Journal of Function Spaces

Unique and Non-Unique Fixed Points and their Applications

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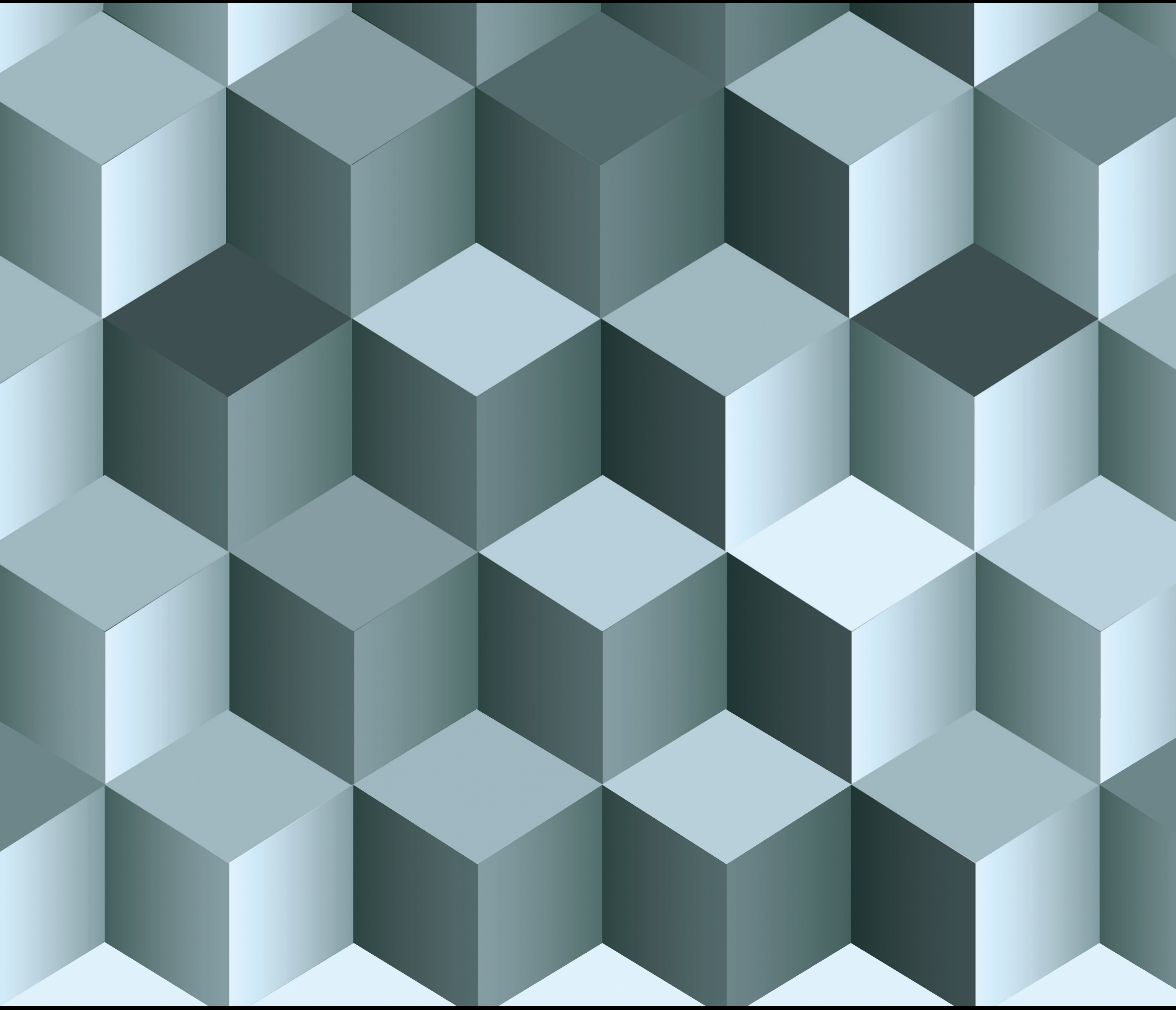

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Research Article

Forbidden Restrictions and the Existence of $P_{\geq 2}$-Factor and $P_{\geq 3}$-Factor

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The existence of factor and fractional factor in network graph in various settings has raised much attention from both mathematicians and computer scientists. It implies the availability of data transmission and network segmentation in certain special settings. In our paper, we consider $P_{\geq 2}$-factor and $P_{\geq 3}$-factor which are two special cases of general $\mathscr{H}$-factor. Specifically, we study the existence of these two kinds of path factor when some subgraphs are forbidden, and several conclusions on the factor-deleted graph, factor critical-covered graph, and factor uniform graph are given with regards to network parameters. Furthermore, we show that these bounds are best in some sense.

1. Introduction

All graphs considered in this work are finite simple graphs. Let $G=(V(G), E(G))$ be a graph, $N_G(v)$ be the neighborhood of vertex v, and $d_G(v)=|N_G(v)|$. Let $\omega(G)$ be the number of connected components in G and $i(G)=|\{v \in V(G)\colon d_G(v)=0\}|$. For the commonly used notations and terminologies, please refer to book [1] by Bondy and Mutry.

Let $n \geq 2$ and $P_{\geq n}$ be a path with at least n vertices. A $P_{\geq n}$-factor is a spanning subgraph of G such that each component is isomorphic to $P_{\geq n}$. A graph G is a $(P_{\geq n}, m)$-factor-deleted graph if removing any m edges from G, the resting subgraph still admits $P_{\geq n}$-factor. For $P_{\geq 2}$-factor, Akiyama et al. [2] demonstrated the following characteristic for its existence.

Lemma 1. *A graph G permits a $P_{\geq 2}$-factor if and only if $2|X| \geq i(G-X)$ established for arbitrary vertex subset X of G.*

More recent results on graph factors in various settings can be referred to Gao et al. [3, 4], Wang and Zhang and Zhou et al. [5–10], and Zhu et al. [11, 12].

A graph R is factor-critical if deleting any vertex v, the resulting subgraph has a perfect matching. A graph G is called a sun if it is isomorphic to K_1, K_2, or the corona of a factor-critical graph, and the last class of sun is a big sun. Let sun(G) be the number of sun components of G. Kaneko [13] and Kano et al. [14] revealed that sun components can describe the existence of $P_{\geq 3}$-factor, i.e, a graph G admits a $P_{\geq 3}$-factor if and only if $2|S| \geq \mathrm{sun}(G-S)$ for any vertex subset S of G.

Zhang and Zhou [15] introduced the concept of $P_{\geq n}$-factor-covered graph, i.e., a graph G is $P_{\geq n}$-factor covered if for any edge e, there is a $P_{\geq n}$-factor containing e. Moreover, they obtained the following two conclusions for $P_{\geq n}$-factor-covered graph when $n=2$ or 3.

Lemma 2 (Zhang and Zhou [15]). *A connected graph G is a $P_{\geq 2}$-factor-covered graph if and only if*

$$i(G-S) \leq 2|S| - \varepsilon_1(S), \tag{1}$$

for any vertex subset S of G, where

$$\varepsilon_1(S) = \begin{cases} 2, & \text{if } S \text{ is not an independent set}, \\ 1, & S \text{ is independent, and there exists a} \\ & \text{nontrivial component of } G - S, \\ 0, & \text{otherwise}. \end{cases} \quad (2)$$

Lemma 3 (Zhang and Zhou [15]). *Assume G as a connected graph. Then, G is a $P_{\geq 3}$-factor-covered graph if and only if*

$$sun(G - S) \leq 2|S| - \varepsilon_2(S), \quad (3)$$

for any $S \subseteq V(G)$, where

$$\varepsilon_2(S) = \begin{cases} 2, & \text{if } S \text{ is not an independent set}, \\ 1, & S \text{ is independent and there exists a} \\ & \text{nonsun component of } G - S, \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

The concept of factor-covered graph can be further extended to factor-critical-covered graph. A graph G is $(P_{\geq n}, k)$-factor-critical covered if deleted any k vertices from G, and the resting subgraph is still a $P_{\geq n}$-factor-covered graph.

In computer data communication networks, there are three main indices to test the robustness and vulnerability of networks, and also, there are some variables of these parameters.

(i) Chvátal [16] firstly introduced toughness where $t(G) = +\infty$ if G is complete; otherwise

$$t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} \,|\, \omega(G - S) \geq 2 \right\}. \quad (5)$$

Enomoto et al. [17] introduced a variant of toughness by revising the denominator to $\omega(G - S) - 1$ and denoted it by $\tau(G)$. That is to say, $\tau(G) = +\infty$ if G is a complete graph; and

$$\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S) - 1} \,|\, \omega(G - S) \geq 2 \right\}, \quad (6)$$

for noncomplete graph.

(ii) Isolated toughness was introduced by Yang et al. [18] as follows: if G is a complete graph, then $I(G) = +\infty$; elsewise

$$I(G) = \min \left\{ \frac{|S|}{i(G - S)} \,|\, S \subset V(G), i(G - S) \geq 2 \right\}. \quad (7)$$

Similar to τ, Zhang and Liu [19] introduced a variant of isolated toughness by revising the denominator to $i(G - S) - 1$, denoted by $I'(G)$: $I'(G) = +\infty$ for a complete graph G, and

$$I'(G) = \min \left\{ \frac{|S|}{i(G - S) - 1} \,|\, S \subset V(G), i(G - S) \geq 2 \right\}, \quad (8)$$

for others.

(iii) Binding number is defined by Woodall [20] which is formulated by

$$\text{bind}(G) = \min \left\{ \frac{|N_G(X)|}{|X|} \,|\, \varnothing \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}. \quad (9)$$

The main contributions of this article are three folded: (1) the relationships between $(P_{\geq 2}, m)$-factor-deleted graph and the above three parameters are studied; (2) toughness conditions for $(P_{\geq 2}, k)$-factor-critical covered and $(P_{\geq 3}, k)$-factor-critical covered graph are given; (3) toughness bounds for a graph to be $P_{\geq 2}$-factor uniform graph and $P_{\geq 3}$-factor uniform graph are determined. The main conclusions and detailed proofs are manifested in the next section, and then, in the third section, we present the sharpness of these bounds.

2. Main Results and Proofs

The purpose of this section is to present the main theorems and detailed proofs.

2.1. Bounds for $(P_{\geq 2}, m)$-Factor-Deleted Graphs

Theorem 4. *Let m be a positive integer and G be an $(m + 1)$-edge-connected graph. If $t(G) > m/m + 1$ (resp. $\tau(G) > 1$) then G is a $(P_{\geq 2}, m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly we have $|V(G)| \geq m + 2$. □

For arbitrary edge subset $E' = \{e_1, \cdots, e_m\}$ with m edges, let $G' = G - E'$, and we have $V(G') = V(G)$ and $E(G') = E(G) - E'$. We verify the theorem by means of proving that G' admits $P_{\geq 2}$-factor. In contrast, we assume G' has no $P_{\geq 2}$-factor, and hence, in view of Lemma 1, there is a subset S of $V(G')$ satisfying

$$i\left(G' - S\right) \geq 2|S| + 1. \quad (10)$$

If $|S| = 0$, then $i(G') \geq 1$ by (1) which contradicts to G is $(m + 1)$-edge-connected and $|V(G)| \geq m + 2$. Therefore, we infer $|S| \geq 1$ and $i(G' - S) \geq 2|S| + 1 \geq 3$. Deleting one edge from $G - S$, the number of its components adds most 1, thus $\omega(G' - S) = \omega(G - E - S) \leq \omega(G - S) + m$.

We divide $E' = \{e_i\}_{i=1}^{m}$ into three classes E'_1, E'_2, and E'_3.

If $e_i \in E'$ is a unique edge in K_2 which is a component in $G-S$, then $e_i \in E_1'$.

If $e_i \in E'$ and $e_i \in E(G-S)$, one of end vertex of e_i (say v_i) meets $d_{G-S}(v_i) \geq 2$, then $e_i \in E_2'$.

Otherwise, $e_i \in E'$ and at least one of its end vertices in S, then $e_i \in E_3'$.

We have $|E_1'| + |E_2'| \leq m$ and $|E_1'|, |E_2'| \in \{0, \cdots, m\}$. Select one vertex in each edge in E_2' with larger degree in $G-S$ and denote X by the set of these vertices. Thus, $|X| \leq |E_2'|$.

According to

$$\frac{m}{m+1} < t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{\omega\left(G'-S\right)-m} \leq \frac{|S|}{i\left(G'-S\right)-m} \leq \frac{|S|}{2|S|+1-m}, \tag{11}$$

or accordingly

$$1 < \tau(G) \leq \frac{|S|}{\omega(G-S)-1} \leq \frac{|S|}{\omega\left(G'-S\right)-m-1} \leq \frac{|S|}{i\left(G'-S\right)-m-1} \leq \frac{|S|}{2|S|-m}, \tag{12}$$

we get $|S| \in \{1, \cdots, m-1\}$.

For $t(G)$, we have

$$\begin{aligned} \frac{m}{m+1} < t(G) &\leq \frac{|S \cup X|}{\omega(G-S\cup X)} = \frac{|S|+|X|}{\omega\left(G'-S\cup X\right)-\left|E_1'\right|} \\ &\leq \frac{|S|+|X|}{\omega\left(G'-S\right)-\left|E_1'\right|} \leq \frac{|S|+\left|E_2'\right|}{i\left(G'-S\right)-\left|E_1'\right|} \\ &\leq \frac{|S|+m-\left|E_1'\right|}{i\left(G'-S\right)-\left|E_1'\right|} \leq \frac{|S|+m-\left|E_1'\right|}{2|S|+1-\left|E_1'\right|}. \end{aligned} \tag{13}$$

Let $f(|E_1'|) = (|S| + m - |E_1'|)/(2|S| + 1 - |E_1'|)$ be a function with regard to $|E_1'|$. We have

$$f'\left(\left|E_1'\right|\right) = \frac{\left(2|S|+1-\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)' - \left(2|S|+1-\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|+1-\left|E_1'\right|\right)^2} = \frac{m-1-|S|}{\left(2|S|+1-\left|E_1'\right|\right)^2} \geq 0. \tag{14}$$

Hence, $f(|E_1'|)$ is a monotonically increasing function and $\max\{f(|E_1'|)\} = f(m)$. We get

$$\begin{aligned} \frac{m}{m+1} < t(G) &\leq \frac{|S|}{2|S|+1-m} = \frac{1}{2} + \frac{m/2-1/2}{2|S|+1-m} \\ &\leq \frac{1}{2} + \frac{m/2-1/2}{2+1-m} = \frac{-m^2+4m-2}{2}, \end{aligned} \tag{15}$$

which implies $m = 2$.

If $m = 2$, then $|S| = 1$ and $i(G'-S) \geq 2|S| + 1 = 3$. If $\omega(G-S) \geq 2$, then $2/3 = m/(m+1) < t(G) \leq |S|/\omega(G-S) \leq 1/2$, a contradiction. Hence, $G-S$ is a connected graph, and there are at least 3 isolated vertices after removing 2 edges from $G-S$. That is to say, $G = K_1 \vee P_3$ which contradicts to G is a 3-edge-connected graph.

For $\tau(G)$, we have

$$\begin{aligned} 1 < \tau(G) &\leq \frac{|S \cup X|}{\omega(G-S\cup X)-1} = \frac{|S|+|X|}{\omega\left(G'-S\cup X\right)-\left|E_1'\right|-1} \\ &\leq \frac{|S|+|X|}{\omega\left(G'-S\right)-\left|E_1'\right|-1} \leq \frac{|S|+\left|E_2'\right|}{i\left(G'-S\right)-\left|E_1'\right|-1} \\ &\leq \frac{|S|+m-\left|E_1'\right|}{i\left(G'-S\right)-\left|E_1'\right|-1} \leq \frac{|S|+m-\left|E_1'\right|}{2|S|+1-\left|E_1'\right|-1} \\ &= \frac{|S|+m-\left|E_1'\right|}{2|S|-\left|E_1'\right|}. \end{aligned} \tag{16}$$

Let $g(|E_1'|) = (|S| + m - |E_1'|)/(2|S| - |E_1'|)$ be a function with regard to $|E_1'|$. We obtain

$$g'\left(\left|E_1'\right|\right) = \frac{\left(2|S|-\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)' - \left(2|S|-\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|-\left|E_1'\right|\right)^2} = \frac{m-|S|}{\left(2|S|-\left|E_1'\right|\right)^2} > 0. \tag{17}$$

Hence, $g(|E_1'|)$ is a monotonically increasing function and $\max\{g(|E_1'|)\}=g(m)$. We get

$$1<\tau(G)\le\frac{|S|}{2|S|-m}=\frac{1}{2}+\frac{m}{2(2|S|-m)}\le\frac{1}{2}+\frac{m}{2(2-m)}=\frac{1}{2-m},\tag{18}$$

which implies $m=2$.

If $m=2$, then $|S|=1$ and $i(G'-S)\ge 2|S|+1=3$. If $\omega(G-S)\ge 2$, then $1<\tau(G)\le|S|/\omega(G-S)-1\le 1$, a contradiction. Hence, $G-S$ is a connected graph, and there are at least three isolated vertices after removing two edges from $G-S$. That is to say, $G=K_1\vee P_3$ which contradicts to G that is a 3-edge-connected graph.

Hence, the proof of result is completed.

Theorem 5. *Let m be a positive integer and G be an $(m+1)$-edge-connected graph. If $I(G)>2m/(m+1)$ (resp. $I'(G)>2$), then, G is a $(P_{\ge 2},m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly, we have $|V(G)|\ge m+2$. □

For arbitrary edge subset $E'=\{e_1,\cdots,e_m\}$ with m edges, let $G'=G-E'$, and we have $V(G')=V(G)$ and $E(G')=E(G)-E'$. We check the correctness of Theorem 5 via proving G' permits $P_{\ge 2}$-factor. If not, we assume G' has no $P_{\ge 2}$-factor, and hence ,using Lemma 1, there is a subset S of $V(G')$ satisfying (1).

If $|S|=0$, then $i(G')\ge 1$ by (1) which contradicts to G being $(m+1)$-edge-connected and $|V(G)|\ge m+2$. Therefore, we infer $|S|\ge 1$ and $i(G'-S)\ge 2|S|+1\ge 3$. Deleting one edge from $G-S$, the number of its isolated vertices adds most 2; thus, $i(G'-S)=i(G-E-S)\le i(G-S)+2m$.

We divide E' into three classes E_1', E_2', and E_3' as described in Theorem 4, and hence, $|E_1'|+|E_2'|\le m$ and $|E_1'|,|E_2'|\in\{0,\cdots,m\}$. Also, we use the same way to select vertex set X, and thus, $|X|\le|E_2'|$.

For $I(G)$, we have

$$\frac{2m}{m+1}<I(G)\le\frac{|S\cup X|}{i(G-S\cup X)}=\frac{|S|+|X|}{i\left(G'-S\cup X\right)-2\left|E_1'\right|}\le\frac{|S|+|X|}{i\left(G'-S\right)-2\left|E_1'\right|}\tag{19}$$

Reset $f(|E_1'|)=(|S|+m-|E_1'|)/(2|S|+1-2|E_1'|)$ be a function with regard to $|E_1'|$. We acquire

$$f'\left(\left|E_1'\right|\right)=\frac{\left(2|S|+1-2\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)'-\left(2|S|+1-2\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|+1-2\left|E_1'\right|\right)^2}=\frac{2m-1}{\left(2|S|+1-\left|E_1'\right|\right)^2}>0.\tag{20}$$

Hence, $f(|E_1'|)$ is a monotonically increasing function and $\max\{f(|E_1'|)\}=f(m)$. Thus, we get

$$\frac{2m}{m+1}<I(G)\le\frac{|S|}{2|S|+1-2m}=\frac{1}{2}+\frac{m-1/2}{2|S|+1-m}\le\frac{1}{2}+\frac{m-1/2}{2+1-2m}=\frac{1}{3-2m},\tag{21}$$

a contradiction.

For $I'(G)$, we have

$$2<I'(G)\le\frac{|S\cup X|}{i(G-S\cup X)-1}=\frac{|S|+|X|}{i\left(G'-S\cup X\right)-2\left|E_1'\right|-1}\le\frac{|S|+|X|}{i\left(G'-S\right)-2\left|E_1'\right|-1}\le\frac{|S|+m-\left|E_1'\right|}{2|S|+1-2\left|E_1'\right|-1}=\frac{|S|+m-\left|E_1'\right|}{2|S|-2\left|E_1'\right|}.\tag{22}$$

Reset $g(|E_1'|)=(|S|+m-|E_1'|)/(2|S|-2|E_1'|)$ be a function with regard to $|E_1'|$. We acquire

$$g'\left(\left|E_1'\right|\right)=\frac{\left(2|S|-2\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)'-\left(2|S|-2\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|-2\left|E_1'\right|\right)^2}=\frac{2m}{\left(2|S|-\left|E_1'\right|\right)^2}>0.\tag{23}$$

Hence, $g(|E_1'|)$ is a monotonically increasing function and $\max\{g(|E_1'|)\} = f(m)$. Thus, we get

$$2 < I'(G) \le \frac{|S|}{2|S| - 2m} = \frac{1}{2} + \frac{m}{2|S| - 2m} \tag{24}$$

a contradiction if $m \ge 2$.

Specially, if $m = 1$, then

$$\begin{aligned} 2 < I'(G) &\le \frac{|S|}{i(G - S) - 1} \le \frac{|S|}{i\left(G' - S\right) - 2m - 1} \\ &\le \frac{|S|}{2|S| + 1 - 3} = \frac{|S|}{2|S| - 2}, \end{aligned} \tag{25}$$

which implies $|S| = 1$. In this case, $i(G' - S) \ge 3$ leads to $i(G - S) \ge i(G' - S) - 2m \ge 1$ which contradicts to G being a 2-edge-connected graph.

Hence, the proof of this result is completed.

Theorem 6. *Let m be a positive integer and G be an $(m+1)$-edge-connected graph. If $\mathrm{bind}(G) > 3/2$, then, G is a $(P_{\ge 2}, m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly, $|V(G)| \ge m + 2$. □

Let $G' = G - E'$ for arbitrary edge subset E' with m edges, and we have $V(G') = V(G)$ and $E(G') = E(G) - E'$. Assume that G' has no $P_{\ge 2}$-factor, and hence, in view of Lemma 1, there is a subset S of $V(G')$ satisfying (1).

If $|S| = 0$, then, $i(G') \ge 1$ by (1) which contradicts to G being $(m+1)$-edge-connected and $|V(G)| \ge m + 2$. Therefore, we infer $|S| \ge 1$ and $i(G' - S) \ge 2|S| + 1 \ge 3$. Deleting one edge from $G - S$, the number of its isolated components adds most 2, thus, $i(G' - S) = i(G - E - S) \le i(G - S) + 2m$.

Note that there are at least 3 isolated vertices after removing m edges from $G - S$. Also, since $\delta(G) \ge \lambda(G) \ge m + 1$, we get $|S| \ge m + 1 - m/i(G' - S) \ge m + 1 - m/(2|S| + 1)$, i.e., $m \le (2|S| + 1)(|S| - 1)/2|S|$. Let X be the vertex set of these isolated vertices in $G' - S$. If $N_G(X) \ne V(G)$, we acquire

$$\begin{aligned} \frac{3}{2} < \mathrm{bind}(G) &\le \frac{|N_G(X)|}{|X|} \le \frac{|S| + 2m}{i\left(G' - S\right)} \le \frac{|S| + 2m}{2|S| + 1} \\ &\le \frac{|S| + 2((2|S| + 1)(|S| - 1)/2|S|)}{2|S| + 1} \\ &= \frac{3}{2} - \frac{1}{|S|} - \frac{1}{2(2|S| + 1)} < \frac{3}{2}, \end{aligned} \tag{26}$$

a contradiction.

Now, we consider $N_G(X) \ne V(G)$. If there is a vertex v in $G - S$ meeting $d_{G-S}(v) = 1$, then, set $uv \in E(G - S)$ and $u \in X$ since $N_G(X) \ne V(G)$. We yield

$$\begin{aligned} \frac{3}{2} < \mathrm{bind}(G) &\le \frac{|N_G(X - \{u\})|}{|X - \{u\}|} \le \frac{|S| + 2m - 1}{i\left(G' - S\right) - 1} \\ &\le \frac{|S| + 2m - 1}{2|S| + 1 - 1} = \frac{|S| + 2m - 1}{2|S|} \\ &\le \frac{|S| + 2((2|S| + 1)(|S| - 1)/2|S|) - 1}{2|S|} \\ &= \frac{3|S|^2 - 2|S| - 1}{2|S|^2} < \frac{3}{2}, \end{aligned} \tag{27}$$

a contradiction.

If each vertex in X has a degree at least 2 in $G - S$, then, we can get the contradiction similar to what discussed above.

Hence, the proof of result is completed.

2.2. Toughness Conditions for $(P_{\ge 2}, k)$-Factor-Critical Covered and $(P_{\ge 3}, k)$-Factor-Critical Covered Graph

Theorem 7. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \ge k + 1$. If $(G) > (k + 2)/3$ (resp.$\tau(G) > (k + 2)/2$), then, G is a $(P_{\ge 2}, k)$-factor critical covered graph.*

Proof. If G is complete, the result follows from $\delta(G) \ge \kappa(G) \ge k + 1$. In what follows, we consider noncomplete graph. □

For any $U \subseteq V(G)$ with $|U| = k$, set $G' = G - U$. To demonstrate G is $(P_{\ge 2}, k)$-factor critical covered, it is enough to prove G' is $P_{\ge 2}$-factor covered. Otherwise, suppose G' is not $P_{\ge 2}$-factor covered; then, according to Lemma 2, there is a vertex subset S of G' such that

$$i\left(G' - S\right) \ge 2|S| - \varepsilon_1(S) + 1. \tag{28}$$

The following discussion is divided into three cases in terms of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, $\varepsilon_1(S) = 0$ and $i(G') \ge 1$ by (2), which contradicts to $\delta(G) \ge \kappa(G) \ge k + 1$.

Case 2. $|S| = 1$.

We consider the following two subcases.

Case 3. $G' - S$ has no nontrivial component.

We infer $\varepsilon_1(S) = 0$ and $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 = 3$. By means of the definition of toughness, we deduce

$$\frac{k + 2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} \le \frac{k + 1}{3}, \tag{29}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} \le \frac{k+1}{2}, \quad (30)$$

a contradiction.

Case 4. $G' - S$ has a nontrivial component.

We yield $\varepsilon_1(S) = 1$, $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 = 2$, and $\omega(G' - S) \ge 3$. Using the definition of toughness, we have

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} \le \frac{k+1}{3}, \quad (31)$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} \le \frac{k+1}{2}, \quad (32)$$

a contradiction.

Case 5. $|S| \ge 2$.

We acquire $\varepsilon_1(S) \le 2$ and $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 \ge 3$. In light of the definition of toughness, we obtain

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{k+|S|}{\omega\left(G' - S\right)} \\ &\le \frac{k+|S|}{i\left(G' - S\right)} \le \frac{k+|S|}{2|S| - \varepsilon_1(S) + 1} \le \frac{k+|S|}{2|S| - 1} \\ &= \frac{1}{2} + \frac{k+1/2}{2|S| - 1} \le \frac{1}{2} + \frac{k+1/2}{2 \times 2 - 1} = \frac{k+2}{3}, \end{aligned} \quad (33)$$

or

$$\begin{aligned} \frac{k+2}{2} < \tau(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{k+|S|}{\omega\left(G' - S\right) - 1} \\ &\le \frac{k+|S|}{i\left(G' - S\right) - 1} \le \frac{k+|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \\ &\le \frac{k+|S|}{2|S| - 2} = \frac{1}{2} + \frac{k+1}{2|S| - 2} \le \frac{1}{2} + \frac{k+1}{2 \times 2 - 2} = \frac{k+2}{2}, \end{aligned} \quad (34)$$

a contradiction.

Therefore, the result follows.

Theorem 8. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \ge k+1$ and $|V(G)| \ge k+3$. If $(G) > (k+2)/3$ (resp.$\tau(G) > (k+2)/2$), then, G is a $(P_{\ge 3}, k)$-factor critical covered graph.*

Proof. If G is a complete graph, then, the result follows from $|V(G)| \ge k+3$. We only consider noncomplete graph in what follows.

For any $U \subseteq V(G)$ with k vertices, let $G' = G - U$, and we aim to prove G' is $P_{\ge 3}$-factor covered. On the contrary, G is not a $P_{\ge 3}$-factor covered graph, and then, by Lemma 3, there is a subset S of $V(G')$ meeting

$$\text{sun}\left(G' - S\right) \ge 2|S| - \varepsilon_2(S) + 1. \quad (35)$$

□

The following discussion is divided into three cases by means of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, we summarize $\varepsilon_2(S) = 0$ and $\text{sun}(G') \ge 1$ by (3). Using $\kappa(G) \ge k+1$ and $|U| = k$, we get $\text{sun}(G') = \omega(G') = 1$. Since $|V(G)| \ge k+3$, we confirm that G' is a big sun. Let R be the factor-critical graph of G' with $|V(R)| \ge 3$ and $v \in V(R)$ be a vertex in R. Using the definition of toughness, we obtain

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup (V(R) - \{v\})|}{\omega(G - U \cup (V(R) - \{v\}))} \\ &= \frac{k + |V(R)| - 1}{|V(R)|} = 1 + \frac{k-1}{|V(R)|} \\ &\le 1 + \frac{k-1}{3} = \frac{k+2}{3}, \end{aligned} \quad (36)$$

or

$$\begin{aligned} \frac{k+2}{2} < \tau(G) &\le \frac{|U \cup (V(R) - \{v\})|}{\omega(G - U \cup (V(R) - \{v\})) - 1} \\ &= \frac{k + |V(R)| - 1}{|V(R)| - 1} = 1 + \frac{k}{|V(R)| - 1} \\ &\le 1 + \frac{k}{3-1} = \frac{k+2}{2}, \end{aligned} \quad (37)$$

a contradiction.

Case 2. $|S| = 1$.

If there is a nonsun component of $G' - S$, we have $\varepsilon_2(S) = 1$, $\text{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 = 2$ by (3), and $\omega(G' - S) \ge \text{sun}(G' - S) + 1$. Directly using the definition of toughness, we yield

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} \\ &\le \frac{k+1}{\text{sun}\left(G' - S\right) + 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1 + 1} = \frac{k+1}{3}, \end{aligned} \quad (38)$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} \le \frac{k+1}{\mathrm{sun}\left(G' - S\right) + 1 - 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1} = \frac{k+1}{2}, \tag{39}$$

a contradiction.

If there is no nonsun component of $G' - S$, we get $\varepsilon_2(S) = 0$, $\mathrm{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 = 3$ by (3), and $\omega(G' - S) = \mathrm{sun}(G' - S)$. In light of the definition of toughness, we infer

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} = \frac{k+1}{\mathrm{sun}\left(G' - S\right)} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1} = \frac{k+1}{3}, \tag{40}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} = \frac{k+1}{\mathrm{sun}\left(G' - S\right) - 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1 - 1} = \frac{k+1}{2}, \tag{41}$$

a contradiction.

Case 3. $|S| \ge 2$.

In this case, we acquire $\varepsilon_2(S) \le 2$ and $\mathrm{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 \ge 3$ in terms of (3). We verify

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} \le \frac{k+|S|}{\mathrm{sun}\left(G' - S\right)} \le \frac{k+|S|}{2|S| - \varepsilon_2(S) + 1} \le \frac{k+|S|}{2|S| - 1} = \frac{1}{2} + \frac{k+1/2}{2|S| - 1} \le \frac{1}{2} + \frac{k+1/2}{2 \times 2 - 1} = \frac{k+2}{3}, \tag{42}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} \le \frac{k+|S|}{\mathrm{sun}\left(G' - S\right) - 1} \le \frac{k+|S|}{2|S| - \varepsilon_2(S) + 1 - 1} \le \frac{k+|S|}{2|S| - 2} = \frac{1}{2} + \frac{k+1}{2|S| - 2} \le \frac{1}{2} + \frac{k+1}{2 \times 2 - 2} = \frac{k+2}{2}, \tag{43}$$

a contradiction.

Hence, Theorem 8 is verified.

Theorem 9. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \ge k+1$. If $(G) > (k+1)/2(resp. I'(G) > k+1)$, then, G is a $(P_{\ge 2}, k)$-factor critical covered graph.*

Proof. If G is complete, we check the theorem using $\delta(G) \ge \kappa(G) \ge k+1$. Hence, we only consider noncomplete graph in the following contents. □

For any $U \subseteq V(G)$ with $|U| = k$, set $G' = G - U$. To demonstrate G that is $(P_{\ge 2}, k)$-factor critical covered, it is enough to prove G' is $P_{\ge 2}$-factor covered. Otherwise, suppose G' is not $P_{\ge 2}$-factor covered; then, using Lemma 2, there is a vertex subset S of G' satisfying (2).

The following discussion is divided into three cases in terms of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, we get contradiction as we discussed in Theorem 7.

Case 2. $|S| = 1$.

We consider the following two subcases.

Case 3. $G' - S$ has no nontrivial component.

We infer $\varepsilon_1(S) = 0$ and $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 = 3$. By means of the definition of isolated toughness, we deduce

$$\frac{k+1}{2} < I(G) \le \frac{|U \cup S|}{i(G - U \cup S)} \le \frac{k+1}{3}, \tag{44}$$

or

$$k+1 < I'(G) \le \frac{|U \cup S|}{i(G - U \cup S) - 1} \le \frac{k+1}{2}, \tag{45}$$

a contradiction.

Case 4. $G' - S$ has nontrivial component.

We yield $\varepsilon_1(S)=1$ and $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1=2$. Using the definition of isolated toughness, we have

$$\frac{k+1}{2}<I(G)\leq\frac{|U\cup S|}{i(G-U\cup S)}\leq\frac{k+1}{2}, \tag{46}$$

or

$$k+1<I'(G)\leq\frac{|U\cup S|}{i(G-U\cup S)-1}\leq k+1, \tag{47}$$

a contradiction.

Case 5. $|S|\geq 2$.

We acquire $\varepsilon_1(S)\leq 2$ and $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 3$. We can get the contradiction using the similar derivation to Theorem 7.

Therefore, we get the desired result.

Theorem 10. *Let $k\in N$ and G be a graph with $\kappa(G)\geq k+1$ and $|V(G)|\geq k+3$. If $I(G)>(k+3)/2$ (resp. $I'(G)>k+3$), then, G is a $(P_{\geq 3},k)$-factor critical covered graph.*

Proof. If G is a complete graph, the result is hold from $|V(G)|\geq k+3$. We only discuss noncomplete graph in the following context.

For any $U\subseteq V(G)$ with k vertices, let $G'=G-U$, and we aim to prove G' is $P_{\geq 3}$-factor covered. On the contrary, G is not a $P_{\geq 3}$-factor covered graph; then, using Lemma 3, there is a subset S of $V(G')$ satisfying (3).

The following discussion is divided into three cases according to how many elements in S. □

Case 1. $|S|=0$.

In this case, similar to what's discussed in Theorem 8, we have $\varepsilon_2(S)=0$ and $\text{sun}(G')=\omega(G')=1$, and G' is a big sun. Let R be the factor-critical of G' with $|V(R)|\geq 3$. Using the definition of $I(G)$, we obtain

$$\begin{aligned}\frac{k+3}{2}<I(G)&\leq\frac{|U\cup V(R)|}{i(G-U\cup V(R))}=\frac{k+|V(R)|}{|V(R)|}\\&=1+\frac{k}{|V(R)|}\leq 1+\frac{k}{3}=\frac{k+3}{3},\end{aligned} \tag{48}$$

or

$$\begin{aligned}k+3<I'(G)&\leq\frac{|U\cup V(R)|}{i(G-U\cup V(R))-1}\\&=\frac{k+|V(R)|}{|V(R)|-1}=1+\frac{k+1}{|V(R)|-1}\\&\leq 1+\frac{k+1}{3-1}=\frac{k+3}{2},\end{aligned} \tag{49}$$

a contradiction.

Case 2. $|S|=1$.

We have $\varepsilon_2(S)\leq 1$. Suppose that there are K_1's, bK_2's, and c big sun components $H_1,\cdots,H_c$ with $|V(H_i)|\geq 6$ in $G'-S$. Hence, $a+b+c=\text{sun}(G'-S)\geq 2|S|-\varepsilon_2(S)+1\geq 2$ by (3). We select one vertex from each K_2 and choose vertex set of factor-critical subgraph of every big sun and then denote X by the vertex set of all these selected vertices. We infer $|X|=b+\sum_{i=1}^{c}|V(H_i)|/2$ and $i(G-U\cup S\cup X)\geq 2$. In terms of the definition of isolated toughness, we yield

$$\begin{aligned}\frac{k+3}{2}<I(G)&\leq\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)}\\&\leq\frac{k+1+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)}.\end{aligned} \tag{50}$$

It implies

$$\begin{aligned}2k+2&>(k+3)a+(k+1)b+(k+1)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\&\geq(k+3)a+(k+1)b+(3k+3)c\\&\geq(k+1)(a+b+c)\geq 2k+2,\end{aligned} \tag{51}$$

a contradiction.

For $I'(G)$, we have

$$\begin{aligned}k+3<I'(G)&\leq\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)-1}\\&=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)-1}\leq\frac{k+1+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)-1}.\end{aligned} \tag{52}$$

It implies

$$\begin{aligned}2k+4&>(k+3)a+(k+2)b+(k+2)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\&\geq(k+3)a+(k+2)b+(3k+6)c\\&\geq(k+2)(a+b+c)\geq 2k+4,\end{aligned} \tag{53}$$

a contradiction.

Case 3. $|S|\geq 2$.

In this case, we acquire $\varepsilon_2(S)\leq 2$ and $a+b+c=\text{sun}(G'-S)\geq 2|S|-\varepsilon_2(S)+1\geq 3$ in terms of (3). Let X be vertex subset defined as Case 2. We verify

$$\begin{aligned}\frac{k+3}{2}<I(G)&\leq\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)}\\&\leq\frac{k+|S|+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)},\end{aligned} \tag{54}$$

that is,

$$
\begin{aligned}
2k+2|S| &> (3+k)a+(k+1)b+(k+1)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\
&\geq (3+k)a+(k+1)b+(3k+3)c\\
&\geq (k+1)(a+b+c)\geq (k+1)(2|S|-\varepsilon_2(S)+1)\\
&\geq (k+1)(2|S|-1).
\end{aligned} \tag{55}
$$

It is implies that $|S|<(3k+1)/2k\leq 2$ since $k\geq 1$, a contradiction.

For $I'(G)$, we confirm

$$
\begin{aligned}
k+3<I'(G) &\leq \frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)-1}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)-1}\\
&\leq \frac{k+|S|+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)-1},
\end{aligned} \tag{56}
$$

which means,

$$
\begin{aligned}
2k+3+|S| &> (3+k)a+(k+2)b+(k+2)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\
&\geq (3+k)a+(k+2)b+(3k+6)c\\
&\geq (k+2)(a+b+c)\geq (k+2)(2|S|-\varepsilon_2(S)+1)\\
&\geq (k+2)(2|S|-1).
\end{aligned} \tag{57}
$$

It implies that $|S|<(3k+5)/(2k+3)\leq 2$, a contradiction.

Hence, Theorem 10 is verified.

Note that $k\neq 0$ in Theorem 10. From Zhou et al. [21], we know that G is a $P_{\geq 3}$-factor covered graph if $I(G)>5/3$, and 5/3 is tight.

2.3. Toughness Conditions for Factor Uniform Graph. A graph G is a $P_{\geq n}$-factor uniform graph if for any two edges e_1 and e_2, G admits a $P_{\geq n}$-factor including e_1 and excluding e_2. Zhou and Sun [?] studied the binding number condition for $P_{\geq 2}$-factor uniform graph and $P_{\geq 3}$-factor uniform graph. In this section, we research on other two parameters: toughness and isolated toughness. The idea to prove the following results is based on the observation that G is $P_{\geq n}$-factor uniform if $G-e$ is $P_{\geq n}$-covered for any $e\in E(G)$.

Theorem 11. *Let G be a 2-edge-connected graph. If $(G)>1$ (resp.$\tau(G)>2$), then, G is a $P_{\geq 2}$-factor uniform graph.*

Proof. For any $e=uv$, $G'=G-e$ is connected since G is 2-edge-connected graph. To confirm Theorem 11, we need to verify that G' is $P_{\geq 2}$-factor covered. If not, we assume that G' is not $P_{\geq 2}$-factor covered. Using Lemma 2, there is a vertex subset S of G' satisfying

$$
i\left(G'-S\right)\geq 2|S|-\varepsilon_1(S)+1. \tag{58}
$$

Furthermore, we have $i(G-S)\leq i(G'-S)\leq i(G-S)+2$. We consider three cases according to the value of $|S|$. □

Case 1. If $|S|=0$.

We obtain $i(G')\geq 1$ which contradicts $\lambda(G)\geq 2$.

Case 2. If $|S|=1$.

Then, $\varepsilon_1(S)\leq 1$ and $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 2$. If $i(G-S)\geq 2$, then

$$
1<t(G)\leq \frac{|S|}{\omega(G-S)}\leq \frac{|S|}{i(G-S)}\leq \frac{1}{2}, \tag{59}
$$

or

$$
2<\tau(G)\leq \frac{|S|}{\omega(G-S)-1}\leq \frac{|S|}{i(G-S)-1}\leq 1, \tag{60}
$$

a contradiction.

If $i(G-S)=1$, then, $e=uv\in E(G-S)$ and $\omega(G-S)\geq 2$. We infer

$$
1<t(G)\leq \frac{|S|}{\omega(G-S)}\leq \frac{1}{2}, \tag{61}
$$

or

$$
2<\tau(G)\leq \frac{|S|}{\omega(G-S)-1}\leq 1, \tag{62}
$$

a contradiction.

If $i(G-S)=0$, then, K_2 is a component in $G-S$ and $e=uv\in E(K_2)$. If there is another component in $G-S$ except K_2, then, $\omega(G-S)\geq 2$, and we get the contradiction similar to the derivation above. If $\omega(G-S)=1$, then, $G\cong K_3$ since G is 2-edge-connected graph. Special for K_3, we yield $t(K_3)=\tau(K_3)=+\infty$, $G'=P_3$ which is a $P_{\geq 2}$-factor covered graph. Hence, K_3 satisfies the condition of theorem which is a $P_{\geq 2}$-factor uniform graph.

Case 3. If $|S|\geq 2$.

Then, $\varepsilon_1(S)\leq 2$, $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 3$ and $i(G-S)\geq i(G'-S)-2\geq 1$.

Notice that if $i(G-S)\neq i(G'-S)$, then, $e\in E(G-S)$ and $\omega(G-S)\geq i(G-S)+1\geq i(G'-S)-2+1=i(G'-S)-1$. If $i(G-S)=i(G'-S)$, then, $\omega(G-S)\geq i(G-S)=i(G'-S)$. Combining the above two cases, we have $\omega(G-S)\geq i(G'-S)-1$.

If $i(G-S)\geq 2$, then

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{i\left(G'-S\right)-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-1}\leq\frac{|S|}{2|S|-2},\end{aligned}\tag{63}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|S|}{\omega(G-S)-1}\leq\frac{|S|}{i\left(G'-S\right)-1-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-2}\leq\frac{|S|}{2|S|-3}.\end{aligned}\tag{64}$$

It implies $|S|<2$, a contradiction.

If $i(G-S)=1$, then, using the fact that $i(G'-S)\geq 3$, we confirm that K_1 and K_2 are components in $G-S$, $e=uv\in E(K_2)$, and $i(G'-S)=i(G-S)+2=3$. We acquire

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{i(G-S)+1}=\frac{|S|}{i\left(G'-S\right)-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-1}\leq\frac{|S|}{2|S|-2},\end{aligned}\tag{65}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|S|}{\omega(G-S)-1}\leq\frac{|S|}{i(G-S)+1-1}\\&=\frac{|S|}{i\left(G'-S\right)-2}\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-2}\leq\frac{|S|}{2|S|-3}.\end{aligned}\tag{66}$$

Again, in both situation we get $|S|<2$, which leads to a contradiction.

Theorem 12. *Let G be a 2-edge-connected graph. If* $(G)>1$ *(resp.*$\tau(G)>2$*), then, G is a* $P_{\geq 3}$*-factor uniform graph.*

Proof. For any $e=uv\in E(G)$, $G'=G-e$ is connected, and we only need to prove that G' is $P_{\geq 3}$-factor covered. On the contrary, G' is not $P_{\geq 3}$-factor covered, and we can find a subset S of $V(G')$ such that

$$\operatorname{sun}\left(G'-S\right)\geq 2|S|-\varepsilon_2(S)+1.\tag{67}$$

The following discussion is divided into three cases according to the value of $|S|$. □

Case 1. $|S|=0$.

Then, $\varepsilon_2(S)=0$ and $\operatorname{sun}(G')\geq 1$ by (67). It implies sun $(G')=1$, and G' is a big sun with at least six vertices. Moreover, G is a graph constructed by adding an edge in a big sun. Let R be the factor-critical of G' and $x\in V(R)$. We have

$$1<t(G)\leq\frac{|V(R)\setminus\{x\}|}{\omega(G-V(R)\setminus\{x\})}\leq\frac{|R|-1}{|R|-1}=1,\tag{68}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|V(R)\setminus\{x\}|}{\omega(G-V(R)\setminus\{x\})-1}\\&\leq\frac{|R|-1}{|R|-2}=1+\frac{1}{|R|-2}\leq 1+\frac{1}{3-2}=2,\end{aligned}\tag{69}$$

a contradiction.

Case 2. $|S|=1$.

Then, $\varepsilon_2(S)\leq 1$ and $\operatorname{sun}(G'-S)\geq 2$ by (67). If $\omega(G-S)\geq 2$, then

$$1<t(G)\leq\frac{|S|}{\omega(G-S)}\leq\frac{1}{2},\tag{70}$$

or

$$2<\tau(G)\leq\frac{|S|}{\omega(G-S)-1}\leq 1,\tag{71}$$

a contradiction. If $\omega(G-S)=1$, then, $e\in E(G-S)$, and it produces two sun components after deleting e from $G-S$. If $G-S$ isomorphic to K_2, then, $G\cong K_3$ which is a $P_{\geq 3}$-factor uniform graph. Otherwise, $|V(G-S)|\geq 3$, and there are at least two vertices having degree 1 in $G-S$. Let $xy\in E(G-S)$ such that $d_{G-S}(x)=1$. We acquire $1<t(G)\leq|S\cup\{y\}|/\omega(G-S\cup\{y\})\leq 1$ or $2<\tau(G)\leq|S\cup\{y\}|/\omega(G-S\cup\{y\})-1\leq 2$, a contradiction.

Case 3. $|S|\geq 2$.

In this case, $\varepsilon_2(S)\leq 2$, $\operatorname{sun}(G'-S)\geq 3$ by (67), $\operatorname{sun}(G-S)\geq\operatorname{sun}(G'-S)-2\geq 1$, and $\omega(G-S)\geq 2$. If $\operatorname{sun}(G-S)=\operatorname{sun}(G'-S)$ or $\operatorname{sun}(G-S)=\operatorname{sun}(G'-S)-1$, we deduce

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{\operatorname{sun}(G-S)}\\&\leq\frac{|S|}{\operatorname{sun}\left(G'-S\right)-1}\leq\frac{|S|}{2|S|-\varepsilon_2(S)+1-1}\\&\leq\frac{|S|}{2|S|-2}=\frac{1}{2}+\frac{1}{2|S|-2}\leq\frac{1}{2}+\frac{1}{2\times 2-2}=1,\end{aligned}\tag{72}$$

or

$$\begin{aligned}2<\tau(G) &\le \frac{|S|}{\omega(G-S)-1} \le \frac{|S|}{\mathrm{sun}(G-S)-1} \\ &\le \frac{|S|}{\mathrm{sun}\left(G'-S\right)-1-1} \le \frac{|S|}{2|S|-\varepsilon_2(S)+1-2} \\ &\le \frac{|S|}{2|S|-3} = \frac{1}{2}+\frac{3}{2(2|S|-3)} \le \frac{1}{2}+\frac{3}{2(2\times 2-3)} = 2,\end{aligned} \tag{73}$$

a contradiction.

If $\mathrm{sun}(G-S)=\mathrm{sun}(G'-S)-2$, then, edge $e=uv$ belongs to a nonsun component W, while removing e will produce two sun components. It means at least one of u and v is a cut vertex of component W, and without loss of generality, we set u as a cut vertex in W. Hence, we get

$$\begin{aligned}1<t(G) &\le \frac{|S\cup\{u\}|}{\omega(G-S\cup\{u\})} \le \frac{|S\cup\{u\}|}{\omega(G-S)+1} \\ &\le \frac{|S|+1}{\mathrm{sun}(G-S)+2} \le \frac{|S|+1}{\mathrm{sun}\left(G'-S\right)-2+2} \\ &\le \frac{|S|+1}{2|S|-\varepsilon_2(S)+1} \le \frac{|S|+1}{2|S|-1} = \frac{1}{2}+\frac{3}{2(2|S|-1)} \\ &\le \frac{1}{2}+\frac{3}{2(2\times 2-1)} = 1,\end{aligned} \tag{74}$$

or

$$\begin{aligned}2<\tau(G) &\le \frac{|S\cup\{u\}|}{\omega(G-S\cup\{u\})-1} \le \frac{|S\cup\{u\}|}{\omega(G-S)+1-1} \\ &\le \frac{|S|+1}{\mathrm{sun}(G-S)+1} \le \frac{|S|+1}{\mathrm{sun}\left(G'-S\right)-2+1} \\ &\le \frac{|S|+1}{2|S|-\varepsilon_2(S)+1-1} \le \frac{|S|+1}{2|S|-2} \\ &= \frac{1}{2}+\frac{1}{|S|-1} \le \frac{1}{2}+\frac{1}{2-1} = \frac{3}{2},\end{aligned} \tag{75}$$

a contradiction.

Thus, the proof of Theorem 12 is completed.

Theorem 13. *Let G be a 2-edge-connected graph. If* $(G)>(|V(G)|-2)/2$ *(resp.* $I'(G)>|V(G)|-2$*), then, G is a* $P_{\ge 2}$*-factor uniform graph.*

Proof. Clearly, we have $|V(G)|\ge 3$. For any $e=uv$, $G'=G-e$ is connected since G is a 2-edge-connected graph. Similar as Theorem 11, we only need to verify that G' is $P_{\ge 2}$-factor covered. In contrast, suppose that G' is not $P_{\ge 2}$-factor covered. In terms of Lemma 2, there is a vertex subset S of G' that meets (58). Furthermore, $i(G'-S)\in\{i(G-S), i(G-S)+1, i(G-S)+2\}$.

We consider three cases in view of the value of $|S|$. □

Case 1. $|S|=0$.

We get $i(G')\ge 1$ which contradicts to $\lambda(G)\ge 2$.

Case 2. $|S|=1$.

Then, $\varepsilon_1(S)\le 1$ and $i(G'-S)\ge 2|S|-\varepsilon_1(S)+1\ge 2$. If $i(G-S)\ge 2$, then

$$\frac{|V(G)|-2}{2} < I(G) \le \frac{|S|}{i(G-S)} \le \frac{1}{2}, \tag{76}$$

or

$$|V(G)|-2 < I'(G) \le \frac{|S|}{i(G-S)-1} \le 1, \tag{77}$$

a contradiction.

If $i(G-S)=1$, then, $e=uv\in E(G-S)$ and assume $d_{G-S}(u)\ge d_{G-S}(v)=1$. We infer

$$\frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})} \le 1, \tag{78}$$

or

$$|V(G)|-2 < I'(G) \le \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})-1} \le 2, \tag{79}$$

a contradiction.

If $i(G-S)=0$, then, K_2 is a component in $G-S$ and $e=uv\in E(K_2)$. If there is another component in $G-S$ except K_2, then denote this component by W. Select $w\in V(W)$ such that w has a minimum degree in $G-S$ among all vertices in W. Hence, $i(G-S\cup\{u\}\cup N_{G-S}(w))\ge 2$ and

$$\begin{aligned}\frac{|V(G)|-2}{2} < I(G) &\le \frac{|S\cup\{u\}\cup N_{G-S}(w)|}{i(G-S\cup\{u\}\cup N_{G-S}(w))} \\ &\le \frac{2+|V(W)|-1}{2} = \frac{1+|V(W)|}{2} \\ &\le \frac{1+|V(G)|-3}{2} = \frac{|V(G)|-2}{2},\end{aligned} \tag{80}$$

or

$$\begin{aligned}|V(G)|-2 < I'(G) &\le \frac{|S\cup\{u\}\cup N_{G-S}(w)|}{i(G-S\cup\{u\}\cup N_{G-S}(w))-1} \\ &\le \frac{2+|V(W)|-1}{2-1} = 1+|V(W) \\ &\le 1+|V(G)|-3 = |V(G)|-2,\end{aligned} \tag{81}$$

a contradiction. If $\omega(G-S)=1$, then, G becomes K_3. As discussed in Theorem 11, K_3 meets the condition of Theorem 13 that is a $P_{\ge 2}$-factor uniform graph.

Case 3. $|S|\ge 2$.

Then, $\varepsilon_1(S) \le 2$, $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 \ge 3$ and $i(G - S) \ge i(G' - S) - 2 \ge 1$. We consider the following subcases in light of the value of $i(G - S)$.

Case 4. $i(G - S) \ge 2$.

If $i(G - S) = i(G' - S)$, then $|V(G)| \ge 4$,

$$\begin{aligned} 1 &\le \frac{|V(G)| - 2}{2} < I(G) \le \frac{|S|}{i(G - S)} \le \frac{|S|}{i\left(G' - S\right)} \\ &\le \frac{|S|}{2|S| - \varepsilon_1(S) + 1} \le \frac{|S|}{2|S| - 1} = \frac{1}{2} + \frac{1}{2(2|S| - 1)} \\ &\le \frac{1}{2} + \frac{1}{2(2 \times 2 - 1)} = \frac{2}{3}, \end{aligned} \tag{82}$$

or

$$\begin{aligned} 2 &\le |V(G)| - 2 < I'(G) \le \frac{|S|}{i(G - S) - 1} \\ &\le \frac{|S|}{i\left(G' - S\right) - 1} \le \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \\ &\le \frac{|S|}{2|S| - 2} = \frac{1}{2} + \frac{1}{2|S| - 2} \le \frac{1}{2} + \frac{1}{2 \times 2 - 2} = 1, \end{aligned} \tag{83}$$

a contradiction.

If $i(G - S) \ne i(G' - S)$, then, $|V(G)| \ge 6$,

$$\begin{aligned} 2 &\le \frac{|V(G)| - 2}{2} < I(G) \le \frac{|S|}{i(G - S)} \le \frac{|S|}{i\left(G' - S\right) - 2} \\ &\le \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 2} \le \frac{|S|}{2|S| - 3} \\ &= \frac{1}{2} + \frac{3}{2(2|S| - 3)} \le \frac{1}{2} + \frac{3}{2(2 \times 2 - 3)} = 2, \end{aligned} \tag{84}$$

a contradiction. For $\tau(G)$, if $|S| \ge 3$, then

$$\begin{aligned} 4 &\le |V(G)| - 2 < I'(G) \le \frac{|S|}{i(G - S) - 1} \\ &\le \frac{|S|}{i\left(G' - S\right) - 2 - 1} \le \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 3} \\ &\le \frac{|S|}{2|S| - 4} = \frac{1}{2} + \frac{2}{2|S| - 4} \le \frac{1}{2} + \frac{2}{2 \times 3 - 4} = \frac{3}{2}, \end{aligned} \tag{85}$$

a contradiction. If $|S| = 2$, we can easily check that $4 \le |V(G)| - 2 < I'(G) \le |S|/i(G - S) - 1 \le 2$, a contradiction.

Case 5. $i(G - S) = 1$.

Since $i(G' - S) \ge 3$, we confirm that K_1 and K_2 are components in $G - S$, $e = uv \in E(K_2)$ and $i(G' - S) = i(G - S) + 2 = 3$. Using $|V(G)| \ge 5$, we acquire

$$\begin{aligned} \frac{3}{2} &\le \frac{|V(G)| - 2}{2} < I(G) \le \frac{|S \cup \{u\}|}{i(G - S \cup \{u\})} \\ &= \frac{|S| + 1}{i(G - S) + 1} = \frac{|S| + 1}{i\left(G' - S\right) - 2 + 1} \\ &\le \frac{|S| + 1}{2|S| - \varepsilon_1(S) + 1 - 1} \le \frac{|S| + 1}{2|S| - 2} \\ &= \frac{1}{2} + \frac{2}{2|S| - 2} \le \frac{1}{2} + \frac{2}{2 \times 2 - 2} = \frac{3}{2}, \end{aligned} \tag{86}$$

or

$$\begin{aligned} 3 &\le |V(G)| - 2 < I'(G) \le \frac{|S \cup \{u\}|}{i(G - S \cup \{u\}) - 1} \\ &= \frac{|S| + 1}{i(G - S) + 1 - 1} = \frac{|S| + 1}{i\left(G' - S\right) - 2} \\ &\le \frac{|S| + 1}{2|S| - \varepsilon_1(S) + 1 - 2} \le \frac{|S| + 1}{2|S| - 3} \\ &= \frac{1}{2} + \frac{5}{2(2|S| - 3)} \le \frac{1}{2} + \frac{5}{2(2 \times 2 - 3)} = 3, \end{aligned} \tag{87}$$

a contradiction.

Thus, we confirm that Theorem 13 is established.

Theorem 14. *Let G be a 2-edge-connected graph. If* $(G) > (|V(G)| - 2)/2$ *(resp.*$I'(G) > |V(G)| - 2$*), then, G is a* $P_{\ge 3}$*-factor uniform graph.*

Proof. For any $e = uv \in E(G)$, $G' = G - e$ is connected, and we only need to prove that G' is $P_{\ge 3}$-factor covered. On the contrary, G' is not $P_{\ge 3}$-factor covered. Then, there exists a subset S of $V(G')$ satisfying (67). □

Let a, b, c be the number of K_1 components, K_2 components, and big sun components in $G - S$, respectively. Let $H_1, \cdots, H_c$ be big sun components in $G - S$ with $|V(H_i)| \ge 6$. Choosing one vertex from each K_2 component in $G - S$ and let X be the set of these vertices. Set R_i as the factor-critical subgraph of H_i and $Y = \cup_{i=1}^{c} V(R_i)$. We have $|X| = b$, $|Y| = \sum_{i=1}^{c} |H_i|/2$ and $a + b + c = \mathrm{sun}(G - S) \ge \mathrm{sun}(G' - S) - 2$. The following discussion is divided into three cases according to the value of $|S|$.

Case 1. $|S| = 0$.

Then, $\varepsilon_2(S) = 0$ and $\mathrm{sun}(G') \ge 1$ by (67). It implies $\mathrm{sun}(G') = 1$, G' is a big sun with at least six vertices, and $|V(G)| \ge 6$. Moreover, G is a graph constructed by adding an edge in a big sun. Let R be the factor-critical of G'. We obtain

$$2 \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|V(R)|}{i(G-V(R))} \le \frac{|R|}{|R|-1} = 1 + \frac{1}{|R|-1} \le 1 + \frac{1}{3-1} = \frac{3}{2}, \quad (88)$$

or

$$4 \le |V(G)|-2 < I'(G) \le \frac{|V(R)|}{i(G-V(R))-1} \le \frac{|R|}{|R|-2} = 1 + \frac{2}{|R|-2} \le 1 + \frac{2}{3-2} = 3, \quad (89)$$

a contradiction.

Case 2. $|S|=1$.

In this case, $\varepsilon_2(S) \le 1$, $\mathrm{sun}(G'-S) \ge 2$ by (67), and $a=0$ since $|S|=1$ and G is 2-edge-connected.

Case 3. $\mathrm{sun}(G-S) = \mathrm{sun}(G'-S)$.

We get $i(G-S\cup X\cup Y) = b + \sum_{i=1}^{c}|H_i|/2 \ge b+3c \ge b+c = \mathrm{sun}(G-S) = \mathrm{sun}(G'-S) \ge 2$ and $|V(G)| \ge 4$ (if $|V(G)|=3$, then $G \cong K_3$, $G-S$ isomorphic to K_2 which contradicts to $\mathrm{sun}(G-S) = \mathrm{sun}(G'-S) \ge 2$).

If $|V(G)| \ge 6$, using the definition of isolated toughness, we have

$$2 \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)} \le \frac{1+b+\sum_{i=1}^{c}(|H_i|/2)}{b+\sum_{i=1}^{c}(|H_i|/2)}, \quad (90)$$

which implies $b+\sum_{i=1}^{c}|H_i|/2 < 1$, a contradiction. For $I'(G)$, we yield

$$4 \le |V(G)|-2 < I'(G) \le \frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)-1} \le \frac{1+b+\sum_{i=1}^{c}(|H_i|/2)}{b+\sum_{i=1}^{c}(|H_i|/2)-1}, \quad (91)$$

which implies $3b+3\sum_{i=1}^{c}|H_i|/2 < 5$, contradicting to $b+c \ge 2$.

If $|V(G)|=5$, then, $c=a=0$ and

$$\frac{3}{2} = \frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup X|}{i(G-S\cup X)} \le \frac{1+b}{b}, \quad (92)$$

or

$$3 = |V(G)|-2 < I'(G) \le \frac{|S\cup X|}{i(G-S\cup X)-1} \le \frac{1+b}{b-1}, \quad (93)$$

which implies $b<2$ which contradicts to $b=\mathrm{sun}(G-S) = \mathrm{sun}(G'-S) \ge 2$.

If $|V(G)|=4$, then, $c=a=0$ and $b=1$ contradicting to $b=\mathrm{sun}(G-S)=\mathrm{sun}(G'-S) \ge 2$.

Case 4. $\mathrm{sun}(G-S) = \mathrm{sun}(G'-S)-1$.

In this case, $\mathrm{sun}(G-S) \ge 1$ since $\mathrm{sun}(G'-S) \ge 2$.

Claim 1. If K_2 is one of components in $G-S$, then, $e \in E(K_2)$.

Proof. Suppose K_2 is a component in $G-S$ and $e \in E(K_2)$ is exactly a deleted edge, set $u \in V(K_2)$. If $G-S$ is isomorphic to K_2, then, G is isomorphic to K_3 which is clearly a $P_{\ge 3}$-factor uniform graph. □

If there is a K_1 component in $G-S$, then, $|V(G)| \ge 4$ and

$$1 \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})} \le \frac{2}{2} = 1, \quad (94)$$

or

$$2 \le |V(G)|-2 < I'(G) \le \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})-1} \le \frac{2}{2-1} = 2, \quad (95)$$

a contradiction.

If there is another K_2 component or big sun component in $G-S$ (say W), then, there is a vertex x in W such that $d_{G-S}(x)=1$ and assume $xy \in E(G-S)$. We have $|V(G)| \ge 5$ and

$$\frac{3}{2} \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup\{u,y\}|}{i(G-S\cup\{u,y\})} \le \frac{3}{2}, \quad (96)$$

or

$$3 \le |V(G)|-2 < I'(G) \le \frac{|S\cup\{u,y\}|}{i(G-S\cup\{u,y\})-1} \le \frac{3}{2-1} = 3, \quad (97)$$

a contradiction.

If there exists a nonsun component in $G-S$ (say M), the,n we select $x \in V(M)$ with its degree in $G-S$ as small as possible. We infer

$$\frac{|V(G)|-2}{2} < I(G) \le \frac{|S\cup\{u\}\cup(V(M)-\{x\})|}{i(G-S\cup\{u\}\cup(V(M)-\{x\}))} \le \frac{|V(G)|-2}{2}, \quad (98)$$

or

$$|V(G)|-2 < I'(G) \le \frac{|S\cup\{u\}\cup(V(M)-\{x\})|}{i(G-S\cup\{u\}\cup(V(M)-\{x\}))-1} \le |V(G)|-2, \quad (99)$$

a contradiction.

Hence, the claim is hold.

From Claim 1, we see that there is a nonsun component W in $G-S$ with $|V(W)|\geq 3$ (and hence, $|V(G)|\geq 5$), delete edge $e=uv$ from W, and then, it produces a new sun component in $G-S$. Thus, there is a vertex x in W with $d_{G-S}(x)=1$, and set $xy\in E(G-S)$. Note that $\mathrm{sun}(G-S)\geq 1$, if K_1 is a component in $G-S$, then, we yield

$$\frac{3}{2}\leq \frac{|V(G)|-2}{2}<I(G)\leq \frac{|S\cup\{y\}|}{i(G-S\cup\{y\})}\leq \frac{2}{2}=1, \quad (100)$$

or

$$3\leq |V(G)|-2<I'(G)\leq \frac{|S\cup\{y\}|}{i(G-S\cup\{y\})-1}\leq \frac{2}{2-1}=2, \quad (101)$$

a contradiction. If K_2 or a big sun is a component in $G-S$ (denote this sun component by M), then, there is a vertex x' in M with $d_{G-S}(x')=1$, and set $x'y'\in E(G-S)$. We acquire

$$\frac{3}{2}\leq \frac{|V(G)|-2}{2}<I(G)\leq \frac{\left|S\cup\left\{y,y'\right\}\right|}{i\left(G-S\cup\left\{y,y'\right\}\right)}\leq \frac{3}{2}, \quad (102)$$

or

$$3\leq |V(G)|-2<I'(G)\leq \frac{\left|S\cup\left\{y,y'\right\}\right|}{i\left(G-S\cup\left\{y,y'\right\}\right)-1}\leq \frac{3}{2-1}=3, \quad (103)$$

a contradiction.

Case 3. $\mathrm{sun}(G-S)=\mathrm{sun}(G'-S)-2$.

In this case, there is a nonsun component W in $G-S$, and it produces two sun components after deleting $e=uv$ from W. Thus, there are at least two vertices $x,x'\in V(W)$ such that $d_{G-S}(x)=d_{G-S}(x')=1$. Set $xy,x'y'\in E(W)$ and note that y and y' are allowed to be the same vertex (if $W\cong P_3$). If $W\cong P_3$, then, $y=y'$, $|V(G)|\geq 4$, and

$$1\leq \frac{|V(G)|-2}{2}<I(G)\leq \frac{|S\cup\{y\}|}{i(G-S\cup\{y\})}\leq \frac{2}{2}=1, \quad (104)$$

or

$$2\leq |V(G)|-2<I'(G)\leq \frac{|S\cup\{y\}|}{i(G-S\cup\{y\})-1}\leq \frac{2}{2-1}=2, \quad (105)$$

a contradiction. Otherwise, $|V(G)|\geq 5$, and

$$\frac{3}{2}\leq \frac{|V(G)|-2}{2}<I(G)\leq \frac{\left|S\cup\left\{y,y'\right\}\right|}{i\left(G-S\cup\left\{y,y'\right\}\right)}\leq \frac{3}{2}, \quad (106)$$

or

$$3\leq |V(G)|-2<I'(G)\leq \frac{\left|S\cup\left\{y,y'\right\}\right|}{i\left(G-S\cup\left\{y,y'\right\}\right)-1}\leq \frac{3}{2-1}=3, \quad (107)$$

a contradiction.

Case 4. $|S|\geq 2$.

In this case, $\varepsilon_2(S)\leq 2$, $a+b+c=\mathrm{sun}(G'-S)\geq 3$ by (67), and $\mathrm{sun}(G-S)\geq \mathrm{sun}(G'-S)-2\geq 1$. We have $|V(G)|\geq 5$,

$$\begin{aligned}\frac{3}{2}&\leq \frac{|V(G)|-2}{2}<I(G)\\ &\leq \frac{|S\cup X\cup Y|}{i(G-S\cup X\cup Y)}\leq \frac{|S|+b+\sum_{i=1}^{c}(|H_i|/2)}{b+\sum_{i=1}^{c}(|H_i|/2)}.\end{aligned} \quad (108)$$

Then, the rest proof process is consistent with the part of Theorem 4 and Theorem 5 in Gao et al. [22], and we will not repeat here.

Hence, the proof of Theorem 14 is finished.

3. Sharpness

In this section, we present some counterexamples to verify that the bounds of parameters in theorems in the second section are tight.

3.1. Sharpness of Theorem 4-Theorem 6. We manifest that (1) $\lambda(G)\geq m+1$ and $t(G)>m/(m+1)$ or $\tau(G)>1$ in Theorem 4 cannot change to $\lambda(G)\geq m$ and $t(G)=m/(m+1)$ (or $\tau(G)=1$); (2) $\lambda(G)\geq m+1$ and $I(G)>2m/(m+1)$ or $I'(G)>2$ in Theorem 5 cannot change to $\lambda(G)\geq m$ and $I(G)=2m/(2m+1)$ (or $I'(G)=1$); (3) $\lambda(G)\geq m+1$ and $\mathrm{bind}(G)>3/2$ in Theorem 6 cannot change to $\lambda(G)\geq m$ and $\mathrm{bind}(G)=3/2$.

Let $G=K_m\vee(mK_2\cup K_1)$. Taking one vertex from each K_2 and denote X by the set of these vertices, we have

$$\begin{aligned}t(G)&=\frac{|V(K_m)|}{\omega(G-V(K_m))}=\frac{m}{m+1},\\ \tau(G)&=\frac{|V(K_m)|}{\omega(G-V(K_m))-1}=\frac{m}{m+1-1}=1,\\ I(G)&=\frac{|V(K_m)\cup X|}{i(G-V(K_m)-X)}=\frac{2m}{m+1},\\ I'(G)&=\frac{|V(K_m)\cup X|}{i(G-V(K_m)-X)-1}=\frac{2m}{m+1-1}=2,\\ bind(G)&=\frac{|N_G(V(mK_2))|}{|V(mK_2)|}=\frac{3m}{2m}=\frac{3}{2}.\end{aligned} \quad (109)$$

Set $E' = E(mK_2)$ and $G' = G - E' = K_m \vee ((2m+1)K_1)$. Then, $|E'| = m$, and by setting $S = K_m$, we have

$$i\left(G' - S\right) = 2m + 1 > 2m = 2|S|. \tag{110}$$

Thus, G' has no $P_{\geq 2}$-factor, and accordingly, G is not a $(P_{\geq 2}, m)$-factor-deleted graph.

3.2. Sharpness of Theorem 7 and Theorem 8. We show that the toughness bounds in Theorem 7 and Theorem 8 are best. Consider $G = K_{k+2} \vee (3K_1)$, and we have $\kappa(G) = k + 2$, $t(G) = (k+2)/3$ and $\tau(G) = (k+2)/2$. Set $U \subseteq V(G)$ with $|U| = k$, and let $G' = G - U = K_2 \vee (3K_1)$. Take $S = K_2$ in G', then, we have $\varepsilon_1(S) = \varepsilon_2(S) = 2$,

$$\begin{aligned} i\left(G' - S\right) &= 3 > 2 = 2|S| - \varepsilon_1(S), \\ \text{sun}\left(G' - S\right) &= 3 > 2 = 2|S| - \varepsilon_2(S). \end{aligned} \tag{111}$$

Hence, according to Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $(P_{\geq 2}, k)$-factor critical covered graph. Moreover, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $(P_{\geq 3}, k)$-factor critical covered graph.

3.3. Sharpness of Theorem 9. We depict that the isolated toughness bounds in Theorem 9 for a graph to be $(P_{\geq 2}, k)$-factor critical covered are best. Consider $G = K_{k+1} \vee (2K_1 \cup K_t)$ where t is enough large, and we have $\kappa(G) = k + 1$, $I(G) = (k + 1/2)(k+1)/2$ and $I'(G) = k + 1$. Set $U \subseteq V(G)$ with $|U| = k$, and let $G' = G - U = K_1 \vee (2K_1 \cup K_t)$. Set S as the first K_1 in G', then, we have $\varepsilon_1(S) = 1$ and

$$i\left(G' - S\right) = 2 > 1 = 2|S| - \varepsilon_1(S). \tag{112}$$

Hence, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $(P_{\geq 2}, k)$-factor critical covered graph.

3.4. Sharpness of Theorem 10. The isolated toughness conditions in Theorem 10 are tight. Consider $G = K_{k+1} \vee (2K_2 \cup G'')$ where G'' is connected but not a sun. Set $U \subset V(K_{k+1})$ with $|U| = k$, $G' = G - U = K_1 \vee (2K_2 \cup G'')$, and $S = K_1$ in G'. Selecting one vertex from each K_2 in the $2K_2$ part and denoting X by the set of these two vertices, we confirm

$$\begin{aligned} I(G) &= \frac{|U \cup S \cup X|}{i(G - U \cup S \cup X)} = \frac{|U| + |S| + |X|}{i\left(G' - S \cup X\right)} \\ &= \frac{k+1+2}{2} = \frac{3+k}{2}, \\ I'(G) &= \frac{|U \cup S \cup X|}{i(G - U \cup S \cup X) - 1} \\ &= \frac{|U| + |S| + |X|}{i\left(G' - S \cup X\right) - 1} = \frac{k+1+2}{2-1} = k + 3. \end{aligned} \tag{113}$$

On the other hand, $\varepsilon_2(S) = 1$ since G'' is a nonsun component of $G' - S$ and

$$\text{sun}\left(G' - S\right) = 2 > 1 = 2|S| - \varepsilon_2(S). \tag{114}$$

In view of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $(P_{\geq 3}, k)$-factor critical covered graph.

3.5. Sharpness of Theorem 11. The toughness bounds in Theorem 11 are tight. Consider $G = K_2 \vee (K_1 \cup K_2)$ which is 2-edge-connected graph with $t(G) = 1$ and $\tau(G) = 2$. Select $e \in E(K_1 \cup K_2)$ and set $G' = G - e = K_2 \vee (3K_1)$. Let $S = V(K_2) \subseteq V(G')$. We have $\varepsilon_1(S) = 2$ and

$$i\left(G' - S\right) = 3 > 2 = 2|S| - \varepsilon_1(S). \tag{115}$$

Therefore, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $P_{\geq 2}$-factor uniform graph.

3.6. Sharpness of Theorem 12. The isolated toughness bounds in Theorem 12 are sharp. Consider $G = K_2 \vee (2K_2)$ which is a 2-edge-connected graph. We have $t(G) = 1$ and $\tau(G) = 2$. Let $e \in E(2K_2)$, $G' = G - e = K_2 \vee (K_2 \cup 2K_1)$, and S be the vertex set of first K_2 in G'. We infer $\varepsilon_2(S) = 2$ and

$$\text{sun}\left(G' - S\right) = 3 > 2 = 2|S| - \varepsilon_2(S). \tag{116}$$

Hence, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $P_{\geq 3}$-factor uniform graph.

3.7. Sharpness of Theorem 13 and Theorem 14. To show the isolated toughness bounds in Theorem 13 and Theorem 14 that are sharp, we consider $G = K_1 \vee (K_2 \cup K_t)$ where t is a large number. Select one vertex from K_2 and $t - 1$ vertices from K_t and denote X by the vertex subset of these vertices. We have

$$\begin{aligned} I(G) &= \frac{|V(K_1) \cup X|}{i(G - V(K_1) \cup X)} \\ &= \frac{1+t}{2} = \frac{1 + |V(G)| - 3}{2} = \frac{|V(G)| - 2}{2}, \\ I'(G) &= \frac{|V(K_1) \cup X|}{i(G - V(K_1) \cup X) - 1} \\ &= \frac{1+t}{2-1} = 1 + |V(G)| - 3 = |V(G)| - 2. \end{aligned} \tag{117}$$

On the other hand, let $e \in E(K_2)$ and $G' = G - e = K_1 \vee (2K_1 \cup K_t)$. Let S be the vertex set of first K_1 in G', and then, we have $\varepsilon_1(S) = \varepsilon_2(S) = 1$,

$$\begin{aligned} i\left(G' - S\right) &= 2 > 1 = 2|S| - \varepsilon_1(S), \\ sun\left(G' - S\right) &= 2 > 1 = 2|S| - \varepsilon_2(S). \end{aligned} \tag{118}$$

Therefore, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $P_{\geq 2}$-factor uniform graph. Also, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $P_{\geq 3}$-factor uniform graph.

4. Open Problems

The restrictions in factor critical graphs can be further extended to more general ones. For instance, a graph G is a $(P_{\geq n}, k, m)$-factor critical covered graph if removing any k vertices from G, the resting subgraph is still a $(P_{\geq n}, m)$-factor covered graph (that is, if for any $E \subseteq E(G)$ with $|E| = m$, G has a $P_{\geq n}$-factor containing all the edges in E, and then, G is called a $(P_{\geq n}, m)$-factor covered graph). The biggest obstacle to solve these problems is lacking of necessary and sufficient condition for $(P_{\geq n}, m)$-factor covered graph. Hence, as the first step, we need to expand the results on $P_{\geq 2}$-factor covered graph and $P_{\geq 3}$-factor covered graph determined by Zhang and Zhou [15] to necessary and sufficient condition of $(P_{\geq 2}, m)$-factor covered graph and $(P_{\geq 3}, m)$-factor covered graph. These problems are worthy of deep study in the future.

Data Availability

This work is a pure theoretical contribution, and no data are contained in the paper.

Conflicts of Interest

All authors declare no conflict of interests in publishing this work.

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Research Article

On a Unique Solution of a T-Maze Model Arising in the Psychology and Theory of Learning

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In the form of a T, a T-maze is an experimental design in which each trial consists of decisions between two or more options. It contains choices with particular kinds of symmetries that have gained considerable attention in psychology and learning theories. One of the simplest mazes utilized by rats is the T-maze since it requires just a single point of preference. At a T-maze base, the mouse chooses to turn right or left to get food. This paper aims at analyzing the rat's behavior in such circumstances and proposing a suitable mathematical model for it. The existence and uniqueness of a solution to the proposed T-maze model are investigated by using the appropriate fixed point method.

1. Introduction

Mathematical psychology is an approach to psychological study focused on mathematical modeling of perceptual, thinking, cognitive, and motor processes. The mathematical methods are used to develop more reliable theories and thus produce more rigorous empirical validations. The biggest issue with today's application of mathematics to psychological problems and most likely for some time to come is modeling these problems.

In an animal or human being, the learning phase may often be viewed as a series of choices between multiple possible reactions. Even in basic repetitive experiments under strictly regulated conditions, preference sequences are typically volatile, suggesting that the probability governs the choice of responses. It is also helpful to identify structural adjustments in the series of alternatives that reflect changes in trial-to-trial outcome probabilities. From this perspective, most of the learning analysis explains the probability of a trial-to-test occurrence that describes a stochastic mechanism.

In modern mathematical learning experiments, the researchers concluded that a basic learning experiment was compatible with any stochastic process. It is not a new idea (see [1] for a summary of its history). After 1950, two critical features emerged mainly in the research initiated by Bush, Estes, and Mosteller. In the first instance, the learning method egalitarian essence was a core feature of the developed model. Second, these frameworks were studied and applied in areas that did not conceal their quantitative aspects.

Several studies (Estes and Straughan [2], Grant et al. [3], Humphreys [4], and Jarvik [5]) on human actions in probability–learning scenarios have produced results aligned with the so–called event-matching hypothesis that the allocation of incentives would mirror the asymptotic distribution of answers in a two-choice setting. Conflicting findings have been reported in other studies. For example, if subjects choose the correct option in most trials, then, it would accelerate the probability close to 1 (for the detail, see [1, 6]).

Turab and Sintunavarat [7, 8] presented a functional equation to analyze Bush and Wilson's experimental study on a paradise fish [9], in which they offered the fish two options for swimming. As the starting gate was raised, swimmers had two options: swim on the right-hand side or the left-hand side of the tank's far end.

Recently, in [10], the authors discussed a particular type of traumatic avoidance learning experiment of normal dogs proposed by Solomon and Wynne [11]. They examined 30 mongrel dogs weighing between 9 and 13 kg and observed a particular form of emotional resistance performed in a tiny box with a steel grid floor. Turab and Sintunavarat [10] analyzed the dogs' behavior in such situations and proposed a mathematical model and also presented the existence of solutions of such model by using the fixed point technique.

On the other hand, the genesis of the fixed point theory was primarily for the use of successive approximations to prove the existence and uniqueness of solutions, primarily of differential and integral equations, in the second half of the nineteenth century. It is indeed a beautiful blend of pure and applied analysis, topology, and geometry. Picard's work demonstrates the fundamental concepts of a fixed point theoretic perspective. However, it is attributed to the Polish mathematician Banach for abstracting the fundamental ideas into a framework applicable to a wide variety of applications beyond ordinary differential and integral equations (see [12]). It has been generalized and extended in various directions (for the detail, see [13–16]). For more details about the fixed point theory and its applications in different spaces, we refer the reader to [17–22].

In this paper, we present a specific type of psychological learning theory experiment related to the T-maze model proposed by Brunswik and Stanley in [23, 24], and suggest a mathematical model that is appropriate for it. The existence and uniqueness of the proposed model's solution are investigated by using the suitable fixed point theorem. Later on, to check the proposed model's validity, we shall highlight some particular aspects of the T-maze model under the experimenter-subject controlled events. In the end, we raise an open problem for the interested readers.

2. A T–Maze Experiment Proposed by Brunswik and Stanley

A T-maze [23, 24] is a unique design that has gotten much attention in the past few years. It is a classic maze for rats since it has only one choice point. While experimental design modifications and generalizations have been used with mice and other subjects, we shall concentrate on the primary form of the open maze used with rats.

In Figure 1, a schematic of the apparatus can be seen. At the starting position, **s**, a rat is put and it runs to the point of preference, **w**. After that, the rat moves to one of the two aim bins, **A** and **B**, where it may get food.

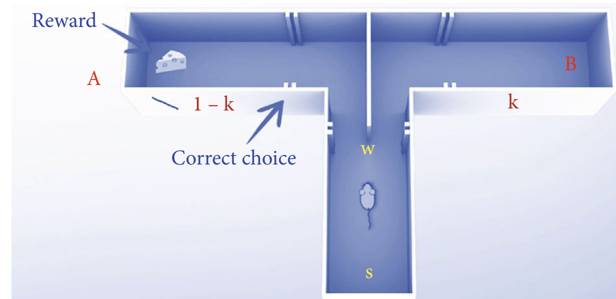

Figure 1: The behavior of a rat in a T-maze experiment.

Table 1: Alternative definitions of experimental events in a T-maze experiment depending on the placement of the food and chosen side.

Responses	Outcomes
A : left turn	$\mathbf{O}_1$: food side (reward)
B : left turn	$\mathbf{O}_1$: food side (reward)
A : right turn	$\mathbf{O}_2$: nonfood side (no reward)
B : right turn	$\mathbf{O}_2$: nonfood side (no reward)

Here, the experimental trials constitute the series of behavior. With the same rat, the process is usually performed several times. A very elaborate course of action is the overall activity of a rat during an experimental study. The rat is in a particular stimulation position when it crosses the labyrinth and is in two potential stimulus conditions after reaching the preference stage. Of course, this overall activity on a test may be divided up and appropriate measurements or indexes used by each part can be used. For example, we could ask about the starting location latency, **s** , or the running momentum between **s** and the **w**-point of preference. However, it seems to us that the part of the rat's conduct that is unique to this experiment is the behavior at the option stage, **w**.

In the study that follows, we only consider a rat's choice of the path on a trial instead of any other actions it may exhibit. The rat arrives at the decision point at a complete experimental study where the stimulation factor population is kept unchanged from trial to trial. Corresponding to the target box achieved, **A** or **B**, two groups of responses are listed in Table 1. One and only one of these response groups take part in each study trial. Then, an experimental study compares to a trial as described in [12]—a chance to select between alternatives that are mutually exclusive and exhaustive.

The condition of the organism on a specific test, according to the model, is fully specified by a probability **k** that the rat will go to goal box **A** and a probability **1 − k** that it will go to goal box **B**. We have complete information about the

Table 2: The movement of a rat and its corresponding events.

Response	Outcomes	Events
A	$\mathbf{O_1}$	$\mathbf{E_1}$
B	$\mathbf{O_1}$	$\mathbf{E_2}$
A	$\mathbf{O_2}$	$\mathbf{E_3}$
B	$\mathbf{O_2}$	$\mathbf{E_4}$

learning process model when these probabilities are recognized for each trial. These probabilities can be estimated from the proportion of turns made by a single rat on several trials to goal box **A** or from the proportion of a population of rats that go on a specific trial to goal box **B**.

3. Mathematical Modeling of the Proposed T-Maze Experiment

In the above experiment, the significant interest lies in the behavior of a rat; turn left or right, "**A**" or "**B**," and get the food which is dependent on where the food is placed and the movement of a rat towards that compartment. In our view, if a rat chooses the food side, there would be an occurrence of alternative $\mathbf{O_1}$, and if a rat made a move to the other side, then, there will be an occurrence of alternative $\mathbf{O_2}$. According to the mathematical point of view, there would be four possibilities of events, depending on the movement of the rat and the placement of the food. These events are listed in Table 2.

The probability of the responses **A** and **B** are x and $(1-x)$, respectively, where $x \in [0,1]$. The experimental pattern asks for the outcomes of the responses (whether the rat gets the food or not), trials' fixed proportion of $\varsigma \in [0,1]$. Therefore, we get the event probabilities stated below (see Table 3).

Let us assume that $\vartheta_1, \vartheta_2 \in (0,1)$ are the learning–rate parameters and their values measure the ineffectiveness of the events $\mathbf{E_1}-\mathbf{E_4}$ in altering the response probability. Also, $\lambda_k \in [0,1]$, where $k=1,2$ is the constant of the corresponding event $\mathbf{E_1}-\mathbf{E_4}$. If ςx is the probability of response **A** with outcome $\mathbf{O_1}$ on some trial and **A** is fulfilled, the new probability of **A** with outcome $\mathbf{O_1}$ is $\vartheta_1 x + (1-\vartheta_1)\lambda_1$, and if **A** is achieved with outcome $\mathbf{O_2}$, then, the new probability will be $\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)$ with the event probability $(1-\varsigma)x$. Similarly, if **B** is performed with outcomes $\mathbf{O_1}$ and $\mathbf{O_2}$, then, the new probabilities of **B** are $\vartheta_2 x + (1-\vartheta_2)\lambda_2$ and $\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)$, having event probabilities of occurrence $(1-x)\varsigma$ and $(1-\varsigma)(1-x)$, respectively. For the four events $\mathbf{E_1}-\mathbf{E_4}$, we can define the transition operators $P_1, P_2, P_3, P_4 : [0,1] \longrightarrow [0,1]$ as

$$\begin{aligned} P_1 x &= \vartheta_1 x + (1-\vartheta_1)\lambda_1, \\ P_2 x &= \vartheta_2 x + (1-\vartheta_2)\lambda_2, \\ P_3 x &= \vartheta_1 x + (1-\vartheta_1)(1-\lambda_1), \\ P_4 x &= \vartheta_2 x + (1-\vartheta_2)(1-\lambda_2), \end{aligned} \tag{1}$$

for all $x \in [0,1]$ and $0 < \vartheta_1, \vartheta_2 < 1$.

Table 3: Corresponding probabilities of the four events.

Event	Probability of occurrence
$\mathbf{E_1}$	ςx
$\mathbf{E_2}$	$(1-x)\varsigma$
$\mathbf{E_3}$	$(1-\varsigma)x$
$\mathbf{E_4}$	$(1-\varsigma)(1-x)$

It can be observed that a rat following such description, in the long run, will stop giving feedback to one of the responses and react only with the other (with probability one). Now, giving $x, \vartheta_1, \vartheta_2, \lambda_1, \lambda_2$, what is the probability that the rat stops providing **B**'s, that is, consumed by **A**? We define such probability by $P(x, \vartheta_1, \vartheta_2, \lambda_1, \lambda_2)$ as a function of x, and it depends on the path as well as the responses and outcomes. After one trial, the rat has a new probability shown in (1) depending on the events $\mathbf{E_1}-\mathbf{E_4}$ with the respective probabilities of occurrence. Thus, if its first trial is **A** with outcomes $\mathbf{O_1}$ and $\mathbf{O_2}$, its new probability of consumption by **A** will be $P(\vartheta_1 x + (1-\vartheta_1)\lambda_1, \vartheta_1, \lambda_1)$ and $P(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1), \vartheta_1, \lambda_1)$, respectively. But, if the first trial is **B** with outcomes $\mathbf{O_1}$ and $\mathbf{O_2}$, then, the new probability of absorption by **B** will be $P(\vartheta_2 x + (1-\vartheta_2)\lambda_2, \vartheta_2, \lambda_2)$ and $P(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2), \vartheta_2, \lambda_2)$, respectively.

By considering the above transition operators with their corresponding probabilities and events given in Table 3, we have the following functional equation

$$\begin{aligned} P(x, \vartheta_1, \vartheta_2, \lambda_1, \lambda_2) &= \varsigma x P(\vartheta_1 x + (1-\vartheta_1)\lambda_1, \vartheta_1, \lambda_1) \\ &\quad + (1-x)\varsigma P(\vartheta_2 x + (1-\vartheta_2)\lambda_2, \vartheta_2, \lambda_2) \\ &\quad + (1-\varsigma)x P(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1), \vartheta_1, \lambda_1) \\ &\quad + (1-\varsigma)(1-x) P(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2), \vartheta_2, \lambda_2). \end{aligned} \tag{2}$$

In the progression, the following noted result will be needed.

$$d(\mathcal{M}\ell, \mathcal{M}m) \le \varpi d(\ell, m), \tag{3}$$

for some $\varpi < 1$ and for all $\ell, m \in \mathcal{S}$. Then, $\mathcal{M}$ has precisely one fixed point. Moreover, the Picard iteration $\{\ell_n\}$ in $\mathcal{S}$ which is defined by $\ell_n = \mathcal{M}\ell_{n-1}$ for all $n \in \mathbb{N}$, where $\ell_0 \in \mathcal{S}$, converges to the unique fixed point of $\mathcal{M}$.

Theorem 1. *(Banach fixed point theorem [12]). Let $(\mathcal{S}, d)$ be a complete metric space and $\mathcal{M} : \mathcal{S} \longrightarrow \mathcal{S}$ be a mapping defined by*

4. Main Results

Let $\mathcal{A} = [0,1]$. Throughout this article, we denote by $\mathcal{B}$ the class of all continuous real-valued functions $\mathcal{W} : \mathcal{A} \longrightarrow \mathbb{R}$ such that $\mathcal{W}(0) = 0$ and

$$\sup_{x \ne y} \frac{|\mathcal{W}(x) - \mathcal{W}(y)|}{|x-y|} < \infty. \tag{4}$$

It is easy to see that $(\mathscr{B}, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined by

$$\|\mathscr{W}\| = \sup_{x\neq y} \frac{|\mathscr{W}(x) - \mathscr{W}(y)|}{|x-y|}, \tag{5}$$

for all $\mathscr{W} \in \mathscr{B}$.

For the computational convenience, we define an operator $\mathscr{W} : \mathscr{A} \longrightarrow \mathbb{R}$ and write functional equation (2) as

$$\begin{aligned}\mathscr{W}(x) = {} & \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + (1-x)\varsigma \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) \\ & + (1-\varsigma)x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) \\ & + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)),\end{aligned} \tag{6}$$

where $0 < \vartheta_1, \vartheta_2 < 1$ and $\lambda_1, \lambda_2 \in \mathscr{A}$. Our objective is to investigate the existence and uniqueness of a solution to functional equation (6) by using the fixed point technique. We begin with the following outcome.

Theorem 2. *Let $0 < \vartheta_1, \vartheta_2 < 1$ and $\lambda_1, \lambda_2, \varsigma \in \mathscr{A}$ such that $\varpi < 1$, where*

$$\begin{aligned}\varpi := {} & |(2\varsigma - 1)((1-\vartheta_1)\lambda_1 + (1-\vartheta_2)\lambda_2) + (1-\varsigma) \\ & \cdot ((1-\vartheta_1) + (1-\vartheta_2)) + 2(\vartheta_1 + \vartheta_2)|.\end{aligned} \tag{7}$$

If there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$\begin{aligned}(Z\mathscr{W})(x) = {} & \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma(1-x) \\ & \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma) \\ & \cdot x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma) \\ & \cdot (1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)),\end{aligned} \tag{8}$$

for all $x \in \mathscr{A}$, then, Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Proof. Let $\mathscr{W}_1, \mathscr{W}_2 \in \Lambda$. For each distinct points $x, y \in \mathscr{A}$, we obtain

$$\begin{aligned}
& \frac{|(Z\mathscr{W}_1 - Z\mathscr{W}_2)(x) - (Z\mathscr{W}_1 - Z\mathscr{W}_2)(y)|}{|x-y|} \\
& = \left|\frac{1}{x-y}\left[\varsigma x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1)\right.\right. \\
& \quad + \varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)\lambda_2) \\
& \quad + (1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) \\
& \quad + (1-\varsigma)(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2) \\
& \quad \cdot (1-\lambda_2)) - \varsigma y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1) \\
& \quad - \varsigma(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) \\
& \quad - (1-\varsigma)y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) \\
& \quad \left.\left. - (1-\varsigma)(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))\right]\right| \\
& = \left|\frac{1}{x-y}\left[\varsigma x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1) - \varsigma x(\mathscr{W}_1 - \mathscr{W}_2)\right.\right. \\
& \quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1) + \varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 x + (1-\vartheta_2)\lambda_2) - \varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2) + (1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma)(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma)(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) + \varsigma x(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1) - \varsigma y(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1) + \varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2) - \varsigma(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2) \\
& \quad \cdot \lambda_2) + (1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1) \\
& \quad \cdot (1-\lambda_1)) - (1-\varsigma)y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1) \\
& \quad \cdot (1-\lambda_1)) + (1-\varsigma)(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2) \\
& \quad \cdot (1-\lambda_2)) - (1-\varsigma)(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2) \\
& \quad \left.\left. \cdot (1-\lambda_2))\right]\right| = \left|\frac{1}{x-y}\left[\varsigma x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1) - \varsigma x\right.\right. \\
& \quad \left. \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1)\right] + \frac{1}{x-y}\left[\varsigma(1-x)\right. \\
& \quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)\lambda_2) - \varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \left. \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2)\right] + \frac{1}{x-y}\left[(1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2)\right. \\
& \quad \cdot (\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma)x(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \left. \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1))\right] + \frac{1}{x-y}\left[(1-\varsigma)(1-x)\right. \\
& \quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma)(1-x) \\
& \quad \left. \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))\right] + \frac{1}{x-y} \\
& \quad \cdot \left[\varsigma x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1) - \varsigma y(\mathscr{W}_1 - \mathscr{W}_2)\right. \\
& \quad \left. \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1)\right] + \frac{1}{x-y}\left[\varsigma(1-x)(\mathscr{W}_1 - \mathscr{W}_2)\right. \\
& \quad \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2) - \varsigma(1-y)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \left. \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2)\right] + \frac{1}{x-y}\left[(1-\varsigma)\right. \\
& \quad \cdot x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma) \\
& \quad \left. \cdot y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1))\right] + \frac{1}{x-y}\left[(1-\varsigma)\right. \\
& \quad \cdot (1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma) \\
& \quad \left.\left. \cdot (1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))\right]\right| \le \vartheta_1 \varsigma x \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2 \varsigma(1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_1(1-\varsigma)x \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2(1-\varsigma)(1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| + |\varsigma \\
& \quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1) - \varsigma(\mathscr{W}_1 - \mathscr{W}_2)(0)| + |\varsigma \\
& \quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) - \varsigma \\
& \quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(0)| + |(1-\varsigma)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1) \\
& \quad \cdot (1-\lambda_1)) - (1-\varsigma)(\mathscr{W}_1 - \mathscr{W}_2)(0)| + |(1-\varsigma)(\mathscr{W}_1 - \mathscr{W}_2) \\
& \quad \cdot (\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma)(\mathscr{W}_1 - \mathscr{W}_2)(0)| = \vartheta_1 \varsigma x \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2 \varsigma(1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_1 \\
& \quad \cdot (1-\varsigma)x\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2(1-\varsigma)(1-x) \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \varsigma(\vartheta_1 y + (1-\vartheta_1)\lambda_1) \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \varsigma(\vartheta_2 y + (1-\vartheta_2)\lambda_2) \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + (1-\varsigma)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + (1-\varsigma)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) \\
& \quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| \le \varpi \|\mathscr{W}_1 - \mathscr{W}_2\|,
\end{aligned} \tag{9}$$

where ϖ is defined in (7). This gives that

$$d(Z\mathscr{W}_1, Z\mathscr{W}_2) = \|Z\mathscr{W}_1 - Z\mathscr{W}_2\| \le \varpi\|\mathscr{W}_1 - \mathscr{W}_2\| = \varpi d(\mathscr{W}_1, \mathscr{W}_2). \tag{10}$$

It follows from $0 < \varpi < 1$ that Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.□ □

We get the following conclusion from Theorem 2 about the uniqueness of a solution to functional equation (6).

Theorem 3. *Functional equation (6) has a unique solution provided that $\varpi < 1$, where ϖ is defined in (7), and there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by*

$$(Z\mathscr{W})(x) = \varsigma x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma(1-x) \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma) \cdot x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma) \cdot (1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)), \tag{11}$$

for all $x \in \mathscr{A}$. Moreover, the iteration $\{\mathscr{W}_n\}$ in Λ is defined by

$$(\mathscr{W}_n)(x) = \varsigma x\mathscr{W}_{n-1}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma(1-x) \cdot \mathscr{W}_{n-1}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma) \cdot x\mathscr{W}_{n-1}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma)(1-x)\mathscr{W}_{n-1}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)), \tag{12}$$

for all $n \in \mathbb{N}$, where $\mathscr{W}_0 \in \Lambda$, converges to the unique solution of functional equation (6) in the sense of the metric d induced by $\|\cdot\|$.

Proof. We derive the result of this theorem by combining the Banach fixed point theorem with Theorem 3.□ □

5. A Certain Case of a T-Maze Experiment with Experimenter-Subject-Controlled Events

It has been highlighted that the examination of any experiment is based on suppositions, which are assembled about the subject. Experiments are classified as contingent and noncontingent, based on the occurrences of the result.

In the previous models on imitation problems such as T-maze experiments with fish, dogs, and humans (see [7, 10, 25]), it was already mentioned that such experiments required a contingent approach; the result of the trials was entirely dependent on the subject's choice. Such types of models required experimenter-subject-controlled events. The two responses **A** and **B** along with outcomes $\mathbf{O}_1$ and $\mathbf{O}_2$ are choosing the right or left side or pushing the right or left button, which coincides with rewarding and nonrewarding or choosing the correct and incorrect side, respectively. Now, we define the probabilities ς_1 and ς_2 which indicate the conditional probability of outcomes $\mathbf{O}_1$ and $\mathbf{O}_2$ of the given alternatives **A** and **B**, respectively, such that

$$\varsigma_1 + \varsigma_2 = 1. \tag{13}$$

Table 4: Four events under conditional probabilities of occurrence.

Events	Outcomes	Transition operators	Probability of occurrence
A	$\mathbf{O}_1$	$P_1x = \vartheta_1 x + (1-\vartheta_1)\lambda_1$	$\varsigma_1 x$
B	$\mathbf{O}_1$	$P_2x = \vartheta_2 x + (1-\vartheta_2)\lambda_2$	$\varsigma_2(1-x)$
A	$\mathbf{O}_2$	$P_3x = \vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)$	$(1-\varsigma_1)x$
B	$\mathbf{O}_2$	$P_4x = \vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)$	$(1-\varsigma_2)(1-x)$

With such conditions, we have Table 4.

We have the following functional equation from the data given above

$$\mathscr{W}(x) = \varsigma_1 x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x) \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1) \cdot x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2)(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)), \tag{14}$$

where $\mathscr{W} : \mathscr{A} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \vartheta_1, \vartheta_2 < 1$, and $\lambda_1, \lambda_2, \varsigma_1, \varsigma_2 \in \mathscr{A}$ with $\varsigma_1 + \varsigma_2 = 1$. We shall begin with the following outcome.

Theorem 4. *Let $0 < \vartheta_1, \vartheta_2 < 1$ and $\lambda_1, \lambda_2, \varsigma_1, \varsigma_2 \in \mathscr{A}$ such that $\varpi^{\mathring{a}} < 1$, where*

$$\varpi^* := \left| \begin{array}{l} (2\lambda_1 - 1)(\varsigma_1(1-\vartheta_1)) + (2\lambda_2 - 1)(\varsigma_2(1-\vartheta_2)) \\ + ((1-\vartheta_1)(1-\lambda_1) + (1-\vartheta_2)(1-\lambda_2)) + 2(\vartheta_1 + \vartheta_2) \end{array} \right|. \tag{15}$$

If there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$(Z\mathscr{W})(x) = \varsigma_1 x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x) \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1) \cdot x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2)(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)), \tag{16}$$

for all $x \in \mathscr{A}$, then, Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Proof. Let $\mathscr{W}_1, \mathscr{W}_2 \in \Lambda$. For each distinct points $x, y \in \mathscr{A}$, we obtain

$$
\begin{aligned}
&\frac{|(Z\mathscr{W}_1 - Z\mathscr{W}_2)(x) - (Z\mathscr{W}_1 - Z\mathscr{W}_2)(y)|}{|x-y|} \\
&= \Big| \frac{1}{x-y} [\varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)\lambda_2) \\
&\quad + (1-\varsigma_1)x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2)(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)) - \varsigma_1 y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1) \\
&\quad \cdot \lambda_1) - \varsigma_2(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) - (1-\varsigma_1) \\
&\quad \cdot y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_2)(1-y) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))] \Big| \\
&= \Big| \frac{1}{x-y} [\varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1) - \varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2) \\
&\quad \cdot \lambda_2) - \varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1) \\
&\quad \cdot x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_1) \\
&\quad \cdot x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2)(1-x) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma_2)(1-x) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) + \varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1) - \varsigma_1 y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1) \\
&\quad + \varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) - \varsigma_2(1-y) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1)x(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_1)y(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot \vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2)(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_2)(1-y)(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1))] \Big| \\
&= \Big| \frac{1}{x-y} [\varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)\lambda_1) - \varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)\lambda_1)] + \frac{1}{x-y} [\varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_2 x + (1-\vartheta_2)\lambda_2) - \varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2)] \\
&\quad + \frac{1}{x-y} [(1-\varsigma_1)x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_1) \\
&\quad \cdot x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1))] + \frac{1}{x-y} [(1-\varsigma_2)(1-x) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma_2)(1-x) \\
&\quad \cdot (\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))] + \frac{1}{x-y} [\varsigma_1 x(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y \\
&\quad + (1-\vartheta_1)\lambda_1) - \varsigma_1 y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1)] + \frac{1}{x-y} \\
&\quad \cdot [\varsigma_2(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) - \varsigma_2(1-y)(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_2 y + (1-\vartheta_2)\lambda_2)] + \frac{1}{x-y} [(1-\varsigma_1)x(\mathscr{W}_1 - \mathscr{W}_2) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_1)y(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1) \\
&\quad \cdot (1-\lambda_1))] + \frac{1}{x-y} [(1-\varsigma_2)(1-x)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) \\
&\quad - (1-\varsigma_2)(1-y)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))] \Big| \le \vartheta_1 \varsigma_1 x \|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2 \varsigma_2 (1-x) \\
&\quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_1(1-\varsigma_1)x\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2(1-\varsigma_2)(1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| + \\
&\quad \cdot |\varsigma_1(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)\lambda_1) - \varsigma_1(\mathscr{W}_1 - \mathscr{W}_2)(0)| + |\varsigma_2(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)\lambda_2) \\
&\quad - \varsigma_2(\mathscr{W}_1 - \mathscr{W}_2)(0)| + |(1-\varsigma_1)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1)) - (1-\varsigma_1)(\mathscr{W}_1 - \mathscr{W}_2)(0)| \\
&\quad + |(1-\varsigma_2)(\mathscr{W}_1 - \mathscr{W}_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2)) - (1-\varsigma_2)(\mathscr{W}_1 - \mathscr{W}_2)(0)| = \vartheta_1 \varsigma_1 x \|\mathscr{W}_1 - \mathscr{W}_2\| \\
&\quad + \vartheta_2 \varsigma_2 (1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_1(1-\varsigma_1)x\|\mathscr{W}_1 - \mathscr{W}_2\| + \vartheta_2(1-\varsigma_2)(1-x)\|\mathscr{W}_1 - \mathscr{W}_2\| \\
&\quad + \varsigma_1(\vartheta_1 y + (1-\vartheta_1)\lambda_1)\|\mathscr{W}_1 - \mathscr{W}_2\| + \varsigma_2(\vartheta_2 y + (1-\vartheta_2)\lambda_2)\|\mathscr{W}_1 - \mathscr{W}_2\| + (1-\varsigma_1) \\
&\quad \cdot (\vartheta_1 y + (1-\vartheta_1)(1-\lambda_1))\|\mathscr{W}_1 - \mathscr{W}_2\| + (1-\varsigma_2)(\vartheta_2 y + (1-\vartheta_2)(1-\lambda_2))\|\mathscr{W}_1 - \mathscr{W}_2\| \le \varpi^\star \\
&\quad \cdot \|\mathscr{W}_1 - \mathscr{W}_2\|,
\end{aligned} \tag{17}
$$

where $\varpi^\star$ is defined in (15). This gives that

$$d(Z\mathscr{W}_1, Z\mathscr{W}_2) = \|Z\mathscr{W}_1 - Z\mathscr{W}_2\| \le \varpi^\star \|\mathscr{W}_1 - \mathscr{W}_2\| = \varpi^\star d(\mathscr{W}_1, \mathscr{W}_2). \tag{18}$$

It follows from $0 < \varpi^\star < 1$ that Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.□ □

We get the following conclusion from Theorem 4 about the uniqueness of a solution to functional equation (14).

Theorem 5. *Functional equation (14) has a unique solution provided that $\varpi^\star < 1$, where $\varpi^\star$ is given in (15), and there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by*

$$
\begin{aligned}
(Z\mathscr{W})(x) = {} & \varsigma_1 x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x) \\
& \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1) \\
& \cdot x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2) \\
& \cdot (1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)),
\end{aligned} \tag{19}
$$

for all $x \in \mathscr{A}$. Moreover, the iteration $\{\mathscr{W}_n\}$ in Λ which is defined by

$$
\begin{aligned}
(\mathscr{W}_n)(x) = {} & \varsigma_1 x \mathscr{W}_{n-1}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma_2(1-x) \\
& \cdot \mathscr{W}_{n-1}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma_1) \\
& \cdot x\mathscr{W}_{n-1}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda_1)) + (1-\varsigma_2) \\
& \cdot (1-x)\mathscr{W}_{n-1}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)),
\end{aligned} \tag{20}
$$

for all $n \in \mathbb{N}$, where $\mathscr{W}_0 \in \Lambda$, converges to the unique solution of functional equation (14) in the sense of the metric d induced by $\|\cdot\|$.

Proof. We derive the result of this theorem by combining the Banach fixed point theorem with Theorem 4.□ □

6. Some Particular Aspects of the Proposed T-Maze Model

In this section, we have discussed some particular cases of the proposed T-maze model.

6.1. Events with Equal Lambda Conditions. This condition (sometimes called commutative condition) says that the transition operators $P_1 - P_4$ (none of them is an identity operator) have the same lambda conditions ($\lambda_1 = \lambda = \lambda_2$). These conditions reduce our transition operators (1) to

$$
\begin{aligned}
P_1 x &= \vartheta_1 x + (1-\vartheta_1)\lambda, \\
P_2 x &= \vartheta_2 x + (1-\vartheta_2)\lambda, \\
P_3 x &= \vartheta_1 x + (1-\vartheta_1)(1-\lambda), \\
P_4 x &= \vartheta_2 x + (1-\vartheta_2)(1-\lambda).
\end{aligned} \tag{21}
$$

Now, we can write our functional equation (6) as

$$
\begin{aligned}
\mathscr{W}(x) = {} & \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda) + \varsigma(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda) \\
& + (1-\varsigma)x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)(1-\lambda)) \\
& + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda)),
\end{aligned} \tag{22}
$$

where $\mathscr{W} : \mathscr{A} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \vartheta_1, \vartheta_2 < 1$ and $\lambda, \varsigma \in \mathscr{A}$. The following conclusions are drawn as a result of Theorem 3.

Corollary 6. *Let $0 < \vartheta_1, \vartheta_2 < 1$ and $\lambda, \varsigma \in \mathscr{A}$ with*

$$|((1-\varsigma)(1-\lambda)+\lambda\varsigma)((1-\vartheta_1)+(1-\vartheta_2))+2(\vartheta_1+\vartheta_2)| < 1. \tag{23}$$

If there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$\begin{aligned}(Z\mathscr{W})(x) = {}& \varsigma x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1)\lambda)+\varsigma(1-x) \\ & \cdot \mathscr{W}(\vartheta_2 x+(1-\vartheta_2)\lambda)+(1-\varsigma) \\ & \cdot x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1)(1-\lambda))+(1-\varsigma) \\ & \cdot (1-x)\mathscr{W}(\vartheta_2 x+(1-\vartheta_2)(1-\lambda)),\end{aligned} \tag{24}$$

for all $x \in \mathscr{A}$, then, Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Corollary 7. *Functional equation (22) has a unique solution provided that*

$$|((1-\varsigma)(1-\lambda)+\lambda\varsigma)((1-\vartheta_1)+(1-\vartheta_2))+2(\vartheta_1+\vartheta_2)| < 1. \tag{25}$$

Also, there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$\begin{aligned}(Z\mathscr{W})(x) = {}& \varsigma x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1)\lambda)+\varsigma(1-x) \\ & \cdot \mathscr{W}(\vartheta_2 x+(1-\vartheta_2)\lambda)+(1-\varsigma) \\ & \cdot x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1)(1-\lambda))+(1-\varsigma) \\ & \cdot (1-x)\mathscr{W}(\vartheta_2 x+(1-\vartheta_2)(1-\lambda)),\end{aligned} \tag{26}$$

for all $x \in \mathscr{A}$. Moreover, the iteration $\{\mathscr{W}_n\}$ in Λ which is defined by

$$\begin{aligned}(\mathscr{W}_n)(x) = {}& \varsigma x\mathscr{W}_{n-1}(\vartheta_1 x+(1-\vartheta_1)\lambda)+\varsigma(1-x) \\ & \cdot \mathscr{W}_{n-1}(\vartheta_2 x+(1-\vartheta_2)\lambda)+(1-\varsigma) \\ & \cdot x\mathscr{W}_{n-1}(\vartheta_1 x+(1-\vartheta_1)(1-\lambda)) \\ & +(1-\varsigma)(1-x)\mathscr{W}_{n-1}(\vartheta_2 x+(1-\vartheta_2)(1-\lambda)),\end{aligned} \tag{27}$$

for all $n \in \mathbb{N}$, where $\mathscr{W}_0 \in \Lambda$, converges to the unique solution the functional equation (22) in the sense of the metric d induced by $\|\cdot\|$.

6.2. Extinction of an Operant Response. In some cases, non-food side responses (turning right or left) frequently by the mouse decrease the probability of that event towards an asymptote to zero. In this situation, we have $\lambda_1 = 0 = \lambda_2$. These conditions reduce our operators (1) to

$$\begin{aligned}P_1 x &= \vartheta_1 x, \\ P_2 x &= \vartheta_2 x, \\ P_3 x &= \vartheta_1 x+(1-\vartheta_1), \\ P_4 x &= \vartheta_2 x+(1-\vartheta_2).\end{aligned} \tag{28}$$

Now, we can write our functional equation (6) as

$$\begin{aligned}\mathscr{W}(x) = {}& \varsigma x\mathscr{W}(\vartheta_1 x)+\varsigma(1-x)\mathscr{W}(\vartheta_2 x)+(1-\varsigma) \\ & \cdot x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1))+(1-\varsigma)(1-x) \\ & \cdot \mathscr{W}(\vartheta_2 x+(1-\vartheta_2)),\end{aligned} \tag{29}$$

where $\mathscr{W} : \mathscr{A} \longrightarrow \mathbb{R}$ is an unknown function such that, $0 < \vartheta_1, \vartheta_2 < 1$, and $\varsigma \in \mathscr{A}$. We have the following corollaries of Theorem 3.

Corollary 8. *For $0 < \vartheta_1, \vartheta_2 < 1$ and $\varsigma \in \mathscr{A}$ with*

$$|(1-\varsigma)((1-\vartheta_1)+(1-\vartheta_2))+2(\vartheta_1+\vartheta_2)| < 1. \tag{30}$$

If there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$\begin{aligned}(Z\mathscr{W})(x) = {}& \varsigma x\mathscr{W}(\vartheta_1 x)+\varsigma(1-x)\mathscr{W}(\vartheta_2 x)+(1-\varsigma) \\ & \cdot x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1))+(1-\varsigma)(1-x) \\ & \cdot \mathscr{W}(\vartheta_2 x+(1-\vartheta_2)),\end{aligned} \tag{31}$$

for all $x \in \mathscr{A}$, then, Z is a Banach contraction mapping with the metric d induced by $\|\cdot\|$.

Corollary 9. *Functional equation (29) has a unique solution provided that*

$$|(1-\varsigma)((1-\vartheta_1)+(1-\vartheta_2))+2(\vartheta_1+\vartheta_2)| < 1. \tag{32}$$

Also, there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$\begin{aligned}(Z\mathscr{W})(x) = {}& \varsigma x\mathscr{W}(\vartheta_1 x)+\varsigma(1-x)\mathscr{W}(\vartheta_2 x) \\ & +(1-\varsigma)x\mathscr{W}(\vartheta_1 x+(1-\vartheta_1))+(1-\varsigma) \\ & \cdot (1-x)\mathscr{W}(\vartheta_2 x+(1-\vartheta_2)),\end{aligned} \tag{33}$$

for all $x \in \mathscr{A}$. Moreover, the iteration $\{\mathscr{W}_n\}$ in Λ which is defined by

$$\begin{aligned}(\mathscr{W}_n)(x) = {}& \varsigma x\mathscr{W}_n(\vartheta_1 x)+\varsigma(1-x)\mathscr{W}_n(\vartheta_2 x)+(1-\varsigma) \\ & \cdot x\mathscr{W}_n(\vartheta_1 x+(1-\vartheta_1))+(1-\varsigma)(1-x) \\ & \cdot \mathscr{W}_n(\vartheta_2 x+(1-\vartheta_2)),\end{aligned} \tag{34}$$

for all $n \in \mathbb{N}$, where $\mathscr{W}_0 \in \Lambda$, converges to the unique solution *of functional equation (29) in the sense of the metric d induced by* $\|\cdot\|$.

Similarly, if the mouse chooses the food side repeatedly, then, the probability of that specific event will increase. Thus, we have $\lambda_1 = 1 = \lambda_2$. *In this situation, our four operators (1) will be*

$$\begin{aligned} P_1 x &= \vartheta_1 x + (1-\vartheta_1), \\ P_2 x &= \vartheta_2 x + (1-\vartheta_2), \\ P_3 x &= \vartheta_1 x, \\ P_4 x &= \vartheta_2 x. \end{aligned} \tag{35}$$

Now, we can write our functional equation (6) as

$$\begin{aligned} \mathscr{W}(x) &= \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)) + \varsigma(1-x) \\ &\quad \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)) + (1-\varsigma) x \mathscr{W}(\vartheta_1 x) \\ &\quad + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x), \end{aligned} \tag{36}$$

where $\mathscr{W}: \mathscr{A} \longrightarrow \mathbb{R}$ *is an unknown function,* $0 < \vartheta_1, \vartheta_2 < 1$, *and* $\varsigma \in \mathscr{A}$. *Now, we have the following corollaries of Theorem 3.*

Corollary 10. *For* $0 < \vartheta_1, \vartheta_2 < 1$ *and* $\varsigma \in \mathscr{A}$ *with*

$$|\varsigma((1-\vartheta_1) + (1-\vartheta_2)) + 2(\vartheta_1 + \vartheta_2)| < 1. \tag{37}$$

If there exists a closed subset Λ *of* $\mathscr{B}$ *such that* Λ *is Z invariant, that is,* $Z(\Lambda) \subseteq \Lambda$, *where Z is the operator from* Λ *defined for each* $\mathscr{W} \in \Lambda$ *by*

$$\begin{aligned} (Z\mathscr{W})(x) &= \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)) + \varsigma(1-x) \\ &\quad \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)) + (1-\varsigma) x \mathscr{W}(\vartheta_1 x) \\ &\quad + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x), \end{aligned} \tag{38}$$

for all $x \in \mathscr{A}$, *then, Z is a Banach contraction mapping with the metric d induced by* $\|\cdot\|$.

Corollary 11. *Functional equation (36) has a unique solution provided that*

$$|\varsigma((1-\vartheta_1) + (1-\vartheta_2)) + 2(\vartheta_1 + \vartheta_2)| < 1. \tag{39}$$

Also, there exists a closed subset Λ *of* $\mathscr{B}$ *such that* Λ *is Z invariant, that is,* $Z(\Lambda) \subseteq \Lambda$, *where Z is the operator from* Λ *defined for each* $\mathscr{W} \in \Lambda$ *by*

$$\begin{aligned} (Z\mathscr{W})(x) &= \varsigma x \mathscr{W}(\vartheta_1 x + (1-\vartheta_1)) + \varsigma(1-x) \\ &\quad \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)) + (1-\varsigma) \\ &\quad \cdot x \mathscr{W}(\vartheta_1 x) + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x), \end{aligned} \tag{40}$$

for all $x \in \mathscr{A}$. *Moreover, the iteration* $\{\mathscr{W}_n\}$ *in* Λ *which is defined by*

$$\begin{aligned} (\mathscr{W}_n)(x) &= \varsigma x \mathscr{W}_n(\vartheta_1 x + (1-\vartheta_1)) + \varsigma(1-x) \\ &\quad \cdot \mathscr{W}_n(\vartheta_2 x + (1-\vartheta_2)) + (1-\varsigma) x \mathscr{W}_n(\vartheta_1 x) \\ &\quad + (1-\varsigma)(1-x)\mathscr{W}_n(\vartheta_2 x), \end{aligned} \tag{41}$$

for all $n \in \mathbb{N}$, *where* $\mathscr{W}_0 \in \Lambda$, *converges to the unique solution of functional equation (36) in the sense of the metric d induced by* $\|\cdot\|$.

6.3. Attraction towards the Specific Choice. In some specific cases, it is possible that the mouse always follows the $\mathbf{O_1}$ outcome and never choose $\mathbf{O_2}$. For such a case, we choose $\lambda_1 = 1$. Similarly, if the mouse chooses $\mathbf{O_2}$ again and again, then, the probability of that event should turn towards zero. It means that $\lambda_2 = 0$. These conditions reduce our four operators (1) to

$$\begin{aligned} P_1 x &= \vartheta_1 x + (1-\vartheta_1), \\ P_2 x &= \vartheta_2 x, \\ P_3 x &= \vartheta_1 x, \\ P_4 x &= \vartheta_2 x + (1-\vartheta_2). \end{aligned} \tag{42}$$

Now, we can write our functional equation (6) as

$$\begin{aligned} \mathscr{W}(x) &= \varsigma x \mathscr{W}(\vartheta_1 x + 1 - \vartheta_1) + \varsigma(1-x)\mathscr{W}(\vartheta_2 x) \\ &\quad + (1-\varsigma) x \mathscr{W}(\vartheta_1 x) + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x + 1 - \vartheta_2), \end{aligned} \tag{43}$$

where $\mathscr{W}: \mathscr{A} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \vartheta_1, \vartheta_2 < 1$, and $\varsigma \in \mathscr{A}$. We have the following results of Theorem 3.

Corollary 12. *For* $0 < \vartheta_1, \vartheta_2 < 1$ *and* $\varsigma \in \mathscr{A}$ *with*

$$|(2\varsigma - 1)(1-\vartheta_1) + (1-\varsigma)((1-\vartheta_1) + (1-\vartheta_2)) + 2(\vartheta_1 + \vartheta_2)| < 1. \tag{44}$$

If there exists a closed subset Λ *of* $\mathscr{B}$ *such that* Λ *is Z invariant, that is,* $Z(\Lambda) \subseteq \Lambda$, *where Z is the operator from* Λ *defined for each* $\mathscr{W} \in \Lambda$ *by*

$$\begin{aligned} (Z\mathscr{W})(x) &= \varsigma x \mathscr{W}(\vartheta_1 x + 1 - \vartheta_1) + \varsigma(1-x)\mathscr{W}(\vartheta_2 x) \\ &\quad + (1-\varsigma) x \mathscr{W}(\vartheta_1 x) + (1-\varsigma) \\ &\quad \cdot (1-x)\mathscr{W}(\vartheta_2 x + 1 - \vartheta_2), \end{aligned} \tag{45}$$

for all $x \in \mathscr{A}$, *then Z is a Banach contraction mapping with the metric d induced by* $\|\cdot\|$.

Corollary 13. *The functional equation (43) has a unique solution provided that*

$$|(2\varsigma - 1)(1-\vartheta_1) + (1-\varsigma)((1-\vartheta_1) + (1-\vartheta_2)) + 2(\vartheta_1 + \vartheta_2)| < 1, \tag{46}$$

and there exists a closed subset Λ of $\mathscr{B}$ such that Λ is Z invariant, that is, $Z(\Lambda) \subseteq \Lambda$, where Z is the operator from Λ defined for each $\mathscr{W} \in \Lambda$ by

$$(Z\mathscr{W})(x) = \varsigma x\mathscr{W}(\vartheta_1 x + 1 - \vartheta_1) + \varsigma(1-x)\mathscr{W}(\vartheta_2 x) + (1-\varsigma)x\mathscr{W}(\vartheta_1 x) + (1-\varsigma)(1-x) \cdot \mathscr{W}(\vartheta_2 x + 1 - \vartheta_2), \tag{47}$$

for all $x \in \mathscr{A}$. Moreover, the iteration $\{\mathscr{W}_n\}$ in Λ which is defined by

$$(\mathscr{W}_n)(x) = \varsigma x\mathscr{W}_n(\vartheta_1 x + 1 - \vartheta_1) + \varsigma(1-x)\mathscr{W}_n(\vartheta_2 x) + (1-\varsigma)x\mathscr{W}_n(\vartheta_1 x) + (1-\varsigma)(1-x) \cdot \mathscr{W}_n(\vartheta_2 x + 1 - \vartheta_2), \tag{48}$$

for all $n \in \mathbb{N}$, where $\mathscr{W}_0 \in \Lambda$, converges to the unique solution of functional equation (36) in the sense of the metric d induced by $\|\cdot\|$.

7. Conclusion

In an animal or a human being, the learning phase may also be analyzed through a sequence of choices between multiple possible answers. The choice sequence is usually unpredictable, even in basic experiments conducted under highly regulated conditions, indicating that probabilities govern the selection of responses. Thus, it is helpful to think of the sequential changes in a sequence of choices in response probabilities from trial to trial. In this paper, we have discussed a particular type of stochastic process related to the T-maze experiment [23, 24], which plays a vital role in observing the behavior of the mouse in a two-choice situation. We analyzed the rat's behavior in such situations and proposed a mathematical model for it. The existence and uniqueness of a solution to the proposed model have been investigated by using the Banach fixed point theorem. To observe the flexibility of the T-maze model, we examined it under the experimenter-subject-controlled events. Furthermore, the proposed model depends only on the contingent reinforcement behavior of rats in which a rat gets the reward for choosing the food side. However, in general, a natural question arises, which we present here to make this interaction more interesting.

7.1. Question. What happens if a mouse does not move to any side (left or right) on a specific trial k and remains sticking to its starting position?

Moreover, one of the critical issues in functional equations is to find out its stability regarding Hyers-Ulam- and Hyers-Ulam-Rassias-type stability (see for the detail, [26–30]). We leave the stability question to the following functional equation as an open problem:

$$\mathscr{W}(x) = \varsigma x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1)\lambda_1) + \varsigma(1-x) \cdot \mathscr{W}(\vartheta_2 x + (1-\vartheta_2)\lambda_2) + (1-\varsigma)x\mathscr{W}(\vartheta_1 x + (1-\vartheta_1) \cdot (1-\lambda_1)) + (1-\varsigma)(1-x)\mathscr{W}(\vartheta_2 x + (1-\vartheta_2)(1-\lambda_2)), \tag{49}$$

where $\mathscr{W} : \mathscr{A} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \vartheta_1, \vartheta_2 < 1$, and $\lambda_1, \lambda_2, \varsigma \in \mathscr{A}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Research Article

Wardowski's Contraction and Fixed Point Technique for Solving Systems of Functional and Integral Equations

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In this manuscript, some tripled fixed point results are presented in the framework of complete metric spaces. Furthermore, Wardowski's contraction was mainly applied to discuss some theoretical results with and without a directed graph under suitable assertions. Moreover, some consequences and supportive examples are derived to strengthen the main results. In the last part of the paper, the obtained theoretical results are used to find a unique solution to a system of functional and integral equations.

1. Introduction

Mathematics is one of the most important ways to understand things that happen around us. Mathematics has been divided into branches, and with its help, one can analyze other sciences. Integral and differential equations are very important tools that can be used to build patterns in order to understand the models that happen around us. The fixed point theory also plays a crucial role in integral and differential equations.

A commonly used tool that has a major role in nonlinear analysis is the fixed point technique, which was given by the well-known scientist Banach. The famous "Banach Contraction Principle" [1] can be announced as follows.

Theorem 1. *Assume that* (Θ, ϖ) *is a complete metric space (CMS) and Q is a self-mapping defined on it, such that for all* $l^1, l^2 \in \Theta$ *and* $\varrho \in [0, 1)$*, the following inequality holds:*

$$\varpi\left(Ql^1, Ql^2\right) \leq \varrho\varpi\left(l^1, l^2\right). \tag{1}$$

Then, there exists a unique fixed point (FP) of Q and the sequence $\{Q^n l^1_\circ\}_{n\in\mathbb{N}}$ *converges to it, for all* $l^1_\circ \in \Theta$.

There are more generalizations of the inequality (1) either by replacing the contraction condition or by using more general spaces. For more results, see [2–4].

We construct the present paper as follows: in Section 1, we recall the background of our work; in Section 2, we give essential results, which are useful for understanding the aim of the paper; in Section 3, we discuss the existence of tripled fixed point (TFP) results via π-contraction mappings in CMS with and without a directed graph (DG); in Section 4, we prove the existence of a solution of different types of tripled systems of functional integral equations; and in Section 5, illustrative examples are given to support our study.

2. Preliminaries

In 2012, a new type of contraction was given by Wardowski, called Wardowski's contraction or π-contraction (see [5]). He generalized the condition in Banach's theorem and stated the following definition.

Definition 2 (see [5]). Assume that (Θ, ϖ) is a metric space and Q is a self-mapping defined on it. We say Q is π-contraction, if there is $\pi \in \mathrm{F}$ and $\ell \in (0,+\infty)$ such that

$$\varpi\left(Ql^1, Ql^2\right) > 0 \text{ implies } \ell + \pi\left(\varpi\left(Ql^1, Ql^2\right)\right) \leq \pi\left(\varpi\left(l^1, l^2\right)\right) \forall l^1, l^2 \in \Theta, \tag{2}$$

where z is the family of all functions $\pi : (0,+\infty) \longrightarrow R$ such that the conditions below hold:

(π_i): for each $l^1, l^2 \in \mathbb{R}^+$, if $l^1 < l^2$, then $\pi(l^1) < \pi(l^2)$; i.e., π is strictly increasing.

(π_{ii}): $\lim_{n \longrightarrow \infty} l_n^1 = 0$ if and only if $\lim_{n \longrightarrow \infty} \pi(l_n^1) = -\infty$, where $\{l_n^1\}_{n \in N}$ is a sequence of positive numbers.

(π_{iii}): $\lim_{l^1 \longrightarrow 0^+} (l^1)^\mu \pi(l^1) = 0$ for each $\mu \in (0, 1)$.

By the inequality (2), the same author introduced some examples of various contractions as follows: for all $l^1, l^2 \in \Theta$ with $\upsilon > 0$ and $Ql^1 \neq Ql^2$,

(i) $\pi_1(\upsilon) = \ln(\upsilon)$, $\varpi(Ql^1, Ql^2)/\varpi(l^1, l^2) \leq e^{-\ell}$

(ii) $\pi_2(\upsilon) = \ln(\upsilon) + \upsilon$, $\quad \varpi(Ql^1, Ql^2) e^{\varpi(Ql^1, Ql^2)} \leq \varpi(l^1, l^2) e^{\varpi(l^1, l^2) - \ell}$

(iii) $\pi_3(\upsilon) = -1/\sqrt{\upsilon}$, $\quad \varpi(Ql^1, Ql^2)\left(1 + \ell\sqrt{\varpi(l^1, l^2)}\right)^2 \leq \varpi(l^1, l^2)$

(iv) $\pi_4(\upsilon) = \ln(\upsilon^2 + \upsilon)$, $\varpi(Ql^1, Ql^2)(1 + \varpi(Ql^1, Ql^2))) \leq e^{-\ell} \varpi(l^1, l^2)(1 + \varpi(l^1, l^2))$

where all functions $\{\pi_n : n = 1, 2, 3, 4\} \in \mathrm{F}$..

Remark 3. The inequality (2) implies that Q is a contractive mapping, that is,

$$\varpi\left(Ql^1, Ql^2\right) < \varpi\left(l^1, l^2\right), \tag{3}$$

for all $l^1, l^2 \in \Theta$ such that $Ql^1 \neq Ql^2$. Hence, every π-contraction is continuous.

Remark 4 (see [6]). Let $\pi(\upsilon) = -1/\sqrt[\zeta]{\upsilon}$, where $\zeta > 1$ and $\upsilon > 0$. Then, $\pi \in \mathrm{F}$.

Wardowski states his theorem as follows.

Theorem 5 (see [6]). *Assume that the mapping Q satisfies the contraction condition (2) on a CMS (Θ, ϖ). Then, there is a unique fixed point of Q and $\{Q^n l_\circ^1\}_{n \in \mathbb{N}}$ converges to the fixed point for all $l_\circ^1 \in \Theta$..*

For two mappings, this theorem has been generalized by Isik [7] as follows.

Lemma 6 (see [7]). *Suppose that (Θ, ϖ) is a CMS and Q and R are self-mappings defined on it. If there is $\ell > 0$ and $\pi \in \mathrm{F}$ such that*

$$\ell + \pi\left(\varpi\left(Ql^1, Rl^2\right)\right) \leq \pi\left(\varpi\left(l^1, l^2\right)\right), \tag{4}$$

for all $l^1, l^2 \in \Theta$ and $\min\{\varpi(Ql^1, Rl^2), \varpi(l^1, l^2)\} > 0$, then there exists a unique common fixed point of Q and R.

A number of papers related to π-contraction and related fixed point theorems in the setting of various spaces were published. See, for example, [8–10].

In the paper [11], the concept of the coupled fixed point (CFP) was presented and studied. In partially ordered metric spaces and abstract spaces, some main results in this direction have been considered. See [12, 13].

Definition 7. Assume that $\Theta \neq \varnothing$ and $Q, R : \Theta \times \Theta \longrightarrow \Theta$ are given mappings; then, the pair $(l^1, l^2) \in \Theta \times \Theta$ is called

(i) CFP of Q if $Q(l^1, l^2) = l^1$ and $Q(l^2, l^1) = l^2$

(ii) a common CFP of Q and R, if $Q(l^1, l^2) = R(l^1, l^2) = l^1$ and $Q(l^2, l^1) = R(l^2, l^1) = l^2$

Using the generalized notion of CFP, Berinde and Borcut [14] defined the notion of a tripled fixed point (TFP) for self-mappings and established some interesting consequences in partially ordered metric spaces. Many other research results were given in this direction, for different spaces and different types of mappings. For additional results, see [4, 15–17].

Definition 8. Assume that$\Theta \neq \varnothing$ and $Q, R : \Theta^3 \longrightarrow \Theta$ (where $\Theta^3 = \Theta \times \Theta \times \Theta$) are given mappings; then, the pair $(l^1, l^2, l^3) \in \Theta^3$ is called a TFP of Q if $Q(l^1, l^2, l^3) = l^1$, $Q(l^2, l^3, l^1) = l^2$, and $Q(l^3, l^1, l^2) = l^3$..

Here, the symbol Ω refers to the set of all TFPs of the mapping Q, that is,

$$\Omega = \left\{\left(l^1, l^2, l^3\right) \in \Theta^3 : Q\left(l^1, l^2, l^3\right) = l^1, Q\left(l^2, l^3, l^1\right) = l^2, \text{and } Q\left(l^3, l^1, l^2\right) = l^3\right\}. \tag{5}$$

In [18], Jachymski used the following notations.

Assume that (Θ, ϖ) is a MS and Y is the diagonal of the Cartesian product $\Theta \times \Theta$. Consider $\sqsupset = (\triangle(\sqsupset), \nabla(\sqsupset))$ a directed graph (DG), where $\triangle(\sqsupset)$ is the set of vertices that coincides with Θ and $\nabla(\sqsupset)$ is the set of edges that contains all loops, i.e., $\nabla(\sqsupset) \supseteq Y$.

The two definitions below were introduced by Chaobankoh and Charoensawa [19].

Definition 9 (see [19]). A mapping $Q : \Theta^3 \longrightarrow \Theta$ is called edge-preserving if

$$\left(\left(l^1, u^1\right), \left(l^2, u^2\right), \left(l^3, u^3\right)\right) \in \nabla(\sqsupset), \tag{6}$$

implies

$$\left[\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right), \left(Q\left(l^2, l^3, l^1\right), Q\left(u^2, u^3, u^1\right)\right), \left(Q\left(l^3, l^1, l^2\right), Q\left(u^3, u^1, u^2\right)\right)\right] \in \nabla(\sqsupset). \tag{7}$$

Definition 10 (see [19]). A mapping $Q : \Theta^3 \longrightarrow \Theta$ is called $\sqsupset$-continuous for each $(l^1, l^2, l^3) \in \Theta^3$ and for any sequence $\{m_j\}_{j\in\mathbb{N}}$ of positive integers with

$$\begin{aligned} Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right) &\longrightarrow l^1, \\ Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right) &\longrightarrow l^2, \\ Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right) &\longrightarrow l^3, \end{aligned} \tag{8}$$

as $j \longrightarrow \infty$, and

$$\begin{aligned} &\left(Q\left(\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), \left(u^1_{m_j+1}, u^2_{m_j+1}, u^3_{m_j+1}\right)\right), Q\left(\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right),\right.\right. \\ &\left.\left(u^2_{m_j+1}, u^3_{m_j+1}, u^1_{m_j+1}\right)\right), Q\left(\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right),\right. \\ &\left.\left.\left(u^3_{m_j+1}, u^1_{m_j+1}, u^2_{m_j+1}\right)\right)\right) \in \nabla(\sqsupset). \end{aligned} \tag{9}$$

Then, for $j \longrightarrow \infty$, we have

$$\begin{aligned} Q\left(Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right), Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right)\right) &\longrightarrow Q\left(l^1, l^2, l^3\right), \\ Q\left(Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right), Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right), Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right)\right) &\longrightarrow Q\left(l^2, l^3, l^1\right), \\ Q\left(Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right), Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right)\right) &\longrightarrow Q\left(l^3, l^1, l^2\right). \end{aligned} \tag{10}$$

Definition 11 (see [18]). Let (Θ, ϖ) be a CMS and $\sqsupset$ be a directed graph. A triple $(\Theta, \varpi, \sqsupset)$ has the property (K) if for any sequence $\{l_m\}_{m\in\mathbb{N}} \subset \Theta$ with $\lim_{n\longrightarrow\infty} l_m = l$ and $(l_m, l_{m+1}) \in \nabla(\sqsupset)$, for $n \in \mathbb{N}$, we get $(l_m, l) \in \nabla(\sqsupset)$.

3. Tripled Fixed Point Technique

Let us start this section by giving the following lemma, which is useful in the proof of the main result.

Lemma 12. *Let (Θ, ϖ) be a CMS and Θ^3 be a Cartesian product. Define a distance $\varpi_{\max}$ by*

$$\begin{aligned} &\varpi_{\max}\left(\left(l^1, l^2, l^3\right), \left(u^1, u^2, u^3\right)\right) \\ &\quad = \max\left\{\varpi\left(l^1, u^1\right), \varpi\left(l^2, u^2\right), \varpi\left(l^3, u^3\right)\right\}. \end{aligned} \tag{11}$$

Then, $(\Theta^3, \varpi_{\max})$ is also CMS.

Proof. The proof of the lemma is obvious. □

Furthermore, let us give the first main theorem of this section.

Theorem 13. *Assume that (Θ, ϖ) is a CMS and $Q, R : \Theta^3 \longrightarrow \Theta$ are continuous mappings. If there is $\ell > 0$ and $\pi \in \mathrm{F}$ such that $\varpi((l^1, l^2, l^3), (u^1, u^2, u^3)) > 0$ implies*

$$\begin{aligned} &\ell + \pi\left(\varpi\left(Q\left(l^1, l^2, l^3\right), R\left(u^1, u^2, u^3\right)\right)\right) \\ &\quad \le \pi\left(\max\left\{\varpi\left(l^1, u^1\right), \varpi\left(l^2, u^2\right), \varpi\left(l^3, u^3\right)\right\}\right), \end{aligned} \tag{12}$$

for each $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, then Q and R have a unique common TFP.

Proof. Define the mappings $M^*, H^* : \Theta^3 \longrightarrow \Theta^3$ by

$$\begin{aligned} M^*\left(l^1, l^2, l^3\right) &= \left(Q\left(l^1, l^2, l^3\right), Q\left(l^2, l^3, l^1\right), Q\left(l^3, l^1, l^2\right)\right), \\ H^*\left(l^1, l^2, l^3\right) &= \left(R\left(l^1, l^2, l^3\right), R\left(l^2, l^3, l^1\right), R\left(l^3, l^1, l^2\right)\right). \end{aligned} \tag{13}$$

Next, for a CMS Θ^3 (see Lemma 12), we shall show that M^* and H^* justify the inequality (4). For $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, let

$$\begin{aligned} &\varpi_{\max}\left(M^*\left(l^1, l^2, l^3\right), H^*\left(u^1, u^2, u^3\right)\right) \\ &\quad = \varpi_{\max}\left(\left(Q\left(l^1, l^2, l^3\right), Q\left(l^2, l^3, l^1\right), Q\left(l^3, l^1, l^2\right)\right),\right. \\ &\qquad \left.\left(R\left(u^1, u^2, u^3\right), R\left(u^2, u^3, u^1\right), R\left(u^3, u^1, u^2\right)\right)\right) \\ &\quad = \max\left\{\varpi\left(Q\left(l^1, l^2, l^3\right), R\left(u^1, u^2, u^3\right)\right), \varpi\left(\left(Q\left(l^2, l^3, l^1\right),\right.\right.\right. \\ &\qquad \left.\left.R\left(u^2, u^3, u^1\right)\right), \varpi\left(Q\left(l^3, l^1, l^2\right), R\left(u^3, u^1, u^2\right)\right)\right\} > 0. \end{aligned} \tag{14}$$

Here, if we put

$$\begin{aligned} D_{QR} = \max&\left\{\varpi\left(Q\left(l^1, l^2, l^3\right), R\left(u^1, u^2, u^3\right)\right), \varpi\left(Q\left(l^2, l^3, l^1\right),\right.\right. \\ &\left.\left.R\left(u^2, u^3, u^1\right)\right), \varpi\left(Q\left(l^3, l^1, l^2\right), R\left(u^3, u^1, u^2\right)\right)\right\}, \end{aligned} \tag{15}$$

then three cases will be discussed for $\varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3)) > 0$ as follows:

($\bigstar_i$): if $D_{QR} = \varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3))$, then, by relation (12), we obtain

$$\begin{aligned} \ell + \pi(\varpi_{\max}(M^*(l^1, l^2, l^3), H^*(u^1, u^2, u^3))) &= \ell + \pi(\varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3))) \\ &\leq \pi(\max\{\varpi(l^1, l^2, l^3), \varpi(u^1, u^2, u^3)\}) \\ &= \pi(\max\{\varpi(l^1, u^1), \varpi(l^2, u^2), \varpi(l^3, u^3)\}). \end{aligned} \tag{16}$$

($\bigstar_{ii}$): if $D_{QR} = \varpi(Q(l^2, l^3, l^1), R(u^2, u^3, u^1))$, then, by (12), we have

$$\begin{aligned} \ell + \pi(\varpi_{\max}(M^*(l^1, l^2, l^3), H^*(u^1, u^2, u^3))) &= \ell + \pi(\varpi(Q(l^2, l^3, l^1), R(u^2, u^3, u^1))) \\ &\leq \pi(\max\{\varpi(l^2, l^3, l^1), \varpi(u^2, u^3, u^1)\}) \\ &= \pi(\max\{\varpi(l^2, u^2), \varpi(l^1, u^1)\}) \\ &\leq \pi(\max\{\varpi(l^1, u^1), \varpi(l^2, u^2), \varpi(l^3, u^3)\}). \end{aligned} \tag{17}$$

($\bigstar_{iii}$): if $D_{QR} = \varpi(Q(l^3, l^1, l^2), R(u^3, u^1, u^2))$, it follows from (12) that

$$\begin{aligned} \ell + \pi(\varpi_{\max}(M^*(l^1, l^2, l^3), H^*(u^1, u^2, u^3))) &= \ell + \pi(\varpi(Q(l^3, l^1, l^2), R(u^3, u^1, u^2))) \\ &\leq \pi(\max\{\varpi(l^3, l^1, l^2), \varpi(u^3, u^1, u^2)\}) \\ &= \pi(\max\{\varpi(l^3, u^3), \varpi(l^2, u^2), \varpi(l^1, u^1)\}). \end{aligned} \tag{18}$$

The above cases prove that the condition (4) is fulfilled. Then, M^* and H^* have a unique common FP $(l'^1, l'^2, l'^3) \in \Theta^3$. This means

$$\begin{aligned} (l'^1, l'^2, l'^3) &= M^*(l'^1, l'^2, l'^3) = (Q(l'^1, l'^2, l'^3), Q(l'^2, l'^3, l'^1), Q(l'^3, l'^1, l'^2)), \\ (l'^1, l'^2, l'^3) &= H^*(l'^1, l'^2, l'^3) = (R(l'^1, l'^2, l'^3), R(l'^2, l'^3, l'^1), R(l'^3, l'^1, l'^2)). \end{aligned} \tag{19}$$

Hence,

$$\begin{aligned} Q(l'^1, l'^2, l'^3) &= R(l'^1, l'^2, l'^3) = l'^1, \\ Q(l'^2, l'^3, l'^1) &= R(l'^2, l'^3, l'^1) = l'^2, \\ Q(l'^3, l'^1, l'^2) &= R(l'^3, l'^1, l'^2) = l'3. \end{aligned} \tag{20}$$

Therefore, $(l'1, l'2, l'3)$ is a common TFP of Q and R. The uniqueness follows immediately from the definition of M^* and H^*. □

A pivotal result follows below by letting $Q = R$ in Theorem 13.

Corollary 14. *Assume that (Θ, ϖ) is a CMS and $Q : \Theta^3 \longrightarrow \Theta$ is a continuous mapping. If there is $\ell > 0$ and $\pi \in$ F such that $\varpi((l^1, l^2, l^3), (u^1, u^2, u^3)) > 0$ implies*

$$\begin{aligned} \ell + \pi(\varpi(Q(l^1, l^2, l^3), Q(u^1, u^2, u^3))) \\ \leq \pi(\max\{\varpi(l^1, u^1), \varpi(l^2, u^2), \varpi(l^3, u^3)\}), \end{aligned} \tag{21}$$

for all $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, then Q has a unique TFP.

Now, we will discuss the existence and uniqueness of a TFP in a CMS with a directed graph.Following the paper [19], we define the set $(\Theta^3)_Q$ by

$$\begin{aligned} (\Theta^3)_Q = \{(l^1, l^2, l^3) \in \Theta^3 : (l^1, Q(l^1, l^2, l^3)), (l^2, Q(l^2, l^3, l^1)), \\ (l^3, Q(l^3, l^1, l^2)) \in \nabla(\sqsupset)\}. \end{aligned} \tag{22}$$

Proposition 15. *Let $Q : \Theta^3 \longrightarrow \Theta$ be an edge-preserving mapping; then, for all $n \in \mathbb{N}$,*

($\dagger_1$): $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset) \Rightarrow [(Q^n(l^1, l^2, l^3), Q^n(u^1, u^2, u^3)), (Q^n(l^2, l^3, l^1), Q^n(u^2, u^3, u^1)), (Q^n(l^3, l^1, l^2), Q^n(u^3, u^1, u^2))] \in \nabla(\sqsupset)$.

($\dagger_2$): $(l^1, l^2, l^3) \in (\Theta^3)_Q \Rightarrow [(Q^n(l^1, l^2, l^3), Q^{n+1}(l^1, l^2, l^3)), (Q^n(l^2, l^3, l^1), Q^{n+1}(l^2, l^3, l^1)), (Q^n(l^3, l^1, l^2), Q^{n+1}(l^3, l^1, l^2))] \in \nabla(\sqsupset)$.

($\dagger_3$): $(l^1, l^2, l^3) \in (\Theta^3)_Q \Rightarrow (Q^n(l^1, l^2, l^3), Q^n(l^2, l^3, l^1), Q^n(l^3, l^1, l^2)) \in (\Theta^3)_Q$.

Proof. ($\dagger_1$): consider $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset)$. Because R is a preserving mapping, we get $(Q(l^1, l^2, l^3), Q(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. Using the same property, we can write $(Q^2(l^1, l^2, l^3), Q^2(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. It follows that, by induction, $(Q^n(l^1, l^2, l^3), Q^n(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. In the same manner, we can prove $(Q^n(l^2, l^3, l^1), Q^n(u^2, u^3, u^1)) \in \nabla(\sqsupset)$ and $(Q^n(l^3, l^1, l^2), Q^n(u^3, u^1, u^2)) \in \nabla(\sqsupset)$.

($\dagger_2$): assume that

$$\begin{aligned} (l^1, l^2, l^3) \in (\Theta^3)_Q : (l^1, Q(l^1, l^2, l^3)), (l^2, Q(l^2, l^3, l^1)), \\ (l^3, Q(l^3, l^1, l^2)) \in \nabla(\sqsupset). \end{aligned} \tag{23}$$

By ($\dagger_1$), we get

$$\begin{aligned} (Q^n(l^1, l^2, l^3), Q^{n+1}(l^1, l^2, l^3)) = (Q^n(l^1, l^2, l^3), \\ Q^n(Q(l^1, l^2, l^3), Q(l^2, l^3, l^1), Q(l^3, l^1, l^2))) \in \nabla(\sqsupset). \end{aligned} \tag{24}$$

Similarly, one can show that $(Q^n(l^2, l^3, l^1), Q^{n+1}(l^2, l^3, l^1)) \in \nabla(\sqsupset)$ and $(Q^n(l^3, l^1, l^2), Q^{n+1}(l^3, l^1, l^2)) \in \nabla(\sqsupset)$.($\dagger_3$): from ($\dagger_2$), we get

$$\begin{aligned} (Q^n(l^1, l^2, l^3), Q(Q^n(l^1, l^2, l^3), Q^n(l^2, l^3, l^1), Q^n(l^3, l^1, l^2))) \\ = (Q^n(l^1, l^2, l^3), Q^{n+1}(l^1, l^2, l^3)) \in \nabla(\sqsupset), \end{aligned} \tag{25}$$

which is equivalent to $(Q^n(l^1, l^2, l^3), Q^n(l^2, l^3, l^1), Q^n(l^3, l^1, l^2)) \in (\Theta^3)_Q$. □

Definition 16. We say $Q : \Theta^3 \longrightarrow \Theta$ is a $\pi^{\sqsupset}$-rational contraction mapping ($\pi^{\sqsupset}$-RCM) if

($\heartsuit_i$): Q is edge-preserving.

($\heartsuit_{ii}$): there is a positive $\ell > 0$ such that

$$\ell + \pi\left(\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right)\right) \leq \pi\left(\frac{\varpi\left(l^1, u^1\right) + \varpi\left(l^2, u^2\right) + \varpi\left(l^3, u^3\right)}{3}\right), \quad (26)$$

for all $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset)$, with $\varpi(Q(l^1, l^2, l^3), Q(u^1, u^2, u^3)) > 0$.

Lemma 17. *Assume that (Θ, ϖ) is a MS and $Q : \Theta^3 \longrightarrow \Theta$ is a $\pi^{\sqsupset}$-RCM with a DG $\sqsupset$. Then, for each $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset)$, we have*

$$\pi\left(\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right)\right) \leq \pi\left(\frac{\varpi\left(l^1, u^1\right) + \varpi\left(l^2, u^2\right) + \varpi\left(l^3, u^3\right)}{3}\right) - n\ell. \quad (27)$$

Proof. Let $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset)$. Because Q is edge-preserving, we have

$$\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right) \in \nabla(\sqsupset). \quad (28)$$

It follows from Proposition 15 ($\dagger_1$) that $(Q^n(l^1, l^2, l^3), Q^n(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. Because Q is a $\pi^{\sqsupset}$-RCM, one can obtain

$$\begin{aligned} \pi\left(\varpi\left(Q^2\left(l^1, l^2, l^3\right), Q^2\left(u^1, u^2, u^3\right)\right)\right) &= \pi\left(\varpi\left(Q\left[Q\left(l^1, l^2, l^3\right), Q\left(l^2, l^3, l^1\right), Q\left(l^3, l^1, l^2\right)\right], Q\left[Q\left(u^1, u^2, u^3\right), Q\left(u^2, u^3, u^1\right), Q\left(u^3, u^1, u^2\right)\right]\right)\right) \\ &\leq \pi\left(\frac{\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right) + \varpi\left(Q\left(l^2, l^3, l^1\right), Q\left(u^2, u^3, u^1\right)\right) + \varpi\left(Q\left(l^3, l^1, l^2\right), Q\left(u^3, u^1, u^2\right)\right)}{3}\right) \\ &\quad - \ell \leq \pi\left(\frac{\varpi\left(l^1, u^1\right) + \varpi\left(l^2, u^2\right) + \varpi\left(l^3, u^3\right)}{3}\right) - 2\ell. \end{aligned} \quad (29)$$

Moreover, we have the same result if $(Q^n(l^2, l^3, l^1), Q^n(u^2, u^3, u^1)) \in \nabla(\sqsupset)$ or $(Q^n(l^3, l^1, l^2), Q^n(u^3, u^1, u^2)) \in \nabla(\sqsupset)$. Therefore, the conclusion follows using mathematical induction. □

Lemma 18. *Let $Q : \Theta^3 \longrightarrow \Theta$ be a $\pi^{\sqsupset}$-RCM on a CMS (Θ, ϖ) with a DG $\sqsupset$. Then, for each $(l^1, l^2, l^3) \in (\Theta^3)_Q$, there is $(l'1, l'2, l'3) \in \Theta^3$ such that $Q^n(l^1, l^2, l^3)_{n\in\mathbb{N}} \longrightarrow l'^1$, $Q^n(l^2, l^3, l^1)_{n\in\mathbb{N}} \longrightarrow l'^2$, and $Q^n(l^3, l^1, l^2)_{n\in\mathbb{N}} \longrightarrow l'^3$, as .*

Proof. Suppose that $(l^1, l^2, l^3) \in (\Theta^3)_Q$; then,

$$\left(l^1, Q\left(l^1, l^2, l^3\right)\right), \left(l^2, Q\left(l^2, l^3, l^1\right)\right), \left(l^3, Q\left(l^3, l^1, l^2\right)\right) \in \nabla(\sqsupset). \quad (30)$$

Set $u^1 = Q(l^1, l^2, l^3)$, $u^2 = Q(l^2, l^3, l^1)$, and $u^3 = Q(l^3, l^1, l^3)$ in the contractive condition of Lemma 17 and put

$$\mathfrak{J}\left(l^1, l^1, l^2\right) = \frac{\varpi\left(l^1, Q\left(l^1, l^2, l^3\right)\right) + \varpi\left(l^2, Q\left(l^2, l^3, l^1\right)\right) + \varpi\left(l^3, Q\left(l^3, l^1, l^2\right)\right)}{3}. \quad (31)$$

Then, we have

$$\pi\left(\varpi\left(Q^n\left(l^1, l^2, l^3\right), Q^n\left(Q\left(l^1, l^2, l^3\right), Q\left(l^2, l^3, l^1\right), Q\left(l^3, l^1, l^2\right)\right)\right)\right) \leq \pi\left(\mathfrak{J}\left(l^1, l^1, l^2\right)\right) - n\ell, , \quad (32)$$

or equivalently,

$$\pi\left(\varpi\left(Q^n\left(l^1, l^2, l^3\right), Q^{n+1}\left(l^1, l^2, l^3\right)\right)\right) \leq \pi\left(\mathfrak{J}\left(l^1, l^1, l^2\right)\right) - n\ell. \quad (33)$$

As $n \longrightarrow \infty$ in (33), we can write

$$\lim_{n\longrightarrow\infty} \pi\left(\varpi\left(Q^n\left(l^1, l^2, l^3\right), Q^{n+1}\left(l^1, l^2, l^3\right)\right)\right) = -\infty. \quad (34)$$

Applying condition (π_{ii}), we have

$$\lim_{n\longrightarrow\infty} \varpi\left(Q^n\left(l^1, l^2, l^3\right), Q^{n+1}\left(l^1, l^2, l^3\right)\right) = 0. \quad (35)$$

Using the same steps, we can write

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \varpi\left(Q^n\left(l^2, l^3, l^1\right), Q^{n+1}\left(l^2, l^3, l^1\right)\right) = 0, \\ &\lim_{n\longrightarrow\infty} \varpi\left(Q^n\left(l^3, l^1, l^2\right), Q^{n+1}\left(l^3, l^1, l^2\right)\right) = 0. \end{aligned} \quad (36)$$

From condition (π_{iii}) to (35), there exists $\mu \in (0,1)$ such that

$$\lim_{n\longrightarrow\infty}\left(\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\right)^{\mu}\cdot\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)=0. \tag{37}$$

For all $n\in\mathbb{N}$, the inequality (33) yields

$$\begin{aligned}&\left(\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\right)^{\mu}\\&\times\left[\pi\left(\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\right)-\pi\left(\mathfrak{F}\left(l^1,l^1,l^2\right)\right)\right]\\&\le -n\left(\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\right)^{\mu}\ell\le 0.\end{aligned} \tag{38}$$

Take into account (35) and (37), and taking $n\longrightarrow\infty$ in (38), we get

$$\lim_{n\longrightarrow\infty}\left(\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\right)^{\mu}=0. \tag{39}$$

By (39), there is $n_\circ\in\mathbb{N}$, such that $n(\varpi(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)))^{\mu}\le 1$, for all $n\ge n_\circ$, or

$$\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\le\frac{1}{n^{1/\mu}},\quad \text{for all } n\ge n_\circ. \tag{40}$$

Using (40), for $m>n\ge n_\circ$, we get

$$\begin{aligned}\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^m\left(l^1,l^2,l^3\right)\right)&\le\varpi\left(Q^n\left(l^1,l^2,l^3\right),Q^{n+1}\left(l^1,l^2,l^3\right)\right)\\&+\cdots+\varpi\left(Q^{m-1}\left(l^1,l^2,l^3\right),Q^m\left(l^1,l^2,l^3\right)\right)\le\sum_{n\ge n_\circ}^{\infty}\frac{1}{n^{1/\mu}}.\end{aligned} \tag{41}$$

The convergence series $\sum_{n\ge n_\circ}^{\infty}1/n^{1/\mu}$ leads to $\lim_{n,m\longrightarrow\infty}\varpi(Q^n(l^1,l^2,l^3),Q^m(l^1,l^2,l^3))=0$. Moreover, we can write

$$\begin{aligned}&\lim_{n,m\longrightarrow\infty}\varpi\left(Q^n\left(l^2,l^3,l^1\right),Q^m\left(l^2,l^3,l^1\right)\right)=0,\\&\lim_{n,m\longrightarrow\infty}\varpi\left(Q^n\left(l^3,l^1,l^2\right),Q^m\left(l^3,l^1,l^2\right)\right)=0.\end{aligned} \tag{42}$$

This implies that $\{Q^n(l^1,l^2,l^3)\}_{n\in\mathbb{N}}$, $\{Q^n(l^2,l^3,l^1)\}_{n\in\mathbb{N}}$, and $\{Q^n(l^3,l^1,l^2)\}_{n\in\mathbb{N}}$ are Cauchy sequences in Θ. The completeness of (Θ,ϖ) tells us that there is $(l'1,l'2,l'3)\in\Theta^3$ such that $Q^n(l^1,l^2,l^3)_{n\in\mathbb{N}}\longrightarrow {l'}^1$, $Q^n(l^2,l^3,l^1)_{n\in\mathbb{N}}\longrightarrow {l'}^2$, and $Q^n(l^3,l^1,l^2)_{n\in\mathbb{N}}\longrightarrow {l'}^3$, as $n\longrightarrow\infty$. Then, the conclusion follows. □

Theorem 19. *Assume that $Q:\Theta^3\longrightarrow\Theta$ is a $\pi^{\sqsupset}$-RCM on a CMS (Θ,ϖ) with a DG $\sqsupset$. Let*

(a) Q be $\sqsupset$-continuous

(b) the triple $(\Theta,\varpi,\sqsupset)$ satisfy the property (K) and π be continuous

Then, $\Omega\ne\varnothing$ if and only if $(\Theta^3)_Q\ne\varnothing$.

Proof. Let $\Omega\ne\varnothing$; then, there is $(l'1,l'2,l'3)\in\Omega$ so that $(l'1,Q(l'1,l'2,l'3))=(l'1,l'1)\in Y\subset\nabla(\sqsupset)$, $(l'2,Q(l'2,l'3,l'1))=(l'2,l'2)\in Y\subset\nabla(\sqsupset)$, and $(l'3,Q(l'3,l'1,l'2))=(l'3,l'3)\in Y\subset\nabla(\sqsupset)$. So, $(l'1,Q(l'1,l'2,l'3)),(l'2,Q(l'2,l'3,l'1)),(l'3,Q(l'3,l'1,l'2))\in\nabla(\sqsupset)$; this yields $(\Theta^3)_Q\ne\varnothing$.

Conversely, suppose that $(\Theta^3)_Q\ne\varnothing$; this means that there is $(l^1,l^2,l^3)\in(\Theta^3)_Q$ such that

$$\left(l^1,Q\left(l^1,l^2,l^3\right)\right),\left(l^2,Q\left(l^2,l^3,l^1\right)\right),\left(l^3,Q\left(l^3,l^1,l^2\right)\right)\in\nabla(\sqsupset). \tag{43}$$

Considering a positive integer sequence $\{n_i\}_{i\in\mathbb{N}}$, by Proposition 15 $(\dagger_2)$, we obtain

$$\left(Q^{n_i}\left(l^1,l^2,l^3\right),Q^{n_i+1}\left(l^1,l^2,l^3\right)\right)\in\nabla(\sqsupset). \tag{44}$$

Applying Lemma 18 to (44), there are $l'1,l'2,l'3\in\Theta$ such that

$$\begin{aligned}&\lim_{i\longrightarrow\infty}Q^{n_i}\left(l^1,l^2,l^3\right)=l'1,\\&\lim_{i\longrightarrow\infty}Q^{n_i}\left(l^2,l^3,l^1\right)=l'2,\\&\lim_{i\longrightarrow\infty}Q^{n_i}\left(l^3,l^1,l^2\right)=l'3.\end{aligned} \tag{45}$$

(a) Let Q be $\sqsupset$-continuous; then, we get

$$\begin{aligned}&Q\left(Q^{n_i}\left(l^1,l^2,l^3\right),Q^{n_i}\left(l^2,l^3,l^1\right),\right.\\&\left.Q^{n_i}\left(l^3,l^1,l^2\right)\right)\longrightarrow Q\left(l'1,l'2,l'3\right),\quad \text{as } i\longrightarrow\infty.\end{aligned} \tag{46}$$

From triangle inequality, it follows

$$\begin{aligned}&\varpi\left(Q\left(l'1,l'2,l'3\right),l'1\right)\\&\le\varpi\left(Q\left(l'1,l'2,l'3\right),Q^{n_i+1}\left(l^1,l^2,l^3\right)\right)\\&+\varpi\left(Q^{n_i+1}\left(l^1,l^2,l^3\right),l'1\right).\end{aligned} \tag{47}$$

The continuity of Q and (45) leads to $\varpi(Q(l'1,l'2,l'3),l'1)=0$, i.e., $Q(l'1,l'2,l'3)=l'1$. Similarly, one can show that $Q(l'2,l'3,l'1)=l'2$ and $Q(l'3,l'1,l'2)=l'3$. Hence, a triple $(l'1,l'2,l'3)$ is a TFP of Q and $\Omega\ne 0$.

(b) If a triple $(\Theta,\varpi,\sqsupset)$ satisfies the property (K), then we get

$$\varpi\left(Q^{n}\left(l^{1}, l^{2}, l^{3}\right), l'^{1}\right) \in \nabla(\sqsupset). \quad (48)$$

Again, by the triangle inequality, we have

$$\begin{aligned}\varpi\left(Q\left(l'^{1}, l'^{2}, l'^{3}\right), l'^{1}\right) &\leq \varpi\left(Q\left(l'^{1}, l'^{2}, l'^{3}\right), Q^{n+1}\left(l^{1}, l^{2}, l^{3}\right)\right)\\&\quad+\varpi\left(Q^{n+1}\left(l^{1}, l^{2}, l^{3}\right), l'^{1}\right)\\&\leq \varpi\left(Q\left(l'^{1}, l'^{2}, l'^{3}\right), Q\left(Q^{n}\left(l^{1}, l^{2}, l^{3}\right), Q^{n}\left(l^{2}, l^{3}, l^{1}\right), Q^{n}\left(l^{3}, l^{1}, l^{2}\right)\right)\right)\\&\quad+\varpi\left(Q^{n+1}\left(l^{1}, l^{2}, l^{3}\right), l'^{1}\right).\end{aligned} \quad (49)$$

Using mapping π yields

$$\begin{aligned}&\pi\left(\varpi\left(Q\left(l'^{1}, l'^{2}, l'^{3}\right), l'^{1}\right)\right)-\varpi\left(Q^{n+1}\left(l^{1}, l^{2}, l^{3}\right), l'^{1}\right)\big)\\&\leq \pi\left(\varpi\left(Q\left(l'^{1}, l'^{2}, l'^{3}\right), Q\left(Q^{n}\left(l^{1}, l^{2}, l^{3}\right), Q^{n}\left(l^{2}, l^{3}, l^{1}\right), Q^{n}\left(l^{3}, l^{1}, l^{2}\right)\right)\right)\right)\\&\leq \pi\left(\frac{\varpi\left(t'^{1}, Q^{n}\left(l^{1}, l^{2}, l^{3}\right)\right)+\varpi\left(l'^{2}, Q^{n}\left(l^{2}, l^{3}, l^{1}\right)\right)+\varpi\left(l'^{3}, Q^{n}\left(l^{3}, l^{1}, l^{2}\right)\right)}{3}\right)-\tau.\end{aligned} \quad (50)$$

As $n \longrightarrow \infty$ in (50), we obtain that $\pi(\varpi(Q(l'^{1}, l'^{2}, l'^{3}), l'^{1})) \leq -\infty$, that is, $\varpi(Q(l'^{1}, l'^{2}, l'^{3}), l'^{1})=0$, i.e., $Q(l'^{1}, l'^{2}, l'^{3})=l'^{1}$. Similarly, one can prove that $Q(l'^{2}, l'^{3}, l'^{1})=l'^{2}$ and $Q(l'^{3}, l'^{1}, l'^{2})=l'^{3}$. So $(l'^{1}, l'^{2}, l'^{3}) \in \Omega$. □

4. Applications

The fixed point theory is a very important tool in nonlinear analysis, due to its applications in various domains (see [20, 21]).

