each, with 3.7-day period). As discussed above, IMBH is likely to form in dense clusters containing young massive stars. IMBH is believed to power ultraluminous X-ray sources (ULXs). IMBH can capture companion stars, which provide accreting material to sustain ULX. These stars may be blue giants, white dwarfs, and so forth.

Tidal forces can rip giants and the material can fall into the IMBH. To rip apart these stars the mass of the black hole is around the mass of the IMBH [6]. The energy released during this process will be the binding energy of the stars which is given by

$$E_{\mathrm{BE}} = \frac{3}{5}\frac{GM^2}{R} \approx 10^{42}\ \mathrm{J}. \tag{8}$$

Several ULXs are identified in the Antenna, a pair of colliding galaxies, producing several stars in dense cluster. Stellar collisions lead to formation of so-called megastars (~$10^3\,\mathrm{M}_\Theta$), which collapses on short time scales to form IMBH. Clusters in M82 have such stellar densities. Binary and multiple IMBH can also form.

## 2. Black Body Considerations of IMBH

We have already obtained the Eddington luminosity as

$$L_{\mathrm{Edd}} = \frac{4\pi c G M m_P}{\sigma_T}. \tag{9}$$

By considering the black hole emission to obey black body radiation, we can use Stefan's law to relate the luminosity to the temperature, $T$, as

$$\frac{4\pi c G m_P M}{\sigma_T} = \sigma A T_{\max}^4, \tag{10}$$

where $\sigma$ is Stefan's constant and area of the black hole and it is given by

$$A = f\left(4\pi R_S^2\right), \tag{11}$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius and $f$ indicates the size of the ambient gas around the black hole.

Making use of this in (10) we get

$$T_{\max}^4 = \left(\frac{c^5}{G}\frac{m_P}{f\sigma_T\sigma}\right)M \Longrightarrow T_{\max} \propto M^{-1/4}. \tag{12}$$

This implies that the temperature of the IMBH (accretion disk) of mass of the order of 500 to $10^4\,\mathrm{M}_\Theta$ and $f$ ranging from 10 to 100 is

$$T_{\max} \approx 3\times 10^6\,\mathrm{K}. \tag{13}$$

From Wien's law the corresponding wavelength is given by

The corresponding wavelength is

$$\lambda = \frac{\left(2.898\times 10^{-3}\,\mathrm{Km}\right)}{T} \approx 10^{-9}\,\mathrm{m}. \tag{14}$$

This lies in the soft X-ray region of the spectrum [7].

During the earlier epochs of the universe, the density was much larger; hence, the ambient density is also larger by the same factor of $(1+z)^3$.

The present density of the universe is of the order of one proton per cubic metre. As we will see in Section 5, the maximum redshift up to which we can detect supermassive black hole is of the order of $z = 12$.

The number density at this epoch is given by

$$n = 1\,\mathrm{p/m^3}(1+z)^3 \approx 10^3\,\mathrm{p/m^3}. \tag{15}$$

The temperature corresponding to this redshift and this number density is of the order of

$$T \approx 2\times 10^7\,\mathrm{K}. \tag{16}$$

And the corresponding wavelength is of the order of $10^{-10}$ m.

This falls in the X-ray region of the spectrum.

This wavelength is further red-shifted by a factor of $(1+z)$. Hence, the observed wavelength will be of the order of $7\times 10^{-8}$ m.

This lies in the UV region of the spectrum.

## 3. Jets from the Black Hole

One of the manifestations of this accretion energy release is the production of so-called jets: the collimated beams of matter that are expelled from the innermost regions of accretion disks. These jets shine particularly brightly at radio frequencies. In rotating black holes, the matter forms a disk due to the mechanical forces present. In a Schwarzschild black hole, the matter would be drawn in equally from all directions and thus would form an omnidirectional accretion cloud rather than disk. Jets form in Kerr black holes (rotating black holes) that have an accretion disk [8].

Black holes convert a specific fraction of accretion energy into radiation, which is traced by the X-ray luminosity and jet kinetic energy, which is traced by the radio-emission luminosity [9]. The matter is funnelled into a disk-shaped torus by the black hole's spin and magnetic fields, but, in the very narrow regions over the black hole's poles, matter can be energized to extremely high temperatures and speeds, escaping the black hole in the form of high-speed jets. Inferred jet velocities close to the speed of light suggest that jets are formed within a few gravitational radii of the event horizon of the black hole.

The horizon for the Kerr black hole is given by

$$r = m \pm \sqrt{m^2 - a^2}. \tag{17}$$

Here, $m = GM/c^2$ is the geometric mass and $a = J_{\max}/Mc^2$ is the geometric angular momentum.

From the condition that $r$ should be real, the limiting case is given by $m = a$.

From this, the maximum angular momentum is given by

$$J_{\max} = \frac{M^2 G}{c}. \tag{18}$$

From the classical expression for the angular momentum associated with a jet of length $l$, assuming the particles to be travelling at near speed of light, the expression becomes

$$J = mcl. \tag{19}$$

Considering a conical jet with base radius $r$ and density $\rho$, the mass of the jet is given by

$$m = \frac{1}{3}\pi r^2 l\rho. \tag{20}$$

Then the angular momentum becomes

$$J = \frac{1}{3}\pi l^2 r^2 c\rho. \tag{21}$$

From the geometry of the jet, we can relate the length of the jet to the radius $r$ as $r = l\tan 5^\circ$. Here, we have assumed the small opening angle of the jet to be $5^\circ$.

Length of the jet is

$$l = \left(\frac{3GM^2}{\pi\rho c^2(\tan 5)^2}\right)^{1/4}. \tag{22}$$

For typical densities of the ambient gas and for the IMBH of mass $10^4\,\mathrm{M}_\Theta$, the length is of the order of 20 pc.

## 4. Evolution of a Star Cluster and Possible Scenario for IMBH Formation

One of the possible models for the formation of an IMBH is the collapse of a cluster of stars [10]. The collapsed core can accrete matter ejected during the formation. If a collection of a thousand 10 solar mass stars in a volume of a parsec cube collapses, ejecting 30% of its mass, then the total mass of the ambient gas is given by

$$nm_P = 0.3 \times 10^3 \times 10\,\mathrm{M}_\Theta \approx 6 \times 10^{33}\,\mathrm{kg}. \tag{23}$$

The accretion rate is given by (4) as

$$\dot{m} = \frac{16\pi n m_P G^2 M^2}{c^3}. \tag{24}$$

If 30% of the mass of each star is ejected, then this implies an accretion rate of $\dot{m} \approx 4 \times 10^7$ kg/s. And the corresponding (Eddington) luminosity is

$$L_{\mathrm{Edd}} = \frac{GM\dot{m}}{\sigma_T R_S^2 n} \approx 10^{24}\,\mathrm{W}. \tag{25}$$

Dynamical friction is related to loss of momentum and kinetic energy of moving bodies through a gravitational interaction with surrounding matter in space. The effect must exist if the principle of conservation of energy and momentum is valid since any gravitational interaction between two or more bodies corresponds to elastic collisions between those bodies. For example, when a heavy body moves through a cloud of lighter bodies, the gravitational interaction between this body and the lighter bodies causes the lighter bodies to accelerate and gain momentum and kinetic energy.

Since energy and momentum are conserved, this body has to lose a part of its momentum and energy equal to the sums of all momenta and energies gained by the light bodies. Because of the loss of momentum and kinetic energy of the body under consideration, the effect is called dynamical friction. Of course the mechanism works the same way for all masses of interacting bodies and for any relative velocities between them.

However, while in the above case the most probable outcome is the loss of momentum and energy by the body under consideration, in the general case it might be either loss or gain. In a case when the body under consideration is gaining momentum and energy, the same physical mechanism is called sling effect.

The full Chandrasekhar dynamical friction formula for the change in velocity of the object involves integrating over the phase space density of the field of matter [11]. By assuming a constant density, though, a simplified equation for the force from dynamical friction, $f_d$, is given as

$$f_d \approx C\frac{(GM)^2\rho}{v_M^2}, \tag{26}$$

where $G$ is the gravitational constant, $M$ is the mass of the moving object, $\rho$ is the density, and $v_M$ is the velocity of the object in the frame in which the surrounding matter was initially at rest.

In this equation, $C$ is not a constant but depends on how $v_M$ compares to the velocity dispersion of the surrounding matter. The greater is the density of the surrounding media, the stronger will be the force from dynamical friction. Similarly, the force is proportional to the square of the mass of the object. The force is also proportional to the inverse square of the velocity. This means the fractional rate of energy loss drops rapidly at high velocities.

Dynamical friction is, therefore, unimportant for objects that move relativistically, such as photons. Dynamical friction is particularly important in the formation of planetary systems and interactions between galaxies.

During the formation of planetary systems, dynamical friction between the protoplanet and the protoplanetary disk causes energy to be transferred from the protoplanet to the disk. This results in the inward migration of the protoplanet.

When galaxies interact through collisions, dynamical friction between stars causes matter to sink toward the centre of the galaxy and for the orbits of stars to be randomised. The dynamical friction comes into effect in the evolution of cluster between the ambient gas and dust and the central IMBH.

If an IMBH of mass $M_{\mathrm{BH}}$ is moving with a velocity of $v_b$ in a uniform background of "fixed" lighter stars of equal masses $m$. Then as the IMBH moves, a star approaching with impact parameter $b$, will have a velocity change given by:

$$\Delta v \approx a\Delta t, \tag{27}$$

where $\Delta t$ is the encounter duration and $a$ is the acceleration. They are given by

$$ma = \frac{GmM_{\mathrm{BH}}}{b^2}; \qquad \Delta t = \frac{b}{v_b}. \tag{28}$$

The change in velocity becomes

$$\Delta v \approx a\Delta t \approx \frac{GM_{\mathrm{BH}}}{b^2}\frac{b}{v_b}. \tag{29}$$

The kinetic energy gained by the star corresponding to this change in velocity is given by

$$\Delta E = \frac{1}{2}m(\Delta v)^2 \approx \frac{1}{2}m\left(\frac{GM_{\mathrm{BH}}}{bv_b}\right)^2. \tag{30}$$

If $n$ is the number density of the stars, then the number of encounters with impact parameter between $b + \Delta b$ and $b$ is given by

$$\Delta N \approx n\left(v_{\mathrm{BH}}\Delta t\right)\Delta\left(\pi b^2\right). \tag{31}$$

The total change in velocity is given by

$$\frac{dv_{\mathrm{BH}}}{dt} \approx \frac{1}{M_{\mathrm{BH}}v_{\mathrm{BH}}}\int\frac{dE}{dt}dN \approx \frac{\pi G^2 n M_{\mathrm{BH}}}{v_{\mathrm{BH}}^2}\int_{b_1}^{b_2}\frac{db}{b}, \tag{32}$$

where $b_1$, the lower bound on $b$, is given for the case where the gravitational energy is of the order of the kinetic energy of the black hole. That is,

$$\frac{GmM_{\mathrm{BH}}}{b_1} \approx \frac{1}{2}mv_{\mathrm{BH}}^2. \tag{33}$$

And the upper limit $b_2$ is the size of the system. Let $\int_{b_1}^{b_2}(db/b) = \ln\Lambda$.

Then the total change in velocity of the black hole is given by

$$\frac{dv_{\mathrm{BH}}}{dt} \approx \frac{\pi G^2 n M_{\mathrm{BH}}}{v_{\mathrm{BH}}^2}\ln\Lambda. \tag{34}$$

In the above discussion we have assumed that the stars are stationary. This need not be true. If the stars have a velocity dispersion of $\sigma$ and the black hole is moving at a slower velocity than this, that is, $v_{\mathrm{BH}} \ll \sigma$, then the dynamical friction force is given by

$$F_{\mathrm{DF}} = ma_{\mathrm{DF}} \approx \frac{-3\pi G^2 nmM_{\mathrm{BH}}^2 \ln\Lambda}{\left(\sqrt{2}\sigma\right)^3} v_{\mathrm{BH}} = -\gamma v_{\mathrm{BH}}, \tag{35}$$

where

$$\gamma = \frac{-3\pi G^2 nmM_{\mathrm{BH}}^2 \ln\Lambda}{\left(\sqrt{2}\sigma\right)^3}. \tag{36}$$

In the case of the velocity of the black hole being faster than $\sigma$, the dynamical friction will be reduced and the black hole will slide through the cluster.

The black hole is also subjected to a gravitational force due to the star cluster, which is given by

$$\nabla^2\phi(r) = 4\pi G\rho(r) = 4\pi Gmn(r), \tag{37}$$

where $n(r)$ is the density distribution of stars and $m$ is the typical mass of the star (assuming all stars are of the same mass).

For a constant density, $\rho(r) = \rho_0$, the potential is given by

$$\phi(r) = -2\pi G\rho_0\left(R^2 - \frac{1}{3}r^2\right), \tag{38}$$

where $R$ is the radius of the cluster.

The gravitational force on the black hole is given by

$$F_g = -M_{\mathrm{BH}}\nabla\phi(r) = -\frac{4}{3}\pi G\rho_0 M_{\mathrm{BH}} r = -kr. \tag{39}$$

(The particle inside a homogenous gravitational system performs simple harmonic motion!)

The equation of motion of the black hole in the star cluster is given by

$$M_{\mathrm{BH}}\frac{d^2r}{dt^2} + kr + \gamma\frac{dr}{dt} = 0. \tag{40}$$

This corresponds to a damped oscillator. And the solution is given by [12]

$$r = M_{\mathrm{BH}}\exp\left(-\frac{\gamma t}{2M_{\mathrm{BH}}}\right)\cos\left(\left(\sqrt{\frac{k}{M_{\mathrm{BH}}}}\right)t + \gamma\right). \tag{41}$$

The black hole undergoes a damped oscillation in the star cluster. The damping time corresponding to the system is given by

$$t = \frac{M_{\mathrm{BH}}}{\gamma}, \tag{42}$$

where

$$\gamma = \frac{-3\pi G^2 nmM_{\mathrm{BH}}^2 \ln\Lambda}{\left(\sqrt{2}\sigma\right)^3}. \tag{43}$$

In the case of the system M82, which harbours an IMBH of mass in the range of 500 to $10^4\ \mathrm{M}_\Theta$, the black hole is not found at the centre but displaced by about one kilo-parsec from the centre.