Before stating the main results of this section, we need the following lemma.

Lemma 20 (see [22]). *Assume that* $\varphi_{\ell}^{\zeta}:[0,\infty) \longrightarrow [0,\infty)$ *is a function defined by*

$$\varphi_{\ell}^{\zeta}(e)=\frac{e}{\left(1+\ell \sqrt[\zeta]{e}\right)^{\zeta}}, \quad (51)$$

for $\zeta>1$ *and* $\ell>0$. *Then,*

(i) $\varphi_{\ell}^{\zeta}(e)$ *is strictly increasing*

(ii) $\varphi_{\ell}^{\zeta}(0)=0$ *and* $\varphi_{\ell}^{\zeta}(e)$ *is a concave function*

(iii) *for* $e, r \in [0,\infty)$, $|\varphi_{\ell}^{\zeta}(r)-\varphi_{\ell}^{\zeta}(e)| \leq \varphi_{\ell}^{\zeta}(|r-e|)$

4.1. System of Tripled Functional Equations. The fixed point technique contributes to the study of dynamic programming, which is considered an essential tool in optimization problems such as the study of dynamic economic models. This technique has been studied by many researchers to give a unique solution to a system of functional equations via suitable contraction conditions in various spaces. For more results, we refer to Bhakta and Mitra [23], Liu [24], Pathak et al. [25], Zhang [26], and Bellman and Lee [27].

Consider a system of tripled functional equations below:

$$\begin{cases} z\left(l^{1}\right)=\sup_{l^{2} \in D}\left\{c\left(l^{1}, l^{2}\right)+J\left(l^{1}, l^{2}, z\left(o\left(l^{1}, l^{2}\right)\right), b\left(o\left(l^{1}, l^{2}\right)\right), a\left(o\left(l^{1}, l^{2}\right)\right)\right)\right\},\\ b\left(l^{1}\right)=\sup_{l^{2} \in D}\left\{c\left(l^{1}, l^{2}\right)+J\left(l^{1}, l^{2}, b\left(o\left(l^{1}, l^{2}\right)\right), a\left(o\left(l^{1}, l^{2}\right)\right), z\left(o\left(l^{1}, l^{2}\right)\right)\right)\right\},\\ a\left(l^{1}\right)=\sup_{l^{2} \in D}\left\{c\left(l^{1}, l^{2}\right)+J\left(l^{1}, l^{2}, a\left(o\left(l^{1}, l^{2}\right)\right), z\left(o\left(l^{1}, l^{2}\right)\right), b\left(o\left(l^{1}, l^{2}\right)\right)\right)\right\},\end{cases} \quad (52)$$

where S and D are state and decision spaces, respectively, $l^{1} \in S$, $o: S \times D \longrightarrow S$, $c: S \times D \longrightarrow \mathbb{R}$, and $J: S \times D \times \mathbb{R}^{3} \longrightarrow \mathbb{R}$.

We denote the set of all bounded real-valued functions on a nonempty set S, by A_{S}. Define

$$\|\nu\|=\sup_{l^{1} \in S}\left|\nu\left(l^{1}\right)\right|, \quad (53)$$

for any $\nu \in A_{S}$. Moreover, on A_{S}, define a distance as follows:

$$\varpi(r, u)=\sup_{l^{1} \in S}\left|r\left(l^{1}\right)-u\left(l^{1}\right)\right|, \quad (54)$$

for all $r, u \in A_{S}$. Clearly, the pair (A_{S}, ϖ) is a CMS.

Problem (52) will be considered via the two hypotheses below:

($\ddagger_{i}$): the functions $c: S \times D \longrightarrow \mathbb{R}$ and $J: S \times D \times \mathbb{R}^{3} \longrightarrow \mathbb{R}$ are bounded.

($\ddagger_{ii}$): for all $l^{1} \in S$, $l^{2} \in D$, and $z, b, e, z^{*}, b^{*}, e^{*} \in \mathbb{R}$, for $\zeta > 1$ and $\ell>0$, we have

$$\begin{aligned}&\left|J\left(l^{1}, l^{2}, z, b, a\right)-J\left(l^{1}, l^{2}, z^{*}, b^{*}, a^{*}\right)\right|\\&\leq \frac{\max\left\{\left|z-z^{*}\right|,\left|b-b^{*}\right|,\left|a-a^{*}\right|\right\}}{\left(1+\ell \sqrt[\zeta]{\max\left\{\left|z-z^{*}\right|,\left|b-b^{*}\right|,\left|a-a^{*}\right|\right\}}\right)^{\zeta}}.\end{aligned} \quad (55)$$

Theorem 21. *Using the hypotheses* ($\ddagger_{i}$) *and* ($\ddagger_{ii}$) *on* $A_{S} \times A_{S}$, *the problem (52) has a unique bounded common solution.*

Proof. On the space A_{S}, let us define an operator Q as follows:

$$\begin{aligned}Q(z, b, a)\left(l^{1}\right)=\sup_{l^{2} \in D}&\left\{c\left(l^{1}, l^{2}\right)+J\left(l^{1}, l^{2}, z\left(o\left(l^{1}, l^{2}\right)\right),\right.\right.\\&\left.\left.b\left(o\left(l^{1}, l^{2}\right)\right), a\left(o\left(l^{1}, l^{2}\right)\right)\right)\right\},\end{aligned} \quad (56)$$

for each $(z, b, a) \in A_{S}$ and $l^{1} \in S$. The boundedness of the functions c and J assures that the mapping Q is well defined.

Suppose that $(z, b, a), (z^{*}, b^{*}, a^{*}) \in A_{S} \times A_{S}$, and take

$$\begin{aligned}\chi_{z^{*}, b^{*}, a^{*}}^{z, b, a}=\max&\left\{\left|z\left(o\left(l^{1}, l^{2}\right)\right)-z^{*}\left(o\left(l^{1}, l^{2}\right)\right)\right|,\right.\\&\left|b\left(o\left(l^{1}, l^{2}\right)\right)-b^{*}\left(o\left(l^{1}, l^{2}\right)\right)\right|,\\&\left.\left|a\left(o\left(l^{1}, l^{2}\right)\right)-a^{*}\left(o\left(l^{1}, l^{2}\right)\right)\right|\right\},\end{aligned}$$

$$\begin{aligned}\vartheta^{z,b,a}_{z^*,b^*,a^*} = \max\{&\|z(o(l^1,l^2)) - z^*(o(l^1,l^2))\|,\\ &\|b(o(l^1,l^2)) - b^*(o(l^1,l^2))\|,\\ &\|a(o(l^1,l^2)) - a^*(o(l^1,l^2))\|\}.\end{aligned} \tag{57}$$

Then, by hypothesis $(\ddagger_{ii})$, we have

$$\begin{aligned}&\varpi(Q(z,b,a),Q(z^*,b^*,a^*))\\ &= \sup_{l^1\in S}\left|Q(z,b,a)(l^1) - Q(z^*,b^*,a^*)(l^1)\right|\\ &= \sup_{l^1\in S}\left|\sup_{l^2\in D}\{c(l^1,l^2) + J(l^1,l^2,z(o(l^1,l^2)),b(o(l^1,l^2)),a(o(l^1,l^2)))\}\right.\\ &\quad\left.-\sup_{l^2\in D}\{c(l^1,l^2) + J(l^1,l^2,z^*(o(l^1,l^2)),b^*(o(l^1,l^2)),a^*(o(l^1,l^2)))\}\right|\\ &= \sup_{l^1\in S}\left\{\sup_{l^2\in D}\left|J(l^1,l^2,z(o(l^1,l^2)),b(o(l^1,l^2)),a(o(l^1,l^2)))\right.\right.\\ &\quad\left.\left.-J(l^1,l^2,z^*(o(l^1,l^2)),b^*(o(l^1,l^2)),a^*(o(l^1,l^2)))\right|\right\}\\ &\le \sup_{l^1\in S}\left\{\sup_{l^2\in D}\left(\frac{\chi^{z,b,a}_{z^*,b^*,a^*}}{\left(1+\ell\sqrt[\zeta]{\chi^{z,b,a}_{z^*,b^*,a^*}}\right)^\zeta}\right)\right\}\\ &\le \sup_{l^1\in S}\left(\frac{\vartheta^{z,b,a}_{z^*,b^*,a^*}}{\left(1+\ell\sqrt[\zeta]{\vartheta^{z,b,a}_{z^*,b^*,a^*}}\right)^\zeta}\right)\\ &\le \frac{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}{\left(1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}\right)^\zeta},\end{aligned} \tag{58}$$

where the nondecreasing character of φ^ζ_ℓ was used (Lemma 20). Then,

$$\begin{aligned}&\varpi(Q(z,b,e),Q(z^*,b^*,e^*))\\ &\quad\le \frac{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(e,e^*)\}}{\left(1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(e,e^*)\}}\right)^\zeta}.\end{aligned} \tag{59}$$

Taking $\sqrt[\zeta]{\ }$ on both sides, we have

$$\begin{aligned}&\sqrt[\zeta]{\varpi(Q(z,b,e),Q(z^*,b^*,a^*))}\\ &\quad\le \frac{\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}}{1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}},\end{aligned} \tag{60}$$

or equivalently,

$$\begin{aligned}&\frac{1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}}{\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}}\\ &\quad\le \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a),Q(z^*,b^*,a^*))}},\end{aligned} \tag{61}$$

yields

$$\begin{aligned}&\frac{1}{\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}} + \ell\\ &\quad\le \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a),Q(z^*,b^*,a^*))}},\end{aligned} \tag{62}$$

and this leads to

$$\begin{aligned}&\ell - \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a),Q(z^*,b^*,a^*))}}\\ &\quad\le -\frac{1}{\sqrt[\zeta]{\max\{\varpi(z,z^*),\varpi(b,b^*),\varpi(a,a^*)\}}}.\end{aligned} \tag{63}$$

This confirms that the inequality (21) of Corollary 14 holds with $\pi(v) = (-1/\sqrt[\zeta]{v}) \in \mathrm{F}$ (Remark 4). Then, it follows that the operator Q has a unique TFP. At the same time, it is a unique bounded solution of the problem (52) on $A_S \times A_S$. □

4.2. Tripled System of the First Type of Integral Equations. In this subsection, the theoretical results of Corollary 14 will be applied to discuss the existence and uniqueness of a solution of an integral equation tripled system. Let us consider the following system:

$$\begin{cases} l^1(e) = k(e) + \int_0^1 Y(e,r,l^1(r),l^2(r),l^3(r))dr,\\ l^2(e) = k(e) + \int_0^1 Y(e,r,l^2(r),l^3(r),l^1(r))dr,\\ l^3(e) = k(e) + \int_0^1 Y(e,r,l^3(r),l^1(r),l^2(r))dr,\end{cases} \tag{64}$$

where $k(e)$ is defined for all $e \in [0,1]$.

Consider $C[0,1]$, the set of all real continuous functions defined on $[0,1]$, and together with the distance defined above, we can notice that $(C[0,1],\varpi)$ is a CMS.

Now, we discuss the problem (64) according to the assumptions below:

($\spadesuit_1$): $k : [0,1] \longrightarrow \mathbb{R}$ is a continuous function.

($\spadesuit_2$): $Y : [0,1]\times[0,1]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function verifying

$$\begin{aligned}&\left|Y(e,r,l^1(r),l^2(r),l^3(r)) - Y(e,r,l'^1(r),l'^2(r),l'^3(r))\right|\\ &\quad\le \frac{\max\{|l^1-l'^1|,|l^2-l'^2|,|l^3-l'^3|\}}{\left(1+\ell\sqrt[\zeta]{\max\{|l^1-l'^1|,|l^2-l'^2|,|l^3-l'^3|\}}\right)^\zeta},\end{aligned} \tag{65}$$

for each $e, r \in [0,1]$ and $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \mathbb{R}$, and $\ell > 0$ and $\zeta > 1$.

Furthermore, let us present the main theorem of this subsection.

Theorem 22. *There is a unique solution of system (64) $(l'^1, l'^2, l'^3) \in (C[0,1])^3$, as long as the conditions $(\spadesuit_1)$ and $(\spadesuit_2)$ are satisfied.*

Proof. Define a mapping Q on $C[0,1]$ as follows:

$$Q(l^1, l^2, l^3)(e) = k(e) + \int_0^1 Y\left(e, r, l^1(r), l^2(r), l^3(r)\right)dr, \quad (66)$$

for all $l^1, l^2, l^3 \in C[0,1]$. In virtue of $(\spadesuit_1)$ and $(\spadesuit_2)$, we conclude that $Q(l^1, l^2, l^3) \in C[0,1]$ for each $l^1, l^2, l^3 \in C[0,1]$. Thus, we can write

$$Q : (C[0,1])^3 \longrightarrow C[0,1]. \quad (67)$$

Let $\varpi(Q(l^1, l^2, l^3), Q(l'^1, l'^2, l'^3)) > 0$; then, for $e \in [0,1]$, we get

$$\begin{aligned}
&\left|Q(l^1, l^2, l^3)(e) - Q\left(l'^1, l'^2, l'^3\right)(t)\right| \\
&= \left|\int_0^1 Y\left(e, r, l^1(r), l^2(r), l^3(r)\right)dr - \int_0^1 Y\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)d\right| \\
&\le \int_0^1 \left|Y\left(e, r, l^1(r), l^2(r), l^3(r)\right) - Y\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)\right|dr \\
&\le \int_0^1 \left(\frac{\max\left|l^1(r) - l'^1(r)\right|, \left|l^2(r) - l'^2(r)\right|, \left|l^3(r) - l'^3(r)\right|\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\left|l^1(r) - l'^1(r)\right|, \left|l^2(r) - l'^2(r)\right|, \left|l^3(r) - l'^3(r)\right|\right\}}\right)^\zeta}\right)dr \\
&\le \int_0^1 \left(\frac{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}\right)^\zeta}\right)dr \\
&= \frac{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}\right)^\zeta},
\end{aligned} \quad (68)$$

where the nondecreasing characters of φ_ℓ^ζ were used (Lemma 20). Thus,

$$\begin{aligned}
&\varpi\left(Q(l^1, l^2, l^3), Q\left(l'^1, l'^2, l'^3\right)\right) \\
&\le \frac{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}\right)^\zeta}.
\end{aligned} \quad (69)$$

By the same approach used at the inequalities (60)–(62), we get

$$\begin{aligned}
&\ell - \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1, l^2, l^3), Q\left(l'^1, l'^2, l'^3\right)\right)}} \\
&\le - \frac{1}{\sqrt[\zeta]{\max\left\{\varpi\left(l^1(r), l'^1(r)\right), \varpi\left(l^2(r), l'^2(r)\right), \varpi\left(l^3(r), l'^3(r)\right)\right\}}}.
\end{aligned} \quad (70)$$

Hence, the hypotheses of Corollary 14 are fulfilled on $\pi(\upsilon) = (-1/\sqrt[\zeta]{\upsilon}) \in \digamma$ (Remark 4). There is a unique TFP of the mapping Q. In other words, there is $(l'^1, l'^2, l'^3) \in (C[0,1])^3$ such that

$$\begin{cases}
l'^1(e) = \Gamma\left(l_\circ'^1, l_\circ'^2, l_\circ'^3\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^1(r), l_\circ'^2(r), l_\circ'^3(r)\right)dr, \\
l'^2(e) = \Gamma\left(l_\circ'^2, l_\circ'^3, l_\circ'^1\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^2(r), l_\circ'^3(r), l_\circ'^1(r)\right)dr, \\
l'^3(e) = \Gamma\left(l_\circ'^3, l_\circ'^1, l_\circ'^2\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^3(r), l_\circ'^1(r), l_\circ'^2(r)\right)dr.
\end{cases} \quad (71)$$

□

4.3. Tripled System of the Second Type of Integral Equations. Let us consider the following type of system of integral equations:

$$\begin{cases}
l^1(e) = \int_0^M W(e, r)X\left(e, r, l^1(r), l^2(r), l^3(r)\right)dr, \\
l^2(e) = \int_0^M W(e, r)X\left(e, r, l^2(r), l^3(r), l^1(r)\right)dr, \\
l^3(e) = \int_0^M W(e, r)X\left(e, r, l^3(r), l^1(r), l^2(r)\right)dr,
\end{cases} \quad (72)$$

where $e, r \in [0, M]$ with $M > 0$.

This subsection is devoted to discussing the influence of the theoretical results of a DG for solving this new type of system of integral equations.

Let $\Pi = C([0, M], \mathbb{R}^n)$ endowed with $\|l^1\| = \max_{0\le e\le M}|l^1(e)|$ for all $l^1 \in \Pi$. Moreover, define a partial order on a graph $\sqsupset$ as follows, for all $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \Pi$ and $e \in [0, M]$,

$$\begin{aligned}
l^1 \le l'^1 &\Leftrightarrow l^1(e) \le l'^1(e), \\
l^2 \le l'^2 &\Leftrightarrow l^2(e) \le l'^2(e), \\
l^3 \le l'^3 &\Leftrightarrow l^3(e) \le l'^3(e).
\end{aligned} \quad (73)$$

Thus, $(\Pi, \|.\|)$ is a CMS equipped with a directed graph $\sqsupset$. Let $(\Pi, \|.\|, \sqsupset)$ be a triple with the property (K) and

$$\begin{aligned}
(\Pi^3 = \Pi \times \Pi \times \Pi)_Q = \{&\left(l^1, l^2, l^3\right) \in \Pi^3 : l^1 \le Q\left(l^1, l^2, l^3\right), l^2 \\
&\le Q\left(l^2, l^3, l^1\right), \text{and } l^3 \le Q\left(l^3, l^1, l^2\right)\}.
\end{aligned} \quad (74)$$

We can state the main theorem.

Theorem 23. *There is at least one solution of the problem (72), if the assumptions below are fulfilled:*

$(\blacktriangleright_1)$: *the functions $X : [0, M] \times [0, M] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $W : [0, M] \times [0, M] \longrightarrow \mathbb{R}^n$ are continuous such that*

$$\int_0^M W(e,r)dr \leq \frac{M}{\ell}, \tag{75}$$

for all $e, r \in [0, M]$ *and* $\ell > 0$.

($\blacktriangleright_2$): *for all* $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \mathbb{R}^n$ *with* $l^1 \leq l'^1$, $l^2 \leq l'^2$, *and* $l^3 \leq l'^3$, *we have*

$$X\left(e, r, l^1(r), l^2(r), l^3(r)\right) \leq X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right), \tag{76}$$

for all $e, r \in [0, M]$.

($\blacktriangleright_3$): *there are* $\ell > 0$ *and* $\zeta > 1$ *so that*

$$\begin{aligned}&\left|X\left(e, r, l^1(r), l^2(r), l^3(r)\right) - X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)\right| \\ &\leq \frac{\ell}{M} \frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}},\end{aligned} \tag{77}$$

for any $e, r \in [0, M]$.

($\blacktriangleright_4$): *there is* $(l^1_\circ, l^2_\circ, l^3_\circ) \in \Pi^3$ *such that*

$$\begin{cases} l^1_\circ(e) = \int_0^M W(e,r) X\left(e, r, l^1_\circ(r), l^2_\circ(r), l^3_\circ(r)\right) dr, \\ l^2_\circ(e) = \int_0^M W(e,r) X\left(e, r, l^2_\circ(r), l^3_\circ(r), l^1_\circ(r)\right) dr, \\ l^3_\circ(e) = \int_0^M W(e,r) X\left(e, r, l^3_\circ(r), l^1_\circ(r), l^2_\circ(r)\right) dr, \end{cases} \tag{78}$$

where $e \in [0, M]$.

Proof. Let the mapping $Q : \Pi^3 \longrightarrow \Pi$ defined by

$$\begin{aligned}&Q\left(l^1, l^2, l^3\right)(e) \\ &= \int_0^M W(e,r) X\left(e, r, l^1(r), l^2(r), l^3(r)\right) dr, \quad e \in [0, M].\end{aligned} \tag{79}$$

Next, we show that Q is $\sqsupset$-edge-preserving. Let $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \Pi$ with $l^1 \leq l'^1$, $l^2 \leq l'^2$, and $l^3 \leq l'^3$. Then, we get

$$\begin{aligned}Q\left(l^1, l^2, l^3\right)(e) &= \int_0^M W(e,r) X\left(e, r, l^1(r), l^2(r), l^3(r)\right) dr \\ &\leq \int_0^M W(e,r) X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right) dr \\ &= Q\left(l'^1, l'^2, t'^3\right)(e).\end{aligned} \tag{80}$$

Using the same steps, we can write $Q(l^2, l^3, l^1)(e) \leq Q(l'^2, l'3, l'^1,)(e)$ and $Q(l^3, l^1, l^2)(e) \leq Q(l'^3, l'^1, l'^2)(e)$, for all $e \in [0, M]$.

Next, from ($\blacktriangleright_4$), it follows

$$\begin{aligned}\left(\Pi^3\right)_Q = \{&\left(l^1, l^2, l^3\right) \in \Pi^3 : l^1 \leq Q\left(l^1, l^2, l^3\right), l^2 \leq Q\left(l^2, l^3, l^1\right), \\ &\text{and } l^3 \leq Q\left(l^3, l^1, l^2\right)\} \neq \varnothing.\end{aligned} \tag{81}$$

Ultimately,

$$\begin{aligned}&\left|Q\left(l^1, l^2, l^3\right)(e) - Q\left(l'^1, l'^2, l'^3\right)(e)\right| \\ &\leq \int_0^M W(e,r)\left|X\left(e, r, l^1(r), l^2(r), l^3(r)\right) - X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)\right| dr \\ &\leq \int_0^M W(e,r)\left(\frac{\ell}{M}\frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}}\right) dr \\ &\leq \frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right.}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}} \\ &\leq \frac{\left((1/3)\left[\left\|l^1 - l'^1\right\| + \left\|l^2 - l'^2\right\| + \left\|l^3 - l'^3\right\|\right]\right)}{\left(1 + \ell\sqrt[\zeta]{(1/3)\left[\left\|l^1 - l'^1\right\| + \left\|l^2 - l'^2\right\| + \left\|l^3 - l'^3\right\| \right|}\right)^{\zeta}} \\ &= \frac{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}{\left(1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}\right)^{\zeta}},\end{aligned} \tag{82}$$

where the nondecreasing characters of φ^{ζ}_{ℓ} were used (Lemma 20). Thus,

$$\begin{aligned}&\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(l'^1, l'^2, l'^3\right)\right) \\ &\quad\leq \frac{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}{\left(1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}\right)^{\zeta}}.\end{aligned} \tag{83}$$

Taking $\sqrt[\zeta]{}$ on both sides, we get

$$\begin{aligned}&\sqrt[\zeta]{\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(l'^1, l'^2, l'^3\right)\right)} \\ &\quad\leq \frac{\sqrt[\zeta]{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}}{1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}},\end{aligned} \tag{84}$$

or equivalently,

$$\frac{1+\ell\sqrt[\zeta]{\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3}}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}} \le \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}}, \tag{85}$$

or

$$\frac{1}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}}+\ell \le \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}}. \tag{86}$$

This leads to

$$\ell-\frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}} \le -\frac{1}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}}. \tag{87}$$

Hence, Q is $\pi^{\beth}$-RCM with $\pi(\upsilon)=(-1/\sqrt[\zeta]{\upsilon})\in \mathrm{F}$ (Remark 4). So, it follows from Theorem 19 that the mapping Q has a TFP, which is a solution of the problem (72). □

5. Examples

In this section, some important examples satisfying theoretical consequences are presented, with the role to strengthen our results.

Example 1. Assume that $\Theta=[0,\infty)$ and $\varpi(l^1,l^2)=|l^1-l^2|$. Clearly, (Θ,ϖ) is a CMS. Define $Q,R:\Theta^3\longrightarrow\Theta$ by

$$Q\left(l^1,l^2,l^3\right)=\begin{cases}\dfrac{l^1-4l^2+l^3}{5}, l^1+l^3\ge 4l^2,\end{cases}$$
$$R\left(l^1,l^2,l^3\right)=\begin{cases}\dfrac{l^1-l^2+l^3}{5}, l^1+l^3\ge l^2,\end{cases} \tag{88}$$

for all $l^1,l^2,l^3\in\Theta$.

Moreover, from the definition, Q and R are continuous. Let $\pi:\mathbb{R}^+\longrightarrow\mathbb{R}$ be a function defined by $\pi(\upsilon)=\ln(\upsilon)$ for $\upsilon>0$. To verify the inequality (12) of Theorem 13, we consider the following cases:

($\bullet_1$): if $l^1+l^3\ge 4l^2$ and $l'^1+l'^3\ge l'^2$, we can write $Q(l^1,l^2,l^3)=(l^1-4l^2+l^3)/5$ and $R(l'^1,l'^2,l'^3)=(l'^1-l'^2+l'^3)/5$; then,

$$\begin{aligned}\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'^2,l'^3\right)\right)&=\left|\frac{l^1-4l^2+l^3}{5}-\frac{l'^1-l'^2+l'^3}{5}\right|\\&=\left|\frac{l^1-l'^1}{5}+\frac{l'^2-4l^2}{5}+\frac{l^3-l'^3}{5}\right|\le\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l'^2-4l^2}{5}\right|\\&\quad+\left|\frac{l^3-l'^3}{5}\right|\le\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l^2-l'^2}{5}\right|+\left|\frac{l^3-l'^3}{5}\right|\\&\le\frac{3}{5}\max\left\{\left|l^1-l'^1\right|,\left|l^2-l'^2\right|,\left|l^3-l'^3\right|\right\}\\&=\frac{3}{5}\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}.\end{aligned} \tag{89}$$

Taking π into account, we can write

$$\begin{aligned}&\ln\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'^2,l'^3\right)\right)\right)\\&\le\ln\left(\frac{3}{5}\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right)\\&=\ln\left(\frac{3}{5}\right)+\ln\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right),\end{aligned} \tag{90}$$

or

$$\begin{aligned}&\ln\left(\frac{5}{3}\right)+\ln\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)\right)\\&\le\ln\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right),\end{aligned} \tag{91}$$

which leads to

$$\begin{aligned}&\ell+\pi\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)\right)\\&\le\pi\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right).\end{aligned} \tag{92}$$

($\bullet_2$): if $l^1+l^3\ge 4l^2$ and $l'^1+l'^3<l'^2$, we have $Q(l^1,l^2,l^3)=(l^1-4l^2+l^3)/5$ and $R(l'^1,l'^2,l'^3)=0$; then,

$$\begin{aligned}\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)&=\left|\frac{l^1-4l^2+l^3}{5}-0\right|\\&\le\left|\frac{\left(l^1-l'^1\right)+\left(l'^2-l^2\right)+\left(l^3-l'^3\right)+\left(l'^1+l'^3-l'^2\right)}{5}\right|\\&\le\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l'^2-4l^2}{5}\right|+\left|\frac{l^3-l'^3}{5}\right|,\end{aligned} \tag{93}$$

since for all $i, j, k \in \Theta$, $i + j + k \leq 3 \max \{i, j, k\}$, one can get

$$\begin{aligned} &\varpi\left(Q(l^1, l^2, l^3), R\left(l'^1, l'2, l'3\right)\right) \\ &\leq \frac{3}{5} \max \left\{\left|\frac{l^1 - l'^1}{5}\right| + \left|\frac{l'^2 - 4l^2}{5}\right| + \left|\frac{l^3 - l'^3}{5}\right|\right\} \\ &\leq \frac{3}{5} \max \left\{\varpi\left(l^1, l'^1\right), \varpi\left(l^2, l'^2\right), \varpi\left(l^3, l'^3\right)\right\}, \end{aligned} \tag{94}$$

and by the same manner of $(\bullet_1)$, we get (92).

$(\bullet_3)$: if $l^1 + l^3 < 4l^2$ and $l'^1 + l'3 \geq l'^2$, then $Q(l^1, l^2, l^3) = 0$ and $R(l'^1, l'^2, l'^3) = (l'^1 - l'^2 + l'^3)/5$. Hence, by the same method of $(\bullet_2)$, we obtain (92).

$(\bullet_4)$: if $l^1 + l^3 < 4l^2$ and $l'^1 + l'^3 < l'^2$, we get $Q(l^1, l^2, l^3) = 0$ and $R(l'^1, l'^2, l'^3) = 0$; it is trivial.

It follows from $(\bullet_1) - (\bullet_4)$ that the inequality (12) of Theorem 13 with $\ell = \ln(5/3) > 0$ is verified.

Then, $(0, 0, 0) \in \Theta^3$ is a unique common TFP of Q and R.

Example 2. We consider the following tripled system of functional equations:

$$\begin{cases} z(l^1) = \sup\limits_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\dfrac{1}{1 + (l^1)^2} + \dfrac{1}{1 + e^{l^2}} + \dfrac{1}{3} \dfrac{|z(o)|}{\left(1 + 8\sqrt[3]{|z(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|b(o)|}{\left(1 + 5\sqrt[3]{|b(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|a(o)|}{\left(1 + 4\sqrt[3]{|a(o)|}\right)^3} \right) \right\}, \\ b(l^1) = \sup\limits_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\dfrac{1}{1 + (l^1)^2} + \dfrac{1}{1 + e^{l^2}} + \dfrac{1}{3} \dfrac{|b(o)|}{\left(1 + 8\sqrt[3]{|b(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|a(o)|}{\left(1 + 5\sqrt[3]{|a(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|z(o)|}{\left(1 + 4\sqrt[3]{|z(o)|}\right)^3} \right) \right\}, \\ a(l^1) = \sup\limits_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\dfrac{1}{1 + (l^1)^2} + \dfrac{1}{1 + e^{l^2}} + \dfrac{1}{3} \dfrac{|a(o)|}{\left(1 + 8\sqrt[3]{|a(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|z(o)|}{\left(1 + 5\sqrt[3]{|z(o)|}\right)^3} + \dfrac{1}{3} \dfrac{|b(o)|}{\left(1 + 4\sqrt[3]{|b(o)|}\right)^3} \right) \right\}, \end{cases} \tag{95}$$

for $l^1 \in [0, 1]$.

It is clear that the system (95) is a special form of system (52) with $S = [0, 1]$ and $D = \mathbb{R}$. The condition $(\ddagger_i)$ of Theorem 21 is clear. For $(\ddagger_{ii})$, we can write

$$\begin{aligned} &\left|J(l^1, l^2, z(o(l^1, l^2)), b(o(l^1, l^2)), a(o(l^1, l^2)))\right. \\ &\quad \left.- J(l^1, l^2, z^*(o(l^1, l^2)), b^*(o(l^1, l^2)), a^*(o(l^1, l^2)))\right| \\ &\leq \frac{1}{3}\left|\frac{|z(o)|}{\left(1 + 8\sqrt[3]{|z(o)|}\right)^3} - \frac{|z^*(o)|}{\left(1 + 8\sqrt[3]{|z^*(o)|}\right)^3}\right| \\ &\quad + \frac{1}{3}\left|\frac{|b(o)|}{\left(1 + 5\sqrt[3]{|b(o)|}\right)^3} - \frac{|b^*(o)|}{\left(1 + 5\sqrt[3]{|b^*(o)|}\right)^3}\right| \\ &\quad + \frac{1}{3}\left|\frac{|a(o)|}{\left(1 + 4\sqrt[3]{|a(o)|}\right)^3} - \frac{|a^*(o)|}{\left(1 + 4\sqrt[3]{|a^*(o)|}\right)^3}\right| \\ &= \frac{1}{3}\left|\varphi_8^3(|z(o)|) - \varphi_8^3(|z^*(o)|)\right| + \frac{1}{3}\left|\varphi_5^3(|b(o)|) - \varphi_5^3(|b^*(o)|)\right| \\ &\quad + \frac{1}{3}\left|\varphi_4^3(|a(o)|) - \varphi_5^3(|a^*(o)|)\right| \leq \frac{1}{3}\varphi_8^3(||z(o)| - |z^*(o)||) \\ &\quad + \frac{1}{3}\varphi_5^3(||b(o)| - |b^*(o)||) + \frac{1}{3}\varphi_4^3(||a(o)| - |a^*(o)||) \\ &\leq \frac{1}{3}\varphi_8^3(|z(o) - z^*(o)|) + \frac{1}{3}\varphi_5^3(|b(o) - b^*(o)|) \\ &\quad + \frac{1}{3}\varphi_4^3(|a(o) - a^*(o)|) \leq \frac{1}{3}\varphi_8^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\quad + \frac{1}{3}\varphi_5^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\quad + \frac{1}{3}\varphi_4^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\leq 3 \times \frac{1}{3}\varphi_4^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &= \frac{\max\{|z - z^*|, |b - b^*|, |a - a^*|\}}{\left(1 + 4\sqrt[3]{\max\{|z - z^*|, |b - b^*|, |a - a^*|\}}\right)^3}, \end{aligned} \tag{96}$$

where Lemma 20 is used. Hence, $(\ddagger_{ii})$ is satisfied with $\ell = 4$ and $\zeta = 3$. According to Theorem 21, the system (95) has a unique solution in $A_S \times A_S$.

Example 3. Suppose the following tripled system of integral equations:

$$\begin{cases} l^1(w) = e^w + \int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+10\sqrt[5]{l^1(r)}\right)^5} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+7\sqrt[5]{|l^2(r)|}\right)^5} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+6\sqrt[5]{|l^3(r)|}\right)^5} \right) dr, \\ l^2(w) = e^w + \int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+10\sqrt[5]{l^2(r)}\right)^5} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+7\sqrt[5]{|l^3(r)|}\right)^5} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+6\sqrt[5]{|l^1(r)|}\right)^5} \right) dr, \\ l^3(w) = e^w + \int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+10\sqrt[5]{l^3(r)}\right)^5} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+7\sqrt[5]{|l^1(r)|}\right)^5} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+6\sqrt[5]{|l^2(r)|}\right)^5} \right) dr, \end{cases} \quad (97)$$

for $w \in [0, 1]$.

Again, system (97) is a special case of system (64), where $k(w) = e^w$.

It is obvious that the condition ($♠_1$) of Theorem 22 holds. For the condition ($♠_2$), we get

$$\begin{aligned} &\left| Y\left(w, r, l^1(r), l^2(r), l^3(r)\right) - Y\left(w, r, l'^1(r), l'^2(r), l'^3(r)\right) \right| \\ &\le \frac{1}{3}\left| \frac{|l^1(r)|}{\left(1+10\sqrt[5]{|l^1(r)|}\right)^5} - \frac{|l'^1(r)|}{\left(1+10\sqrt[5]{|l'^1(r)|}\right)^5} \right| \\ &\quad + \frac{1}{3}\left| \frac{|l^2(r)|}{\left(1+7\sqrt[5]{|l^2(r)|}\right)^5} - \frac{|l'^2(r)|}{\left(1+7\sqrt[5]{|l'^2(r)|}\right)^5} \right| \\ &\quad + \frac{1}{3}\left| \frac{|l^3(r)|}{\left(1+6\sqrt[5]{|l^3(r)|}\right)^5} - \frac{|l'^3(r)|}{\left(1+6\sqrt[5]{|l'^3(r)|}\right)^5} \right| \\ &= \frac{1}{3}\left|\varphi_{10}^5\left(|l^1(r)|\right) - \varphi_{10}^5\left(|l'^1(r)|\right)\right| \\ &\quad + \frac{1}{3}\left|\varphi_7^5\left(|l^2(r)|\right) - \varphi_7^5\left(|l'^2(r)|\right)\right| \\ &\quad + \frac{1}{3}\left|\varphi_6^5\left(|l^3(r)|\right) - \varphi_7^5\left(|l'^3(r)|\right)\right| \\ &\le \frac{1}{3}\varphi_{10}^5\left(\left||l^1(r)| - |l'^1(r)|\right|\right) + \frac{1}{3}\varphi_7^5\left(\left||l^2(r)| - |l'^2(r)|\right|\right) \\ &\quad + \frac{1}{3}\varphi_6^5\left(\left||l^3(r)| - |l'^3(r)|\right|\right) \le \frac{1}{3}\varphi_{10}^5\left(\left|l^1(r) - l'^1(r)\right|\right) \\ &\quad + \frac{1}{3}\varphi_7^5\left(\left|l^2(r) - l'^2(r)\right|\right) + \frac{1}{3}\varphi_6^5\left(\left|l^3(r) - l'^3(r)\right|\right) \\ &\le \frac{1}{2}\varphi_{10}^5\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &\quad + \frac{1}{2}\varphi_7^5\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &\quad + \frac{1}{3}\varphi_6^5\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &\le 3 \times \frac{1}{3}\varphi_6^5\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &= \frac{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1+6\sqrt[5]{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^5}, \end{aligned} \quad (98)$$

where Lemma 20 was used. Hence, ($♠_2$) holds with $\ell = 6$ and $\zeta = 5$. According to Corollary 14, system (97) has a unique solution $(l'^1, l'2, l'^3) \in (C[0, 1])^3$.

6. Conclusions

The present paper is dedicated to the study of the existence and uniqueness of tripled fixed points in a CMS with and without a directed graph. Common tripled fixed point results are given too. Moreover, some applications of the main results in solving different types of tripled equation systems are presented. Then, using our main results, we study the existence and uniqueness of a solution of some systems of tripled functional and integral equations used in the study of dynamic programming. To sustain our results, the last part of the paper is dedicated to some illustrative examples. Our results come to improve some results from the related literature and give new directions in the study of economic phenomena, using the tripled fixed point technique.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Additional Points

Open Questions. (1) A new research direction can be considered the existence of fixed points in the case of multivalued operators. Which conditions can be imposed in order to obtain the uniqueness of the fixed point for the multivalued operators' case? (2) Moreover, the case of coincidence fixed points and the case of coupled fixed points can be considered for further research proposals.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

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Research Article

Semianalytical Solutions of Some Nonlinear-Time Fractional Models Using Variational Iteration Laplace Transform Method

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In this work, we combined two techniques, the variational iteration technique and the Laplace transform method, in order to solve some nonlinear-time fractional partial differential equations. Although the exact solutions may exist, we introduced the technique VITM that approximates the solutions that are difficult to find. Even a single iteration best approximates the exact solutions. The fractional derivatives being used are in the Caputo-Fabrizio sense. The reliability and efficiency of this newly introduced method is discussed in details from its numerical results and their graphical approximations. Moreover, possible consequences of these results as an application of fixed-point theorem are placed before the experts as an open problem.

1. Introduction and Preliminaries

Almost all the phenomena in science and engineering are naturally modeled in the form of nonlinear differential equations, like Korteweg-de Vries equation [1, 2], nonlinear Schrödinger equation [3–5], alternating current power flow model [6], Richards equation for unsaturated water flow [7–10], Burger equation [11], and gravitational general theory [12].

Recently, the above-mentioned and other nonlinear model equations are solved by using more than one semianalytical and numerical methods, like the Laplace transform method (LTM) [13, 14], variational iteration method (VIM) [15, 16], Newton-Raphson formula (NRF) [17], Adomian decomposition method (ADM) [18], homotopy analysis method (HAM) [19, 20], homotopy perturbation method (HPM) [21], spectral collocation technique [22] and the equation presented in [23].

Nowadays, the techniques of fractional calculus are being employed successfully for better understanding of complex natural phenomena, which not only agree with the ordinary calculus techniques but also give the best results and understanding of the phenomena. Laplace transform is a powerful tool, which has been used in the past decades to solve the ODEs with constant and variable coefficients as well as to solve PDEs. Similarly, in these days, the variational iteration technique, developed by the Chinese mathematician He [15], is also a reliable technique (which was originally developed to solve differential and integrodifferential equations) to solve PDEs. The main drawback of the variational iteration method is that one may have difficulty in calculating the Lagrange multiplier. Currently, much attention is being paid in combining more than one technique to solve a model especially nonlinear models, to get better and rapid results. In this direction the work has been started, and it is observed that the results obtained by combining more than one technique are much better than that of a single technique as discussed in [24, 25].

In the current paper, two techniques, variational iteration technique and the Laplace transform, are being utilized, and the combined technique, the variational iteration transform method (VITM), is employed to handle the nonlinear fractional order partial differential equations, like the Korteweg-de Vries equation [26], Schrödinger equation [27], and Burger equation [28]. The rapid convergence of the method proves that it is a more reliable technique now more than ever than the existing one to solve FPDEs, and it introduces a new significant improvement. In the proceeding sections, method description along with the validity of the results obtained by the technique is presented.

In the study of the fractional differential equations, the Caputo-Fabrizio fractional derivative [29] will be considered. The Caputo-Fabrizio fractional derivative is the most recent fractional derivative which is more effective than the other fractional derivatives present in the literature, in dealing with the initial value problems. First, let us recall the some definitions from the area of fractional calculus.

(1) *Riemann-Liouville Fractional Derivative*. The Riemann-Liouville fractional derivative of a function $f(t)$ is defined to be

$$D^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\varepsilon+1} \int_a^t (t-\tau)^{\varepsilon-\alpha} f(\tau) d\tau, \quad (1)$$

where $\varepsilon \le \alpha < \varepsilon + 1$ or

$$D^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \frac{d^{\kappa}}{dt^{\kappa}} \int_a^t (t-\tau)^{\kappa-\alpha-1} f(\tau) d\tau, \quad (2)$$

where $\kappa - 1 \le \alpha < \kappa$. Both κ and ε are integers.

(2) *Caputo's Fractional Derivative*. Caputo's fractional derivative of $f(t)$ is given by

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(t-\tau)}{(\tau-a)^{\alpha}} d\tau, \quad (3)$$

such that $\alpha \in [0, 1]$.

(3) *Caputo-Fabrizio Derivative*. Let us recall one of the most recent definitions of the fractional derivative Caputo-Fabrizio derivative, as follows. Let $\mathscr{F}(t) \in H^1(a, b), b > a$; then, the Caputo-Fabrizio time fractional derivative of $\mathscr{F}(t)$ is defined as

$$D_t^{\alpha} \mathscr{F}(t) = \frac{\mathscr{M}(a)}{(1-\alpha)} \int_a^t \mathscr{F}'(t) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \quad (4)$$

where $\alpha \epsilon [0, 1]$ and $\mathscr{M}(a)$ is a normalization function that is $\mathscr{M}(0) = \mathscr{M}(1) = 1$.

(4) *Laplace Transform*. Let $f(t)$ be a function; then, its Laplace transform is defined as

$$\mathscr{L}\{f(t)\} = \mathscr{F}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad (5)$$

and the Laplace transform of $f'(t)$ is given by

$$\mathscr{L}\left\{D_t^{(\alpha+n)} f'(t)\right\} = \frac{s^{(n+1)}\mathscr{L}[f(t)] - s^n f(0) - s^{n-1} f'(0) - \cdots - f^{(n)}(0)}{s + \alpha(1-s)}. \quad (6)$$

2. Methodology of VITM

This section is devoted to present the methodology of the proposed technique. Then, let us consider the general time fractional partial differential equation.

$$D_t^{\alpha} \mathscr{X}(\varphi, t) + L(\mathscr{X}(\varphi, t)) + N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t)) = \mathscr{F}(\varphi, t), \quad (7)$$

subject to

$$\mathscr{X}(\varphi, 0) = \mathscr{X}_0, \quad (8)$$

where $L(\mathscr{X}(\varphi, t))$, $N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t))$, and $\mathscr{F}(\varphi, t)$ are linear, nonlinear, and known functions, respectively. Also D_t^{α} is in the Caputo-Fabrizio sense.

Further, we apply the variational iteration method on the above equation. Then, we found the following iterative form:

$$\mathscr{X}_{n+1}(\varphi, t) = \mathscr{X}_n(\varphi, t) + \lambda\{D_t^{\alpha} \mathscr{X}(\varphi, t) + L(\mathscr{X}(\varphi, t)) + N(\mathscr{X}(\varphi, t), \mathscr{Y}(\varphi, t), \mathscr{Z}(\varphi, t)) - \mathscr{F}(\varphi, t)\}. \quad (9)$$

Also, if we apply the Laplace transform on this equation, we transform the variable t, to the new variable s, such that

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) + \lambda \mathscr{L}\left\{D_t^{\alpha} \mathscr{X}_n(\varphi, s) + L\tilde{\mathscr{X}}_n(\varphi, s) + N\left(\tilde{\mathscr{X}}_n(\varphi, s), \tilde{\mathscr{Y}}_n(\varphi, s), \tilde{\mathscr{Z}}_n(\varphi, s)\right) - \mathscr{F}(\varphi, s)\right\}, \quad (10)$$

where $\tilde{\mathscr{X}}_n(\varphi, t)$, etc. are restricted values, which means

$$\delta \tilde{\mathscr{X}}_n(\varphi, t) = 0. \quad (11)$$

Using the following relations:

$$\mathscr{L}\{D^{\alpha} \mathscr{X}_n(\varphi, t)\} = s^{\alpha} \mathscr{X}_n(\varphi, s) - s^{\alpha-1} \mathscr{X}_n(\varphi, 0),$$
$$\mathscr{L}\{\delta D^{\alpha} \mathscr{X}_n(\varphi, t)\} = s^{\alpha} \delta \mathscr{X}_n(\varphi, s) - s^{\alpha-1} \delta \mathscr{X}_n(\varphi, 0), \quad (12)$$

where

$$\delta \mathscr{X}_n(\varphi, 0) = 0. \quad (13)$$

Then, we have

$$\mathscr{L}\{\delta D^{\alpha}\mathscr{X}_n(\varphi, t)\} = s^{\alpha}\delta\mathscr{X}_n(\varphi, s). \tag{14}$$

The optimization conditions,

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi, s)}{\delta\tilde{\mathscr{X}}_n(\varphi, s)} = 0, \quad \delta\widetilde{\mathscr{X}_n} = 0, \delta\widetilde{\mathscr{Y}_n} = 0, \delta\widetilde{\mathscr{Z}_n} = 0, \tag{15}$$

give

$$0 = 1 + \lambda\left\{\frac{s^{\alpha}\delta\tilde{\mathscr{X}}_n(\varphi, s)}{\delta\tilde{\mathscr{X}}_n(\varphi, s)}\right\}. \tag{16}$$

The above equation implies $\lambda = -1/s^{\alpha}$.

On substituting in equation (10), we obtain

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) - \frac{1}{s^{\alpha}}\mathscr{L}\Big\{D_t^{\alpha}\mathscr{X}_n(\varphi, s) + L\tilde{\mathscr{X}}_n(\varphi, s) + N\left(\tilde{\mathscr{X}}_n(\varphi, s), \tilde{\mathscr{Y}}_n(\varphi, s), \tilde{\mathscr{Z}}_n(\varphi, s)\right) - \mathscr{F}(\varphi, t)\Big\}. \tag{17}$$

The inverse Laplace transform gives

$$\mathscr{X}_{n+1}(\varphi, t) = \mathscr{X}_n(\varphi, t) - \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathscr{L}\{D_t^{\alpha}\mathscr{X}_n(\varphi, s) + L\mathscr{X}_n(\varphi, s) + N(\mathscr{X}_n(\varphi, s), \mathscr{Y}_n(\varphi, s), \mathscr{Z}_n(\varphi, s)) - \mathscr{F}(\varphi, s)\}\right\}. \tag{18}$$

Substituting $n = 0, 1, 2, \cdots$, we find the following successive approximations:

$$\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \cdots. \tag{19}$$

3. Applications of VITM on Various FODE Types

In this section, we present and apply VITM on some important FODEs from related literature. Then, our first application takes into consideration the most general time fractional form of the Korteweg-de Vries equation (see [24]).

$$D_t^{\alpha}\mathscr{X} + \alpha_1\mathscr{X}\mathscr{X}_{\varphi} + \beta_1\mathscr{X}_{\varphi\varphi\varphi} = 0, \quad 0 < \alpha \leq 1, \tag{20}$$

subject to

$$\mathscr{X}(\varphi, 0) = \mathscr{X}_0 = \frac{a}{\cosh^2\beta_1\varphi}, \tag{21}$$

where

$$\alpha_1 = \frac{c_0}{2k^2}(\varepsilon c\lambda_3), \tag{22}$$

is the nonlinear parameter and

$$\beta_1 = \frac{c_0 h^2}{6}, \tag{23}$$

is the dispersion parameter.

Applying the variational iteration and Laplace transform, we get

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) + \mathscr{L}\left\{\lambda\left\{D_t^{\alpha}\tilde{\mathscr{X}}_n(\varphi, s) + \alpha_1\tilde{\mathscr{X}}_n(\varphi, s)\frac{\partial\tilde{\mathscr{X}}_n(\varphi, s)}{\partial\varphi} + \beta_1\frac{\partial^3\tilde{\mathscr{X}}_n(\varphi, s)}{\partial\varphi^3}\right\}\right\}. \tag{24}$$

Also, substituting the following relation

$$\mathscr{L}\{D_t^{\alpha}\mathscr{X}_n(\varphi, t)\} = s^{\alpha}\mathscr{X}_n(\varphi, s) - s^{\alpha-1}\mathscr{X}_n(\varphi, 0), \tag{25}$$

and optimality conditions, etc., we get the next results:

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi, s)}{\delta\tilde{\mathscr{X}}_n(\varphi, s)} = 0, \; \delta\widetilde{\mathscr{X}_n} = 0, \lambda = \frac{-1}{s^{\alpha}}. \tag{26}$$

By substitution, we have

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) - \frac{1}{s^{\alpha}}\mathscr{L}\left\{\alpha_1\mathscr{X}_n(\varphi, s)\frac{\partial\mathscr{X}_n(\varphi, s)}{\partial\varphi} + \beta_1\frac{\partial^3\mathscr{X}n(\varphi, s)}{\partial\varphi^3}\right\}. \tag{27}$$

Applying the inverse Laplace transform on the above equation and simplifying, we get

$$\mathscr{X}_{n+1}(\varphi, t) = \mathscr{X}_n(\varphi, t) - \mathscr{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathscr{L}\left\{D_t^{\alpha}\mathscr{X}_n(\varphi, s) + \alpha_1\mathscr{X}_n(\varphi, s)\frac{\partial\mathscr{X}_n(\varphi, s)}{\partial\varphi} + \beta_1\frac{\partial^3\mathscr{X}n(\varphi, s)}{\partial\varphi^3}\right\}\right\}. \tag{28}$$

For $n = 0, 1, 2, \cdots$, the following approximations are obtained: $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \cdots$, such as

$$\begin{aligned}\mathscr{X}_1(\varphi, t) = {} & a\mathrm{sech}^2\beta_1\varphi - \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & \cdot \{-2a^2\alpha_1\beta_1 \sec h^4\beta_1\varphi \tan h\beta_1\varphi \\ & + \beta_1\{16a\beta_1{}^3 \sec h^4\beta_1\varphi \tan h\beta_1\varphi \\ & - 8a\beta_1{}^3 \sec h^2\beta_1\varphi \tan h^3\beta_1\varphi\}\},\end{aligned}$$

$$\mathcal{X}_2(\varphi,t)=a\sec h^2\beta_1\varphi-\frac{t^\alpha}{\Gamma(1+\alpha)}\cdot\left\{-2a^2\alpha_1\beta_1\sec h^4\beta_1\varphi\tan h\beta_1\varphi+\beta_1\left\{16a\beta_1{}^3\sec h^4\beta_1\varphi\tan h\beta_1\varphi-8a\beta_1{}^3\sec h^2\beta_1\varphi\tan h^3\beta_1\varphi\right\}\right\}-\frac{2a^3t^{2\alpha}\alpha_1{}^2\beta_1{}^2\sec h^8\beta_1\varphi}{\Gamma(1+2\alpha)}+\cdots,\tag{29}$$

and so on.

The solution $\mathcal{X}(\varphi,t)$ can be found as

$$\mathcal{X}(\varphi,t)=\lim_{i\longrightarrow\infty}\mathcal{X}_i.\tag{30}$$

As particular examples, let us consider further some versions of time fractional equations.

Example 1. The first particular example is a simple time fractional Korteweg-de Vries equation (see [18]).

$$D_t^\alpha\mathcal{X}-6\mathcal{X}\mathcal{X}_\varphi+\mathcal{X}_{\varphi\varphi\varphi}=0\,;\mathcal{X}(\varphi,0)=6\varphi.\tag{31}$$

Application of the proposed VITM step by step gives

$$\mathcal{X}_{n+1}(\varphi,s)=\mathcal{X}_n(\varphi,s)+\mathcal{L}\left\{\lambda\left\{D_t^\alpha\widetilde{\mathcal{X}}_n(\varphi,s)-6\widetilde{\mathcal{X}}_n(\varphi,s)\frac{\partial\widetilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi}+\frac{\partial^3\widetilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi^3}\right\}\right\}.\tag{32}$$

The optimality conditions, etc. give the following results:

$$\frac{\delta\tilde{\mathcal{X}}_{n+1}(\varphi,s)}{\delta\tilde{\mathcal{X}}_n(\varphi,s)}=0,\delta\widetilde{\mathcal{X}}_n=0,\ \lambda=\frac{-1}{s^\alpha}\tag{33}$$

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi,t)=\mathcal{X}_n(\varphi,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha\mathcal{X}_n(\varphi,s)-6\mathcal{X}_n(\varphi,s)\frac{\partial\mathcal{X}_n(\varphi,s)}{\partial\varphi}+\frac{\partial^3\mathcal{X}n(\varphi,s)}{\partial\varphi^3}\right\}\right\}.\tag{34}$$

For $n=0,1,2,\cdots$, we get the approximations $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\cdots$, such as

$$\mathcal{X}_1(\varphi,t)=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)},$$
$$\mathcal{X}_2(\varphi,t)=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)}-216\varphi t^\alpha\left\{\frac{-1+t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)}-\frac{72t^\alpha}{\Gamma(1+2\alpha)}-\frac{1296t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}.\tag{35}$$

The solution $\mathcal{X}(\varphi,t)$ can be found as

$$\mathcal{X}(\varphi,t)=\lim_{i\longrightarrow\infty}\mathcal{X}_i,\tag{36}$$

that is,

$$\mathcal{X}(\varphi,t)=6\varphi+\frac{216\varphi t^\alpha}{\Gamma(1+\alpha)}-216\varphi t^\alpha\left\{\frac{-1+t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)}-\frac{72t^\alpha}{\Gamma(1+2\alpha)}-\frac{1296t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}+\cdots.\tag{37}$$

For $\alpha=1$, it turns out to be

$$\mathcal{X}(\varphi,t)=6\varphi+216\varphi t+7776\varphi t^2+\cdots,\tag{38}$$

which is the expansion of the exact solution $\mathcal{X}(\varphi,t)=6(\varphi)/(1-36t)$ that confirms the validity of the proposed VITM (see [18]).

Next, let us give a graphical representation of the approximated solution $\mathcal{X}(\varphi,t)$, for different values of α using Mathematica. Moreover, we will also give a graphical 3D representation for the exact solution $\mathcal{X}(\varphi,t)=6(\varphi)/(1-36t)$ (Figure 1(b)). In this way, we show how the proposed technique approaches the exact solution; see Figures 2(a), 2(b), and 1(a), which are the approximations of Figure 1(b). The scale for all the four figures is $-50\geq\varphi\leq50$ and $-50\geq t\leq50$.

Example 2. Let us consider another version of time fractional version of Korteweg-de Vries equation (see [18]).

$$D_t^\alpha\mathcal{X}-6\mathcal{X}\mathcal{X}_\varphi+\mathcal{X}_{\varphi\varphi\varphi}=0,\mathcal{X}(\varphi,0)=-2\sec h^2\varphi.\tag{39}$$

Applying VITM step by step, we obtain

$$\mathcal{X}_{n+1}(\varphi,s)=\mathcal{X}_n(\varphi,s)+\mathcal{L}\left\{\lambda\left\{D_t^\alpha\tilde{\mathcal{X}}_n(\varphi,s)-6\tilde{\mathcal{X}}_n(\varphi,s)\frac{\partial\tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi}+\frac{\partial^3\tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi^3}\right\}\right\}.\tag{40}$$

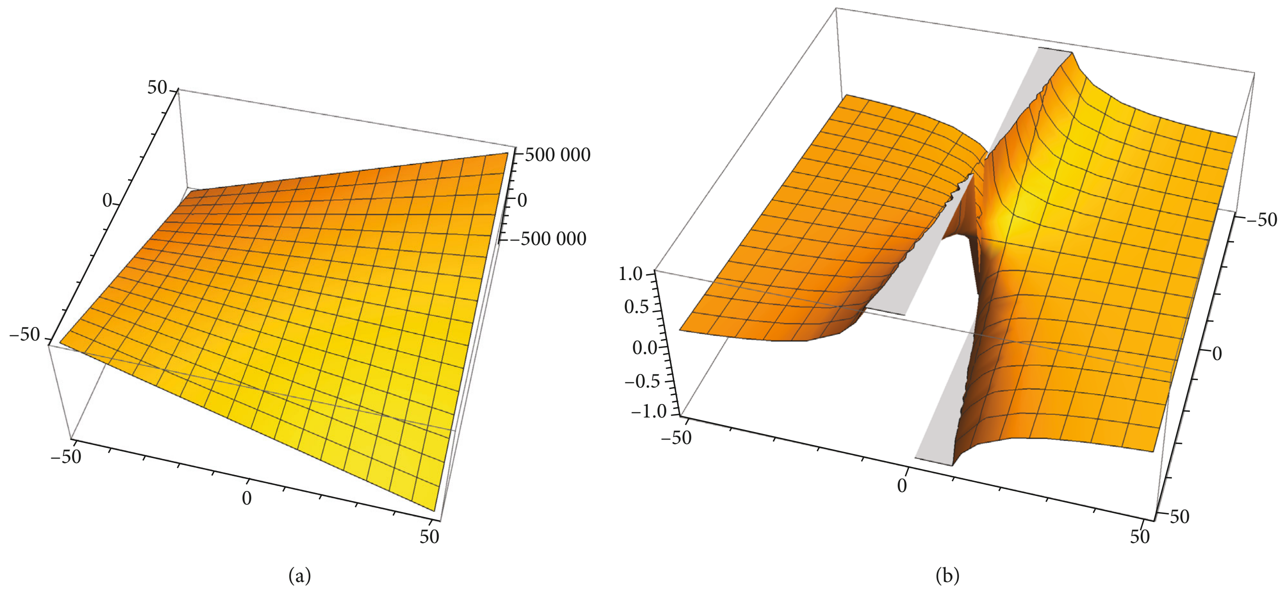


FIGURE 1: 3D representation of X for $\alpha = 1$ (a) and of the exact solution (b).

Using optimality conditions, etc., we get the following results:

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi, s)}{\delta\tilde{\mathscr{X}}_n(\varphi, s)} = 0,\ \delta\tilde{\mathscr{X}}_n = 0,\ \lambda = \frac{-1}{s^\alpha}. \tag{41}$$

Substitution and inverse Laplace transform implies

$$\mathscr{X}_{n+1}(\varphi, t) = \mathscr{X}_n(\varphi, t) - \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{D_t^\alpha \mathscr{X}_n(\varphi, s) - 6\mathscr{X}_n(\varphi, s)\frac{\partial \mathscr{X}_n(\varphi, s)}{\partial\varphi} + \frac{\partial^3 \mathscr{X}n(\varphi, s)}{\partial\varphi^3}\right\}\right\}. \tag{42}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \cdots$, such as

$$\mathscr{X}_1(\varphi, t) = -2\sec h^2\varphi - \frac{t^\alpha\{16\sec h^4\varphi\tan h\varphi + 16\sec h^2\varphi\tan h^3\varphi\}}{\Gamma(1+\alpha)},$$
$$\mathscr{X}_2(\varphi, t) = -2\sec h^2\varphi - \frac{t^\alpha\{16\sec h^4\varphi\tan h\varphi + 16\sec h^2\varphi\tan h^3\varphi\}}{\Gamma(1+\alpha)} - \frac{16t^{-1+\alpha}\sec h^2\varphi + \cdots}{\Gamma(1+\alpha)^2\Gamma(2\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)}. \tag{43}$$

The solution $\mathscr{X}(\varphi, t)$ can be found as

$$\mathscr{X}(\varphi, t) = \lim_{i\longrightarrow\infty}\mathscr{X}_i. \tag{44}$$

Then,

$$\mathscr{X}(\varphi, t) = -2\sec h^2\varphi - \frac{t^\alpha\{16\sec h^4\varphi\tan h\varphi + 16\sec h^2\varphi\tan h^3\varphi\}}{\Gamma(1+\alpha)} - \frac{16t^{-1+\alpha}\sec h^2\varphi + \cdots}{\Gamma(1+\alpha)^2\Gamma(2\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \cdots. \tag{45}$$

For $\alpha = 1$, we have

$$\mathscr{X}(\varphi, t) = -2\sec h^2\varphi - t\{16\sec h^4\varphi\tan h\varphi + 16\sec h^2\varphi\tan h^3\varphi\} + \cdots, \tag{46}$$

which is the expansion of the exact solution, $\mathscr{X}(\varphi, t) = -2\sec h^2(\varphi - 4t)$ (see [18]).

Example 3. Consider the simple time fractional Burgers equation (see [18]).

$$D_t^\alpha \mathscr{X} + \mathscr{X}\mathscr{X}_\varphi + \mathscr{X}_{\varphi\varphi} = 0, \mathscr{X}(\varphi, 0) = \varphi. \tag{47}$$

Applying the proposed VITM step by step, we get

$$\mathscr{X}_{n+1}(\varphi, s) = \mathscr{X}_n(\varphi, s) + \mathscr{L}\left\{\lambda\left\{D_t^\alpha\widetilde{\mathscr{X}}_n(\varphi, s) + \widetilde{\mathscr{X}}_n(\varphi, s)\frac{\partial\widetilde{\mathscr{X}}_n(\varphi, s)}{\partial\varphi} + \frac{\partial^2\widetilde{\mathscr{X}}_n(\varphi, s)}{\partial\varphi^2}\right\}\right\}. \tag{48}$$

The optimality conditions, etc. give the following results:

$$\frac{\delta\tilde{\mathscr{X}}_{n+1}(\varphi, s)}{\delta\widetilde{\mathscr{X}}_n(\varphi, s)} = 0,\ \delta\widetilde{\mathscr{X}}_n = 0, \lambda = \frac{-1}{s^\alpha}. \tag{49}$$

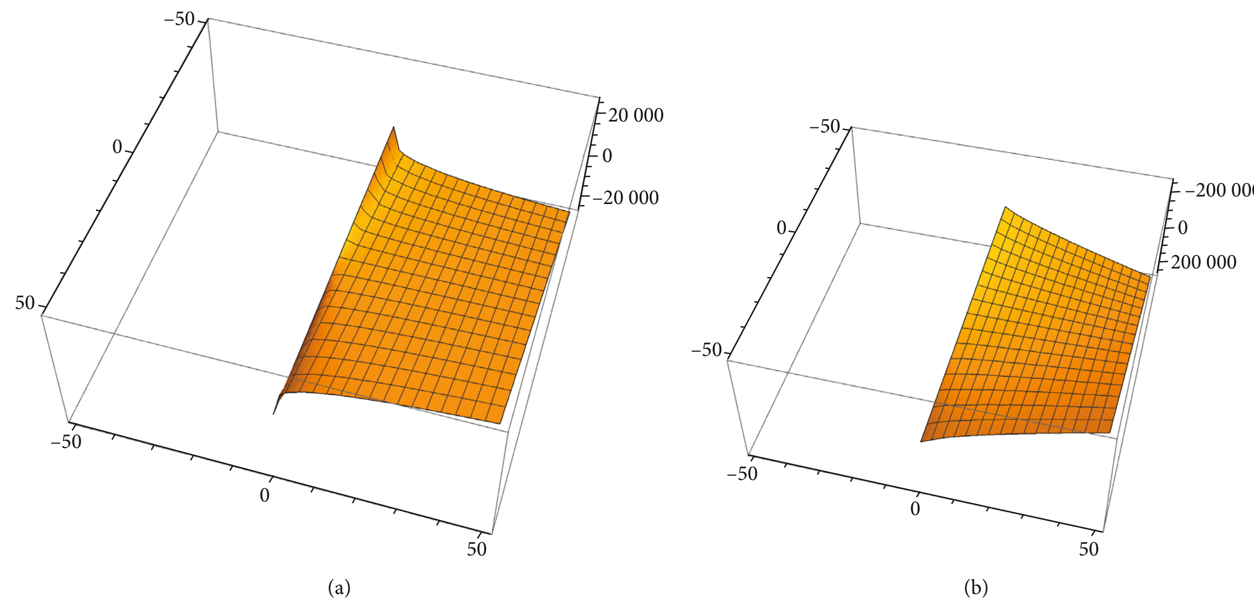


Figure 2: 3D representation of X for $\alpha = 0.2$ (a) and $\alpha = 0.8$ (b).

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi,t) = \mathcal{X}_n(\varphi,t) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha \mathcal{X}_n(\varphi,s) + \mathcal{X}_n(\varphi,s)\frac{\partial \mathcal{X}_n(\varphi,s)}{\partial\varphi} + \frac{\partial^2 \mathcal{X}n(\varphi,s)}{\partial\varphi^2}\right\}\right\}. \tag{50}$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as

$$\begin{aligned}\mathcal{X}_1(\varphi,t) &= \varphi - \frac{\varphi t^\alpha}{\Gamma(1+\alpha)},\\ \mathcal{X}_2(\varphi,t) &= \varphi + \frac{\varphi t^\alpha}{\Gamma(1+\alpha)} - \varphi t^\alpha\left\{\frac{1 - t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)} - \frac{2t^\alpha}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}.\end{aligned} \tag{51}$$

The solution $\mathcal{X}(\varphi,t)$ can be found as

$$\mathcal{X}(\varphi,t) = \lim_{i\longrightarrow\infty}\mathcal{X}_i. \tag{52}$$

It means

$$\mathcal{X}(\varphi,t) = \varphi + \frac{\varphi t^\alpha}{\Gamma(1+\alpha)} - \varphi t^\alpha\left\{\frac{1 - t^{-1+\alpha}\alpha\Gamma\alpha/\Gamma(2\alpha)}{\Gamma(1+\alpha)} - \frac{2t^\alpha}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right\}. \tag{53}$$

For $\alpha = 1$, we have

$$\mathcal{X}(\varphi,t) = \varphi - \varphi t + \varphi t^2 + \cdots, \tag{54}$$

which resembles with the expansion of the exact solution $\mathcal{X}(\varphi,t) = \varphi/(1+t)$, confirming the validity of the proposed VITM (see [18]).

Further, let us draw some approximations of $\mathcal{X}(\varphi,t) = \varphi - \varphi t + \varphi t^2 + \cdots$, for different values of α. Then, see Figures 3(a), 3(b), 4(a), and 4(b), which are the approximations of Figure 5.

Example 4. Let us consider another time fractional version of Burgers equation (see [18]) as follows:

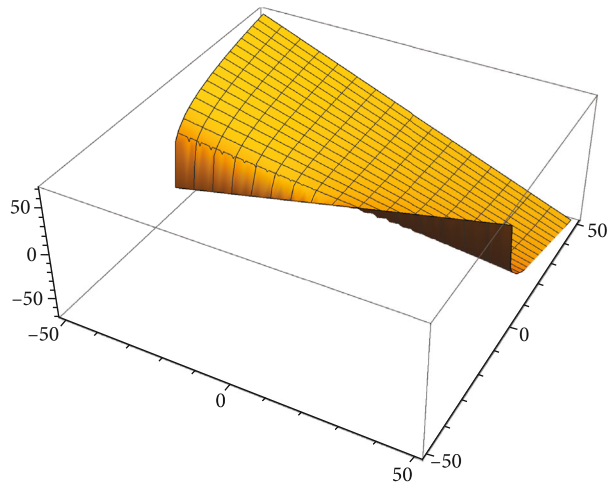
$$D_t^\alpha \mathcal{X} + \mathcal{X}\mathcal{X}_\varphi + \mathcal{X}_{\varphi\varphi} = 0\,;\, \mathcal{X}(\varphi,0) = 2\tan\varphi. \tag{55}$$

Applying the variational iteration and Laplace transform, we get

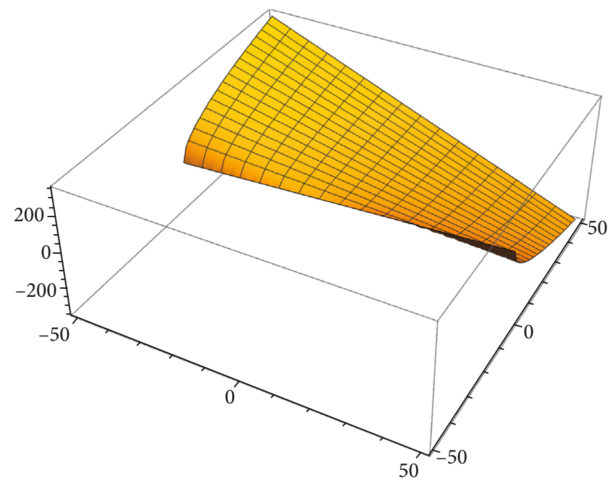
$$\mathcal{X}_{n+1}(\varphi,s) = \mathcal{X}_n(\varphi,s) + \mathcal{L}\left\{\lambda\left\{D_t^\alpha \tilde{\mathcal{X}}_n(\varphi,s) + \tilde{\mathcal{X}}_n(\varphi,s)\frac{\partial \tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi} + \frac{\partial^2 \tilde{\mathcal{X}}_n(\varphi,s)}{\partial\varphi^2}\right\}\right\}. \tag{56}$$

Using optimality conditions, etc., we obtain the following results:

$$\frac{\delta\tilde{\mathcal{X}}_{n+1}(\varphi,s)}{\delta\tilde{\mathcal{X}}_n(\varphi,s)} = 0,\ \delta\tilde{\mathcal{X}}_n = 0,\ \lambda = \frac{-1}{s^\alpha}. \tag{57}$$



(a)



(b)

FIGURE 3: 3D representation of X for $\alpha = 0.2$ (a) and $\alpha = 0.5$ (b).

Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{D_t^\alpha \mathcal{X}_n(\varphi, s) + \mathcal{X}_n(\varphi, s)\frac{\partial \mathcal{X}_n(\varphi, s)}{\partial \varphi} + \frac{\partial^2 \mathcal{X}_n(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \quad (58)$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as

$$\begin{aligned}
\mathcal{X}_1(\varphi, t) &= 2\tan\varphi - \frac{8t^\alpha \sec^2\varphi \tan\varphi}{\Gamma(1+\alpha)}, \\
\mathcal{X}_2(\varphi, t) &= 2\tan\varphi - \frac{8t^\alpha \sec^2\varphi \tan\varphi}{\Gamma(1+\alpha)} \\
&\quad - \frac{8t^{-1+\alpha}\sec^2\varphi \tan\varphi}{\Gamma(2\alpha)\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+3\alpha)} \\
&\quad \times \{-t^\alpha \alpha\Gamma(\alpha)\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha) \\
&\quad + t\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha) \\
&\quad + 4t^\alpha \sec^2\varphi(-4+\cos 2\varphi)\Gamma(1+\alpha)^2\Gamma(1+3\alpha) \\
&\quad - 8t^{2\alpha}\sec^4 x(-2+\cos 2x)\Gamma(1+2\alpha)^2\}.
\end{aligned} \quad (59)$$

The solution is in the series form, such as

$$\mathcal{X}(\varphi, t) = \lim_{i\longrightarrow\infty} \mathcal{X}_i, \quad (60)$$

which turns out to be the expansion of the exact solution $\mathcal{X}(\varphi, t) = -2\tan\varphi$ for $\alpha = 1$, as discussed earlier (see [18]).

Example 5. Let us consider the time fractional version of the nonlinear simple Schrödinger equation [18] as follows

$$iD_t^\alpha \mathcal{X} + \mathcal{X}_{\varphi\varphi} - 2|\mathcal{X}|^2\mathcal{X} = 0, \mathcal{X}(\varphi, 0) = e^{i\varphi}. \quad (61)$$

Applying the variational iteration and Laplace transform, we obtain

$$\mathcal{X}_{n+1}(\varphi, s) = \mathcal{X}_n(\varphi, s) + \mathcal{L}\left\{\lambda\left\{iD_t^\alpha \tilde{\mathcal{X}}_n(\varphi, s) - 2\,|\,\widetilde{\mathcal{X}^2}_n(\varphi, s)\,|\,\tilde{\mathcal{X}}_n(\varphi, s) + \frac{\partial^2 \tilde{\mathcal{X}}_n(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \quad (62)$$

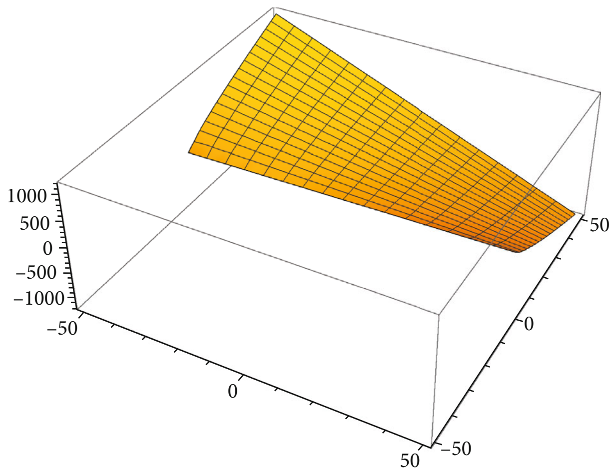
Using optimality conditions, etc., we get the following results:

$$\frac{\delta \tilde{\mathcal{X}}_{n+1}(\varphi, s)}{\delta \tilde{\mathcal{X}}_n(\varphi, s)} = 0,\ \delta \tilde{\mathcal{X}}_n = 0,\ \lambda = \frac{-1}{s^\alpha}. \quad (63)$$

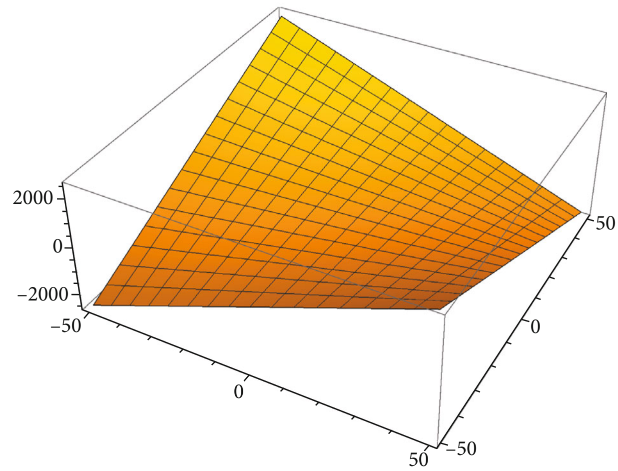
Substitution and inverse Laplace transform implies

$$\mathcal{X}_{n+1}(\varphi, t) = \mathcal{X}_n(\varphi, t) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{iD_t^\alpha \mathcal{X}_n(\varphi, s) - 2|\mathcal{X}_n(\varphi, s)|^2\mathcal{X}_n(\varphi, s) + \frac{\partial^2 \mathcal{X}_n(\varphi, s)}{\partial \varphi^2}\right\}\right\}. \quad (64)$$

For $n = 0, 1, 2, \cdots$, we get the approximations $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \cdots$, such as



(a)



(b)

Figure 4: 3D representation of X for $\alpha = 0.8$ (a) and $\alpha = 1$ (b).

$$\mathcal{X}_1(\varphi, t) = e^{i\varphi} + \frac{2e^{3i\varphi}t^\alpha}{\Gamma(1+\alpha)},$$

$$\mathcal{X}_2(\varphi, t) = e^{i\varphi} + \frac{2e^{3i\varphi}t^\alpha}{\Gamma(1+\alpha)} - 2e^{3i\varphi}t^\alpha \left\{ \frac{-1 + it^{-1+\alpha}\alpha\Gamma(\alpha)/\Gamma(2\alpha)}{\Gamma(1+\alpha)} - \frac{6e^{2\alpha}t^\alpha}{\Gamma(1+2\alpha)} - \frac{12e^{4\alpha}t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{8e^{6\alpha}t^{3\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+4\alpha)} \right\}. \quad (65)$$

The solution is in the series form, such as

$$\mathcal{X}(\varphi, t) = \lim_{i\longrightarrow\infty} \mathcal{X}_i, \quad (66)$$

which turns out to be the exact solution $\mathcal{X}(\varphi, t) = e^{i(\varphi+t)}$ for $\alpha = 1$ (see [18]).

Open question: as an application of the VITM on nonlinear-time fractional differential equations towards fixed-point theorem. One can obtain some approximations of the solution $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \cdots$. Moreover, it can be asked whether these approximations of the solution are equivalent with the iterations of a sequence of successive approximations which are convergent to a fixed point or not? What are the minimum hypotheses imposed which lead us to the existence and the uniqueness of a fixed point in this case?

4. Discussions and Concluding Remarks

The proposed method VITM being the combination of two basic techniques, VIM and Laplace transform, is understandable by just having the formal knowledge of *advanced calculus*; indeed, it is understandable even for the reader who has no strong background and base in *calculus of variations*. It is simple and can be easily applied as compared to the more traditional VIM for fractional differential equations.

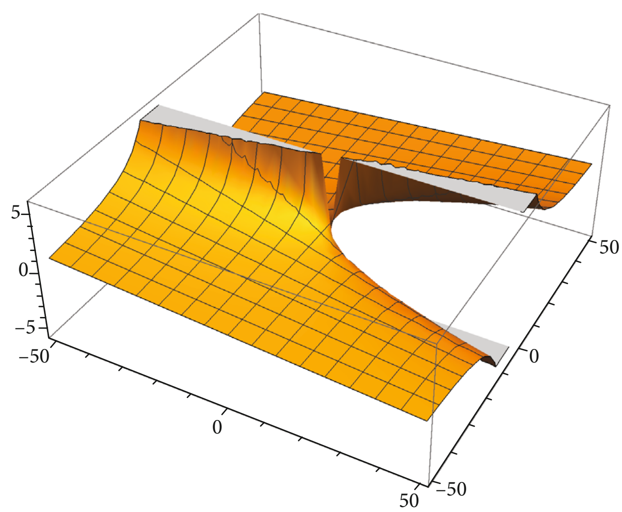


Figure 5: 3D representation of the exact solution.

Using the proposed method in the present paper, we study the convergence for some nonlinear fractional order partial differential equations as the Korteweg-de Vries equation, Schrödinger equation, and Burger equation. The rapid convergence of the method proves that it is a more reliable technique now more than ever than the existing ones to solve FPDEs, and it introduces a new significant improvement. The reliability and efficiency of this simple and newly introduced method is discussed by giving some numerical results and their graphical approximations.

Moreover, we propose an interesting connection between the approximations of the solution $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \cdots$ and the iterations of a sequence (convergent or not), from the area of fixed-point theory.

The main advantage of this proposed variational method together with Laplace transform helps to speed up the computational work and may easily be applied to nonlinear dynamical systems using software like Mathematica™, MATLAB™, and Maple™.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing of this article. All authors have read and agreed to the last version of the manuscript.

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Research Article

Nonunique Fixed Point Results via Kannan F-Contraction on Quasi-Partial b-Metric Space

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This paper is aimed at acquainting with a new Kannan F-expanding type mapping by the approach of Wardowski in the complete metric space. We establish some fixed point results for Kannan F-expanding type mapping and F-contractive type mappings which satisfy F-contraction conditions. Additionally, some new results are given which generalize several results present in the literature. Moreover, some applications and examples are provided to show the practicality of our results.

1. Introduction and Preliminaries

In 1922, Banach [1] commenced one of the most essential and notable results called the Banach contraction principle, i.e., let P be a self-mapping on a nonempty set X and d be a complete metric, if there exists a constant $k \in [0, 1)$ such that

$$d(Pu, Pv) \leq k\, d(u, v), \tag{1}$$

for all $u, v \in X$. Then, it has a unique fixed point in X. Due to its significance, in 1968, Kannan [2] introduced a different intuition of the Banach contraction principle which removes the condition of continuity, i.e., for all $u, v \in [0, 1/2]$, there exists a constant $\rho \in [0, 1)$ such that

$$d(Pu, Pv) \leq \rho[d(u, Pu) + d(v, Pv)]. \tag{2}$$

On the other hand, the notion of metric space has been generalized in several directions, and the abovementioned contraction principle has been enhanced in the new settings by considering the concept of convergence of functions. In 1989, Bakhtin [3] introduced the notion of b-metric space which was revaluated by Czerwik [4] in 1993.

Definition 1. A b-metric space on a nonempty set X is a function $d : X \times X \longrightarrow [0,\infty)$ such that for all $u, v, w \in X$ and for some real number $s \geq 1$, it satisfies the following:

$(M1)$ If $d(u, v) = 0$, then $u = v$
$(M2)$ $d(u, v) = d(v, u)$
$(M3)$ $d(u, w) \leq s[d(u, v) + d(v, w)]$

Then, the pair (X, d, s) is called the b-metric space. Motivated by this, many researchers [5–8] generalized the concept of metric spaces and established on the existence of fixed points in the setting of b-metric space keeping in mind that, unlike standard metric, b-metric is not necessarily continuous due to the modified triangle inequality. In general, a b-metric does not induce a topology on X.

Partial metric space is one of the attempts to generalize the notion of the metric space. In 1994, Matthews [9] introduced the notion of a partial metric space in which $d(u, u)$ are no longer necessarily zero.

Definition 2. A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow [0,\infty)$ such that for all $u, v, w \in X$, it satisfies the following:

$(PM1)$ If $p(u, u) = p(u, v) = p(v, v)$, then $u = v$
$(PM2)$ $p(u, u) \leq p(u, v)$

$(PM3)$ $p(u, v) = p(v, u)$
$(PM4)$ $p(u, v) \le p(u, w) + p(w, v) - p(w, w)$

Then, the pair (X, p) is called the partial metric space.

Definition 3. Let (X, p) be a partial metric space. Then, several topological concepts for partial metric space can be easily defined as follows:

(1) A sequence $\{u_n\}$ in the partial metric space (X, p) converges to the limit u if $p(u, u) = \lim_{n \longrightarrow \infty} p(u, u_n)$

(2) It is said to be a Cauchy sequence if $\lim_{n \longrightarrow \infty} p(u_n, u_m)$ exists and is finite

(3) A partial metric space (X, p) is called complete if every Cauchy sequence $\{u_n\}$ in X converges with respect to τ_p, to a point $u \in X$ such that $p(u, u) = \lim_{n \longrightarrow \infty} p(u_n, u_m)$

For more details, see, for example, [10–12], and the related references therein. The following definition gives room for the lack of symmetry in the spaces under study. In 2013, Karapinar et al. [13] introduced quasi-partial metric space that satisfies the same axioms as metric spaces.

Definition 4. A quasi-partial metric on a nonempty set X is a function $q : X \times X \longrightarrow \mathbb{R}^+$ that satisfies the following:

$(QPM1)$ If $q(u, u) = q(u, v) = q(v, v)$, then $u = v$
$(QPM2)$ $q(u, u) \le q(u, v)$
$(QPM3)$ $q(u, u) \le q(v, u)$
$(QPM4)$ $q(u, v) + q(w, w) \le q(u, w) + q(w, v)$for all $u, v, w \in X$. Then, the pair (X, q) is called quasi-partial metric space.

Later on, Gupta and Gautam [14, 15] introduced quasi-partial b-metric space.

Definition 5. A quasi-partial b-metric on a nonempty set X is a function $qp_b : X \times X \longrightarrow [0,\infty)$ such that for some real number $\rho \ge 1$, it satisfies the following:

(QPb_1) If $qp_b(u, u) = qp_b(u, v) = qp_b(v, v)$, then $u = v$ (indistancy implies equality)
(QPb_2) $qp_b(u, u) \le qp_b(u, v)$ (small self-distances)
(QPb_3) $qp_b(u, u) \le qp_b(v, u)$ (small self-distances)
(QPb_4) $qp_b(u, v) + qp_b(w, w) \le \rho\{qp_b(u, w) + qp_b(w, v)\}$ (triangularity)

for all $u, v, w \in X$. The infimum over all reals $\rho \ge 1$ satisfying (QPb_4) is called the coefficient of (X, qp_b) and represented by $R(X, qp_b)$.

Lemma 6. *Let (X, qp_b) be a quasi-partial b-metric space. Then, the following hold*:

(i) If $qp_b(u, v) = 0$, then $u = v$

(ii) If $u \neq v$, then $qp_b(u, v) > 0$ and $qp_b(v, u) > 0$

Definition 7. Let (X, qp_b) be a quasi-partial b-metric. Then,

(i) a sequence $\{u_n\} \subset X$ converges to $u \in X$ if and only if

$$qp_b(u, u) = \lim_{n \longrightarrow \infty} qp_b(u, u_n) = \lim_{n \longrightarrow \infty} qp_b(u_n, u) \tag{3}$$

(ii) a sequence $\{u_n\} \subset X$ is called a Cauchy sequence if and only if

$$\lim_{n,m \longrightarrow \infty} qp_b(u_n, u_m) \& \lim_{m,n \longrightarrow \infty} qp_b(u_m, u_n) \text{ exist} \tag{4}$$

(iii) the quasi-partial b-metric space (X, qp_b) is said to be complete if every Cauchy sequence $\{u_n\} \subset X$ converges with respect to τqp_b to a point $u \in X$ such that

$$qp_b(u, u) = \lim_{n,m \longrightarrow \infty} qp_b(u_n, u_m) = \lim_{m,n \longrightarrow \infty} qp_b(u_m, u_n) \tag{5}$$

(iv) a mapping $f : X \longrightarrow X$ is said to be continuous at $u_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(u_0, \delta)) \subset B(f(u_0), \varepsilon)$

The extensive application of the Banach contraction principle has motivated many researchers to study the possibility of its generalization. A great number of generalizations of this famous result have appeared in the literature. In 2012, Wardowski [16] established a new notion of F-contraction and proved the fixed point theorem which generalized the Banach contraction principle.

Definition 8 (see [16]). Let (X, d) be a metric space, and there exists a mapping $F : (0,\infty) \longrightarrow \mathbb{R}$ which satisfies the following condition:

(F_1) F is strictly increasing
(F_2) For any sequence $\{x_n\}_{n\in N}$, $\lim_{n \longrightarrow \infty} x_n = 0$ if and only if $\lim_{n \longrightarrow \infty} F(x_n) = -\infty$
(F_3) $\lim_{x \longrightarrow 0^+} x^k F(x) = 0$ for some $k \in (0, 1)$

Then, a mapping $P : X \longrightarrow X$ is said to be Wardowski F-contraction if $d(Pu, Pv) > 0$ implies

$$\delta + F(d(Pu, Pv)) \le F(d(u, v)) \tag{6}$$

for all $u, v \in X$

Theorem 9 (see [16]). *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an F-contraction. Then, T has a unique fixed point $x^* \in X$, and for every $x \in X$, the sequence $\{T^n x\}_{n\in N}$ converges to x^*.*

In 2012, Samet et al. [17] established the class of α-admissible mappings as follows.

Definition 10 (see [17]). Let $\alpha : X \times X \longrightarrow [0,\infty)$ be given mapping where X is a nonempty set. A self-mapping T is called α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1. \tag{7}$$

Motivated by this, Aydi et al. [18] extended the notion of F-contraction and prove the following result.

Theorem 11 (see [18]). *Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is said to be a modified F-contraction via α-admissible mappings. Suppose that*

(i) T is α-admissible

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

(iii) T is continuous

Then, T has a fixed point. In 2015, Kumam et al. [19] generalized the contraction condition by adding four new values $d(T^2x, x), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)$ and introduced F-Suzuki contraction mappings in complete metric space. The Suzuki-type generalization can be said to have many applications, as in computer science, game theory, biosciences, and in other areas of mathematical sciences such as in dynamic programming, integral equations, and data dependence. Recently, Wardowski [20] proposed the replacement of the positive constant δ in equation (6) by a function ϕ and relaxed the conditions on F.

Definition 12 (see [20]). Let (X, d) be a metric space, $F : (0,\infty) \longrightarrow \mathbb{R}$ and $\phi : (0,\infty) \longrightarrow (0,\infty)$ satisfy the following:

(1) F is strictly increasing, i.e., $x < y$ implies $F(x) < F(y)$ for all $x, y \in (0,\infty)$

(2) $\lim_{\alpha \longrightarrow 0^+} F(\alpha) = -\infty$

(3) $\liminf_{\alpha \longrightarrow s^+} \phi(\alpha) > 0$ for all $s > 0$

A mapping $T : X \longrightarrow X$ is called an (ϕ, F)-contraction on (X, d) if

$$\phi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{8}$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Consider a function $F_B : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by $F_B(u) = \ln u$. Note that with $F = F_B$, the F-contraction reduces to a Banach contraction. Therefore, the Banach contractions are a particular case of F-contractions. Meanwhile, there exist F-contractions which are not Banach contractions.

The concept of an F-contraction has been generalized in many directions (see, e.g., [21–24]), and as an extension, engaging work was done by many authors [25–34], which enhanced this field. In 2015, Cosentino et al. [35] extended the concept of F-contraction in metric space to F-contraction in b-metric space by introducing the following condition with continuation of Definition 7.

(F_4) For some $\delta > 0$ and any sequence $\{x_n\}$, we have

$$\begin{aligned} \delta + F(sx_n) &\leq F(x_{n-1}), \\ \delta + F(s^n x_n) &\leq F(s^{n-1} x_{n-1}) \end{aligned} \tag{9}$$

for all $n \in N, s \in \mathbb{R}$

In 2017, Gornicki [36] established F-expanding type mappings.

Definition 13. Let (X, d) be a metric space. A mapping $P : X \longrightarrow X$ is called F-expanding if for all $u, v \in X$ and $\delta > 0$, we have

$$d(u, v) > 0 \Rightarrow F(d(Pu, Pv)) \geq F(d(u, v)) + \delta. \tag{10}$$

The concept of F-expanding type mappings was redefined as Kannan F-expanding type mappings by Goswami et al. [37].

Definition 14 (see [37]). A mapping $P : X \longrightarrow X$ is said to be Kannan F-expanding type mapping if there exists $\Delta > 0$ such that $d(u, Pu)d(v, Pv) \neq 0$ implies

$$\Delta + F(sd(u, v)) \leq \frac{1}{2}\{F(d(u, Pu)) + F(d(v, Pv))\}, \tag{11}$$

and $d(u, Pu)d(v, Pv) = 0$ implies

$$\Delta + F(sd(u, v)) \leq \frac{1}{2}\{F(d(u, Pv)) + F(d(v, Pu))\} \tag{12}$$

for all $u, v \in X$. Following this direction, we have established a new type of mapping, i.e., Kannan F-expanding type mapping, and proved some fixed point results for F-contractive type mappings as well as Kannan F-expanding type mappings in the setting of quasi-partial b-metric space without using the continuity of mapping. Also, we attain the non-unique fixed point in quasi-partial b-metric space which lacks symmetry property.

The main motive behind this study is that today, this field of research has vast literature. The significance of the Kannan type mapping is that it characterizes completeness which the Banach contraction does not; also, it does not require continuous mapping. In this paper, some examples and applications for the solution of a certain integral equation and the existence of a bounded solution of the functional equation are also given to represent the practicality of the results obtained. The application shows the role of fixed point theorems in dynamic programming, which is used in computer programming and optimization.

The future aspect of this study is to prove the existence of a unique fixed point in Kannan F-expanding type mapping. Another field of research can be the existence of a common fixed point for the same. The notion of interpolative F

-contraction as well as interpolation for Kannan F-expanding type mapping can also be future studies concerning the present manuscript.

2. Fixed Point for F-Contractive Type Mappings

In this section, the existence of a fixed point for F-contractive type mappings in a quasi-partial b-metric space is obtained.

Definition 15. For a quasi-partial b-metric space (X, qp_b), a mapping $P : X \longrightarrow X$ is said to be an F-contractive type mapping if there exists $\delta > 0$ such that, if $qp_b(u, Pu)qp_b(v, Pv) \neq 0$, then

$$\delta + F(\rho qp_b(Pu, Pv)) \leq \frac{1}{3}[F(qp_b(u, v)) + F(qp_b(u, Pv)) + F(qp_b(v, Pu))] - F(qp_b(w, Pw)), \tag{13}$$

and if $qp_b(u, Pu)qp_b(v, Pv) = 0$, then

$$\delta + F(\rho qp_b(Pu, Pv)) \leq \frac{1}{3}[F(qp_b(u, v)) + F(qp_b(u, Pu)) + F(qp_b(v, Pv))] - F(qp_b(w, Pw)) \tag{14}$$

for all $u, v, w \in X$ and $\rho \geq 1$.

Definition 16. Let (X, qp_b) be a quasi-partial b-metric space. A self-mapping P on X is called an F-contraction if there exist $\tau \in \mathbb{R}^+$ such that

$$\tau + F(\rho qp_b(Pu, Pv)) \leq F(qp_b(u, v)), \tag{15}$$

for all $u, v \in X$ with $qp_b(Pu, Pv) > 0$.

Example 17. Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be given by $F(u) = \log u$. Here, F satisfies (F1)-(F3) for any $k \in (0, 1)$. Each mapping $P : X \longrightarrow X$ satisfying Definition 16 is an F-contraction such that

$$qp_b(Pu, Pv) \leq \rho e^{-\tau} qp_b(u, v) \tag{16}$$

for all $u, v \in X, Pu \neq Pv$.

It is clear that for $u, v \in X$ such that $Pu = Pv$, the previous inequality also holds, and hence, P is a contraction as shown in Figure 1.

Example 18. Consider a function $F(u) = -1/\sqrt{u}, u > 0$ where F satisfies (F1)-(F3) for any $k \in (1/2, 1)$. In this case, a mapping $P : X \longrightarrow X$ satisfies

$$\rho qp_b(Pu, Pv) \leq \frac{1}{\left(1 + \tau\sqrt{qp_b(u, v)}\right)^2} qp_b(u, v) \tag{17}$$

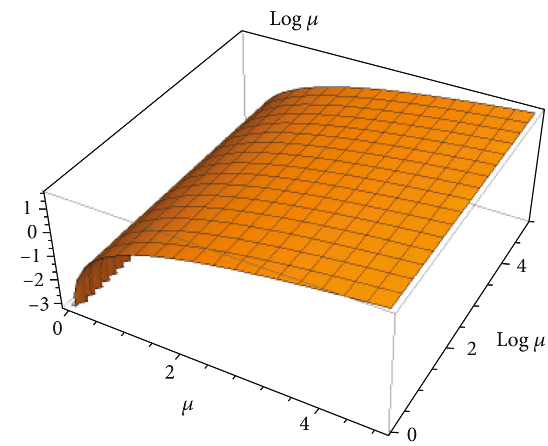

FIGURE 1: The graphical surface represents a 3-D view of F-contractive function $F(u) = \log u$ where $u \in (0, 5), v \in (0, 5)$ and $w \in \mathbb{R}^+$.

for all $u, v \in X, Pu \neq Pv$. Hence, P is a contraction as shown in Figure 2.

Theorem 19. *Let (X, qp_b) be a quasi-partial b-metric space and $P : X \longrightarrow X$ be an F-contractive type mapping. Then, P has a unique fixed point $u^* \in X$, and for every $u_0 \in X$, a sequence $\{P^n x_0\}_{n\in\mathbb{N}}$ is convergent to u^*.*

Proof. Let u_0 be an arbitrary and fixed point in X, and we assume a sequence $\{u_n\}_{n\in N} \subset X$ such that $u_{n+1} = Pu_n, n = 0, 1, \cdots$. To prove P has a fixed point, we need to show that if $u_{n_0+1} = u_{n_0}$, then $Pu_{n_0} = u_{n_0}$ for all $n_0 \in \mathbb{N}$. Suppose that $u_{n+1} \neq u_n$ for every $n \in \mathbb{N}$, then $qp_b(u_{n+1}, u_n) > 0$, and using equation (6), we have

$$F(qp_b(u_{n+1}, u_n)) \leq \rho F(qp_b(u_n, u_{n-1})) - \delta \leq \rho F(qp_b(u_{n-1}, u_{n-2})) - 2\delta \leq \vdots \leq \rho F(qp_b(u, u_0)) - n\delta, \tag{18}$$

which implies

$$\lim_{n\longrightarrow\infty} \rho F(qp_b(u_{n+1}, u_n)) = -\infty. \tag{19}$$

Using (F_2), we get

$$\lim_{n\longrightarrow\infty} \rho\, qp_b(u_{n+1}, u_n) = 0. \tag{20}$$

Also, using (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\longrightarrow\infty} qp_b(u_{n+1}, u_n)^k F(qp_b(u_{n+1}, u_n)) = 0. \tag{21}$$

Let us denote $qp_b(u_{n+1}, u_n)$ by α_n. From inequality (18), the following holds

$$F(\alpha_n) - \alpha_n^k F(\alpha_0)) \leq \rho \left(\alpha_n^k (F(\alpha_0) - n\delta) - \alpha_n^k F(\alpha_0)\right) = -\rho\, \alpha_n^k\, n\delta \leq 0, \tag{22}$$

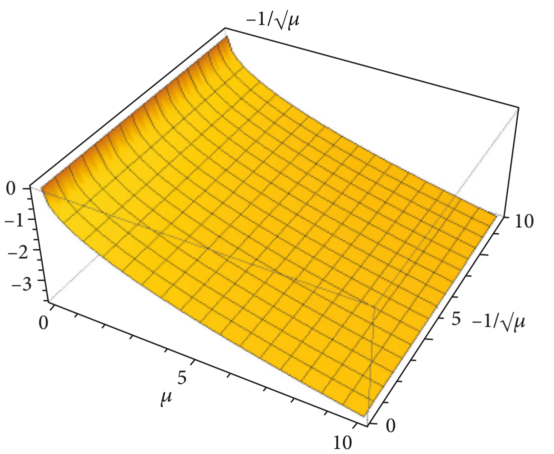

FIGURE 2: The graphical surface represents a 3-D view of F-contractive function $F(u) = -1/\sqrt{u}$ where $u \in (0, 10)$, $v \in (0, 10)$ and $w \in \mathbb{R}^+$.

which implies

$$\lim_{n \longrightarrow \infty} n\alpha_n^k = 0. \tag{23}$$

Also, if there exists $n_1 < n \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$, we have

$$\rho\, \alpha_n \leq n^{-1/k}. \tag{24}$$

To prove $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let us consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. From the definition of quasi-partial b-metric space and equation (24), we have

$$qp_b(u_m, u_n) \leq \rho(\alpha_{m-1} + \alpha_{m-2} + \cdots + \alpha_n) \leq \rho \sum_{i=n}^{\infty} \alpha_i \leq \rho \sum_{i=n}^{\infty} i^{-1/k}. \tag{25}$$

Using the convergence of series, we get that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $u^* \in X$ such that $\lim_{n\longrightarrow\infty} u_n = u^*$, and the continuity of P implies

$$qp_b(Pu^*, u^*) = \rho \lim_{n\longrightarrow\infty} qp_b(Pu_n, u_n) = \rho qp_b(u_{n+1}, u_n) = 0. \tag{26}$$

Hence, P has a unique fixed point.□ □

Theorem 20. *For a quasi-partial b-metric space (X, qp_b), we say X is complete if for every closed subset Y of X, $P : Y \longrightarrow Y$ is an F-contractive type mapping having a fixed point.*

Proof. Suppose that there does not exist any Cauchy sequence in X which has a convergent subsequence and we have a sequence

$$\theta(u_n) = \inf\{qp_b(u_n, u_m) : m > n\} > 0 \tag{27}$$

for all $n \in \mathbb{N}$ where $\theta(u_n) \leq \theta(u_m)$ for $m \geq n$. Also, we consider a subsequence $\{u_{n_k}\}$ such that

$$\rho qp_b(u_i, u_j) < a\theta(u_{n_{k-1}}), \tag{28}$$

for any a with $0 < a < 1$ and for all $i, j \geq n_k$. Then, $Y = \{u_{n_k} : k \in \mathbb{N}\}$ is a closed subset of X. Define $P : X \longrightarrow X$ by

$$Pu_{n_k} = u_{n_k+1} \tag{29}$$

for all $k \in \mathbb{N}$, which implies P has no fixed point. Now,

$$\rho qp_b(Pu, Pv) = \rho qp_b(Pu_{n_k}, Pu_{n_k+1}) = \rho qp_b(u_{n_k+1}, Pu_{n_{k+i+1}}) < a\theta(u_{n_k}). \tag{30}$$

By definition,

$$\begin{aligned} \theta(u_{n_k}) &\leq \rho qp_b(u_{n_k}, u_{n_{k+i}}) = \rho qp_b(u, v) \leq \rho qp_b(u_{n_k}, u_{n_k+1}) \\ &= \rho qp_b(u, Pu) \leq \theta(u_{n_{k+i}}) \leq \rho qp_B(u_{n_{k+i}}, u_{n_{k+i+1}}) \\ &= \rho qp_b(v, Pv), \end{aligned} \tag{31}$$

which implies

$$\begin{aligned} \delta + F(\rho qp_b(Pu, Pv)) \leq \frac{1}{3}\{&F(qp_b(u, v)) + F(qp_b(u, Pu)) \\ &+ F(qp_b(v, Pv))\} - (qp_b(w, Pw)) \end{aligned} \tag{32}$$

for some $\delta > 0$. Hence, it proves that P is an F-contractive type mapping on a closed subset of X which has no fixed point. Thus, this is a contradiction and X is complete.□ □

Theorem 21. *Let (X, qp_b) be a quasi-partial b-metric space and $P : X \times C(X)$ be a closed F-contraction. Then, P has a fixed point.*

Proof. Let $u_0 \in X$ be an arbitrary point of X, and we have $u_1 \in Pu_0$. If $u_1 = u_0$, then u_1 is a fixed point of P, and hence, the proof is completed. Now, assume that $u_1 \neq u_0$. Since P is a F-contraction, there exists $u_2 \in Pu_1$ such that

$$\tau + F(\rho qp_b(u_1, u_2)) \leq F(M(u_0, u_1)), \tag{33}$$

where

$$M(u_0, u_1) = \max\left\{qp_b(u_0, u_1), qp_b(u_0, Pu_0), qp_b(u_1, Pu_1), \frac{1}{2}[qp_b(u_0, Pu_1) + qp_b(u_1, Pu_0)]\right\}, \tag{34}$$

and $u_2 \neq u_1$. Also, there exists $u_3 \in Pu_2$ such that

$$\tau + F(\rho qp_b(u_2, u_3)) \leq F(M(u_1, u_2)), \tag{35}$$

and $u_3 \neq u_2$. With the recurrence of the same process, we get

$$\tau + F(\rho qp_b(u_n, u_{n+1})) \leq F(M(u_{n-1}, u_n)) \tag{36}$$

for all $n \in \mathbb{N}$. It implies

$$\tau + F(\rho qp_b(u_n, u_{n+1})) \le F(M(u_{n-1}, u_n)). \tag{37}$$

Assume that $qp_{b_n} = qp_b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By equation (37), we have

$$F\left(\rho qp_{b_n}\right) \le F\left(\rho qp_{b_{n-1}}\right) \le \cdots < \le F\left(\rho qp_{b_0}\right) - n\tau \tag{38}$$

for all $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, property (F_2) implies

$$\lim_{n \longrightarrow \infty} F\left(\rho qp_{b_n}\right) = -\infty. \tag{39}$$

Let $k \in (0, 1)$ such that

$$\lim_{n \longrightarrow \infty} qp_{b_n}^k F\left(\rho qp_{b_n}\right) = 0. \tag{40}$$

By equation (38), the following holds

$$\begin{aligned} & qp_{b_n}^k F\left(\rho qp_{b_n}\right) - qp_{b_n}^k F\left(\rho qp_{b_0}\right) \\ & \le qp_{b_n}^k \left(F\left(\rho qp_{b_0}\right) - n\tau\right) - qp_{b_n}^k F\left(\rho qp_{b_0}\right) = -n\tau qp_{b_n}^k \le 0 \end{aligned} \tag{41}$$

for all $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, we get

$$\lim_{n \longrightarrow \infty} n qp_{b_n}^k = 0. \tag{42}$$

This implies $\lim_{n \longrightarrow \infty} n^{1/k} qp_{b_n} = 0$ and $\sum_{n=1}^{+\infty} qp_{b_n}$ is convergent. Hence, $\{u_n\}$ is Cauchy sequence. Since X is complete, there exists $x \in X$ such that $u_n = x$ as $n \longrightarrow +\infty$. Since P is closed, $(u_n, u_{n+1}) \longrightarrow (x, x)$, we get $x \in Px$, and hence, x is the fixed point of P.□ □

Corollary 22. *Let (X, qp_b) be a quasi-partial b-metric space and $P : X \longrightarrow C(X)$ be an upper semicontinuous F-contraction. Then, P has a fixed point.*

Example 23. Consider the quasi-partial b-metric space (X, qp_b) where $X = \{0, 2, 4, \cdots\}$ and $qp_b : X \times X \longrightarrow [0,\infty)$ is given by

$$qp_b(u, v) = \begin{cases} u + v, & u \ne v, \\ 0, & u = v, \end{cases} \tag{43}$$

which is also shown in Figure 3, and $P : X \longrightarrow C(X)$ is defined by

$$P(u) = \begin{cases} \{0\}, & u \in [0, 1], \\ \{0, 2, \cdots, 2u - 2\}, & u \ge 4. \end{cases} \tag{44}$$

Now, we show that P satisfies Definition 16, where $\rho = 2$, $\tau = 2$ and $F(u) = \log u + u$ for each $u \in \mathbb{R}^+$. Let for all $u, v \in X$ with $v \in Pu$, we have $w = 0 \in Pv$. Here, $qp_b(v, w) > 0$ iff

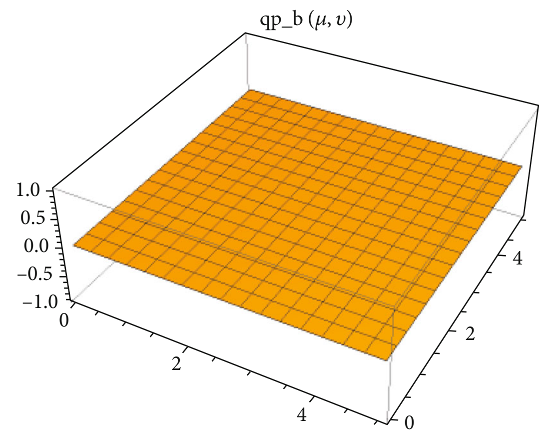

Figure 3: 3-D view. The plane in yellow represents the quasi-partial b-metric space defined by $qp_b : X \times X \longrightarrow [0,\infty)$ and $qp_b(u, v) = u + v$ when $u \ne v$ for all $u, v \in X$.

$x \ge 4$ and $w > 0$. If it is true, then

$$qp_b(v, w) = v < u + v = qp_b(u, v). \tag{45}$$

This implies

$$\begin{aligned} qp_b(v, w) - M(u, v) \le qp_b(v, w) - qp_b(u, v) \le -4, \\ \frac{qp_b(v, w)}{M(u, v)} e^{qp_b(u,v) - M(u,v)} \le e^{-2}. \end{aligned} \tag{46}$$

Hence,

$$2 + F(\rho qp_b(v, w)) \le F(M(u, v)) \tag{47}$$

for all $u, v \in X$ and $qp_b(v, w) > 0$. Then, by Theorem 21, P has a fixed point.

3. Fixed Point for Kannan F-Expanding Type Mapping

In this section, we prove the fixed point results for Kannan F-expanding type mappings in a quasi-partial b-metric space.

Definition 24. Let us consider a mapping $P : X \longrightarrow X$; it is said to be Kannan F-expanding type mapping if there exists $\Delta > 0$ such that $qp_b(u, Pu) qp_b(v, Pv) \ne 0$ implies

$$\begin{aligned} \Delta + F(\rho qp_b(u, v)) \le \frac{1}{2}[F(qp_b(u, Pu)) + F(qp_b(v, Pv))] \\ - F(qp_b(Pw, Pw)), \end{aligned} \tag{48}$$

and $qp_b(u, Pu)qp_b(v, Pv) = 0$ implies

$$\Delta + F(\rho qp_b(u, v)) \leq \frac{1}{2}[F(qp_b(u, Pv)) + F(qp_b(Pv, Pu))] - F(qp_b(Pw, Pw)) \quad (49)$$

for all $u, v, w \in X$.

Lemma 25. *Let (X, qp_b) be a quasi-partial b-metric space and $P : X \longrightarrow X$ be surjective. Then, there exists a mapping $P^* : X \longrightarrow X$ such that $P \circ P^*$ is the identity map on X.*

Proof. For any point $u \in X$, let $v_u \in X$ be any point such that $Pv_u = u$. Let $P^*u = v_u$ for all $u \in X$. Then, $(P \circ P^*)(u) = P(P^*u) = Pv_u = u$ for all $u \in X$.☐ ☐

Theorem 26. *Let (X, qp_b) be a quasi-partial b-metric space and $P : X \longrightarrow X$ be surjective and a Kannan F-expanding type mapping. Then, P has a unique fixed point $\gamma \in X$.*

Proof. Assume that there exists a mapping $P^* : X \longrightarrow X$ such that $P \circ P^*$ is the identity map on X. Let u, v be arbitrary points of X such that $u \neq v$ and $x = P^*u, y = P^*v$ which also implies that $x \neq y$. Applying equation (48) on x, y, we have

$$\Delta + F(\rho qp_b(x, y)) \leq \frac{1}{2}[F(qp_b(x, Px)) + F(qp_b(y, Py))] - F(qp_b(z, Pz)) \quad (50)$$

for $qp_b(x, Px)qp_b(y, Py) \neq 0$ and

$$\Delta + F(\rho qp_b(x, y)) \leq \frac{1}{2}[F(qp_b(x, Py)) + F(qp_b(y, Px))] - F(qp_b(z, Pz)) \quad (51)$$

for $qp_b(x, Px)qp_b(y, Py) = 0$. Since $Px = P(P^*(u)) = u$ and $Py = P(P^*(v)) = v$, we get

$$\Delta + F(\rho qp_b(P^*u, P^*v)) \leq \frac{1}{2}[F(qp_b(u, P^*u)) + F(qp_b(v, P^*v))] - F(qp_b(w, P^*w)) \quad (52)$$

for $qp_b(u, Pu)qp_b(v, Pv) \neq 0$ and

$$\Delta + F(\rho qp_b(P^*u, P^*v)) \leq \frac{1}{2}[F(qp_b(u, P^*v)) + F(qp_b(v, P^*u))] - F(qp_b(w, P^*w)) \quad (53)$$

for $qp_b(u, Pu)qp_b(v, Pv) = 0$, which implies P^* is Kannan F-contractive type mapping. Also, we know that P^* has a unique fixed point $\gamma \in X$, and for every $u_0 \in X$, the sequence $\{P^{*n}u_0\}$ converges to γ. In particular, γ is also a fixed point of P since $P^*\gamma = \gamma$ implies that

$$P\gamma = P(P^*\gamma) = \gamma. \quad (54)$$

Finally, if $\gamma_0 = P\gamma_0$ is another fixed point, then from equation (49),

$$\Delta + F(\rho qp_B(\gamma, \gamma_0)) \leq \frac{1}{2}[F(qp_b(\gamma, P\gamma_0)) + F(qp_b(\gamma_0, P\gamma))] - F(qp_b(v, Pv)), \quad (55)$$

which is not possible, and hence, P has a unique fixed point.☐ ☐

4. Applications of F-Contraction

In this section, we discuss the applications of the results obtained to prove the existence of the solution of an integral equation and a functional equation.

4.1. Existence of Solution of Integral Equation. Now, we study the existence of solution of the following Volterra type integral equation

$$u(x) = \int_0^x f(x, y, u(y))dy + g(x), \quad (56)$$

$x \in [0, \sigma]$ where $\sigma > 0$. Let $C([0, \sigma], \mathbb{R})$ denote space of all continuous functions on $[0, \sigma]$, and for an arbitrary $u \in C([0, \sigma], \mathbb{R})$, we define

$$||u||_\tau = \sup_{x \in [0,\sigma]}\{|u(x)|e^{-\tau x}\}, \quad (57)$$

where $\tau > 0$ is taken arbitrary. Clearly, $(C([0, \sigma], \mathbb{R}), ||\cdot||_\tau)$ is endowed with quasi-partial b-metric defined by

$$qp_b(u, v) = \sup_{x \in [0,\sigma]}\{|u(x) - v(x)| e^{-\tau x}\} \quad (58)$$

for all $u, v \in C([0, \sigma], \mathbb{R})$ is a Banach space and

$$u \leq v \Leftrightarrow u(x) \leq v(x) \quad (59)$$

for all $x \in [0, \sigma]$.

Theorem 27. *Let us consider that for the integral equation (56), the following conditions are satisfied:*

(i) f and g are continuous where $f : [0, \sigma] \times [0, \sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [0, \sigma] \longrightarrow \mathbb{R}$

(ii) $f(x, y, .)$: $\mathbb{R} \longrightarrow \mathbb{R}$ is increasing

(iii) $u_0(x) \leq \int_0^x f(x, y, u_0(y))dy + g(x)$ for some $u_0 \in C([0, \sigma], \mathbb{R})$

(iv) There exists $\tau \in [1, \infty)$ such that

$$|f(x, y, u) - f(x, y, v)| \leq \tau e^{-2\tau}\rho|u - v| \quad (60)$$

for all $x, y \in [0, \sigma]$ and $u, v \in \mathbb{R}$. Then, integral equation (56) has a solution.

Proof. Here, $(C([0,\sigma],\mathbb{R}), qp_b)$ is a quasi-partial b-metric space, where qp_b is the quasi-partial b-metric given by equation (58). Let us define $P : C([0,\sigma],\mathbb{R}) \longrightarrow C([0,\sigma],\mathbb{R})$ by

$$P(u)(x) = \int_0^x f(x, y, u(y))dy + g(x), x \in [0, \sigma]. \tag{61}$$

From (iv) we have,

$$\begin{aligned} |P(u)(x) - P(v)(x)| &\le \int_0^x |f(x,y,u(y)) - f(x,y,v(y))|dy \\ &\le \int_0^x \tau e^{-2\tau}\rho|u(y)-v(y)|dy \\ &= \int_0^x \tau e^{-2\tau}\rho|u(y)-v(y)|e^{-\tau y}e^{\tau y}dy \\ &\le \int_0^x \tau e^{-2\tau}e^{\tau y}\rho|u(y)-v(y)|e^{-\tau y}dy \\ &\le \tau e^{-2\tau}\rho\|u-v\|_\tau\int_0^x e^{\tau y}dy \\ &\le \tau e^{-2\tau}\rho\frac{1}{\tau}\|u-v\|_\tau e^{\tau x}. \end{aligned} \tag{62}$$

It implies

$$|P(u)(x) - P(v)(x)|e^{-\tau x} \le e^{-2\tau}\rho\|u-v\|_\tau \tag{63}$$

or

$$qp_b(P(u), P(v)) \le e^{-2\tau}\rho qp_b(u, v). \tag{64}$$

Taking logarithm in both sides, we get

$$\ln\left(qp_b(P(u), P(v))\right) \le \ln\left(e^{-2\tau}\rho qp_b(u,v)\right), \tag{65}$$

which on solving reduces to

$$2\tau + \ln\left(qp_b(P(u), P(v))\right) \le \ln\left(\rho qp_b(u,v)\right). \tag{66}$$

Now, we observe that the function $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $F(u) = \log u$ for each $u \in C([0,\sigma],\mathbb{R})$ is F-contraction. Clearly, from (iii), we have

$$u_0 \le P(u_0), \tag{67}$$

and hence, Theorem 19 applies to P, which has a fixed point $u^* \in C([0,\sigma],\mathbb{R})$. Hence, u^* is a solution of integral equation (56).□ □

4.2. Existence of Bounded Solutions of Functional Equations. Fixed point theory is widely used in the field of dynamic programming which is the most commonly used tool for mathematical optimization. With this approach, the problem of the dynamic programming process reduces to solving the functional equations.

Let us consider that U and V are Banach spaces, $W \subset U$ is a state space, i.e., the set of the initial state of process, and $D \subset V$ is a decision space, i.e., the set of possible actions that are allowed for the process.

Here, we will prove the existence of the bounded solution of the following functional equation:

$$\phi(u) = \sup_{x\in D}\{f(u,v) + g(u, v + \phi(\tau(u,v)))\}, \tag{68}$$

where $\tau : W \times D, f : W \times D \longrightarrow \mathbb{R}, g : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$. Let $B(W)$ denote the set of all bounded real valued functions W and for an arbitrary $\alpha \in B(W)$, define $\|\alpha\| = \sup_{x\in W}|\alpha(x)|$. Clearly, $(B(w), \|\cdot\|)$ endowed with quasi-partial b-metric defined by

$$qp_b(\alpha, \beta) = \sup_{x\in W}|\alpha(x) - \beta(x)| \tag{69}$$

for all $\alpha, \beta \in B(W)$ is a Banach space. Thus, if we consider a Cauchy sequence $\{\alpha_n\}$ in $B(W)$, then $\{\alpha_n\}$ converges uniformly to a function, let α^* that is bounded and so $\alpha^* \in B(w)$. Also, we have $P : B(W) \times B(W)$ defined by

$$P(\alpha)(x) = \sup_{y\in D}\{f(x,y) + g(x,y,\alpha(\tau(x,y)))\} \tag{70}$$

for all $\alpha \in B(W)$ and $x \in W$. Hence, P is well defined if f and g are bounded.

Theorem 28. *Let $P : B(W) \longrightarrow B(W)$ be an upper semicontinuous operator defined by (70), and we assume that the following conditions are satisfied:*

(i) *f and g are bounded and continuous where $f : W \times D \longrightarrow \mathbb{R}$ and $g : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$*

(ii) *There exists $\tau \in \mathbb{R}^+$ such that*

$$|g(x,y,\alpha(x)) - g(x,y,\beta(x))| \le \frac{\rho|\alpha-\beta|}{\left(1+\tau\sqrt{|\alpha-\beta|}\right)^2} \tag{71}$$

for all $\alpha, \beta \in B(W), x \in W, y \in D, \rho \ge 1$. Then, the functional equation (68) has a bounded solution.

Proof. Clearly, $(B(W), qp_b)$ is a quasi-partial b-metric given by equation (69). Let σ be an arbitrary positive number, $x \in W$, $\alpha_1, \alpha_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$P(\alpha_1)(x) < f(x,y_1) + g(x,y_1,\alpha_1(\tau(x,y_1))) + \sigma, \tag{72}$$

$$P(\alpha_2)(x) < f(x,y_2) + g(x,y_2,\alpha_2(\tau(x,y_2))) + \sigma, \tag{73}$$

$$P(\alpha_1)(x) < f(x,y_2) + g(x,y_2,\alpha_1(\tau(x,y_2))) + \sigma, \tag{74}$$

$$P(\alpha_2)(x) < f(x,y_1) + g(x,y_1,\alpha_2(\tau(x,y_1))) + \sigma. \tag{75}$$

From equations (72) and (75),

$$\begin{aligned} P(\alpha_1)(x) - P(\alpha_2)(x) &< g(x, y_1, \alpha_1(\tau(x, y_1))) - g(x, y_1, \alpha_2(\tau(x, y_1))) + \sigma \\ &\le |g(x, y_1, \alpha_1(\tau(x, y_1))) - g(x, y_1, \alpha_2(\tau(x, y_1)))| + \sigma \\ &\le \frac{\rho \mid \alpha_1 - \alpha_2 \mid}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2 \mid}\right)^2} + \sigma. \end{aligned} \tag{76}$$

It implies,

$$P(\alpha_1)(x) - P(\alpha_2)(x) \le \frac{\rho \mid \alpha_1 - \alpha_2 \mid}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2 \mid}\right)^2} + \sigma. \tag{77}$$

Similarly, from equations (73) and (74),

$$P(\alpha_2)(x) - P(\alpha_1)(x) \le \frac{\rho \mid \alpha_1 - \alpha_2 \mid}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2 \mid}\right)^2} + \sigma. \tag{78}$$

From equations (77) and (78), we get

$$|P(\alpha_1)(x) - P(\alpha_2)(x)| \le \frac{\rho|\alpha_1 - \alpha_2|}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2|}\right)^2} + \sigma, \tag{79}$$

i.e.,

$$qp_b(P(\alpha_1), P(\alpha_2)) \le \frac{\rho \mid \alpha_1 - \alpha_2 \mid}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2 \mid}\right)^2} + \sigma. \tag{80}$$

Hence, we conclude that

$$qp_b(P(\alpha_1), P(\alpha_2)) \le \frac{\rho \mid \alpha_1 - \alpha_2 \mid}{\left(1 + \tau\sqrt{|\alpha_1 - \alpha_2 \mid}\right)^2}. \tag{81}$$

Now, we observe that the function $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $F(\alpha) = -1/\sqrt{\alpha}$ for each $\alpha \in W$ is F-contractive function, and hence, operator P is F-contractive.

Since any upper semicontinuous F-contractive function has a fixed point $\alpha^* \in B(W)$, it implies that there exists a bounded solution of functional equation (68).□ □

5. Conclusion

In this manuscript, we established a new type of mappings that is Kannan F-expanding mappings and obtained fixed point theorems for contractive mappings in the framework of quasi-partial b-metric spaces. Moreover, we provided examples that demonstrate the usability of our results. As an application of our result, we also studied a system of integral and functional inclusions. It would be more engaging to work on the obtained results to prove the uniqueness of the fixed point in the future.

Data Availability

This clause is not applicable to this paper.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

A Fractional Order Hepatitis C Mathematical Model with Mittag-Leffler Kernel

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In this paper, a mathematical fractional order Hepatitis C virus (HCV) spread model is presented for an analytical and numerical study. The model is a fractional order extension of the classical model. The paper includes the existence, singularity, Hyers-Ulam stability, and numerical solutions. Our numerical results are based on the Lagrange polynomial interpolation. We observe that the model of fractional order has the same behavior of the solutions as the integer order existing model.

1. Introduction

Hepatitis C is a kind of viral maladies caused by the Hepatitis C virus (HCV), which mostly damages the liver. People generally have minor or no symptoms when they first become infected. Black urine, Fever, yellow-tinged skin, and abdominal pain are all symptoms that might occur. The virus remains in the liver, in roughly 75 percent to 85 percent of patients who are infected. Initially, in the period of a chronic infection, there are usually no symptoms. However, it frequently develops to cirrhosis over time. Hepatitis C, on the other hand, can sometimes lead to liver cancer, blood cancer, and liver failure [1]. The most common way for HCV to spread is by blood-to-blood contact, which is related to injectable usage of drugs, improperly cleaned equipment for medical care, needle stick injuries in health care, and transfusions. The most typical reason for liver transplantation is Hepatitis C, even though virus generally returns after the procedure. Hepatitis C infects an estimated 71 million people (1 percent of the global population) in 2015. Low- and middle-income nations bear the brunt of the health burden, with Africa and Central and East Asia having the greatest rates of prevalence. In 2015, Hepatitis C caused around 167,000 liver cancer deaths and 326,000 cirrhosis deaths. 15th Hepatitis C is a disease that can be transmitted from one person to another. Hepatitis C was first identified in the 1970s, it was thought to be a kind of non-A non-B Hepatitis, and its presence was confirmed in 1989. Only humans and chimps are infected with Hepatitis C; for more details, see [2, 3].

In natural and physical sciences, mathematical and computational tools have been used to investigate phenomena at many scales, ranging from the global human population to individual atoms within a biomolecule. The relevant modeling methodologies span time spans ranging from years to picoseconds, region to region of interest (impacts ranging from evolutionary to atomic), and importance. This exploration will go over some of the most common and useful approaches in mathematics and computing. Differential equations, statistical models, dynamical systems, and game theoretic models are all examples of mathematical models; we refer to [4–10]. These and other types of models can be mixed and matched, resulting in a single model that has a diverse set of abstract structures. Logic models can be used in mathematical models in general. In many instances, the quality of a scientific topic is determined by how well theoretical mathematical models accord with the results of repeated experiments. As better theories are discovered, a lack of concordance between mathematical models that are theoretical and experimental findings frequently leads to significant advancements. Many mathematical models were formulated for the Hepatitis C diseases to understand the dynamics of the diseases and control the spreading of diseases; we refer to [11–14].

Fractional calculus is a discipline of mathematics that investigates the various ways in which the differentiation operator can be defined in terms of real or complex number powers. Because of their ability to include notion of nonlocal operators used to incorporate more complicated natural phenomena into mathematical equations, differential equations have attracted many scholars from practically all disciplines of science, technology, and engineering in recent years. The exponential decay law, the power law, and the extended Mittag-Leffler law were recommended as three dominants in fractional calculus. The kernel Mittag-Leffler function was shown to be more broadly applicable than the power law and exponential decay functions; both Riemann-Liouville and Caputo-Fabrizio are special examples of the Atangana-Baleanu fractional operator; we refer to [15–18]. Many biological models have been studied on fractional operators; Atangana and Alqahtani [19] considered a mathematical model river blindness as in Caputo sense and beta operators. Stability and numerical solutions were obtained for the fractional order model. Gómez-Aguilar et al. [20] examined a cancer model in three dimensions in the sense of Caputo-Fabrizio-Caputo and the novel fractional derivative with Mittag-Leffler kernel. Special solutions were obtained by an iterative process that used the Laplace transform rule, Sumudu-Picard integration approach, and the Adams-Moulton approach. Shah and Bushnaq [21] evaluated an endemic infection model in the fractional sense. Numerical solutions were obtained for the proposed model by combining the Laplace transform with the Adomian decomposition approach. Arfan et al. [22] studied semianalytical solutions for a fractional order COVID-19 model under Caputo derivative; for more details, see [23–26].

Inspired from the above literature, in this paper, we consider a Hepatitis C model in the sense of a fractional order derivative. Furthermore, we investigate the existence and uniqueness of the fractional order model with the help of a fixed point theorem and stability analysis of the fractional order Hepatitis C model. Finally, numerical simulations of the solutions are demonstrated and compared with classical derivatives by using different values of fractional order and parameters. This paper is organized as follows; Section 1: introduction of the paper; Section 2: framework of the model; Section 3: preliminaries; Section 4: existence of solutions; Section 5: numerical data fitting; Section 6: conclusion.

2. Framework of the Model

In this section, we discuss an integer order Hepatitis C model. The population is split into four classes based on their size. $\mathbb{S}$ denotes class of susceptible, $\mathbb{I}$ class of acutely infected, $\mathbb{P}$ class of persistently infected, and $\mathbb{T}$ class of treatment for infection:

$$\begin{cases} \mathbb{D}_t^{\prime}\mathbb{S}(t) = \mu - \lambda\mathbb{S}(t) - \mu\mathbb{S}(t), \\ \mathbb{D}_t^{\prime}\mathbb{I}(t) = (1-\psi)\lambda\mathbb{S}(t) + \psi\lambda(1-\mathbb{I}-\mathbb{P}-\mathbb{T}) - (\mu+\sigma+\varepsilon+d)\mathbb{I}, \\ \mathbb{D}_t^{\prime}\mathbb{P}(t) = \varepsilon\mathbb{I} + \rho\mathbb{T} - (\mu+\delta+\wp)\mathbb{P}, \\ \mathbb{D}_t^{\prime}\mathbb{T}(t) = \wp\mathbb{P} - (\mu+\rho+\theta)\mathbb{T}. \end{cases} \tag{1}$$

With the initial conditions

$$\begin{aligned} \mathbb{S}(0) &\geq 0, \\ \mathbb{I}(0) &\geq 0, \\ \mathbb{P}(0) &\geq 0, \\ \mathbb{T}(0) &\geq 0, \end{aligned} \tag{2}$$

where μ is the rate of natural death, λ is force of infection, ψ is the rate of susceptibility of recovered, σ is the rate of recovery from acute infection, ε is the rate of progression to chronic infection, d is the rate of death due to acute infection, ρ is the rate of treatment failure of chronically infected, δ is the rate of recovery from chronic infection, $\wp$ is the rate of treatment of chronically infected, and θ is the rate of treatment cure. For more details on the existence of infection, endemic equilibrium, reproduction numbers, and stability of endemic equilibrium, see [11–13].

Definition 1 (see [17, 27]). On the basis of Mittag-Leffler kernel and $\psi \in H^*(a,b), b > a$, for $\varrho \in [0,1]$, the *ABC*-fractional differential operator is given as

$$ {}_{a}^{ABC}\mathbb{D}_{\wp}^{\varrho}\psi(\wp) = \frac{\beta(\varrho)}{1-\varrho}\int_a^{\wp}\psi'(s)E_{\varrho}\left[\frac{-\varrho(\wp-s)^{\varrho}}{1-\varrho}\right]ds, \tag{3}$$

where $B(\varrho)$ denote a weighted function which satisfied the main property $\beta(0)=\beta(1)=1$:

$$E_{\rho}(h) = \sum_{r=0}^{\infty}\frac{h^r}{(1+\rho r)}, \quad \rho > 0. \tag{4}$$

Definition 2 (see [17, 27]). On the basis of Mittag-Leffler kernel and $\psi \in H^*(a,b), b > a, \varrho \in [0,1]$, the *ABR*-fractional derivative is defined as

$$ {}_{a}^{ABR}\mathbb{D}_{\wp}^{\varrho}\psi(\wp) = \frac{\beta(\varrho)}{1-\varrho}\frac{d}{d\wp}\int_a^{\wp}\psi(s)E_{\varrho}\left[\frac{-\varrho(\wp-s)^{\varrho}}{1-\varrho}\right]ds. \tag{5}$$

Definition 3 (see [17, 27]). Let $\psi \in H^*(a,b), b > a, 0 < \rho < 1$; the *AB*-integral is given

$$ {}^{AB}{}_{a}\mathbb{I}_{\wp}^{\varrho}\psi(\wp) = \frac{1-\varrho}{\beta(\varrho)}\psi(\wp) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_a^{\wp}\psi(s)(\wp-s)^{\varrho-1}ds. \tag{6}$$

Lemma 4 (see [17]). *Let function ψ, then the AB fractional integral and derivative satisfy the following special character of Newton-Leibniz formula:*

$$ {}_{a}^{AB}\mathbb{I}_{\wp}^{\varrho}\left({}_{a}^{ABC}\mathbb{D}_{\wp}^{\varrho}\psi(\wp)\right)=\psi(\wp)-\psi(a). \tag{7}$$

3. Existence Criteria

Let $\mathscr{B}=\mathscr{I}\times\mathbb{R}^4\longrightarrow\mathbb{R}$, where $\mathscr{I}=[0,T]$, for $0<t<T<\infty$, with a norm defined by $\|(\mathbb{S},\mathbb{I},\mathbb{P},\mathbb{T})\|=\max_{t\in\mathscr{I}}\{|\mathbb{S}|+|\mathbb{I}|+|\mathbb{P}|+|\mathbb{T}|\}$. Then, clearly, $(\mathscr{B},\|\cdot\|)$ is Banach's space. Let us consider system (1) in the sense of fractional order operator:

$$\begin{cases} {}^{ABC}\mathbb{D}_t^{\rho}\mathbb{S}(t)=\mu-\lambda\mathbb{S}(t)-\mu\mathbb{S}(t),\\ {}^{ABC}\mathbb{D}_t^{\rho}\mathbb{I}(t)=(1-\psi)\lambda\mathbb{S}(t)+\psi\lambda(1-\mathbb{I}-\mathbb{P}-\mathbb{T})-(\mu+\sigma+\varepsilon+d)\mathbb{I},\\ {}^{ABC}\mathbb{D}_t^{\rho}\mathbb{P}(t)=\varepsilon\mathbb{I}+\rho\mathbb{T}-(\mu+\delta+\wp)\mathbb{P},\\ {}^{ABC}\mathbb{D}_t^{\rho}\mathbb{T}(t)=\wp\mathbb{P}-(\mu+\rho+\theta)\mathbb{T}. \end{cases} \tag{8}$$

By employing Definition (3) to (8), we have

$$\begin{aligned}\mathbb{S}(t)-\mathbb{S}(0)&=\frac{1-\varrho}{\beta(\varrho)}(\mu-\lambda\mathbb{S}-\mu\mathbb{S}(t))\\&+\frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}(\mu-\lambda\mathbb{S}-\mu\mathbb{S})ds,\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbb{I}(t)-\mathbb{I}(0)&=\frac{1-\varrho}{\beta(\varrho)}((1-\psi)\lambda\mathbb{S}(t)+\psi\lambda(1-I-\mathbb{P}-\mathbb{T})\\&-(\mu+\sigma+\varepsilon+d)\mathbb{I})+\frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}((1-\psi)\lambda\mathbb{S}(t)\\&+\psi\lambda(1-I-\mathbb{P}-\mathbb{T})-(\mu+\sigma+\varepsilon+d)\mathbb{I})ds,\end{aligned} \tag{10}$$

$$\begin{aligned}\mathbb{P}(t)-\mathbb{P}(0)&=\frac{1-\varrho}{\beta(\varrho)}(\varepsilon\mathbb{I}+\rho\mathbb{T}-(\mu+\delta+\wp)\mathbb{P})\\&+\frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}(\varepsilon\mathbb{I}+\rho\mathbb{T}-(\mu+\delta+\wp)\mathbb{P})ds,\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbb{T}(t)-\mathbb{T}(0)&=\frac{1-\varrho}{\beta(\varrho)}(\wp\mathbb{P}-(\mu+\rho+\theta)\mathbb{T})\\&+\frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\rho-1}(\wp\mathbb{P}-(\mu+\rho+\theta)\mathbb{T})ds.\end{aligned} \tag{12}$$

For simplicity in the above Equation (9), we introduce $\mathbb{H}_i$ for $i=1,2,3,4$, given below:

$$\mathbb{H}_1(t,\mathbb{S})=\mu-\lambda\mathbb{S}-\mu\mathbb{S}, \tag{13}$$

$$\mathbb{H}_2(t,\mathbb{I})=(1-\psi)\lambda\mathbb{S}+\psi\lambda(1-I-\mathbb{P}-\mathbb{T})-(\mu+\sigma+\varepsilon+d)\mathbb{I}, \tag{14}$$

$$\mathbb{H}_3(t,\mathbb{P})=\varepsilon\mathbb{I}+\rho\mathbb{T}-(\mu+\delta+\wp)\mathbb{P}, \tag{15}$$

$$\mathbb{H}_4(t,\mathbb{T})=\wp\mathbb{P}-(\mu+\rho+\theta)\mathbb{T}, \tag{16}$$

$$\begin{cases}\chi_1=\lambda-\mu,\\ \chi_2=\mu+\sigma+\varepsilon+d+\psi\lambda,\\ \chi_3=(\mu+\delta+\wp),\\ \chi_4=(\mu+\rho+\theta).\end{cases} \tag{17}$$

For proving our results, we consider the following assumption $\mathbb{B}$. For the below continuous functions $\mathbb{S}(t)$, $\mathbb{S}^*(t)$, $\mathbb{I}(t)$, $\mathbb{I}^*(t)$, $\mathbb{P}(t)$, $\mathbb{P}^*(t)$, $\mathbb{T}(t)$, $T^*(t)$, $\in L[0,1]$ such that $\|\mathbb{S}(t)\|\le\xi_1$, $\|\mathbb{I}(t)\|\le\xi_2$, $\|\mathbb{P}(t)\|\le\xi_3$, $0<\|\mathbb{T}(t)\|\le\xi_4$, there exist three constant $\kappa_i>0$, $i\in\mathbb{N}_1^3$, such that the below hold:

$$\begin{aligned}\|\mathbb{S}(t\|&\le\kappa_1,\\ \|\mathbb{P}(t)-\mathbb{T}(t)\|&\le\kappa_2,\\ \|\mathbb{P}\|&\le\kappa_3.\end{aligned} \tag{18}$$

Theorem 5. *$\mathbb{H}_i$, for $i\in\mathbb{N}_1^5$, satisfy Lipschitz condition if* $\max\{\chi_i$*, for $i=1,2,3,4\}<1$ for χ_i defined in (17).*

Consider for $\mathbb{H}_1$, below

$$\begin{aligned}\|\mathbb{H}_1(t,\mathbb{I})-\mathbb{H}_1(t,\mathbb{I}^*)\|&=\|\mu-\lambda\mathbb{S}(t)-\mu\mathbb{S}(t)-\mu+\lambda\mathbb{S}^*(t)\\&+\mu\mathbb{S}^*(t)\|\le[\lambda-\mu]\|\mathbb{S}-\mathbb{S}^*\|=\chi_1\|\mathbb{S}-\mathbb{S}^*\|.\end{aligned} \tag{19}$$

For $\mathbb{H}_2(t,\mathbb{I})$, we have

$$\begin{aligned}\|\mathbb{H}_2(t,\mathbb{I})-\mathbb{H}_2(t,\mathbb{I}^*)\|&=\|(1-\psi)\lambda\mathbb{S}(t)+\psi\lambda(1-\mathbb{I}-\mathbb{P}-\mathbb{T})\\&-(\mu+\sigma+\varepsilon+d)\mathbb{I}-((1-\psi)\lambda\mathbb{S}(t)+\psi\lambda(1-\mathbb{I}^*-\mathbb{P}-\mathbb{T})\\&-(\mu+\sigma+\varepsilon+d)\mathbb{I}^*)\|\le\psi\lambda\|\mathbb{I}-\mathbb{I}\|+(\mu+\sigma+\varepsilon+d)\|\mathbb{I}-\mathbb{I}^*\|\\&\le(\mu+\sigma+\varepsilon+d+\psi\lambda)\|\mathbb{I}-\mathbb{I}^*\|=\chi_2\|\mathbb{I}-\mathbb{I}^*\|.\end{aligned} \tag{20}$$

$\mathbb{H}_3(t,\mathbb{P})$ implies

$$\begin{aligned}\|\mathbb{H}_3(t,\mathbb{P})-\mathbb{H}_3(t,\mathbb{P}^*)\|&=\|\varepsilon\mathbb{I}+\rho\mathbb{T}\\&-(\mu+\delta+\wp)\mathbb{P}-(\varepsilon\mathbb{I}+\rho\mathbb{T}-(\mu+\delta+\wp)\mathbb{P}^*)\|\\&\le(\mu+\delta+\wp)\|\mathbb{P}-\mathbb{P}^*\|=\chi_3\|\mathbb{P}-\mathbb{P}^*\|.\end{aligned} \tag{21}$$

$\mathbb{H}_4(t,\mathbb{T})$ implies

$$\begin{aligned}\|\mathbb{H}_4(t,\mathbb{T})-\mathbb{T}_4(t,\mathbb{T}^*)\|&=\|\wp\mathbb{P}-(\mu+\rho+\theta)\mathbb{T}\\&-(\wp\mathbb{P}-(\mu+\rho+\theta)-\mathbb{T}^*\|\le(\mu+\rho+\theta)\|\mathbb{T}-\mathbb{T}^*\|\\&=\chi_4\|\mathbb{T}-\mathbb{T}^*\|.\end{aligned} \tag{22}$$

Thus, from (19)(–)(22), we have that $\mathbb{Q}_i$ for $i=1,2,3,4$, satisfying the Lipschitz condition. And this completes the proof. Assuming that $\mathbb{S}(0)=\mathbb{I}(0)=\mathbb{P}(0)=\mathbb{T}(0)=0$, then we have

$$\mathbb{S}(t)=\frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_1(t,\mathbb{S}(t))+\frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}\mathbb{H}_1(s,\mathbb{S})(s)ds, \tag{23}$$

$$\mathbb{I}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_2(t, \mathbb{I}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_2(s, \mathbb{I})(s)ds, \tag{24}$$

$$\mathbb{P}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_3(t, \mathbb{P}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\rho-1}\mathbb{H}_3(s, \mathbb{P})(s)ds, \tag{25}$$

$$\mathbb{T}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_4(t, \mathbb{T}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_4(s, \mathbb{T})(s)ds. \tag{26}$$

For the iterative scheme of the ABC-fractional order HCV model (8), define

$$\begin{aligned}
\mathbb{S}_n(t) &= \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_1(t, \mathbb{S}_{n-1}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_1(s, \mathbb{S}_{n-1})(s)ds,\\
\mathbb{I}_n(t) &= \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_2(t, \mathbb{I}_{n-1}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_2(s, \mathbb{I}_{n-1})(s)ds,\\
\mathbb{P}_n(t) &= \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_3(t, \mathbb{P}_{n-1}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_3(s, \mathbb{P}_{n-1})(s)ds,\\
\mathbb{T}_n(t) &= \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_4(t, \mathbb{T}_{n-1}(t)) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_4(s, \mathbb{T}_{n-1})(s)ds.
\end{aligned} \tag{27}$$

Theorem 6. *The ABC-fractional order HCV model (8) has a solution if*

$$\Delta = \max\{\chi_i\} < 1, \quad i \in \mathbb{N}_1^4. \tag{28}$$

We define the function

$$\begin{aligned}
\mathbb{G}1_n(t) &= \mathbb{S}_{n+1}(t) - \mathbb{S}(t),\\
\mathbb{G}2_n(t) &= \mathbb{I}_{n+1}(t) - \mathbb{I}(t),
\end{aligned} \tag{29}$$

$$\begin{aligned}
\mathbb{G}3_n(t) &= \mathbb{P}_{n+1}(t) - \mathbb{P}(t),\\
\mathbb{G}4_n(t) &= \mathbb{T}_{n+1}(t) - \mathbb{T}(t).
\end{aligned} \tag{30}$$

By the help of (29) and (30), we have

$$\begin{aligned}
\|\mathbb{G}1_n\| &\le \frac{1-\varrho}{\beta(\varrho)}\|\mathbb{H}_1(t, \mathbb{S}_n(t)) - \mathbb{H}_1(t, \mathbb{S}_{n-1}(t))\|\\
&\quad + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\rho-1}\|\mathbb{H}_1(s, \mathbb{S}_n(s))\\
&\quad - \mathbb{H}_1(t, \mathbb{S}_{n-1}(t))\|ds \le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_1\|\mathbb{S}_n - \mathbb{S}_{n-1}\|\\
&\le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]^n \Delta^n\|\mathbb{S}_n - \mathbb{S}_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
\|\mathbb{G}2_n\| &\le \frac{1-\varrho}{\beta(\varrho)}\|\mathbb{H}_2(t, \mathbb{I}_n(t)) - \mathbb{H}_2(t, \mathbb{I}_{n-1}(t))\|\\
&\quad + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\rho-1}\|\mathbb{H}_2(s, \mathbb{I}_n(s))\\
&\quad - \mathbb{H}_2(t, \mathbb{I}_{n-1}(t))\|ds \le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_2\|\mathbb{I}_n - \mathbb{I}_{n-1}\|\\
&\le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]^n \Delta^n\|\mathbb{I}_n - \mathbb{I}_{n-1}\|.
\end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}
\|\mathbb{G}3_n\| &\le \frac{1-\varrho}{\beta(\varrho)}\|\mathbb{H}_3(t, \mathbb{P}_n(t)) - \mathbb{H}_3(t, \mathbb{P}_{n-1}(t))\|\\
&\quad + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\|\mathbb{H}_3(s, \mathbb{P}_n(s))\\
&\quad - \mathbb{H}_3(t, \mathbb{P}_{n-1}(t))\|ds \le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_3\|\mathbb{P}_n - \mathbb{P}_{n-1}\|\\
&\le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]^n \Delta^n\|\mathbb{P}_n - \mathbb{P}_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
\|\mathbb{G}4_n\| &\le \frac{1-\varrho}{\beta(\varrho)}\|\mathbb{H}_4(t, \mathbb{T}_n(t)) - \mathbb{H}_4(t, \mathbb{T}_{n-1}(t))\|\\
&\quad + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\rho-1}\|\mathbb{H}_4(s, \mathbb{T}_n(s))\\
&\quad - \mathbb{H}_4(t, \mathbb{T}_{n-1}(t))\|ds \le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_4\|\mathbb{T}_n - \mathbb{T}_{n-1}\|\\
&\le \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]^n \Delta^n\|\mathbb{T}_n - \mathbb{T}_{n-1}\|,
\end{aligned} \tag{32}$$

which ensure that $\mathbb{G}(t)_n \longrightarrow 0$, $i \in \mathbb{N}_1^4$, as $n \longrightarrow \infty$ for $\Delta < 1$, which completes the proof.

4. Uniqueness Solution

For our suggested model (8), we study the analysis of the uniqueness of solution.

Theorem 7. *The ABC-fractional order Hepatitis C model (8) has a unique solution provided that*

$$\left[\frac{1-\varrho_i}{\beta(\varrho_i)} + \frac{1}{\beta(\varrho_i)\Gamma(\varrho_i)}\right]\chi_i \le 1, \quad i \in \mathbb{N}_1^4. \tag{33}$$

Assume another solution exist $\bar{\mathbb{S}}(t), \bar{\mathbb{I}}(t), \overline{\mathbb{P}}(t), \bar{\mathbb{T}}(t)$, and $\overline{\mathbb{R}}(t)$ such that

$$\bar{\mathbb{S}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_1\left(t, \bar{\mathbb{H}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_1\left(s, \bar{\mathbb{S}}(s)\right)ds,$$
$$\bar{\mathbb{I}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_2\left(t, \bar{\mathbb{I}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_2\left(s, \bar{\mathbb{I}}(s)\right)ds,$$
$$\bar{\mathbb{P}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_3\left(t, \bar{\mathbb{P}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_3\left(s, \bar{\mathbb{P}}(s)\right)ds,$$
$$\bar{\mathbb{T}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_4\left(t, \bar{\mathbb{T}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_4\left(s, \bar{\mathbb{T}}(s)\right)ds. \tag{34}$$

Then,

$$\begin{aligned}\left\|\mathbb{S}(t)-\bar{\mathbb{S}}(t)\right\| &\leq \frac{1-\varrho}{\beta(\varrho)}\left\|\mathbb{H}_1(t,\mathbb{S}(t)) - \mathbb{H}_1\left(t, \bar{\mathbb{S}}(t)\right)\right\| \\ &+ \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\|\mathbb{H}_1(s,\mathbb{S}(s)) \\ &- \mathbb{H}_1\left(t,\bar{\mathbb{S}}(t)\right)\|ds \leq \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_1\left\|\mathbb{S}-\bar{\mathbb{S}}\right\|,\end{aligned} \tag{35}$$

which implies

$$\left[\frac{1-\varrho}{\beta(\varrho)}\chi_1 + \frac{\chi_1}{\beta(\varrho)\Gamma(\varrho)} - 1\right]\left\|\mathbb{S}-\bar{\mathbb{S}}\right\| \geq 0. \tag{36}$$

By (33), (36) holds if $\|\mathbb{S}-\bar{\mathbb{S}}\| = 0$*; this implies* $\mathbb{S}(t) = \bar{\mathbb{S}}(t)$*. With the same procedure, for* $\mathbb{I}$*, we have*

$$\begin{aligned}\left\|\mathbb{I}(t)-\bar{\mathbb{I}}(t)\right\| &\leq \frac{1-\varrho}{\beta(\varrho)}\left\|\mathbb{H}_2(t,\mathbb{I}(t)) - \mathbb{H}_2\left(t, \bar{\mathbb{I}}(t)\right)\right\| \\ &+ \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\|\mathbb{H}_2(s,\mathbb{I}(s)) \\ &- \mathbb{H}_2\left(t,\bar{\mathbb{I}}(t)\right)\|ds \leq \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_2\left\|\mathbb{I}-\bar{\mathbb{I}}\right\|,\end{aligned} \tag{37}$$

which implies

$$\left[\frac{1-\varrho}{\beta(\varrho)}\chi_1 + \frac{\chi_1}{\beta(\varrho)\Gamma(\varrho)} - 1\right]\left\|\mathbb{P}-\bar{\mathbb{P}}\right\| \geq 0. \tag{38}$$

By (33), (38) is true if $\|\mathbb{P}-\bar{\mathbb{P}}(t)\| = 0$*; this implies* $\mathbb{P}(t) = \bar{\mathbb{P}}(t)$*. With the same procedure, for* $\mathbb{T}$*, we have*

$$\begin{aligned}\left\|\mathbb{T}(t)-\bar{\mathbb{T}}(t)\right\| &\leq \frac{1-\varrho}{\beta(\varrho)}\left\|\mathbb{H}_3(t,\mathbb{T}(t)) - \mathbb{H}_3\left(t, \bar{\mathbb{T}}(t)\right)\right\| \\ &+ \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\|\mathbb{H}_3(s,\mathbb{T}(s)) \\ &- \mathbb{H}_3\left(t,\bar{\mathbb{T}}(t)\right)\|ds \leq \left[\frac{1-\varrho}{\beta(\varrho)} + \frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_3\left\|\mathbb{T}-\bar{\mathbb{T}}\right\|,\end{aligned} \tag{39}$$

which implies

$$\left[\frac{1-\varrho}{\beta(\varrho)}\chi_4 + \frac{\chi_4}{\beta(\varrho)\Gamma(\varrho)} - 1\right]\left\|\mathbb{T}-\bar{\mathbb{T}}\right\| \geq 0. \tag{40}$$

By (33), (40) holds if $\|\mathbb{T}-\bar{\mathbb{T}}\| = 0$*, which implies* $\mathbb{T}(t) = \bar{\mathbb{T}}(t)$*. Similarly,* $\mathbb{T}(t) = \bar{\mathbb{T}}(t)$*. Thus, the ABC-fractional order (8) has a unique solution.*

5. Hyers-Ulam Stability

Definition 8. The integral system (23)–(26) is Hyers-Ulam stable if for $\Delta_i > 0$, $i \in \mathbb{N}_1^4$, and $\gamma_i > 0$, $i \in \mathbb{N}_1^5$, such that

$$\left|\mathbb{S}(t) - \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_1(t,\mathbb{S}(t)) - \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_1(s,\mathbb{S}(s))ds\right| \leq \gamma_1,$$
$$\left|\mathbb{I}(t) - \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_2(t,\mathbb{I}(t)) - \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_2(s,\mathbb{I}(s))ds\right| \leq \gamma_2,$$
$$\left|\mathbb{P}(t) - \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_3(t,\mathbb{P}(t)) - \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_3(s,\mathbb{P}(s))ds\right| \leq \gamma_3,$$
$$\left|\mathbb{T}(t) - \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_4(t,\mathbb{T}(t)) - \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_4(s,\mathbb{T}(s))ds\right| \leq \gamma_4. \tag{41}$$

We have $\dot{\mathbb{S}}(t), \dot{\mathbb{I}}(t), \dot{\mathbb{P}}(t), \dot{\mathbb{T}}(t)$ which implies

$$\dot{\mathbb{S}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_1\left(t,\dot{\mathbb{S}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_1\left(s,\dot{\mathbb{S}}(s)\right)ds,$$
$$\dot{\mathbb{I}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_2\left(t,\dot{\mathbb{I}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_2\left(s,\dot{\mathbb{I}}(s)\right)ds,$$
$$\dot{\mathbb{P}}(t) = \frac{1-\rho}{\beta(\rho)}\mathbb{H}_3\left(t,\dot{\mathbb{P}}(t)\right) + \frac{\rho}{\beta(\rho)\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\mathbb{H}_3\left(s,\dot{\mathbb{I}}_H(s)\right)ds,$$
$$\dot{\mathbb{T}}(t) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{H}_4\left(t,\dot{\mathbb{T}}(t)\right) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{H}_4\left(s,\dot{\mathbb{I}}_{TH}(s)\right)ds, \tag{42}$$

such that

$$\left|\mathbb{S}(t) - \dot{\mathbb{S}}(t)\right| \leq \delta_1\gamma_1,$$
$$\left|\mathbb{I}(t) - \dot{\mathbb{I}}(t)\right| \leq \delta_2\gamma_2,$$
$$\left|\mathbb{P}(t) - \mathbb{P}(t)\right| \leq \delta_3\gamma_3,$$
$$\left|\mathbb{T}(t) - \dot{\mathbb{T}}(t)\right| \leq \delta_4\gamma_4. \tag{43}$$

Theorem 9. *Let* $(\mathbb{B})$ *be satisfied. Then, ABC-fractional order HCV model (8) is Hyers-Ulam stable.*

Proof. By Theorem 7, the *ABC*-fractional order HCV model (8) has a unique solution, say $\mathbb{S}(t), \mathbb{I}(t), \mathbb{P}(t), \mathbb{T}(t)$. Let us consider $\dot{\mathbb{S}}(t), \dot{\mathbb{I}}(t), \mathbb{P}(t), \dot{\mathbb{T}}(t)$ to be another solution of (8) satisfying (23)–(26). Then, we have

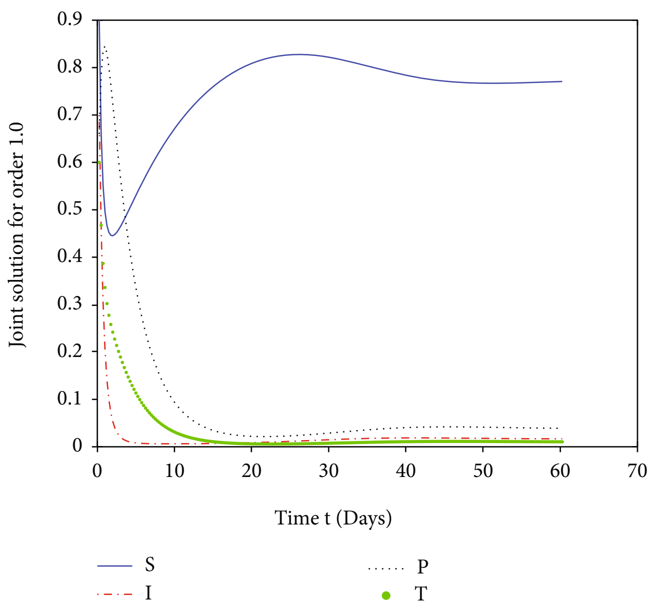

FIGURE 1: The joint solution for $\varrho = 1$.

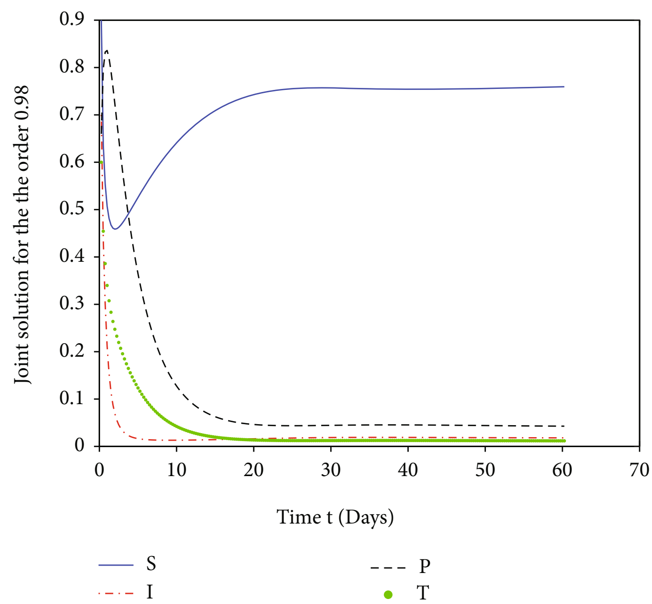

FIGURE 2: The joint solution for $\varrho = 0.98$.

$$\begin{aligned}\left\|\mathbb{S}(t)-\dot{\mathbb{S}}(t)\right\| &\le \frac{1-\varrho}{\beta(\varrho)}\left\|\mathbb{H}_1(t,\mathbb{S}(t))-\mathbb{H}_1\left(t,\dot{\mathbb{S}}(t)\right)\right\| \\ &\quad + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\left\|\mathbb{H}_1(s,\mathbb{S}(s))-\mathbb{H}_1\left(t,\dot{\mathbb{S}}(t)\right)\right\|ds \\ &\le \left[\frac{1-\varrho}{\beta(\varrho)}+\frac{1}{\beta(\varrho)\Gamma(\varrho)}\right]\chi_1\|\mathbb{S}-\dot{\mathbb{S}}\|.\end{aligned} \tag{44}$$

Taking $\gamma_1 = \chi_1$, $\Delta = (1-\varrho)/\beta(\varrho) + \varrho/\beta(\varrho)\Gamma(\varrho)$, this implies

$$\|\mathbb{S}(t)-\dot{\mathbb{S}}(t)\| \le \gamma_1\Delta_1. \tag{45}$$

Similarly, for $\mathbb{I}(t), \dot{\mathbb{I}}(t), \mathbb{P}(t), \dot{\mathbb{P}}(t), \mathbb{T}(t), \dot{\mathbb{T}}(t)$, we have

$$\left\{\left\|\mathbb{I}(t)-\dot{\mathbb{I}}(t)\right\| \le \gamma_2\Delta, \left\|\mathbb{P}(t)-\dot{\mathbb{P}}(t)\right\| \le \gamma_3\Delta, \left\|\mathbb{T}(t)-\dot{\mathbb{T}}(t)\right\| \le \gamma_4\Delta.\right. \tag{46}$$

This implies that system (8) is Hyers-Ulam stable which ultimately ensures the stability of (8). This completes the proof. □

6. Numerical Scheme

We provide the following numerical scheme by the Caputo fractional derivative:

$$\begin{cases} {}_0^{ABC}\mathbb{D}_t^{\varrho}\mathbb{S}(t) = \mathbb{M}_1(t,\mathbb{S}), \\ {}_0^{ABC}\mathbb{D}_t^{\varrho}\mathbb{I}_T(t) = \mathbb{M}_2(t,\mathbb{I}_T), \\ {}_0^{ABC}\mathbb{D}_t^{\varrho}\mathbb{I}_H(t) = \mathbb{M}_3(t,\mathbb{I}_H), \\ {}_0^{ABC}\mathbb{D}_t^{\varrho}\mathbb{I}_{TH}(t) = \mathbb{M}_4(t,\mathbb{I}_{TH}). \end{cases} \tag{47}$$

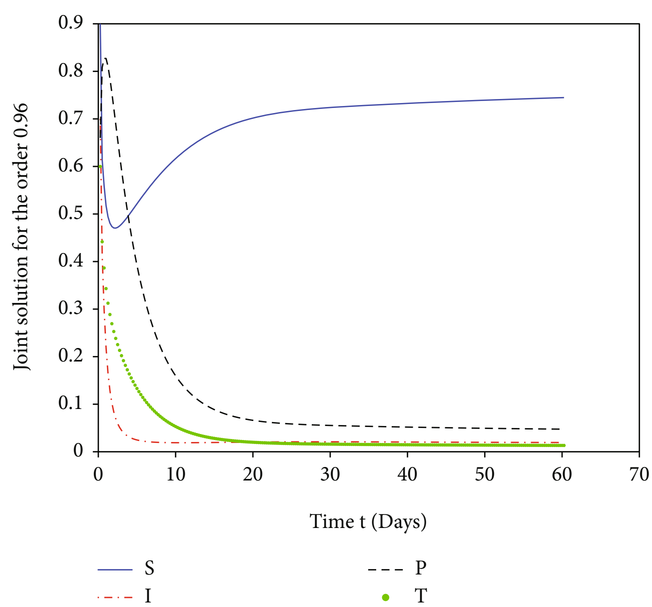

FIGURE 3: The joint solution for $\varrho = 0.96$.

With the help of fractional *AB*-integral operator, (48) gets the following form:

$$\mathbb{S}(t)-\mathbb{S}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_1(t,\mathbb{S}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{M}_1(s,\mathbb{S})ds, \tag{48}$$

$$\mathbb{I}(t)-\mathbb{I}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_2(t,\mathbb{I}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{M}_2(s,\mathbb{I})ds, \tag{49}$$

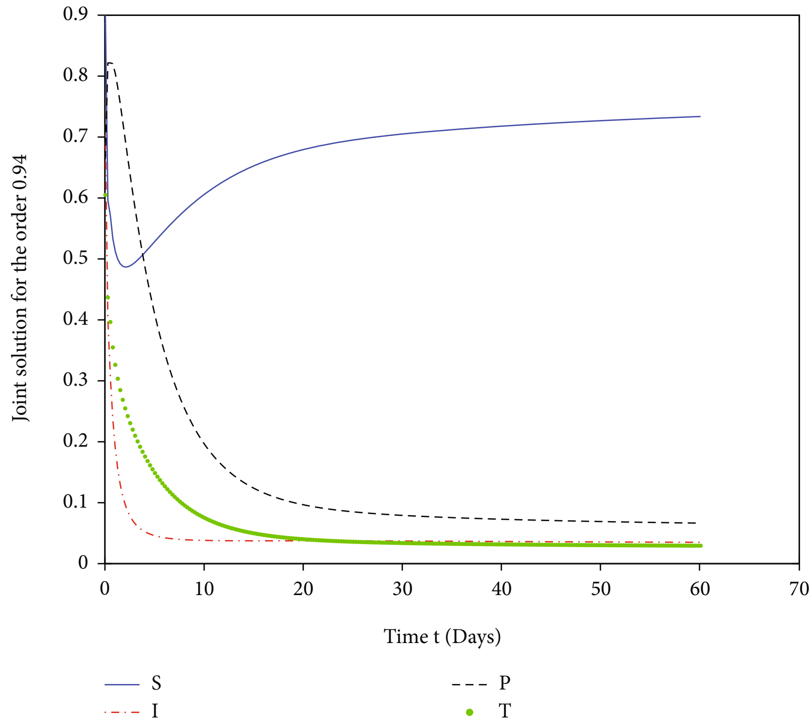

Figure 4: The joint solution for $\varrho = 0.94$.

$$\mathbb{P}(t) - \mathbb{P}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_3(t, \mathbb{P}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\mathbb{M}_3(s, \mathbb{P})ds, \quad (50)$$

$$\mathbb{T}(t) - \mathbb{T}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_4(t, \mathbb{P}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\rho-1}\mathbb{M}_4(s, \mathbb{T})ds. \quad (51)$$

By dividing the assumed interval $[0, t]$ into subintervals by the help of point t_{n+1}, for $n = 0,1,2, \cdots$, we have

$$\mathbb{S}(t_{n+1}) - \mathbb{S}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_1(t_n, \mathbb{S}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \cdot (t_{n+1}-s)^{\varrho-1}\mathbb{M}_1(s, \mathbb{S})ds,$$

$$\mathbb{I}(t_{n+1}) - \mathbb{I}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_2(t_n, \mathbb{I}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \cdot (t_{n+1}-s)^{\varrho-1}\mathbb{M}_2(s, \mathbb{I})ds,$$

$$\mathbb{P}(t_{n+1}) - \mathbb{P}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_3(t_n, \mathbb{P}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \cdot (t_{n+1}-s)^{\varrho-1}\mathbb{M}_3(s, \mathbb{P})ds,$$

$$\mathbb{T}(t_{n+1}) - \mathbb{T}(0) = \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_4(t_n, \mathbb{T}) + \frac{\varrho}{\beta(\varrho)\Gamma(\varrho)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \cdot (t_{n+1}-s)^{\varrho-1}\mathbb{M}_4(s, \mathbb{T})ds. \quad (52)$$

Now, using Lagrange's interpolation, we have

$$\mathbb{S}(t_{n+1}) = \mathbb{S}(0) + \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_1(t_k, \mathbb{S}) + \frac{\varrho}{\mathbb{B}(\varrho)} \times \sum_{k=0}^{n} \cdot \left[\frac{h^{\varrho}\mathbb{M}_1(t_k, \mathbb{S})}{\Gamma(\varrho+2)}((n+1-k)^{\varrho}(n-k+2+\varrho) - (n-k)^{\varrho}(n-k+2+2\varrho)) - \frac{h^{\varrho}\mathbb{Z}_1(t_{k-1}, \mathbb{S})}{\Gamma(\varrho+2)} \times ((n+1-k)^{\varrho} - (n-k)^{\varrho}(n+1-k+\varrho))\right],$$

$$\mathbb{I}(t_{n+1}) = \mathbb{I}(0) + \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_2(t_k, \mathbb{I}) + \frac{\varrho}{\mathbb{B}(\varrho)} \times \sum_{k=0}^{n} \cdot \left[\frac{h^{\varrho}\mathbb{M}_2(t_k, \mathbb{I})}{\Gamma(\varrho+2)}((n+1-k)^{\varrho}(n-k+2+\varrho) - (n-k)^{\varrho}(n-k+2+2\varrho)) - \frac{h^{\varrho}\mathbb{M}_2(t_{k-1}, \mathbb{I})}{\Gamma(\varrho+2)} \times ((n-k+1)^{\varrho} - (n-k)^{\varrho}(n+1-k+\varrho))\right],$$

$$\mathbb{P}(t_{n+1}) = \mathbb{P}(0) + \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_3(t_k, \mathbb{P}) + \frac{\varrho}{\mathbb{B}(\varrho)} \times \sum_{k=0}^{n} \cdot \left[\frac{h^{\varrho}\mathbb{M}_3(t_k, \mathbb{I})}{\Gamma(\varrho+2)}((n+1-k)^{\varrho}(n-k+2+\varrho) - (n-k)^{\varrho}(n-k+2+2\varrho)) - \frac{h^{\varrho}\mathbb{Z}_3(t_{k-1}, \mathbb{I}_H)}{\Gamma(\varrho+2)} \times ((n+1-k)^{\varrho} - (n-k)^{\varrho}(n+1-k+\varrho))\right],$$

Table 1: Shows the parameter values given in Equation (8). The time units are taken in years.

Parameter	Description	Value	Reference
μ	Rate of natural death	0.09	[12]
λ	Force of infection	Assumed	
ψ	Rate of susceptibility of recovered	0.5	[11–13]
σ	Rate of recovery from acute infection	0.02	[12]
ε	Rate of progression to chronic infection	0.7	[12]
d	Rate of death due to acute infection	0.035	[12]
ρ	Rate of treatment failure of chronically infected	0.50	[12]
δ	Rate of recovery from chronic infection	0.012	[12]
$\wp$	Rate of treatment of chronically infected	0.34	[12]
θ	Rate of treatment cure	0.67	[12]

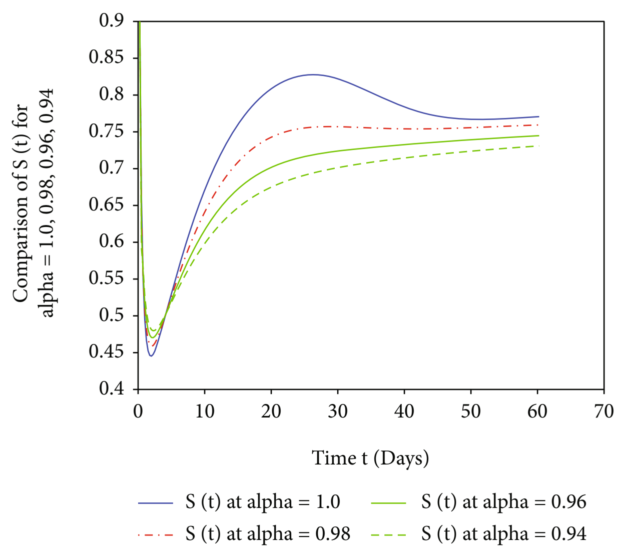

Figure 5: The susceptible class $\mathbb{S}(t)$ for $\varrho = 1.0, \quad 0.98, \quad 0.96, \quad 0.94$.

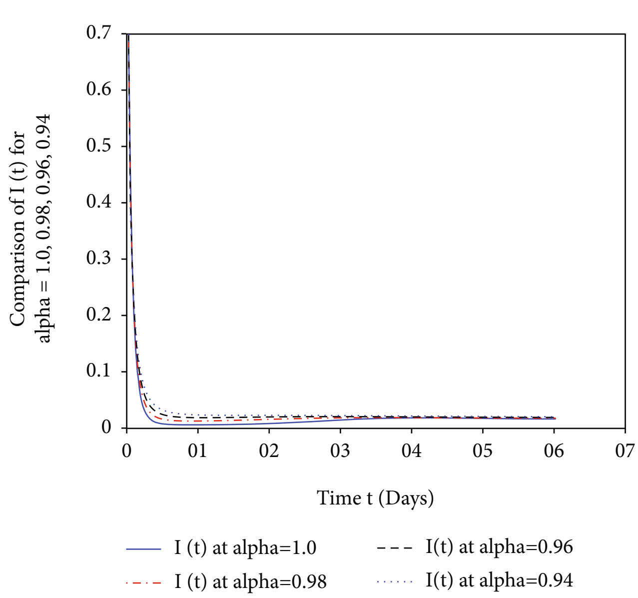

Figure 6: The infected class $\mathbb{I}(t)$ for $\varrho = 1.0, 0.98, 0.96, 0.94$.

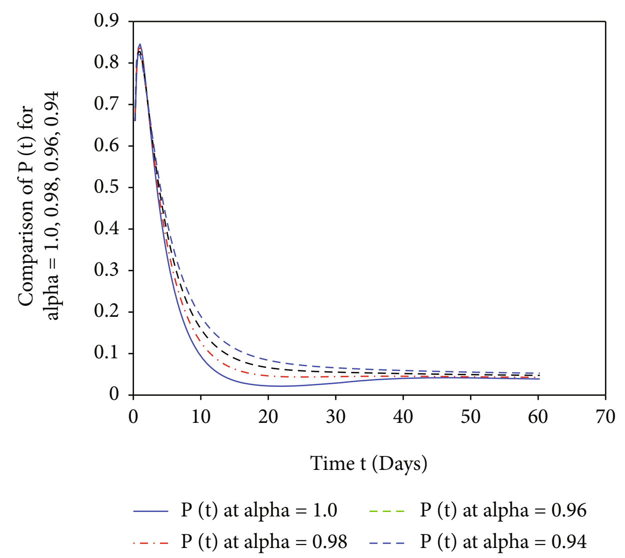

Figure 7: The chronically infected $P(t)$ for $\varrho = 1.0, 0.98, 0.96, 0.94$.

$$\begin{aligned} \mathbb{T}(t_{n+1}) = {} & \mathbb{T}(0) + \frac{1-\varrho}{\beta(\varrho)}\mathbb{M}_4(t_k, \mathbb{T}) + \frac{\varrho}{\mathbb{B}(\varrho)} \times \sum_{k=0}^{n} \\ & \cdot \left[\frac{h^{\varrho}\mathbb{M}_4(t_k, \mathbb{T})}{\Gamma(\varrho+2)}\left((n+1-k)^{\varrho}(n-k+2+\varrho)\right.\right. \\ & \left. - (n-k)^{\varrho}(n-k+2+2\varrho)\right) - \frac{h^{\varrho}\mathbb{M}_4(t_{k-1}, \mathbb{I}_{TH})}{\Gamma(\varrho+2)} \\ & \left. \times \left((n+1-k)^{\varrho+1} - (n-k)^{\varrho}(n+1-k+\varrho)\right)\right]. \end{aligned} \tag{53}$$

7. Computational Results

Here, we present some computational results based on the parametric values defined in [12]. We consider the initial values $\mathbb{S}_0 = 0.90, \mathbb{I}_0 = 0.70, \mathbb{P}_0 = 0.66, \mathbb{T}_0 = 0.60,$ and the parametric values given by $\beta = 1.30, \mu = 0.09, \chi = 0.5, \psi = 0.5, \sigma = 0.5, \varepsilon = 0.7, d = 0.035, \rho = 0.50, \delta = 0.012, \wp = 0.34, \theta = 0.67$. We get the same behavior of the fractional order model as for the integer order.

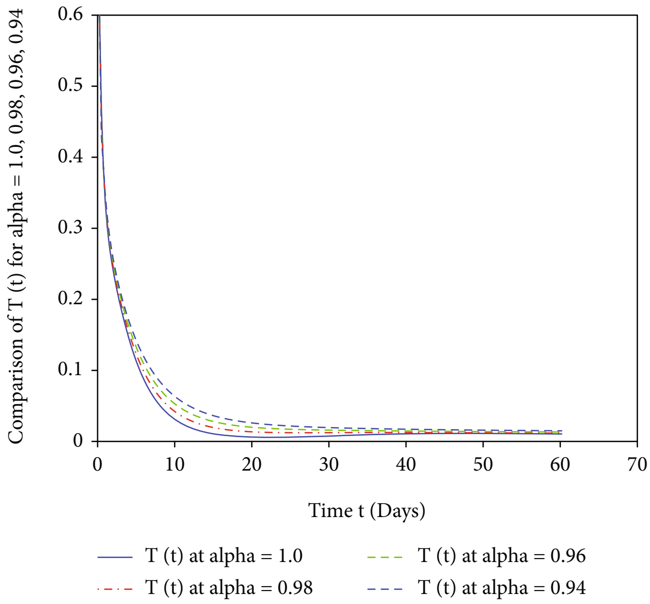

Figure 8: Chronically infected treated class $T(t)$ for $\varrho = 1.0, 0.98, 0.96, 0.94$.

Figures 1–4 represent the numerical solution of the model for the orders $\varrho = 1.0, 0.98, 0.96, 0.94$ and using Table 1 values, respectively. Figure 5 represents the susceptible class for the orders $\varrho = 1.0, 0.98, 0.96, 0.94$ and using Table 1 values, which are increasing with the passage of time and get stability after 60 days of the treatment. The infection class is given in Figure 6, for the fractional orders $\varrho = 1.0, 0.98, 0.96, 0.94$ and using Table 1 values, it has been observed that with the passage of time, the infection is decreased to a certain limit. Figure 7 shows the chronic infection class for $\varrho = 1.0, 0.98, 0.96, 0.94$, and using Table 1 values, finally the chronically infected treated class for $\varrho = 1.0, 0.98, 0.96, 0.94$ is given in Figure 8. This numerical analysis of the fractional order model ensures that the fractional order model is more informative and has the same behavior as the classical model.

8. Conclusions

In this article, we have given a mathematical fractional order Hepatitis C virus (HCV) spread model for an analytical and numerical study. The model is a fractional order extension of the classical model. The paper includes the existence, uniqueness, Hyers-Ulam stability, and a numerical scheme for the computational results. Our numerical results are based on the Lagrange polynomial interpolation. On the basis of the numerical scheme, we have given graphical explanation of the model and its subclasses. For details, Figures 1–4 represent the numerical solution of the model with the fractional orders $\varrho = 1.0, 0.98, 0.96, 0.94$, respectively. Figure 5 represents the susceptible class for the orders $\varrho = 1.0, 0.98, 0.96, 0.94$ which are increasing with the passage of time and get stability after 60 days of treatment. The infection class is given in Figure 6 for the fractional orders $\varrho = 1.0, 0.98, 0.96, 0.94$, and it has been observed that with the passage of time the infection is decreased to a certain limit. Figure 7 shows the chronic infection class for $\varrho = 1.0, 0.98, 0.96, 0.94$, and finally, the chronically infected treated class for $\varrho = 1.0, 0.98, 0.96, 0.94$ is given in Figure 8. Our numerical analysis of the fractional order model ensures that the fractional order model is more informative and has the same behavior as the classical model.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Research Article

Some Existence and Stability Criteria to a Generalized FBVP Having Fractional Composite *p*-Laplacian Operator

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In this paper, we consider a generalized Caputo boundary value problem of fractional differential equation with composite p-Laplacian operator. Boundary value conditions of this problem are of three-point integral type. First, we obtain Green's function in relation to the proposed fractional boundary value problem and then for establishing the existence and uniqueness results, we use topological degree theory and Banach contraction principle. Further, we consider a stability analysis of Ulam-Hyers and Ulam-Hyers-Rassias type. To examine the validity of theoretical results, we provide an illustrative example.

1. Introduction

Fractional calculus, as a generalization of classical ordinary calculus to integrodifferential operators in the noninteger settings, has attracted considerable interest in recent years and has grown rapidly since its introduction. Fractional calculus is now broadly used in several fields such as biology, fluid dynamics, viscoelastic theory, neural networks, and epidemic models; see for instance [1, 2] and references therein.

By using fixed point techniques, a large number of researchers studied the existence-uniqueness properties of solutions for different classes of differential equations in the fractional settings. In 2016, Ntouyas et al. [3] studied two fractional boundary value problems (FBVPs) with three-point boundary conditions and derived the existence results by using the fixed point notion. Similarly, in [4], Boutiara et al. used fixed point theorems to prove the existence results for another FBVP with three-point boundary conditions in the context of the Caputo-Hadamard and Hadamard operators. More recently, Derbazi et al. [5] designed a new FBVP by applying the generalized ψ-operators and proved their desired results via monotone iterative techniques.

As you know, every numerical method must be accurate in order to give desired results which are acceptable for different applications. For this purpose, the analysis of the stability is needed. Various types of stability involving exponential, Lyapunov, and Mittag-Leffler have been studied for different types of problems. The abovementioned types of stability have been improved for many differential equations in both linear and nonlinear fractional cases and their related systems over the last few years. However, the stability of some nonlinear systems undergo unavoidable deficiencies which appear due to the need of predefining Lyapunov function. This is often considered as an uneasy task.

In [6, 7], Ulam and Hyers have initiated the concept of Ulam-Hyers stability. In addition, this notion has been considered for nonlinear fractional differential equations and their related systems. For instance, Abdo et al. [8] investigated the stability criteria for ψ-Hilfer fractional integrodifferential equations and in the same time, Zada et al. [9] derived similar results for impulsive integrodifferential equations with Riemann-Liouville boundary conditions. In [10], Kheiryan and Rezapour considered a new multisingular FBVP and checked its Hyers-Ulam stability.

On the other hand, some properties of solutions of FBVPs including the uniqueness, existence, and stability notions have been investigated with the help of various techniques such as topological degree theory (T-degree theory) and fixed point theory. In this paper, we will apply the existing concepts in T-degree theory, as well as there are a large number of nonlinear mathematical models in engineering and the scientific fields to investigate and analyze dynamical systems. One of the most important nonlinear operators frequently used is the classical p-Laplacian operator. Models with p-Laplacian operators are often used to simulate practical problems such as tides caused by celestial gravity and elastic deformation of beams. Such extensive applications attract the attention of many researchers to study mathematical models having p-Laplacian operators.

Specifically, Ma et al. [11] defined a new multipoint FBVP with p-Laplacian operator and derived the existence and iteration of monotone positive solutions for the given system. Next, Matar et al. [12] studied another p-Laplacian FBVP having Caputo-katugampula fractional derivatives recently. For more details about p-Laplacian fractional boundary value problems, we refer to [13, 14].

Also, to see the importance of existing techniques in T-degree theory, we can point out to a paper published by Shah and Khan [15] on the existence-uniqueness results to a coupled system of FBVPs. Further, Sher et al. [16] implemented a qualitative analysis on a multiterm delay FBVP with the help of the same technique in T-degree theory.

In 2017, Ali et al. [17] studied a coupled fractional structure of a system involving two differential equations with non-integer boundary conditions of integral type which takes the form

$$\begin{cases} {}^{C}D_{0^+}^{r_1} v_1(t) = \phi_1(t, v_2(t)), & \leq t \leq 1, \\ {}^{C}D_{0^+}^{r_1} v_1(t) = \phi_2(t, v_1(t)), & \leq t \leq 1, \\ v_1(0) = 0, & v_1(1) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^K (K-s)^{\alpha-1} p(v_1(s)) ds, \\ v_2(0) = 0, & v_2(1) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^K (K-s)^{\beta-1} p(v_2(s)) ds, \end{cases} \tag{1}$$

where ${}^{C}D_{0^+}^{r_1}(\cdot), {}^{C}D_{0^+}^{r_2}(\cdot)$ stands for the Caputo derivative of orders $1 < r_1 < 2$ and $1 < r_2 < 2$, respectively, and $\phi_1, \phi_2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous functions along with $p, q \in L[0,1]$ which satisfy some certain linear growth conditions. By setting certain particular conditions, they derived their desired existence results using some techniques in T-degree theory. The authors also investigated the Hyers-Ulam stability for the proposed problem.

In [18], Khan et al. studied the existence of solutions and their uniqueness for the proposed coupled fractional structure of a FBVP having the nonlinear operator of p-Laplacian type and integral boundary conditions given by

$$\begin{cases} {}^{C}D_{0^+}^{r_1^*} \varphi_p\left({}^{C}D_{0^+}^{r_1} v_1(t)\right) = \varrho_1(t, v_1, v_2(t)), & 0 \leq t \leq 1, \\ {}^{C}D_{0^+}^{r_2^*} \varphi_p\left({}^{C}D_{0^+}^{r_2} v_1(t)\right) = \varrho_2(t, v_1, v_2(t)), & 0 \leq t \leq 1, \\ \varphi_p\left({}^{C}D_{0^+}^{r_1} v_1\right)(0) = v_1'(0) = 0, & v_1(0) = c_1 \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^K (K-s)^{\alpha-2} v_1(s) ds, \\ \varphi_p\left({}^{C}D_{0^+}^{r_2} v_2\right)(0) = v_2'(0) = 0, & v_2(0) = c_2 \dfrac{1}{\Gamma(\beta-1)} \displaystyle\int_0^K (\eta-s)^{\beta-2} v_2(s) ds, \end{cases} \tag{2}$$

where ${}^{C}D_{0^+}^{r_j^*}(\cdot)$ and ${}^{C}D_{0^+}^{r_j}(\cdot)$ for $j = 1, 2$ denote the Caputo derivative of orders $0 < r_j^* < 1$ and $1 < r_j < 2$. Additionally, $\varrho_1, \varrho_2 : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous, and $\alpha, \beta \geq 1, -1 \leq c_1, c_2 \leq 0$ and $\varphi_p(\theta) = |\theta|^{p-2}\theta$ stand for the p-Laplacian operator such that $1/q + 1/p = 1$. The authors established their desired theorems using the techniques attributed to Leray-Schauder and Banach. Further, the Hyers-Ulam stability was investigated.

In [19] and by means of T-degree theory, Shah and Hussain established sufficient conditions for investigation of the existence of solutions and their stability on the following nonlinear FBVP

$$\begin{cases} {}^{C}D_{0^+}^{r} = \Theta(t, \mu(t)), & 2 < r < 4, \\ \mu(0) = \zeta(t), \quad \mu'(0) = \mu''(0) = 0, & \mu(1) = \nu\mu(\eta), \end{cases} \tag{3}$$

where ${}^{C}D_{0^+}^{r}(\cdot)$ represents the Caputo derivative of order r. Further, $\Theta : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\zeta : [0,1] \longrightarrow \mathbb{R}$ are regarded to be continuous and $\mu, \eta \in (0,1)$.

By considering the existing literatures, we see that all differential equations having a p-Laplacian operator with three-point integral boundary conditions are not well explored by T-degree theory, and even the boundary conditions of integral type cover a wide range of applications which have direct contributions in the theory of fluid mechanics, optimization, and viscoelasticity.

Inspired and motivated by the above fractional systems, we focus on the existence of solutions and establish four classes of Hyres-Ulam stability of a generalized FBVP having p-Laplacian operator with 3-point integral boundary conditions given by

$$\begin{cases} {}^{C}D^{\beta,\delta}\left(\varphi_p\left({}^{C}D^{\chi,\rho} x(t)\right)\right) = \hbar_1(t, x(t)) & (t \in [t_0, K], t_0 \geq 0), \\ x(t_0) + \mu_1 x(K) = \sigma_1 \displaystyle\int_{t_0}^K \hbar_2(s) ds, & (\mu_1 \neq -1), \\ x'(t_0) + \mu_2 x'(K) = \sigma_2 \displaystyle\int_{t_0}^K \hbar_3(s) ds, & \left(\mu_2 \neq -\left(\dfrac{t_0}{K}\right)^{\rho-1}\right), \\ {}^{C}D^{\chi,\rho} x(t_0) = 0, {}^{C}D^{\chi,\rho} x(K) = \upsilon {}^{C}D^{\chi,\rho} x(\eta), & (\eta \in (t_0, K)), \end{cases} \tag{4}$$

so that ${}^{C}D^{\beta,\delta}$ and ${}^{C}D^{\chi,\rho}$ are generalized derivatives in the sense of Caputo, of order $\beta, \chi \in (1,2)$ and $\delta, \rho > 0$. Along with these, $\mu_1, \mu_2, \upsilon, \sigma_1, \sigma_2 \in \mathbb{R}$ with $\upsilon \neq (K^\delta - t_0^\delta/\eta^\delta - t_0^\delta)^{q-1}$ and also $\hbar_2, \hbar_3 : [t_0, K] \longrightarrow \mathbb{R}$ and $\hbar_1 : [t_0, K] \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to be continuous. We emphasize that the proposed FBVP (4) has a novel structure and is designed for the first time in the context of a generalized fractional settings along with the p-Laplacian operator.

2. Auxiliary Preliminaries

The main purpose of this section is to collect some important definitions, primitive lemmas, and theoretical results of generalized fractional integrals and derivatives which are applicable in this paper.

By $\mathscr{C}([t_0, K])$, we mean the category of all continuous real functions defined on $[t_0, K]$ which is simply proved that it is a Banach space along with $\|x\| = \sup_{t\in[t_0,K]}|x(t)|$. Moreover, $AC^l([t_0, K], \mathbb{R}) = \{x : [t_0, K] \longrightarrow \mathbb{R} : x^{(l-1)}(t) \in AC([t_0, K], \mathbb{R})\}$ stands for the space of absolutely continuous functions on $[t_0, K]$ having real values up to $(l-1)$-derivative. Thus, in this regard, we define

$$AC^l_\delta([t_0, K], \mathbb{R}) = \left\{x : [t_0, K] \longrightarrow \mathbb{R} : \left(\delta^{l-1}x\right)(t) \in AC([t_0, K], \mathbb{R}), \delta = \frac{1}{t^{\rho-1}}\frac{d}{dt}\right\}, \quad (5)$$

as a category of functions having absolutely continuous δ^{l-1}-derivatives, and a norm is defined by

$$\|x\|_{C^l_\delta} = \sum_{k=0}^{l-1} \left\| \left(\delta^k x\right)(t)\right\|_C, \quad (6)$$

so that $\delta^k = \overbrace{\delta\delta\cdots\delta}^{k\text{-times}}$.

Definition 1. (see [20]). Let $0 < a, b < +\infty$, $q > 0$ and $x \in \mathscr{X}^p_c(a, b)$, where $\mathscr{X}^p_c(a, b)$ is the space of all Lebesgue measurable complex functions. The integral operator given by

$${}^\rho I^q_{a^+}x(t) = \frac{\rho^{1-q}}{\Gamma(q)}\int_{a^+}^t s^{\rho-1}(t^\rho - s^\rho)^{q-1}x(s)ds, \rho > 0, t > a^+ \quad (7)$$

is named as the generalized Riemann-Liouville integral such that the R.H.S. integral is finite-valued.

Definition 2. (see [20]). The generalized derivative in sense of Caputo for a given function $f \in AC^l_\delta[a, b]$ of order $l-1 < q < l$ with $l = [q] + 1$ is defined by

$${}^C D^{q,\rho}_{a^+}x(t) = {}^\rho I^{l-q}_{a^+}\left(\delta^l f\right)(t) = \frac{\rho^{q-l+1}}{\Gamma(l-q)}\int_{a^+}^t s^{\rho-1}(t^\rho - s^\rho)^{l-q-1} \cdot \left(\delta^l f\right)(s)ds, (\rho > 0, t > a^+) \quad (8)$$

In particular, if $q = l \in \mathbb{N}$,

$${}^C D^{q,\rho}_{a^+}x(t) = \left(\delta^l f\right)(s). \quad (9)$$

Lemma 3. (see [20]). *Let $f \in AC^l_\delta[a, b]$. Then, for every $l-1 < q < l$,*

$${}^\rho I^q_{a^+}\,{}^C D^{q,\rho}_{a^+}x(t) = x(t) - \sum_{k=0}^{l-1}\frac{\left(\delta^k f\right)(a)}{\rho^k k!}(t^\rho - a^\rho)^k. \quad (10)$$

Moreover, for $0 < q < 1$, (10) becomes

$${}^\rho I^q_{a^+}\,{}^C D^{q,\rho}_{a^+}x(t) = x(t) - x(a). \quad (11)$$

Now, we will present a definition of Kuratowski's measure of noncompactness $\xi(\cdot)$ which is constructed by

$$\xi(\mathscr{D}) = \inf\left\{\varepsilon > 0 : \mathscr{D} = \bigcup_{i=1}^n \mathscr{D}_i \text{ and } \mathrm{Diam}(\mathscr{D}_i) \le \varepsilon \text{ for } i = 1, \cdots, n\right\}, \quad (12)$$

where $\mathrm{Diam}(\mathscr{D}_i) = \sup\{|x - y| : x, y \in \mathscr{D}_i\}$ and $\mathscr{D}$ are a bounded subset of the Banach space $\mathscr{C}([t_0, K])$. It is clear that $0 \le \xi(\mathscr{D}) \le \mathrm{Diam}(\mathscr{D}) < +\infty$ [21].

Definition 4. (see [21]). Let $\mathscr{G} : \mathscr{V} \longrightarrow \mathscr{W}$ be bounded and continuous with $\mathscr{V} \subset \mathscr{W}$. Then, $\mathscr{G}$ will be ξ-Lipschitz if $\exists\varsigma \ge 0$ so that

$$\xi(\mathscr{G}(A)) < \varsigma\xi(A), \forall \text{bounded } A \subset \mathscr{V}. \quad (13)$$

As well as $\mathscr{G}$ is named as strict ξ-contraction when $\varsigma < 1$ holds.

Definition 5. (see [21]). A function $\mathscr{G}$ is ξ-condensing if

$$\xi(\mathscr{G}(A)) < \xi(A), \forall A \subset \mathscr{V} \text{ bounded, with } \xi(A) > 0. \quad (14)$$

So, $\xi(\mathscr{G}(A)) \ge \xi(A)$ gives $\xi(A) = 0$. Also, $\mathscr{G} : \mathscr{V} \longrightarrow \mathscr{W}$ is Lipschitz for $\varsigma > 0$ such that

$$\left\|\mathscr{G}(\mu) - \mathscr{G}\left(\mu'\right)\right\| \le \varsigma\|\mu - \mu'\| \text{ for all } \mu, \mu' \in \mathscr{V}. \quad (15)$$

If $\varsigma < 1$, in this case $\mathscr{G}$ is called a strict contraction.

Proposition 6. (see [21]). *$\mathscr{G}$ is ξ-Lipschitz with constant $\varsigma = 0$ iff $\mathscr{G} : \mathscr{V} \longrightarrow \mathscr{W}$ is compact.*

Proposition 7. (see [21]). *A function $\mathscr{G}$ is ξ-Lipschitz with constant ς if and only if $\mathscr{G} : \mathscr{V} \longrightarrow \mathscr{W}$ is Lipschitz with Lipschitz constant ς.*

Theorem 8. (see [22]). *Let $\mathscr{G} : \mathscr{C}[t_0, K] \longrightarrow \mathscr{C}[t_0, K]$ be a ξ-condensing and*

$$\mathbb{M} = \{u \in \mathscr{C}[t_0, K] \colon \theta \in [0, 1] \text{exists so that } u = \theta \mathscr{G}(u)\}. \tag{16}$$

If $\mathbb{M}$ *is a bounded subset contained in* $\mathscr{C}[t_0, K]$, *i.e., a constant* $r > 0$ *exists with* $\mathbb{M} \subset K_r(0)$, *then* $\mathfrak{deg}(I - \theta\mathscr{G}, K_r(0), 0) = 1$ *for all* $\theta \in [0, 1]$. *Therefore,* $\mathscr{G}$ *has a fixed point, and the set* $\mathbb{FIX}(\mathscr{G})$ *belongs to* $K_r(0)$.

Lemma 9. (see [13]). *Consider* φ_p *as an operator in the p-Laplacian settings.*

(p1) For $1 < p \le 2$, *if* $Y_1, Y_2 > 0$ *and* $|Y_1|, |Y_2| \ge \lambda > 0$, *then*

$$\left|\varphi_p(Y_1) - \varphi_p(Y_2)\right| \le (p-1)\lambda^{p-2}|Y_1 - Y_2|. \tag{17}$$

(p2) For $p > 2$, *if* $|Y_1|, |Y_2| \le \lambda^*$, *then*

$$|\varphi_p(Y_1) - \varphi_p(Y_2)| \le (p-1)\lambda^{*p-2}|Y_1 - Y_2|. \tag{18}$$

3. Main Analytical Results

This important section is divided into some subsections. In each part, we shall study desired theorems about different specifications of solutions to the proposed p-Laplacian FBVP (4).

3.1. Existence-Uniqueness Results. We here present the first result which yields the solution of the proposed p-Laplacian FBVP (4) in the equivalent format of integral equations.

Theorem 10. *Let* $\hbar_1, \hbar_2, \hbar_3 \in \mathscr{C}[t_0, K]$, $K > t_0 \ge 0$, $\chi, \beta \in (1, 2)$, *and* $\delta, \rho, \upsilon > 0$. *The p-Laplacian FBVP with given integral composite conditions*

$$\begin{cases} {}^{C}D^{\beta,\delta}\left(\varphi_p\left({}^{C}D^{\chi,\rho}x(t)\right)\right) = \hbar_1(t), \\ x(t_0) + \mu_1 x(K) = \sigma_1 \int_{t_0}^{K} g(s)ds, \\ x'(t_0) + \mu_2 x'(K) = \sigma_2 \int_{t_0}^{K} \hbar_3(s)ds, \\ {}^{C}D^{\chi,\rho}x(t_0) = 0, {}^{C}D^{\chi,\rho}x(K) = \upsilon\, {}^{C}D^{\chi,\rho}x(\eta), \end{cases} \tag{19}$$

has a solution given by

$$x(t) = \int_{t_0}^{K}\left(H_1(t,\varpi)\varphi_q\left(\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s)ds\right) + \frac{\sigma_1\hbar_2(\varpi)}{1+\mu_1} + \frac{\sigma_2\hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2K^{\rho-1}}\left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1}\frac{K^\rho - t_0^\rho}{\rho}\right)\right)\right)d\varpi, \tag{20}$$

such that $H_1(t,\varpi)$ *and* $H_2(t,s)$ *stand for Green's functions defined as follows:*

$$H_1(t,\varpi) = \begin{cases} \frac{\rho^{1-\chi}}{\Gamma(\chi)}(t^\rho - \varpi^\rho)^{\chi-1}\varpi^{\rho-1} - \frac{\mu_1}{1+\mu_1}\frac{\rho^{1-\chi}}{\Gamma(\chi)}(K^\rho - \varpi^\rho)^{\chi-1}\varpi^{\rho-1} + \frac{\mu_2\rho^{2-\chi}}{\left(t_0^{\rho-1} + \mu_2K^{\rho-1}\right)\Gamma(\chi-1)} \times \left(\frac{\mu_1}{1+\mu_1}\frac{K^\rho - t_0^\rho}{\rho} - \frac{t^\rho - t_0^\rho}{\rho}\right)(K^\rho - \varpi^\rho)^{\chi-2}(\varpi K)^{\rho-1}, - \frac{\mu_1}{1+\mu_1}\frac{\rho^{1-\chi}}{\Gamma(\chi)}(K^\rho - \varpi^\rho)^{\chi-1}\varpi^{\rho-1} + \frac{\mu_2\rho^{2-\chi}}{\left(t_0^{\rho-1} + \mu_2K^{\rho-1}\right)\Gamma(\chi-1)} & t_0 \le \varpi \le t \le K, \\ \left(\frac{\mu_1}{1+\mu_1}\frac{K^\rho - t_0^\rho}{\rho} - \frac{t^\rho - t_0^\rho}{\rho}\right)(K^\rho - \varpi^\rho)^{\chi-2}(\varpi K)^{\rho-1}, & t_0 \le t \le \varpi \le K, \end{cases} \tag{21}$$

$$H_2(t,s) = \begin{cases} \frac{\delta^{1-\beta}}{\Gamma(\beta)}\left(t^\delta - s^\delta\right)^{\beta-1}s^{\delta-1} - \frac{\delta^{1-\beta}\left(t^\delta - t_0^\delta\right)}{\left(\left(K^\delta - t_0^\delta\right) - \upsilon^{1/q-1}\left(\eta^\delta - t_0^\delta\right)\right)\Gamma(\beta)}\left(K^\delta - s^\delta\right)^{\beta-1}s^{\delta-1} + \frac{\delta^{1-\beta}\upsilon^{1/q-1}\left(t^\delta - t_0^\delta\right)}{\left(\left(K^\delta - t_0^\delta\right) - \upsilon^{1/q-1}\left(\eta^\delta - t_0^\delta\right)\right)\Gamma(\beta)}\left(\eta^\delta - s^\delta\right)^{\beta-1}s^{\delta-1}, & t_0 \le s \le \eta \le t, \le K, \\ \frac{\delta^{1-\beta}}{\Gamma(\beta)}\left(t^\delta - s^\delta\right)^{\beta-1}s^{\delta-1} - \frac{\delta^{1-\beta}\left(t^\delta - t_0^\delta\right)\left(K^\delta - s^\delta\right)^{\beta-1}s^{\delta-1}}{\left(\left(K^\delta - t_0^\delta\right) - \upsilon^{1/q-1}\left(\eta^\delta - t_0^\delta\right)\right)\Gamma(\beta)}, & t_0 \le \eta \le s \le t, \le K, \\ -\frac{\delta^{1-\beta}\left(t^\delta - t_0^\delta\right)}{\left(\left(K^\delta - t_0^\delta\right) - \upsilon^{1/q-1}\left(\eta^\delta - t_0^\delta\right)\right)\Gamma(\beta)}\left(K^\delta - s^\delta\right)^{\beta-1}s^{\delta-1}, & t_0 \le \eta \le t \le s. \le K. \end{cases} \tag{22}$$

Proof. By utilizing the integral operator ${}^{\delta}I^{\beta}$ on (19), the following relation is produced

$$\varphi_p\left({}^{C}D^{\chi,\rho}x(t)\right) = {}^{\delta}I^{\beta}\hbar_1(t) - a_1 - a_2\left(\frac{t^\delta - t_0^\delta}{\delta}\right), a_1, a_2 \in \mathbb{R}. \tag{23}$$

□

By taking the inverse φ_q of φ_p on above equation, we have

$${}^{C}D^{\chi,\rho}x(t) = \varphi_q\left(\frac{\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{t}\left(t^\delta - s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds - a_1 - a_2\left(\frac{t^\delta - t_0^\delta}{\delta}\right)\right). \tag{24}$$

Using the boundary conditions ${}^{C}D^{\chi,\rho}x(t_0) = 0$ and ${}^{C}D^{\chi,\rho}x(K) = \upsilon\,{}^{C}D^{\chi,\rho}x(\eta)$, we have $a_1 = 0$ and

$$\varphi_q\left(\frac{\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{K}\left(K^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds-a_2\left(\frac{K^\delta-t_0^\delta}{\delta}\right)\right)$$
$$=\upsilon\varphi_q\left(\frac{\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{\eta}\left(\eta^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds-a_2\left(\frac{\eta^\delta-t_0^\delta}{\delta}\right)\right)$$
$$=\varphi_q\left(\frac{\upsilon^{1/q-1}\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{\eta}\left(\eta^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds-a_2\upsilon^{\frac{1}{q-1}}\left(\frac{\eta^\delta-t_0^\delta}{\delta}\right)\right) \tag{25}$$

which is obtained by using the property of p-Laplacian operator. Therefore,

$$a_2\left(\left(\frac{K^\delta-t_0^\delta}{\delta}\right)-\upsilon^{\frac{1}{q-1}}\left(\frac{\eta^\delta-t_0^\delta}{\delta}\right)\right)=\frac{\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{K}\left(K^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds-\frac{\upsilon^{1/q-1}\delta^{1-\beta}}{\Gamma(\beta)}\int_{t_0}^{\eta}\cdot\left(\eta^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds, \tag{26}$$

which implies that

$$a_2=\frac{\delta^{2-\beta}}{\left(\left(K^\delta-t_0^\delta\right)-\upsilon^{1/q-1}\left(\eta^\delta-t_0^\delta\right)\right)\Gamma(\beta)}\int_{t_0}^{K}\cdot\left(K^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds-\frac{\upsilon^{1/q-1}\delta^{2-\beta}}{\left(\left(K^\delta-t_0^\delta\right)-\upsilon^{1/q-1}\left(\eta^\delta-t_0^\delta\right)\right)\Gamma(\beta)}\int_{t_0}^{\eta}\cdot\left(\eta^\delta-s^\delta\right)^{\beta-1}s^{\delta-1}\hbar_1(s)ds. \tag{27}$$

According to (24) and the definition of $H_2(t,s)$, we get

$${}^{C}D^{\chi,\rho}x(t)=\varphi_q\left(\int_{t_0}^{K}H_2(t,s)\hbar_1(s)ds\right). \tag{28}$$

Applying the integral operator ${}^{\rho}I^{\chi}$, we shall write

$$x(t)={}^{\rho}I^{\chi}\varphi_q\left(\int_{t_0}^{K}H_2(t,s)\hbar_1(s)ds\right)-b_1-b_2\frac{t^\rho-t_0^\rho}{\rho}. \tag{29}$$

By virtue of the boundary condition $x(t_0)+\mu_1x(K)=\sigma_1\int_{t_0}^{K}\hbar_2(s)ds$, we have

$$-b_1+\mu_1\left(I^{\chi,\rho}\varphi_q\left(\int_{t_0}^{K}H_2(t,s)\hbar_1(s)ds\right)-b_1-b_2\frac{K^\rho-t_0^\rho}{\rho}\right)=\sigma_1\int_{t_0}^{K}\hbar_2(s)ds, \tag{30}$$

which implies

$$b_1=\frac{\mu_1}{1+\mu_1}\frac{\rho^{1-\chi}}{\Gamma(\chi)}\int_{t_0}^{K}(K^\rho-\varpi^\rho)^{\chi-1}\varpi^{\rho-1}\varphi_q\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi-\frac{\mu_1}{1+\mu_1}\frac{K^\rho-t_0^\rho}{\rho}b_2-\frac{\sigma_1}{1+\mu_1}\int_{t_0}^{K}\hbar_2(s)ds. \tag{31}$$

Therefore, we obtain

$$x'(t)=\frac{\rho^{2-\chi}}{\Gamma(\chi-1)}\int_{t_0}^{t}(t^\rho-\varpi^\rho)^{\chi-2}(t\varpi)^{\rho-1}\varphi_q\cdot\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar(s)ds\right)d\varpi-b_2t^{\rho-1}. \tag{32}$$

Using the boundary condition $x'(t_0)+\mu_2x'(K)=\sigma_2\int_{t_0}^{K}\hbar_3(s)ds$,

$$-b_2t_0^{\rho-1}+\mu_2\frac{\rho^{2-\chi}}{\Gamma(\chi-1)}\int_{t_0}^{K}(K^\rho-\varpi^\rho)^{\chi-2}(\varpi K)^{\rho-1}\varphi_q\cdot\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi-\mu_2b_2K^{\rho-1}=\sigma_2\int_{t_0}^{K}\hbar_3(s)ds. \tag{33}$$

Then,

$$b_2=\frac{\mu_2}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\frac{\rho^{2-\chi}}{\Gamma(\chi-1)}\int_{t_0}^{K}(K^\rho-\varpi^\rho)^{\chi-2}(\varpi K)^{\rho-1}\varphi_q\cdot\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi-\frac{\sigma_2}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\int_{t_0}^{K}\hbar_3(s)ds. \tag{34}$$

Therefore,

$$\begin{aligned}x(t)&=\frac{\rho^{1-\chi}}{\Gamma(\chi)}\int_{t_0}^{t}(t^\rho-\varpi^\rho)^{\chi-1}\varpi^{\rho-1}\varphi_q\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi\\&\quad-\frac{\mu_1}{1+\mu_1}\frac{\rho^{1-\chi}}{\Gamma(\chi)}\int_{t_0}^{K}(K^\rho-\varpi^\rho)^{\chi-1}\varpi^{\rho-1}\varphi_q\\&\quad\cdot\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi+\frac{\mu_2\rho^{2-\chi}}{\left(t_0^{\rho-1}+\mu_2K^{\rho-1}\right)\Gamma(\chi-1)}\\&\quad\cdot\left(\frac{\mu_1}{1+\mu_1}\frac{K^\rho-t_0^\rho}{\rho}-\frac{t^\rho-t_0^\rho}{\rho}\right)\times\int_{t_0}^{K}(K^\rho-\varpi^\rho)^{\chi-2}(\varpi K)^{\rho-1}\varphi_q\\&\quad\cdot\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)d\varpi+\frac{\sigma_1}{1+\mu_1}\int_{t_0}^{K}\hbar_2(\varpi)d\varpi\\&\quad+\frac{\sigma_2}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\left(\frac{t^\rho-t_0^\rho}{\rho}-\frac{\mu_1}{1+\mu_1}\frac{K^\rho-t_0^\rho}{\rho}\right)\int_{t_0}^{K}\hbar_3(\varpi)d\varpi\\&=\int_{t_0}^{K}\left(H_1(t,\varpi)\varphi_q\left(\int_{t_0}^{K}H_2(\varpi,s)\hbar_1(s)ds\right)+\frac{\sigma_1\hbar_2(\varpi)}{1+\mu_1}\right.\\&\quad\left.+\frac{\sigma_2h(\varpi)}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\left(\frac{t^\rho-t_0^\rho}{\rho}-\frac{\mu_1}{1+\mu_1}\frac{K^\rho-t_0^\rho}{\rho}\right)\right)d\varpi,\end{aligned} \tag{35}$$

where $H_1(t,\varpi)$ is defined in (18). This ends the proof.

In this part, we intend to state and prove our required existence-uniqueness theorems. To achieve such an intention and in view of Theorem 10, the solution of the suggested p-Laplacian FBVP (4) is equivalent to a fixed point $x(t)$ of the self–map $\Xi : \mathscr{C}[t_0, K] \longrightarrow \mathscr{C}[t_0, K]$ which is formulated as

$$(\Xi x)(t) = \int_{t_0}^{K} \left(H_1(t, \varpi)\varphi_q \left(\int_{t_0}^{K} H_2(\varpi, s)\hbar_1(s, x(s))ds \right) + \frac{\sigma_1 \hbar_2(\varpi)}{1+\mu_1} + \frac{\sigma_2 \hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2 K^{\rho-1}} \left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1} \frac{K^\rho - t_0^\rho}{\rho} \right) \right) d\varpi, \tag{36}$$

where $H_1(t, \varpi)$ and $H_2(\varpi, s)$ are represented by (21) and (22), respectively. In the sequel, we utilize the following notations:

$$\begin{aligned}
\Delta_1 &= \left(\frac{K^\rho - {t_0}^\rho}{\rho} \right)^\chi \left(\left| \frac{\mu_1}{1+\mu_1} \right| + 1 \right) \left(\frac{1}{\Gamma(\chi+1)} + \frac{|\mu_2| K^{\rho-1}}{\Gamma(\chi) \left| \left(t_0^{\rho-1} + \mu_2 K^{\rho-1} \right) \right|} \right), \\
\Delta_2 &= \left(\frac{K^\delta - {t_0}^\delta}{\delta} \right)^\beta \frac{1}{\Gamma(\beta+1)} + \frac{\left(K^\delta - t_0^\delta \right)}{\left| \left(\left(K^\delta - t_0^\delta \right) - \upsilon^{1/q-1} \left(\eta^\delta - t_0^\delta \right) \right) \right|} \cdot \left(\left(\frac{K^\delta - {t_0}^\delta}{\delta} \right)^\beta \frac{1}{\Gamma(\beta+1)} + \left(\frac{\eta^\delta - {t_0}^\delta}{\delta} \right)^\beta \frac{\upsilon^{1/q-1}}{\Gamma(\beta+1)} \right), \\
\Delta_3 &= \left| \frac{\sigma_1}{1+\mu_1} \right| \hbar_2^* + \frac{|\sigma_2| \hbar_3^*}{\left| t_0^{\rho-1} + \mu_2 K^{\rho-1} \right|} \left(\frac{K^\rho - {t_0}^\rho}{\rho} \right) \left(\left| \frac{\mu_1}{1+\mu_1} \right| + 1 \right),
\end{aligned} \tag{37}$$

and $\omega_1^* = \sup_{t\in[t_0,K]} \{\omega_1(t)\}$, $\omega_2^* = \sup_{t\in[t_0,K]} \{\omega_2(t)\}$, $\Omega_1 = \omega_1^* \Delta_1 \Delta_2^{q-1} + \Delta_3$, $\Omega_2 = \omega_2^* \Delta_1 \Delta_2^{q-1}$, $y_2^* = \sup_{t\in[t_0,K]} | \hbar_2(t) |$, and $\hbar_3^* = \sup_{t\in[t_0,K]} |\hbar_3(t)|$.

Theorem 11. *Let (HP1): the functions $\omega_1, \omega_2 \in \mathscr{C}[t_0, K]$ exist so that $|\hbar_1(t, x)| \le \varphi_p(\omega_1(t) + \omega_2(t)|x(t)|)$ for any $x \in \mathscr{C}[t_0, K]$ and $t \in [t_0, K]$.*

Then, $\Xi : \mathscr{C}[t_0, K] \longrightarrow \mathscr{C}[t_0, K]$ is continuous, and also the growth condition $\|\Xi x\| \le \Omega_1 + \Omega_2 \|x\|$ holds.

Proof. Define a set $\mathfrak{B}_\varepsilon = \{x \in \mathscr{C}[t_0, K]: \|x\| \le \varepsilon\}$ having the boundedness property. In order to prove the continuity of Ξ, we consider x_n as a sequence converging to x in $\mathfrak{B}_\varepsilon$. Then, Lemma 9 yields

$$\begin{aligned}
&|(\Xi x_n)(t) - (\Xi x)(t)| \\
&= \left| \int_{t_0}^{K} \left(H_1(t, \varpi)\varphi_q \left(\int_{t_0}^{K} H_2(\varpi, s)\hbar_1(s, x_n(s))ds \right) d\varpi - \int_{t_0}^{K} \left(H_1(t, \varpi)\varphi_q \left(\int_{t_0}^{K} H_2(\varpi, s)\hbar_1(s, x(s))ds \right) d\varpi \right| \\
&\le \int_{t_0}^{K} |H_1(t, \varpi)| \left| \varphi_q \left(\int_{t_0}^{K} H_2(\varpi, s)\hbar_1(s, x_n(s))ds \right) - \varphi_q \left(\int_{t_0}^{K} H_2(\varpi, s)\hbar_1(s, x(s))ds \right) \right| d\varpi \\
&\le (q-1)\lambda^{q-2} \int_{t_0}^{K} |H_1(t, \varpi)| \cdot \left(\int_{t_0}^{K} |H_2(\varpi, s)||\hbar_1(s, x_n(s)) - \hbar_1(s, x(s))|ds \right) d\varpi.
\end{aligned} \tag{38}$$

According to Lebesgue's dominated convergence theorem and the continuity of the function $\hbar_1$, we get $\|\Xi x_n - \Xi x\| \longrightarrow 0$ when $n \longrightarrow \infty$. Hence, Ξ is continuous. ☐

Now, about the growth condition, by (HP1), we obtain

$$\begin{aligned}
|(\Xi x)(t)| &\le \int_{t_0}^{K} \left(|H_1(t, \varpi)|\varphi_q \left(\int_{t_0}^{K} |H_2(\varpi, s)||\hbar_1(s, x(s))|ds \right) + \left| \frac{\sigma_1 \hbar_2(\varpi)}{1+\mu_1} \right| + \left| \frac{\sigma_2 \hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2 K^{\rho-1}} \left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1} \frac{K^\rho - t_0^\rho}{\rho} \right) \right| \right) d\varpi \\
&\le \int_{t_0}^{K} \left(|H_1(t, \varpi)|\varphi_q \left(\int_{t_0}^{K} |H_2(\varpi, s)|\varphi_p(\omega_1(s) + \omega_2(s)|x(s)|)ds \right) + \left| \frac{\sigma_1 \hbar_2(\varpi)}{1+\mu_1} \right| + \left| \frac{\sigma_2 \hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2 K^{\rho-1}} \left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1} \frac{K^\rho - t_0^\rho}{\rho} \right) \right| \right) d\varpi \\
&\le \int_{t_0}^{K} \left(|H_1(t, \varpi)| {\Delta_2}^{q-1} (\omega_1^* + \omega_2^* \|x\|) + \left| \frac{\sigma_1 \hbar_2(\varpi)}{1+\mu_1} \right| + \left| \frac{\sigma_2 \hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2 K^{\rho-1}} \left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1} \frac{K^\rho - t_0^\rho}{\rho} \right) \right| \right) d\varpi \\
&\le \Delta_1 \Delta_2^{q-1} (\omega_1^* + \omega_2^* \|x\|) + \Delta_3 \le \omega_1^* \Delta_1 \Delta_2^{q-1} + \Delta_3 + \omega_2^* \Delta_1 \Delta_2^{q-1} \|x\| \\
&\le \Omega_1 + \Omega_2 \|x\|.
\end{aligned} \tag{39}$$

Thus, $\|\Xi x\| \le \Omega_1 + \Omega_2 \|x\|$ and this complete the argument.

Theorem 12. *Under hypothesis (HP1), the single-valued operator $\Xi : \mathscr{C}[t_0, K] \longrightarrow \mathscr{C}[t_0, K]$ is ξ-Lipschitz with the constant zero and is compact.*

Proof. In view of Theorem 11, Ξ is bounded. In the subsequent step, we show that Ξ is an equicontinuous operator. Then, by the hypothesis (HP1), for any $x \in \mathfrak{B}_\varepsilon$ and $t_1, t_2 \in [t_0, K]$ subject to $t_1 < t_2$, we have

$$|(\Xi x)(t_2)-(\Xi x)(t_1)| \le \left| \int_{t_0}^{K} \left(H_1(t_2,\varpi)\varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,x(s))ds \right) + \frac{\sigma_2\hbar_3(\varpi)}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\left(\frac{t_2^{\rho}-t_0^{\rho}}{\rho}\right)\right)d\varpi - \int_{t_0}^{K} \cdot \left(H_1(t_1,\varpi)\varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,x(s))ds \right) + \frac{\sigma_2\hbar_3(\varpi)}{t_0^{\rho-1}+\mu_2K^{\rho-1}}\left(\frac{t_1^{\rho}-t_0^{\rho}}{\rho}\right)\right)d\varpi \right| \le \int_{t_0}^{K} \left(|H_1(t_2,\varpi)-H_1(t_1,\varpi)|\varphi_q \cdot \left(\int_{t_0}^{K} |H_2(\varpi,s)|\varphi_p(\omega_1(s)+\omega_2(s)|x(s)|)ds \right) + \frac{|\sigma_2\hbar_3(\varpi)|}{|t_0^{\rho-1}+\mu_2K^{\rho-1}|}\left(\frac{t_2^{\rho}-t_0^{\rho}}{\rho}-\frac{t_1^{\rho}-t_0^{\rho}}{\rho}\right)\right)d\varpi. \tag{40}$$

Clearly, the R.H.S. of (40) goes to zero by taking $t_2 \longrightarrow t_1$, and so $\Xi(\mathfrak{B}_\varepsilon)$ is equicontinuous. Therefore, by virtue of the well-known Arzelá-Ascoli theorem, $\Xi(\mathfrak{B}_\varepsilon)$ is compact, and thus Proposition 6 gives a result stating this fact that Ξ is ξ-Lipschitz with the constant zero. □

Theorem 13. *Under the following hypothesis, i.e.,*

(HP2) A real constant ℓ exists so that for any $\mu_1, \mu_2 \in \mathscr{C}[t_0, K]$ and $t \in [t_0, K]$,

$$|\hbar_1(t,\mu_1)-\hbar_1(t,\mu_2)| \le \ell|\mu_1(t)-\mu_2(t)|. \tag{41}$$

The generalized p-Laplacian FBVP (4) has a unique solution such that

$$\ell(q-1)\lambda^{q-2}\Delta_1\Delta_2 < 1. \tag{42}$$

Proof. Consider Ξ as defined in (36). Then by Lemma 9, we obtain

$$\begin{aligned} |(\Xi x)(t)-(\Xi y)(t)| &\le \left| \int_{t_0}^{K} H_1(t,\varpi)\varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,x(s))ds\right)d\varpi - \int_{t_0}^{K} H_1(t,\varpi)\varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,y(s))ds\right)d\varpi \right| \\ &\le \int_{t_0}^{K} |H_1(t,\varpi)| \left| \varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,x(s))ds\right) - \varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)\hbar_1(s,y(s))ds\right)\right| d\varpi \le (q-1)\lambda^{q-2}\int_{t_0}^{K}|H_1(t,\varpi)| \cdot \left(\int_{t_0}^{K}|H_2(\varpi,s)||\hbar_1(s,x(s))-\hbar y_1(s,y(s))|ds\right)d\varpi \\ &\le \ell(q-1)\lambda^{q-2}\int_{t_0}^{K}|H_1(t,\varpi)|\left(\int_{t_0}^{K}|H_2(\varpi,s)||x(s)-y(s)|ds\right)d\varpi \\ &\le \ell(q-1)\lambda^{q-2}\int_{t_0}^{K}|H_1(t,\varpi)|(\Delta_2\|x-y\|)d\varpi \le \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2\|x-y\|, \end{aligned} \tag{43}$$

for $x, y \in \mathscr{C}[t_0, K]$. So, $\|\Xi x - \Xi y\| \le \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2\|x-y\|$. Hence, in view of the well-known contraction principle due to Banach, we follow that Ξ admits a fixed point uniquely. Thus, the generalized p-Laplacian FBVP (4) involves a solution uniquely. □

Theorem 14. *If hypotheses (HP1) and (HP2) hold, then the generalized p-Laplacian FBVP (4) has a solution such that $\Omega_2 < 1$. Moreover, the set containing solutions of the generalized p -Laplacian FBVP (4) is bounded.*

Proof. According to Theorem 13, Ξ is Lipschitz and by Proposition 7, Ξ is ξ-Lipschitz which yields that Ξ is ξ-condensing. With the aid of Theorem 8, we need to prove that

$$\mathbb{W} = \{x \in \mathscr{C}[t_0, K]: \theta \in [0,1] \text{exists so that } x = \theta\Xi(x)\}, \tag{44}$$

is bounded. For this regard, we suppose that $x \in \mathbb{W}$ for some $\theta \in [0,1]$ and for each $t \in [t_0, K]$. Then, from the growth condition of Ξ derived in Theorem 11, we may write

$$\|x\| = \|\theta\Xi(x)\| \le \Omega_1 + \Omega_2\|x\|. \tag{45}$$

Hence,

$$\|x\| \le \frac{\Omega_1}{1-\Omega_2}, \tag{46}$$

which yields that $\mathbb{W}$ is a bounded set contained in $\mathscr{C}[t_0, K]$. By Theorem 8, one can understand that Ξ involves at least a fixed point which confirms the existence of at least a solution for the proposed generalized p-Laplacian FBVP (4), and hence $\mathbb{W}$ consisting of solutions of the mentioned FBVP (4) is a bounded subset of $\mathscr{C}[t_0, K]$. This ends the proof.□ □

3.2. Analysis of the Stability. In this part, we discuss on four kinds of stability for the generalized p-Laplacian FBVP (4) as follows [6, 7].

Definition 15. The generalized p-Laplacian FBVP (4) is called Ulam-Hyers stable if there is a real number $\mathfrak{C}_{y_1} > 0$ such that for every $\kappa > 0$ and every solution $\tilde{x} \in \mathscr{C}[t_0, K]$ of the inequality

$$\left| {}^{C}D^{\beta,\delta}\left(\varphi_p\left({}^{C}D^{\chi,\rho}\tilde{x}(t)\right)\right) - \hbar_1(t,\tilde{x}(t)) \right| \le \kappa, \tag{47}$$

there is a unique solution $x \in \mathscr{C}[t_0, K]$ of (4) such that

$$\|x-\tilde{x}\| \le \mathfrak{C}_{y_1}\kappa, (t \in [t_0, K]). \tag{48}$$

Definition 16. The generalized p -Laplacian FBVP (4) is called the generalized Ulam-Hyers stable with respect to $\rho \in \mathscr{C}(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ with $\rho(0)=0$, if for each approximate solution $\tilde{x} \in \mathscr{C}[t_0, K]$ of inequality (47), there is a unique solution $x \in \mathscr{C}[t_0, K]$ of (4) so that

$$\|x-\tilde{x}\| \le \rho(\kappa), (t \in [t_0, K]). \tag{49}$$

Definition 17. The generalized p-Laplacian FBVP (4) is called Ulam-Hyers-Rassias stable with respect to $\psi \in \mathscr{C}[t_0, K]$ if there is a real number $\mathfrak{K}_{\hbar_1,\psi} > 0$ such that for every $\kappa > 0$ and approximate solution $\tilde{x} \in \mathscr{C}[t_0, K]$ of the inequality

$$\left| {}^C D^{\beta,\delta}\left(\varphi_p\left({}^C D^{\chi,\rho}\tilde{x}(t)\right)\right) - \hbar_1(t, \tilde{x}(t)) \right| \le \psi(t)\kappa, \tag{50}$$

there is a unique solution $x \in \mathscr{C}[t_0, K]$ of (4) so that

$$\|x - \tilde{x}\| \le \mathfrak{K}_{\Upsilon_1,\psi}\kappa\psi(t), (t \in [t_0, K]). \tag{51}$$

Definition 18. The generalized p-Laplacian FBVP (4) is called the generalized Ulam-Hyers-Rassias stable with respect to $\psi \in \mathscr{C}[t_0, K]$, if there is a real number $\mathfrak{K}_{y_1,\psi} > 0$ such that for each approximate solution $\tilde{x} \in \mathscr{C}[t_0, K]$ of inequality (50), there is a unique solution $x \in \mathscr{C}[t_0, K]$ of (4) such that

$$\|x - \tilde{x}\| \le \mathfrak{K}_{\Upsilon_1,\psi}\psi(t), (t \in [t_0, K]). \tag{52}$$

Remark 19. The function $\tilde{x} \in \mathscr{C}[t_0, K]$ is a solution of (47) if and only if there exists a function $\Psi \in \mathscr{C}[t_0, K]$ such that

$$|\Psi(t)| \le \kappa \text{ for } t \in [t_0, K]$$
$${}^C D^{\beta,\delta}(\varphi_p({}^C D^{\chi,\rho}\tilde{x}(t))) = \hbar_1(t, \tilde{x}(t)) + \Psi(t) \text{ for } t \in [t_0, K]$$

Remark 20. The function $\tilde{x} \in \mathscr{C}[t_0, K]$ is a solution of (50) if and only if there exists a function $\Phi \in \mathscr{C}[t_0, K]$ such that

$$|\Phi(t)| \le \kappa\psi(t) \text{ for } t \in [t_0, K]$$
$${}^C D^{\beta,\delta}(\varphi_p({}^C D^{\chi,\rho}\tilde{x}(t))) = \hbar_1(t, \tilde{x}(t)) + \Phi(t) \text{ for } t \in [t_0, K]$$

Theorem 21. *If the hypothesis (HP2) and the inequality (42) are valid, then the unique solution of the generalized p-Laplacian FBVP (4) is Ulam-Hyers stable and is the generalized Ulam-Hyers stable.*

Proof. Set $\kappa > 0$ and let $\tilde{x} \in \mathscr{C}[t_0, K]$ be the approximate solution of (47) and $x \in \mathscr{C}[t_0, K]$ be the unique solution of the approximate generalized p-Laplacian FBVP

$$\begin{cases} {}^C D^{\beta,\delta}\left(\varphi_p\left({}^C D^{\chi,\rho}\tilde{x}(t)\right)\right) = \hbar_1(t, \tilde{x}(t)) + \Psi(t), & (t \in [t_0, K], t_0 \ge 0), \\ \tilde{x}(t_0) + \mu_1\tilde{x}(K) = \sigma_1\int_{t_0}^{K}\hbar_2(s)ds, & (\mu_1 \ne -1), \\ \tilde{x}'(t_0) + \mu_2\tilde{x}'(K) = \sigma_2\int_{t_0}^{K}\hbar_3(s)ds, & \left(\mu_2 \ne -\left(\dfrac{t_0}{K}\right)^{\rho-1}\right), \\ {}^C D^{\chi,\rho}\tilde{x}(t_0) = 0, {}^C D^{\chi,\rho}\tilde{x}(K) = \upsilon\, {}^C D^{\chi,\rho}\tilde{x}(\eta), & (\eta \in (t_0, K)), \end{cases} \tag{53}$$

with $\upsilon \ne (K^\delta - t_0^\delta/\eta^\delta - t_0^\delta)^{q-1}$. According to Theorem 10, we get

$$\tilde{x}(t) = \int_{t_0}^{K}\left(H_1(t, \varpi)\varphi_q\left(\int_{t_0}^{K}H_2(\varpi, s)(\hbar(s, \tilde{x}(s)) + \Psi(t))ds\right) + \frac{\sigma_1\hbar(\varpi)}{1+\mu_1} + \frac{\sigma_2\hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2K^{\rho-1}}\left(\frac{t^\rho - t_0^\rho}{\rho} - \frac{\mu_1}{1+\mu_1}\frac{K^\rho - t_0^\rho}{\rho}\right)\right)d\varpi, \tag{54}$$

where $H_1(t, \varpi)$ and $H_2(t, s)$ are defined by (21) and (22), respectively. Hence, from Theorem 13, we estimate

$$\begin{aligned} |x(t) - \tilde{x}(t)| &\le \left| \int_{t_0}^{K} H_1(t, \varpi)\varphi_q\left(\int_{t_0}^{K}H_2(\varpi, s)\hbar_1(s, x(s))ds\right)d\varpi \right. \\ &\quad \left. - \int_{t_0}^{\nu}H_1(t, \varpi)\varphi_q\left(\int_{t_0}^{K}H_2(\varpi, s)(\hbar_1(s, \tilde{x}(s)) + \Psi(t))ds\right)d\varpi \right| \\ &\le \int_{t_0}^{K}|H_1(t, \varpi)|\left|\varphi_q\left(\int_{t_0}^{K}H_2(\varpi, s)\hbar_1(s, x(s))ds\right) - \varphi_q\left(\int_{t_0}^{K}H_2(\varpi, s)\hbar_1(s, \tilde{x}(s))ds\right)\right|d\varpi \\ &\quad + \int_{t_0}^{K}|H_1(t, \varpi)|\varphi_q\left(\int_{t_0}^{K}|H_2(\varpi, s)||\Psi(t)|ds\right)d\varpi \\ &\le \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2\|x - \tilde{x}\| + \Delta_1(\Delta_2\kappa)^{q-1}. \end{aligned} \tag{55}$$

Thus,

$$\|x - \tilde{x}\| \le \frac{\Delta_1(\Delta_2\kappa)^{q-1}}{1 - \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2} = \mathfrak{C}_{\Upsilon_1}\kappa, \tag{56}$$

where $\mathfrak{C}_{\Upsilon_1} \coloneqq \Delta_1\Delta_2^{q-1}\kappa^{q-2}/1 - \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2$. This shows that the generalized p-Laplacian FBVP (4) is Ulam-Hyers stable. Along with this, if $\|x - \tilde{x}\| \le \rho(\kappa)$ so that $\rho(0) = 0$, then the solution related to the generalized p-Laplacian FBVP (4) is the generalized Ulam-Hyers stable, and the proof is completed. □

Theorem 22. *Let the hypothesis (HP2) and (42) are valid, and there exists an increasing function $\psi(t) \in \mathscr{C}[t_0, K]$; there exists $\lambda_\psi > 0$ such that $\int_{t_0}^{K}|\psi(s)|ds \le \lambda_\psi\psi(t), \forall t \in [t_0, K]$. Then, the unique solution of the generalized p-Laplacian FBVP (4) is Ulam-Hyers-Rassias stable and thus is the generalized Ulam-Hyers-Rassias stable.*

Proof. Consider $\kappa > 0$ and let $\tilde{x} \in \mathscr{C}[t_0, K]$ be the approximate solution of (50) and $x \in \mathscr{C}[t_0, K]$ be the unique solution of the generalized p-Laplacian FBVP (4). By remark 20, we have

$$\begin{cases} {}^{C}D^{\beta,\delta}\left(\varphi_p\left({}^{C}D^{\chi,\rho}\tilde{x}(t)\right)\right) = \hbar_1(t,\tilde{x}(t)) + \Phi(t), & (t \in [t_0, K], t_0 \geq 0), \\ \tilde{x}(t_0) + \mu_1 \tilde{x}(K) = \sigma_1 \int_{t_0}^{K} \hbar_2(s) ds, & (\mu_1 \neq -1), \\ \tilde{x}'(t_0) + \mu_2 \tilde{x}'(K) = \sigma_2 \int_{t_0}^{K} \hbar_3(s) ds, & \left(\mu_2 \neq -\left(\frac{t_0}{K}\right)^{\rho-1}\right), \\ {}^{C}D^{\chi,\rho}\tilde{x}(t_0) = 0, {}^{C}D^{\chi,\rho}\tilde{x}(K) = \upsilon {}^{C}D^{\chi,\rho}\tilde{x}(\eta), & (\eta \in (t_0, K)), \end{cases} \tag{57}$$

with $\upsilon \neq (K^{\delta} - t_0^{\delta}/\eta^{\delta} - t_0^{\delta})^{q-1}$. In view of Theorem 10, we have

$$\tilde{x}(t) = \int_{t_0}^{K}\left(H_1(t,\varpi)\varphi_q\left(\int_{t_0}^{K} H_2(\varpi,s)(\hbar_1(s,\tilde{x}(s)) + \Phi(s))ds\right) + \frac{\sigma_1\hbar_2(\varpi)}{1+\mu_1} + \frac{\sigma_2\hbar_3(\varpi)}{t_0^{\rho-1} + \mu_2 K^{\rho-1}}\left(\frac{t^{\rho} - t_0^{\rho}}{\rho} - \frac{\mu_1}{1+\mu_1}\frac{K^{\rho} - t_0^{\rho}}{\rho}\right)\right)d\varpi, \tag{58}$$

where $H_1(t,\varpi)$ and $H_2(t,s)$ are defined by (21) and (22), respectively. Hence, we can immediately estimate that

$$\|x - \tilde{x}\| \leq \frac{\Delta_1\left(\Delta_2 \kappa \lambda_{\psi} \psi(t)\right)^{q-1}}{1 - \ell(q-1)\lambda^{q-2}\Delta_1\Delta_2} = \mathfrak{K}_{\hbar,\psi}\kappa\psi(t), \tag{59}$$

where $\mathfrak{K}_{y_1,\psi} \coloneqq \Delta_1 \Delta_2^{q-1}\lambda_{\psi}^{q-1}(\kappa\psi(t))^{q-2}/1 - \ell(q-1)\lambda^{q-2}\Delta_1 \Delta_2$. This proves that the generalized p-Laplacian FBVP (4) is Ulam-Hyers-Rassias stable. Furthermore, if $\kappa = 1$, then the solution of generalized p-Laplacian FBVP (4) is the generalized Ulam-Hyers-Rassias stable, and the proof is completed. □

4. Example

As an application to validate the theoretical results, an illustrative example is given here.

Example 23. Regarding to the given FBVP (4), we provide a special structure of the generalized problem having the fractional composite p-Laplacian as

$$\begin{cases} {}^{C}D^{1.08,0.5}\left(\varphi_{\frac{5}{4}}\left({}^{C}D^{1.09,0.7}x(t)\right)\right) = \frac{|6\ \sin x(t)|}{|12000 \sin x(t)| + 12000} + 0.18e^{t}, (t \in [0,1]) \\ x(0) + 0.1x(1) = 0.8\int_0^1 \left(2e^{3s}+1\right)ds, \\ x'(0) + 0.2x'(1) = 0.9\int_0^1 \cos(s)ds, \\ {}^{C}D^{1.09,0.7}x(0) = 0, {}^{C}D^{1.09,0.7}x(1) = 0.3\,{}^{C}D^{1.09,0.7}x(0.25), \end{cases} \tag{60}$$

in which the following parameters are considered $\beta = 1.08$, $\chi = 1.09$, $\delta = 0.5$, $\rho = 0.7$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\upsilon = 0.3$, $\sigma_1 = 0.8$, $\sigma_2 = 0.9$, $\eta = 0.25$, $\lambda = 1.5$, $p = 5/4$, $q = 5$, $t_0 = 0$, $K = 1$, and $\varphi_p(\theta) = |\theta|^{p-2}\theta$. In addition to these, the continuous functions y_1, y_2, and y_3 are introduced by

$$\hbar_1(t,x(t)) = \frac{|6\ \sin x(t)|}{|12000 \sin x(t)| + 12000} + 0.18e^{t}d, \quad \hbar_2(t) = 2e^{3t} + 1, \hbar_3(t) = \cos(t), \tag{61}$$

for $t \in [0,1]$. By utilizing some of above data, we get $\Delta_1 \simeq = 3.0899$ and $\Delta_2 \simeq 6.4094$. On the other side, for any $x, \hat{x} \in \mathbb{R}$, we can write

$$\begin{aligned} |\hbar_1(t,x(t)) - \hbar_1(t,\hat{x}(t))| &\leq \left|\frac{|6\ \sin x(t)|}{|12000 \sin x(t)| + 12000} - \frac{|6\ \sin \hat{x}(t)|}{|12000 \sin \hat{x}(t)| + 12000}\right| \\ &\leq \frac{6}{12000}\left|\frac{|\sin x(t)|}{|\sin x(t)| + 1} - \frac{|\sin \hat{x}(t)|}{|\sin \hat{x}(t)| + 1}\right| \\ &\leq \frac{6}{12000}|\sin x(t) - \sin \hat{x}(t)| \leq \frac{6}{12000}|x(t) - \hat{x}(t)|, \end{aligned} \tag{62}$$

where $\ell = 6/12000 = 0.0005$ is obtained. Then, since

$$\ell(q-1)\lambda^{q-2}\Delta_1\Delta_2 \simeq 0.1336797342 < 1, \tag{63}$$

thus the conditions of Theorem 13 are satisfied, and so the generalized composite p-Laplacian FBVP (60) has a unique solution. On the other side, since all hypotheses of Theorems 21 and 22 hold, we find out that the given generalized composite p-Laplacian FBVP (60) is Ulam-Hyers and Ulam-Hyers-Rassias stable and thus is stable of their generalized type.

5. Conclusion

Qualitative analysis such as the investigation of the existence, uniqueness, and stability of fractional differential equations is an important and useful task. In this paper, we studied a generalized fractional composite differential equation with p-Laplacian operator equipped with three-point integral boundary value conditions. We used the classical results for this purpose and obtained the relevant Green's function. The existence and uniqueness of solutions were established by means of topological degree theory and Banach contraction principle. Besides, four types of stability in the sense of Ulam-Hyers, Ulam-Hyers-Rassias, and their generalized versions were analyzed. Finally, we provided an illustrative example to validate our results. In the next researches, one can study these qualitative behaviors of solutions for different generalizations of fractional p-Laplacian boundary value problems by means of generalized operators with nonsingular kernels such as Caputo-Fabrizio operators or Atangana-Baleanu operators.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Unique Fixed-Point Results in Fuzzy Metric Spaces with an Application to Fredholm Integral Equations

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This paper aims at proving some unique fixed-point results for different contractive-type self-mappings in fuzzy metric spaces by using the "triangular property of the fuzzy metric". Some illustrative examples are presented to support our results. Moreover, we present an application by resolving a particular case of a Fredholm integral equation of the second kind.

1. Introduction

In 1922, Banach [1] proved a "Banach contraction principle (BCP)," which is stated as "a self-mapping in a complete metric space satisfying a contraction condition has a unique fixed point". This theorem plays a very important role in the theory of fixed points. Many researchers gave generalization and improved the BCP in many directions for single-valued and multivalued mappings in the context of metric spaces by ensuring the existence of fixed point, common fixed point, and coincidence point results with different types of applications, such as differential-type applications, integral-type applications, functional-type applications. In 2004, Ran and Reurings [2] proved a fixed-point theorem in a metric space by using partially ordered sets and they present some applications to matrix equations. While in [3], Nieto and Rodrguez-López extended and improved the result of Ran and Reurings [2] by using increasing mappings and applied the result to get a unique solution for the first-order ordinary differential equation with periodic boundary equations. In 2017, Priskillal and Thangavelu [4] established some fixed-point theorems in complete metric spaces by using ψ-contractive fuzzy mappings with an application to fuzzy differential equations. Some more fixed-point results in the context of metric spaces can be found in [5–16].

In 1965, the theory of fuzzy sets was introduced by Zadeh [17]. Lately, this theory is improved, investigated, and applied in many directions. Among them, we state the theory of fuzzy logic, which is based on the notion of relative graded memberships, as inspired by the processing of human perceptions and cognitions. Fuzzy logic can deal with information arising from computational perceptions and cognitions, that is, uncertain, obscure, imprecise, partly true, or without sharp limits. A fuzzy logic permits the inclusion of vague human assessments in computing problems. The fuzzy logic is extremely useful for many people associated with innovative work including engineering (electrical, chemical, civil, environmental, mechanical, industrial, geological, etc.), mathematics, computer software, earth science, and physics. Some of their findings can be found in [18–25].

The other direction of fuzzy sets is used in topology and analysis by many mathematicians. Subsequently, several authors have applied various forms of general topologies and developed the concept of fuzzy spaces. Kramosil and Michalek [26] developed the concept of a fuzzy metric space (FM-space). Later on, Grabeic [27] extended the BCP and proved a fixed-point result in FM-spaces in the sense of Kramosil and Michalek. George and Veeramani [28] modified the concept of FM-spaces with the help of continuous t-norms and proved some basic properties in this direction. In 2002, Gregori and Sapena [29] proved some contractive-type fixed-point theorems in complete FM-spaces in the sense of Kramosil and Michalek [26] and in the sense of George and Veeramani [28]. Rana et al. [30] established some fixed-point theorems in FM-spaces by using implicit relations. Many authors have introduced the number of fixed-point theorems in FM-spaces by using the concept of compatible maps, implicit relations, weakly compatible maps, and R-weakly compatible maps (see [31–38] and the references therein). Furthermore, Beg and Abbas [39], Popa [40], and Imad et al. [41, 42] obtained some fixed-point and invariant approximation results in FM-spaces. Recently, Li et al. [43] proved some strong coupled fixed-point theorems in FM-spaces with an integral-type application. Later on, Rehman et al. [44] proved some rational fuzzy-contraction theorems in FM-spaces with nonlinear integral-type application.

The purpose of this paper is at obtaining some extended unique fixed-point theorems in FM-spaces without the "assumption that all the sequences are Cauchy" by using the concept of Li et al. [43] and Rehman et al. [44]. We present some illustrative examples and an integral-type application to support our work. By using this concept, one can prove more generalized contractive-type fixed-point and common fixed-point results in FM-spaces with different types of integral equations. Our paper is organized as follows: Section 2 consists of preliminary concepts. In Section 3, we prove some generalized fixed-point results without continuity in FM-spaces and we presented some examples in the support of our obtained results. In Section 4, we consider some generalized Ćirić fuzzy contraction results in complete FM-spaces. In Section 5, we present an application of a particular case of the Fredholm integral equation of the second kind by ensuring the existence of a solution.

2. Preliminaries

The concept of a continuous t-norm is given by Schweizer and Sklar [45].

Definition 1 (see [45]). An operation $* : [0,1]\times[0,1] \longrightarrow [0,1]$ is known as a continuous t-norm if it satisfies the following:

(1) $*$ is commutative, associative, and continuous

(2) $1 * \rho_1 = \rho_1$ and $\rho_1 * \rho_2 \le \rho_3 * \rho_4$, whenever $\rho_1 \le \rho_3$ and $\rho_2 \le \rho_4$, for all $\rho_1, \rho_2, \rho_3, \rho_4 \in [0,1]$

The basic continuous t-norms: the minimum, the Lukasiewicz, and the product t-norms are defined, respectively, as follows:

$$\begin{gathered}\rho_1 * \rho_2 = \min\{\rho_1, \rho_2\},\\ \rho_1 * \rho_2 = \max\{\rho_1 + \rho_2 - 1, 0\},\\ \rho_1 * \rho_2 = \rho_1\rho_2.\end{gathered} \tag{1}$$

Definition 2 (see [28]). A 3-tuple $(W, M_F, *)$ is said to be a FM-space if W is an arbitrary set, $*$ is a continuous t-norm, and M_F is a fuzzy set on $W \times W \times (0,\infty)$ satisfying the following:

(i) $M_F(w,x,t) > 0$ and $M_F(w,x,t) = 1 \Leftrightarrow w = x$

(ii) $M_F(w,x,t) = M_F(w,x,t)$

(iii) $M_F(w,z,t) * M_F(z,x,s) \le M_F(w,x,t+s)$

(iv) $M_F(w,x,t)$: $(0,\infty) \longrightarrow [0,1]$ is continuous for all $w,x,z \in W$ and $t,s > 0$

Definition 3 (see [28, 29]). Let $(W, M_F, *)$ be a FM-space, $w \in W$, and $\{w_i\}$ be a sequence in W. Then,

(i) A sequence $\{w_i\}$ in W is said to be convergent to a point $w \in W$ if $\lim_{i\longrightarrow\infty} M_F(w_i, w, t) = 1$ for $t > 0$

(ii) $\{w_i\}$ is said to be a Cauchy sequence, if for each $0 < \varepsilon < 1$ and $t > 0$, there is $i_0 \in \mathbb{N}$ such that $M_F(w_k, w_i, t) > 1 - \varepsilon$, $\forall k, i \ge i_0$

(iii) $(W, M_F, *)$ is complete, if every Cauchy sequence is convergent in W

(iv) $\{w_i\}$ is known as a fuzzy contractive, if there is $0 < \beta < 1$ so that

$$\begin{aligned}&\frac{1}{M_F(w_i, w_{i+1}, t)} - 1\\ &\quad\le \beta\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right), \quad \text{for } t > 0 \text{ and } i \ge 1\end{aligned} \tag{2}$$

Definition 4 (see [46]). Let $(W, M_F, *)$ be a FM-space. The fuzzy metric M_F is triangular if

$$\frac{1}{M_F(w,x,t)} - 1 \le \left(\frac{1}{M_F(w,y,t)} - 1\right) + \left(\frac{1}{M_F(y,x,t)} - 1\right), \tag{3}$$

for all $w, x, y \in W$ and $t > 0$.

Lemma 5. *A fuzzy metric M_F is triangular.*

Proof. Let $M_F : W \times W \times (0,\infty) \longrightarrow [0,1]$ be a fuzzy metric defined by

$$M_F(w,x,t) = \frac{t}{t+|w-x|}, \quad \text{for } w,x \in W \text{ and } t>0. \tag{4}$$

□

Now, we have

$$\begin{aligned}\frac{1}{M_F(w,x,t)} - 1 &= \frac{|w-x|}{t} = \frac{|w-z+z-x|}{t} \\ &\le \frac{|w-z|}{t} + \frac{|z-x|}{t} \\ &= \left(\frac{1}{M_F(w,z,t)} - 1\right) + \left(\frac{1}{M_F(z,x,t)} - 1\right).\end{aligned} \tag{5}$$

This implies that

$$\begin{aligned}\frac{1}{M_F(w,x,t)} - 1 \le &\left(\frac{1}{M_F(w,z,t)} - 1\right) \\ &+ \left(\frac{1}{M_F(z,x,t)} - 1\right), \quad \text{for } t>0.\end{aligned} \tag{6}$$

Hence, it is proved that a fuzzy metric M_F is triangular.

Lemma 6 (see [46]). *Let $(W, M_F, *)$ be a FM-space. Let $w \in W$ and $\{w_i\}$ be a sequence in W. Then, $w_i \longrightarrow w$ iff $\lim_{i\longrightarrow\infty} M_F(w_i, w, t) = 1$, for $t > 0$.*

Definition 7 (see [29]). Let $(W, M_F, *)$ be a FM-space and $G : W \longrightarrow W$. Then, G is known as a fuzzy contraction, if there is $0 < h < 1$ so that

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le h\left(\frac{1}{M_F(w,x,t)} - 1\right), \tag{7}$$

for all $w, x \in W$ and $t > 0$.

3. Generalized Fixed-Point Results in FM-Spaces

In this section, we consider some generalized contraction theorems on FM-spaces for fixed points (by using the "triangular property of the fuzzy metric").

Theorem 8. *Let $(W, M_F, *)$ be a complete FM-space so that the fuzzy metric M_F is triangular. Let $G : W \longrightarrow W$ satisfy*

$$\begin{aligned}&\frac{1}{M_F(Gw,Gx,t)} - 1 \\ &\le a\left(\frac{1}{M_F(w,x,t)} - 1\right) + b\left(\frac{1}{M_F(w,Gw,t)} - 1\right. \\ &\quad \left. + \frac{1}{M_F(x,Gx,t)} - 1 + \frac{1}{M_F(x,Gw,t)} - 1 + \frac{1}{M_F(w,Gx,t)} - 1\right) \\ &\quad + c\left(\frac{1}{\min\{M_F(w,Gw,t), M_F(x,Gx,t), M_F(x,Gw,t), M_F(w,Gx,t)\}} - 1\right),\end{aligned} \tag{8}$$

for all $w, x \in W$, $a \in (0, 1)$, $b \in [0, 1/4)$, and $c \in [0, 1)$ with $(a + 4b + 2c) < 1$. Then, G has a unique fixed point.

Proof. Fix $w_0 \in W$. Take an iterative sequence $\{w_i\}$ such that $w_{i+1} = Gw_i$ for all $i \ge 0$. Now, by view of (8), we have

Then, we have for $t > 0$,

$$\begin{aligned}\frac{1}{M_F(w_i,w_{i+1},t)} - 1 = \frac{1}{M_F(Gw_{i-1},Gw_i,t)} - 1 &\le a\left(\frac{1}{M_F(w_{i-1},w_i,t)} - 1\right) + b\left(\frac{1}{M_F(w_{i-1},Gw_{i-1},t)} - 1\right. \\ &\quad \left. + \frac{1}{M_F(w_i,Gw_i,t)} - 1 + \frac{1}{M_F(w_i,Gw_{i-1},t)} - 1 + \frac{1}{M_F(w_{i-1},Gw_i,t)} - 1\right) \\ &\quad + c\left(\frac{1}{\min\{M_F(w_{i-1},Gw_{i-1},t), M_F(w_i,Gw_i,t), M_F(w_i,Gw_{i-1},t), M_F(w_{i-1},Gw_i,t)\}} - 1\right) \\ &\le a\left(\frac{1}{M_F(w_{i-1},w_i,t)} - 1\right) + b\left(\frac{1}{M_F(w_{i-1},w_i,t)} - 1 + \frac{1}{M_F(w_i,w_{i+1},t)} - 1 + \frac{1}{M_F(w_{i-1},w_i,t)} - 1\right. \\ &\quad \left. + \frac{1}{M_F(w_i,w_{i+1},t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w_{i-1},w_i,t), M_F(w_{i-1},w_i,t), 1, M_F(w_i,w_{i+1},t)\}} - 1\right) \\ &= a\left(\frac{1}{M_F(w_{i-1},w_i,t)} - 1\right) + b\left(\frac{1}{M_F(w_{i-1},w_i,t)} - 1 + \frac{1}{M_F(w_i,w_{i+1},t)} - 1 + \frac{1}{M_F(w_{i-1},w_i,t)} - 1\right. \\ &\quad \left. + \frac{1}{M_F(w_i,w_{i+1},t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w_{i-1},w_i,t), M_F(w_{i-1},w_i,t), M_F(w_i,w_{i+1},t)\}} - 1\right).\end{aligned} \tag{9}$$

Three possibilities arise:

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \le a\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right) + 2b\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 + \frac{1}{M_F(w_i, w_{i+1}, t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w_{i-1}, w_i, t), M_F(w_i, w_{i+1}, t), M_F(w_{i-1}, w_{i+1}, t)\}} - 1\right). \tag{10}$$

(i) If $M_F(w_{i-1}, w_i, t)$ is the minimum term in $\{M_F(w_{i-1}, w_i, t), M_F(w_i, w_{i+1}, t), M_F(w_{i-1}, w_{i+1}, t)\}$, then, after simplification, (10) can be written as

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \le \frac{a+2b+c}{1-2b}\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right), \quad \text{for } t > 0 \tag{11}$$

(ii) If $M_F(w_i, w_{i+1}, t)$ is the minimum term in $\{M_F(w_{i-1}, w_i, t), M_F(w_i, w_{i+1}, t), M_F(w_{i-1}, w_{i+1}, t)\}$, then again, (10) can be written as

$$\frac{1}{M_F(w_i, w_{i+1}, t) - 1} \le \frac{a+2b}{1-2b-c}\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right), \quad \text{for } t > 0 \tag{12}$$

(iii) If $M_F(w_{i-1}, w_{i+1}, t)$ is the minimum term in $\{M_F(w_{i-1}, w_i, t), M_F(w_i, w_{i+1}, t), M_F(w_{i-1}, w_{i+1}, t)\}$, then again, (10) becomes

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \le \frac{a+2b+c}{1-2b-c}\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right), \quad \text{for } t > 0 \tag{13}$$

Let $\beta \coloneqq \max\{(a+2b+c)/(1-2b), (a+2b)/(1-2b-c), (a+2b+c)/(1-2b-c)\} < 1$. Then, from all cases, we get

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \le \beta\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right), \quad \text{for } t > 0. \tag{14}$$

Similarly,

$$\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \le \beta\left(\frac{1}{M_F(w_{i-2}, w_{i-1}, t)} - 1\right), \quad \text{for } t > 0. \tag{15}$$

Now, from (14) and (15) and by induction, for $t > 0$,

$$\begin{aligned}\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 &\le \beta\left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1\right) \le \beta^2\left(\frac{1}{M_F(w_{i-2}, w_{i-1}, t)} - 1\right) \\ &\le \cdots \le \beta^i\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{16}$$

This yields that

$$\lim_{i \longrightarrow \infty} M_F(w_i, w_{i+1}, t) = 1, \quad \text{for } t > 0. \tag{17}$$

Since M_F is triangular, we have

$$\begin{aligned}\frac{1}{M_F(w_i, w_k, t)} - 1 &\le \left(\frac{1}{M_F(w_i, w_{i+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{i+1}, w_{i+2}, t)} - 1\right) \\ &\quad + \cdots + \left(\frac{1}{M_F(w_{k-1}, w_k, t)} - 1\right) \\ &\le \left(\beta^i + \beta^{i+1} + \cdots + \beta^{k-1}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \\ &\le \left(\frac{\beta^i}{1-\beta}\right)\left(\frac{1}{M_F(w_0, w_1, t)} - 1\right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{18}$$

Thus, $\{w_i\}$ is a Cauchy sequence. Since W is complete, there is $\kappa \in W$ so that

$$\lim_{i \longrightarrow \infty} M_F(\kappa, w_i, t) = 1, \quad \text{for } t > 0. \tag{19}$$

We shall show that $G\kappa = \kappa$. By the triangular property of M_F, we have that

$$\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \le \left(\frac{1}{M_F(\kappa, w_{i+1}, t)} - 1\right) + \left(\frac{1}{M_F(w_{i+1}, G\kappa, t)} - 1\right), \quad \text{for } t > 0. \tag{20}$$

Now, by using (8), (17), and (19), we have

$$\begin{aligned}\frac{1}{M_F(w_{i+1}, G\kappa, t)} - 1 &\le \frac{1}{M_F(Gw_i, G\kappa, t)} - 1 \le a\left(\frac{1}{M_F(w_i, \kappa, t)} - 1\right) + b\left(\frac{1}{M_F(w_i, Gw_i, t)} - 1 + \frac{1}{M_F(\kappa, G\kappa, t)} - 1 \frac{1}{M_F(\kappa, Gw_i, t)} \right.\\ &\quad \left. - 1 + \frac{1}{M_F(w_i, G\kappa, t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w_i, Gw_i, t), M_F(\kappa, G\kappa, t), M_F(\kappa, Gw_i, t), M_F(w_i, G\kappa, t)\}} - 1\right)\\ &= a\left(\frac{1}{M_F(w_i, \kappa, t)} - 1\right) + b\left(\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 + \frac{1}{M_F(\kappa, G\kappa, t)} - 1 \frac{1}{M_F(\kappa, w_{i+1}, t)} - 1\right.\\ &\quad \left. + \frac{1}{M_F(w_i, G\kappa, t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w_i, Gw_{i+1}, t), M_F(\kappa, G\kappa, t), M_F(\kappa, w_{i+1}, t), M_F(w_i, G\kappa, t)\}} - 1\right)\\ &\longrightarrow 2b\left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1\right) + c\left(\frac{1}{\min\{1, M_F(\kappa, G\kappa, t)\}} - 1\right), \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{21}$$

Hence,

$$\begin{aligned}&\lim_{i \longrightarrow \infty} \sup \frac{1}{M_F(w_{i+1}, G\kappa, t)} - 1\\ &\quad \le (2b + c)\left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1\right), \quad \text{for } t > 0.\end{aligned} \tag{22}$$

Equation (22) together with (20) and (19) implies that

$$\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \le (2b + c)\left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1\right), \quad \text{for } t > 0. \tag{23}$$

As $(2b + c) < 1$, one has $M_F(\kappa, G\kappa, t) = 1$. This implies that $G\kappa = \kappa$.

The uniqueness is as follows: let $\kappa^* \in W$ be such that $G\kappa^* = \kappa^*$. Then, in view of (8), we have for $t > 0$

$$\begin{aligned}\frac{1}{M_F(\kappa, \kappa^*, t)} - 1 &\le a\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) + b\left(\frac{1}{M_F(\kappa, G\kappa_i, t)} - 1 + \frac{1}{M_F(\kappa^*, G\kappa^*, t)} - 1 \frac{1}{M_F(\kappa, G\kappa^*, t)} - 1 + \frac{1}{M_F(\kappa^*, G\kappa, t)} - 1\right)\\ &\quad + c\left(\frac{1}{\min\{M_F(\kappa, G\kappa^*, t), M_F(\kappa^*, G\kappa^*, t), M_F(\kappa, G\kappa_i^*, t), M_F(\kappa^*, G\kappa, t)\}} - 1\right)\\ &= a\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) + 2b\left(\frac{1}{M_F(\kappa, G\kappa^*, t)} - 1\right) + c\left(\frac{1}{\min\{1, M_F(\kappa, \kappa^*, t)\}} - 1\right)\\ &= (a + 2b + c)\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \le (a + 2b + c)\left(\frac{1}{M_F(G\kappa, G\kappa^*, t)} - 1\right)\\ &\le (a + 2b + c)^2\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \le \cdots \le (a + 2b + c)^i\left(\frac{1}{M_F(\kappa, \kappa^*, t)} - 1\right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty.\end{aligned} \tag{24}$$

Hence, we get that $M_F(\kappa, \kappa^*, t) = 1$, so $\kappa = \kappa^*$. Thus, G has a unique fixed point in W.

Corollary 9. *Let $(W, M_F, *)$ be a complete FM-space so that the fuzzy metric M_F is triangular. Let $G : W \longrightarrow W$ verify that*

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le a\left(\frac{1}{M_F(w, x, t)} - 1\right) + b\left(\frac{1}{M_F(w, Gw, t)} - 1 + \frac{1}{M_F(x, Gx, t)} - 1 + \frac{1}{M_F(x, Gw, t)} - 1 + \frac{1}{M_F(w, Gx, t)} - 1\right), \tag{25}$$

for all $w, x \in W, t > 0$, $a \in (0, 1)$, and $b \in [0, 1/4)$ with $(a + 2b) < 1$. Then, G has a unique fixed point.

Proof. It follows by putting $c = 0$ in Theorem (8). □

Example 10. Let $W = [0, 1]$ be equipped with a continuous t-norm. Let $M_F : W \times W \times (0,\infty) \longrightarrow [0, 1]$ be a fuzzy metric defined by

$$M_F(w, x, t) = \frac{t}{t + d(w, x)}, \quad \text{where } d(w, x) = \left|\frac{w - x}{3}\right|, \tag{26}$$

for all $w, x \in W$ and $t > 0$. Then, $(W, M_F, *)$ is a complete FM-space. Now, we define the mapping $G : W \longrightarrow W$ by

$$G(w) = \frac{2w}{3} + \frac{4}{15}, \quad \text{for all } w \in [0, 1] \text{ and } t > 0. \tag{27}$$

Then,

$$\begin{aligned}
\frac{1}{M_F(Gw, Gx, t)} - 1 &= \left|\frac{Gw - Gx}{3t}\right| = \left|\frac{2(w - x)}{9t}\right| \le \left|\frac{2(w - x)}{9t}\right| + \frac{1}{15}\left|\frac{2(5w + 5x - 8)}{45t}\right| = \frac{2}{3}\left|\frac{w - x}{3t}\right| \\
&+ \frac{1}{15}\begin{pmatrix} \left|\frac{1}{3t}\left(\frac{5w - 4}{15}\right)\right| + \left|\frac{1}{3t}\left(\frac{5x - 4}{15}\right)\right| \\ \left| + \frac{1}{3t}\left(\frac{15x - 10w - 4}{15}\right)\right| + \left|\frac{1}{3t}\left(\frac{15w - 10x - 4}{15}\right)\right| \end{pmatrix} \\
&= \frac{2}{3}\left|\frac{w - x}{3t}\right| + \frac{1}{15}\left(\left|\frac{1}{3t}\left(w - \frac{2w}{3} - \frac{4}{15}\right)\right| + \left|\frac{1}{3t}\left(x - \frac{2wx}{3} - \frac{4}{15}\right)\right| + \left|\frac{1}{3t}\left(x - \frac{2w}{3} - \frac{4}{15}\right)\right| \right. \\
&\left. + \left|\frac{1}{3t}\left(w - \frac{2x}{3} - \frac{4}{15}\right)\right|\right) = \frac{2}{3}\left(\frac{1}{M_F(w, x, t)} - 1\right) \\
&+ \frac{1}{15}\left(\frac{1}{M_F(w, Gw, t)} - 1 + \frac{1}{M_F(x, Gx, t)} - 1 + \frac{1}{M_F(x, Gw, t)} - 1 + \frac{1}{M_F(w, Gx, t)} - 1\right).
\end{aligned} \tag{28}$$

Hence, all the conditions of Corollary 9 are satisfied with $a = 2/3$ and $b = 1/15$. Hence, the self-mapping G has a unique fixed point, that is, $G(4/5) = 4/5 \in [0, 1]$.

Corollary 11. *Let $(W, M_F, *)$ be a complete FM-space so that the fuzzy metric M_F is triangular. Let $G : W \longrightarrow W$ verify that*

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le a\left(\frac{1}{M_F(w, x, t)} - 1\right) + c\left(\frac{1}{\min\{M_F(w, Gw, t), M_F(x, Gx, t), M_F(x, Gw, t), M(w, Gx, t)\}} - 1\right), \tag{29}$$

for all $w, x \in W, t > 0$, $a \in (0, 1)$, and $c \in [0, 1)$ with $(a + 2c) < 1$. Then, G has a unique fixed point.

Proof. It follows by putting $b = 0$ in Theorem (8). □

Example 12. Let $W = [0,\infty)$ be endowed with a continuous t-norm. Let $M_F : W \times W \times (0,\infty) \longrightarrow [0, 1]$ be a fuzzy metric defined by

$$M_F(w, x, t) = \frac{t}{t + d(w, x)} \text{ where } d(w, x) = |w - x|, \tag{30}$$

for all $w, x \in W$ and $t > 0$. Then, $(W, M_F, *)$ is a complete FM-space. Now, we define a mapping $G : W \longrightarrow W$ by

$$Gw = \begin{cases} \frac{3w}{4} + \frac{1}{2}, & \text{if } w \in [0, 1], \\ \frac{w}{2} + \frac{7}{2}, & \text{if } w \in (1,\infty). \end{cases} \tag{31}$$

We have

$$\begin{aligned}
\frac{1}{M_F(Gw, Gx, t)} - 1 &= \frac{3}{4t}|w - x| = \frac{3}{4t}\left(\frac{1}{M_F(w, x, t)} - 1\right) \le \frac{3}{4}\left(\frac{1}{M_F(w, x, t)} - 1\right) \\
&+ \frac{1}{9}\left(\frac{1}{\min\{M_F(w, Gw, t), M(x, Gx, t), M_F(x, Gw, t), M_F(w, Gx, t)\}} - 1\right).
\end{aligned} \tag{32}$$

Hence, all the conditions of Corollary 11 are satisfied with $a = 3/4$ and $b = 1/9$. Then, the self-mapping G has a unique fixed point, that is, $G(7) = 7$.

4. Ćirić-Type Fuzzy Contraction Results in FM-Spaces

In this section, we define Ćirić-type fuzzy contraction mappings and we present a unique related fixed-point theorem on a complete FM-space.

Definition 13. Let $(W, M_F, *)$ be a complete FM-space. A self-mapping $G : W \longrightarrow W$ is said to be a Ćirić contraction if there is $\alpha \in (0, 1)$ such that

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \leq \alpha \max \left\{ \begin{array}{c} \frac{1}{M_F(w, Gw, t)} - 1, \frac{1}{M_F(x, Gx, t)} - 1 \\ \frac{1}{M_F(x, Gw, t)} - 1, \frac{1}{M_F(w, Gx, t)} - 1 \\ \frac{1}{M_F(w, x, t)} - 1 \end{array} \right\}, \quad (33)$$

for all $w, x \in W$ and $t > 0$. Here, α is called the contractive constant of T.

Theorem 14. *Let a self-mapping $G : W \longrightarrow W$ be a Ćirić contraction in a complete FM-space $(W, M_F, *)$ so that M_F is triangular and (33) satisfies with $2\alpha < 1$. Then, G has a unique fixed point.*

Proof. Fix $w_0 \in W$. Take an iterative sequence $\{w_i\}$ such that $w_{i+1} = Gw_i$ for all $i \geq 0$. Now, by using (33), we have

$$\begin{aligned} \frac{1}{M_F(w_i, w_{i+1}, t)} - 1 &= \frac{1}{M_F(Gw_{i-1}, Gw_i, t)} - 1 \\ &\leq \alpha \max \left\{ \begin{array}{c} \frac{1}{M_F(w_{i-1}, Gw_{i-1}, t)} - 1, \frac{1}{M_F(w_i, Gw_i, t)} - 1 \\ \frac{1}{M_F(w_{i-1}, Gw_i, t)} - 1, \frac{1}{M_F(w_i, Gw_{i-1}, t)} - 1 \\ \frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \end{array} \right\}. \end{aligned} \quad (34)$$

□

After simplification, for $t > 0$, we get

$$\begin{aligned} \frac{1}{M_F(w_i, w_{i+1}, t)} - 1 &\leq \alpha \max \left\{ \frac{1}{M_F(w_{i-1}, w_i, t)} - 1, \frac{1}{M_F(w_i, w_{i+1}, t)} \right. \\ &\left. - 1, \frac{1}{M_F(w_{i-1}, w_{i+1}, t)} - 1 \right\}. \end{aligned} \quad (35)$$

Now, there are three possibilities:

(i) If $(1/M_F(w_{i-1}, w_i, t)) - 1$ is the maximum in $\{(1/M_F(w_{i-1}, w_i, t)) - 1, (1/M_F(w_i, w_{i+1}, t)) - 1, (1/M_F(w_{i-1}, w_{i+1}, t)) - 1\}$, then, from (35), we have

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \leq \alpha \left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \right), \quad \text{for } t > 0 \quad (36)$$

(ii) If $(1/M_F(w_i, w_{i+1}, t)) - 1$ is the maximum in $\{(1/M_F(w_{i-1}, w_i, t)) - 1, (1/M_F(w_i, w_{i+1}, t)) - 1, (1/M_F(w_{i-1}, w_{i+1}, t)) - 1\}$, then, from (35), we have

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \leq \alpha \left(\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \right), \quad \text{for } t > 0, \text{ which is not possible} \quad (37)$$

(iii) If $(1/M_F(w_{i-1}, w_{i+1}, t)) - 1$ is the maximum in $\{(1/M_F(w_{i-1}, w_i, t)) - 1, (1/M_F(w_i, w_{i+1}, t)) - 1, (1/M_F(w_{i-1}, w_{i+1}, t)) - 1\}$, then, from (35), we have

$$\begin{aligned} \frac{1}{M_F(w_i, w_{i+1}, t)} - 1 &\leq \alpha \left(\frac{1}{M_F(w_{i-1}, w_{i+1}, t)} - 1 \right) \\ &\leq \frac{\alpha}{1 - \alpha} \left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \right), \quad \text{for } t > 0 \end{aligned} \quad (38)$$

Let $\delta \coloneqq \max \{\alpha, a/(1 - \alpha)\} < 1$. Using (36) and (38), we have

$$\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \leq \delta \left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \right), \quad \text{for } t > 0. \quad (39)$$

Similarly,

$$\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \le \delta^2 \left(\frac{1}{M_F(w_{i-2}, w_{i-1}, t)} - 1 \right), \quad \text{for } t > 0. \tag{40}$$

Now, from (39) and (40) and by induction, for $t > 0$

$$\begin{aligned} &\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \\ &\le \delta \left(\frac{1}{M_F(w_{i-1}, w_i, t)} - 1 \right) \le \delta^2 \left(\frac{1}{M_F(w_{i-2}, w_{i-1}, t)} - 1 \right) \\ &\le \cdots \le \delta^i \left(\frac{1}{M_F(w_0, w_1, t)} - 1 \right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{41}$$

This yields that

$$\lim_{i \longrightarrow \infty} M_F(w_i, w_{i+1}, t) = 1, \quad \text{for } t > 0. \tag{42}$$

Since M_F is triangular and $k > i$, we have

$$\begin{aligned} &\frac{1}{M_F(w_i, w_k, t)} - 1 \\ &\le \left(\frac{1}{M_F(w_i, w_{i+1}, t)} - 1 \right) + \left(\frac{1}{M_F(w_{i+1}, w_{i+2}, t)} - 1 \right) + \cdots \\ &\quad + \left(\frac{1}{M_F(w_{k-1}, w_k, t)} - 1 \right) \le \left(\delta^i + \delta^{i+1} + \cdots + \delta^{k-1} \right) \\ &\cdot \left(\frac{1}{M_F(w_0, w_1, t)} - 1 \right) \le \left(\frac{\delta^i}{1-\delta} \right) \left(\frac{1}{M_F(w_0, w_1, t)} - 1 \right) \\ &\longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{43}$$

Hence, $\{w_i\}$ is a Cauchy sequence. Since $(W, M_F, *)$ is complete, there is $\kappa \in W$ so that

$$\lim_{i \longrightarrow \infty} M_F(\kappa, w_i, t) = 1, \quad \text{for } t > 0. \tag{44}$$

Now, we have to show that $G\kappa = \kappa$. Since M_F is triangular, one writes

$$\begin{aligned} \frac{1}{M_F(G\kappa, \kappa, t)} - 1 &\le \left(\frac{1}{M_F(G\kappa, w_{i+1}, t)} - 1 \right) \\ &\quad + \left(\frac{1}{M_F(w_{i+1}, \kappa, t)} - 1 \right), \quad \text{for } t > 0. \end{aligned} \tag{45}$$

In view of (33), (42), and (44), we have for $t > 0$

$$\begin{aligned} &\frac{1}{M_F(w_{i+1}, G\kappa, t)} - 1 \\ &\le \frac{1}{M_F(Gw_i, G\kappa, t)} - 1 \\ &\le \alpha \max \left\{ \begin{array}{c} \frac{1}{M_F(w_i, Gw_i, t)} - 1, \frac{1}{M_F(\kappa, G\kappa, t)} - 1 \\ \frac{1}{M_F(w_i, G\kappa, t)} - 1, \frac{1}{M(\kappa, Gw_i, t)} - 1 \\ \frac{1}{M_F(w_i, \kappa, t)} - 1 \end{array} \right\} \\ &= \alpha \max \left\{ \begin{array}{c} \frac{1}{M_F(w_i, w_{i+1}, t)} - 1, \frac{1}{M_F(\kappa, G\kappa, t)} - 1 \\ \frac{1}{M_F(w_i, G\kappa, t)} - 1, \frac{1}{M_F(\kappa, w_{i+1}, t)} - 1 \\ \frac{1}{M_F(w_i, \kappa, t)} - 1 \end{array} \right\} \\ &\longrightarrow \alpha \left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \right), \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{46}$$

Hence,

$$\begin{aligned} &\lim_{i \longrightarrow \infty} \sup \frac{1}{M_F(w_{i+1}, G\kappa, t)} - 1 \\ &\le \alpha \left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \right), \quad \text{for } t > 0. \end{aligned} \tag{47}$$

This together with (45) and (44), we have

$$\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \le \alpha \left(\frac{1}{M_F(\kappa, G\kappa, t)} - 1 \right), \quad \text{for } t > 0. \tag{48}$$

Since $\alpha \in (0, 1)$, one gets $M_F(\kappa, G\kappa, t) = 1$. This implies that $G\kappa = \kappa$.

The uniqueness is as follows: let $\kappa^* \in W$ be such that $G\kappa = \kappa^*$. Using (33), we have

$$\begin{aligned} &\frac{1}{M_F(G\kappa, G\kappa^*, t)} - 1 \\ &\le \alpha \max \left\{ \begin{array}{c} \frac{1}{M_F(\kappa, G\kappa, t)} - 1, \frac{1}{M_F(\kappa^*, G\kappa^*, t)} - 1 \\ \frac{1}{M_F(\kappa, G\kappa^*, t)} - 1, \frac{1}{M_F(\kappa^*, G\kappa, t)} - 1 \\ \frac{1}{M_F(\kappa, \kappa^*, t)} - 1 \end{array} \right\} \end{aligned}$$

$$= \alpha\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right) \le \alpha\left(\frac{1}{M_F(G\kappa,G\kappa^*,t)}-1\right)$$
$$\le \alpha^2\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right) \le \cdots$$
$$\le \alpha^i\left(\frac{1}{M_F(\kappa,\kappa^*,t)}-1\right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{49}$$

We get that $M_F(\kappa,\kappa^*,t)=1$ for $t>0$. This implies that $\kappa=\kappa^*$. Thus, G has a unique fixed point in W.

Example 15. Let $W=[0,\infty)$ be endowed with a continuous t-norm. Let a fuzzy metric $M_F: W\times W\times(0,\infty)\longrightarrow[0,1]$ be defined by

$$M_F(w,x,t)=\frac{t}{t+d(w,x)}, \quad \text{where } d(w,x)=2|w-x|, \tag{50}$$

for all $w,x\in W$ and $t>0$. Then, $(W,M_F,*)$ is a complete FM-space. Now, we define a mapping $G: W\longrightarrow W$ as

$$Gw=\begin{cases}\dfrac{w}{3}+\dfrac{1}{2}, & \text{if } w\in[0,1],\\ \dfrac{7w}{8}+\dfrac{2}{3}, & \text{if } w\in(1,\infty).\end{cases} \tag{51}$$

We have

$$\frac{1}{M_F(Gw,Gx,t)}-1$$
$$=\frac{2}{t}\left|\frac{w}{3}-\frac{x}{3}\right|=\frac{1}{3}\left(\frac{1}{M(w,x,t)}-1\right)$$
$$\le\frac{1}{3}\max\begin{pmatrix}\dfrac{1}{M_F(w,Gw,t)}-1, \dfrac{1}{M_F(x,Gx,t)}-1\\ \dfrac{1}{M_F(x,Gw,t)}-1, \dfrac{1}{M_F(w,Gx,t)}-1\\ \dfrac{1}{M_F(w,x,t)}-1\end{pmatrix}, \tag{52}$$

for $t>0$. Hence, all the conditions of Theorem 14 are satisfied with $\alpha=1/3$ and G has a unique fixed point, that is, $G(16/3)=16/3$.

5. Application

In this section, we present an integral-type equation. Let $W=C([0,\xi],\mathbb{R})$ be the space of all real-valued continuous functions on the interval $[0,\xi]$, where $0<\xi\in\mathbb{R}$. Now, we present a particular case of a Fredholm integral equation (FIE) of the second kind given as follows:

$$w(\tau)=\int_0^\xi K(\tau,\nu,w(\nu))d\nu, \tag{53}$$

where $\tau\in[0,\xi]$ and $K:[0,\xi]\times[0,\xi]\times\mathbb{R}\longrightarrow\mathbb{R}$. The induced metric $d: W\times W\longrightarrow\mathbb{R}$ is defined by

$$d(w,x)=\|w-x\|, \quad \forall w,x\in W. \tag{54}$$

The binary operation $*$, being a continuous t-norm, is defined by $\alpha*\beta=\alpha\beta$ for all $\alpha,\beta\in[0,\xi]$. The standard fuzzy metric $M_F: W\times W\times(0,\infty)\longrightarrow[0,1]$ can be expressed as

$$M_F(w,x,t)=\frac{t}{t+d(w,x)}, \quad \forall w,x\in W, \text{and } t>0. \tag{55}$$

Then easily, we can show that M_F is triangular and $(W,M_F,*)$ is a complete $FM-$space.

Theorem 16. *Assume that there is $\eta\in(0,1)$ so that*

$$\|Gw-Gx\|\le\eta N(G,w,x), \quad \forall w,x\in W, \tag{56}$$

where

$$N(G,w,x)$$
$$=\max\left\{\begin{matrix}\|w-x\|,\|Gw-w\|,\|Gx-x\|,\|Gw-x\|,\|Gx-w\|\\ \|Gw-w\|+\|Gx-x\|+\|Gw-x\|+\|Gx-w\|,\end{matrix}\right\}. \tag{57}$$

Then, the FIE (53) has a unique solution.

Proof. Give $G: W\longrightarrow W$ as

$$Gw(\tau)=\int_0^\xi K(\tau,\nu,w(\nu))d\nu. \tag{58}$$

□

Notice that G is well defined and (53) has a unique solution if and only if G has a unique fixed point in W. Now, we have to show that Theorem 8 is applied to the integral operator G. Then, for all $w,x\in W$, we have the following six cases:

(1) If $\|w-x\|$ is the maximum term in (57), then $N(G,w,x)=\|w-x\|$. Therefore, in view of (55) and (56), we have

$$\frac{1}{M_F(Gw,Gx,t)}-1=\frac{d(Gw,Gx)}{t}\le\eta\frac{N(G,w,x)}{t}$$
$$=\eta\frac{\|w-x\|}{t}=\eta\left(\frac{1}{M_F(w,x,t)}-1\right). \tag{59}$$

This implies that

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le \eta\left(\frac{1}{M_F(w, x, t)} - 1\right), \quad \text{for } t > 0, \tag{60}$$

for all $w, x \in W$ such that $Gw \neq Gx$. The inequality (60) holds if $Gw = Gx$. Thus, the integral operator G satisfies all the conditions of Theorem 8 with $\eta = a$ and $b = c = 0$ in (8). Then, the integral operator G has a unique fixed point, i.e., (53) has a solution in W

(2) If $\|Gw - w\|$ is the maximum term in (57), then, $N(G, w, x) = \|Gw - w\|$. Therefore, using (55) and (56), we have

$$\frac{1}{M_F(Gw, Gx, t)} - 1 = \frac{d(Gw, Gx)}{t} \le \eta \frac{N(G, w, x)}{t} = \eta \frac{\|Gw - w\|}{t} = \eta\left(\frac{1}{M_F(Gw, w, t)} - 1\right). \tag{61}$$

It yields that

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le \eta\left(\frac{1}{M_F(Gw, w, t)} - 1\right), \quad \text{for } t > 0, \tag{62}$$

for all $w, x \in W$ such that $Gw \neq Gx$

(3) If $\|Gx - x\|$ is the maximum term in (57), then $N(G, w, x) = \|Gx - x\|$. Therefore, by (55) and (56), we have

$$\frac{1}{M_F(Gw, Gx, t)} - 1 = \frac{d(Gw, Gx)}{t} \le \eta \frac{N(G, w, x)}{t} = \eta \frac{\|Gx - x\|}{t} = \eta\left(\frac{1}{M(Gx, x, t)} - 1\right). \tag{63}$$

That is,

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le \eta\left(\frac{1}{M_F(Gx, x, t)} - 1\right), \quad \text{for } t > 0, \tag{64}$$

for all $w, x \in W$ such that $Gw \neq Gx$

(4) If $\|Gw - x\|$ is the maximum term in (57), then, $N(G, w, x) = \|Gw - x\|$. Therefore, due to (55) and (56), we have

$$\frac{1}{M_F(Gw, Gx, t)} - 1 = \frac{d(Gw, Gx)}{t} \le \eta \frac{N(G, w, x)}{t} = \eta \frac{\|Gw - x\|}{t} = \eta\left(\frac{1}{M_F(Gw, x, t)} - 1\right). \tag{65}$$

Hence,

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le \eta\left(\frac{1}{M_F(Gw, x, t)} - 1\right), \quad \text{for } t > 0, \tag{66}$$

for all $w, x \in W$ such that $Gw \neq Gx$

(5) If $\|Gx - w\|$ is the maximum term in (57), then, $N(G, w, x) = \|Gx - w\|$. Using (55) and (56), we have

$$\frac{1}{M_F(Gw, Gx, t)} - 1 = \frac{d(Gw, Gx)}{t} \le \eta \frac{N(G, w, x)}{t} = \eta \frac{\|Gx - w\|}{t} = \eta\left(\frac{1}{M_F(Gx, w, t)} - 1\right). \tag{67}$$

It implies that

$$\frac{1}{M_F(Gw, Gx, t)} - 1 \le \eta\left(\frac{1}{M_F(Gx, w, t)} - 1\right), \quad \text{for } t > 0, \tag{68}$$

for all $w, x \in W$ such that $Gw \neq Gx$

The inequalities (62), (64), (66), and (68) hold if $Gw = Gx$. Thus, the integral operator G satisfies all the conditions of Theorem 8 with $\eta = c$ and $a = b = 0$ in (8). The integral operator G has a unique fixed point, i.e., (53) has a solution in W.

(6) If $\|Gw - w\| + \|Gx - x\| + \|Gw - x\| + \|Gx - w\|$ is the maximum term in (57), then, $N(G, w, x) = \|Gw - w\| + \|Gx - x\| + \|Gw - x\| + \|Gx - w\|$. Therefore, from (55) and (56), we have

$$\begin{aligned} &\frac{1}{M_F(Gw, Gx, t)} - 1 \\ &= \frac{d(Gw, Gx)}{t} \le \eta \frac{N(G, w, x)}{t} \\ &= \eta \frac{\|Gw - w\| + \|Gx - x\| + \|Gw - x\| + \|Gx - w\|}{t} \\ &= \eta\left(\frac{1}{M_F(w, Gw, t)} - 1 + \frac{1}{M_F(x, Gx, t)} - 1 \right. \\ &\quad \left. + \frac{1}{M(x, Gw, t)} - 1 + \frac{1}{M_F(w, Gx, t)} - 1\right). \end{aligned} \tag{69}$$

That is,

$$\begin{aligned}
&\frac{1}{M_F(Gw, Gx, t)} - 1 \\
&\leq \eta\left(\frac{1}{M_F(w, Gw, t)} - 1 + \frac{1}{M_F(x, Gx, t)} - 1\right. \\
&\quad \left. + \frac{1}{M_F(x, Gw, t)} - 1 + \frac{1}{M_F(w, Gx, t)} - 1\right), \quad \text{for } t > 0,
\end{aligned} \tag{70}$$

for all $w, x \in W$ such that $Gw \neq Gx$. The inequality (70) holds if $Gw = Gx$. Thus, the integral operator G satisfies all the conditions of Theorem 8 with $\eta = b$ and $a = c = 0$ in (8). The integral operator G has a unique fixed point, i.e., (53) has a solution in W.

Now, we present a special type of example for a particular case of an FIE of a second kind.

Example 17. Take $W = [0, 1]$. If we put $\xi = 1$ in (53), then, we have the following integral equation:

$$\begin{aligned}
w(\tau) &= \int_0^1 \frac{2}{5(\tau + 1 + w(\nu))} d\nu, \quad \text{where } K(\tau, \nu, w(\nu)) \\
&= \frac{2}{5(\tau + 1 + w(\nu))}.
\end{aligned} \tag{71}$$

Equation (71) is a special kind of the integral equation (53), where $\tau \in [0, 1]$. Then,

$$\begin{aligned}
&\|K(\tau, \nu, w(\nu)) - K(\tau, \nu, x(\nu))\| \\
&= \left\| \frac{2}{5(\tau + 1 + w(\nu))} - \frac{2}{5(\tau + 1 + x(\nu))} \right\| \\
&= \frac{2}{5} \left\| \frac{w(\nu) - x(\nu)}{(\tau + 1 + w(\nu))(\tau + 1 + x(\nu))} \right\| \\
&\leq \frac{2}{5} \|w(\nu) - x(\nu)\| = \frac{2}{5} N(G, w, x),
\end{aligned} \tag{72}$$

where $N(G, w, x) = \|w(\nu) - x(\nu)\|$. Now, we have to show that $\|Gw(\tau) - Gx(\tau)\| \leq \eta N(G, w, x)$. From equation (58), we have

$$\begin{aligned}
&\|Gw(\tau) - Gx(\tau)\| \\
&= \left\| \int_0^1 K(\tau, \nu, w(\nu)) d\nu - \int_0^1 K(\tau, \nu, x(\nu)) d\nu \right\| \\
&= \int_0^1 \|K(\tau, \nu, w(\nu)) - K(\tau, \nu, x(\nu))\| d\nu \\
&\leq \int_0^1 \frac{2}{5} \|w(\nu) - x(\nu)\| d\nu = \int_0^1 \frac{2}{5} N(T, w, x) d\nu \\
&= \frac{2}{5} N(G, w, x) \int_0^1 d\nu = \frac{2}{5} N(G, w, x).
\end{aligned} \tag{73}$$

Hence, all conditions of Theorem 16 hold with $\eta = 2/5 < 1$. The integral equation (71) has a unique solution by using Theorem 16.

6. Conclusion

In this paper, we proved variant unique fixed-point results for some generalized contraction-type self-mappings in complete FM-spaces, without continuity and by using the "triangular property of the fuzzy metric" as a basic tool. We presented illustrative examples. Moreover, we provided an application about a particular case of Fredholm integral equation of second kind. In this direction, researchers can prove more fixed-point results in complete FM-spaces without using continuity via different types of applications.

Data Availability

Data sharing is not applicable to this article as no dataset were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors have equally contributed to the final manuscript.

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Research Article

Kannan-Type Contractions on New Extended *b*-Metric Spaces

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This article is focused on the generalization of some fixed point theorems with Kannan-type contractions in the setting of new extended b-metric spaces. An idea of asymptotic regularity has been incorporated to achieve the new results. As an application, the existence of a solution of the Fredholm-type integral equation is presented.

1. Introduction and Preliminaries

The existence of fixed points for some operators has a noteworthy contribution in many branches of applied and pure mathematics. The theory of a fixed point provides very valuable and effective tools in mathematics. It has a wide range of implications in nonlinear analysis and has been established in two directions. One is to change the space under consideration (see the works of Bakhtin [1], Jleli and Samet [2], Karapinar [3], Kamran et al. [4], etc.), and the other is to change the contraction conditions (see the works of Ćirić [5], Popescu [6], Rakotch [7], etc.).

In 1922, the Polish mathematician Banach [8] established a remarkable result relevant to a metric fixed point theory, that is known as the Banach contraction principle (BCP). The work of Banach is well regarded and an adaptable consequence in the theory of fixed points. BCP laid a foundation of research in this field, which is further investigated by many researchers from 1922 till now. One of the prominent generalizations of the BCP was presented by Kannan [9]. Additional works on the existence of (common) fixed points can be seen in [10–14].

In 1989, Bakhtin [1] introduced the notion of b-metric spaces. Later on, the concept of b-metric spaces was further used by Czerwik [15] to establish different fixed point results in b-metric spaces. The study of b-metric spaces endowed an imperative place in the fixed point theory with multiple aspects. Many mathematicians (Abdeljawad et al. [16, 17], Ali et al. [18], Akkouchi [19], Chifu and Karapinar [20], Kadelburg and Radenović [21], Parvaneh et al. [11], Gupta et al. [12], Mlaiki et al. [22], etc.) led the foundation to improve the fixed point theory in b-metric spaces. Another innovative task has been achieved by Kamran et al. [4] in 2017 by introducing the notion of an extended b-metric space, which generalizes the notion of a b-metric space. Some fixed point results are proved in this new setting; see for instance the work presented in [23–25].

The work of Kannan [9] refined the concept of the Banach contraction mapping by introducing a new contraction, now known as Kannan contraction. The Kannan fixed point result has been further extended and generalized in the setup of b-metric spaces [15] and for generalized metric spaces [26].

In 2019, the notion of a new extended b-metric space has been initiated by Aydi et al. [27], where the control function depends on three variables. This fact is new since all precedent control functions depend on two variables. The objective of this work is at investigating Kannan-type contractions in the context of new extended b- metric spaces by extending the main results of Gornicki [28]. For this purpose, some basic concepts are needed in the sequel.

Definition 1 (see [15]). Let X be a nonempty set and $s \geq 1$ be a real number. A function $d_b : X \times X \longrightarrow \mathbb{R}$ is called a b -metric, if it satisfies the following for all $j, \kappa, \ell \in X$:

(b1) $d_b(j, \kappa) \geq 0$

(b2) $d_b(j, \kappa) = 0$, if and only if $j = \kappa$

(b3) $d_b(j, \kappa) = d_b(\kappa, j)$

(b4) $d_b(j, \kappa) \leq s[d_b(j, \ell) + d_b(\ell, \kappa)]$

The pair (X, d_b) is called a b-metric space. If $s = 1$, then, a b-metric space becomes a metric space. In 2017, Kamran et al. [4] generalized the b-metric space setting to an extended b-metric space (in the same direction, see also [29, 30]).

Definition 2 (see [4]). Let X be a nonempty set and $\theta : X \times X \longrightarrow [1,\infty)$ be a function. The map $d_\theta : X \times X \longrightarrow \mathbb{R}$ is called an extended b-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

(i) $d_\theta(j, \kappa) \geq 0$

(ii) $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$

(iii) $d_\theta(j, \kappa) = d_\theta(\kappa, j)$

(iv) $d_\theta(j, \ell) \leq \theta(j, \ell)[d_\theta(j, \kappa) + d_\theta(\kappa, \ell)]$

In 2019, Aydi et al. [27] introduced the notion of new extended b-metric spaces. Here, the control function depends on 3 variables.

Definition 3 (see [27]). Let X be a nonempty set and $\theta : X \times X \times X \longrightarrow [1,\infty)$ be a function. The map $d_\theta : X \times X \longrightarrow \mathbb{R}$ is called a new extended b-metric if for all $j, \kappa, \ell \in X$, it satisfies the following axioms:

(i) $d_\theta(j, \kappa) \geq 0$

(ii) $d_\theta(j, \kappa) = 0$, if and only if $j = \kappa$

(iii) $d_\theta(j, \kappa) = d_\theta(\kappa, j)$

(iv) $d_\theta(j, \ell) \leq \theta(j, \kappa, \ell)[d_\theta(j, \kappa) + d_\theta(\kappa, \ell)]$

The pair (X, d_θ) is named to be a new extended b-metric space. If $\theta(j, \kappa, \ell) = s$ (for $s \geq 1$), we get Definition 1.

Example 4 (see [27]). Let $X = \mathbb{N}$. Define $d_\theta : X \times X \longrightarrow \mathbb{R}$ by

$$d_\theta(j, \kappa) = \begin{cases} 0, & \Longleftrightarrow j = \kappa, \\ \dfrac{1}{j}, & \text{if } j \text{ is even and } \kappa \text{ is odd}, \\ \dfrac{1}{\kappa}, & \text{if } j \text{ is odd and } \kappa \text{ is even}, \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\theta(j, \kappa, \ell) = \begin{cases} 1, & \text{if } j = \ell \text{ and } \kappa \text{ is even or odd}, \\ \dfrac{j\ell}{j + \ell}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are even and } \kappa \text{ odd}, \\ \dfrac{\kappa}{2}, & \text{if } j \neq \ell, j \text{ and } \ell \text{ are odd and } \kappa \text{ even}, \\ \dfrac{3}{2}, & \text{if } j \neq \ell, j, \ell \text{ and } \kappa \text{ are all even or all odd}, \\ \dfrac{j + \kappa(1 + j)}{j(1 + \kappa)}, & \text{if } j \neq \ell, j \text{ is even}, \ell \text{ is odd and } \kappa \text{ is even}, \\ \dfrac{\ell + \kappa(\ell + 1)}{\ell(\kappa + 1)}, & \text{if } j \neq \ell, j \text{ is odd}, \ell \text{ is even and } \kappa \text{ is even}, \\ \dfrac{2 + \ell}{1 + \ell}, & \text{if } j \neq \ell, j \text{ is odd}, \ell \text{ is even and } \kappa \text{ is odd}, \\ \dfrac{j + 1}{j}, & \text{if } j \neq \ell, j \text{ is even}, \ell \text{ is odd and } \kappa \text{ is odd}. \end{cases} \tag{2}$$

Here, (X, d_θ) is a new extended b-metric space.

On the other hand, by taking $j = 2p + 1, \ell = 4p + 1$, and $\kappa = 2p$, we have

$$\frac{d_\theta(j, \ell)}{d_\theta(j, \kappa) + d_\theta(\kappa, \ell)} = \frac{d_\theta(2p + 1, 4p + 1)}{d_\theta(2p + 1, 2p) + d_\theta(2p, 4p + 1)} = p. \tag{3}$$

It is not possible to find $s \geq 1$ so that $(b4)$ holds. Thus, d_θ is not a b-metric on X.

Example 5. Consider $X = \{1, 2, 3\}$. Take $d_\theta : X \times X \longrightarrow [0, \infty)$ as

$$d_\theta(j, \kappa) = (j - \kappa)^2, \tag{4}$$

where $\theta : X \times X \times X \longrightarrow [1,\infty)$ as $\theta(j, \kappa, \ell) = j + \kappa + 2\ell$. Then, (X, d_θ) is a new extended b-metric space.

Proof. The first three conditions are trivially verified. To check the triangular inequality, we proceed as follows:

$$d_\theta(1,2) \le \theta(1,3,2)[d_\theta(1,3)+d_\theta(3,2)] \le (1+3+4)[4+1], \tag{5}$$

so $1<40$. Similarly, we can check the other two pairs.

Therefore, for all j,κ,ℓ in X, $d_\theta(j,\ell) \le \theta(j,\kappa,\ell)[d_\theta(j,\kappa) + d_\theta(\kappa,\ell)]$. □

Definition 6. Let (X, d_θ) be a new extended b -metric space.

(i) A sequence $\{a_n\} \subset X$ is called convergent to $a \in X$ if for $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that $\lim_{n\longrightarrow\infty} d_\theta(a_n, a) < \varepsilon$ for all $n \ge N(\varepsilon)$

(ii) A sequence $\{a_n\} \subset X$ is called Cauchy if for $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that $\lim_{n\longrightarrow\infty} d_\theta(a_n, a_m) < \varepsilon$ for all $n, m \ge N(\varepsilon)$

(iii) The space is called complete if every Cauchy sequence in X is convergent in X

Definition 7. Let (X, d_θ) be a new extended b -metric space. Denote by $\mathscr{B}(a,r) = \{b \in X : d_\theta(b,a) < r\}$ and $\mathscr{B}[a,r] = \{b \in X : d_\theta(b,a) \le r\}$ the open and closed balls in X, respectively.

(i) A subset U of X is called open if for any $u \in U$, there exists an $\varepsilon > 0$ such that $\mathscr{B}(u,\varepsilon) \subset U$

(ii) A subset V of X is called closed if for any $\{v_n\} \subset V$ such that $\lim_{n\longrightarrow\infty} v_n = v$, then $v \in X$

In this paper, we are going to prove some Kannan-type fixed point theorems in the setting of new extended b-metric spaces. Some examples are also provided to make effective the obtained results.

2. Main Results

We define a Kannan-type fixed point contraction on new extended b-metric spaces.

Definition 8. Let (X, d_θ) be a new extended b-metric space. A mapping $T : X \longrightarrow X$ is a Kannan-type contraction if there are $K \in [0, 1/2)$ and $0 \le L < 1$ such that

$$\begin{aligned} d_\theta(T(a), T(b)) \le K[d_\theta(a, T(a)) + d_\theta(b, T(b))] \\ + L d_\theta(b, T(a)), \quad \text{for all } a, b \in X. \end{aligned} \tag{6}$$

Our first main result is as follows:

Theorem 9. *Consider a complete new extended b-metric space (X, d_θ) such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be a mapping such that there are $\xi \in (0, 1/2)$ and $L \in [0, 1)$ so that*

$$\begin{aligned} d_\theta(T(a), T(b)) \le \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))] \\ + L d(b, T(a)), \quad \forall a, b \in X. \end{aligned} \tag{7}$$

Assume that

$$\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1-\xi}{\xi}, \tag{8}$$

for each $a_0 \in X$, such that $a_n = T^n(a_0)$, $n = 1, 2, \cdots$.

Then, T has only one fixed point $v \in X$. Further, for any $a \in X$, the iterative sequence $\{T^n(a)\}$ converges to v, and

$$d_\theta(T(a_n), v) \le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \cdots. \tag{9}$$

Proof. For an arbitrary point $a_0 \in X$, construct the iterative sequence

$$a_{n+1} = T(a_n) = T^{n+1}(a_0). \tag{10}$$

If for some n, $a_n = a_{n+1}$, so a_n is a fixed point of T. Otherwise, assume that $a_n \ne a_{n+1}$ for all $n \ge 0$. Since

$$d_\theta(a_n, a_{n+1}) = d_\theta(T(a_{n-1}), T(a_n)), \tag{11}$$

one writes

$$\begin{aligned} d_\theta(T(a_{n-1}), T(a_n)) \le \xi[d_\theta(a_{n-1}, T(a_{n-1})) + d_\theta(a_n, T(a_n))] \\ + L d_\theta(a_n, T a_{n-1}). \end{aligned} \tag{12}$$

Then,

$$d_\theta(a_n, a_{n+1}) \le \xi[d_\theta(a_{n-1}, a_n) + d_\theta(a_n, a_{n+1})] + L d_\theta(a_n, a_n). \tag{13}$$

That is,

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right) d_\theta(a_{n-1}, a_n). \tag{14}$$

Continuing in this way, we have

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1), \tag{15}$$

$$d_\theta(T(a_{n-1}), T(a_n)) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, T(a_0)). \tag{16}$$

Let $m, n \in \mathbb{N}$ be such that $m > n$. Applying triangular inequality, we get

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)[d_\theta(a_n, a_{n+1}) + d_\theta(a_{n+1}, a_m)] \\ &= \theta(a_n, a_{n+1}, a_m)d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)d_\theta(a_{n+1}, a_m) \\ &\le \theta(a_n, a_{n+1}, a_m)d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)\,[d_\theta(a_{n+1}, a_{n+2}) + d_\theta(a_{n+2}, a_m)] \\ &\le \theta(a_n, a_{n+1}, a_m)d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)d_\theta(a_{n+1}, a_{n+2}) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)\theta(a_{n+2}, a_{n+3}, a_m)\cdots\theta \\ &\quad \cdot (a_{m-2}, a_{m-1}, a_m)d_\theta(a_{m-1}, a_m). \end{aligned} \tag{17}$$

Since

$$d_\theta(a_n, a_{n+1}) \le \left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1)\; n = 0, 1, 2, \cdots, \tag{18}$$

one writes

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right)^n d_\theta(a_0, a_1) + \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\left(\frac{\xi}{1-\xi}\right)^{n+1} d_\theta \\ &\quad \cdot (a_0, a_1) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta \\ &\quad \cdot (a_{n+2}, a_{n+3}, a_m)\cdots\theta(a_{m-2}, a_{m-1}, a_m)\left(\frac{\xi}{1-\xi}\right)^{m-1} d_\theta(a_0, a_1) \\ &\le \Bigg\{\theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m)\cdots\theta(a_{n-1}, a_n, a_m)\theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right)^n + \theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m)\cdots\theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\left(\frac{\xi}{1-\xi}\right)^{n+1} + \cdots + \theta \\ &\quad \cdot (a_1, a_2, a_m)\theta(a_2, a_3, a_m)\cdots\theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)\cdots\theta(a_{m-3}, a_{m-2}, a_m)\theta(a_{m-2}, a_{m-1}, a_m) \\ &\quad \cdot \left(\frac{\xi}{1-\xi}\right)^{m-1}\Bigg\} d_\theta(a_0, a_1). \end{aligned} \tag{19}$$

Since

$$\sup_{m\ge1} \lim_{n,m\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m)\left(\frac{\xi}{1-\xi}\right) < 1, \tag{20}$$

in view of (8), the series $\sum_{n=1}^{\infty}(\xi/(1-\xi))^n \prod_{i=1}^{n}\theta(a_i, a_{i+1}, a_m)$ is convergent for each $m \in \mathbb{N}$ by ratio test.

Let $S = \sum_{n=1}^{\infty}(\xi/(1-\xi))^n \prod_{i=1}^{n}\theta(a_i, a_{i+1}, a_m)$ and

$$S_n = \sum_{j=1}^{n}\left(\frac{\xi}{1-\xi}\right)^j \prod_{i=1}^{j}\theta(a_i, a_{i+1}, a_m). \tag{21}$$

So, for $m > n$, the above inequality implies that

$$d_\theta(a_n, a_m) \le d_\theta(a_0, a_1)(S_{m-1} - S_{n-1}). \tag{22}$$

That is,

$$\lim_{n\longrightarrow\infty} d_\theta(a_n, a_m) = 0. \tag{23}$$

Hence, the sequence $\{a_n\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that $\lim_{n\longrightarrow\infty} a_n = v$.

We claim that v is a fixed point of T. We have

$$d_\theta(T(a_n), T(v)) \le \xi(d_\theta(a_n, T(a_n)) + d_\theta(v, T(v)) + Ld_\theta(v, T(a_n)). \tag{24}$$

That is,

$$d_\theta(a_{n+1}, T(v)) \le \xi(d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v))) + Ld_\theta(v, a_{n+1}). \tag{25}$$

As $n \longrightarrow \infty$, we have in view of the assumption that d_θ is continuous,

$$d_\theta(v, T(v)) \le \xi d_\theta(v, Tv), \tag{26}$$

which holds unless $d_\theta(v, T(v)) = 0$, and so, $T(v) = v$.

The uniqueness is as follows:

Let τ be another fixed point of T. We have

$$\begin{aligned} 0 \le d_\theta(v, \tau) = d_\theta(T(v), T(\tau)) &\le \xi[d_\theta(v, T(v)) + d_\theta(\tau, T(\tau))] \\ &\quad + Ld_\theta(\tau, T(v)). \end{aligned} \tag{27}$$

That is,

$$d_\theta(v, \tau) \le Ld_\theta(v, \tau). \tag{28}$$

It is only possible if $d_\theta(v, \tau) = 0$. Thus, $v \in X$ is the unique fixed point of T. Further, we have

$$\begin{aligned} d_\theta(T(a_{n-1}), T(a_n)) &\le \xi[d_\theta(T(a_{n-2}), T(a_{n-1})) + d_\theta(T(a_{n-1}), T(a_n))] \\ &\quad + Ld_\theta(a_n, a_n). \end{aligned} \tag{29}$$

Then,

$$d_\theta(T(a_{n-1}), T(a_n)) \le \left(\frac{\xi}{1-\xi}\right) d_\theta(T(a_{n-2}), T(a_{n-1})). \tag{30}$$

Also,

$$\begin{aligned} d_\theta(T(a_n), v) &\le \xi[d_\theta(T(a_{n-1}), T(a_n)) + d_\theta(v, T(v))] + Ld_\theta(v, T(a_n)) \\ &\le \xi d_\theta(T(a_{n-1}), T(a_n)) + Ld_\theta(v, T(a_n)). \end{aligned} \tag{31}$$

Using (15),

$$d_\theta(T(a_n), \nu) \le \xi \left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0) + L d_\theta(\nu, T(a_n)). \tag{32}$$

That is,

$$d_\theta(T(a_n), \nu) \le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a_0), a_0), \quad n = 0, 1, 2, \cdots. \tag{33}$$

□

The following examples illustrate Theorem 9. We deal with noncompact sets.

Example 10. Let $X = \ell_\infty$ be the space of all bounded sequences of real numbers, that is,

$$\ell_\infty = \{\eta = \{\eta_n\}: |\eta_n| \le C_\eta, \forall n \in \mathbb{N}\}, \tag{34}$$

where $C_\eta \in \mathbb{R}$ may depend on the sequence η but does not depend on n. Take that

$$\begin{aligned} d_\theta(\eta, \zeta) &= \sup_{n\in\mathbb{N}} |\eta_n - \zeta_n|^2, \\ \eta &= \{\eta_n\}, \\ \zeta &= \{\zeta_n\}, \end{aligned} \tag{35}$$

are in X.

Then, X is a complete new extended b-metric space with $\theta : X \times X \times X \longrightarrow [1,\infty)$ being defined by

$$\theta(\eta, \zeta, \varphi) = \sup_{n\in\mathbb{N}} \frac{|\eta_n + \zeta_n + \varphi_n|}{|\eta_n + \zeta_n + \varphi_n| + 1} + 3. \tag{36}$$

Consider $T : X \longrightarrow X$ given as $T(\eta) = \{(\eta_n - 1)/5\}, \forall n = 1, 2, 3, \cdots$. For each $\eta, \zeta \in X$, we have

$$\begin{aligned} d_\theta(T\eta, T\zeta) &= \sup_{n\in\mathbb{N}} \left|\frac{\eta_n - 1}{5} - \frac{\zeta_n - 1}{5}\right|^2 = \sup_{n\in\mathbb{N}} \left|\frac{\eta_n}{5} - \frac{\zeta_n}{5}\right|^2 \\ &\le \sup_{n\in\mathbb{N}} 2\left|\frac{\eta_n}{5}\right|^2 + \sup_{n\in\mathbb{N}} 2\left|\frac{\zeta_n}{5}\right|^2 = \frac{1}{8}\sup_{n\in\mathbb{N}} 16\left|\frac{\eta_n}{5}\right|^2 \\ &\quad + \frac{1}{8}\sup_{n\in\mathbb{N}} 16\left|\frac{\zeta_n}{5}\right|^2 = \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\eta_n}{5}\right|^2 + \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\zeta_n}{5}\right|^2 \\ &\le \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\eta_n + 1}{5}\right|^2 + \frac{1}{8}\sup_{n\in\mathbb{N}}\left|\frac{4\zeta_n + 1}{5}\right|^2 \\ &= \frac{1}{8}[d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta)]. \end{aligned} \tag{37}$$

Thus, (7) holds with $\xi = 1/8$ and $L \in [0, 1)$. Also, $(1-\xi)/\xi = 7$ and $\theta(\eta, \zeta, \varphi) < 7$ for all $\eta, \zeta, \varphi \in X$. Hence, (8) holds, and by Theorem 9, T has a fixed point.

Example 11. Let $X = C[a, b]$ be the set of all real-valued continuous functions defined on [a, b]. Define

$$d_\theta(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b], \tag{38}$$

and $\theta : X \times X \times X \longrightarrow [1,\infty)$ as

$$\theta(x, y, z) = \sup_{t\in[a,b]} \frac{|x(t)| + |y(t)| + |z(t)|}{|x(t)| + |y(t)| + |z(t)| + 1} + 2. \tag{39}$$

Then, X is a complete new extended b-metric space, consider a mapping $T : X \longrightarrow X$ given as

$$T(\eta(t)) = \frac{\eta(t) - 1}{7}, \quad \forall \eta \in C[a, b]. \tag{40}$$

For all $\eta, \zeta \in X$, we have

$$\begin{aligned} d_\theta(T\eta, T\zeta) &= \sup_{t\in[a,b]} \left|\frac{\eta(t) - 1}{7} - \frac{\zeta(t) - 1}{7}\right|^2 = \sup_{t\in[a,b]} \left|\frac{\eta(t)}{7} - \frac{\zeta(t)}{7}\right|^2 \\ &\le \sup_{t\in[a,b]} 2\left|\frac{\eta(t)}{7}\right|^2 + \sup_{t\in[a,b]} 2\left|\frac{\zeta(t)}{7}\right|^2 = \frac{1}{18}\sup_{t\in[a,b]} 36\left|\frac{\eta(t)}{7}\right|^2 \\ &\quad + \frac{1}{18}\sup_{t\in[a,b]} 36\left|\frac{\zeta(t)}{7}\right|^2 = \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\eta(t)}{7}\right|^2 + \frac{1}{18}\sup_{n\in\mathbb{N}}\left|\frac{6\zeta(t)}{7}\right|^2 \\ &\le \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\eta(t) + 1}{5}\right|^2 + \frac{1}{18}\sup_{t\in[a,b]}\left|\frac{6\zeta(t) + 1}{5}\right|^2 \\ &< \frac{1}{6}[d_\theta(\eta, T\eta) + d_\theta(\zeta, T\zeta)]. \end{aligned} \tag{41}$$

Thus, (7) holds with $\xi = 1/6$ and $L \in [0, 1)$. Also, $(1-\xi)/\xi = 5$ and $\theta(\eta, \zeta, \varphi) < 5$ for all $\eta, \zeta, \varphi \in X$. That is, (8) holds. Since all the conditions of Theorem 9 are satisfied, T has a fixed point.

Example 12. Choose $X = \{1/4, 1/16, \cdots, 1/4^n, \cdots\} \cup \{0, 1\}$. Define $\theta : X \times X \times X \longrightarrow [1,\infty)$ and $d_\theta : X \times X \longrightarrow [0,\infty)$ by

$$\theta(x, y, z) = x + y + z + 2, d_\theta(x, y) = (x - y)^2. \tag{42}$$

Let $T : X \longrightarrow X$ be given as

$$Tu = \begin{cases} \dfrac{1}{4^{n+1}}, & \text{if } u = \dfrac{1}{4^n}, n = 0, 1, 2, 3, \cdots, \\ u, & \text{if } u = 0. \end{cases} \tag{43}$$

Then, for all $x, y \in X$ with neither $x = 0$ nor $y = 0$, we have

$$d_\theta(Tx, Ty) = \left|\frac{1}{4^{n+1}} - \frac{1}{4^{m+1}}\right|^2 \le 2\left|\frac{1}{4^{n+1}}\right|^2 + 2\left|\frac{1}{4^{m+1}}\right|^2 = \frac{2}{9}\left|\frac{3}{4^{n+1}}\right|^2 + \frac{2}{9}\left|\frac{3}{4^{m+1}}\right|^2 = \frac{2}{9}[d_\theta(x, Tx) + d_\theta(y, Ty)]. \quad (44)$$

If $x = 0$ and $y \ne 0$, then

$$d_\theta(Tx, Ty) = \left|0 - \frac{1}{4^{m+1}}\right|^2 = \left|\frac{1}{4^{m+1}}\right|^2 = \frac{2}{9}\left|\frac{3}{4^{m+1}}\right|^2 = \frac{2}{9}[d_\theta(x, Tx) + d_\theta(y, Ty)]. \quad (45)$$

Thus, (7) is satisfied for $\xi = 2/9$ and for each $L \in [0, 1)$. Also, $(1-\xi)/\xi = 7/2$. If $a_0 = 0$, then, for the iterative sequence $a_n = T^n a_0 = 0$ for each $n \in \mathbb{N}$, we have $\lim_{n,m\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) = \lim_{n,m\longrightarrow\infty} a_n + a_{n+1} + a_m + 2 < 7/2$. If $a_0 \ne 0$ (say $a_0 = 1/4^k$ for some $k \in \{0, 1, 2, \cdots\}$); then, for the iterative sequence $a_n = T^n a_0 = 1/4^{k+n}$ for each $n \in \mathbb{N}$, we have $\lim_{n,m\longrightarrow\infty}\theta(a_n, a_{n+1}, a_m) = \lim_{n,m\longrightarrow\infty} a_n + a_{n+1} + a_m + 2 < 7/2$. Hence, Theorem 9 ensures the existence of a fixed point of T.

Remark 13. In the following, we ensure the completeness of the spaces given in precedent examples.

(a) Completeness of ℓ_∞

Let $X = \ell_\infty$ and let $\{x_m\} = \{\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)}, \cdots\}$ be a Cauchy sequence in X. Define a metric on X as $d_\theta(x, y) = \sup_j |\xi_j - \eta_j|^2$, where $x = (\xi_j)$ and $y = (\eta_j)$. Since $\{x_m\}$ is a Cauchy sequence, for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \ge N$,

$$d_\theta(x_n, x_m) = \sup_j \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^2 < \varepsilon \forall m, n \ge N \Longrightarrow \left|\xi_j^{(m)} - \xi_j^{(n)}\right|^2 < \varepsilon, \quad \forall j \text{ and } m, n \ge N. \quad (46)$$

That is,

$$\left|\xi_j^{(m)} - \xi_j^{(n)}\right| < \varepsilon_1, \quad \forall j \text{ and } m, n \ge N, \quad (47)$$

where $\varepsilon_1 = \sqrt{\varepsilon}$ is arbitrary. Hence, for every fixed j, $(\xi_j^{(1)}, \xi_j^{(2)}, \cdots)$ is a Cauchy sequence of complex numbers and it converges, so $\xi_j^{(m)} \longrightarrow \xi_j$ as $m \longrightarrow \infty$. Construct a sequence $x = (\xi_1, \xi_2, \cdots)$ by using these infinitely many limits to show that $x \in \ell_\infty$ and $x_m \longrightarrow x$. From (48) with $n \longrightarrow \infty$, we have

$$\left|\xi_j^{(m)} - \xi_j\right| < \varepsilon_1, \quad m \ge N. \quad (48)$$

Since $x_m = \xi_j^{(m)} \in \ell_\infty$, there is a real number c_m so that $|\xi_j^{(m)}| \le c_m$, $\forall j$. Hence,

$$|\xi_j|^2 = \left|\xi_j - \xi_j^{(m)} + \xi_j^{(m)}\right|^2 \le \left(\left|\xi_j - \xi_j^{(m)}\right| + \left|\xi_j^{(m)}\right|\right)^2 \le 2\left(\left|\xi_j - \xi_j^{(m)}\right|^2 + \left|\xi_j^{(m)}\right|^2\right). \quad (49)$$

That is,

$$\left|\xi_j\right| \le \sqrt{2\varepsilon + 2(c_m)^2}. \quad (50)$$

So, (51) holds for each j. It implies that $\{\xi_j\}$ is a bounded sequence of complex numbers. This leads to $\{\xi_j\} \in \ell_\infty$. Also, from (48), we have

$$d_\theta(x_m, x) = \sup_j \left|\xi_j^{(m)} - \xi_j\right|^2 \le \varepsilon, \quad m > N. \quad (51)$$

This implies that $x_m \longrightarrow x \in \ell_\infty$ (ℓ_∞ is endowed with the new extended metric d_θ).

(b) Completeness of $C[a, b]$

Let X be the function space $C[a, b]$, where $[a, b]$ is any closed interval in $\mathbb{R}$. Define

$$d_\theta(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2, \quad \forall x, y \in C[a, b]. \quad (52)$$

Let $\{x_m\}$ be a Cauchy sequence in $C[a, b]$. Then, for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d_\theta(x_m, x_n) = \sup_{t\in[a,b]} |x_m(t) - x_n(t)|^2 < \varepsilon, \quad \forall m, n \ge N. \quad (53)$$

Hence, for each fixed $t_0 \in [a, b]$, we have

$$|x_m(t_0) - x_n(t_0)|^2 < \varepsilon, \quad \forall m, n \ge N. \quad (54)$$

That is,

$$|x_m(t_0) - x_n(t_0)| < \sqrt{\varepsilon} = \varepsilon_1, \quad (\text{say}) \forall m, n \ge N. \quad (55)$$

This shows that $(x_1(t_0), x_2(t_0), \cdots)$ is a Cauchy sequence of real numbers; hence, it converges. That is, $x_m(t_0) \longrightarrow x(t_0)$ as $m \longrightarrow \infty$. In this way, we can associate to each $t \in [a, b]$ a unique real number $x(t)$. This defines a function x (pointwise) in $[a, b]$. Further, we need to show that $x_m \longrightarrow x \in C[a, b]$. From (53), we have as $n \longrightarrow \infty$

$$\sup_{t\in[a,b]} |x_m(t) - x(t)|^2 < \varepsilon, \quad m \ge N. \quad (56)$$

Hence, for each $t \in [a, b]$ and $m > N$, we have

$$|x_m(t) - x(t)|^2 < \varepsilon, \quad (57)$$

which implies that $|x_m(t) - x(t)| < \varepsilon_1$. Thus, $\{x_m(t)\}$ converges uniformly to $x(t)$ on the interval $[a, b]$. Since $t \longrightarrow x_m(t)$ is continuous and the convergence is uniform; hence,

$x(t)$ is continuous. This leads to the completeness of $C[a, b]$ with the new extended metric d_θ.

Theorem 14. *Let (X, d_θ) be a complete new extended b-metric space where d_θ is a continuous functional and M is a nonempty closed subset of X. Let $T : M \longrightarrow M$ satisfy*

$$d_\theta(T(a), T(b)) \le \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))] + Ld_\theta(b, T(a)), \quad \forall a, b \in M \text{ and } L, \xi \in [0, 1). \tag{58}$$

Assume that there exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < 1$ and $\beta > 0$ such that for an arbitrary $a \in M$, there is $\mu^* \in M$ verifying

$$d_\theta(\mu^*, T(\mu^*)) \le \alpha d_\theta(a, T(a)), d_\theta(\mu^*, a) \le \beta d_\theta(a, T(a)). \tag{59}$$

Also, for an arbitrary $a_0 \in M$, assume that the sequence $\{a_n = T^n(a_0)\}$ verifies

$$\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1}{\alpha}. \tag{60}$$

Then, T has only one fixed point.

Proof. Let a_0 be an arbitrary element of M. Consider the sequence $\{a_n = T^n(a_0)\}$ in M. We have

$$d_\theta(T(a_{n+1}), a_{n+1}) \le \alpha d_\theta(T(a_n), a_n), d_\theta(a_{n+1}, a_n) \le \beta d_\theta(T(a_n), a_n), \quad n = 0, 1, 2, \cdots. \tag{61}$$

Since

$$\begin{gathered} d_\theta(a_{n+1}, a_n) = d_\theta(T(a_n), a_n) \le \beta d_\theta(T(a_n), a_n), \quad n = 0, 1, 2, \cdots, \\ \beta d_\theta(T(a_n), a_n) \le \beta\alpha d_\theta(T(a_{n-1}), a_{n-1}) \le \beta\alpha^2 d_\theta(T(a_{n-2}), a_{n-2}) \le \cdots \le \beta\alpha^n d_\theta(T(a_0), a_0), \end{gathered} \tag{62}$$

we have

$$d_\theta(a_{n+1}, a_n) \le \beta\alpha^n d_\theta(T(a_0), a_0). \tag{63}$$

Let $m, n \in \mathbb{N}$ such that $m > n$. Applying triangular inequality, we get

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)[d_\theta(a_n, a_{n+1}) + d_\theta(a_{n+1}, a_m)] \\ &= \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m) d_\theta \cdot (a_{n+1}, a_m) \le \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)(d_\theta(a_{n+1}, a_{n+2}) + d_\theta(a_{n+2}, a_m)) \\ &\le \theta(a_n, a_{n+1}, a_m) d_\theta(a_n, a_{n+1}) + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m) d_\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}) + \cdots + \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta \\ &\quad \cdot (a_{n+2}, a_{n+3}, a_m) \cdots \theta(a_{m-2}, a_{m-1}, a_m) d_\theta(a_{m-1}, a_m). \end{aligned} \tag{64}$$

Using (63), we get

$$\begin{aligned} d_\theta(a_n, a_m) &\le \theta(a_n, a_{n+1}, a_m)\beta\alpha^n d_\theta(T(a_0), a_0) + \theta(a_n, a_{n+1}, a_m)\theta \\ &\quad \cdot (a_{n+1}, a_{n+2}, a_m)\beta\alpha^{n+1} d_\theta(T(a_0), a_0) + \cdots + \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\theta(a_{n+2}, a_{n+3}, a_m) \cdots \theta \\ &\quad \cdot (a_{m-2}, a_{m-1}, a_m)\, \beta\alpha^{m-1} d_\theta(T(a_0), a_0) \\ &\le [\theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta(a_{n-1}, a_n, a_m)\theta \\ &\quad \cdot (a_n, a_{n+1} a_m)\alpha^n + \theta(a_1, a_2, a_m)\theta(a_2, a_3, a_m) \cdots \theta \\ &\quad \cdot (a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2}, a_m)\alpha^{n+1} + \cdots + \theta(a_1, a_2, a_m)\theta \\ &\quad \cdot (a_2, a_3, a_m) \cdots \theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{n+2} a_m) \cdots \theta \\ &\quad \cdot (a_{m-2}, a_{m-1}, a_m)\alpha^{m-1}]\, \beta d_\theta(T(a_0), a_0). \end{aligned} \tag{65}$$

Since $\sup_{m\ge 1} \lim_{n,m\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m)\alpha < 1$, the series $\sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \theta(a_i, a_{i+1}, a_m)$ is convergent for each $m \in \mathbb{N}$ by ratio test.

Let

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \alpha^n \prod_{i=1}^{n} \theta(a_i, a_{i+1}, a_m), \\ S_n &= \sum_{j=1}^{n} \alpha^j \prod_{i=1}^{j} \theta(a_i, a_{i+1}, a_m). \end{aligned} \tag{66}$$

So, for $m > n$, the above inequality implies that

$$d_\theta(a_n, a_m) \le d_\theta(a_0, a_1)(S_{m-1} - S_{n-1}). \tag{67}$$

Letting $n \longrightarrow \infty$, the sequence $\{a_n\}$ is a Cauchy sequence. By the completeness of M, there is $v \in M$ such that $\lim_{n\longrightarrow\infty} a_n = v$.

We will prove that v is a fixed point of T. By using (58), we get

$$\begin{aligned} & d_\theta(T(a_n), T(v)) \le \xi(d_\theta(a_n, T(a_n)) + d_\theta(v, Tv)) + Ld_\theta(v, T(a_n)) \\ & \Longrightarrow d_\theta(a_{n+1}, T(v)) \le \xi(d_\theta(a_n, a_{n+1}) + d_\theta(v, Tv)) + Ld_\theta(v, a_{n+1}). \end{aligned} \tag{68}$$

As $n \longrightarrow \infty$, we have

$$d_\theta(v, T(v)) \le \xi d_\theta(v, T(v)). \tag{69}$$

Hence, $T(v) = v$.

The uniqueness is as follows:

Assume on contrary that there is $\tau (\ne v) \in M$ so that $T(\tau) = \tau$; then,

$$0 < d_\theta(v, \tau) = d_\theta(T(v), T(\tau)) \le \xi[d_\theta(v, T(v)) + d_\theta(\tau, T(\tau))] + Ld_\theta(\tau, T(v)) = Ld_\theta(\tau, v). \tag{70}$$

That is, $d_\theta(v, \tau) < Ld_\theta(v, \tau)$, which is a contradiction. Thus, $v \in X$ is the unique fixed point of T. □

Remark 15. To prove Theorem 9 in new extended b-metric spaces, by using the following conditions:

$$\begin{aligned} d_\theta(\mu^*, T(\mu^*)) &\le \alpha d_\theta(a, T(a)), \\ d_\theta(\mu^*, a) &\le \beta d_\theta(a, T(a)), \end{aligned} \tag{71}$$

we proceed as follows:

For any $a \in X$, take $\mu^* = T(a)$. Then,

$$\begin{aligned} d_\theta(\mu^*, T(\mu^*)) = d_\theta(T(a), T(\mu^*)) &\le \xi[d_\theta(a, T(a)) + d_\theta(\mu^*, T(\mu^*))] \\ &\quad + L d_\theta(\mu^*, T(a))\,(1-\xi) d_\theta(\mu^*, T(\mu^*)) \\ &\le \xi d_\theta(a, T(a)) d_\theta(\mu^*, T(\mu^*)) \le \left(\frac{\xi}{1-\xi}\right) d_\theta(a, T(a)), \end{aligned} \tag{72}$$

where by assumption $(\xi/1-\xi) < 1$ and $d_\theta(\mu^*, a) = d_\theta(T(a), a)$. Now, for arbitrary $a_0 \in X$, we can inductively define a sequence $\{a_{n+1} = T(a_n)\}$. By Theorem 14, this sequence is convergent. So, $\lim_{n\longrightarrow\infty} a_n = v$. Thus, $T(v) = v$.

Also, for each $a \in X$,

$$\begin{aligned} d_\theta(T(a_{n-1}), T(a_n)) &\le \xi[d_\theta(T(a_{n-2}), T(a_{n-1})) + d_\theta(T(a_{n-1}), T(a_n))] \\ &\quad + L d_\theta(a_n, T(a_{n-1})) \Longrightarrow d_\theta(T(a_{n-1}), T(a_n)) \\ &\le \left(\frac{\xi}{1-\xi}\right) d_\theta(T(a_{n-2}), T(a_{n-1})), \end{aligned}$$

$$\begin{aligned} d_\theta(T(a_n), v) &\le \xi[d_\theta(T(a_{n-1}), T(a_n)) + d_\theta(v, T(v))] + L d_\theta(v, T(a_n)) \\ &\le \xi d_\theta(T(a_{n-1}), T(a_n)) + L d_\theta(v, T(a_n)) \Longrightarrow d_\theta(T(a_n), v) \\ &\le \frac{\xi}{1-L}\left(\frac{\xi}{1-\xi}\right)^n d_\theta(T(a), a), \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{73}$$

Theorem 16. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ satisfy*

$$\begin{aligned} d_\theta(T(a), T(b)) &\le \alpha d_\theta(a, T(a)) + \beta d_\theta(b, T(b)) + \gamma d_\theta(a, b) \\ &\quad + L d_\theta(a, T(b)), \quad \forall a, b \in X, \end{aligned} \tag{74}$$

where α, β, γ, L are nonnegative real numbers such that $\alpha + \beta + \gamma + L < 1$ and $\beta + \gamma > 0$. Assume that for an arbitrary $a_0 \in M$, we have

$$\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \theta(a_n, a_{n+1}, a_m) < \frac{1}{p}, \tag{75}$$

where $p = ((\beta+\gamma)/(1-\alpha))$ and $a_n = T^n(a_0)$.

Then, T has only one fixed point.

Proof. For an arbitrary $a_0 \in X$, take the sequence $\{T^n(a_0)\}$. Substituting $a = T^{n-1}(a_0) = T(a_{n-1}) = a_n$ and $b = T^{n-2}(a_0) = T(a_{n-2}) = a_{n-1}$ in (74), we obtain

$$\begin{aligned} d_\theta(T(a_n), T(a_{n-1})) &\le \alpha d_\theta(T(a_{n-1}), T(a_n)) + \beta d_\theta(T(a_{n-2}), T(a_{n-1})) \\ &\quad + \gamma d_\theta(T(a_{n-1}), T(a_{n-2})) + L d_\theta(T(a_{n-1}), T(a_{n-1})). \end{aligned} \tag{76}$$

That is,

$$(1-\alpha) d_\theta(T(a_n), T(a_{n-1})) \le (\beta+\gamma) d_\theta(T(a_{n-1}), T(a_{n-2})). \tag{77}$$

Thus,

$$d_\theta(T(a_n), T(a_{n-1})) \le \left(\frac{\beta+\gamma}{1-\alpha}\right) d_\theta(T(a_{n-1}), T(a_{n-2})). \tag{78}$$

Moreover,

$$\begin{aligned} d_\theta(T(a_n), T(a_{n-1})) &\le p d_\theta(T(a_{n-1}), T(a_{n-2})) \\ &\le p^2 d_\theta(T(a_{n-2}), T(a_{n-3})) \vdots \\ &\le p^{n-1} d_\theta(T(a_1), T(a_0)), \quad \forall n > 1. \end{aligned} \tag{79}$$

Thus, we reach

$$d_\theta(T(a_n), T(a_{n-1})) \le p^n d_\theta(a_0, a_1), \quad \forall n \in \mathbb{N}. \tag{80}$$

By assumption on the parameters α, β, and γ, one has $p = ((\beta+\gamma)/(1-\alpha)) < 1$.

Following the same steps as given in Theorem 14, one can show that $\{a_n\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that $T^n(a_0) \longrightarrow v$. To prove $T(v) = v$, replace $a = T^n(a_0)$ and $b = v$ in (74). We have

$$\begin{aligned} d_\theta\left(T^{n+1}(a_0), T(v)\right) &\le \alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v)) \\ &\quad + \gamma d_\theta(T^n(a_0), v) + L d_\theta(T^n(a_0), T(v)). \end{aligned} \tag{81}$$

Then,

$$\begin{aligned} d_\theta(a_{n+1}, T(v)) &\le \alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v)) \\ &\quad + \gamma d_\theta(a_n, v) + L d_\theta(Tv, a_{n+1}). \end{aligned} \tag{82}$$

That is,

$$\begin{aligned} \lim_{n\longrightarrow\infty} d_\theta(a_{n+1}, T(v)) &\le \lim_{n\longrightarrow\infty} \left(\alpha d_\theta\left(T^n(a_0), T^{n+1}(a_0)\right) + \beta d_\theta(v, T(v))\right. \\ &\quad \left. + \gamma d_\theta(a_n, v) + L d_\theta(T(v), a_{n+1})\right). \end{aligned} \tag{83}$$

We have $d_\theta(\nu, T(\nu)) \leq (\beta + L)d_\theta(\nu, T(\nu))$, which holds unless $T(\nu) = \nu$.

The uniqueness is as follows:

Let τ and ν be two fixed points, such that $\tau \neq \nu$. Then, using inequality (74), we get

$$d_\theta(T(\tau), T(\nu)) \leq [\alpha d_\theta(\tau, T(\tau)) + \beta d_\theta(\nu, T(\nu)) + \gamma d_\theta(\tau, \nu)] + L d_\theta(T(\nu), \tau) d_\theta(\tau, \nu) \leq [(L + \gamma) d_\theta(\tau, \nu)] 1 \leq L + \gamma, \tag{84}$$

which is a contradiction. Hence, T has only one fixed point. □

Now, we use the concept of an asymptotically regular mapping [31, 32] in new extended b-metric spaces.

Definition 17. Let (X, d_θ) be a new extended b-metric space. A mapping $T : X \longrightarrow X$ satisfying the condition

$$\lim_{n\longrightarrow\infty} d_\theta\left(T^{n+1}a, T^n a\right) = 0, \quad \text{for all } a \in X, \tag{85}$$

is called asymptotically regular.

Example 18. Let $X = \{0\} \cup [1, 3]$. Define $T : X \longrightarrow X$ by $T0 = 1$ and $Tx = 0$, for $0 < x \leq 3$. Consider $d_\theta(x, y) = (x - y)^2$ and $\theta(x, y, z) = (x + y + z + 11)/(x + y + z + 1)$. We claim that T satisfies condition (7). Indeed,

Case 1. If $x = y = 0$, then, (7) gives $0 \leq 2\xi + L$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$.

Case 2. If $x \in [1, 3]$, $y = 0$, then, (7) gives $1 \leq \xi(1 + x^2)$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$.

Case 3. If $x, y \in [1, 3]$, then, (7) implies that $0 \leq \xi(x^2 + y^2) + Ly^2$, which is true for all $\xi \in [0, 1/2)$ and $L \in (0, 1)$. Notice that T is fixed point free. The iterative sequence $\{x_n = T^n 0\}$ is not convergent, so T is not asymptotically regular.