The period corresponding to the oscillation is given by

$$T = \frac{2\pi}{\omega}; \qquad \omega = \sqrt{\frac{k}{M_{\mathrm{BH}}}} \approx 10^{-13}\ \mathrm{s}^{-1}. \tag{44}$$

From (37), we can work out the time taken for the BH to shift by this distance. Using this equation along with (32) and (35), we get the time of the order of $10^6$ years.

For the system under consideration, the number density of the stars in the cluster is $n \approx 10^4/(\mathrm{pc})^3$, typical mass of the star in the cluster is about one solar mass, and the velocity dispersion is of the order of $\sigma \approx 30$ km/s.

We get the damping time for the system as

$$t = \frac{\left(\sqrt{2}\sigma\right)^3}{3\pi G^2 nmM_{\mathrm{BH}}\ln\Lambda} \approx 4\times 10^{12}\ \mathrm{s}. \tag{45}$$

This works out to be of the order of $10^5$ years.

Taking the effects of dynamical friction into consideration, the relaxation time for the system to form the IMBH is given by

$$t = \frac{v^3}{nG^2M_{\mathrm{BH}}^2\ln N} \approx 10^7\ \text{years}. \tag{46}$$

For a denser core, that is, about $10^3$ stars/$(0.01\ \mathrm{pc})^3$, the time taken to form the IMBH is given by

$$t = \frac{v^3}{nG^2M_{\mathrm{BH}}^2\ln N} \approx 4\times 10^{12}\ \mathrm{s} = 10^6\ \text{years}. \tag{47}$$

## 5. Formation of Supermassive Black Hole (SBH): Merger of IMBH

Zwart et al. [13] propound that dense star densities near galactic centres can lead to runaway stellar mergers thus efficiently producing IMBH. Indeed they suggest that the inner most ten parsecs of our galaxy can contain about 50 IMBH each of about thousand solar masses. These can sink towards the core as they interact with the stars. Every time an IMBH has a stellar encounter, the stars' velocity is boosted. This reduces the potential energy making the IMBH sink to

the galactic core. These IMBH can thus ultimately merge with the galaxy's supermassive black hole.

However if there are not enough stars very near to the galaxy's SBH, then the IMBH may stop falling inward, halting at about 0.01 light years from the centre. However if there are several IMBH near the galactic core, they can interact with one another and once every million years an IMBH can merge with the SBH.

It has been suggested that the presence of several IMBH near the galactic centre might explain why there are so many clusters of young stars (like S2, the B type star) in the galactic core, where tidal forces should rip apart the gas clouds from which the stars form.

These stars could have formed much further away and the IMBH could have shepherded them inwards, much like the thin rings of Uranus or Saturn that are kept in place by being "shepherded" by tiny satellites.

This IMBH can merge with the stars in the surrounding volume ($\sim 10^6\ \mathrm{M}_\odot/(1\ \mathrm{pc})^3$) to give a supermassive black hole in the time scale given by

$$t = \frac{v^3}{nG^2M^2 \ln N} \approx 10^{15}\ \mathrm{s} = 10^8\ \text{years}. \tag{48}$$

This is the time scale for relaxing in a cluster of $N$ objects of average mass $M$ moving with a mean velocity of $v$. If $\sigma$ is the average cross section for interaction, $\sigma \approx (GM/v^2)^2$; $(GM/v^2)$ gives the "radius" of the sphere of influence around a given individual object and the mean free path is $1/n\sigma$, and $n$ is the number density of objects. Then the "collision" or "interaction" time scale is given by

$$t \approx \frac{1}{n\sigma v} = \frac{1}{n(GM/v^2)^2 v} = \frac{v^3}{nG^2M^2}. \tag{49}$$

The $\ln N$ term comes from many body effects (i.e., $\int (dN/N)$, the so-called Coulomb logarithm in plasma physics). This leads to (48). Note the sharp $v^3$ dependence as well as $1/nM^2$.

The maximum redshift observed till now is of the order of $z = 6.3$. The age of the universe corresponding to this redshift is given by

$$t = \frac{1}{H_0}\left(\frac{1-(1+z)^{-1/2}}{1+z}\right) \approx 10^9\ \text{yrs}. \tag{50}$$

The above equation is the standard formula for the age of the universe, corresponding to any redshift $z$. Higher $z$ implies younger objects. So $z = 6.3$ corresponds to $\sim 10^9$ years. Now, recent supermassive black holes are claimed to have been observed at $z \approx 10$, hardly $\sim 5 \times 10^8$ years after the big bang [14].

We estimate the time to form the black hole from the accretion time scale, corresponding to Eddington luminosity. This gives a logarithmic time scale of $t_0 \approx c\sigma_T/4\pi G m_p \approx 5 \times 10^8$ years. These are typical time scales for growth of massive black holes in the early universe.

Essentially we have (with $M$ as the initial mass) accretion power $\approx GM\dot{M}/R \leq 4\pi G M m_p c/\sigma_T$, with $R \approx f(GM/c^2)$, $f$ is a numerical parameter, and integrating we get $M = M_0 \exp((4\pi G M m_p/c\sigma_T)t) = M_0 \exp(t/t_0)$. This time can be translated into corresponding redshift.

According to the model suggested above, the time taken for the formation of SBH is of the order of $10^8$ years. The corresponding redshift is of the order of $z = 12$. For redshifts above this limit, as per this model, we should not be able to detect any supermassive black holes.

Other possible ways in which SBH can form are the following. (1) The first is by the merger of two or more IMBH. Moving masses like black holes produce gravitational waves in the fabric of space-time. A more massive moving object will produce more powerful waves, and objects that move very quickly will produce more waves over a certain time period. (2) The second is by accretion of matter by the IMBH in systems such as AGNs and quasars.

Gravitational waves are usually produced in an interaction between two or more compact masses. Such interactions include the binary orbit of two black holes orbiting each other. As the black holes orbit each other, they send out waves of gravitational radiation that reaches the Earth; however, once the waves do get to the Earth, they are extremely weak.

This is because gravitational waves decrease in strength as they move away from the source. Even though they are weak, the waves can travel unobstructed within the fabric of space-time.

From Kepler's third law, the period is related to the separation by

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} R^3. \tag{51}$$

Knowing the period, we can determine the orbital velocity from $vP = 2\pi R$.

Power lost by gravitational waves (quadrupole formula for gravitational waves) is given by (for objects of mass $M$ with separation of $R$)

$$\dot{E}_{\mathrm{GW}} = \frac{32}{5}\frac{G}{c^5} M^2 R^4 \omega^6. \tag{52}$$

This emission results in the objects coming closer. The merger time is obtained by integrating the rate of binding energy change as follows:

$$\frac{GM^2}{R^2}\frac{dR}{dt} = \frac{32}{5}\frac{G}{c^5} M^2 R^4 \omega^6. \tag{53}$$

This implies a merger time for two objects separated by an initial distance $R_0$ of

$$\tau_{\mathrm{mer}} = K R_0^4. \tag{54}$$

With the constant, $K = (5/256)(c^5/G^3)(1/M^3)$, for equal mass objects.

So a larger initial separation would imply a longer merger time. So for the merger time to be $\approx 10^9$ years or less, the initial separation can be precisely calculated.

For two IMBH of mass $M = 10^4\,\mathrm{M}_\Theta$ each, to merge in $\tau_{\mathrm{mer}} \approx 10^9$ years, the distance of separation should be of the order of

$$R = \frac{GM^2}{\tau_{\mathrm{mer}}\dot{E}_{\mathrm{GW}}}, \tag{55}$$

where

$$\dot{E}_{\mathrm{GW}} = \frac{128 v^{10}}{5Gc^5}, \qquad v = \frac{2\pi R}{P},$$
$$P = \left(\frac{4\pi^2}{G(M_1+M_2)}R^3\right)^{1/2} \Longrightarrow v = \sqrt{\frac{2GM}{R}}. \tag{56}$$

And the energy loss due to the gravitational waves emission is given by

$$\dot{E}_{\mathrm{GW}} = \frac{128}{5Gc^5}\left(\frac{2GM}{R}\right)^5. \tag{57}$$

Therefore, the distance of separation is given by

$$R \approx \left(8\times 10^2 \frac{(GM)^3\tau_{\mathrm{mer}}}{c^5}\right)^{1/4} \approx 2\times 10^{12}\,\mathrm{m}. \tag{58}$$

The corresponding orbital frequency is given by

$$f = \frac{v}{2\pi R}. \tag{59}$$

The orbital velocity is of the order of

$$v = \sqrt{\frac{2GM}{R}} \approx 10^6\,\mathrm{m/s}. \tag{60}$$

This implies that the frequency is $f \approx 10^{-7}$ Hz, and hence the period will be given by $P = 10^7\,\mathrm{s} = 1$ year.

For two IMBH with separation of about $10^{-4}$ parsecs, the time taken to merge is of the order of Hubble time. For such a model to produce an SBH, we need about $10^3$ IMBH merging together.

Hence, the model discussed earlier, with the IMBH merging with the surrounding stars, gives a much more efficient way of generating a supermassive black hole.

In the case of accretion of matter by IMBH to form SBH, the increase in mass is exponential with time

$$M = M_0 \exp(kt), \tag{61}$$

where $k^{-1} = t_0 \approx 6\times 10^8$ years is the characteristic time required for the mass to increase $e$-fold, with the accreting disk emitting at maximum luminosity.

For the IMBH to accrete enough matter to become SBH of mass say $10^8\,\mathrm{M}_\Theta$, we have

$$\exp(kt) = \frac{M}{M_0} = 10^4. \tag{62}$$

The corresponding time scale is of the order of

$$t \approx 5\times 10^9 \text{ years}. \tag{63}$$

This implies that the mass will have to increase $\approx e^{10}$-fold by accretion for the IMBH to become an SBH. Hence, even this model does not provide an efficient way of formation of SBH from an IMBH.

IMBH would have formed in the early universe. Owing to low metal content, the earliest stars would have been very massive, a few hundred solar masses [15]. Such stars would end up in a pair-instability supernova (around oxygen-neon burning temperature of 2 billion degrees) and would collapse into a black hole (if their mass exceeds 250 solar masses).

Oxygen-neon burning occurs at $\sim 2\times 10^9$ K. At this stage, temperatures are high enough for the electron-positron pair production processes ($\gamma \rightarrow e^+ + e^-$). At $\sim 6\times 10^9$ K, it is maximal. Now if the nature of the black body radiation changes from photons to electron-positron pairs (which are fermions) the energy density is no longer $aT^4$ but $(7/8)aT^4$. So once the pairs are forming, the radiation pressure drops by $(1/8)aT^4$. So there is less support against gravitational collapse or gravitational contraction.

The balance between the two (or the "stability condition") can be expressed as

$$\frac{1}{8}aT^4 \geq \frac{GM^2}{R^4}. \tag{64}$$

Now, $T \sim 2\times 10^9$ K, $R \sim 2\times 10^8$ m (appropriate density for oxygen-neon burning); this implies a limiting mass $M$ of 250 solar masses, beyond which the gravitational energy density is higher. This result in the context of population III supermassive stars is discussed by Heger and Woosley [16].

Signatures of such explosions of supermassive stars at $z \approx 10$ (when the universe was only half a billion years old and eleven times smaller) could be sought with future space telescopes [17, 18].

Observations of galaxy core show correlation between black hole masses and the spheroidal (bulge) component of the host galaxy. Thus, the primordial low mass galaxies (blue galaxies) would host low mass central black holes, which just correspond to the IMBH mass (about $10^{-5}$ to $10^{-6}$ the galaxy mass). There is a well-known relation between black hole mass and spheroidal bulge component mass of the host galaxy [19].

Typically, the black hole mass is $10^{-3}$ the bulge mass. Primordial galaxies (so-called blue galaxies as seen, e.g., in Hubble deep field) have a lower mass $\sim 10^8$ solar mass. This would give a central black hole mass of $\sim 10^4$ solar mass, corresponding to a typical IMBH.

Merger of these primeval galaxies would lead to larger galaxies and cluster and the IMBH would also merge and sink to the core forming an SBH (like, e.g., M87 has a $3\times 10^9\,\mathrm{M}_\Theta$ black hole).

## 6. Concluding Remarks

We have discussed several aspects of the expected characteristics of IMBH like their luminosity, accretion rate, formation, and so forth. They are likely to have formed in the early universe, around $z = 12$. This redshift corresponds to a minimal time scale for formation (and growth) of a massive

black hole. Beyond $z = 12$, the mass would not be large enough, as not enough time would have elapsed for the growth of black holes by accretion or merger. Supermassive black holes are not likely to form from merger of IMBH. Other possible scenarios are also discussed.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Electrostatics in the Surroundings of a Topologically Charged Black Hole in the Brane

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We determine the expression for the electrostatic potential generated by a point charge held stationary in the topologically charged black hole spacetime arising from the Randall-Sundrum II braneworld model. We treat the static electric point charge as a linear perturbation on the black hole background and an expression for the electrostatic multipole solution is given: PACS: 04.70.-s, 04.50.Gh, 11.25.-w, 41.20.-q, 41.90.+e.

## 1. Introduction

The idea of our universe as a brane embedded in a higher dimensional spacetime has recently attracted attention. According to the braneworld scenario, the physical fields (electromagnetic, Yang-Mills, etc.) in our 4-dimensional universe are confined to the three-brane and only gravity propagates in the bulk spacetime. One of the most interesting scenarios is Randall-Sundrum II model in which it is considered a $\mathbb{Z}_2$-symmetric, 5-dimensional, asymptotically anti-de-Sitter bulk [1] and our brane is identified as a domain wall. The 5-dimensional metric can be written in the general form $ds^2 = e^{-F(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$ and due to the appearance of the warp factor, it reproduces a large hierarchy between the scale of particle physics and gravity. Moreover, even if the fifth dimension is uncompactified, standard 4-dimensional gravity on the brane is reproduced. However, due to the correction terms coming from the extradimensions, significant deviations from general relativity may occur at high energies [2–4].