Theorem 19. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be an asymptotically regular self mapping such that there is $\xi < 1$ so that*

$$d_\theta(T(a), T(b)) \leq \xi[d_\theta(a, T(a)) + d_\theta(b, T(b))], \quad \forall a, b \in X. \tag{86}$$

Then, T has only one fixed point $\nu \in X$.

Proof. Let $a \in X$ and take $a_n = T^n(a)$ be defined inductively. Let $m, n \in \mathbb{N}$ such that $m > n$; then, according to asymptotic regularity,

$$d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) \leq \xi\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{87}$$

Thus, the sequence $\{T^n(a)\}$ is a Cauchy sequence. By the completeness of X, there is $\nu \in X$ such that

$$\lim_{n\longrightarrow\infty} T^n(a) = \nu. \tag{88}$$

To prove that ν is a fixed point of T, we proceed as follows:

$$d_\theta(T(a_n), T(\nu)) \leq \xi(d_\theta(a_n, T(a_n)) + d_\theta(\nu, T(\nu))). \tag{89}$$

That is,

$$d_\theta(T(a_n), T(\nu)) \leq \xi(d_\theta(a_n, a_{n+1}) + d_\theta(\nu, T(\nu))). \tag{90}$$

Taking limit $n \longrightarrow \infty$ and using the asymptotic regularity of T, one obtains

$$d_\theta(\nu, T(\nu)) \leq \xi d_\theta(\nu, T(\nu)), \tag{91}$$

which implies that $T(\nu) = \nu$.

To prove uniqueness, let τ be another fixed point of T. We have

$$d_\theta(\nu, \tau) = d_\theta(T(\nu), T(\tau)) \leq \xi(d_\theta(\nu, T(\nu)) + d_\theta(\tau, T(\tau))). \tag{92}$$

This is true unless $d_\theta(\nu, \tau) = 0$, and so, $\nu = \tau$. Hence, ν is the only fixed point of T. Further, for each $a \in X$, the iterative sequence $\{T^n(a)\}$ converges to ν. □

Remark 20. It is noteworthy that if the mapping is asymptotically regular, then, the condition on $\theta(a_n, a_{n+1}, a_m)$ can be relaxed.

Theorem 21. *Let (X, d_θ) be a complete new extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ be an asymptotically regular mapping such that there is $0 < S < 1$ so that*

$$d_\theta(T(a), T(b)) \leq S[d_\theta(a, T(a)) + d_\theta(b, T(b)) + d_\theta(a, b)], \quad \forall a, b \in X. \tag{93}$$

Then, T has only one fixed point $\nu \in X$ provided that

$$\lim_{n\longrightarrow\infty} \frac{S + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)} \tag{94}$$

exists for $a_n = T^n(a)$, $m > n$ and a is arbitrary in X.

Proof. Let $a \in X$ and take $a_n = T^n(a)$ defined inductively. Let $m, n \in \mathbb{N}$ such that $m > n$; then, using (93), we have

$$\begin{aligned} d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) &\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right) + d_\theta\left(T^n(a), T^m(a)\right)\right] \\ &= S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + Sd_\theta\left(T^n(a), T^m(a)\right) \\ &\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + S\theta(a_n, a_{n+1}, a_m) \\ &\quad \cdot \left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^{n+1}(a), T^m(a)\right)\right] = S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] \\ &\quad + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^n(a), T^{n+1}(a)\right) + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^{n+1}(a), T^m(a)\right) \\ &\le S\left[d_\theta\left(T^n(a), T^{n+1}(a)\right) + d_\theta\left(T^m(a), T^{m+1}(a)\right)\right] + S\theta(a_n, a_{n+1}, a_m) d_\theta\left(T^n(a), T^{n+1}(a)\right) \\ &\quad + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m)\left[d_\theta\left(T^{n+1}(a), T^{m+1}(a)\right) + d_\theta\left(T^{m+1}(a), T^m(a)\right)\right] d_\theta \\ &\quad \cdot \left(T^{n+1}(a), T^{m+1}(a)\right) \le \left(\frac{S + S\theta(a_n, a_{n+1}, a_m)}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}\right) d_\theta\left(T^n(a), T^{n+1}(a)\right) \\ &\quad + \left(\frac{S + S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}{1 - S\theta(a_n, a_{n+1}, a_m)\theta(a_{n+1}, a_{m+1}, a_m))}\right) d_\theta\left(T^{m+1}(a), T^m(a)\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{95}$$

Thus, the sequence $\{T^n(a)\}$ is a Cauchy sequence. By the completeness of X, there is $v \in X$ such that

$$\lim_{n\longrightarrow\infty} T^n(a) = v. \tag{96}$$

Now, by using triangular inequality and (93), we get

$$d_\theta(T(a_n), T(v)) \le S[d_\theta(a_n, T(a_n)) + d_\theta(v, T(v)) + d_\theta(a_n, v)], \tag{97}$$

so

$$d_\theta(a_{n+1}, T(v)) \le S[d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v)) + d_\theta(a_n, v)]. \tag{98}$$

At the limit,

$$\lim_{n\longrightarrow\infty} d_\theta(a_{n+1}, T(v)) \le \lim_{n\longrightarrow\infty} S[d_\theta(a_n, a_{n+1}) + d_\theta(v, T(v)) + d_\theta(a_n, v)]. \tag{99}$$

Thus, $d_\theta(v, T(v)) \le Sd_\theta(v, T(v))$, which is possible only if $T(v) = v$.

The uniqueness is as follows:

Suppose that there is $\tau(\neq v) \in M$ so that $T(\tau) = \tau$, then

$$d_\theta(T(\tau), T(v)) \le S[d_\theta(\tau, T(\tau)) + d_\theta(v, T(v)) + d_\theta(\tau, v)]1 \le S, \tag{100}$$

which a contradiction. Hence, T has only one fixed point. Thus, for each $a \in X$, $\{T^n(a)\}$ converges to v. □

3. Application

Let $X = C[a, b]$ be the set of all real-valued continuous functions on $[a, b]$, and let $d_\theta : X \times X \longrightarrow [0,\infty)$ be defined as

$$\begin{aligned} d_\theta(x, y) &= \sup_{t\in[a,b]} |x(t) - y(t)|^2, \quad \text{with } \theta(x, y, z) \\ &= \sup_{t\in[a,b]} \frac{|x(t)| + |y(t)| + |z(t)|}{|x(t)| + |y(t)| + |z(t)| + 1} + 2. \end{aligned} \tag{101}$$

One can easily verify that X is a complete new extended b-metric space. Consider the Fredholm integral equation

$$x(t) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \text{for all } t \in [a, b], \tag{102}$$

where $f : [a, b] \longrightarrow \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

Theorem 22. *Let $X = C[a, b]$ and let the operator $I : X \longrightarrow X$ be defined by*

$$I(x(t)) = \int_a^b K(t, \tau, x(\tau))d\tau + f(t), \quad \textit{for all } t \in [a, b]. \tag{103}$$

Assume that the following condition holds for each $x, y \in X$

$$\begin{aligned} &|K(t, \tau, x(\tau)) - K(t, \tau, y(\tau))|^2 \\ &\le \frac{\xi}{2(b-a)} |x(\tau) - I(x(\tau)) + y(\tau) - I(y(\tau))|^2, \end{aligned} \tag{104}$$

for all $t, \tau \in [a, b]$, where $\xi \in [0, 1/2)$. Then, the integral equation (102) has a solution, provided that for every iterative sequence $\{x_n = I^n x_0\}$, for each $x_0 \in X$, we have

$$\sup_{m\geq 1}\lim_{n\longrightarrow\infty}\theta(x_n, x_{n+1}, x_m) < \frac{1-\xi}{\xi}. \tag{105}$$

Proof. It is required to prove that the operator I satisfies the conditions of Theorem 9. For this, we will use the following inequality for $\beta > 1$:

$$\left(\frac{a+b}{2}\right)^{\beta} \leq \frac{a^{\beta}+b^{\beta}}{2}. \tag{106}$$

For $x, y \in X$, consider

$$\begin{aligned}\sup_{t\in[a,b]}|I(x(t))-I(y(t))|^2 &= \left|\int_a^b K(t,\tau,x(\tau))d\tau + f(t) - \int_a^b K(t,\tau,y(\tau))d\tau - f(t)\right|^2 \\ &= \left|\int_a^b (K(t,\tau,x(\tau)) - K(t,\tau,y(\tau)))d\tau\right|^2 \\ &\leq \int_a^b |K(t,\tau,x(\tau)) - K(t,\tau,y(\tau))|^2 d\tau \\ &\leq \int_a^b \frac{\xi}{2(b-a)}|x(\tau)-I(x(\tau))+y(\tau)-I(y(\tau))|^2 d\tau \\ &\leq \frac{\xi}{2(b-a)}\int_a^b 2\left[|x(\tau)-I(x(\tau))|^2 + |y(\tau)-I(y(\tau))|^2\right]d\tau \\ &\leq \xi \sup_{t\in[a,b]}\left[|x(t)-I(x(t))|^2 + \sup_{t\in[a,b]}|y(t)-I(y(t))|^2\right].\end{aligned} \tag{107}$$

That is,

$$d_\theta(Ix, Iy) \leq \xi[d_\theta(x, Tx) + d_\theta(y, Ty)], \quad \forall x, y \in X. \tag{108}$$

This implies that (7) holds for $L = 0$. Hence, by Theorem 9, the operator I has a fixed point, provided that for every iterative sequence $x_n = I^n x_0$, for each $x_0 \in X$, we have $\sup_{m\geq 1}\lim_{n\longrightarrow\infty}\theta(x_n, x_{n+1}, x_m)<(1-\xi)/\xi$, that is, the Fredholm integral equation (102) has a solution. □

4. Conclusion

(i) The idea of new extended b-metric spaces was elaborated with examples

(ii) Some results involving Kannan-type contractions on new extended b-metric spaces are provided

(iii) Results presented by Gornicki [28] are generalized and modified

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

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Research Article

Some Coincidence and Common Fixed-Point Results on Cone b_2-Metric Spaces over Banach Algebras with Applications to the Infinite System of Integral Equations

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In this article, common fixed-point theorems for self-mappings under different types of generalized contractions in the context of the cone b_2-metric space over the Banach algebra are discussed. The existence results obtained strengthen the ones mentioned previously in the literature. An example and an application to the infinite system of integral equations are also presented to validate the main results.

1. Introduction and Preliminaries

Gähler [1] proposed the definition of 2-metric spaces as a generalization of an ordinary metric space. He defined that $d(s, m, z)$ geometrically represents the area of a triangle with vertices $s, m, z \in \aleph$. 2-metric is not a continuous function of its variables. This was one of the key drawbacks of the 2-metric space while an ordinary metric is a continuous function.

Keeping these drawbacks in mind, Dhage [2], in his PhD thesis, proposed a concept of the D-metric space as a generalized version of the 2-metric space. He also defined an open ball in such spaces and studied other topological properties of the mentioned structure. According to him, $D(s, m, z)$ represented the perimeter of a triangle. He stated that the D-metric induced a Hausdorff topology and in the D-metric space, the family of all open balls forms a basis for such topology.

Later, Mustafa and Sims [3] illustrate that the topological structure of Dhage's D-metric is invalid. Then, they revised the D-metric and expanded the notion of a metric in which each triplet of an arbitrary set is given a real number called as the G-metric space [4].

In addition, the definition of the D^*-metric space is proposed by Sedghi et al. [5] as an updated version of Dhage's D-metric space. Later, they analyzed and found that G-metric and D^*-metric have shortcomings. Later, they proposed a new simplified sturcture called the S-metric space [6].

On the other hand, by swapping the real numbers with the ordered Banach space and established cone metric space, Huang and Zhang [7] generalized the notion of a metric space and demonstrated some fixed-point results of contractive maps using the normality condition in such spaces. Rezapour and Hamlbarani [8] subsequently ignored the normality assumption and obtained some generalizations of the Huang and Zhang [7] results. However, it should be noted that the equivalence between cone metric spaces and metric spaces has been developed in recent studies by some scholars in the context of the presence of fixed points in the mapping involved. Liu and Xu [9] proposed the concept of a cone metric space over the Banach algebra in order to solve these shortcomings by replacing the Banach space with the

Banach algebra. This became an interesting discovery in the study of fixed-point theory since it can be shown that cone metric spaces over the Banach algebra are not equal to metric spaces in terms of the presence of the fixed points of mappings. Among these generalizations, by generalizing the cone 2-metric spaces [10] over the Banach algebra and b_2-metric spaces [11], Fernandez et al. [12] examined cone b_2-metric spaces over the Banach algebra with the constant $b \geq 1$. In the setting of the new structure, they proved some fixed-point theorems under different types of contractive mappings and showed the existence and uniqueness of a solution to a class of system of integral equations as an application.

Recently, in 2020, Islam et al. [13] initiated the notion of the cone b_2-metric space over the Banach algebra with constant $b \succcurlyeq e$ which is a generalization of the definition of Fernandez et al. [12]. They proved some fixed-point theorems under α-admissible Hardy-Rogers contractions which generalize many of the results from the existence literature, and as an application, they proved results which guarantee the existence of solution of an infinite system of integral equations.

In 1973, Hardy and Rogers [14] proposed a new definition of mappings called the contraction of Hardy-Rogers that generalizes the theory of the Banach contraction and the theorem of Reich [15] in a metric space setting. For other related work about the concept of Hardy-Rogers contractions, see, for instance, [16, 17] and the references therein.

We recollect certain essential notes, definitions required, and primary results consistent with the literature.

Definition 1 (see [18]). Consider $\widehat{\mathscr{U}}$ the Banach algebra which is real, and the multiplication operation is defined under the below properties (for all $s, m, z \in \widehat{\mathscr{U}}, \rho \in \mathbb{R}$):

(a_1) $(sm)z = s(mz)$

(a_2) $s(m+z) = sm + sz$ and $(s+m)z = sz + mz$

(a_3) $\rho(sm) = (\rho s)m = s(\rho m)$

(a_4) $\|sm\| \leq \|s\|\|m\|$

Unless otherwise stated, we will assume in this article that $\widehat{\mathscr{U}}$ is a real Banach algebra. If $s \in \widehat{\mathscr{U}}$ occurs, we call e the unit of $\widehat{\mathscr{U}}$, so that $es = se = s$. We call $\widehat{\mathscr{U}}$ a unital in this case. If an inverse element $m \in \widehat{\mathscr{U}}$ exists, the element $s \in \widehat{\mathscr{U}}$ is said to be invertible, so that $sm = ms = e$. The inverse of s in such case is unique and is denoted by s^{-1}. We require the following propositions in the sequel.

Proposition 2 (see [18]). *Consider the unital Banach algebra $\widehat{\mathscr{U}}$ with unit e, and let $s \in \widehat{\mathscr{U}}$ be the arbitrary element. If the spectral radius $r(s) < 1$, i.e.,*

$$r(s) = \lim_{n \longrightarrow \infty} \|s^n\|^{1/n} = \inf \|s^n\|^{1/n} < 1, \tag{1}$$

then $e - s$ is invertible. In fact,

$$(e-s)^{-1} = \sum_{k=1}^{\infty} s^k. \tag{2}$$

Remark 3. We see from [18] that $r(s) \leq \|s\|$ for all $s \in \widehat{\mathscr{U}}$ with unit e.

Remark 4 (see [19]). In Proposition 2, by replacing "$r(s) < 1$" by $\|s\| \leq 1$, then the conclusion remains the same.

Remark 5 (see [19]). If $r(s) < 1$, then $\|s^n\| \longrightarrow 0$ as $(n \longrightarrow \infty)$.

Definition 6. Consider the Banach algebra $\widehat{\mathscr{U}}$ with unit element e, zero element $\theta_{\widehat{\mathscr{U}}}$, and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Then, $\mathscr{C}_{\widehat{\mathscr{U}}} \subset \widehat{\mathscr{U}}$ is a cone in $\widehat{\mathscr{U}}$ if:

(b_1) $e \in \mathscr{C}_{\widehat{\mathscr{U}}}$

(b_2) $\mathscr{C}_{\widehat{\mathscr{U}}} + \mathscr{C}_{\widehat{\mathscr{U}}} \subset \mathscr{C}_{\widehat{\mathscr{U}}}$

(b_3) $\lambda \mathscr{C}_{\widehat{\mathscr{U}}} \subset \mathscr{C}_{\widehat{\mathscr{U}}}$ for all $\lambda \geq 0$

(b_4) $\mathscr{C}_{\widehat{\mathscr{U}}} \cdot \mathscr{C}_{\widehat{\mathscr{U}}} \subset \mathscr{C}_{\widehat{\mathscr{U}}}$

(b_5) $\mathscr{C}_{\widehat{\mathscr{U}}} \cap (-\mathscr{C}_{\widehat{\mathscr{U}}}) = \{\theta_{\widehat{\mathscr{U}}}\}$

Define the partial order relation $\preccurlyeq$ in $\widehat{\mathscr{U}}$ w.r.t $\mathscr{C}_{\widehat{\mathscr{U}}}$ by $s \preccurlyeq m$ if and only if $m - s \in \mathscr{C}_{\widehat{\mathscr{U}}}$; also, $s \prec m$ if $s \preccurlyeq m$ but $s \neq m$ while $s \ll m$ stands for $m - s \in \operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}}$, where $\operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}}$ is the interior of $\mathscr{C}_{\widehat{\mathscr{U}}}$. $\mathscr{C}_{\widehat{\mathscr{U}}}$ is solid if $\operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$.

If there is $M > 0$ such that for all $s, m \in \mathscr{C}_{\widehat{\mathscr{U}}}$, we have

$$\theta_{\widehat{\mathscr{U}}} \preccurlyeq s \preccurlyeq m \text{ implies } \|s\| \leq M\|m\|, \tag{3}$$

then $\mathscr{C}_{\widehat{\mathscr{U}}}$ is normal. If M is least and positive in the above, then it is the normal constant of $\mathscr{C}_{\widehat{\mathscr{U}}}$ [7].

Definition 7 (see [7, 9]). Let $d : \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ and $\aleph \neq \varnothing$ be the mapping:

(c_1) For all $s, m \in \aleph$, $d(s, m) \succcurlyeq \theta_{\widehat{\mathscr{U}}}$ and $d(s, m) = \theta_{\widehat{\mathscr{U}}}$ if and only if $s = m$

(c_2) For all $s, m \in \aleph$, $d(s, m) = d(m, s)$

(c_3) For all $s, m, z \in \aleph$, $d(s, z) \preccurlyeq d(s, m) + d(m, z)$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ with cone metric d is a cone metric space.

In [20], over the Banach algebra with constant $b \geq 1$, the cone b-metric space is introduced as a generalization of the cone metric space over the Banach algebra while in Mitrovic and Hussain [16], over the Banach algebra with parameter $b \succcurlyeq e$, the concept of cone b-metric spaces is introduced.

Definition 8 (see [16]). Let $d : \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ and $\aleph \neq \varnothing$ be the mapping:

(e_1) For all $s, m \in \aleph$, $\theta_{\widehat{\mathscr{U}}} \preccurlyeq d(s, m)$ and $d(s, m) = \theta_{\widehat{\mathscr{U}}}$ if and only if $s = m$

(e_2) For all $s, m \in \aleph$, $d(s, m) = d(m, s)$

(e_3) There is $b \in \mathscr{C}_{\widehat{\mathscr{U}}}$, $b \succcurlyeq e$, and for all $s, m, z \in \aleph$, $d(s, z) \preccurlyeq b[d(s, m) + d(m, z)]$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ with cone b-metric d is a cone b-metric space. Note that if we take $b = e$, then it reduces to the cone metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Definition 9 (see [1]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \mathbb{R}^+$ and $\aleph \neq \varnothing$:

(f_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq 0$, if at least two of s, m, z are not equal

(f_2) $d(s, m, z) = 0$ if and only if at least two of s, m, z are equal

(f_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(f_4) $d(s, m, z) \leq d(s, m, t) + d(s, z, t) + d(m, z, t)$ for all $s, m, z, t \in \aleph$

Then, $(\aleph, d)$ is a 2-metric space with 2-metric d.

Definition 10 (see [12]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$, $\aleph \neq \varnothing$, and $b \geq 1$ be a real number:

(g_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq \theta_{\widehat{\mathscr{U}}}$ if at least two of s, m, z are not equal

(g_2) $d(s, m, z) = \theta_{\widehat{\mathscr{U}}}$ if and only if at least two of s, m, z are equal

(g_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(g_4) $d(s, m, z) \preccurlyeq b[d(s, m, t) + d(s, z, t) + d(m, z, t)]$ for all $s, m, z, t \in \aleph$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ with parameter $b \geq 1$ is a cone b_2-metric space. By taking $b = 1$, it became a cone 2-metric space. We refer the reader to [21] for other details about the cone 2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Islam et al. [13] initiated the concept of the cone b_2-metric space over the Banach algebra with parameter $b \succcurlyeq e$.

Definition 11 (see [13]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ and $\aleph \neq \varnothing$:

(h_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq \theta_{\widehat{\mathscr{U}}}$ if at least two of s, m, z are not equal

(h_2) $d(s, m, z) = \theta_{\widehat{\mathscr{U}}}$ iff at least two of s, m, z are equal

(h_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(h_4) $d(s, m, z) \preccurlyeq b[d(s, m, t) + d(s, z, t) + d(m, z, t)]$ for all $s, m, z, t \in \aleph$ with $b \in \mathscr{C}_{\widehat{\mathscr{U}}}$, $b \succcurlyeq e$

Then, $(\aleph, d)$ over the Banach algebra with parameter $b \succcurlyeq e$ is a cone b_2-metric space. By taking $b = e$, it reduces to a cone 2-metric space.

Example 12 (see [13]). Let $\widehat{\mathscr{U}} = C^1_{\mathbb{R}}[0, 1]$. For each $h(t) \in \widehat{\mathscr{U}}$, $\|h(t)\| = \|h(t)\|_\infty + \|h'(t)\|_\infty$. Then, $\widehat{\mathscr{U}}$ is a Banach algebra with unit $e = 1$ as a constant function, and multiplication is defined pointwise. Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{h(t) \in \widehat{\mathscr{U}} \mid h(t) \geq 0, t \in [0, 1]\}$. Then, $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone in $\widehat{\mathscr{U}}$. Let $\aleph = \{(k, 0) \in \mathbb{R}^2 \mid \mid 0 \leq k \leq 1\} \cup \{(0, 1)\}$. For all $S, M, Z \in \aleph$, define $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ as

$$d(S, M, Z) = \begin{cases} d(P(S, M, Z)), & P \text{ denotes permutations}, \\ \Delta \cdot h(t), & \text{otherwise}, \end{cases} \tag{4}$$

where Δ is the square of the area of the triangle formed by S, M, Z and $h : [0, 1] \longrightarrow \mathbb{R}$ defined by $h(t) = e^t$. Consider

$$\begin{aligned} d((s, 0), (m, 0), (0, 1)) \cdot e^t \preccurlyeq\, & d((s, 0), (m, 0), (z, 0)) \cdot e^t \\ & + d((s, 0), (z, 0), (0, 1)) \cdot e^t \\ & + d((z, 0), (m, 0), (0, 1)) \cdot e^t. \end{aligned} \tag{5}$$

That is, $1/4(s - m)^2 \cdot e^t \preccurlyeq 1/4((s - z)^2 + (z - m)^2) \cdot e^t$, showing that d is not a cone 2-metric, because $-(3/16)e^t \in \mathscr{C}_{\widehat{\mathscr{U}}}$ for $0 \leq s, m, z \leq 1$ with $z = 1/2$, $m = 0$, and $s = 1$. But for $b \geq 2 \in \mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Definition 13 (see [13]). Consider that $(\aleph, d)$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$ with $b \succcurlyeq e$, and let $\{s_n\}$ be a sequence in $(\aleph, d)$; then,

(i_1) $\{s_n\}$ is said to converge to $s \in \aleph$ if for every $c \gg \theta_{\widehat{\mathscr{U}}}$ there is $N \in \mathbb{N}$ such that $d(s_n, s, a) \ll c$ for all $n \geq N$. That is,

$$\lim_{n \longrightarrow \infty} s_n = s \text{ (or) } s_n \longrightarrow s \,(n \longrightarrow \infty). \tag{6}$$

(i_2) If for every $c \gg \theta_{\widehat{\mathscr{U}}}$ there exists $N \in \mathbb{N}$ such that $d(s_n, s_m, a) \ll c$ for all $m, n \geq N$, then we say that $\{s_n\}$ is a Cauchy sequence

(i_3) $(\aleph, d)$ is complete if every Cauchy sequence is convergent in $\aleph$

Definition 14 (see [22]). Let a sequence $\{s_n\}$ be in $\widehat{\mathscr{U}}$; then, sequence $\{s_n\}$ is a c -sequence, if for each $c \gg \theta_{\widehat{\mathscr{U}}}$ there is $N \in \mathbb{N}$ such that $s_n \ll c$ for all $n > N$.

Lemma 15 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Also, consider $\{s_n\}$ a c -sequence in $\widehat{\mathscr{U}}$ and $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ where k is arbitrary; then, $\{ks_n\}$ is a c -sequence.*

Lemma 16 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Let $\{s_n\}$ and $\{z_n\}$ be c-sequences in $\widehat{\mathscr{U}}$. Then, for arbitrary $\eta, \zeta \in \mathscr{C}_{\widehat{\mathscr{U}}}$, we have $\{\eta s_n + \zeta z_n\}$ which is also a c-sequence.*

Lemma 17 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Let $\{s_n\} \subset \mathscr{C}_{\widehat{\mathscr{U}}}$ such that $\|s_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then, $\{s_n\}$ is a c-sequence.*

Lemma 18 (see [19]). *Let $\widehat{\mathscr{U}}$ be the Banach algebra, e be their unit element, and $m, s \in \widehat{\mathscr{U}}$. If m, s commutes, then*

(k_1) $r(s + m) \leq r(s) + r(m)$

(k_2) $r(sm) \leq r(s)r(m)$

Lemma 19 (see [19]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and $s \in \widehat{\mathscr{U}}$. If $0 \leq r(s) < 1$, then*

$$r\left((e - s)^{-1}\right) \leq (1 - r(s))^{-1}. \tag{7}$$

Lemma 20 (see [24]). *Consider the Banach algebra $\widehat{\mathscr{U}}$, e is their unit element, and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$. Let $L \in \widehat{\mathscr{U}}$ and $s_n = L^n$. If $r(L) < 1$, then $\{s_n\}$ is a c-sequence.*

Lemma 21 (see [25]). *Let $\mathscr{C}_{\widehat{\mathscr{U}}} \subset \widehat{\mathscr{U}}$ be a cone.*

(l_1) If $s, m \in \widehat{\mathscr{U}}$, $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$, and $m \preccurlyeq s$, then $km \preccurlyeq ks$
(l_2) If $m, k \in \mathscr{C}_{\widehat{\mathscr{U}}}$, $r(k) < 1$, and $m \preccurlyeq km$, then $m = 0$
(l_3) For any $n \in \mathbb{N}$, $r(k^n) < 1$ with $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ and $r(k) < 1$

Lemma 22 (see [26]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$.

(n_1) If $m, s, z \in \widehat{\mathscr{U}}$ and $m \preccurlyeq s \ll z$, then $m \ll s$
(n^2) If $m \in \mathscr{C}_{\widehat{\mathscr{U}}}$ and $m \ll c$ for $c \gg \theta_{\widehat{\mathscr{U}}}$, then $m = \theta_{\widehat{\mathscr{U}}}$

Lemma 23 (see [21]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$. *Let $m \in \widehat{\mathscr{U}}$, and suppose that $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ is an arbitrary given vector such that $m \ll c$ for any $\theta_{\widehat{\mathscr{U}}} \ll c$, then $km \ll c$ for any $\theta_{\widehat{\mathscr{U}}} \ll c$.*

Lemma 24 (see [27]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$. *Let $\theta_{\widehat{\mathscr{U}}} \preccurlyeq m \ll c$ for each $\theta_{\widehat{\mathscr{U}}} \ll c$; then, $m = \theta_{\widehat{\mathscr{U}}}$.*

Lemma 25 (see [27]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$. *If $\|s_n\| \longrightarrow 0$ as $(n \longrightarrow \infty)$, then for each $c \gg \theta_{\widehat{\mathscr{U}}}$, there is $N \in \mathbb{N}$ with $n > N$, such that $s_n \ll c$.*

Definition 26 (see [28]). Let g and g be self-maps of a set $\aleph$. If $m = gs = fs$ for some $s \in \aleph$, then for g and f, s is known as a coincidence point and m is known as a point of coincidence of g and f.

Definition 27 (see [29]). The mappings $g, f : \aleph \longrightarrow \aleph$ are said to be weakly compatible, whenever $fs = gs$ and $fgs = gfs$ for any $s \in \aleph$.

Lemma 28 (see [28]). *Let the mappings g and f be weakly compatible self-maps of a set $\aleph$. If g and f have a unique point of coincidence $m = fs = gs$, then m is the unique common fixed point of g and f.*

2. Main Results

In this section, in the framework of the cone b_2-metric space over Banach algebras with parameter $b \succcurlyeq e$, we prove some common fixed-point results.

Proposition 29. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be the complete cone b_2-metric space and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$ be a cone in $\widehat{\mathscr{U}}$. If a sequence $\{s_n\}$ in $\aleph$ converges to $s \in \aleph$, then we have the following:*

(i) $\{d(s_n, s, a)\}$ is a c-sequence for all $a \in \aleph$

(ii) For any $\alpha \in \mathbb{N}$, $\{d(s_n, s_{n+\alpha}, a)\}$ is a c-sequence for all $a \in \aleph$

Proof. Since the proof is easy, so we left it.

Now, we here state and prove our first main results which generalize and extend many of the conclusions from the existence literature. □

Theorem 30. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \emptyset$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$\begin{aligned} d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq\, & \vartheta_1 d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\ & + \vartheta_2 d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) \\ & + \vartheta_3 d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right) \\ & + \vartheta_4 d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) \\ & + \vartheta_5 d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right), \end{aligned} \tag{8}$$

where $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5 \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $\sum_{w=1}^{3} r(\vartheta_w) + 2r(\vartheta_4)r(b) + 2r(\vartheta_5)r(b) < 1$, $r(\vartheta_2)r(b) + r(\vartheta_5)r(b^2) < 1$, $r(\vartheta_3)r(b) + r(\vartheta_4)r(b^2) < 1$, and $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, b$ commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Set $E_i^{\eta_i} = S_{2i+1}$, $F_j^{\eta_j} = T_{2j+2}$, $G_k^{\eta_k} = I_{2k+3}$, and $H_l^{\eta_l} = J_{2l+2}$, $i, j, k, l \geq 1$. Then, by (8), we have

$$\begin{aligned} d\left(S_{2i+1}(s), T_{2j+2}(m), a\right) \preccurlyeq\, & \vartheta_1 d(I_{2k+3}(s), J_{2l+2}(m), a) \\ & + \vartheta_2 d(I_{2k+3}(s), S_{2i+1}(s), a) \\ & + \vartheta_3 d\left(J_{2l+2}(m), T_{2j+2}(m), a\right) \\ & + \vartheta_4 d\left(I_{2k+3}(s), T_{2j+2}(m), a\right) \\ & + \vartheta_5 d(J_{2l+2}(m), S_{2i+1}(s), a). \end{aligned} \tag{9}$$

Choose $s_0 \in \aleph$ to be arbitrary. Since $E_i(\aleph) \subseteq H_l(\aleph)$ and $F_j(\aleph) \subseteq G_k(\aleph)$ for each $i, j, k, l \geq 1$, there exists $s_1, s_2 \in \aleph$ such that $S_1(s_0) = J_2(s_1)$ and $T_2(s_1) = I_3(s_2)$. Continuing this process, we can define $\{s_n\}$ by $S_{2n+1}(s_{2n}) = J_{2n+2}(s_{2n+1})$ and $T_{2n+2}(s_{2n+1}) = I_{2n+3}(s_{2n+2})$.

Denote $\mu_{2n} = J_{2n+2}(s_{2n+1}) = S_{2n+1}(s_{2n})$ and $\mu_{2n+1} = I_{2n+3}(s_{2n+2}) = T_{2n+2}(s_{2n+1})$ for $n = 0, 1, 2, \cdots$. Now, we show that $\{\mu_n\}$ is a Cauchy sequence.

From (9), we know that

$$
\begin{aligned}
d(\mu_{2n},\mu_{2n+1},a) &= d(S_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),a)\\
&\preccurlyeq \vartheta_1 d(I_{2n+1}(s_{2n}),J_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_2 d(I_{2n+1}(s_{2n}),S_{2n+1}(s_{2n}),a)\\
&\quad+\vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_4 d(I_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(s_{2n}),a)\\
&= \vartheta_1 d(\mu_{2n-1},\mu_{2n},a)+\vartheta_2 d(\mu_{2n-1},\mu_{2n},a)\\
&\quad+\vartheta_3 d(\mu_{2n},\mu_{2n+1},a)+\vartheta_4 d(\mu_{2n-1},\mu_{2n+1},a)\\
&\quad+\vartheta_5 d(\mu_{2n},\mu_{2n},a)\preccurlyeq\vartheta_1 d(\mu_{2n-1},\mu_{2n},a)\\
&\quad+\vartheta_2 d(\mu_{2n-1},\mu_{2n},a)+\vartheta_3 d(\mu_{2n},\mu_{2n+1},a)\\
&\quad+\vartheta_4 b d(\mu_{2n-1},\mu_{2n+1},\mu_{2n})\\
&\quad+\vartheta_4 b d(\mu_{2n-1},\mu_{2n},a)\\
&\quad+\vartheta_4 b d(\mu_{2n},\mu_{2n+1},a).
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}) &= d(S_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),\mu_{2n-1})\\
&\le \vartheta_1 d(I_{2n+1}(s_{2n}),J_{2n+2}(s_{2n+1}),\mu_{2n-1})\\
&\quad+\vartheta_2 d(I_{2n+1}(s_{2n}),S_{2n+1}(s_{2n}),\mu_{2n-1})\\
&\quad+\vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),\mu_{2n-1})\\
&\quad+\vartheta_4 d(I_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),\mu_{2n-1})\\
&\quad+\vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(s_{2n}),\mu_{2n-1})\\
&= \vartheta_1 d(\mu_{2n-1},\mu_{2n},\mu_{2n-1})\\
&\quad+\vartheta_2 d(\mu_{2n-1},\mu_{2n},\mu_{2n-1})\\
&\quad+\vartheta_3 d(\mu_{2n},\mu_{2n+1},\mu_{2n-1})\\
&\quad+\vartheta_4 d(\mu_{2n-1},\mu_{2n+1},\mu_{2n-1})\\
&\quad+\vartheta_5 \mathrm{d}(\mu_{2n},\mu_{2n},\mu_{2n-1})\\
&\preccurlyeq \vartheta_3 d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}).
\end{aligned} \tag{11}
$$

It means that $d(\mu_{2n},\mu_{2n+1},\mu_{2n-1})=\theta_{\widehat{\mathscr{U}}}$. Therefore, (10) becomes

$$
(e-\vartheta_3-\vartheta_4 b)d(\mu_{2n},\mu_{2n+1},a)\preccurlyeq(\vartheta_1+\vartheta_2+\vartheta_4 b)d(\mu_{2n-1},\mu_{2n},a), \tag{12}
$$

that is,

$$
d(\mu_{2n},\mu_{2n+1},a)\preccurlyeq L_1 d(\mu_{2n-1},\mu_{2n},a), \tag{13}
$$

where $L_1=(e-\vartheta_3-\vartheta_4 b)^{-1}(\vartheta_1+\vartheta_2+\vartheta_4 b)$. Similarly, it is not difficult to show that

$$
\begin{aligned}
d(\mu_{2n+2},\mu_{2n+1},a) &= d(S_{2n+3}(s_{2n+2}),T_{2n+2}(s_{2n+1}),a)\\
&\preccurlyeq \vartheta_1 d(I_{2n+3}(s_{2n+2}),J_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_2 d(I_{2n+3}(s_{2n+2}),S_{2n+3}(s_{2n+2}),a)\\
&\quad+\vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_4 d(I_{2n+3}(s_{2n+2}),T_{2n+2}(s_{2n+1}),a)\\
&\quad+\vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+3}(s_{2n+2}),a)\\
&= \vartheta_1 d(\mu_{2n+1},\mu_{2n},a)+\vartheta_2 d(\mu_{2n+1},\mu_{2n+2},a)\\
&\quad+\vartheta_3 d(\mu_{2n},\mu_{2n+1},a)+\vartheta_4 d(\mu_{2n+1},\mu_{2n+1},a)\\
&\quad+\vartheta_5 d(\mu_{2n},\mu_{2n+2},a)\preccurlyeq\vartheta_1 d(\mu_{2n+1},\mu_{2n},a)\\
&\quad+\vartheta_2 d(\mu_{2n+1},\mu_{2n+2},a)+\vartheta_3 d(\mu_{2n},\mu_{2n+1},a)\\
&\quad+\vartheta_5 b d(\mu_{2n},\mu_{2n+2},\mu_{2n+1})\\
&\quad+\vartheta_5 b d(\mu_{2n},\mu_{2n+1},a)\\
&\quad+\vartheta_5 b d(\mu_{2n+1},\mu_{2n+2},a).
\end{aligned} \tag{14}
$$

As $d(\mu_{2n},\mu_{2n+2},\mu_{2n+1})=\theta_{\widehat{\mathscr{U}}}$, therefore (14) becomes

$$
(e-\vartheta_2-\vartheta_5 b)d(\mu_{2n+2},\mu_{2n+1},a)\preccurlyeq(\vartheta_1+\vartheta_3+\vartheta_5 b)d(\mu_{2n},\mu_{2n+1},a), \tag{15}
$$

that is,

$$
d(\mu_{2n+2},\mu_{2n+1},a)\preccurlyeq L_2 d(\mu_{2n},\mu_{2n+1},a), \tag{16}
$$

where $L_2=(e-\vartheta_2-\vartheta_5 b)^{-1}(\vartheta_1+\vartheta_3+\vartheta_5 b)$. Set $K=L_1L_2$, and using inequalities (13) and (16), we deduce that

$$
\begin{aligned}
d(\mu_{2n+2},\mu_{2n+1},a) &\preccurlyeq L_2 d(\mu_{2n+1},\mu_{2n},a)\\
&\preccurlyeq L_2L_1 d(\mu_{2n},\mu_{2n-1},a)\\
&\preccurlyeq L_2L_1L_2 d(\mu_{2n-1},\mu_{2n-2},a)\\
&= L_2K d(\mu_{2n-1},\mu_{2n-2},a)\\
&\preccurlyeq\cdots\preccurlyeq L_2K^n d(\mu_1,\mu_0,a),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
d(\mu_{2n+3},\mu_{2n+2},a) &\preccurlyeq L_1 d(\mu_{2n+2},\mu_{2n+1},a)\\
&\preccurlyeq L_1L_2K^n d(\mu_1,\mu_0,a)\\
&= K^{n+1} d(\mu_1,\mu_0,a).
\end{aligned} \tag{18}
$$

In this case, for all $t<n$, similar to (17) and (18), we have

$$
\begin{aligned}
d(\mu_{2n+1},\mu_{2n},\mu_{2t+1}) &= L_2K^{n-t-1}d(\mu_{2t+1},\mu_{2t+2},\mu_{2t+1})=\theta_{\widehat{\mathscr{U}}},\\
d(\mu_{2n-1},\mu_{2n},\mu_{2t+1}) &= K^{n-t-1}d(\mu_{2t+1},\mu_{2t+2},\mu_{2t+1})=\theta_{\widehat{\mathscr{U}}},\\
d(\mu_{2n+1},\mu_{2n},\mu_{2t}) &= K^{n-t}d(\mu_{2t+1},\mu_{2t},\mu_{2t})=\theta_{\widehat{\mathscr{U}}},\\
d(\mu_{2n-1},\mu_{2n},\mu_{2t}) &= L_2K^{n-t-1}d(\mu_{2t+1},\mu_{2t},\mu_{2t})=\theta_{\widehat{\mathscr{U}}}.
\end{aligned} \tag{19}
$$

Therefore, for all $m<n$, $a\in\aleph$, and using the above inequalities, we have

$$\begin{aligned}
d(\mu_{2n+1},\mu_{2m+1},a) &\preccurlyeq bd(\mu_{2n+1},\mu_{2m+1},\mu_{2n}) \\
&\quad + bd(\mu_{2n+1},\mu_{2n},a) \\
&\quad + bd(\mu_{2n},\mu_{2m+1},a) \\
&= bd(\mu_{2n+1},\mu_{2n},a) + bd(\mu_{2n},\mu_{2m+1},a) \\
&\preccurlyeq bd(\mu_{2n+1},\mu_{2n},a) + b^2 d(\mu_{2n},\mu_{2m+1},\mu_{2n-1}) \\
&\quad + b^2 d(\mu_{2n},\mu_{2n-1},a) \\
&\quad + b^2 d(\mu_{2n-1},\mu_{2m+1},a) \\
&= bd(\mu_{2n+1},\mu_{2n},a) + b^2 d(\mu_{2n},\mu_{2n-1},a) \\
&\quad + b^2 d(\mu_{2n-1},\mu_{2m+1},a).
\end{aligned} \tag{20}$$

Continuing this process, we get

$$\begin{aligned}
d(\mu_{2n+1},\mu_{2m+1},a) &\le bd(\mu_{2n+1},\mu_{2n},a) + b^2 d(\mu_{2n},\mu_{2n-1},a) \\
&\quad + b^3 d(\mu_{2n-1},\mu_{2n-2},a) \\
&\quad + b^4 d(\mu_{2n-2},\mu_{2n-3},a) \\
&\quad + \cdots + b^{n-m-4} d(\mu_{2m+5},\mu_{2m+4},a) \\
&\quad + b^{n-m-3} d(\mu_{2m+4},\mu_{2m+3},a) \\
&\quad + b^{n-m-2} d(\mu_{2m+3},\mu_{2m+2},a) \\
&\quad + b^{n-m-1} d(\mu_{2m+2},\mu_{2m+1},a) \\
&\le bK^n d(\mu_1,\mu_0,a) + b^2 L_2 K^{n-1} d(\mu_1,\mu_0,a) \\
&\quad + b^3 K^{n-1} d(\mu_1,\mu_0,a) \\
&\quad + b^4 L_2 K^{n-2} d(\mu_1,\mu_0,a) \\
&\quad + \cdots + b^{n-m-4} K^{m+2} d(\mu_1,\mu_0,a) \\
&\quad + b^{n-m-3} L_2 K^{m+1} d(\mu_1,\mu_0,a) \\
&\quad + b^{n-m-2} K^{m+1} d(\mu_1,\mu_0,a) \\
&\quad + b^{n-m-1} L_2 K^{m} d(\mu_1,\mu_0,a) \\
&\le \big(bK^n + b^2 L_2 K^{n-1} + b^3 K^{n-1} \\
&\quad + b^4 L_2 K^{n-2} + \cdots + b^{n-m-4} K^{m+2} \\
&\quad + b^{n-m-3} L_2 K^{m+1} + b^{n-m-2} K^{m+1} \\
&\quad + b^{n-m-1} L_2 K^{m}\big) d(\mu_1,\mu_0,a),
\end{aligned}$$

$$\begin{aligned}
d(\mu_{2n+1},\mu_{2m+1},a) &= \big(bK^n + b^3 K^{n-1} + \cdots + b^{n-m-4} K^{m+2} \\
&\quad + b^{n-m-2} K^{m+1} + b^2 L_2 K^{n-1} \\
&\quad b^4 L_2 K^{n-2} + \cdots + b^{n-m-3} L_2 K^{m+1} \\
&\quad + b^{n-m-1} L_2 K^{m}\big) d(\mu_1,\mu_0,a) \\
&= K^{m+1} b^{n-m-2} \big(e + b^{-2}K \\
&\quad + \cdots + b^{m-n+3} K^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&\quad + L_2 K^m b^{n-m-1} \big(e + b^{-2}K \\
&\quad + \cdots + b^{m-n+3} K^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&= \left(K^{m+1} b^{n-m-2} \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \right. \\
&\quad \left. + L_2 K^m b^{n-m-1} \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \right) d(\mu_1,\mu_0,a).
\end{aligned} \tag{21}$$

That is,

$$\begin{aligned}
d(\mu_{2n+1},\mu_{2m+1},a) &\preccurlyeq \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \big(K^{m+1} b^{n-m-2} \\
&\quad + L_2 K^m b^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&\preccurlyeq \left(e - b^{-2}K\right)^{-1} K^m \big(K b^{n-m-2} \\
&\quad + L_2 b^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&= MK^m d(\mu_1,\mu_0,a),
\end{aligned} \tag{22}$$

where $M = (e - b^{-2}K)^{-1} (K b^{n-m-2} + L_2 b^{n-m-1})$.

Similarly, we have

$$\begin{aligned}
d(\mu_{2n},\mu_{2m+1},a) &\le bd(\mu_{2n},\mu_{2n-1},a) \\
&\quad + b^2 d(\mu_{2n-1},\mu_{2n-2},a) \\
&\quad + b^3 d(\mu_{2n-2},\mu_{2n-3},a) \\
&\quad + b^4 d(\mu_{2n-3},\mu_{2n-4},a) \\
&\quad + \cdots + b^{n-m-4} d(\mu_{2m+5},\mu_{2m+4},a) \\
&\quad + b^{n-m-3} d(\mu_{2m+4},\mu_{2m+3},a) \\
&\quad + b^{n-m-2} d(\mu_{2m+3},\mu_{2m+2},a) \\
&\quad + b^{n-m-1} d(\mu_{2m+2},\mu_{2m+1},a) \\
&\le \big(bL_2 K^{n-1} + b^3 L_2 K^{n-2} \\
&\quad + \cdots + b^{n-m-4} L_2 K^{m+2} + b^{n-m-2} L_2 K^{m+1} \\
&\quad + b^2 K^{n-1} + b^4 K^{n-2} + \cdots + b^{n-m-3} K^{m+2} \\
&\quad + b^{n-m-1} K^m\big) d(\mu_1,\mu_0,a) \\
&\le b^{n-m-2} L_2 K^{m+1} \big(e + b^{-2}K \\
&\quad + \cdots + b^{m-n+3} K^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&\quad + b^{n-m-1} K^m \big(e + b^{-2}K \\
&\quad + \cdots + b^{m-n+3} K^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&\le \left(b^{n-m-2} L_2 K^{m+1} \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \right. \\
&\quad \left. + b^{n-m-1} K^m \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \right) d(\mu_1,\mu_0,a) \\
&\le \sum_{r=0}^{\infty} \left(b^{-2}K\right)^r \big(b^{n-m-2} L_2 K^{m+1} \\
&\quad + b^{n-m-1} K^m\big) d(\mu_1,\mu_0,a) \\
&\le \left(e - b^{-2}K\right)^{-1} K^m \big(b^{n-m-2} L_2 K \\
&\quad + b^{n-m-1}\big) d(\mu_1,\mu_0,a) \\
&= NK^m d(\mu_1,\mu_0,a),
\end{aligned} \tag{23}$$

where $N = (e - b^{-2}K)^{-1} (b^{n-m-2} L_2 K + b^{n-m-1})$.

Similar to the above, one can easily get that

$$\begin{aligned} d(\mu_{2n},\mu_{2m},a) &\preccurlyeq OK^m d(\mu_1,\mu_0,a),\\ d(\mu_{2n+1},\mu_{2m},a) &\preccurlyeq PK^m d(\mu_1,\mu_0,a), \end{aligned} \tag{24}$$

where $O=(e-b^{-2}K)^{-1}(b^{n-m}L_2+b^{n-m-1}K)$ and $P=(e-b^{-2}K)^{-1}(b^{n-m-1}+b^{n-m-2}L_2)$.

From Lemmas 18 and 19, we have that

$$\begin{aligned} r(K) &= r(L_1L_2)\le r(L_1)\times r(L_2)\\ &= r\left[(e-\vartheta_3-\vartheta_4 b)^{-1}(\vartheta_1+\vartheta_2+\vartheta_4 b)\right]\\ &\quad\times r\left[(e-\vartheta_2-\vartheta_5 b)^{-1}(\vartheta_1+\vartheta_3+\vartheta_5 b)\right]\\ &\le r\left((e-\vartheta_3-\vartheta_4 b)^{-1}\right)\times r(\vartheta_1+\vartheta_2+\vartheta_4 b)\\ &\quad\times r\left((e-\vartheta_2-\vartheta_5 b)^{-1}\right)\times r(\vartheta_1+\vartheta_3+\vartheta_5 b)\\ &\le \frac{r(\vartheta_1)+r(\vartheta_2)+r(\vartheta_4)r(b)}{1-r(\vartheta_3)-r(\vartheta_4)r(b)}\\ &\quad\times\frac{r(\vartheta_1)+r(\vartheta_3)+r(\vartheta_5)r(b)}{1-r(\vartheta_2)-r(\vartheta_5)r(b)}<1. \end{aligned} \tag{25}$$

Since $r(K)<1$, therefore in the light of Remark 5 and Lemma 25, $\|K^m\|\longrightarrow 0$ as $(m\longrightarrow\infty)$, and so, for every $c\in \operatorname{int}\mathscr{C}_{\widehat{\mathscr{U}}}$, there exists $n_0\in\mathbb{N}$ such that $K^m\ll c$ for all $n>n_0$; that is, the sequence $\{K^m\}$ is a c-sequence. By Lemma 15, the sequences $\{MK^m d(\mu_1,\mu_0,a)\}$, $\{NK^m d(\mu_1,\mu_0,a)\}$, $\{OK^m d(\mu_1,\mu_0,a)\}$, and $\{PK^m d(\mu_1,\mu_0,a)\}$ are also c-sequences. Therefore, for any $c\in\widehat{\mathscr{U}}$ with $\theta_{\widehat{\mathscr{U}}}\ll c$, there exists $n_1\in\mathbb{N}$ such that, for any $n>m>n_1$, we have $d(\mu_n,\mu_m,a)\ll c$ for all $n>n_1$ and for all $a\in\aleph$. Thus, from Lemma 24, it means that $d(\mu_n,\mu_m,a)=\theta_{\widehat{\mathscr{U}}}$. This implies that $\{\mu_n\}$ is a Cauchy sequence in $\aleph$.

If $H_l(\aleph)$ is complete for each $l=1,2,3,\cdots$, there exists $q\in H_l(\aleph)$ such that

$$\mu_{2n}=J_{2n+2}(s_{2n+1})=S_{2n+1}(s_{2n})\longrightarrow q\ (n\longrightarrow\infty). \tag{26}$$

So we can find a $p\in\aleph$ such that $J_{2n+2}(p)=q$ (if $E_i(\aleph)$ is complete for each $i=1,2,3,\cdots$, there exists $q\in E_i(\aleph)\subseteq H_l(\aleph)$; then, the conclusion remains the same). Now, we show that $T_{2n+2}(p)=q$. By (9), we have

$$\begin{aligned} d(T_{2n+2}(p),q,a) &\preccurlyeq bd(T_{2n+2}(p),q,S_{2n+1}(s_{2n}))\\ &\quad+bd(T_{2n+2}(p),S_{2n+1}(s_{2n}),a)\\ &\quad+bd(S_{2n+1}(s_{2n}),q,a)\\ &\preccurlyeq bd(T_{2n+2}(p),q,S_{2n+1}(s_{2n}))\\ &\quad+bd(S_{2n+1}(s_{2n}),q,a)\\ &\quad+b\vartheta_1 d(I_{2n+1}(s_{2n}),J_{2n+2}(s_{2n+1}),a)\\ &\quad+b\vartheta_2 d(I_{2n+1}(s_{2n}),S_{2n+1}(s_{2n}),a)\\ &\quad+b\vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(p),a)\\ &\quad+b\vartheta_4 d(I_{2n+1}(s_{2n}),T_{2n+2}(p),a)\\ &\quad+b\vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(s_{2n}),a), \end{aligned}$$

$$\begin{aligned} d(T_{2n+2}(p),q,a) &= bd(T_{2n+2}(p),q,\mu_{2n})+bd(\mu_{2n},q,a)\\ &\quad+b\vartheta_1 d(\mu_{2n-1},q,a)+b\vartheta_2 d(\mu_{2n-1},\mu_{2n},a)\\ &\quad+b\vartheta_3 d(q,T_{2n+2}(p),a)\\ &\quad+b\vartheta_4 d(\mu_{2n-1},T_{2n+2}(p),a)\\ &\quad+b\vartheta_5 d(q,\mu_{2n},a)\\ &\preccurlyeq bd(T_{2n+2}(p),q,\mu_{2n})+bd(\mu_{2n},q,a)\\ &\quad+b\vartheta_1 d(\mu_{2n-1},q,a)+b\vartheta_2 d(\mu_{2n-1},\mu_{2n},a)\\ &\quad+b\vartheta_3 d(q,T_{2n+2}(p),a)\\ &\quad+b^2\vartheta_4 d(\mu_{2n-1},T_{2n+2}(p),q)\\ &\quad+b^2\vartheta_4 d(\mu_{2n-1},q,a)\\ &\quad+b^2\vartheta_4 d(q,T_{2n+2}(p),a)+b\vartheta_5 d(q,\mu_{2n},a). \end{aligned} \tag{27}$$

That is,

$$\begin{aligned} &\left(e-b\vartheta_3-b^2\vartheta_4\right)d(T_{2n+2}(p),q,a)\\ &\quad\preccurlyeq bd(\mu_{2n},T_{2n+2}(p),a)+b^2\vartheta_4 d(\mu_{2n-1},T_{2n+2}(p),q)\\ &\qquad+(b+b\vartheta_5)d(\mu_{2n},q,a)+\left(b\vartheta_1+b^2\vartheta_4\right)d(\mu_{2n-1},q,a)\\ &\qquad+b\vartheta_2 d(\mu_{2n-1},\mu_{2n},a). \end{aligned} \tag{28}$$

Therefore, it follows from Proposition 29 and Lemmas 15 and 16 that

$$\left(e-b\vartheta_3-b^2\vartheta_4\right)d(T_{2n+2}(p),q,a)\preccurlyeq z_n, \tag{29}$$

where $\{z_n\}$ is a c-sequence in $\mathscr{C}_{\widehat{\mathscr{U}}}$. In addition, from Proposition 2 and

$$r\left(b\vartheta_3+b^2\vartheta_4\right)\le r(b)r(\vartheta_3)+r\left(b^2\right)r(\vartheta_4)<1, \tag{30}$$

it means that $e-(b\vartheta_3+b^2\vartheta_4)$ is invertible. In this case, we have

$$\left(e-b\vartheta_3-b^2\vartheta_4\right)d(T_{2n+2}(p),q,a)\ll c, \tag{31}$$

for any $c\gg\theta_{\widehat{\mathscr{U}}}$, which together with Lemma 23 implies that $\theta_{\widehat{\mathscr{U}}}\preccurlyeq d(T_{2n+2}(p),q,a)\ll c$, for any $a\in\aleph$, $n\in\mathbb{N}$, and $c\gg\theta_{\widehat{\mathscr{U}}}$ as $(e-(b\vartheta_3+b^2\vartheta_4))$ is invertible. Therefore, by Lemma 24, we have $d(T_{2n+2}(p),q,a)=\theta_{\widehat{\mathscr{U}}}$ for any $n\in\mathbb{N}$. Namely, $T_{2n+2}(p)=q$ for any $n\in\mathbb{N}$. That is, $T_{2n+2}(p)=q=J_{2n+2}(p)$.

At the same time, as $q=T_{2n+2}(p)\in F_j(\aleph)\subseteq G_k(\aleph)$, there exists $u\in\aleph$ such that $I_{2n+3}(u)=q$.

Now, we show that $S_{2n+1}(u) = q$. From (9), we have that

$$\begin{aligned} d(S_{2n+1}(u), q, a) &= d(S_{2n+1}(u), T_{2n+2}(p), a) \\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(u), J_{2n+2}(p), a) \\ &\quad + \vartheta_2 d(I_{2n+3}(u), S_{2n+1}(u), a) \\ &\quad + \vartheta_3 d(J_{2n+2}(p), T_{2n+2}(p), a) \\ &\quad + \vartheta_4 d(I_{2n+3}(u), T_{2n+2}(p), a) \\ &\quad + \vartheta_5 d(J_{2n+2}(p), S_{2n+1}(u), a) \\ &= \vartheta_1 d(q, q, a) + \vartheta_2 d(q, S_{2n+1}(u), a) \\ &\quad + \vartheta_3 d(q, q, a) + \vartheta_4 d(q, q, a) \\ &\quad + \vartheta_5 d(q, S_{2n+1}(u), a). \end{aligned} \tag{32}$$

That is,

$$d(S_{2n+1}(u), q, a) \preccurlyeq (\vartheta_2 + \vartheta_5) d(S_{2n+1}(u), q, a). \tag{33}$$

Hence, by Lemma 20, we know that $d(S_{2n+1}(u), q, a) = \theta_{\widehat{\mathcal{U}}}$, and so $S_{2n+1}(u) = q$. Therefore, $S_{2n+1}(u) = I_{2n+3}(u) = q$ and $T_{2n+2}(p) = J_{2n+2}(p) = q$.

Next, if we assume $G_k(\aleph)$ is complete for each $k = 1, 2, 3, \cdots$, there exists $q \in G_k(\aleph)$ such that

$$\mu_{2n+1} = I_{2n+3}(s_{2n+2}) = T_{2n+2}(s_{2n+1}) \longrightarrow q \text{ as } (n \longrightarrow \infty). \tag{34}$$

So, we can find $u \in \aleph$ such that $I_{2n+3}(u) = q$ (if $F_j(\aleph)$ is complete for each $j = 1, 2, 3, \cdots$, there exists $q \in F_j(\aleph) \subseteq G_k(\aleph)$; then, the conclusion remains the same).

Now, we show that $S_{2n+1}(u) = q$. By (9), we get that

$$\begin{aligned} d(S_{2n+1}(u), q, a) &\preccurlyeq b d(S_{2n+1}(u), q, T_{2n+2}(s_{2n+1})) \\ &\quad + b d(S_{2n+1}(u), T_{2n+2}(s_{2n+1}), a) \\ &\quad + b d(T_{2n+2}(s_{2n+1}), q, a) \\ &\preccurlyeq b d(S_{2n+1}(u), q, T_{2n+2}(s_{2n+1})) \\ &\quad + b d(T_{2n+2}(s_{2n+1}), q, a) \\ &\quad + b\vartheta_1 d(I_{2n+3}(u), J_{2n+2}(s_{2n+1}), a) \\ &\quad + b\vartheta_2 d(I_{2n+3}(u), S_{2n+1}(u), a) \\ &\quad + b\vartheta_3 d(J_{2n+2}(s_{2n+1}), T_{2n+2}(s_{2n+1}), a) \\ &\quad + b\vartheta_4 d(I_{2n+3}(u), T_{2n+2}(s_{2n+1}), a) \\ &\quad + b\vartheta_5 d(J_{2n+2}(s_{2n+1}), S_{2n+1}(u), a) \\ &\preccurlyeq b d(S_{2n+1}(u), q, \mu_{2n+1}) + b d(\mu_{2n+1}, q, a) \\ &\quad + b\vartheta_1 d(q, \mu_{2n}, a) + b\vartheta_2 d(q, S_{2n+1}(u), a) \\ &\quad + b\vartheta_3 d(\mu_{2n}, \mu_{2n+1}, a) + b\vartheta_4 d(q, \mu_{2n+1}, a) \\ &\quad + b^2\vartheta_5 d(\mu_{2n}, S_{2n+1}(u), q) \\ &\quad + b^2\vartheta_5 d(\mu_{2n}, q, a) + b^2\vartheta_5 d(q, S_{2n+1}(u), a). \end{aligned} \tag{35}$$

That is,

$$\begin{aligned} &\left(e - b\vartheta_3 - b^2\vartheta_4\right) d(T_{2n+2}(p), q, a) \\ &\quad \leq b d(\mu_{2n}, T_{2n+2}(p), a) + b^2\vartheta_4 d(\mu_{2n-1}, T_{2n+2}(p), q) \\ &\qquad + (b + b\vartheta_5) d(\mu_{2n}, q, a) + \left(b\vartheta_1 + b^2\vartheta_4\right) d(\mu_{2n-1}, q, a) \\ &\qquad + b\vartheta_2 d(\mu_{2n-1}, \mu_{2n}, a). \end{aligned} \tag{36}$$

Therefore, it follows from Proposition 29 and Lemmas 15 and 16 that

$$\left(e - b\vartheta_2 - b^2\vartheta_5\right) d(S_{2n+1}(u), q, a) \preccurlyeq z_n^*, \tag{37}$$

where $\{z_n^*\}$ is a c-sequence in $\mathscr{C}_{\widehat{\mathcal{U}}}$. In addition, from Proposition 2 and

$$r\left(b\vartheta_3 + b^2\vartheta_4\right) \leq r(b) r(\vartheta_3) + r\left(b^2\right) r(\vartheta_4) < 1, \tag{38}$$

it means that $e - \left(b\vartheta_2 + b^2\vartheta_5\right)$ is invertible. In this case, we have

$$\left(e - b\vartheta_2 - b^2\vartheta_5\right) d(S_{2n+1}(u), q, a) \ll c, \tag{39}$$

for any $c \gg \theta_{\widehat{\mathcal{U}}}$, which together with Lemma 23 implies that $\theta_{\widehat{\mathcal{U}}} \preccurlyeq d(S_{2n+1}(u), q, a) \ll c$ for any $a \in \aleph$, $n \in \mathbb{N}$, and $c \gg \theta_{\widehat{\mathcal{U}}}$ as $\left(e - \left(b\vartheta_2 + b^2\vartheta_5\right)\right)$ is invertible. Therefore, by Lemma 24, we have $d(S_{2n+1}(u), q, a) = \theta_{\widehat{\mathcal{U}}}$ for any $n \in \mathbb{N}$. Namely, $S_{2n+1}(u) = q$ for any $n \in \mathbb{N}$. That is, $S_{2n+1}(u) = q = I_{2n+3}(u)$.

At the same time, as $q = S_{2n+1}(u) \in E_i(\aleph) \subseteq H_l(\aleph)$, there exists $p \in \aleph$ such that $J_{2n+2}(p) = q$. Now, we show that $T_{2n+2}(p) = q$. From (9), we have

$$\begin{aligned} d(T_{2n+2}(p), q, a) &= d(S_{2n+1}(u), T_{2n+2}(p), a) \\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(u), J_{2n+2}(p), a) \\ &\quad + \vartheta_2 d(I_{2n+3}(u), S_{2n+1}(u), a) \\ &\quad + \vartheta_3 d(J_{2n+2}(p), T_{2n+2}(p), a) \\ &\quad + \vartheta_4 d(I_{2n+3}(u), T_{2n+2}(p), a) \\ &\quad + \vartheta_5 d(J_{2n+2}(p), S_{2n+1}(u), a) \\ &= \vartheta_1 d(q, q, a) + \vartheta_2 d(q, q, a) \\ &\quad + \vartheta_3 d(q, T_{2n+2}(p), a) \\ &\quad + \vartheta_4 d(q, T_{2n+2}(p), a) + \vartheta_5 d(q, q, a). \end{aligned} \tag{40}$$

That is,

$$d(T_{2n+2}(p), q, a) \preccurlyeq (\vartheta_3 + \vartheta_4) d(T_{2n+2}(p), q, a). \tag{41}$$

Hence, by Lemma 20, we know that $d(T_{2n+2}(p), q, a) = \theta_{\widehat{\mathcal{U}}}$, and so $T_{2n+2}(p) = q$. Therefore, $T_{2n+2}(p) = J_{2n+2}(p) = q$ and $S_{2n+1}(u) = I_{2n+3}(u) = q$.

Finally, we show that S_{2i+1} and I_{2k+3}, T_{2j+2}, and J_{2l+2} have a unique point of coincidence in $\aleph$. Assume that there is another point $z \in \aleph$ such that $T_{2n+2}(x) = J_{2n+2} = z$; then,

$$\begin{aligned} d(q,z,a) &= d(S_{2n+1}(u), T_{2n+2}(x), a) \\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(u), J_{2n+2}(x), a) \\ &\quad + \vartheta_2 d(I_{2n+3}(u), S_{2n+1}(u), a) \\ &\quad + \vartheta_3 d(J_{2n+2}(x), T_{2n+2}(x), a) \\ &\quad + \vartheta_4 d(I_{2n+3}(u), T_{2n+2}(x), a) \\ &\quad + \vartheta_5 d(J_{2n+2}(x), S_{2n+1}(u), a) \\ &= \vartheta_1 d(q,z,a) + \vartheta_2 d(q,q,a) + \vartheta_3 d(z,z,a) \\ &\quad + \vartheta_4 d(q,z,a) + \vartheta_5 d(z,q,a). \end{aligned} \tag{42}$$

That is,

$$d(q,z,a) \preccurlyeq (\vartheta_1 + \vartheta_4 + \vartheta_5) d(q,z,a). \tag{43}$$

Hence, by Lemma 20, we have that $d(q,z,a) = \theta_{\widehat{\mathcal{U}}}$, and so $q = z$; that is, q is the unique point of coincidence of T_{2j+2} and J_{2l+2}.

Similarly, we also have q which is the unique point of coincidence of S_{2i+1} and I_{2k+3} by induction.

So, according to Lemma 28, q is the unique common fixed point of $\{S_{2i+1}, I_{2k+3}\}$ and $\{T_{2j+2}, J_{2l+2}\}$ for each $i, j, k, l = 1, 2, 3, \cdots$. Therefore, q is the unique common fixed point of $S_{2i+1}, I_{2k+3}, T_{2j+2}$, and J_{2l+2}.

Now, it is left to show that q is the unique common fixed point of $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$.

As $q = S_{2n+1}(q) = E_n^{\eta_n}(q)$, so we have $E_n(q) = E_n(E_n^{\eta_n}(q)) = E_n^{\eta_n}(E_n(q)) = S_{2n+1}(q)$, that is, $S_{2n+1}(E_n(q)) = E_n(q)$. But $S_{2n+1}(q) = q$ is unique; therefore, $E_n(q) = q$ for $n = 1, 2, 3, \cdots$.

Also, as $q = T_{2n+2}(q) = F_n^{\eta_n}(q)$, so we have $F_n(q) = F_n(F_n^{\eta_n}(q)) = F_n^{\eta_n}(F_n(q)) = T_{2n+2}(q)$, that is, $T_{2n+2}(F_n(q)) = F_n(q)$. But $T_{2n+2}(q) = q$ is unique; therefore, $F_n(q) = q$ for $n = 1, 2, 3, \cdots$. Similarly, $G_n(q) = q$ and $H_n(q) = q$ for $n = 1, 2, 3, \cdots$. Thus, the four families of mappings $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point. □

Remark 31. Theorem 30 of this paper extends and improves Theorem 2.1 of [30] from cone metric spaces to cone b_2-metric spaces; also, it extends and improves Theorem 3.2 of [17] and Theorem 3.1 of [31] from one family and two families, respectively, to four families of mappings.

We obtain a series of new common fixed-point results using Theorem 30 for four families of mappings in the context of cone b_2-metric spaces over Banach algebras, which generalize and improve many known results from the existence literature.

Corollary 32. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \pm e$ and $\mathscr{C}_{\widehat{\mathcal{U}}} \neq \varnothing$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$\begin{aligned} d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) &\preccurlyeq \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\ &\quad + \beta\left[d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) + d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right)\right] \\ &\quad + \gamma\left[d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) + d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right)\right], \end{aligned} \tag{44}$$

where $\alpha, \beta, \gamma \in \mathscr{C}_{\widehat{\mathcal{U}}}$ with $r(\alpha) + r(\beta) + 2r(\gamma)r(b) < 1$, $r(\beta)r(b) + r(\gamma)r(b^2) < 1$, and α, β, γ, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \alpha$, $\vartheta_2 = \vartheta_3 = \beta$, $\vartheta_4 = \vartheta_5 = \gamma$ in Theorem 30. □

Corollary 33. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \pm e$ and $\mathscr{C}_{\widehat{\mathcal{U}}} \neq \varnothing$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$\begin{aligned} d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) &\preccurlyeq \vartheta_1 d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\ &\quad + \vartheta_2 d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) \\ &\quad + \vartheta_3 d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right), \end{aligned} \tag{45}$$

where $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathscr{C}_{\widehat{\mathcal{U}}}$ with $r(\vartheta_1) + r(\vartheta_2) + r(\vartheta_3) < 1$, $r(\vartheta_2)r(b) + r(\vartheta_3)r(b) < 1$, and $\vartheta_1, \vartheta_2, \vartheta_3, b$ commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Taking $\vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathcal{U}}}$ in Theorem 30, one can get the desired result. □

Remark 34. We can have Theorem 3.1 in [21], when $\{E_i\}_{i=1}^{\infty}$ and $\{F_j\}_{j=1}^{\infty}$ are the same mapping and $\{G_k\}_{k=1}^{\infty}$ and $\{H_l\}_{l=1}^{\infty}$ are the identity mappings. Therefore, Theorem 3.1 of [21] is a special case of Corollary 33. Also, Corollary 33 of this paper generalizes Theorem 2.1 of [10] from the cone 2-metric space to the cone b_2-metric space and extends Theorem 6.1 in [12].

Corollary 35. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \pm e$ and $\mathcal{C}_{\widehat{\mathcal{U}}} \neq \emptyset$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) + \beta\left[d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) + d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right)\right], \quad (46)$$

where $\alpha, \beta \in \mathcal{C}_{\widehat{\mathcal{U}}}$ with $r(\alpha) + 2r(\beta) < 1$, $2r(\beta)r(b) < 1$, and α, β, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ is a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. One can the result taking $\vartheta_1 = \alpha, \vartheta_2 = \vartheta_3 = \beta$, and $\vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathcal{U}}}$ in Theorem 30. □

Remark 36. Corollary 35 of this paper extends Theorem 6 in [32]; therefore, Theorem 6 in [32] is a special case of Corollary 35.

Corollary 37. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathcal{C}_{\widehat{\mathcal{U}}} \neq \emptyset$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) + \beta d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) + \gamma d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right), \quad (47)$$

where $\alpha, \beta, \gamma \in \mathcal{C}_{\widehat{\mathcal{U}}}$ with $r(\alpha) + 2r(\beta)r(b) + 2r(\gamma)r(b) < 1$, $r(\beta)r(b^2) + r(\gamma)r(b^2) < 1$, and α, β, γ, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. One can get the result taking $\vartheta_1 = \alpha, \vartheta_2 = \vartheta_3 = \theta_{\widehat{\mathcal{U}}}$ and $\vartheta_4 = \beta, \vartheta_5 = \gamma$ in Theorem 30. □

Corollary 38. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathcal{C}_{\widehat{\mathcal{U}}} \neq \emptyset$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) + l d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right), \quad (48)$$

where $k, l \in \mathcal{C}_{\widehat{\mathcal{U}}}$ with $r(k) + r(l) < 1$, $r(k)r(b) + r(l)r(b) < 1$, and k, l, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathcal{U}}}, \vartheta_2 = k, \vartheta_3 = l$ in Theorem 30. □

Corollary 39. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathcal{C}_{\widehat{\mathcal{U}}} \neq \emptyset$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) + l d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right), \quad (49)$$

where $k, l \in \mathcal{C}_{\widehat{\mathcal{U}}}$ with $2r(k)r(b) + 2r(l)r(b) < 1$, $r(k)r(b^2) + r(l)r(b^2) < 1$, and k, l, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \vartheta_2 = \vartheta_3 = \theta_{\widehat{\mathcal{U}}}, \vartheta_4 = k, \vartheta_5 = l$ in Theorem 30. □

Corollary 40. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathcal{C}_{\widehat{\mathcal{U}}} \neq \emptyset$ be a cone in $\widehat{\mathcal{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right), \quad (50)$$

where $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(k) < 1$ and k, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = k$, $\vartheta_2 = \vartheta_3 = \vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathscr{U}}}$ in Theorem 30. □

From the above corollary, we obtain the following.

Corollary 41. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a complete cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$ be the family of self-mapping on $\aleph$. For all $i \in \mathbb{N}$ and for all $s, m, z \in \aleph$,*

$$d(E_i(s), E_i(m), a) \preccurlyeq k d(s, m, a), \tag{51}$$

where $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(k) < 1$ and k, b commute. Then, $\{E_i\}_{i=1}^{\infty}$ have a unique common fixed point.

Proof. Taking $\eta_n = 1$, $E_i = F_j$, and G_k, H_l which are identity mappings in Corollary 40, then we can obtain the required result. □

We finish this section with an example that will demonstrate the consequence of Theorem 30.

Example 42. Let $\widehat{\mathscr{U}} = \mathbb{R}^2$. For each $(s_1, s_2) \in \widehat{\mathscr{U}}$, $\|(s_1, s_2)\| = |s_1| + |s_2|$. The multiplication is defined by $sm = (s_1, s_2)(m_1, m_2) = (s_1 m_1, s_1 m_2 + s_2 m_1)$. Then, $\widehat{\mathscr{U}}$ is a Banach algebra with unit element $e = (1, 0)$. Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{(s_1, s_2) \in \mathbb{R}^2 \mid s_1, s_2 \geq 0\}$. Then, $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone in $\widehat{\mathscr{U}}$.

Let $\aleph = \{(s, 0) \in \mathbb{R}^2 \mid s \geq 0\} \cup \{(0, 2)\} \subset \mathbb{R}^2$ and define $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ as follows:

$$d(S, M, Z) = \begin{cases} (0,0), & \text{if atleast two of } S, M, Z \text{ are equal}, \\ d(P(S, M, Z)), & P \text{ denotes permutations}, \\ (\Delta, \Delta), & \text{otherwise}, \end{cases} \tag{52}$$

where Δ is the square of the area of the triangle S, M, Z. We have

$$\begin{aligned} d((s,0),(m,0),(0,2)) \preccurlyeq\, & d((s,0),(m,0),(z,0)) \\ & + d((s,0),(z,0),(0,2)) \\ & + d((z,0),(m,0),(0,2)). \end{aligned} \tag{53}$$

That is, $(s-m)^2 \preccurlyeq (s-z)^2 + (z-m)^2$, which shows that d is not a cone 2-metric, because $(-9/2, -9/2) \in \mathscr{C}_{\widehat{\mathscr{U}}}$ for $s, m, z \geq 0$ with $s = 5$, $m = 0$, and $z = 1/2$. But for the parameter $b = (2, 0) \succcurlyeq e$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Now, we define mappings $E_i : \aleph \longrightarrow \aleph (i = 1, 2, 3, \cdots)$ by

$$\begin{aligned} E_i((s,0)) &= \left(\left(\frac{1}{6}\right)^{1/(2i-1)} \left(\frac{3}{2}\right)^{1/(2i-1)} \frac{s}{s^{2i-2}}, 0 \right), \\ E_i((0,2)) &= (0,0). \end{aligned} \tag{54}$$

We have

$$\begin{aligned} E_i^{2i-1}((s,0)) &= E_i^{2i-2}(E_i((s,0))) \\ &= E_i^{2i-2}\left(\left(\frac{1}{6}\right)^{1/(2i-1)} \left(\frac{3}{2}\right)^{1/(2i-1)} \frac{s}{s^{2i-2}}, 0 \right) \\ &= E_i^{2i-3}\left(\left(\frac{1}{6}\right)^{2/(2i-1)} \left(\frac{3}{2}\right)^{2/(2i-1)} \frac{s^2}{s^{2i-2}}, 0 \right) \\ &= E_i^{2i-4}\left(\left(\frac{1}{6}\right)^{3/(2i-1)} \left(\frac{3}{2}\right)^{3/(2i-1)} \frac{s^3}{s^{2i-2}}, 0 \right) \\ &= \cdots\cdots \\ &= E_i\left(\left(\frac{1}{6}\right)^{(2i-2)/(2i-1)} \left(\frac{3}{2}\right)^{(2i-2)/(2i-1)} \frac{s^{2i-2}}{s^{2i-2}}, 0 \right) \\ &= \left(\left(\frac{1}{6}\right)^{(2i-1)/(2i-1)} \left(\frac{3}{2}\right)^{(2i-1)/(2i-1)} \frac{s^{2i-1}}{s^{2i-2}}, 0 \right) \\ &= \left(\frac{1}{4} s, 0 \right). \end{aligned} \tag{55}$$

We define mappings $G_k : \aleph \longrightarrow \aleph$ $(k = 1, 2, 3, \cdots)$ by

$$\begin{aligned} G_k((s,0)) &= \left(\left(\frac{1}{3}\right)^{1/(2k-1)} \left(\frac{1}{2}\right)^{-1/(2k-1)} \frac{s}{s^{2k-2}}, 0 \right), \\ G_k((0,2)) &= (0,0). \end{aligned} \tag{56}$$

We have

$$\begin{aligned} G_k^{2k-1}((s,0)) &= G_k^{2k-2}(G_k((s,0))) \\ &= G_k^{2k-2}\left(\left(\frac{1}{3}\right)^{1/(2k-1)} \left(\frac{1}{2}\right)^{-1/(2k-1)} \frac{s}{s^{2k-2}}, 0 \right) \\ &= G_k^{2k-3}\left(\left(\frac{1}{3}\right)^{2/(2i-1)} \left(\frac{1}{2}\right)^{-2/(2k-1)} \frac{s^2}{s^{2k-2}}, 0 \right) \\ &= G_k^{2k-4}\left(\left(\frac{1}{3}\right)^{3/(2k-1)} \left(\frac{1}{2}\right)^{-3/(2k-1)} \frac{s^3}{s^{2k-2}}, 0 \right) \\ &= \cdots\cdots \\ &= G_k\left(\left(\frac{1}{3}\right)^{(2k-2)/(2k-1)} \left(\frac{1}{2}\right)^{(-2k-2)/(2k-1)} \frac{s^{2k-2}}{s^{2k-2}}, 0 \right), \end{aligned}$$

$$\begin{aligned} G_k^{2k-1}((s,0)) &= \left(\left(\frac{1}{3}\right)^{(2k-1)/(2k-1)} \left(\frac{1}{2}\right)^{(-2k+1)/(2k-1)} \frac{s^{2k-1}}{s^{2k-2}}, 0 \right) \\ &= \left(\frac{2}{3} s, 0 \right). \end{aligned} \tag{57}$$

Similarly, we define mappings $F_j, H_l : \aleph \longrightarrow \aleph$ $(j, l = 1, 2, 3, \cdots)$ by

$$\begin{gathered} F_j((s,0)) = \left(\left(\frac{2}{3}\right)^{1/(2j-1)}\left(\frac{3}{10}\right)^{1/(2j-1)}\frac{s}{s^{2j-2}},0\right), \\ F_j((0,2)) = (0,0), \\ H_l((s,0)) = \left(\left(\frac{2}{3}\right)^{1/(2l-1)}\left(\frac{1}{2}\right)^{-1/(2l-1)}\frac{s}{s^{2l-2}},0\right), \\ H_l((0,2)) = (0,0). \end{gathered} \tag{58}$$

Then, it is not difficult to show that $F_j^{2j-1}((s,0)) = ((1/5)s,0)$ and $H_l^{2l-1}((s,0)) = ((1/3)s,0)$. Choose $\vartheta_1 = (1/10,0)$, $\vartheta_2 = \vartheta_3 = (1/8,0)$, and $\vartheta_4 = \vartheta_5 = (1/16,0)$. Clearly,

$$\begin{aligned} &\sum_{w=1}^{3} r(\vartheta_w) + 2r(\vartheta_4)r(b) + 2r(\vartheta_5)r(b) \\ &\quad = \frac{1}{10} + \frac{1}{8} + \frac{1}{8} + 2\left(\frac{1}{16}\right)2 + 2\left(\frac{1}{16}\right)2 = \frac{34}{40} < 1, \end{aligned} \tag{59}$$

also $r(\vartheta_2)r(b) + r(\vartheta_5)r(b^2) = 2(1/8) + 4(1/16) = 1/2 < 1$ and $r(\vartheta_3)r(b) + r(\vartheta_4)r(b^2) = 2(1/8) + 4(1/16) = 1/2 < 1$.

Now, considering the contractive condition (8), we have

$$\begin{aligned} d\left(\left(\frac{s}{4},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \preccurlyeq{}& \left(\frac{1}{10},0\right)d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{3},0\right),(0,2)\right) \\ &+ \left(\frac{1}{8},0\right)d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{4},0\right),(0,2)\right) \\ &+ \left(\frac{1}{8},0\right)d\left(\left(\frac{s}{3},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \\ &+ \left(\frac{1}{16},0\right)d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \\ &+ \left(\frac{1}{16},0\right)d\left(\left(\frac{s}{3},0\right),\left(\frac{s}{4},0\right),(0,2)\right), \end{aligned} \tag{60}$$

that is,

$$\begin{aligned} \left(\left(\frac{s}{4}-\frac{s}{5}\right)^2,\left(\frac{s}{4}-\frac{s}{5}\right)^2\right) \preccurlyeq{}& \left(\frac{1}{10},0\right)\left(\left(\frac{2s}{3}-\frac{s}{3}\right)^2,\left(\frac{2s}{3}-\frac{s}{3}\right)^2\right) \\ &+ \left(\frac{1}{8},0\right)\left(\left(\frac{2s}{3}-\frac{s}{4}\right)^2,\left(\frac{2s}{3}-\frac{s}{4}\right)^2\right) \\ &+ \left(\frac{1}{8},0\right)\left(\left(\frac{s}{3}-\frac{s}{5}\right)^2,\left(\frac{s}{3}-\frac{s}{5}\right)^2\right) \\ &+ \left(\frac{1}{16},0\right)\left(\left(\frac{2s}{3}-\frac{s}{5}\right)^2,\left(\frac{2s}{3}-\frac{s}{5}\right)^2\right) \\ &+ \left(\frac{1}{16},0\right) + \left(\left(\frac{s}{3}-\frac{s}{4}\right)^2,\left(\frac{s}{3}-\frac{s}{4}\right)^2\right) \\ ={}& \left(\frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2,\frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2\right) \\ &+ \left(\frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2,\frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2\right) \\ &+ \left(\frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2,\frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2\right) \\ &+ \left(\frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2,\frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2\right) \\ &+ \left(\frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2,\frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2\right), \end{aligned} \tag{61}$$

which means that

$$\begin{aligned} \left(\frac{s}{4}-\frac{s}{5}\right)^2 \preccurlyeq{}& \frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2 + \frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2 \\ &+ \frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2 + \frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2 \\ &+ \frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2, \left(\frac{s}{20}\right)^{2\circ}\frac{1}{10}\left(\frac{s}{3}\right)^2 \\ &+ \frac{1}{8}\left(\frac{5s}{12}\right)^2 + \frac{1}{8}\left(\frac{2s}{15}\right)^2 \\ &+ \frac{1}{16}\left(\frac{7s}{15}\right)^2 + \frac{1}{16}\left(\frac{s}{12}\right)^2, \end{aligned} \tag{62}$$

that is,

$$\frac{s^2}{400} \preccurlyeq \frac{s^2}{90} + \frac{25s^2}{1152} + \frac{4s^2}{1800} + \frac{49s^2}{3600} + \frac{s^2}{2304}, \tag{63}$$

which shows that $s^2/400 \preccurlyeq 2827s^2/57600$, and so $(2827s^2/57600) - (s^2/400) \in \mathscr{C}_{\widehat{\mathscr{U}}}$, which is true for all $s \geq 0$. Hence, condition (8) is true for all $s, m, a \in \aleph$ and $i, j, k, l \geq 1$, where $\eta_i = 2i-1$, $\eta_j = 2j-1$, $\eta_k = 2k-1$, and $\eta_l = 2l-1$. All other conditions of Theorem 30 are satisfied. By Theorem 30, E_i, F_j, G_k, and H_l have a unique common fixed point $(0,0)$ for all $i, j, k, l \geq 1$.

3. Application to the Infinite System of Integral Equations

We give here a couple of auxiliary facts that will be needed in our further considerations. Let $\widehat{\mathscr{U}} = \mathbb{R}^2$ with norm $\|.\|_{\widehat{\mathscr{U}}}$ be a real Banach algebra. Let $I = [0,T]$, and denote by $C(I,\widehat{\mathscr{U}})$ the space consisting of all continuous functions defined on interval I with values in the Banach algebra $\widehat{\mathscr{U}}$. The space $C(I,\widehat{\mathscr{U}})$ will be equipped with $\|s\| = \max\{\|s(a)\|_{\widehat{\mathscr{U}}} : a \in I\}$.

Let $\aleph = C(I,\widehat{\mathscr{U}})$ and define $d : \aleph^3 \longrightarrow \widehat{\mathscr{U}}$ by

$$d(s(t),m(t),z) = [\min\{|s(t)-m(t)|,|m(t)-z|,|s(t)-z|\}]^p, \tag{64}$$

where $p \geq 1$ and for all $s(t), m(t), z \in \aleph$. Then, $(\aleph, d)$ is a complete cone b_2-metric space over the Banach algebra.

We consider the infinite system of integral equations of the form

$$s_i(t) = g_i(t) + \int_0^T M_i(t,w)f_i(w,s(w))dw, \tag{65}$$

where $i = 1, 2, 3, \cdots$. Let $E_i : \aleph \longrightarrow \aleph$. We redefine the above infinite system of integral equations as

$$E_i(s_i(t)) = g_i(t) + \int_0^T M_i(t, w) f_i(w, s(w)) dw, \qquad (66)$$

for all $s_i(t) \in \aleph$ and $t, w \in I$. Clearly, by using Corollary 41, the existence of solution to (65) is equivalent to the existence of a common fixed point of E_i.

We assume that

(i) $g_i : I \longrightarrow \mathbb{R}$ are continuous for each $i = 1, 2, 3, \cdots$

(ii) $M_i : I \times \mathbb{R} \longrightarrow [0,+\infty)$ are continuous and $\int_0^T M_i(t, w) dw \le 1$ for each $i = 1, 2, 3, \cdots$

(iii) $f_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous for each $i = 1, 2, 3, \cdots$ such that

$$|f_i(w, s(w)) - f_i(w, m(w))| \le v^{1/p}[\min\{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}], \qquad (67)$$

for all $s(w), m(w), z \in \aleph$ and $0 \le v < 1$.

Theorem 43. *Under the assumptions (i)–(iii), the infinite system of integral equations (65) has a unique solution in* $\aleph$.

Proof. Take $\widehat{\mathscr{U}} = \mathbb{R}^2$ with norm $\|s\| = \|(s_1, s_2)\| = |s_1| + |s_2|$, and multiplication is defined by the following way:

$$sm = (s_1, s_2)(m_1, m_2) = (s_1 m_1, s_1 m_2 + s_2 m_1). \qquad (68)$$

Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{(s_1, s_2) \in \widehat{\mathscr{U}} : s_1, s_2 \ge 0\}$. It is clear that $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a normal cone and $\widehat{\mathscr{U}}$ is a Banach algebra with unit element $e = (1, 0)$.

Consider the family of mapping $E_i : \aleph \longrightarrow \aleph$ defined by (66). Let $s_i(t), m_i(t), z \in \aleph$. From (64), we deduce that

$$\begin{aligned} d(E_i(s_i(t)), E_i(m_i(t)), z) &= \max_{a\in[0,T]} [\min\{|E_i(s_i(t)) - E_i(m_i(t))|, |E_i(s_i(t)) - z|, |E_i(m_i(t)) - z|\}]^p \\ &\preccurlyeq \left(\max_{a\in[0,T]} |E_i(s_i(t)) - E_i(m_i(t))|\right)^p \\ &= \left(\max_{a\in[0,T]} \left|\int_0^T M_i(t, w) f_i(w, s(w)) dw - \int_0^T M_i(t, w) f_i(w, m(w)) dw\right|\right)^p \\ &= \left(\max_{a\in[0,T]} \left|\int_0^T M_i(t, w)[f_i(w, s(w)) - f_i(w, m(w))] dw\right|\right)^p \\ &\preccurlyeq \left(\max_{a\in[0,T]} \int_0^T M_i(t, w) |f_i(w, s(w)) - f_i(w, m(w))| dw\right)^p \\ &\preccurlyeq \left(\max_{a\in[0,T]} \int_0^T M_i(t, w) v^{1/p} [\min\{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}] dw\right)^p \\ &\preccurlyeq \left(\int_0^T \left(\max_{a\in[0,T]} M_i(t, w)\right) v^{1/p} \cdot \left(\max_{a\in[0,T]} [\min\{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}]^p\right)^{1/p} dw\right)^p \\ &\le \left(\int_0^T \left(\max_{a\in[0,T]} M_i(t, w)\right) v^{1/p} \cdot (d(s_i(t), m_i(t), z))^{1/p} dw\right)^p. \end{aligned} \qquad (69)$$

Therefore,

$$d(E_i(s_i(t)), E_i(m_i(t)), z) \preccurlyeq v d(s_i(t), m_i(t), z). \qquad (70)$$

Now, all the assumptions of Corollary 41 are fulfilled and the family of mapping E_i has a unique common fixed point in $\aleph$, which means that the infinite system of integral equations (65) has a unique solution in $\aleph$. ☐

Data Availability

Data are available upon request or included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors make equal contributions and read and supported the last original copy.

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Research Article

Extragradient Method for Fixed Points in CAT(0) Spaces

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This paper is dedicated to construct a viscosity extragradient algorithm for finding fixed points in a CAT(0) space. The mappings we consider are nonexpansive. Strong convergence of the algorithm is obtained. The results established in this work extend and improve some recent discovers in the literature.

1. Introduction

Let X be a CAT(0) space and K be any closed convex subset of X. Let $\phi : K \times K \longrightarrow \mathbb{R}$ be a bifunction with $\phi(v, v) = 0$ for all $v \in K$. The main equilibrium problem is to get $r \in K$ satisfying

$$\phi(r, w) \geq 0, \quad \text{for all } w \in K. \tag{1}$$

Finding solutions for equilibrium problems aids in solving problems in other areas of science like physics, optimization, and economics. The set of all solutions of the equilibrium problem is denoted by $\mathrm{EP}(\phi)$, i.e.,

$$\mathrm{EP}(\phi) = \{r \in K : \phi(r, w) \geq 0, \forall w \in K\}. \tag{2}$$

Numerous iterative algorithms for monotome equilibrium have been investigated previously; for finding the solutions, see [1, 2]. Here, we will find the iterative algorithm for pseudomonotone bifunction. The bifunction ϕ is said to be pseudomonotone if

$$\phi(v, w) \geq 0 \longrightarrow \phi(w, v) \leq 0, \quad \text{for all } v, w \in K. \tag{3}$$

An extragradient method, to solve a pseudomonotone equilibrium problem in $\mathbb{R}^n$, was introduced in [3]. The method of extragradient is as follows: given $v_0 \in K$, find successively w_n and v_{n+1} by

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \dfrac{1}{2} \|w - v_n\|^2 \right\}, \\ v_{n+1} = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(w_n, w) + \dfrac{1}{2} \|w - v_n\|^2 \right\}, \end{cases} \tag{4}$$

where $\{\lambda_n\} \subset (0,\infty)$ and ϕ is such that the Lipschitz (type) condition holds. In [4], the following algorithm was introduced by Anh for finding a fixed point of a nonexpansive mapping T which is also the solution of the equilibrium problem for pseudomonotone bifunction ϕ in a Hilbert space:

$$\begin{cases} w_n = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(v_n, w) + \dfrac{1}{2} \|w - v_n\|^2 \right\}, \\ v_{n+1} = \operatorname{argmin}_{w \in K} \left\{ \lambda_n \phi(w_n, w) + \dfrac{1}{2} \|w - v_n\|^2 \right\}, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\lambda_n\} \subset (0, 1]$ and ϕ is such that the Lipschitz (type) condition holds. The convergence (strong) of $\{v_n\}$ with $\lim_{n \longrightarrow \infty} \|v_{n+1} - v_n\| = 0$ and under certain considerations on $\{\alpha_n\}, \{\lambda_n\} \in (0, 1]$.

In 2012, an algorithm for hybrid projection was considered by Vuong et al. [5] for

$$\begin{cases} w_n = \operatorname{argmin}_{w\in K}\left\{\lambda_n \phi(v_n, w) + \frac{1}{2}\|w - v_n\|^2\right\}, \\ v_{n+1} = \operatorname{argmin}_{w\in K}\left\{\lambda_n \phi(v_n, w) + \frac{1}{2}\|w - v_n\|^2\right\}, \\ s_n = \alpha_n v_0 + (1-\alpha_n[\beta_n z + (1-\beta_n)Sz_n], \\ K_n = \{z \in K : \|s_n - z\| \le \|v_n - z\|\}, \\ D_n = \{z \in K : \langle v_n - z, v_0 - v_n\rangle \ge 0\}, \\ v_{n+1} = P_{K_n \cap D_n} v_0, \end{cases} \quad (6)$$

where $\{\alpha_n\}, \{\lambda_n\} \in (0,1]$ and ϕ is such that the Lipschitz (type) condition holds. The convergence proved was strong.

The Armijo-type method for pseudomonotone equilibrium problems was formulated in [6] in the setting of Hilbert spaces. After that, the authors in [7] presented the convergence of weak and strong types for the algorithms in order to solve the equilibrium problem. The admirable outcomes are due to Dinh and Kim in which there is no restriction on monotonicity of the bifunction. The current results of equilibrium problems are given for pseudomonotone type. For further references, see [8, 9]. To the current knowledge, the authors modified the "hybrid projection algorithm" in order to get convergence of strong type for iterative algorithms of equilibrium problems of pseudomonotone type [10, 11].

The aim of this paper is to construct an extragradient algorithm of viscosity type for finding the same element for the solution set of a pseudomonotone equilibrium problem and fixed point set of a nonexpansive mapping in the framework of a CAT(0) space and derive its strong convergence.

2. Definitions and Known Results

In this section, we present basic definitions and known results. The notions that are not defined in this paper can be seen in [12–14].

Throughout this paper, (X, d) denotes the geodesic metric space with a geodesic triangle $\Delta(v_1, v_2, v_3)$ in (X, d), where v_1, v_2, and v_3 represent the vertices of Δ in (X, d). We will represent (X, d) as X henceforth. A *comparison triangle* in X is a triangle $\bar{\Delta}(v_1, v_2, v_3) \coloneqq \Delta(\bar{v_1}, \bar{v_2}, \bar{v_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(v_i, v_j) = d(v_i, v_j)$ for $i, j = 1, 2, 3$.

A geodesic space is called a CAT(0) space, if all the geodesic triangles satisfy the following comparison axiom

$$d(v, w) \le d_{\mathbb{E}^2}(\bar{v}, \bar{w}). \quad (7)$$

Let $v, w \in X$, and by Lemma 2.1(iv) of [15] for each $s \in [0,1]$, there exists a unique point $z \in [v, w]$ such that

$$d(v, z) = sd(v, w), \quad (8)$$

$$d(w, z) = (1-s)d(v, w), \quad (9)$$

for all $v, w \in \Delta(v_1, v_2, v_3)$ with $\bar{v}, \bar{w}$ the corresponding points of v, w in $\bar{\Delta}(v_1, v_2, v_3)$.

Lemma 1 (see [16]). *Let X be a CAT(0) space. Then the following assertions are true:*

(i) For any $v, w, z \in X$ and $s \in [0, 1]$,

$$d((1-s)v \oplus sy, z) \le (1-s)d(v, z) + sd(w, z) \quad (10)$$

(ii) For any $v, w, z \in X$ and $s \in [0, 1]$,

$$d^2((1-s)v \oplus sy, z) \le (1-s)^2 d(v, z) + sd^2(w, z) - s(1-s)d^2(v, w) \quad (11)$$

Let (X, d) be a uniquely geodesic metric space; that is, for each $v, w \in X$, there exists a unique isometry $c : [0, d(v, w)] \longrightarrow X$ such that $c(0) = v$ and $c(d(v, w)) = w$, and in this case, we write $[v, w] = \{c(t)\colon t \in [0, d(x, y)]\}$. For each $s \in [0, 1]$, we write $(1-s)v \oplus sw$ for the element $z \in [v, w]$ such that $d(z, v) = sd(v, w)$ and $d(z, w) = (1-s)d(v, w)$.

Hadamard spaces are the complete CAT(0) spaces; for details, see [16]. If v, w_1, w_2 are points of a CAT(0) space and w_0 is the midpoint of the segment $[w_1, w_2]$, which we will denote by $w_1/2 \oplus w_2/2$, then the CAT(0) inequality gives

$$d^2\left(v, \frac{w_1}{2} \oplus \frac{w_2}{2}\right) \le \frac{1}{2}d^2(v, w_1) + \frac{1}{2}d^2(v, w_2) - \frac{1}{4}d^2(w_1, w_2). \quad (12)$$

This inequality is called the (CN) inequality of Bruhat and Tits [16]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [16], page 163). Berg and Nikolaev [17] introduced the idea of the quasilinearization as follows: Let us denote the pair $(\alpha, \beta) \in X \times X$ by $\overleftarrow{\alpha\beta}$ and call it a vector. Then, quasilinearization is defined as a map

$$\langle .,. \rangle\colon (X \times X) \times (X \times X) \longrightarrow \mathbb{R} \quad (13)$$

defined as

$$\left\langle \overrightarrow{\alpha\beta}, \overrightarrow{v\vartheta} \right\rangle = \frac{1}{2}\left(d^2(\alpha, \vartheta) + d^2(\beta, v) - d^2(\alpha, v) - d^2(\beta, \vartheta)\right). \quad (14)$$

It is easy to see that $\langle \overrightarrow{\alpha\beta}, \overrightarrow{v\vartheta} \rangle = \langle \overrightarrow{v\vartheta}, \overrightarrow{\alpha\beta} \rangle$, $\langle \overrightarrow{\alpha\beta}, \overrightarrow{v\vartheta} \rangle = -\langle \overrightarrow{\beta\alpha}, \overrightarrow{v\vartheta} \rangle$, and $\langle \overrightarrow{\alpha x}, \overrightarrow{v\vartheta} \rangle + \langle \overrightarrow{x\beta}, \overrightarrow{v\vartheta} \rangle = \langle \overrightarrow{\alpha\beta}, \overrightarrow{v\vartheta} \rangle$ for all $\alpha, \beta, v, \vartheta \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\left\langle \overleftarrow{\alpha\beta}, \overleftarrow{\nu\vartheta} \right\rangle \leq d(\alpha, \beta) d(\nu, \vartheta) \tag{15}$$

for all $\alpha, \beta, \nu, \vartheta \in X$. It is well known [17] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 1976, Lim gave the definition of Δ-convergence in a general metric space case (see [18]). He named this type of convergence as strong Δ-convergence, proving that CAT(0) spaces provide a natural framework for the Lim concept and, in the setting of Δ-convergence, provide many properties of the usual notion of weak convergence in Banach spaces. Let us recall this type of convergence for the case of CAT(0) spaces, as follows.

Definition 2. Let (X, d) be a CAT(0) space. A sequence $\{\nu_n\}$ in X is said to Δ-converge to $\nu \in X$ if and only if ν is the unique asymptotic center of all subsequences of $\{\nu_n\}$. In this case, we write $\Delta - \lim_{n\longrightarrow\infty} \nu_n = \nu$, and ν is called the Δ-limit of $\{\nu_n\}$.

Note that the asymptotic center $\mathbb{A}(\{\nu_n\})$ of $\{\nu_n\}$ is the set

$$\mathbb{A}(\{\nu_n\}) \coloneqq \{\nu \in X : r(\nu, \{\nu_n\}) = r(\{\nu_n\})\}, \tag{16}$$

and the asymptotic radius $r(\{\nu_n\})$ of $\{x_n\}$ is given by

$$r(\{\nu_n\}) \coloneqq \inf_{\nu \in X} \{r(x, \{\nu_n\})\}, \tag{17}$$

where $r(\nu, \{\nu_n\}) \coloneqq \limsup_{n\longrightarrow\infty} d(\nu_n, \nu)$, for $\{\nu_n\}$ is a bounded sequence in X.

Ahmadi Kakavandi and Amini introduced in [19] another variant of weak convergence in complete CAT(0) spaces, taking into account the concept of quasilinearization.

Definition 3 (see [19]). Let (X, d) be a complete CAT(0) space. A sequence $\{\nu_n\}$ in X is said to w-converge to an element $\nu \in X$ if for each $w \in X$, $\lim_{n\longrightarrow\infty} \langle \overrightarrow{\nu\nu_n}, \overrightarrow{\nu w} \rangle = 0$.

Obviously, the convergence in the metric implies w-convergence, and the w-convergence implies Δ-convergence (see Proposition 2.5 in [19]). But the converse is not true (see [20]). The following result proves an explicit connection between w-convergence and Δ-convergence.

Theorem 4 (see [20]). *Let (X, d) be a complete CAT(0) space. Then a sequence $\{\nu_n\}$ in X Δ-converges to $\nu \in X$ if and only if, for every $w \in X$, $\limsup_{n\longrightarrow\infty} \langle \overrightarrow{\nu\nu_n}, \overrightarrow{\nu w} \rangle \leq 0$.*

Further, let us recall some definitions for the case of a CAT(0) space.

Definition 5. Let X be a CAT(0) space and $T : X \longrightarrow X$ be a mapping. Then T is called nonexpansive if

$$d(T(\nu), T(w)) \leq d(\nu, w), \quad \text{for every } \nu, w \in K. \tag{18}$$

Definition 6. Let X be a CAT(0) space and $T : X \longrightarrow X$ be a mapping. Then T is called a contraction if

$$d(T(\nu), T(w)) \leq \theta d(\nu, w), \quad \text{for every } \nu, w \in K, \theta \in [0, 1). \tag{19}$$

Throughout this paper, we denote by $\mathrm{Fix}(T) = \{\nu \in C \mid T\nu = \nu\}$ the set of fixed points of T.

Concerning the convexity in CAT(0) spaces, we remark that, in this type of spaces, angles exist in a strong sense, the distance function is convex, and one has both uniform convexity and orthogonal projection onto convex subsets. Moreover, CAT(0) spaces turn up to represent a real framework for convexity theory.

Remark 7. Considering the CAT(0) space case, a subset $C \subset X$ is said to be convex if C includes every geodesic segment joining any two of its points, i.e., $(1-t)x \oplus ty \in C$, for every $x, y \in C$ and $t \in (0, 1)$.

Definition 8. Let X be a CAT(0) space. A function $f : X \longrightarrow (-\infty, +\infty]$ is said to be convex if

$$f(t\nu \oplus (1-t)w) \leq tf(\nu) + (1-t)f(w), \tag{20}$$

for all $\nu, w \in X$, and $t \in (0, 1)$.

Let X be a CAT(0) space and $K \subset X$ a convex and closed subset. Further, let a bifunction $\phi : K \times K \longrightarrow \mathbb{R}$; then ϕ is called

(1) ς-strong monotone $(\varsigma < 0)$ on K if $\forall \nu, w \in K$, we have

$$\phi(\nu, w) + \phi(w, \nu) \leq \varsigma d^2(\nu, w) \tag{21}$$

(2) monotone on K if for each $\nu, w \in K$, one has

$$\phi(\nu, w) + \phi(w, \nu) \leq 0 \tag{22}$$

(3) pseudomonotone on K if for each $\nu, w \in K$, one has

$$\phi(\nu, w) \geq 0 \Rightarrow \phi(w, \nu) \leq 0 \tag{23}$$

The above bifunction ϕ is Lipschitz-type continuous on

K, if there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$\phi(v,z) \le \phi(v,w) + \phi(w,z) + c_1 d^2(v,w) + c_2 d^2(w,z), \quad \text{for every } v,w \in K. \tag{24}$$

In [19], Kakavandi and Amini define the notion of a subdifferential of a function as follows.

Definition 9 (see [19]). Let $f : X \longrightarrow (-\infty,+\infty]$ be a proper function with efficient domain $\mathcal{D}(f) = \{v : f(v) < +\infty\}$, then the subdifferential of f is the multifunction $\partial f : X \rightrightarrows X^*$ defined by

$$\partial f(v^*) = \left\{v^* : f(z) - f(v) \ge \left\langle v^*, \overrightarrow{vz}\right\rangle, \text{for all } z \in X\right\}, \tag{25}$$

when $v \in \mathcal{D}(f)$ and $\partial f(v) = \varnothing$.

We mention that X^* denote the notion of dual space of the metric space (X,d). For more details, see [19].

In [21], Georgiou and Papadopoulos gave a strong discussion concerning the convergence types and topologies on function spaces. Then, let us consider X, Y, Z as three topological spaces. A mapping $f : X \longrightarrow Y$ into a is called *weakly continuous* at $v \in X$ if for every open neighbourhood V of $f(v)$ there exists an open neighbourhood U of v such that $f(U) \subseteq \mathrm{Cl}(V)$. $\mathrm{Cl}(V)$ denotes the closure of V. The mapping f is weakly continuous on X, if it is weakly continuous at each point of X. In the following, $\mathrm{WC}(Y,Z)$ denotes the set of all weakly continuous maps of Y into Z. If τ is a topology on the set $\mathrm{WC}(Y,Z)$, then the corresponding topological space is denoted by $\mathrm{WC}_\tau(Y,Z)$.

In this conditions, we can recall the notion of *jointly weakly continuous* function.

Definition 10. A topology τ on $\mathrm{WC}(Y,Z)$ is called weakly jointly continuous if for every X, the weak continuity of a map $G : X \longrightarrow \mathrm{WC}_\tau(Y,Z)$ implies the weak continuity of the map $\tilde{G} : X \times Y \longrightarrow Z$.

For more details and results concerning the topology and the convergence types in function spaces, see [21–23].

Further, taking into account the previous notions, let us consider the following properties of ϕ:

(1) $\phi(v,v) = 0$ for all $v \in K$, and ϕ is taken pseudomonotone on the subset K

(2) ϕ is taken to be Lipschitz-type continuous on the subset K

(3) For all $v \in K$, $w \longrightarrow \phi(v,w)$ is subdifferentiable and convex

(4) $\phi(v,w)$ is taken jointly weakly continuous on $K \times K$

With conditions (6), (7), (8), and (10), the set $\mathrm{EP}(\phi)$ is convex and closed.

Lemma 11 (see [24]). *Let X be a CAT(0) space. Assume that $EP(\phi) \ne \varnothing$ and $v \in K$. Further, let $w, s \in K$ be solutions of*

$$w = argmin_{z\in K}\left\{\lambda\phi(v,w) + \frac{1}{2}d^2(v,z)\right\}, \tag{26}$$

$$s = argmin_{a\in K}\left\{\lambda\phi(v,w) + \frac{1}{2}d^2(w,a)\right\}, \tag{27}$$

strongly convex problems, where $\lambda > 0$, then

$$\begin{aligned} \lambda[\phi(v,z) - \phi(v,w)] &\ge \left\langle \overrightarrow{wv}, \overrightarrow{wz}\right\rangle, \quad z \in K, \\ d^2(s,x) &\le d^2(v,x) - (1-2\lambda c_1)d^2(v,w) \\ &\quad - (1-2\lambda c_2)d^2(s,x), \quad \forall x \in EP(\phi). \end{aligned} \tag{28}$$

Lemma 12 (the demiclosedness principle). *Let K be a nonempty closed convex subset of the CAT(0) space X and $T : K \longrightarrow K$ such that*

$$\begin{aligned} v_n &\rightharpoonup v \in K, \\ d(v_n, Tv_n) &\longrightarrow 0. \end{aligned} \tag{29}$$

Then, $v = Tv$. (Here, $\longrightarrow$ (respectively, $\rightharpoonup$) denotes strong (respectively, weak) convergence.) The notion of weak convergence is the same as defined in [25].

Lemma 13 (see [26]). *Let X be a CAT(0) space and K be any closed convex subset of X. For each point $v \in X$, there exists a unique nearest point of K, denoted by $P_K v$, such that $d(v, P_K v) \le d(v,w)$ for all $w \in K$. Such a P_K is metric projection onto K from X. Then,*

(1) for all $v \in X$ and $z \in K$, $z = P_K v$ iff

$$\left\langle \overrightarrow{vz}, \overrightarrow{zw}\right\rangle \ge 0, \quad \text{for all } w \in K \tag{30}$$

(2) for all $v \in X$ and $z \in K$, it holds

$$d^2(P_K v, z) \le d^2(v,z) - d^2(P_K v, v) \tag{31}$$

For more information, see Section 3 of [27].

Lemma 14. *Let X be a complete CAT(0) space. For all $v, w \in X$ and $0 = [(0,1)] \in X$, the following hold:*

(1) $d^2(v \oplus w, 0) \le d^2(v,0) + 2\langle \overrightarrow{w0}, \overrightarrow{v0}\rangle + 2d^2(w,0)$

(2) $d^2(sv \oplus (1-s)w, 0) = sd^2(v,0) + (1-s)d^2(w,0) - s(1-s)d^2(v,w)$ for all $s \in [0,1]$

Consider $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}, \{\alpha_n\} \subset (0, 1/(2-\rho)), \{\beta_n\} \subset (0,1)$, and $v_0 \in K$. Consider $n = 0$.

Algorithm 19

Lemma 15 (see [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\beta_n)a_n + \delta_n, \quad \text{for all } n \ge 0, \tag{32}$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence with

(1) $\sum_{n=0}^{\infty}\beta_n = \infty$

(2) $\limsup_{n\longrightarrow\infty}\delta_n/\beta_n \le 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Another interesting result useful in the proof of our main results is the following.

Lemma 16 (see [29]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that there exists subsequences $\{n_j\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$ as $k \longrightarrow \infty$, and the following properties are satisfied all (sufficiently large) number $k \in \mathbb{N}$:*

$$\begin{aligned} a_{m_k} &< a_{m_k+1}, \\ a_k &< a_{m_k+1}. \end{aligned} \tag{33}$$

In fact, $m_k = \max\{j \le k : a_j \le a_{j+1}\}$.

Example 17. Let $X = R^n (n \ge 2)$ and $K = \{(v_1, v_2, \cdots, v_n): v_i \ge 1, i = 1, 2, \cdots, n\}$. Consider a bifunction $f : K \times K \longrightarrow R$ defined as $f(v, w) = 2(w_n - v_n)d(v, w)$ for each $v, w \in K$. Then we have the inequality

$$\begin{aligned} f(v,w) = 2(w_n - v_n)d(v,w) &\le 2(w_n - r_n)d(v,r) \\ &+ 2(w_n - r_n)d(w,r) + 2(r_n - v_n)d(v,r) \\ &+ 2(r_n - v_n)d(r,w) \le 2d(w,r)d(v,r) \\ &+ 2d(r,v)d(r,w) + f(v,r) + f(r,w) \le 2d^2(v,r) \\ &+ 2d^2(r,w) + f(v,r) + f(r,w). \end{aligned} \tag{34}$$

Then, f is Lipschitz-type continuous on K with $c_1 = 2 = c_2$.

3. Strong Convergence of the Proposed Algorithm

Suppose that a nonexpansive mapping $T : K \longrightarrow K$ and a bifunction $\phi : K \times K \longrightarrow R$ satisfy the conditions (6)–(10) and $\mathrm{Fix}(T) \cap \mathrm{EP}(\phi) \ne \varnothing$. Further, let $p : K \longrightarrow K$ be a ρ-contraction. Because $P_{\mathrm{Fix}(T)\cap\mathrm{EP}(\phi)}p$ is a contraction mapping on K, we have $\widehat{q} \in K$, such that $\widehat{q} = P_{\mathrm{Fix}(T)\cap\mathrm{EP}(\phi)}p(\widehat{q})$.

Before presenting the first result of this section, let us recall a crucial theorem given by Lim in [18], which is used in the proof of our main results.

Theorem 18 ([18], Theorem 5.2). *Every bounded sequence $\{x_n\}$ in a Hadamard space (X, d) has a Δ-convergent subsequence.*

The following algorithm is useful in finding a common element of a solution set of pseudomonotone equilibrium problem on ϕ and fixed point set of T.

Step 1.

$$\begin{cases} w_n = \operatorname{argmin}_{w\in K}\left\{\lambda_n\phi(v_n, w) + \dfrac{1}{2}d^2(w, v_n)\right\}, \\ v_n = \operatorname{argmin}_{s\in K}\left\{\lambda_n\phi(w_n, s) + \dfrac{1}{2}d^2(s, v_n)\right\}. \end{cases} \tag{35}$$

Step 2. If $w_n = v_n = Tv_n$, then $v_n \in \mathrm{Fix}(T) \cap \mathrm{EP}(\phi)$, stop the process; otherwise, go to Step 3.

Step 3. Generate $v_{n+1} = \alpha_n p(v_n) \oplus (1-\alpha_n)(\beta_n v_n \oplus (1-\beta_n)T s_n)$. Set $n = n+1$ and go to Step 1. Obviously, if $w_n = v_n$ for $n \in N$, by using (26), it gives $\phi(v_n, z) \ge 0$ for all $z \in K$ and $v_n \in \mathrm{EP}(\phi)$. It gives $v_n \in \mathrm{Fix}(T) \cap \mathrm{EP}(\phi)$: from $v_n = Tv_n$. Further, for convergence of the algorithm, let us consider that Step 2 is not true for $n \in N$.

Lemma 20. *Consider $\{v_n\}$ to be bounded sequence. If $d(v_n, w_n) \longrightarrow 0$, $d(v_n, s_n) \longrightarrow 0$, $d(s_n, Ts_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then*

$$\limsup_{n\longrightarrow\infty}\left\langle \overrightarrow{v_n\widehat{q}}, \overrightarrow{p(\widehat{q})\widehat{q}} \right\rangle \le 0. \tag{36}$$

Proof. We consider the following bounded sequence $\{v_n\}$. Using Theorem 18, there exists $\{v_{n_k}\}$ a subsequence of $\{v_n\}$ such that $\{v_{n_k}\}$ weakly converges to $v \in K$ with

$$\limsup_{n\longrightarrow\infty}\left\langle \overrightarrow{v_n\widehat{q}}, \overrightarrow{p(\widehat{q})\widehat{q}} \right\rangle = \lim_{k\longrightarrow\infty}\left\langle \overrightarrow{v_{n_k}\widehat{q}}, \overrightarrow{p(\widehat{q})\widehat{q}} \right\rangle = \left\langle \overrightarrow{v\widehat{q}}, \overrightarrow{p(\widehat{q})\widehat{q}} \right\rangle. \tag{37}$$

Since $d(v_n, w_n) \longrightarrow 0$, we have $w_{n_k} \rightharpoonup v$.

By (26) with $v = v_{n_k}$ and $w = w_{n_k}$, we have

$$\lambda\left[\phi\left(v_{n_k}, z\right) - \phi\left(v_{n_k}, w_{n_k}\right)\right] \ge \left\langle \overrightarrow{w_{n_k}v_{n_k}}, \overrightarrow{yz} \right\rangle, \quad \text{for all } z \in K. \tag{38}$$

Taking $k \longrightarrow \infty$ on $\{\lambda_n\}$, (6), and (10), we achieve

$$\phi(v, z) \geq 0 \text{ for all } z \in K. \tag{39}$$

This gives $v \in \mathrm{EP}(\phi)$. Also $s_{n_k} \rightharpoonup v$.

By Lemma 12 with $d(v_n, w_n) \longrightarrow 0$, we obtain $v \in \mathrm{Fix}(T)$. Then $v \in \mathrm{Fix}(T) \cap \mathrm{EP}(\phi)$. Using Lemma 13 with (36), the conclusion follows. □

Theorem 21. *Consider the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the conditions:*

$$\begin{gathered}\lim_{n\longrightarrow\infty} \alpha_n = 0,\\ \sum_{n=0}^{\infty} \alpha_n = \infty,\\ 0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1,\end{gathered} \tag{40}$$

then the sequence $\{v_n\}$ strongly converges to $\hat{q}$.

Proof. Consider $l_n = \beta_n v_n \oplus (1-\beta_n) T s_n \forall n \in \mathbb{N}$. Lemma 11 follows

$$\begin{aligned} d^2(l_n, \hat{q}) &\leq \beta_n d^2(v_n, \hat{q}) + (1-\beta_n) d^2(Ts_n, \hat{q}) \leq \beta_n d^2(v_n, \hat{q}) \\ &\quad + (1-\beta_n) d^2(s_n, \hat{q}) \leq \beta_n d^2(v_n, \hat{q}) + (1-\beta_n)\left[d^2(v_n, \hat{q})\right. \\ &\quad \left. - (1-\lambda_n c_1) d^2(v_n, w_n) - (1-\lambda_n c_2) d^2(v_n, s_n)\right] \\ &\quad \leq \beta_n d^2(v_n, \hat{q}) + (1-\beta_n)(1-\lambda_n c_1) d^2(v_n, w_n) \\ &\quad - (1-\beta_n)(1-\lambda_n c_2) d^2(w_n, s_n) \leq d^2(v_n, \hat{q}). \end{aligned} \tag{41}$$

Then

$$\begin{aligned} d(v_{n+1}, \hat{q}) &\leq \alpha_n d(p(v_n), \hat{q}) + (1-\alpha_n) d(l_n, \hat{q}) \\ &\leq \alpha_n d(p(v_n), p(\hat{q})) + \alpha_n d(p(\hat{q}), \hat{q}) \\ &\quad + (1-\alpha_n) d(v_n, \hat{q}) \leq \alpha_n \rho d(v_n, \hat{q}) \\ &\quad + \alpha_n d(p(\hat{q}), \hat{q}) + (1-\alpha_n) d(v_n, \hat{q}) \\ &= (1-\alpha_n(1-\rho))(v_n, \hat{q}) + \alpha_n d(p(\hat{q}), \hat{q}) \\ &\leq \max\left\{\frac{(p(\hat{q}), \hat{q})}{1-\rho}, d(v_n, \hat{q})\right\} \cdots \leq \max\left\{\frac{d(p(\hat{q}), \hat{q})}{(1-\rho)}, d(v_0, \hat{q})\right\}. \end{aligned} \tag{42}$$

Hence, $\{v_n\}$ is bounded; then, $\{p(v_n)\}$, $\{w_n\}$, and $\{s_n\}$ are bounded too. On the other hand, by (41), we have

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &\leq \alpha_n d^2(p(v_n), \hat{q}) + (1-\alpha_n) d^2(l_n, \hat{q}) \\ &\leq \alpha_n d^2(p(v_n), \hat{q}) + (1-\alpha_n) \\ &\quad \cdot \left[d^2(v_n, \hat{q}) - (1-\beta_n)(1-\lambda_n c_1) d^2(v_n, w_n)\right. \\ &\quad \left. + (1-\lambda_n c_2) d^2(w_n, s_n)\right]. \end{aligned} \tag{43}$$

Let us consider

$$\begin{aligned} M = \sup &\left\{\left|d^2(p(v_n), \hat{q}) - d^2(v_n, \hat{q})\right| + (1-\beta_n)[(1-\lambda_n c_1)\right. \\ &\left. \cdot d^2(v_n, w_n) + (1-\lambda_n c_2) d^2(w_n, s_n)] : n \in \mathbb{N}\right\}. \end{aligned} \tag{44}$$

Combining (43) and (44), we get

$$\begin{aligned} &(1-\beta_n)(1-\lambda_n K_1) d^2(v_n, w_n) + (1-\lambda_n c_2) d^2(w_n, s_n)] \\ &\quad \leq d^2(v_n, \hat{q}) - d^2(v_{n+1}, \hat{q}) + \alpha_n M. \end{aligned} \tag{45}$$

By Lemma 14 and (40), we have

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &= d^2(\alpha_n(p(v_n) - \hat{q}) \oplus (1-\alpha_n)(l_n - \hat{q}), 0) \\ &\leq (1-\alpha_n)^2 d^2(l_n, \hat{q}) + 2\alpha_n \left\langle \overrightarrow{p(v_n)\hat{q}} \overrightarrow{v_{n+1}\hat{q}} \right\rangle \\ &\leq (1-\alpha_n)^2 d^2(v_n, \hat{q}) + 2\alpha_n \left\langle \overrightarrow{p(v_n)p(\hat{q})}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \\ &\quad + 2\alpha_n \left\langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \leq (1-\alpha_n)^2 d^2(v_n, \hat{q}) \\ &\quad + 2\alpha_n d(p(v_n), p(\hat{q})) d(v_{n+1}, \hat{q}) \\ &\quad + 2\alpha_n \left\langle \overrightarrow{p(v_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \leq (1-\alpha_n)^2 d^2(v_n, \hat{q}) \\ &\quad + 2\alpha_n \rho d(v_n, \hat{q}) d(v_{n+1}, \hat{q}) + 2\alpha_n \left\langle \overrightarrow{p(q_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \\ &\leq (1-\alpha_n)^2 d^2(v_n, \hat{q}) + \alpha_n \rho \left[d^2(v_n, \hat{q}) + d^2(v_{n+1}, \hat{q})\right. \\ &\quad \left. + 2\alpha_n \left\langle \overrightarrow{p(q_n)\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle\right. . \end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned} d^2(v_{n+1}, \hat{q}) &\leq \frac{(1-\alpha_n)^2 + \alpha_n \rho}{1-\alpha_n \rho} d^2(v_n, \hat{q}) + \frac{2\alpha_n}{1-\alpha_n \rho} \left\langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \\ &\leq \frac{1 - 2\alpha_n + \alpha_n \rho}{1-\alpha_n \rho} d^2(v_n, \hat{q}) + \frac{\alpha_n^2}{1-\alpha_n \rho} d^2(v_n, \hat{q}) \\ &\quad + \frac{2\alpha_n}{1-\alpha_n \rho} \left\langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle \leq \left(1 - \frac{2(1-\rho)\alpha_n}{1-\alpha_n \rho}\right) \\ &\quad \cdot d^2(v_n, \hat{q}) + \frac{2(1-\rho)\alpha_n^2}{1-\alpha_n} M_0 + \frac{2\alpha_n}{1-\alpha_n \rho} \left\langle \overrightarrow{p(\hat{q})\hat{q}}, \overrightarrow{v_{n+1}\hat{q}} \right\rangle, \end{aligned} \tag{47}$$

where $M_0 = \sup\{d^2(v_n, \hat{q}) : n \in \mathbb{N}\}$. Taking $\gamma_n = 2(1-\rho)\alpha_n / 1 - \alpha_n \rho$ for all $n \in \mathbb{N}$. As $\{\alpha_n\} \subset (0, 1/(2-\rho))$, then $\{\gamma_n\} \subset (0, 1)$. By the conditions on $\{\alpha_n\}$, we get

$$\begin{gathered}\lim_{n\longrightarrow\infty} \gamma_n = 0,\\ \sum_{n=0}^{\infty} \gamma_n = \infty.\end{gathered} \tag{48}$$

Then, we have

$$d^2(v_{n+1},\widehat{q}) \le (1-\gamma_n)d^2(v_n,\widehat{q}) + \frac{\alpha_n\gamma_n M_0}{2(1-\rho)} + \frac{\gamma_n}{1-\rho}\left\langle \overrightarrow{p(\widehat{q})\widehat{q}}, \overrightarrow{v_{n+1}\widehat{q}} \right\rangle. \tag{49}$$

The rest of the proof will be divided into two parts. □

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(v_n,\widehat{q})\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{d(v_n,\widehat{q})\}$ is convergent. This together with the hypothesis on $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and (45) gives

$$\lim_{n\longrightarrow\infty} d(v_n,w_n) = \lim_{n\longrightarrow\infty} d(v_n,s_n) = 0. \tag{50}$$

On the other hand, by Lemmas 11 and 14, we have

$$\begin{aligned} d^2(v_{n+1},\widehat{q}) &= d^2(\alpha_n(p(v_n)-\widehat{q}) \oplus (1-\alpha_n)[\beta_n v_n \oplus (1-\beta_n) \\ &\cdot Ts_n - \widehat{q}],0) \le \alpha_n d^2(p(v_n),\widehat{q}) + (1-\alpha_n) \\ &\cdot d^2(\beta_n v_n \oplus (1-\beta_n)Ts_n - \widehat{q},0) \le \alpha_n d^2(p(v_n),\widehat{q}) \\ &+ (1-\alpha_n)\left[\beta_n d^2(v_n,\widehat{q}) + (1-\beta_n)d^2(Ts_n,\widehat{q})\right. \\ &\left.- \beta_n(1-\beta_n)d^2(v_n,Ts_n)\right] \le \alpha_n d^2(p(v_n),\widehat{q}) \\ &+ (1-\alpha_n)\left[\beta_n d^2(v_n,\widehat{q}) + (1-\beta_n)d^2(v_n,\widehat{q})\right. \\ &\left.- \beta_n(1-\beta_n)d^2(v_n,Ts_n)\right] \le \alpha_n d^2(p(v_n),\widehat{q}) \\ &+ (1-\alpha_n)\beta_n d^2(v_n,\widehat{q}) - \beta_n(1-\beta_n) \\ &\cdot (1-\alpha_n)d^2(v_n,Ts_n). \end{aligned} \tag{51}$$

Hence,

$$\beta_n(1-\beta_n)(1-\alpha_n)d^2(v_n,Ts_n) \le d^2(v_n,\widehat{q}) - d^2(v_{n+1},\widehat{q}) + \alpha_n d^2(p(v_n),\widehat{q}). \tag{52}$$

Then, $\{d(v_n,\widehat{q})\}$ converges. Also, $\{\alpha_n\} \subset (0, 1/(2-\rho))$, and conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ gives

$$\lim_{n\longrightarrow\infty} d(v_n,Ts_n) = 0. \tag{53}$$

Combining (49) and (52), we get

$$d(s_n,Ts_n) \le d(s_n,v_n) + d(v_n,Ts_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{54}$$

Lemma 20 and (49) and (53) give

$$\limsup_{n\longrightarrow\infty}\left\langle \overrightarrow{p(q_n)\widehat{q}}, \overrightarrow{v_{n+1}\widehat{q}} \right\rangle \le 0. \tag{55}$$

The conclusion follows from Lemma 15 and (48), (49), and (54).

Case 2. Let $\{v_{n_i}\}$ be a subsequence of $\{v_n\}$ with

$$d\left(v_{n_i},\widehat{q}\right) < d\left(v_{n_i},\widehat{q}\right) \forall i \in \mathbb{N}. \tag{56}$$

Then, by Lemma 16, there exists a subsequence $\{m_k\}$ such that $m_k \longrightarrow \infty$:

$$\begin{aligned} &d\left(v_{n_k},\widehat{q}\right) < d\left(v_{m_k+1},\widehat{q}\right), \\ &d(v_k,\widehat{q}) < d\left(v_{n_k+1},\widehat{q}\right), \\ &\forall k \in \mathbb{N}. \end{aligned} \tag{57}$$

The above expression with (44) concludes

$$\begin{aligned} &\left(1-\beta_{m_k}\right)\left[\left(1-\lambda_{m_k}c_1\right)d^2\left(v_{m_k},w_{m_k}\right) + \left(1-\lambda_{m_k}c_2\right)d^2\left(w_{m_k},s_{m_k}\right)\right] \\ &\le d^2\left(v_{m_k},\widehat{q}\right) - d^2\left(v_{m_k+1},\widehat{q}\right) + \alpha_{m_k}M \le \alpha_{m_k}M \forall k \in \mathbb{N}. \end{aligned} \tag{58}$$

By the hypothesis on $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$, it follows that

$$\lim_{k\longrightarrow\infty} d^2\left(v_{m_k},w_{m_k}\right) = d^2\left(w_{m_k},s_{m_k}\right) = 0. \tag{59}$$

Using (51), we have

$$\begin{aligned} &\beta_{m_k}\left(1-\beta_{m_k}\right)\left[\left(1-\alpha_{m_k}\right)d^2\left(v_{m_k},Ts_{m_k}\right) \le d^2\left(v_{m_k},\widehat{q}\right)\right. \\ &\quad - d^2\left(v_{m_k+1},\widehat{q}\right) + \alpha_{m_k}d^2\left(p\left(v_{m_k}\right),\widehat{q}\right) \\ &\quad \le \alpha_{m_k}d^2\left(p\left(v_{m_k}\right),\widehat{q}\right) \longrightarrow 0, \text{as } k \longrightarrow \infty. \end{aligned} \tag{60}$$

By the hypothesis on $\{\alpha_n\}$ and $\{\beta_n\}$, it follows that

$$\lim_{k\longrightarrow\infty} d\left(v_{m_k},Ts_{m_k}\right) = 0. \tag{61}$$

In a similar way as Case 1, we arrive at

$$\limsup_{k\longrightarrow\infty}\left\langle \overrightarrow{p\left(v_{m_k}+1\right)\widehat{q}}, \overrightarrow{p(\widehat{q})\widehat{q}} \right\rangle \le 0. \tag{62}$$

Note that

$$d^2\left(v_{m_{k+1}},\widehat{q}\right) \le \left(1-\alpha_{m_k}\right)d^2\left(v_{m_k},\widehat{q}\right) + 2\alpha_{m_k}\left\langle \overrightarrow{p(\widehat{q})\widehat{q}}, \overrightarrow{v_{m_k}+1\widehat{q}} \right\rangle. \tag{63}$$

Since

$$d\left(v_{m_k},\widehat{q}\right) < d\left(v_{m_k+1},\widehat{q}\right) \text{ for all } k \in \mathbb{N}, \tag{64}$$

we have

$$\begin{aligned} \alpha_{m_k}d^2\left(v_{m_k},\widehat{q}\right) &\le d^2\left(v_{m_k},\widehat{q}\right) - d^2\left(v_{m_k+1},\widehat{q}\right) + 2\alpha_{m_k}\left\langle \overrightarrow{p(\widehat{q})\widehat{q}}, \overrightarrow{v_{m_k+1}\widehat{q}} \right\rangle \\ &+ 2\alpha_{m_k}\left\langle \overrightarrow{p(\widehat{q})\widehat{q}}, \overrightarrow{v_{m_k+1}\widehat{q}} \right\rangle. \end{aligned} \tag{65}$$

As $\{\alpha_{m_k}\} > 0$, we have

$$d^2(v_{m_k}, \widehat{q}) \le \left\langle \overrightarrow{p(\widehat{q})\widehat{q}}, \overrightarrow{v_{m_k+1}\widehat{q}} \right\rangle. \quad (66)$$

From (61), it follows that $d(v_{m_k}, \widehat{q}) \longrightarrow 0$ as $k \longrightarrow \infty$. By the above arguments, using relations (61) and (62) and conditions of $\{\alpha_n\}$, we obtain

$$d(v_{m_k+1}, \widehat{q}) \longrightarrow 0, \quad (67)$$

with $k \longrightarrow \infty$.

As $d(v_k, \widehat{q}) < d(v_{m_k}, \widehat{q})$ for all $k \in \mathbb{N}$, we get $v_k \longrightarrow \widehat{q}$ when $k \longrightarrow \infty$.

Corollary 22. *Consider a CAT(0) space X and nonempty, convex, and closed subset K of X. Further, consider a bifunction* $\phi : K \times K \longrightarrow R$ *that fulfils conditions (6)–(10) and a nonexpensive mapping* $T : K \longrightarrow K$ *with* $Fix(T) \cap EP(\phi) \neq \emptyset$. *Then, define a sequence* $\{v_n\}$ *as follows: for any* $u, v_0 \in K$, *consider*

$$\begin{cases} w_n = argmin_{w\in K}\left\{\lambda_n \phi(v_n, w) + \dfrac{1}{2}d^2(w, v_n)\right\}, \\ s_n = argmin_{w\in K}\left\{\lambda_n \phi(v_n, s) + \dfrac{1}{2}d^2(s, v_n)\right\}, \\ v_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\beta_n v_n \oplus (1\beta_n) T s_n), n \in \mathbb{N}. \end{cases} \quad (68)$$

Initially, we choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ *with* $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}, \{\alpha_n\} \subset (0, 1/(2-\rho)), \{\beta_n\} \subset (0,1)$, *and take* $v_0 \in K$. *Set* $n = 0$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *as in Theorem 21, then* $\{v_n\}$ *converges strongly to* $\widehat{q} = P_{Fix(T)\cap EP(\phi)} u$.

Corollary 23. *Consider a CAT(0) space X and nonempty, convex, and closed subset K of X. Further consider a bifunction* $\phi : K \times K \longrightarrow R$ *that fulfils conditions (6)–(10) such that* $EP(\phi) \neq \phi$, *the sequence* $\{v_n\}$ *with* $u, v_0 \in K$ *and*

$$\begin{cases} w_n = argmin_{w\in K}\left\{\lambda_n \phi(v_n, w) + \dfrac{1}{2}d^2(w, v_n)\right\}, \\ s_n = argmin_{s\in K}\left\{\lambda_n \phi(s_n, w) + \dfrac{1}{2}d^2(s, v_n)\right\}, \\ v_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\beta_n v_n \oplus (1-\beta_n) T s_n), n \in \mathbb{N}, \end{cases} \quad (69)$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ *with* $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. *If the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are same as in Theorem 21, then the* $\{v_n\}$ *converges strongly to* $\widehat{q} = P_{EP(\phi)} u$.

Example 24. Let $H = \mathbb{R}^3$ and $K = \{(v_1, v_2, v_3): v_1, v_2, v_3 = 0\}$. Let $\phi(v, w) = (w_3 - v_3)\|v\|$ for *all* $v = (v1, v2, v3)$, $w = (w_1, w_2, w_3) \in K$. Then ϕ is Lipschitz-type continuous with the constants $c_1 = c_2 = 1$.

Table 1: Some v_n and $\|v_n - w_n\|$ generated by Algorithm 19.

Iteration (n)	v_1^n	v_2^n	v_3^n	$\|v_n - w_n\|$
1	2.00000	2.00000	2.00000	1.29903
2	1.90000	1.88000	0.98071	0.88973
3	1.85250	1.82360	0.36780	0.36780
4	1.82162	1.78713	0.12097	0.12096
5	1.79886	1.76032	0.03746	0.03746
6	1.78087	1.73920	0.01116	0.01116
7	1.76026	1.72181	0.00323	0.00324
8	1.75341	1.70704	0.00092	0.00092
9	1.74245	1.69424	0.00026	0.00026
10	1.73277	1.68295	0.00007	0.00007
11	1.72411	1.67285	0.00001	0.00002
12	1.71627	1.66373	0.00001	0.00001
13	1.70912	1.65541	0.00000	0.00000

It is easy to see that ϕ satisfies conditions (6)–(10). Let $Tv = (v_1, v_2, \arctan(0.1v_3))$ and $p(v) = (0.5v_1, 0.4v_2, 0.2v_3)$ for all $v = (v_1, v_2, v_3) \in K$. It follows that T is a nonexpansive mapping and p is a contraction. Take the initial point $v_1 = (2,2,2)$ and put the sequences $\alpha_n = 1/10n$, $\beta_n = 1/4 + 4n$, and $\gamma_n = 1/4 + 1/8n$ for all $n \in \mathbb{N}$. It is easy to see that

$$\mathrm{Fix}(T) \cap \mathrm{EP}(\phi) = \{(v_1, v_2, 0): v_1, v_2 = 0\}. \quad (70)$$

We give some v_n and $\|v_n - w_n\|$ by Table 1: from Table 1, we see that after 13 iterations $v_{13} = (1.70912, 1.65541, 0.00000) \in \mathrm{Fix}(T) \cap \mathrm{EP}(\phi)$.

Corollary 25. *Consider a CAT(0) space X and nonempty, convex, and closed subset K of X. Further consider a bifunction* $\phi : K \times K \longrightarrow R$ *that fulfils conditions (6)–(10) and a nonexpensive mapping* $T : K \longrightarrow K$ *with* $Fix(T) \cap EP(\phi) \neq \emptyset$. *Then define a sequence* $\{v_n\}$ *as follows.*

For any $u, v_0 \in K$, *consider*

$$\begin{cases} w_n = argmin_{w\in K}\left\{\lambda_n \phi(v_n, w) + \dfrac{1}{2}d^2(w, v_n)\right\}, \\ s_n = argmin_{w\in K}\left\{\lambda_n \phi(v_n, s) + \dfrac{1}{2}d^2(s, v_n)\right\}, \\ v_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\beta_n v_n \oplus (1\beta_n) T s_n), n \in \mathbb{N}. \end{cases} \quad (71)$$

Initially choose $\{\lambda_n\} \subset [\delta_1, \delta_2]$ *with* $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}, \{\alpha_n\} \subset (0, 1/(2-\rho)), \{\beta_n\} \subset (0,1)$ *and take* $v_0 \in K$. *Set* $n = 0$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *as in Theorem 21.*

Then $\{v_n\}$ *converges strongly to* $\widehat{q} = P_{Fix(T)\cap EP(\phi)} u$.

Corollary 26. *Consider a CAT(0) space X and nonempty, convex, and closed subset K of X. Further consider a bifunction* $\phi : K \times K \longrightarrow R$ *that fulfils conditions (6)–(10) such that*

$EP(\phi) \neq \phi$, the sequence $\{v_n\}$ with $u, v_0 \in K$ and

$$\begin{cases} w_n = argmin_{w\in K}\left\{\lambda_n\phi(v_n, w) + \frac{1}{2}d^2(w, v_n)\right\}, \\ s_n = argmin_{s\in K}\left\{\lambda_n\phi(s_n, w) + \frac{1}{2}d^2(s, v_n)\right\}, \\ v_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\beta_n v_n \oplus (1-\beta_n)Ts_n), n \in \mathbb{N}, \end{cases} \tag{72}$$

where $\{\lambda_n\} \subset [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{1/2c_1, 1/2c_2\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. If the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are the same as in Theorem 21, then the sequence $\{v_n\}$ converges strongly to $\hat{q} = P_{EP(\phi)}u$.

Remark 27. Our Theorem 21 is an analog of Theorem 3.1 of Wang et al. [30] for the Hilbert space case.

Remark 28. Our results can be extended to any CAT(κ) space with $\kappa \leq 0$, since any CAT(κ') space is a CAT(κ) space, for any $\kappa' > \kappa$ (see [17]).

4. Conclusions

It is well known that finding solutions for equilibrium problems play an important role in solving problems in other areas of science like physics, optimization, and economics. In this paper, we construct a new viscosity extragradient algorithm in order to find fixed points in a CAT(0) space for the case of nonexpansive mappings. Also, we prove a strong convergence of the proposed algorithm, and we give some examples to support our results.

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

There are no competing interests among the authors.

Authors' Contributions

Yu-Pei Lv and Farman Ali contributed equally to this work.

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Research Article

Fixed Point, Data Dependence, and Well-Posed Problems for Multivalued Nonlinear Contractions

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The aim of the paper is to discuss data dependence, existence of fixed points, strict fixed points, and well posedness of some multivalued generalized contractions in the setting of complete metric spaces. Using auxiliary functions, we introduce Wardowski type multivalued nonlinear operators that satisfy a novel class of contractive requirements. Furthermore, the existence and data dependence findings for these multivalued operators are obtained. A nontrivial example is also provided to support the results. The results generalize, improve, and extend existing results in the literature.

1. Introduction and Preliminaries

Let $(\mathfrak{Z}, d)$ be a metric space (in short MS). The set of all nonempty subsets of $\mathfrak{Z}$ is denoted by $P(\mathfrak{Z})$, the set of all nonempty closed subsets of $\mathfrak{Z}$ is denoted by $CL(\mathfrak{Z})$, the set of all nonempty closed and bounded subsets of $\mathfrak{Z}$ is denoted by $CB(\mathfrak{Z})$, and the set of all nonempty compact subsets of X is denoted by $K(\mathfrak{Z})$. It is obvious that $CB(\mathfrak{Z})$ includes $K(\mathfrak{Z})$. For $U, V \in CB(\mathfrak{Z})$, define $H : CB(\mathfrak{Z}) \times CB(\mathfrak{Z}) \longrightarrow [0, \infty)$ by

$$H(U, V) = \max \left\{ \sup_{u \in U} D(u, V), \sup_{v \in V} D(v, U) \right\}, \quad (1)$$

where $D(u, V) = \inf \{d(u, v) : v \in V\}$. Such a function H is called the Pompei-Hausdorff metric induced by d, for more details, see, e.g., [1].

Lemma 1 [2]. *Let $(\mathfrak{Z}, d)$ be a MS and $A, B \in CL(\mathfrak{Z})$ with $H(A, B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) < hH(A, B)$.*

If $\Omega : \mathfrak{Z} \longrightarrow P(\mathfrak{Z})$ is a multivalued operator, then an element $\varpi \in \mathfrak{Z}$ is called a fixed point for Ω if $x \in \Omega\varpi$. The symbol fix $\Omega = \{\varpi \in \mathfrak{Z} : x \in \Omega\varpi\}$ denotes the fixed point set of Ω. On the other hand, a strict fixed point for Ω is an element $\varpi \in \mathfrak{Z}$ with the property $\{x\} = \Omega\varpi$. The set of all strict fixed points of Ω is denoted by SFix Ω.

Banach's contraction principle [3] is the most fundamental result in metric fixed point theory. Since then, many authors have extended and generalized Banach's contraction principle in many ways. Extensions of Banach's contraction principle have spawned a wealth of literature. (see [13, 29]). One of an attractive and important generalization is given by Wardowski in [10]. He introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction.

Definition 2 [10]. Let $(\mathfrak{Z}, d)$ be a *MS*. A mapping $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is said to be F -contraction if there exists $\tau > 0$ such that

$$d(\Omega\varpi, \Omega\omega) > 0 \text{ implies } \tau + F(d(\Omega\varpi, \Omega\omega)) \leq F(d(\varpi, \omega)), \quad (2)$$

for all $x, y \in X$, where $F : (0,\infty) \longrightarrow \mathbb{R}$ is a function satisfying

$(F1)$ F is strictly increasing

($F2$) For all sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\longrightarrow\infty} t_n = 0$, if and only if $\lim_{n\longrightarrow\infty} F(t_n) = -\infty$

($F3$) There exists $0 < k < 1$ such that $\lim_{k\longrightarrow 0^+} t^k F(t) = 0$

We denote by $\Delta(F)$ the collection of all functions $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying ($F1$), ($F2$), and ($F3$). Also, define

$$\Delta(\bigcirc *) = \{F \in \Delta(F) \mid F \text{ satifies } (F4)\}, \tag{3}$$

where

($F4$) $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Theorem 3 [10]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ be a F-contraction. Then, Ω has a unique fixed point $\varpi^* \in \mathfrak{Z}$ and for every $\varpi_0 \in \mathfrak{Z}$, a picard sequence $\{T^n \varpi_0\}_{n\in\mathbb{N}}$ converges to ϖ^*.*

Further, Turinici [11] is replaced ($F2$) by the following condition: ($F2'$) $\lim_{t\longrightarrow 0^+} F(t) = -\infty$.

Note that, in general, $F \in \Delta(F)$ is not continuous. However, by ($F1$) and the properties of the monotone functions, we have the following proposition.

Proposition 4 [11]. *Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying ($F1$) and ($F2$), and then there exists a countable subset $\Lambda(F) \subseteq (0, 1)$ such that*

$$F(t-0) = F(t) = F(t+0) \text{foreach}\, t \in (0, 1) \setminus \Lambda(F). \tag{4}$$

Lemma 5 [11]. *Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying ($F1$) and ($F2'$). Then, for each sequence $\{t_n\}$ in $(0, 1)$,*

$$F(t_n) \longrightarrow -\infty \Rightarrow t_n \longrightarrow 0. \tag{5}$$

After this, many authors generalized the F-contraction in several ways (see [12–22] and references therein). In 2015, Klim and Wardowski [23] extended the concept of F-contractive mappings to the case of nonlinear F-contractions and proved fixed point theorems via the dynamic processes. In 2017, Wardowski [24] omitted one of the conditions of F-contraction and introduced nonlinear F-contraction).

Definition 6 [24]. A mapping $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is said to be a (φ, F) -contraction (or nonlinear F-contraction), if there exists $F \in \mathscr{F}$ and a function $\varphi : (0,\infty) \longrightarrow (0,\infty)$ satisfying

(H1) $\liminf_{s\longrightarrow t^+} \varphi(s) > 0$, forall $t \geq 0$.

(H2) $\varphi(d(\varpi, \omega)) + F(d(\Omega\varpi, \Omega\omega)) \leq F(d(\varpi, \omega))$, for all $\varpi, \omega \in \mathfrak{Z}$ such that $\Omega\varpi \neq \Omega\omega$

Theorem 7 [24]. *Let $(\mathfrak{Z}, d)$ be a complete MS and let $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ be a (φ, F) -contraction. Then, Ω has a unique fixed point in $\mathfrak{Z}$.*

Very recently, Iqbal and Rizwan [25] considered a rich class of functions and generalized Definition 6 to obtain some new fixed point theorems for nonlinear F-contractions involving generalized distance. On unifying the concept of Wardowski [10], Nadler [9] and Altun et al. [26] gave the concept of multivalued F-contraction as follows.

Definition 8 [26]. Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow CB(\mathfrak{Z})$ be a mapping. Then, Ω is a multivalued F -contraction, if there exists $\tau > 0$ and $F \in \Delta(F)$ such that for all $\varpi, \omega \in \mathfrak{Z}$,

$$H(\Omega\varpi, \Omega\omega) > 0 \Rightarrow \tau + F(H(\Omega\varpi, \Omega\omega)) \leq F((d(\varpi, \omega)). \tag{6}$$

Theorem 9 [26]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued F -contraction, and then Ω has a fixed point in $\mathfrak{Z}$.*

Afterwards, Olgun et al. [27] proved the nonlinear case of Theorem 9 as follows.

Theorem 10 [27]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$, if there exists $F \in \Delta(F)$ and $\varphi : (0, 1) \longrightarrow (0, 1)$ satisfying*

$$\begin{gathered} \liminf_{s\longrightarrow t^+} \varphi(s) > 0 \, for \, all \varpi, \omega \in \mathfrak{Z}, \\ \varphi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \leq F(d(\varpi, \omega)). \end{gathered} \tag{7}$$

Then, Ω has a fixed point in $\mathfrak{Z}$.

For more directions for nonlinear F-contractions, consult [28, 29] and references there in. Next, we denote by $\mathscr{P}$ the set of all continuous mappings $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfying the following conditions:

(ρ_1) $\rho(1, 1, 1, 2, 0) \in (0, 1]$;

(ρ_2) ρ is subhomogeneous; that is, for all $(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \in [0,\infty)^5$ and $\alpha \geq 0$, we have $\rho(\alpha\varpi_1, \alpha\varpi_2, \alpha\varpi_3, \alpha\varpi_4, \alpha\varpi_5) \leq \alpha\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)$

(ρ_3) ρ is nondecreasing function; that is, for $\varpi_i, \omega_i \in \mathbb{R}^+$, $\varpi_i \leq \omega_i$, $i = 1, \cdots, 5$, we have $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \leq \rho(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$

If $\varpi_i, \omega_i \in \mathbb{R}^+$, $\varpi_i < \omega_i$, $i = 1, \cdots, 4$, then $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, 0) < \rho(\omega_1, \omega_2, \omega_3, \omega_4, 0)$ and $\rho(\varpi_1, \varpi_2, \varpi_3, 0, \varpi_5) < \rho(\omega_1, \omega_2, \omega_3, 0, \omega_5)$.

Also, define

$$\mathbb{P} = \{\rho \in \mathscr{P} | \rho(1, 0, 0, 1, 1) \in (0, 1]\}. \tag{8}$$

Note that $\mathbb{P} \subseteq \mathscr{P}$.

Example 1. Define $\rho_1 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_1(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1 + \xi x_5, \tag{9}$$

where $\xi \in (0, 1)$. Then, $\rho_1 \in \mathscr{P}$. Since $\rho_1(1, 0, 0, 1, 1) = 1 + \xi > 1$, so $\rho_1 \notin \mathbb{P}$. Also, note that $\rho_1(1, 1, 1, 0, 2) = 1 + 2\xi > 1$, so $\rho_1 \notin \mathfrak{R}$ [30].

Example 2. Define $\rho_2 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_2(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = g \max \left\{ \varpi_1, \frac{1}{2}(\varpi_2 + \varpi_3), \frac{1}{2}(\varpi_4 + \varpi_5) \right\}, \tag{10}$$

where $g \in (0, 1)$. Then, $\rho_2 \in \mathbb{P}$.

Example 3. Define $\rho_3 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_3(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = a\varpi_1 + b(\varpi_2 + \varpi_3) + c(\varpi_4 + \varpi_5), \tag{11}$$

where $a + 2b + 2c < 1$. Then, $\rho_3 \in \mathscr{P}$.

Now, we prove the following Lemma.

Lemma 11. *If $\rho \in \mathscr{P}$ and $u, v \in [0,\infty)$ are such that*

$$u \le \max \{\rho(v, v, u, v + u, 0), \rho(v, v, u, 0, v + u), \rho(v, u, v, v + u, 0), \rho(v, u, v, 0, v + u)\}, \tag{12}$$

then $u \le v$.

Proof. Without loss of generality, we can suppose that $u \le \rho(v, v, u, v + u, 0)$. If $v < u$, then

$$u \le \rho(v, v, u, v + u, 0) < \rho(u, u, u, 2u, 0) \le u\rho(1, 1, 1, 2, 0) \le u, \tag{13}$$

a contradiction. Thus, $u \le v$. □

Now, consider following examples.

Example 4. Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$F(t) = \frac{-1}{t} \text{ for } all\, t \in (0,\infty). \tag{14}$$

Then, F satisfies $(F1)$, $(F2')$, and F that is continuous but does not satisfies $(F3)$.

Example 5. Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$F(t) = \begin{cases} \dfrac{-1}{t} & \text{if } t \in (0, 1) \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Then, F satisfies $(F1)$ and $(F2')$ but F is not continuous.

Example 6. [31] Let $F : (0,\infty) \longrightarrow \mathbb{R}$ *be* a function defined by

$$F(t) = -\frac{1}{(t + [t])^{\ell}}, \tag{16}$$

where $[t]$ denotes the integral part of Ω and $\ell \in (0, 1/a)$, $a > 1$. Then, F satisfies $(F1)$, $(F2')$, and $(F3)$ but F is not continuous.

Examples 4–6 clearly show that there exist some functions $F : (0,\infty) \longrightarrow \mathbb{R}$ which does not satisfy the condition of continuity, $(F1)$, $(F2)$, and $(F3)$ at a time. By getting inspiration from this, in this paper, we prove fixed point results for contractive conditions involving functions F, not necessarily continuous and belongs to $\Delta(F)$ by taking support of a continuous function from $\mathscr{P}$. Our results generalize many results appearing recently in the literature including Altun et al. [32], Olgun et al., [27] Sgroi and Vetro [33], Vetro [34], Wardowski [24], and Wardowski and Dung [35].

For convenience, we set Φ, the collection of all functions $\chi : (0,\infty) \longrightarrow (0,\infty)$ satisfying

$$\lim_{s \longrightarrow t^+} \inf \chi(s) > 0 \,\text{forall}\, t \ge 0. \tag{17}$$

Theorem 12. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing real valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ and $(F3)$ such that the following conditions hold:*

(N1) $F_1(c) \le F_2(c)$ for all $c > 0$

(N2) For all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathscr{P}$, $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\begin{aligned} &\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \\ &\quad \le F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \tag{18}$$

Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Assume that $\varpi_1 \in \Omega\varpi_1$; otherwise, ϖ_1 is a fixed point of Ω, and the proof is complete. Then, $D(\varpi_1, \Omega\varpi_1) > 0$ and consequently $H(\Omega\varpi_0, \Omega\varpi_1) > 0$. Compactness of $\Omega\varpi_1$ ensures the existence of $\varpi_2 \in \Omega\varpi_1$, such that $d(\varpi_1, \varpi_2) = D(\varpi_1, \Omega\varpi_1)$. From (N1) and (N2), we get

$$\begin{aligned} F_1(d(\varpi_1, \varpi_2)) &= F_1(D(\varpi_1, \Omega\varpi_1)) \le F_1(H(\Omega\varpi_0, \Omega\varpi_1)) \le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\ &\le F_1(\rho(d(\varpi_0, \varpi_1), D(\varpi_0, \Omega\varpi_0), D(\varpi_1, \Omega\varpi_1), D(x_0, \Omega\varpi_1), D(\varpi_1, \Omega\varpi_0))) \\ &\quad - \chi(d(\varpi_0, \varpi_1)) < F_1(\rho(d(\varpi_0, \varpi_1), d(x_0, \varpi_1), d(\varpi_1, \varpi_2), d(\varpi_0, \varpi_2), 0)). \end{aligned} \tag{19}$$

□

Since F_1 is an nondecreasing function, (19) with (ρ_3) implies that

$$\begin{aligned} d(\varpi_1, \varpi_2) &< \rho(d(\varpi_0, \varpi_1), d(x_0, \varpi_1), d(\varpi_1, \varpi_2), d(x_0, \varpi_1), 0) \\ &\le \rho(d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), d(\varpi_1, \varpi_2), d(\varpi_0, \varpi_1) + d(\varpi_1, \varpi_2), 0). \end{aligned} \tag{20}$$

By using Lemma 11, (20) implies

$$d(\varpi_1, \varpi_2) < d(\varpi_0, \varpi_1). \tag{21}$$

Next, arguing as previous, we get $\varpi_3 \in \Omega\varpi_2$, such that $d(\varpi_2, \varpi_3) = D(\varpi_2, \Omega\varpi_2)$ with $D(x_2, \Omega\varpi_2) > 0$. Also, by using Lemma 11, from (N1) and (N2), we obtain

$$d(\varpi_2, \varpi_3) < d(\varpi_1, \varpi_2). \tag{22}$$

Continuing in the same manner, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ satisfying $d(\varpi_n, \varpi_{n+1}) = D(\varpi_n, \Omega\varpi_n)$ with $D(\varpi_n, \Omega\varpi_n) > 0$ and

$$d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n), \tag{23}$$

for all $n \in \mathbb{N}$. (23) implies that $\{d(\varpi_n, \varpi_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Hence, from (N1) and (N2), we get

$$\begin{aligned}
&\chi(d(\varpi_n, \varpi_{n+1})) + F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), D(\varpi_n, \Omega\varpi_n), D(\varpi_{n+1}, \Omega\varpi_{n+1}), D(\varpi_n, \Omega\varpi_{n+1}), D(\varpi_{n+1}, \Omega\varpi_n))) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2}), d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{n+2}), 0)) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), 2d(\varpi_n, \varpi_{n+1}), 0)) \\
&\quad \le F_1(d(\varpi_n, \varpi_{n+1})\rho(1, 1, 1, 2, 0)) \le F_1(d(\varpi_n, \varpi_{n+1})) = F_1(D(\varpi_n, \Omega\varpi_n)) \\
&\quad \le F_1(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) \le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)).
\end{aligned} \tag{24}$$

Thus, for all $n \in \mathbb{N}$,

$$F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) - \chi(d(\varpi_n, \varpi_{n+1})). \tag{25}$$

Since $\chi \in \Phi$, there exists $h > 0$ and $n_0 \in \mathbb{N}$ such that $\chi(d(\varpi_n, \varpi_{n+1})) > h$, for all $n \ge n_0$. From (25), we obtain

$$\begin{aligned}
F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) &\le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) - \chi(d(\varpi_n, \varpi_{n+1})) \\
&\le F_2(H(\Omega\varpi_{n-2}, \Omega\varpi_{n-1})) - \chi(d(\varpi_{n-1}, \varpi_n)) \\
&\quad - \chi(d(\varpi_n, \varpi_{n+1})) \vdots \\
&\le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - \sum_{i=1}^{n} \chi(d(\varpi_i, \varpi_{i+1})) \\
&= F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - \sum_{i=1}^{n_0-1} \chi(d(\varpi_i, \varpi_{i+1})) \\
&\quad - \sum_{i=n_0}^{n} \chi(d(\varpi_i, \varpi_{i+1})) \\
&\le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - (n - n_0)h, n \ge n_0.
\end{aligned} \tag{26}$$

Taking $n \longrightarrow \infty$ in (26), we get $F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} H(\Omega\varpi_n, \Omega\varpi_{n+1}) = 0, \tag{27}$$

which further implies that

$$\lim_{n\longrightarrow\infty} d(\varpi_n, \varpi_{n+1}) = \lim_{n\longrightarrow\infty} D(\varpi_n, \Omega\varpi_n) \le \lim_{n\longrightarrow\infty} H(\Omega\varpi_{n-1}, \Omega\varpi_n) = 0. \tag{28}$$

Now from $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\longrightarrow\infty} (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) = 0. \tag{29}$$

Then, from (26), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \\
&\quad - (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\
&\quad \le (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k (F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - (n - n_0)h) \\
&\quad - (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\
&\quad = -(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k (n - n_0)h \le 0.
\end{aligned} \tag{30}$$

Taking limit $n \longrightarrow \infty$, in (30) and using (27) and (29), we have

$$\lim_{n\longrightarrow\infty} n(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k = 0. \tag{31}$$

Observe that from (31), there exist $sn_1 \in \mathbb{N}$ such that $n(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k \le 1$ for all $n \ge n_1$. Thus, for all $n \ge n_1$, we have

$$H(\Omega\varpi_n, \Omega\varpi_{n+1}) \le \frac{1}{n^{1/k}} \text{ for all } n \ge n_1, \tag{32}$$

which further implies that

$$d(\varpi_n, \varpi_{n+1}) = D(\varpi_n, \Omega\varpi_n) \le H(\Omega\varpi_{n-1}, \Omega\varpi_n) \le \frac{1}{n^{1/k}} \text{ for all } n \ge n_1. \tag{33}$$

Now, in order to show that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n > n_1$. From (33), we get

$$d(\varpi_m, \varpi_n) \le \sum_{i=n}^{m-1} d(\varpi_i, \varpi_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \tag{34}$$

As a result of the above and the series' convergence, $\sum_{i=n}^{\infty} (1/i^{1/k})$, we receive that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence. Since $\mathfrak{Z}$ is a complete space, so there exists $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{35}$$

Now,

$$\begin{aligned}
F_1(H(\Omega\varpi, \Omega\omega)) &\le F_2(H(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega) \\
&\le F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))).
\end{aligned} \tag{36}$$

Since F_1 is nondecreasing function, we obtain for all ϖ

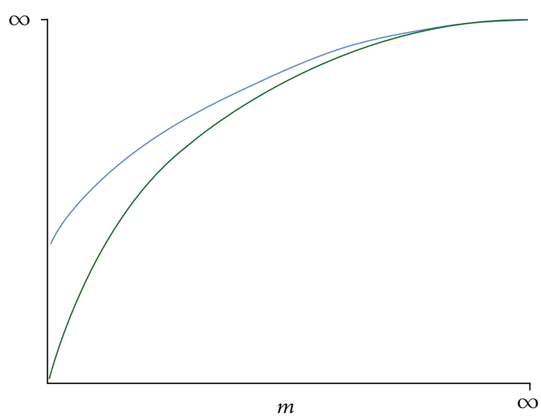

Figure 1: Graph of Inequality (44).

, $\omega \in \mathfrak{Z}$.

$$H(\Omega\varpi, \Omega\omega) \le \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)). \tag{37}$$

We claim that ϖ^* is fixed point of $\mathfrak{Z}$. On contrary, suppose that $D(\varpi^*, \Omega\varpi^*) > 0$ and by equation (37), we have

$$\begin{aligned} D(\varpi^*, \Omega\varpi^*) &\le d(\varpi^*, \varpi_{n+1}) + D(\varpi_{n+1}, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(x^*, \Omega\varpi_n)) \\ &\le d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), d(\varpi_n, \varpi_{n+1}), \\ &\quad D(\varpi^*, \Omega\varpi^*), d(\varpi_n, \varpi^*) \\ &\quad + D(\varpi^*, \Omega\varpi^*), d(\varpi^*, \varpi_{n+1})). \end{aligned} \tag{38}$$

Passing to limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$D(\varpi^*, \Omega\varpi^*) < \rho(0, 0, D(\varpi^*, \Omega\varpi^*), 0 + D(\varpi^*, \Omega\varpi^*), 0), \tag{39}$$

which implies by Lemma 1 that

$$0 < D(\varpi^*, \Omega\varpi^*) < 0, \tag{40}$$

which is a contradiction. Hence, $D(\varpi^*, \Omega\varpi^*) = 0$. Since $\Omega\varpi^*$ is closed, therefore, $\varpi^* \in \Omega\varpi^*$.

Remark 13. By defining $F_1 = F_2 = F$ and $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1$ in Theorem 12, we get back Theorem 2.3 of [27].

Example 7. Consider $\mathfrak{Z} = \{u_n = n(n+1)/2 : n \in \mathbb{N}\}$, and then $\mathfrak{Z}$ is complete MS with metric $d(u, v) = |u - v|$. Define functions $F_1, F_2 : (0,\infty) \longrightarrow \mathbb{R}$ by

$$F_1(u) = \begin{cases} -\dfrac{1}{u} & \text{if } u \in (0, 1) \\ u & \text{if } u \in [1,\infty) \end{cases} \tag{41}$$

and $F_2(u) = \ln u + u$ for all $u \in (0,\infty)$, respectively. Then F_1 is nondecreasing, F_2 satisfy the conditions $(F2')$ and $(F3)$ and $F_1(u) \le F_2(u)$ for all $u > 0$.

Next, define $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$, $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ and $\chi : (0,\infty) \longrightarrow (0,\infty)$ by

$$\begin{aligned} \Omega\varpi &= \begin{cases} \{u_1\} & \text{if } \varpi = u_1, \\ \{u_1, u_2\} & \text{if } \varpi = u_n, n \ge 2, \end{cases} \\ \rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) &= \varpi_1 + \xi\varpi_5, \xi \in (0, 1) \end{aligned} \tag{42}$$

and $\chi(t) = 1/t$ for all $t \in (0,\infty)$, respectively. Then $\chi \in \Phi$ and $\rho \in \mathscr{P}$ (see Example 1). Observe that

$$m, n \in \mathbb{N}, H(\Omega u_m, \Omega u_n) > 0 \Leftrightarrow (m > 2 \text{and}\, n = 1). \tag{43}$$

Assume that $H(\Omega\varpi, \Omega\omega) > 0$, and then $m > 2$ and $n = 1$. From Figure 1, it is clear that

$$\frac{2}{|m^2 + m - 2|} + \ln\left|\frac{|m^2 - m - 2|}{2}\right| + \frac{|m^2 - m - 2|}{2} \le \frac{|m^2 + m - 2|}{2}, \tag{44}$$

$H(\Omega u_m, \Omega u_1) = |u_{m-1} - 1|$ and $D(u_1, \Omega u_m) = 0$. Which further implies that

$$\begin{aligned} \chi(d(u_m, u_1)) + F_2(H(\Omega u_m, \Omega u_1)) &= \frac{1}{|u_m - u_1|} + F_2(|u_{m-1} - 1|) \\ &= \frac{2}{|m^2 + m - 2|} + \ln\left|\frac{|m^2 - m - 2|}{2}\right| + \frac{|m^2 - m - 2|}{2} \\ &\le \frac{|m^2 + m - 2|}{2} = d(u_m, u_1) + \xi D(u_1, \Omega u_m) \\ &= F_1(d(u_m, u_1) + \xi D(u_1, \Omega u_m)) \\ &= F_1(\rho(d(u_m, u_1), D(u_1, \Omega u_1), D(u_m, \Omega u_m), D(u_m, \Omega u_1), \\ &\quad D(u_1, \Omega u_m))). \end{aligned} \tag{45}$$

All hypothesis of Theorem 12 are satisfied and fix $\Omega = \{u_1, u_2\}$.

Observe the following in Example 7:

(i) F_1 is not continuous at 1

(ii) $F_1 \ne F_2$

(iii) $\rho \notin \mathfrak{R}$

(iv) $\rho \notin \mathbb{P}$.

Corollary 14. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing real valued function F_1 on $(0,\infty)$ and a real valued function F_2 on $(0,\infty)$ satisfying condition $(F2')$ and $(F3)$ such that (N1) and the following condition holds:*

$H(\Omega\varpi, \Omega\omega)) > 0$ implies $\chi(d(\varpi,\omega)) + F_2(H(\Omega\varpi,\Omega\omega)) \leq F_1(M(\varpi,\omega))$ for all $\varpi, \omega \in \mathfrak{Z}$, where

$$M(\varpi,\omega) = \max\left\{d(\varpi,\omega), D(\varpi,\Omega\varpi), D(\omega,\Omega\omega), \frac{D(x,\Omega\omega)+D(\omega,\Omega\varpi)}{2}\right\}. \tag{46}$$

Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5) = \max\left\{\varpi_1,\varpi_2,\varpi_3,\frac{\varpi_4+\varpi_5}{2}\right\}. \tag{47}$$

Then, $\rho \in \mathscr{P}$ and result follow from Theorem 12. □

Remark 15. Corollary 14 generalizes and improves Theorem 2.4 of [35]. In fact, by taking $F_1 = F_2$ and by defining $\Omega\varpi = \{\varpi\}$ for all $\varpi, \omega \in \mathfrak{Z}$ and $\chi(t) = \tau > 0$ for all $t \in (0,\infty)$ in Corollary 14, then we find Theorem 2.4 of [35]. Corollary 14 shows that condition $(F2)$ can be replaced by $(F2')$ and the strictness of the monotonicity of F is not necessary.

Corollary 16. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exist $\chi \in \Phi$, a non decreasing real valued function F_1 on $(0,\infty)$ and a real valued function F_2 on $(0,\infty)$ satisfying condition $(F2')$ and $(F3)$ such that (N1) and the following condition holds:*

$H(\Omega\varpi, \Omega\omega)) > 0$ implies $\chi(d(\varpi,\omega)) + F_2(H(\Omega\varpi,\Omega\omega)) \leq F_1(N(\varpi,\omega))$ for all $\varpi, \omega \in \mathfrak{Z}$, where

$$N(\varpi,\omega) = ad(\varpi,\omega) + bD(\varpi,\Omega\varpi) + cD(\omega,\Omega\omega) + e[D(\varpi,\Omega\omega) + D(\omega,\Omega\varpi)], \tag{48}$$

$a, b, c, e \geq 0$ and $a+b+c+2e < 1$. Then Fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5) = a\varpi_1 + b\varpi_2 + c\varpi_3 + e[\varpi_4+\varpi_5], \tag{49}$$

where $a, b, c, e > 0$ and $a+b+c+2e < 1$. Then $\rho \in \mathscr{P}$ and result follows from Theorem 12. □

Now, in the next Theorem, we replace the condition $(F3)$ of F_2 by continuity of F_1 in hypothesis of Theorem 12 and obtain another fixed point result.

Theorem 17. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0,\infty)$ and a real valued function F_2 on $(0,\infty)$ satisfying condition $(F2')$ such that following conditions hold:*

(N1) $F_1(c) \leq F_2(c)$ for all $c > 0$

(N2) $H(\Omega\varpi,\Omega\omega)) > 0$ implies

$$\begin{aligned}&\chi(d(\varpi,\omega)) + F_2(H(\Omega\varpi,\Omega\omega))\\ &\quad\leq F_1(\rho(d(\varpi,\omega), D(\varpi,\Omega\varpi), D(\omega,\Omega\omega), D(\varpi,\Omega\omega), D(\omega,\Omega\varpi)))\end{aligned} \tag{50}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathbb{P}$.

Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Then, as in proof of Theorem 12, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ satisfying $d(\varpi_n,\varpi_{n+1}) = D(\varpi_n,\Omega\varpi_n)$ with $D(\varpi_n,\Omega\varpi_n) > 0$,

$$d(\varpi_n,\varpi_{n+1}) < d(\varpi_{n-1},\varpi_n), \text{for } all\, n \in \mathbb{N}, \tag{51}$$

$$F_2(H(\Omega\varpi_n,\Omega\varpi_{n+1})) \leq F_2(H(\Omega\varpi_0,\Omega\varpi_1)) - (n-n_0)h, n \geq n_0. \tag{52}$$

Taking $n \longrightarrow \infty$ in (52), we get $F_2(H(\Omega\varpi_n,\Omega\varpi_{n+1})) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} H(\Omega\varpi_n,\Omega\varpi_{n+1}) = 0, \tag{53}$$

which further implies that

$$\lim_{n\longrightarrow\infty} d(\varpi_n,\varpi_{n+1}) = \lim_{n\longrightarrow\infty} D(\varpi_n,\Omega\varpi_n) \leq \lim_{n\longrightarrow\infty} H(\Omega\varpi_{n-1},\Omega\varpi_n) = 0. \tag{54}$$

Next, we claim that

$$\lim_{n,m\longrightarrow\infty} d(\varpi_n,\varpi_m) = 0. \tag{55}$$

If (55) is not true, then there exists $\delta > 0$ such that for all $r \geq 0$, there exists $m_k > n_k > r$

$$d(\varpi_n,\varpi_m) > \delta. \tag{56}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\lambda_{r_0} = d(\varpi_{n-1},\varpi_n) < \delta \text{ forall } n \geq r_0. \tag{57}$$

Consider two subsequences $\{\varpi_{n_k}\}$ and $\{\varpi_{m_k}\}$ of $\{\varpi_n\}$ satisfying

$$r_0 \leq n_k \leq m_k + 1 \text{ and } d\left(\varpi_{m_k},\varpi_{n_k}\right) > \delta \text{ forall } k > 0. \tag{58}$$

Observe that

$$d\left(\varpi_{m_k-1},\varpi_{n_k}\right) \leq \delta \text{ forall } k, \tag{59}$$

where m_k is chosen as minimal index for which (59) is satisfied. Also, note that because of (58) and (59), the case $n_{k+1} \le n_k$ is impossible. Thus, $n_{k+2} \le m_k$ for all k. It implies

$$n_k + 1 < m_k < m_k + 1 \text{ for all } k. \tag{60}$$

Using triangle inequality and by (58) and (59), we have

$$\delta < d(\varpi_{m_k}, \varpi_{n_k}) \le d(\varpi_{m_k}, \varpi_{m_k-1}) + d(\varpi_{m_k-1}, \varpi_{n_k}) \le \lambda_{m_k} + \delta. \tag{61}$$

Letting limit $k \longrightarrow \infty$ in (61) and using (53), we get

$$\lim_{k\longrightarrow\infty} d(\varpi_{m_k}, \varpi_{n_k}) = \delta. \tag{62}$$

Now, by using (53) and (62), we obtain

$$\lim_{k\longrightarrow\infty} d(\varpi_{m_k+1}, \varpi_{n_k+1}) = \delta. \tag{63}$$

Then, from $(N1)$, $(N2)$, and monotonicity of F_1, we get

$$\begin{aligned}
&\chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(d(\varpi_{m_{k+1}}, \varpi_{n_{k+1}})) \\
&\quad = \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(D(\varpi_{m_{k+1}}, \Omega\varpi_{n_k})) \\
&\quad \le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\
&\quad \le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_2(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\
&\quad \le F_1(\rho(d(\varpi_{m_k}, \varpi_{n_k}), D(\varpi_{m_k}, \Omega\varpi_{m_k}), D(\varpi_{n_k}, \Omega\varpi_{n_k}), \\
&\qquad D(\varpi_{m_k}, \Omega\varpi_{n_k}), D(\varpi_{n_k}, \Omega\varpi_{m_k}))) \\
&\quad \le F_1(\rho(d(\varpi_{m_k}, \varpi_{n_k}), d(\varpi_{m_k}, \varpi_{m_{k+1}}), d(\varpi_{n_k}, \varpi_{n_{k+1}}), \\
&\qquad d(\varpi_{n_{k+1}}, \varpi_{n_k}) + d(\varpi_{n_k}, \varpi_{m_k}), d(\varpi_{n_k}, \varpi_{n_{k+1}}) + d(\varpi_{n_{k+1}}, \varpi_{m_{k+1}}))).
\end{aligned} \tag{64}$$

Since F_1 is continuous, so by passing the limit $k \longrightarrow \infty$, using equations (62) and (63), we have

$$\lim_{k\longrightarrow\infty} \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(\delta) \le F_1(\rho(\delta, 0, 0, \delta, \delta)) \le F_1(\delta\rho(1, 0, 0, 1, 1)). \tag{65}$$

Now, since $\rho \in \mathbb{P}$, we have $\rho(1, 0, 0, 1, 1) \in (0, 1]$; so, (65) implies

$$\lim_{s\longrightarrow\delta^+} \inf \phi(s) \le 0, \tag{66}$$

which is a contradiction to (17). Hence, (55) holds, which implies that $\{\varpi_n\}$ is a Cauchy sequence. Completeness of $\mathfrak{Z}$ ensures the existence of $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{67}$$

By following the same steps as in the proof of Theorem 12, we get $\varpi^* \in \Omega\varpi^*$. This completes the proof. □

Corollary 18. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ such that (N1) and the following condition holds:*

$$\begin{aligned}
&H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \\
&\qquad \le F_1(\wp_1(d(\varpi, \omega) + \wp_2 D(\varpi, \Omega\varpi) + \wp_3 D(\omega, \Omega\omega) \\
&\qquad\quad + \wp_4 D(\varpi, \Omega\omega) + \wp_5 D(\omega, \Omega\varpi) \text{ for all } \varpi, \omega \in \mathfrak{Z},
\end{aligned} \tag{68}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$ and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \tag{69}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$ and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, $\rho \in \mathbb{P}$ and result follow from Theorem 17. □

Remark 19. Corollary 18 improves Theorem 1 of [35]. In fact, by taking $F_1 = F_2$ and by defining $\Omega\varpi = \{\varpi\}$ *for all* ϖ, $\omega \in \mathfrak{Z}$ in Corollary 18, then we are back to Theorem 1 of [34]. In Corollary 18, condition $(F2)$ is weakened to the condition $(F2')$.

Next, we consider $\Omega\varpi$ that are closed subsets of $\mathfrak{Z}$ instead of compact subsets for all $\mathfrak{Z}$ and obtain the following theorems.

Theorem 20. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(X)$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that following holds:*

(G1) $F(c) \le L(c)$ for all $c > 0$

(G2) $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\begin{aligned}
&\chi(d(\varpi, \omega)) + L(H(\Omega\varpi, \Omega\omega)) \\
&\qquad \le F(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))),
\end{aligned} \tag{70}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathscr{P}$.

Then, fix (Ω) is nonempty.

Proof. Let $x_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Assume that $\varpi_1 \in \Omega\varpi_1$; otherwise, ϖ_1 is a fixed point of Ω, and the proof is complete. Then, since $\Omega\varpi_1$ is closed, $D(\varpi_1, \Omega\varpi_1) > 0$ and consequently, $H(\Omega\varpi_0, \Omega\varpi_1) > 0$. Due to $(F4)$, we obtain

$$F(D(\varpi_1, \Omega\varpi_1)) = \inf_{z\in\Omega\varpi_1} F(d(\varpi_1, z)). \tag{71}$$

□

Then, (71) with (G1) and (G2) gives

$$\begin{aligned}\inf_{z\in\Omega\varpi_1} F(d(\varpi_1,z)) &= F(D(\varpi_1,\Omega\varpi_1)) \le F(H(\Omega\varpi_0,\Omega\varpi_1))\\ &\le L(H(\Omega\varpi_0,\Omega\varpi_1))\\ &\le F(\rho(d(\varpi_0,\varpi_1),D(\varpi_0,\Omega\varpi_0),D(\varpi_1,\Omega\varpi_1),\\ &\quad D(\varpi_0,\Omega\varpi_1),D(\varpi_1,\Omega\varpi_0))) - \chi(d(\varpi_0,\varpi_1))\\ &< F(\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_2),0)).\end{aligned} \tag{72}$$

Thus, there exists $x_2 \in \Omega\varpi_1$ such that

$$F(d(\varpi_1,\varpi_2)) < F(\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1),0)). \tag{73}$$

Since F is an nondecreasing function, (73) with (ρ_3) implies that

$$\begin{aligned}d(\varpi_1,\varpi_2) &< \rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1),0)\\ &\le \rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2),0).\end{aligned} \tag{74}$$

By using Lemma 11, (74) implies

$$d(\varpi_1,\varpi_2) < d(\varpi_0,\varpi_1). \tag{75}$$

Next, arguing as previous, we get $\varpi_3 \in \Omega\varpi_2$ with $D(\varpi_2, \Omega\varpi_2) > 0$. Also, by using Lemma 11, from (G1) and (G2), we obtain

$$d(\varpi_2,\varpi_3) < d(\varpi_1,\varpi_2) \tag{76}$$

Continuing in the same manner, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ with $D(\varpi_n,\Omega\varpi_{n+1}) > 0$ and

$$d(\varpi_n,\varpi_{n+1}) < d(\varpi_{n-1},\varpi_n), \tag{77}$$

for all $n \in \mathbb{N}$. (77) implies that $\{d(\varpi_n,\varpi_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Hence, from $(F4)$, (G1), and (G2), we get

$$\begin{aligned}\inf_{z\in\Omega\varpi_n} F(d(\varpi_n,z)) &= F(D(\varpi_n,\Omega\varpi_n)) \le F(H(\Omega\varpi_{n-1},\Omega\varpi_n))\\ &\le L(H(\Omega\varpi_{n-1},\Omega\varpi_n)) \le F(\rho(d(\varpi_{n-1},\varpi_n),D(\varpi_{n-1},\Omega\varpi_{n-1}),\\ &\quad D(\varpi_n,\Omega\varpi_n),D(\varpi_{n-1},\Omega\varpi_n),D(\varpi_n,\Omega\varpi_{n-1})))\\ &\quad - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(\rho(d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),d(\varpi_n,\varpi_{n+1}),\\ &\quad d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_{n+1}),0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(\rho(d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),\\ &\quad 2d(\varpi_{n-1},\varpi_n),0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(d(\varpi_{n-1},\varpi_n)\rho(1,1,1,2,0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)).\end{aligned} \tag{78}$$

Thus, for all $n \in \mathbb{N}$,

$$\inf_{z\in\Omega\varpi_n} F(d(\varpi_n,z)) \le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)). \tag{79}$$

Thus, from (79), there exists $\varpi_{n+1} \in \Omega\varpi_n$ such that

$$F(d(\varpi_n,\varpi_{n+1})) \le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)). \tag{80}$$

Since $\chi \in \Phi$, there exists $h > 0$ and $n_0 \in \mathbb{N}$ such that $\chi(d(\varpi_n,\varpi_{n+1})) < h$, for all $n \ge n_0$. From (80), we obtain

$$\begin{aligned}F(d(\varpi_n,\varpi_{n+1})) &\le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F((\varpi_{n-2},\varpi_{n-1})) - \chi(d(\varpi_{n-2},\varpi_{n-1})) - \chi(d(\varpi_{n-1},\varpi_n)) \vdots\\ &\le F(d(\varpi_0,\varpi_1)) - \sum_{i=1}^{n-1}\chi(d(\varpi_{i-1},\varpi_i))\\ &= F(d(\varpi_0,\varpi_1)) - \sum_{i=1}^{n_0-1}\chi(d(\varpi_{i-1},\varpi_i)) - \sum_{i=n_0}^{n-1}\chi(d(\varpi_{i-1},\varpi_i))\\ &= F(d(\varpi_0,\varpi_1)) - (n-n_0)h, n \ge n_0.\end{aligned} \tag{81}$$

Taking $n \longrightarrow \infty$ in (81), we get $F(d(\varpi_{n-1},\varpi_n)) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} d(\varpi_{n-1},\varpi_n) = 0, \tag{82}$$

Now, from $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\longrightarrow\infty} (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_{n-1},\varpi_n)) = 0. \tag{83}$$

Then, from (81), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&(d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_{n-1},\varpi_n)) - (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_0,\varpi_1))\\ &\le (d(\varpi_{n-1},\varpi_n))^k (F(d(\varpi_{n-1},\varpi_n)) - (n-n_0)h)\\ &\quad - (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_0,\varpi_1)))\\ &= -(d(\varpi_{n-1},\varpi_n))^k (n-n_0)h \le 0.\end{aligned} \tag{84}$$

Taking limit $n \longrightarrow \infty$, in (84) and using (82) and (83), we have

$$\lim_{n\longrightarrow\infty} n(d(\varpi_{n-1},\varpi_n))^k = 0. \tag{85}$$

Observe that from (85), there exists $n_1 \in \mathbb{N}$ such that $n(d(\varpi_{n-1},\varpi_n))^k \le 1$ for all $n \ge n_1$. Thus, for all $n \ge n_1$, we have

$$d(\varpi_{n-1},\varpi_n) \le \frac{1}{n^{1/k}} \text{ for all } n \ge n_1, \tag{86}$$

Now, in order to show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n > n_1$. From (86), we get

$$d(\varpi_m, \varpi_n) \leq \sum_{i=n}^{m-1} d(\varpi_i, \varpi_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \quad (87)$$

As a result of the above and the series' convergence, $\sum_{i=n}^{\infty}(1/i^{1/k})$, we receive that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence. Since $\mathfrak{Z}$ is a complete space, so there exists $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} x_n = \varpi^*. \quad (88)$$

Now,

$$F(\mathrm{H}(\Omega\varpi, \Omega\omega)) \leq L(\mathrm{H}(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + L(H(\Omega\varpi, \Omega\omega) \\ \leq F(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \quad (89)$$

Since F is nondecreasing function, we obtain

$$H(\Omega\varpi, \Omega\omega) \leq \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)), \text{ for all } \varpi, \omega \in \mathfrak{Z}. \quad (90)$$

We claim that ϖ^* is a fixed point of $\mathfrak{Z}$. On contrary, suppose that $D(\varpi^*, \Omega\varpi^*) > 0$ and by equation (90), we have

$$\begin{aligned} D(\varpi^*, \Omega\varpi^*) &\leq d(\varpi^*, \varpi_{n+1}) + D(\varpi_{n+1}, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(\varpi^*, \Omega\varpi_n)) \\ &\leq d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), d(\varpi_n, \varpi_{n+1}), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \varpi^*) + D(\varpi^*, \Omega\varpi^*), \\ &\quad d(\varpi^*, \varpi_{n+1})). \end{aligned} \quad (91)$$

Passing to limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$D(\varpi^*, \Omega\varpi^*) < \rho(0, 0, D(\varpi^*, \Omega\varpi^*), 0 + D(\varpi^*, \Omega\varpi^*), 0), \quad (92)$$

which implies by Lemma 1 that

$$0 < D(x^*, \Omega\varpi^*) < 0, \quad (93)$$

which is a contradiction. Hence, $D(\varpi^*, \Omega\varpi^*) = 0$. Since $\Omega\varpi^*$ is closed, therefore, $\varpi^* \in \Omega\varpi^*$.

Corollary 21. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition holds:*

$$H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \\ \leq L(\wp_1(d(\varpi, \omega) + \wp_2 D(\varpi, \Omega\varpi) + \wp_3 D(\omega, \Omega\omega) \\ + \wp_4 D(\varpi, \Omega\omega) + \wp_5 D(\omega, \Omega\varpi) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \quad (94)$$

where $\wp_i \geq 0$ and $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \quad (95)$$

where $\wp_i \geq 0$ and $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$. Then. $\rho \in \mathscr{P}$ and result follow from Theorem 20. ☐

Corollary 22. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition holds:*

$$H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \\ \leq L(d(\varpi, \omega) + \lambda D(\omega, \Omega\varpi)) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \quad (96)$$

where $\lambda \geq 0$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1 + \lambda\varpi_5, \quad (97)$$

where $\lambda \geq 0$. Then, $\rho \in \mathscr{P}$ and result follow from Theorem 20. ☐

Remark 23. Corollary 21 generalizes Theorem 24 of [33]. Indeed, by considering $L = F$ and by defining $\chi(t) = 2\tau$ for all $\varpi, \omega \in \mathfrak{Z}$, where $\tau > 0$ in Corollary 21, we obtain Theorem 32 of [33]. Also, by considering $L = F$ and by defining $\chi(t) = \tau$ for all $\varpi, \omega \in \mathfrak{Z}$, where $\tau > 0$ in Corollary 22, we get back the Theorem 24 of [32].

Theorem 24. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that following holds:*

(G1) $F(c) \leq L(c)$ for all $c > 0$

(G2) $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\chi(d(\varpi,\omega)) + L(H(\Omega\varpi,\Omega\omega)) \le F(\rho(d(\varpi,\omega), D(\varpi,\Omega\varpi), D(\omega,\Omega\omega), D(\varpi,\Omega\omega), D(\omega,\Omega\varpi))), \tag{98}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathbb{P}$.
Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Then, as in proof of Theorem 20, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ with $D(\varpi_n, \Omega\varpi_{n+1}) > 0$,

$$d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n), \text{for all } n \in \mathbb{N}, \tag{99}$$

$$F(d(\varpi_{n-1}, \varpi_n)) \le F(d(\varpi_0, \varpi_1)) - (n - n_0)h, n \ge n_0. \tag{100}$$

Taking $n \longrightarrow \infty$ in (100), we get $F(d(\varpi_{n-1}, \varpi_n)) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} d(\varpi_{n-1}, \varpi_n) = 0. \tag{101}$$

Next, we claim that

$$\lim_{n,m\longrightarrow\infty} d(\varpi_n, \varpi_m) = 0. \tag{102}$$

If (102) is not true, then there exists $\delta > 0$ such that for all $r \ge 0$, there exists $m_k > n_k > r$

$$d(\varpi_n, \varpi_m) > \delta. \tag{103}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\lambda_{r_0} = d(\varpi_{n-1}, \varpi_n) < \delta \text{ forall } n \ge r_0. \tag{104}$$

Consider two subsequences $\{\varpi_{n_k}\}$ and $\{\varpi_{m_k}\}$ of $\{x_n\}$; then, as is proof of Theorem 17, we get

$$\begin{aligned} &\lim_{k\longrightarrow\infty} d(\varpi_{m_k}, \varpi_{n_k}) = \delta, \\ &\lim_{k\longrightarrow\infty} d(\varpi_{m_k+1}, \varpi_{n_k+1}) = \delta. \end{aligned} \tag{105}$$

Then, from (G1), (G2), and monotonicity of F, we get

$$\begin{aligned} &\chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(d(\varpi_{m_{k+1}}, \varpi_{n_{k+1}})) \\ &= \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(D(\varpi_{m_{k+1}}, \Omega\varpi_{n_k})) \\ &\le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\ &\le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + L(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\ &\le F(\rho(d(\varpi_{m_k}, \varpi_{n_k}), D(\varpi_{m_k}, \Omega\varpi_{m_k}), D(\varpi_{n_k}, \Omega\varpi_{n_k}), \\ &\quad D(\varpi_{m_k}, \Omega\varpi_{n_k}), D(\varpi_{n_k}, \Omega\varpi_{m_k}))) \\ &\le F(\rho(d(\varpi_{m_k}, \varpi_{n_k}), d(\varpi_{m_k}, \varpi_{m_{k+1}}), d(\varpi_{n_k}, \varpi_{n_{k+1}}), d(\varpi_{n_{k+1}}, \varpi_{n_k}) \\ &+ d(\varpi_{n_k}, \varpi_{m_k}), d(\varpi_{n_k}, \varpi_{n_{k+1}}) + d(\varpi_{n_{k+1}}, \varpi_{m_{k+1}}))). \end{aligned} \tag{106}$$

Since F is continuous, so by passing the limit $k \longrightarrow \infty$, using equations (105) and (106), we have

$$\lim_{k\longrightarrow\infty} \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(\delta) \le F(\rho(\delta, 0, 0, \delta, \delta)) \le F(\delta\rho(1, 0, 0, 1, 1)). \tag{107}$$

Now, since $\rho \in \mathbb{P}$, we have $\rho(1, 0, 0, 1, 1) \in (0, 1]$; so, (107) implies

$$\lim_{s\longrightarrow\delta^+} \inf \phi(s) \le 0, \tag{108}$$

which is a contradiction to (17). Hence, (102) holds, which implies that $\{\varpi_n\}$ is Cauchy sequence. Completeness of $\mathfrak{Z}$ ensures the existence of $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{109}$$

By following the same steps as in the proof of Theorem 20, we get $\varpi^* \in \Omega\varpi^*$. This completes the proof. □

Corollary 25. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition hold:*

$$\begin{aligned} &H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi,\omega)) + F(H(\Omega\varpi,\Omega\omega)) \\ &\le L(\wp_1(d(\varpi,\omega) + \wp_2 D(\varpi,\Omega\varpi) + \wp_3 D(\omega,\Omega\omega) \\ &+ \wp_4 D(\varpi,\Omega\omega) + \wp_5 D(\omega,\Omega\varpi) \text{ for all } \varpi,\omega \in \mathfrak{Z}, \end{aligned} \tag{110}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$, and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \tag{111}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$, and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, $\rho \in \mathbb{P}$ and result follow from Theorem 24. □

If we restrict $\lambda = 0$ in Corollary 22, then ρ defined in the proof of Corollary 22 also satisfies $\rho(1, 0, 0, 1, 1) = 1$ and hence $\rho \in \mathbb{P}$. Consequently, from Theorem 24, we get

Corollary 26. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following* *condition hold:*

$$H(\Omega\varpi,\Omega\omega))>0\ \textit{implies}\ \chi(d(\varpi,\omega))+F(H(\Omega\varpi,\Omega\omega)) \leq L(d(\varpi,\omega)+\lambda D(\omega,\Omega\varpi))\ \textit{for all}\ \varpi,\omega\in\mathfrak{Z}, \tag{112}$$

where $\lambda\geq 0$. Then, fix Ω is nonempty.

2. Data Dependence

Let $\mathfrak{Z}$, $\mathfrak{V}$ be two nonempty sets and $\Omega:\mathfrak{Z}\longrightarrow P(\mathfrak{V})$. Denote by $G(\Omega)$, the graph of the multivalued operator is Ω. A multivalued operator $\Omega:\mathfrak{Z}\longrightarrow P(\mathfrak{V})$ is said to be closed if $G(\Omega)$ is a closed set in $\mathfrak{Z}\times\mathfrak{V}$. A selection for Ω is a singlevalued operator $\Omega:\mathfrak{Z}\longrightarrow\mathfrak{V}$ such that $t(\varpi)\in\Omega(\varpi)$, for each $\varpi\in\mathfrak{Z}$. Mo t and Petrusel in [36] discussed some basic problems including data dependence of the fixed point theory for a new type contractive multivalued operator. In [37], Rus et al. gave an important abstract notion as follows:

Definition 27. Let $(\mathfrak{Z},d)$ be a MS and $\Omega:\mathfrak{Z}\longrightarrow CL(\mathfrak{Z})$ a multivalued operator. Then, Ω is a multivalued weakly Picard operator (briefly MWP operator) if for all $\varpi\in\mathfrak{Z}$ and $\omega\in\Omega\varpi$, there exists a sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ such that

(i) $\varpi_0=\varpi$, $\varpi_1=\omega$

(ii) $\varpi_{n+1}\in\Omega\varpi_n$, for all $n\in\mathbb{N}$

(iii) The sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ is convergent, and its limit is a fixed point of Ω

A sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) in Definition 27 is also called a sequence of successive approximations of Ω starting from ϖ_0. Now, we present the main result of this section.

Theorem 28. *Let $(\mathfrak{Z},d)$ be a MS and $\Omega_1,\Omega_2:\mathfrak{Z}\longrightarrow K(\mathfrak{Z})$ be two multivalued operators. Assume that there exists $\chi\in\Phi$, a nondecreasing real-valued function F_1 on $(0,\infty)$ and a real valued function F_2 on $(0,\infty)$ satisfying condition $(F2')$ and $(F3)$ such that Ω_i satisfies (N1) and (N2) for all $i\in\{1,2\}$:*

(i) There exists $\lambda>0$ such that $H(\Omega_1(\varpi),\Omega_2(\varpi))\leq\lambda$, for all $\varpi\in\mathfrak{Z}$

(ii) Then

(iii) Fix $(\Omega_i)\in CL(\mathfrak{Z})$ and $i\in\{1,2\}$

(iv) Ω_1 and Ω_2 are MWP operators and

$$H(Fix(\Omega_1),Fix(\Omega_2))\leq\frac{\lambda}{1-\max\{\rho_1(1,1,1,2,0),\rho_2(1,1,1,2,0)\}}. \tag{113}$$

Proof. (a) By Theorem 12, we have that fix $(\Omega_i)\neq\varnothing$, for $i\in\{1,2\}$. Next, we prove that the fixed point set of multivalued operators Ω_i is closed for $i\in\{1,2\}$. For this, let $\{w_n\}$ be a sequence in fix (Ω_i) such that $w_n\longrightarrow w$ as $n\longrightarrow\infty$. Then,

$$F_1(H(\Omega\varpi,\Omega\omega))\leq F_2(H(\Omega\varpi,\Omega\omega))<\phi(d(\varpi,\omega))+F_2(H(\Omega\varpi,\Omega\omega) \leq F_1(\rho(d(\varpi,\omega),D(\varpi,\Omega\varpi),D(\omega,\Omega\omega),D(\varpi,\Omega\omega),D(\omega,\Omega\varpi))). \tag{114}$$

Since F_1 is nondecreasing function, we obtain for all ϖ, $\omega\in\mathfrak{Z}$

$$H(\Omega\varpi,\Omega\omega)\leq\rho(d(\varpi,\omega),D(\varpi,\Omega\varpi),D(\omega,\Omega\omega),D(\varpi,\Omega\omega),D(\omega,\Omega\varpi)). \tag{115}$$

Suppose that $D(w,\Omega w)>0$, then we have

$$\begin{aligned}D(w,\Omega w)&\leq d(w,w_{n+1})+D(w_{n+1},\Omega w)<d(w,w_{n+1})+H(\Omega w_n,\Omega w)\\&<d(w,w_{n+1})+\rho(d(w_n,w),D(w_n,\Omega w_n),D(w,\Omega w),\\&\quad D(w_n,\Omega w),D(w,\Omega w_n))\\&\leq d(w,w_{n+1})+\rho(d(w_n,w),d(w_n,w_{n+1}),D(w,Tw),\\&\quad d(w_n,w)+D(w,Tw),d(w,w_{n+1})).\end{aligned} \tag{116}$$

Passing to limit as $n\longrightarrow\infty$ in the above inequality, we obtain

$$D(w,\Omega w)<\rho(0,0,D(w,\Omega w),0+D(w,\Omega w),0), \tag{117}$$

which implies by Lemma 1 that

$$0<D(w,Tw)<0, \tag{118}$$

which is a contradiction. Hence, $D(w,\Omega w)=0$. Since Ωw is closed, so $w\in\Omega w$.

(b) From the proof of Theorem 12, we immediately get that Ω_i operators are MWP operators for $i\in\{1,2\}$. Now, we will show that $H(\text{Fix}(\Omega_1),\text{Fix}(\Omega_2))\leq\lambda/1-\max\{\rho_1(1,1,1,2,0),\rho_2(1,1,1,2,0)\}$. For this purpose, Let $q>1$, $\varpi_0\in Fix(\Omega_1)$, be arbitrary. Then, there exists $\varpi_1\in\Omega_2\varpi_0$ such that $d(\varpi_0,\varpi_1)=D(\varpi_0,\Omega_2\varpi_0)$ and $d(\varpi_0,\varpi_1)\leq qH(\Omega_1\varpi_0,\Omega_2\varpi_0)$. Next, for $\varpi_1\in\Omega_2\varpi_0$, there exists $\varpi_2\in\Omega_2\varpi_1$ such that $d(\varpi_0,\varpi_1)=D(\varpi_0,\Omega_2\varpi_0)$ and $d(\varpi_1,\varpi_2)\leq qH(\Omega_2\varpi_0,\Omega_2\varpi_1)$. Then, by using (3.1), we get $d(\varpi_1,\varpi_2)\leq d(\varpi_0,\varpi_1)$ and

$$\begin{aligned}d(\varpi_1,\varpi_2)&\leq qH(\Omega_2\varpi_0,\Omega_2\varpi_1)\\&\leq q\rho(d(\varpi_0,\varpi_1),D(\varpi_0,\Omega_2\varpi_0),D(\varpi_1,\Omega_2\varpi_1),\\&\quad D(\varpi_0,\Omega_2\varpi_1),D(\varpi_1,\Omega_2\varpi_0))\\&\leq q\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,x_2),d(\varpi_0,\varpi_1)\\&\quad +d(\varpi_1,x_2),0))\\&\leq q\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),2d(\varpi_0,\varpi_1),0))\\&\leq q\rho(1,1,1,2,0)d(\varpi_0,\varpi_1).\end{aligned} \tag{119}$$

Inductively, we will obtain a sequence of successive approximations for Ω_2 starting from ϖ_0, satisfying the following:

$$d(\varpi_n, \varpi_{n+1}) \le (q\rho_1(1,1,1,2,0))^n d(\varpi_0, \varpi_1), n \in \mathbb{N}, \quad (120)$$

which further implies for each $n \in \mathbb{N}$,

$$d(\varpi_n, \varpi_{n+m}) \le \frac{(q\rho_1(1,1,1,2,0))^n}{1 - q\rho_1(1,1,1,2,0)} d(\varpi_0, \varpi_1). \quad (121)$$

Letting $n \longrightarrow \infty$, we get that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $(\mathfrak{Z}, d)$, and so it converges to an element $u \in \mathfrak{Z}$. As in the proof of Theorem 12, we get that $u \in \text{Fix}\,(\Omega_2)$. From (121), letting $m \longrightarrow \infty$ to get

$$d(\varpi_n, u) \le \frac{(q\rho_1(1,1,1,2,0))^n}{1 - q\rho_1(1,1,1,2,0)} d(\varpi_0, \varpi_1), \text{for each } n \in \mathbb{N}. \quad (122)$$

Putting $n = 0$, we get that

$$d(\varpi_0, u) \le \frac{1}{1 - q\rho_1 o(1,1,1,2,0)} d(\varpi_0, \varpi_1) \le \frac{q\lambda}{1 - q\rho_1(1,1,1,2,0)}. \quad (123)$$

By interchanging the roles of Ω_1 and Ω_2, we obtain that for each $u_0 \in Fix\,(\Omega_2)$, there exists $x \in Fix\,(\Omega_1)$ such that

$$d(u_0, \varpi) \le \frac{1}{1 - q\rho_2(1,1,1,0,2)} d(u_0, u_1) \le \frac{q\lambda}{1 - q\rho_2(1,1,1,0,2)}. \quad (124)$$

Hence, $H(\text{Fix}\,(\Omega_1), \text{Fix}\,(\Omega_2)) \le q\lambda/1 - \max\{q\rho_1(1,1,1,2,0), q\rho_2(1,1,1,2,0)$, and letting $q \longrightarrow 1$, we get the conclusion. ☐

3. Strict Fixed Points and Well Posedness

Firstly, we define the notions of well posedness of a fixed point problem.

Definition 29 [38, 39]. Let $(\mathfrak{Z}, d)$ be a MS, $\mathfrak{B} \in P(\mathfrak{Z})$, and $\Omega : \mathfrak{B} \longrightarrow C(\mathfrak{Z})$ be a multivalued operator. Then, the fixed point problem is well posed for Ω with respect to D if

(a_1) $Fix\,\Omega = \{\varpi^*$;

(b_1) If $\varpi_n \in \mathfrak{B}$, $n \in \mathbb{N}$, and $D(\varpi_n, \Omega\varpi_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\varpi_n \xrightarrow{d} \varpi^* \in Fix\,\Omega$ as $n \longrightarrow \infty$

Definition 30 [38, 39]. Let $(\mathfrak{Z}, d)$ be a MS, $\mathfrak{B} \in P(\mathfrak{Z})$, and $\Omega : \mathfrak{B} \longrightarrow C(\mathfrak{Z})$ be a multivalued operator. Then, the fixed point problem is well posed for Ω with respect to H if

(a_2) $S\text{Fix}\,\Omega = \{\varpi^*\}$

(b_2) If $\varpi_n \in \mathfrak{B}$, $n \in \mathbb{N}$, and $H(\varpi_n, \Omega\varpi_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\varpi_n \xrightarrow{d_n} \varpi^* \in \text{SFix}\,\Omega$ as $n \longrightarrow \infty$

Remark 31. Note that if the fixed point problem is well posed for Ω with respect to D and $\text{Fix}\,\Omega = \text{SFix}\,\Omega$, then the fixed point problem is well posed for Ω with respect to H.

Theorem 32. *Let $(\mathfrak{Z}, d)$ be a MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued operators. Assume that*

(1) There exist $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0,\infty)$ and a real-valued function F_2 on $(0,\infty)$ satisfying condition $(F2')$ such that (N1) and (N2) hold for $\rho \in \mathscr{P}$ with $\rho(1,0,0,1,1) \in ($ $0, 1)$,

$$SFix\,\Omega \ne \varnothing. \quad (125)$$

Then,

(a) $Fix\,\Omega = SFix\,\Omega = \{\varpi^*\}$;

(b) The fixed point problem is well posed for Ω with respect to H

Proof. (a) By Theorem 17, we have that fix $(\Omega) \ne \varnothing$. Next, We will prove that $\text{Fix}\,\Omega = \{\varpi^*\}$. From (N1) and (N2), we get that

$$\begin{aligned} F_1(H(\Omega\varpi, \Omega\omega)) &\le F_2(H(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega) \\ &\le F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \quad (126)$$

Since F_1 is nondecreasing function, we obtain for all $\varpi, \omega \in \mathfrak{Z}$,

$$H(\Omega\varpi, \Omega\omega) \le \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)). \quad (127)$$

Let $u \in Fix\,\Omega$, with $u \ne \varpi^*$; then $D(\varpi^*, \Omega u) > 0$, and we have

$$\begin{aligned} D(\varpi^*, \Omega u) &= H(\Omega\varpi^*, \Omega u) \\ &\le \rho(d(\varpi^*, u), D(\varpi^*, \Omega\varpi^*), D(u, \Omega u), D(\varpi^*, \Omega u), D(u, \Omega\varpi^*)) \\ &\le \rho(d(\varpi^*, u), 0, 0, d(\varpi^*, u), d(u, \varpi^*)) \le d(\varpi^*, u)\rho(1,0,0,1,1). \end{aligned} \quad (128)$$

Since $\rho(1,0,0,1,1) \in (0,1)$, above inequality implies that

$$d(\varpi^*, u) = D(\varpi^*, \Omega u) < d(\varpi^*, u), \quad (129)$$

which is a contradiction. Hence, $d(\varpi^*, u) = 0$; so, $\varpi^* = u$.

(b) Let $\varpi_n \in \mathfrak{B}$, $n \in \mathbb{N}$, such that

$$\lim_{n\longrightarrow\infty} D(\varpi_n, \Omega\varpi_n) = 0. \quad (130)$$

We claim that

$$\lim_{n\longrightarrow\infty} d(\varpi_n, \varpi^*) = 0, \quad (131)$$

where $\varpi^* \in \text{Fix}\,\Omega$. If (131) is not true, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, we have that

$$d(\varpi_n, \varpi^*) > \varepsilon. \tag{132}$$

On the other hand, from (130), there exists $n_\varepsilon \in \mathbb{N} \setminus 0$ such that $D(\varpi_n, \Omega\varpi_n) < \varepsilon$ for each $n > n_\varepsilon$. Hence, for each $n > n_\varepsilon$, we get

$$\begin{aligned} d(\varpi_n, \varpi^*) &= D(\varpi_n, \Omega\varpi^*) = D(\varpi_n, \Omega\varpi_n) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &\le D(\varpi_n, \Omega\varpi_n) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(\varpi^*, \Omega\varpi_n)) \\ &\le D(\varpi_n, \Omega\varpi_n) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad d(\varpi^*, \varpi^*), d(\varpi_n, \varpi^*), d(\varpi^*, \varpi_n) + D(\varpi_n, \Omega\varpi_n)). \end{aligned} \tag{133}$$

Since $\rho(1, 0, 0, 1, 1) \in (0, 1)$, so by passing the limit $n \longrightarrow \infty$, we obtain $d(x_n, \varpi^*) \longrightarrow 0$ as $n \longrightarrow \infty$, a contradiction. Consequently, proof is complete by Remark 31. □

4. Conclusion

In the theory of set-valued dynamic systems, fixed points and strict fixed points of multivalued operators are essential notions. A rest point of the dynamic system can be read as a fixed point for the multivalued map Ω, whereas a strict fixed point for Ω can be viewed as the system's endpoint. We have made a contribution in this approach by establishing some basic problems in multivalued fixed point and strict fixed point theory. We have proved several existence and data dependence results for multivalued nonlinear mappings satisfying a new class of contractive conditions via auxiliary functions. The obtained outcomes are backed up by a nontrivial example. The findings add to and expand on some of the most recent results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Point Results for Multivalued Mappings with Applications

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In this manuscript, using the concept of multivalued contractions, some new Banach- and Caristi-type fixed point results are established in the context of metric spaces. For the reliability of the presented results, some examples and applications to Volterra integral type inclusion are also studied. The established results unify and generalize some existing results from the literature.

1. Introduction and Preliminaries

Volterra integral equations appear in different scientific applications such as in the spread of epidemics, semiconductor devices, and population dynamics. Also, the dynamics of multispan uniform continuous beams subjected to a moving load is one of the best applications of Volterra integral equations. A universal method for finding sorption uptake curves of fluid multicomponent mixtures in porous solid at variable and constant concentration of mixture components on the basis of the Volterra integral equation has been proposed. The fixed point theory for multivalued mappings is a serious subject of set-valued analysis. In Banach spaces, several well-known fixed point theorems of single-valued mappings such as Banach and Schauder have been extended to multivalued mappings. There are a lot of applications for multivalued mappings such as optimal control theory, differential inclusions, game theory, and many branches in physics.

The Banach contraction principle is used in a variety of fields of mathematics. This technique has many applications in studying the existence of solutions for nonlinear Volterra integral equations, and nonlinear integrodifferential equations in Banach spaces.

Recently, it has been widely spread. For example, in abstract spaces, the Fredholm integral equation introduced by Fredholm [1] and the solutions of Fredholm- and Volterra-type integral equations have been discussed analytically by Rezan et al. [2–4], Rus [5], Aydi et al. [6], Karapinar et al. [7], and Hammad and De la Sen [8, 9] and numerically by Panda et al. [10, 11] and Berenguer et al. [12].

Throughout this paper, the symbols $N(M)$, $C(M)$, $CB(M)$, and $CP(M)$ refer to the family of all nonempty, nonempty closed, nonempty closed bounded, and nonempty compact subset of M, respectively.

For multivalued contraction mappings, we have the following.

Definition 1 (see [13]). Suppose that (M, d) is a metric space. Denote by $CB(M)$, the set of all nonempty closed bounded subsets of M. The Pompeiu-Hausdorff metric $P_{HM} : CB(M) \times CB(M) \longrightarrow [0,\infty)$ induced by the distance d is formulated as follows: for all $A, B \in CB(M)$,

$$P_{HM}(A, B) = \max \left\{ \sup_{s\in B} d(s, A), \sup_{t\in A} d(t, B) \right\}, \qquad (1)$$

where $d(s, A) = \inf_{t\in A} d(s, t)$. Also, the pair $(CB(M), P_{HM})$ is called a generalized Hausdorff distance induced by d.

In 1969, the stipulation of Banach in single-valued mappings was modified to multivalued mappings by Nadler [13] as follows.

Theorem 2. *Let (M, d) be a complete metric space (CMS) and T be a multivalued mapping on M so that $T(s)$ is a nonempty closed, bounded subset of M. If for each $s \in M$ there is $c \in (0\,1)$ so that*

$$P_{HM}(T(s), T(t)) \le cd(s, t), \quad \forall s, t \in M, \tag{2}$$

then, T has a fixed point in M.

In the literature, via abstract spaces, some authors obtained nice fixed point results for contractive mappings under certain conditions, for example, Hussain et al. [14] prove the existence of several fixed point results in ordinary and partially order metric spaces by studying the notion of Geraghty-type contractive mapping via simulation function along with C-class function. In integral type, Branciari [15] introduced some common fixed point results under general contractive conditions.

In Meir-Keeler type, Agarwal et al. [16] obtained some exciting fixed point results. In Menger probabilistic metric spaces, Chauhan et al. [17] discussed a hybrid coincidence and common fixed point theorem under a strict contractive condition with an application. Via the notion of α-admissible mapping, the existence of fixed point theorems under w-distance mappings with an application is presented by Kutbi and Sintunavarat [18], and the others concerned with studying the notion of multivalued mapping and its contributions in fixed point theory such as Nadler [13], Ali and Kamran [19], Aubin and Siegel [20], Covitz and Nadler [21], Hot [22], and Ali et al. [23].

Du and Karapinar [24] introduce the concept of a Caristi-type cyclic map and present a new convergence theorem and a best proximity point theorem for Caristi-type cyclic maps. Petrusel and Sîntămarian [25] obtained a new result in the link of single-valued and multivalued Caristi-type mappings. Hussain et al. [26] introduce the notion of Suzuki-type multivalued contraction with simulation functions and then set up some new fixed point and data dependence results for these types of contraction mappings. Karapinar [27] used lower semicontinuous mappings to generalize Caristi-Kirk's fixed point theorem on partial metric spaces. Abdeljawad and Karapinar [28] generalize Cristi-Kirik's fixed point theorem to Cone metric spaces using Cone-valued lower semicontinuous maps. The relation between Caristi's result and its restriction to the function verifying Caristi's stipulations with continuous real functions is explained by Jachymski [29]. Khojasteh et al. [30] introduce the idea that many known fixed point theorems can easily be derived from the Caristi theorem. Also, Karapinar et al. [31] proposed a new fixed point theorem which is inspired from both Caristi and Banach.

In 2013, Ali et al. [32] initiated the idea of generalized α_*-admissible mappings. Via this concept, fixed point consequences to generalized Mizoguchi's fixed point theorem are derived. For single-valued mappings, Caristi [33] introduced an important theorem in fixed point field and called it the "Caristi fixed point theorem." This theorem was generalized to multivalued mappings in Banach spaces by Feng and Liu [34] as follows:

Theorem 3. *Assume that (M, d) is a CMS and $T : M \longrightarrow N(M)$ is a multivalued mapping so that $T(s)$ is a closed subset of M, if there is a constant $c \in (0, 1)$ so that*

$$d(t, T(t)) \le cd(s, t), \tag{3}$$

for all $s \in M$ and $t \in I_b^s$, where

$$I_b^s = \{t \in T(s)\colon bd(s, t) \le d(s, T(s)), b \in (0, 1)\}, \tag{4}$$

then, there is a fixed point of T in M with $c < b$ and $d(s, T(s))$ is lower semicontinuous.

Theorem 4. *Assume that (M, d) is a CMS and $T : M \longrightarrow C(M)$ is a multivalued mapping so that $T(s)$ is a nonempty subset of M. Let $\phi : M \longrightarrow \mathbb{R}$ be a lower semicontinuous and bounded from below function and $p : [0,\infty) \longrightarrow 0,\infty)$ be a nondecreasing, continuous, and subadditive function so that $p^{-1}(\{0\}) = \{0\}$. If for each $s \in X$, there is an element $t \in T(s)$ verifying*

$$p(d(s, t)) \le \phi(s) - \phi(t), \tag{5}$$

then, the mapping T has a fixed point on M.

Recently, Isik et al. [35] generalized Banach contraction and Caristi's fixed point theorem for a single-valued map as the following:

Theorem 5. *Let (M, d) be a CMS and $T : M \longrightarrow M$ be a self-continuous mapping, if there is the mapping $\varphi : [0,\infty] \longrightarrow [0,\infty]$ so that $\lim_{s\longrightarrow 0^+} \varphi(s) = 0, \varphi(0) = (0)$, and*

$$d(Ts, Tt) \le \varphi(d(s, t)) - \varphi(d(Ts, Tt)), \quad \forall s, t \in M, \tag{6}$$

then, there is a unique fixed point of T.

Theorem 6. *Assume that (M, d) be a CMS and $T : M \longrightarrow M$ is a self-mapping. Let W be a set of mappings $f : \mathbb{R} \longrightarrow (0, \infty)$ so that the hypotheses in the following hold:*

(1) f is continuous and strictly increasing

(2) For each sequence $a_n \subseteq \mathbb{R}^+, \lim_{n\longrightarrow\infty} a_n = 0$ if and only if $\lim_{n\longrightarrow\infty} f(a_n) = 1$

(3) For all $a, b \in \mathbb{R}, f(a + b) \le f(a)f(b)$

If the function $\phi : M \longrightarrow R$ is lower semicontinuous and bounded from as follows so that

$$f(d(s, Ts)) \le \frac{f(\phi(s))}{f(\phi(Ts))}, \quad \forall s \in M, \tag{7}$$

then, T has a fixed point.

Definition 7 (see [36]). A function $p : [0,\infty) \longrightarrow 0,\infty)$ is called subadditive if $p(a + b) \le p(a) + p(b)$ for every $a, b \in 0,\infty)$.

Definition 8 (see [13]). Suppose that $T : M \longrightarrow N(M)$ be a multivalued mapping, a point $s \in M$ is called a fixed point of T if $s \in T(M)$.

Definition 9 (see [37]). Let (M, d) be a metric space A and B be nonempty subsets of M. Then, a set-valued mapping $T : A \cup B \longrightarrow A \cup B$ is called a set-valued cyclic map if $T(A) \subset B$ and $T(B) \subset A$, where $T(A) = \cup\{Tx : x \in A\}$.

The lemma in the following is very useful in the sequel.

Lemma 10 (Zorn's lemma). *Let S be a partially order set. If every totally ordered subset of S has an upper bound then S contains a maximal element.*

Lemma 11 (see [10]). *Assume that A and B are closed and bounded subset of M and let $a \in A$. For each positive $\eta > 0$, then there is $b \in B$ so that $d(a, b) \le P_{HM}(A, B) + \eta$. Moreover, if B is a compact then there is $b \in B$ so that $d(a, b) \le P_{HM}(A, B)$.*

The focus of this work is extending Theorems 3, 4, 5, and 6 for multivalued mappings via generalized contractive conditions. An example and application for the existence of solution of Volterra integral inclusion are also given.

2. Fixed Point Results

We begin this section with the first main result.

Theorem 12. *Let (M, d) be a complete metric space and $T : M \longrightarrow C(M)$ be a multivalued mapping. If there is a nondecreasing mapping $\varphi : [0\infty) \longrightarrow 0\infty)$ so that $\lim_{t \longrightarrow 0^+} \varphi(t) = 0$, $\varphi(0) = 0$ and for all $s \in M$, there is $t \in I_b^s$, so that*

$$d(t, Tt) \le \varphi(d(s, t)) - \varphi\left(\frac{d(t, Tt)}{b}\right), \tag{8}$$

where $I_b^s = \{t \in T(s): bd(s, t) \le d(s, T(s)), b \in (0, 1)\}$ and $d(s, Ts)$ is lower semicontinuous; then, T has a fixed point in M.

Proof. Let $s_\circ$ be an arbitrary element in M. Since I_b^s is nonempty for any $b \in (0, 1)$, we can build a sequence $\{s_n\}$ with $s_{n+1} \in I_b^{s_n}$ for each $n \in \mathbb{N} \cup \{0\}$ so that

$$d(s_{n+1}, Ts_{n+1}) \le \varphi(d(s_n, s_{n+1})) - \varphi\left(\frac{1}{b} d(s_{n+1}, Ts_{n+1})\right). \tag{9}$$

Since $s_{n+2} \in I_b^{s+1}$, then, we can write

$$d(s_{n+1}, s_{n+2}) \le \frac{1}{b} d(s_{n+1}, Ts_{n+1}). \tag{10}$$

Because the mapping φ is nondecreasing, so by (9), we have

$$0 \le d(s_{n+1}, Ts_{n+1}) \le \varphi(d(s_n, s_{n+1})) - \varphi(d(s_{n+1}, s_{n+2})). \tag{11}$$

This implies that the sequence $\{\varphi(d(s_n, s_{n+1}))\}$ is nonincreasing. Since φ is bounded as follows, then there is $k \in \mathbb{R}$ so that $\varphi(d(s_n, s_{n+1})) \longrightarrow k$ as $n \longrightarrow \infty$.

Now, for $m, n \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned} d(s_n, s_m) &\le \sum_{i=n}^{m-1} d(s_i, s_{i+1}) \le \frac{1}{b} \sum_{i=n}^{m-1} d(s_i, Ts_i) \\ &\le \frac{1}{b} \sum_{i=n}^{m-1} [\varphi(d(s_{i-1}, s_i)) - \varphi(d(s_i, s_{i+1}))] \\ &\le \frac{1}{b} (\varphi(d(s_{n-1}, s_n)) - \varphi(d(s_{m-1}, s_m))). \end{aligned} \tag{12}$$

Passing the limit in the above inequality as $n, m \longrightarrow \infty$, one can write

$$\lim_{n,m \longrightarrow \infty} d(s_n, s_m) = 0, \tag{13}$$

this proves that $\{s_n\}$ is a Cauchy sequence. The completeness of M leads to there is $s \in M$ so that $\lim_{n \longrightarrow \infty} d(s_n, s) = 0$. Since $d(s_n, Ts_n)$ is lower semicontinuous, decreasing, φ is nondecreasing and by (9), we obtain that

$$\begin{aligned} 0 \le d(s_{n+1}, Ts_{n+1}) &= \varphi(d(s_n, s_{n+1})) - \varphi\left(\frac{1}{b} d(s_{n+1}, Ts_{n+1})\right) \\ &= \varphi\left(\frac{1}{b} d(s_n, Ts_n)\right) - \varphi\left(\frac{1}{b} d(s_{n+1}, Ts_{n+1})\right), \end{aligned} \tag{14}$$

or, equivalently,

$$d(s_{n+1}, Ts_{n+1}) < d(s_n, Ts_n), \tag{15}$$

this implies

$$d(s_{n+1}, Ts_{n+1}) < d(s_n, Ts_n), \tag{16}$$

therefore, $d(s_n, Ts_n)$ converges to zero. Again, because $d(s_n, Ts_n)$ is lower semicontinuous, then, we have

$$0 \le d(s, Ts) \le \lim_{n \longrightarrow \infty} d(s_n, Ts_n) = 0. \tag{17}$$

Hence, $d(s, Ts) = 0$; also, the closed property of $T(s)$ implies that $s \in Ts$. ☐

Remark 13. Theorem 12 is more general than Theorem 3, because if T verifies the stipulation of Theorem 3, then for each $s \in M$, there is $t \in I_b^s$ so that

$$d(t, Tt) \le cd(s, t) \le \frac{c}{1 + c/b - \sqrt{c/b}} d(s, t), \tag{18}$$

this insinuate that

$$\left(1+\frac{c}{b}-\sqrt{\frac{c}{b}}\right)d(t,Tt)\le cd(s,t), \tag{19}$$

this equivalent to

$$\left(1-\sqrt{\frac{c}{b}}\right)d(t,Tt)+\left(\frac{c}{b}\right)d(t,Tt)\le cd(s,t), \tag{20}$$

yields

$$d(t,Tt)\le \frac{cd(s,t)}{1-\sqrt{c/b}}-\frac{c}{1-\sqrt{c/b}}\left(\frac{d(t,Tt)}{b}\right). \tag{21}$$

Setting $\varphi(p)=cp/(1-\sqrt{c/b})$, we have

$$d(t,Tt)\le \varphi(d(s,t))-\varphi\left(\frac{d(t,Tt)}{b}\right). \tag{22}$$

Moreover, $d(s,Ts)$ is lower semicontinuous; therefore, T has a fixed point by Theorem 12.

If we put $\varphi(p)=p^2$ for all $p\ge 0$ in Theorem 12, we get the result as follows.

Corollary 14. *Suppose that (M,d) is a CMS and $T:M\longrightarrow C(M)$ is a multivalued mapping. If for any $s\in M$ there is $t\in I_b^s$ so that*

$$d(t,Tt)\le (d(s,t))^2-\left(\frac{d(t,Tt)}{b}\right)^2, \tag{23}$$

then, there exists a fixed point of T.

Theorem 15. *Let (M,d) be a CMS and $T:M\longrightarrow CP(M)$ be a multivalued mapping, where $CP(M)$ is a compact subset of M. If there exists a nondecreasing mapping $\varphi:[0,\infty)\longrightarrow 0,\infty)$ so that $\lim_{t\longrightarrow 0}\varphi(t)=0$, $\varphi(0)=0$, and*

$$P_{HM}(Ts,Tt)\le \varphi(d(s,t))-\varphi(P_{HM}(Ts,Tt)),\quad \forall s,t\in M, \tag{24}$$

then, there is a fixed point of T in M.

Proof. Consider $s_\circ$ is an arbitrary point of M. According to Lemma 11, construct a sequence $\{s_n\}$ with $s_{n+1}\in Ts_n$ for all $n\in N$ so that

$$\begin{aligned}0\le d(s_n,s_{n+1})&\le P_{HM}(Ts_{n-1},Ts_n)\\&\le \varphi(d(s_{n-1},s_n))-\varphi(P_{HM}(Ts_{n-1},Ts_n))\\&\le \varphi(d(s_{n-1},s_n))-\varphi(d(s_n,s_{n+1})),\end{aligned} \tag{25}$$

this implies that $\{\varphi(d(s_n,s_{n+1}))\}$ is a nonincreasing sequence, as φ is bounded from the following so there is $r\in\mathbb{R}$ so that $\lim_{n\longrightarrow\infty}\varphi(d(s_n,s_{n+1}))=r$.

To show that $\{s_n\}$ is a Cauchy sequence, let $m,n\in N$ with $m>n$ and applying Lemma 11, we have

$$\begin{aligned}d(s_n,s_m)&\le \sum_{i=n}^{m-1}d(s_i,s_{i+1})\\&\le \sum_{i=n}^{m-1}P_{HM}(Ts_{i-1},Ts_i)\\&\le \sum_{i=n}^{m-1}[\varphi(d(s_{i-1},s_i))-\varphi(d(s_i,s_{i+1}))]\\&=[\varphi(d(s_{n-1},s_n))-\varphi(d(s_{m-1},s_m))].\end{aligned} \tag{26}$$

As $n,m\longrightarrow\infty$, we get

$$\lim_{n,m\longrightarrow\infty}d(s_n,s_m)=0. \tag{27}$$

Thus, $\{s_n\}$ is a Cauchy sequence. The completeness of M leads to there is $s_\circ\in M$ so that $\lim_{n\longrightarrow\infty}s_n=s_\circ$. Thus, the sequence Ts_n converges to $Ts_\circ$. Since $s_{n+1}\in Ts_n$ for each $n\in N$, this implies that $s_\circ\in Ts_\circ$. This finishes the proof. ☐

Example 1. Let

$$\begin{aligned}M&=\left\{s_i=\frac{i(i+1)}{2}\,i\in N\right\},\\d(s,t)&=|s-t|.\end{aligned} \tag{28}$$

It is obvious that the pair (M,d) is complete metric space. Define a multivalued mapping T by

$$T(s)=\begin{cases}s_1, & \text{if } s=s_1\wedge h\in Mg,\\ \{s_1,s_2,\cdots,s_{i-1}\}, & \text{if } s=s_i \text{ and } i>1.\end{cases} \tag{29}$$

Now, we verify that the multivalued mapping T satisfied the condition of Theorem 15 with $\varphi(p)=p.\exp(p)$. Note that $P_{HM}(Ts_j,Ts_i)>0$ iff $(i=1 \text{ and } j>2)$ or $(j>i>1)$ so we have the following two cases.

Case 1. If $j>2$ and $i=1$, we have

$$\begin{aligned}\frac{P_{HM}(Ts_j,Ts_1)\left(1+e^{P_{HM}(Ts_j,Ts_1)}\right)}{d(s_j,s_1)e^{d(s_j,s_1)}}&\le\frac{P_{HM}(Ts_j,Ts_1)\left(2e^{P_{HM}(Ts_j,Ts_1)}\right)}{d(s_j,s_1)e^{d(s_j,s_1)}}\\&=\frac{(s_{j-1}-s_1)(2e^{s_{j-1}-s_1})}{(s_j-s_1)(e^{s_j-s_1})}\\&=\frac{(s_{j-1}-s_1)(2e^{s_{j-1}-s_j})}{(s_j-s_1)}\\&=\frac{(j^2-j-2)(2e^{-j})}{(j^2-j-2)}<2e^{-j}<2e^{-1}<1.\end{aligned} \tag{30}$$

That is,

$$\frac{P_{HM}(Ts_j, Ts_1)\left(1+e^{P_{HM}(Ts_j,Ts_1)}\right)}{d(s_j, s_1)e^{d(s_j,s_1)}} < 1, \tag{31}$$

which implies that

$$P_{HM}(Ts_j, Ts_1) \le d(Ts_j, Ts_1)e^d(s_j, s_1) - d(Ts_j, Ts_1)e^d(Ts_j, Ts_1). \tag{32}$$

Case 2. If $j > i > 1$, then, we have

$$\begin{aligned} \frac{P_{HM}(Ts_j, Ts_i)\left(1+e^{P_{HM}(Ts_j,Ts_i)}\right)}{d(s_j, s_i)e^{d(s_j,s_i)}} &\le \frac{P_{HM}(Ts_j, Ts_i)\left(2e^{P_{HM}(Ts_j,Ts_i)}\right)}{d(s_j, s_i)e^{d(s_j,s_i)}} \\ &= \frac{(s_{j-1}-s_{i-1})(2e(s_{j-1}-s_{i-1}))}{(s_j-s_i)e(s_j-s_i)} \\ &= \frac{(s_{j-1}-s_{i-1})(2e(s_{j-1}-s_{i-1}-s_j+s_i))}{(s_j-s_i)} \\ &= \frac{(j+i-1)(2e^{i-j})}{(j+i+1)} \le 2e^{i-j} \le 2e^{-1} < 1, \end{aligned} \tag{33}$$

that is,

$$\frac{P_{HM}(Ts_j, Ts_i)\left(1+e^{P_{HM}(Ts_j,Ts_i)}\right)}{d(s_j, s_i)e^{d(s_j,s_i)}} \le 1, \tag{34}$$

therefore,

$$P_{HM}(Ts_j, Ts_i) < d(s_j, s_i)e^d(s_j, s_i) - P_{HM}(Ts_j, Ts_i)e^{P_{HM}(Ts_j,Ts_i)}. \tag{35}$$

Hence, the condition of Theorem 15 is satisfied. Also, $s_1 = 1$ is a fixed point of T.

Example 2. Suppose that $M = [0,\infty), d(s, t) = |s-t|$ for all $s, t \in M$. It is obvious that the pair (M, d) is a CMS. Define a multivalued mapping T by

$$T(s) = \left\{\frac{s}{2}\right\}, \tag{36}$$

then, we have

$$P_{HM}(Ts, Tt) = \frac{1}{2}|s-t| = \frac{1}{2}d(s, t), \tag{37}$$

$$d^2(s, t) - P^2_{HM}(Ts, Tt) = d^2(s, t) - \frac{1}{4}d^2(s, t) = \frac{3}{4}d^2(s, t), \tag{38}$$

it follows from (37) and (38) that

$$P_{HM}(Ts, Tt) \le d^2(s, t) - P^2_{HM}(Ts, Tt), \quad \forall s, t \in M. \tag{39}$$

Hence, T satisfies the condition of Theorem 15 with $\varphi(p) = p^2$.

Remark 16. Theorem 15 upgrades the Nadler fixed point result in a finite-dimensional space. Indeed, if T holds condition of Nadler's theorem, then for $s, t \in M$, we have

$$P_{HM}(Ts, Tt) \le cd(s, t) \le \frac{cd(s, t)}{1+c-\sqrt{c}}, \tag{40}$$

or, equivalently,

$$(1-\sqrt{c})P_{HM}(Ts, Tt) + cP_{HM}(Ts, Tt) \le cd(s, t), \tag{41}$$

yields

$$P_{HM}(Ts, Tt) \le \frac{c}{1-\sqrt{c}}d(s, t) - \frac{c}{1-\sqrt{c}}P_{HM}(Ts, Tt). \tag{42}$$

Taking $\varphi(p) = cp/(1-\sqrt{c})$, we get

$$P_{HM}(Ts, Tt) \le \varphi(d(s, t)) - \varphi(P_{HM}(Ts, Tt)), \tag{43}$$

moreover, for each $s \in M, T(s) \in CB(M)$ but M is a finite-dimensional space; therefore, $T(s)$ is compact, and hence, by Theorem 15, there is a fixed point of T.

Let Ω be the set of all mappings $f : R \longrightarrow (0\infty)$ having the same stipulations (1)–(3) of Theorem 6. Note: by property (2), of Theorem 6, we have $f(a) = 1$ if and only if $a = 0$.

Example 3. All the functions in the following belong to Ω.

(i) $f_1(s) = 1 + \tanh s$

(ii) $f_2(s) = e^s$

(iii) $f_3(s) = \begin{cases} 1 + \ln(1+s), & \text{if } s \in 0, \infty) \\ e^s, & \text{if } s \in (-\infty, 0] \end{cases}$

The lemmas in the following help us to supplement the theoretical results.

Lemma 17. *Assume that (M, d) is a CMS and $T : M \longrightarrow N(M)$ is a multivalued mapping. Let $\phi : M \longrightarrow R$ be a function defined in Theorem 6 and $f \in \Omega$. Define the relation "≤" on M so that*

$$s \le t \Leftrightarrow f((d(s, t))) \le \frac{f(\phi(s))}{f(\phi(t))}, \tag{44}$$

then, the relation "≤" is a partial order on M and M is a partial order space.

Proof. Since $f(a)=1$ iff $a=0$, then, we have

$$f(d(s,s))=f(0)=1=\frac{f(\phi(s))}{f(\phi(s))}, \tag{45}$$

this shows that $s\leq s$.

If $s\leq t$ and $t\leq s$, then

$$\begin{aligned} f(d(s,t)) &\leq \frac{f(\phi(s))}{f(\phi(t))},\\ f(d(t,s)) &\leq \frac{f(\phi(t))}{f(\phi(s))}, \end{aligned} \tag{46}$$

because $d(t,s)=d(s,t)$; thus, $f(d(t,s))=1$. Moreover, $f(a)=1$ iff $a=0$; therefore, $d(t,s)=0$; this implies that $s=t$.

Finally, if $s\leq t$ and $t\leq u$, then

$$\begin{aligned} f(d(s,t)) &\leq \frac{f(\phi(s))}{f(\phi(t))},\\ \text{and} f(d(t,u)) &\leq \frac{f(\phi(t))}{f(\phi(u))}, \end{aligned} \tag{47}$$

it follows from $f(a+b)\leq f(a).f(b)$ that

$$f(d(s,u))\leq f(d(s,t)+d(t,u))\leq f(d(s,t).f(d(t,u)))\leq \frac{f(\phi(s))}{f(\phi(u))}. \tag{48}$$

From the above results, we conclude that "≤" is a partial order on M. □

Lemma 18. *Let the pair (M,d) be a CMS and $T:M\longrightarrow N(M)$ be a multivalued mapping. Consider $\phi:M\longrightarrow R$ be a function defined in Theorem 6, $f\in\Omega$ and $\eta:[0,\infty)\longrightarrow 0,\infty)$ is a nondecreasing, continuous, and subadditive function that ensures that $\eta^{-1}(\{0\})=\{0\}$. Define the relation "≤" on M by*

$$s\leq t \Leftrightarrow f(\eta(d(s,t)))\leq \frac{f(\phi(s))}{f(\phi(t))}, \tag{49}$$

then, the relation "≤" is a partial on M and M is a partial order space.

Proof. Since $f(a)=1$ iff $a=0$ and $\eta^{-1}(\{0\})=\{0\}$, then, we get

$$f(\eta(d(s,s)))=f(\eta(0))=f(0)=1=\frac{f(\phi(s))}{f(\phi(s))}, \tag{50}$$

this implies that $s\leq s$.

If $s\leq t$ and $t\leq s$, then

$$\begin{aligned} f(\eta(d(s,t))) &\leq \frac{f(\phi(s))}{f(\phi(t))},\\ f(\eta(d(t,s))) &\leq \frac{f(\phi(t))}{f(\phi(s))}. \end{aligned} \tag{51}$$

Since $d(t,s)=d(s,t)$, thus $f(d(t,s))=1$. Moreover, $f(a)=1$ iff $a=0$ and $\eta^{-1}(\{0\})=\{0\}$; therefore, $d(t,s)=0$, i.e., $s=t$.

Again, if $s\leq t$ and $t\leq u$, then

$$\begin{aligned} f(\eta(d(s,t))) &\leq \frac{f(\phi(s))}{f(\phi(t))},\\ f(\eta(d(t,u))) &\leq \frac{f(\phi(t))}{f(\phi(u))}, \end{aligned} \tag{52}$$

it follows from $f(a+b)\leq f(a).f(b)$, and η is nondecreasing and subadditive that

$$\begin{aligned} f(\eta(d(s,u))) &\leq f(\eta(d(s,t)+d(t,u)))\\ &\leq f(\eta(d(s,t))+\eta(d(t,u)))\\ &\leq f(\eta(d(s,t))).f(\eta(d(t,u)))\leq \frac{f(\phi(s))}{f(\phi(u))}, \end{aligned} \tag{53}$$

and this completes the required. So, "≤" is a partial order on M. □

Now, we can state and prove the next main theorems.

Theorem 19. *Let (M,d) be a CMS, $T:M\longrightarrow N(M)$ be a multivalued mapping, $\phi:M\longrightarrow R$ be a function defined as Theorem 6, and $f\in\Omega$. If for all $s\in M$, there is $t\in T(s)$ so that*

$$f(d(s,t))\leq \frac{f(\phi(s))}{f(\phi(t))}, \tag{54}$$

then, the mapping T has a fixed point in M.

Proof. Since M is a partial order space, then we need to prove only that M has a maximal element. Suppose that $\{s_\alpha\}_{\alpha\in I}$ is increasing sequence in M, that is, for $\alpha,\beta\in I$ with $\alpha\leq\beta$ then $s_\alpha\leq s_\beta$ from (54), we have $\{\phi(s_\alpha)\}_{\alpha\in I}$ is decreasing, since ϕ is bounded as follows then $\inf_{\alpha\in I}\phi(s_\alpha)$ holds. Assume that α_n is increasing sequence in I that ensures

$$\lim_{n\longrightarrow\infty}\phi(s_{\alpha_n})=\inf_{\alpha\in I}\phi(s_\alpha)=r. \tag{55}$$

Now, for $m,n\in\mathbb{N}$ with $m\geq n$, then $\alpha_n\leq\alpha_m$ and $\{s_{\alpha_n}\}\leq\{s_{\alpha_m}\}$.

Applying (54), we get

$$f\left(d\left(s_{\alpha_n}, s_{\alpha_m}\right)\right) = f\left(\sum_{i=n}^{m-1} d(s_i, s_{i+1})\right) \le \Pi_{i=n}^{m-1} f(d(s_i, s_{i+1})) \le \Pi_{i=n}^{m-1} \frac{f(\phi(s_i))}{f(\phi(s_{i+1}))} = \frac{f(\phi(s_n))}{f(\phi(s_m))}. \tag{56}$$

As $m, n \longrightarrow \infty$ in the above inequality, we have $\lim_{n,m \longrightarrow \infty} f d(s_{\alpha_n}, s_{\alpha_m}) = 1$. Since f is continuous so, we obtain

$$\lim_{n,m \longrightarrow \infty} d\left(s_{\alpha_n}, s_{\alpha_m}\right) = 0. \tag{57}$$

Hence, $\{s_{\alpha_n}\}$ is a Cauchy sequence in M. Since M is complete then there exists $s \in M$ so that

$\{s_{\alpha_n}\} \longrightarrow s$ as $n \longrightarrow \infty$. Thus, s is an upper bound for $\{s_{\alpha_n}\}$ for each $n \ge 1$. Now, we want to show that s is also an upper bound for $\{s_\alpha\}$. Suppose that $\beta \in I$ with $s_{\alpha_n} \le s_\beta$ for each $n \ge 1$. Then, by (54), we obtain that $\phi(s_\beta) \le \phi(s_{\alpha_n})$ for each $n \ge 1$. Also, from (54), we have

$$\phi\left(s_\beta\right) = \inf_{\alpha \in I} \phi(s_\alpha) = r. \tag{58}$$

Since

$$f\left(d\left(s_{\alpha_n}, s_\beta\right)\right) \le \frac{f\left(\phi\left(s_{\alpha_n}\right)\right)}{f\left(\phi\left(s_\beta\right)\right)}, \tag{59}$$

then, by taking the limit as $n \longrightarrow \infty$, we get $\lim_{n \longrightarrow \infty} s_{\alpha_n} = s_\beta$. Hence, for each $\alpha \in I$, there is $n \ge 1$ so that $s_\alpha \le s_{\alpha_n}$, also $s_{\alpha_n} \le s$, yields $s_\alpha \le s$ for each $\alpha \in I$. Thus, s is an upper bound for $\{s_\alpha\}_{\alpha \in I}$. By Zorn's lemma, M has a maximal element $s_\circ$. The condition $t_\circ \in T(s_\circ)$ implies that

$$f(d(s_\circ, t_\circ)) \le \frac{f(\phi(s_\circ))}{f(\phi(t_\circ))}, \tag{60}$$

this shows that $s_\circ \le t_\circ$. Since $s_\circ$ is maximal, therefore, $s_\circ = t_\circ$. Therefore, $s_\circ \in T(s_\circ)$. This ends the proof. □

Theorem 20. *Let (M, d) be a CMS, $T : M \longrightarrow N(M)$ be a multivalued mapping, $\phi : M \longrightarrow R$ be a function defined as Theorem 6, $f \in \Omega$, and $\eta : [0,\infty) \longrightarrow 0, \infty)$ be a nondecreasing continuous subadditive function such that $\eta^{-1}(\{0\}) = \{0\}$. If for any $s \in M$, there is $t \in T(s)$ so that*

$$f(\eta(d(s, t))) \le \frac{f(\phi(s))}{f(\phi(t))}, \tag{61}$$

then, there is a fixed point of T in M.

Proof. By the same manner of the proof of Theorem 19, we can easily show that there is a maximal point $s_\circ$ of partial order space M, and by hypothesis, there is $t_\circ \in T(s_\circ)$ that ensures

$$f(\eta(d(s_\circ, t_\circ))) \le \frac{f(\phi(s_\circ))}{f(\phi(t_\circ))}. \tag{62}$$

This implies that $s_\circ \le t_\circ$. As $s_\circ$ is a maximal element of M, therefore, $s_\circ = t_\circ$; hence, $s_\circ \in T(s_\circ)$. This completes the proof. □

3. Supportive Application

Here, we use Theorem 15 with $\varphi(p) = p^2$ to determine the existence of a solution to the Volterra-type integral inclusion of the following form:

$$s(q) \in \int_a^q W(q, j, s(j))dj + g(q), q \in [a, b], \tag{63}$$

where $W : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow P_{cv}(\mathbb{R})$, $P_{cv}(\mathbb{R})$ refers to the class of nonempty compact and convex subset of $\mathbb{R}$ and $g : [a, b] \longrightarrow \mathbb{R}$ is a continuous function.

Let $M = C([a, b], \mathbb{R})$ be the space of all continuous real-valued functions on $[a, b]$. Define the distance

$$d(s, t) = \sup_{q \in [a,b]} |s(q) - t(q)|, \tag{64}$$

for all $s \in M$. It is clear that the pair (M, d) is a metric space.

We shall consider Problem (63) under the hypotheses as follows:

(1) $W : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow P_{cv}(\mathbb{R})$ and $g : [a, b] \longrightarrow \mathbb{R}$ are continuous functions, as well as W is compact and convex

(2) There is a continuous function $l : M \longrightarrow 0\infty)$ so that

$$P_{HM}(W(q, j, s(j)), W(q, j, t(j))) \le l(j) \left(\frac{-1 + \sqrt{1 + 4(s(j) - t(j))^2}}{2} \right), \tag{65}$$

for each $q, j \in [a, b]$ and $s, t \in M$

(3) We have $\sup_{t \in [a,b]} \int_a^t l(j)dj \le 1$

Now, our main theorem in this part becomes valid for showing.

Theorem 21. *Under the assumptions (1)–(3), Problem (63) has a solution on M.*

Proof. Define the multivalued mapping $T : C([a, b], R) \longrightarrow CP(C([a, b], R))$ by

$$Ts(q) = \left\{ u \in M : u \in \int_a^q W(q, j, s(j))dj + g(q), q \in [a, b] \right\}. \tag{66}$$

□

The unique solution of Problem (63) is equivalent to find a fixed point of T in M.

It is obvious that Ts is compact because for any $s \in M$, $W(.,.,s)$ is lower semicontinuous. For clarification, let $W_s = W(q, j, s(j))$, for each $q, j \in [a, b]$. Thus, $W_s : [a, b] \times [a, b] \longrightarrow P_{cv}(\mathbb{R})$, by Michael selection theorem [38], there is a continuous function $m_s : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ so that $m_s(q, j) \in W_s(q, j)$. It follows that $\int_a^q m_s(q, j)dj + g(q) \in Ts(q)$; this implies that Ts is nonempty. Hence, it is compact.

Proof. Suppose that $s, u \in M$ such that $u \in Ts$. Then, for each $q, j \in [a, b]$,$m_s(q, j) \in W_s(q, j)$ and

$$u(q) = \int_a^q m_s(q, j)dj + g(q), \quad \forall q \in [a, b]. \tag{67}$$

Now, by condition (1), there exist $v_t(q, j) \in W_t(q, j)$ such that

$$|m_s(q, j) - v_t(q, j)| \le l(j) \frac{-1 + \sqrt{1 + 4|s(j) - t(r)|^2}}{2}, \quad \forall q, j \in [a, b]. \tag{68}$$

Let us define the multivalued operator O by

$$O(q, j) = W_t(q, j) \cap \left\{ w \in \mathbb{R} : |m_s(q, s) - w| \le q(j) \frac{-1 + \sqrt{1 + 4|s(j) - t(j)|^2}}{2} \right\}, \quad \forall q, j \in [a, b]. \tag{69}$$

Since T is lower semicontinuous, then there exist $m_t : [a, b] \times [a, b] \longrightarrow R$ such that for all $q, j \in [a, b]$, $m_t(q, j) \in W_t(q, j)$. Thus, for any $r \in M$,

$$r(q) = \int_a^q m_t(q, j)dj + g(q) \in \int_a^q W(q, j, t(j))dj + g(q), q \in [a, b]. \tag{70}$$

By condition (3), we can write

$$\begin{aligned} |u(q) - r(q)| &\le \int_a^q |m_s(q, j) - m_t(q, j)|dj \\ &\le \int_a^q l(j) \frac{-1 + \sqrt{1 + 4|s(j) - t(j)|^2}}{2} dj \\ &\le \int_a^q l(j) \frac{-1 + \sqrt{1 + 4)\|s(j) - t(j)\|^2}}{2} dj \\ &= \frac{-1 + \sqrt{1 + 4\|s(j) - t(j)\|^2}}{2} \int_a^q l(j)dj \\ &\le \left(\frac{-1 + \sqrt{1 + 4\|s(j) - t(j)\|^2}}{2} \right) \left(\sup \int_a^q l(j)dj \right) \\ &\le \frac{-1 + \sqrt{1 + 4\|s(j) - t(j)\|^2}}{2}. \end{aligned} \tag{71}$$

This implies that

$$d(u(q), r(q)) \le \frac{-1 + \sqrt{1 + 4[d(s(j), t(j))]^2}}{2}. \tag{72}$$

Now, by exchanging the rule of s and t, we have

$$P_{HM}(Ts, Tt) \le \frac{-1 + \sqrt{1 + 4[d(s(j), t(j))]^2}}{2}, \quad \forall s, t \in M, \tag{73}$$

yielding

$$(1 + 2P_{HM}(Ts, Tt))^2 \le 1 + 4[d(s, t)]^2 \tag{74}$$

is equivalent to

$$P_{HM}(Ts, Tt) + [P_{HM}(Ts, Tt)]^2 \le [d(s, t)]^2. \tag{75}$$

Therefore,

$$P_{HM}(Ts, Tt) \le [d(s, t)]^2 - [P_{HM}(Ts, Tt)]^2, \quad \forall s, t \in M. \tag{76}$$

Therefore, all the conditions of Theorem 15 are fulfilled. Therefore, the operator T has a fixed point which is a solution to Problem (63). □

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Research Article

On a Couple of Nonlocal Singular Viscoelastic Equations with Damping and General Source Terms: Blow-Up of Solutions

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Under some given conditions, we prove the explosion result of the solution of the system of nonlocal singular viscoelastic with damping and source terms on general case. This current study is a general case of the previous work of Boulaaras.

1. Introduction

During the last decades, many nonlocal problems of deterministic and parabolic partial differential equations have been studied. These equations and their systems represent the modeling of many physical phenomena related to time. These constraints can be data measured directly at the boundary or give integral boundary conditions (for instance, see [1–25]).

In this work, we investigate the blow-up of the following system of nonlinear damping term:

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \displaystyle\int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x \mathrm{d}s + |u_t|^{m-1}u_t = f_1(u,v), Q, \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \displaystyle\int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x \mathrm{d}s + |v_t|^{m-1}v_t = f_2(u,v), Q, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,\alpha), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), \quad x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0, \displaystyle\int_0^\alpha xu(x,t)\mathrm{d}x = \int_0^\alpha xv(x,t)\mathrm{d}x = 0, \end{cases} \tag{1}$$

where$f_1(u,v), f_2(u,v)$: $R^2 \longrightarrow R$given by

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}, \\ f_2(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}, \end{aligned} \tag{2}$$

with $a, b \in R$, $r \geq -1$ (we get $a = b = 1$), $Q = (0,\alpha) \times (0,T)$, $\alpha < \infty$, $T < \infty$, and

$$g_1(.), g_2(.) : R^+ \longrightarrow R^+ \tag{3}$$

are given functions which will be specified later. The motivation of our work is because of some results regarding the following research paper: in [12], under some conditions suitable for the relaxation function, the author explained that solutions with initial negative energy explode in a finite time if $p > m$ and continue to find if $m \geq p$, for the following studied problem:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u + u_t|u_t|^{m-2} = |u|^{p-2}u, \quad \text{in } \Omega \times (0,\infty), \\ u = 0 x \in \partial\Omega, \quad t \in (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{4}$$

In [4], the author studied a model describing the movement of a flexible two-dimensional viscous body on the unit disk (i.e., radial solutions) and by using some density arguments and some prior estimates, the authors demonstrated the existence and uniqueness of a generalized solution to the following problem:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x \mathrm{d}s = f(x,t,u,u_x), & \text{in } Q, u_x(1,t)=0, \\ \int_0^1 xu(x,t)\mathrm{d}x = 0, & t \in (0,T), \\ u(x,0) = \varphi(x), & \\ u_t(x,0) = \psi(x), & x \in (0,1), \end{cases} \tag{5}$$

where

$$Q = (0,1) \times (0,T) \tag{6}$$

and f is the right-hand side that satisfied the Lipschitzian condition. Recently, in [3], the authors demonstrated the decay result of energy for a small enough initial data together with the explosion result of large initial data of the following singular problem:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x \mathrm{d}s = |u|^{p-2}u, \\ u(a,t)=0, \int_0^a xu(x,t)\mathrm{d}x = 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \end{cases} \tag{7}$$

That is, they obtained the blow-up properties of local solution by Georgiev-Todorova method with nonpositive initial energy. More work followed up on similar nonlocal singular viscoelastic equations and systems in [8, 9].

In this work, we continue the study on system (1). According to some given conditions, we prove the explosion result of the solution of the system of nonlocal singular viscoelastic with damping and source terms on general case, where we begin by giving basic definitions and theories about the function spaces we need, and then, we mention the theorem of local existence. Finally, we announce and prove the main result of our studied problem in (1).

1.1. Preliminaries. In this section, we introduce some functional spaces and give some lemma's need for the remaining of this paper. Let $L_x^p = L_x^p((0,\alpha))$ be the weighed Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^\alpha x|u|^p \mathrm{d}x\right)^{1/p}. \tag{8}$$

$H = L_x^2((0,\alpha))$is, in particular, the Hilbert space of square integral functions having the finite norm

$$\|u\|_H = \left(\int_0^\alpha xu^2 \mathrm{d}x\right)^{1/2}. \tag{9}$$

$V = V_x^1((0,\alpha))$is the Hilbert space equipped with the norm

$$\begin{gathered} \|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2\right)^{1/2}, \\ V_0 = \{u \in V : u(\alpha) = 0\}. \end{gathered} \tag{10}$$

Lemma 1 (Poincare-type inequality). *For any $u \in V_0$,*

$$\int_0^\alpha xu^2 dx \le C_P \int_0^\alpha xu_x^2 dx, \tag{11}$$

where C_P is some positive constant.

Remark 2. It is clear that $\|u\|_{V_0} = \|u_x\|_H$ defines an equivalent norm on V_0.

Lemma 3. *For any $u \in V_0$ and $2 < p < 4$, we have*

$$\|u\|_{L_x^p(0,\alpha)}^p \le C_* \|u_x\|_{H=L_x^2(0,\alpha)}^p, \tag{12}$$

where C_* is a constant depending on α and p only. For the g_1 and g_2 functions, assumptions are as follows:(G1): $g_1(.)$, $g_2(.)$: $R^+ \longrightarrow R^+$ are two differentiable and nonincreasing functions with

$$\begin{gathered} g_1(t) \ge 0, 1 - \int_0^\infty g_1(s)ds = I_1 \ge 0, \\ g_2(t) \ge 0, 1 - \int_0^\infty g_2(s)ds = I_2 \ge 0, \end{gathered} \tag{13}$$

(G2): For all $t \ge 0$,

$$\begin{gathered} g_1(t) \ge 0, \\ g_1'(t) \le 0, \\ g_2(t) \ge 0, \\ g_2'(t) \le 0. \end{gathered} \tag{14}$$

(G3): $r \ge -1$.

Theorem 4. *Suppose that (G1), (G2), and (G3) hold. Then, for all $(u_0, v_0) \in V_0^2$ and all $(u_1, v_1) \in H^2$, problem ((1)) admits a unique local solution (u, v):*

$$u, v \in C((0,T); V_0) \cap C^1((0,T); H), \tag{15}$$

for $T > 0$ small enough.

Lemma 5. *Assume that (G1), (G2), and (G3) hold and (u, v) is a solution of problem((1)); then, the energy functional*

$$E(t) = \frac{1}{2}\int_0^\alpha xu_t^2\,dx + \frac{1}{2}\int_0^\alpha xv_t^2\,dx + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_0^\alpha xu_x^2\,dx + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\int_0^\alpha xv_x^2dx + \frac{1}{2}(g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ v_x)(t) - \int_0^\alpha xF(u,v)dx, \quad (16)$$

where

$$(g_1 \circ u_x)(t) = \int_0^\alpha\int_0^t xg_1(t-s)|u_x(x,s) - u_x(x,t)|^2 dsdx,$$
$$(g_2 \circ v_x)(t) = \int_0^\alpha\int_0^t xg_2(t-s)|v_x(x,s) - v_x(x,t)|^2 dsdx,$$
$$F(u,v) = \frac{1}{2(r+2)}\left[\left|u+v\right|^{2(r+2)} + 2\left|uv\right|^{r+2}\right]. \quad (17)$$

Remark 6. Multiplying the first equation in((1))by xu_t and the second equation in((1))by xv_t integrating over $(0,\alpha)$, we obtain the following equation:

$$\frac{d}{dt}[E(t)] = -\int_0^\alpha x|u_t|^{m+1}\,\mathrm{d}x - \int_0^\alpha x|v_t|^{m+1}\,\mathrm{d}x, = -\left[\|u_t\|_{L_x^{m+1}}^{m+1} + \|v_t\|_{L_x^{m+1}}^{m+1}\right]. \quad (18)$$

The definition of the norm is as follows:

$$\left[\|u_t\|_{L_x^{m+1}}^{m+1} + \|v_t\|_{L_x^{m+1}}^{m+1}\right] \geq 0. \quad (19)$$

From here,

$$-\left[\|u_t\|_{L_x^{m+1}}^{m+1} + \|v_t\|_{L_x^{m+1}}^{m+1}\right] \leq 0. \quad (20)$$

Thus,

$$\frac{d}{dt}[E(t)] = -\left[\|u_t\|_{L_x^{m+1}}^{m+1} + \|v_t\|_{L_x^{m+1}}^{m+1}\right] \leq 0. \quad (21)$$

Lemma 7. *There exist c_0 and c_1 positive constants such that*

$$\frac{c_0}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(r+2)}\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right). \quad (22)$$

Lemma 8. *If $2 \leq s \leq p$,*

$$\|u\|_{L_x^p}^s \leq C\left(\|u_x\|_H^2 + \|u\|_{L_x^p}^p\right). \quad (23)$$

2. Blow-Up of Solution

In this section, we shall deal with the blow-up behavior of solutions for problem (1). We derive the blow-up properties of solutions of problem (1) with nonpositive initial energy by the method given in [1].

Theorem 9. *Assume that (G1), (G2), and (G3) hold. $E(0) < 0$ and*

$$\int_0^\infty g_i(s)ds < \frac{r+1}{r+1+1/(4(r+2))}, i = 1, 2. \quad (24)$$

Then, the solution of problem (1) blows up in finite time.

Proof. Since $(d/dt)[E(t)] = E'(t) \leq 0$,

$$E(t) \leq E(0) < 0, \quad \forall t \geq 0. \quad (25)$$

We define $H(t) = -E(t)$; then,

$$0 < H(0) \leq H(t) = -E(t), \quad \forall t \geq 0. \quad (26)$$

□

We obviously substitute $E(t)$ in (26); then,

$$0 < H(0) \leq H(t) = -\frac{1}{2}\int_0^\alpha xu_t^2\,\mathrm{d}x - \frac{1}{2}\int_0^\alpha xv_t^2\,\mathrm{d}x - \frac{1}{2}\left(1 - \int_0^t g_1(s)\mathrm{d}s\right)\int_0^\alpha xu_x^2\mathrm{d}x - \frac{1}{2}\left(1 - \int_0^t g_2(s)\mathrm{d}s\right)\int_0^\alpha xv_x^2\mathrm{d}x - \frac{1}{2}(g_1 \circ u_x)(t) - \frac{1}{2}(g_2 \circ v_x)(t) + \int_0^\alpha xF(u,v)\mathrm{d}x. \quad (27)$$

From (22) and (27),

$$0 < H(0) \leq H(t) \leq \int_0^\alpha xF(u,v)\,\mathrm{d}x \leq \frac{c_1}{2(r+2)}\left[\int_0^\alpha x|u|^{2(r+2)}\mathrm{d}x + \int_0^\alpha x|v|^{2(r+2)}\mathrm{d}x\right] = \frac{c_1}{2(r+2)}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right]. \quad (28)$$

Thus,

$$H(t) \leq \frac{c_1}{2(r+2)}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right]. \quad (29)$$

Equation (29) will then be used as an important data for proof of the theorem. Now, we define

$$L(t)=H^{1-\sigma}(t)+\varepsilon\left(\int_0^\alpha xuu_t\mathrm{d}x+\int_0^\alpha xvv_t\mathrm{d}x\right) \quad (30)$$

for ε small enough and

$$0<\sigma\le\min\left\{\frac{2(r+2)-\mathrm{m}}{2m(r+2)},\frac{2(r+2)-m}{2m(r+2)},\frac{2r+2}{4(r+2)}\right\}. \quad (31)$$

By differentiating (30), using (1) and $H'(t)=\|u_t\|_{L_x^{m+1}}^{m+1}+\|v_t\|_{L_x^{m+1}}^{m+1}$, we obtain

$$\begin{aligned}L'(t)&=(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|_{L_x^{m+1}}^{m+1}+\|v_t\|_{L_x^{m+1}}^{m+1}\right]+\varepsilon\int_0^\alpha xu_t^2\mathrm{d}x\\&+\varepsilon\int_0^\alpha xv_t^2\mathrm{d}x-\varepsilon\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\int_0^\alpha xu_x^2\mathrm{d}x\\&-\varepsilon\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\int_0^\alpha xv_x^2\mathrm{d}x+\varepsilon\int_0^\alpha\int_0^t xg_1\\&\cdot(t-s)u_x(x,t)[u_x(x,s)-u_x(x,t)]\mathrm{d}s\mathrm{d}x+\varepsilon\int_0^\alpha\int_0^t xg_2\\&\cdot(t-s)v_x(x,t)[v_x(x,s)-v_x(x,t)]\mathrm{d}s\mathrm{d}x\\&-\varepsilon\int_0^\alpha xuu_t|u_t|^{m-1}\mathrm{d}x-\varepsilon\int_0^\alpha xvv_t|v_t|^{m-1}\mathrm{d}x\\&+\varepsilon2(r+2)\int_0^\alpha xF(u,v)\mathrm{d}x.\end{aligned} \quad (32)$$

By using Young inequality and from $H(t)=-E(t)$,

$$\begin{aligned}\int_0^\alpha xF(u,v)\,\mathrm{d}x&=H(t)+\frac{1}{2}\int_0^\alpha xu_t^2\,\mathrm{d}x+\frac{1}{2}\int_0^\alpha xv_t^2\,\mathrm{d}x\\&+\frac{1}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\int_0^\alpha xu_x^2\mathrm{d}x\\&+\frac{1}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\int_0^\alpha xv_x^2\mathrm{d}x\\&+\frac{1}{2}(g_1\circ u_x)(t)+\frac{1}{2}(g_2\circ v_x)(t),\end{aligned} \quad (33)$$

we obtain

$$\begin{aligned}L'(t)&\ge(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|_{L_x^{m+1}}^{m+1}+\|v_t\|_{L_x^{m+1}}^{m+1}\right]\\&+\varepsilon(r+3)\int_0^\alpha xu_t^2\mathrm{d}x+\varepsilon(r+3)\int_0^\alpha xv_t^2\mathrm{d}x\\&+\varepsilon\left[(r+1)-\left((r+1)+\frac{1}{4\theta}\right)\int_0^t g_1(s)\mathrm{d}s\right]\int_0^\alpha xu_x^2\mathrm{d}x\\&+\varepsilon\left[(r+1)-\left((r+1)+\frac{1}{4\theta}\right)\int_0^t g_2(s)\mathrm{d}s\right]\int_0^\alpha xv_x^2\mathrm{d}x\\&+\varepsilon(r-\theta+2)(g_1\circ u_x)(t)+\varepsilon(r-\theta+2)(g_2\circ v_x)(t)+\varepsilon2(r+2)H(t)\\&-\varepsilon\int_0^\alpha xuu_t|u_t|^{m-1}\mathrm{d}x-\varepsilon\int_0^\alpha xvv_t|v_t|^{m-1}\mathrm{d}x.\end{aligned} \quad (34)$$

where

$$\begin{aligned}\alpha_3&=r-\theta+2>0\Rightarrow r+2>\theta>0,\\\alpha_4&=r-\theta+2>0\Rightarrow r+2>\theta>0,\\\alpha_1&=\left[(r+1)-\left((r+1)+\frac{1}{4\theta}\right)\int_0^t g_1(s)\mathrm{d}s\right]>0,\\\alpha_2&=\left[(r+1)-\left((r+1)+\frac{1}{4\theta}\right)\int_0^t g_2(s)\mathrm{d}s\right]>0.\end{aligned} \quad (35)$$

From (34),

$$\begin{aligned}L'(t)&\ge(1-\sigma)H^{-\sigma}(t)\left[\|u_t\|_{L_x^{m+1}}^{m+1}+\|v_t\|_{L_x^{m+1}}^{m+1}\right]\\&+\varepsilon(r+3)\int_0^\alpha xu_t^2\mathrm{d}x+\varepsilon(r+3)\int_0^\alpha xv_t^2\mathrm{d}x+\varepsilon\alpha_1\int_0^\alpha xu_x^2\mathrm{d}x\\&+\varepsilon\alpha_2\int_0^\alpha xv_x^2\mathrm{d}x+\varepsilon\alpha_3(g_1\circ u_x)(t)+\varepsilon\alpha_4(g_2\circ v_x)(t)\\&+\varepsilon2(r+2)H(t)-\varepsilon\int_0^\alpha xuu_t\left|u_t\right|^{m-1}\mathrm{d}x-\varepsilon\int_0^\alpha xvv_t\left|v_t\right|^{m-1}\mathrm{d}x,\end{aligned} \quad (36)$$

$$\begin{aligned}H(t)=-E(t)&=-\frac{1}{2}\int_0^\alpha xu_t^2\,\mathrm{d}x-\frac{1}{2}\int_0^\alpha xv_t^2\,\mathrm{d}x\\&-\frac{1}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\int_0^\alpha xu_x^2\mathrm{d}x-\frac{1}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\int_0^\alpha xv_x^2\mathrm{d}x\\&-\frac{1}{2}(g_1\circ u_x)(t)-\frac{1}{2}(g_2\circ v_x)(t)+\int_0^\alpha xF(u,v)\mathrm{d}x,\end{aligned} \quad (37)$$

and for $a_5<\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,2(r+2)\}$,

$$\begin{aligned}\varepsilon2(r+2)H(t)&=\varepsilon(a_5+(2(r+2)-a_5))H(t)\\&=\varepsilon a_5H(t)+\varepsilon(2(r+2)-a_5)H(t)\\&=-\frac{\varepsilon}{2}a_5\int_0^\alpha xu_t^2\mathrm{d}x-\frac{\varepsilon}{2}a_5\int_0^\alpha xv_t^2\mathrm{d}x\\&-\frac{\varepsilon}{2}a_5\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\int_0^\alpha xu_x^2\mathrm{d}x\\&-\frac{\varepsilon}{2}a_5\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\int_0^\alpha xv_x^2\mathrm{d}x\\&-\frac{\varepsilon}{2}a_5(g_1\circ u_x)(t)-\frac{\varepsilon}{2}a_5(g_2\circ v_x)(t)\\&+\varepsilon a_5\int_0^\alpha xF(u,v)\mathrm{d}x+\varepsilon((2(r+2)-a_5))H(t).\end{aligned} \quad (38)$$

To estimate the last term in (36), we apply the three-parameter Young inequality: $a,b\ge0,(1/r)+(1/q)=1$ $,ab\le(\delta^r/r)a^r+(\delta^{-q}b^q/q),\forall\delta>0$. We take

$$r=m+1\ v\ eq=\frac{m+1}{m}, \quad (39)$$

in this case:

$$-\varepsilon\int_0^\alpha xuu_t|u_t|^{m-1}\mathrm{d}x\ge-\varepsilon\frac{\delta_1^{m+1}}{m+1}\|u\|_{L_x^{m+1}}^{m+1}-\varepsilon\frac{m}{m+1}\delta_1^{-((m+1)/m)}\|u_t\|_{L_x^{m+1}}^{m+1}. \quad (40)$$

Similarly

$$-\varepsilon\int_0^\alpha xvv_t|v_t|^{m-1}\mathrm{d}x \ge -\varepsilon\frac{\delta_2^{m+1}}{m+1}\|v\|_{L_x^{m+1}}^{m+1} - \varepsilon\frac{m}{m+1}\delta_2^{-((m+1)/m)}\|v_t\|_{L_x^{m+1}}^{m+1}. \tag{41}$$

Substituting (38), (40), and (41) into (36), by organizing, we obtain

$$\begin{aligned}L'(t) \ge{}& \left[(1-\sigma)H^{-\sigma}(t) - \frac{m}{m+1}\varepsilon\delta_1^{-((m+1)/m)}\right]\|u_t\|_{L_x^{m+1}}^{m+1} \\ &+ \left[(1-\sigma)H^{-\sigma}(t) - \frac{m}{m+1}\varepsilon\delta_2^{-((m+1)/m)}\right]\|v_t\|_{L_x^{m+1}}^{m+1} \\ &+ \varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha xu_t^2\mathrm{d}x + \varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha xv_t^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_1 - \frac{a_5}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\right)\int_0^\alpha xu_x^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_2 - \frac{a_5}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\right)\int_0^\alpha xv_x^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_3 - \frac{a_5}{2}\right)(g_1\circ u_x)(t) + \varepsilon\left(\alpha_4 - \frac{a_5}{2}\right)(g_2\circ v_x)(t) \\ &+ \varepsilon a_5\int_0^\alpha xF(u,v)\mathrm{d}x + \varepsilon((2(r+2)-a_5))H(t) \\ &- \varepsilon\frac{\delta_1^{m+1}}{m+1}\|u\|_{L_x^{m+1}}^{m+1} - \varepsilon\frac{\delta_2^{m+1}}{m+1}\|v\|_{L_x^{m+1}}^{m+1}.\end{aligned} \tag{42}$$

Since integration in estimate (40) and (41) is performed over the space, the parameter δ_1 and δ_2 can be a function of time; we get them as follows:

$$\delta_1^{-((m+1)/m)} = k_1H^{-\sigma}(t) \Rightarrow \delta_1^{m+1} = k_1^{-m}H^{\sigma m}(t), \tag{43}$$

$$\delta_2^{-((m+1)/m)} = k_2H^{-\sigma}(t) \Rightarrow \delta_2^{m+1} = k_2^{-m}H^{\sigma m}(t), \tag{44}$$

where $k_1 > 0$ and $k_2 > 0$ are sufficiently large constants to be specified further. By using (43) and (44) in (42), we have

$$\begin{aligned}L'(t) \ge{}& \left((1-\sigma) - \frac{m}{m+1}\varepsilon k_1\right)H^{-\sigma}(t)\|u_t\|_{L_x^{m+1}}^{m+1} \\ &+ \left((1-\sigma) - \frac{m}{m+1}\varepsilon k_2\right)H^{-\sigma}(t)\|v_t\|_{L_x^{m+1}}^{m+1} \\ &+ \varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha xu_t^2\mathrm{d}x + \varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha xv_t^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_1 - \frac{a_5}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\right)\int_0^\alpha xu_x^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_2 - \frac{a_5}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\right)\int_0^\alpha xv_x^2\mathrm{d}x \\ &+ \varepsilon\left(\alpha_3 - \frac{a_5}{2}\right)(g_1\circ u_x)(t) + \varepsilon\left(\alpha_4 - \frac{a_5}{2}\right)(g_2\circ v_x)(t) \\ &+ \varepsilon a_5\int_0^\alpha xF(u,v)\mathrm{d}x + \varepsilon((2(r+2)-a_5))H(t) \\ &- \frac{\varepsilon k_1^{-m}}{m+1}H^{\sigma m}(t)\|u\|_{L_x^{m+1}}^{m+1} - \frac{\varepsilon k_2^{-m}}{m+1}H^{\sigma m}(t)\|v\|_{L_x^{m+1}}^{m+1}.\end{aligned} \tag{45}$$

To estimate the last two terms in (45), we use (29); then,

$$\frac{\varepsilon k_1^{-m}}{m+1}(H(t))^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_1^{-m}c_2^{\sigma m}}{m+1}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right]^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1}. \tag{46}$$

On the other hand, since $r > \max\{m, m\}$ from $L_x^{2(r+2)^\circ}$ L_x^{m+1},

$$\|u\|_{L_x^{m+1}}^{m+1} \le C\|u\|_{L_x^{2(r+2)}}^{m+1} \le C\left[\|u\|_{L_x^{2(r+2)}} + \|v\|_{L_x^{2(r+2)}}\right]^{m+1}. \tag{47}$$

Substituting (47) into (46),

$$\frac{\varepsilon k_1^{-m}}{m+1}(H(t))^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_1^{-m}c_2^{\sigma m}C}{m+1}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right]^{\sigma m} \cdot \left[\|u\|_{L_x^{2(r+2)}} + \|v\|_{L_x^{2(r+2)}}\right]^{m+1}. \tag{48}$$

By using

$$a, b \ge 0, \quad 1 \le p < \infty, \quad a^p + b^p \le (a+b)^p, \tag{49}$$

we can estimate the following:

$$\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right] \le \left[\|u\|_{L_x^{2(r+2)}} + \|v\|_{L_x^{2(r+2)}}\right]^{2(r+2)}. \tag{50}$$

Consequently, we have

$$\frac{\varepsilon k_1^{-m}}{m+1}(H(t))^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_1^{-m}c_2^{\sigma m}C}{m+1}\left[\|u\|_{L_x^{2(r+2)}} + \|v\|_{L_x^{2(r+2)}}\right]^{2(r+2)\sigma m+m+1}. \tag{51}$$

Similarly

$$\frac{\varepsilon k_2^{-m}}{m+1}(H(t))^{\sigma m}\|v\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_2^{-m}c_2^{\sigma m}C}{m+1}\left[\|u\|_{L_x^{2(r+2)}} + \|v\|_{L_x^{2(r+2)}}\right]^{2(r+2)\sigma m+m+1}. \tag{52}$$

By using (51) and (52),

$$a, b \ge 0, \quad 1 \le p < \infty, (a+b)^p \le c\left(a^p + b^p\right), \quad \left(c = 2^{p-1}\right), \tag{53}$$

for $c \cdot C = C'$; we have

$$\frac{\varepsilon k_1^{-m}}{m+1}(H(t))^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_1^{-m}c_2^{\sigma m}C'}{m+1}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1}\right], \tag{54}$$

$$\frac{\varepsilon k_2^{-m}}{m+1}(H(t))^{\sigma m}\|v\|_{L_x^{m+1}}^{m+1} \le \frac{\varepsilon k_2^{-m}c_2^{\sigma m}C'}{m+1}\left[\|u\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1}\right]. \tag{55}$$

From (31),

$$
\begin{gathered}
2(r+2)\sigma m+m+1 \leq 2(r+2)\left(\frac{2(r+2)-m}{2m(r+2)}\right)m+m+1=2(r+2)+1, \\
r \geq -1, \\
2(r+2) \geq 2, \\
2(r+2)\sigma m+m+1 \geq 2.
\end{gathered}
\tag{56}
$$

From here,

$$
2 \leq 2(r+2)\sigma m+m+1 \leq 2(r+2)+1. \tag{57}
$$

Thus, by applying (23), we obtain

$$
\begin{aligned}
\|u\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1} &\leq \|u_x\|_H^2+\|u\|_{L_x^{2(r+2)}}^{2(r+2)}, \\
\|v\|_{L_x^{2(r+2)}}^{2(r+2)\sigma m+m+1} &\leq \|v_x\|_H^2+\|v\|_{L_x^{2(r+2)}}^{2(r+2)}.
\end{aligned}
\tag{58}
$$

Substituting these inequalities in (54) and (55), in this case,

$$
\frac{\varepsilon k_1^{-m}}{m+1}(H(t))^{\sigma m}\|u\|_{L_x^{m+1}}^{m+1} \leq \frac{\varepsilon k_1^{-m} c_2^{\sigma m} C'}{m+1}\left[\|u_x\|_H^2+\|u\|_{L_x^{2(r+2)}}^{2(r+2)}+\|v_x\|_H^2+\|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right], \tag{59}
$$

$$
\frac{\varepsilon k_2^{-m}}{m+1}(H(t))^{\sigma m}\|v\|_{L_x^{m+1}}^{m+1} \leq \frac{\varepsilon k_2^{-m} c_2^{\sigma m} C'}{m+1}\left[\|u_x\|_H^2+\|u\|_{L_x^{2(r+2)}}^{2(r+2)}+\|v_x\|_H^2+\|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right]. \tag{60}
$$

With the combination of (59) and (60), we obtain

$$
\begin{aligned}
&-\frac{\varepsilon k_1^{-m}}{m+1}H^{\sigma m}(t)\|u\|_{L_x^{m+1}}^{m+1}-\frac{\varepsilon k_2^{-m}}{m+1}H^{\sigma m}(t)\|v\|_{L_x^{m+1}}^{m+1} \\
&\quad \geq \left[-\frac{\varepsilon k_1^{-m} c_2^{\sigma m} C'}{m+1}-\frac{\varepsilon k_2^{-m} c_2^{\sigma m} C'}{m+1}\right]\left(\|u\|_{L_x^{2(r+2)}}^{2(r+2)}+\|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right) \\
&\quad +\left[-\frac{\varepsilon k_1^{-m} c_2^{\sigma m} C'}{m+1}-\frac{\varepsilon k_2^{-m} c_2^{\sigma m} C'}{m+1}\right]\left(\|u_x\|_H^2+\|v_x\|_H^2\right).
\end{aligned}
\tag{61}
$$

Finally,

$$
\begin{gathered}
\|u_x\|_H^2=\int_0^\alpha x u_x^2 \mathrm{d}x, \ \|v_x\|_H^2=\int_0^\alpha x v_x^2 \mathrm{d}x, \\
\|u\|_{L_x^{2(r+2)}}^{2(r+2)}=\int_0^\alpha x|u|^{2(r+2)}\mathrm{d}x, \ \|v\|_{L_x^{2(r+2)}}^{2(r+2)}=\int_0^\alpha x|v|^{2(r+2)}\mathrm{d}x, \\
c' \cdot\left(\int_0^\alpha x|u|^{2(r+2)}\mathrm{d}x+\int_0^\alpha x|v|^{2(r+2)}\mathrm{d}x\right) \leq \int_0^\alpha x F(u,v)\mathrm{d}x, \ \left(\frac{c_0}{2(r+2)}=c'\right),
\end{gathered}
\tag{62}
$$

and by considering (61), thus by organizing (45), we have

$$
\begin{aligned}
L'(t) \geq & \left((1-\sigma)-\frac{m}{m+1}\varepsilon k_1\right)H^{-\sigma}(t)\|u_t\|_{L_x^{m+1}}^{m+1} \\
& +\left((1-\sigma)-\frac{m}{m+1}\varepsilon k_2\right)H^{-\sigma}(t)\|v_t\|_{L_x^{m+1}}^{m+1} \\
& +\varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha x u_t^2 \mathrm{d}x+\varepsilon\left((r+3)-\frac{a_5}{2}\right)\int_0^\alpha x v_t^2 \mathrm{d}x \\
& +\varepsilon\left[\left(\alpha_1-\frac{a_5}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\right)-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right] \\
& \cdot\int_0^\alpha x u_x^2 \mathrm{d}x+\varepsilon\left[\left(\alpha_2-\frac{a_5}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\right)-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right] \\
& \cdot\int_0^\alpha x v_x^2 \mathrm{d}x+\varepsilon\left[c' a_5-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right]\int_0^\alpha x|u|^{2(r+2)}\mathrm{d}x \\
& +\varepsilon\left[c' a_5-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right]\int_0^\alpha x|v|^{2(r+2)}\mathrm{d}x \\
& +\varepsilon\left[\alpha_3-\frac{a_5}{2}\right](g_1 \circ u_x)(t)+\varepsilon\left[\alpha_4-\frac{a_5}{2}\right](g_2 \circ u_x)(t)+\varepsilon[2(r+2)-a_5]H(t),
\end{aligned}
\tag{63}
$$

which introduce the constant

$$
\begin{aligned}
\gamma=\varepsilon \cdot \min & \left\{(r+3)-\frac{a_5}{2},\left[\left(\alpha_1-\frac{a_5}{2}\left(1-\int_0^t g_1(s)\mathrm{d}s\right)\right)-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right],\right. \\
& \left[\left(\alpha_2-\frac{a_5}{2}\left(1-\int_0^t g_2(s)\mathrm{d}s\right)\right)-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right], \\
& \left.\left[c' a_5-\frac{k_1^{-m}c_2^{\sigma m}C'}{m+1}-\frac{k_2^{-m}c_2^{\sigma m}C'}{m+1}\right],\left[\alpha_3-\frac{a_5}{2}\right],\left[\alpha_4-\frac{a_5}{2}\right],[2(r+2)-a_5]\right\}.
\end{aligned}
\tag{64}
$$

Taking sufficiently large $k_1>0$ and $k_2>0$ for the positive constant γ, we simplify (63)

$$
\begin{aligned}
L'(t) \geq & \left((1-\sigma)-\frac{m}{m+1}\varepsilon k_1\right)H^{-\sigma}(t)\|u_t\|_{L_x^{m+1}}^{m+1} \\
& +\left((1-\sigma)-\frac{m}{m+1}\varepsilon k_2\right)H^{-\sigma}(t)\|v_t\|_{L_x^{m+1}}^{m+1} \\
& +\varepsilon\gamma\left[\int_0^\alpha x u_t^2 \mathrm{d}x+\int_0^\alpha x v_t^2 \mathrm{d}x+\int_0^\alpha x u_x^2 \mathrm{d}x+\int_0^\alpha x v_x^2 \mathrm{d}x\right. \\
& +\int_0^\alpha x|u|^{2(r+2)}\mathrm{d}x+\int_0^\alpha x|v|^{2(r+2)}\mathrm{d}x+(g_1 \circ u_x)(t) \\
& \left.+(g_2 \circ v_x)(t)+H(t)\right].
\end{aligned}
\tag{65}
$$

For fixed $k_1>0$, $k_2>0$, and $\gamma>0$, we choose $\varepsilon>0$ so small that the following inequality holds:

$$
\begin{aligned}
&\left((1-\sigma)-\frac{m}{m+1}\varepsilon k_1\right) \geq 0, \\
&\left((1-\sigma)-\frac{m}{m+1}\varepsilon k_2\right) \geq 0.
\end{aligned}
\tag{66}
$$

Moreover, we assume that the initial data satisfy the estimate

$$
L(0)=H^{1-\sigma}(0)+\varepsilon\left(\int_0^\alpha x u_0 u_1 \mathrm{d}x+\int_0^\alpha x v_0 v_1 \mathrm{d}x\right)>0. \tag{67}
$$

Then, from (65), we obtain the following inequality:

$$L'(t) \geq \varepsilon\gamma\left[\int_0^\alpha xu_t^2\,dx + \int_0^\alpha xv_t^2\,dx + \int_0^\alpha xu_x^2\,dx + \int_0^\alpha xv_x^2\,dx + \int_0^\alpha x|u|^{2(r+2)}dx + \int_0^\alpha x|v|^{2(r+2)}dx + (g_1 \circ u_x)(t) + (g_2 \circ v_x)(t) + H(t)\right]. \quad (68)$$

On the other hand, in Equation (30), we take the$1/(1-\sigma)\cdot$powerof each side

$$[L(t)]^{1/(1-\sigma)} = \left[H^{1-\sigma}(t) + \varepsilon\left(\int_0^\alpha xuu_t dx + \int_0^\alpha xvv_t dx\right)\right]^{1/(1-\sigma)}. \quad (69)$$

Twice by applying the following inequality to (69)

$$a, b \geq 0, 1 \leq p < \infty, (a+b)^p \leq 2^{p-1}\left(a^p + b^p\right), \quad (70)$$

we have

$$\begin{aligned}[L(t)]^{1/(1-\sigma)} &\leq 2^{\sigma/(1-\sigma)}\left[H(t) + \varepsilon^{1/(1-\sigma)}\left|\int_0^\alpha xuu_t\,dx + \int_0^\alpha xvv_t\,dx\right|^{1/(1-\sigma)}\right] \\ &\leq 2^{\sigma/(1-\sigma)}\left[H(t) + \varepsilon^{1/(1-\sigma)}2^{\sigma/(1-\sigma)}\left(\left|\int_0^\alpha xuu_t dx\right|^{1/(1-\sigma)} + \left|\int_0^\alpha xvv_t dx\right|^{1/(1-\sigma)}\right)\right] \\ &\leq C\left[H(t) + \left|\int_0^\alpha xuu_t dx\right|^{1/(1-\sigma)} + \left|\int_0^\alpha xvv_t dx\right|^{1/(1-\sigma)}\right],\end{aligned} \quad (71)$$

where $C > 0$. Now, to estimate the last two terms in (71), we, respectively, apply Holder inequality, $L_x^{2(r+2)\circ} L_x^H$, and Young inequality; thus,

$$\begin{aligned}\left|\int_0^\alpha xuu_t\,dx\right|^{1/(1-\sigma)} &\leq \|u\|_H^{1/(1-\sigma)}\|u_t\|_H^{1/(1-\sigma)} \leq C\|u\|_{L_x^{2(r+2)}}^{1/(1-\sigma)}\|u_t\|_H^{1/(1-\sigma)} \\ &\leq C\left(\|u\|_{L_x^{2(r+2)}}^{\theta/(1-\sigma)} + \|u_t\|_H^{\mu/(1-\sigma)}\right).\end{aligned} \quad (72)$$

Similarly,

$$\left|\int_0^\alpha xvv_t dx\right|^{1/(1-\sigma)} \leq C\left(\|v\|_{L_x^{2(r+2)}}^{\theta/(1-\sigma)} + \|v_t\|_H^{\mu/(1-\sigma)}\right), \quad (73)$$

where $(1/\theta) + (1/\mu) = 1$. In these inequalities by collecting side by side, we obtain

$$\left|\int_0^\alpha xuu_t\,dx\right|^{1/(1-\sigma)} + \left|\int_0^\alpha xvv_t\,dx\right|^{1/(1-\sigma)} \leq C\left(\|u\|_{L_x^{2(r+2)}}^{\theta/(1-\sigma)} + \|u_t\|_H^{\mu/(1-\sigma)} + \|v\|_{L_x^{2(r+2)}}^{\theta/(1-\sigma)} + \|v_t\|_H^{\mu/(1-\sigma)}\right). \quad (74)$$

We choose $\mu = 2(1-\sigma)$, to get

$$\theta = \frac{2(1-\sigma)}{1-2\sigma} \leq 2(r+2), \quad (75)$$

then

$$\left|\int_0^\alpha xuu_t\,dx\right|^{1/(1-\sigma)} + \left|\int_0^\alpha xvv_t\,dx\right|^{1/(1-\sigma)} \leq C\left(\|u\|_{L_x^{2(r+2)}}^{2/(1-2\sigma)} + \|u_t\|_H^2 + \|v\|_{L_x^{2(r+2)}}^{2/(1-2\sigma)} + \|v_t\|_H^2\right). \quad (76)$$

By applying (23), we can write

$$\begin{aligned}\|u\|_{L_x^{2(r+2)}}^{2/(1-2\sigma)} &\leq C\left(\|u_x\|_H^2 + \|u\|_{L_x^{2(r+2)}}^{2(r+2)}\right), \\ \|v\|_{L_x^{2(r+2)}}^{2/(1-2\sigma)} &\leq C\left(\|v_x\|_H^2 + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right).\end{aligned} \quad (77)$$

From here, we obtain

$$\left|\int_0^\alpha xuu_t\,dx\right|^{1/(1-\sigma)} + \left|\int_0^\alpha xvv_t\,dx\right|^{1/(1-\sigma)} \leq C\left(\|u_t\|_H^2 + \|v_t\|_H^2 + \|u_x\|_H^2 + \|v_x\|_H^2 + \|u\|_{L_x^{2(r+2)}}^{2(r+2)} + \|v\|_{L_x^{2(r+2)}}^{2(r+2)}\right). \quad (78)$$

Thus, by considering (78) and the following in (71),

$$\begin{aligned}&\|u_t\|_H^2 = \int_0^\alpha xu_t^2 dx, \|v_t\|_H^2 = \int_0^\alpha xv_t^2 dx, \\ &\|u_x\|_H^2 = \int_0^\alpha xu_x^2 dx, \|v_x\|_H^2 = \int_0^\alpha xv_x^2 dx, \\ &\|u\|_{L_x^{2(r+2)}}^{2(r+2)} = \int_0^\alpha x|u|^{2(r+2)}dx, \|v\|_{L_x^{2(r+2)}}^{2(r+2)} = \int_0^\alpha x|v|^{2(r+2)}dx, \\ &(g_1 \circ u_x)(t) \geq 0, (g_2 \circ v_x)(t) \geq 0,\end{aligned} \quad (79)$$

we obtain

$$[L(t)]^{1/(1-\sigma)} \leq C\left[H(t) + \int_0^\alpha xu_t^2\,dx + \int_0^\alpha xv_t^2\,dx + \int_0^\alpha xu_x^2\,dx + \int_0^\alpha xv_x^2 dx + \int_0^\alpha x|u|^{2(r+2)}dx + \int_0^\alpha x|v|^{2(r+2)}dx + (g_1 \circ u_x)(t) + (g_2 \circ v_x)(t)\right]. \quad (80)$$

Finally, by combining (68) and (80), we obtain the following ordinary differential inequality:

$$L'(t) \geq \lambda L^{1/(1-\sigma)}(t) \forall t \geq 0, \quad (81)$$

obviously, where $\lambda > 0$ is a constant depending only C, ε, and

γ. This differential inequality integration over $(0, t)$ gives

$$L^{1/(1-\sigma)}(t) \geq \frac{1}{L^{-\sigma/(1-\sigma)}(0) - \lambda(\sigma/(1-\sigma))t}, \tag{82}$$

where we choose

$$t \leq T^* = \frac{1-\sigma}{\lambda\sigma L^{\sigma/(1-\sigma)}(0)}. \tag{83}$$

Hence,

$$\lim_{t \longrightarrow T^{*-}} L(t) \longrightarrow \infty. \tag{84}$$

3. Conclusions

The purpose of this paper is to study the explosion result of the solution of the system of nonlocal singular viscoelastic with damping and source terms on general case. This current study is a general case of the previous work of Boulaaras in ([5]). In the next work, we will try to obtain the same result for the two-dimensional problem that allows a reasonable description of the phenomenon occurring in a three-dimensional domain. Then, we will try to prove uniqueness results of the weak solution.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

An Approach of Lebesgue Integral in Fuzzy Cone Metric Spaces via Unique Coupled Fixed Point Theorems

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In the theory of fuzzy fixed point, many authors have been proved different contractive type fixed point results with different types of applications. In this paper, we establish some new fuzzy cone contractive type unique coupled fixed point theorems (FP-theorems) in fuzzy cone metric spaces (FCM-spaces) by using "the triangular property of fuzzy cone metric" and present illustrative examples to support our main work. In addition, we present a Lebesgue integral type mapping application to get the existence result of a unique coupled FP in FCM-spaces to validate our work.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [1]. Later on, in 1975, Kramosil and Michalek [2] introduced the concept of fuzzy metric spaces (FM-space); they present some structural properties of FM-space. In 1988, Grabiec [3] used the concept of Kramosil and Michalek [2] and proved two fixed point theorems (FP-theorems) of "Banach and Edelstein contraction mapping theorems on complete and compact FM-spaces, respectively." After that, the idea of FM-space given by Kramosil and Michalek [2] was modified by George and Veeramani [4], and they proved that every metric induces a fuzzy metric and also proved some fundamental properties and Baire's theorem for FM-spaces. In 2002, Gregory and Sapena [5] proved some contractive type FP-theorems in FM-spaces. Roldan et al. [6] presented some new FP-results in FM-spaces, while Jleli et al. [7] proved some results by using cyclic (ψ, ϕ)-contractions in Kaleva-Seikkala's type fuzzy metric spaces. Kiany and Harandi [8] presented the concept of set-valued fuzzy-contractive type maps and proved some FP and end point results in FM-spaces. Latterly, Rehman et al. [9] gave out the notion of rational type fuzzy contraction for FP in complete FM-spaces with an application. Some more related FP-results can be found in [10–15].

Indeed, Huang and Zhang [16] rediscovered the idea of Banach-valued metric space. Indeed, many mathematicians proposed it; but it becomes popular after Huang and Zhang's study. By adopting the theory that the underlying cone is normal, they demonstrated the convergence properties and some FP-theorems. Rezapour and Hamlbarani [17], in 2008, proved FP-theorems without assuming the cone's normality, while in [18] Karapinar proved some Ćirić-type non-unique FP-theorems on cone metric spaces. After that, many others contributed their ideas to the problem of FP-findings in cone metric spaces. A few of their FP-findings can be found (e.g., see [19–22]).

In 2015, Oner et al. [23] gave the idea of fuzzy cone metric space (FCM-space), and they also presented some fundamental properties and "a single-valued Banach contraction theorem for FP with the assumption that all the sequences are Cauchy." After that, Rehman and Li [24] settled some generalized fuzzy cone contractive type FP-results neglecting that "all the sequences are Cauchy" in complete FCM-space. And later, Jabeen et al. [25] presented some common FP-theorems for three self-mappings, by taking into consideration the idea of weakly compatible in FCM-spaces with an integral type application. Chen et al. [26], in 2020, gave the idea of coupled fuzzy cone contractive-type mappings. They proved "some coupled FP-theorems in FCM-spaces with non-linear integral type application." Latterly, Rehman and Aydi [27], in 2021, presented the concept of rational type fuzzy cone contraction mappings in FCM-spaces. They used "the triangular property of fuzzy metric" as a fundamental tool and proved some common FP-theorems and give an application.

Guo and Lakshmikantham [28] proved "coupled FP-results for the nonlinear operator with applications". Later, Bhaskar and Lakshmikantham [29] present some coupled FP-theorems in the context of partially ordered metric spaces, and this work is also presented by Lakshmikantham and Ciric [30]. In the year 2010, Sedghi et al. [31] proved some common coupled FP-results for commuting mappings in fuzzy metric spaces.

In this paper, we present some unique coupled FP-findings in FCM-spaces by taking the idea of Guo and Lakshmikantham [28] and Chen et al. [26]. Furthermore, we have also presented an application of the two Lebesgue Integral Equations (LIE) for a common solution to uphold our work. This paper is organized as follows: Section 2 consists of preliminaries. In Section 3, we establish some unique coupled FP-results in FCM-spaces with illustrative examples. In Section 4, we present an application of Lebesgue integral mapping to get the existence result of unique coupled FP in FCM-spaces to hold up our main work. In Section 5, we discuss the conclusion of our work.

2. Preliminaries

Definition 1 [32]. A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ would be a continuous t-norm if $*$ fulfils the following conditions:

(i) $*$ is associative and commutative

(ii) $*$ is continuous

(iii) $1 * \alpha = \alpha, \forall \alpha \in [0, 1]$

(iv) $\alpha_1 * \alpha_2 \leq \alpha_3 * \alpha_4$ whenever $\alpha_1 \leq \alpha_3$ and $\alpha_2 \leq \alpha_4$, for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$

Throughout the complete paper, ζ-norm represents a continuous t-norm.

Definition 2 [16]. Let E be a real Banach space and ϑ be the zero element of E , and C is a subset of E. Then, C is called a cone if,

(i) C is closed and nonempty, and $C \neq \{\vartheta\}$

(ii) $\alpha_1, \alpha_2 \in R, \alpha_1, \alpha_2 \geq 0$ and $a, b \in C$, then $\alpha_1 a + \alpha_2 b \in C$

(iii) both $a \in C$ and $-a \in C$ and then $a = \vartheta$

A partial ordering on a given cone $C \subset E$ is defined by $a \preceq b \Leftrightarrow b - a \in$ C. $a \prec b$ stands for $a \preceq b$ and $a \neq b$, while $a \ll b$ stands for $b - a \in \operatorname{int}(C)$. In this paper, all cones have nonempty interior.

Definition 3 [4]. A 3-tuple $(A, \mathfrak{M}_c, *)$ is said to be a FM-space if A is any set, $*$ is a ζ-norm, and $\mathfrak{M}_c$ is a fuzzy set on $A^2 \times (0,\infty)$ satisfying

(i) $\mathfrak{M}_c(a, b, \zeta) > 0$

(ii) $\mathfrak{M}_c(a, b, \zeta) = 0 \Leftrightarrow a = b$

(iii) $\mathfrak{M}_c(a, b, \zeta) = \mathfrak{M}_c(b, a, \zeta)$

(iv) $\mathfrak{M}_c(a, c, \zeta) * \mathfrak{M}_c(c, b, s) \leq \mathfrak{M}_c(a, b, \zeta + s)$

(v) $\mathfrak{M}_c(a, b, .): (0,\infty) \longrightarrow [0, 1]$is continuous,$\forall a, b, c \in A$ and $\zeta, s > 0$

Definition 4 [23]. A 3-tuple $(A, \mathfrak{M}_c, *)$ is said to be a FCM-space if C is a cone of E, *A is an arbitrary set, $*$ is a ζ -norm, and $\mathfrak{M}_c$ is a fuzzy set on $A^2 \times \operatorname{int}(C)$ satisfying*

(i) $\mathfrak{M}_c(a, b, \zeta) > 0$

(ii) $\mathfrak{M}_c(a, b, \zeta) = 0 \Leftrightarrow a = b$

(iii) $\mathfrak{M}_c(a, b, \zeta) = \mathfrak{M}_c(b, a, \zeta)$

(iv) $\mathfrak{M}_c(a, c, \zeta) * \mathfrak{M}_c(c, b, s) \leq \mathfrak{M}_c(a, b, \zeta + s)$

(v) $\mathfrak{M}_c(a, b, .)$: $\operatorname{int}(C) \longrightarrow [0, 1]$ is continuous, $\forall a, b, c \in A$, and $\zeta, s \gg \vartheta$

Definition 5 [23]. Let a 3-tuple $(A, \mathfrak{M}_c, *)$ be a FCM-space, $b_1 \in A$, which is a sequence $\{b_j\}$ *in A*

(i) It converges to$b$ if $\alpha_3 \in (0, 1)$ and $\zeta \gg \vartheta$; there is $j_1 \in \mathcal{N}$ such that $\mathfrak{M}_c(b_j, b, \zeta) > 1 - \alpha_3$, for $j \geq j_1$, or we write it as $\lim_{j \longrightarrow \infty} b_j = b_1$ or $b_j \longrightarrow b_1$ as $j \longrightarrow \infty$

(ii) It is a Cauchy sequence if $\alpha_3 \in (0, 1)$ and $\zeta \gg \vartheta$; there is $j_1 \in \mathcal{N}$ such that $\mathfrak{M}_c(b_j, b_i, \zeta) > 1 - \alpha_3$, for $j, i \geq j_1$

(iii) $(A, \mathfrak{M}_c, *)$ is complete if every Cauchy sequence is convergent in A

(iv) It is fuzzy cone contractive if $\exists \alpha \in (0, 1)$ and fulfilling

$$\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \leq \alpha \left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 \right), \quad (1)$$
$$\text{for } \zeta \gg \vartheta, j \geq 1.$$

Lemma 6 [23]. *Let $(A, \mathfrak{M}_c, *)$ be a FCM-space, and let a sequence $\{b_j\}$ in A converge to a point $b \in A \Leftrightarrow \mathfrak{M}_c(b_j, b, \zeta)$ which converges to 1 as $j \longrightarrow \infty$, for $\zeta \gg \vartheta$.*

Definition 7 [24]. Let $(A, \mathfrak{M}_c, *)$ be a FCM-space. The FCM $\mathfrak{M}_c$ is triangular, if

$$\frac{1}{\mathfrak{M}_c(a, b, \zeta)} - 1 \leq \left(\frac{1}{\mathfrak{M}_c(a, c, \zeta)} - 1\right) + \left(\frac{1}{\mathfrak{M}_c(c, b, \zeta)} - 1\right), \quad \forall a, c, b \in A, \zeta \gg \vartheta. \tag{2}$$

Definition 8 [23]. Let $(A, \mathfrak{M}_c, *)$ be a FCM-space and $\Gamma : A \longrightarrow A$. Then, Γ is said to be fuzzy cone contractive if $\exists$ $\alpha_1 \in (0, 1)$ such that

$$\frac{1}{\mathfrak{M}_c(\Gamma b, \Gamma c, \zeta)} - 1 \leq \alpha_1\left(\frac{1}{\mathfrak{M}_c(b, c, \zeta)} - 1\right), \quad \forall b, c \in A, \zeta \gg \vartheta. \tag{3}$$

Definition 9. Let (b, c) be an element in $A \times A$. Then, it is called coupled FP of a mapping $\Gamma : A \times A \longrightarrow A$ if

$$\begin{aligned} \Gamma(b, c) &= b, \\ \Gamma(c, b) &= c. \end{aligned} \tag{4}$$

Now, in the following, we prove some unique couple FP-theorems in FCM-spaces with examples to support our main work. Furthermore, we present an application of Lebesgue integral contractive type mapping to prove a unique coupled FP-theorem in FCM-spaces.

3. Main Results

Now, we present our first main result.

Theorem 10. *Let $\Gamma : A \times A \longrightarrow A$ be a mapping on complete FCM-spaces $(A, \mathfrak{M}_c, *)$ in which $\mathfrak{M}_c$ is triangular and satisfies the inequality:*

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(\kappa, \varrho), \zeta)} - 1 \\ &\quad \leq \alpha_1\left(\frac{1}{\mathfrak{M}_c(a, \kappa, \zeta)} - 1\right) + \alpha_2\left(\frac{1}{N(\Gamma, (a, b), (\kappa, \varrho), \zeta)} - 1\right), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\frac{1}{N(\Gamma, (a, b), (\kappa, \varrho), \zeta)} - 1 \\ &\quad = \left(\frac{1}{\mathfrak{M}_c(a, \Gamma(a, b), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(\kappa, \Gamma(\kappa, \varrho), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a, \Gamma(\kappa, \varrho), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(\kappa, \Gamma(a, b), \zeta)} - 1\right), \end{aligned} \tag{6}$$

$\forall a, b, \kappa, \varrho \in A$, $\zeta \gg \vartheta$, $\alpha_1 \in [0, 1)$, and $\alpha_2 \geq 0$ with $(\alpha_1 + 4\alpha_2) < 1$. Then, Γ has a unique couple FP in A.

Proof. Any $a_0, b_0 \in A$; we define sequences $\{a_j\}$ and $\{b_j\}$ in A such that

$$\begin{aligned} \Gamma(a_j, b_j) &= a_{j+1}, \\ \Gamma(b_j, a_j) &= b_{j+1}, \quad \text{for } j \geq 0. \end{aligned} \tag{7}$$

□

Now from (5) for $\zeta \gg \vartheta$, we have

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \\ &\quad = \frac{1}{\mathfrak{M}_c(\Gamma(a_{j-1}, b_{j-1}), \Gamma(a_j, b_j), \zeta)} - 1 \\ &\quad \leq \alpha_1\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right) \\ &\qquad + \alpha_2\left(\frac{1}{N(\Gamma, (a_{j-1}, b_{j-1}), (a_j, b_j), \zeta)} - 1\right), \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\frac{1}{N(\Gamma, (a_{j-1}, b_{j-1}), (a_j, b_j), \zeta)} - 1 \\ &\quad = \left(\frac{1}{\mathfrak{M}_c(a_{j-1}, \Gamma(a_{j-1}, b_{j-1}), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_j, \Gamma(a_j, b_j), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_{j-1}, \Gamma(a_j, b_j), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_j, \Gamma(a_{j-1}, b_{j-1}), \zeta)} - 1\right) \\ &\quad = \left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_{j-1}, a_{j+1}, \zeta)} - 1\right) \\ &\quad \leq 2\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1\right). \end{aligned} \tag{9}$$

Now from (8) and (9), for $\zeta \gg \vartheta$,

$$\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \leq \alpha\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right) + 2\alpha_2\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1\right). \quad (10)$$

We get, after simplification,

$$\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \leq \lambda\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta, \quad (11)$$

where $\lambda = (\alpha_1 + 2\alpha_2)/(1 - 2\alpha_2) < 1$. Similarly,

$$\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \leq \lambda\left(\frac{1}{\mathfrak{M}_c(a_{j-2}, a_{j-1}, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta. \quad (12)$$

Now, from (11) and (12) and by induction, for $\zeta \gg \vartheta$,

$$\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \leq \lambda\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right) \leq \lambda^2\left(\frac{1}{\mathfrak{M}_c(a_{j-2}, a_{j-1}, \zeta)} - 1\right) \leq \cdots \leq \lambda^j\left(\frac{1}{\mathfrak{M}_c(a_0, a_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty. \quad (13)$$

It shows that the sequence $\{a_j\}$ is a fuzzy cone contractive; therefore,

$$\lim_{j \longrightarrow \infty} \mathfrak{M}_c(a_j, a_{j+1}, \zeta) = 1, \quad \text{for } \zeta \gg \vartheta. \quad (14)$$

Now for $i > j$ and for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\frac{1}{\mathfrak{M}_c(a_j, a_i, \zeta)} - 1 &\leq \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1\right) + \left(\frac{1}{\mathfrak{M}_c(a_{j+1}, a_{j+2}, \zeta)} - 1\right) + \cdots + \left(\frac{1}{\mathfrak{M}_c(a_{i-1}, a_i, \zeta)} - 1\right) \\ &\leq \lambda^j\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) + \lambda^{j+1}\left(\frac{1}{\mathfrak{M}_c(a_0, a_1, \zeta)} - 1\right) + \cdots + \lambda^{i-1}\left(\frac{1}{\mathfrak{M}_c(a_0, a_1, \zeta)} - 1\right) \\ &= \left(\lambda^j + \lambda^{j+1} + \cdots + \lambda^{i-1}\right)\left(\frac{1}{\mathfrak{M}_c(a_0, a_1, \zeta)} - 1\right) \\ &= \frac{\lambda^j}{1-\lambda}\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.\end{aligned} \quad (15)$$

Hence, the sequence $\{a_j\}$ is Cauchy. Now for sequence $\{b_j\}$ and from (5), for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 &= \frac{1}{\mathfrak{M}_c(\Gamma(b_{j-1}, a_{j-1}), \Gamma(b_j, a_j), \zeta)} - 1 \\ &\leq \alpha_1\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) + \alpha_2\left(\frac{1}{N(\Gamma, (b_{j-1}, a_{j-1}), (b_j, a_j), \zeta)} - 1\right),\end{aligned} \quad (16)$$

where

$$\begin{aligned}\frac{1}{N(\Gamma, (b_{j-1}, a_{j-1}), (b_j, a_j), \zeta)} - 1 &= \left(\frac{1}{\mathfrak{M}_c(b_{j-1}, \Gamma(b_{j-1}, a_{j-1}), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_j, \Gamma(b_j, a_j), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_{j-1}, \Gamma(b_j, a_j), \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_j, \Gamma(b_{j-1}, a_{j-1}), \zeta)} - 1\right) \\ &= \left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_{j-1}, b_{j+1}, \zeta)} - 1\right) \\ &\leq 2\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right).\end{aligned} \quad (17)$$

Now from (16) and (17), for $\zeta \gg \vartheta$,

$$\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \le \alpha_1\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) + 2\alpha_2\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 + \frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right). \quad (18)$$

We get, after simplification,

$$\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \le \lambda\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta, \quad (19)$$

where $\lambda = (\alpha_1 + 2\alpha_2)/(1 - 2\alpha_2) < 1$. Similarly,

$$\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 \le \lambda\left(\frac{1}{\mathfrak{M}_c(b_{j-2}, b_{j-1}, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta. \quad (20)$$

Now, from (19) and (20) and by induction, for $\zeta \gg \vartheta$,

$$\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \le \lambda\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) \le \lambda^2\left(\frac{1}{\mathfrak{M}_c(b_{j-2}, b_{j-1}, \zeta)} - 1\right) \le \cdots \le \lambda^j\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty \quad (21)$$

It shows that the sequence $\{b_j\}$ is a fuzzy cone contractive; therefore,

$$\lim_{j\longrightarrow\infty} \mathfrak{M}_c(b_j, b_{j+1}, \zeta) = 1, \quad \text{for } \zeta \gg \vartheta. \quad (22)$$

Now for $i > j$ and for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\frac{1}{\mathfrak{M}_c(b_j, b_i, \zeta)} - 1 &\le \left(\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right) + \left(\frac{1}{\mathfrak{M}_c(b_{j+1}, b_{j+2}, \zeta)} - 1\right) + \cdots + \left(\frac{1}{\mathfrak{M}_c(b_{i-1}, h_i, \zeta)} - 1\right) \\ &\le \lambda^j\left(\frac{1}{\mathfrak{M}_c(b_o, b_1, \zeta)} - 1\right) + \lambda^{j+1}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) + \cdots + \lambda^{i-1}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \\ &= \left(\lambda^j + \lambda^{j+1} + \cdots + \lambda^{i-1}\right)\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \\ &= \frac{\lambda^j}{1-\lambda}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.\end{aligned} \quad (23)$$

Hence, the sequence $\{b_j\}$ is Cauchy. Since A is complete and $\{a_j\}, \{b_j\}$ are Cauchy sequences in A, so $\exists a, b \in A$ such that $a_j \longrightarrow a$ and $b_j \longrightarrow b$ as $j \longrightarrow \infty$ or this can be written as $\lim_{j\longrightarrow\infty} a_j = a$ and $\lim_{j\longrightarrow\infty} b_j = b$. Therefore,

$$\lim_{j\longrightarrow\infty} \mathfrak{M}_c(a_j, a, \zeta) = 1, \quad \lim_{j\longrightarrow\infty} \mathfrak{M}_c(b_j, b, \zeta) = 1, \quad \text{for } \zeta \gg \vartheta. \quad (24)$$

Hence,

$$\lim_{j\longrightarrow\infty} a_{j+1} = \lim_{j\longrightarrow\infty} \Gamma(a_j, b_j) = \Gamma\left(\lim_{j\longrightarrow\infty} a_j, \lim_{j\longrightarrow\infty} b_j\right) \Rightarrow \Gamma(a, b) = a. \quad (25)$$

Similarly,

$$\lim_{j\longrightarrow\infty} b_{j+1} = \lim_{j\longrightarrow\infty} \Gamma(b_j, a_j) = \Gamma\left(\lim_{j\longrightarrow\infty} b_j, \lim_{j\longrightarrow\infty} a_j\right) \Rightarrow \Gamma(b, a) = b. \quad (26)$$

Regarding its uniqueness, suppose (a_1, b_1) and (b_1, a_1) are another couple fixed point pairs in $A \times A$ such that $\Gamma(a_1, b_1) = a_1$ and $\Gamma(b_1, a_1) = b_1$. Now, from (5), for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1 &= \frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(a_1, b_1), \zeta)} - 1 \\ &\le \alpha_1\left(\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1\right) + \alpha_2\left(\frac{1}{N(\Gamma, (a, b), (a_1, b_1), \zeta)} - 1\right),\end{aligned} \quad (27)$$

where

$$\begin{aligned}
&\frac{1}{N(\Gamma,(a,b),(a_1,b_1),\zeta)}-1\\
&=\left(\frac{1}{\mathfrak{M}_c(a,\Gamma(a,b),\zeta)}-1+\frac{1}{\mathfrak{M}_c(a_1,\Gamma(a_1,b_1),\zeta)}-1\right.\\
&\quad\left.+\frac{1}{\mathfrak{M}_c(a,\Gamma(a_1,b_1),\zeta)}-1+\frac{1}{\mathfrak{M}_c(a_1,\Gamma(a,b),\zeta)}-1\right)\\
&=\left(\frac{1}{\mathfrak{M}_c(a,a,\zeta)}-1+\frac{1}{\mathfrak{M}_c(a_1,a_1,\zeta)}-1+\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}\right.\\
&\quad\left.-1+\frac{1}{\mathfrak{M}_c(a_1,a,\zeta)}-1\right)=2\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}\right).
\end{aligned}\tag{28}$$

Now from (27) and for $\zeta\gg\vartheta$,

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\\
&\le\alpha_1\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\right)+2\alpha_2\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\right)\\
&=(\alpha_1+2\alpha_2)\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\right)\\
&=(\alpha_1+2\alpha_2)\left(\frac{1}{\mathfrak{M}_c(\Gamma(a,b),\Gamma(a_1,b_1),\zeta)}-1\right)\\
&\le(\alpha_1+2\alpha_2)^2\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\right)\le\cdots\\
&\le(\alpha_1+2\alpha_2)^j\left(\frac{1}{\mathfrak{M}_c(a,a_1,\zeta)}-1\right)\\
&\longrightarrow 0,\quad \text{as } j\longrightarrow\infty,
\end{aligned}\tag{29}$$

where $(\alpha_1+2\alpha_2)<1$. Hence, we have $\mathfrak{M}_c(a,a_1,\zeta)=1$ for $\zeta\gg\vartheta$, $\Rightarrow a=a_1$.

Similarly, again from (5), for $\zeta\gg\vartheta$, we have

$$\begin{aligned}
\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1&=\frac{1}{\mathfrak{M}_c(\Gamma(b,a),\Gamma(b_1,a_1),\zeta)}-1\\
&\le\alpha_1\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)\\
&\quad+\alpha_2\left(\frac{1}{N(\Gamma,(b,a),(b_1,a_1),\zeta)}-1\right),
\end{aligned}\tag{30}$$

where

$$\begin{aligned}
&\frac{1}{N(\Gamma,(b,a),(b_1,a_1),\zeta)}-1\\
&=\left(\frac{1}{\mathfrak{M}_c(b,\Gamma(b,a),\zeta)}-1+\frac{1}{\mathfrak{M}_c(b_1,\Gamma(b_1,a_1),\zeta)}-1\right.\\
&\quad\left.+\frac{1}{\mathfrak{M}_c(b,\Gamma(b_1,a_1),\zeta)}-1+\frac{1}{\mathfrak{M}_c(b_1,\Gamma(b,a),\zeta)}-1\right)\\
&=\left(\frac{1}{\mathfrak{M}_c(b,b,\zeta)}-1+\frac{1}{\mathfrak{M}_c(b_1,b_1,\zeta)}-1+\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}\right.\\
&\quad\left.-1+\frac{1}{\mathfrak{M}_c(b_1,b,\zeta)}-1\right)=2\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}\right).
\end{aligned}\tag{31}$$

Now from (30) and for $\zeta\gg\vartheta$,

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\\
&\le\alpha_1\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)+2\alpha_2\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)\\
&=(\alpha_1+2\alpha_2)\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)\\
&=(\alpha_1+2\alpha_2)\left(\frac{1}{\mathfrak{M}_c(\Gamma(b,a),\Gamma(b_1,a_1),\zeta)}-1\right)\\
&\le(\alpha_1+2\alpha_2)^2\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)\le\cdots\\
&\le(\alpha_1+2\alpha_2)^j\left(\frac{1}{\mathfrak{M}_c(b,b_1,\zeta)}-1\right)\longrightarrow 0,\quad \text{as } j\longrightarrow\infty.
\end{aligned}\tag{32}$$

Hence, we have $\mathfrak{M}_c(b,b_1,\zeta)=1$ for $\zeta\gg\vartheta$, $\Rightarrow b=b_1$.

Corollary 11. *Let $\Gamma:A\times A\longrightarrow A$ be a mapping on complete FCM-spaces $(A,\mathfrak{M}_c,*)$ in which $\mathfrak{M}_c$ is triangular and satisfies*

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(\Gamma(a,b),\Gamma(\kappa,\varrho),\zeta)}-1\\
&\le\alpha_1\left(\frac{1}{\mathfrak{M}_c(a,\kappa,\zeta)}-1\right)+\alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(a,\Gamma(a,b),\zeta)}-1\right)\right.\\
&\quad\left.+\left(\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(\kappa,\varrho),\zeta)}-1\right)\right],
\end{aligned}\tag{33}$$

for all $a,b,\kappa,\varrho\in A$, $\zeta\gg\vartheta$, $\alpha_1\in[0,1)$, and $\alpha_2\ge 0$ with $(\alpha_1+2\alpha_2)<1$. Then, Γ has a unique couple FP in A.

Corollary 12. *Let $\Gamma:A\times A\longrightarrow A$ be a mapping on complete FCM-spaces $(A,\mathfrak{M}_c,*)$ in which $\mathfrak{M}_c$ is triangular and satisfies*

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(\Gamma(a,b),\Gamma(\kappa,\varrho),\zeta)}-1\\
&\le\alpha_1\left(\frac{1}{\mathfrak{M}_c(a,\kappa,\zeta)}-1\right)+\alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(a,\Gamma(\kappa,\varrho),\zeta)}-1\right)\right.\\
&\quad\left.+\left(\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(a,b),\zeta)}-1\right)\right],
\end{aligned}\tag{34}$$

for all $a, b, \kappa, \varrho \in A$, $\zeta \gg \vartheta$, $\alpha_1 \in [0, 1)$, and $\alpha_2 \geq 0$ with $(\alpha_1 + 2\alpha_2) < 1$. Then, Γ has a unique couple FP in A.

Example 1. $A = (0,\infty)$, $*$ is a ζ-norm, and $\mathfrak{M}_c : A^2 \times (0,\infty) \longrightarrow [0, 1]$ is defined as

$$\mathfrak{M}_c(a, b, \zeta) = \frac{\zeta}{\zeta + d(a, b)}, \quad d(a, b) = |a - b|, \tag{35}$$

for all $a, b \in A$ and $\zeta > 0$. Then, it is easy to verify that $\mathfrak{M}_c$ is triangular and $(A, \mathfrak{M}_c, *)$ is a complete FCM-space. We define

$$\Gamma(g, h) = \begin{cases} \dfrac{a - b}{12}, & \text{if } a, b \in [0, 1], \\ \dfrac{2a + 2b - 2}{3}, & \text{if } a, b \in [1,\infty). \end{cases} \tag{36}$$

Now from (5), for $\zeta \gg \vartheta$, we have

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(\kappa, \varrho), \zeta)} - 1 \\ &= \left(\frac{1}{\mathfrak{M}_c((a-b)/12, (\kappa-\varrho)/12, \zeta)} - 1 \right) \\ &= \frac{1}{\zeta} \left(d\left(\frac{a-b}{12}, \frac{\kappa-\varrho}{12} \right) \right) = \frac{1}{12\zeta} |a - b - \kappa + \varrho| \\ &\leq \frac{1}{12\zeta} [|(a-\kappa) + (a - (a-b)) + (\kappa - (\kappa - \varrho)) \\ &\quad + (a - (\kappa - \varrho)) + (\kappa - (a-b)) |] \\ &\leq \frac{1}{12\zeta} |a - \kappa| + \frac{1}{12\zeta} (|a - (a-b)| + |\kappa - (\kappa - \varrho)| \\ &\quad + |a - (\kappa - \varrho)| + |\kappa - (a-b)|) \\ &= \frac{1}{12} \left(\frac{1}{\mathfrak{M}_c(a, \kappa, \zeta)} - 1 \right) + \frac{1}{12} \left(\left(\frac{1}{\mathfrak{M}_c(a, \Gamma(a, b), \zeta)} - 1 \right) \right. \\ &\quad + \left(\frac{1}{\mathfrak{M}_c(\kappa, \Gamma(\kappa, \varrho), \zeta)} - 1 \right) + \left(\frac{1}{\mathfrak{M}_c(a, \Gamma(\kappa, \varrho), \zeta)} - 1 \right) \\ &\quad \left. + \left(\frac{1}{\mathfrak{M}_c(\kappa, \Gamma(a, b), \zeta)} - 1 \right) \right) = \frac{1}{12} \left(\frac{1}{\mathfrak{M}_c(a, \kappa, \zeta)} - 1 \right) \\ &\quad + \frac{1}{12} \left(\frac{1}{N(\Gamma, (a, b), (\kappa, \varrho), \zeta)} - 1 \right), \quad \text{for } \zeta \gg \vartheta \end{aligned} \tag{37}$$

It is easy to verify that conditions of Theorem 10 are satisfied with $\alpha_1 = \alpha_2 = 1/12$. Then, Γ has unique coupled FP for $a = 2$ and $b = 2$.

$$\Gamma(a, b) = \Gamma(2, 2) = \frac{2(2) + 2(2) - 2}{3} = 2 \Rightarrow \Gamma(2, 2) = 2. \tag{38}$$

Theorem 13. *Let $\Gamma : A \times A \longrightarrow A$ be a mapping in a complete FCM-space $(A, \mathfrak{M}_c, *)$ in which $\mathfrak{M}_c$ is triangular and satisfies*

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(\kappa, \varrho), \zeta)} - 1 \\ &\leq \alpha_1 \left(\frac{1}{\mathfrak{M}_c(a, \kappa, \zeta)} - 1 \right) + \alpha_2 \left[\left(\frac{1}{\mathfrak{M}_c(a, \Gamma(a, b), \zeta)} - 1 \right) \right. \\ &\quad \left. + \left(\frac{1}{\mathfrak{M}_c(\kappa, \Gamma(\kappa, \varrho), \zeta)} - 1 \right) \right] \\ &\quad + \alpha_3 \left[\left(\frac{1}{\mathfrak{M}_c(\kappa, \Gamma(a, b), \zeta) * \mathfrak{M}_c(\kappa, \Gamma(\kappa, \varrho), \zeta)} - 1 \right) \right], \end{aligned} \tag{39}$$

for all $a, b, \kappa, \varrho \in A$, $\zeta \gg \vartheta$ $\alpha_1 \in [0, 1)$, and $\alpha_2, \alpha_3 \geq 0$ with $(\alpha_1 + 2\alpha_2 + \alpha_3) < 1$. Then, Γ has a unique couple FP in A.

Proof. Any $a_0, b_0 \in A$, and we define sequence $\{a_j\}$ by

$$\begin{aligned} \Gamma(a_j, b_j) &= a_{j+1}, \\ \Gamma(b_j, a_j) &= b_{j+1}, \quad \text{for } j \geq 0. \end{aligned} \tag{40}$$

□

Now, from (39), for $\zeta \gg \vartheta$, we have

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \\ &= \frac{1}{\mathfrak{M}_c(\Gamma(a_{j-1}, b_{j-1}), \Gamma(a_j, b_j), \zeta)} - 1 \\ &\leq \alpha_1 \left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \right) \\ &\quad + \alpha_2 \left[\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, \Gamma(a_{j-1}, b_{j-1}), \zeta)} - 1 \right) \right. \\ &\quad \left. + \left(\frac{1}{\mathfrak{M}_c(a_j, \Gamma(a_j, b_j), \zeta)} - 1 \right) \right] \\ &\quad + \alpha_3 \left(\frac{1}{\mathfrak{M}_c(a_j, \Gamma(a_{j-1}, b_{j-1}), \zeta) * \mathfrak{M}_c(a_j, \Gamma(a_j, b_j), \zeta)} - 1 \right) \\ &= \alpha_1 \left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \right) + \alpha_2 \left[\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \right) \right. \\ &\quad \left. + \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \right) \right] \\ &\quad + \alpha_3 \left(\frac{1}{\mathfrak{M}_c(a_j, a_j, \zeta) * \mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \right) \\ &= \alpha_1 \left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \right) + \alpha_2 \left[\left(\frac{1}{\mathfrak{M}_c(a_{j-1}), a_j, \zeta)} - 1 \right) \right. \\ &\quad \left. + \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \right) \right] + \alpha_3 \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \right). \end{aligned} \tag{41}$$

We get, after simplification,

$$\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \le \delta\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta, \tag{42}$$

where $\delta = (\alpha_1 + \alpha_2)/(1 - \alpha_2 - \alpha_3) < 1$. Similarly,

$$\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1 \le \delta\left(\frac{1}{\mathfrak{M}_c(a_{j-2}, a_{j-1}, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta. \tag{43}$$

Now, from (42) and (43) and by induction, for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \\
&\quad \le \delta\left(\frac{1}{\mathfrak{M}_c(a_{j-1}, a_j, \zeta)} - 1\right) \le \delta^2\left(\frac{1}{\mathfrak{M}_c(a_{j-2}, a_{j-1}, \zeta)} - 1\right) \\
&\quad \le \cdots \le \delta^j\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{44}$$

Hence, the sequence $\{a_j\}$ is fuzzy cone contractive; therefore,

$$\lim_{j \longrightarrow \infty} \mathfrak{M}_c(a_j, a_{j+1}, \zeta) = 1 \, \zeta \gg \vartheta. \tag{45}$$

Now for $i > j$ and for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(a_j, a_i, \zeta)} - 1 \\
&\quad \le \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1\right) + \left(\frac{1}{\mathfrak{M}_c(a_{j+1}, a_{j+2}, \zeta)} - 1\right) \\
&\qquad + \cdots + \left(\frac{1}{\mathfrak{M}_c(a_{i-1}, a_i, \zeta)} - 1\right) \\
&\quad \le \delta^j\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) + \delta^{j+1}\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \\
&\qquad + \cdots + \delta^{i-1}\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \\
&\quad = \left(\delta^j + \delta^{j+1} + \cdots + \delta^{i-1}\right)\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \\
&\quad = \frac{\delta^j}{1 - \delta}\left(\frac{1}{\mathfrak{M}_c(a_o, a_1, \zeta)} - 1\right) \longrightarrow 0, \text{ as } j \longrightarrow \infty.
\end{aligned} \tag{46}$$

Hence, the sequence $\{a_j\}$ is Cauchy. Now for sequence $\{b_j\}$, again from (39), for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \\
&\quad = \frac{1}{\mathfrak{M}_c(\Gamma(b_{j-1}, a_{j-1}), \Gamma(b_j, a_j), \zeta)} - 1 \\
&\quad \le \alpha_1\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) \\
&\qquad + \alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, \Gamma(b_{j-1}, a_{j-1}, \zeta)} - 1\right)\right. \\
&\qquad \left. + \left(\frac{1}{\mathfrak{M}_c(b_j, \Gamma(b_j, a_j), \zeta)} - 1\right)\right] \\
&\qquad + \alpha_3\left(\frac{1}{\mathfrak{M}_c(b_j, \Gamma(b_{j-1}, a_{j-1}), \zeta) * \mathfrak{M}_c(b_j, \Gamma(b_j, a_j), \zeta)} - 1\right) \\
&\quad = \alpha_1\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) + \alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right)\right. \\
&\qquad \left. + \left(\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right)\right] \\
&\qquad + \alpha_3\left(\frac{1}{\mathfrak{M}_c(b_j, b_j, \zeta) * \mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right) \\
&\quad = \alpha_1\left(\frac{1}{\mathfrak{M}_c(h_{j-1}, b_j, \zeta)} - 1\right) + \alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right)\right. \\
&\qquad \left. + \left(\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right)\right] + \alpha_3\left(\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1\right).
\end{aligned} \tag{47}$$

We get, after simplification,

$$\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \le \delta\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta, \tag{48}$$

where the value of δ is the same as in (42). Similarly,

$$\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1 \le \delta\left(\frac{1}{\mathfrak{M}_c(b_{j-2}, b_{j-1}, \zeta)} - 1\right), \quad \text{for } \zeta \gg \vartheta. \tag{49}$$

Now, from (48) and (49) and by induction, for $\zeta \gg \vartheta$, we have that

$$\begin{aligned}
&\frac{1}{\mathfrak{M}_c(b_j, b_{j+1}, \zeta)} - 1 \\
&\quad \le \delta\left(\frac{1}{\mathfrak{M}_c(b_{j-1}, b_j, \zeta)} - 1\right) \le \delta^2\left(\frac{1}{\mathfrak{M}_c(b_{j-2}, b_{j-1}, \zeta)} - 1\right) \\
&\quad \le \cdots \le \delta^j\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{50}$$

Hence, the sequence $\{b_j\}$ is fuzzy cone contractive; therefore,

$$\lim_{j\longrightarrow\infty} \mathfrak{M}_c\left(b_j, b_{j+1}, \zeta\right) = 1, \quad \text{for } \zeta \gg \vartheta. \tag{51}$$

Now for $i > j$, for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
\frac{1}{\mathfrak{M}_c\left(b_j, b_i, \zeta\right)} - 1
&\le \left(\frac{1}{\mathfrak{M}_c\left(b_j, b_{j+1}, \zeta\right)} - 1\right) + \left(\frac{1}{\mathfrak{M}_c\left(b_{j+1}, b_{j+2}, \zeta\right)} - 1\right) \\
&\quad + \cdots + \left(\frac{1}{\mathfrak{M}_c\left(b_{i-1}, b_i, \zeta\right)} - 1\right) \\
&\le \delta^j\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) + \delta^{j+1}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \\
&\quad + \cdots + \delta^{i-1}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \\
&= \left(\delta^j + \delta^{j+1} + \cdots + \delta^{i-1}\right)\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \\
&= \frac{\delta^j}{1-\delta}\left(\frac{1}{\mathfrak{M}_c(b_0, b_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{52}$$

Hence, the sequence $\{b_j\}$ is Cauchy. Since A is complete and $\{a_j\}$ and $\{b_j\}$ are Cauchy sequences in A, $\exists a, b \in A$ such that $a_j \longrightarrow a$ and $b_j \longrightarrow b$ as $j \longrightarrow \infty$, or this can be written as $\lim_{j\longrightarrow\infty} a_j = a$ and $\lim_{j\longrightarrow\infty} b_j = b$. Therefore,

$$\begin{aligned}
\lim_{j\longrightarrow\infty} a_{j+1} &= \lim_{j\longrightarrow\infty} \Gamma\left(a_j, b_j\right) = \Gamma\left(\lim_{j\longrightarrow\infty} a_j, \lim_{j\longrightarrow\infty} b_j\right) \\
&\Rightarrow \Gamma(a, b) = a.
\end{aligned} \tag{53}$$

Similarly,

$$\begin{aligned}
\lim_{j\longrightarrow\infty} b_{j+1} &= \lim_{j\longrightarrow\infty} \Gamma\left(b_j, a_j\right) = \Gamma\left(\lim_{j\longrightarrow\infty} b_j, \lim_{j\longrightarrow\infty} a_j\right) \\
&\Rightarrow \Gamma(b, a) = b.
\end{aligned} \tag{54}$$

Regarding its uniqueness, let (a_1, b_1) and (b_1, a_1) be another couple fixed point pairs in $A \times A$ such that $\Gamma(a_1, b_1) = a_1$ and $\Gamma(b_1, a_1) = b_1$. Now, from (39), for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1
&= \left(\frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(a_1, b_1), \zeta)} - 1\right) \\
&\le \alpha_1\left(\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1\right) + \alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(a, \Gamma(a, b), \zeta) - 1} - 1\right)\right. \\
&\quad \left. + \left(\frac{1}{\mathfrak{M}_c(a_1, \Gamma(a_1, b_1), \zeta)} - 1\right)\right] \\
&\quad + \alpha_3\left(\frac{1}{\mathfrak{M}_c(a_1, \Gamma(a, b), \zeta) * \mathfrak{M}_c(a_1, \Gamma(a_1, b_1), \zeta)} - 1\right) \\
&= (\alpha_1 + \alpha_3)\left(\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1\right) \\
&= (\alpha_1 + \alpha_3)\left(\frac{1}{\mathfrak{M}_c(\Gamma(a, b), \Gamma(a_1, b_1), \zeta)} - 1\right) \\
&\le (\alpha_1 + \alpha_3)^2\left(\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1\right) \le \cdots \\
&\le (\alpha_1 + \alpha_3)^j\left(\frac{1}{\mathfrak{M}_c(a, a_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{55}$$

Hence, we get that $\mathfrak{M}_c(a, a_1, \zeta) = 1, \Rightarrow a = a_1$. Similarly, again from (39), for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
\frac{1}{\mathfrak{M}_c(b, b_1, \zeta)} - 1
&= \left(\frac{1}{\mathfrak{M}_c(\Gamma(b, a), \Gamma(b_1, a_1), \zeta)} - 1\right) \\
&\le \alpha_1\left(\frac{1}{\mathfrak{M}_c(b, b_1, \zeta)} - 1\right) + \alpha_2\left[\left(\frac{1}{\mathfrak{M}_c(b, \Gamma(b, a), \zeta) - 1} - 1\right)\right. \\
&\quad \left. + \left(\frac{1}{\mathfrak{M}_c(b_1, \Gamma(b_1, a_1), \zeta)} - 1\right)\right] \\
&\quad + \alpha_3\left(\frac{1}{\mathfrak{M}_c(b_1, \Gamma(b, a), \zeta) * \mathfrak{M}_c(b_1, \Gamma(b_1, a_1), \zeta)} - 1\right) \\
&= (\alpha_1 + \alpha_3)\left(\frac{1}{\mathfrak{M}_c(b, b_1, \zeta)} - 1\right) \\
&= (\alpha_1 + \alpha_3)\left(\frac{1}{\mathfrak{M}_c(\Gamma(b, a), \Gamma(b_1, a_1), \zeta)} - 1\right) \\
&\le (\alpha_1 + \alpha_3)^2\left(\frac{1}{\mathfrak{M}_c(b, b_1, \zeta)} - 1\right) \le \cdots \\
&\le (\alpha_1 + \alpha_3)^j\left(\frac{1}{\mathfrak{M}_c(b, b_1, \zeta)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{56}$$

Hence, we get that $\mathfrak{M}_c(b, b_1, \zeta) = 1$ for $\zeta \gg \vartheta, \Rightarrow b = b_1$.