As is well known, in general relativity the exterior spacetime of a spherical compact object is described by Schwarzschild solution. In the braneworld scenario, the high energy corrections to the energy density together with Weyl stresses from bulk gravitons imply that the exterior metric of a spherical compact object on the brane is no longer described by Schwarzschild metric. In fact, black hole solutions in the braneworld model are particularly interesting because they have considerably richer physical aspects than black holes in general relativity [5–11]. The first solutions describing static and spherically symmetric exterior vacuum solutions of the braneworld model were proposed by Dadhich et al. [12] and Germani and Maartens [13] and later they were generalised by Chamblin et al. [14] and revisited by Sheykhi and Wang [15]. This kind of solutions carries a topological charge arising from the bulk Weyl tensor and the line element resembles Reissner-Nördstrom solution, with the tidal Weyl parameter playing the role of the electric charge. In order to obtain this solution, there was the null energy condition imposed on the three-brane for a bulk having nonzero Weyl curvature.

On the other hand, the generation of an electromagnetic field by static sources in black hole backgrounds has been considered in several papers beginning with the studies of Copson [16], Cohen and Wald [17], and Hanni and Ruffini [18] where they discussed the electric field of a point charge in Schwarzschild background. Afterwards, Petterson [19] and later Linet [20] studied the magnetic field of a current loop surrounding a Schwarzschild black hole. More recently, similar studies have been performed in other background

geometries including charged black holes of general relativity and black holes with Brans-Dicke modifications or with conical defects [21–25].

In this paper we are interested in obtaining an expression for the electrostatic potential generated by a point charge held stationary in the region outside the event horizon of a braneworld black hole. There are several vacuum solutions of the spherically symmetric static gravitational field equations on the brane with arbitrary parameters which depend on properties of the bulk or that are simply put in by using general physical considerations. At the present, it is theoretically not known whether these parameters should be universal over all braneworld black holes, or whether each separate black hole may have different values of them. Similarly, there is not a single complete solution in the sense that the metric in the bulk is uniquely known. Since this situation is unsatisfactory from a theoretical point of view, it may be useful to investigate more closely the observational effects of the black hole properties. Specifically, we consider the effects due to the projections of the Weyl tensor and how they specify the corrections on the electrostatic potential. Since the generic form of the Weyl tensor in the full 5-dimensional spacetime is yet unknown, the effects of known solutions must be studied on a case-by-case basis. Although one can, in principle, constrain the projections, this only yields very mild constraints on the 5-dimensional Weyl tensor [26, 27]. Therefore we decided to derive the electrostatic potential generated by a point charge held stationary in the outside region of the event horizon of the particular solution in the Randall-Sundrum braneworld model obtained by Dadhich et al. [12] and revisited by Sheykhi and Wang [15]. In this case, tidal charge is arising via gravitational effects from the fifth dimension; that is, it is arising from the projection onto the brane of free gravitational field effects in the bulk, and is this term the one that will modify the electrostatic potential produced by the particle. We obtain the corrections, due to the topological charge arising from the bulk Weyl tensor, to the electrostatic potential and deduce the necessary correction to incorporate Gauss's law. Finally we also present the solution of Maxwell equations in the form of series of multipoles.

## 2. The Topologically Charged Black Hole in the Braneworld

The gravitational field on the brane is described by the Gauss and Codazzi equations of 5-dimensional gravity [2],

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$ is the 4-dimensional Einstein tensor and $\kappa_5$ is the 5-dimensional gravity coupling constant,

$$\kappa_5^4 = \left(8\pi G_5\right)^2 = \frac{48\pi G}{\tau}, \tag{2}$$

with $G_5$ the gravitational constant in five dimensions and $\Lambda$ is the 4-dimensional cosmological constant that is given in terms of the 5-dimensional cosmological constant $\Lambda_5$ and the brane tension $\tau$ by

$$\Lambda = \frac{\kappa_5^2}{2}\left(\Lambda_5 + \frac{\kappa_5^2}{6}\tau^2\right). \tag{3}$$

$T_{\mu\nu}$ is the stress-energy tensor of matter confined on the brane, $\Pi_{\mu\nu}$ is a quadratic tensor in the stress-energy tensor given by

$$\begin{aligned}\Pi_{\mu\nu} &= \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\sigma} T_{\nu}^{\sigma} + \frac{1}{8} g_{\mu\nu}\left(T_{\alpha\beta} T^{\alpha\beta} - \frac{1}{3} T^2\right)\end{aligned} \tag{4}$$

with $T = T_{\sigma}^{\sigma}$, and $E_{\mu\nu}$ is the projection of the 5-dimensional bulk Weyl tensor $C_{ABCD}$ on the brane ($E_{\mu\nu} = \delta_{\mu}^{A} \delta_{\nu}^{B} C_{ABCD} n^A n^B$ with $n^A$ the unit normal to the brane). $E_{\mu\nu}$ encompasses the nonlocal bulk effect and it is traceless, $E_{\sigma}^{\sigma} = 0$.

Considering the Randall-Sundrum scenario with

$$\Lambda_5 = -\frac{\kappa_5^2}{6}\tau^2 \tag{5}$$

which implies

$$\Lambda = 0, \tag{6}$$

the four-dimensional Gauss and Codazzi equations for an arbitrary static spherically symmetric object have been completely solved on the brane, obtaining a black hole type solution of the field equations (1) with $T_{\mu\nu} = 0$, given by the line element [12, 14, 15, 28]

$$ds^2 = h(r)\, dt^2 - \frac{dr^2}{h(r)} - r^2 d\Omega^2, \tag{7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and

$$h(r) = 1 - \frac{2GM}{r} + \frac{\beta}{r^2}. \tag{8}$$

For this model, the expression of the projected Weyl tensor transmitting the tidal charge stresses from the bulk to the brane is

$$E_t^t = E_r^r = -E_\theta^\theta = -E_\varphi^\varphi = \frac{\beta}{r^4}. \tag{9}$$

This result shows that the parameter $\beta$ can be interpreted as a tidal charge associated with the bulk Weyl tensor and therefore, there is no restriction on it to take positive as well as negative values (other interpretations consider $\beta$ as a five-dimensional mass parameter as discussed in [14]). The induced metric in the domain wall presents horizons at the radii (taking $G = 1$). Consider the following:

$$r_{\pm} = M \pm \sqrt{M^2 - \beta}. \tag{10}$$

For $\beta \geq 0$ there is a direct analogy to the Reissner-Nördstrom solution, showing two horizons that, as in general relativity, both lie inside the Schwarzschild radius $2M$; that is,

$$0 \leq r_- \leq r_+ \leq r_s. \tag{11}$$

Clearly, there is an upper limit on $\beta$, namely,

$$0 \leq \beta \leq M^2. \tag{12}$$

However, there is nothing to stop us choosing $\beta$ to be negative. This intriguing new possibility is impossible in Reissner-Nördstrom case and leads to only one horizon, $r_*$, lying outside the corresponding Schwarzschild radius,

$$r_* = M + \sqrt{M^2 + |\beta|} > 2M. \tag{13}$$

In this case, the single horizon has a greater area that its Schwarzschild counterpart. Thus, one concludes that the effect of the bulk producing a negative $\beta$ is to strengthen the gravitational field outside the black hole (obviously it also increases the entropy and decreases the Hawking temperature but these facts are not important in this paper).

## 3. The Electrostatic Field of a Point Particle

Copson [16] and Linet [20] found the electrostatic potential in a closed form of a point charge at rest outside the horizon of a Schwarzschild black hole and that the multipole expansion of this potential coincides with the one given by Cohen and Wald [17] and by Hanni and Ruffini [18]. In this section we will investigate this problem in the background of the topologically charged black hole (7).

According to the braneworld scenario, we will consider that the physical fields are confined to the three brane and if the electromagnetic field of the point particle is assumed to be sufficiently weak so its gravitational effect is negligible, and the Einstein-Maxwell equations reduce to Maxwell's equations confined in the curved background of the brane (7). These are written as

$$\nabla_\rho F^{\rho\mu} = 4\pi J^\mu, \tag{14}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15}$$

with $A^\mu$ the electromagnetic vector potential and $J^\mu$ the current density. Considering that the test charge $q$ is held stationary at the point $(r_0, \theta_0, \varphi_0)$, the associated current density $J^i$ vanishes while the charge density $J^0$ is given by

$$J^0 = \frac{q}{r_0^2 \sin\theta} \delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(\varphi - \varphi_0). \tag{16}$$

From now on, we will assume that the charge $q$ is held outside the black hole; that is, $r_0 > r_+$ in the case $\beta \geq 0$ and $r_0 > r_*$ in the case $\beta < 0$. We will not consider here the possibility of having the charge inside the event horizon because; as well known, this construction leads to the description of a charged black hole of the Reissner-Nördstrom type [15].

The spatial components of the potential vanish, $A^i = 0$, while the temporal component $A^0 = \phi$ is determined by the $\mu = 0$ component of (14), giving the differential equation:

$$\begin{aligned} &\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) \\ &\quad + \left(1 - \frac{2M}{r} + \frac{\beta}{r^2}\right)^{-1} \\ &\quad \times \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right] \\ &= -r^2 4\pi J^0. \end{aligned} \tag{17}$$

In order to solve this equation we will perform the substitutions

$$\begin{aligned} r &= \tilde{r} + r_-, \\ \phi(r, \theta, \varphi) &= \frac{r - r_-}{r} \tilde{\phi}(r - r_-, \theta, \varphi) \end{aligned} \tag{18}$$

which turn the differential equation into

$$\begin{aligned} &\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}^2 \frac{\partial \tilde{\phi}}{\partial \tilde{r}}\right) \\ &\quad + \left(1 - \frac{2m}{\tilde{r}}\right)^{-1} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\tilde{\phi}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\tilde{\phi}}{\partial\varphi^2}\right] \\ &= -\frac{4\pi\tilde{q}}{\sin\theta} \delta(\tilde{r} - \tilde{r}_0)\, \delta(\theta - \theta_0)\, \delta(\varphi - \varphi_0), \end{aligned} \tag{19}$$

where $m = \sqrt{M^2 - \beta}$ and $\tilde{q} = q\tilde{r}_0/(\tilde{r}_0 + r_-)$. Note that, in the extremal case $m = 0$, or equivalently $M^2 = \beta$, (19) becomes Laplace's equation in Minkowski spacetime. On the other hand, when $m \neq 0$ (including values with $0 \leq \beta < M^2$ as well as $\beta < 0$), (19) is formally identical to the partial differential equation for the electrostatic potential in the Schwarzschild spacetime. Hence, assuming $\theta_0 = 0$, $\varphi_0 = 0$ and proceeding in analogy with Copson [16], we find that the solution of (17), after using the substitutions (18), is

$$\phi_C(r, \theta) = \frac{q}{r_0 r} \frac{(r - M)(r_0 - M) - (M^2 - \beta)\cos\theta}{\sqrt{(r - M)^2 + (r_0 - M)^2 - 2(r - M)(r_0 - M)\cos\theta - (M^2 - \beta)\sin^2\theta}}. \tag{20}$$

This solution describes, as stated before, the potential for a charge $q$ situated at the point $(r_0, 0, 0)$ and it is regular for any value of $r$ outside the event horizon, except at the position of the point charge. However, as shown by Linet [20] and Léauté and Linet [21], it is easy to see that this solution includes another source. Note that, for $r \to \infty$, the potential $\phi_C$ takes the asymptotic form

$$\phi_C(r,\theta) \longrightarrow \frac{q}{r}\left(1 - \frac{M}{r_0}\right), \tag{21}$$

and consequently, by virtue of Gauss's theorem, there is a second charge with value $-qM/r_0$. Solution (20) has only the source $q$ outside the horizon and thus the second charge must lie inside the horizon. Moreover, since the only electric field which is regular outside the horizon is spherical symmetric [29], the electrostatic potential for our physical system will be of the form

$$\phi(r,\theta) = \phi_C(r,\theta) + \frac{q}{r}\frac{M}{r_0}. \tag{22}$$

The electrostatic solution can be analysed for all values $r$. Considering this equation, it is clear that when the charge $q$ is approaching the outer horizon $r_+$ in the case $\beta \geq 0$ or the horizon $r_*$ in the case $\beta < 0$, the electrostatic potential $\phi$ tends to become spherically symmetric, recovering a charged black hole of the Reissner-Nördstrom type as stated before.

When $r_0 > 2M$, there is only one charge $q$ in the case $\beta \geq 0$ as well as in the case $\beta < 0$. However, when $r_+ \leq r_0 < 2M$ in the case $\beta \geq 0$, there are also two other charges $q_1 = q(1-2M/r_0)$ and $q_2 = -q_1$ located at $(r = 2M - r_0, \theta = \pi)$ and $r = 0$, respectively. This behaviour is obviously not present for $\beta < 0$ because $r_* > 2M$ as shown in Section 2 of this paper.