Corollary 14. *Let $\Gamma : A \times A \longrightarrow A$ be a mapping on complete FCM-spaces $(A, \mathfrak{M}_c, *)$ in which $\mathfrak{M}_c$ is triangular and satisfies*

$$\frac{1}{\mathfrak{M}_c(\Gamma(a,b),\Gamma(\kappa,\varrho),\zeta)}-1 \le \alpha_1\left(\frac{1}{\mathfrak{M}_c(a,\kappa,\zeta)}-1\right) + \alpha_3\left[\left(\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(a,b),\zeta)*\mathfrak{M}_c(\kappa,\Gamma(\kappa,\varrho),\zeta)}-1\right)\right], \tag{57}$$

for all $a, b, \kappa, \varrho \in A$, $\zeta \gg \vartheta$, $\alpha_1 \in (0,1)$, and $\alpha_3 \ge 0$ with $(\alpha_1 + \alpha_3) < 1$. Then, Γ has a unique couple FP.

Example 2. $A = (0,\infty)$, $*$ is a ζ-norm, and $\mathfrak{M}_c : A \times A \times (0,\infty) \longrightarrow [0,1]$ is defined as

$$\mathfrak{M}_c(a,b,\zeta) = \frac{\zeta}{\zeta + d(a,b)}, \quad d(a,b) = |a-b|, \tag{58}$$

for all $a, b \in A$ and $\zeta > 0$. Then, it is easy to verify that $\mathfrak{M}_c$ is triangular and $(A,\mathfrak{M}_c,*)$ is a complete FCM-space. We define

$$\ddot{F}(a,b) = \begin{cases} \dfrac{a-b}{8}, & \text{if } a,b \in [0,1], \\ \dfrac{2a+2b-3}{3}, & \text{if } a,b \in [1,\infty). \end{cases} \tag{59}$$

Now from (39), for $\zeta \gg \vartheta$, we have

$$\begin{aligned} &\frac{1}{\mathfrak{M}_c(\Gamma(a,b),\Gamma(\kappa,\varrho),\zeta)}-1 \\ &= \left(\frac{1}{\mathfrak{M}_c((a-b)/8,(\kappa-\varrho)/8,\zeta)}-1\right) \\ &= \frac{1}{\zeta}\left(d\left(\frac{a-b}{8},\frac{\kappa-\varrho}{8}\right)\right) = \frac{1}{8\zeta}(|a-b-\kappa+\varrho|) \\ &\le \frac{1}{8\zeta}[|a-\kappa|+|(a-(a-b))+(\kappa-(\kappa-\varrho))|] \\ &\le \frac{1}{8\zeta}|a-\kappa| + \frac{1}{8\zeta}[|a-(a-b)|+|\kappa-(\kappa-\varrho)|] \\ &= \frac{1}{8}\left(\frac{1}{\mathfrak{M}_c(a,\kappa,\zeta)}-1\right) + \frac{1}{8}\left[\left(\frac{1}{\mathfrak{M}_c(a,\Gamma(a,b),\zeta)}-1\right) + \left(\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(\kappa,\varrho),\zeta)}-1\right)\right], \quad \text{for } \zeta \gg \vartheta. \end{aligned} \tag{60}$$

It is easy to verify that all the conditions of Theorem 13 are satisfied with $\alpha_1 = \alpha_2 = 1/8$ and $\alpha_3 = 0$. Then, Γ has unique coupled FP.

$$\Gamma(a,b) = \Gamma(3,3) = \frac{2(3)+2(3)-3}{3} = 3 \Rightarrow \Gamma(3,3) = 3. \tag{61}$$

4. Application

In this section, we present an application on Lebesgue integral (LI) mapping to support our main work. In 2002, Branciari proved the following result on complete metric space for unique FP (see [33]):

Theorem 15. *Let (A,d) be a complete metric space, $\alpha \in (0,1)$, and $\Gamma : A \longrightarrow A$ a mapping such that for each $a, b \in A$,*

$$\int_0^{d(\Gamma a,\Gamma b)} \varphi(s)ds \le \alpha \int_0^{d(a,b)} \varphi(s)ds, \tag{62}$$

where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $[0,\infty)$) and for each $\tau > 0$,

$$\int_0^{\tau} \varphi(s)ds > 0. \tag{63}$$

Then, Γ has a unique FP $u \in A$ such that for any $a \in A$, $\lim_{j\longrightarrow\infty} \Gamma^j a = u$.

Now, we are in the position to use the above concept and to prove a unique coupled FP-theorem in FCM-spaces.

Theorem 16. *Let $\Gamma : A \times A \longrightarrow A$ be a mapping on complete FCM-spaces $(A,\mathfrak{M}_c,*)$ in which $\mathfrak{M}_c$ is triangular and satisfies*

$$\begin{aligned} &\int_0^{((1/(\mathfrak{M}_c(\Gamma(a,b),\Gamma(\kappa,\varrho),\zeta)))-1)} \varphi(s)ds \\ &\le \alpha_1 \int_0^{((1/(\mathfrak{M}_c(a,\kappa,\zeta)))-1)} \varphi(s)ds \\ &+ \alpha_2 \int_0^{((1/(N(\Gamma,(a,b),(\kappa,\varrho),\zeta)))-1)} \varphi(s)ds, \end{aligned} \tag{64}$$

where

$$\begin{aligned} &\frac{1}{N(\Gamma,(a,b),(\kappa,\varrho),\zeta)}-1 \\ &= \left(\frac{1}{\mathfrak{M}_c(a,\Gamma(a,b),\zeta)}-1+\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(\kappa,\varrho),\zeta)}-1 + \frac{1}{\mathfrak{M}_c(a,\Gamma(\kappa,\varrho),\zeta)}-1+\frac{1}{\mathfrak{M}_c(\kappa,\Gamma(a,b),\zeta)}-1\right), \end{aligned} \tag{65}$$

for all $a, b, \kappa, \varrho \in A$, $\zeta \gg \vartheta$, $\alpha_1 \in (0,1)$, and $\alpha_2 \ge 0$ with $(\alpha_1 + 4\alpha_2) < 1$ and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $[0,\infty)$) and for each $\tau > 0$,

$$\int_0^{\tau} \varphi(s)ds > 0. \tag{66}$$

Then, Γ has a unique couple FP in A.

Proof. Any $a_o, b_o \in A$; we define sequences $\{a_j\}$ and $\{b_j\}$ in A such that

$$\Gamma(a_j, b_j) = a_{j+1}, \quad \Gamma(b_j, a_j) = b_{j+1}, \quad \text{for } j \geq 0. \tag{67}$$

□

Now from (64) and from the proof of Theorem 10, for $\zeta \gg \vartheta$, we have that

$$\begin{aligned}\int_0^{((1/(\mathfrak{M}_c(a_j,a_{j+1},\zeta)))-1)} \varphi(s)ds &= \int_0^{((1/(\mathfrak{M}_c(\Gamma(a_{j-1},b_{j-1}),\Gamma(a_j,b_j),\zeta)))-1)} \varphi(s)ds \\ &\leq \lambda \int_0^{((1/(\mathfrak{M}_c(a_{j-1},a_j,\zeta)))-1)} \varphi(s)ds,\end{aligned} \tag{68}$$

where $\lambda = (\alpha_1 + 2\alpha_2)/(1 - 2\alpha_2) < 1$. Similarly, again by using the same arguments, we have

$$\int_0^{((1/(\mathfrak{M}_c(a_{j-1},a_j,\zeta)))-1)} \varphi(s)ds \leq \lambda \int_0^{((1/(\mathfrak{M}_c(a_{j-2},a_{j-1},\zeta)))-1)} \varphi(s)ds, \quad \text{for } \zeta \gg \vartheta. \tag{69}$$

Now, from (68) and (69) and by induction, for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\int_0^{((1/(\mathfrak{M}_c(a_j,a_{j+1},\zeta)))-1)} \varphi(s)ds &\leq \lambda \int_0^{((1/(\mathfrak{M}_c(a_{j-1},a_j,\zeta)))-1)} \varphi(s)ds \\ &\leq \lambda^2 \int_0^{((1/(\mathfrak{M}_c(a_{j-2},a_{j-1},\zeta)))-1)} \varphi(s)ds \leq \cdots \\ &\leq \lambda^j \int_0^{((1/(\mathfrak{M}_c(a_0,a_1,\zeta)))-1)} \varphi(s)ds \longrightarrow 0, \quad \text{as } j \longrightarrow \infty,\end{aligned} \tag{70}$$

which shows that the sequence $\{a_j\}$ is a fuzzy cone contractive, therefore

$$\begin{aligned}&\lim_{j\longrightarrow\infty} \int_0^{((1/(\mathfrak{M}_c(a_j,a_{j+1},\zeta)))-1)} \varphi(s)ds \\ &= 0 \Rightarrow \lim_{j\longrightarrow\infty} \left(\frac{1}{\mathfrak{M}_c(a_j, a_{j+1}, \zeta)} - 1 \right) \\ &= 0, \quad \text{for } \zeta \gg \vartheta.\end{aligned} \tag{71}$$

Hence, we get that

$$\lim_{j\longrightarrow\infty} \mathfrak{M}_c(a_j, a_{j+1}, \zeta) = 1, \quad \text{for } \zeta \gg \vartheta. \tag{72}$$

Now for $i > j$ and for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\int_0^{((1/(\mathfrak{M}_c(a_j,a_i,\zeta)))-1)} \varphi(s)ds &\leq \int_0^{((1/(\mathfrak{M}_c(a_j,a_{j+1},\zeta)))-1)} \varphi(s)ds \\ &\quad + \int_0^{((1/(\mathfrak{M}_c(a_{j+1},a_{j+2},\zeta)))-1)} \varphi(s)ds + \cdots \\ &\quad + \int_0^{((1/(\mathfrak{M}_c(a_{i-1},a_i,\zeta)))-1)} \varphi(s)ds \\ &\leq \lambda^j \int_0^{((1/(\mathfrak{M}_c(a_o,a_1,\zeta)))-1)} \varphi(s)ds \\ &\quad + \lambda^{j+1} \int_0^{((1/(\mathfrak{M}_c(a_o,a_1,\zeta)))-1)} \varphi(s)ds + \cdots \\ &\quad + \lambda^{i-1} \int_0^{((1/(\mathfrak{M}_c(a_0,a_1,\zeta)))-1)} \varphi(s)ds \\ &= \left(\lambda^j + \lambda^{j+1} + \cdots + \lambda^{i-1}\right) \int_0^{((1/(\mathfrak{M}_c(a_0,a_1,\zeta)))-1)} \varphi(s)ds \\ &= \frac{\lambda^j}{1-\lambda} \int_0^{((1/(\mathfrak{M}_c(a_0,a_1,\zeta)))-1)} \varphi(s)ds \\ &\longrightarrow 0, \quad \text{as } j \longrightarrow \infty.\end{aligned} \tag{73}$$

We get that

$$\begin{aligned}&\lim_{j\longrightarrow\infty} \int_0^{((1/(\mathfrak{M}_c(a_j,a_i,\zeta)))-1)} \varphi(s)ds \\ &= 0 \Rightarrow \lim_{j\longrightarrow\infty} \left(\frac{1}{\mathfrak{M}_c(a_j, a_i, \zeta)} - 1 \right) = 0, \quad \text{for } \zeta \gg \vartheta.\end{aligned} \tag{74}$$

Hence proved that the sequence $\{a_j\}$ is Cauchy. Now for sequence $\{b_j\}$ from (64) and from the proof of Theorem 10, for $\zeta \gg \vartheta$, we have

$$\begin{aligned}\int_0^{((1/(\mathfrak{M}_c(b_j,b_{j+1},\zeta)))-1)} \varphi(s)ds &= \int_0^{((1/(\mathfrak{M}_c(\Gamma(b_{j-1},a_{j-1}),\Gamma(b_j,a_j),\zeta)))-1)} \varphi(s)ds \\ &\leq \lambda \int_0^{((1/(\mathfrak{M}_c(b_{j-1},b_j,\zeta)))-1)} \varphi(s)ds,\end{aligned} \tag{75}$$

where $\lambda = (\alpha_1 + 2\alpha_2)/(1 - 2\alpha_2) < 1$. Similarly, again by using the same arguments, we have

$$\int_0^{((1/(\mathfrak{M}_c(b_{j-1},b_j,\zeta)))-1)} \varphi(s)ds \leq \lambda \int_0^{((1/(\mathfrak{M}_c(b_{j-2},a_{j-1},\zeta)))-1)} \varphi(s)ds, \quad \text{for } \zeta \gg \vartheta. \tag{76}$$

Now, from (75) and (76) and by induction, for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
&\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_j,b_{j+1},\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&\leq \lambda\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_{j-1},b_j,\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&\leq \lambda^2\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_{j-2},b_{j-1},\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&\leq \cdots \leq \lambda^j\int_0^{((1/(\mathfrak{M}_c(b_0,b_1,\zeta)))-1)}\varphi(s)ds \\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty,
\end{aligned} \tag{77}$$

which shows that the sequence $\{b_j\}$ is fuzzy cone contractive; therefore,

$$\lim_{j\longrightarrow\infty}\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_j,b_{j+1},\zeta\right)\right)\right)-1\right)}\varphi(s)ds = 0 \Rightarrow \lim_{j\longrightarrow\infty}\left(\frac{1}{\mathfrak{M}_c\left(b_j,b_{j+1},\zeta\right)}-1\right) = 0, \quad \text{for } \zeta \gg \vartheta. \tag{78}$$

Hence, we get that

$$\lim_{j\longrightarrow\infty}\mathfrak{M}_c\left(b_j,b_{j+1},\zeta\right) = 1, \quad \text{for } \zeta \gg \vartheta. \tag{79}$$

Now for $i > j$ and for $\zeta \gg \vartheta$, we have

$$\begin{aligned}
&\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_j,b_i,\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&\leq \int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_j,b_{j+1},\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&\quad \times + \int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(b_{j+1},b_{j+2},\zeta\right)\right)\right)-1\right)}\varphi(s)ds+\cdots \\
&\quad + \int_0^{((1/(\mathfrak{M}_c(b_{i-1},b_i,\zeta)))-1)}\varphi(s)ds \\
&\leq \lambda^j\int_0^{((1/(\mathfrak{M}_c(b_o,b_1,\zeta)))-1)}\varphi(s)ds \\
&\quad + \lambda^{j+1}\int_0^{((1/(\mathfrak{M}_c(b_o,b_1,\zeta)))-1)}\varphi(s)ds+\cdots \\
&\quad +\lambda^{i-1}\int_0^{((1/(\mathfrak{M}_c(b_o,b_1,\zeta)))-1)}\varphi(s)ds \\
&= \left(\lambda^j+\lambda^{j+1}+\cdots+\lambda^{i-1}\right)\int_0^{((1/(\mathfrak{M}_c(b_o,b_1,\zeta)))-1)}\varphi(s)ds \\
&= \frac{\lambda^j}{1-\lambda}\int_0^{((1/(\mathfrak{M}_c(b_o,b_1,\zeta)))-1)}\varphi(s)ds \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{80}$$

We get that

$$\begin{aligned}
&\lim_{j\longrightarrow\infty}\int_0^{\left(\left(1/\left(\mathfrak{M}_c\left(h_j,h_i,\zeta\right)\right)\right)-1\right)}\varphi(s)ds \\
&= 0 \Rightarrow \lim_{j\longrightarrow\infty}\left(\frac{1}{\mathfrak{M}_c\left(h_j,h_i,\zeta\right)}-1\right) \\
&= 0, \quad \text{for } \zeta \gg \vartheta.
\end{aligned} \tag{81}$$

Hence, it was proved that the sequence $\{b_j\}$ is Cauchy. Since A is complete and $\{a_j\}, \{b_j\}$ are Cauchy sequences in A, so $\exists a, b \in A$ such that $a_j \longrightarrow a$ and $b_j \longrightarrow b$ as $j \longrightarrow \infty$ or this can be written as $\lim_{j\longrightarrow\infty} a_j = a$ and $\lim_{j\longrightarrow\infty} b_j = b$. Therefore,

$$\begin{aligned}
&\lim_{j\longrightarrow\infty}\mathfrak{M}_c\left(a_j,a,\zeta\right) = 1, \\
&\lim_{j\longrightarrow\infty}\mathfrak{M}_c\left(b_j,b,\zeta\right) = 1, \quad \text{for } \zeta \gg \vartheta.
\end{aligned} \tag{82}$$

Hence,

$$\begin{aligned}
&\lim_{j\longrightarrow\infty} a_{j+1} = \lim_{j\longrightarrow\infty}\Gamma\left(a_j,b_j\right) = \Gamma\left(\lim_{j\longrightarrow\infty} a_j, \lim_{j\longrightarrow\infty} b_j\right) \\
&\Rightarrow \Gamma(a,b) = a.
\end{aligned} \tag{83}$$

Similarly,

$$\begin{aligned}
&\lim_{j\longrightarrow\infty} b_{j+1} = \lim_{j\longrightarrow\infty}\Gamma\left(b_j,a_j\right) = \Gamma\left(\lim_{j\longrightarrow\infty} b_j, \lim_{j\longrightarrow\infty} a_j\right) \\
&\Rightarrow \Gamma(b,a) = b.
\end{aligned} \tag{84}$$

Regarding its uniqueness, suppose (a_1, b_1) and (b_1, a_1) are another couple fixed point pairs in $A \times A$ such that $\Gamma(a_1, b_1) = a_1$ and $\Gamma(b_1, a_1) = b_1$. Now, from (64) and from the proof of Theorem 10, for $\zeta \gg \vartheta$, we have that

$$\begin{aligned}
&\int_0^{((1/(\mathfrak{M}_c(a,a_1,\zeta)))-1)}\varphi(s)ds \\
&= \int_0^{((1/(\mathfrak{M}_c(\Gamma(a,b),\Gamma(a_1,b_1),\zeta)))-1)}\varphi(s)ds \\
&\leq \left(\alpha_1+2\alpha_2\right)\int_0^{((1/(\mathfrak{M}_c(a,a_1,\zeta)))-1)}\varphi(s)ds \\
&= \left(\alpha_1+2\alpha_2\right)\int_0^{((1/(\mathfrak{M}_c(\Gamma(a,b),\Gamma(a_1,b_1),\zeta)))-1)}\varphi(s)ds \\
&\leq \left(\alpha_1+2\alpha_2\right)^2\int_0^{((1/(\mathfrak{M}_c(a,a_1,\zeta)))-1)}\varphi(s)ds \leq \cdots \\
&\leq \left(\alpha_1+2\alpha_2\right)^j\int_0^{((1/(\mathfrak{M}_c(a,a_1,\zeta)))-1)}\varphi(s)ds \\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{85}$$

Hence, we get that $\mathfrak{M}_c(a, a_1, \zeta) = 1$ for $\zeta \gg \vartheta$; this implies $a = a_1$.

Similarly, again from (64)and from the proof of Theorem 10, for $\zeta \gg \vartheta$, we have that

$$\begin{aligned}
&\int_0^{((1/(\mathfrak{M}_c(b,b_1,\zeta)))-1)} \varphi(s)ds \\
&= \int_0^{((1/(\mathfrak{M}_c(\Gamma(b,a),\Gamma(b_1,a_1),t^*)))-1)} \varphi(s)ds \\
&\le (\alpha_1 + 2\alpha_2)\int_0^{((1/(\mathfrak{M}_c(b,b_1,\zeta)))-1)} \varphi(s)ds \\
&= (\alpha_1 + 2\alpha_2)\int_0^{((1/(\mathfrak{M}_c(\Gamma(b,a),\Gamma(b_1,a_1),\zeta)))-1)} \varphi(s)ds \\
&\le (\alpha_1 + 2\alpha_2)^2\int_0^{((1/(\mathfrak{M}_c(b,b_1,\zeta)))-1)} \varphi(s)ds \\
&\le \cdots \le (\alpha_1 + 2\alpha_2)^j\int_0^{((1/(\mathfrak{M}_c(b,b_1,\zeta)))-1)} \varphi(s)ds \\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{86}$$

Hence, we get that $\mathfrak{M}_c(b, b_1, \zeta) = 1$ for $\zeta \gg \vartheta$; this implies $b = b_1$.

5. Conclusion

We presented the concept of coupled FP-results in FCM-spaces and prove some unique coupled FP-theorems under the modified contractive type conditions by using "the triangular property of fuzzy cone metric." We presented examples in support of our result. Further, we presented an application of Lebesgue integral mapping to uplift our main work. With the help of this new concept, one can prove more modified and general contractive type coupled FP-results with different types of integral contractive type of conditions and applications in complete FCM-spaces.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analysed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Research Article

General Solution and Stability of Additive-Quadratic Functional Equation in IRN-Space

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The investigation of the stabilities of various types of equations is an interesting and evolving research area in the field of mathematical analysis. Recently, there are many research papers published on this topic, especially additive, quadratic, cubic, and mixed type functional equations. We propose a new functional equation in this study which is quite different from the functional equations already dealt in the literature. The main feature of the equation dealt in this study is that it has three different solutions, namely, additive, quadratic, and mixed type functions. We also prove that the stability results hold good for this equation in intuitionistic random normed space (briefly, IRN-space).

1. Introduction

The theory of random normed spaces (RN-spaces) is important as a generalization of the deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the tools to study the geometry of nuclear physics and have important applications in quantum particle physics.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. Aoki generalized the result of Hyers [3] for approximate additive mappings and by Rassias [4] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\varepsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Th.M. Rassias' theorem was got by Gavruta [5], who replaced $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. For additional information regarding the outcomes about such issues, the related background in [6–12] can be examined. Absorbing new outcomes concerning mixed-type functional equations has as of late been acquired by Najati et al. [13–15], Jun and Kim [16, 17], and Park [18–22].

The functional equations

$$f(x+y)=f(x)+f(y), \tag{1}$$

and

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{2}$$

are called the additive and quadratic functional equations, respectively. Every solution of the additive and quadratic functional equations is said to be additive mapping and quadratic mapping, respectively.

As of late, Zhang [23] examined the cubic functional equation in intuitionistic random space. The stability of various equations in RN-spaces has been as of late concentrated in Alsina [24], Eshaghi Gordji et al. [25, 26], Mihet and Radu [27–29], and Saadati et al. [30]. Xu et al. [31–33] presented the various mixed types of functional equations investigated in Intuitionistic fuzzy normed spaces, quasi Banach spaces, and random normed spaces. Also, Shu et al. [33–35] discussed various differential equations to study the Hyers-Ulam stability, which provides a wide view of this stability problem.

In this present work, we introduce a new mixed type additive-quadratic functional equation

$$\begin{aligned}\varphi\left(\sum_{i=1}^{m} a^i v_i\right) + \sum_{i=1}^{m}\varphi\left(-a^i v_i + \sum_{j=1;i\neq j}^{m} a^j v_j\right) &= (m-3)\sum_{i=1}^{m}\varphi\left(a^i v_i + a^j v_j\right) - \left(m^2-5m+2\right) \\ &\cdot \sum_{i=1}^{m} a^{2i}\left[\frac{\varphi(v_i)+\varphi(-v_i)}{2}\right] - \left(m^2-5m+4\right) \\ &\cdot \sum_{i=1}^{m} a^{i}\left[\frac{\varphi(v_i)-\varphi(-v_i)}{2}\right],\end{aligned} \tag{3}$$

where a is a fixed integer and $m \geq 5$ and investigate the Ulam-Hyers stability results of this mixed type additive-quadratic functional equation in an intuitionistic random normed space.

So far various forms of additive and quadratic functional equations are considered in this research field to obtain their stability results through different methods. For the first time, a new mixed additive-quadratic functional equation is proposed in this paper, and its stability results are proved in an intuitionistic random normed space.

This type of functional equation can be of use in solving many physical problems and also has significant relevance in various scientific fields of research and study. In particular, additive-quadratic functional equations have applications in electric circuit theory, physics, and relations connecting the harmonic mean and arithmetic mean of several values. Providing a wealth of essential insights and new concepts in the field of functional equations.

2. Preliminaries

We recall the following ideas and conceptions of IRN-spaces in [36–41].

Definition 1 (see [42]). A mapping $\mu : \mathbb{R} \longrightarrow [0,1]$ is said to be a measure distribution function, if μ is left continuous on $\mathbb{R}$, non-decreasing, $\inf_{t\in\mathbb{R}}\mu(t) = 0$, and $\sup_{t\in\mathbb{R}}\mu(t) = 1$.

Definition 2 (see [42]). A mapping $\nu : \mathbb{R} \longrightarrow [0,1]$ is said to be a non-measure distribution function, if ν is right continuous on $\mathbb{R}$, non-increasing, $\sup_{t\in\mathbb{R}}\nu(t) = 1$, and $\inf_{t\in\mathbb{R}}\nu(t) = 0$.

Lemma 3 (see [43, 44]). *Let L^* be a set with an operator $\leq_{L^*}$ is defined by*

$$\begin{aligned}L^* &= \left\{(v_1, v_2)\colon (v_1, v_2) \in [0,1]^2 \text{ and } v_1 + v_2 \leq 1\right\}, \\ (v_1, v_2) \leq_{L^*} (w_1, w_2) &\Leftrightarrow v_1 \leq w_1, v_2 \geq w_2, (v_1, v_2), (w_1, w_2) \in L^*.\end{aligned} \tag{4}$$

Then, the pair $(L^*, \leq_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Typically, a triangular norm (t-norm) $* = \Phi$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $\Phi : [0,1]^2 \longrightarrow [0,1]$ satisfying $\Phi(1,p) = 1 * p = p$ for every $p \in [0,1]$, and a triangular conorm (t-conorm) $Y = \diamond$ is defined as an increasing, commutative, associative mapping $Y : [0,1]^2 \longrightarrow [0,1]$ satisfying $Y(0,p) = 0 \diamond p = p$ for all $p \in [0,1]$.

By using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 4 (see [44]). *A triangular norm (t-norm) on L^* is a mapping $\Phi : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:*

(i) Boundary condition

i.e., $\Phi(p, 1_{L^*}) = p, \forall p \in L^*$;

(ii) Commutativity

i.e., $\Phi(p,q) = \Phi(q,p), \forall (p,q) \in (L^*)^2$;

(iii) Associativity

i.e., $\Phi(p, \Phi(q,r)) = \Phi(\Phi(p,q), r), \forall (p,q,r) \in (L^*)^3$;

(iv) Monotonicity

i.e., $p \leq_{L^*} p'$ and $q \leq_{L^*} q' \Rightarrow \Phi(p,q) \leq_{L^*} \Phi(p', q')$ for all $(p, p', q, q') \in (L^*)^4$.

If $(L^*, \leq_{L^*}, \Phi)$ is an Abelian topological monoid with unit 1_{L^*}, then Φ is called a continuous t-norm.

Definition 5 (see [42]). *A negator on L^* is any decreasing mapping N from L^* to L^* satisfying $N(1_{L^*}) = 0_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(p)) = p$ for all $p \in L^*$, then N is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$.*

N_s denotes the *standard negator* on $[0,1]$ defined by

$$N_s(p) = 1 - p, \tag{5}$$

for all $p \in [0,1]$.

Definition 6 (see [23]). Let μ and ν be measure and nonmeasure distribution functions from $V \times (0,+\infty)$ to $[0,1]$ such that

$$\mu_p(t) + \nu_p(t) \leq 1, p \in V, t > 0. \tag{6}$$

The triple $(V, I_{\mu,\nu}, \Phi)$ is said to be an intuitionistic random normed space if a vector space V, continuous t-representable Φ, and a mapping $I_{\mu,\nu} : V \times (0,+\infty) \longrightarrow L^*$ holds the following conditions: for all $p, q \in V$ and $t_1, t_2 > 0$

$$\begin{gathered} I_{\mu,\nu}(p, 0) = 0_{L^*}, \\ I_{\mu,\nu}(p, t_1) = 1_{L^*} \Leftrightarrow p = 0, \\ I_{\mu,\nu}(\alpha p, t_1) = I_{\mu,\nu}\left(p, \frac{t_1}{|\alpha|}\right), \forall \alpha \neq 0, \\ I_{\mu,\nu}(p + q, t_1 + t_2) \geq_{L^*} \Phi\left(I_{\mu,\nu}(p, t_1), I_{\mu,\nu}(q, t_2)\right). \end{gathered} \tag{7}$$

Thus, $I_{\mu,\nu}$ is called an intuitionistic random norm. Hence,

$$I_{\mu,\nu}(p, t_1) = \left(\mu_p(t_1), \nu_p(t_1)\right). \tag{8}$$

Example 1 (see [42]). *Let $(V, \|\cdot\|)$ be a normed space. Let $\Phi(p, q) = (p_1 q_1, \min(p_2 + q_2, 1))$ for all $p = (p_1, p_2)$, $q = (q_1, q_2) \in L^*$ and let μ, ν be measure and non-measure distribution functions defined by*

$$I_{\mu,\nu}(\nu, \varepsilon) = (\mu_\nu(\varepsilon), \nu_\nu(\varepsilon)) = \left(\frac{\varepsilon}{\varepsilon + \|\nu\|}, \frac{\|\nu\|}{\varepsilon + \|\nu\|}\right), \forall \varepsilon \in R^+. \tag{9}$$

Then, $(V, I_{\mu,\nu}, \Phi)$ is an IRN-space.

Definition 7 (see [42]). *Let $(V, I_{\mu,\nu}, \Phi)$ be an IRN-space.*

(i) A sequence $\{p_m\}$ in $(V, I_{\mu,\nu}, \Phi)$ is known as a Cauchy sequence if, for some $\delta > 0$ and $t > 0$, there is an $m_0 \in \mathbb{N}$ satisfies

$$I_{\mu,\nu}(p_m - p_n, t) \geq_{L^*} (N_s(\delta), \delta), m, n \geq m_0. \tag{10}$$

(ii) The sequence $\{p_m\}$ is convergent to any point $p \in V$ if $I_{\mu,\nu}(p_m - p, t) \longrightarrow 1_{L^*}$ as $m \longrightarrow \infty$ for all $t > 0$

(iii) An intuitionistic random normed space $(V, I_{\mu,\nu}, \Phi)$ is known as complete if every Cauchy sequence in V is convergent to a point $p \in V$

3. Solution of the Functional Equation (3)

In this section, let us consider V and W are two real vector spaces.

Theorem 8. *If an odd mapping $\varphi : V \longrightarrow W$ satisfies the functional equation (3) for all $v_1, v_2, \cdots, v_m \in V$, then the function φ is additive.*

Proof. In the view of the oddness of φ, we have $\varphi(-v) = -\varphi(v)$ for all $v \in V$. Using the oddness property, the functional equation (3) reduces as

$$\begin{aligned} &\varphi\left(\sum_{i=1}^{m} a^i v_i\right) + \sum_{i=1}^{m} \varphi\left(-a^i v_i + \sum_{j=1; i \neq j}^{m} a^j v_j\right) \\ &\quad = (m-3) \sum_{i=1}^{m} \varphi\left(a^i v_i + a^j v_j\right) - \left(m^2 - 5m + 4\right) \sum_{i=1}^{m} a^i \varphi(v_i), \end{aligned} \tag{11}$$

for all $v_1, v_2, \cdots, v_m \in V$. Now, replacing $(v_1, v_2, \cdots, v_m)$ by $(0, 0 \cdots, 0)$ in (11), we get $\varphi(0) = 0$. Interchanging $(v_1, v_2, \cdots, v_m)$ with $(v, 0, 0, \cdots, 0)$ in (11), we get

$$\varphi(av) = a\varphi(v), v \in V. \tag{12}$$

Again interchanging v with av in (12), we have

$$\varphi\left(a^2 v\right) = a^2 \varphi(v), \tag{13}$$

for all $v \in V$. Replacing v by av in (13), we obtain

$$\varphi\left(a^3 v\right) = a^3 \varphi(v), v \in V. \tag{14}$$

From the equalities (12)–(14), we can generalize the results for any nonnegative integer m as

$$\varphi(a^m v) = a^m \varphi(v), v \in \mathrm{V}. \tag{15}$$

Similarly, we have

$$\varphi\left(\frac{v}{a^m}\right) = \frac{1}{a^m} \varphi(v), v \in V. \tag{16}$$

Replacing $(v_1, v_2, \cdots, v_m)$ by $((x/a), (y/a^2), 0, \cdots, 0)$ in (11), we have

$$\varphi(x + y) = \varphi(x) + \varphi(y), x, y \in V. \tag{17}$$

Hence, the function φ is additive. □

Theorem 9. *If an even mapping $\varphi : V \longrightarrow W$ satisfies the functional equation (3) for all $v_1, v_2, \cdots, v_m \in V$, then the function φ is quadratic.*

Proof. Since, in the view of evenness of φ, we have $\varphi(-v) = \varphi(v)$ for all $v \in V$. Now, the functional equation (3) reduces as

$$\begin{aligned} &\varphi\left(\sum_{i=1}^{m} a^i v_i\right) + \sum_{i=1}^{m} \varphi\left(-a^i v_i + \sum_{j=1; i \neq j}^{m} a^j v_j\right) \\ &\quad = (m-3) \sum_{i=1}^{m} \varphi\left(a^i v_i + a^j v_j\right) - \left(m^2 - 5m + 2\right) \sum_{i=1}^{m} a^{2i} \varphi(v_i). \end{aligned} \tag{18}$$

for all $v_1, v_2, \cdots, v_m \in V$. Now, replacing $(v_1, v_2, \cdots, v_m)$ by $(0, 0, \cdots, 0)$ in (18), we obtain $\varphi(0) = 0$. Interchanging $(v_1, v_2, \cdots, v_m)$ with $(v, 0, 0, \cdots, 0)$ in (18), we obtain

$$\varphi(av) = a^2 \varphi(v), v \in V. \tag{19}$$

Replacing v by av in (19), we reach

$$\varphi\left(a^2 v\right) = a^4\varphi(v), v \in V. \tag{20}$$

Switching v by av in (20), we get

$$\varphi\left(a^3 v\right) = a^6\varphi(v), v \in V. \tag{21}$$

From (19)–(21), we can generalize the results for any nonnegative integer m as

$$\varphi(a^m v) = a^{2m}\varphi(v), v \in V. \tag{22}$$

Similarly, we have

$$\varphi\left(\frac{v}{a^m}\right) = \frac{1}{a^{2m}}\varphi(v), v \in V. \tag{23}$$

Replacing $(v_1, v_2,\cdots,v_m)$ by $((x/a), (y/a^2), 0,\cdots,0)$ in (18), we obtain

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + 2\varphi(y), x, y \in V. \tag{24}$$

Hence, the function φ is quadratic. □

Theorem 10. *If a mapping $\varphi : V \longrightarrow W$ satisfies $\varphi(0) = 0$ and satisfies the functional equation (3) for all $v_1, v_2, \cdots, v_m \in V$ if and only if there exists a symmetric biadditive mapping $Q : V \times V \longrightarrow W$ and a additive mapping $A : V \longrightarrow W$ satisfies $\varphi(v) = Q(v, v) + A(v)$ for all $v \in V$.*

Proof. Let a mapping $\varphi : V \longrightarrow W$ with $\varphi(0) = 0$ satisfies the functional equation (3). We divide the function φ into the odd part and even part as

$$\varphi_o(v) = \frac{\varphi(v) - \varphi(-v)}{2}, \varphi_e(v) = \frac{\varphi(v) + \varphi(-v)}{2}, v \in V, \tag{25}$$

respectively. Clearly, $\varphi(v) = \varphi_e(v) + \varphi_o(v)$ for all $v \in V$. □

It is easy to prove that φ_o and φ_e satisfies the functional equation (3). By Theorems 8 and 9, we conclude that φ_o and φ_e are additive and quadratic, respectively. Then, there exist a symmetric biadditive mapping $Q : V \times V \longrightarrow W$ which satisfies $\varphi_e(v) = Q(v, v)$ and an additive mapping $A : V \longrightarrow W$ which satisfies $\varphi_o(v) = A(v)$ for all $v \in V$. Hence, $\varphi(v) = Q(v, v) + A(v)$ for all $v \in V$.

Conversely, suppose that there exists a symmetric biadditive mapping $Q : V \times V \longrightarrow W$ and an additive mapping $A : V \longrightarrow W$ and satisfies $\varphi(v) = Q(v, v) + A(v)$ for all $v \in V$. It is easy to prove that the mappings $v \mapsto Q(v, v)$ and $A : V \longrightarrow W$ satisfy the functional equation (3). Hence, the mapping $\varphi : V \longrightarrow W$ satisfies the functional equation (3).

For our notational convenience, we can define a mapping $\varphi : V \longrightarrow W$ by

$$\begin{aligned} D\varphi(v_1, v_2,\cdots,v_m) &= \varphi\left(\sum_{i=1}^{m} a^i v_i\right) + \sum_{i=1}^{m}\varphi\left(-a^i v_i + \sum_{j=1;i\neq j}^{m} a^j v_j\right) \\ &- (m-3)\sum_{i=1}^{m}\varphi\left(a^i v_i + a^j v_j\right) \\ &+ \left(m^2 - 5m + 2\right)\sum_{i=1}^{m} a^{2i}\left[\frac{\varphi(v_i) + \varphi(-v_i)}{2}\right] \\ &+ \left(m^2 - 5m + 4\right)\sum_{i=1}^{m} a^{i}\left[\frac{\varphi(v_i) - \varphi(-v_i)}{2}\right], \end{aligned} \tag{26}$$

for all $v_1, v_2, \cdots, v_m \in V$.

In the following sections, we consider V is a linear space, $(V, I_{\mu',v'}, Y)$ is an intuitionistic random normed space and $(W, I_{\mu,v}, Y)$ is a complete intuitionistic random normed space.

4. Stability Results for Even Case

Theorem 11. *Let $\alpha, \beta : V^m \longrightarrow D^+$, where $\alpha(v_1, v_2,\cdots,v_m)$ is denoted by $\alpha_{v_1,v_2,\cdots,v_m}$, $\beta(v_1, v_2,\cdots,v_m)$ is denoted by $\beta_{v_1,v_2,\cdots,v_m}$ and $\alpha_{v_1,v_2,\cdots,v_m}(\varepsilon), \beta_{v_1,v_2,\cdots,v_m}(\varepsilon)$ is denoted by $\Theta_{\alpha,\beta}(v_1, v_2,\cdots, v_m, \varepsilon)$, be a mapping such that*

$$\lim_{w\longrightarrow\infty}\Theta_{\alpha,\beta}\left(a^w v_1, a^w v_2,\cdots,a^w v_m, a^{2w}\varepsilon\right) = 1_{L^*}, \tag{27}$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, and

$$\lim_{w\longrightarrow\infty}\Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}v, 0,\cdots,0, a^{2w+i}\varepsilon\right)\right) = 1_{L^*}, \tag{28}$$

for all $v \in V$ and all $\varepsilon > 0$. If an even mapping $\varphi : V \longrightarrow W$ with $\varphi(0) = 0$ satisfies

$$I_{\mu,v}(D\varphi(v_1, v_2,\cdots,v_m), \varepsilon) \geq_{L^*} \Theta_{\alpha,\beta}(v_1, v_2,\cdots,v_m, \varepsilon), \tag{29}$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, then there exists a unique quadratic mapping $Q_2 : V \longrightarrow W$ such that

$$\begin{aligned} &I_{\mu,v}(\varphi(v) - Q_2(v), \varepsilon) \\ &\quad\geq_{L^*}\Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{i-1}v, 0,\cdots,0, a^i\left(m^2 - 5m + 2\right)\varepsilon\right)\right), \end{aligned} \tag{30}$$

for all $v \in V$ and all $\varepsilon > 0$.

Proof. Replacing $(v_1, v_2,\cdots,v_m)$ by $(v, 0,\cdots,0)$ in (29), we have

$$\begin{aligned} &I_{\mu,v}\left(\left(m^2 - 5m + 2\right)\varphi(av) - \left(m^2 - 5m + 2\right)a^2\varphi(v), \varepsilon\right) \\ &\quad\geq_{L^*}\Theta_{\alpha,\beta}(v, 0,\cdots,0, \varepsilon), \end{aligned} \tag{31}$$

for all $\nu \in V$ and all $\varepsilon > 0$. From inequality (31), we get

$$I_{\mu,\nu}\left(\frac{\varphi(a\nu)}{a^2} - \varphi(\nu), \varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}\left(\nu, 0, \cdots, 0, a^2\left(m^2 - 5m + 2\right)\varepsilon\right), \quad \nu \in V, \varepsilon > 0. \tag{32}$$

Interchanging ν with $a\nu$ in (32), we obtain

$$I_{\mu,\nu}\left(\frac{\varphi\left(a^2\nu\right)}{a^4} - \frac{\varphi(a\nu)}{a^2}, \varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}\left(a\nu, 0, \cdots, 0, a^{2(2)}\left(m^2 - 5m + 2\right)\varepsilon\right), \nu \in V, \varepsilon > 0. \tag{33}$$

Replacing ν by $a^{l-1}\nu$ and divide by a^{2l} in (33), we conclude that

$$I_{\mu,\nu}\left(\frac{\varphi\left(a^{l+1}\nu\right)}{a^{2(l+1)}} - \frac{\varphi\left(a^l\nu\right)}{a^{2l}}, \varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}\left(a^l\nu, 0, \cdots, 0, a^{2(l+1)}\left(m^2 - 5m + 2\right)\varepsilon\right), \tag{34}$$

for all $\nu \in V$ and all $\varepsilon > 0$. Thus,

$$I_{\mu,\nu}\left(\frac{\varphi(a^w\nu)}{a^{2w}} - \varphi(\nu), \varepsilon\right) \geq_{L^*} \Phi_{l=0}^{w-1}\left(I_{\mu,\nu}\left(\frac{\varphi\left(a^{l+1}\nu\right)}{a^{2(l+1)}} - \frac{\varphi\left(a^l\nu\right)}{a^{2l}}, \sum_{l=0}^{w-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \tag{35}$$

for all $\nu \in V$ and all $\varepsilon > 0$. To prove the convergence of the sequence $\{\varphi(a^w\nu)/a^{2w}\}$, replacing ν by $a^k\nu$ in (35), we obtain

$$I_{\mu,\nu}\left(\frac{\varphi\left(a^{w+k}\nu\right)}{a^{2(w+k)}} - \frac{\varphi\left(a^k\nu\right)}{a^{2(k)}}, \varepsilon\right) \geq_{L^*} \Phi_{l=k}^{w+k-1}\left(I_{\mu,\nu}\left(\frac{\varphi\left(a^{l+1}\nu\right)}{a^{2(l+1)}} - \frac{\varphi\left(a^l\nu\right)}{a^{2(l)}}, \sum_{l=k}^{w+k-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \tag{36}$$

for all $\nu \in V$ and all $\varepsilon > 0$ and all $k, w \geq 0$. Since the R.H.S of the inequality (36) tends to 1_{L^*} as $w, k \longrightarrow \infty$, the sequence $\{\varphi(a^w\nu)/a^{2w}\}$ is a Cauchy sequence in $(W, I_{\mu,\nu}, Y)$. Since $(W, I_{\mu,\nu}, Y)$ is a complete IRN-space, this sequence converges to some point $Q_2(\nu) \in W$. So one can define the mapping $Q_2 : V \longrightarrow W$ by

$$Q_2(\nu) = \lim_{w\longrightarrow\infty}\frac{\varphi(a^w\nu)}{a^{2w}}, \tag{37}$$

for all $\nu \in V$. Letting $k = 0$ in (36), we obtain

$$I_{\mu,\nu}\left(\frac{\varphi(a^w\nu)}{a^{2w}} - \varphi(\nu), \varepsilon\right) \geq_{L^*} \Phi_{l=0}^{w-1}\left(I_{\mu,\nu}\left(\frac{\varphi\left(a^{l+1}\nu\right)}{a^{2(l+1)}} - \frac{\varphi\left(a^l\nu\right)}{a^{2(l)}}, \sum_{l=0}^{w-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \tag{38}$$

for all $\nu \in V$ and all $\varepsilon > 0$. Taking the limit $w \longrightarrow \infty$ in (38), we get

$$I_{\mu,\nu}(\varphi(\nu) - Q_2(\nu), \varepsilon) \geq_{L^*} \Phi_{l=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{l-1}\nu, 0, \cdots, 0, a^l\left(m^2 - 5m + 2\right)\varepsilon\right)\right), \tag{39}$$

for all $\nu \in V$ and all $\varepsilon > 0$.

Next, we prove that the function Q_2 is quadratic. Replacing $(\nu_1, \nu_2, \cdots, \nu_m)$ by $(a^w\nu_1, a^w\nu_2, \cdots, a^w\nu_m)$ in (29), we obtain

$$I_{\mu,\nu}\left(\frac{1}{a^{2w}}D\varphi(a^w\nu_1, a^w\nu_2, \cdots, a^w\nu_m), \varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}\left(a^w\nu_1, a^w\nu_2, \cdots, a^w\nu_m, a^{2w}\varepsilon\right), \tag{40}$$

for all $\nu_1, \nu_2, \cdots, \nu_m \in V$ and all $\varepsilon > 0$. Taking the limit as $w \longrightarrow \infty$, we find that $I_{\mu,\nu}(DQ_2(\nu_1, \nu_2, \cdots, \nu_m), \varepsilon) = 1_{L^*}$ for all $\nu_1, \nu_2, \cdots, \nu_m \in V$ and all $\varepsilon > 0$, which implies $DQ_2(\nu_1, \nu_2, \cdots, \nu_m) = 0$. Thus, the function Q_2 satisfies the functional equation (3). Hence, $Q_2 : V \longrightarrow W$ is a quadratic mapping. Passing to the limit as $w \longrightarrow \infty$ in (35), we have (30).

Finally, to show the uniqueness of Q_2 subject to (30), consider that there exists an another quadratic function Q_2' which satisfies the inequality (30). Clearly, $Q_2(a^w\nu) = a^{2w}Q_2(\nu)$ and $Q_2'(a^w\nu) = a^{2w}Q_2'(\nu)$ for all $\nu \in V$ and $w \in \mathbb{N}$, from (30) and (28) that

$$\begin{aligned} I_{\mu,\nu}\left(Q_2(\nu) - Q_2'(\nu), \varepsilon\right) &\geq_{L^*} I_{\mu,\nu}\left(Q_2(a^w\nu) - Q_2'(a^w\nu), a^{2w}\varepsilon\right) \\ &\geq_{L^*} \Phi\left(I_{\mu,\nu}\left(Q_2(a^w\nu) - \varphi(a^w\nu), \frac{a^{2w}}{2}\varepsilon\right), I_{\mu,\nu}\left(\varphi(a^w\nu) - Q_2'(a^w\nu), \frac{a^{2w}}{2}\varepsilon\right)\right) \\ &\geq_{L^*} \Phi\left(\Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}\nu, 0, \cdots, 0, a^{2w+i}\left(m^2 - 5m + 2\right)\frac{\varepsilon}{2}\right)\right), \Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}\nu, 0, \cdots, 0, a^{2w+i}\left(m^2 - 5m + 2\right)\frac{\varepsilon}{2}\right)\right)\right), \end{aligned} \tag{41}$$

for all $\nu \in V$ and all $\varepsilon > 0$. By taking $w \longrightarrow \infty$ in (41), we show the uniqueness of Q_2. This ends the proof of the uniqueness, as desired. □

Corollary 12. *If an even mapping $\varphi : V \longrightarrow W$ satisfies*

$$I_{\mu,\nu}(D\varphi(v_1, v_2,\cdots,v_m, \varepsilon) \geq_{L^*} I_{\mu',\nu'}\left(\sum_{i=1}^{m} v_i, \varepsilon\right), \quad (42)$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, and

$$\lim_{w\longrightarrow\infty} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{w+i-1}v, a^{2w+i}\left(m^2 - 5m + 2\right)\varepsilon\right)\right) = 1_{L^*}, \quad (43)$$

for all $v \in V$ and all $\varepsilon > 0$, then there exists a unique quadratic mapping $Q_2 : V \longrightarrow W$ such that

$$I_{\mu,\nu}(\varphi(v) - Q_2(v), \varepsilon) \geq_{L^*} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{i-1}v, a^{i}\left(m^2 - 5m + 2\right)\varepsilon\right)\right), \quad (44)$$

for all $v \in V$ and all $\varepsilon > 0$.

Proof. By taking $\Theta_{\alpha,\beta}(v_1, v_2,\cdots,v_m, \varepsilon) = I_{\mu',\nu'}(\sum_{i=1}^{m} v_i, \varepsilon)$ in Theorem 11, we obtain our desired result. ☐

5. Stability Results for Odd Case

Theorem 13. *Let $\alpha, \beta : V^m \longrightarrow D^+$, where $\alpha(v_1, v_2,\cdots,v_m)$ is denoted by $\alpha_{v_1,v_2,\cdots,v_m}$, $\beta(v_1, v_2,\cdots,v_m)$ is denoted by $\beta_{v_1,v_2,\cdots,v_m}$ and $\alpha_{v_1,v_2,\cdots,v_m}(\varepsilon), \beta_{v_1,v_2,\cdots,v_m}(\varepsilon)$ is denoted by $\Theta_{\alpha,\beta}(v_1, v_2,\cdots, v_m, \varepsilon)$, be a mapping such that*

$$\lim_{w\longrightarrow\infty} \Theta_{\alpha,\beta}(a^w v_1, a^w v_2,\cdots,a^w v_m, a^w\varepsilon) = 1_{L^*}, \quad (45)$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, and

$$\lim_{w\longrightarrow\infty} \Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}v, 0,\cdots,0, a^w\varepsilon\right)\right) = 1_{L^*}, \quad (46)$$

for all $v \in V$ and all $\varepsilon > 0$. If an odd mapping $\varphi : V \longrightarrow W$ with $\varphi(0) = 0$ satisfies

$$I_{\mu,\nu}(D\varphi(v_1, v_2,\cdots,v_m), \varepsilon) \geq_{L^*} \Theta_{\alpha,\beta}(v_1, v_2,\cdots,v_m, \varepsilon), \quad (47)$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, then there exist a unique additive mapping $A_1 : V \longrightarrow W$ such that

$$\begin{aligned} &I_{\mu,\nu}(\varphi(v) - A_1(v), \varepsilon) \\ &\quad \geq_{L^*} \Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{i-1}v, 0,\cdots,0, \left(m^2 - 5m + 4\right)\varepsilon\right)\right), \end{aligned} \quad (48)$$

for all $v \in V$ and all $\varepsilon > 0$.

Proof. Replacing $(v_1, v_2,\cdots,v_m)$ by $(v, 0,\cdots,0)$ in (47), we obtain

$$\begin{aligned} &I_{\mu,\nu}\left(\left(m^2 - 5m + 4\right)\varphi(av) - \left(m^2 - 5m + 4\right)a\varphi(v), \varepsilon\right) \\ &\quad \geq_{L^*} \Theta_{\alpha,\beta}(v, 0,\cdots,0, \varepsilon), \end{aligned} \quad (49)$$

for all $v \in V$ and all $\varepsilon > 0$. From inequality (49), we get

$$I_{\mu,\nu}\left(\frac{\varphi(av)}{a} - \varphi(v), \varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}\left(v, 0,\cdots,0, a\left(m^2 - 5m + 4\right)\varepsilon\right), \quad (50)$$

for all $v \in V$ and all $\varepsilon > 0$. Replacing v by av in the above inequality (50), we have

$$\begin{aligned} &I_{\mu,\nu}\left(\frac{\varphi(a^2v)}{a^2} - \frac{\varphi(av)}{a}, \varepsilon\right) \\ &\quad \geq_{L^*} \Theta_{\alpha,\beta}\left(av, 0,\cdots,0, a^2\left(m^2 - 5m + 4\right)\varepsilon\right), \end{aligned} \quad (51)$$

for all $v \in V$ and all $\varepsilon > 0$. Replacing v by $a^{l-1}v$ in (51), we conclude that

$$\begin{aligned} &I_{\mu,\nu}\left(\frac{\varphi(a^{l+1}v)}{a^{l+1}} - \frac{\varphi(a^l v)}{a^l}, \varepsilon\right) \\ &\quad \geq_{L^*} \Theta_{\alpha,\beta}\left(a^l v, 0,\cdots,0, a^{(l+1)}\left(m^2 - 5m + 4\right)\varepsilon\right), \end{aligned} \quad (52)$$

for all $v \in V$ and all $\varepsilon > 0$. Thus,

$$\begin{aligned} &I_{\mu,\nu}\left(\frac{\varphi(a^w v)}{a^w} - \varphi(v), \varepsilon\right) \\ &\quad \geq_{L^*} \Phi_{l=0}^{w-1}\left(I_{\mu,\nu}\left(\frac{\varphi(a^{l+1}v)}{a^{(l+1)}} - \frac{\varphi(a^l v)}{a^l}, \sum_{l=0}^{w-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \end{aligned} \quad (53)$$

for all $v \in V$ and all $\varepsilon > 0$. To prove the convergence of the sequence $\{\varphi(a^w v)/a^w\}$, replacing v by $a^k v$ in (53), we obtain

$$\begin{aligned} &I_{\mu,\nu}\left(\frac{\varphi(a^{w+k}v)}{a^{(w+k)}} - \frac{\varphi(a^k v)}{a^{(k)}}, \varepsilon\right) \\ &\quad \geq_{L^*} \Phi_{l=k}^{w+k-1}\left(I_{\mu,\nu}\left(\frac{\varphi(a^{l+1}v)}{a^{(l+1)}} - \frac{\varphi(a^l v)}{a^{(l)}}, \sum_{l=k}^{w+k-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \end{aligned} \quad (54)$$

for all $v \in V$ and all $\varepsilon > 0$ and all $k, w \geq 0$. Since the R.H.S of the inequality (54) tends to 1_{L^*} as $w, k \longrightarrow \infty$, the sequence $\{\varphi(a^w v)/a^w\}$ is a Cauchy sequence in $(W, I_{\mu,\nu}, Y)$. Since $(W, I_{\mu,\nu}, Y)$ is a complete IRN-space, this sequence converges to some point $A_1(v) \in W$. So one can define the mapping $A_1 : V \longrightarrow W$ by

$$A_1(v) = \lim_{w\longrightarrow\infty} \frac{\varphi(a^w v)}{a^w}, \quad (55)$$

for all $v \in V$. Letting $k=0$ in (54), we obtain

$$I_{\mu,\nu}\left(\frac{\varphi(a^w v)}{a^w}-\varphi(v),\varepsilon\right) \geq_{L^*} \Phi_{l=0}^{w-1}\left(I_{\mu,\nu}\left(\frac{\varphi(a^{l+1}v)}{a^{(l+1)}}-\frac{\varphi(a^l v)}{a^{(l)}},\sum_{l=0}^{w-1}\frac{\varepsilon}{a^{l+1}}\right)\right), \tag{56}$$

for all $v \in V$ and all $\varepsilon > 0$. Taking the limit as $w \longrightarrow \infty$ in (56), we get

$$I_{\mu,\nu}(\varphi(v)-A_1(v),\varepsilon) \geq_{L^*} \Phi_{l=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{l-1}v,0,\cdots,0,\left(m^2-5m+4\right)\varepsilon\right)\right), \tag{57}$$

for all $v \in V$ and all $\varepsilon > 0$.

Next, we want to prove that the function A_1 is additive. Replacing $(v_1, v_2,\cdots,v_m)$ by $(a^w v_1, a^w v_2,\cdots,a^w v_m)$ in (47), we obtain

$$I_{\mu,\nu}\left(\frac{1}{a^w}D\varphi(a^w v_1, a^w v_2,\cdots,a^w v_m),\varepsilon\right) \geq_{L^*} \Theta_{\alpha,\beta}(a^w v_1, a^w v_2,\cdots,a^w v_m, a^w\varepsilon), \tag{58}$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$. Taking the limit as $w \longrightarrow \infty$, we find that $I_{\mu,\nu}(DA_1(v_1, v_2,\cdots,v_m),\varepsilon)=1_{L^*}$ for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, which implies $DA_1(v_1, v_2,\cdots,v_m)=0$. Thus, A_1 satisfies the functional equation (3). Hence, the function $A_1 : V \longrightarrow W$ is additive. Passing to the limit as $w \longrightarrow \infty$ in (53), we have (48).

Finally, to show the uniqueness of the additive function A_1 subject to (48), consider that there exists another additive function A_1' which satisfies the inequality (48). Evidently, $A_1(a^w v)=a^w A_1(v)$ and $A_1'(a^w v)=a^w A_1'(v)$ for all $v \in V$ and $w \in \mathbb{N}$, from (48) and (46) that

$$\begin{aligned}
I_{\mu,\nu}\left(A_1(v)-A_1'(v),\varepsilon\right) &\geq_{L^*} I_{\mu,\nu}\left(A_1(a^w v)-A_1'(a^w v),a^w\varepsilon\right)\\
&\geq_{L^*} \Phi\left(I_{\mu,\nu}\left(A_1(a^w v)-\varphi(a^w v),\frac{a^w}{2}\varepsilon\right), I_{\mu,\nu}\left(\varphi(a^w v)-A_1'(a^w v),\frac{a^w}{2}\varepsilon\right)\right)\\
&\geq_{L^*} \Phi\left(\Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}v,0,\cdots,0,a^w\left(m^2-5m+4\right)\frac{\varepsilon}{2}\right)\right),\right.\\
&\left.\quad \Phi_{i=1}^{\infty}\left(\Theta_{\alpha,\beta}\left(a^{w+i-1}v,0,\cdots,0,a^w\left(m^2-5m+4\right)\frac{\varepsilon}{2}\right)\right)\right),
\end{aligned} \tag{59}$$

for all $v \in V$ and all $\varepsilon > 0$. By taking the limit $w \longrightarrow \infty$ in (59), we show the uniqueness of A_1. □

Corollary 14. *If an odd mapping $\varphi : V \longrightarrow W$ satisfies*

$$I_{\mu,\nu}(D\varphi(v_1, v_2,\cdots,v_m),\varepsilon) \geq_{L^*} I_{\mu',\nu'}\left(\sum_{i=1}^{m} v_i,\varepsilon\right), \tag{60}$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, and

$$\lim_{w\longrightarrow\infty} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{w+i-1}v, a^w\left(m^2-5m+4\right)\varepsilon\right)\right)=1_{L^*}, \tag{61}$$

for all $v \in V$ and all $\varepsilon > 0$. Then, there exists a unique additive mapping $A_1 : V \longrightarrow W$ such that

$$I_{\mu,\nu}(\varphi(v)-A_1(v),\varepsilon) \geq_{L^*} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{i-1}v,\left(m^2-5m+4\right)\varepsilon\right)\right), \tag{62}$$

for all $v \in V$ and all $\varepsilon > 0$.

Proof. By taking $\Theta_{\alpha,\beta}(v_1, v_2,\cdots,v_m,\varepsilon)=I_{\mu',\nu'}(\sum_{i=1}^{m} v_i,\varepsilon)$ in Theorem 13, we obtain our desired result. □

6. Stability Results for Mixed Case

Theorem 15. *Let $\alpha, \beta : V^m \longrightarrow D^+$ be mappings satisfying (27), (28), (45), and (46) for all $v_1, v_2, \cdots, v_m, v \in V$ and all $\varepsilon > 0$. If a mapping $\varphi : V \longrightarrow W$ with $\varphi(0)=0$ satisfies (29) for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, then there exist a unique quadratic mapping $Q_2 : V \longrightarrow W$ and a unique additive mapping $A_1 : V \longrightarrow W$ satisfying (3) and*

$$\begin{aligned}
I_{\mu,\nu}(\varphi(v)-Q_2(v)-A_1(v),\varepsilon) &\geq_{L^*} \Phi\Big(\Phi\big(\Phi_{i=1}^{\infty}\left(a^{i-1}v,0,\cdots,0,a^i\left(m^2-5m+2\right)\varepsilon\right)\big),\\
&\quad \Phi_{i=1}^{\infty}\left(-a^{i-1}v,0,\cdots,0,a^i\left(m^2-5m+2\right)\varepsilon\right)\big)\\
&\quad \cdot\big(\Phi\big(\Phi_{i=1}^{\infty}\left(a^{i-1}v,0,\cdots,0,\left(m^2-5m+4\right)\varepsilon\right)\big),\\
&\quad \Phi_{i=1}^{\infty}\left(-a^{i-1}v,0,\cdots,0,\left(m^2-5m+4\right)\varepsilon\right)\big)\big)\Big),
\end{aligned} \tag{63}$$

for all $v \in V$.

Proof. Let $\varphi_e(v)=\varphi(v)+\varphi(-v)/2$ for all $v \in V$. Thus, $\varphi_e(0)=0$, $\varphi_e(-v)=\varphi_e(v)$ and for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$,

$$\begin{aligned}
I_{\mu,\nu}(D\varphi_e(v_1, v_2,\cdots,v_m),\varepsilon) &= I_{\mu,\nu}\left(\frac{D\varphi(v_1, v_2,\cdots,v_m)+D\varphi(-v_1,-v_2,\cdots,-v_m)}{2},\varepsilon\right),\\
&\geq_{L^*} \Phi\left(I_{\mu,\nu}(D\varphi(v_1, v_2,\cdots,v_m),\varepsilon), I_{\mu,\nu}(D\varphi(-v_1,-v_2,\cdots,-v_m),\varepsilon)\right),\\
&\geq_{L^*} \Phi\left(\Theta_{\alpha,\beta}(v_1, v_2,\cdots,v_m,\varepsilon),\Theta_{\alpha,\beta}(-v_1,-v_2,\cdots,-v_m,\varepsilon)\right).,
\end{aligned} \tag{64}$$

By Theorem 11, there exists a quadratic mapping $Q_2 : V \longrightarrow W$ such that

$$\begin{aligned}
I_{\mu,\nu}(\varphi_e(v)-Q_2(v),\varepsilon) &\geq_{L^*} \Phi\big(\Phi_{i=1}^{\infty}\left(a^{i-1}v,0,\cdots,0,a^i\left(m^2-5m+2\right)\varepsilon\right),\\
&\quad \Phi_{i=1}^{\infty}\left(-a^{i-1}v,0,\cdots,0,a^i\left(m^2-5m+2\right)\varepsilon\right)\big),
\end{aligned} \tag{65}$$

for all $v \in V$ and all $\varepsilon > 0$. □

On the other hand, let $\varphi_o(v) = \varphi(v) - \varphi(-v)/2$ for all $v \in V$. Then $\varphi_o(0) = 0$, $\varphi_o(-v) = -\varphi_o(v)$. By Theorem 13, there exists a additive mapping $A_1 : V \longrightarrow W$ satisfies

$$\begin{aligned} I_{\mu,\nu}(\varphi_0(v) - A_1(v), \varepsilon) &\geq_{L^*} \Phi\left(\Phi_{i=1}^{\infty}\left(a^{i-1}v, 0,\cdots,0, \left(m^2 - 5m + 4\right)\varepsilon\right),\right. \\ &\quad \left.\Phi_{i=1}^{\infty}\left(-a^{i-1}v, 0,\cdots,0, \left(m^2 - 5m + 4\right)\varepsilon\right)\right), \end{aligned} \tag{66}$$

for all $v \in V$ and all $\varepsilon > 0$. From inequalities (65) and (66), we obtain our desired result (64).

Corollary 16. *If a mapping $\varphi : V \longrightarrow W$ satisfies*

$$I_{\mu,\nu}(D\varphi(v_1, v_2,\cdots,v_m, \varepsilon) \geq_{L^*} I_{\mu',\nu'}\left(\sum_{i=1}^{m} v_i, \varepsilon\right), \tag{67}$$

for all $v_1, v_2, \cdots, v_m \in V$ and all $\varepsilon > 0$, and

$$\begin{aligned} &\lim_{w\longrightarrow\infty} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{w+i-1}v, a^{2w+i}\left(m^2 - 5m + 2\right)\varepsilon\right)\right) = 1_{L^*}, \\ &\lim_{w\longrightarrow\infty} \Phi_{i=1}^{\infty}\left(I_{\mu',\nu'}\left(a^{w+i-1}v, a^{w}\left(m^2 - 5m + 4\right)\varepsilon\right)\right) = 1_{L^*}, \end{aligned} \tag{68}$$

for all $v \in V$ and all $\varepsilon > 0$. Then, there exists a unique quadratic mapping $Q_2 : V \longrightarrow W$ and a unique additive mapping $A_1 : V \longrightarrow W$ such that

$$\begin{aligned} I_{\mu,\nu}(\varphi(v) - Q_2(v) - A_1(v), \varepsilon) &\geq_{L^*} \Phi\left(\Phi\left(\Phi_{i=1}^{\infty}\left(a^{i-1}v, a^i\left(m^2 - 5m + 2\right)\varepsilon\right),\right.\right. \\ &\quad \left.\Phi_{i=1}^{\infty}\left(-a^{i-1}v, a^i\left(m^2 - 5m + 2\right)\varepsilon\right)\right) \\ &\quad \cdot \left(\Phi\left(\Phi_{i=1}^{\infty}\left(a^{i-1}v, \left(m^2 - 5m + 4\right)\varepsilon\right),\right.\right. \\ &\quad \left.\left.\left.\Phi_{i=1}^{\infty}\left(-a^{i-1}v, \left(m^2 - 5m + 4\right)\varepsilon\right)\right)\right)\right), \end{aligned} \tag{69}$$

for all $v \in V$ and all $\varepsilon > 0$.

7. Conclusion

In this paper, we introduced a new mixed type of additive-quadratic functional equation, and we applied the Hyers direct technique to investigate the Hyers-Ulam stability of the mixed type of additive-quadratic functional equation. Moreover, we have derived its general solution. The main objective of this work has been discussed: In Section 4, we have proved its Ulam-Hyers stability for the even case; in Section 5, examined Ulam-Hyers stability for odd case, and in Section 6, investigated Ulam-Hyers stability for the mixed cases, respectively, in intuitionistic random normed space.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Ćirić-Prešić Operators in Metric Spaces

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We show that the Prešić type operators of several variables can be regarded as an operator of a single variable and the fixed point problem of Prešić type can be regarded as a classical fixed point problem. We extend the recent result of Ćirić and Prešić by using the classical approach of Prešić. The key of the proof is based on the mappings introduced by Kada, Suzuki, and Takahashi. We also discuss the convergence problems of recursive real sequences and the Volterra integral equations as an application of our result.

Dedicated to the memory of Professor Wataru Takahashi

1. Introduction

Let (X, d) be a metric space and $k \geq 1$ be a fixed integer. Suppose that $T : X^k \longrightarrow X$ is given. The following two problems were studied by Prešić [1]:

(i) Find $\hat{x} \in X$ such that

$$\hat{x} = T(\hat{x}, \hat{x}, \cdots, \hat{x}). \tag{1}$$

(ii) For any given $x_1, x_2, \cdots, x_k \in X$, if we define

$$x_{n+k} = T(x_n, x_{n+1}, \cdots, x_{n+k-1}) \quad \text{for all } n \geq 1, \tag{2}$$

then what can we say about the convergence of the sequence $\{x_n\}$?

This problem is very interesting and was further studied by many authors, for example, see [2–4].

In 1965, Prešić [1] proved the following interesting theorem which can be regarded as a generalization of the classical fixed point theorem proposed by Banach [5]. Let k be a positive integer and let $q_1, q_2, \cdots, q_k$ be nonnegative real numbers such that $q_1 + q_2 + \cdots + q_k < 1$. Let $\mathscr{P}(q_1, q_2, \cdots, q_k)$ be a family of mappings $T : X^k \longrightarrow X$ such that

$$d(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1})) \leq \sum_{i=1}^{k} q_i d(u_i, u_{i+1}) \quad \text{for all } u_1, u_2, \cdots, u_{k+1} \in X. \tag{3}$$

Theorem P1. Let (X, d) be a complete metric space. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\liminf_{n \longrightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathscr{P}(q_1, q_2, \cdots, q_k)$ such that

$$d(T_n(u_1, u_2, \cdots, u_k), T_{n+1}(u_1, u_2, \cdots, u_k)) \leq \alpha_n \quad \text{for all } u_1, u_2, \cdots, u_k \in X \text{ and for all } n \geq 1. \tag{4}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \cdots, x_k \in X$ are arbitrary and

$$x_{n+k} = T_n(x_n, x_{n+1}, \cdots, x_{n+k-1}) \quad \text{for all } n \geq 1. \tag{5}$$

Then, the following statements are true.

(a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n\longrightarrow\infty} x_n$

(b) There exists an operator $T \in \mathscr{P}(q_1, q_2, \cdots, q_k)$ such that $\{T_n\}_{n=1}^{\infty}$ converges to T uniformly; that is, $\lim_{n\longrightarrow\infty} \sup_{x\in X} d(T_n x, Tx) = 0$

(c) The element $\hat{x}$ is the only one satisfying Equation (1)

Theorem P1, where $k = 1$, is nothing but Banach's fixed point theorem [5].

Remark 1. It was shown in [6] that the condition $\liminf_{n\longrightarrow\infty}(\alpha_{n+1}/\alpha_n) = 1$ is superfluous.

As a consequence of Theorem P1, we have the following result.

Theorem P2. Let (X, d) be a complete metric space and let k be a positive integer. Let $q_1, q_2, \cdots, q_k$ be nonnegative real numbers such that $q_1 + q_2 + \cdots + q_k < 1$. Let $T \in \mathscr{P}(q_1, q_2, \cdots, q_k)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \cdots, x_k \in X$ are arbitrary and Equation (2) holds. Then, the following statements are true.

(a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n\longrightarrow\infty} x_n$

(b) The element $\hat{x}$ is the only one satisfying Equation (1).

Ćirić and Prešić [2] proved the following improvement of Theorem P2.

Theorem CP. Let (X, d) be a complete metric space. Let k be a positive integer and let q be a positive real number such that $q < 1$. Let $T : X^k \longrightarrow X$ be such that

$$\begin{aligned} &d(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1})) \\ &\quad \leq q \max\{d(u_i, u_{i+1}) : i = 1, 2, \cdots, k\} \\ &\qquad \text{for all } u_1, u_2, \cdots, u_{k+1} \in X. \end{aligned} \tag{6}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \cdots, x_k \in X$ are arbitrary and Equation (2) holds. Then, the following statements are true.

(a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n\longrightarrow\infty} x_n$ and Equation (1) holds.

(b) If, in addition

$$d(T(u, u, \cdots, u), T(v, v, \cdots, v)) < d(u, v) \quad \text{for all } u, v \in X, \tag{7}$$

then the element $\hat{x}$ is the only one satisfying Equation (1).

Remark 2. It is clear that if $T : X^k \longrightarrow X$ satisfies Expression (3), then it satisfies Expressions (6) and (7).

Proof. Suppose that $T : X^k \longrightarrow X$ satisfies Expression (3) with nonnegative constants $q_1, q_2, \cdots, q_k$ such that $q_1 + q_2 + \cdots + q_k < 1$. To show that T satisfies Expression (6) with $q := \sum_{i=1}^{k} q_i$, let $u_1, u_2, \cdots, u_{k+1} \in X$. It follows that

$$\begin{aligned} &d(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1})) \\ &\quad \leq \sum_{i=1}^{k} q_i d(u_i, u_{i+1}) \leq q \max\{d(u_i, u_{i+1}) : i = 1, 2, \cdots, k\}. \end{aligned} \tag{8}$$

To see that T satisfies Equation (7), let $u, v \in X$. It follows that

$$\begin{aligned} &d(T(u, u, \cdots, u), T(v, v, \cdots, v)) \\ &\quad \leq d(T(u, u, \cdots, u), T(v, u, \cdots, u)) \\ &\qquad + d(T(v, u, \cdots, u), T(v, v, u, \cdots, u)) + \cdots \\ &\qquad + d(T(v, v, \cdots, v, u), T(v, v, \cdots, v)) \\ &\quad \leq (q_1 + q_2 + \cdots + q_k) d(u, v) < d(u, v). \end{aligned} \tag{9}$$

This completes the proof. □

It is natural to ask the following:

Question: *Is it possible to generalize Theorem CP by using the approach of Theorem P1?*

In this paper, we answer the question above by considering a wider class of mappings than those satisfying Expression (6). The class of mappings studied in this paper is motivated by the one in work of Kada et al. [7]. Some progress on these mappings can be found in [8].

Theorem KST (see [7]). Let (X, d) be a complete metric space. Let $r \in [0, 1)$ be given and let $T : X \longrightarrow X$ be a mapping such that the following conditions hold:

(1) **(KST1)** $d(Tx, T^2x) \leq r d(x, Tx)$ for all $x \in X$

(2) **(KST2)** If $y \neq Ty$, then $\inf\{d(x, Tx) + d(x, y) : x \in X\} > 0$

Then, every sequence $\{x_n\}_{n=1}^{\infty}$ with arbitrary x_1 and $x_{n+1} := Tx_n$ for all $n \geq 1$ converges to a fixed point of T.

2. Main Results

2.1. On the Fixed Point Problem of Prešić Type. We first show that the fixed point problem of Prešić type is equivalent to the classical fixed point problem. Suppose that X is a nonempty set and $k \geq 1$ is an integer. The *fixed point problem of Prešić type* for a given mapping $T : X^k \longrightarrow X$ is to find $u \in X$ such that

$$u = T(u, u, \cdots, u). \tag{10}$$

We denote by $\mathrm{PFix}(T)$ the set of all solutions of the problem above. We show that this problem is connected to the classical fixed point problem. To simplify the notation of the following result, we write $\mathbf{X} := X^k$ and $\dot{x} := (x, x, \cdots, x) \in \mathbf{X}$ where $x \in X$.

Theorem 3. *Suppose that X is a nonempty set and $k \geq 1$ is an integer. Suppose that $T : X^k \longrightarrow X$ is given. Then, there exists a mapping $\mathbf{T} : \mathbf{X} \longrightarrow \mathbf{X}$ such that the following statements hold.*

(a) *If $u \in PFix(T)$, then $\dot{u} \in Fix(\mathbf{T})$*

(b) *If $(u_1, u_2, \cdots, u_k) \in Fix(\mathbf{T})$, then $u_1 = u_2 = \cdots = u_k \in PFix(T)$*

Proof. Define $\mathbf{T} : \mathbf{X} \longrightarrow \mathbf{X}$ by

$$\mathbf{Tu} := (u_2, \cdots, u_k, T(u_1, \cdots, u_k)), \tag{11}$$

where $\mathbf{u} := (u_1, \cdots, u_k) \in \mathbf{X}$.

The statement (a) is trivial. We prove the statement (b). Suppose that $(u_1, u_2, \cdots, u_k) \in \mathrm{Fix}(\mathbf{T})$. It follows that

$$(u_1, u_2, \cdots, u_k) = (u_2, \cdots, u_k, T(u_1, \cdots, u_k)). \tag{12}$$

This implies that

$$u_1 = u_2 = \cdots = u_k = T(u_1, \cdots, u_k). \tag{13}$$

This completes the proof. □

2.2. On the Class of Mappings. In this subsection, we discuss the following classes of mappings. The first two classes are from Theorem P and Theorem CP and the last one from Theorem KST.

Suppose that $X := (X, d)$ is a metric space and $k \geq 1$ is an integer. Suppose that $r, q, q_1, \cdots, q_k$ are nonnegative real numbers. Define

$$\begin{aligned}
&\mathcal{F}(X, d, q_1, \cdots, q_k) := \left\{T : X^k \longrightarrow X \text{ satisfies } (3)\right\},\\
&\mathcal{G}(X, d, q) := \left\{T : X^k \longrightarrow X \text{ satisfies } (6)\right\},\\
&\mathcal{H}(X, d, r) := \{T : X \longrightarrow X \text{ satisfies Conditions (KST1)and (KST2)}\}.
\end{aligned} \tag{14}$$

Remark 4. Suppose that $X := (X, d)$ is a metric space and $k \geq 1$ is an integer. The following statements are true.

(a) If $q_1, \cdots, q_k$ are nonnegative real numbers, then $\mathcal{F}(X, d, q_1, \cdots, q_k) \subset \mathcal{G}(X, d, q)$, where $q := \sum_{i=1}^{k} q_i$

(b) If q is a nonnegative real number and $T \in \mathcal{G}(X, d, q)$, then $\mathbf{T} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$, where $\mathbf{X} := X^k$, $r = q^{1/k}$, $\mathbf{T}$ is defined by Equation (11) and

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) := \max\left\{q^{-i/k} d(x_i, y_i) : i = 1, \cdots, k\right\}$$
$$\text{for all } \mathbf{x} := (x_1, \cdots, x_k) \in \mathbf{X} \text{ and } \mathbf{y} := (y_1, \cdots, y_k) \in \mathbf{X}. \tag{15}$$

Proof. The statement (a) is trivial. We prove the statement (b). Let $T \in \mathcal{G}(X, d, q)$. It is clear that $\mathbf{d}$ is a metric on $\mathbf{X}$. We now prove that $\mathbf{T} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$. To see that $\mathbf{T}$ satisfies Condition (KST1), let $\mathbf{u} := (u_1, \cdots, u_k) \in \mathbf{X}$. We write $u_{k+1} := T(u_1, \cdots, u_k)$ and $u_{k+2} := T(u_2, \cdots, u_{k+1})$. It follows from $T \in \mathcal{G}(X, d, q)$ that

$$d(u_{k+1}, u_{k+2}) \leq q \max\{d(u_i, u_{i+1}) : i = 1, \cdots, k\}. \tag{16}$$

Note that

$$\begin{aligned}
\mathbf{Tu} &= (u_2, \cdots, u_{k+1}),\\
\mathbf{T}^2\mathbf{u} &= (u_3, \cdots, u_{k+2}).
\end{aligned} \tag{17}$$

Put $r := 1/s \in (0, 1)$ where $s = q^{-1/k}$. It follows from Equation (6) that

$$\begin{aligned}
\mathbf{d}(\mathbf{Tu}, \mathbf{T}^2\mathbf{u}) &= \max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k\right\}\\
&= \max\left\{\begin{matrix}\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\},\\ s^k d(u_{k+1}, u_{k+2})\end{matrix}\right\}\\
&= \max\left\{\begin{matrix}\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\},\\ s^k d(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1}))\end{matrix}\right\}\\
&\leq \max\left\{\begin{matrix}\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\},\\ s^k q \max\{d(u_i, u_{i+1}) : i = 1, \cdots, k\}\end{matrix}\right\}\\
&= \max\left\{\begin{matrix}\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\},\\ \max\{d(u_i, u_{i+1}) : i = 1, \cdots, k\}\end{matrix}\right\}\\
&= \max\left\{\begin{matrix}\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\},\\ d(u_1, u_2), \max\{d(u_i, u_{i+1}) : i = 2, \cdots, k\}\end{matrix}\right\}\\
&= \max\left\{d(u_1, u_2), \max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\}\right\}\\
&= \frac{1}{s}\max\left\{s d(u_1, u_2), s\max\left\{s^i d(u_{i+1}, u_{i+2}) : i = 1, \cdots, k-1\right\}\right\}\\
&= r\max\left\{s d(u_1, u_2), \max\left\{s^i d(u_i, u_{i+1}) : i = 2, \cdots, k\right\}\right\}\\
&= r\max\left\{s^i d(u_i, u_{i+1}) : i = 1, \cdots, k\right\}\\
&= r\max\left\{q^{-i/k} d(u_i, u_{i+1}) : i = 1, \cdots, k\right\} = r\mathbf{d}(\mathbf{u}, \mathbf{Tu}).
\end{aligned} \tag{18}$$

Finally, we show that $\mathbf{T}$ satisfies Condition (KST2). To see this, let $\{\mathbf{u}_n\}$ be a sequence in $\mathbf{X}$ and let $\mathbf{u} \in \mathbf{X}$ be such that $\lim_{n \longrightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{Tu}_n) = \lim_{n \longrightarrow \infty} \mathbf{d}(\mathbf{u}_n, \mathbf{u}) = 0$. For each $n \in \mathbb{N}$, we write $\mathbf{u}_n := (u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)})$ and $\mathbf{u} := (u_1, u_2, \cdots, u_k)$. It follows from the definition of $\mathbf{d}$ that

$$\lim_{n\longrightarrow\infty} d\left(u_i^{(n)}, u_{i+1}^{(n)}\right) = \lim_{n\longrightarrow\infty} d\left(u_i^{(n)}, u_i\right) = 0, \quad \text{for all } i = 1, 2, \cdots, k, \text{ where } u_{k+1}^{(n)} := T\left(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)}\right) \text{ for all } n \geq 1. \tag{19}$$

In particular, we have $u_1 = u_2 = \cdots = u_k$. This implies that $\mathbf{u} = (u_1, u_1, \cdots, u_1)$ and $\lim_{n\longrightarrow\infty} d(u_1, T(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)})) = 0$. We consider

$$\begin{aligned} d&\left(T\left(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)}\right), T(u_1, u_1, \cdots, u_1)\right) \\ &\leq d\left(T\left(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)}\right), T\left(u_2^{(n)}, u_3^{(n)}, \cdots, u_k^{(n)}, u_1\right)\right) \\ &\quad + d\left(T\left(u_2^{(n)}, u_3^{(n)}, \cdots, u_k^{(n)}, u_1\right), T\left(u_3^{(n)}, \cdots, u_k^{(n)}, u_1, u_1\right)\right) \\ &\quad + \cdots + d\left(T\left(u_k^{(n)}, u_1, \cdots, u_1\right), T(u_1, u_1, \cdots, u_1)\right) \\ &\leq q \max\left\{d\left(u_k^{(n)}, u_1\right), d\left(u_i^{(n)}, u_{i+1}^{(n)}\right): i = 1, \cdots, k-1\right\} \\ &\quad + q \max\left\{d\left(u_k^{(n)}, u_1\right), d\left(u_i^{(n)}, u_{i+1}^{(n)}\right): i = 2, \cdots, k-1\right\} \\ &\quad + \cdots + qd\left(u_k^{(n)}, u_1\right). \end{aligned} \tag{20}$$

Hence, $\lim_{n\longrightarrow\infty} d(T(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)}), T(u_1, u_1, \cdots, u_1)) = 0$. Since

$$\lim_{n\longrightarrow\infty} d\left(T\left(u_1^{(n)}, u_2^{(n)}, \cdots, u_k^{(n)}\right), u_1\right) = \lim_{n\longrightarrow\infty} d\left(u_{k+1}^{(n)}, u_1\right) = 0, \tag{21}$$

we have $u_1 = T(u_1, u_1, \cdots, u_1)$. Hence, $\mathbf{u} = \mathbf{T}\mathbf{u}$. This completes the proof. □

Remark 5. The classes $\mathcal{F}(X, d, q_1, \cdots, q_k)$ and $\mathcal{G}(X, d, q)$ are closed under the pointwise convergence, that is, if $\{T_n\}$ is a sequence in $\mathcal{F}(X, d, q_1, \cdots, q_k)$ ($\mathcal{G}(X, d, q)$, respectively) and there exists a mapping $T : X^k \longrightarrow X$ such that

$$\begin{aligned} &\lim_{n\longrightarrow\infty} d(T_n(u_1, u_2, \cdots, u_k), T(u_1, u_2, \cdots, u_k)) \\ &\quad = 0 \quad \text{for all } (u_1, u_2, \cdots, u_k) \in X^k, \end{aligned} \tag{22}$$

then $T \in \mathcal{F}(X, d, q_1, \cdots, q_k)$ ($T \in \mathcal{G}(X, d, q)$, respectively).

Proof. Suppose that $\{T_n\}$ is a sequence in $\mathcal{G}(X, d, q)$ and there exists a mapping $T : X \longrightarrow X$ such that

$$\begin{aligned} &\lim_{n\longrightarrow\infty} d(T_n(u_1, u_2, \cdots, u_k), T(u_1, u_2, \cdots, u_k)) \\ &\quad = 0 \quad \text{for all } (u_1, u_2, \cdots, u_k) \in X^k. \end{aligned} \tag{23}$$

We prove that $T \in \mathcal{G}(X, d, q)$. To see this, let $u_1, u_2, \cdots, u_k, u_{k+1} \in X$. Note that

$$\begin{aligned} &\lim_{n\longrightarrow\infty} d(T(u_1, u_2, \cdots, u_k), T_n(u_1, u_2, \cdots, u_k)) \\ &\quad = \lim_{n\longrightarrow\infty} d(T_n(u_2, u_3, \cdots, u_{k+1}), T(u_2, u_3, \cdots, u_{k+1})) = 0. \end{aligned} \tag{24}$$

It follows that

$$\begin{aligned} d&(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1})) \\ &\leq d(T(u_1, u_2, \cdots, u_k), T_n(u_1, u_2, \cdots, u_k)) \\ &\quad + d(T_n(u_1, u_2, \cdots, u_k), T_n(u_2, u_3, \cdots, u_{k+1})) \\ &\quad + d(T_n(u_2, u_3, \cdots, u_{k+1}), T(u_2, u_3, \cdots, u_{k+1})) \\ &\leq d(T(u_1, u_2, \cdots, u_k), T_n(u_1, u_2, \cdots, u_k)) \\ &\quad + q \max\{d(u_i, u_{i+1}): i = 1, 2, \cdots, k\} \\ &\quad + d(T_n(u_2, u_3, \cdots, u_{k+1}), T(u_2, u_3, \cdots, u_{k+1})). \end{aligned} \tag{25}$$

In particular,

$$\begin{aligned} &d(T(u_1, u_2, \cdots, u_k), T(u_2, u_3, \cdots, u_{k+1})) \\ &\quad \leq q \max\{d(u_i, u_{i+1}): i = 1, 2, \cdots, k\}. \end{aligned} \tag{26}$$

This implies that $T \in \mathcal{G}(X, d, q)$. The case that $\{T_n\}$ is a sequence in $\mathcal{F}(X, d, q_1, q_2, \cdots, q_k)$ can be done similarly. □

2.3. A Generalization of Theorem KST and Its Consequences. The following result is analogous to Theorem P1 with a wider class of mappings.

Theorem 6. *Let (X, d) be a complete metric space and $r \in [0, 1)$ be given. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{H}(X, d, r)$ such that*

$$d(T_n x, T_{n+1} x) \leq \alpha_n \quad \text{for all } x \in X. \tag{27}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1 \in X$ is arbitrary and $x_{n+1} := T_n x_n$ for all $n \geq 1$. Then, the following statements are true.

(a) *There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n\longrightarrow\infty} x_n$*

(b) *There exists an operator $T : X \longrightarrow X$ such that $\{T_n\}$ converges to T uniformly*

(c) *If T satisfies Condition (KST2), then $\hat{x} \in Fix(T)$*

Lemma 7. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be two sequences of non-negative real numbers such that $\sum_{n=1}^{\infty}\alpha_n < \infty$. Let $r \in (0, 1)$ be given. If*

$$s_{n+1} \leq rs_n + \alpha_n \quad \text{for all } n \geq 1, \tag{28}$$

then $\sum_{n=1}^{\infty} s_n < \infty$.

Proof. Note that, for each $m \geq 1$, we have $\sum_{n=2}^{m+1} s_n \leq r\sum_{n=1}^{m} s_n + \sum_{n=1}^{m}\alpha_n$. In particular, $(1-r)\sum_{n=2}^{m} s_n \leq (1-r)\sum_{n=2}^{m} s_n + s_{m+1} \leq rs_1 + \sum_{n=1}^{m}\alpha_n \leq rs_1 + \sum_{n=1}^{\infty}\alpha_n$ for all $m \geq 2$. Hence, the conclusion follows. ☐

Proof of Theorem 6. First, we note that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(T_n x_n, T_{n+1}x_{n+1}) \leq d(T_n x_n, T_n x_{n+1}) \\ &\quad + d(T_n x_{n+1}, T_{n+1}x_{n+1}) \\ &= d\left(T_n x_n, T_n^2 x_n\right) + d(T_n x_{n+1}, T_{n+1}x_{n+1}) \\ &\leq rd(x_n, T_n x_n) + \alpha_n \\ &= rd(x_n, x_{n+1}) + \alpha_n. \end{aligned} \tag{29}$$

It follows from Lemma 7 that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. In particular, $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n\longrightarrow\infty} x_n = \hat{x}$ for some $\hat{x} \in X$ by the completeness of X. Hence, (a) is proved.

We now prove (b). For each $x \in X$ and for each $n, k \geq 1$, we note that

$$d(T_n x, T_{n+k}x) \leq \sum_{j=n}^{n+k-1} d\left(T_j x, T_{j+1}x\right) \leq \sum_{j=n}^{n+k-1} \alpha_j \leq \sum_{j=n}^{\infty} \alpha_j. \tag{30}$$

It follows that $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence and hence $\lim_{n\longrightarrow\infty} T_n x$ exists. We then define $T : X \longrightarrow X$ by

$$Tx \coloneqq \lim_{n\longrightarrow\infty} T_n x. \tag{31}$$

It follows then that

$$\begin{aligned} d(T_n x, Tx) &\leq \lim_{k\longrightarrow\infty} d(T_n x, T_{n+k}x) \leq \lim_{k\longrightarrow\infty} \sum_{j=n}^{n+k-1} d\left(T_j x, T_{j+1}x\right) \\ &\leq \lim_{k\longrightarrow\infty} \sum_{j=n}^{n+k-1} \alpha_j = \sum_{j=n}^{\infty} \alpha_j \quad \text{for all } x \in X \text{ and for all } n \geq 1. \end{aligned} \tag{32}$$

To see that $\lim_{n\longrightarrow\infty} \sup_{x\in X} d(T_n x, Tx) = 0$, let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty}\alpha_n < \infty$, there exists an integer M such that $\sum_{j=n}^{\infty}\alpha_j < \varepsilon$ for all $n \geq M$. For each $x \in X$ and $n \geq M$, we have

$$d(T_n x, Tx) \leq \sum_{j=n}^{\infty} \alpha_j < \varepsilon. \tag{33}$$

That is, $\sup_{x\in X} d(T_n x, Tx) \leq \varepsilon$ for all $n \geq M$. Hence,

$$\lim_{n\longrightarrow\infty} \sup_{x\in X} d(T_n x, Tx) = 0. \tag{34}$$

Finally, we assume that T satisfies (KST2). Note that $\lim_{n\longrightarrow\infty} d(x_n, \hat{x}) = 0$. Moreover,

$$\begin{aligned} \lim_{n\longrightarrow\infty} d(x_n, Tx_n) &\leq \lim_{n\longrightarrow\infty} (d(x_n, T_n x_n) + d(T_n x_n, Tx_n)) \\ &\leq \lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) + \lim_{n\longrightarrow\infty} \sum_{j=n}^{\infty} \alpha_j = 0. \end{aligned} \tag{35}$$

It follows from Condition (KST2) that $\hat{x} = T\hat{x}$.

We are now ready to give an affirmative answer of the problem in the introduction. In fact, we can generalize Theorem CP by using the approach of Theorem P1.

Theorem 8. *Let (X, d) be a complete metric space and $k \geq 1$ be a fixed integer. Let q be a positive real number such that $q < 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\alpha_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{G}(X, d, q)$ such that*

$$\begin{aligned} &d(T_n(u_1, u_2, \cdots, u_k), T_{n+1}(u_1, u_2, \cdots, u_k)) \\ &\quad \leq \alpha_n \quad \text{for all } u_1, u_2, \cdots, u_k \in X \text{ and for all } n \geq 1. \end{aligned} \tag{36}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $x_1, x_2, \cdots, x_k \in X$ are arbitrary and

$$x_{n+k} \coloneqq T_n(x_n, x_{n+1}, \cdots, x_{n+k-1}) \quad \text{for all } n \geq 1. \tag{37}$$

Then, the following statements are true.

(a) There exists an element $\hat{x} \in X$ such that $\hat{x} = \lim_{n\longrightarrow\infty} x_n$

(b) There exists an operator $T \in \mathcal{G}(X, d, q)$ such that $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T and $\hat{x} \in PFix(T)$

(c) $PFix(T) = \{\hat{x}\}$, that is, $\hat{x}$ is the only solution of the fixed point problem of Prešić type for T, provided that

$$d(T(u, u, \cdots, u), T(v, v, \cdots, v)) < d(u, v) \quad \text{for all } u, v \in X. \tag{38}$$

Proof. Suppose that $\{\alpha_n\}$, $\{T_n\}$, and $\{x_n\}$ are given as in the statement of the theorem. Let $\mathbf{X}$ and $\mathbf{d}$ be defined as in Theorem 3 and Remark 4. For each $n \geq 1$, we define $\mathbf{T}_n : \mathbf{X} \longrightarrow \mathbf{X}$ by

$$\mathbf{T}_n\mathbf{u} \coloneqq (u_2, \cdots, u_k, T_n(u_1, \cdots, u_k)), \tag{39}$$

where $\mathbf{u} \coloneqq (u_1, \cdots, u_k) \in \mathbf{X}$. By Remark 4, we have $\mathbf{T}_n \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$ for all $n \geq 1$. Define $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{X}$ by

$$\mathbf{x}_n := (x_n, x_{n+1}, \cdots, x_{n+k-1}) \quad \text{for all } n \geq 1. \tag{40}$$

We prove that $\mathbf{x}_{n+1} := \mathbf{T}_n\mathbf{x}_n$ for each $n \geq 1$. To see this, we have

$$\begin{aligned}\mathbf{x}_{n+1} &= (x_{n+1}, x_{n+2}, \cdots, x_{n+k}) \\ &= (x_{n+1}, x_{n+2}, \cdots, T_n(x_n, x_{n+1}, \cdots, x_{n+k-1})) \\ &= \mathbf{T}_n\mathbf{x}_n.\end{aligned} \tag{41}$$

Next, we prove that

$$\mathbf{d}(\mathbf{T}_n\mathbf{u}, \mathbf{T}_{n+1}\mathbf{u}) \leq \frac{\alpha_n}{q} \quad \text{for all } n \geq 1 \text{ and } \mathbf{u} \in \mathbf{X}. \tag{42}$$

In fact, for each $n \geq 1$ and $\mathbf{u} := (u_1, \cdots, u_k) \in \mathbf{X}$, we have

$$\begin{aligned}\mathbf{d}(\mathbf{T}_n\mathbf{u}, \mathbf{T}_{n+1}\mathbf{u}) &= \mathbf{d}((u_2, u_3, \cdots, u_k, T_n(u_1, \cdots, u_k)), \\ &\quad \cdot (u_2, u_3, \cdots, u_k, T_{n+1}(u_1, \cdots, u_k))) \\ &= q^{-1} d(T_n(u_1, \cdots, u_k), T_{n+1}(u_1, \cdots, u_k)) \\ &\leq \frac{\alpha_n}{q}.\end{aligned} \tag{43}$$

It follows from Theorem 6 that there exists an element $\widehat{\mathbf{x}} := (\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_k) \in \mathbf{X}$ such that $\widehat{\mathbf{x}} = \lim_{n\longrightarrow\infty}\mathbf{x}_n$. In particular, we have $\lim_{n\longrightarrow\infty}x_n = \widehat{x}_1 = \widehat{x}_2 = \cdots = \widehat{x}_k$. It also follows from Theorem 6 that there exists an operator $\mathbf{S} : \mathbf{X} \longrightarrow \mathbf{X}$ such that $\{\mathbf{T}_n\}$ converges uniformly to $\mathbf{S}$. Note that, for each $(u_1, u_2, \cdots, u_k) \in \mathbf{X}$, we have

$$\begin{aligned}\mathbf{S}(u_1, u_2, \cdots, u_k) &= \lim_{n\longrightarrow\infty} \mathbf{T}_n(u_1, u_2, \cdots, u_k) \\ &= \lim_{n\longrightarrow\infty} (u_2, u_3, \cdots, u_k, T_n(u_1, u_2, \cdots, u_k)) \\ &= \left(u_2, u_3, \cdots, u_k, \lim_{n\longrightarrow\infty} T_n(u_1, u_2, \cdots, u_k)\right).\end{aligned} \tag{44}$$

We now define $T : \mathbf{X} \longrightarrow X$ (from $\mathbf{S}$) by

$$T(u_1, u_2, \cdots, u_k) := v_k \overset{\text{def}}{\Longleftrightarrow} \mathbf{S}(u_1, u_2, \cdots, u_k) := (v_1, v_2, \cdots, v_k). \tag{45}$$

In particular, we have

(i) $T(u_1, u_2, \cdots, u_k) = \lim_{n\longrightarrow\infty} T_n(u_1, u_2, \cdots, u_k)$,

(ii) $\mathbf{S}(u_1, u_2, \cdots, u_k) = (u_2, u_3, \cdots, u_k, T(u_1, u_2, \cdots, u_k))$ for all $(u_1, u_2, \cdots, u_k) \in \mathbf{X}$.

It follows from each $T_n \in \mathcal{G}(X, d, q)$ and Remark 5 that $T \in \mathcal{G}(X, d, q)$. Hence $\mathbf{S} \in \mathcal{H}(\mathbf{X}, \mathbf{d}, r)$. In particular, $\mathbf{S}$ satisfies Condition KST2 and hence $\widehat{\mathbf{x}} = \mathbf{S}\widehat{\mathbf{x}}$ by Theorem 6. This implies that

$$(\widehat{x}_1, \widehat{x}_1, \cdots, \widehat{x}_1) = \mathbf{S}(\widehat{x}_1, \widehat{x}_1 \cdots, \widehat{x}_1) = (\widehat{x}_1, \widehat{x}_1, \cdots, \widehat{x}_1, T(\widehat{x}_1, \widehat{x}_1 \cdots, \widehat{x}_1)). \tag{46}$$

Hence,

$$T(\widehat{x}_1, \widehat{x}_1 \cdots, \widehat{x}_1) = \widehat{x}_1. \tag{47}$$

The uniqueness of the fixed point of Prešić type is obvious if the additional hypothesis is assumed. ☐

3. Applications

We finally discuss some applications of our result.

3.1. Some Convergence Problem of Recursive Real Sequences. We reconsider the following example studied by Ćirić and Prešić [2, Example 1] and give some remark.

Example 9. Let $X := [0, 1] \cup [2, 3]$ be a metric space endowed with the usual metric d and $T : X^2 \longrightarrow X$ be defined by

$$T(x, y) := \begin{cases} \dfrac{x+y}{4} & \text{if } (x, y) \in [0, 1]^2, \\ 1 + \dfrac{x+y}{4} & \text{if } (x, y) \in [2, 3]^2, \\ \dfrac{x+y}{4} - \dfrac{1}{2} & \text{if} (x, y) \in ([0, 1] \times [2, 3]) \cup ([2, 3] \times [1, 2]). \end{cases} \tag{48}$$

It was proved in [2] that

$$d(T(x, y), T(y, z)) \leq \frac{1}{2} \max \{d(x, y), d(y, z)\} \quad \text{for all } x, y \in X \text{ and } z := T(x, y). \tag{49}$$

The author of [2] claimed that $T \in \mathcal{G}(X, d, 1/2)$. We note that $T \notin \mathcal{G}(X, d, q)$ for all $0 < q < 1$. In fact, let $x := 3$, $y := 2$, and $z := 1$; it follows that

$$\begin{aligned}d(T(x, y), T(y, z)) &= d\left(\frac{9}{4}, \frac{1}{4}\right) \\ &= 2 \text{ while } \max \{d(x, y), d(y, z)\} \\ &= 1.\end{aligned} \tag{50}$$

Using our approach, let $\mathbf{X} := X^2$ and

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) := \max \left\{\sqrt{2}d(x_1, x_2), 2d(y_1, y_2)\right\}, \tag{51}$$

where $\mathbf{x} := (x_1, x_2) \in \mathbf{X}$ and $\mathbf{y} := (y_1, y_2) \in \mathbf{X}$. Moreover, let $\mathbf{T} : \mathbf{X} \longrightarrow \mathbf{X}$ be defined by

$$\mathbf{Tx} := (x_2, T(x_1, x_2)), \tag{52}$$

where $\mathbf{x} := (x_1, x_2) \in \mathbf{X}$. We can follow the proof in Remark 4 to show that

$$\mathbf{d}\left(\mathbf{Tx}, \mathbf{T}^2\mathbf{x}\right) \leq \frac{1}{\sqrt{2}}\mathbf{d}(\mathbf{x}, \mathbf{Tx}) \quad \text{for all } \mathbf{x} \in \mathbf{X}. \tag{53}$$

Next, we show that $\mathbf{T}$ satisfies Condition (KST2). Note

Table 1: A numerical experiment for Example 10.

x_1	0.5	1	1	2	3
x_2	1	1	2	3	3
x_3	0.375	0.5	0.25	2.25	2.5
x_4	0.34375	0.375	0.0625	2.3125	2.375
x_5	0.1796875	0.21875	0.078125	2.140625	2.21875
x_6	0.130859375	0.1484375	0.03515625	2.11328125	2.1484375
x_7	0.077636719	0.091796875	0.028320313	2.063476563	2.091796875
x_8	0.052124023	0.060058594	0.015869141	2.044189453	2.060058594
x_9	0.032440186	0.037963867	0.011047363	2.026916504	2.037963867
x_{10}	0.021141052	0.024505615	0.006729126	2.017776489	2.024505615
x_{11}	0.013395309	0.015617371	0.004444122	2.011173248	2.015617371
x_{12}	0.00863409	0.010030746	0.002793312	2.007237434	2.010030746
x_{13}	0.00550735	0.006412029	0.001809359	2.004602671	2.006412029
x_{14}	0.00353536	0.004110694	0.001150668	2.002960026	2.004110694
x_{15}	0.002260678	0.002630681	0.000740007	2.001890674	2.002630681
x_{16}	0.001449009	0.001685344	0.000472669	2.001212675	2.001685344
x_{17}	0.000927422	0.001079006	0.000303169	2.000775837	2.001079006
x_{18}	0.000594108	0.000691087	0.000193959	2.000497128	2.000691087
x_{19}	0.000380382	0.000442523	0.000124282	2.000318241	2.000442523
x_{20}	0.000243623	0.000283403	$7.95603E-05$	2.000203842	2.000283403

that $\mathrm{PFix}(T)=\{0,2\}$ so $\mathrm{Fix}(\mathbf{T})=\{(0,0),(2,2)\}$. Let $\{\mathbf{u}_n\}$ be a sequence in $\mathbf{X}$ and let $\mathbf{u}\in\mathbf{X}$ be such that $\lim_{n\longrightarrow\infty}\mathbf{d}(\mathbf{u}_n,\mathbf{T}\mathbf{u}_n)=\lim_{n\longrightarrow\infty}\mathbf{d}(\mathbf{u}_n,\mathbf{u})=0$. For each $n\in\mathbb{N}$, we write $\mathbf{u}_n:=(u_1^{(n)},u_2^{(n)})$ and $\mathbf{u}:=(u_1,u_2)$. It follows from the definition of $\mathbf{d}$ that

$$\begin{aligned}\lim_{n\longrightarrow\infty} d\left(u_1^{(n)},u_2^{(n)}\right)&=\lim_{n\longrightarrow\infty} d\left(u_2^{(n)},T\left(u_1^{(n)},u_2^{(n)}\right)\right)\\&=\lim_{n\longrightarrow\infty} d\left(u_1^{(n)},u_1\right)\\&=\lim_{n\longrightarrow\infty} d\left(u_2^{(n)},u_2\right)=0.\end{aligned}\tag{54}$$

In particular, we have $u_1=u_2$. This implies that $\mathbf{u}=(u_1,u_1)$ and $\lim_{n\longrightarrow\infty}d(u_1,T(u_1^{(n)},u_2^{(n)}))=0$. We consider the following two cases.

Case 1. $u_1\in[0,1]$. In this case, we may assume that $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are sequences in $[0,1]$ and hence $T(u_1^{(n)},u_2^{(n)})=(u_1^{(n)}+u_2^{(n)})/4$. This implies that $u_1=(u_1+u_2)/4=u_1/2$ and hence $u_1=0$. Then, $\mathbf{u}:=(u_1,u_2)=(u_1,u_1)=(0,0)\in\mathrm{Fix}(\mathbf{T})$.

Case 2. $u_1\in[2,3]$. In this case, we may assume that $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are sequences in $[2,3]$ and hence $T(u_1^{(n)},u_2^{(n)})=1+(u_1^{(n)}+u_2^{(n)})/4$. This implies that $u_1=1+(u_1+u_2)/4=1+u_1/2$ and hence $u_1=2$. Then $\mathbf{u}:=(u_1,u_2)=(u_1,u_1)=(2,2)\in\mathrm{Fix}(\mathbf{T})$.

Hence, $\mathbf{T}\in\mathscr{H}(\mathbf{X},\mathbf{d},1/\sqrt{2})$. In particular, we can apply this example to our Theorem 8. Note that, since $\mathrm{PFix}(T)=\{0,2\}$, the condition (7) cannot be omitted for the uniqueness of the solution as claimed in [2].

We now discuss the following convergence problem of real sequences inspired by [6].

Example 10. Suppose that $\{x_n\}$ is a real sequence satisfying the following recursive relation: $x_1,x_2\in[0,1]\cup[2,3]$ and for each $n\geq 1$

$$x_{n+2}:=\begin{cases}\dfrac{x_n+x_{n+1}}{4} & \text{if } (x_n,x_{n+1})\in[0,1]^2,\\ 1+\dfrac{x_n+x_{n+1}}{4} & \text{if } (x_n,x_{n+1})\in[2,3]^2,\\ \dfrac{x_n+x_{n+1}}{4}-\dfrac{1}{2} & \text{if } (x_n,x_{n+1})\in([0,1]\times[2,3])\cup([2,3]\times[0,1]).\end{cases}\tag{55}$$

It is clear that this example is related to the preceding one. Table 1 shows the numerical experiment regarding to the problem with respect to the initial inputs x_1 and x_2. For example, if $x_1:=0.5$ and $x_2:=1$, then $\lim_{n\longrightarrow\infty}x_n=0$; if $x_1:=2$ and $x_2:=3$, then $\lim_{n\longrightarrow\infty}x_n=2$.

3.2. Volterra Type Integral Equations

We discuss a further application of our Theorem 8 in the context of Volterra type integral equations.

Theorem 12. *Suppose that $X := C([0, T], \mathbb{R})$ is the space of continuous real-valued functions defined on an interval $[0, T]$, where $T > 0$, equipped with the supremum metric d defined by*

$$d(u, v) := \sup \{|u(t) - v(t)|: t \in [0, T]\} \quad \text{for all } u, v \in X. \tag{56}$$

Suppose that $\lambda, \mu : [0, T] \times \mathbb{R} \longrightarrow [0,\infty)$ and $f, g : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that

(i) f and g satisfy the Lipschitz condition for the second argument with constants α and β, respectively, that is,

$$\left|f(t, s) - f\left(t, s'\right)\right| \leq \alpha|s - s'| \text{ and } |g(t, s) - g\left(t, s'\right)| \leq \beta|s - s'| \quad \text{for all } t \in [0, T] \text{ and for all } s, s' \in \mathbb{R}, \tag{57}$$

(ii) $q := \alpha \max_{t\in[0,T]} \int_0^T \lambda(t, s)ds + \beta \max_{t\in[0,T]} \int_0^T \mu(t, s)ds < 1$.

Suppose that $\{h_n\}$ is a sequence in X such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ where $\alpha_n := d(h_n, h_{n+1})$ for all $n \geq 1$. If $u_1, u_2 \in X$ and for each $n \geq 1$

$$\begin{aligned} u_{n+2}(t) := h_n(t) &+ \int_0^T \lambda(t, s)f(s, u_n(s))ds \\ &+ \int_0^T \mu(t, s)g(s, u_{n+1}(s))ds (t \in [0, T]), \end{aligned} \tag{58}$$

then $u_n \in X$ for all $n \geq 3$ and there exists an element $\widehat{u} \in X$ such that $\lim_{n\longrightarrow\infty} d(u_n, \widehat{u}) = 0$ and $\widehat{u}$ is a unique solution of the following Volterra type integral equation

$$\begin{aligned} u(t) = h(t) &+ \int_0^T \lambda(t, s)f(s, u(s))ds \\ &+ \int_0^T \mu(t, s)g(s, u(s))ds (t \in [0, T]), \end{aligned} \tag{59}$$

where $h(t) := \lim_{n\longrightarrow\infty} h_n(t)$.

Proof. Now, for each $n \geq 1$, we define $F_n : X \times X \longrightarrow X$, by for each $u, v \in X$,

$$\begin{aligned} F_n(u, v)(t) := h_n(t) &+ \int_0^T \lambda(t, s)f(s, u(s))ds \\ &+ \int_0^T \mu(t, s)g(s, v(s))ds (t \in [0, T]). \end{aligned} \tag{60}$$

To apply our Theorem 8, it is sufficient to prove that

(a) $F_n \in \mathcal{G}(X, d, q)$ for all $n \geq 1$

(b) $d(F_n(u_1, u_2), F_{n+1}(u_1, u_2)) = \alpha_n$ for all $n \geq 1$ and for all $u_1, u_2 \in X$

We now prove the statements (a) and (b).

(a) Let $n \geq 1$ and let $u_1, u_2, u_3 \in X$. Then,

$$\begin{aligned} d(F_n(u_1, u_2), F_n(u_2, u_3)) &= \sup_{t\in[0,T]} |F_n(u_1, u_2)(t) - F_n(u_2, u_3)(t)| \\ &\leq \sup_{t\in[0,T]} \int_0^T \lambda(t, s)|f(s, u_1(s)) - f(s, u_2(s))|ds \\ &\quad + \sup_{t\in[0,T]} \int_0^T \mu(t, s)|g(s, u_2(s)) - g(s, u_3(s))|ds \\ &\leq \left(\alpha \max_{t\in[0,T]} \int_0^T \lambda(t, s)ds)\right) d(u_1, u_2) \\ &\quad + \left(\beta \max_{t\in[0,T]} \int_0^T \mu(t, s)ds\right) d(u_2, u_3) \\ &\leq q \max \{d(u_1, u_2), d(u_2, u_3)\}. \end{aligned} \tag{61}$$

(b) Let $n \geq 1$ and let $u_1, u_2 \in X$. Then,

$$\begin{aligned} d(F_n(u_1, u_2), F_{n+1}(u_1, u_2)) &= \sup_{t\in[0,T]} |F_n(u_1, u_2)(t) - F_{n+1}(u_1, u_2)(t)| \\ &= \sup_{t\in[0,T]} |h_n(t) - h_{n+1}(t)| \\ &= d(h_n, h_{n+1}) \\ &= \alpha_n. \end{aligned} \tag{62}$$

Note that

$$u_{n+2}(t) = F_n(u_n(t), u_{n+1}(t))(t \in [0, T]). \tag{63}$$

It follows from our Theorem 8, where $k = 2$, there exists an element $\widehat{u} \in X$ such that

$$\lim_{n\longrightarrow\infty} d(u_n, \widehat{u}) = 0. \tag{64}$$

Moreover,

$$\widehat{u} = F(\widehat{u}, \widehat{u})(\Longleftrightarrow \widehat{u} \text{ is a solution of } (11)), \tag{65}$$

where $F : X \times X \longrightarrow X$ is defined by

$$\begin{aligned} F(u, v)(t) := h(t) &+ \int_0^T \lambda(t, s)f(s, u(s))ds \\ &+ \int_0^T \mu(t, s)g(s, v(s))ds\, (t \in [0, T]), \end{aligned} \tag{66}$$

and $h(t) = \lim_{n\longrightarrow\infty} h_n(t)$. Finally, to show that $\widehat{u}$ is a unique solution of the Volterra type integral Equation (59), we show that $d(F(u, u), F(v, v)) < d(u, v)$ for all $u, v \in X$. Let $u, v \in X$. Then,

$$\begin{aligned}
d(F(u,u),F(v,v)) &= \sup_{t\in[0,T]}|F(u,u)(t)-F(v,v)(t)|\\
&\le \sup_{t\in[0,T]}\int_0^T \lambda(t,s)|f(s,u(s))-f(s,v(s))|ds\\
&\quad+\sup_{t\in[0,T]}\int_0^T \mu(t,s)|g(s,u(s))-g(s,v(s))|ds\\
&\le\left(\alpha\max_{t\in[0,T]}\int_0^T\lambda(t,s)ds)\right)d(u,v)\\
&\quad+\left(\beta\max_{t\in[0,T]}\int_0^T\mu(t,s)ds\right)d(u,v)\\
&= qd(u,v)<d(u,v).
\end{aligned} \tag{67}$$

This completes the proof. □

4. Conclusion

We show that the fixed point problem of Prešić type (with respect to several variables) can be regarded a fixed point problem (of a single variable) by using a product space approach. With an appropriate metric on the product space, the Ćirić-Prešić operator can be regarded a mapping studied by Kada et al. in the product space. In particular, we deduce the fixed point result under a weaker assumption. We apply our result for the convergence problems of recursive real sequences and the Volterra type integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. They read and approved the final manuscript.

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Research Article

Existence and Numerical Analysis of Imperfect Testing Infectious Disease Model in the Sense of Fractional-Order Operator

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In the present paper, we study a mathematical model of an imperfect testing infectious disease model in the sense of the Mittage-Leffler kernel. The Banach contraction principle has been used for the existence and uniqueness of solutions of the suggested model. Furthermore, a numerical method equipped with Lagrangian polynomial interpolation has been utilized for the numerical outcomes. Diagramming and discussion are used to clarify the effects of related parameters in the fractional-order imperfect testing infectious disease model.

1. Introduction

The aggregate of human microbiota is called human microbiome, including viruses, bacteria, protists, archaea, and fungi. These microbiomes live in or on our body including the skin, placenta, mammary glands, seminal fluid, ovarian follicles, uterus, lung, oral mucosa, conjunctiva, saliva, biliary, and gastrointestinal tract [1]. A number of infectious maladies are caused by these mirobiome such as influenza, malaria, dengue, Ebola, COVID-19, HIV/AIDS, rabies, syphilis, yellow fever, hepatitis, Zika virus infection, and tuberculosis [2]. Yearly, 9.2 million people died due to infectious diseases [3, 4]. Due to this life-threatening situation, health departments spend more money to reduce the outbreak of infectious maladies. Several techniques were applied to minimize the exposure of infectious diseases, such as prevention measures, screening, testing diagnostics, education, and counseling. Among all of these techniques, testing diagnostics is a very useful tool to identify the infected individuals and susceptible. For the laboratory test, a sample is required such as a stool, tissue, cerebrospinal fluid, genital area, mucus from the nose, blood, sputum, urine, stool, and throat. There are two main types after testing, germ-negative and germ-positive. If the individual can identify as germ-positive then a proper treatment begins for the disease. Sometimes, the test results suffer due to test imperfections. These effects come from sensitivity and specificity, which might be useful when trying to mitigate an epidemic.

Mathematics plays an effective rule in modeling to understand the dynamical behavior of complex phenomena in the life sciences. By mathematical models, one can easily know the basic properties of the complex system such as severity, clinical features, structures, risk analysis, evaluated interventions, and various transmission forms of viruses have been studied along with different dimensions, see for more details, [5–7]. Bernoulli [8] for the first time formulated a mathematical model for infectious diseases and analyzed the impact of prevention smallpox vaccines and life tables. After that, numbers of models have been systemized for infectious maladies such as control strategies for tuberculosis [9], sexually transmitted infections [10], human immunodeficiency virus [11], control of foot and mouth disease epidemic in the UK in

2001 [12], Middle East respiratory syndrome corona virus (MERS-CoV) [13], and severe acute respiratory syndrome (SARS) [14].

Fractional calculus is a generalization of classical calculus. Fractional calculus has lately gained popularity due to its remarkable properties, nonsingular, nonlocal, and memory and filter effects. Researchers of various disciplines are applying the fractional-order operators to real-life phenomena to study the behavior of the models theoretically and numerically. Atangana [15] studied Markovian and non-Markovian, Gaussian and non-Gaussian, and random and nonrandom properties of the fractional derivative, providing numerical examples. Bonyah et al. [16] formulated a human African trypanosomiasis model consisting of an AB-fractional operator and provide numerical solutions. Khan et al. [17] provided an HIV-TB model including AB-fractional derivative and analyzed the model for well-posedness, stability analysis, and numerical solutions. Koca in [18] studied the AB-fractional spread Ebola virus model for the existence of solutions and illustrated the results numerically. Khan et al. [19] considered the AB-fractional-order HIV/AIDS model and applied the fixed point theorem for the existence results and studied the stability analysis. Atangana [20] analyzed the numerical approximations for fractional differentiation based on the Riemann-Liouville definition, from the power law kernel to the generalized Mittag-Leffler law via the exponential decay law.

In this paper, we investigate the dynamical behavior of the fractional-order ITI disease model. The integer-order derivative of the model is replaced by fractal fractional operator in the sense of the Mittage-Leffler kernel. To study the existence of solutions and numerical simulations for the fractional-order ITI disease model. The paper is organized as follows: in Section 2, the definition of fractal fractional operators is shown. In Section 3, the model formulation is discussed. In Section 4, existence and uniqueness of solutions are shown. In Section 5, the numerical scheme is discussed. In Section 6, the numerical discussion and data fitting is discussed. In Section 7, the conclusion is discussed.

2. Preliminaries

Here, we will discuss some definitions which are utilized in the main proof of this study [21–24].

Definition 1 (see [22]). Let $F(t)$ be a continuous and fractal differential in the open interval (a, b) with $0 < n-1 < \sigma \leq n$; then, the fractal fractional operator $0 < n-1 < \epsilon \leq n$ in the sense of Caputo with power law is characterized as

$$
{}_{a}^{FFP}\mathbf{D}_{t}^{\epsilon,\sigma}F(t) = \frac{1}{\Gamma[n-\epsilon]}\int_a^t \frac{dF(z)}{dz^\sigma}(t-z)^{n-\epsilon-1}dz,
$$
where
$$
\frac{dF(z)}{dz^\sigma} = \lim_{t-z}\frac{F(t)-F(z)}{t^\sigma - z^\sigma}. \tag{1}
$$

The generalized form is given as

$$
{}_{a}^{FFP}\mathbf{D}_{t}^{\epsilon,\sigma}F(t) = \frac{1}{\Gamma[n-\epsilon]}\int_a^t \frac{\mathrm{d}^\theta F(z)}{dz^\sigma}(t-z)^{n-\epsilon-1}dz,
$$
where
$$
\frac{d^\theta F(z)}{dz^\sigma} = \lim_{t-z}\frac{F^\theta(t)-F^\theta(z)}{t^\sigma - z^\sigma}, \text{ where } 0<\theta\leq 1. \tag{2}
$$

Definition 2 (see [22]). Let $F(t)$ be a continuous and fractal differential in the open interval (a, b) with $0 < \sigma \leq 1$; then, the fractal fractional operator $0 < \epsilon \leq 1$, in the sense of Caputo with the exponential decay kernel, is characterized as

$$
{}_{a}^{FFP}\mathbf{D}_{t}^{\epsilon,\sigma}F(t) = \frac{\wp(\epsilon)}{[1-\epsilon]}\int_a^t \frac{dF(z)}{dz^\sigma}\exp\left[-\frac{\epsilon}{1-\epsilon}(t-z)\right]dz. \tag{3}
$$

The generalized form given as

$$
{}_{a}^{FFP}\mathbf{D}_{t}^{\epsilon,\sigma,\theta}F(t) = \frac{\wp(\epsilon)}{[1-\epsilon]}\int_a^t \frac{d^\theta F(z)}{dz^\sigma}\exp\left[-\frac{\epsilon}{1-\epsilon}(t-z)\right]dz, 0<\theta\leq 1, \tag{4}
$$

where $\wp(t)$ is the normalization function such that $\wp(0) = 1 = \wp(1)$.

Definition 3 (see [22, 23]). Let $F(t)$ be a continuous and fractal differential in the open interval (a, b) with $0 < \sigma \leq 1$; then, the fractal fractional operator $0 < \epsilon \leq 1$, in the sense of Caputo with the generalized Mittage-Leffler kernel, is characterized as

$$
{}_{a}^{FFM}\mathbf{D}_{t}^{\epsilon,\sigma}F(t) = \frac{AB(\epsilon)}{[1-\epsilon]}\int_a^t \frac{dF(z)}{dz^\sigma}E_\epsilon\left[-\frac{\epsilon}{1-\epsilon}(t-z)^\epsilon\right]dz,
$$
$$
AB(\epsilon) = 1-\epsilon+\frac{\epsilon}{\Gamma(\epsilon)}. \tag{5}
$$

The generalized form is given as

$$
{}_{a}^{FFM}\mathbf{D}_{t}^{\epsilon,\sigma,\theta}F(t) = \frac{AB(\epsilon)}{[1-\epsilon]}\int_a^t \frac{d^\theta F(z)}{dz^\sigma}E_\epsilon\left[-\frac{\epsilon}{1-\epsilon}(t-z)^\epsilon\right]dz, 0<\theta\leq 1, \tag{6}
$$

where $\wp(t)$ is normalization function such that $\wp(0) = 1 = \wp(1)$.

Definition 4 (see [21, 22]). Assume that $F(t)$ is a continuous and fractal differential in the open interval (a, b) with then the fractal fractional integral $0 < \epsilon \leq 1$, in the sense of the power law kernel, is characterized as

$$
{}_{a}^{FFP}\mathbf{I}_{0,t}^{\epsilon,\sigma}F(t) = \frac{1}{\Gamma(\epsilon)}\int_0^t (t-z)^{\epsilon-1}z^{1-\sigma}F(z)dz. \tag{7}
$$

Definition 5 (see [21, 22]). Let $F(t)$ be a continuous and fractal differential in the open interval (a, b) with then the fractal fractional integral $0 < \epsilon \leq 1$, in the sense of the exponential kernel, is characterized as

$$ {}_{a}^{FFE}\mathbf{I}_{0,t}^{\epsilon,\sigma}F(t) = \frac{\sigma(1-\epsilon)t^{\sigma-1}F(t)}{\wp(\epsilon)} + \frac{\epsilon\sigma}{\wp(\epsilon)}\int_0^t z^{\epsilon-1}F(z)dz. \quad (8) $$

Definition 6 (see [21, 22]). Let $F(t)$ be a continuous and fractal differential in the open interval (a, b) with then the fractal fractional integral $0 < \epsilon \leq 1$, in the sense of the Mittage-Leffler kernel, is characterized as

$$ {}_{a}^{FFM}\mathbf{I}_{0,t}^{\epsilon,\sigma}F(t) = \frac{\sigma(1-\epsilon)t^{\sigma-1}F(t)}{AB(\epsilon)} + \frac{\epsilon\sigma}{AB(\epsilon)}\int_0^t (t-z)^{\epsilon-1}z^{\epsilon-1}F(z)dz. \quad (9) $$

3. Model Formulation

In this section, we will study the dynamics of the ordinary differential equations of the infectious disease model formulated in the reference therein [25–27], which is leveraged from the SIR system. This model has two main components, $\mathbf{S}_m(t)$, a stand for rate of suspectable individual tested, $\mathbf{I}_m(t)$, infected cases which is tested positive and started treatment. While $\mathbf{S}(t)$ denotes susceptible, $\mathbf{I}(t)$ denotes infected and $\mathbf{R}(t)$ denotes the class of recovered individuals. The following SSIIR model:

$$ \begin{aligned} \dot{\mathbf{S}}(t) &= \eta\mathbf{S}_m(t) + \mu - \theta\mathbf{S}(t) - \beta\mathbf{S}(t)(\mathbf{I}(t) + \mathbf{I}_m(t)), \\ \dot{\mathbf{S}}_m(t) &= \theta\mathbf{S}(t) - (\eta+\mu)\mathbf{S}_m(t) - \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ \dot{\mathbf{I}}(t) &= \beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)) - (\gamma+\alpha+\mu)\mathbf{I}(t) + \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ \dot{\mathbf{I}}_m(t) &= \alpha\mathbf{I}(t) - \mu\mathbf{I}_m(t) - \gamma_m\mathbf{I}_m(t), \\ \dot{\mathbf{R}}(t) &= \gamma\mathbf{I}(t) - \mu\mathbf{R}(t) + \gamma_m\mathbf{I}_m(t), \end{aligned} \quad (10) $$

where β denotes the rate of infected susceptible individuals, $\theta = \kappa(1-\lambda)$ denotes the rate of susceptible individuals that are tested and deemed incorrectly, and β_m denotes the rate of effectively infected individuals. For this model, $\beta_m < \beta$ is assumed, μ denotes total population, κ denotes the rate of converging from susceptible to θ susceptible-infected-deemed, ψ is the sensitivity, $\alpha = \kappa\psi$ rate of treatment, γ rate of recovered individuals, and γ_m denotes the rate of recovered infected-undertreatment that $\gamma < \gamma_m$ assumed in the model.

The predominant of this paper is to study the existence of results and numerical analysis of fractal fractional-order ITI disease model. In the upcoming section, we are going to produce existence of solution for the model (10) and later on the uniqueness of solution is our interest. For these, we need to define the following Banach's space. Consider $\mathscr{Y} = \mathscr{I} \times \mathbb{R}^5 \longrightarrow \mathbb{R}$, where $\mathscr{I} = [0, \tau]$, for $0 < t < \tau < \infty$, with a norm defined by $\|(\mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})\| = \max_{t\in\mathscr{I}}\{|\mathbf{S}| + |\mathbf{S}_m| + |\mathbf{I}| + |\mathbf{I}_m| + |\mathbf{R}\,|\}$.

Then, clearly $(\mathscr{Y}, \|\|)$ is a Banach space.

4. Existence and Uniqueness of Solutions

In this section, the fixed point theorem is used to investigate the existence and uniqueness of the results for the fractional-order ITI disease model. The integer operator of the model (10) is replaced by a fractal fractional operator

$$ \begin{aligned} {}^{ABR}\mathbb{D}_{0,t}^{\epsilon}(\mathbf{S}(t)) &= \eta\mathbf{S}_m(t) + \mu - \theta\mathbf{S}(t) - \beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ {}^{ABR}\mathbb{D}_{0,t}^{\epsilon}(\mathbf{S}_m(t)) &= \theta\mathbf{S}(t) - (\eta+\mu)\mathbf{S}_m(t) - \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ {}^{ABR}\mathbb{D}_{0,t}^{\epsilon}(\mathbf{I}(t)) &= \beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)) - (\gamma+\alpha+\mu)\mathbf{I}(t) \\ &\quad + \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ {}^{ABR}\mathbb{D}_{0,t}^{\epsilon}(\mathbf{I}_m(t)) &= \alpha\mathbf{I}(t) - \mu\mathbf{I}_m(t) - \gamma_m\mathbf{I}_m(t), \\ {}^{ABR}\mathbb{D}_{0,t}^{\epsilon}(\mathbf{R}(t)) &= \gamma\mathbf{I}(t) - \mu\mathbf{R}(t) + \gamma_m\mathbf{I}_m(t). \end{aligned} \quad (11) $$

with initial boundary value conditions

$$ \mathbf{S}(0) = \mathbf{S}_0, \mathbf{S}_m(0) = \mathbf{S}_{m0}, \mathbf{I}(0) = \mathbf{I}_0, \mathbf{I}_m(0) = \mathbf{I}_{\mathrm{m}0}, \mathbf{R}(0) = \mathbf{R}0. \quad (12) $$

For simplicity, we write the system:

$$ \begin{aligned} \Lambda_1(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \eta\mathbf{S}_m(t) + \mu - \theta\mathbf{S}(t) - \beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ \Lambda_2(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \theta\mathbf{S}(t) - (\eta+\mu)\mathbf{S}_m(t) \\ &\quad - \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ \Lambda_3(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)) - (\gamma+\alpha+\mu)\mathbf{I}(t) \\ &\quad + \beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \\ \Lambda_4(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \alpha\mathbf{I}(t) - \mu\mathbf{I}_m(t) - \gamma_m\mathbf{I}_m(t), \\ \Lambda_5(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \gamma\mathbf{I}(t) - \mu\mathbf{R}(t) + \gamma_m\mathbf{I}_m(t). \end{aligned} \quad (13) $$

Furthermore, the above system (11) can be written as

$$ {}^{ABR}D_{0,t}(\Phi(t)) = \Psi(t, \Phi(t)), \quad (14) $$

where $\Phi(t)$ and Ψ stand for

$$ \Phi(t) = \begin{cases} \mathbf{S}(t) \\ \mathbf{S}_m(t) \\ \mathbf{I}(t) \\ \mathbf{I}_m(t) \\ \mathbf{R}(t) \end{cases} \text{and } \psi\{t, \Phi(t)\} = \begin{cases} \Lambda_1(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}), \\ \Lambda_2(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}), \\ \Lambda_3(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}), \\ \Lambda_4(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}), \\ \Lambda_5(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}). \end{cases} \quad (15) $$

By applying Definition (3), to (14), we get the following form:

$$ \frac{AB(\epsilon)}{1-\epsilon}\frac{d}{dt}\int_0^t \Psi(\omega, \Phi(\omega))E_\epsilon\left[-\frac{\epsilon}{1-\epsilon}(t-z)^\epsilon\right]dz = zt^{z-1}\Psi(t, \Phi(t)). \quad (16) $$

Now, by employing Definition (6), with (16), we get the following form:

$$\Psi(t)=\Psi(0)+\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\Psi(t,\Phi(t))+\frac{z\epsilon}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\Psi(\omega,\Phi(\omega))\omega^{z-1}dz. \tag{17}$$

Let us consider here:

$$\mathscr{B}_a^q=\mathscr{H}_n(t_n)\times\overline{C_0(\Phi_0)}, \tag{18}$$

where $\mathscr{H}_n=[t_{n-a},t_{n+a}]$ and $\overline{C_0(\Phi_0)}=[t_0-b,t_0+b]$. Now, assume $\sup t\in\mathscr{B}_a^q\|\Psi\|=\mathscr{P}$.

Let us define a norm:

$$\|\Omega\|_\infty=\sup_{t\in\mathscr{B}_a^q}|\Omega(t)|. \tag{19}$$

Now, consider operator $\Xi:\mathscr{G}[\mathscr{H}_n(t_n),C_b(t_n)]\longrightarrow\mathscr{G}[\mathscr{H}_n(b),C_b(t_n)]$ such that

$$\Xi\Phi(t)=\Phi_0+\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\Psi(t,\Phi(t))+\frac{z\epsilon}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\Psi(\omega,\Phi(\omega))\omega^{z-1}dz. \tag{20}$$

First, we will show $\|\Xi\Phi(t)-\Phi_0\|<q$. For this, we have

$$\begin{aligned}\|\Xi\Phi(t)-\Phi_0\|&\le\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\|\Psi(t,\Phi(t))\|+\frac{z\epsilon}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\|\Psi(\omega,\Phi(\omega))\|\omega^{z-1}dz,\\&\le\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\mathscr{P}+\frac{z\epsilon}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\omega^{z-1}dz.\end{aligned} \tag{21}$$

Consider $\omega=tv$ and putting in Equation (21), then we get the following:

$$\|\Xi\Phi(t)-\Phi_0\|\le\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\mathscr{P}+\frac{z\epsilon\mathscr{P}}{AB(\epsilon)\Gamma(\epsilon)}\omega^{z+\epsilon-u}\mathscr{Q}(z,\epsilon), \tag{22}$$

which yields

$$\|\Xi\Phi(t)-\Phi_0\|\le q\longrightarrow\mathscr{P}<\frac{q\mathscr{Q}(z,\epsilon)AB(\epsilon)\Gamma(\epsilon)}{(1-\epsilon)\Gamma(\alpha)zt^{z-1}+1-\epsilon zt^{z+\epsilon-u}}. \tag{23}$$

Now, consider for any $\Phi_1,\Phi_2\in\mathscr{G}[\mathscr{H}_n(t_n),C_b(t_n)]$, then, we have

$$\begin{aligned}\|\Xi\Phi_1-\Xi\Phi_2\|\le{}&\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\|\Psi(t,\Phi_1(t))-\Psi(t,\Phi_2(t))\|\\&+\frac{z\epsilon}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\|\Psi(t,\Phi_1(t))-\Psi(t,\Phi_2(t))\|\omega^{z-1}dz.\end{aligned} \tag{24}$$

As Ξ is a contraction, then we have

$$\begin{aligned}\|\Xi\Phi_1-\Xi\Phi_2\|&\le\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\mathscr{X}\|\Phi_1-\Phi_2\|_\infty+\frac{z\epsilon\mathscr{X}}{AB(\epsilon)\Gamma(\epsilon)}\int_0^t(t-z)^{\epsilon-1}\|\Phi_1-\Phi_2\|_\infty\omega^{z-1}dz.\\&\le\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\mathscr{X}\|\Phi_1-\Phi_2\|_\infty+\frac{z\epsilon\mathscr{X}}{AB(\epsilon)\Gamma(\epsilon)}\|\Phi_1-\Phi_2\|_\infty t^{\epsilon+z-3}\mathscr{Q}(z,\epsilon),\\\|\Xi\Phi_1-\Xi\Phi_2\|&\le\left[\frac{1-\epsilon}{AB(\epsilon)}zt^{z-1}\mathscr{X}+\frac{z\epsilon\mathscr{X}}{AB(\epsilon)\Gamma(\epsilon)}t^{\epsilon+z-3}\mathscr{Q}(z,\epsilon)\right]\|\Phi_1-\Phi_2\|_\infty\end{aligned} \tag{25}$$

Therefore, Ξ is a contraction if

$$\|\Xi\Phi_1-\Xi\Phi_2\|_\infty\le\|\Phi_1-\Phi_2\|. \tag{26}$$

Then, we have

$$\mathscr{X}<\frac{1}{((1-\epsilon)/(AB(\epsilon)))zt^{z-1}+(z\epsilon/AB(\epsilon)\Gamma(\epsilon))t^{\epsilon+z-3}\mathscr{Q}(z,\epsilon)}, \tag{27}$$

such that

$$\mathscr{P}<\frac{1}{((1-\epsilon)/(AB(\epsilon)))zt^{z-1}+(z\epsilon/AB(\epsilon)\Gamma(\epsilon))t^{\epsilon+z-3}\mathscr{Q}(z,\epsilon)}. \tag{28}$$

Hence, by necessary condition, the proposed fractional-order ITI disease model (11) has a unique solution.

5. Numerical Scheme

We consider the ITI disease model (10), in the sense of the fractal fractional Mittag-Leffler Kernel

$${}^{\mathrm{ABR}}\mathbb{D}_{0,t}^{\epsilon,\sigma}(\mathbf{S}(t))=\eta\mathbf{S}_m(t)+\mu-\theta\mathbf{S}(t)-\beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \tag{29}$$

$${}^{\mathrm{ABR}}\mathbb{D}_{0,t}^{\epsilon,\sigma}(\mathbf{S}_m(t))=\theta\mathbf{S}(t)+(\eta+\mu)\mathbf{S}_m(t)-\beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \tag{30}$$

$${}^{\mathrm{ABR}}\mathbb{D}_{0,t}^{\epsilon,\sigma}(\mathbf{I}(t))=\beta\mathbf{S}(t)(\mathbf{I}(t)+\mathbf{I}_m(t))-(\gamma+\alpha+\mu)\mathbf{I}(t)+\beta_m\mathbf{S}_m(t)(\mathbf{I}(t)+\mathbf{I}_m(t)), \tag{31}$$

$${}^{\mathrm{ABR}}\mathbb{D}_{0,t}^{\epsilon,\sigma}(\mathbf{I}_m(t))=\alpha\mathbf{I}(t)-\mu\mathbf{I}_m(t)-\gamma_m\mathbf{I}_m(t), \tag{32}$$

$$ {}^{\mathrm{ABR}}\mathbb{D}_{0,t}^{\epsilon,\sigma}(\mathbf{R}(t)) = \gamma \mathbf{I}(t) - \mu \mathbf{R}(t) - \gamma_m \mathbf{I}_m(t). \quad (33) $$

For simplicity,

$$ \begin{aligned} \Lambda_1(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \eta \mathbf{S}_m(t) + \mu - \theta \mathbf{S}(t) \\ &\quad - \beta \mathbf{S}(t)(\mathbf{I}(t) + \mathbf{I}_m(t)), \\ \Lambda_2(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \theta \mathbf{S}(t) - (\eta + \mu)\mathbf{S}_m(t) \\ &\quad - \beta_m \mathbf{S}_m(t)(\mathbf{I}(t) + \mathbf{I}_m(t)), \\ \Lambda_3(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \beta \mathbf{S}(t)(\mathbf{I}(t) + \mathbf{I}_m(t)) - (\gamma + \alpha + \mu)\mathbf{I}(t) \\ &\quad + \beta_m \mathbf{S}_m(t)(\mathbf{I}(t) + \mathbf{I}_m(t)), \\ \Lambda_4(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \alpha \mathbf{I}(t) - \mu \mathbf{I}_m(t) - \gamma_m \mathbf{I}_m(t), \\ \Lambda_5(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) &= \gamma \mathbf{I}(t) - \mu \mathbf{R}(t) - \gamma_m \mathbf{I}_m(t). \end{aligned} \quad (34) $$

By applying the Atangan-Baleanu integral operator to Equation (29), which deduced to the following form:

$$ \mathbf{S}(t) = \mathbf{S}(0) + \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_1(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^t (t-z)^{\varepsilon-1} \Lambda_1(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (35) $$

$$ \mathbf{S}_m(t) = \mathbf{S}_m(0) + \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_1(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^t (t-z)^{\varepsilon-1} \Lambda_2(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (36) $$

$$ \mathbf{I}(t) = \mathbf{I}(0) + \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_3(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^t (t-z)^{\varepsilon-1} \Lambda_3(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (37) $$

$$ \mathbf{I}_m(t) = \mathbf{I}_m(0) \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_3(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^t (t-z)^{\varepsilon-1} \Lambda_4(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (38) $$

$$ \mathbf{R}(t) = \mathbf{R}(0) + \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_5(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R})}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^t (t-z)^{\varepsilon-1} \Lambda_5(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz. \quad (39) $$

For the numerical scheme fitting $t = t_{n+1}$ in Equation (35), which deduced to the below form:

$$ \mathbf{S}^{n+1}(t) = \mathbf{S}^0 + \frac{\sigma t^{\sigma-1}(1-\varepsilon)\Lambda_1(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_{n+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_1(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (40) $$

$$ \mathbf{S}_m^{n+1}(t) = \mathbf{S}_m^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_2(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_{n+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_2(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (41) $$

$$ \mathbf{I}^{n+1}(t) = \mathbf{I}^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_3(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_{n+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_3(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (42) $$

$$ \mathbf{I}_m^{n+1}(t) = \mathbf{I}_m^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_4(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_{n+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_4(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \quad (43) $$

$$ \mathbf{R}^{n+1}(t) = \mathbf{R}^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_5(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \int_0^{t_{n+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_5(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz. \quad (44) $$

By approximating the integral in the above system (40), then we get

$$ \begin{aligned} \mathbf{S}^{n+1}(t) &= \mathbf{S}^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_1(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \sum_{r=0}^{q} \int_{t_r}^{t_{r+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_1(t, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \\ \mathbf{S}_m^{n+1}(t) &= \mathbf{S}_m^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_2(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \sum_{r=0}^{q} \int_{t_r}^{t_{r+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_2(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \\ \mathbf{I}^{n+1}(t) &= \mathbf{I}^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_3(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} \\ \mathbf{I}_m^{n+1}(t) &= \mathbf{I}_m^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_4(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \sum_{r=0}^{q} \int_{t_r}^{t_{r+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_4(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz, \\ \mathbf{R}^{n+1}(t) &= \mathbf{R}^0 + \frac{\sigma t_n^{\sigma-1}(1-\varepsilon)\Lambda_5(t_n, \mathbf{S}^n, \mathbf{S}_m^n, \mathbf{I}^n, \mathbf{I}_m^n, \mathbf{R}^n)}{AB(\varepsilon)} + \frac{\varepsilon\sigma}{AB(\varepsilon)\Gamma(\varepsilon)} \sum_{r=0}^{q} \int_{t_r}^{t_{r+1}} (t_{n+1}-z)^{\varepsilon-1} \Lambda_5(z, \mathbf{S}, \mathbf{S}_m, \mathbf{I}, \mathbf{I}_m, \mathbf{R}) z^{\sigma-1} dz. \end{aligned} \quad (45) $$

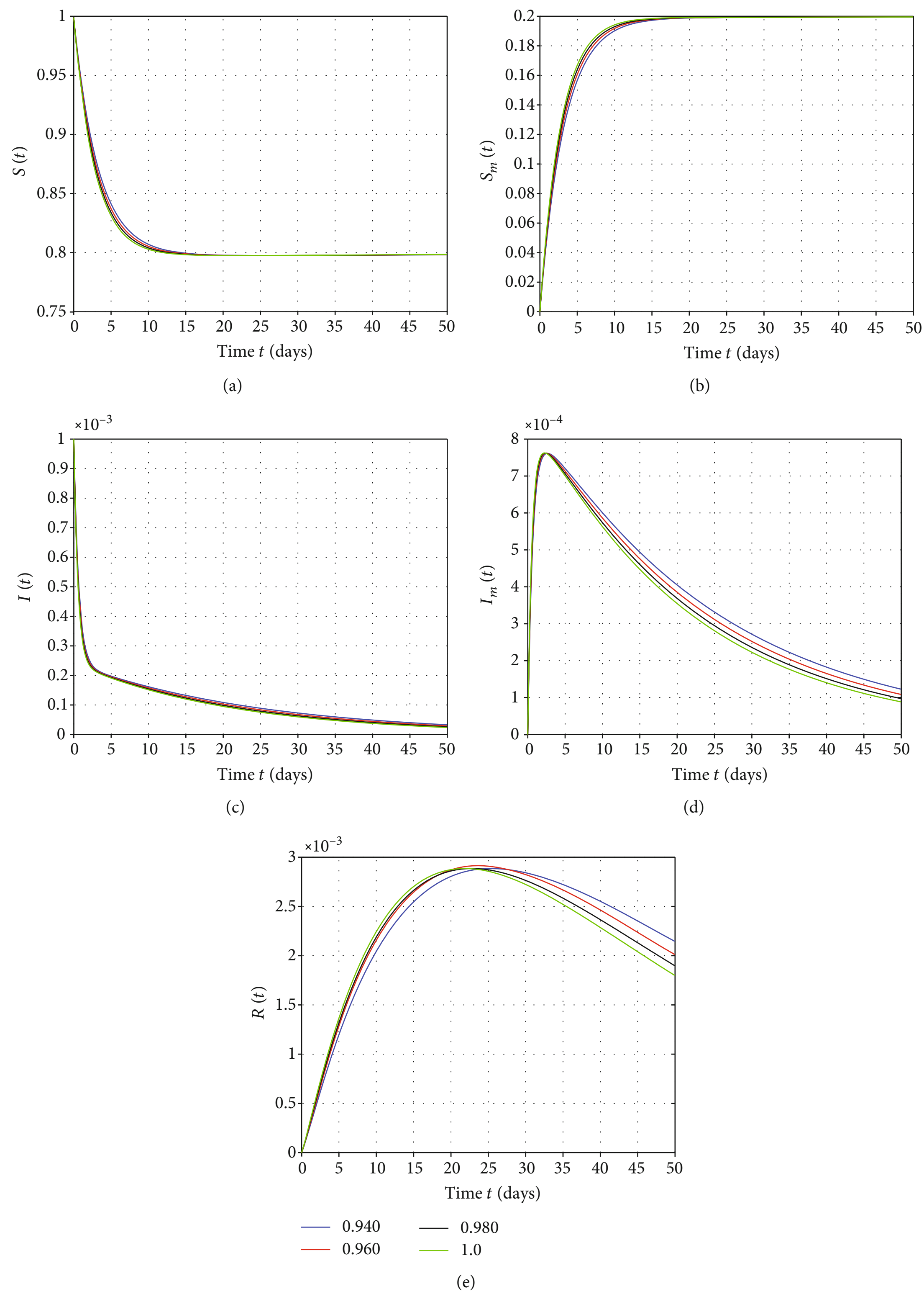

Figure 1: Numerical solution of (10) for $\kappa = 0.1$ and different values of ϵ fractional order.

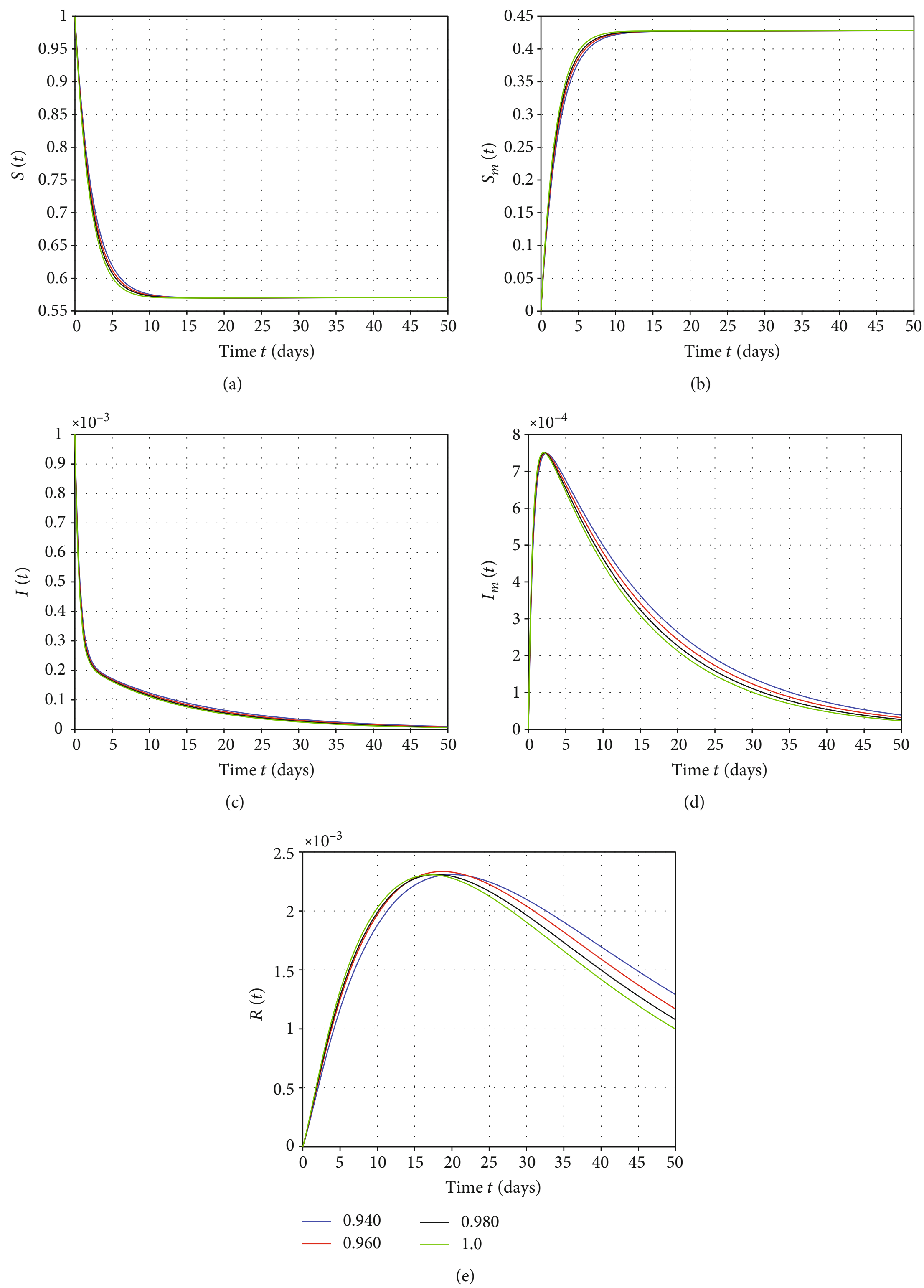

Figure 2: Numerical solution of (10) for $\kappa = 0.3$ and different values of ϵ fractional order.

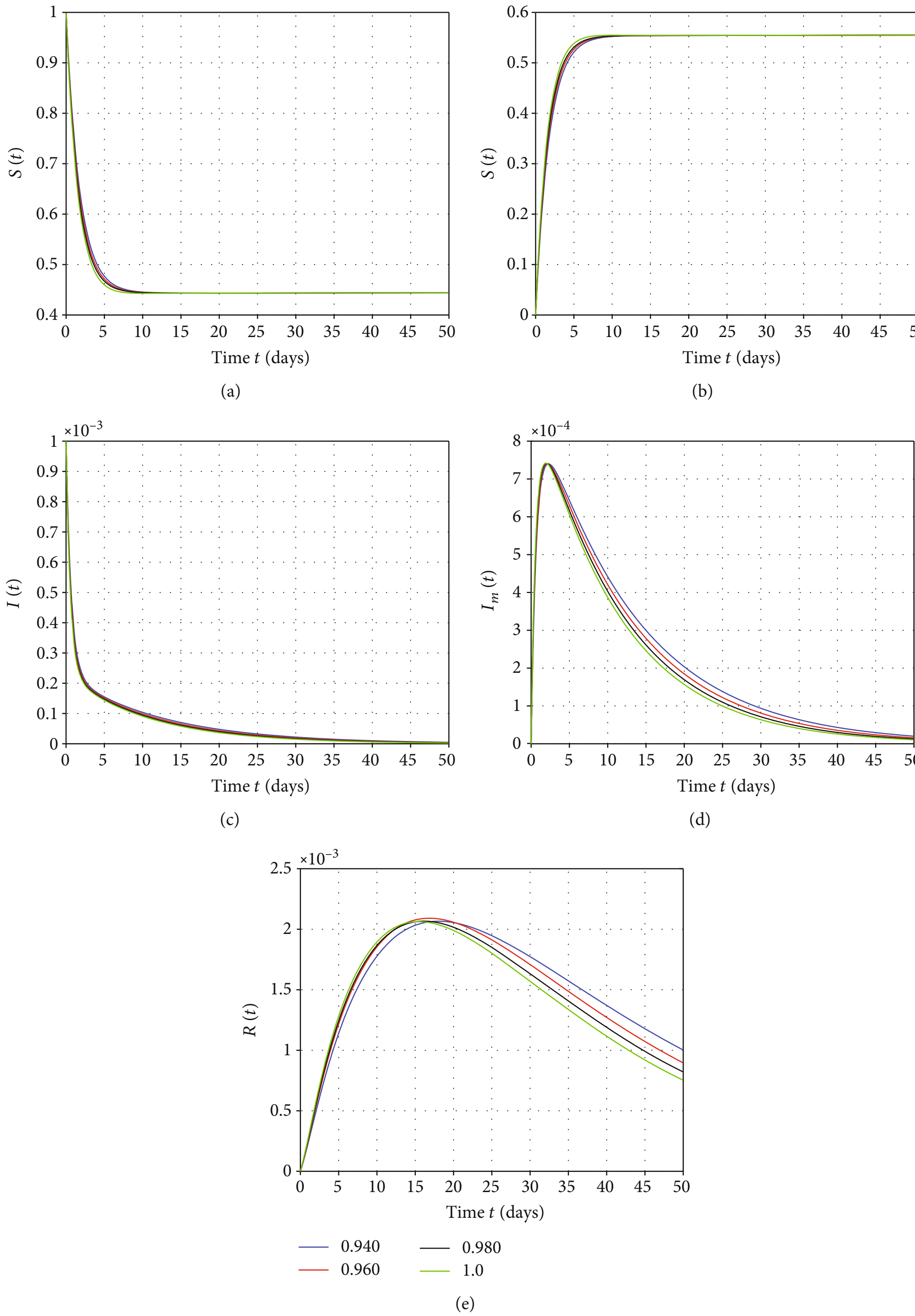

FIGURE 3: Numerical solution of system (10) for $\kappa = 0.5$ and different values of ϵ fractional order.

Now, by applying the Lagrangian interpolation polynomial piecewise which yields the following form:

$$
\begin{aligned}
\mathbf{S}^{n+1}(t) &= \mathbf{S}^0 \\
&+ \frac{\sigma t_n^{\sigma-1}(1-\epsilon)\Lambda_1(t_n,\mathbf{S}^n,\mathbf{S}_m^n,\mathbf{I}^n,\mathbf{I}_m^n,\mathbf{R}^n)}{AB(\epsilon)} \\
&+ \frac{\sigma(\blacktriangle t)^\epsilon}{AB(\epsilon)\Gamma(\epsilon+2)} \\
&\cdot \sum_{r=0}^{q}\left[t_r^{\sigma-1}\Lambda_1(t_r,\mathbf{S}^r,\mathbf{S}_m^r,\mathbf{I}^r,\mathbf{I}_m^r,\mathbf{R}^r)\right. \\
&\times((q+1-r)^\epsilon(q-r+2+\epsilon)-(q-r)^\epsilon(p-r+2+2\epsilon)) \\
&- t_{r-1}^{\sigma-1}\Lambda_1\left(t_r,\mathbf{S}^{r-1},\mathbf{S}_m^{r-1},\mathbf{I}^{r-1},\mathbf{I}_m^{r-1},\mathbf{R}^{r-1}\right) \\
&\times\left.\left((q-r+1)^{\epsilon+1}-(p-r)^\epsilon(p-r+1+\epsilon)\right)\right], \\
\mathbf{S}_m^{n+1}(t) &= \mathbf{S}_m^0 \\
&+ \frac{\sigma t_n^{\sigma-1}(1-\epsilon)\Lambda_2(t_n,\mathbf{S}^n,\mathbf{S}_m^n,\mathbf{I}^n,\mathbf{I}_m^n,\mathbf{R}^n)}{AB(\epsilon)} \\
&+ \frac{\sigma(\blacktriangle t)^\epsilon}{AB(\epsilon)\Gamma(\epsilon+2)} \\
&\cdot \sum_{r=0}^{q}\left[t_r^{\sigma-1}\Lambda_2(t_r,\mathbf{S}^r,\mathbf{S}_m^r,\mathbf{I}^r,\mathbf{I}_m^r,\mathbf{R}^r)\right. \\
&\times((q+1-r)^\epsilon(q-r+2+\epsilon)-(q-r)^\epsilon(p-r+2+2\epsilon)) \\
&- t_{r-1}^{\sigma-1}\Lambda_2\left(t_r,\mathbf{S}^{r-1},\mathbf{S}_m^{r-1},\mathbf{I}^{r-1},\mathbf{I}_m^{r-1},\mathbf{R}^{r-1}\right) \\
&\times\left.\left((q-r+1)^{\epsilon+1}-(p-r)^\epsilon(p-r+1+\epsilon)\right)\right], \\
\mathbf{I}^{n+1}(t) &= \mathbf{I}^0 \\
&+ \frac{\sigma t_n^{\sigma-1}(1-\epsilon)\Lambda_3(t_n,\mathbf{S}^n,\mathbf{S}_m^n,\mathbf{I}^n,\mathbf{I}_m^n,\mathbf{R}^n)}{AB(\epsilon)} \\
&+ \frac{\sigma(\blacktriangle t)^\epsilon}{AB(\epsilon)\Gamma(\epsilon+2)} \\
&\cdot \sum_{r=0}^{q}\left[t_r^{\sigma-1}\Lambda_3(t_r,\mathbf{S}^r,\mathbf{S}_m^r,\mathbf{I}^r,\mathbf{I}_m^r,\mathbf{R}^r)\right. \\
&\times((q+1-r)^\epsilon(q-r+2+\epsilon)-(q-r)^\epsilon(p-r+2+2\epsilon)) \\
&- t_{r-1}^{\sigma-1}\Lambda_2\left(t_r,\mathbf{S}^{r-1},\mathbf{S}_m^{r-1},\mathbf{I}^{r-1},\mathbf{I}_m^{r-1},\mathbf{R}^{r-1}\right) \\
&\times\left.\left((q-r+1)^{\epsilon+1}-(p-r)^\epsilon(p-r+1+\epsilon)\right)\right], \\
\mathbf{I}_m^{n+1}(t) &= \mathbf{I}_m^0 \\
&+ \frac{\sigma t_n^{\sigma-1}(1-\epsilon)\Lambda_4(t_n,\mathbf{S}^n,\mathbf{S}_m^n,\mathbf{I}^n,\mathbf{I}_m^n,\mathbf{R}^n)}{AB(\epsilon)} \\
&+ \frac{\sigma(\blacktriangle t)^\epsilon}{AB(\epsilon)\Gamma(\epsilon+2)} \\
&\cdot \sum_{r=0}^{q}\left[t_r^{\sigma-1}\Lambda_4(t_r,\mathbf{S}^r,\mathbf{S}_m^r,\mathbf{I}^r,\mathbf{I}_m^r,\mathbf{R}^r)\right. \\
&\times((q+1-r)^\epsilon(q-r+2+\epsilon)-(q-r)^\epsilon(p-r+2+2\epsilon)) \\
&- t_{r-1}^{\sigma-1}\Lambda_4\left(t_r,\mathbf{S}^{r-1},\mathbf{S}_m^{r-1},\mathbf{I}^{r-1},\mathbf{I}_m^{r-1},\mathbf{R}^{r-1}\right) \\
&\times\left.\left((q-r+1)^{\epsilon+1}-(p-r)^\epsilon(p-r+1+\epsilon)\right)\right], \\
\mathbf{R}^{n+1}(t) &= \mathbf{R}^0 \\
&+ \frac{\sigma t^{\sigma-1}(1-\epsilon)\Lambda_5(t_n,\mathbf{S}^n,\mathbf{S}_m^n,\mathbf{I}^n,\mathbf{I}_m^n,\mathbf{R}^n)}{AB(\epsilon)} \\
&+ \frac{\sigma(\blacktriangle t)^\epsilon}{AB(\epsilon)\Gamma(\epsilon+2)} \\
&\cdot \sum_{r=0}^{q}\left[t_r^{\sigma-1}\Lambda_5(t_r,\mathbf{S}^r,\mathbf{S}_m^r,\mathbf{I}^r,\mathbf{I}_m^r,\mathbf{R}^r)\right. \\
&\times((q+1-r)^\epsilon(q-r+2+\epsilon)-(q-r)^\epsilon(p-r+2+2\epsilon)) \\
&- t_{r-1}^{\sigma-1}\Lambda_5\left(t_r,\mathbf{S}^{r-1},\mathbf{S}_m^{r-1},\mathbf{I}^{r-1},\mathbf{I}_m^{r-1},\mathbf{R}^{r-1}\right) \\
&\times\left.\left((q-r+1)^{\epsilon+1}-(p-r)^\epsilon(p-r+1+\epsilon)\right)\right].
\end{aligned} \tag{46}
$$

6. Discussion and Numerical Results

A numerical scheme utilized to obtain the approximate solutions for the fractal fractional-order ITI disease model. Different scenarios have been discussed for the fractal fractional-order ITI disease model by choosing different parametric values and testing rates for the model. We observed that as the testing rate κ increasing; then, the incidence decreased effectively. We apply the aforementioned iterative scheme for the numerical analysis to demonstrate graphically the fractal fractional ITI disease model. To examine the dynamical behavior of the model, we choose suitable constant values for the parameters used in the model.

Figure 1 shows the effect of testing rate κ increasing and different order values of fractal fractional operator. By choosing the parametric values involved in the fractional-order ITI disease model such that $\kappa = 0.1$, $\beta = 0.15$, $\beta_m = 0.1$, $\gamma = 0.1$, $\gamma_m = 0.15$, $\mu = 0.003$, and $\eta = 0.1$ and assuming initial conditions for $\mathbf{S}$, $\mathbf{S}_m$,$\mathbf{I}$, $\mathbf{I}_m$ and $\mathbf{R}$. (c) and (d) show the infected class decreasing as the κ value increases.

Figure 2 shows the effect of testing rate κ increasing and different order values of fractal fractional operator. By choosing the parametric values involved in the fractional order ITI disease model such that $\kappa = 0.3$, $\beta = 0.15$, $\beta_m = 0.1$, $\gamma = 0.1$, $\gamma_m = 0.15$, $\mu = 0.003$, and $\eta = 0.1$ and assuming initial conditions for $\mathbf{S}$, $\mathbf{S}_m$, $\mathbf{I}$, $\mathbf{I}_m$, and $\mathbf{R}$. (c) and (d) show the infected class decreasing as the κ value increasing.

Figure 3 shows the effect of testing rate κ increasing and different order values of fractal fractional operator. By choosing the parametric values involved in the fractional order ITI disease model such that $\kappa = 0.5$, $\beta = 0.15$, $\beta_m = 0.1$, $\gamma = 0.1$, $\gamma_m = 0.15$, $\mu = 0.003$, and $\eta = 0.1$ and assuming initial conditions for $\mathbf{S}$, $\mathbf{S}_m$, $\mathbf{I}$, $\mathbf{I}_m$, and $\mathbf{R}$. (c) and (d) shows the infected class decreasing as the κ value increases.

7. Conclusion

In this article, we study the theoretical and numerical aspects of the fractal fractional ITI disease model in the sense of the Mittag-leffler kernel. The existence of a solution is derived with the help of a fixed point theorem for the proposed model. Different scenarios have been investigated for fractal fractional-order ITI disease models by choosing different parametric values and testing rates for the model. We observed that as different fractional orders for and testing rate κ increase, then the infected class decreased effectively. The numerical approximate solutions achieved by Lagrangian polynomial piecewise interpolation iterative method. Furthermore, one can study the stability analysis fractal fractional-order ITI disease model by using various types of approaches.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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Research Article

On Some Interpolative Contractions of Suzuki Type Mappings

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The goal of this study is to propose a new interpolative contraction mapping by using an interpolative approach in the setting of complete metric spaces. We present some fixed point theorems for interpolative contraction operators using w-admissible maps which satisfy Suzuki type mappings. In addition, some results are given. These results generalize several new results present in the literature. Moreover, examples are provided to show the suitability of our given results.

1. Introduction

In 1922, Banach [1] proved his famous remarkable fixed-point theorem; the result is known as the Banach contraction principle, which states that "Let $(\mathscr{K}, \mathrm{d})$ be a complete metric space and $\mathrm{S} : \mathscr{K} \longrightarrow \mathscr{K}$ be a contraction, then S has a unique fixed point." The Banach contraction principle is one of the essential and most valuable theorems of analysis and is accepted as the main results of metric fixed-point theory. In the last century, the fixed point and its applications have been the subject of research by many authors in the literature, since it provides useful tools to solve many complex problems that have applications in different sciences like computer science, engineering, data science, physics, economics, game theory, and biosciences [2–7]. Due to several applications of "fixed point theory," researchers were motivated to further generalize it in different directions, by generalizing the contractive conditions underlying the space concept of completeness.

The background literature on the famous Banach contraction principle has been extended in various comprehensive directions by many researchers. One of the exciting generalizations was given by Kannan [8], which characterize the completeness of underlying metric spaces. Kannan introduced the following theorem.

Theorem 1. *[8] Let $(\mathscr{K}, d)$ be a complete metric space. A mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ is said to be a Kannan contraction if there exists $\lambda \in [0, 1/2)$ such that*

$$d(Sv, St) \leq \lambda(d(v, Sv) + d(t, St)), \tag{1}$$

for all $v, t \in \mathscr{K} \setminus Fix(S)$. Then, S posses a unique fixed point.