Finally, in order to write formally the electrostatic potential of a charge $q$ located at the arbitrary point with coordinates $(r_0, \theta_0, \varphi_0)$, we simply replace the term $\cos\theta$ by the function $\lambda(\theta,\varphi) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$. This gives the general solution

$$\phi(r,\theta,\varphi) = \frac{q}{r_0 r}\frac{(r-M)(r_0-M) - \left(M^2-\beta\right)\lambda(\theta,\varphi)}{\sqrt{(r-M)^2 + (r_0-M)^2 - 2(r-M)(r_0-M)\lambda(\theta,\varphi) - (M^2-\beta)\left[1-\lambda^2(\theta,\varphi)\right]}} + \frac{q}{r}\frac{M}{r_0}. \tag{23}$$

*3.1. Multipole Expansion.* If the angular part of the potential in (19) is expanded in terms of Legendre polynomials in $\cos\theta$, one may obtain the well-known electric multipole solutions. Using again the substitutions (18), the radial parts of the two linearly independent multipole solutions of (17) are

$$g_l(r) = \begin{cases} 1 & \text{for } l = 0 \\ \dfrac{2^l l!\,(l-1)!\left(M^2-\beta\right)^{l/2}}{(2l)!}\dfrac{r^2-2Mr+\beta}{r}\dfrac{dP_l}{dr}\left(\dfrac{r-M}{\sqrt{M^2-\beta}}\right) & \text{for } l = 1, 2, \ldots, \end{cases} \tag{24}$$

$$f_l(r) = -\frac{(2l+1)!}{2^l(l+1)!\,l!\left(M^2-\beta\right)^{(l+1)/2}}\frac{r^2-2Mr+\beta}{r}\frac{dQ_l}{dr}\left(\frac{r-M}{\sqrt{M^2-\beta}}\right) \quad \text{for } l = 0, 1, 2, \ldots, \tag{25}$$

where $P_l$ and $Q_l$ are the two types of Legendre functions. These functions satisfy the following:

(1) for $l = 0$, $g_0(r) = 1$ and $f_0(r) = 1/r$;

(2) for all values of $l$, as $r \to \infty$, the leading term of $g_l(r)$ is $r^l$ while the leading term of $f_l(r)$ is $r^{-(l+1)}$;

(3) as $r \to r_+$, $g_l(r) \to 0$ (for $l \neq 0$) while $f_l(r) \to$ finite constant. However, $df_l/dr$ blows up for $l \neq 0$ when $r \to r_+$.

The above properties let us infer that only the set of solutions (25) has the correct behaviour at infinity and only the multipole term $l = 0$ does not produce a divergent field at the horizon $r = r_+$. This set of solutions reproduces Israel's result [29] when $\beta = 0$ (i.e., for the Schwarzschild black hole).

As is well known, the gravitational field modifies the electrostatic interaction of a charged particle in such a way that the particle experiences a finite self-force [22, 23, 30–34] whose origin comes from the spacetime curvature associated with the gravitational field. However, even in the absence of curvature, it was shown that a charged point particle [35, 36] or a linear charge distribution [37] placed at rest may become subject to a finite repulsive electrostatic self-force (see also [24]). In these references, the origin of the force is the distortion in the particle field caused by the lack of global flatness of the spacetime of a cosmic string. Therefore, one may conclude that the modifications of the electrostatic potential come from two contributions: one of geometric origin and the other of a topological one. In a forthcoming paper we will show that, considering the topologically charged black hole

described by the line element (7) as the background metric, both kinds of contributions appear in the electrostatic self-force of a charged particle.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Energy Loss of a Heavy Particle Near 3D Rotating Hairy Black Hole

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We consider rotating black hole in 3 dimensions with a scalar charge and discuss energy loss of heavy particle moving near the black hole horizon. We find that drag force was increased by scalar charge while it was decreased due to the rotation of black hole. We also study quasnormal modes.

## 1. Introduction

The lower dimensional theories may be used as toy models to study some fundamental ideas which yield to better understanding of higher dimensional theories, because they are easier to study [1]. Moreover, these are useful for application of AdS/CFT correspondence [2–5]. This paper is indeed an application of AdS/CFT correspondence to probe moving charged particle near the three-dimensional black holes which are recently introduced by Xu et al. [6, 7] where charged black holes with a scalar hair in (2 + 1) dimensions and rotating hairy black hole in (2 + 1) dimensions are constructed, respectively. Here, we are interested in the case of rotating black hole with a scalar hair in (2 + 1) dimensions. Recently, a charged rotating hairy black hole in 3 dimensions corresponding to infinitesimal black hole parameters was constructed [8]. Also, thermodynamics of such systems is recently studied in [9, 10]. We consider this background in AdS side as a dual picture of a QCD model as CFT side.

In this paper, we would like to study the motion of a heavy charged particle near the black hole horizon and calculate the energy loss. The energy loss of moving heavy charged particle through a thermal medium is known as the drag force. One can consider a moving heavy particle (such as charm and bottom quarks) near the black hole horizon with the momentum $P$, mass $m$, and constant velocity $v$, which is influenced by an external force $F$. So, one can write the equation of motion as $\dot{P} = F - \zeta P$, where in the nonrelativistic motion $P = mv$, and in the relativistic motion $P = mv/\sqrt{1-v^2}$; also $\zeta$ is called the friction coefficient. In order to obtain drag force, one can consider two special cases. The first case is the constant momentum which yields to obtain $F = (\zeta m)v$ for the norelativistic case. In this case, the drag force coefficient $(\zeta m)$ will be obtained. In the second case, the external force is zero, so one can find $P(t) = P(0)\exp(-\zeta t)$. In other words, by measuring the ratio $\dot{P}/P$ or $\dot{v}/v$, one can determine friction coefficient $\zeta$ without any dependence on mass $m$. These methods lead us to obtain the drag force for a moving heavy particle. The moving heavy particle in context of QCD has dual picture in the string theory in which an open string is attached to the D-brane and stretched to the horizon of the black hole. Therefore, we can apply AdS/CFT correspondence to probe a charged particle (such as a quark) moving through 3D hairy black hole background.

Similar studies are already performed in several backgrounds [11–22]. Most of them considered $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang-Mills plasma with asymptotically AdS geometries. Also [20] considered 4D Kerr-AdS black holes. All of the mentioned studies used $AdS_5/CFT_4$ correspondence. Now, we are going to consider the same problem in a rotating hairy 3D background and use $AdS_3/CFT_2$ correspondence [23–25].

This paper is organized as follows. In the next section, we review rotating hairy black hole in (2 + 1) dimensions. In Section 3, we obtain equation of motion and in Section 4,

we try to obtain solution and discuss about drag force. In Section 5, we give linear analysis and discuss quasinormal modes. Finally, in Section 6, we summarized our results.

## 2. Rotating Hairy Black Hole in (2 + 1) Dimensions

Rotating hairy black hole in (2 + 1) dimensions is described by the following action:

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\left[R - g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{R}{8}\phi^2 - 2V(\phi)\right], \quad (1)$$

which yields to the following line element [1]:

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\psi + \omega(r)\,dt\right)^2, \quad (2)$$

where

$$f(r) = 3\beta + \frac{2\beta B}{r} + \frac{(3r+2B)^2a^2}{r^4} + \frac{r^2}{l^2}, \quad (3)$$

where $a$ is a rotation parameter related to the angular momentum of the solution and $l$ is related to the cosmological constant via $\Lambda = -1/l^2$. $\beta$ is integration constants depending on the black hole mass:

$$\beta = -\frac{M}{3}, \quad (4)$$

and scalar charge $B$ is related to the scalar field as

$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}}. \quad (5)$$

Also, one can obtain

$$\omega(r) = -\frac{(3r+2B)\,a}{r^3},$$
$$V(\phi) = \frac{2}{l^2} + \frac{1}{512}\left[\frac{1}{l^2} + \frac{\beta}{B^2}\right]\phi^6. \quad (6)$$

Ricci scalar of this model is given by

$$R = -\frac{6r^6 + 36Bl^2a^2r + 30l^2a^2B^2}{l^2r^6}. \quad (7)$$

We can see that Ricci scalar is singular at the origin.

Black hole horizon, which is obtained by $f(r) = 0$, may be written as follows:

$$r_h = \frac{4l}{2C}\left(1 + \sqrt{1 - \frac{BC}{3l^2}}\right), \quad (8)$$

where we defined

$$C \equiv \frac{2BM}{27a^2} - \frac{3l}{B}. \quad (9)$$

## 3. The Equations of Motion

The moving heavy particle near the black hole may be described by the following Nambu-Goto action:

$$S = -\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma\sqrt{-G}, \quad (10)$$

where $T_0 = 1/2\pi\alpha'$ is the string tension. The coordinates $\tau$ and $\sigma$ are corresponding to the string world-sheet. Also, $G_{ab}$ is the induced metric on the string world-sheet with determinant $G$ obtained as follows:

$$G = -1 - r^2 f(r)\left(x'\right)^2 + \frac{r^2}{f(r)}(\dot{x})^2, \quad (11)$$

where we used static gauge in which $\tau = t$, $\sigma = r$, and the string only extends in one direction $x(r,t)$. Then, the equation of motion is obtained as follows:

$$\partial_r\left(\frac{r^2 f(r)\,x'}{\sqrt{-G}}\right) - \frac{r^2}{f(r)}\partial_t\left(\frac{\dot{x}}{\sqrt{-G}}\right) = 0. \quad (12)$$

We should obtain canonical momentum densities associated with the string as follows:

$$\begin{aligned}
\pi^0_\psi &= \frac{1}{2\pi\alpha'\sqrt{-G}}\frac{r^2}{f(r)}\dot{x},\\
\pi^0_r &= -\frac{1}{2\pi\alpha'\sqrt{-G}}\frac{r^2}{f(r)}\dot{x}x',\\
\pi^0_t &= -\frac{1}{2\pi\alpha'\sqrt{-G}}\left(1 + r^2 f(r)\left(x'\right)^2\right),\\
\pi^1_\psi &= \frac{1}{2\pi\alpha'\sqrt{-G}}r^2 f(r)\,x',\\
\pi^1_r &= -\frac{1}{2\pi\alpha'\sqrt{-G}}\left(1 - \frac{r^2}{f(r)}\dot{x}^2\right),\\
\pi^1_t &= \frac{1}{2\pi\alpha'\sqrt{-G}}r^2 f(r)\,\dot{x}x'.
\end{aligned} \quad (13)$$

The simplest solution of the equation of motion is static string described by $x$ = constant with total energy of the form,

$$E = -\int_{r_h}^{r_m}\pi^0_t\,dr = \frac{1}{2\pi\alpha'}\left(r_h - r_m\right) = M_{\text{rest}}, \quad (14)$$

where $r_m$ is an arbitrary location of D-brane. As we expected, the energy of static particle is interpreted as the remaining mass.

## 4. Time Dependent Solution

In the general case, we can assume that the particle moves with constant speed $\dot{x} = v$; in that case, the equation of motion (12) reduces to

$$\partial_r\left(\frac{r^2 f(r)\,x'}{\sqrt{-G}}\right) = 0, \quad (15)$$

where

$$G = -1 - r^2 f(r)\left(x'\right)^2 + \frac{r^2}{f(r)}v^2. \tag{16}$$

Equation (15) gives the following expression:

$$\left(x'\right)^2 = \frac{C^2\left(r^2v^2 - f(r)\right)}{r^2 f(r)^2\left(C^2 - r^2 f(r)\right)}, \tag{17}$$

where $C$ is an integration constant which will be determined by using reality condition of $\sqrt{-G}$. Therefore, we yield to the following canonical momentum densities:

$$\begin{aligned} \pi_\psi^1 &= -\frac{1}{2\pi\alpha'}C, \\ \pi_t^1 &= \frac{1}{2\pi\alpha'}Cv. \end{aligned} \tag{18}$$

These give us loosing energy and momentum through an endpoint of string:

$$\begin{aligned} \frac{dP}{dt} &= \pi_\psi^1|_{r=r_h} = -\frac{1}{2\pi\alpha'}C, \\ \frac{dE}{dt} &= \pi_t^1|_{r=r_h} = \frac{1}{2\pi\alpha'}Cv. \end{aligned} \tag{19}$$

As we mentioned before, reality condition of $\sqrt{-G}$ gives us constant $C$. The expression $\sqrt{-G}$ is real for $r = r_c > r_h$. In the case of small $v$, one can obtain

$$r_c = r_h + \frac{r^2v^2}{f(r)'}|_{r=r_h} + \mathcal{O}\left(v^4\right), \tag{20}$$

which yields to

$$C = vr_h^2 + \mathcal{O}\left(v^3\right). \tag{21}$$

Therefore, we can write drag force as follows:

$$\frac{dP}{dt} = -\frac{vr_h^2}{2\pi\alpha'} + \mathcal{O}\left(v^3\right). \tag{22}$$

We draw drag force in terms of velocity and in agreement with the previous works such as [11–22]; the value of drag force increased by $v$. In Figure 1, we can see behavior of drag force with rotation parameter and scalar charge. It is shown that the scalar charge increases the value of drag force but the increasing rotational parameter decreases the value of the drag force.

## 5. Linear Analysis

Motion of string yields to small perturbation after late time due to the drag force. In that case, the speed of particle is infinitesimal and one can write $G \approx -1$. Also, we assume that $x = e^{-\mu t}$, where $\mu$ is the friction coefficient. Therefore, one can rewrite the equation of motion as follows:

$$\frac{f(r)}{r^2}\partial_r\left(r^2 f(r)x'\right) = \mu^2 x. \tag{23}$$

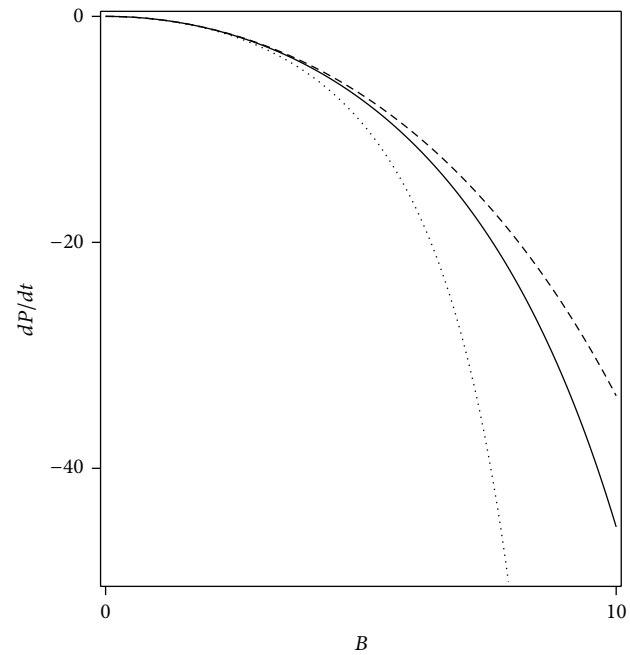


Figure 1: Drag force in terms of $B$ for $M = 1$, $l = 1$, and $v = 0.1$; $a = 1.8$ (dotted line), $a = 3$ (solid line), and $a = 4.2$ (dashed line).