The Kannan theorem has been generalized in different aspects by many authors; one of the crucial generalizations was given by Karapinar in [9]. Karapinar [9] introduced the notion of an interpolative Kannan contraction mapping and proved the following:

A mapping $\mathrm{S} : \mathscr{K} \longrightarrow \mathscr{K}$ on $(\mathscr{K}, \mathrm{d})$ a complete metric space such that

$$\mathrm{d}(\mathrm{S}v, \mathrm{S}t) \leq \kappa[\mathrm{d}(v, \mathrm{S}v)]^{\alpha} \cdot [\mathrm{d}(t, \mathrm{S}t)]^{1-\alpha}, \tag{2}$$

where $\kappa \in [0, 1)$ and $\alpha \in (0, 1)$, for each $v, t \in \mathscr{K} \setminus \mathrm{Fix}(\mathrm{S})$. Then, S has a unique fixed point in $\mathscr{K}$. Subsequently, Karapinar et al. [10] introduced the following notion of interpolative Ciric-Reich-Rus contractions.

Theorem 2 (see [10]). *Let $(\mathscr{K}, p)$ be a partial metric space. The mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ is called an interpolative Ciric-*

Reich-Rus contraction if there exist $\lambda \in [0, 1)$ and positive reals $\beta, \alpha > 0$, with $\beta + \alpha < 1$, such that

$$p(Sv, St) \le \lambda\left([p(v,t)]^{\beta} \cdot [p(v, Sv)]^{\alpha} \cdot [p(t, St)]^{1-\alpha-\beta}\right), \quad (3)$$

for each $v, t \in \mathscr{K} \setminus Fix(S)$. Then, the mapping S has a fixed point in $\mathscr{K}$.

Afterward, this concept has been extended in different aspects, for example, [11–15].

Let Ψ be the set of all nondecreasing self-mappings ψ on $[0, \infty)$ such that $\sum_{r=1}^{\infty} \psi^r(z) < \infty$ for every $z > 0$. Notice that for $\psi \in \Psi$, we have $\psi(0) = 0$ and $\psi(z) < z$ for all $z > 0$ (see [16, 17]).

The concept of w-orbital admissible mappings was introduced by Popescu as a clarification of the concept of α-admissible mappings of Samet et al. [18].

Definition 3 (see [19]). *Let S be a self-map defined on $\mathscr{K}$ and $w : \mathscr{K} \times \mathscr{K} \longrightarrow [0,\infty)$ be a function. S is said to be an w-orbital admissible if for all $v \in \mathscr{K}$, we have*

$$w(v, Sv) \ge 1 \Rightarrow w\left(Sv, S^2 v\right) \ge 1. \quad (4)$$

In our appointed theorems, if the continuity of the involved contractive mappings is removed, to handle this defect, it is necessary that $(\mathscr{K}, d)$ be w-regular.

$(\mathfrak{R})$ A space $(\mathscr{K}, d)$ is defined as w-regular, if $\{v_r\}$ is a sequence in $\mathscr{K}$ such that $w(v_r, v_{r+1}) \ge 1$ for each r and $v_r \longrightarrow \omega \in \mathscr{K}$ as $r \longrightarrow \infty$, then $w(v_r, \omega) \ge 1$ for all r.

Some curious results in this sense are found in the works in [20–24].

Another most interesting Banach contraction principle generalization was given by Suzuki [25, 26]. He introduced a weaker notion of contraction and discussed the existence of some new fixed point theorems. Besides the famous theorem, Suzuki generalized also the results of Nemytzki [27] and Edelstein [28] for compact metric spaces. One of the recently popular topics in fixed point theory is addressing the existence of fixed points of Suzuki type mappings. As with many generalizations of the famous Banach theorems, Suzuki type generalization can be said to have many applications, such as in computer science [29], game theory [30], and biosciences [31] and in other areas of mathematical sciences such as in dynamic programming, integral equations, data dependence, and homotopy [32, 33]. Subsequently, Popescu [34] has modified the nonexpansiveness situation with the weaker C-condition presented by Suzuki. Accordingly, the existence of fixed points of maps satisfying the C-condition has been extensively studied (see [35–38]). Karapınar et al. [39] introduced the definition of a nonexpansive mapping satisfying the C-condition:

Definition 4. A mapping S on a metric space $(\mathscr{K}, d)$ satisfies the C-condition if

$$\frac{1}{2} d(v, Sv) \le d(v, t) \Rightarrow d(Sv, St) \le d(v, t), \quad (5)$$

for each $v, t \in \mathscr{K}$.

2. Main Results

We start the section with the following essential definition:

Definition 5. Let $(\mathscr{K}, d)$ be a metric space. A mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ is called an w-ψ-interpolative Kannan contraction of Suzuki type if there exist $\psi \in \Psi$, $w : \mathscr{K} \times \mathscr{K} \longrightarrow [0,\infty)$, and a real number $\beta \in [0, 1)$, such that

$$\frac{1}{2} d(v, Sv) \le d(v, t) \Rightarrow w(v, t) d(Sv, St) \le \psi\left([d(v, Sv)]^{\beta} \cdot [d(t, St)]^{1-\beta}\right), \quad (6)$$

for each $v, t \in \mathscr{K} \setminus \mathrm{Fix}(S)$.

Theorem 6. *Let $(\mathscr{K}, d)$ be a complete metric space and $S : \mathscr{K} \longrightarrow \mathscr{K}$ be an w-ψ-interpolative Kannan contraction of the Suzuki type. Suppose that S is an w-orbital admissible mapping and $w(v_0, Sv_0) \ge 1$ for some $v_0 \in \mathscr{K}$. Then, S has a fixed point in $\mathscr{K}$ provided that at least one of the following conditions holds:*

(a) $(\mathscr{K}, d)$ is w-regular

(b) S is continuous

(c) S^2 is continuous and $w(v, Sv) \ge 1$ where $v \in Fix(S^2)$

Proof. Let $v_0 \in \mathscr{K}$ such that $w(v_0, Sv_0) \ge 1$ and $\{v_r\}$ be the sequence constructed by $S^r(v_0) = v_r$ for each positive integer r. Assuming that for some $r_0 \in \mathbb{N}$, $v_{r_0} = v_{r_{0+1}}$, we get $v_{r_0} = Sv_{r_0}$, so v_{r_0} is a fixed point of S. Then, $v_r \ne v_{r+1}$ for each positive integer r. As S is w-orbital admissible, $w(v_0, Sv_0) = w(v_0, v_1) \ge 1$ implies that $w(v_1, Sv_1) = w(v_1, v_2) \ge 1$. Similarly, continuing this process, we have

$$w(v_r, v_{r+1}) \ge 1. \quad (7)$$

Thereupon, choosing $v = v_{r-1}$ and $t = Sv_{r-1}$ in (6), we get

$$\begin{aligned} \frac{1}{2} d(v_{r-1}, Sv_{r-1}) &= \frac{1}{2} d(v_{r-1}, v_r) \le d(v_{r-1}, v_r) \\ &\Rightarrow d(v_r, v_{r+1}) \le w(v_{r-1}, v_r) d(Sv_{r-1}, Sv_r) \\ &\le \psi\left([d(v_{r-1}, Sv_{r-1})]^{\beta} \cdot [d(v_r, Sv_r)]^{1-\beta}\right) \\ &= \psi\left([d(v_{r-1}, v_r)]^{\beta} \cdot [d(v_r, v_{r+1})]^{1-\beta}\right) \\ &< [d(v_{r-1}, v_r)]^{\beta} \cdot [d(v_r, v_{r+1})]^{1-\beta}, \end{aligned} \quad (8)$$

whence it follows that

$$[d(v_r, v_{r+1})]^\beta < [d(v_{r-1}, v_r)]^\beta, \tag{9}$$

or equivalent

$$d(v_r, v_{r+1}) < d(v_{r-1}, v_r). \tag{10}$$

Thus, on the one hand, it follows that the sequence $\{d(v_{r-1}, v_r)\}$ is a nonincreasing sequence with positive terms, so there exists $l \geq 0$ such that $\lim_{r\longrightarrow\infty} d(v_{r-1}, v_r) = l$. On the other hand, combining (8) and (10) and keeping in mind that the function ψ is nondecreasing, we obtain

$$d(v_r, v_{r+1}) \leq \psi(d(v_{r-1}, v_r)) \leq \psi^2(d(v_{r-2}, v_{r-1})) \leq \cdots \leq \psi^r(d(v_0, v_1)). \tag{11}$$

Now, applying the triangle inequality and using (11), for all $j \geq 1$, we get

$$\begin{aligned} d\left(v_r, v_{r+j}\right) &\leq d(v_r, v_{r+1}) + d(v_{r+1}, v_{r+2}) + \cdots + d\left(v_{r+j-1}, v_{r+j}\right) \\ &\leq \psi^r(d(v_0, v_1) + \psi^{r+1}(d(v_0, v_1) + \cdots + \psi^{r+j-1} \\ &\quad \cdot (d(v_0, v_1) = \sum_{m=r}^{r+j-1} \psi^m(d(v_0, v_1) = P_{r+j-1} - P_{r-1}, \end{aligned} \tag{12}$$

where $P_k = \sum_{m=0}^{k} \psi^m(d(v_0, v_1))$. But, $\psi \in \Psi$, the series $\sum_{m=0}^{\infty} \psi^m(d(v_0, v_1))$ is convergent, so there exists a positive real number P such that $\lim_{k\longrightarrow\infty} P_k = P$. Consequently, letting r, $j \longrightarrow \infty$ in the above inequality, we get

$$d\left(v_r, v_{r+j}\right) \longrightarrow 0. \tag{13}$$

Therefore, $\{v_r\}$ is a Cauchy sequence, and taking into account the completeness of the space $(\mathscr{K}, \mathrm{d})$, it follows that there exists $\omega \in \mathscr{K}$ such that

$$\lim_{r\longrightarrow\infty} v_r = \omega, \tag{14}$$

and we claim that this ω is a fixed point of S.

In case that the assumption (a) holds, we have $w(v_r, \omega) \geq 1$, and we claim that

$$\frac{1}{2} d(v_r, Sv_r) \leq d(v_r, \omega) \tag{15}$$

or

$$\frac{1}{2} d(Sv_r, S(Sv_r)) \leq d(Sv_r, \omega), \tag{16}$$

for every $r \in \mathbb{N}$. Supposing

$$\begin{aligned} \frac{1}{2} d(v_r, Sv_r) &> d(v_r, \omega), \\ \frac{1}{2} d(Sv_r, S(Sv_r)) &> d(Sv_r, \omega), \end{aligned} \tag{17}$$

on the account of the triangle inequality, we have

$$\begin{aligned} d(v_r, v_{r+1}) = d(v_r, Sv_r) &\leq d(v_r, \omega) + d(\omega, Sv_r) \\ &< \frac{1}{2} d(v_r, Sv_r) + \frac{1}{2} d(Sv_r, S(Sv_r)) \\ &= \frac{1}{2} d(v_r, v_{r+1}) + \frac{1}{2} d(v_{r+1}, v_{r+2}) \\ &\leq \frac{1}{2} d(v_r, v_{r+1}) + \frac{1}{2} d(v_r, v_{r+1}) = d(v_r, v_{r+1}), \end{aligned} \tag{18}$$

which is a contradiction. Thereupon, for every $r \in \mathbb{N}$, either

$$\frac{1}{2} d(v_r, Sv_r) \leq d(v_r, \omega), \tag{19}$$

or

$$\frac{1}{2} d(Sv_r, S(Sv_r)) \leq d(Sv_r, \omega), \tag{20}$$

holds. In the case that (19) holds, we obtain

$$\begin{aligned} d(v_{r+1}, S\omega) &\leq w(v_r, \omega) d(Sv_r, S\omega) \leq \psi\left[d([d(v_r, Sv_r)]^\beta \cdot [d(\omega, S\omega)]^{1-\beta}\right. \\ &= \psi\left[[d(v_r, v_{r+1})]^\beta \cdot [d(\omega, S\omega)]^{1-\beta} < [d(v_r, v_{r+1})]^\beta \cdot [d(\omega, S\omega)]^{1-\beta}.\right. \end{aligned} \tag{21}$$

If the second condition, (20), holds, we have

$$\begin{aligned} d(v_{r+2}, S\omega) &\leq w(v_{r+1}, \omega) d\left(S^2 v_r, S\omega\right) \\ &\leq \psi\left[d\left(\left[d\left(Sv_r, S^2 v_r\right)\right]^\beta \cdot [d(\omega, S\omega)]^{1-\beta}\right.\right. \\ &= \psi[[d(v_{r+1}, v_{r+2})]^\beta \cdot [d(\omega, S\omega)]^{1-\beta} \\ &< [d(v_{r+1}, v_{r+2})]^\beta \cdot [d(\omega, S\omega)]^{1-\beta}. \end{aligned} \tag{22}$$

Therefore, letting $r \longrightarrow \infty$ in (21) and (22), we get that $d(\omega, S\omega) = 0$, that is, $\omega = S\omega$.

In the case that the assumption (b) is true, that is, the mapping S is continuous,

$$S\omega = \lim_{r\longrightarrow\infty} Sv_r = \lim_{r\longrightarrow\infty} v_{r+1} = \omega. \tag{23}$$

If the last assumption, (c), holds, as above, we have $S^2\omega = \lim_{r\longrightarrow\infty} S^2 v_r = \lim_{r\longrightarrow\infty} v_{r+2} = \omega$ and we want to show that also $S\omega = \omega$. Supposing on the contrary, that $\omega \neq S\omega$, since

$$\frac{1}{2} d\left(S\omega, S^2\omega\right) = \frac{1}{2} d(S\omega, \omega) \leq d(S\omega, \omega), \tag{24}$$

by (6), we get

$$d(\omega, S\omega) \le w(S\omega, \omega)d(S^2\omega, S\omega) \le \psi\left(\left[d(S\omega, S^2\omega)\right]^{\beta} \cdot \left[d(\omega, S\omega)^{1-\beta}\right]\right) < \left[d(S\omega, \omega)\right]^{\beta} \cdot \left[d(\omega, S\omega)^{1-\beta}\right] = d(S\omega, \omega), \tag{25}$$

which is a contradiction. Consequently, $\omega = S\omega$, that is, ω is a fixed point of the mapping S. □

Example 7. Let $\mathscr{K} = [0, 3]$ and $d : \mathscr{K} \times \mathscr{K} \longrightarrow [0,+\infty)$ be the usual distance on $\mathbb{R}$. Consider the mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ be defined as

$$Sv = \begin{cases} \frac{4}{5}, & \text{if } v \in [0, 1], \\ \frac{1}{3}, & \text{if } v \in (1, 2], \\ \frac{\sqrt[3]{v^3 + 6v + \ln(4 - v)}}{5}, & \text{if } v \in (2, 3]. \end{cases} \tag{26}$$

Let also $w : \mathscr{K} \times \mathscr{K} \longrightarrow [0,\infty)$, where

$$w(v, t) = \begin{cases} 2, & \text{if } v, t \in [0, 1], \\ v^2 + t^2, & \text{if } v, t \in (1, 2), \\ 1, & \text{if } v = 0, t = 3, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

We remark that the space is not regular since, for example, considering the sequence $\{v_r\}$, with $v_r = (r + 8)/(2r + 4)$ we have $v_r \longrightarrow 1/2$ as $r \longrightarrow \infty$, $w(v_r, v_{r+1}) = v_r^2 + v_{r+1}^2 \le 1$, but $w(v_r, 1/2) = 0$. On the other hand, the mapping S is not continuous, but since $S^2 = 4/5$, we have that S^2 is a continuous mapping. Let the function $\psi \in \Psi$ defined as $\psi(z) = z/3$ and we choose $\beta = 1/9$. Thus, we have to check that (6) holds. We have to consider the following cases:

(1) For $v, t \in [0, 1]$, respectively, $v, t \in (1, 2)$, we have $d(Sv, St) = 0$, so (6) holds

(2) For $v = 0$ and $t = 3$

$$\begin{aligned} \frac{1}{2}d(0, S0) = \frac{2}{5} < 3 = d(0, 3) &\Rightarrow w(0, 3)d(S0, S3) \\ &= 0, 088621339 \le 0, 678785019 \\ &= \frac{1}{3}(0.8)^{1/9} \cdot (2, 288621339)^{8/9} \\ &= \psi\left(\left[d(0, S0)\right]^{1/9} \cdot \left[d(3, S3)\right]^{8/9}\right) \end{aligned} \tag{28}$$

(3) All other cases are not interesting because $w(v, t) = 0$

Consequently, the assumptions of Theorem 6 being satisfied, it follows that the mapping S has a fixed point, which is $v = 4/5$.

Corollary 8. *Let $(\mathscr{K}, d)$ be a complete metric space and S be a self-mapping on $\mathscr{K}$, such that,*

$$\frac{1}{2}d(v, Sv) \le d(v, t) \text{ implies } d(Sv, St) \le \psi\left(\left[d(v, Sv)\right]^{\beta} \cdot \left[d(t, St)\right]^{1-\beta}\right), \tag{29}$$

for each $v, t \in \mathscr{K} \setminus Fix(S)$, where $\psi \in \Psi$ and $\beta \in [0, 1)$. Then, S possesses a fixed point in $\mathscr{K}$.

Proof. Theorem 6 is sufficient to get $w(v, t) = 1$ for the proof. □

Moreover, taking $\psi(z) = z\kappa$, with $\kappa \in [0, 1)$ in Corollary (8), we obtain the following consequence.

Corollary 9. *Let $(\mathscr{K}, d)$ be a complete metric space and S be a self-mapping on $\mathscr{K}$, such that*

$$\frac{1}{2}d(v, Sv) \le d(v, t) \text{ implies } d(Sv, St) \le \kappa\left[d(v, Sv)\right]^{\beta} \cdot \left[d(t, St)\right]^{1-\beta}, \tag{30}$$

for each $v, t \in \mathscr{K} \setminus Fix(S)$, where $\beta \in [0, 1)$. Then, the mapping S possesses a fixed point in $\mathscr{K}$.

Definition 10. Let $(\mathscr{K}, d)$ be a metric space. The mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ is called an w - ψ -interpolative Ćirić-Reich-Rus contraction of Suzuki type if there exist $\psi \in \Psi$, $w : \mathscr{K} \times \mathscr{K} \longrightarrow [0,\infty)$, and positive reals $\beta, \alpha > 0$, with $\beta + \alpha < 1$, such that

$$\begin{aligned} \frac{1}{2}\mathrm{d}(v, Sv) \le \mathrm{d}(v, t) &\Rightarrow w(v, t)\mathrm{d}(Sv, St) \\ &\le \psi\left(\left[\mathrm{d}(v, t)\right]^{\beta} \cdot \left[\mathrm{d}(v, Sv)\right]^{\alpha} \cdot \left[\mathrm{d}(t, St)\right]\right)^{1-\alpha-\beta} \end{aligned} \tag{31}$$

for each $v, t \in \mathscr{K} \setminus \mathrm{Fix}(S)$.

Theorem 11. *Let $(\mathscr{K}, d)$ be a complete metric space and the mapping $S : \mathscr{K} \longrightarrow \mathscr{K}$ be an w - ψ -interpolative Ćirić-Reich-Rus contraction of the Suzuki type. Suppose that S is w -orbital admissible and $w(v_0, Sv_0) \ge 1$ for some $v_0 \in \mathscr{K}$. If $(\mathscr{K}, d)$ is w -regular or either*

(1) S is continuous or

(2) S^2 is continuous and $w(S\omega, \omega) \le 1$ for any $v \in Fix(S^2)$, then the mapping S has a fixed point in $\mathscr{K}$

Proof. Let $v_0 \in \mathscr{K}$ satisfy $w(v_0, Sv_0) \ge 1$ and $\{v_r\}$ be the sequence defined by $S^r(v_0) = v_r$ for each positive integer r. If $v_{r_0} = v_{r_{0+1}}$ for some $r_0 \in \mathbb{N}$, we get $v_{r_0} = Sv_{r_0}$, that means v_{r_0} is a fixed point of S. Then, we can assume that $v_r \ne v_{r+1}$ for each positive integer r. Moreover, due to the assumption that S is w-orbital admissible, as in the previous proof, we have

$$w(v_r, v_{r+1}) \geq 1. \quad (32)$$

By letting $v = v_{r-1}$ and $t = Sv_{r-1} = v_r$ in (31), we obtain

$$\begin{aligned}
\frac{1}{2}d(v_{r-1}, Sv_{r-1}) &= \frac{1}{2}d(v_{r-1}, v_r) \\
&\leq d(v_{r-1}, v_r) \Rightarrow d(v_r, v_{r+1}) \\
&\leq w(v_{r-1}, v_r)d(Sv_{r-1}, Sv_r) \\
&\leq \psi\left([d(v_{r-1}, v_r)]^\beta \cdot [d(v_{r-1}, Sv_{r-1})]^\alpha \cdot [d(v_r, Sv_r)]^{1-\alpha-\beta}\right) \\
&= \psi\left([d(v_{r-1}, v_r)]^\beta \cdot [d(v_{r-1}, v_r)]^\alpha \cdot [d(v_r, v_{r+1})]^{1-\alpha-\beta}\right),
\end{aligned} \quad (33)$$

then, using $\psi(z) < z$ for every $z > 0$.

$$d(v_r, v_{r+1}) \leq [d(v_{r-1}, v_r)]^{\beta+\alpha} \cdot [d(v_r, v_{r+1})]^{1-\alpha-\beta}, \quad (34)$$

or equivalent

$$[d(v_r, v_{r+1})]^{\alpha+\beta} < [d(v_{r-1}, v_r)]^{\alpha+\beta}. \quad (35)$$

So,

$$d(v_r, v_{r+1}) < d(v_{r-1}, v_r), \quad (36)$$

for every r ∈ ℕ. Therefore, the positive sequence $\{d(v_{r-1}, v_r)\}$ is decreasing. Eventually, by (33), we have

$$d(v_r, v_{r+1}) \leq \psi(d(v_{r-1}, v_r)), \quad (37)$$

and by repeating this process, we find that

$$d(v_r, v_{r+1}) \leq \psi^r(d(v_0, v_1)). \quad (38)$$

We assert that $\{v_r\}$ is a fundamental sequence in $(\mathscr{K}, \mathrm{d})$. Thus, using the triangle inequality with (38), we can write

$$\begin{aligned}
d(v_r, v_{r+l}) &\leq d(v_r, v_{r+1}) + d(v_{r+1}, v_{r+2}) + \cdots + d(v_{r+l-1}, v_{r+l}) \\
&\leq \psi^r d(v_0, v_1) + \psi^{r+1}(d(v_0, v_1)) + \cdots + \psi^{r+l-1}(d(v_0, v_1)) \\
&\leq \sum_{k=r}^{\infty} \psi^k(d(v_0, v_1)).
\end{aligned} \quad (39)$$

Taking $r \longrightarrow \infty$ in (39), we deduce that $\{v_r\}$ is a fundamental sequence in $(\mathscr{K}, \mathrm{d})$, and using the completeness $(\mathscr{K}, \mathrm{d})$, there exists $\omega \in \mathscr{K}$ such that

$$\lim_{r \longrightarrow \infty} \mathrm{d}(v_r, \omega) = 0. \quad (40)$$

We claim that the point ω is a fixed point of S. In the case of the space $(\mathscr{K}, \mathrm{d})$ being w-regular and $\{v_r\}$ verifies (32), that is, $w(v_r, v_{r+1}) \geq 1$ for every $r \in \mathbb{N}$, we get $w(v_r, \omega) \geq 1$. On the other hand, we know (see the proof of Theorem 6) that either

$$\frac{1}{2}d(v_r, Sv_r) \leq d(v_r, \omega), \quad (41)$$

or

$$\frac{1}{2}d(Sv_r, S(Sv_r)) \leq d(Sv_r, \omega), \quad (42)$$

holds, for every $r \in \mathbb{N}$. If (41) is holds, we obtain

$$\begin{aligned}
d(v_{r+1}, S\omega) &\leq w(v_r, \omega)d(Sv_r, S\omega) \\
&\leq \psi[d(v_r, \omega)]^\beta \cdot [d(v_r, Sv_r)]^\alpha \cdot [d(\omega, S\omega)]^{1-\alpha-\beta}, \\
&= \psi[d(v_r, \omega)]^\beta \cdot [d(v_r, v_{r+1})]^\alpha \cdot [d(\omega, S\omega)]^{1-\alpha-\beta} \\
&< d(v_r, \omega)]^\beta \cdot [d(v_r, v_{r+1})]^\alpha \cdot [d(\omega, S\omega)]^{1-\alpha-\beta}.
\end{aligned} \quad (43)$$

Letting $r \longrightarrow \infty$ in the above inequality, we get that $d(\omega, S\omega) = 0$, that is, $\omega = S\omega$. If the second condition (42) is true, we get that ω is a fixed point S by a similar argument.

Furthermore, if the w-regular of $(\mathscr{K}, \mathrm{d})$ is removed and, instead, S is continuous, we get that S has a fixed point in $\mathscr{K}$, because

$$\omega = \lim_{r \longrightarrow \infty} v_{r+1} = \lim_{r \longrightarrow \infty} Sv_r = S\left(\lim_{r \longrightarrow \infty} v_r\right) = S\omega. \quad (44)$$

Finally, if the mapping S is such that S^2 is continuous, we easily obtain $S^2\omega = \omega$. Supposing that $S\omega \neq \omega$, since $w(S\omega, \omega) \leq 1$ for any $v \in \mathrm{Fix}(S^2)$ and $(1/2)d(S\omega, S^2\omega) = (1/2)d(S\omega, \omega) \leq d(S\omega, \omega)$, we have

$$\begin{aligned}
d(\omega, S\omega) = d\left(S^2\omega, S\omega\right) &\leq w(S\omega, \omega)d\left(S^2\omega, S\omega\right) \\
&\leq \psi\left([d(S\omega, \omega)]^\alpha \cdot \left[d\left(S\omega, S^2\omega\right)\right]^\beta \cdot [d(\omega, S\omega)]^{1-\alpha-\beta}\right) \\
&< [d(S\omega, \omega)]^\alpha \cdot [d(S\omega, \omega)]^\beta \cdot [d(\omega, S\omega)]^{1-\alpha-\beta} = [d(\omega, S\omega)].
\end{aligned} \quad (45)$$

That is a contradiction. Thereupon, $S\omega = \omega$. □

Example 12. Let $\mathscr{K} = \{0, 1/8, 1/4, 1/2, 1\}$, $d : \mathscr{K} \times \mathscr{K} \longrightarrow [0, +\infty)$, $d(v, t) = |v - t|$ *, and* $S : \mathscr{K} \longrightarrow \mathscr{K}$ *, where* $S0 = S(1/2) = 1/8$, $S(1/8) = 1/2$ *, and* $S1 = S(1/4) = 1/4$ *. Consider the function* $w : \mathscr{K} \times \mathscr{K} \longrightarrow [0,\infty)$,

$$w(v, t) = \begin{cases} 1, & \text{if } (v, t) \in \left\{(0, 1), \left(0, \frac{1}{2}\right)\right\}, \\ 3, & \text{if } (v, t) \in \mathscr{K} \cup \left\{\frac{1}{4}\right\}, \\ t + 1, & \text{if } (v, t) \in \left\{\frac{1}{4}\right\} \cup \mathscr{K}, \\ 0, & \text{otherwise}, \end{cases} \quad (46)$$

let $\psi \in \Psi$, $\psi(z) = 2z/5$, and the real constants $\alpha = \beta = 1/3$.

Taking into account the definition of the function w, the only interesting situations are for $v = 0, y = 1/2$, respectively, $v = 0, y = 1$. For the first case, we have

$$\begin{aligned}\frac{1}{2}d(0, S0) &= \frac{1}{2}d\left(0, \frac{1}{8}\right) = \frac{1}{16} < \frac{1}{2} = d\left(0, \frac{1}{2}\right) \\ &\Rightarrow w\left(0, \frac{1}{2}\right)d\left(S0, S\frac{1}{2}\right) = 0 \\ &\leq \psi\left(\left[d\left(0, \frac{1}{2}\right)\right]^{1/3} \cdot [d(0, S0)]^{1/3}\right) \cdot \left[d\left(\frac{1}{2}, S\frac{1}{2}\right)\right]^{1/3}\right).\end{aligned} \tag{47}$$

For the second case,

$$\begin{aligned}\frac{1}{2}d(0, S0) &= \frac{1}{2}d\left(0, \frac{1}{8}\right) = \frac{1}{16} < 1 = d(0, 1) \Rightarrow w(0, 1)d(S0, S1) \\ &= \frac{1}{8} \leq 0, 158740105 = \frac{2}{5} \cdot \left(\frac{1}{8}\right)^{1/3} \cdot \left(\frac{3}{4}\right)^{1/3} \\ &= \psi\left([d(0, 1)]^{1/3} \cdot [d(0, S0)]^{1/3} \cdot [d(1, S1)]^{1/3}\right).\end{aligned} \tag{48}$$

Definition 13. Let $(\mathcal{K}, d)$ be a metric space. The mapping $S : \mathcal{K} \longrightarrow \mathcal{K}$ is called an ψ -interpolative Ćirić-Reich-Rus contraction of Suzuki type if there exist $\psi \in \Psi$ and the constants $\beta, \alpha > 0$, with $\beta + \alpha < 1$, such that

$$\begin{aligned}\frac{1}{2}d(v, Sv) \leq d(v, t) &\Rightarrow d(Sv, St) \\ &\leq \psi\left([d(v, t)]^{\beta} \cdot [d(v, Sv)]^{\alpha} \cdot [d(t, St)]^{1-\alpha-\beta}\right),\end{aligned} \tag{49}$$

for each $v, t \in \mathcal{K} \setminus \mathrm{Fix}(S)$.

Theorem 14. *Let $(\mathcal{K}, d)$ be a complete metric space and the mapping $S : \mathcal{K} \longrightarrow \mathcal{K}$ be an ψ -interpolative Ćirić-Reich-Rus contraction of the Suzuki type. Then, the mapping S has a fixed point in $\mathcal{K}$.*

Proof. Put $w(v, t) = 1$ in Theorem 11. □

Definition 15. Let $(\mathcal{K}, d)$ be a metric space. A mapping $S : \mathcal{K} \longrightarrow \mathcal{K}$ is called an interpolative Ćirić-Reich-Rus contraction of the Suzuki type if there exist $\kappa \in [0, 1)$ and positive reals $\beta, \alpha > 0$, with $\beta + \alpha < 1$, such that

$$\begin{aligned}\frac{1}{2}\mathrm{d}(v, Sv) \leq \mathrm{d}(v, t) \Rightarrow \mathrm{d}(Sv, St) \leq \kappa[\mathrm{d}(v, t)]^{\beta} \cdot [\mathrm{d}(v, Sv)]^{\alpha} \\ \cdot [\mathrm{d}(t, St)]^{1-\alpha-\beta},\end{aligned} \tag{50}$$

for each $v, t \in \mathcal{K} \setminus \mathrm{Fix(S)}$.

Theorem 16. *Let $(\mathcal{K}, d)$ be a complete metric space and $S : \mathcal{K} \longrightarrow \mathcal{K}$ be an interpolative Ćirić-Reich-Rus contraction of the Suzuki type. Therefore, S has a fixed point in $\mathcal{K}$.*

Proof. Put $\psi(z) = \kappa z$, for all $z > 0$, in Theorem 14. □

3. Conclusions

In this manuscript, we introduce new concepts on completeness of w-ψ-interpolative Kannan contraction of Suzuki type and w-ψ-interpolative Ćirić-Reich-Rus contraction of Suzuki type mappings in metric space. We prove the existence of some fixed point theorems for mappings these concepts. Further, we obtain some fixed point results and give examples to show that the new results are applicable. Interpolation contraction, which is generalized from the Kannan type contraction, is a new and interesting contraction in fixed point theory, and different interpolation contractions of Suzuki type studies can be obtained by combining it with a Suzuki type contraction in the future. Additionally, these proposed contractions can be generalized in other well-known spaces and can give new fixed point results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

Both authors contributed equally to this work. Both authors have read and approved the final manuscript.

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Research Article

Multiple Positive Solutions of Second-Order Nonlinear Difference Systems with Repulsive Singularities

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We study the existence of positive solutions for second-order nonlinear repulsive singular difference systems with periodic boundary conditions. Our nonlinearity may be singular in its dependent variable. The proof of the main result relies on a fixed point theorem in cones and a nonlinear alternative principle of Leray-Schauder; the result is applicable to the case of a weak singularity as well as the case of a strong singularity. An example is given; some recent results in the literature are improved and generalized.

1. Introduction

Difference systems are widely used in modeling real-life phenomena [1] and references therein. In this paper, we establish the existence positive solutions for the following nonlinear difference systems:

$$-\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=f(n,x(n))+e(n), \quad (1)$$

with the boundary conditions:

$$x(0)=x(T), p(0)\Delta x(0)=p(T)\Delta x(T), \quad (2)$$

where $q(n)=\text{diag}\,(q_1(n),q_2(n),\cdots,qN(n))$, $p(n)=\text{diag}\,(p_1(n),p_2(n),\cdots,pN(n))$, $e=(e_1,e_2,\cdots,e_N)T$, and $f=(f_1,f_2,\cdots,fN)T$, $N\geq 1$. By a periodic solution, we mean a function $x=(x_1,x_2,\cdots,x_N)T$, solving (1) and (2) and such that $x(n)\neq 0$ for all n. We call boundary condition (2) the periodic boundary conditions which are important representatives of nonseparated boundary conditions. For convenience, we denote by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ the sets of all integer numbers, natural numbers, and real numbers, respectively. For $a,b\in\mathbb{Z}$, let $\mathbb{Z}(a)=\{a,a+1,\cdots\}$, $\mathbb{Z}[a,b]=\{a,a+1,\cdots,b\}$ when $a\leq b$. As usual, Δ denotes the forward difference operator defined by

$$\Delta x(n)=x(n+1)-x(n). \quad (3)$$

In particular, the nonlinearity $f(x,x(n))$: $\mathbb{N}\times\mathbb{R}^N\setminus\{0\}\longrightarrow\mathbb{R}^N$ may have a repulsive singularity at $x=0$, from the physical explanation, which means that $\lim_{x\longrightarrow 0} f_i(n,x)=+\infty$, uniformly in $n\in\mathbb{Z}[1,T]$, $i=1,2,\cdots,N$.

Such repulsive singularity appears in many problems of applications such as the Brillouin focusing systems and nonlinear elasticity [2].

System (1) can be viewed as a discretization of the following more general class of the Sturm singular second-order differential system:

$$-(p(t)y0)0+q(t)y=f(t,y)+e(t). \quad (4)$$

Such systems, even in case $p\equiv 1$, where they are referred to as being of Klein-Gordon or Schrödinger type, appear in many scientific areas including fluid mechanics, gas dynamics, and quantum field theory. During the last few decades, the study of the existence of periodic solutions for singular differential equations has deserved the attention of many researchers [3–11]. Tracing back to 1987, Lazer and Solimini

[5] investigated the singular model:

$$x'' + \frac{h(t)}{x^\lambda} = g(t), \tag{5}$$

where $\lambda > 0, h, g$ are T-periodic functions and the mean value of g is negative, $\bar{g} < 0$. One of the common conditions to guarantee the existence of positive periodic solution is a so-called strong force condition (corresponds to the case $\lambda \geq 1$ in (5)) [11, 12]. For example, if we consider the system:

$$\ddot{x} + \nabla V_x(t, x) = f(t), \tag{6}$$

with $V(t, x) = 1/|x|^a$; the strong force condition holds for $\alpha \geq 2$. On the other hand, the existence of positive periodic solutions of the singular differential equations has been established with a weak force condition (corresponds to the case $0 < \lambda < 1$ in (5)) [13–15].

From then on, some classical tools have been used to study singular differential equations in the literature, including the degree theory [6, 11, 16], the method of the upper and lower solutions [8, 17], Schauder's fixed point theorem [14], some fixed point theorems in cones for completely continuous operators [13, 18], and a nonlinear Leray-Schauder alternative principle [19].

For the existence of periodic solutions of difference equations, some results have been obtained using the variational methods or the topological methods [1, 20–25]. For example, by minimax principle, Guo and Yu [23] discussed the existence of periodic solutions for difference equation:

$$-\Delta^2 x(n-1) + f(n, x(n)) = 0, \tag{7}$$

where the nonlinearity f is of superlinear or sublinear growth at infinity. Based on the method of the upper and lower solutions, Atici and Cabada [21] studied the existence of periodic solutions for difference equation:

$$-\Delta^2 x(n-1) + q(n)x(n) = f(n, x(n)). \tag{8}$$

In [26], Zhou and Liu investigated the following autonomous difference equations:

$$\Delta^2 x(n-1) + f(x(n)) = 0. \tag{9}$$

By Conley index theory, the author showed that the suitable assumptions of asymptotically linear nonlinear are enough to guarantee the existence of periodic solutions.

In this paper, we establish two different existence results of positive periodic solutions for (1) and (2) and proof of the existence of positive solutions; the first one is based on an application of a nonlinear alternative of Leray-Schauder, which has been used by many authors [19, 27, 28] and references therein; the second one is based on a fixed point theorem in cones. Our main motivation is to obtain new existence results for positive periodic solutions of the system:

$$\begin{cases} -\Delta[p_1(n-1)\Delta x(n-1)] + q_1(n)x(n) = \left(x^2+y^2\right)^{-a/2} + \mu\left(x^2+y^2\right)^{\beta/2} + e_1(n), \\ -\Delta[p_2(n-1)\Delta y(n-1)] + q_2(n)y(n) = \left(x^2+y^2\right)^{-a/2} + \mu\left(x^2+y^2\right)^{\beta/2} + e_2(n). \end{cases} \tag{10}$$

Here, we emphasize that the new results are applicable to the case of a strong singularity as well as the case of a weak singularity and that e does not need to be positive.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results. We will use the notation $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_i \geq 0$ for each $i = 1, 2, \cdots, N\}$, for $x = (x_1, \cdots, x_N), y = (y_1, \cdots, y_N)$, we write $x \geq y$, if $x - y = (x_1 - y_1, \cdots, x_N - Y_N) \in \mathbb{R}_+^N$. We say that a function $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is nondecreasing if $\varphi(x) \geq \varphi(y)$ for $x, y \in \mathbb{R}^N$ with $x \geq y$. For a given function p defined on $\mathbb{Z}[0, T]$, we denote its maximum and minimum by p^* and p_*, respectively.

2. Preliminaries

For $i = 1, 2, \cdots, N$, let us denote by $\varphi_i(n)$ and $\psi_i(n)$ the solutions of the corresponding homogeneous equations:

$$-\Delta[p_i(n-1)\Delta x(n-1)] + q_i(n)x(n) = 0, n \in \mathbb{Z}[1, T], \tag{11}$$

satisfying the initial conditions:

$$\varphi_i(0) = \varphi_i(1) = 0\,;\psi_i(0) = 0, p_i(0)\psi_i(1) = 1. \tag{12}$$

Let

$$D_i = \varphi_i(T) + p_i(T)\Delta\psi_i(T) - 2. \tag{13}$$

Throughout this paper, we always assume that

(H) For each $i = 1, 2, \cdots, N, p_i(n) > 0, q_i(n) \geq 0, q_i(\cdot)6 \not\equiv 0, n \in Z[1, T]$

Lemma 1 (see [29]). *If (H) holds, then $D_i > 0$.*

Lemma 2 (see [29]). *Assume (H) holds. For the solution of the problem:*

$$\begin{cases} -\Delta[p_i(n-1)\Delta x(n-1)] + q_i(n)x(n) = e_i(n), n \in \mathbb{Z}[1,T], \\ x(0) = x(T), p_i(0)\Delta x(0) = p_i(T)\Delta x(T), \end{cases} \tag{14}$$

the formula

$$x(n) = \sum_{s=1}^{T} G_i(n,s)e_i(s), \tag{15}$$

holds, where

$$G_i(n,s) = \frac{\psi_i(T)}{D_i}\varphi_i(n)\varphi_i(s) - \frac{p_i(T)\Delta\varphi_i(T)}{D_i}\psi_i(n)\psi_i(s) + \begin{cases} \frac{p_i(T)\Delta\psi_i(T)-1}{D_i}\varphi_i(n)\psi_i(s) - \frac{\varphi_i(T)-1}{D_i}\varphi_i(s)\psi_i(n), & 0 \le s \le n \le T+1, \\ \frac{p_i(T)\Delta\psi_i(T)-1}{D_i}\varphi_i(s)\psi_i(n) - \frac{\varphi_i(T)-1}{D_i}\varphi_i(n)\psi_i(s), & 0 \le n \le s \le T+1, \end{cases} \tag{16}$$

is the Green's function; the number D_i is defined by (13).

Lemma 3 (see [29]). *Under condition (H), the Green's function $G_i(n,s)$ of the boundary value problem (14) is positive, i.e., $G_i(n,s) > 0$ for $n, s \in Z[0,T]$.*

We denote

$$A_i = \min_{n,s\in Z[0,T]} G_i(n,s), B_i = \max_{n,s\in Z[0,T]} G_i(n,s), \sigma = A_i/B_i. \tag{17}$$

Obviously, $B_i > A_i > 0$ and $0 < \sigma_i < 1$.

Remark 4. If $p_i(t) = 1, q_i(t) = \alpha > 0$, then Green's function $G_i(n,s)$ of the boundary value problem (14) has the form:

$$G_i(n,s) = \begin{cases} \frac{\beta^{n-s} + \beta^{s-n+N}}{(\beta-\beta^{-1})(\beta^n - 1)}, & 0 \le s \le n \le T+1, \\ \frac{\beta^{s-n} + \beta^{n-s+N}}{(\beta-\beta^{-1})(\beta^n - 1)}, & 0 \le n \le s \le T+1, \end{cases} \tag{18}$$

where $\beta = (\alpha + 2 + \sqrt{\alpha(\alpha+2)})/2$. If n is even, a direct calculation shows that

$$\begin{aligned} A_i &= \frac{2\beta^{T/2}}{(\beta-\beta^{-1})\left(\beta^T - 1\right)}, \\ B_i &= \frac{1+\beta^T}{(\beta-\beta^{-1})\left(\beta^T - 1\right)}, \\ \sigma_i &= \frac{2\beta^{T/2}}{1+\beta^T} < 1. \end{aligned} \tag{19}$$

3. Main Results

In this section, we state and prove the new existence results for (1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is needed, which can be found in [30].

Lemma 5. *Assume Ω is a relatively compact subset of a convex set E in a normed space X. Let $\mathscr{A} : \bar{\Omega} \longrightarrow E$ be a compact map with $0 \in \Omega$. Then, one of the following two conclusions holds:*

(i) *T has at least one fixed point in Ω*

(ii) *There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda\mathscr{A}u$*

Let

$$X_1 = \{x : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R} \mid x(0) = x(T), p(0)\Delta x(0) = p(T)\Delta x(T)\}. \tag{20}$$

Then, X_1 is a Banach space with the norm

$$\|x\| = \max_{n\in Z[1,T]} x(n). \tag{21}$$

We take

$$X = X_1 \times X_1 \times \cdots \times X_1 (N \text{ copies}), \tag{22}$$

with the norm

$$|x| = \max\{\|x_1\|, \|x_2\|, \cdots, \|x_N\|\}. \tag{23}$$

Define

$$\gamma_i(n) = \sum_{s=1}^{T} G_i(n,s)e_i(s), \tag{24}$$

which corresponds to the unique solution of (14), and the *operator* $\mathscr{A} : X \longrightarrow X$ *by* $\mathscr{A}_x = (\mathscr{A}_1 x, \mathscr{A}_2 x, \cdots, \mathscr{A}_N x)^T$, *where*

$$(\mathscr{A}_i x)(n) = \sum_{s=1}^{T} G_i(n,s) f_i(s, x(s) + \gamma(s)), i = 1, 2, \cdots, N. \quad (25)$$

Now, we present the first existence result of the positive solution to problem (1).

Theorem 6. *Suppose that condition (H) holds and* $\gamma_* \geq 0$. *Furthermore, we assume that*

(H$_1$*) For each constant* $L > 0$, *there exists a function* $\varphi L(n) > 0$ *for all* $n \in \mathbb{Z}[1, T]$ *such that each component* f_i *of* f *satisfies* $f_i(n,x) \geq \varnothing_L(n)$ *for all* $(n, |x|) \in \mathbb{Z}[1,T] \times (0, L]$

(H$_2$*) For each component* f_i *of* f, *there exist nonnegative functions* $g_i(x)$, $h_i(x)$, *and* $k_i(n)$ *such that*

$$0 \leq f_i(n,x) \leq \{g_i(x) + h_i(x)\} k_i(n) \text{ for all } (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}_+^N\{0\}, \quad (26)$$

and $g_i(x) > 0$ *is nonincreasing and* $h_i(x)/g_i(x)$ *is nondecreasing in* x

(H$_3$*) There exists a positive number* r *such that* $\sigma r + \gamma_* > 0$ *and*

$$\frac{r}{g_i(\gamma_*, \cdots, \gamma_*, \sigma_i r + \gamma_*, \gamma_*, \cdots, \gamma_*)\{1 + ((h_i(r + \gamma^*, \cdots, r + \gamma^*))/(g_i(r + \gamma^*, \cdots, r + \gamma^*)))\}} > K_i^*, \quad (27)$$

for all $i = 1, 2, \cdots, N$. *Here,*

$$\begin{aligned} K_i(n) &= \sum_{s=1}^{T} G_i(n,s) k_i(s), \\ \sigma &= \min_{i=1,2,\cdots,N} \{\sigma_i\}, \\ \gamma^* &= \min_{i,n} \gamma(n), \\ \gamma^* &= \max_{i,n} \gamma(n). \end{aligned} \quad (28)$$

Then, (1) and (2) has at least one positive periodic solution x with $x(n) > \gamma(n)$ for all $n \in Z[0,T]$ and $0 < |x - \gamma| < r$.

Proof. We first show that

$$-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f(n, x(n) + \gamma(n)), \quad (29)$$

together with (2) has a positive solution x satisfying $x(n) + \gamma(n)$ for $n \in Z[0,T]$ and $0 < |x| < r$. If this is true, it is easy to see that $u(n) = x(n) + \gamma(n)$ will be a positive solution of (1) and (2) with $0 < |u - \gamma| < r$ since

$$\begin{aligned} -\Delta[p(n-1)\Delta u(n-1)] + q(n)u(n) &= -\Delta[p(n-1)\Delta(x(n-1) \\ &+ \gamma(n-1))] + q(n)(x(n) + \gamma(n)) = f(n, x(n) + \gamma(n)) \\ &+ e(n) = f(n, u(n)) + e(n). \end{aligned} \quad (30)$$

Since (H$_3$) holds, let $J_0 = \{j_0, j_0 + 1, \cdots\}$, we can choose $j_0 \in \{1, 2, \cdots\}$ such that $1/J_0 \leq \sigma r + \gamma_*$ and

$$g_i(\gamma_*, \cdots, \gamma_*, \sigma_i r + \gamma_*, \gamma_*, \cdots, \gamma_*)\left\{1 + \frac{h_i(r + \gamma^*, \cdots, r + \gamma^*)}{g_i(r + \gamma^*, \cdots, r + \gamma^*)}\right\} K_i^* + \frac{1}{j_0} < r, \quad (31)$$

for all $i = 1, 2, \cdots, N$.

Fix $j \in J_0$. Consider the family of systems

$$\begin{aligned} -\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) &= \lambda f^j(n, x(n) + \gamma(n)) \\ &+ \frac{q(n)}{j}, n \in \mathbb{Z}[1, N], \end{aligned} \quad (32)$$

where $\lambda \in [0,1]$ and for each $i = 1, 2, \cdots, N$,

$$f_i^j(n,x) = \begin{cases} f_i(n,x), & \text{if } x \geq \dfrac{1}{j}, \\ f_i\left(n, x_1, \cdots, x_{i-1}, \dfrac{1}{j}, x_i + 1, \cdots, x_N\right), & \text{if } x \leq \dfrac{1}{j}. \end{cases} \quad (33)$$

Problem (29) and (2) are equivalent to the following fixed point problem:

$$x_i(n) = \lambda \sum_{s=1}^{T} G_i(n,s) f_i^j(s, x(s) + \gamma(s)) + \frac{1}{j} = \lambda\left(\mathscr{A}_i^j x\right)(n) + \frac{1}{j}, \quad (34)$$

for each $i = 1, 2, \cdots, N$, here, we used the fact

$$\sum_{s=1}^{T} G_i(n,s) q_i(s) \equiv 1, i = 1, 2, \cdots, N. \quad (35)$$

We claim that any fixed point x of (34) for any $\lambda \in [0,1]$ must satisfy $|x| \neq r$. Otherwise, assume that x is a fixed point of (34) for some $\lambda \in [0,1]$ such that $|x| = r$. Without loss of generality, we assume that $|x_l| = r$ for some $l = 1, 2, \cdots, N$.

Thus, we have

$$\begin{aligned}
x_l(n) - \frac{1}{j} &= \lambda \sum_{s=1}^{N} G_l(n,s) f_l^j(n, x(s) + \gamma(s)) ds \geq \lambda A_l \sum_{s=1}^{T} f_l^j(n, x(s) \\
&\quad + \gamma(s)) ds = \sigma_l B_l \lambda \sum_{s=1}^{T} f_l^j(n, x(s) \\
&\quad + \gamma(s)) ds \geq \sigma_l \max_n \left\{ \lambda \sum_{s=1}^{T} G_l(n,s) f_l^j(n, x(s) + \gamma(s)) ds \right\} \\
&= \sigma_l \left\| x_l - \frac{1}{j} \right\|.
\end{aligned} \tag{36}$$

Hence, for all $n \in Z[1,T]$, we have

$$x_l(n) \geq \sigma_l \left\| x_l - \frac{1}{j} \right\| + \frac{1}{j} \geq \sigma_l \left(\|x_l\| - \frac{1}{j} \right) + \frac{1}{j} \geq \sigma_l r. \tag{37}$$

Therefore,

$$x_l(n) + \gamma_l(n) \geq \sigma_l r + \gamma_* > \frac{1}{j}. \tag{38}$$

Using (34), we have from condition (H_2), for all $n \in Z[1, T]$,

$$\begin{aligned}
x_l(n) &= \lambda \sum_{s=1}^{T} G_l(n,s) f_l^j(s, x(s) + \gamma(s)) + \frac{1}{j} \\
&= \lambda \sum_{s=1}^{T} G_l(n,s) f_l(s, x(s) + \gamma(s)) + \frac{1}{j} \\
&\leq \sum_{s=1}^{T} G_l(n,s) f_l(s, x(s) + \gamma(s)) + \frac{1}{j} \\
&\leq \sum_{s=1}^{T} G_l(n,s) k_l(s) g_l(x(s) + \gamma(s)) \left\{ 1 + \frac{h_1(x(s) + \gamma(s))}{g_1(x(s) + \gamma(s))} \right\} \\
&\leq g_l(\gamma_*, \cdots, \gamma_*, \sigma_l r + \gamma_*, \gamma_*, \cdots, \gamma_*) \\
&\quad \cdot \left\{ 1 + \frac{h_l(r + \gamma^*, \cdots, r + \gamma^*)}{g_l(r + \gamma^*, \cdots, r + \gamma^*)} \right\} K_l^* + \frac{1}{j_0}.
\end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned}
r = |x_l| &\leq g_l(\gamma_*, \cdots, \gamma_*, \sigma_l r + \gamma_*, \gamma_*, \cdots, \gamma_*) \\
&\quad \cdot \left\{ 1 + \frac{h_l(r + \gamma^*, \cdots, r + \gamma^*)}{g_l(r + \gamma^*, \cdots, r + \gamma^*)} \right\} K_l^* + \frac{1}{j_0}.
\end{aligned} \tag{40}$$

This is a contradiction to the choice of j_0, and the claim is proved.

From this claim, the nonlinear alternative of Leray-Schauder guarantees that

$$x(n) = \left(\mathscr{A}^j x\right)(n) + \frac{1}{j}, \tag{41}$$

has a fixed point, denoted by $x^j(n)$, in $B_r = \{x \in X : |x| < r\}$, i.e.,

$$-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f^j(n, x(n) + \gamma(n)) + \frac{q(n)}{j}, \tag{42}$$

has a periodic solution x^j with $|x^j| < r$.

Next, we claim that these solutions $x^j(n) + \gamma(n)$ have a uniform positive lower bound, that is, there exists a constant $\delta > 0$, independent of $j \in J_0$, such that

$$\min_{i,n} \left\{ x_i^j(n) + \gamma_i(n) \right\} \geq \delta, \tag{43}$$

for all $j \in J_0$. To see this, we know from (H_1) that there exists a continuous function $\phi_r + \gamma^*(n)$ such that each component f_i of f satisfies $f_i(n,x) \geq \phi_{r+\gamma^*}(n)$ for all $(n, |x|) \in \mathbb{Z}[1,T] \times (0, r + \gamma^*]$. Now, let $x^{r+\gamma^*}(n)$ be the unique solution to

$$-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = \Phi(n), \tag{44}$$

with (2), here $\Phi(n) = (\phi_{r+\gamma^*}(n), \cdots, \phi_{r+\gamma^*}(n))^T$. Then, we have

$$x_i^{r+\gamma^*}(n) + \gamma_i(n) = \sum_{s=1}^{T} G_i(n,s) \phi_{r+\gamma^*}(s) + \gamma_i(n) \geq \Phi_* + \gamma_* > 0 \tag{45}$$

for each $i = 1, \cdots, N$, here

$$\Phi_* = \inf_n \Phi_i(n), \; \Phi_i(n) = \sum_{s=1}^{T} G_i(n,s) \phi_{r+\gamma^*}(s). \tag{46}$$

Next, we show that (43) holds for $\delta = \Phi_* + \gamma_* > 0$. To see this, for each $i = 1, \cdots, N$, since $x_i^j(n) + \gamma_i(n) \leq r + \gamma^*$ and $x_i^j(n) + \gamma_* \geq 1/j$, we have

$$\begin{aligned}
x_i^j(n) + \gamma_i(n) &= \sum_{s=1}^{T} G_i(n,s) f_i^j\left(s, x^j(s) + \gamma(s)\right) + \gamma_i(n) \\
&\quad + \frac{1}{j} \geq \sum_{s=1}^{T} G_i(n,s) \varnothing_{r+\gamma^*} + \gamma_i(n) \geq \Phi_* + \gamma_* \coloneqq \delta.
\end{aligned} \tag{47}$$

The fact $|x(n)| < r$ and (43) show that for each $i = 1, 2, \cdots, N$, $\{x_i^j\}_{j \in J_0}$ is a bounded family on $\mathbb{Z}[1,T]$. Moreover, we have

$$x_i^j(0) = x_i^j(T), p_i(0)\Delta x_i^j(0) = p_i(T)\Delta x_i^j(T), \tag{48}$$

which implies that

$$x_i^j(T+1) = \frac{p_i(0)}{p_i(T)} \Delta x_i^j(0) + x_i^j(T), j \in J_0. \tag{49}$$

Thus, the Arzela–Ascoli theorem guarantees that $\{x_i^j\}$ $j \in J_0$ has a subsequence, $\{x_i^{jk}\} j_k \in J_0, k \in N$ converging uniformly on $\mathbb{Z}[0, T+1]$ to a function x_i. Let $x = (x_1, \cdots, x_N)$, $x(n)$ satisfies $\delta \le x_i(n) + \gamma_i(n) < r + \gamma^*$ for all $n \in \mathbb{Z}[1, T]$ and $i = 1, \cdots, N$. Moreover, x_i^{jk} satisfies the integral equation:

$$x_i^{jk}(n) = \sum_{s=1}^{T} G_i(n, s) f_i\left(s, x^{jk}(s) + \gamma(s)\right) + \frac{1}{jk}, i = 1, \cdots, N. \tag{50}$$

Letting $k \longrightarrow \infty$, we arrive at

$$x_i(n) = \sum_{s=1}^{T} G_i(n, s) f_i(s, x(s) + \gamma(s)), i = 1, 2, \cdots, N, \tag{51}$$

here, we have used the fact that $f(n, x)$ is with respect to (n, x) with $n \in \mathbb{Z}[1, T]$ and $x > 0$ satisfying $\delta \le |x| \le r + \gamma^*$. Therefore, x is a positive periodic solution of (1) and satisfies $0 < |x| \le r$

Corollary 7. *Assume that (H) holds, $\alpha > 0$, $\beta \ge 0$. Then, for each e_1, e_2 with $\gamma_* \ge 0$, we have*

(i) if $\beta < 1$, then (10) has at least one positive periodic solution for each $\mu > 0$

(ii) if $\beta \ge 1$, then (10) has at least one positive periodic solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant

Proof. We will apply Theorem 6. To this end, assumption (H_1) is fulfilled by $\varnothing_L = (\sqrt{2L})^{-\alpha}$. If we take

$$\begin{aligned} g_1(x, y) = g_2(x, y) &= \left(x^2 + y^2\right)^{-\alpha/2}, \\ h_1(x, y) = h_2(x, y) &= \mu\left(x^2 + y^2\right)^{\beta/2}, \end{aligned} \tag{52}$$

and $k_1(n) = k_2(n) = 1$, then (H_2) is satisfied.

Let

$$\omega_1(n) = \sum_{s=1}^{T} G_1(n, s), \omega_2(n) = \sum_{s=1}^{T} G_2(n, s). \tag{53}$$

Then, the existence condition (H_3) becomes

$$\mu < \frac{r\left[(\sigma_i r + \gamma_*)^2 + \gamma_*^2\right]^{2/\alpha} - \omega_i^*}{2^{(\alpha+\beta)/2}(r + \gamma^*)^{\alpha+\beta}}, i = 1, 2, \tag{54}$$

for some $r > 0$. So, (10) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \min_{i=1,2} \sup_{r>0} \frac{r\left[(\sigma_i r + \gamma_*)^2 + \gamma_*^2\right]^{2/\alpha} - \omega_i^*}{2^{(\alpha+\beta)/2}(r + \gamma^*)^{\alpha+\beta}}, i = 1, 2. \tag{55}$$

Note that $\mu_1 = \infty$ if $\beta < 1$ and $\mu_1 < \infty$ if $\beta \ge 1$. We have (i) and (ii).

In more general, we can obtain the following result.

Corollary 8. *Assume that (H) holds and there exist functions $a, \hat{a}, b, \hat{b}$ and $\alpha, \beta > 0$ such that, for $i = 1, 2, \cdots, N$,*

$$\frac{\alpha(n)}{|x|^\alpha} + b(n)|x|^\beta \le f_i(n, x) \le \frac{\hat{\alpha}(n)}{|x|^\alpha} + \mu\hat{b}(n)|x|^\beta. \tag{56}$$

Then, for each e with $\gamma_* \ge 0$, we have

(i) if $\beta < 1$, then (10) has at least one positive periodic solution for each $\mu > 0$

(ii) if $\beta \ge 1$, then (10) has at least one positive periodic solution for each $0 < \mu < \mu_2$, where μ_2 is some positive constant

By using a fixed point theorem for compact maps on conical shells [31], we established the second positive periodic solution for (1). Recall that a compact operator means an operator which transforms every bounded set into a relatively compact set and introducing the definition of a cone.

Definition 9. Let X be a Banach space and let K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$

(ii) $u, -u \in K$ implies $u = 0$

Lemma 10 (see [31]). *Let X be a Banach space and K a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \in \Omega_2$. Let*

$$\Phi : K \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K \tag{57}$$

be a continuous and completely continuous operator such that

(i)

$$\|\Phi x\| \le \|x\| \text{ for } x \in K \cap \partial\Omega_1 \tag{58}$$

(ii) There exist $\psi \in K \setminus \{0\}$ such that $x \ne \Phi x + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$

Then, F has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The same conclusion remains valid if (i) holds on $K \cap \partial\Omega_2$, and (ii) holds on $K \cap \partial\Omega_1$.

Define

$$K = \left\{x = (x_1, \cdots, x_N) \in X : \min_{0 \le n \le T} x_i(n) \ge \sigma_i \|x_i\|\right\} \text{ for all } n \in Z[0, T], i = 1, \cdots, N\}. \quad (59)$$

Then, one can readily verify that K is a cone in X.

Theorem 11. *Suppose conditions (H), (H_1)–(H_3) hold. Furthermore, assume that the following two conditions are satisfied:*

(H_4) There exist continuous, nonnegative functions $g^1(x)$, $h^1(x)$ and $k^1(n)$ such that

$$f_i(n, x) \ge \left\{g_i^1(x) + h_i^1(x)\right\} k_i^1(n) \text{ for all } (n, x) \in [0, T] \times \mathbb{R}_+^n\{0\}, \quad (60)$$

where $g_i^1(x) > 0$ is nonincreasing and $h_i^1(x)/g_i^1(x)$ is nondecreasing in x

(H_5) There exists $R > r$ such that

$$\frac{\sigma R}{g_i^1(R + \gamma^*, \cdots, R + \gamma^*)\left\{1 + \left((h_i^1(\sigma_1 R + \gamma_*, \cdots, \sigma_n R + \gamma_*))/(g_i^1(\sigma_1 R + \gamma_*, \cdots, \sigma_n R + \gamma_*))\right)\right\}} \le K_{i*}. \quad (61)$$

Then, problems (1) and (2) have another one positive periodic solution $\tilde{x}$ with $r < |\tilde{x} - \gamma| \le R$.

Proof. Let $\mathscr{A}x = (\mathscr{A}_1 x, \cdots, \mathscr{A}_N x)^T$, $\mathscr{A}_i x$ is given by (25), then, it is easy to verify that $\mathscr{A}$ is well defined and maps X into K. Moreover, $\mathscr{A}$ is continuous and completely continuous, and let K be a cone in X defined by (59). Define the

$$\Omega_1 = \{x \in X : |x| < r\}, \Omega_2 = \{x \in X : |x| < R\}. \quad (62)$$

As in the proof of Theorem 3.1, we only need to show that (29) has a positive periodic solution $u \in X$ with $u(n) + \gamma(n) > 0$ and $r < |u| \le R$. We claim that

(i)

$$|\mathscr{A}x| \le |x| \text{ for } x \in K \cap \partial\Omega_1 \quad (63)$$

(ii) There exist $\psi \in K \setminus \{0\}$ such that $x \ne \mathscr{A}x + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$

We start with (i). In fact, if $x \in K \cap \partial\Omega_1$, then $|x| = r$ and $\sigma_i r + \gamma_* \le x_i(n) + \gamma(n) \le r + \gamma^*$ for all $t \in [0, T]$. Fix $i \in \{1, 2, \cdots, N\}$, thus, we have

$$\begin{aligned}(\mathscr{A}_i x)(t) &= \sum_{s=1}^{T} G_i(n, s) f_i(s, x(s) + \gamma(s)) \le \sum_{s=1}^{T} G_i(n, s) k_i(s) g_i(x(s) + \gamma(s))\left\{1 + \frac{h_i(x(s) + \gamma(s))}{g_i(x(s) + \gamma(s))}\right\} \le g_i(x(s) \\ &\quad + \gamma(s))\left\{1 + \frac{h_i(x(s) + \gamma(s))}{g_i(x(s) + \gamma(s))}\right\} \sum_{s=1}^{T} G_i(n, s) k_i(s) \\ &\le g_i(\gamma_*, \cdots, \gamma_*, \sigma_i r + \gamma_*, \gamma_*, \cdots, \gamma_*) \cdot \left\{1 + \frac{h_i(r + \gamma^*, \cdots, r + \gamma^*)}{g_i(r + \gamma^*, \cdots, r + \gamma^*)}\right\} K_{i*} < r = |x|.\end{aligned} \quad (64)$$

Therefore, $\|\mathscr{A}_i x\| \le |x|$ for each $i = 1, 2, \ldots, N$. This implies that (i) holds.

Next, we show that (ii) holds. Let $\psi \equiv (1, \cdots, 1)$, then $\psi \in K \setminus \{0\}$. Suppose that there exists $x \in K \cap \partial\Omega_2$ and $\lambda > 0$ such that $x = \mathscr{A}x + \lambda\psi$. Since $x \in K \cap \partial\Omega_2$, then $\sigma_i R + \gamma_* \le x_i(n) + \gamma(n) \le R + \gamma^*$ for all $n \in Z[0, T]$. As a result, it follows from (H_4) and (H_5) that, for all $n \in Z[0, T]$,

$$\begin{aligned}x_i(n) &= (\mathscr{A}_i x)(n) + \lambda = \sum_{s=1}^{T} G_i(n, s) f_i(s, x(s) + \gamma(s)) ds + \lambda \\ &\ge \sum_{s=1}^{T} G_i(n, s) k_i^l(s) g_1^l(x(s) + \gamma(s))\left\{1 + \frac{h_i^l(x(s) + \gamma(s))}{g_i^l(x(s) + \gamma(s))}\right\} \\ &\quad + \lambda \ge g_i^l(x(s) + \gamma(s))\left\{1 + \frac{h_i^l(x(s) + \gamma(s))}{g_i^l(x(s) + \gamma(s))}\right\} \sum_{s=1}^{T} G_i(n, s) k_i^l(s) \\ &\quad + \lambda \ge g_i^l(R + \gamma^*, \cdots, R + \gamma^*)\left\{1 + \frac{h_i^l(\sigma_1 R + \gamma_*, \cdots, \sigma_N R + \gamma_*)}{g_i^l(\sigma_1 R + \gamma_*, \cdots, \sigma_N R + \gamma_*)}\right\} K_{i*} \\ &\quad + \lambda \ge \sigma R + \lambda.\end{aligned} \quad (65)$$

Hence, $\min_{0 \le n \le T} x_i(n) > \sigma R$; this is a contradiction and we prove the claim.

Now, Lemma 3.7 guarantees that $\mathscr{A}$ has at least one fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r \le |x| \le R$.

Let us consider again the example (10) in Corollary 7 for the superlinear case.

Corollary 12. *Assume in (10) that $p_i, q_i (i = 1, 2)$ satisfy (H), for each e_1, e_2 with $\gamma_* \ge 0, \beta > 1$. Then, for each μ with $0 < \mu < \mu_1$, where μ_1 is given as in Corollary 7, problem (10) has at least two different positive solutions. To verify*

(H_4), one may take

$$g_1^1(x,y) = g_2^1(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{-\alpha/2}, \quad h_2^1(x,y) = \frac{1}{2}\left(x^2+y^2\right)^{\beta/2}, \tag{66}$$

and $k^1(n)_1 = k_2^1(n) = 1$. If $\beta > 1$, then the existence condition (H_5) becomes

$$\mu \geq \frac{2^{(\alpha+2)/2}(R+\gamma *)^{\alpha}\sigma R - 2\omega_{i*}}{\left[(\sigma_1 R+\gamma_*)^2 + (\sigma_2 R+\gamma_*)^2\right]^{(\alpha+\beta)/2}\omega_{i*}}, i = 1, 2. \tag{67}$$

Since $\beta > 1$, the right-hand side goes to 0 as $R \longrightarrow +\infty$. Thus, for any given $0 < \mu < \mu 1$, it is always possible to find such $R \gg r$ that (67) is satisfied. Thus, (10) has an additional positive periodic solution $\tilde{x}$.

Remark 13. We emphasize that our results are applicable to the case of a strong singularity as well as the case of a weak singularity since we only need $\alpha > 0$. Moreover, e does not need to be positive. In fact, using the assumption that the Green function is positive, one may readily verify that $\gamma_* \geq 0$ is equivalent to the $\sum_{i=1}^{N} e_i(n) \geq 0$, $i = 1, 2, \cdots, N$.

Let us consider the 2-dimensional system

$$\begin{cases} -\Delta^2 x(n-1) + x(n) = \dfrac{\alpha(n)}{|x|^{\alpha}} + \mu b(n)|x|^{\beta} + e_1(n), \\ -\Delta^2 y(n-1) + y(n) = \dfrac{\alpha(n)}{|x|^{\alpha}} + \mu b(n)|x|^{\beta} + e_2(n), \end{cases} \tag{68}$$

with

$$e_i(n) = n(d_i - n), d_i \in \mathbb{R}, i = 1, 2. \tag{69}$$

Example 1. Assume that $\alpha > 0$, $\beta > 1$, $a(n)$, and $b(n)$ are positive functions, $e_i(n)$, $i = 1, 2$ are given by (69) with

$$d_1 + 2d_2 \geq 5. \tag{70}$$

Then, the results in Corollary 12 hold.

Proof. We only need show $\gamma_* \geq 0$, which is equivalent to

$$\sum_{i=1}^{2} e_i(n) \geq 0, \tag{71}$$

Since $d_1 + 2d_2 \geq 5$, a direct computation show that

$$\sum_{i=1}^{2} e_i(n) \geq 0 \sum_{i=1}^{2} = n(d_i - n) \geq 0. \tag{72}$$

4. Conclusions

In this paper, we study the periodic problem for nonlinear difference systems with a singularity of repulsive type in the case of $\gamma_* \geq 0$. The proofs of main results are based on a nonlinear alternative principle of Leray-Schauder and a fixed point theorem in cones. It is interesting that the singularity f is applicable to the case of a weak singularity as well as the case of a strong singularity. In the next research, we will continue to study the periodic problem to the difference systems like (10) where f may have attractive singularity at $x = 0$, and whether the condition $\gamma_* \geq 0$ can be removed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Final Value Problem for Parabolic Equation with Fractional Laplacian and Kirchhoff's Term

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In this paper, we study a diffusion equation of the Kirchhoff type with a conformable fractional derivative. The global existence and uniqueness of mild solutions are established. Some regularity results for the mild solution are also derived. The main tools for analysis in this paper are the Banach fixed point theory and Sobolev embeddings. In addition, to investigate the regularity, we also further study the nonwell-posed and give the regularized methods to get the correct approximate solution. With reasonable and appropriate input conditions, we can prove that the error between the regularized solution and the search solution is towards zero when δ tends to zero.

1. Introduction

The aim of this study is to investigate the final value for the space fractional diffusion equation

$$\begin{cases} \dfrac{^{\mathscr{C}}\partial^{\alpha}}{\partial t^{\alpha}} v(x,t) + (\|\nabla v\|_{L^2})(-\Delta)^{\beta} v(x,t) = F(x,t), & x \in \Omega, t \in (0,T), \\ v(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ v(x,T) = f(x), & x \in \Omega, \end{cases} \quad (1)$$

where the symbol $^{\mathscr{C}}\partial^{\alpha} v(t)/\partial t^{\alpha}$ is called the conformable derivative which is defined clearly in Section 2. Here, $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with the smooth boundary $\partial\Omega$, and $T > 0$ is a given positive number. The function F represents the external forces or the advection term of a diffusion phenomenon, etc., and the function f is the final datum which will be specified later.

The applications of the conformable derivative are interested in various models such as the harmonic oscillator, the damped oscillator, and the forced oscillator (see, e.g., [1]), electrical circuits (see, e.g., [2]), chaotic systems in dynamics (see, e.g., [3]), and quantum mechanics (see, e.g., [4]). From the paper, see, e.g., [5], we must confirm that the study of the ODE problem with the conformable derivative is very different from the study of the PDE problem with a conformable derivative. Results and research methods of the well-posedness for the ODE and PDE model are not the same and are completely different. The following two remarks confirm what we have just pointed out.

Remark 1. Let us first discuss conformable ODEs. Let v be the functions whose domain of its value is $\mathbb{R}$. If $\alpha = 1$, $^{\mathscr{C}}\partial^{\alpha}/\partial t^{\alpha}$ becomes the classical derivative. If $0 < \alpha < 1$, by the paper of [6], we know that the relation between the conformable derivative and the classical derivative by the following lemma.

Lemma 2. *If $v : [0,T] \longrightarrow \mathbb{R}$, then a conformable derivative of order α at $s > 0$ of v exists if and only if it is differentiable at s, and the following equality is true:*

$$\frac{^{\mathscr{C}}\partial^{\alpha} v(s)}{\partial s^{\alpha}} = s^{1-\alpha} \frac{\partial v(s)}{\partial s}. \quad (2)$$

Remark 3. In the following, we mention the PDEs with conformable derivative where D is a Sobolev space, such as $L^2(\Omega)$, $W^{\gamma,p}(\Omega)$, and $D(A^\gamma)$. When we study the PDE model, we often do with a multivariable function $v : (0, T) \longrightarrow D$, where D is a Sobolev space. This means that, for each t, $v(t)$ can take on values in many classes of spaces with $D_1 \hookrightarrow D \hookrightarrow D_2 \cdots$. Some illustrated examples given in [5] say that (2) may be not true on Sobolev spaces.

Let us mention some recent works on diffusion equations with a conformable derivative, for example, [2, 5, 7–16]. Some interesting papers on fractional diffusion equations can be found in [17–24] and the references therein.

When $\alpha = 1$, the main equation of Problem (1) appears in many population dynamics. By the work of Chipot and Lovat [25], we know that the diffusion coefficient B is dependent on the entire population in the domain instead of local density; that is, the moves are guided by considering the global state of the vehicle. The function u is a descriptive population density (e.g., bacteria) spread. According to article [26], we find that model (1) is a type of Kirchhoff equation, arising in vibration theory; see, for example, [27].

(i) This paper is the first study on the final value problem for a diffusion equation with a Kirchhoff-type equation and conformable derivative. Since our models are nonlinear, in order to establish the existence and uniqueness of solutions, we have to use the Banach contracting mapping theorem combined with some techniques to evaluate inequality and some Sobolev embeddings. One of the most difficult points is finding the appropriate functional spaces for the solution

(ii) The second result is to investigating the regularized solution for our problem. We show the ill-posedness of the problem and give Fourier regularization. The most difficult thing that we have to overcome is finding the appropriate space, to prove that the regularized solution converges with the exact solution

It can be said that our article is one of the first results, giving a general and comprehensive picture, considering both the frequency and the inaccuracy of Kirchhoff's diffusion equation with fractional time and space derivative. Using complex and interoperable assessment techniques, we find the right keys and tools to achieve both of our goals.

This paper is organized as follows. In Section 3, we present the existence of the backward Problem (1) with the simple case $F = 0$. In the appropriate terms of the terminal data f, we show that the mild solution of (1) in the case $\beta < 1$ converges with the mild solution of the same problem in the case $\beta = 1$ when $\beta \longrightarrow 1^-$. Finally, in Section 4, we consider a backward problem with an inhomogeneous source term. The first part of this section discusses the existence of a mild solution under the appropriate conditions of the source function F. Furthermore, we also give an example, which shows that the problem is not stable, and then look for the approximate solution. Using the Fourier truncation method, we involve the regularized solution. Convergence error between the regularized solution and the correct solution has also been established, with some suitable conditions of input value data.

2. Preliminaries

2.1. Conformable Derivative Model. Let the function $v : [0, \infty) \longrightarrow D$, where D is a Banach space.

If for each $t > 0$, the limitation

$$\frac{{}^{\mathscr{C}}\partial^\alpha v(t)}{\partial t^\alpha} := \lim_{\varepsilon \longrightarrow 0} \frac{v\left(t + \varepsilon t^{1-\alpha}\right) - v(t)}{\varepsilon} \text{ in } D, \tag{3}$$

finitely exists, then it is called the conformable derivative of order $\alpha \in (0, 1]$ of v. We can refer the reader to [6, 8, 14, 28, 29].

We introduce fractional powers of $\mathscr{A}$ as follows:

$$D(\mathscr{A}^v) = \left\{ g \in L^2(\Omega) : \sum_{j=1}^{\infty} \left|\left\langle g, w_j \right\rangle\right|^2 \lambda_j^{2v} < \infty \right\}. \tag{4}$$

The space $D(\mathscr{A}^v)$ is a Banach space in the following with the corresponding norm:

$$\|g\|_{D(\mathscr{A}^v)} := \left(\sum_{j=1}^{\infty} \left|\left\langle g, w_j \right\rangle\right|^2 \lambda_j^{2v} \right)^{1/2}, g \in D(\mathscr{A}^v). \tag{5}$$

The information for negative fractional power $\mathscr{A}^{-v}$ can be provided by [30]. For any $\theta > 0$, we introduce the following Hölder continuous space of exponent θ

$$\begin{aligned} &C^\theta([0, T] ; \mathscr{B}) \\ &= \left\{ v \in C([0, T] ; \mathscr{B}) : \sup_{0 \le s < t \le T} \frac{\|v(.,t) - v(.,s)\|_{\mathscr{B}}}{|t - s|^\theta} < \infty \right\}, \end{aligned} \tag{6}$$

corresponding to the following norm:

$$\|v\|_{C^\theta([0,T] : \mathscr{B})} = \sup_{0 \le s < t \le T} \frac{\|v(.,t) - v(.,s)\|_{\mathscr{B}}}{|t - s|^\theta}. \tag{7}$$

For any $0 < \theta < 1$, let us introduce the following space:

$$\begin{aligned} &\mathscr{C}^\theta((0, T] ; \mathscr{B}) \\ &= \left\{ v \in C\left((0, T] ; L^2(\Omega)\right) : \sup_{0 < t \le T} t^\theta \|v(.,t)\|_{\mathscr{B}} < \infty \right\}, \end{aligned} \tag{8}$$

corresponding to the norm $\|v\|_{\mathscr{C}^\theta((0,T];\mathscr{B})} := \sup_{0<t\le T} t^\theta \|v(.,t)\|_{\mathscr{B}}$.

Let us define the space as follows:

$$\mathbb{X}_{\beta,\alpha}(\Omega) = \left\{ g \in L^2(\Omega), \sum_{j=1}^{\infty} \lambda_j^{2+2\beta} \cdot \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \langle g, w_j \rangle^2 < \infty \right\}. \quad (9)$$

3. Backward Problem for Homogeneous Case

In this section, we consider the final value problem for the homogeneous equation with a space fractional derivative as follows:

$$\begin{cases} \frac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha} v(x,t) + B(\|\nabla v\|_{L^2})(-\Delta)^\beta v(x,t) = 0, & x \in \Omega, t \in (0,T), \\ v(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ v(x,T) = f(x), & x \in \Omega, \end{cases} \quad (10)$$

where $0 < M_0 \le B(\xi) \le M_1$ and $\xi \in [0,T]$. The following theorem states the existence and uniqueness of the solution of Problem (10).

Theorem 4. *Let $f \in \mathbb{X}_{\beta,\alpha}(\Omega)$. Then, Problem (10) has a unique mild solution $v \in C([0,T]; H^1(\Omega))$ which satisfies that*

$$v(x,t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(.,s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (11)$$

Furthermore, this solution is not stable in the L^2 norm.

Proof. We express a mild solution of (10) by Fourier series as follows:

$$v(x,t) = \sum_{j=1}^{\infty} \langle v(.,t), w_j \rangle w_j(x). \quad (12)$$

It follows from Problem (10) and the equality $\langle (-\Delta)^\beta v(.,t), w_j \rangle = \lambda_j^\beta \langle v(.,t), w_j \rangle$ that

$$\begin{cases} \frac{{}^{\ell}\partial^\alpha}{\partial t^\alpha} \langle v(.,t), w_j \rangle + \lambda_j^\beta B(\|\nabla v\|_{L^2}) \langle v(.,t), w_j \rangle = 0, & t \in (0,T), \\ \langle u(.,0), w_j \rangle = \langle u_0, w_j \rangle. \end{cases} \quad (13)$$

Note that this formula

$$\frac{{}^{\ell}\partial^\alpha}{\partial t^\alpha} \langle u(.,t), w_j \rangle = t^{1-\alpha} \frac{\partial}{\partial t} \langle u(.,t), w_j \rangle, \quad (14)$$

is correct; we get that

$$\frac{\partial}{\partial t} \langle v(.,t), w_j \rangle + \lambda_j^\beta t^{\alpha-1} B(\|\nabla v\|_{L^2}) \langle v(.,t), w_j \rangle = 0. \quad (15)$$

Multiply both sides of equation (15) by the quantity $\exp\left(\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(.,s)\|_{L^2}) ds\right)$, we reach the following assertion:

$$\frac{\partial}{\partial t}\left(\langle v(.,t), w_j \rangle \exp\left(\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|v(.,s)\|_{L^2}) ds \right)\right) = 0, \quad (16)$$

where we have used the fact that

$$\frac{\partial}{\partial t}\left(\exp\left(\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(.,s)\|_{L^2}) ds \right)\right) = \exp\left(\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(.,s)\|_{L^2}) ds \right) t^{\alpha-1} \lambda_j^\beta B(\|\nabla v(.,t)\|_{L^2}). \quad (17)$$

Integrating the two sides of the latter equation 0 to t, we obtain the following confirmation:

$$\langle v(.,t), w_j \rangle \exp\left(\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(.,s)\|_{L^2}) ds \right) = \langle v(.,0), w_j \rangle. \quad (18)$$

It yields that

$$\langle v(.,t), w_j \rangle = \exp\left(-\int_0^{t\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(.,s)\|_{L^2}) ds \right) \langle v(.,0), w_j \rangle. \quad (19)$$

Therefore, we find that

$$v(x,t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(.,s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (20)$$

For $v \in L^\infty(0,T; H^1(\Omega))$, we consider the following function:

$$\mathcal{Q}(v)(x,t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(.,s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (21)$$

We shall prove by induction if $w_1, w_2 \in L^\infty(0,T; H^1(\Omega))$, then

$$\left\|\mathcal{Q}^k(w_1)-\mathcal{Q}^k(w_2)\right\|_{L^2(\Omega)} \le \left(\frac{\left(\|f\|^2_{\mathbb{X}_{\beta,\alpha(\Omega)}}K_b^2((T^\alpha-t^\alpha)/\alpha)\right)^{1/2}}{k!}\right) \cdot \|w_1-w_2\|_{L^\infty(0,T;L^2(\Omega))} \quad \forall q\le 1. \tag{22}$$

For $m=1$, using the inequality $|e^a-e^b|\le|a-b|\max(e^a,e^b)$ for any $a,b\in\mathbb{R}$, we have

$$\begin{aligned}
&\|\mathcal{Q}(w_1)-\mathcal{Q}(w_2)\|^2_{H^1(\Omega)} \\
&= \sum_{j=1}^\infty \lambda_j^{2+2\beta}\left[\exp\left(\lambda_j^\beta\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B(\|\nabla w_1(.,s)\|_{L^2})ds\right) - \exp\left(\lambda_j^\beta\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B(\|\nabla w_2(.,s)\|_{L^2})ds\right)\right]^2\langle f,w_j\rangle^2 \\
&\le \sum_{j=1}^\infty \lambda_j^{2-2\beta}\left|\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B(\|\nabla w_1(.,s)\|_{L^2})ds - \int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B(\|\nabla w_2(.,s)\|_{L^2})ds\right|^2 \cdot \exp\left(\frac{2T^\alpha-2t^\alpha}{\alpha}M_1\lambda_j\right)\langle f,w_j\rangle^2 \\
&\le K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}\|\nabla(w_1-w_2)(.,s)\|^2_{L^2(\Omega)}ds \\
&\le K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\frac{T^\alpha-t^\alpha}{\alpha}\|w_1-w_2\|^2_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned} \tag{23}$$

Assume that (22) holds for $m=k$. We show that (22) holds for $m=k+1$. Indeed, we have

$$\begin{aligned}
&\left\|\mathcal{Q}^{k+1}(w_1)-\mathcal{Q}^{k+1}(w_2)\right\|^2_{H^1(\Omega)} \\
&= \sum_{j=1}^\infty \lambda_j^2\left[\exp\left(\lambda_j^\beta\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B\left(\left\|\nabla\mathcal{Q}^k(w_1)(.,s)\right\|_{L^2}\right)ds\right) - \exp\left(\lambda_j^\beta\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B\left(\left\|\nabla\mathcal{Q}^k(w_2)(.,s)\right\|_{L^2}\right)ds\right)\right]^2\langle f,w_j\rangle^2 \\
&\le \|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}K_b^2\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}\left\|\mathcal{Q}^k(w_1)(.,s)-\mathcal{Q}^k(w_2)(.,s)\right\|^2_{H^1(\Omega)}ds \\
&\le \|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}K_b^2\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}\frac{\left(\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}K_b^2((T^\alpha-t^\alpha)/\alpha)\right)^k}{k!}ds \\
&\le \frac{\left(\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}K_b^2((T^\alpha-t^\alpha)/\alpha)\right)^{k+1}}{(k+1)!}\|w_1-w_2\|^2_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned} \tag{24}$$

By the theory of the induction principle, (22) holds for all $w_1,w_2\in L^\infty(0,T;H^1(\Omega))$. Since the fact that

$$\lim_{k\to+\infty}\left(\frac{\left(\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}K_b^2((T^\alpha-t^\alpha)/\alpha)\right)^k}{k!}\right)^{1/2} = 0, \tag{25}$$

there exists a positive integer number k_0 such that $\mathcal{Q}^{k_0}$ is a contraction. It follows that the equation $\mathcal{Q}^{k_0}v=v$ has a unique solution $v\in L^\infty(0,T;H^1(\Omega))$. It is easy to see that v is also a fixed point of $\mathcal{Q}$. □

Theorem 5. *Assume that $f\in X\beta_{+\gamma,\alpha}(\Omega)$ for any $\gamma>\beta$. Let us choose v_0 such that*

$$2K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\frac{T^{\alpha(1-\delta)}|C_\delta|^2v_0^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}<1. \tag{26}$$

Let any $0<\varepsilon<\gamma-\beta$. Then, there exists $C_\varepsilon>0$ such that

$$\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_{v_0}(0,T;H^1(\Omega))} \le \frac{C_\varepsilon(1-\beta)^\varepsilon\|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}}{\sqrt{1-2K_b^2\|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}\left(T^{\alpha(1-\delta)}|C_\delta|^2v_0^{-2\delta}/\alpha^{1-\delta}(1-\delta)\right)}}, \tag{27}$$

where

$$L^\infty_{v_0}(0,T;H^1(\Omega)) = \left\{v\in L^\infty(0,T;H^1(\Omega)),\ \|v\|_{L^\infty_{v_0}(0,T;H^1(\Omega))} = \sup_{0\le t\le T}e^{v_0(t-T)}\|v(t)\|_{H^1(\Omega)}<\infty\right\}. \tag{28}$$

Proof. Let $v_{\alpha,\beta}$ be the solution of Problem (11). Let w_α be the solution to Problem (11) with $\beta=1$. Then, we get

$$\begin{aligned}
v_{\alpha,\beta}(x,t) &= \sum_{j=1}^\infty \exp\left(\lambda_j^\beta\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B\left(\left\|\nabla v_{\alpha,\beta}(.,s)\right\|_{L^2}\right)ds\right)\cdot\langle f,w_j\rangle w_j(x), \\
w_\alpha(x,t) &= \sum_{j=1}^\infty \exp\left(\lambda_j\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}B(\|\nabla w_\alpha(.,s)\|_{L^2})ds\right)\cdot\langle f,w_j\rangle w_j(x).
\end{aligned} \tag{29}$$

We have

$$\begin{aligned} v_{\alpha,\beta}(x,t) - w_\alpha(x,t) &= \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla v_{\alpha,\beta}(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x) \\ &\quad - \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x) \\ &\quad + \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x) \\ &\quad - \sum_{j=1}^{\infty} \exp\left(\lambda_j \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x) \\ &= \mathscr{D}_1 + \mathscr{D}_2, \end{aligned} \tag{30}$$

whereby

$$\begin{aligned} \mathscr{D}_1 &= \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x), \\ \mathscr{D}_2 &= \sum_{j=1}^{\infty} \exp\left(\lambda_j \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right) \langle f, w_j\rangle w_j(x). \end{aligned} \tag{31}$$

The term $\mathscr{D}_1$ is bounded by

$$\begin{aligned} \|\mathscr{D}_1\|_{H^1(\Omega)}^2 &\leq \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \\ &\quad \cdot \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla v_{\alpha,\beta}(.,s)\|_{L^2}\right) ds - \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right)^2 \langle f, w_j\rangle^2 \\ &\leq K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \left\|\nabla\left(v_{\alpha,\beta} - w_\alpha\right)(\cdot,s)\right\|_{L^2(\Omega)}^2 ds. \end{aligned} \tag{32}$$

The term $\mathscr{D}_2$ is estimated as follows:

$$\begin{aligned} \|\mathscr{D}_2\|_{H^1(\Omega)}^2 &\leq \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \left(\lambda_j^\beta - \lambda_j\right)^2 \\ &\quad \cdot \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B\left(\|\nabla w_\alpha(.,s)\|_{L^2}\right) ds\right)^2 \langle f, w_j\rangle^2 \\ &\leq M_1^2 \left(\frac{T^\alpha - t^\alpha}{\alpha}\right)^2 \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \\ &\quad \cdot \left(\lambda_j^\beta - \lambda_j\right)^2 \langle f, w_j\rangle^2. \end{aligned} \tag{33}$$

Consider the following subset:

$$A_1 = \left\{j \in \mathbb{N}, \lambda_j \leq 1\right\}, \quad A_2 = \left\{j \in \mathbb{N}, \lambda_j > 1\right\}. \tag{34}$$

If $j \in A_1$, then using the inequality $1 - e^{-z} \leq C_\varepsilon z^\varepsilon$, we get

$$\lambda_j^\beta - \lambda_j = \lambda_j^\beta\left(1 - \lambda_j^{1-\beta}\right) = \lambda_j^\beta C_\varepsilon (1-\beta)^\varepsilon \lambda_j^{-\varepsilon} = C_\varepsilon (1-\beta)^\varepsilon \lambda_j^{\beta-\varepsilon}, \tag{35}$$

which allows us to obtain

$$\begin{aligned} &\sum_{j \in A_1} \lambda_j^{2\beta} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \left(\lambda_j^\beta - \lambda_j\right)^2 \langle f, w_j\rangle^2 \\ &\quad \leq |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2-2\varepsilon} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \langle f, w_j\rangle^2. \end{aligned} \tag{36}$$

If $j \in A_2$, then using the inequality $1 - e^{-z} \leq C_\varepsilon z^\varepsilon$, we find

$$\left|\lambda_j^\beta - \lambda_j\right| = \lambda_j\left(1 - \lambda_j^{\beta-1}\right) = \lambda_j C_\varepsilon (1-\beta)^\varepsilon \lambda_j^\varepsilon = C_\varepsilon (1-\beta)^\varepsilon \lambda_j^{\beta+\varepsilon}. \tag{37}$$

Hence, we obtain

$$\begin{aligned} &\sum_{j \in A_2} \lambda_j^{2\beta+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \left(\lambda_j^\beta - \lambda_j\right)^2 \langle f, w_j\rangle^2 \\ &\quad \leq |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2+2\varepsilon} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \langle f, w_j\rangle^2. \end{aligned} \tag{38}$$

Combining (36) and (38), we find that

$$\begin{aligned} &\lambda_j^{2\beta} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \left(\lambda_j^\beta - \lambda_j\right)^2 \langle f, w_j\rangle^2 \\ &\quad \leq C|C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2\varepsilon+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \langle f, w_j\rangle^2. \end{aligned} \tag{39}$$

Let us choose $0 < \varepsilon < \gamma - \beta$. Then, we follow from (33) and the latter equality that

$$\begin{aligned}\|\mathscr{D}_2\|^2 &\le CM_1^2\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)^2 |C_\varepsilon|^2(1-\beta)^{2\varepsilon}\sum_{j\in A_1}\lambda_j^{4\beta+2\varepsilon+2} \\ &\quad\cdot \exp\left(\frac{2T^\alpha M_1\lambda_j^\beta}{\alpha}\right)\langle f, w_j\rangle^2 \\ &\le CM_1^2\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)^2 |C_\varepsilon|^2(1-\beta)^{2\varepsilon}\sum_{j\in A_1}\lambda_j^{2\beta+2\gamma2} \\ &\quad\cdot \exp\left(\frac{2T^\alpha M_1\lambda_j^\beta}{\alpha}\right)\langle f, w_j\rangle^2 \\ &= CM_1^2\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)^2 |C_\varepsilon|^2(1-\beta)^{2\varepsilon}\|f\|^2_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}.\end{aligned} \tag{40}$$

This above inequality together with (32) and (3) yields that

$$\begin{aligned} &e^{2\nu(t-T)}\left\|v_{\alpha,\beta}(.,t)-w_\alpha(.,t)\right\|^2_{H^1(\Omega)} \\ &\quad\le 2e^{2\nu(t-T)}\|\mathscr{D}_1\|^2_{H^1(\Omega)} + 2e^{2\nu(t-T)}\|\mathscr{D}_2\|^2_{H^1(\Omega)} \\ &\quad\le 2K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)} e^{2\nu(t-T)}\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}\left\|v_{\alpha,\beta}(.,s)-w_\alpha(.,s)\right\|^2_{H^1(\Omega)}ds \\ &\qquad + 2CM_1^2 e^{2\nu(t-T)}\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)^2|C_\varepsilon|^2(1-\beta)^{2\varepsilon}\|f\|^2_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}.\end{aligned} \tag{41}$$

It is easy to get that

$$\begin{aligned} &e^{2\nu(t-T)}\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}\left\|v_{\alpha,\beta}(.,s)-w_\alpha(.,s)\right\|^2_{H^1(\Omega)}ds \\ &\quad= \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)}e^{2\nu(s-T)}\left\|v_{\alpha,\beta}(.,s)-w_\alpha(.,s)\right\|^2_{H^1(\Omega)}ds \\ &\quad\le \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)}ds\right)\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_\nu(0,T;H^1(\Omega))}.\end{aligned} \tag{42}$$

It follows from the inequality

$$e^{-z} \le C_\delta z^{-\delta}, \quad 0<\delta<1, \tag{43}$$

we get

$$\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)}ds \le |C_\delta|^2\nu^{-2\delta}\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}(s-t)^{-\delta}ds. \tag{44}$$

The inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ leads to

$$\begin{aligned}\int_{t^\alpha/\alpha}^{T^\alpha/\alpha}(s-t)^{-\delta}ds &= \frac{(T^\alpha/\alpha - t)^{1-\delta} - (t^\alpha/\alpha - t)^{1-\delta}}{1-\delta} \\ &\le \frac{(T^\alpha/\alpha - t^\alpha/\alpha)^{1-\delta}}{1-\delta} \le \frac{T^{\alpha(1-\delta)}}{\alpha^{1-\delta}(1-\delta)},\end{aligned} \tag{45}$$

which allows us to get immediately that

$$\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)}ds \le \frac{T^{\alpha(1-\delta)}|C_\delta|^2\nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}. \tag{46}$$

It follows from (41) and (42) that for any $t\in[0,T]$

$$\begin{aligned} &e^{2\nu(t-T)}\left\|v_{\alpha,\beta}(.,t)-w_\alpha(.,t)\right\|^2_{H^1(\Omega)} \\ &\quad\le 2K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\frac{T^{\alpha(1-\delta)}|C_\delta|^2\nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_\nu(0,T;H^1(\Omega))} \\ &\qquad + 2CM_1^2\left(\frac{T^\alpha}{\alpha}\right)^2|C_\varepsilon|^2(1-\beta)^{2\varepsilon}\|f\|^2_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}.\end{aligned} \tag{47}$$

Since the right-hand side of (47) is independent of t, we deduce that

$$\begin{aligned} &\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_\nu(0,T;H^1(\Omega))} \\ &\quad\le 2K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\frac{T^{\alpha(1-\delta)}|C_\delta|^2\nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_\nu(0,T;H^1(\Omega))} \\ &\qquad + 2CM_1^2\left(\frac{T^\alpha}{\alpha}\right)^2|C_\varepsilon|^2(1-\beta)^{2\varepsilon}\|f\|^2_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}.\end{aligned} \tag{48}$$

Then, we find that

$$\begin{aligned} &\left\|v_{\alpha,\beta}-w_\alpha\right\|^2_{L^\infty_{\nu_0}(0,T;H^1(\Omega))} \\ &\quad\le \frac{2CM_1^2(T^\alpha/\alpha)^2|C_\varepsilon|^2(1-\beta)^{2\varepsilon}\|f\|^2_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}}{1-2K_b^2\|f\|^2_{\mathbb{X}_{\beta,\alpha}(\Omega)}\left(T^{\alpha(1-\delta)}|C_\delta|^2\nu_0^{-2\delta}/\alpha^{1-\delta}(1-\delta)\right)}.\end{aligned} \tag{49}$$

□

4. Backward Problem for Inhomogeneous Case

In this section, we consider the final value problem for homogeneous equation as follows:

$$\begin{cases}\dfrac{\partial^\alpha}{\partial t^\alpha}u(x,t) + B(\|\nabla u\|_{L^2})(-\Delta)^\beta u(x,t) = F(x,t), & x\in\Omega, t\in(0,T), \\ u(x,t)=0, & x\in\partial\Omega, t\in(0,T), \\ u(x,T)=0, & x\in\Omega,\end{cases} \tag{50}$$

where F is defined later.

4.1. Existence and Uniqueness of the Mild Solution. In this subsection, we state the existence and uniqueness of the mild solution. In order to give the main results, we require the condition F which belongs to the space $L^{2p}(0, T; X\beta, \alpha(\Omega))$.

Theorem 6. *Let $0<\beta\le 1$ and F be the source function that belongs to $L^{2p}(0, T; X\beta, \alpha(\Omega))$ for any $1<p<1/(1-\alpha)$. Let B be the functions which satisfy $M_0 \le B(z) \le M_1$, $z \in [0, T]$ and*

$$|B(z_1) - B(z_2)| \le K_b |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}. \tag{51}$$

Then, Problem (50) has a unique mild solution $u \in L^{\infty}_{\mu_0}(0, T; H^1(\Omega))$, where μ_0 is small enough. The function u satisfies that

$$u(x,t) = -\sum_{j=1}^{\infty}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla u(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds\right) w_j. \tag{52}$$

Furthermore, this solution is not stable in the L^2 norm.

Proof. By a simple calculation, we get the following equality:

$$\left\langle v(.,t), w_j\right\rangle = \exp\left(-\int_0^{t^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,s)\|_{L^2}\right)ds\right)\left\langle u(.,0), w_j\right\rangle + \int_0^t s^{\alpha-1}\exp\left(-\int_{s^\alpha/\alpha}^{t^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds. \tag{53}$$

By letting $t=T$ and noting that $v(x,T)=0$, we find that

$$\exp\left(-\int_0^{T^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,s)\|_{L^2}\right)ds\right)\left\langle u(.,0), w_j\right\rangle + \int_0^T s^{\alpha-1}\exp\left(-\int_{s^\alpha/\alpha}^{T^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds = 0. \tag{54}$$

Therefore, we obtain

$$\left\langle u(.,0), w_j\right\rangle = -\int_0^T s^{\alpha-1}\exp\left(\int_0^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds. \tag{55}$$

Combining (53) and (55), we deduce that

$$\begin{aligned}\left\langle v(.,t), w_j\right\rangle &= \int_0^t s^{\alpha-1}\exp\left(-\int_{s^\alpha/\alpha}^{t^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds - \int_0^T s^{\alpha-1}\exp \cdot \left(-\int_{s^\alpha/\alpha}^{t^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right)\left\langle F(.,s), w_j\right\rangle ds \\ &= -\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle \Big| ds.\end{aligned} \tag{56}$$

Let us denote by $L^{\infty}_{\mu}(0, T; \mathbb{V})$ the functional subspace of $L^{\infty}(0, T; \mathbb{V})$ corresponding to the norm

$$\|g\|_{L^{\infty}_{\mu}(0,T;\mathbb{V})} := \max_{0\le t\le T}\|\exp\left(\mu(t-T)\right)g(.,t)\|_{\mathbb{V}}, \quad \forall g \in L^{\infty}_{\mu}(0, T; \mathbb{V}), \tag{57}$$

where

$$\mathbb{V} = \left\{\mu \in \mathbb{R}, g \in L^2(\Omega), \sum_{j=1}^{\infty}\exp\left(\mu(t-T)\right)\left\langle g, w_j\right\rangle^2 < \infty\right\}. \tag{58}$$

Set the following function:

$$\mathscr{P}v(t) = -\sum_{j=1}^{\infty}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right) \cdot \left\langle F(.,s), w_j\right\rangle ds\right) w_j, \tag{59}$$

and we let

$$\mathscr{M}(s,t,j,w) = \exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla w(.,r)\|_{L^2}\right)dr\right). \tag{60}$$

So, using Parseval's equality, we get that

$$\begin{aligned}\|\mathscr{P}w_1 - \mathscr{P}w_2\|^2_{H^1(\Omega)} &= \sum_{j=1}^{\infty}\lambda_j\left(\int_t^T s^{\alpha-1}\left(\mathscr{M}(s,t,j,w_1) - \mathscr{M}(s,t,j,w_2)\right)\left\langle F(.,s), w_j\right\rangle ds\right)^2 \\ &\le \sum_{j=1}^{\infty}\lambda_j\left(\int_t^T s^{\alpha-1}ds\right)\left(\int_0^t s^{\alpha-1}\left(\mathscr{M}(s,t,j,w_1) - \mathscr{M}(s,t,j,w_2)\right)^2\left\langle F(.,s), w_j\right\rangle^2 ds\right).\end{aligned} \tag{61}$$

Using the inequality $|e^a - e^b| \le |a-b| \max(e^a, e^b)$, we continue to treat the term $\mathscr{M}(s,t,j,w_1) - \mathscr{M}(s,t,j,w_2)$ as follows:

$$\begin{aligned}
&|\mathscr{M}(s,t,j,w_1) - \mathscr{M}(s,t,j,w_2)| \\
&\quad = \left|\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B\left(\|\nabla w_1(.,r)\|_{L^2}\right) dr\right)\right. \\
&\qquad \left. - \exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B\left(\|\nabla w_2(.,r)\|_{L^2}\right) dr\right)\right| \\
&\quad \le \exp\left(\frac{T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) K_b \lambda_j^\beta \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \|u(.,r) - v(.,r)\|_{H^1} dr.
\end{aligned} \tag{62}$$

Therefore, applying the Hölder inequality, we get

$$\begin{aligned}
&|\mathscr{M}(s,t,j,v_1) - \mathscr{M}(s,t,j,v_2)|^2 \\
&\quad \le \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) |K_b|^2 \lambda_j^{2\beta} \frac{T^{2\alpha}}{\alpha^2} \\
&\qquad \cdot \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \|v_1(.,r) - v_2(.,r)\|_{H^1(\Omega)}^2 ds.
\end{aligned} \tag{63}$$

Inserting (61) and (63) yields the following inequality:

$$\begin{aligned}
&\exp(2\mu(t-T)) \|\mathscr{P}v_1 - \mathscr{P}v_2\|_{H^1(\Omega)}^2 \\
&\quad \le |K_b|^2 \frac{T^{3\alpha}}{\alpha^3} \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} e^{2\mu(t-s)} e^{2\mu(s-T)} \|v_1(.,s) - v_2(.,s)\|_{L^2} ds\right) \\
&\qquad \times \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} s^{\alpha-1} \left(\sum_{j=1}^\infty \lambda_j^{2\beta+2} \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \left\langle F(.,s), w_j\right\rangle^2\right) ds\right) \\
&\quad \le \frac{K_b^2 T^{3\alpha}}{3\alpha} \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \exp(2\mu(t-s)) ds\right) \\
&\qquad \times \left(\max_{0\le s\le T} \exp(2\mu(s-T)) \|v_1(.,s) - v_2(.,s)\|_{H^1(\Omega)}^2\right) \\
&\qquad \cdot \left(\int_0^t s^{\alpha-1} \|F(.,s)\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 ds\right).
\end{aligned} \tag{64}$$

Take any $\delta \in (0,1)$. By a similar explanation as (46), we find that

$$\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \exp(2\mu(t-s)) ds \le \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}. \tag{65}$$

By applying the Hölder inequality, we also obtain that

$$\begin{aligned}
&\int_0^t s^{\alpha-1} \|F(.,s)\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 ds \\
&\quad \le \left(\int_0^t s^{(\alpha-1)p^*} ds\right)^{1/p^*} \left(\int_0^t \|F(.,s)\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^{2p} ds\right)^{1/p} \\
&\quad \le \frac{p-1}{\alpha p - 1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0,T;\mathbb{X}_{\beta,\alpha}(\Omega))}^2.
\end{aligned} \tag{66}$$

From some observations as above, we deduce that

$$\begin{aligned}
&\exp(2\mu(t-T)) \left\|\mathscr{P}_{v_1} - \mathscr{P}_{v_2}\right\|_{H^1(\Omega)}^2 \\
&\quad \le \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \\
&\qquad \cdot \|F\|_{L^{2p}(0,T;X_{\beta,\alpha}(\Omega))}^2 \|v_1 - v_2\|_{L_\mu^\infty(0,T;H^1(\Omega))}^2.
\end{aligned} \tag{67}$$

Since the right-hand side of the latter estimate is independent of t, we find that

$$\begin{aligned}
&\left\|\mathscr{P}_{v_1} - \mathscr{P}_{v_2}\right\|_{L_\mu^\infty(0,T;H^1(\Omega))}^2 \\
&\quad \le \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \\
&\qquad \cdot \|F\|_{L^{2p}(0,T;X_{\beta,\alpha}(\Omega))}^2 \|v_1 - v_2\|_{L_\mu^\infty(0,T;H^1(\Omega))}^2.
\end{aligned} \tag{68}$$

Let us choose μ_0 such that

$$\mu_0^{2\delta} > \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0,T;\mathbb{X}_{\beta,\alpha}(\Omega))}^2. \tag{69}$$

Then, we can conclude that $\mathscr{P}$ is a contraction mapping in the space $L_{\mu_0}^\infty(0,T;H^1(\Omega))$. Next, we continue to show that if $v \in L_{\mu_0}^\infty(0,T;H^1(\Omega))$, then $\mathscr{P}v \in L_{\mu_0}^\infty(0,T;H^1(\Omega))$. If $v_1 = 0$, then

$$\mathscr{P}v_1(t) = \sum_{j=1}^\infty \left(\int_t^T s^{\alpha-1} \left\langle F(.,s), w_j\right\rangle ds\right) w_j. \tag{70}$$

Hence, from Parseval's equality, we find that

$$\begin{aligned}
\|\mathscr{P}v_1\|_{H^1(\Omega)}^2 &= \sum_{j=1}^\infty \lambda_j^2 \left(\int_t^T s^{\alpha-1} \left\langle F(.,s), w_j\right\rangle ds\right)^2 \\
&\le \left(\int_t^T s^{\alpha-1} ds\right) \left(\sum_{j=1}^\infty \int_t^T s^{\alpha-1} \lambda_j^2 \left\langle F(.,s), w_j\right\rangle^2 ds\right) \\
&\le \frac{T^\alpha}{\alpha} \int_t^T s^{\alpha-1} \|F(.,s)\|_{H^1(\Omega)}^2 ds \\
&\le \frac{T^\alpha}{\alpha} \left(\int_0^t s^{(\alpha-1)p^*} ds\right)^{1/p^*} \left(\int_0^t \|F(.,s)\|_{H^1(\Omega)}^{2p} ds\right)^{1/p} \\
&\le \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0,T;H^1(\Omega))}^2.
\end{aligned} \tag{71}$$

This says that $\mathscr{P}v_1$ belongs to the space $L_{\mu_0}^\infty(0,T;H^1(\Omega))$. Using (68), we arrive at the confirmation that $\mathscr{P}v$ belongs to $L_{\mu_0}^\infty(0,T;H^1(\Omega))$ if $v \in L_{\mu_0}^\infty(0,T;H^1(\Omega))$. For any $m \in \mathbb{N}$, let u_m be the function that satisfies the following integral equation:

$$u_m(x,t) = -\sum_{j=1}^{\infty}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla u_m(.,r)\|_{L^2}\right)dr\right)\cdot\left\langle F(.,s), w_j\right\rangle ds\right)w_j(x). \tag{72}$$

Let us assume that

$$F_m(x,t) = \frac{1}{\lambda_m}\sum_{j=1}^{\infty} w_j(x). \tag{73}$$

It is not difficult to verify that $F_m \in L^\infty(0,T;\mathbb{X}_{\beta,\alpha}(\Omega))$, so we get that $F_m \in L^{2p}(0,T;\mathbb{X}_{\beta,\alpha}(\Omega))$.

Using Theorem 6, we conclude that equation (72) has a unique solution $u_m \in L^\infty(0,T;H^1(\Omega))$. By the fact that $B(z) \ge M_0 \forall z \in \mathbb{R}$, we obtain the following estimate:

$$\begin{aligned}\|u_m(\cdot,t)\|_{L^2(\Omega)}^2 &= \frac{1}{\lambda_m^2}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla u_m(.,r)\|_{L^2}\right)dr\right)\right)^2\\ &\ge \frac{\left(\int_t^T s^{\alpha-1}\exp\left(M_0\lambda_j^\beta((s^\alpha - t^\alpha)/\alpha)\right)ds\right)^2}{\lambda_m^2}.\end{aligned} \tag{74}$$

The estimate is true for all $t \in [0,T]$, so it is easy to see that

$$\begin{aligned}\|u_m\|_{L^\infty(0,T;L^2(\Omega))} &\ge \frac{\int_0^T s^{\alpha-1}\exp\left(M_0\lambda_j^\beta(s^\alpha/\alpha)\right)ds}{\lambda_m}\\ &\ge \frac{\exp\left(\lambda_m^\beta M_0(T^\alpha/\alpha)\right)}{M_0\lambda_m^{\beta+1}}.\end{aligned} \tag{75}$$

When m tends to $+\infty$, we can check that $\|f_m\|_{L^2(\Omega)} = 1/\lambda_m$ go to zero when $m \longrightarrow +\infty$ and

$$\begin{aligned}&\lim_{m\longrightarrow+\infty}\|u_m\|_{C([0,T];L^2(\Omega))}\\ &= \lim_{m\longrightarrow+\infty}\frac{\exp\left(\lambda_m^\beta M_0(T^\alpha/\alpha)\right)}{M_0\lambda_m^{\beta+1}} = +\infty.\end{aligned} \tag{76}$$

This shows that Problem (50) is ill-posed in the sense of Hadamard in the L^2-norm. □

4.2. Fourier Truncation Method. In this section, we will provide a regularized solution and solve the problem by the Fourier truncation method as follows:

$$u^{N,\delta}(x,t) = -\sum_{j=1}^{N}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\left\|\nabla u^{N,\delta}(.,r)\right\|_{L^2}\right)dr\right)\cdot\left\langle F^\delta(.,s), w_j\right\rangle ds\right)w_j(x). \tag{77}$$

Here, $N := N(\delta)$ goes to infinity as δ tends to zero which is called a parameter regularization. The function F is disturbed by the observed data $F^\delta \in L^\infty(0,T;L^2(\Omega))$ provided by

$$\left\|F^\delta - F\right\|_{L^\infty(0,T;L^2(\Omega))} \le \delta. \tag{78}$$

The main results of this subsection are given by the theorem below.

Theorem 7. *Let $\nu > 0$ such that F belongs to the space $L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))$. Let F^δ be as above. Let us assume that Problem (50) has a unique mild solution $u \in L^\infty(0,T;D(\mathscr{A}^{\nu+\theta}))$ for $\theta > 0$. Let us choose N such that*

$$\lim_{\delta\longrightarrow 0}\lambda_N^\nu \exp\left(\frac{T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\delta = 0, \lim_{\delta\longrightarrow 0}\lambda_N = +\infty. \tag{79}$$

Here $\nu \ge 1/2$. Then, there exists a positive $\bar{\mu}$ large enough such that Problem has a unique solution $v^{N,\delta} \in L^\infty_{\bar{\mu}}(0,T;D(\mathscr{A}^\nu))$. Moreover, we have the following estimate:

$$\begin{aligned}&\left\|v^{N,\delta} - u\right\|^2_{L^\infty_{\bar{\mu}}(0,T;D(\mathscr{A}^\nu))}\\ &\le \frac{6T^{2\alpha}}{\alpha^2}\lambda_N^{2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\delta^2\\ &\quad + 6\lambda_N^{-2\theta}\|u\|^2_{L^\infty(0,T;D(\mathscr{A}^{\nu+\theta}))}.\end{aligned} \tag{80}$$

Remark 8. Since $\lambda_N \sim N^{2/d}$, we can choose a natural number N such that

$$\lambda_N = \left(\frac{\alpha(1-b)\log(1/b)}{T^\alpha M_1}\right)^{1/\beta}. \tag{81}$$

Proof. Part 1: prove that the nonlinear integral equation (77) has a unique mild solution.

Let any $v \in C([0,T];H^1(\Omega))$, we denote by the following function

$$\mathscr{G}(v)(x,t) = -\sum_{j=1}^{N}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\|\nabla v(.,r)\|_{L^2}\right)dr\right)\cdot\left\langle F^\delta(.,s), w_j\right\rangle ds\right)w_j(x). \tag{82}$$

By applying Parseval's equality, we follow from (82) that

$$\begin{aligned}
&\|\mathcal{G}(v_1)(.,t)-\mathcal{G}(v_2)(.,t)\|^2_{D(\mathcal{A}^v)}\\
&= \sum_{j=1}^{N}\lambda_j^{2v}\left(\int_t^T s^{\alpha-1}(\mathcal{M}(s,t,j,v_1)\right.\\
&\qquad\left.-\mathcal{M}(s,t,j,v_2))\langle F(.,s),w_j\rangle ds\right)^2\\
&\le \frac{T^\alpha}{\alpha}\lambda_N^{2v}\sum_{j=1}^{N}\left(\int_0^t s^{\alpha-1}(\mathcal{M}(s,t,j,w_1)\right.\\
&\qquad\left.-\mathcal{M}(s,t,j,w_2))^2\left\langle F^\delta(.,s),w_j\right\rangle^2 ds\right).
\end{aligned} \tag{83}$$

If $1 \le j \le N$, then we have in view of (63) that

$$\begin{aligned}
&|\mathcal{M}(s,t,j,v_1)-\mathcal{M}(s,t,j,v_2)|^2\\
&\le \exp\left(\frac{2T^\alpha M_1\lambda_j^\beta}{\alpha}\right)|K_b|^2\lambda_j^{2\beta}\frac{T^{2\alpha}}{\alpha^2}\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\|v_1(.,r)-v_2(.,r)\|^2_{H^1(\Omega)}ds\\
&\le \exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta}K_b^2\frac{T^{2\alpha}}{\alpha^2}\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\|v_1(.,r)-v_2(.,r)\|^2_{H^1(\Omega)}ds.
\end{aligned} \tag{84}$$

The above two observations (65) lead to

$$\begin{aligned}
&\exp(2\mu(t-T))\|\mathcal{G}(v_1)(.,t)-\mathcal{G}(v_2)(.,t)\|^2_{D(\mathcal{A}^v)}\\
&\le |K_b|^2\frac{T^{3\alpha}}{\alpha^3}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2v}\\
&\quad\cdot\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}e^{2\mu(t-s)}e^{2\mu(s-T)}\|v_1(.,s)-v_2(.,s)\|_{H^1}ds\right)\\
&\quad\cdot\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}s^{\alpha-1}\left(\sum_{j=1}^{\infty}\left\langle F^\delta(.,s),w_j\right\rangle^2\right)ds\right)\\
&\le |K_b|^2\frac{T^{3\alpha}}{\alpha^3}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2v}\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}\\
&\quad\cdot\left(\max_{0\le s\le T}\exp(2\mu(s-T))\|v_1(.,s)-v_2(.,s)\|^2_{H^1(\Omega)}\right)\\
&\quad\cdot\left(\int_0^t s^{\alpha-1}\left\|F^\delta(.,s)\right\|^2_{L^2(\Omega)}ds\right)\\
&\le |K_b|^2\frac{T^{4\alpha}}{\alpha^4}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2v}\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}\\
&\quad\cdot\left\|F^\delta\right\|^2_{L^\infty(0,T;L^2(\Omega))}\|v_1-v_2\|^2_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned} \tag{85}$$

Because the right-hand side of the latter estimate is independent of t and noting the Sobolev embedding $D(\mathcal{A}^v)\hookrightarrow H^1(\Omega)$, we arrive at

$$\begin{aligned}
&\|\mathcal{G}(v_1)-\mathcal{G}(v_2)\|^2_{L^\infty_\mu(0,T;D(\mathcal{A}^v))}\\
&\le |K_b|^2\frac{T^{4\alpha}}{\alpha^4}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2v}\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}\\
&\quad\cdot\left\|F^\delta\right\|^2_{L^\infty(0,T;L^2(\Omega))}\|v_1-v_2\|^2_{L^\infty_\mu(0,T;D(\mathcal{A}^v))}.
\end{aligned} \tag{86}$$

Let us choose μ_1 such that

$$\begin{aligned}
&|K_b|^2\frac{T^{4\alpha}}{\alpha^4}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2v}\frac{T^{\alpha(1-\delta)}|C_\delta|^2}{\alpha^{1-\delta}(1-\delta)}\\
&\quad\cdot\left\|F^\delta\right\|^2_{L^\infty(0,T;L^2(\Omega))}<\mu_1^{2\delta}.
\end{aligned} \tag{87}$$

It is easy to see that $\mathcal{G}$ is a contracting mapping on the space $L^\infty_{\mu_1}(0,T;D(\mathcal{A}^v))$. Therefore, we can conclude that there exists a uniqueness solution $v^{N,\delta}$ for Problem (77).

Next, we continue to give the upper bound of the term $\|v^{N,\delta}(\cdot,t)-u(\cdot,t)\|^2_{D(\mathcal{A}^v)}$. First, we have

$$\begin{aligned}
&v^{N,\delta}(x,t)-u(x,t)\\
&=-\sum_{j=1}^{N}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\left\|\nabla v^{N,\delta}(.,r)\right\|_{L^2}\right)dr\right)\right.\\
&\quad\left.\cdot\left\langle F^\delta(.,s)-F(.,s),w_j\right\rangle ds\right)w_j(x)\\
&\quad+\sum_{j=1}^{N}\left(\int_t^T s^{\alpha-1}\left(\mathcal{M}\left(s,t,j,v^{N,\delta}\right)\right.\right.\\
&\quad\left.\left.-\mathcal{M}(s,t,j,u)\right)\langle F(.,s),w_j\rangle ds\right)w_j(x)\\
&\quad+\sum_{j=1}^{N}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B(\|\nabla u(.,r)\|_{L^2})dr\right)\right.\\
&\quad\left.\cdot\langle F(.,s),w_j\rangle ds\right)w_j.
\end{aligned} \tag{88}$$

The above equality and Parseval's equality allow us to get that

$$\begin{aligned}
&\left\|v^{N,\delta}(\cdot,t)-u(\cdot,t)\right\|^2_{D(\mathcal{A}^v)}\\
&\le \underbrace{3\sum_{j=1}^{N}\lambda_j^{2v}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\left\|\nabla v^{N,\delta}(.,r)\right\|_{L^2}\right)dr\right)\left\langle F^\delta(.,s)-F(.,s),w_j\right\rangle ds\right)^2}_{J_1}\\
&\quad+\underbrace{3\sum_{j=1}^{N}\lambda_j^{2v}\left(\int_t^T s^{\alpha-1}\left(\mathcal{M}\left(s,t,j,v^{N,\delta}\right)-\mathcal{M}(s,t,j,u)\right)\langle F(.,s),w_j\rangle ds\right)^2}_{J_2}\\
&\quad+\underbrace{3\sum_{j=N+1}^{\infty}\lambda_j^{2v}\left(\int_t^T s^{\alpha-1}\exp\left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B(\|\nabla u(.,r)\|_{L^2})dr\right)\langle F(.,s),w_j\rangle ds\right)^2}_{J_3}.
\end{aligned} \tag{89}$$

Since the condition $\mathscr{B}(z) \le M_1 \forall z \in \mathbb{R}$ and applying Hölder inequality, the quantity J_1 is bounded by

$$\begin{aligned} J_1 &\le 3\lambda_N^{2\nu}\left(\int_t^T s^{\alpha-1}\right)\sum_{j=1}^N\left(\int_t^T s^{\alpha-1}\exp\left(2\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\lambda_j^\beta B\left(\left\|\nabla v^{N,\delta}(.,r)\right\|_{L^2}\right)dr\right)\right. \\ &\quad \cdot\left\langle F^\delta(.,s)-F(.,s),w_j\right\rangle^2 ds \\ &\le \frac{3T^\alpha}{\alpha}\lambda_N^{2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\left(\int_0^t s^{\alpha-1}\left\|F^\delta(.,s)-F\right\|_{L^2(\Omega)}^2 ds\right) \\ &\le \frac{3T^\alpha}{\alpha}\lambda_N^{2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\left(\int_t^T s^{\alpha-1}ds\right)\left\|F^\delta-F\right\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ &\le \frac{3T^{2\alpha}}{\alpha^2}\lambda_N^{2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\delta^2, \end{aligned} \tag{90}$$

where we have used the fact that $\|F^\delta - F\|_{L^\infty(0,T;L^2(\Omega))} \le \delta$.

The quantity J_2 is estimated as follows:

$$\begin{aligned} &\exp(2\mu(t-T))J_2 \\ &\le 3\exp(2\mu(t-T))\left(\int_t^T s^{\alpha-1}ds\right) \\ &\quad\cdot\left[\sum_{j=1}^N\int_t^T s^{\alpha-1}\lambda_j^{2\nu}\left(\mathscr{M}\left(s,t,j,v^{N,\delta}\right)\right.\right. \\ &\quad\left.\left. -\mathscr{M}(s,t,j,u)\right)^2\left\langle F(.,s),w_j\right\rangle^2 ds\right]. \end{aligned} \tag{91}$$

We have in view of (63) that

$$\begin{aligned} &\left|\mathscr{M}\left(s,t,j,v^{N,\delta}\right)-\mathscr{M}(s,t,j,u)\right|^2 \\ &\le \exp\left(\frac{2T^\alpha M_1\lambda_j^\beta}{\alpha}\right)\lambda_j^{2\beta}\frac{K_b^2T^{2\alpha}}{\alpha^2} \\ &\quad\cdot\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}\left\|v^{N,\delta}(.,r)-u(.,r)\right\|_{D(\mathscr{A}^\nu)}^2 dr. \end{aligned} \tag{92}$$

This leads to the following estimate:

$$\begin{aligned} &\exp(2\mu(t-T))J_2 \\ &\le \frac{3K_b^2T^{3\alpha}}{\alpha^3}\int_{t^\alpha/\alpha}^{s^\alpha/\alpha}e^{2\mu(t-s)} \\ &\quad\cdot e^{2\mu(s-T)}\left\|v^{N,\delta}(.,r)-u(.,r)\right\|_{D(\mathscr{A}^\nu)}^2 \\ &\quad\cdot dr\int_0^T s^{\alpha-1}\left(\sum_{j=1}^\infty\lambda_j^{2\beta+2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_j^\beta}{\alpha}\right)\left\langle F(.,s),w_j^2\right\rangle\right)ds \\ &\le \frac{3K_b^2T^{4\alpha}}{\alpha^4}\|F\|_{L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))}^2\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \\ &\quad\cdot\left\|v^{N,\delta}-u\right\|_{L_\mu^\infty(0,T;D(\mathscr{A}^\nu))}. \end{aligned} \tag{93}$$

The term J_3 is estimated as follows:

$$\begin{aligned} J_3 &= 3\sum_{j=N+1}^\infty\lambda_j^{2\nu}\left\langle u(.,t),w_j\right\rangle^2 \\ &= 3\sum_{j=N+1}^\infty\lambda_j^{-2\theta}\lambda_j^{2\nu+2\theta}\left\langle u(.,t),w_j\right\rangle^2 \\ &\le 3\lambda_N^{-2\theta}\|u\|_{L^\infty(0,T;D(\mathscr{A}^{\nu+\theta}))}^2. \end{aligned} \tag{94}$$

Combining (89), (90), (93), and (94), we find that

$$\begin{aligned} &\exp(2\mu(t-T))\left\|v^{N,\delta}(\cdot,t)-u(\cdot,t)\right\|_{D(\mathscr{A}^\nu)}^2 \\ &\le \exp(2\mu(t-T))(J_1+J_2+J_3) \\ &\le \frac{3T^{2\alpha}}{\alpha^2}\lambda_N^{2\nu}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\delta^2+\frac{3K_b^2T^{4\alpha}}{\alpha^4} \\ &\quad\cdot\|F\|_{L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))}^2\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \\ &\quad\cdot\left\|v^{N,\delta}-u\right\|_{L_\mu^\infty(0,T;D(\mathscr{A}^\nu))}+3\lambda_N^{-2\theta}\|u\|_{L^\infty(0,T;D(\mathscr{A}^{\nu+\theta}))}^2. \end{aligned} \tag{95}$$

We choose $\bar{\mu}$ such that both the following inequalities are satisfied:

$$\begin{aligned} &\frac{3K_b^2T^{4\alpha}}{\alpha^4}\|F\|_{L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))}^2\frac{T^{\alpha(1-\delta)}|C_\delta|^2}{\alpha^{1-\delta}(1-\delta)} \le \frac{1}{2}\bar{\mu}^{2\delta}, \\ &|K_b|^2\frac{T^{4\alpha}}{\alpha^4}\exp\left(\frac{2T^\alpha M_1\lambda_N^\beta}{\alpha}\right)\lambda_N^{2\beta+2\nu}\frac{T^{\alpha(1-\delta)}|C_\delta|^2}{\alpha^{1-\delta}(1-\delta)} \\ &\quad\cdot\left\|F^\delta\right\|_{L^\infty(0,T;L^2(\Omega))}^2 < \bar{\mu}^{2\delta}. \end{aligned} \tag{96}$$

Some observations above give us the following confirmation:

$$\begin{aligned} &\left\|v^{N,\delta}-u\right\|_{L_\mu^\infty(0,T;D(\mathscr{A}^\nu))}^2 \\ &\le \frac{(3T^{2\alpha}/\alpha^2)\lambda_N^{2\nu}\exp\left(2T^\alpha M_1\lambda_N^\beta/\alpha\right)\delta^2+3\lambda_N^{-2\theta}\|u\|_{L^\infty(0,T;D(\mathscr{A}^{\nu+\theta}))}^2}{1-\left(3K_b^2T^{4\alpha}/\alpha^4\right)\|F\|_{L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))}^2\left(T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}/\alpha^{1-\delta}(1-\delta)\right)}. \end{aligned} \tag{97}$$

Since the fact that

$$1-\frac{3K_b^2T^{4\alpha}}{\alpha^4}\|F\|_{L^\infty(0,T;\mathbb{X}_{\beta+\nu,\alpha}(\Omega))}^2\frac{T^{\alpha(1-\delta)}|C_\delta|^2\mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \ge \frac{1}{2}, \tag{98}$$

We easily obtain the desired result (80). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to the work. The four authors read and approved the final manuscript.

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Research Article

A Complete Model of Crimean-Congo Hemorrhagic Fever (CCHF) Transmission Cycle with Nonlocal Fractional Derivative

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Crimean-Congo hemorrhagic fever is a common disease between humans and animals that is transmitted to humans through infected ticks, contact with infected animals, and infected humans. In this paper, we present a boxed model for the transmission of Crimean-Congo fever virus. With the help of the fixed-point theory, our proposed system model is investigated in detail to prove its unique solution. Given that the Caputo fractional-order derivative preserves the system's historical memory, we use this fractional derivative in our modeling. The equilibrium points of the proposed system and their stability conditions are determined. Using the Euler method for the Caputo fractional-order derivative, we calculate the approximate solutions of the fractional system, and then, we present a numerical simulation for the transmission of Crimean-Congo hemorrhagic fever.

1. Introduction

Crimean-Congo hemorrhagic fever is a common disease between human and livestock. The virus that causes this disease is one of the most important Arthropod-Borne viruses of the Bunyaviridae family, and it is a genus of Nairovirus that can cause severe and deadly disease in humans, but it is not associated with any specific clinical sign in livestock. The most common vector is a tick called Hyalomma, but it is also transmitted by other ticks [1]. The average mortality rate among infected people is 30 percent [2].

The first known case of the disease was recorded in 1942 in the Crimean region of the former Soviet Union. The virus that caused the disease was also isolated from the blood of a feverish patient in 1956 in the Democratic Republic of the Congo. The relationship between these two reported places of disease and the attention to the main symptoms of the disease (fever and bleeding) has led to the choice of the current name of the disease (Crimean-Congo hemorrhagic fever) (see [3, 4]). The disease has been reported in more than 31 countries in Africa, Asia, and Eastern Europe [5].

Numerous serological studies have confirmed infections in animals, especially domestic animals such as cattle, sheep, and goats that may occur as feverish reactions. Infection in animals occurs through the bite of ticks infected with the Crimean-Congo hemorrhagic fever virus [6]. Crimean-Congo hemorrhagic fever virus can also infect a wide range of wild animals. Among wild mammals, rabbits have been an important reservoir of the virus in the European part of the former Soviet Union and Bulgaria. In Asia, hedgehogs, rats, and particular species of rabbits are the reservoir of this virus [7].

The most important ways of getting infected with the Crimean-Congo fever virus are as follows: the person is getting bitten by infected ticks, the contact of scratched or injured skin of a person's body with the contents of infected crushed ticks, the contact of damaged skin or human mucosa with infected animal blood or secretions, and the contact with blood and other secretions of the infected person, as well as the contact with infected surgical instruments [8–10]. Because Crimean-Congo hemorrhagic fever is more likely to come from the contact with an infected animal or human or bite by infected ticks are transmitted to humans so hunters, farmers, ranchers, health personnel, and those contact with infected animals and humans due to occupations more likely to be infected.

Clinical signs and the course of this disease include four stages:

(i) Incubation Period: After a tick bite, the incubation period usually lasts 1-3 days and reaches a maximum of 1 day. The incubation period following the contact with infected tissues or blood is usually 5-6 days, and the maximum time is 13 days [11]

(ii) Prehaemorrhagy: In 80 percent of cases, Crimean-Congo hemorrhagic fever infections are asymptomatic. People in whom the disease has clinical manifestations, the onset of symptoms is sudden, and it lasts about 1 up to 7 days (average 3 days). The initial symptoms are severe headache, fever, chills, joint pain, muscle cramps, dizziness, pain and stiffness of the neck, eye pain, and fear of light. Nausea, vomiting, diarrhea, abdominal pain, loss of appetite, swelling and redness of the face, decreased heart rate, and low blood pressure have also been reported [12]

(iii) Haemorrhagy: The bleeding phase is short and usually starts on days 3 to 5 and lasts 1 to 11 days (average 4 days). Bleeding in the mucosa, hematoma, bleeding gums and nose, bleeding from the uterus, bloody sputum, and bleeding from the conjunctiva and ears are the symptoms of the disease at this stage. Bleeding from various organs worsens the patient's condition so that the patient may die in the second week of severe bleeding, intravascular coagulation, liver failure, and dehydration [13]

(iv) Convalescence Period: Between days 7 and 20, the fever stops, and then the bleeding stops. From the tenth day, when the skin lesions fade, patients gradually recover. Most patients are discharged from the hospital in the third to sixth week after the onset of illness when blood and urine tests return to be normal [11]

Biological and mathematical researchers have conducted research studies to model the transmission of Crimean-Congo fever. Kashkynbayev et al. have used an SI Model to study tick-borne diseases, including Crimean-Congo fever [14]. Ergena et al. have used an SIR Model to study the dynamic of tuberculosis and Crimean-Congo fever as epidemic diseases [15]. Switkes et al. have used the deterministic system of nonlinear differential equations to model the transmission of Crimean-Congo haemorrhagic fever with host immunity [16].

In recent years, extensive studies [17–19] have been conducted on the mathematical analysis of fractional derivatives and integrals. The fractional-order derivative is nonlocal and includes the historical and long-term memory effect of the system, and this is one of its most important advantages over the integer-order derivative, which helps to model natural phenomena better [20–23].

By the expansion of fractional differential calculus, researchers in many branches of science have turned to use the fractional differential equation system in their research. Mathematical modeling of the spread of viruses and the transmission of infectious diseases using systems of fractional differential equations are considered as one of the topics that has attracted the attention of researchers in recent decades [24]. Almeida et al. [25] proposed an epidemiologial MSEIR model formulated in the sense of Caputo fractional derivative. Baleanu et al. [26, 27] formulated new models of the HIV-1 infection of CD4+ T-cell and human liver via Caputo-Fabrizio fractional derivative. In addition, Rezapour et al. [28, 29] introduced new models for the spread of AH1N1 influenza and the transmission of Zika virus between humans and mosquitoes via Caputo-Fabrizio and Cuputo fractional derivatives, respectively. Singh analyzed the fractional blood alcohol model with a composite fractional derivative [30], and Singh et al. investigated the fractional fish farm model and fractional model of guava for biological pest control, [31, 32]. Also, Ghanbari et al. presented an efficient numerical method for the fractional model of allelopathic stimulatory phytoplankton species [33].

In this article, we model the complete Crimean-Congo fever transmission cycle between humans, animals, and ticks, which in previous articles, researchers have only modeled a part of the cycle. Due to the effect of fractional derivative memory and good results obtained in recent years from fractional mathematical modeling, in this study, we use the fractional-order differential equation system to model the Crimean-Congo fever transmission.

The structure of this paper is organized as follows: In Section 2, some basic definitions and concepts of fractional calculus are recalled. A fractional-order mathematical model for the Crimean-Congo fever transmission cycle is formulated in Section 3. In Section 4, with the help of the fixed-point theory, our proposed system (10) is proven to have a unique solution. The approximate solution of the fractional differential equation system (10) is obtained numerically, and a numerical simulation for the transmission of the Crimean-Congo fever virus is also provided in Section 5. In Section 6, we conclude our research work.

2. Preliminary Results and Definitions

In the current section, we recall the two definitions of the fractional-order derivative and corresponding integral of each one. A concept of the Laplace transform of fractional derivative is also discussed.

Definition 1 [34]. For an integrable function w, the Caputo derivative of fractional order $\vartheta \in (0,1)$ is given by

$$^{C}D^{\vartheta}w(t)=\frac{1}{\Gamma(m-\vartheta)}\int_0^t \frac{w^{(m)}(v)}{(t-v)^{\vartheta-m+1}}dv, \quad m[\vartheta]+1. \quad (1)$$

The Gamma function, denoted by $\Gamma(.)$, is defined as:

$$\Gamma(\vartheta)=\lim_{m\longrightarrow\infty}\frac{m!m^{\vartheta}}{\vartheta(\vartheta+1)(\vartheta+2)\cdots(\vartheta+m)}. \quad (2)$$

Also, the corresponding fractional integral of order ϑ with Re $(\vartheta)>0$ is given by

$$^{C}I^{\vartheta}w(t)=\frac{1}{\Gamma(\vartheta)}\int_0^t (t-v)^{\vartheta-1}w(v)dv. \quad (3)$$

Definition 2 ([35, 36]). For $w\in H^1(c,d)$ and $d>c$, the Caputo-Fabrizio derivative of fractional order $\vartheta\in(0,1)$ for w is given by

$$^{CF}D^{\vartheta}w(t)=\frac{M(\vartheta)}{(1-\vartheta)}\int_c^t \exp\left(\frac{-\vartheta}{1-\vartheta}(t-v)\right)w'(v)dv, \quad (4)$$

where $t\geq 0$, $M(\vartheta)$ is a normalization function that depends on ϑ and $M(0)=M(1)=1$. If $w\notin H^1(c,d)$ and $0<\vartheta<1$, this derivative for $w\in L^1(-\infty,d)$ as given by

$$^{CF}D^{\vartheta}w(t)=\frac{\vartheta M(\vartheta)}{(1-\vartheta)}\int_{-\infty}^d (w(t)-w(v))\exp\left(\frac{-\vartheta}{1-\vartheta}(t-v)\right)dv. \quad (5)$$

Also, the corresponding CF fractional integral is presented by

$$^{CF}I^{\vartheta}w(t)=\frac{2(1-\vartheta)}{(2-\vartheta)M(\vartheta)}w(t)+\frac{2\vartheta}{(2-\vartheta)M(\vartheta)}\int_0^t w(v)dv. \quad (6)$$

The Laplace transform is one of the most important tools in solving differential equations, which has different definitions in fractional calculus. The next definition presents the Laplace transform of the Caputo fractional-order derivative.

Definition 3 [34]. The Laplace transform of Caputo Fractional differential operator of order ϑ is given by

$$L\left[^{C}D^{\vartheta}w(t)\right](s)=s^{\vartheta}Lw(t)-\sum_{i=0}^{m-1}s^{\vartheta-i-1}w^{(i)}(0), \quad m-1<\vartheta\leq m\in N. \quad (7)$$

This can also be obtained in the following form:

$$L\left[^{C}D^{\vartheta}w(t)\right]=\frac{s^mL[w(t)]-s^{m-1}w(0)-s^{m-1}w'(0)-\cdots-w^{(m-1)}}{s^{m-\vartheta}}. \quad (8)$$

3. Model Formulation

Mathematical models are considered as one of the most important tools in the study of disease transmission. In this section, we present a fractional-order mathematical model for the Crimean-Congo fever transmission cycle.

Crimean-Congo haemorrhagic fever (CCHF) is a feverish hemorrhagic disease that is mostly transmitted by ticks. Although the virus is specific to animals, single infection, and epidemic cases of CCHF also occurred in humans. To model the transmission of this viral disease, we consider the population of transmitting ticks N_k, the population of livestock and wild animals N_l, and the human population N_h. We divide the tick population into two groups and denote susceptible ticks with S_k and infected ticks with I_k. In the previous section, we have mentioned that livestock and some wild animals can also be infected with this disease and be a virus reservoir, which we divide into two groups, susceptible group S_l and infected group I_l. Like the previous two populations, we divide the human population into two susceptible S_h and infected I_h groups. Susceptible ticks are infected through infected ticks at the effective contact rate β_1 and through infected animals at the effective contact rate β_2. Infected ticks transmit the virus to susceptible animals at the effective contact rate β_3 when they feed on animal body. Crimean-Congo fever virus is transmitted to humans in three ways. The virus is transmitted to humans through infected ticks at the effective contact rate β_4, through the blood and blood products of an infected animal at the effective contact rate β_5, and through the blood and bloody mucosa of infected human at the effective contact rate β_6. We also consider the recruitment rate of ticks, animals, and humans as Λ_k, Λ_l, and Λ_h, respectively. The natural mortality rates of ticks, animals, and humans are d_k, d_l, and d_h, respectively.

Based on the provided explanations, we present the Crimean-Congo fever transfer model with the system of differential equations as follows:

$$\begin{cases}\frac{dS_k}{dt}=\Lambda_k-\beta_1S_k(t)I_k(t)-\beta_2S_k(t)I_l(t)-d_kS_k(t),\\ \frac{dI_k}{dt}=\beta_1S_k(t)I_k(t)+\beta_2S_k(t)I_l(t)-d_kI_k(t),\\ \frac{dS_l}{dt}=\Lambda_l-\beta_3S_l(t)I_l(t)-d_lS_l(t),\\ \frac{dI_l}{dt}=\beta_3S_l(t)I_l(t)-d_lI_l(t),\\ \frac{dS_h}{dt}=\Lambda_h-\beta_4S_h(t)I_k(t)-\beta_5S_h(t)I_l(t)-\beta_6S_h(t)I_h(t)-d_hS_h(t),\\ \frac{dI_h}{dt}=\beta_4S_h(t)I_k(t)+\beta_5S_h(t)I_l(t)+\beta_6S_h(t)I_h(t)-d_hI_h(t),\end{cases} \quad (9)$$

where all of the initial conditions $S_k(0)=S_{0k}, I_k(0)=I_{0k}, S_l(0)=S_{0l}, I_l(0)=I_{0l}, S_h(0)=S_{0h}$, and $I_h(0)=I_{0h}$ are positive.

The fractional-order system (FDEs) is related to systems with memory, history, or nonlocal effects which exist in the many biological systems that show the realistic biphasic decline behavior of infection or diseases but at a slower rate. In the above integer-order system, since the internal memory effects of the biological system of CCHF are not included, it is better that we extend the proposed ordinary model to a fractional model. In this alternative, the equality of the dimensions of both sides of the equation is disturbed, and we use an auxiliary parameter σ, with the dimension of sec., to solve this problem ([37]). Thus, the fractional-order model for the Crimean-Congo haemorrhagic fever (CCHF) is given as follows:

$$\begin{cases} \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}S_k(t)=\Lambda_k-\beta_1S_k(t)I_k(t)-\beta_2S_k(t)I_l(t)-d_kS_k(t),\\ \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}I_k(t)=\beta_1S_k(t)I_k(t)+\beta_2S_k(t)I_l(t)-d_kI_k(t),\\ \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}S_l(t)=\Lambda_l-\beta_3S_l(t)I_k(t)-d_lS_l(t),\\ \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}I_l(t)=\beta_3S_l(t)I_k(t)-d_lI_l(t),\\ \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}S_h(t)=\Lambda_h-\beta_4S_h(t)I_k(t)-\beta_5S_h(t)I_l(t)-\beta_6S_h(t)I_h(t)-d_hS_h(t),\\ \sigma^{\vartheta-1}{}^{C}D_t^{\vartheta}I_h(t)=\beta_4S_h(t)I_k(t)+\beta_5S_h(t)I_l(t)+\beta_6S_h(t)I_h(t)-d_hI_h(t), \end{cases} \tag{10}$$

where $t\ge 0$ and $0<\vartheta<1$.

3.1. Nonnegative Solution. To show the nonnegativity of solutions, we claim that $M=\{(S_k,I_k,S_l,I_l,S_h,I_h)\in R_6^+ : N_k(t)\le(\Lambda_k/d_k), N_l(t)\le(\Lambda_l/d_l), N_h(t)\le(\Lambda_h/d_h)\}$ is the feasibility region of system (10). To prove this claim, we consider the following Lemma.

Lemma 4. *The closed set M with respect to the fractional system (10) is positively invariant.*

Proof. We first add two relations in the system (10) to obtain the fractional derivative of the total population of ticks. So,

$$\sigma^{\vartheta-1}{}^{C}D_t^{v}N_k(t)=\Lambda_k-d_kN_k(t), \tag{11}$$

where $N_k(t)=S_k(t)+I_k(t)$. We apply the Laplace transform to the parties of the above relation, then

$$N_k(t)=N_k(0)E_\vartheta\left(-d_k\sigma^{1-\vartheta}t^\vartheta\right)+\int_0^t \Lambda_k\sigma^{1-\vartheta}\eta^{\vartheta-1}E_{\vartheta,\vartheta}\left(-d_k\sigma^{1-\eta}\eta^\vartheta\right)d\eta. \tag{12}$$

In the above equation, $N_k(0)$ is the initial size of ticks population, and the terms $E_\vartheta, E_{\vartheta,\vartheta}$ are the Mittag-Leffler functions which are defined by

$$E_\vartheta(w)=\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(1+n\vartheta)}, E_{\vartheta,\vartheta}(w)=\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(\vartheta+n\vartheta)},\quad \vartheta>0. \tag{13}$$

By simplifying the relations, we conclude that

$$\begin{aligned} N_k(t)&=N_k(0)E_\vartheta\left(-d_k\sigma^{1-\vartheta}t^\vartheta\right)+\int_0^t\Lambda_k\sigma^{1-\vartheta}\eta^{\vartheta-1}\\ &\cdot\sum_{n=0}^{\infty}\frac{(-1)^n d_k^n\sigma^{n(1-\vartheta)}\eta^{n\vartheta}}{\Gamma(n\vartheta+\vartheta)}d\eta=\frac{\Lambda_k\sigma^{1-\vartheta}}{d_k\sigma^{1-\vartheta}}\\ &+E_\vartheta\left(-d_k\sigma^{1-\vartheta}t^\vartheta\right)\left(N_k(0)-\frac{\Lambda_k\sigma^{1-\vartheta}}{d_k\sigma^{1-\vartheta}}\right),\\ &=\frac{\Lambda_k}{d_k}+E_\vartheta\left(-d_k\sigma^{1-\vartheta}t^\vartheta\right)\left(N_k(0)-\frac{\Lambda_k}{d_k}\right). \end{aligned} \tag{14}$$

Now, if $N_k(0)\le(\Lambda_k/d_k)$, then for $t>0$, $N_k(t)\le(\Lambda_k/d_k)$. At the same way for N_l and N_h, we can prove that if $N_l(0)\le(\Lambda_l/d_l)$ and $N_h(0)\le(\Lambda_h/d_h)$, then $N_l(t)\le(\Lambda_l/d_l)$ and $N_h(t)\le(\Lambda_h/d_h)$. Thus, the closed set M with respect to fractional model (2) is positively invariant. □

3.2. Equilibrium Points. In the current section, we determine the equilibrium points of the system (10) and the basic reproduction number. We present the necessary conditions for the stability of the system at the equilibrium point. To determine the equilibrium points, we set the equations to zero in system (10),

$$\begin{aligned} {}^{C}D^{\vartheta}S_k(t)&={}^{C}D^{\vartheta}I_k(t)={}^{C}D^{\vartheta}S_l(t)={}^{C}D^{\vartheta}I_l(t)={}^{C}D^{\vartheta}S_h(t)\\ &={}^{C}D^{\vartheta}I_h(t)=0, \end{aligned} \tag{15}$$

We solve the resulting algebraic equations and determine the equilibrium point of the system. The disease-free equilibrium point, denoted by E_0, is obtained as: $E_0=((\Lambda_k/d_k),0,(\Lambda_l/d_l),0,(\Lambda_h/d_h),0)$. The second equilibrium point, called the endemic equilibrium point, is obtained as $E^*=(S_k^*,I_k^*,S_l^*,I_l^*,S_h^*,I_h^*)$,

$$\begin{aligned} S_k^*&=\frac{\Lambda_k}{\beta_1I_k^*+\beta_2I_l^*+d_k}, S_l^*=\frac{\Lambda_l}{\beta_3I_k^*+d_l}, S_h^*\\ &=\frac{\Lambda_h}{\beta_4I_k^*+\beta_5I_l^*+\beta_6I_h^*+d_h}. \end{aligned} \tag{16}$$

When the basic reproduction number is greater than one, and the spread of the disease continues, the endemic equilibrium point is defined. To obtain the basic reproduction number, we use the next generation method [38]. We consider the matrix form of the system (10) as follows:

$$^{C}D^{\vartheta}v(t)=F(v(t))-V(v(t)), \tag{17}$$

where

$$F(\nu(t)) = \sigma^{1-\vartheta} \begin{bmatrix} \beta_1 S_k(t) I_k(t) + \beta_2 S_k(t) I_l(t) \\ \beta_3 S_l(t) I_k(t) \\ \beta_4 S_h(t) I_k(t) + \beta_5 S_h(t) I_l(t) + \beta_6 S_h(t) I_h(t) \end{bmatrix},$$

$$V(\nu(t)) = \sigma^{1-\vartheta} \begin{bmatrix} d_k I_k(t) \\ d_l I_l(t) \\ d_h I_h(t) \end{bmatrix}. \tag{18}$$

By calculating the Jacobian matrix for F and V at the disease-free equilibrium point, we obtain:

$$J_F(E_0) = \sigma^{1-\vartheta} \begin{bmatrix} \dfrac{\beta_1 \Lambda_k}{d_k} & \dfrac{\beta_2 \Lambda_k}{d_k} & 0 \\ \dfrac{\beta_3 \Lambda_l}{d_l} & 0 & 0 \\ \dfrac{\beta_4 \Lambda_h}{d_h} & \dfrac{\beta_5 \Lambda_h}{d_h} & \dfrac{\beta_6 \Lambda_h}{d_h} \end{bmatrix}, \tag{19}$$

$$J_v(E_0) = \sigma^{1-\vartheta} \begin{bmatrix} d_k & 0 & 0 \\ 0 & d_l & 0 \\ 0 & 0 & d_h \end{bmatrix}.$$

The basic reproduction number R_0 is defined as the eigenvalue of next generation matrix of system (10), $R_0 = \rho(FV^{-1})$. We obtain: $R_0 = \max(R_h, R_{kl})$,

$$R_h = \frac{\beta_6 \Lambda_h}{d_h^2}, R_{kl} = \frac{\beta_1 \Lambda_k d_l + \sqrt{\beta_1^2 \Lambda_k^2 d_l^2 + 4\beta_2 \beta_3 \Lambda_k \Lambda_l d_k^2}}{2 d_l d_k^2}. \tag{20}$$

In commonly used infection models, when $R_0 > 1$, the infection will be able to start spreading in a population, but not if $R_0 < 1$.

3.3. Stability of Equilibrium Points. To determine the necessary conditions for the stability of the disease-free equilibrium point, we investigate the roots of the characteristic equation of system (10). The Jacobian matrix of the system (10) is

$$J = \sigma^{1-\vartheta} \times \begin{bmatrix} -\beta_1 I_k - \beta_2 I_l - d_k & -\beta_1 S_k & 0 & -\beta_2 S_k & 0 & 0 \\ \beta_1 I_k + \beta_2 I_l & \beta_1 S_k - d_k & 0 & \beta_2 S_k & 0 & 0 \\ 0 & -\beta_3 S_l & -\beta_3 I_k - d_l & 0 & 0 & 0 \\ 0 & \beta_3 S_l & \beta_3 I_k & -d_l & 0 & 0 \\ 0 & -\beta_4 S_h & 0 & -\beta_5 S_h & -\beta_4 I_k - \beta_5 I_l - \beta_6 I_h - d_h & -\beta_h S_h \\ 0 & \beta_4 S_h & 0 & \beta_5 S_h & \beta_4 I_k + \beta_5 I_l + \beta_6 I_h & \beta_6 S_h - d_h \end{bmatrix}. \tag{21}$$

Then, the Jacobian matrix at E_0 is obtained as:

$$J(E_0) = \sigma^{1-\vartheta} \begin{bmatrix} -d_k & -\beta_1 \dfrac{\Lambda_k}{d_k} & 0 & -\beta_2 \dfrac{\Lambda_k}{d_k} & 0 & 0 \\ 0 & \beta_1 \dfrac{\Lambda_k}{d_k} - d_k & 0 & \beta_2 \dfrac{\Lambda_k}{d_k} & 0 & 0 \\ 0 & -\beta_3 \dfrac{\Lambda_l}{d_l} & -d_l & 0 & 0 & 0 \\ 0 & \beta_3 \dfrac{\Lambda_l}{d_l} & 0 & -d_l & 0 & 0 \\ 0 & -\beta_4 \dfrac{\Lambda_h}{d_h} & 0 & -\beta_5 \dfrac{\Lambda_h}{d_h} & -d_h & -\beta_h \dfrac{\Lambda_h}{d_h} \\ 0 & \beta_4 \dfrac{\Lambda_h}{d_h} & 0 & \beta_5 \dfrac{\Lambda_h}{d_h} & 0 & \beta_6 \dfrac{\Lambda_h}{d_h} - d_h \end{bmatrix}. \tag{22}$$

In the following theorem, we determine the necessary conditions for the stability of the disease-free equilibrium point.

Theorem 5. *The disease-free equilibrium point E^0 is locally asymptotically stable if $R_0 < 1$.*

Proof. The characteristic equation of matrix $J(E_0)$ is obtained as follows:

$$(d_l + \lambda)(d_k + \lambda)(d_h + \lambda)\left(\beta_6 \frac{\Lambda_h}{d_h} - d_h - \lambda\right) \cdot \left[(d_l + \lambda)\left(\beta_1 \frac{\Lambda_k}{d_k} - d_k - \lambda\right) - \beta_2 \beta_3 \frac{\Lambda_k \Lambda_l}{d_k d_l}\right] = 0. \tag{23}$$

Therefore, the eigenvalues of the Jacobin matrix are $\lambda_1 = -d_l, \lambda_2 = -d_k, \lambda_3 = -d_h, \lambda_4 = (\beta_6 \Lambda_h / d_h) - d_h$, and the roots of the following equation are:

$$\lambda^2 - \lambda\left(\frac{\beta_1 \Lambda_k}{d_k} - d_k - d_l\right) - \frac{\beta_1 \Lambda_k d_l}{d_k} + d_l d_k + \beta_2 \beta_3 \frac{\Lambda_k \Lambda_l}{d_k d_l} = 0. \tag{24}$$

The three roots λ_1, λ_2, and λ_3 are negative. If $R_0 < 1$, then $R_h = (\beta_6 \Lambda_h / d_h^2) < 1$, we obtain $\lambda_4 < 0$. It also follows from $R_0 < 1$ that $R_{kl} < 1$, then we conclude by simplifying $\beta_1 \Lambda_k < d_k^2$. In Equation (24), which is a quadratic equation, we have:

$$P = \frac{-\beta_1 \Lambda_k d_l}{d_k} + d_l d_k + \beta_2 \beta_3 \frac{\Lambda_k \Lambda_l}{d_k d_l}, S = \frac{\beta_1 \Lambda_k}{d_k} - d_k - d_l, \tag{25}$$

since $\beta_1 \Lambda_k < d_k^2$ then $P > 0, S < 0$ so Equation (24) has 2 negative roots. Therefore, all of the eigenvalues are negative, and the disease-free equilibrium point is locally asymptotically stable. □

4. Existence of Unique Solution

In the current section, using the fixed-point theory, we prove that system (10) has a unique solution. Fixed-point theory is essential in proving the existence of a solution to the proposed system where adequate conditions are provided by fixed-point theorems such that a unique fixed point exists for a given function. To achieve this goal, we prove that kernels are satisfied under the Lipschitz condition, and they are contraction. Then, the existence of solution to the proposed system is constructed via fixed-point theorem. From the Lipschitz condition, the uniqueness of our obtained solution is proven when the obtained condition is satisfied.

First, we consider system (10) in the following compact form:

$$\begin{cases} \sigma^{\vartheta-1C}D_t^{\vartheta}S_k(t) = R_1(t, S_k(t)), \\ \sigma^{\vartheta-1C}D_t^{\vartheta}I_k(t) = R_2(t, I_k(t)), \\ \sigma^{\vartheta-1C}D_t^{\vartheta}S_l(t) = R_3(t, S_l(t)), \\ \sigma^{\vartheta-1C}D_t^{\vartheta}I_l(t) = R_4(t, I_l(t)), \\ \sigma^{\vartheta-1C}D_t^{\vartheta}S_h(t) = R_5(t, S_h(t)), \\ \sigma^{\vartheta-1C}D_t^{\vartheta}I_h(t) = R_6(t, I_h(t)). \end{cases} \quad (26)$$

We apply the fractional-order integral to the parties of the above equations, so

$$\begin{cases} S_k(t) - S_k(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_1(\mu, S_k)(t-\mu)^{\vartheta-1}d\mu, \\ I_k(t) - I_k(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_2(\mu, I_k)(t-\mu)^{\vartheta-1}d\mu, \\ S_l(t) - S_l(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_3(\mu, S_l)(t-\mu)^{\vartheta-1}d\mu, \\ I_l(t) - I_l(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_4(\mu, I_l)(t-\mu)^{\vartheta-1}d\mu, \\ S_h(t) - S_h(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_5(\mu, S_h)(t-\mu)^{\vartheta-1}d\mu, \\ I_h(t) - I_h(0) = \dfrac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\displaystyle\int_0^t R_6(\mu, I_h)(t-\mu)^{\vartheta-1}d\mu. \end{cases} \quad (27)$$

In the following, we prove that the kernels $R_j, j = 1, 2, 3, 4, 5, 6$ are satisfied in the Lipschitz condition, and they are contraction.

Theorem 6. *Kernel R_1 is satisfied in Lipschitz condition and contraction if we have:*

$$0 \le \beta_1 z_1 + \beta_2 z_2 + d_k < 1. \quad (28)$$

Proof. We can write for S_k and S_{1k},

$$\begin{aligned} &\|R_1(t, S_k) - R_1(t, S_{1k})\| \\ &\quad = \|-\beta_1 I_k(S_k - S_{1k}) - \beta_2 I_l(S_k - S_{1k}) - d_k(S_k - S_{1k})\|, \\ &\quad \le \beta_1\|I_k\|\|S_k - S_k\| + \beta_2\|I_l\|\|S_k - S_{1k}\| + d_k\|S_k - S_{1k}\|, \\ &\quad \le (\beta_1\|I_k\| + \beta_2\|I_l\| + d_k)\|S_k - S_{1k}\|, \\ &\quad \le (\beta_1 z_1 + \beta_2 z_2 + d_k)\|S_k - S_{1k}\|. \end{aligned} \quad (29)$$

Consider $e_1 = \beta_1 z_1 + \beta_2 z_2 + d_k$, where $\|I_k(t)\| \le z_1$ and $\|I_l\| \le z_2$, are bounded functions. We get:

$$\|R_1(t, S_k) - R_1(t, S_{1k})\| \le e_1\|S_k(t) - S_{1k}(t)\|, \quad (30)$$

if $0 \le \beta_1 z_1 + \beta_2 z_2 + d_k < 1$, then the kernel R_1 is satisfied in Lipschitz condition, and it is contraction. □

In a similar way, we can show that the kernels $R_j, j = 2, 3, 4, 5, 6$ are satisfied in the Lipschitz condition as follows:

$$\begin{cases} \|R_2(t, I_k) - R_2(t, I_{1k})\| \le e_2\|I_k(t) - I_{1k}(t)\|, \\ \|R_3(t, S_l) - R_3(t, S_{1l})\| \le e_3\|S_l(t) - S_{1l}(t)\|, \\ \|R_4(t, I_l) - R_4(t, I_{1l})\| \le e_4\|I_l(t) - I_{1l}(t)\|, \\ \|R_5(t, S_h) - R_3(t, S_{1h})\| \le e_5\|S_h(t) - S_{1h}(t)\|, \\ \|R_6(t, I_h) - R_4(t, I_{1h})\| \le e_6\|I_h(t) - I_{1h}(t)\|, \end{cases} \quad (31)$$

so that $e_2 = \beta_1 z_4 + d_k, e_3 = \beta_3 z_1 + d_l, e_4 = d_l, e_5 = \beta_4 z_1 + \beta_5 z_2 + \beta_6 z_3 + d_h, e_6 = \beta_6 z_6 + d_h$ are bounded functions where $\|I_h(t)\| \le z_3$, $\|S_k(t)\| \le z_4$, $\|S_l(t)\| \le z_5$, and $\|S_h(t)\| \le z_6$. Also, if $0 \le e_j < 1, j = 2, 3, 4, 5, 6$, then R_j are contraction for $j = 2, 3, 4, 5, 6$.

Based on system (27), we define:

$$A_{1n}(t) = S_{nk}(t) - S_{(n-1)k}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\int_0^t \left(R_1\left(\mu, S_{(n-1)k}\right) - R_1\left(\mu, S_{(n-2)k}\right)\right)(t-\mu)^{\vartheta-1}d\mu,$$

$$A_{2n}(t) = I_{nk}(t) - I_{(n-1)k}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\int_0^t \left(R_2\left(\mu, I_{(n-1)k}\right) - R_2\left(\mu, I_{(n-2)k}\right)\right)(t-\mu)^{\vartheta-1}d\mu,$$

$$A_{3n}(t) = S_{nl}(t) - S_{(n-1)l}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\int_0^t \left(R_3\left(\mu, S_{(n-1)l}\right) - R_3\left(\mu, S_{(n-2)l}\right)\right)(t-\mu)^{\vartheta-1}d\mu,$$

$$A_{4n}(t) = I_{nl}(t) - I_{(n-1)l}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)}\int_0^t \left(R_4\left(\mu, I_{(n-1)l}\right) - R_4\left(\mu, I_{(n-2)l}\right)\right)(t-\mu)^{\vartheta-1}d\mu,$$

$$A_{5n}(t) = S_{nh}(t) - S_{(n-1)h}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \left(R_5\left(\mu, S_{(n-1)h}\right) - R_5\left(\mu, S_{(n-2)h}\right)\right)(t-\mu)^{\vartheta-1} d\mu,$$

$$A_{6n}(t) = I_{nh}(t) - I_{(n-1)h}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \left(R_6\left(\mu, I_{(n-1)h}\right) - R_6\left(\mu, I_{(n-2)h}\right)\right)(t-\mu)^{\vartheta-1} d\mu, \quad (32)$$

where $S_k(0) = S_{0k}, I_k(0) = I_{0k}, S_l(0) = S_{0l}, I_l(0) = I_{0l}, S_h(0) = S_{0h}$, and $I_h(0) = I_{0h}$ are initial conditions. The norm of A_{1n} in the above system is expressed as follows:

$$\|A_{1n}(t)\| = \left\|S_{nk}(t) - S_{(n-1)k}(t)\right\| \le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \left\|R_1\left(\mu, S_{(n-1)k}\right) - R_1\left(\mu, S_{(n-2)k}\right)\right)(t-\mu)^{\vartheta-1}\right\| d\mu. \quad (33)$$

According to the Lipschitz condition (30), we conclude

$$\|A_{1n}(t)\| \le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 \int_0^t \left\|A_{1(n-1)}(\mu)\right\| d\mu. \quad (34)$$

Similarly, we can prove that

$$\begin{aligned}
\|A_{2n}(t)\| &\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_2 \int_0^t \left\|A_{2(n-1)}(\mu)\right\| d\mu,\\
\|A_{3n}(t)\| &\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_3 \int_0^t \left\|A_{3(n-1)}(\mu)\right\| d\mu,\\
\|A_{4n}(t)\| &\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_4 \int_0^t \left\|A_{4(n-1)}(\mu)\right\| d\mu,\\
\|A_{5n}(t)\| &\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_5 \int_0^t \left\|A_{5(n-1)}(\mu)\right\| d\mu,\\
\|A_{6n}(t)\| &\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_6 \int_0^t \left\|A_{6(n-1)}(\mu)\right\| d\mu.
\end{aligned} \quad (35)$$

Therefore, we get

$$\begin{aligned}
S_{nk}(t) &= \sum_{i=1}^n A_{1i}(t), I_{nk}(t) = \sum_{i=1}^n A_{2i}(t), S_{nl}(t) = \sum_{i=1}^n A_{3i}(t),\\
I_{nl}(t) &= \sum_{i=1}^n A_{4i}(t), S_{nh}(t) = \sum_{i=1}^n A_{5i}(t), I_{nh}(t) = \sum_{i=1}^n A_{6i}(t).
\end{aligned} \quad (36)$$

In the next theorem, we prove the existence of solution by the fixed-point theorem.