We assume outgoing boundary conditions near the black hole horizon and use the following approximation:

$$(4\pi T)^2\left(r - r_h\right)\partial_r\left(r - r_h\right)x' = \mu^2 x, \tag{24}$$

which suggests the following solutions:

$$x = c\left(r - r_h\right)^{-\mu/4\pi T}, \tag{25}$$

where $T$ is the black hole temperature. In the case of infinitesimal $\mu$, we can use the following expansion:

$$x = x_0 + \mu^2 x_1 + \cdots. \tag{26}$$

Inserting this equation in the relation (24) gives $x_0 =$ constant, and

$$x_1' = \frac{A}{r^2 f(r)}\int_{r_h}^{r_m}\frac{r^2}{f(r)}dr, \tag{27}$$

where $A$ is a constant. Assuming near horizon limit enables us to obtain the following solution:

$$x_1 \approx \frac{A}{4\pi T r_h^2\left(r - r_h\right)}\left(-r_m + \frac{r_h^2}{4\pi T}\ln\left(r - r_h\right)\right). \tag{28}$$

Comparing (25) and (27) gives the following quasinormal mode condition:

$$\mu = \frac{r_h^2}{r_m}. \tag{29}$$

It is interesting to note that these results recover drag force (22) for infinitesimal speed. In Figure 2, we can see behavior of $\mu$ with rotational parameter and scalar charge. We find that scalar charge increases the value of friction coefficient, but the effect of rotation decreases $\mu$.

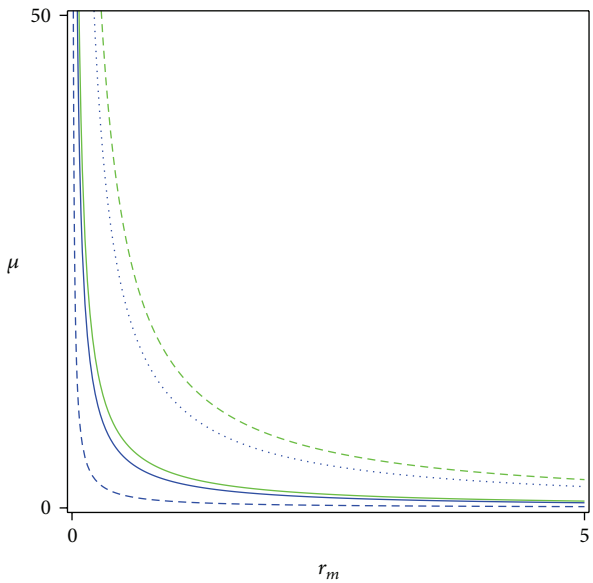


FIGURE 2: $\mu$ in terms of $r_m$: $B = 0.5$ and $a = 2$ (blue dashed line), $B = 1$ and $a = 2$ (blue solid line), $B = 2$ and $a = 2$ (blue dotted line), and $B = 1$ and $a = 0.2$ (green dashed line), and $B = 1$ and $a = 0.4$ (green solid line).

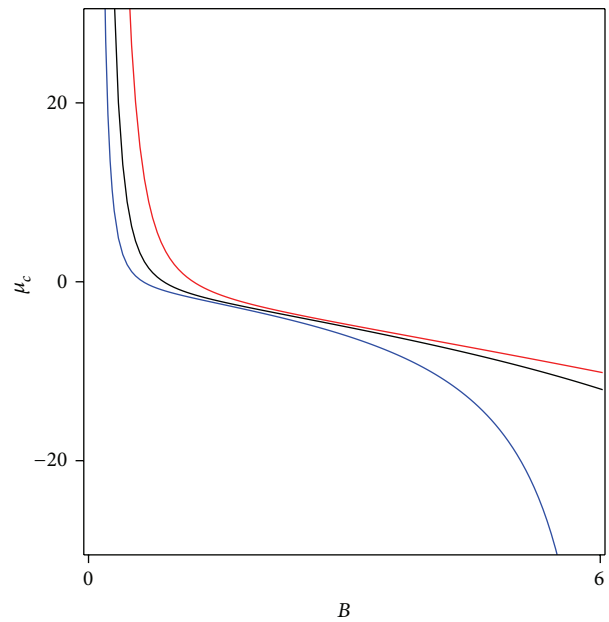


FIGURE 3: $\mu_c$ in terms of $B$ for $M = 1$ and $l = 1$; $a = 1$ (blue line), $a = 2$ (black line), and $a = 4$ (red line).

*5.1. Low Mass Limit.* Low mass limit means that $r_m \to r_h$, and we use the following assumptions:

$$f(r) \approx 4\pi T (r - r_h),$$
$$r^2 = r_h^2 + 2r_h (r - r_h) + \cdots, \tag{30}$$

so, by using relation (23) we can write

$$x(r) = (r - r_h)^{-\mu/4\pi T} (1 + (r - r_h) A + \cdots). \tag{31}$$

Then, we can obtain constant $A$ as follows:

$$A = \frac{\mu}{2\pi T r_h - \mu r_h}. \tag{32}$$

It tells that $\mu = 2\pi T$ yields to divergence; therefore we called this a critical behavior of the friction coefficient and found that

$$\mu_c = \frac{3r_h^6 + BMl^2 r_h^3 - 27a^2 l^2 r_h^2 - 54Ba^2 l^2 r_h - 24B^2 a^2 l^2}{r_h^5}. \tag{33}$$

Figure 3 shows behavior of critical friction coefficient with the black hole parameters.

## 6. Conclusions

In this paper, we considered rotating 3D black hole together with a scalar charge as a background where a charged particle moves with speed $v$ and then calculated drag force. We used motivation of AdS/CFT correspondence and string theory method to study motion of charged particle. This is indeed in the context of $AdS_3/CFT_2$ where drag force on moving heavy particle is calculated. Numerically, we found that the scalar charge increases the value of drag force but rotational parameter decreases the value of the drag force. Therefore, in order to have the most free motion we need to increase $a$ and decrease $B$. It means that $a$ and $B$ may cancel the effect of each other on the drag force. We can find critical values of scalar charge and rotational parameters in which the value of drag force will be infinite as

$$a_c = \sqrt{\frac{2M}{81l}} B_c. \tag{34}$$

Then, we studied quasinormal modes and obtained friction coefficient $\mu$ which was enhanced by the black hole charge and reduced by rotation. Quasinormal mode analysis also reproduced drag force at slow velocities. It is also possible to study dispersion relations which again reproduce the drag force which was obtained in (22). For the future work, we will consider charged rotating 3D hairy black hole and study drag force.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# Hawking Radiation-Quasi-Normal Modes Correspondence and Effective States for Nonextremal Reissner-Nordström Black Holes

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It is known that the nonstrictly thermal character of the Hawking radiation spectrum harmonizes Hawking radiation with black hole (BH) quasi-normal modes (QNM). This paramount issue has been recently analyzed in the framework of both Schwarzschild BHs (SBH) and Kerr BHs (KBH). In this assignment, we generalize the analysis to the framework of *nonextremal* Reissner-Nordström BHs (RNBH). Such a generalization is important because in both Schwarzschild and Kerr BHs an absorbed (emitted) particle has only mass. Instead, in RNBH the particle has charge as well as mass. In doing so, we expose that, for the RNBH, QNMs can be naturally interpreted in terms of quantum levels for both particle emission and absorption. Conjointly, we generalize some concepts concerning the RNBH's "effective states."

## 1. Introduction

A RNBH of mass $M$ is identical to a SBH of mass $M$ except that a RNBH has the *nonzero* charge quantity $Q$. In this paper, we are interested in RNBHs with the *nonextremal* constraint $M > Q$ [1]. The quantity $Q$ is the physical mechanism for the RNBH's *dual* horizons from (1) in [1]:

$$r_{\pm} = R_{\pm_{\mathrm{RNBH}}}(M,Q) = M \pm \sqrt{M^2 - Q^2}, \tag{1}$$

because the RNBH outer (event) horizon radius $R_{+_{\mathrm{RNBH}}}(M,Q)$ and the *RNBH inner (Cauchy) horizon radius* $R_{-_{\mathrm{RNBH}}}(M,Q)$ are clearly functions of both $M$ and $Q$, not just $M$, as in the well known case of the *SBH horizon radius*

$$r_s = R_{\mathrm{SBH}}(M) = 2M. \tag{2}$$

Energy conservation plays a fundamental role in BH radiance [2] because the emission or absorption of Hawking quanta with mass $m$ and energy-frequency $\omega$ causes a BH of mass $M$ to undergo a transition between *discrete* energy spectrum levels [3–7], where

$$E = m = \omega = \Delta M \tag{3}$$

for $G = c = k_B = \hbar = 1/4\pi\epsilon_0 = 1$ (Planck units). Given that emission and absorption are *reverse* processes for the quantized energy spectrum conservation [3–7], we consider this pair of transitions as being *equal in magnitude* but *opposite in direction* from the neutral radius perspective of $r_0 = (r_+ + r_-)/2$.

It is known that the countable character of successive emissions of Hawking quanta which is a consequence of

the nonstrictly thermal character of the Hawking radiation spectrum (see [3–12]) generates a natural correspondence between Hawking radiation and BH QNMs [3–7]. Moreover, it has also been shown that QNMs can be naturally interpreted in terms of quantum levels, where the emission or absorption of a particle is interpreted as a transition between two distinct levels on the *discrete* energy spectrum [3–7]. The thermal spectrum correction is an imperative adjustment to the physical interpretation of BH QNMs because these results are important to realize the underlying unitary quantum gravity theory [3–7]. Hod's intriguing works [13, 14] suggested that BH QNMs carry principle information regarding a BH's horizon area quantization. Hod's influential conjecture was later refined and clarified by Maggiore [15]. Moreover, it is also believed that QNMs delve into the microstructure of spacetime [16].

To make sense of the state space for the energy spectrum states and the underlying BH perturbation field states, an *effective framework* based on the *nonstrictly thermal* behavior of Hawking's framework began to emerge [3–7]. In the midst of this superceding BH effective framework [3–7], the BH effective state concept was originally introduced for KBHs in [6] and subsequently applied through Hawking's periodicity arguments [17, 18] to the BH tunneling mechanism's nonstrictly black body spectrum [7]. The effective state is meaningful to BH physics and thermodynamics research because one needs additional features and knowledge to consider in future experiments and observations.

In this paper, our objective is to apply the nonstrictly thermal BH effective framework of [3–7] to nonextremal RNBHs. Thus, upon recalling that a RNBH of mass $M$ is identical to a SBH of mass $M$ except that a RNBH has the charge $Q$, we prepare for our BH QNM investigation by reviewing relevant portions of the SBH effective framework [3–7] for quantities related to SBH states and transitions in Section 2. Then in Section 3, we launch our RNBH QNM exploration by introducing a RNBH effective framework for quantities pertaining to RNBH states and transitions. Finally, we conclude with a brief comparison between the fundamental SBH and RNBH results in Section 4 followed by the recapitulation in Section 5.

## 2. Schwarzschild Black Hole Framework: Background and Review

*2.1. Schwarzschild Black Hole States and Transitions.* Here, we recall some quantities that characterize the SBH.

First, consider a SBH of initial mass $M$, when the SBH emits or absorbs a quantum of energy-frequency $\omega$ (for particle mass $m$ and SBH mass change $\Delta M$, such that $m = \omega = \Delta M$) to achieve a final mass of $M-\omega$ or $M+\omega$, respectively, for the SBH mass-energy transition between states in state space. Thus, we follow [3–5], where the *SBH initial and final horizon area* are

$$\begin{aligned} A_{\mathrm{SBH}}(M) &= 16\pi M^2 = 4\pi R_{\mathrm{SBH}}^2(M), \\ A_{\mathrm{SBH}}(M\pm\omega) &= 16\pi(M\pm\omega)^2 = 4\pi R_{\mathrm{SBH}}^2(M\pm\omega), \end{aligned} \tag{4}$$

respectively, for the *SBH area quanta number*

$$N_{\mathrm{SBH}}(M,\omega) = \frac{A_{\mathrm{SBH}}(M)}{\left|\Delta A_{\mathrm{SBH}}(M,\omega)\right|}, \tag{5}$$

such that the *SBH horizon area change* for the corresponding mass change $\Delta M$ is

$$\begin{aligned} \Delta A_{\mathrm{SBH}}(M,\omega) &= A_{\mathrm{SBH}}(M\pm\omega) - A_{\mathrm{SBH}}(M) \\ &= 32\pi M\omega + O\left(\omega^2\right) \sim 32\pi M\Delta M \\ &= 32\pi M\Delta E, \end{aligned} \tag{6}$$

because the transition's minus (−) and plus (+) signs depend on emission and absorption, respectively. Next, in [3–5], the *Bekenstein-Hawking SBH initial and final entropy* are

$$\begin{aligned} S_{\mathrm{SBH}}(M) &= \frac{A_{\mathrm{SBH}}(M)}{4}, \\ S_{\mathrm{SBH}}(M\pm\omega) &= \frac{A_{\mathrm{SBH}}(M\pm\omega)}{4}, \end{aligned} \tag{7}$$

respectively, where the corresponding *SBH entropy change* is

$$\Delta S_{\mathrm{SBH}}(M,\omega) = \frac{\Delta A_{\mathrm{SBH}}(M,\omega)}{4}. \tag{8}$$

Subsequently, the *SBH initial and final total entropy* are [3–5]

$$\begin{aligned} S_{\mathrm{SBH-total}}(M) &= S_{\mathrm{SBH}}(M) - \ln S_{\mathrm{SBH}}(M) \\ &\quad + \frac{3}{2A_{\mathrm{SBH}}(M)}, \\ S_{\mathrm{SBH-total}}(M\pm\omega) &= S_{\mathrm{SBH}}(M\pm\omega) - \ln S_{\mathrm{SBH}}(M\pm\omega) \\ &\quad + \frac{3}{2A_{\mathrm{SBH}}(M\pm\omega)}, \end{aligned} \tag{9}$$

respectively. Additionally, the *SBH initial and final Hawking temperature* are [3–5]

$$\begin{aligned} T_{H_{\mathrm{SBH}}}(M) &= \frac{1}{8\pi M}, \\ T_{H_{\mathrm{SBH}}}(M\pm\omega) &= \frac{1}{8\pi(M\pm\omega)}, \end{aligned} \tag{10}$$

respectively. Therefore, the quantum transition's *SBH emission tunneling rate* is [3–5]

$$\begin{aligned} \Gamma_{\mathrm{SBH}}(M,\omega) &\sim \exp\left[-8\pi M\omega\left(1-\frac{\omega}{2M}\right)\right] \\ &\sim \exp\left[-\frac{\omega}{T_{H_{\mathrm{SBH}}}(M)}\left(1-\frac{\omega}{R_{\mathrm{SBH}}(M)}\right)\right] \\ &\sim \exp\left[+\Delta S_{\mathrm{SBH}}(M,\omega)\right]. \end{aligned} \tag{11}$$

*2.2. Schwarzschild Black Hole Effective States and Transitions.* Here, we recall some effective quantities that characterize the SBH.

Given that $M$ is the mass state *before* and $M \pm \omega$ is the mass state *after* the quantum transition, the *SBH effective mass* and *SBH effective horizon* are, respectively, identified in [3–5] as

$$M_E(M,\omega) = \frac{M + (M \pm \omega)}{2} = M \pm \frac{\omega}{2}, \qquad R_{E_{\mathrm{SBH}}}(M,\omega) = 2M_E(M,\omega), \tag{12}$$

which are *average quantities* between the two states *before* and *after* the process [3–5]. Consequently, using (4) and (12) we define the *SBH effective horizon area* as

$$A_{E_{\mathrm{SBH}}}(M,\omega) \equiv \frac{A_{\mathrm{SBH}}(M) + A_{\mathrm{SBH}}(M \pm \omega)}{2} = 16\pi M_E^2(M,\omega) = 4\pi R_{E_{\mathrm{SBH}}}^2(M,\omega), \tag{13}$$

which is the average of the SBH's initial and final horizon areas. Subsequently, utilizing (7), the Bekenstein-Hawking *SBH effective entropy* is defined as

$$S_{E_{\mathrm{SBH}}}(M,\omega) \equiv \frac{S_{\mathrm{SBH}}(M) + S_{\mathrm{SBH}}(M \pm \omega)}{2}, \tag{14}$$

and consequently employs (13) and (14) to define the *SBH effective total entropy* as

$$S_{E_{\mathrm{SBH-total}}}(M,\omega) \equiv S_{E_{\mathrm{SBH}}}(M,\omega) - \ln S_{E_{\mathrm{SBH}}}(M,\omega) + \frac{3}{2A_{E_{\mathrm{SBH}}}(M,\omega)}. \tag{15}$$

Thus, employing (3) and (10), the *SBH effective temperature* is [3–5]

$$T_{E_{\mathrm{SBH}}}(M,\omega) = \left(\frac{T_{H_{\mathrm{SBH}}}^{-1}(M) + T_{H_{\mathrm{SBH}}}^{-1}(M \pm \omega)}{2}\right)^{-1} = \left(8\pi\left[\frac{M + M \pm \omega}{2}\right]\right)^{-1} = \frac{1}{4\pi(2M \pm \omega)} = \frac{1}{8\pi M_E(M,\omega)}, \tag{16}$$

which is the inverse of the average value of the inverses of the initial and final Hawking temperatures. Consequently, (16) lets one rewrite (11) to define the *SBH effective emission tunneling rate* (in the Boltzmann-like form) as [3–5]

$$\Gamma_{E_{\mathrm{SBH}}}(M,\omega) \sim \exp\left[-\frac{\omega}{T_{E_{\mathrm{SBH}}}(M,\omega)}\right] = \exp\left[+\Delta S_{E_{\mathrm{SBH}}}(M,\omega)\right], \tag{17}$$

such that (14) defines the *SBH effective entropy change* as

$$\Delta S_{E_{\mathrm{SBH}}}(M,\omega) = S_{\mathrm{SBH}}(M \pm \omega) - S_{\mathrm{SBH}}(M) = \frac{\Delta A_{E_{\mathrm{SBH}}}(M,\omega)}{4} \tag{18}$$

because the *SBH effective horizon area change* is

$$\Delta A_{E_{\mathrm{SBH}}}(M,\omega) = 16\pi M_E(M,\omega)\,\omega \tag{19}$$

and the *SBH effective area quanta number* is

$$N_{E_{\mathrm{SBH}}}(M,\omega) = \frac{A_{E_{\mathrm{SBH}}}(M,\omega)}{\Delta A_{E_{\mathrm{SBH}}}(M,\omega)}. \tag{20}$$

*2.3. Effective Application of Quasi-Normal Modes to the Schwarzschild Black Hole.* Here, we recall how the SBH perturbation field QNM states can be applied to the SBH effective framework.

The quasi-normal frequencies (QNFs) are typically labeled as $\omega_{nl}$, where $l$ is the angular momentum quantum number [3–5, 15, 19]. Thus, for each $l$, such that $l \geq 2$ for gravitational perturbations, there is a countable sequence of QNMs labeled by the overtone number $n$, which is a natural number [3–5, 15].

Now $|\omega_n|$ is the damped harmonic oscillator's proper frequency that is defined as [3–5, 15]

$$|\omega_n| = (\omega_0)_n = \sqrt{\omega_{n_{\mathbb{R}}}^2 + \omega_{n_{\mathbb{I}}}^2}. \tag{21}$$

Maggiore [15] articulated that the establishment $|\omega_n| = \omega_{n_{\mathbb{R}}}$ is only correct for the very long-lived and lowly excited QNMs approximation $|\omega_n| \gg \omega_{n_{\mathbb{I}}}$, whereas for a lot of BH QNMs, such as those that are highly excited, the opposite limit is correct [3–5, 15]. Therefore, the $\omega$ parameter in (12)–(20) is substituted for the $|\omega_n|$ parameter [3–5] because we wish to employ BH QNFs. When $n$ is large, the SBH QNFs become independent of $l$ and thereby exhibit the nonstrictly thermal structure [3–5]

$$\omega_n = \ln 3 \times T_{E_{\mathrm{SBH}}}(M,|\omega_n|) + 2\pi i\left(n + \frac{1}{2}\right) \times T_{E_{\mathrm{SBH}}}(M,|\omega_n|) + \mathcal{O}\left(n^{-1/2}\right) = \frac{\ln 3}{4\pi[2M - |\omega_n|]} + \frac{2\pi i}{4\pi[2M - |\omega_n|]} \times \left(n + \frac{1}{2}\right) + \mathcal{O}\left(n^{-1/2}\right) = \frac{\ln 3}{8\pi M_E(M,|\omega_n|)} + \frac{2\pi(n + 1/2)}{8\pi M_E(M,|\omega_n|)} i + \mathcal{O}\left(n^{-1/2}\right), \tag{22}$$

where

$$m_n \equiv \omega_{n_{\mathbb{R}}} = \frac{\ln 3}{8\pi M_E(M,|\omega_n|)}, \qquad p_n \equiv \omega_{n_{\mathbb{I}}} = \frac{2\pi}{8\pi M_E(M,|\omega_n|)}\left(n + \frac{1}{2}\right). \tag{23}$$

Thus, when referring to highly excited QNMs one gets $|\omega_n| \approx p_n$ [3–5], where the quantized levels differ from [15] because they are not equally spaced in exact form. Therefore, according to [3–5], we have

$$\begin{aligned} |\omega_n| &= \frac{\sqrt{(\ln 3)^2 + 4\pi^2(n+1/2)^2}}{8\pi M_E(M, |\omega_n|)} \\ &= T_{E_{\mathrm{SBH}}}(M, |\omega_n|)\sqrt{(\ln 3)^2 + 4\pi^2\left(n+\frac{1}{2}\right)^2}, \end{aligned} \tag{24}$$

which is solved to yield

$$|\omega_n| = M - \sqrt{M^2 - \frac{\sqrt{(\ln 3)^2 + 4\pi^2(n+1/2)^2}}{4\pi}} \tag{25}$$

when we obey $|\omega_n| < M$ because a BH cannot emit more energy than its total mass.

## 3. Reissner-Nordström Black Hole Framework: An Introduction

We note that for this framework we consider the *RNBH event horizon* features, which are derived from the $R_{+_{\mathrm{RNBH}}}(M, Q)$ in (1).

*3.1. Reissner-Nordström Black Hole States and Transitions.* Here, we recall some quantities that characterize the RNBH.

First, consider a RNBH of initial mass $M$ and initial charge $Q$. Using (1), we define the *RNBH initial event horizon area* as

$$A_{+_{\mathrm{RNBH}}}(M, Q) = 4\pi\left(M + \sqrt{M^2 - Q^2}\right)^2 = 4\pi R^2_{+_{\mathrm{RNBH}}}(M, Q), \tag{26}$$

the *Bekenstein-Hawking RNBH initial entropy* as

$$S_{+_{\mathrm{RNBH}}}(M, Q) = \frac{A_{+_{\mathrm{RNBH}}}(M, Q)}{4}, \tag{27}$$

and the *RNBH initial electrostatic potential* as

$$\Phi_+(M, Q) = \frac{Q}{4\pi R_{+_{\mathrm{RNBH}}}(M, Q)} = \frac{Q}{4\pi\left(M + \sqrt{M^2 - Q^2}\right)}. \tag{28}$$

Consequently, (17) of [2] identifies the *RNBH initial Hawking temperature* as

$$\begin{aligned} T_{+H_{\mathrm{RNBH}}}(M, Q) &= \frac{\sqrt{M^2 - Q^2}}{2\pi\left(M + \sqrt{M^2 - Q^2}\right)^2} \\ &= \frac{R_{+_{\mathrm{RNBH}}}(M, Q) - R_{-_{\mathrm{RNBH}}}(M, Q)}{A_{+_{\mathrm{RNBH}}}(M, Q)}. \end{aligned} \tag{29}$$

Second, consider when the RNBH emits or absorbs a quantum of energy-frequency $\omega$ with charge $q$ to achieve a final mass of $M - \omega$ or $M + \omega$ and a final charge of $Q - q$ or $Q + q$, respectively, for the RNBH mass-energy transition between states in state space. For this, all we need to do is replace the RNBH's mass and charge parameters in (26) and (29). Thus, (26) establishes the *RNBH final event horizon area* as

$$\begin{aligned} &A_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q) \\ &\quad = 4\pi R^2_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q) \\ &\quad = 4\pi\left((M \pm \omega) + \sqrt{(M \pm \omega)^2 - (Q \pm q)^2}\right)^2. \end{aligned} \tag{30}$$

Equation (27) presents the *Bekenstein-Hawking RNBH final entropy* as

$$S_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q) = \frac{A_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q)}{4}, \tag{31}$$

and (28) defines the *RNBH final electrostatic potential* as

$$\begin{aligned} &\Phi_+(M \pm \omega, Q \pm q) \\ &\quad = \frac{Q}{4\pi R_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q)} \\ &\quad = \frac{Q}{4\pi\left((M \pm \omega) + \sqrt{(M \pm \omega)^2 - (Q \pm q)^2}\right)} \end{aligned} \tag{32}$$

for usage in (29) of [20], where it is proposed that the *RNBH adiabatic invariant* is

$$\begin{aligned} I_{+_{\mathrm{RNBH}}}(M, \omega, Q, q) &= \int \frac{\omega - \Phi_+(M \pm \omega, Q)\, q}{\omega} \\ &= \int \frac{\Delta M - \Phi_+(M \pm \Delta M, Q)\, \Delta Q}{\Delta M} \end{aligned} \tag{33}$$

because $\Delta Q = q$. Hence, (29) identifies the *RNBH final Hawking temperature* as

$$\begin{aligned} &T_{+H_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q) \\ &\quad = \frac{R_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q) - R_{-_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q)}{A_{+_{\mathrm{RNBH}}}(M \pm \omega, Q \pm q)} \\ &\quad = \frac{\sqrt{(M \pm \omega)^2 - (Q \pm q)^2}}{2\pi\left((M \pm \omega) + \sqrt{(M \pm \omega)^2 - (Q \pm q)^2}\right)^2}. \end{aligned} \tag{34}$$

Next, upon generalizing (16) in [2] and the work [21], we define the *RNBH tunneling rate* as

$$\begin{aligned} &\Gamma_{+_{\mathrm{RNBH}}}(M, \omega, Q, q) \\ &\quad \sim \exp\Big[-4\pi\Big(2\omega\left(M \pm \frac{\omega}{2}\right) \\ &\qquad - (M \pm \omega)\sqrt{(M \pm \omega)^2 - (Q \pm q)^2} \\ &\qquad + M\sqrt{M^2 - Q^2}\Big)\Big] \\ &\quad \sim \exp\left[\Delta S_{+_{\mathrm{RNBH}}}(M, \omega, Q, q)\right], \end{aligned} \tag{35}$$

where we utilize (30) to define the *Bekenstein-Hawking RNBH entropy change* as

$$\Delta S_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)=\frac{\Delta A_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)}{4}, \tag{36}$$

such that the *RNBH event horizon area change* is

$$\begin{aligned}\Delta A_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)&=A_{+_{\mathrm{RNBH}}}(M\pm\omega,Q\pm q)\\&\quad-A_{+_{\mathrm{RNBH}}}(M,Q)\end{aligned} \tag{37}$$

so we can define the *RNBH event horizon area quanta number* as

$$N_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)=\frac{A_{+_{\mathrm{RNBH}}}(M,Q)}{\left|\Delta A_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)\right|}. \tag{38}$$

*3.2. Reissner-Nordström Black Hole Effective States and Transitions.* Here, we define some effective quantities that characterize the RNBH.

The *RNBH effective mass* is equivalent to the SBH effective mass component of (12), which is

$$M_E(M,\omega)\equiv\frac{M+(M\pm\omega)}{2}. \tag{39}$$

Next, we define the *RNBH effective charge* as

$$Q_E(Q,q)\equiv\frac{Q+(Q\pm q)}{2}, \tag{40}$$

which is the average of the RNBH's initial charge $Q$ and final charge $Q\pm q$. From this, (1), (39), and (40) are used to define the corresponding *RNBH effective event horizon* and *RNBH effective Cauchy horizon* as

$$\begin{aligned}r_{\pm E}&\equiv R_{\pm E_{\mathrm{RNBH}}}(M,\omega,Q,q)\equiv M_E(M,\omega)\\&\quad\pm\sqrt{M_E^2(M,\omega)-Q_E^2(Q,q)},\end{aligned} \tag{41}$$

with respect to the energy conservation and pair production neutrality of (39). Next, we employ (26), (39), and (41) to define the *RNBH effective event horizon area* as

$$\begin{aligned}&A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\\&\quad\equiv 4\pi R^2_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\\&\quad\equiv 4\pi\left(M_E(M,\omega)+\sqrt{M_E^2(M,\omega)-Q_E^2(Q,q)}\right)^2,\end{aligned} \tag{42}$$

which is then used to define the *RNBH effective entropy* as

$$S_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\equiv\frac{A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)}{4}. \tag{43}$$

Afterwards, we use (28) and (42) to define the *RNBH effective electrostatic potential* as

$$\begin{aligned}&\Phi_{+E}(M,\omega,Q,q)\\&\quad\equiv\frac{Q_E(Q,q)}{4\pi R_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)}\\&\quad\equiv\frac{Q_E(Q,q)}{4\pi\left(M_E(M,\omega)+\sqrt{M_E^2(M,\omega)-Q_E^2(Q,q)}\right)}\end{aligned} \tag{44}$$

so we can utilize the $T_{E_{\mathrm{SBH}}}(M,\omega)$ in (16) along with (39), (40), and (44) to define the *RNBH effective adiabatic invariant* as

$$\begin{aligned}&I_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\\&\quad\equiv\int\frac{dM_E(M,\omega)-\Phi_{+E}(M,\omega,Q,q)\,dQ_E(Q,q)}{T_{E_{\mathrm{SBH}}}(M,\omega)}.\end{aligned} \tag{45}$$

At this point, (16) and (35) let us introduce and define the *RNBH effective temperature* as

$$\begin{aligned}&T_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\\&\quad\equiv\frac{\sqrt{(M\pm\omega/2)^2-(Q\pm q/2)^2}}{2\pi\left[(M\pm\omega/2)+\sqrt{(M\pm\omega/2)^2-(Q\pm q/2)^2}\right]^2}\\&\quad\equiv\frac{\sqrt{M_E^2(M,\omega)-Q_E^2(Q,q)}}{2\pi\left(M_E(M,\omega)+\sqrt{M_E^2(M,\omega)-Q_E^2(Q,q)}\right)^2}\\&\quad\equiv\frac{R_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)-R_{-E_{\mathrm{RNBH}}}(M,\omega,Q,q)}{A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)},\end{aligned} \tag{46}$$

which authorizes us to exercise (36) and (46) to rewrite (35) to define the RNBH effective tunneling rate as

$$\begin{aligned}\Gamma_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)&\sim\exp\left[\frac{\pm\omega}{T_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)}\right]\\&\sim\exp\left[\Delta S_{+_{\mathrm{RNBH}}}(M,\omega,Q,q)\right],\end{aligned} \tag{47}$$

such that the *RNBH effective entropy change* is defined as

$$\Delta S_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\equiv\frac{\Delta A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)}{4} \tag{48}$$

for the *RNBH effective event horizon area change*

$$\Delta A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\equiv\frac{2\omega q+Q^3\pi}{(M^2-Q^2)^{3/2}} \tag{49}$$

and the *RNBH effective event horizon area quanta number*

$$N_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\equiv\frac{A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)}{\left|\Delta A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)\right|}. \tag{50}$$

## 4. Effective Application of Quasi-Normal Modes to the Reissner-Nordström Black Hole

Here, we explain how the RNBH perturbation field QNM states can be applied to the RNBH effective framework.

Similarly to SBH QNFs, the RNBH QNFs become independent of $l$ for large $n$ [22]. Thus, for large $n$, we have two families of the QNM:

$$\omega_n = \ln 3 \times T_{+H_{\mathrm{SBH}}}(M,Q) - 2\pi\left(n+\frac{1}{2}\right) i \times T_{+H_{\mathrm{SBH}}}(M,Q) + \frac{qQ}{R_{+_{\mathrm{SBH}}}(M,Q)}, \tag{51}$$

$$\omega_n = \ln 2 \times T_{+H_{\mathrm{RNBH}}}(M,Q) - 2\pi\left(n+\frac{1}{2}\right) i \times T_{+H_{\mathrm{RNBH}}}(M,Q) + \frac{qQ}{R_{+_{\mathrm{RNBH}}}(M,Q)} = \frac{\ln 2\sqrt{M^2-Q^2}}{2\pi\left(M+\sqrt{M^2-Q^2}\right)^2} - \frac{(n+1/2)\sqrt{M^2-Q^2}}{\left(M+\sqrt{M^2-Q^2}\right)^2} i + \frac{qQ}{R_{+_{\mathrm{RNBH}}}(M,Q)}. \tag{52}$$

Now the approximation of (51) and (52) is only relevant under the assumption that the BH radiation spectrum is strictly thermal [3–5] because they both use the Hawking temperature $T_{+H_{\mathrm{RNBH}}}$ in (29). Hence, to operate in compliance with [3–5] and thereby account for the thermal spectrum deviation of (35), we opt to select the (52) case and upgrade it by effectively replacing its $T_{H_{\mathrm{RNBH}}}$ in (29) with the $T_{+E_{\mathrm{RNBH}}}$ in (46). Therefore, the corrected expression for the RNBH QNFs of (52) which encodes the nonstrictly thermal behavior of the radiation spectrum is defined as

$$\begin{aligned}\omega_n &\equiv \ln 2 \times T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) - 2\pi\left(n+\frac{1}{2}\right) i \times T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) + \frac{qQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)} \\ &\equiv \frac{\ln 2\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}}{2\pi\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2} - \frac{(n+1/2)\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}}{\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2} i \\ &\quad + \frac{qQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)}.\end{aligned} \tag{53}$$

From (39), (41), and (46) we define the effective quantities associated with the QNMs as

$$M_E(M,|\omega_n|) \equiv \frac{M+(M-|\omega_n|)}{2}, \tag{54}$$

$$\begin{aligned} r_{\pm E} &\equiv R_{\pm E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) = \\ &= M_E(M,|\omega_n|) \pm \sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)},\end{aligned} \tag{55}$$

$$\begin{aligned} T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) &\equiv \frac{\sqrt{(M-|\omega_n|/2)^2-(Q-q/2)^2}}{2\pi\left[(M-|\omega_n|/2)+\sqrt{(M-|\omega_n|/2)^2-(Q-q/2)^2}\right]^2} \\ &= \frac{\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}}{2\pi\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2} \\ &= \frac{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)-R_{-E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)}{A_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)},\end{aligned} \tag{56}$$

respectively, for the quantum overtone number $n$ in (53). Hence, (53) lets us rewrite the SBH case of (23) to present the RNBH case

$$\begin{aligned} m_n &\equiv \ln 2 \times T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) + \frac{eQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)} \\ &= \frac{\ln 2\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}}{2\pi\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2} + \frac{qQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)}, \\ p_n &\equiv -2\pi\left(n+\frac{1}{2}\right) \times T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) \\ &= -\frac{(n+1/2)\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}}{\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2}.\end{aligned} \tag{57}$$

Thus, we recall that if $|\omega_n| \approx p_n$, then we are referring to highly excited QNMs [3–5]. Therefore, the SBH case of (24) becomes the RNBH case

$$\begin{aligned}|\omega_n| &\equiv \frac{\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\sqrt{(\ln 2)^2-4\pi^2(n+1/2)^2}}{2\pi\left(M_E(M,|\omega_n|)+\sqrt{M_E^2(M,|\omega_n|)-Q_E^2(Q,q)}\right)^2} \\ &\quad + \frac{qQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)} = T_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q) \\ &\quad \times \sqrt{(\ln 2)^2-4\pi^2\left(n+\frac{1}{2}\right)^2} + \frac{qQ_E(Q,q)}{R_{+E_{\mathrm{RNBH}}}(M,|\omega_n|,Q,q)}.\end{aligned} \tag{58}$$

Hence, upon considering (40) and (54), one can rewrite (58) as

$$\begin{aligned}|\omega_n| &\equiv \frac{\sqrt{(M-|\omega_n|/2)^2-(Q-q/2)^2}\sqrt{(\ln 2)^2-4\pi^2(n+1/2)^2}}{2\pi\left[(M-|\omega_n|/2)+\sqrt{(M-|\omega_n|/2)^2-(Q-q/2)^2}\right]^2} \\ &\quad + \frac{q(Q-q/2)}{(M-|\omega_n|/2)+\sqrt{(M-|\omega_n|/2)^2-(Q-q/2)^2}},\end{aligned} \tag{59}$$

where the solution of (59) in terms of $|\omega_n|$ will be the answer of $|\omega_n|$. Therefore, given a quantum transition between the levels $n$ and $n-1$, we define $|\Delta\omega_{n,n-1}| \equiv |\omega_n - \omega_{n-1}|$ where (41)–(45) are rewritten as

$$\begin{aligned}r_{\pm E} &\equiv R_{\pm E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv M_E(M,|\Delta\omega_{n,n-1}|) \\ &\quad \pm \sqrt{M_E^2(M,|\Delta\omega_{n,n-1}|)-Q_E^2(Q,q)}, \\ A_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv 4\pi R^2_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv 4\pi\left(M_E(M,|\Delta\omega_{n,n-1}|)\right. \\ &\quad \left.+\sqrt{M_E^2(M,|\Delta\omega_{n,n-1}|)-Q_E^2(Q,q)}\right)^2, \\ S_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \equiv \frac{A_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)}{4}, \\ \Phi_{+E}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv \frac{Q_E(Q,q)}{4\pi R_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)} \\ &\equiv \frac{Q_E(Q,q)}{4\pi\left(M_E(M,|\Delta\omega_{n,n-1}|)+\sqrt{M_E^2(M,|\Delta\omega_{n,n-1}|)-Q_E^2(Q,q)}\right)}, \\ I_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &= \int \frac{dM_E(M,|\Delta\omega_{n,n-1}|)-\Phi_{+E}(M,|\Delta\omega_{n,n-1}|,Q,q)\,dQ_E(Q,q)}{T_{E_{\mathrm{SBH}}}(M,|\Delta\omega_{n,n-1}|)},\end{aligned} \tag{60}$$

and (47)–(50) become

$$\begin{aligned}\Gamma_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\sim \exp\left[\frac{\pm|\Delta\omega_{n,n-1}|}{T_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)}\right] \\ &\sim \exp\left[\Delta S_{+_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)\right], \\ \Delta S_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv \frac{\Delta A_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)}{4}, \\ \Delta A_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \equiv \frac{2|\Delta\omega_{n,n-1}|q+\pi Q^3}{(M^2-Q^2)^{3/2}}, \\ N_{+E_{\mathrm{RNBH}}}&(M,|\Delta\omega_{n,n-1}|,Q,q) \\ &\equiv \frac{A_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)}{\left|\Delta A_{+E_{\mathrm{RNBH}}}(M,|\Delta\omega_{n,n-1}|,Q,q)\right|},\end{aligned} \tag{61}$$

respectively.

## 5. A Brief Comparison

Here, we will show that the SBH results of Section 2 are in fundamental agreement with the RNBH results of Section 3 for small $Q$, where we recall that the RNBH of mass $M$ is identical to a SBH of mass $M$ except that a RNBH has the nonzero charge quantity $Q$.

First, for small $Q$, the SBH's $T_{E_{\mathrm{SBH}}}(M,\omega)$ of (16) is related to the RNBH's $T_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)$ of (46) as

$$\begin{aligned}T_{+E_{\mathrm{RNBH}}}&(M,\omega,Q,q) \\ &\equiv T_{E_{\mathrm{SBH}}}(M,\omega) - \frac{3q^2Q^2}{8(2m\pm\omega)^5\pi} + \mathcal{O}\left(Q^4,q^4\right).\end{aligned} \tag{62}$$

Second, for small $Q$, the SBH's $A_{E_{\mathrm{SBH}}}(M,\omega)$ of (13) complies with the RNBH's $A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)$ of (42) as

$$A_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q) \equiv A_{E_{\mathrm{SBH}}}(M,\omega) - 8\pi Q^2 + \mathcal{O}\left(Q^4\right). \tag{63}$$

Third, for small $Q$, the SBH's $S_{E_{\mathrm{SBH}}}(M,\omega)$ of (14) corresponds with the RNBH's $S_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q)$ of (43) as

$$S_{+E_{\mathrm{RNBH}}}(M,\omega,Q,q) \equiv S_{E_{\mathrm{SBH}}}(M,\omega) - 2\pi Q^2 + \mathcal{O}\left(Q^4\right). \tag{64}$$

Fourth, for small $Q$, the SBH's QNF $|\omega_n|$ of (24) is consistent with the RNBH's QNF $|\omega_n|$ of (58) and (59) as

$$\begin{aligned}
|\omega_n| &\equiv \frac{\sqrt{\ln 2^2 - 4\pi^2(n+1/2)^2}}{4\left(2M - |\omega_n|\right)\pi} + \frac{qQ}{2M - |\omega_n|} \\
&= \Big(\Big(3\Big(16\pi M^2 - 16\pi|\omega_n|M + 4\pi|\omega_n|^2 \\
&\qquad + \sqrt{-(\ln 2 + \pi + 2\pi n)(-\ln 2 + \pi + 2\pi n)}\Big)\Big) \\
&\qquad \times\left(8\left(2M - |\omega_n|\right)^5\pi\right)^{-1}\Big)Q^2q^2 \\
&= \frac{q^2}{4M - |\omega_n|} + \mathcal{O}\left(Q^4, q^4\right),
\end{aligned} \tag{65}$$

which can be applied to (62)-(63) by replacing the $\omega$ parameter with the pertinent $|\omega_n|$. Hence, (62)–(65) indicate that in general the SBH results of Section 2 are fundamentally consistent with the RNBH results of Section 3 for small $Q$. Moreover, in (65) for large $n$, the result is consistent with the SBH because $\ln 2$ is negligible, but for small $n$ there is an argument between scientists regarding $\ln 2$ and $\ln 3$ because these refer to the two distinct QNM families of (51) and (52).

Here, we provide the physical answer of (65) for the case of emission by using the fact that $Q$ is small, so the term which includes $Q^2$ is also very small and therefore negligible:

$$\begin{aligned}
(\omega_0)_n &\equiv |\omega_n| \approx M \\
&\quad - \sqrt{M^2 + \frac{q^2}{2} - Qq - \frac{1}{4\pi}\sqrt{\ln 2^2 - 4\pi^2\left(n + \frac{1}{2}\right)^2}}.
\end{aligned} \tag{66}$$

Thus, by setting $(\omega_0)_n \equiv |\omega_n|$ we obtain

$$\begin{aligned}
\Delta M_n &\equiv -\Delta\omega_{n,n-1} = (\omega_0)_{n-1} - (\omega_0)_n \\
&\equiv \sqrt{M^2 + \frac{q^2}{2} - Qq - \frac{1}{4\pi}\sqrt{(\ln 2)^2 + 4\pi^2\left(n + \frac{1}{2}\right)^2}} \\
&\quad - \sqrt{M^2 + \frac{q^2}{2} - Qq - \frac{1}{4\pi}\sqrt{(\ln 2)^2 + 4\pi^2\left(n - \frac{1}{2}\right)^2}}
\end{aligned} \tag{67}$$

for an emission involving quantum levels $n$ and $n-1$, which becomes

$$\begin{aligned}
\Delta M_n &\approx \sqrt{M^2 + \frac{q^2}{2} - Qq - \frac{1}{2}\left(n + \frac{1}{2}\right)} \\
&\quad - \sqrt{M^2 + \frac{q^2}{2} - Qq - \frac{1}{2}\left(n - \frac{1}{2}\right)}
\end{aligned} \tag{68}$$

for large $n$.

## 6. Conclusion Remarks

We began our paper by summarizing some basic similarities and differences between SBHs and RNBHs in terms of charge and horizon radii. Moreover, we briefly explored the Parikh-Wilczek statement that explains how energy conservation and pair production [2, 23] are fundamentally related to such BHs. For a BH's discrete energy spectrum, the emission or absorption of a particle yields a transition between two distinct levels, where particle emission and absorption are reverse processes [3–7]. For this, we touched on the important issue that the nonstrictly thermal character of Hawking's radiation spectrum generates a natural correspondence between Hawking's radiation and BH QNMs, because these structures exemplify features of the BH's energy spectrum [3–5], which has been recently generalized to the emerging concept of a BH's effective state [6, 7].

Next, we prepared for our nonextremal RNBH QNM investigation by first reviewing relevant portions of the SBH effective framework [3–5] in Section 2. There, we listed the noneffective and effective quantities for SBH states and transitions, with direct application to the QNM characterization and framework of [3–5]. Subsequently, in Section 3, we identified some existing noneffective quantities and introduced new effective quantities for RNBH states and transitions so we could apply the BH framework of [3–5] to implement a RNBH framework. These results are crucial because the effective quantities in [3–5] have been achieved for the stable four-dimensional RNBH solution in Einstein's general relativity—now effective frameworks exist for the SBH, KBH, and (nonextremal) RNBH solutions.

Ultimately, the RNBH effective quantities permitted us to utilize both the KBH's effective state concept [6, 7] and the BH QNMs [3–5] to construct a foundation for the RNBH's effective state in this developing BH effective framework. The RNBH effective state concept is meaningful because, as scientists who wish to demystify the BH paradigm, we need additional features and knowledge to consider in future experiments and observations.

Finally, we stress that the nonstrictly thermal behavior of the Hawking radiation spectrum has been recently used to construct two very intriguing proposals to solve the BH information loss paradox. The first one received the First Award in the 2013 Gravity Research Foundation Essay Competition [12]. The latter won the Community Rating at the 2013 FQXi Essay Contest—It from Bit or Bit from It [24]. We are working to extend this second approach to the RNBH framework [25].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# A Little Quantum Help for Cosmic Censorship and a Step Beyond All That

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The hypothesis of cosmic censorship (CCH) plays a crucial role in classical general relativity, namely, to ensure that naked singularities would never emerge, since it predicts that whenever a singularity is formed an event horizon would always develop around it as well, to prevent the former from interacting directly with the rest of the Universe. Should this not be so, naked singularities could eventually form, in which case phenomena beyond our understanding and ability to predict could occur, since at the vicinity of the singularity both predictability and determinism break down even at the classical (e.g., nonquantum) level. More than 40 years after it was proposed, the validity of the hypothesis remains an open question. We reconsider CCH in both its weak and strong versions, concerning point-like singularities, with respect to the provisions of Heisenberg's uncertainty principle. We argue that the shielding of the singularities from observers at infinity by an event horizon is also quantum mechanically favored, but ultimately it seems more appropriate to accept that singularities never actually form in the usual sense; thus no naked singularity danger exists in the first place.

## 1. Introduction

Singularities, conceived as spacetime regions, where curvature (as described by scalar invariant quantities like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$) blows up to exceed any possible upper bound, are one of the most problematic notions in Physics. After all, strictly speaking, if the spacetime metric is ill-behaved at a certain point, then the latter should not be considered as part of that spacetime in the first place. Nevertheless, it is this metric that we rely on to try to describe the properties of that point. We overcome this incoherence by considering an augmented spacetime that contains such singular points as ideal boundary points attached to the ordinary, well-behaved manifold. Since the spacetime structure breaks down at singularities while, at the same time, physical laws presuppose space and time to develop and manifest themselves, naked singularities would be sources of lawlessness, absurdity, and uncontrollable information, therefore an anathema for our perception of the Universe. Even worse, Hawking and Penrose have shown that the emergence of singularities is inevitable in a very large class of universe types, where sufficiently reasonable conditions are satisfied (the theorem actually goes as follows: let $M$, $g_{ab}$ be a time-oriented spacetime satisfying the following conditions. (A) $R_{ab}V^aV^b \geq 0$ for any nonspace-like $V^a$. (B) The time-like and null generic conditions are fulfilled. (C) There is no closed time-like curve. (D) At least one of the following holds: (Da) there exists a compact achronal set without edge; (Db) there exists a trapped surface; (Dc) there is a $p \in M$ such that the expansion of the future directed null geodesics through $p$ becomes negative along each geodesic. Then $M$, $g_{ab}$ contains at least one incomplete time-like or null geodesic) [1]. Since all of them are redeemed in our Universe too, singularities are expected with certainty to form in the latter as well. In order to deal with these "monsters," Penrose proposed the famous cosmic censorship hypothesis (CCH) [2]. The weak version of the hypothesis (w-CCH) suggests that observers at infinity can never directly see a singularity, the latter being at all times clothed by an absolute event horizon, whereas its strong version (s-CCH) states that an observer cannot have any direct interaction with a singularity at any time or place [3]. Because of cosmic censorship, then, a naked singularity should never occur except, conceivably, for some special configurations, which are not expected to occur in an actual

astrophysical circumstance (for a thorough and enlightening presentation of the issues concerning singularities and CCH see [4–6]).

It should be noted here that CCH does not stem from some well-established physical law or mathematical theorem. Rather, it is a convenient hypothesis that, considering the catastrophic impact of the alternative, we gladly accept as (probably) true. Soon after it was proposed, it was declared as one of the most important open questions in classical general relativity [3], whose derivation remains obscure until now (see [7–13] for reviews on the work done sofar). Initially, arguments supporting the idea were based largely on geometry and issues concerning causality, usually expressed in terms of TIFs and TIPs (terminal indecomposable futures/pasts, resp.) [14] (see again [6]). Penrose was able to derive inequalities involving black hole masses and horizon radii [15] in support of his hypothesis, which interestingly enough were shown to hold true in a series of different situations [16–21]. Moreover, CCH was proven to be valid in various specific spacetimes [22–28]. At the same time, sceptics were trying to construct counterexamples in which naked singularities could emerge [29–37]. However, the majority of those examples presupposed very special and idealized conditions to hold (thus least possible to occur in a realistic universe) to hold, so the credibility of CCH was far from being fatally undermined by them. Soon it was evident that a quantum treatment was necessary. Besides, the problematic way we describe singularities represents much more our lack of understanding their true nature, namely, the laws of quantum gravity that presumably take over when radii of spacetime curvature of the order of Planck length are attained, rather than their actual behavior. It was proposed (and hoped) by many scientists that the inclusion of quantum phenomena in the picture of gravitational collapse would be the answer to all our difficulties to cope with singularities. In fact, quantum mechanics has been proven very successful in resolving many of the counterexample gedankenexperiments in favor of CCH. More specifically it was used to show that it is impossible to overspin or overcharge a maximal Kerr black hole to produce a naked singularity (a procedure first considered in [38]) [39–41]. In this framework, recent results on the correspondence between Hawking radiation and black holes quasinormal modes [42–45] look to be particularly interesting since they stress too the need for a quantum mechanical approach to the black hole properties if we are to gain a deeper understanding of the latter.

Following this line of thinking we try here to engage quantum mechanics in the treatment of point-like singularities lying at a finite distance (as opposed to singularities lying at infinity or thunderbolts). The key idea proposed is to make appeal to Heisenberg's uncertainty principle,

$$\Delta x \cdot \Delta p \geq 1 \quad \text{(in natural units where } G = c = \hbar = 1) \tag{1}$$

which we consider the most fundamental feature of every natural system, and check the constraints it imposes, concerning the properties of systems that involve singularities.

## 2. Weak Censorship Revisited

Point-like singularities are expected to form because of the unstoppable collapse of matter that occurs when a too large mass is concentrated in a too small volume. The volume of these singularities would effectively tend to zero by definition; thus they should occupy a single point of spacetime. In the case of a naked singularity, an observer at infinity (i.e., at sufficiently large distance away from it so as to be in an asymptotically flat region of spacetime) would in principle be able to determine its position with arbitrarily high accuracy by, for example, direct observation. When we make a measurement with uncertainty $\Delta x \to 0$ concerning the position of a quantum system, however, the uncertainty principle states that we have to end up with complete lack of knowledge about its momentum (i.e., $\Delta p \to \infty$), therefore about its energy as well. We argue, though, that this is not the case with naked singularities. Since in principle they can be of arbitrarily large mass, one reasonably expects that the actual procedure of determining their position could not change their momentum significantly. Furthermore, even though quantum gravity is necessary to describe the singularity *per se*, it is legitimate to anticipate that general relativity is sufficiently accurate to describe spacetime at macroscopic distances away from it. Then, it would be possible to "weigh" the singularity by observing potential gravitational lensing effects or through measuring the trajectory, speed, and acceleration of test bodies that get attracted by it and so forth. This way the mass/energy of the singularity would be known with uncertainty at most of the order of the mass itself ($\Delta M \sim M$). All these mean that the existence of a naked singularity, apart from all other undesired consequences, would also violate the uncertainty principle. The conundrum gets settled when the provisions of the w-CCH are taken into account. The existence of an event horizon of radius $r_h \sim M$, which emerges because of the warping of the spacetime continuum by the singularity mass itself and thus exists in every kind of black hole type, means that the actual position of the singularity can be determined with uncertainty at least $\Delta x \sim r_h$. Then we get from (1) that $\Delta p \gtrsim 1/M$ and consequently we find for the singularity energy the inequality $E \gtrsim 1/M^3$, which obviously is perfectly compatible with measuring its mass/energy with $\Delta M \sim M$. In this sense w-CCH not only is necessary to make general relativity self-consistent, but has a strong quantum support as well.

## 3. Strong Censorship Revisited

What about s-CCH then? It is not hard to imagine a situation where a very large and massive system is in question (e.g., the central region of a galaxy); a trapped surface has already formed while observers living on a planet within the trapped region exist and expect quantum mechanics to hold at all times until they crash into the singularity that will develop some time in their future. Even though the soon-to-form singularity would remain unseen by observers at infinity (so w-CCH is satisfied), an observer inside the horizon would actually encounter a naked singularity (being at the same time at a significantly large distance away from it).

All arguments presented in the paragraphs above hold true for this observer too, so a paradox a rises. The s-CCH is established to resolve the paradox by predicting that an observer would never actually see the singularity, but since it does not provide us with a mechanism capable of deterring this interaction, it looks more like the expression of a hope than a constraint imposed by some physical law. The only way out, then, is to admit that the notion of unstoppable collapse is wrong and, consequently, no point-like singularity is formed at all. Quantum effects should get so enhanced, at Planck scales, that they would manage to counterbalance the gravitational contracting forces to stop the collapse and prevent singularities from forming in the way we consider them to do today (e.g., the confinement of matter in an ever-decreasing volume, which means that it would acquire an ever-increasing momentum/energy, according, once again, to the uncertainty principle, so that it would end up behaving like a highly energetic gas whose pressure would constantly grow to counterbalance eventually the contraction, is a plausible mechanism to be explored in a work to come).

This approach, namely, the expectation that no singularity forms eventually, finds good support from a very interesting result by Geroch which crudely goes as follows: when a manifold admits a Cauchy surface (as is the case for the majority of physically reasonable spacetimes), then it also admits a global time function $t$ that increases along every future-oriented time-like curve, which can be chosen so that every $t = \text{const.}$ surface is a Cauchy one. However, Cauchy surfaces cannot intersect the singularity and thus there is no time at which the singularity exists [46].

To sum up, revisiting CCH on the grounds of the uncertainty principle, we arrive at the conclusion that w-CCH should hold true. However, since, by itself, it is insufficient to make the overall picture self-consistent, it is needed that s-CCH also applies. Yet the latter in its turn imposes so strict restrictions; that is, as a way out, one quite naturally arrives to admit that singularities never emerge in the usual sense, rendering CCH, in all its versions, unnecessary in the first place.

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