Theorem 7. *The Crimean-Congo fever transmission fractional-order model (10) has a solution, if there exists t_ε such that*

$$\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} t_\varepsilon e_i < 1. \quad (37)$$

Proof. By Equation (34) and Equation (46), we obtain

$$\begin{aligned}
\|A_{1n}(t)\| &\le \|S_{nk}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 t\right]^n, \|A_{2n}(t)\| \le \|I_{nk}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_2 t\right]^n,\\
\|A_{3n}(t)\| &\le \|S_{nl}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_3 t\right]^n, \|A_{4n}(t)\| \le \|I_{nl}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_4 t\right]^n,\\
\|A_{5n}(t)\| &\le \|S_{nh}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_5 t\right]^n, \|A_{6n}(t)\| \le \|I_{nh}(0)\| \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_6 t\right]^n.
\end{aligned} \quad (38)$$

The above relations show that the system has a continuous solution. Now, it is sufficient to show that the above functions construct the solution for the fractional-order model (10). We consider the following relations:

$$\begin{aligned}
S_k(t) - S_k(0) &= S_{nk}(t) - \mathrm{U}_{1n}(t), I_k(t) - I_k(0) = I_{nk}(t) - \mathrm{U}_{2n}(t),\\
S_l(t) - S_l(0) &= S_{nl}(t) - \mathrm{U}_{3n}(t), I_l(t) - I_l(0) = I_{nl}(t) - \mathrm{U}_{4n}(t),\\
S_h(t) - S_h(0) &= S_{nh}(t) - \mathrm{U}_{5n}(t), I_h(t) - I_h(0) = I_{nh}(t) - \mathrm{U}_{6n}(t).
\end{aligned} \quad (39)$$

The norm of $\mathrm{U}_{1n}(t)$ is obtained as follows:

$$\begin{aligned}
\|\mathrm{U}_{1n}(t)\| &= \left\|\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \left(R_1(\mu, S_k) - R_1\left(\mu, S_{(n-1)k}\right)\right) d\mu\right\|\\
&\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \left\|R_1(\mu, S_k) - R_1\left(\mu, S_{(n-1)k}\right)\right\| d\mu\\
&\le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 \left\|S_k - S_{(n-1)k}\right\| t.
\end{aligned} \quad (40)$$

By continuing this repetitive method, we conclude:

$$\|\mathrm{U}_{1n}(t)\| \le \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} t\right]^{n+1} e_1^{n+1} k. \quad (41)$$

At t_ε, we have

$$\|\mathrm{U}_{1n}(t)\| \le \left[\frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} t_\varepsilon\right]^{n+1} e_1^{n+1} k. \quad (42)$$

If we take limit on the recent relation as n approaches to ∞, it results $\|\mathrm{U}_{1n}(t)\| \longrightarrow 0$. Similarly, we conclude that $\|\mathrm{B}_{jn}(t)\| \longrightarrow 0, j = 2, 3, 4, 5, 6$, and the proof is complete. □

To show the uniqueness of the solution of CCHF model, we consider that the fractional-order system (10) has another solution such as $(S_{1k}(t), I_{1k}(t), S_{1l}(t), I_{1l}(t), S_{1h}(t), I_{1h}(t)$, then for S_k, S_{1k} can be written as:

$$S_k(t) - S_{1k}(t) = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t (R_1(\mu, S_k) - R_1(\mu, S_{1k}))d\mu. \quad (43)$$

We take the norm on the above equation, so

$$\|S_k(t) - S_{1k}(t)\| = \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} \int_0^t \|R_1(\mu, S_k) - R_1(\mu, S_{1k})\| d\mu. \quad (44)$$

By Lipschitz condition (30), we obtain:

$$\|S_k(t) - S_{1k}(t)\| \le \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 t \|S_k(t) - S_{1k}(t)\|. \quad (45)$$

Thus,

$$\|S_k(t) - S_{1k}(t)\| \left(1 - \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 t\right) \le 0. \quad (46)$$

Theorem 8. *The solution of fractional-order system (10) is unique when the following condition is met:*

$$1 - \frac{\sigma^{1-\vartheta}}{\Gamma(\vartheta)} e_1 t > 0. \quad (47)$$

Proof. Assume that the condition (46) holds, in which case we conclude from (46) and (47) that $\|S_k(t) - S_{1k}(t)\| = 0$, and this shows that $S_h(t) = S_{1h}(t)$. In the same way, similar relationships can be reached for I_k, S_l, I_l, S_h, I_h. This completes the proof. □

5. Numerical Simulation and Discussion

In this section, we first obtain the approximate solution of the fractional differential equation system (10) by a numerical method, and then, we present a numerical simulation for the transmission of the Crimean-Congo fever virus.

5.1. Numerical Method. We use the fractional Euler method for Caputo derivative [39] to obtain the approximate solutions of the Crimean-Congo hemorrhagic fever virus transmission model. First, we consider the compact form of the system (10) as follows:

$$\sigma^{\vartheta-1} {}^C D_t^\vartheta \varphi(t) = Q(t, \varphi(t)), \varphi(0) = \varphi_0, 0 \le t \le T < \infty, \quad (48)$$

where $\varphi = (S_k, I_k, S_l, I_l, S_h, I_h) \in R_+^6$, $\varphi_0 = (S_{0k}, I_{0k}, S_{0l}, I_{0l}, S_{0h}, I_{0h})$, and $Q(t)$ is a continuous real vector function that is satisfied in the Lipschitz condition as follows:

$$\|Q(\varphi_1(t)) - Q(\varphi_2(t))\| \le m \|\varphi_1(t) - \varphi_2(t)\|, m > 0. \quad (49)$$

We apply the fractional-order integral operator corresponding to the Caputo fractional-order derivative on both sides of Equation (48), so

$$\varphi(t) = \sigma^{1-\vartheta} \left[\varphi_0 + I^\vartheta Q(\varphi(t))\right], 0 \le t \le T < \infty. \quad (50)$$

Set $r = (T - 0)/N$ and $t_n = nr$, where $t \in [0, T]$ and N is a natural number and $n = 0, 1, 2, \cdots, N$. Let φ_n be the approximation of $\varphi(t)$ at $t = t_n$. By the fractional Euler method ([39]), we get:

$$\varphi_{n+1} = \sigma^{1-\vartheta} \left[\varphi_0 + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} Q\left(t_p, \varphi_p\right)\right], p = 0, 1, 2, \cdots, N-1, \quad (51)$$

where

$$\omega_{n+1,p} = (n+1-p)^\vartheta - (n-p)^\vartheta, p = 0, 1, 2, \cdots, n. \quad (52)$$

The obtained scheme is stable. Details of the stability analysis are given in Theorem (3.1) of [39]. According to the explanations provided, the answer of the system is obtained as follows:

$$\begin{aligned}
S_{(n+1)k} &= \sigma^{1-\vartheta} \left[S_{0k} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_1\left(t_p, \varphi_p\right)\right], \\
I_{(n+1)k} &= \sigma^{1-\vartheta} \left[I_{0k} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_2\left(t_p, \varphi_p\right)\right], \\
S_{(n+1)l} &= \sigma^{1-\vartheta} \left[S_{0l} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_3\left(t_p, \varphi_p\right)\right], \\
I_{(n+1)l} &= \sigma^{1-\vartheta} \left[I_{0l} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_4\left(t_p, \varphi_p\right)\right], \\
S_{(n+1)h} &= \sigma^{1-\vartheta} \left[S_{0h} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_5\left(t_p, \varphi_p\right)\right], \\
I_{(n+1)h} &= \sigma^{1-\vartheta} \left[I_{0h} + \frac{r^\vartheta}{\Gamma(\vartheta+1)} \sum_{p=0}^{n} \omega_{n+1,p} y_6\left(t_p, \varphi_p\right)\right],
\end{aligned} \quad (53)$$

so that $\omega_{n+1,p} = (n+1-p)^\vartheta - (n-p)^\vartheta$ and the functions y_j for $j = 0, 1, \cdots, 6$ are expressed as:

$$\begin{aligned}
y_1(t, \varphi(t)) &= \Lambda_k - \beta_1 S_k(t) I_k(t) - \beta_2 S_k(t) I_l(t) - d_k S_k(t), \\
y_2(t, \varphi(t)) &= \beta_1 S_k(t) I_k(t) + \beta_2 S_k(t) I_l(t) - d_k I_k(t), \\
y_3(t, \varphi(t)) &= \Lambda_l - \beta_3 S_l(t) I_k(t) - d_l S_l(t), \\
y_4(t, \varphi(t)) &= \beta_3 S_l(t) I_k(t) - d_l I_l(t), \\
y_5(t, \varphi(t)) &= \Lambda_h - \beta_4 S_h(t) I_k(t) - \beta_5 S_h(t) I_h(t) \\
&\quad - \beta_6 S_h(t) I_h(t) - d_h S_h(t), \\
y_6(t, \varphi(t)) &= \beta_4 S_h(t) I_k(t) + \beta_5 S_h(t) I_l(t) \\
&\quad + \beta_6 S_h(t) I_h(t) - d_h I_h(t).
\end{aligned} \quad (54)$$

5.2. Simulation. In the present subsection, we present a numerical simulation to investigate the transmission of

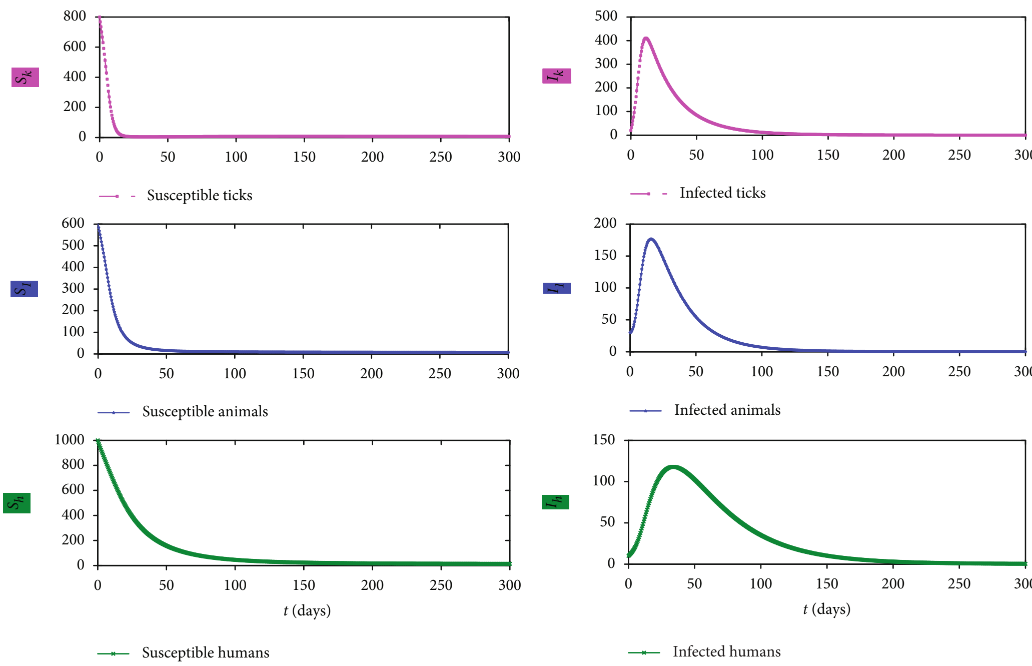

FIGURE 1: Plots of the results of model (10) with $R_0 < 1$ for $\vartheta = 0.98$.

Crimean-Congo fever virus based on the amount of reproduction number. Also, we compare the results of the integer-order and fractional-order models.

To perform the desired simulation, in two cases, we consider different values for the parameters. In the first case, we assume: $\beta_1 = 0.5 \times 10^{-4}$, $\beta_2 = 0.3 \times 10^{-4}$, $\beta_3 = 0.1 \times 10^{-3}$, $\beta_4 = 0.03 \times 10^{-4}$, $\beta_5 = 0.4 \times 10^{-4}$, $\beta_6 = 0.7 \times 10^{-4}$, $\Lambda_k = 0.6$, $\Lambda_l = 0.3$, $\Lambda_h = 0.6$, $d_k = 0.09$, $d_l = 0.07$, $d_h = 0.007$, $\sigma = 0.99$. We also consider the initial values as $S_k = 800$, $I_k = 20$, $S_l = 600$, $I_l = 30$, $S_h = 1000$, $I_h = 10$.

Using the above parameters, we obtain $R_h = 0.0233$, $R_{kl} = 0.00597$; thus, $R_0 = 0.0233 < 1$. Figure 1 shows the results of model (10) for all six groups for $\vartheta = 0.98$. In this case, $R_0 < 1$, and Figure 1 shows that over time, the number of susceptible people is decreased, and the number of infected people is increased, but in less than 20 days, the number of infected people is decreased and eventually reaches zero, and the spread of the disease stops. In this case, $S(t)$ and $I(t)$ converge to the disease-free equilibrium point E_0.

In the second case, we assume that the disease transmission rate increases from the susceptible group to the infected group, and the transmission rates are equal to $\beta_1 = 0.5 \times 10^{-3}$, $\beta_2 = 0.3 \times 10^{-3}$, $\beta_3 = 0.1 \times 10^{-2}$, $\beta_4 = 0.7 \times 10^{-3}$, $\beta_5 = 0.4 \times 10^{-3}$, $\beta_6 = 0.7 \times 10^{-3}$. With these transfer rates, the value of the reproduction number is equal to $R_0 = 1.435 > 1$. Figure 2 shows the results of model (10) for the six groups studied in this case. Over time, the population of susceptible groups decreases and the population of infected groups increases, and finally, after 100 days, the population of infected groups decreases and converges to the endemic equilibrium point. As the rate of disease transmission increases, the value of R_0 increases, and we observe that the disease does not go away and its spread continues.

In this work, we have used the fractional-order derivative for modeling. In order to investigate the effect of derivation order, we have drawn the model results for infected groups with derivatives with integer-order $\vartheta = 1$ and fractional-order $\vartheta = 0.98$ in Figure 3. Figure 3 shows that the results of model (10) are similar for the integer-order and the Caputo fractional order, and a small change in the order of derivation has no effect on the overall trend of the results in terms of ascending and descending, but the resulting numerical values are different.

5.3. The Reproduction Number Sensitivity Analysis. We investigate the effect of parameters in Crimean-Congo hemorrhagic fever fractional model (10) on reproduction number using the method introduced by [40]. For this simulation, we use the parameters in the first case of the previous subsection. Since R_0 is defined as $R_0 = \max(R_h, R_{kl})$, therefore, we analyze the sensitivity of R_0 in two cases.

First, if $R_0 = R_h$, by the mentioned method, we have $S_{\beta_6} = (\partial R_0/\partial \beta_6)(\beta_6/R_0) = 1 > 0$, $S_{\Lambda_h} = (\partial R_0/\partial \Lambda_h)(\Lambda_h/R_0) = 1 > 0$, $S_{d_h} = (\partial R_0/\partial d_h)(d_h/R_0) = -2\beta_6\Lambda_h/d_h^3 = -1.55 < 0$. Figure 4 shows the sensitivity of R_0 with respect to each of the parameters. As you can see, changing each of the parameters of model (10) that is involved in R_h changes the reproduction number. The reproduction number is directly related to parameters β_6, Λ_h and inversely related to parameter d_h. From an epidemiological point of view, whenever the reproduction number decreases, then the spread of the disease is controlled. Given that β_6 has the most positive effect on the R_0, so to control the spread of the disease, β_6 should be reduced through the reduced communication of infected and susceptible humans.

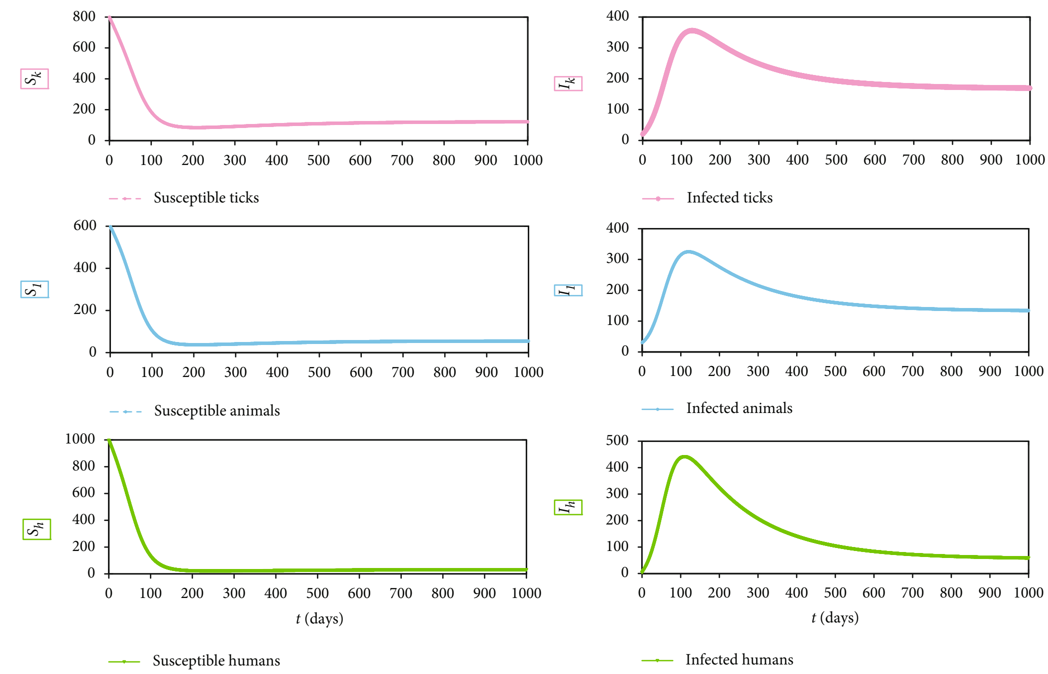

Figure 2: Plots of the results of model (10) with $R_0 > 1$ for $\vartheta = 0.98$.

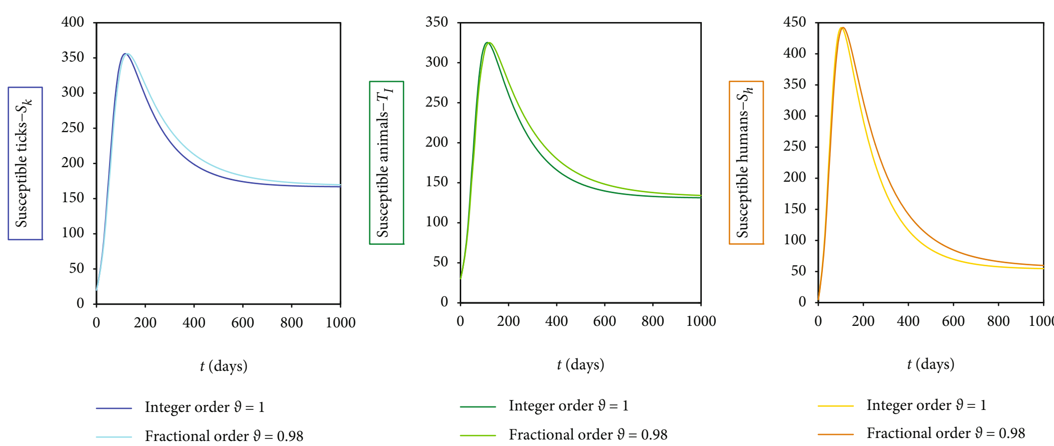

Figure 3: Plots of the results of model (10) for infected groups with integer-order $\vartheta = 1$ and fractional-order $\vartheta = 0.98$ in the case $R_0 > 1$.

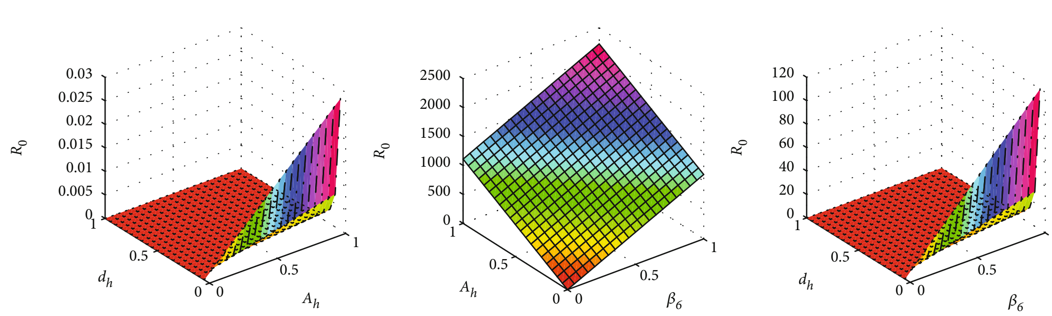

Figure 4: The graphs show the effect of model parameters on the reproduction number for the case $R_0 = R_h$.

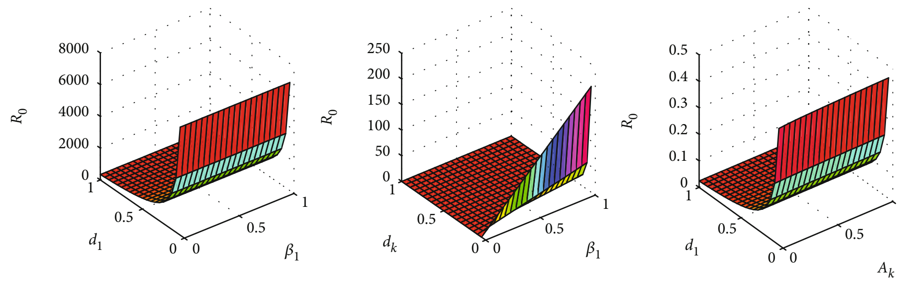

Figure 5: The graphs show the effect of model parameters on the reproduction number for the case $R_0 = R_{kl}$.

In the latter case if $R_0 = R_{kl}$, we obtain the same equations as above

$$S_{\Lambda_k} = \frac{\partial R_0}{\partial \Lambda_k}\frac{\Lambda_k}{R_0} = 0.724 > 0, S_{\beta_1} = \frac{\partial R_0}{\partial \beta_1}\frac{\beta_1}{R_0} = 0.4486 > 0,$$
$$S_{d_l} = \frac{\partial R_0}{\partial d_l}\frac{d_l}{R_0} = -0.5513 < 0,$$

$$S_{\beta_2} = \frac{\partial R_0}{\partial \beta_2}\frac{\beta_2}{R_0} = 0.275 > 0, S_{\beta_3} = \frac{\partial R_0}{\partial \beta_3}\frac{\beta_3}{R_0} = 0.276 > 0,$$
$$S_{d_k} = \frac{\partial R_0}{\partial d_k}\frac{d_k}{R_0} = -1.448 < 0. \tag{55}$$

Figure 5 shows the sensitivity of R_0 with respect to each of the parameters. The reproduction number is directly related to parameters $\beta_1, \beta_2, \beta_3$, and Λ_k and inversely related to parameter d_l, d_k. Among the mentioned parameters, parameters β_1, β_2, and β_3 can be controlled, and all of which have a positive effect on causality, so to reduce the amount of reproduction number, it is enough to reduce the rate of disease transmission between ticks, animals, and humans.

6. Conclusion

In this work, we have presented a box model using the Caputo fractional-order derivative by taking into account the transmission of the Crimean-Congo hemorrhagic fever virus between ticks, animals (domestic and wild), and humans. We have calculated the feasible region and the equilibrium points of the system (10), and we have determined the necessary conditions for the stability of the equilibrium point. In the last section, using the Euler method for the Caputo fractional derivative, we have obtained the approximate solution of system (10), and then, we have provided a numerical simulation for the transmission of Crimean-Congo hemorrhagic fever virus. In two cases: $R_0 < 1$ and $R_0 > 1$, the results of the model have been plotted for the six groups in the model, which clearly show that in the case $R_0 < 1$, the transmission of the disease stops after a while, and the results of the system converge to the disease-free equilibrium point. We have increased the rate of disease transmission among the groups, and in this case, the results for $R_0 > 1$ show that the disease continues endemically, and also, the results converge to the endemic equilibrium point. The results of the model are compared with two types of derivatives of integer-order and fractional-order, and the result of comparison shows that changing the type of derivative with close order has no effect on the overall trend of the results but the obtained numerical values are different.

Later, we have investigated the effect of each of the model parameters on R_0, and the results show that the disease transmission rates among the groups have a positive effect on the value of R_0; therefore, to control the spread of Crimean-Congo hemorrhagic fever, the disease transmission rate should be reduced by reducing contact between different groups.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Iterative Approximation of Fixed Points by Using F Iteration Process in Banach Spaces

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We connect the F iteration process with the class of generalized α-nonexpansive mappings. Under some appropriate assumption, we establish some weak and strong convergence theorems in Banach spaces. To show the numerical efficiency of our established results, we provide a new example of generalized α-nonexpansive mappings and show that its F iteration process is more efficient than many other iterative schemes. Our results are new and extend the corresponding known results of the current literature.

1. Introduction and Preliminaries

Once an existence of a solution for an operator equation is established then in many cases, such solution cannot be obtained by using ordinary analytical methods. To overcome such cases, one needs the approximate value of this solution. To do this, we first rearrange the operator equation in the form of fixed-point equation. We apply the most suitable iterative algorithm on the fixed point equation, and the limit of the sequence generated by this most suitable algorithm is in fact the value of the desired fixed point for the fixed point equation and the solution for the operator equation. The Banach Fixed Point Theorem [1] (BFPT, for short) suggests the elementry Picard iteration $w_{t+1} = \mathcal{G} w_t$ in the case of contraction mappings. Since for the class of nonexpansive mappings, Picard iterates do not always converge to a fixed point of a certain nonexpansive mapping, we, therefore use some other iterative processes involving different steps and set of parameters. Among the other things, Mann [2], Ishikawa [3], Noor [4], S iteration of Agarwal et al. [5], SP iteration of Phuengrattana and Suantai [6], S^* iteration of Karahan and Ozdemir [7], Normal-S [8], Picard-Mann hybrid [9], Krasnoselskii-Mann [10], Abbas [11], Thakur [12], and Picard-S [13] are the most studied iterative processes. In 2018, Ullah and Arshad introduced M [14] iteration process for Suzuki mappings and proved that it converges faster than all of these iteration processes.

Very recently, Ali and Ali [15] introduced the novel iteration process, namely, F iterative scheme for generalized contractions as follows:

$$\begin{cases} w_1 \in \mathcal{P}, \\ u_t = G((1-\alpha_t)w_t + \alpha_t G w_t), \\ v_t = G u_t, \\ w_{t+1} = G v_t, t \geq 1, \end{cases} \tag{1}$$

where $\alpha_t \in (0,1)$.

They showed that the F iteration (1) is stable and has a better rate of convergence when compared with the other iterations in the setting of generalized contractions.

Definition 1. Let $\mathcal{G} : \mathcal{P} \to \mathcal{P}$. Then $\mathcal{G}$ is said to be

(i) nonexpansive provided that $\|\mathcal{G}p' - \mathcal{G}p''\| \le \|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$

(ii) endowed with condition (C) provided that $1/2\|p' - \mathcal{G}p'\| \le \|p' - p''\|$ implies $\|\mathcal{G}p' - \mathcal{G}p''\| \le \|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$

(iii) generalized α-nonexpansive provided that $1/2\|p' - \mathcal{G}p'\| \le \|p' - p''\|$ implies $\|\mathcal{G}p' - \mathcal{G}p''\| \le \alpha\|p' - \mathcal{G}p''\| + \alpha\|p'' - \mathcal{G}p'\| + (1-2\alpha)\|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$ and $\alpha \in [0, 1)$

(iv) endowed with condition I [16] if one has a nondecreasing function f such that $f(0) = 0$ and $f(a) > 0$ at $a > 0$ and $\|p' - \mathcal{G}p'\| \ge f(d(p', F_{\mathcal{G}})$ for all $p' \in \mathcal{P}$

In 1965, Browder [17] and Gohde [18] are in a uniformly convex Banach space (UCBS), while Kirk [19] in a reflexive Banach space (RBS) established an existence of fixed point for nonexpansive maps. In 2008, Suzuki [20] showed that the class of maps endowed with condition (C) is weaker than the notion of nonexpansive maps and proved some related fixed point theorems in Banach spaces. In 2017, Pant and Shukla [21] proved that the notion of generalized α-nonexpansive maps is weaker than the notion of maps endowed with condition (C). They proved some convergence theorems using Agarwal iteration [5] for these maps. Very recently, Ullah et al. [22] used M iteration for finding fixed points of generalized α-nonexpansive maps in Banach spaces. In this paper, we show under some conditions that F iteration converges better to a fixed point of generalized α-nonexpansive map as compared to the leading M iteration and hence many other iterative schemes.

Definition 2. Select a Banach space $\mathcal{J}$ such that $\mathcal{P} \subseteq \mathcal{J}$ is nonempty and $\{w_t\} \subseteq \mathcal{J}$ is bounded. We set for fix $j \in \mathcal{J}$ the following.

(a_1) asymptotic radius of the bounded sequence $\{w_t\}$ at the point j by $r(j, \{w_t\}) := \limsup_{t\to\infty}\|j - w_t\|$;

(a_2) asymptotic radius of the bounded sequence $\{w_t\}$ with the connection of $\mathcal{P}$ by $r(\mathcal{P}, \{w_t\}) = \inf\{r(j, \{w_t\}) : j \in \mathcal{P}\}$;

(a_3) asymptotic center of the bounded sequence $\{w_t\}$ with the connection of $\mathcal{P}$ by $A(\mathcal{P}, \{w_t\}) = \{j \in \mathcal{P} : r(j, \{w_t\}) = r(\mathcal{P}, \{w_t\})\}$.

It is worth mentioning that $A(\mathcal{P}, \{w_t\})$ has a cardinality equal to one in the case of UCBS and nonempty convex in the case of weak compactness and convexity of $\mathcal{P}$ (see [23, 24]).

Definition 3 (see [25]). A Banach space $\mathcal{J}$ is called with Opial's condition in the case when every sequence $\{w_t\} \subseteq \mathcal{J}$ which is weakly convergent to $j \in \mathcal{J}$, then one has the following

$$\limsup_{t\to\infty}\|w_t - j\| < \limsup_{t\to\infty}\|w_t - j'\| \text{for each} j' \in \mathcal{P} - \{j\}. \quad (2)$$

Pant and Shukla [21] observed the following facts about generalized α-nonexpansive operators.

Proposition 4. *If $\mathcal{J}$ is a Banach space such that $\mathcal{P} \subseteq \mathcal{J}$ is closed and nonempty, then for $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ and $\alpha \in [0, 1)$, the following hold*

(i) If $\mathcal{G}$ is endowed with condition (C), then $\mathcal{G}$ is generalized α-nonexpansive

(ii) If $\mathcal{G}$ is generalized α-nonexpansive endowed with a nonempty fixed point, then $\|\mathcal{G}p' - p^*\| \le \|p' - p^*\|$ for $p' \in \mathcal{P}$ and p^* is a fixed point of $\mathcal{G}$

(iii) If $\mathcal{G}$ is generalized α-nonexpansive, then $F_{\mathcal{G}}$ is closed. Furthermore, when the underlying space $\mathcal{J}$ is strictly convex and the set $\mathcal{P}$ is convex, then the set $F_{\mathcal{G}}$ is also convex

(iv) If $\mathcal{G}$ is generalized α-nonexpansive, then for every choice of $p', p'' \in \mathcal{P}$

$$\left\|p' - \mathcal{G}p''\right\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\left\|p' - \mathcal{G}p'\right\| + \left\|p' - p''\right\|. \quad (3)$$

(v) If the underlying space $\mathcal{J}$ is with Opial condition, the operator $\mathcal{G}$ is generalized α-nonexpansive, $\{w_t\}$ is weakly convergent to l and $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$, then $l \in F_{\mathcal{G}}$

We now state an interesting property of a UCBS from [26].

Lemma 5. *Suppose $\mathcal{J}$ is any UCBS. Choose $0 < r \le \alpha_t \le s < 1$ and $\{w_t\}, \{x_t\} \subseteq \mathcal{J}$ such that $\limsup_{t\to\infty}\|w_t\| \le q$, $\limsup_{t\to\infty}\|x_t\| \le q$, and $\lim_{t\to\infty}\|\alpha_t w_t + (1-\alpha_t)x_t\| = q$ for some $q \ge 0$. Then, consequently, $\lim_{t\to\infty}\|w_t - x_t\| = 0$.*

2. Main Results

We first provide a very basic lemma.

Lemma 6. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator satisfying with $F_{\mathcal{G}} \ne \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1), then, consequently, one has $\lim_{t\to\infty}\|w_t - p^*\|$ always exists for every taken $p^* \in F_{\mathcal{G}}$.*

Proof. We may take any $p^* \in F_{\mathcal{G}}$. Using Proposition 4(ii), we see that

$$\begin{aligned}\|u_t - p^*\| &= \|\mathcal{G}((1-\alpha_t)w_t + \alpha_t\mathcal{G}w_t) - p^*\| \le \|(1-\alpha_t)w_t \\ &\quad + \alpha_t\mathcal{G}w_t - p^*\| \le (1-\alpha_t)\|w_t - p^*\| + \alpha_t\|\mathcal{G}w_t \\ &\quad - p^*\| \le (1-\alpha_t)\|w_t - p^*\| + \alpha_t\|w_t - p^*\| \le \|w_t - p^*\|.\end{aligned} \quad (4)$$

This implies that

$$\|w_{t+1}-p^*\| = \|\mathcal{G}v_t - p^*\| \le \|v_t - p^*\| = \|\mathcal{G}u_t - p^*\| \le \|u_t - p^*\| \le \|w_t - p^*\|. \tag{5}$$

Consequently, $\|w_{t+1}-p^*\| \le \|w_t - p^*\|$, that is, $\{\|w_t - p^*\|\}$ is bounded as well as nonincreasing. This follows that $\lim_{t\to\infty}\|w_t - p^*\|$ exists for each $p^* \in F_{\mathcal{G}}$.

We now provide the necessary and sufficient requirements for the existence of fixed points for any given generalized nonexpansive mappings in a Banach space.

Theorem 7. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G}: \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator and $\{w_t\}$ is a sequence of F iterates (1). Then, $F_{\mathcal{G}} \neq \emptyset$ if and only if $\{w_t\}$ is bounded and $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$.*

Proof. Suppose that $F_{\mathcal{G}} \neq \emptyset$ and $p^* \in F_{\mathcal{G}}$. Take any $p^* \in F_{\mathcal{G}}$, and so applying Lemma 6, we have $\lim_{t\to\infty}\|w_t - p^*\|$ exists and $\{w_t\}$ is bounded. Suppose that this limit is equal to some ε, that is,

$$\lim_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{6}$$

As we have established in the proof of Lemma 6 that

$$\|u_t - p^*\| \le \|w_t - p^*\|. \tag{7}$$

This together with (6) gives that

$$\limsup_{t\to\infty}\|u_t - p^*\| \le \limsup_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{8}$$

Since p^* is in the set $F_{\mathcal{G}}$, so we may apply Proposition 4(ii) to obtain the following

$$\|\mathcal{G}w_t - p^*\| \le \|w_t - p^*\|, \Rightarrow \limsup_{t\to\infty}\|\mathcal{G}w_t - p^*\| \le \limsup_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{9}$$

Now, if we look in the proof of Lemma 6, we can see the following

$$\|w_{t+1} - p^*\| \le \|u_t - p^*\| \Rightarrow \varepsilon = \liminf_{t\to\infty}\|w_{t+1} - p^*\| \le \liminf_{t\to\infty}\|u_t - p^*\|. \tag{10}$$

From (8) and (10), we have

$$\varepsilon = \lim_{t\to\infty}\|u_t - p^*\|. \tag{11}$$

By (11) and (1), one has

$$\begin{aligned}\varepsilon &= \lim_{t\to\infty}\|u_t - p^*\| = \lim_{t\to\infty}\|\mathcal{G}((1-\alpha_t)w_t + \alpha_t\mathcal{G}w_t) - p^*\| \le \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*) + \alpha_t(\mathcal{G}w_t - p^*)\| \\ &\le \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*)\| + \lim_{t\to\infty}\|\alpha_t(\mathcal{G}w_t - p^*)\| \\ &\le \lim_{t\to\infty}(1-\alpha_t)\|w_t - p^*\| + \lim_{t\to\infty}\alpha_t\|w_t - p^*\| = \lim_{t\to\infty}\|w_t - p^*\| \le \varepsilon.\end{aligned} \tag{12}$$

If and only if

$$\varepsilon = \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*) + \alpha_t(\mathcal{G}w_t - p^*)\|. \tag{13}$$

One can now apply the Lemma 5, to obtain

$$\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0. \tag{14}$$

Conversely, we want to show that the set $F_{\mathcal{G}}$ is nonempty under the assumptions that $\{w_t\}$ is bounded such that $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$. We may choose a point $p^* \in A(\mathcal{P}, \{w_t\})$. If we apply Proposition 4(iv), then one can observe the following

$$\begin{aligned}r(\mathcal{G}p^*, \{w_t\}) &= \limsup_{t\to\infty}\|w_t - \mathcal{G}p^*\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{t\to\infty}\|\mathcal{G}w_t - w_t\| \\ &\quad + \limsup_{t\to\infty}\|w_t - p^*\| = \limsup_{t\to\infty}\|w_t - p^*\| = r(p^*, \{w_t\}).\end{aligned} \tag{15}$$

We observed that $\mathcal{G}p^* \in A(\mathcal{P}, \{w_t\}$. By using the facts that this set has only element in the case of UCBS $\mathcal{J}$, one concludes $\mathcal{G}p^* = p^*$, accordingly the set $F_{\mathcal{G}}$ is nonempty.

The weak convergence of F iteration is established as follows.

Theorem 8. *Suppose $\mathcal{J}$ is any UCBS with Opial condition and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G}: \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges weakly to a fixed point of $\mathcal{G}$.*

Proof. By Theorem 7, the given sequence $\{w_t\}$ is bounded. Since $\mathcal{J}$ is UCBS, $\mathcal{J}$ is RBS. Therefore, some one construct a weakly convergent sequence of $\{w_t\}$. We may assume that $\{w_{t_r}\}$ be this subsequence having weak limit $x_1 \in \mathcal{P}$. If we apply Theorem 7 on this subsequence, we obtain $\lim_{r\to\infty}\|w_{t_r} - \mathcal{G}w_{t_r}\| = 0$. Thus, by Proposition 4(v), one has $x_1 \in F_{\mathcal{G}}$. It is sufficient to show that $\{w_t\}$ converges weakly to x_1. In fact, if $\{w_t\}$ does not converge weakly to x_1. Then, there exists a subsequence $\{w_{t_s}\}$ of $\{w_t\}$ and $x_2 \in \mathcal{P}$ such that $\{w_{t_s}\}$ converges weakly to x_2 and $x_2 \neq x_1$. Again by Proposition 4(v), $x_2 \in F_{\mathcal{G}}$. By Lemma 6 together with Opial property, we have

$$\lim_{t\to\infty}\|x_n - l_1\| = \lim_{r\to\infty}\left\|w_{t_r} - x_1\right\| < \lim_{r\to\infty}\left\|w_{t_r} - x_2\right\| = \lim_{t\to\infty}\|w_t - x_2\| = \lim_{s\to\infty}\left\|w_{t_s} - x_2\right\| < \lim_{s\to\infty}\left\|w_{t_s} - x_1 \mid \right\| = \lim_{t\to\infty}\|w_t - x_1\|. \quad (16)$$

This is a contradiction. So, we have $x_1 = x_2$. Thus, $\{w_t\}$ converges weakly to $x_1 \in F_{\mathcal{G}}$.

Now we provide some strong convergence results.

Theorem 9. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and compact. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. Since the domain $\mathcal{P}$ is a compact subset of $\mathcal{J}$ and $\{w_t\} \subseteq \mathcal{P}$. It follows that a subsequence $\{w_{t_r}\}$ of $\{w_t\}$ exists such that $\lim_{r\to\infty}\|w_t - p^{**}\| = 0$ for some $p^{**} \in \mathcal{P}$. In the view of Theorem 7, $\lim_{r\to\infty}\|\mathcal{P}w_{t_r} - w_{t_r}\| = 0$. Applying Proposition 4(iv), one has

$$\left\|w_{t_r} - \mathcal{G}p^{**}\right\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\left\|w_{t_r} - \mathcal{G}w_{t_r}\right\| + \left\|w_{t_r} - p^{**}\right\|. \quad (17)$$

Hence, if we let $r \to \infty$, then $\mathcal{G}p^{**} = p^{**}$. The fact that p^{**} is the strong limit of $\{w_t\}$ now follows from the existence of $\lim_{t\to\infty}\|w_t - p^{**}\|$.

Theorem 10. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1) and* $\liminf_{t\to\infty} d(w_t, F_{\mathcal{G}}) = 0$. *Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. By using Lemma 6, one has $\lim_{t\to\infty}\|w_t - p^*\|$ exists, for every fixed point of $\mathcal{G}$. It follows that $\lim_{t\to\infty} d(w_t, F_{\mathcal{G}})$ exists. Accordingly

$$\lim_{t\to\infty} d(w_n, F_{\mathcal{G}}) = 0. \quad (18)$$

The above limit provides us two subsequence $\{w_{t_r}\}$ and $\{p_r\}$ of $\{w_t\}$ and $F_{\mathcal{G}}$, respectively, in the following way

$$\left\|w_{t_r} - p_r\right\| \leq \frac{1}{2^r} \quad \text{for each} \quad r \geq 1. \quad (19)$$

By looking into the proof of Lemma 6, we see that $\{w_t\}$ is nonicreasing, therefore

$$\left\|w_{t_{r+1}} - p_r\right\| \leq \left\|w_{t_r} - p_r\right\| \leq \frac{1}{2^r}. \quad (20)$$

It follows that

$$\|p_{r+1} - p_r\| \leq \left\|p_{r+1} - w_{t_{r+1}}\right\| + \left\|w_{t_{r+1}} - p_r\right\| \leq \frac{1}{2^{r+1}} + \frac{1}{2^r} \leq \frac{1}{2^{r-1}} \to 0, asr \to \infty. \quad (21)$$

Consequently, we obtained that $\lim_{r\to\infty}\|p_{r+1} - p_r\| = 0$ which show that $\{p_r\}$ is Cauchy sequence in $F_{\mathcal{G}}$ and so it converges to an element p^{**}. Applying Proposition 4(iii), $F_{\mathcal{G}}$ is closed and so $p^{**} \in F_{\mathcal{G}}$. By Lemma 6, $\lim_{t\to\infty}\|w_t - p^{**}\|$ exists and hence p^{**} is the strong limit of $\{w_t\}$.

Theorem 11. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator satisfying condition I with $F_{\mathcal{G}} \neq \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. Keeping Theorem 7 in mind, one can write

$$\liminf_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0. \quad (22)$$

From the definition of condition (I), we see that

$$\|w_t - \mathcal{G}w_t\| \geq f(d(w_t, F_{\mathcal{G}})). \quad (23)$$

Applying (22) on (23), we have

$$\liminf_{t\to\infty} f(d(w_t, F_{\mathcal{G}})) = 0. \quad (24)$$

It follows that

$$\liminf_{t\to\infty} d(w_t, F_{\mathcal{G}}) = 0. \quad (25)$$

Now applying Theorem 10, $\{w_t\}$ is strongly convergent to a fixed point of $\mathcal{G}$.

3. Example

To support the main results, we provide an example of generalized α-nonexpansive mappings, which is not endowed with condition (C). Using this example, we compare F with other iterations in the setting of generalized α-nonexpansive mappings.

Example 12. We take a set $\mathcal{P} = [7, 13]$ and set a self map on $\mathcal{G}$ by the following rule:

$$\mathcal{G}p' = \begin{cases} \dfrac{p'+7}{2} & \text{if} \quad p' < 13, \\ 7 \quad \text{if} \quad z = 13. \end{cases} \quad (26)$$

We show that $\mathcal{G}$ is generalized α-nonexpansive having $\alpha = 1/2$, but not Suzuki mapping. This example thus exceeds the class of Suzuki mappings.

Case I. When $p' = 13 = p''$, we have

$$\frac{1}{2}|p' - \mathcal{G}p''| + \frac{1}{2}|p'' - \mathcal{G}p'| + \left(1 - 2\left(\frac{1}{2}\right)\right)|p' - p''| \geq 0 = |\mathcal{G}p' - \mathcal{G}p''|. \quad (27)$$

Table 1: Numerical data generated by F, M, Picard-S, S, Ishikawa, and Mann iterative approximation schemes for the self map given in Example 12.

	F	M	Picard-S	S	Ishikawa	Mann
1	7.9	7.9	7.9	7.9	7.9	7.9
2	7.06468750	7.12937500	7.16284375	7.3256875	7.3931875	7.51750000
3	7.00464941	7.01859766	7.02946454	7.11785816	7.17177379	7.29756250
4	7.00033418	7.00267341	7.00533124	7.04264992	7.07504367	7.17109844
5	7.00002402	7.00038430	7.00096462	7.01543394	7.03278471	7.09838160
6	7.00000173	7.00005524	7.00017454	7.00558516	7.01432282	7.05656942
7	7.00000012	7.00000794	7.00003158	7.00202113	7.00625728	7.03252742
8	7.00000001	7.00000114	7.00000571	7.0007314	7.00273365	7.01870326
9	7	7.00000016	7.00000103	7.00026467	7.00119426	7.01075438
10	7	7.00000002	7.00000019	7.00009578	7.00052174	7.00618377
11	7	7	7.00000003	7.00003466	7.00022794	7.00355567
12	7	7	7.00000001	7.00001254	7.00009959	7.00204451
13	7	7	7	7.00000454	7.00004350	7.00117559
14	7	7	7	7.00000164	7.00001901	7.00067597
15	7	7	7	7.00000059	7.00000830	7.00038868
16	7	7	7	7.00000022	7.00000363	7.00022349
17	7	7	7	7.00000008	7.00000158	7.00012851
18	7	7	7	7.00000003	7.00000069	7.00007389
19	7	7	7	7.00000001	7.00000030	7.00004249
20	7	7	7	7	7.00000013	7.00002443
21	7	7	7	7	7.00000006	7.00001405
22	7	7	7	7	7.00000003	7.00000808
23	7	7	7	7	7.00000001	7.00000464
24	7	7	7	7	7	7.00000260

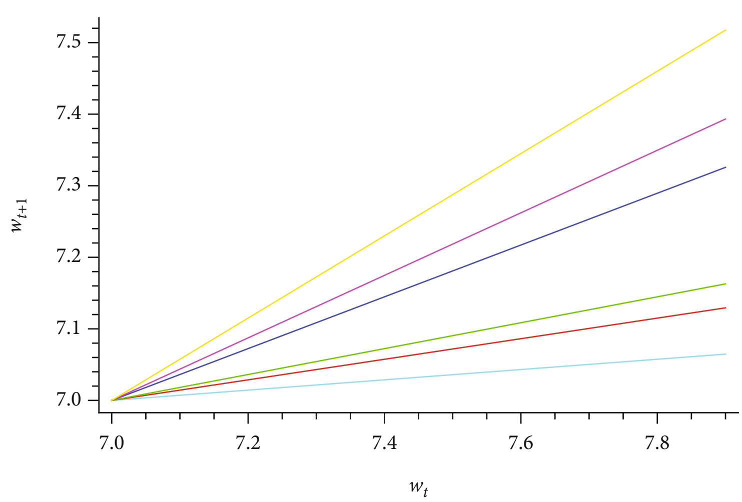

Figure 1: Convergence analysis view of F (cyan), M (red), Picard-S (green), S (blue), Ishikawa (magenta), and Mann (yellow) iteration process for the mapping given in Example 12.

Case II. Choose $p', p'' < 13$, we have

$$\begin{aligned}
&\frac{1}{2}\|p' - \mathcal{G}p''\| + \frac{1}{2}\|p'' - \mathcal{G}p'\| \\
&\quad + \left(1 - 2\left(\frac{1}{2}\right)\right)\|p' - p''\| \\
&= \frac{1}{2}\left\|p'' - \left(\frac{p'+7}{2}\right)\right\| + \frac{1}{2}\left\|p' - \left(\frac{p''+7}{2}\right)\right\| \\
&\geq \frac{1}{2}\left\|\left(p'' - \left(\frac{p'+7}{2}\right)\right) - \left(p' - \left(\frac{p''+7}{2}\right)\right)\right\| \\
&= \frac{1}{2}\left\|\frac{2p'' - p' - 7 - 2p' + p'' + 7}{2}\right\| \\
&= \frac{1}{2}\left\|\frac{3p'' - 3p'}{2}\right\| = \frac{3}{4}\|p' - p''\| \geq \frac{1}{2}\|p' - p''\| \\
&= \|\mathcal{G}p' - \mathcal{G}p''\|.
\end{aligned} \tag{28}$$

Case III. When $p' = 13$ and $p'' < 13$, we have

$$\begin{aligned}
&\frac{1}{2}\|p' - \mathcal{G}p''\| + \frac{1}{2}\|p'' - \mathcal{G}p'\| \\
&\quad + \left(1 - 2\left(\frac{1}{2}\right)\right)\|p' - p''\| \\
&= \frac{1}{2}\|p' - 7\| + \frac{1}{2}\left\|p'' - \left(\frac{p'+7}{2}\right)\right\| \geq \frac{1}{2}\|p' - 7\| \\
&= \left\|\frac{p'-7}{2}\right\| = \|\mathcal{G}p' - \mathcal{G}p''\|.
\end{aligned} \tag{29}$$

Consequently, $\|\mathcal{G}p' - \mathcal{G}p''\| \leq 1/2 \mid p' - \mathcal{G}p'' \mid +1/2 \mid p'' - \mathcal{G}p' \mid +(1 - 2(1/2))|p' - p''|$ for every two points $p', p'' \in \mathcal{G}$. Now if one chooses $p' = 11.8$ and $p'' = 13$, we must have $|p' - p''| = 1.2, |\mathcal{G}p' - \mathcal{G}p''| = 2.4$ and $1/2 \mid p' - \mathcal{G}p' \mid = 1.2$. It has been observed, $1/2 \mid p' - \mathcal{G}p' \mid \leq \mid p' - p'' \mid$ and $|\mathcal{G}p' - \mathcal{G}p'' \mid > \mid p' - p'' \mid$. Thus, $\mathcal{G}$ exceeded the class of Suzuki mappings.

We now compare the effectiveness of the iterative scheme F [15] with the leading M [14] and Picard [13] and the elementry S [5], Ishikawa [3] and Mann [2] approximation scheme. We may take $\alpha_t = 0.85$ and $\beta_t = 0.65$. For the strating $w_1 = 7.9$, we can see some values in Table 1. Furthermore, Figure 1 provides information about the behavior of the leading schemes. Clearly, F iterative scheme is more effective than the other schemes in the general context of generalized α-nonexpansive maps.

Remark 13. In the view of the above discussion, we noted that the main theorems and outcome of this paper improved and extended the main results of Ullah and Arshad [14] from Suzuki mappings to generalized α-nonexpansive mappings and from the setting of M iteration to the more general setting of F iteration process. Moreover, the main results of this paper improved the results of Ali and Ali [15] from the setting of contractions to the general context of generalized α-nonexpansive mappings. We have also improved the results of Ullah et al. [22] in the sense of better rate of convergence.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Solving Integral Equations by Common Fixed Point Theorems on Complex Partial *b*-Metric Spaces

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In this paper, we prove some common fixed point theorems for rational contraction mapping on complex partial *b*-metric space. The presented results generalize and expand some of the literature's well-known results. We also explore some of the applications of our key results.

1. Introduction

Introduced in 1989 by Bakhtin [1] and Czerwick [2], the concept of *b*-metric spaces provided a framework to extend the results already known in the classical setting of metric spaces. The concept of complex valued metric spaces was introduced in 2011 by Azam et al. [3] and given some common fixed point theorems under the condition of contraction. Rao et al. [4] introduced the definition of complex valued *b*-metric spaces in 2013 and provided a scheme to expand the results, as well as proved a common fixed point theorem under contraction. In 2017, Dhivya and Marudai [5] introduced the concept of complex partial metric space and suggested a plan to expand the results, as well as proved common fixed point theorems under the rational expression contraction condition. Gunaseelan [6, 7] presented the concept of complex partial *b*-metric space in 2019, as well as proved the fixed point theorem under the contractive condition. Many researchers have studied some intriguing concepts and applications and have shown significant results [7–23]. In this paper, we prove some common fixed point theorems for rational contraction mapping on complex partial *b*-metric space.

2. Preliminaries

Let $\mathfrak{C}$ be the set of complex numbers and $1, 2, 3 \in \mathfrak{C}$. Define a partial order $\leq$ on $\mathfrak{C}$ as follows: $1 \leq 2$ if and only if $\mathscr{R}(1) \leq \mathscr{R}(2)$ and $\mathscr{I}(1) \leq \mathscr{I}(2)$.

Consequently, one can infer that $1 \leq 2$ if one of the following conditions is satisfied:

(i) $\mathscr{R}(1) = \mathscr{R}(2)$, $\mathscr{I}(1) < \mathscr{I}(2)$

(ii) $\mathscr{R}(1) < \mathscr{R}(2)$, $\mathscr{I}(1) = \mathscr{I}(2)$

(iii) $\mathscr{R}(1) < \mathscr{R}(2)$, $\mathscr{I}(1) < \mathscr{I}(2)$

(iv) $\mathscr{R}(1) = \mathscr{R}(2)$, $\mathscr{I}(1) = \mathscr{I}(2)$

In particular, we write $1 \lesssim 2$ if $1 \neq 2$ and one of (i), (ii), and (iii) is satisfied and we write $1 \prec 2$ if only (iii) is satisfied. Notice that

(a) If $0 \leq 1 \lesssim 2$, then $|1| < |2|$

(b) If $1 \leq 2$ and $2 \prec 3$, then $1 \prec 3$

(c) If $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma$, then $\eta_1 \leq \gamma_1$ for all $0 \leq 1 \in \mathfrak{C}$

Here, $\mathfrak{C}_+ (= \{(\aleph, \mathfrak{y}) \mid \aleph, \mathfrak{y} \geq 0, \aleph, \mathfrak{y} \in \mathbb{R}_+\})$ denote the set of nonnegative complex numbers.

Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 1 [6]. A complex partial b-metric on a nonvoid set $\mathfrak{W}$ is a function $\varphi_{cb} : \mathfrak{W} \times \mathfrak{W} \longrightarrow \mathbb{C}_+$ such that for all $\aleph, \vartheta, \mathfrak{Z} \in \mathfrak{W}$,

(i) $0 \leq \varphi_{cb}(\aleph, \aleph) \leq \varphi_{cb}(\aleph, \vartheta)$(*small self-distances*)

(ii) $\varphi_{cb}(\aleph, \vartheta) = \varphi_{cb}(\vartheta, \aleph)$(*symmetry*)

(iii) $\varphi_{cb}(\aleph, \aleph) = \varphi_{cb}(\aleph, \vartheta) = \varphi_{cb}(\vartheta, \vartheta) \Leftrightarrow \aleph = \vartheta$(*equality*)

(iv) $\exists$ a real number $s \geq 1$ and s is an independent of $\aleph, \vartheta, \mathfrak{Z}$ such that $\varphi_{cb}(\aleph, \vartheta) \leq s[\varphi_{cb}(\aleph, \mathfrak{Z}) + \varphi_{cb}(\mathfrak{Z}, \vartheta)] - \varphi_{cb}(\mathfrak{Z}, \mathfrak{Z})$(*triangularity*)

A complex partial b-metric space is a pair $(\mathfrak{W}, \varphi_{cb})$ such that $\mathfrak{W}$ is a nonvoid set and φ_{cb} is the complex partial b-metric on $\mathfrak{W}$. The number s is called the coefficient of $(\mathfrak{W}, \varphi_{cb})$.

Definition 2 [6]. Let $(\mathfrak{W}, \varphi_{cb})$ be a complex partial b-metric space with coefficient s. Let $\{\aleph_n\}$ be any sequence in $\mathfrak{W}$ and $\aleph \in \mathfrak{W}$. Then,

(1) The sequence $\{\aleph_n\}$ is said to converge to $\aleph$, if $\lim_{n\longrightarrow\infty}\varphi_{cb}(\aleph_n, \aleph) = \varphi_{cb}(\aleph, \aleph)$

(2) The sequence $\{\aleph_n\}$ is said to be Cauchy sequence in $(\mathfrak{W}, \varphi_{cb})$ if $\lim_{n,m\longrightarrow\infty}\varphi_{cb}(\aleph_n, \aleph_m)$ exists and is finite

(3) $(\mathfrak{W}, \varphi_{cb})$ is said to be a complete complex partial b-metric space if for every Cauchy sequence $\{\aleph_n\}$ in $\mathfrak{W}$, there exists $\aleph \in \mathfrak{W}$ such that $\lim_{n,m\longrightarrow\infty}\varphi_{cb}(\aleph_n, \aleph_m) = \lim_{n\longrightarrow\infty}\varphi_{cb}(\aleph_n, \aleph) = \varphi_{cb}(\aleph, \aleph)$.

Definition 3. Let $\mathfrak{S}$ and $\mathfrak{T}$ be self-mappings of nonempty set $\mathfrak{W}$. A point $\aleph \in \mathfrak{W}$ is called a common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$ if $\aleph = \mathfrak{S}\aleph = \mathfrak{T}\aleph$.

In 2019, Gunaseelan [6] proved some fixed point theorems on complex partial b-metric space as follows.

Theorem 4. *Let $(\mathfrak{W}, \varphi_{cb})$ be any complete complex partial b -metric space with coefficient $s \geq 1$ and $\mathfrak{S} : \mathfrak{W} \longrightarrow \mathfrak{W}$ be a mapping satisfying*

$$\varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{S}\vartheta) \leq \lambda \max \{\varphi_{cb}(\aleph, \vartheta), \varphi_{cb}(\aleph, \mathfrak{S}\aleph), \varphi_{cb}(\vartheta, \mathfrak{S}\vartheta)\}, \tag{1}$$

for all $\aleph, \vartheta \in \mathfrak{W}$, where $\lambda \in [0, 1/s]$. Then, $\mathfrak{S}$ has a unique fixed point $\aleph^* \in \mathfrak{W}$ and $\varphi_{cb}(\aleph^*, \aleph^*) = 0$.

Inspired by the study made by Gunaseelan [6], here, we prove some common fixed point theorems for rational contraction mapping on complex partial b-metric space with an application.

3. Main Results

In this section, we will give our main result of this paper, where some common fixed point theorems for rational contraction mapping on complex partial b-metric space are given.

Theorem 5. *Let $(\mathfrak{W}, \varphi_{cb})$ be a complete complex partial b-metric space with the coefficient $s \geq 1$ and $\mathfrak{S}, \mathfrak{T} : \mathfrak{W} \longrightarrow \mathfrak{W}$ be mappings satisfying*

$$\varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) \leq a_1\varphi_{cb}(\aleph, \vartheta) + \frac{a_2\varphi_{cb}(\aleph, \mathfrak{S}\aleph)\varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)}{1 + \varphi_{cb}(\aleph, \mathfrak{T}\vartheta) + \varphi_{cb}(\aleph, \vartheta)}, \tag{2}$$

for all $\aleph, \vartheta \in \mathfrak{W}$, where a_1, a_2 are nonnegative reals with $a_1 + sa_2 < 1$. Then, $\mathfrak{S}$ and $\mathfrak{T}$ have a unique common fixed point in $\mathfrak{W}$.

Proof. Let $\aleph_0$ be arbitrary point in $\mathfrak{W}$, and define a sequence $\{\aleph_\mathfrak{n}\}$ in $\mathfrak{W}$ such that

$$\aleph_{2\mathfrak{n}+1} = \mathfrak{S}\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2} = \mathfrak{T}\aleph_{2\mathfrak{n}+1}, \quad \forall \mathfrak{n} \geq 0. \tag{3}$$

Next, show that the sequence $\{\aleph_\mathfrak{n}\}$ is Cauchy. By using (3), we get

$$\begin{aligned}\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2}) &= \varphi_{cb}(\mathfrak{S}\aleph_{2\mathfrak{n}}, \mathfrak{T}\aleph_{2\mathfrak{n}+1}) \\ &\leq a_1\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1}) \\ &\quad + \frac{a_2\varphi_{cb}(\aleph_{2\mathfrak{n}}, \mathfrak{S}\aleph_{2\mathfrak{n}})\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})}{1 + \varphi_{cb}(\aleph_{2\mathfrak{n}}, \mathfrak{T}\aleph_{2\mathfrak{n}+1}) + \varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})},\end{aligned} \tag{4}$$

so that

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})| &\leq a_1|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})| \\ &\quad + \frac{a_2|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})||\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|}{|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2})| + |\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|}.\end{aligned} \tag{5}$$

By the hypothesis of theorem, we get

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})| &\leq s\{|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}})|+|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+2})|\}\\ &\quad -|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}})||\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})|\\ &\leq s\{|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}})|+|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+2})|\}.\end{aligned} \tag{6}$$

Hence,

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})| &\leq \mathrm{a}_1|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+1})|+s\mathrm{a}_2|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+1})|\\ &=(\mathrm{a}_1+s\mathrm{a}_2)|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+1})||\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})|\\ &\leq(\mathrm{a}_1+s\mathrm{a}_2)|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+1})|.\end{aligned} \tag{7}$$

Similarly,

$$|\varphi_{cb}(\aleph_{2\mathfrak{n}+2},\aleph_{2\mathfrak{n}+3})|\leq(\mathrm{a}_1+s\mathrm{a}_2)|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})|, \tag{8}$$

since $\mathrm{a}_1+s\mathrm{a}_2<1$. Therefore, with $\delta=\mathrm{a}_1+s\mathrm{a}_2<1$ and for all $\mathfrak{n}\geq 0$, consequently, we have

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})| &\leq \delta|\varphi_{cb}(\aleph_{2\mathfrak{n}},\aleph_{2\mathfrak{n}+1})|\leq\delta^2|\varphi_{cb}(\aleph_{2\mathfrak{n}-1},\aleph_{2\mathfrak{n}})|\ \vdots\\ &\leq\delta^{2\mathfrak{n}+1}|\varphi_{cb}(\aleph_0,\aleph_1)|.\end{aligned} \tag{9}$$

That is,

$$\begin{aligned}|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})| &\leq\delta|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\leq\delta^2|\varphi_{cb}(\aleph_{\mathfrak{n}-1},\aleph_{\mathfrak{n}})|\ \vdots\\ &\leq\delta^{\mathfrak{n}+1}|\varphi_{cb}(\aleph_0,\aleph_1)|.\end{aligned} \tag{10}$$

For any $\mathfrak{m}>\mathfrak{n}$, $\mathfrak{m},\mathfrak{n}\in\mathbb{N}$, we have

$$\begin{aligned}|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{m}})| &\leq s\{|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|+|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{m}})|\}\\ &\quad -|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+1})|\leq s\{|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\quad +|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{m}})|\}\leq s|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\quad +s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})|+s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{m}})|\\ &\quad -|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{n}+2})|\leq s|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\quad +s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})|+s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{m}})|\\ &\leq s|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|+s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})|\\ &\quad +s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{n}+3})|+s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+3},\aleph_{\mathfrak{m}})|\\ &\quad -|\varphi_{cb}(\aleph_{\mathfrak{n}+3},\aleph_{\mathfrak{n}+3})|\leq s|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\quad +s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})|+s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{n}+3})|\\ &\quad +s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+3},\aleph_{\mathfrak{m}})|\ \vdots\leq s|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}+1})|\\ &\quad +s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1},\aleph_{\mathfrak{n}+2})|+s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2},\aleph_{\mathfrak{n}+3})|\\ &\quad +\cdots s^{\mathfrak{m}-\mathfrak{n}-2}|\varphi_{cb}(\aleph_{\mathfrak{m}-3},\aleph_{\mathfrak{m}-2})|\\ &\quad +s^{\mathfrak{m}-\mathfrak{n}-1}|\varphi_{cb}(\aleph_{\mathfrak{m}-2},\aleph_{\mathfrak{m}-1})|\\ &\quad +s^{\mathfrak{m}-\mathfrak{n}}|\varphi_{cb}(\aleph_{\mathfrak{m}-1},\aleph_{\mathfrak{m}})|.\end{aligned} \tag{11}$$

From (10), we get

$$\begin{aligned}|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{m}})| &\leq s\delta^{\mathfrak{n}}|\varphi_{cb}(\aleph_0,\aleph_1)|+s^2\delta^{\mathfrak{n}+1}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &\quad +s^3\delta^{\mathfrak{n}+2}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &\quad +\cdots+s^{\mathfrak{m}-\mathfrak{n}-2}\delta^{\mathfrak{m}-3}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &\quad +s^{\mathfrak{m}-\mathfrak{n}-1}\delta^{\mathfrak{m}-2}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &\quad +s^{\mathfrak{m}-\mathfrak{n}}\delta^{\mathfrak{m}-1}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &=\sum_{i=1}^{\mathfrak{m}-\mathfrak{n}}s^i\delta^{i+\mathfrak{n}-1}|\varphi_{cb}(\aleph_0,\aleph_1)|.\end{aligned} \tag{12}$$

Hence,

$$\begin{aligned}|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{m}})| &\leq\sum_{i=1}^{\mathfrak{m}-\mathfrak{n}}s^{i+\mathfrak{n}-1}\delta^{i+\mathfrak{n}-1}|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &=\sum_{t=\mathfrak{n}}^{\mathfrak{m}-1}s^t\delta^t|\varphi_{cb}(\aleph_0,\aleph_1)|\leq\sum_{t=\mathfrak{n}}^{\infty}(s\delta)^t|\varphi_{cb}(\aleph_0,\aleph_1)|\\ &=\frac{(s\delta)^{\mathfrak{n}}}{1-s\delta}|\varphi_{cb}(\aleph_0,\aleph_1)|,\end{aligned} \tag{13}$$

and hence,

$$|\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{m}})|\leq\frac{(s\delta)^{\mathfrak{n}}}{1-s\delta}|\varphi_{cb}(\aleph_0,\aleph_1)|\longrightarrow 0\quad\text{as }\mathfrak{m},\mathfrak{n}\longrightarrow\infty. \tag{14}$$

Thus, $\{\aleph_{\mathfrak{n}}\}$ is a Cauchy sequence in $\mathfrak{W}$. Since $\mathfrak{W}$ is complete, there exists some $\mathfrak{u}\in\mathfrak{W}$ such that $\aleph_{\mathfrak{n}}\longrightarrow\mathfrak{u}$ as $\mathfrak{n}\longrightarrow\infty$ and

$$\varphi_{cb}(\mathfrak{u},\mathfrak{u})=\lim_{n\longrightarrow\infty}\varphi_{cb}(\mathfrak{u},\aleph_{\mathfrak{n}})=\lim_{n\longrightarrow\infty}\varphi_{cb}(\aleph_{\mathfrak{n}},\aleph_{\mathfrak{n}})=0. \tag{15}$$

Assume on the contrary that there exists $\mathfrak{z}\in\mathfrak{W}$ such that

$$|\varphi_{cb}(\mathfrak{u},\mathfrak{S}\mathfrak{u})|=|\mathfrak{z}|>0. \tag{16}$$

By using the triangular inequality and (2), we obtain

$$\begin{aligned}\mathfrak{z} &=\varphi_{cb}(\mathfrak{u},\mathfrak{S}\mathfrak{u})\\ &\leq s\{\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+\varphi_{cb}(\aleph_{2\mathfrak{n}+2},\mathfrak{S}\mathfrak{u})\}-\varphi_{cb}(\aleph_{2\mathfrak{n}+2},\aleph_{2\mathfrak{n}+2})\\ &\leq s\{\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+\varphi_{cb}(\aleph_{2\mathfrak{n}+2},\mathfrak{S}\mathfrak{u})\}\\ &=s\{\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+\varphi_{cb}(\mathfrak{T}\aleph_{2\mathfrak{n}+1},\mathfrak{S}\mathfrak{u})\}\\ &\leq s\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+s\mathrm{a}_1\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+1})\\ &\quad +\frac{s\mathrm{a}_2\varphi_{cb}(\mathfrak{u},\mathfrak{S}\mathfrak{u})\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\mathfrak{T}\aleph_{2\mathfrak{n}+1})}{1+\varphi_{cb}(\mathfrak{u},\mathfrak{T}\aleph_{2\mathfrak{n}+1})+\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+1})}\\ &=s\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+s\mathrm{a}_1\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+1})\\ &\quad +\frac{s\mathrm{a}_2\varphi_{cb}(\mathfrak{u},\mathfrak{S}\mathfrak{u})\varphi_{cb}(\aleph_{2\mathfrak{n}+1},\aleph_{2\mathfrak{n}+2})}{1+\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+2})+\varphi_{cb}(\mathfrak{u},\aleph_{2\mathfrak{n}+1})},\end{aligned} \tag{17}$$

which implies that

$$\begin{aligned}|\mathfrak{z}| &= |\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})|\\ &\le s|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2})| + sa_1|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+1})|\\ &\quad + \frac{sa_2|\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})||\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|}{1+\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2})|+|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+1})|}.\end{aligned} \tag{18}$$

As $\mathfrak{n} \longrightarrow \infty$ in (18), we obtain that $|\mathfrak{z}| = |\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})| \le 0$, a contradiction with (16). Therefore $|\mathfrak{z}| = 0$. Hence, $\mathfrak{S}\mathfrak{u} = \mathfrak{u}$. Similarly, we obtain $\mathfrak{T}\mathfrak{u} = \mathfrak{u}$.

Assume that $\mathfrak{u}^*$ is another common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$. Then,

$$\begin{aligned}\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*) &= \varphi_{cb}(\mathfrak{S}\mathfrak{u}, \mathfrak{T}\mathfrak{u}^*)\\ &\le a_1\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*) + \frac{a_2\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})\varphi_{cb}(\mathfrak{u}^*, \mathfrak{T}\mathfrak{u}^*)}{1+\varphi_{cb}(\mathfrak{u}, \mathfrak{T}\mathfrak{u}^*)+\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)},\end{aligned} \tag{19}$$

so that

$$\begin{aligned}|\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)| &\le a_1|\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)|\\ &\quad + \frac{a_2|\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})||\varphi_{cb}(\mathfrak{u}^*, \mathfrak{T}\mathfrak{u}^*)|}{1+|\varphi_{cb}(\mathfrak{u}, \mathfrak{T}\mathfrak{u}^*)|+|\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)|}\\ &\le a_1|\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)|.\end{aligned} \tag{20}$$

Hence, $\mathfrak{u} = \mathfrak{u}^*$, which proves the uniqueness. This completes the proof of the theorem. □

Theorem 6. *Let* $(\mathfrak{W}, \varphi_{cb})$ *be a complete complex partial b-metric space with the coefficient* $s \ge 1$ *and* $\mathfrak{S}, \mathfrak{T} : \mathfrak{W} \longrightarrow \mathfrak{W}$ *be mappings satisfying*

$$\begin{aligned}\varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) &\le a_1\varphi_{cb}(\aleph, \vartheta) + a_2\varphi_{cb}(\aleph, \mathfrak{T}\vartheta)\\ &\quad + a_3[\varphi_{cb}(\aleph, \mathfrak{S}\aleph) + \varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)],\end{aligned} \tag{21}$$

for all $\aleph, \vartheta \in \mathfrak{W}$*, where* a_1, a_2*, and* a_3 *are nonnegative reals with* $a_1 + 2sa_2 + 2a_3 < 1$*. Then,* $\mathfrak{S}$ *and* $\mathfrak{T}$ *have a unique common fixed point in* $\mathfrak{W}$*.*

Proof. Let $\aleph_0$ be arbitrary point in $\mathfrak{W}$, and define a sequence $\{\aleph_\mathfrak{n}\}$ in $\mathfrak{W}$ such that

$$\aleph_{2\mathfrak{n}+1} = \mathfrak{S}\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2} = \mathfrak{T}\aleph_{2\mathfrak{n}+1}, \quad \forall \mathfrak{n} \ge 0. \tag{22}$$

Next, show that the sequence $\{\aleph_\mathfrak{n}\}$ is Cauchy. By using (22), we get

$$\begin{aligned}\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2}) &= \varphi_{cb}(\mathfrak{S}\aleph_{2\mathfrak{n}}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})\\ &\le a_1\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1}) + a_2\varphi_{cb}(\aleph_{2\mathfrak{n}}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})\\ &\quad + a_3[\varphi_{cb}(\aleph_{2\mathfrak{n}}, \mathfrak{S}\aleph_{2\mathfrak{n}}) + \varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})]\\ &= a_1\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1}) + a_2\varphi_{cb}(\aleph_{2\mathfrak{n}}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})\\ &\quad + a_3[\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1}) + \varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})],\end{aligned} \tag{23}$$

so that

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})| &\le a_1|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|\\ &\quad + a_2|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2})|\\ &\quad + a_3[|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|\\ &\quad + |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})\,|\,].\end{aligned} \tag{24}$$

By the notion of complex partial b-metric space, we get

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2})| &\le s\{|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})| + |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|\}\\ &\quad - |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+1})||\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+2})|\\ &\le s\{|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})| + |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|\}.\end{aligned} \tag{25}$$

Hence,

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})| &\le a_1|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|\\ &\quad + sa_2\{|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})| + |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|\}\\ &\quad + a_3[|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})| + |\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|]\\ &\le \left(\frac{a_1 + sa_2 + a_3}{1 - sa_2 - a_3}\right)|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|.\end{aligned} \tag{26}$$

Similarly,

$$|\varphi_{cb}(\aleph_{2\mathfrak{n}+2}, \aleph_{2\mathfrak{n}+3})| \le \left(\frac{a_1 + sa_2 + a_3}{1 - sa_2 - a_3}\right)|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|. \tag{27}$$

Set $\delta = (a_1 + sa_2 + a_3)/(1 - sa_2 - a_3)$. Since $a_1 + 2sa_2 + 2a_3 < 1$ and for all $\mathfrak{n} \ge 0$, consequently, we have

$$\begin{aligned}|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})| &\le \delta|\varphi_{cb}(\aleph_{2\mathfrak{n}}, \aleph_{2\mathfrak{n}+1})|\\ &\le \delta^2|\varphi_{cb}(\aleph_{2\mathfrak{n}-1}, \aleph_{2\mathfrak{n}})| \vdots\\ &\le \delta^{2\mathfrak{n}+1}|\varphi_{cb}(\aleph_0, \aleph_1)|.\end{aligned} \tag{28}$$

That is,

$$\begin{aligned}|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| &\le \delta|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})|\\ &\le \delta^2|\varphi_{cb}(\aleph_{\mathfrak{n}-1}, \aleph_\mathfrak{n})| \vdots\\ &\le \delta^{\mathfrak{n}+1}|\varphi_{cb}(\aleph_0, \aleph_1)|.\end{aligned} \tag{29}$$

For any $\mathfrak{m} > \mathfrak{n}$, $\mathfrak{m}, \mathfrak{n} \in \mathbb{N}$, we have

$$\begin{aligned}
|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_\mathfrak{m})| &\leq s\{|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})|+|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_\mathfrak{m})|\} \\
&\quad - |\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+1})| \\
&\leq s\{|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})|+|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_\mathfrak{m})|\} \\
&\leq s|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})| + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| \\
&\quad + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_\mathfrak{m})| - |\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_{\mathfrak{n}+2})| \\
&\leq s|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})| + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| \\
&\quad + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_\mathfrak{m})| \\
&\leq s|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})| + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| \\
&\quad + s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_{\mathfrak{n}+3})| + s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+3}, \aleph_\mathfrak{m})| \\
&\quad - |\varphi_{cb}(\aleph_{\mathfrak{n}+3}, \aleph_{\mathfrak{n}+3})| \\
&\leq s|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})| + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| \\
&\quad + s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_{\mathfrak{n}+3})| + s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+3}, \aleph_\mathfrak{m})| \vdots \\
&\leq s|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_{\mathfrak{n}+1})| + s^2|\varphi_{cb}(\aleph_{\mathfrak{n}+1}, \aleph_{\mathfrak{n}+2})| \\
&\quad + s^3|\varphi_{cb}(\aleph_{\mathfrak{n}+2}, \aleph_{\mathfrak{n}+3})| \\
&\quad + \cdots s^{\mathfrak{m}-\mathfrak{n}-2}|\varphi_{cb}(\aleph_{\mathfrak{m}-3}, \aleph_{\mathfrak{m}-2})| \\
&\quad + s^{\mathfrak{m}-\mathfrak{n}-1}|\varphi_{cb}(\aleph_{\mathfrak{m}-2}, \aleph_{\mathfrak{m}-1})| \\
&\quad + s^{\mathfrak{m}-\mathfrak{n}}|\varphi_{cb}(\aleph_{\mathfrak{m}-1}, \aleph_\mathfrak{m})|.
\end{aligned} \tag{30}$$

From (29), we get

$$\begin{aligned}
|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_\mathfrak{m})| &\leq s\delta^\mathfrak{n}|\varphi_{cb}(\aleph_0, \aleph_1)| + s^2\delta^{\mathfrak{n}+1}|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&\quad + s^3\delta^{\mathfrak{n}+2}|\varphi_{cb}(\aleph_0, \aleph_1)| + \cdots + s^{\mathfrak{m}-\mathfrak{n}-2}\delta^{\mathfrak{m}-3}|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&\quad + s^{\mathfrak{m}-\mathfrak{n}-1}\delta^{\mathfrak{m}-2}|\varphi_{cb}(\aleph_0, \aleph_1)| + s^{\mathfrak{m}-\mathfrak{n}}\delta^{\mathfrak{m}-1}|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&= \sum_{i=1}^{\mathfrak{m}-\mathfrak{n}} s^i\delta^{i+\mathfrak{n}-1}|\varphi_{cb}(\aleph_0, \aleph_1)|.
\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}
|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_\mathfrak{m})| &\leq \sum_{i=1}^{\mathfrak{m}-\mathfrak{n}} s^{i+\mathfrak{n}-1}\delta^{i+\mathfrak{n}-1}|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&= \sum_{t=\mathfrak{n}}^{\mathfrak{m}-1} s^t\delta^t|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&\leq \sum_{t=\mathfrak{n}}^{\infty} (s\delta)^t|\varphi_{cb}(\aleph_0, \aleph_1)| \\
&= \frac{(s\delta)^\mathfrak{n}}{1 - s\delta}|\varphi_{cb}(\aleph_0, \aleph_1)|,
\end{aligned} \tag{32}$$

and hence,

$$|\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_\mathfrak{m})| \leq \frac{(s\delta)^\mathfrak{n}}{1 - s\delta}|\varphi_{cb}(\aleph_0, \aleph_1)| \longrightarrow 0 \quad \text{as } \mathfrak{m}, \mathfrak{n} \longrightarrow \infty. \tag{33}$$

Thus, $\{\aleph_\mathfrak{n}\}$ is a Cauchy sequence in $\mathfrak{W}$. Since $\mathfrak{W}$ is complete, there exists some $\mathfrak{u} \in \mathfrak{W}$ such that $\aleph_\mathfrak{n} \longrightarrow \mathfrak{u}$ as $\mathfrak{n} \longrightarrow \infty$ and

$$\varphi_{cb}(\mathfrak{u}, \mathfrak{u}) = \lim_{n\longrightarrow\infty}\varphi_{cb}(\mathfrak{u}, \aleph_\mathfrak{n}) = \lim_{n\longrightarrow\infty}\varphi_{cb}(\aleph_\mathfrak{n}, \aleph_\mathfrak{n}) = 0. \tag{34}$$

Assume on the contrary that there exists $\mathfrak{z} \in \mathfrak{W}$ such that

$$|\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})| = |\mathfrak{z}| > 0. \tag{35}$$

By using the triangular inequality and (21), we obtain

$$\begin{aligned}
\mathfrak{z} &= \varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u}) \\
&\leq s\{\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) + \varphi_{cb}(\aleph_{2\mathfrak{n}+2}, \mathfrak{S}\mathfrak{u})\} - \varphi_{cb}(\aleph_{2\mathfrak{n}+2}, \aleph_{2\mathfrak{n}+2}) \\
&\leq s\{\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) + \varphi_{cb}(\aleph_{2\mathfrak{n}+2}, \mathfrak{S}\mathfrak{u})\} \\
&= s\{\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) + \varphi_{cb}(\mathfrak{S}\mathfrak{u}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})\} \\
&\leq s\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) + s\mathrm{a}_1\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+1}) + s\mathrm{a}_2\varphi_{cb}(\mathfrak{u}, \mathfrak{T}\aleph_{2\mathfrak{n}+1}) \\
&\quad + s\mathrm{a}_3[\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u}) + \varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \mathfrak{T}\aleph_{2\mathfrak{n}+1})] \\
&= s\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) + s\mathrm{a}_1\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+1}) + s\mathrm{a}_2\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2}) \\
&\quad + s\mathrm{a}_3[\mathfrak{z} + \varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})],
\end{aligned} \tag{36}$$

which implies that

$$\begin{aligned}
|\mathfrak{z}| &= |\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})| \\
&\leq s|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2})| + s\mathrm{a}_1|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+1})| \\
&\quad + s\mathrm{a}_2|\varphi_{cb}(\mathfrak{u}, \aleph_{2\mathfrak{n}+2})| + s\mathrm{a}_3[|\mathfrak{z}|+|\varphi_{cb}(\aleph_{2\mathfrak{n}+1}, \aleph_{2\mathfrak{n}+2})|].
\end{aligned} \tag{37}$$

As $\mathfrak{n} \longrightarrow \infty$ in (37), we obtain that $|\mathfrak{z}| = |\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u})| \leq 0$, a contradiction with (35). Therefore, $|\mathfrak{z}| = 0$. Hence, $\mathfrak{S}\mathfrak{u} = \mathfrak{u}$. Similarly, we obtain $\mathfrak{T}\mathfrak{u} = \mathfrak{u}$.

Assume that $\mathfrak{u}^*$ is another common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$. Then,

$$\begin{aligned}
\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*) &= \varphi_{cb}(\mathfrak{S}\mathfrak{u}, \mathfrak{T}\mathfrak{u}^*) \\
&\leq \mathrm{a}_1\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*) + \mathrm{a}_2\varphi_{cb}(\mathfrak{u}, \mathfrak{T}\mathfrak{u}^*) \\
&\quad + \mathrm{a}_3[\varphi_{cb}(\mathfrak{u}, \mathfrak{S}\mathfrak{u}) + \varphi_{cb}(\mathfrak{u}^*, \mathfrak{T}\mathfrak{u}^*)],
\end{aligned} \tag{38}$$

which implies that $|\varphi_{cb}(\mathfrak{u}, \mathfrak{u}^*)| \leq 0$, a contradiction. So $\mathfrak{u} = \mathfrak{u}^*$, which proves the uniqueness. □

Example 1. Let $\mathfrak{W} = \{1, 2, 3, 4\}$ be endowed with the order $\aleph \leq \vartheta$ if and only if $\vartheta \leq \aleph$. Then, $\leq$ is a partial order in $\mathfrak{W}$. Define the complex partial b-metric space $\varphi_{cb} : \mathfrak{W} \times \mathfrak{W} \longrightarrow \mathbb{C}_+$ as follows (Table 1):

Table 1

$(\aleph, \vartheta)$	$\varphi_{cb}(\aleph, \vartheta)$
(1,1), (2,2)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (3,3)	e^{2ix}
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	$9e^{2ix}$

It is easy to verify that $(\mathfrak{W}, \varphi_{cb})$ is a complete complex partial b-metric space with the coefficient $s \geq 1$ for $x \in [0, \pi/2]$. Define $\mathfrak{S}, \mathfrak{T} : \mathfrak{W} \longrightarrow \mathfrak{W}$ by $\mathfrak{S}\aleph = 1$,

$$\mathfrak{T}(\aleph) = \begin{pmatrix} 1 & \text{if } \aleph \in \{1, 2, 3\}, \\ 2 & \text{if } \aleph = 4. \end{pmatrix} \tag{39}$$

Let $\lambda_1 = 1/9$ and $\lambda_2 = 1/8$; we consider the following cases:

(1) If $\aleph = 1$ and $\vartheta \in \mathfrak{W} - \{4\}$, then $\mathfrak{S}(\aleph) = \mathfrak{T}(\vartheta) = 1$ and the conditions of Theorem 5 are satisfied

(2) If $\aleph = 1$ and $\vartheta = 4$, then $\mathfrak{S}\aleph = 1$, $\mathfrak{T}\vartheta = 2$,

$$\begin{aligned} \varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) = e^{2ix} &\leq 9\lambda_1 e^{i2x} \\ &= \lambda_1 \varphi_{cb}(\aleph, \vartheta) + \lambda_2 \frac{\varphi_{cb}(\aleph, \mathfrak{S}\aleph)\varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)}{1 + \varphi_{cb}(\aleph, \mathfrak{T}\vartheta) + \varphi_{cb}(\aleph, \vartheta)} \end{aligned} \tag{40}$$

(3) If $\aleph = 2$ and $\vartheta = 4$, then $\mathfrak{S}\aleph = 1$, $\mathfrak{T}\vartheta = 2$

$$\begin{aligned} \varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) = e^{2ix} &\leq \left(1 + \frac{1}{8}\right) e^{i2x} \\ &= \lambda_1 9 e^{i2x} + \lambda_2 \frac{e^{i2x} 9 e^{i2x}}{1 + 9e^{i2x}} \\ &= \lambda_1 \varphi_{cb}(\aleph, \vartheta) + \lambda_2 \frac{\varphi_{cb}(\aleph, \mathfrak{S}\aleph)\varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)}{1 + \varphi_{cb}(\aleph, \mathfrak{T}\vartheta) + \varphi_{cb}(\aleph, \vartheta)} \end{aligned} \tag{41}$$

(4) If $\aleph = 3$ and $\vartheta = 4$, then $\mathfrak{S}\aleph = 1$, $\mathfrak{T}\vartheta = 2$

$$\begin{aligned} \varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) = e^{2ix} &\leq \left(1 + \frac{9}{80}\right) e^{i2x} \\ &= \lambda_1 9 e^{i2x} + \lambda_2 \frac{e^{i2x} 9 e^{i2x}}{1 + 10e^{i2x}} \\ &= \lambda_1 \varphi_{cb}(\aleph, \vartheta) + \lambda_2 \frac{\varphi_{cb}(\aleph, \mathfrak{S}\aleph)\varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)}{1 + \varphi_{cb}(\aleph, \mathfrak{T}\vartheta) + \varphi_{cb}(\aleph, \vartheta)} \end{aligned} \tag{42}$$

(5) If $\aleph = 4$ and $\vartheta = 4$, then $\mathfrak{S}\aleph = 1$, $\mathfrak{T}\vartheta = 2$

$$\begin{aligned} \varphi_{cb}(\mathfrak{S}\aleph, \mathfrak{T}\vartheta) = e^{2ix} &\leq \left(1 + \frac{9}{16}\right) e^{i2x} \\ &= \lambda_1 9 e^{i2x} + \lambda_2 \frac{9e^{i2x} 9 e^{i2x}}{1 + 9e^{i2x} + 9e^{i2x}} \\ &= \lambda_1 \varphi_{cb}(\aleph, \vartheta) + \lambda_2 \frac{\varphi_{cb}(\aleph, \mathfrak{S}\aleph)\varphi_{cb}(\vartheta, \mathfrak{T}\vartheta)}{1 + \varphi_{cb}(\aleph, \mathfrak{T}\vartheta) + \varphi_{cb}(\aleph, \vartheta)} \end{aligned} \tag{43}$$

Moreover for $a_1 + sa_2 < 1$, the conditions of Theorem 5 are satisfied. Therefore, 1 is the unique common fixed point of $\mathfrak{S}$ and $\mathfrak{T}$.

4. Application

Consider the following systems of integral equations:

$$w(s) = b(s) + \int_a^b T_1(s, p, w(p))dp, \tag{44}$$

$$w(s) = b(s) + \int_a^b T_2(s, p, w(p))dp, \tag{45}$$

where

(1) [label = ()]

(2) $w(s)$ is an unknown variable for each $s \in J = [a, b]$, $b > a \geq 0$

(3) $b(s)$ is the deterministic free term defined for $s \in [a, b]$

(4) $T_1(s, p)$ and $T_2(s, p)$ are deterministic kernels defined for $s, p \in J = [a, b]$

In this section, we present an existence theorem for a common solution to (44) and (45) that belongs to $\mathfrak{W} = (C(J), \mathbb{R}^n)$ (the set of continuous functions defined on J) by using the obtained result in Theorem 5. We consider the continuous mappings $\mathfrak{S}, \mathfrak{T} : \mathfrak{W} \longrightarrow \mathfrak{W}$ given by

$$\begin{aligned} \mathfrak{S}w(s) &= b(s) + \int_a^b T_1(s, p, w(p))dp, \quad w \in \mathfrak{W}, s \in J, \\ \mathfrak{T}w(s) &= b(s) + \int_a^b T_2(s, p, w(p))dp, \quad z \in \mathfrak{W}, s \in J. \end{aligned} \tag{46}$$

Then, the existence of a common solution to the integral equations (44) and (45) is equivalent to the existence of a common fixed point of T_1 and T_2. It is well known that $\mathfrak{W}$, endowed with the metric φ_{cb} defined by

$$\varphi_{cb}(w, z) = |w(s) - z(s)|^2 + 2, \tag{47}$$

for all $w, z \in \mathfrak{W}$, is a complete complex partial b-metric space. $\mathfrak{W}$ can also be equipped with the partial order $\leq$ given by

$$w, z \in \mathfrak{W}, \quad w \leq z \text{ if and only } w(s) \geq z(s), \text{ for all } s \in J. \tag{48}$$

Further, let us consider that a system of integral equation as (44) and (45) under the following condition holds:

(1) $T_1, T_2 : J \times J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous functions satisfying

$$|T_1(s,p,w(p)) - T_2(s,p,z(p))| \le \sqrt{\frac{\varphi_{cb}(w,z)}{(b-a)e^t} - \frac{2}{b-a}}, \quad \forall t > 0. \tag{49}$$

Theorem 7. *Let $(C(J), \mathbb{R}^n, \varphi_{cb})$ be a complete complex partial b-metric space; then, the systems (44) and (45) under condition (3) have a unique common solution.*

Proof. For $w, z \in (C(J), \mathbb{R}^n)$ and $s \in J$, we define the continuous mappings $\mathfrak{S}, \mathfrak{T} : \mathfrak{W} \longrightarrow \mathfrak{W}$ by

$$\begin{aligned} \mathfrak{S}w(s) &= b(s) + \int_a^b T_1(s,p,w(p))dp, \\ \mathfrak{T}w(s) &= b(s) + \int_a^b T_2(s,p,w(p))dp. \end{aligned} \tag{50}$$

Then, we have

$$\begin{aligned} \varphi_{cb}(\mathfrak{S}w(s), \mathfrak{T}z(s)) &= |\mathfrak{S}w(s) - \mathfrak{T}z(s)|^2 + 2 \\ &= \int_a^b |T_1(s,p,w(p)) - T_2(s,p,z(p))|^2 dp + 2 \\ &\le \int_a^b \left(\frac{\varphi_{cb}(w,z)}{(b-a)e^t} - \frac{2}{b-a}\right) dp + 2 \\ &= \frac{\varphi_{cb}(w,z)}{e^t} = \lambda_1 \varphi_{cb}(w,z) \\ &= \lambda_1 \varphi_{cb}(w,z) + \lambda_2 \frac{\varphi_{cb}(w,\mathfrak{S}w)\varphi_{cb}(z,\mathfrak{T}z)}{1 + \varphi_{cb}(w,\mathfrak{T}z) + \varphi_{cb}(w,z)}. \end{aligned} \tag{51}$$

Hence, all the conditions of Theorem 5 are satisfied for $a_1 + sa_2 (= 0) < 1$, with $t > 0$. Therefore, the system of integral equations (44) and (45) has a unique common solution.

5. Conclusion

In this paper, we proved some common fixed point theorems for rational contraction mapping on complex partial b-metric space. An illustrative example and application on complex partial b-metric space is given.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Some Fixed Point Results of Kannan Maps on the Nakano Sequence Space

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In the recent past, some researchers studied some fixed point results on the modular variable exponent sequence space $(\ell_{r(.)})_{\psi}$, where $\psi(v)=\sum_{a=0}^{\infty}(1/r_a)|v_a|^{r_a}$ and $r_a \geq 1$. They depended on their proof that the modular ψ has the Fatou property. But we have explained that this result is incorrect. Hence, in this paper, the concept of the premodular, which generalizes the modular, on the Nakano sequence space such as its variable exponent in $(1, \infty)$ and the operator ideal constructed by this sequence space and s-numbers is introduced. We construct the existence of a fixed point of Kannan contraction mapping and Kannan nonexpansive mapping acting on this space. It is interesting that several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of summable equations are introduced. The novelty lies in the fact that our main results have improved some well-known theorems before, which concerned the variable exponent in the aforementioned space.

1. Introduction

Ideal operators and summability theorems are awfully invaluable in mathematical models and have large executions, for example, the fixed point theory, geometry of Banach spaces, normal series theory, approximation theory, and ideal transformations. For added evidence, see [1–4]. By $\mathfrak{R}^{\mathscr{N}}$, ℓ_{∞}, ℓ_r, and c_0, we denote the spaces of all, bounded, r-absolutely summable and null sequences of real numbers. We indicate the space of all bounded linear operators from a Banach space Z into a Banach space M by $\mathscr{L}(Z, M)$, and if $Z=M$, we inscribe $\mathscr{L}(Z)$ and $e_d=\{0, 0, \cdots, 1, 0, 0, \cdots\}$, while 1 displays at the dth place, for all $d \in \mathscr{N}=\{0, 1, 2, \cdots\}$.

Definition 1 [5]. An s-number function is a map detailed on $\mathscr{L}(Z, M)$ which sorts every map $W \in \mathscr{L}(Z, M)$ a nonnegative scaler sequence $(s_d(W))_{d=0}^{\infty}$ overbearing that the next setting encompasses

(a) $\|W\|=s_0(W) \geq s_1(W) \geq s_2(W) \geq \cdots \geq 0$, for all $W \in \mathscr{L}(Z, M)$

(b) $s_{l+d-1}(W_1+W_2) \leq s_l(W_1)+s_d(W_2)$, for every W_1, $W_2 \in \mathscr{L}(Z, M)$, and l, $d \in \mathscr{N}$

(c) Ideal property: $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for every $W \in \mathscr{L}(Z_0, Z)$, $Y \in \mathscr{L}(Z, M)$, and $V \in \mathscr{L}(M, M_0)$, where Z_0 and M_0 are discretionary Banach spaces

(d) For $W \in \mathscr{L}(Z, M)$ and $\gamma \in \mathfrak{R}$, one has $s_d(\gamma W)=|\gamma| s_d(W)$

(e) Rank property: assume rank $(W) \leq d$, then $s_d(W)=0$, for each $W \in \mathscr{L}(Z, M)$

(f) Norming property: $s_{l \geq a}(I_a)=0$ or $s_{l<a}(I_a)=1$, where I_a mirrors the unit map on the a-dimensional Hilbert space ℓ_2^a

The dth approximation number, established by $\alpha_d(W)$, is defined as

$$\alpha_d(W)=\inf\{\|W-Y\|: Y \in \mathscr{L}(Z, M) \text{ and } \operatorname{rank}(Y) \leq d\}. \quad (1)$$

Notation 2. The sets S_A,$S_A(Z, M)$, S_A^{app}, and $S_A^{\mathrm{app}}(Z, M)$ (cf. [6]) indicate

$$S_A := \{S_A(Z, M)\}, \tag{2}$$

where $S_A(Z, M) := \{W \in \mathscr{L}(Z, M): (s_d(W))_{d=0}^{\infty} \in A\}$. Also, $S_A^{\mathrm{app}} := \{S_A^{\mathrm{app}}(Z, M)\}$, where $S_A^{\mathrm{app}}(Z, M) := \{W \in \mathscr{L}(Z, M): (\alpha_d(W))_{d=0}^{\infty} \in A\}$.

Suppose that $r = (r_a) \in \mathfrak{R}^{+\mathscr{N}}$, the Nakano sequence space defined and studied in [7–9] is denoted by $\ell(r) = \{v = (v_a) \in \mathfrak{R}^{\mathscr{N}} : \phi(\mu v)<\infty, \text{for some } \mu > 0\}$, when $\phi(v) = \sum_{a=0}^{\infty} |v_a|^{r_a}$.

The space $(\ell(r), \|.\|)$, where $\|v\| = \inf\{\kappa > 0 : \phi(v/k) \leq 1\}$ and $r_a \geq 1$, for all $a \in \mathscr{N}$, is a Banach space. If $(r_a) \in \ell_{\infty}$, then,

$$\begin{aligned}
\ell(r) &= \left\{v = (v_a) \in \mathfrak{R}^{\mathscr{N}} : \phi(\mu v)<\infty, \text{for some } \mu > 0\right\} \\
&= \left\{v = (v_a) \in \mathfrak{R}^{\mathscr{N}} : \inf_a |\mu|^{r_a} \sum_{a=0}^{\infty} |v_a|^{r_a} \leq \sum_{a=0}^{\infty} |\mu v_a|^{r_a}<\infty, \text{for some } \mu > 0\right\} \\
&= \left\{v = (v_a) \in \mathfrak{R}^{\mathscr{N}} : \sum_{a=0}^{\infty} |v_a|^{r_a}<\infty\right\} \\
&= \left\{v = (v_a) \in \mathfrak{R}^{\mathscr{N}} : \phi(\mu v)<\infty, \text{for any } \mu > 0\right\}.
\end{aligned} \tag{3}$$

The vector spaces $\ell(r)$ are contained in the variable exponent spaces $L_{(r)}$. In the second half of the twentieth century, it was assumed that these variable exponent spaces provided the proper framework for the mathematical components of numerous issues for which the classical Lebesgue spaces were insufficient. Because of the importance of these spaces and their surroundings, they have become a well-known and environmentally friendly tool in the treatment of a variety of conditions; currently, the region of $L_{(r)}(\Omega)$ spaces is a prolific subject of research, with ramifications extending into a wide range of mathematical specialties (see [10]). The mathematical description of the hydrodynamics of non-Newtonian fluids provides an impetus for learning about variable exponent Lebesgue spaces, $L_{(r)}$ (see [11, 12]). Applications of non-Newtonian fluids, known as electrorheological, vary from their use in army science to civil engineering and orthopedics. Faried and Bakery provided the theory of the pre-quasioperator ideal, which is more general than the quasioperator ideal, in [6]. In [7], Bakery and Abou Elmatty explained the sufficient (not necessary) setting on $\ell(r)$ so that $S_{\ell(r)}$ generated a simple Banach pre-quasioperator ideal. The pre-quasioperator ideal $S_{\ell(r)}^{\mathrm{app}}$ is strictly restricted to different powers. It was a small pre-quasioperator ideal. Because of the booklet of the Banach fixed point theorem [13], many mathematicians have worked on many developments. Kannan [14] gave an example of a class of mappings with the same fixed point actions as contractions, though that fails to be continuous. The only attempt to describe Kannan operators in modular vector spaces was once made in reference [15]. Bakery and Mohamed [16] explored the concept of the pre-quasinorm on the Nakano sequence space such that its variable exponent in $(0, 1]$. They explained the sufficient conditions on it, equipped with the definite pre-quasinorm to generate pre-quasi-Banach and closed space, and examined the Fatou property of different pre-quasinorms on it. Moreover, they showed the existence of a fixed point of Kannan pre-quasinorm contraction maps on it and on the pre-quasi-Banach operator ideal constructed by s-numbers which belong to this sequence space. For more details on Kannan's fixed point theorems, see [17–24]. The aim of this paper is to examine the concept of the pre-quasinorm on $\ell(r)$ with a variable exponent in $(1, \infty)$. We study the sufficient conditions on $\ell(r)$ equipped with the definite pre-quasinorm to form pre-quasi-Banach and closed (sss), the existence of a fixed point of Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach (sss), $(\ell(r))_{\phi}$ satisfies the property (R), and $(\ell(r))_{\phi}$ has the ϕ-normal structure property. The existence of a fixed point of Kannan pre-quasinorm nonexpansive mapping in the pre-quasi-Banach (sss) has been given. Finally, we examine the idea of Kannan pre-quasinorm contraction mapping in the pre-quasioperator ideal. As well, the existence of a fixed point of Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach operator ideal $S_{(\ell(r))_{\phi}}$ has been introduced. Finally, some illustrative examples and applications to the existence of solutions of summable equations are given.

2. Definitions and Preliminaries

By $[0,\infty)^{\mathfrak{A}}$, we denote the space of all functions $\phi : \mathfrak{A} \longrightarrow [0,\infty)$. Nakano [25] introduced the concept of modular vector spaces.

Definition 3. Suppose that $\mathfrak{A}$ is a vector space. A function $\phi \in [0,\infty]^{\mathfrak{A}}$ is called modular if the next conditions hold

(a) For $v \in \mathfrak{A}$, $v = \theta \Longleftrightarrow \phi(v) = 0$ with $\phi(v) \geq 0$, where θ is the zero vector of $\mathfrak{A}$

(b) $\phi(\eta v) = \phi(v)$ holds, for all $v \in \mathfrak{A}$ and $|\eta| = 1$

(c) The inequality $\phi(\alpha v + (1 - \alpha)t) \leq \phi(v) + \phi(t)$ satisfies, for all $v, t \in \mathfrak{A}$ and $\alpha \in [0, 1]$

The concept of premodular vector spaces is more general than modular vector spaces.

Definition 4 [2]. The linear space of sequences $\mathfrak{A}$ is called a special space of sequences (sss), if

(a) $\{e_a\}_{a \in \mathscr{N}} \subseteq \mathfrak{A}$

(b) $\mathfrak{A}$ is solid, i.e., assume that $v = (v_a) \in \mathfrak{R}^{\mathscr{N}}$, $t = (t_a) \in \mathfrak{A}$, and $|v_a| \leq |t_a|$, for each $a \in \mathscr{N}$, and then $v \in \mathfrak{A}$

(c) $(v_{[a/2]})_{a=0}^{\infty} \in \mathfrak{A}$, where $[a/2]$ indicates the integral part of $a/2$, in case $(v_a)_{a=0}^{\infty} \in \mathfrak{A}$

Definition 5 [6]. A subclass $\mathfrak{A}_\phi$ of $\mathfrak{A}$ is named a premodular (sss), if there is $\phi \in [0,\infty)^{\mathfrak{A}}$, it satisfies the next setting:

(i) For $v \in \mathfrak{A}$, $v = \theta \Longleftrightarrow \phi(v) = 0$ with $\phi(v) \geq 0$

(ii) For some $B \geq 1$, the inequality $\phi(\eta v) \leq B|\eta|\phi(v)$ holds, for all $v \in \mathfrak{A}$ and $\eta \in \mathfrak{R}$

(iii) For some $J \geq 1$, the inequality $\phi(v + t) \leq J(\phi(v) + \phi(t))$ satisfies, for all $v, t \in \mathfrak{A}$

(iv) For $a \in \mathcal{N}$ and $|v_a| \leq |t_a|$, we have $\phi((v_a)) \leq \phi((t_a))$

(v) The inequality, $\phi((v_a)) \leq \phi((v_{[a/2]})) \leq J_0\phi((v_a))$ includes, for some $J_0 \geq 1$

(vi) Let F be the space of finite sequences, then $\bar{F} = \mathfrak{A}_\phi$

(vii) we have $\varsigma > 0$ such that $\phi(\beta, 0, 0, 0, \cdots) \geq \varsigma|\beta|\phi(1, 0, 0, 0, \cdots)$, for all $\beta \in \mathfrak{R}$

This is an example of a premodular vector space but not a modular vector space.

Example 6. The function $\phi(v) = (\sum_{a\in\mathcal{N}}|v_a|^{a+1/3a+4})^4$ is a pre-modular (not a modular) on the vector space $\ell((a + 1/3a + 4)_{a=0}^{\infty})$. Since for all $v, t \in \ell((a + 1/3a + 4)_{a=0}^{\infty})$, we have

$$\phi\left(\frac{v+t}{2}\right) = \left(\sum_{a\in\mathcal{N}}\left|\frac{v_a + t_a}{2}\right|^{a+1/3a+4}\right)^4 \leq 4(\phi(v) + \phi(t)). \quad (4)$$

Definition 7 [26]. Let $\mathfrak{A}$ be a (sss). The function $\phi \in [0,\infty)^{\mathfrak{A}}$ is named a pre-quasinorm on $\mathfrak{A}$, if it provides the following setting:

(i) For $v \in \mathfrak{A}$, $v = \theta \Longleftrightarrow \phi(v) = 0$ with $\phi(v) \geq 0$

(ii) For some $B \geq 1$, the inequality $\phi(\eta v) \leq B|\eta|\phi(v)$ holds, for all $v \in \mathfrak{A}$ and $\eta \in \mathfrak{R}$

(iii) For some $J \geq 1$, the inequality $\phi(v + t) \leq J(\phi(v) + \phi(t))$ satisfies, for all $v, t \in \mathfrak{A}$

Theorem 8 [26]. *Let $\mathfrak{A}$ be a premodular (sss), and then, it is pre-quasinormed (sss).*

Theorem 9 [26]. *$\mathfrak{A}$ is a pre-quasinormed (sss), if it is quasi-normed (sss).*

Definition 10 [3]. Suppose that $\mathcal{L}$ is the class of all bounded linear operators within any two arbitrary Banach spaces. A subclass $\mathcal{U}$ of $\mathcal{L}$ is called an operator ideal, if every element $\mathcal{U}(Z, M) = \mathcal{U} \cap \mathcal{L}(Z, M)$ satisfies the next conditions:

(i) $I_\Gamma \in \mathcal{U}$, where Γ describes the Banach space of one dimension

(ii) The space $\mathcal{U}(Z, M)$ is linear over $\mathfrak{R}$

(iii) If $W \in \mathcal{L}(Z_0, Z)$, $X \in \mathcal{U}(Z, M)$, and $Y \in \mathcal{L}(M, M_0)$, then, $YXW \in \mathcal{U}(Z_0, M_0)$, where Z_0 and M_0 are normed spaces (see [27, 28])

This is the concept of the pre-quasioperator ideal which is added in general to the quasioperator ideal.

Definition 11 [6]. A function $\phi \in [0,\infty)^{\mathcal{U}}$ is called a pre-quasinorm on the ideal $\mathcal{U}$ if the next conditions hold:

(1) Let $W \in \mathcal{U}(Z, M)$, $\phi(W) \geq 0$, and $\phi(W) = 0$, if and only if, $W = 0$

(2) We have $D \geq 1$ so as to $\phi(\eta W) \leq D|\eta|\phi(W)$, for every $W \in \mathcal{U}(Z, M)$ and $\eta \in \mathfrak{R}$

(3) We have $J \geq 1$ so that $\phi(W_1 + W_2) \leq J[\phi(W_1) + \phi(W_2)]$, for each $W_1, W_2 \in \mathcal{U}(Z, M)$

(4) We have $\sigma \geq 1$ if $W \in \mathcal{L}(Z_0, Z)$, $X \in \mathcal{U}(Z, M)$, and $Y \in \mathcal{L}(M, M_0)$, and then, $\phi(YXW) \leq \sigma\|Y\|\phi(X)\|W\|$

Theorem 12 [29]. *Assuming that $\mathfrak{A}_\phi$ is a pre-modular (sss), then, $\phi(W) = \phi(s_a(W))_{a=0}^{\infty}$ is a pre-quasinorm on $S_{\mathfrak{A}_\phi}$.*

Theorem 13 [7]. *Let Z and M be Banach spaces and $\mathfrak{A}_\phi$ be a premodular (sss), and then, $(S_{\mathfrak{A}_\phi}, \phi)$ is a pre-quasi-Banach operator ideal, such that $\phi(W) = \phi((s_a(W))_{a=0}^{\infty})$.*

Theorem 14 [6]. *ϕ is a pre-quasinorm on the ideal $\mathcal{U}$, if ϕ is a quasinorm on the ideal $\mathcal{U}$.*

Lemma 15. *The given inequalities will be used in the sequel:*

(i) Let $r \geq 2$, and for every $v, t \in \mathfrak{R}$ [30], then

$$\left|\frac{v+t}{2}\right|^r + \left|\frac{v-t}{2}\right|^r \leq \frac{1}{2}(|v|^r + |t|^r) \quad (5)$$

(ii) Assume that $1 < r \leq 2$, and for all $v, t \in \mathfrak{R}$ so that $|v| + |t| \neq 0$ [31], then

$$\left|\frac{v+t}{2}\right|^r + \frac{r(r-1)}{2}\left|\frac{v-t}{|v|+|t|}\right|^{2-r}\left|\frac{v-t}{2}\right|^r \leq \frac{1}{2}(|v|^r + |t|^r) \quad (6)$$

(iii) Suppose that $r_a \geq 1$ and $v_a, t_a \in \mathfrak{R}$, for every $a \in \mathcal{N}$, then, $|v_a + t_a|^{r_a} \leq 2^{K-1}(|v_a|^{r_a} + |t_a|^{r_a})$, where $K = \sup_a r_a$ [32]

3. Pre-Quasinormed (sss)

We explain the sufficient setting of $\ell(r)$ equipped with a pre-quasinorm ϕ to generate pre-quasi-Banach and closed (sss). The Fatou property of a pre-quasinorm ϕ on $\ell(r)$ has been given.

Definition 16.

(a) The function ϕ on $\ell(r)$ is named ϕ convex, if $\phi(\omega v + (1-\omega)t) \le \omega\phi(v) + (1-\omega)\phi(t)$, for all $\omega \in [0,1]$ and $v, t \in \ell(r)$

(b) $\{v_a\}_{a\in\mathcal{N}} \subseteq (\ell(r))_\phi$ is ϕ convergent to $v \in (\ell(r))_\phi$, if and only if, $\lim_{a\longrightarrow\infty}\phi(v_a - v) = 0$. If the ϕ limit exists, then it is unique

(c) $\{v_a\}_{a\in\mathcal{N}} \subseteq (\ell(r))_\phi$ is ϕ Cauchy, when $\lim_{a,b\longrightarrow\infty}\phi(v_a - v_b) = 0$

(d) $\Lambda \subset (\ell(r))_\phi$ is ϕ closed, if for every ϕ-converges $\{v_a\}_{a\in\mathcal{N}} \subset \Lambda$ to v, then $v \in \Lambda$

(e) $\Lambda \subset (\ell(r))_\phi$ is ϕ bounded, when $\delta_\phi(\Lambda) = \sup\{\phi(v - t): v, t \in \Lambda\} < \infty$

(f) The ϕ ball of radius $d \ge 0$ and center v, for all $v \in (\ell(r))_\phi$, is detailed as

$$\mathscr{B}_\phi(v, d) = \left\{t \in (\ell(r))_\phi : \phi(v - t) \le d\right\} \tag{7}$$

(g) A pre-quasinorm ϕ on $\ell(r)$ provides the Fatou property, if for all sequence $\{t^a\} \subseteq (\ell(r))_\phi$ with $\lim_{a\longrightarrow\infty}\phi(t^a - t) = 0$ and any $v \in (\ell(r))_\phi$, then $\phi(v - t) \le \sup_j \inf_{a\ge j}\phi(v - t^a)$

Note that the Fatou property implies the ϕ closedness of the ϕ balls.

Theorem 17. $(\ell(r))_\phi$*, where* $\phi(v) = [\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$*, for each* $v \in \ell(r)$*, is a premodular (sss), if* $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ *is increasing with* $r_0 > 1$*.*

Proof. To begin with, we have to show that $\ell(r)$ is a (sss):

(1) Assume $v, t \in \ell(r)$. As (r_a) is bounded, we get

$$\begin{aligned}\phi(v+t) &= \left[\sum_{a=0}^{\infty}|v_a + t_a|^{r_a}\right]^{1/K} \le \left[\sum_{a=0}^{\infty}|v_a|^{r_a}\right]^{1/K} + \left[\sum_{a=0}^{\infty}|t_a|^{r_a}\right]^{1/K}\\ &= \phi(v) + \phi(t) < \infty.\end{aligned} \tag{8}$$

Hence, $v + t \in \ell(r)$.

And suppose that $\eta \in \mathfrak{R}$ and $v \in \ell(r)$. Since (r_a) is bounded, we obtain

$$\phi(\eta v) = \left[\sum_{a=0}^{\infty}|\eta v_a|^{r_a}\right]^{1/K} \le \sup_a|\eta|^{r_a/K}\left[\sum_{a=0}^{\infty}|v_a|^{r_a}\right]^{1/K} \le D|\eta|\phi(v) < \infty. \tag{9}$$

So, $\eta v \in \ell(r)$. Therefore, by using equations (8) and (9), we have that $\ell(r)$ is linear. Also, $e_a \in \ell(r)$, for every $a \in \mathcal{N}$, as $\phi(e_a) = [\sum_{j=0}^{\infty}|e_a(j)|^{r_j}]^{1/K} = 1$

(2) Suppose $|v_a| \le |t_a|$, for every $a \in \mathcal{N}$ and $t \in \ell(r)$. We have

$$\phi(v) = \left[\sum_{a=0}^{\infty}|v_a|^{r_a}\right]^{1/K} \le \left[\sum_{a=0}^{\infty}|t_a|^{r_a}\right]^{1/K} = \phi(t) < \infty. \tag{10}$$

Then, $v \in \ell(r)$.

(3) Assuming that $(v_a) \in \ell(r)$ and (r_a) is an increasing sequence, we have

$$\begin{aligned}\phi\left(\left(v_{[a/2]}\right)\right) &= \left[\sum_{a=0}^{\infty}\left|v_{[a/2]}\right|^{r_a}\right]^{1/K} = \left[\sum_{a=0}^{\infty}|v_a|^{r_{2a}} + \sum_{a=0}^{\infty}|v_a|^{r_{2a+1}}\right]^{1/K}\\ &\le 2^{1/K}\left[\sum_{a=0}^{\infty}|v_a|^{r_a}\right]^{1/K} = 2^{1/K}\phi((v_a)).\end{aligned} \tag{11}$$

Then, $(v_{[a/2]}) \in \ell(r)$. As well, we prove that the functional ϕ on $\ell(r)$ is a premodular:

(i) Clearly, $\phi(v) \ge 0$ and $\phi(v) = 0 \Longleftrightarrow v = \theta$

(ii) We have $D = \max\{1, \sup_a|\eta|^{(r_a/K)-1}\} \ge 1$ such that $\phi(\eta v) \le D|\eta|\phi(v)$, for every $v \in \ell(r)$ and $\eta \in \mathfrak{R}$

(iii) We have $J \ge 1$ so that $\phi(v + t) \le J(\phi(v) + \phi(t))$, for every $v, t \in \ell(r)$

(iv) Evidently, from (101)

(v) From (104), we have $J_0 = 2^{1/K} \ge 1$

(vi) Evidently, $\bar{F} = \ell(r)$

(vii) We have $0 < \varsigma \le |\beta|^{(r_0/K)-1}$, for $\beta \ne 0$ or $\varsigma > 0$, for $\beta = 0$ so that

$$\phi(\beta, 0, 0, 0, \cdots) \ge \varsigma|\beta|\phi(1, 0, 0, 0, \cdots) \tag{12}$$

□

Theorem 18. *Let $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ be an increase with $r_0 > 1$, and then, $(\ell(r))_\phi$ is a pre-quasi-Banach (sss), where $\phi(v) = [\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$, for all $v \in \ell(r)$.*

Proof. Suppose that the setup is satisfied. From Theorem 17, the space $(\ell(r))_\phi$ is a premodular (sss). By Theorem 8, the space $(\ell(r))_\phi$ is a pre-quasinormed (sss). To explain that $(\ell(r))_\phi$ is a pre-quasi-Banach (sss), suppose that $v^p = (v_a^p)_{a=0}^{\infty}$ is a Cauchy sequence in $(\ell(r))_\phi$. Therefore, for all $\varepsilon \in (0,1)$, there is $p_0 \in \mathcal{N}$ so that for every $p, q \geq p_0$, we have

$$\phi(v^p - v^q) = \left[\sum_{a=0}^{\infty}|v_a^p - v_a^q|^{r_a}\right]^{1/K} < \varepsilon. \tag{13}$$

Hence, for $p, q \geq p_0$, and $a \in \mathcal{N}$, we have $|v_a^p - v_a^q| < \varepsilon$. Hence, (v_a^q) is a Cauchy sequence in $\mathfrak{R}$, for fixed $a \in \mathcal{N}$, which gives $\lim_{q\longrightarrow\infty} v_a^q = v_a^0$, for constant $a \in \mathcal{N}$. So, $\phi(v^p - v^0) < \varepsilon$, for all $p \geq p_0$. Conclusively, to prove that $v^0 \in \ell(r)$, one has $\phi(v^0) = \phi(v^0 - v^p + v^p) \leq \phi(v^p - v^0) + \phi(v^p) < \infty$. Hence, $v^0 \in \ell(r)$. This gives that $(\ell(r))_\phi$ is a pre-quasi-Banach (sss). □

Theorem 19. *Assuming that $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$, then, $(\ell(r))_\phi$ is a pre-quasiclosed (sss), where $\phi(v) = [\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$, for all $v \in \ell(r)$.*

Proof. Let the setup be satisfied. From Theorem 17, the space $(\ell(r))_\phi$ is a premodular (sss). By Theorem 8, the space $(\ell(r))_\phi$ is a pre-quasinormed (sss). To prove that $(\ell(r))_\phi$ is a pre-quasiclosed (sss), let $v^p = (v_a^p)_{a=0}^{\infty} \in (\ell(r))_\phi$ and $\lim_{p\longrightarrow\infty}\phi(v^p - v^0) = 0$; then, for each $\varepsilon \in (0,1)$, there is $p_0 \in \mathcal{N}$ such that for every $p \geq p_0$, one can see

$$\varepsilon > \phi\left(v^p - v^0\right) = \left[\sum_{a=0}^{\infty}\left|v_a^p - v_a^0\right|^{r_a}\right]^{1/K}. \tag{14}$$

Therefore, for $p \geq p_0$ and $a \in \mathcal{N}$, we have $|v_a^p - v_a^0| < \varepsilon$. Hence, (v_a^p) is a convergent sequence in $\mathfrak{R}$, for constant $a \in \mathcal{N}$. So, $\lim_{p\longrightarrow\infty} v_a^p = v_a^0$, for fixed $a \in \mathcal{N}$. Finally, to show that $v^0 \in \ell(r)$, one has

$$\phi\left(v^0\right) = \phi\left(v^0 - v^p + v^p\right) \leq \phi\left(v^p - v^0\right) + \phi\left(v^p\right) < \infty. \tag{15}$$

Hence, $v^0 \in \ell(r)$. This implies that $(\ell(r))_\phi$ is a pre-quasiclosed (*sss*). □

Theorem 20. *The function $\phi(v) = [\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$ satisfies the Fatou property, if $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$, for every $v \in \ell(r)$.*

Proof. Assume that the setup is verified and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. As the space $(\ell(r))_\phi$ is a pre-quasiclosed space, then, $t \in (\ell(r))_\phi$. Hence, for all $v \in (\ell(r))_\phi$, we have

$$\begin{aligned}\phi(v - t) &= \left[\sum_{a=0}^{\infty}|v_a - t_a|^{r_a}\right]^{1/K} \leq \left[\sum_{a=0}^{\infty}\left|v_a - t_a^b\right|^{r_a}\right]^{1/K} \\ &+ \left[\sum_{a=0}^{\infty}\left|t_a^b - t_a\right|^{r_a}\right]^{1/K} \leq \sup_j \inf_{b\geq j}\phi\left(v - t^b\right).\end{aligned} \tag{16}$$

□

Theorem 21. *The function $\phi(v) = \sum_{a=0}^{\infty}|v_a|^{r_a}$ does not verify the Fatou property, for every $v \in \ell(r)$, if $(r_a) \in \ell_\infty$ and $r_a > 1$, for each $a \in \mathcal{N}$.*

Proof. Assume that the setting is verified and $\{t^b\} \subseteq (\ell(r))_\phi$ with $\lim_{b\longrightarrow\infty}\phi(t^b - t) = 0$. As the space $(\ell(r))_\phi$ is a pre-quasiclosed space, then, $t \in (\ell(r))_\phi$. Then, for all $v \in (\ell(r))_\phi$, one can see

$$\begin{aligned}\phi(v - t) &= \sum_{a=0}^{\infty}|v_a - t_a|^{r_a} \leq 2^{\sup_a r_a - 1}\left[\sum_{a=0}^{\infty}\left|v_a - t_a^b\right|^{r_a} + \sum_{a=0}^{\infty}\left|t_a^b - t_a\right|^{r_a}\right] \\ &\leq 2^{\sup_a r_a - 1}\sup_j \inf_{b\geq j}\phi\left(v - t^b\right).\end{aligned} \tag{17}$$

Therefore, ϕ does not verify the Fatou property. □

Similarly as Theorems 17 and 19 under the conditions $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$, it is easy to prove that the space $(\ell_{r(.)})_\psi$, which is studied in [33], is a pre-quasiclosed (sss), where $\psi(v) = \sum_{a=0}^{\infty} 1/r_a |v_a|^{r_a}$.

Theorem 22. *The function $\psi(v) = [\sum_{a=0}^{\infty} 1/r_a|v_a|^{r_a}]^{1/K}$ satisfies the Fatou property, if $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$, for every $v \in (\ell_{r(.)})_\psi$.*

Proof. Assume that the setup is verified and $\{t^b\} \subseteq (\ell_{r(.)})_\psi$ with $\lim_{b\longrightarrow\infty}\psi(t^b - t) = 0$. As the space $(\ell_{r(.)})_\psi$ is a pre-quasiclosed space, then, $t \in (\ell_{r(.)})_\psi$. Hence, for all $v \in (\ell_{r(.)})_\psi$, we have

$$\begin{aligned}\psi(v - t) &= \left[\sum_{a=0}^{\infty}\frac{1}{r_a}|v_a - t_a|^{r_a}\right]^{1/K} \leq \left[\sum_{a=0}^{\infty}\frac{1}{r_a}\left|v_a - t_a^b\right|^{r_a}\right]^{1/K} \\ &+ \left[\sum_{a=0}^{\infty}\frac{1}{r_a}\left|t_a^b - t_a\right|^{r_a}\right]^{1/K} \leq \sup_j \inf_{b\geq j}\psi\left(v - t^b\right).\end{aligned} \tag{18}$$

Theorem 23. *The function $\psi(v)=\sum_{a=0}^{\infty}1/r_a|v_a|^{r_a}$ does not verify the Fatou property, for every $v\in(\ell_{r(.)})_\psi$, if $(r_a)\in\ell_\infty$ and $r_a>1$, for each $a\in\mathcal{N}$.*

Proof. Assume that the setting is confirmed and $\{t^b\}\subseteq(\ell_{r(.)})_\psi$ with $\lim_{b\longrightarrow\infty}\psi(t^b-t)=0$. As the space $(\ell_{r(.)})_\psi$ is a pre-quasiclosed space, then, $t\in(\ell_{r(.)})_\psi$. Then, for all $v\in(\ell_{r(.)})_\psi$, one can see

$$\begin{aligned}\psi(v-t)&=\sum_{a=0}^{\infty}\frac{1}{r_a}|v_a-t_a|^{r_a}\le 2^{\sup_a r_a-1}\\&\cdot\left[\sum_{a=0}^{\infty}\frac{1}{r_a}\left|v_a-t_a^b\right|^{r_a}+\sum_{a=0}^{\infty}\frac{1}{r_a}\left|t_a^b-t_a\right|^{r_a}\right]\\&\le 2^{\sup_a r_a-1}\sup_j\inf_{b\ge j}\psi\left(v-t^b\right).\end{aligned}\tag{19}$$

Therefore, ψ does not verify the Fatou property. □

Example 24. The function $\phi(v)=\sum_{a\in\mathcal{N}}|v_a|^{3a+2/a+1}$ is a pre-quasinorm (not a quasinorm) on $\ell((3a+2/a+1)_{a=0}^{\infty})$.

Example 25. The function $\phi(v)=\sqrt[3]{\sum_{a\in\mathcal{N}}|v_a|^{3a+2/a+1}}$ is a pre-quasinorm (not a norm) on $\ell((3a+2/a+1)_{a=0}^{\infty})$.

Example 26. The function $\phi(v)=\sqrt[r]{\sum_{a\in\mathcal{N}}|v_a|^{r}}$ is a pre-quasinorm, quasi norm, and not a norm on ℓ_r, for $0<r<1$.

Example 27. For $(r_a)\in[1,\infty)^{\mathcal{N}}$, the function $\phi(v)=\inf\{\kappa>0:\sum_{a\in\mathcal{N}}|v_a/k|^{r_a}\le 1\}$ is a pre-quasinorm (a quasinorm and a norm) on $\ell(r)$.

4. Kannan Prequasi ϕ Contraction Mapping

In this section, we will define Kannan ϕ-Lipschitzian mapping in the pre-quasinormed (sss). We study the sufficient setting on $(\ell(r))_\phi$ constructed with definite pre-quasinorm so that there is one and only one fixed point of Kannan pre-quasinorm contraction mapping.

Definition 28. An operator $W:\mathfrak{A}_\phi\longrightarrow\mathfrak{A}_\phi$ is named a Kannan ϕ-Lipschitzian, if there is $\xi\ge 0$, such that

$$\phi(Wv-Wt)\le\xi(\phi(Wv-v)+\phi(Wt-t)),\tag{20}$$

for every $v,t\in\mathfrak{A}_\phi$. The operator W is named

(1) Kannan ϕ contraction, if $\xi\in[0,1/2)$

(2) Kannan ϕ nonexpansive, if $\xi=1/2$

An element $v\in\mathfrak{A}_\phi$ is called a fixed point of W, when $W(v)=v$.

In fact, the authors of reference [33] in Theorem 1 proved that the Kannan modular contraction mapping on a nonempty modular-closed subset of the modular space $(\ell_{r(.)})_\psi$, where $\psi(v)=\sum_{a=0}^{\infty}1/r_a|v_a|^{r_a}$ and $r_a\ge 1$, for all $a\in\mathcal{N}$, has a unique fixed point. They depended on their proof that the modular ψ has the Fatou property. But from Theorem 23, this result is incorrect. We have improved it in the next theorem.

Theorem 29. *Let $(r_a)_{a\in\mathcal{N}}\in\ell_\infty$ be an increase with $r_0>1$ and $W:(\ell(r))_\phi\longrightarrow(\ell(r))_\phi$ be Kannan ϕ contraction mapping, where $\phi(v)=[\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$, for every $v\in\ell(r)$, so W has a unique fixed point.*

Proof. Assume that the conditions are verified. For all $v\in\ell(r)$, then, $W^pv\in\ell(r)$. Since W is a Kannan ϕ contraction mapping, we have

$$\begin{aligned}\phi\left(W^{p+1}v-W^pv\right)&\le\xi\left(\phi\left(W^{p+1}v-W^pv\right)+\phi\left(W^pv-W^{p-1}v\right)\right)\\&\Rightarrow\phi\left(W^{p+1}v-W^pv\right)\le\frac{\xi}{1-\xi}\phi\left(W^pv-W^{p-1}v\right)\\&\le\left(\frac{\xi}{1-\xi}\right)^2\phi\left(W^{p-1}v-W^{p-2}v\right)\le\\&\le\left(\frac{\xi}{1-\xi}\right)^p\phi(Wv-v).\end{aligned}\tag{21}$$

Therefore, for every $p,q\in\mathcal{N}$ with $q>p$, then, we have

$$\begin{aligned}\phi(W^pv-W^qv)&\le\xi\left(\phi\left(W^pv-W^{p-1}v\right)+\phi\left(W^qv-W^{q-1}v\right)\right)\\&\le\xi\left(\left(\frac{\xi}{1-\xi}\right)^{p-1}+\left(\frac{\xi}{1-\xi}\right)^{q-1}\right)\phi(Wv-v).\end{aligned}\tag{22}$$

Hence, $\{W^pv\}$ is a Cauchy sequence in $(\ell(r))_\phi$. Since the space $(\ell(r))_\phi$ is a pre-quasi-Banach space, so, there is $t\in(\ell(r))_\phi$ so that $\lim_{p\longrightarrow\infty}W^pv=t$. To show that $Wt=t$, as ϕ has the Fatou property, we get

$$\begin{aligned}\phi(Wt-t)&\le\sup_i\inf_{p\ge i}\phi\left(W^{p+1}v-W^pv\right)\\&\le\sup_i\inf_{p\ge i}\left(\frac{\xi}{1-\xi}\right)^p\phi(Wv-v)=0.\end{aligned}\tag{23}$$

So, $Wt=t$. Then, t is a fixed point of W. To prove that the fixed point is unique, assume that we have two different fixed points $b,t\in(\ell(r))_\phi$ of W. Therefore, one can see

$$\phi(b-t)\le\phi(Wb-Wt)\le\xi(\phi(Wb-b)+\phi(Wt-t))=0.\tag{24}$$

Hence, $b=t$. □

Corollary 30. *Suppose that* $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ *is increasing with* $r_0 > 1$ *and* $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ *is Kannan* ϕ *contraction mapping, where* $\phi(v) = [\sum_{a=0}^{\infty}|v_a|^{r_a}]^{1/K}$, *for every* $v \in \ell(r)$, *then* W *has unique fixed point* b *with* $\phi(W^p v - b) \le \xi(\xi/1-\xi)^{p-1}$ $\phi(Wv - v)$.

Proof. Assume that the setup is verified. By Theorem 29, there is a unique fixed point b of W. Therefore, one can see

$$\begin{aligned}\phi(W^p v - b) &= \phi(W^p v - Wb) \le \xi\left(\phi\left(W^p v - W^{p-1} v\right) + \phi(Wb - b)\right)\\ &= \xi\left(\frac{\xi}{1-\xi}\right)^{p-1}\phi(Wv - v).\end{aligned} \tag{25}$$

□

Definition 31. Let $\mathfrak{A}_\phi$ be a pre-quasinormed (sss), $W : \mathfrak{A}_\phi \longrightarrow \mathfrak{A}_\phi$ and $b \in \mathfrak{A}_\phi$. The operator W is named ϕ sequentially continuous at b, if and only if $\lim_{a\longrightarrow\infty}\phi(v_a - b) = 0$, then $\lim_{a\longrightarrow\infty}\phi(Wv_a - Wb) = 0$.

Theorem 32. *If* $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ *is increasing with* $r_0 > 1$ *and* $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$, *where* $\phi(v) = \sum_{a=0}^{\infty}|v_a|^{r_a}$, *for every* $v \in \ell(r)$, *the point* $g \in (\ell(r))_\phi$ *is the only fixed point of* W, *if the next settings are verified:*

(a) *W is Kannan ϕ contraction mapping*

(b) *W is ϕ sequentially continuous at $g \in (\ell(r))_\phi$*

(c) *We have $v \in (\ell(r))_\phi$ such that the sequence of iterates $\{W^p v\}$ has a subsequence $\{W^{p_i} v\}$ converges to g*

Proof. If the settings are satisfied, let g be not a fixed point of W, and then, $Wg \ne g$. By the setups (b) and (c), one can see

$$\lim_{p_i\longrightarrow\infty}\phi(W^{p_i}v - g) = 0 \text{ and } \lim_{p_i\longrightarrow\infty}\phi\left(W^{p_i+1}v - Wg\right) = 0. \tag{26}$$

Since the operator W is Kannan ϕ contraction, we have

$$\begin{aligned}0 < \phi(Wg - g) &= \phi\left(\left(Wg - W^{p_i+1}v\right) + \left(W^{p_i}v - g\right) + \left(W^{p_i+1}v - W^{p_i}v\right)\right)\\ &\le 2^{2\sup_i r_i - 2}\phi\left(W^{p_i+1}v - Wg\right) + 2^{2\sup_i r_i - 2}\phi(W^{p_i}v - g)\\ &\quad + 2^{\sup_i r_i - 1}\xi\left(\frac{\xi}{1-\xi}\right)^{p_i-1}\phi(Wv - v).\end{aligned} \tag{27}$$

Since $p_i \longrightarrow \infty$, we get a contradiction. Hence, g is a fixed point of W. To show that the fixed point g is unique, suppose that we have two different fixed points $g, b \in (\ell(r))_\phi$ of W. Therefore, we have

$$\phi(g - b) \le \phi(Wg - Wb) \le \xi(\phi(Wg - g) + \phi(Wb - b)) = 0. \tag{28}$$

So, $g = b$. □

Example 33. Let $W : (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$, where $\phi(v) = \sqrt{\sum_{a\in\mathcal{N}}|v_a|^{2a+3/a+2}}$, for all $v \in \ell((2a + 3/a + 2)_{a=0}^{\infty})$ and

$$W(v) = \begin{cases}\dfrac{v}{4}, & \phi(v) \in [0, 1),\\[2mm] \dfrac{v}{5}, & \phi(v) \in [1, \infty).\end{cases} \tag{29}$$

Since for all $v_1, v_2 \in (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [0, 1)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{4} - \frac{v_2}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(\phi\left(\frac{3v_1}{4}\right) + \phi\left(\frac{3v_2}{4}\right)\right)\\ &= \frac{1}{\sqrt[4]{27}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{30}$$

For all $v_1, v_2 \in (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [1, \infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{5} - \frac{v_2}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(\phi\left(\frac{4v_1}{5}\right) + \phi\left(\frac{4v_2}{5}\right)\right)\\ &= \frac{1}{\sqrt[4]{64}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{31}$$

For all $v_1, v_2 \in (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1) \in [0, 1)$ and $\phi(v_2) \in [1, \infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{4} - \frac{v_2}{5}\right) \le \frac{1}{\sqrt[4]{27}}\phi\left(\frac{3v_1}{4}\right) + \frac{1}{\sqrt[4]{64}}\phi\left(\frac{4v_2}{5}\right)\\ &\le \frac{1}{\sqrt[4]{27}}\left(\phi\left(\frac{3v_1}{4}\right) + \phi\left(\frac{4v_2}{5}\right)\right)\\ &= \frac{1}{\sqrt[4]{27}}(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)).\end{aligned} \tag{32}$$

Therefore, the map W is Kannan ϕ contraction mapping, since ϕ satisfies the Fatou property. By Theorem 29, the map W has a unique fixed point $\theta \in (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$.

Let $\{v^{(n)}\} \subseteq (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$ be such that $\lim_{n\longrightarrow\infty}\phi(v^{(n)} - v^{(0)}) = 0$, where $v^{(0)} \in (\ell((2a + 3/a + 2)_{a=0}^{\infty}))_\phi$ with $\phi(v^{(0)}) = 1$. Since the pre-quasinorm ϕ is continuous, we have

$$\lim_{n\longrightarrow\infty}\phi\left(Wv^{(n)} - Wv^{(0)}\right) = \lim_{n\longrightarrow\infty}\phi\left(\frac{v^{(n)}}{4} - \frac{v^{(0)}}{5}\right) = \phi\left(\frac{v^{(0)}}{20}\right) > 0. \quad (33)$$

Hence, W is not ϕ sequentially continuous at $v^{(0)}$. So, the map W is not continuous at $v^{(0)}$.

If $\phi(v) = \sum_{a\in\mathcal{N}}|v_a|^{2a+3/a+2}$, for all $v \in \ell((2a+3/a+2)_{a=0}^{\infty})$. Since for all $v_1, v_2 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [0,1)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{4} - \frac{v_2}{4}\right) \le \frac{2}{\sqrt{27}}\left(\phi\left(\frac{3v_1}{4}\right) + \phi\left(\frac{3v_2}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)\right).\end{aligned} \quad (34)$$

For all $v_1, v_2 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1), \phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{5} - \frac{v_2}{5}\right) \le \frac{1}{4}\left(\phi\left(\frac{4v_1}{5}\right) + \phi\left(\frac{4v_2}{5}\right)\right) \\ &= \frac{1}{4}\left(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)\right).\end{aligned} \quad (35)$$

For all $v_1, v_2 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $\phi(v_1) \in [0,1)$ and $\phi(v_2) \in [1,\infty)$, we have

$$\begin{aligned}\phi(Wv_1 - Wv_2) &= \phi\left(\frac{v_1}{4} - \frac{v_2}{5}\right) \le \frac{2}{\sqrt{27}}\phi\left(\frac{3v_1}{4}\right) + \frac{1}{4}\phi\left(\frac{4v_2}{5}\right) \\ &\le \frac{2}{\sqrt{27}}\left(\phi\left(\frac{3v_1}{4}\right) + \phi\left(\frac{4v_2}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(\phi(Wv_1 - v_1) + \phi(Wv_2 - v_2)\right).\end{aligned} \quad (36)$$

Therefore, the map W is Kannan ϕ contraction mapping and

$$W^p(v) = \begin{cases} \dfrac{v}{4^p}, & \phi(v) \in [0,1), \\ \dfrac{v}{5^p}, & \phi(v) \in [1,\infty). \end{cases} \quad (37)$$

It is clear that W is ϕ sequentially continuous at $\theta \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ and $\{W^p v\}$ has a subsequence $\{W^{p_i} v\}$ that converges to θ. By Theorem 32, the point $\theta \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ is the only fixed point of W.

Example 34. Let $W(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi \longrightarrow (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$, where $\phi(v) = \sum_{a\in\mathcal{N}}|v_a|^{2a+3/a+2}$, for all $v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ and

$$W(v) = \begin{cases} \dfrac{1}{4}(e_1 + v), & v_0 \in \left(-\infty, \dfrac{1}{3}\right), \\ \dfrac{1}{3}e_1, & v_0 = \dfrac{1}{3}, \\ \dfrac{1}{4}e_1, & v_0 \in \left(\dfrac{1}{3}, \infty\right). \end{cases} \quad (38)$$

Since for all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $v_0, t_0 \in (-\infty, 1/3)$, we have

$$\begin{aligned}\phi(Wv - Wt) &= \phi\left(\frac{1}{4}(v_0 - t_0, v_1 - t_1, v_2 - t_2, \cdots)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\phi\left(\frac{3v}{4}\right) + \phi\left(\frac{3t}{4}\right)\right) \\ &\le \frac{2}{\sqrt{27}}\left(\phi(Wv - v) + \phi(Wt - t)\right).\end{aligned} \quad (39)$$

For all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $v_0, t_0 \in (1/3, \infty)$, then, for any $\varepsilon > 0$, we have

$$\phi(Wv - Wt) = 0 \le \varepsilon(\phi(Wv - v) + \phi(Wt - t)). \quad (40)$$

For all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $v_0 \in (-\infty, 1/3)$ and $t_0 \in (1/3, \infty)$, we have

$$\begin{aligned}\phi(Wv - Wt) &= \phi\left(\frac{v}{4}\right) \le \frac{1}{\sqrt{27}}\phi\left(\frac{3v}{4}\right) = \frac{1}{\sqrt{27}}\phi(Wv - v) \\ &\le \frac{1}{\sqrt{27}}\left(\phi(Wv - v) + \phi(Wt - t)\right).\end{aligned} \quad (41)$$

Therefore, the map W is Kannan ϕ contraction mapping. It is clear that W is ϕ sequentially continuous at $(1/3)e_1 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ and there is $v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $v_0 \in (-\infty, 1/3)$ such that the sequence of iterates $\{W^p v\} = \{\sum_{n=1}^{p}(1/4^n)e_1 + (1/4^p)v\}$ has a subsequence $\{W^{p_i} v\} = \{\sum_{n=1}^{p_i}(1/4^n)e_1 + (1/4^{p_i})v\}$ converges to $(1/3)e_1$. By Theorem 32, the map W has one fixed point $(1/3)e_1 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$. Note that W is not continuous at $(1/3)e_1 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$.

If $\phi(v) = \sqrt{\sum_{a\in\mathcal{N}}|v_a|^{2a+3/a+2}}$, for all $v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$. Since for all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$ with $v_0, t_0 \in (-\infty, 1/3)$, we have

$$\begin{aligned}\phi(Wv - Wt) &= \phi\left(\frac{1}{4}((v_0 - t_0, v_1 - t_1, v_2 - t_2, \cdots)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\phi\left(\frac{3v}{4}\right) + \phi\left(\frac{3t}{4}\right)\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(\phi(Wv - v) + \phi(Wt - t)\right).\end{aligned} \quad (42)$$

For all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_{\phi}$ with $v_0, t_0 \in (1/3, \infty)$, then, for any $\varepsilon > 0$, we have

$$\phi(Wv - Wt) = 0 \leq \varepsilon(\phi(Wv - v) + \phi(Wt - t)). \tag{43}$$

For all $v, t \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_{\phi}$ with $v_0 \in (-\infty, 1/3)$ and $t_0 \in (1/3, \infty)$, we have

$$\begin{aligned}\phi(Wv - Wt) &= \phi\left(\frac{v}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\phi\left(\frac{3v}{4}\right) = \frac{1}{\sqrt[4]{27}}\phi(Wv - v)\\ &\leq \frac{1}{\sqrt[4]{27}}(\phi(Wv - v) + \phi(Wt - t)).\end{aligned} \tag{44}$$

Therefore, the map W is Kannan ϕ contraction mapping. Since ϕ satisfies the Fatou property. By Theorem 29, the map W has a unique fixed point $(1/3)e_1 \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_{\phi}$.

5. Pre-Quasinormed Uniform Convexity

In this part, we investigate the uniform convexity (UUC 2) defined in [35] of the pre-quasinormed (sss) $(\ell(r))_{\phi}$.

Definition 35 [10, 34]. We define the following uniform convexity-type behavior of the pre-quasinorm ϕ:

(1) Assume that $a > 0$ and $b > 0$ [35]. Indicate that

$$\mathbb{H}_1(a, b) = \left\{(v, t): v, t \in \mathfrak{A}_{\phi}, \phi(v) \leq a, \phi(t) \leq b, \phi(v - t) \geq ab\right\}. \tag{45}$$

When $\mathbb{H}_1(a, b) \neq \varnothing$, we put

$$\mathrm{H}_1(a, b) = \inf\left\{1 - \frac{1}{a}\phi\left(\frac{v + t}{2}\right): (v, t) \in \mathbb{H}_1(a, b)\right\}. \tag{46}$$

When $\mathbb{H}_1(a, b) = \varnothing$, we put $\mathrm{H}_1(a, b) = 1$. The function ϕ investigates the uniform convexity (UC) if for every $a > 0$ and $b > 0$, we have $\mathrm{H}_1(a, b) > 0$. Note that for all $a > 0$, then, $\mathbb{H}_1(a, b) \neq \varnothing$, for very small $b > 0$

(2) The function ϕ provides (UUC) if for every $p \geq 0$ and $b > 0$, there is $\beta_1(p, b)$ based on p and $b > 0$ so that [36]

$$\mathrm{H}_1(a, b) > \beta_1(p, b) > 0, \quad \text{for } a > p \tag{47}$$

(3) Suppose that $a > 0$ and $b > 0$. Indicate [36]

$$\mathbb{H}_2(a, b) = \left\{(v, t): v, t \in \mathfrak{A}_{\phi}, \phi(v) \leq a, \phi(t) \leq a, \phi\left(\frac{v - t}{2}\right) \geq ab\right\}. \tag{48}$$

When $\mathbb{H}_2(a, b) \neq \varnothing$, we put

$$\mathrm{H}_2(a, b) = \inf\left\{1 - \frac{1}{a}\phi\left(\frac{v + t}{2}\right): (v, t)(v, t) \in \mathbb{H}_2(a, b)\right\}. \tag{49}$$

When $\mathbb{H}_2(a, b) = \varnothing$, we put $\mathrm{H}_2(a, b) = 1$. The function ϕ supports (UC 2) if for all $a > 0$ and $b > 0$, we have $\mathrm{H}_2(a, b) > 0$. Observe that for each $a > 0$, $\mathbb{H}_2(a, b) \neq \varnothing$, for very small $b > 0$

(4) The function ϕ satisfies (UUC 2) if for every $p \geq 0$ and $b > 0$, there is $\beta_2(p, b)$ based on p and $b > 0$ so that [36]

$$\mathrm{H}_2(a, b) > \beta_2(p, b) > 0, \quad \text{for } a > p. \tag{50}$$

(5) The function ϕ is strictly convex (SC), if for each v, $t \in \mathfrak{A}_{\phi}$ so that $\phi(v) = \phi(t)$ and $\phi(v + t/2) = (\phi(v) + \phi(t))/2$, we get $v = t$ [35]

Here and after, we will need the following notation: $\phi_U(v) = [\sum_{m \in U}|v_m|^{r_m}]^{1/K}$, for each $U \subset \mathcal{N}$ and $v \in (\ell(r))_{\phi}$. When $U = \varnothing$, we set $\phi_U(v) = 0$.

Theorem 36. *The pre-quasinorm ϕ on $\ell(r)$ is (UUC 2), where $\phi(v) = [[\sum_{d=0}^{\infty}|v_d|^{r_d}]^{1/K}$, for every $v \in \ell(r)$, if $(r_a)_{a \in \mathcal{N}} \in \ell_{\infty}$ is an increasing with $r_0 > 1$.*

Proof. Supposing that the setting is verified, $a > p \geq 0$ and $b > 0$. Let $v, t \in \ell(r)$ such that

$$\begin{aligned}\phi(v) &\leq a,\\ \phi(t) &\leq a,\\ \phi\left(\frac{v - t}{2}\right) &\geq ab.\end{aligned} \tag{51}$$

From the definition of ϕ, one has

$$\begin{aligned}ab \leq \phi\left(\frac{v - t}{2}\right) &= \left[\sum_{d=0}^{\infty}\left|\frac{v_d - t_d}{2}\right|^{r_d}\right]^{1/K} \leq \left[2^{-r_0}\sum_{d=0}^{\infty}|v_d - t_d|^{r_d}\right]^{1/K}\\ &\leq 2^{\frac{-r_0}{K}}\left(\left[\sum_{d=0}^{\infty}|v_d|^{r_d}\right]^{\frac{1}{K}} + \left[\sum_{d=0}^{\infty}|t_d|^{r_d}\right]^{\frac{1}{K}}\right)\\ &= 2^{\frac{-r_0}{K}}(\phi(v) + \phi(t)) \leq 2a.\end{aligned} \tag{52}$$

This implies that $b \leq 2$. So, set $P = \{d \in \mathcal{N} : r_d \geq 2\}$ and $Q = \{d \in \mathcal{N} : 1 < r_d < 2\} = \mathcal{N} \setminus P$. For every $w \in \ell(r)$, we have

$\phi^K(w)=\phi_P^K(w)+\phi_Q^K(w)$. By using the setting, we obtain $\phi_P((v-t)/2)\geq ab/2$ or $\phi_Q((v-t)/2)\geq ab/2$. Suppose first that $\phi_P((v-t)/2)\geq ab/2$. Using Lemma 15, one can see

$$\phi_P^K\left(\frac{v+t}{2}\right)+\phi_P^K\left(\frac{v-t}{2}\right)\leq\frac{\phi_P^K(v)+\phi_P^K(t)}{2}. \quad (53)$$

This gives

$$\phi_P^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_P^K(v)+\phi_P^K(t)}{2}-\left(\frac{ab}{2}\right)^K. \quad (54)$$

Since

$$\phi_Q^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_Q^K(v)+\phi_Q^K(t)}{2}, \quad (55)$$

one has

$$\phi^K\left(\frac{v+t}{2}\right)\leq\frac{\phi^K(v)+\phi^K(t)}{2}-\left(\frac{ab}{2}\right)^K\leq a^K\left(1-\left(\frac{b}{2}\right)^K\right). \quad (56)$$

This implies

$$\phi\left(\frac{v+t}{2}\right)\leq a\left(1-\left(\frac{b}{2}\right)^K\right)^{1/K}. \quad (57)$$

Next, assume that $\phi_Q((v-t)/2)\geq ab/2$. Put $B=b/4$,

$$Q_1=\{d\in Q:|v_d-t_d|\leq B(|v_d|+|t_d|)\},\qquad Q_2=Q\setminus Q_1. \quad (58)$$

As the power function is convex and $B\leq 1$. Hence,

$$\begin{aligned}\phi_{Q_1}^K\left(\frac{v-t}{2}\right)&\leq\sum_{d\in Q_1}B^{r_d}\left|\frac{|v_d|+|t_d|}{2}\right|^{r_d}\\&\leq\left(\frac{B}{2}\right)^{r_0}\left(\phi_{Q_1}^K(v)+\phi_{Q_1}^K(t)\right)\\&\leq\frac{B}{2}\left(\phi_Q^K(v)+\phi_Q^K(t)\right)\leq Ba^K.\end{aligned} \quad (59)$$

As $\phi_Q((v-t)/2)\geq ab/2$, we have

$$\phi_{Q_2}^K\left(\frac{v-t}{2}\right)=\phi_Q^K\left(\frac{v-t}{2}\right)-\phi_{Q_1}^K\left(\frac{v-t}{2}\right)\geq a^K\left(\left(\frac{b}{2}\right)^K-\left(\frac{b}{4}\right)^K\right). \quad (60)$$

For all $d\in Q_2$, one can see

$$r_0-1\leq r_d(r_d-1),\qquad B\leq B^{2-r_d}\leq\left|\frac{v_d-t_d}{|v_d|+|t_d|}\right|^{2-r_d}. \quad (61)$$

By Lemma 15, we have

$$\left|\frac{v_d+t_d}{2}\right|^{r_d}+\frac{(r_0-1)B}{2}\left|\frac{v_d-t_d}{2}\right|^{r_d}\leq\frac{1}{2}\left(|v_d|^{r_d}+|t_d|^{r_d}\right). \quad (62)$$

So,

$$\phi_{Q_2}^K\left(\frac{v+t}{2}\right)+\frac{(r_0-1)B}{2}\phi_{Q_2}^K\left(\frac{v-t}{2}\right)\leq\frac{\phi_{Q_2}^K(v)+\phi_{Q_2}^K(t)}{2}, \quad (63)$$

this gives

$$\phi_{Q_2}^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_{Q_2}^K(v)+\phi_{Q_2}^K(t)}{2}-\frac{(r_0-1)}{2}\left(\frac{b}{4}\right)^{1+K}a^K\left(2^K-1\right). \quad (64)$$

As

$$\phi_{Q_1}^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_{Q_1}^K(v)+\phi_{Q_1}^K(t)}{2}, \quad (65)$$

we have

$$\phi_Q^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_Q^K(v)+\phi_Q^K(t)}{2}-\frac{(r_0-1)}{2}\left(\frac{b}{4}\right)^{1+K}a^K\left(2^K-1\right). \quad (66)$$

As

$$\phi_P^K\left(\frac{v+t}{2}\right)\leq\frac{\phi_P^K(v)+\phi_P^K(t)}{2}, \quad (67)$$

we get

$$\phi\left(\frac{v+t}{2}\right)\leq a\left[1-\frac{(r_0-1)}{2}\left(\frac{b}{4}\right)^{1+K}\left(2^K-1\right)\right]^{1/K}. \quad (68)$$

Obviously,

$$1<r_0\leq K<2^K\Rightarrow 0<\frac{r_0-1}{2^K-1}<1. \quad (69)$$

If we set

$$\beta_2(p,b)=\min\left(1-\left(1-\left(\frac{b}{2}\right)^K\right)^{1/K},1-\left[1-\frac{(r_0-1)}{2}\left(\frac{b}{4}\right)^{1+K}\left(2^K-1\right)\right]^{1/K}\right). \quad (70)$$

Hence, one has $H_2(a,b) > \beta_2(p,b) > 0$ and we deduce that ϕ is (UUC 2). □

In fact, the authors of reference [37] in Proposition 3.5 proved that if $\inf_{d\in\mathcal{N}} r_d > 1$, then, the modular space $(\ell_{r(.)})_{\psi}$, where $\psi(v)=\sum_{d=0}^{\infty}(1/r_d)|v_d|^{r_d}$, has the property ($R$). They depended on their proof that the modular ψ has the Fatou property. But from Theorem 23, this result is incorrect. Consequently, all the related results in the two references [33, 37] are incorrect. In this part, we investigate the property (R) of the pre-quasinormed (sss) $(\ell(r))_{\phi}$.

Theorem 37. *Let $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ be an increase with $r_0 > 1$, then*

(1) The space $(\ell(r))_{\phi}$ is a pre-quasi-Banach (sss), where $\phi(v)=[\sum_{j=0}^{\infty}|v_j|^{r_j}]^{1/K}$, for every $v\in\ell(r)$

(2) Suppose that Λ is a nonempty ϕ closed and ϕ convex subset of $(\ell(r))_{\phi}$. Assume that $v\in(\ell(r))_{\phi}$ so that

$$d_{\phi}(v,\Lambda)=\inf\{\phi((v-t)v-t):\ t\in\Lambda\}<\infty. \tag{71}$$

Then we have a unique $\lambda\in\Lambda$ so that $d_{\phi}(v,\Lambda)=\phi(v-\lambda)$

(3) $(\ell(r))_{\phi}$ verifies the property (R), i.e., for each decreasing sequence $\{\Lambda_j\}_{j\in\mathcal{N}}$ of ϕ closed and ϕ convex nonempty subsets of $(\ell(r))_{\phi}$ such that $\sup_{j\in\mathcal{N}} d_{\phi}(v,\Lambda_j)<\infty$, for some $v\in(\ell(r))_{\phi}$, so we have $\bigcap_{j\in\mathcal{N}}\Lambda_j\neq\varnothing$

Proof. Assume that the setting is verified. The proof of (100) comes from Theorem 18. To prove (101), suppose that $v\notin\Lambda$ as Λ is ϕ closed. Therefore, we have $A:=d_{\phi}(v,\Lambda)>0$. So, for every $p\in\mathcal{N}$, there is $t_p\in\Lambda$ such that $\phi(v-t_p)<A(1+(1/p))$. Assume that $\{t_p/2\}$ is not ϕ Cauchy. So there is a subsequence $\{t_{f(p)}/2\}$ and $b_0>0$ such that $\phi((t_{f(p)}-t_{f(q)})/2)\geq b_0$, for all $p>q\geq 0$. More, we have $H_2(A(1+(1/p)),b_0/2A)>\xi:=\beta_2(A(1+(1/p)),b_0/2A)>0$, for each $p\in\mathcal{N}$. As

$$\begin{aligned}\max\left(\phi\left(v-t_{f(p)}\right),\phi\left(v-t_{f(q)(q)}\right)\right)&\leq A\left(1+\frac{1}{f(q)}\right),\\ \phi\left(\frac{t_{f(p)}-t_{f(q)}}{2}\right)\geq b_0&\geq A\left(1+\frac{1}{f(q)}\right)\frac{b_0}{2A}.\end{aligned} \tag{72}$$

For all $p>q\geq 0$, one has

$$\phi\left(v-\frac{t_{f(p)}+t_{f(q)}}{2}\right)\leq A\left(1+\frac{1}{f(q)}\right)(1-\xi). \tag{73}$$

Hence,

$$A=d_{\phi}(v,\Lambda)\leq A\left(1+\frac{1}{f(q)}\right)(1-\xi), \tag{74}$$

for every $q\in\mathcal{N}$. If we set $q\longrightarrow\infty$, we have

$$0<A\leq A\left(1+\frac{1}{f(q)}\right)(1-\xi)<A. \tag{75}$$

This implies a contradiction. Therefore, $\{t_p/2\}$ is ϕ Cauchy. Since $(\ell(r))_{\phi}$ is ϕ complete, hence, $\{t_p/2\}\phi$ converges to some t. For every $q\in\mathcal{N}$, we get the sequence $\{(t_p+t_q)/2\}\phi$ converges to $t+(t_q/2)$. As Λ is ϕ closed and ϕ convex, one has $t+(t_q/2)\in\Lambda$. Clearly, $t+(t_q/2)\phi$ converges to $2t$; this implies that $2t\in\Lambda$. By setting $\lambda=2t$ and using Theorem 20, as ϕ verifies the Fatou property, we have

$$\begin{aligned}d_{\phi}(v,\Lambda)&\leq\phi(v-\lambda)\leq\sup_i\inf_{q\geq i}\phi\left(v-\left(t+\frac{t_q}{2}\right)\right)\\&\leq\sup_i\inf_{q\geq i}\sup_i\inf_{p\geq i}\phi\left(v-\frac{t_p+t_q}{2}\right)\\&\leq\frac{1}{2}\sup_i\inf_{q\geq i}\sup_i\inf_{p\geq i}\left[\phi\left(v-t_p\right)+\phi\left(v-t_q\right)\right]\\&=d_{\phi}(v,\Lambda).\end{aligned} \tag{76}$$

Hence, $\phi(v-\lambda)=d_{\phi}(v,\Lambda)$. As the function ϕ is (UUC 2), then, it is (SC), which gives the uniqueness of λ. To prove (104), suppose $v\notin\Lambda_{p_0}$, for some $p_0\in\mathcal{N}$. As $(d_{\phi}(v,\Lambda_p))_{p\in\mathcal{N}}\in\ell_\infty$ is increasing, set $\lim_{p\longrightarrow\infty}d_{\phi}(v,\Lambda_p)=A$, if $A>0$. Otherwise $v\in\Lambda_p$, for every $p\in\mathcal{N}$. From (2), there is one point $t_p\in\Lambda_p$ such that $d_{\phi}(v,\Lambda_p)=\phi(v-t_p)$, for all $p\in\mathcal{N}$. A consistent proof will show that $\{t_p/2\}\phi$ converges to some $t\in(\ell(r))_{\phi}$. Since $\{\Lambda_p\}$ are ϕ convex, decreasing, and ϕ closed, we have $2t\in\cap_{p\in\mathcal{N}}\Lambda_p$. □

In this part, we investigate the ϕ normal structure property of the pre-quasinormed (sss) $(\ell(r))_{\phi}$.

Definition 38. $(\ell(r))_{\phi}$ verifies the ϕ normal structure property if for every nonempty ϕ bounded, ϕ convex, and ϕ closed subset Λ of $(\ell(r))_{\phi}$ that are not decreased to one point, there is $v\in\Lambda$ such that

$$\sup_{t\in\Lambda}\phi(v-t)<\delta_{\phi}(\Lambda):=\sup\{\phi(v-t)(v-t):\ v,t\in\Lambda\}<\infty. \tag{77}$$

Theorem 39. *Let $(r_a)_{a\in\mathcal{N}}\in\ell_\infty$ be an increase with $r_0>1$; then, $(\ell(r))_{\phi}$ has the ϕ normal structure property, where $\phi(v)=[\sum_{p=0}^{\infty}|v_p|^{r_p}]^{1/K}$, for all $v\in\ell(r)$.*

Proof. Suppose that the setting is verified. Theorem 36 implies that ϕ is (UUC 2). Assume that Λ is a ϕ bounded, ϕ

convex, and ϕ closed subset of $(\ell(r))_\phi$ that are not decreased to one point. Hence, $\delta_\phi(\Lambda) > 0$. Set $A = \delta_\phi(\Lambda)$. Suppose that $v, t \in \Lambda$ such that $v \neq t$. Hence $\phi((v-t)/2) = b > 0$. For every $\lambda \in \Lambda$, we have $\phi(v-\lambda) \le A$ and $\phi(t-\lambda) \le A$. Since Λ is ϕ convex, we have $(v+t)/2 \in \Lambda$. So,

$$\phi\left(\frac{v+t}{2} - \lambda\right) = \phi\left(\frac{(v-\lambda)+(t-\lambda)}{2}\right) \le A\left(1 - H_2\left(A, \frac{b}{A}\right)\right), \tag{78}$$

for every $\lambda \in \Lambda$. Hence,

$$\sup_{\lambda\in\Lambda}\phi\left(\frac{v+t}{2} - \lambda\right) \le A\left(1 - H_2\left(A, \frac{b}{A}\right)\right) < A = \delta_\phi(\Lambda). \tag{79}$$

6. Kannan ϕ Nonexpansive Mapping on $(\ell(r))_\phi$

We examine here the sufficient conditions on the pre-quasi normed (sss) $(\ell(r))_\phi$ so that the Kannan pre-quasinorm nonexpansive mapping on it has a fixed point.

Lemma 40. *Let the pre-quasinormed (sss) $(\ell(r))_\phi$ verify the (R) property and the ϕ quasinormal property. Assume that Λ is a nonempty ϕ bounded, ϕ convex, and ϕ closed subset of $(\ell(r))_\phi$. Suppose that $W : \Lambda \longrightarrow \Lambda$ is a Kannan ϕ nonexpansive mapping. For $a > 0$, assume that $G_a = \{v \in \Lambda : \phi(v - W(v)) \le a\} \neq \varnothing$. Set*

$$\Lambda_a = \bigcap\left\{\mathscr{B}_\phi(p,q) \colon W(G_a) \subset \mathscr{B}_\phi(p,q)\right\} \cap \Lambda. \tag{80}$$

Then, $\Lambda_a \neq \varnothing$, ϕ convex, and ϕ closed subset of Λ and $W(\Lambda_a) \subset \Lambda_a \subset G_a$ and $\delta_\phi(\Lambda_a) \le a$.

Proof. As $W(G_a) \subset \Lambda_a$, this implies that $\Lambda_a \neq \varnothing$. Since the ϕ balls are ϕ convex and ϕ closed, so Λ_a is a ϕ closed and ϕ convex subset of Λ. To prove that $\Lambda_a \subset G_a$, suppose that $v \in \Lambda_a$. If $\phi(v - W(v)) = 0$, we have $v \in G_a$. Otherwise, let $\phi(v - W(v)) > 0$. Set

$$p = \sup\{\phi(W(w) - W(v)) \colon w \in G_a\}. \tag{81}$$

By using the definition of p, then, $W(G_a) \subset \mathscr{B}_\phi(W(v), p)$. Hence, $\Lambda_a \subset \mathscr{B}_\phi(W(v), p)$; this implies that $\phi(v - W(v)) \le p$. Suppose that $b > 0$. Hence, there is $w \in G_a$ such that $p - b \le \phi(W(w) - W(v))$. Hence,

$$\begin{aligned}\phi(v - W(v)) - b &\le p - b \le \phi(W(w) - W(v)) \\ &\le \frac{1}{2}(\phi(v - W(v))(v - W(v)) + \phi((w - W(w))) \\ &\le \frac{1}{2}(\phi(v - W(v)) + a).\end{aligned} \tag{82}$$

Since b is an arbitrary positive, we have $\phi(v - W(v)) \le a$, so one has $v \in G_a$. As $W(G_a) \subset \Lambda_a$, we have $W(\Lambda_a) \subset W(G_a) \subset \Lambda_a$; this investigates that Λ_a is W invariant, consequent to prove that $\delta_\phi(\Lambda_a) \le a$. As

$$\phi(W(v) - W(t)) \le \frac{1}{2}(\phi(v - W(v))) + \phi(t - W(t)), \tag{83}$$

for each $v, t \in G_a$. Assume that $v \in G_a$. So $W(G_a) \subset \mathscr{B}_\phi(W(v), a)$. The definition of Λ_a implies $\Lambda_a \subset \mathscr{B}_\phi(W(v), a)$. So, $W(v) \in \bigcap_{t\in\Lambda_a}\mathscr{B}_\phi(t, a)$. Hence, we have $\phi(t - w) \le a$; for every $t, w \in \Lambda_a$, this gives $\delta_\phi(\Lambda_a) \le a$. This finishes the proof. □

Theorem 41. *Picking up the pre-quasinormed (sss) $(\ell(r))_\phi$ verifies the ϕ quasinormal property and the (R) property. Assume that Λ is a nonempty, ϕ convex, ϕ closed, and ϕ bounded subset of $(\ell(r))_\phi$. Suppose that $W : \Lambda \longrightarrow \Lambda$ is a Kannan ϕ nonexpansive mapping. Then, W has a fixed point.*

Proof. Set $a_0 = \inf\{\phi(v - W(v)) \colon v \in \Lambda\}$ and $a_p = a_0 + 1/p$, for all $p \ge 1$. From the definition of a_0, we have $G_{a_p} = \{v \in \Lambda : \phi(v - W(v)) \le a_p\} \neq \varnothing$, for all $p \ge 1$. Let Λ_{a_p} be studied as in Lemma 40. Obviously, $\{\Lambda_{a_p}\}$ is a decreasing sequence of nonempty ϕ bounded, ϕ closed, and ϕ convex subsets of Λ. The property (R) gives that $\Lambda_\infty = \bigcap_{p\ge1}\Lambda_{a_p} \neq \varnothing$. Supposing that $v \in \Lambda_\infty$, we get $\phi(v - W(v)) \le a_p$, for each $p \ge 1$. Assuming that $p \longrightarrow \infty$, one has $\phi(v - W(v)) \le a_0$; this implies that $\phi(v - W(v)) = a_0$. Hence, $G_{a_0} \neq \varnothing$. We obtain $a_0 = 0$. Otherwise, $a_0 > 0$; this implies that W fails to have a fixed point. Suppose Λ_{a_0} as defined in Lemma 40. Since W fails to have a fixed point and Λ_{a_0} is W invariant, hence, Λ_{a_0} has more than one point; this gives, $\delta_\phi(\Lambda_{a_0}) > 0$. By the ϕ quasinormal property, there is $v \in \Lambda_{a_0}$ such that

$$\phi(v - t) < \delta_\phi(\Lambda_{a_0}) \le a_0, \tag{84}$$

for each $t \in \Lambda_{a_0}$. By Lemma 40, one has $\Lambda_{a_0} \subset G_{a_0}$. By definition of Λ_{a_0}, then, $W(v) \in G_{a_0} \subset \Lambda_{a_0}$. Evidently, this implies that

$$\phi(v - W(v)) < \delta_\phi(\Lambda_{a_0}) \le a_0. \tag{85}$$

This contradicts the definition of a_0. Hence, $a_0 = 0$ which gives that any point in G_{a_0} is a fixed point of W, i.e., W has a fixed point in Λ. □

Using Theorems 37, 39, and 41, we have the following corollary:

Corollary 42. *Let $(r_a)_{a\in\mathscr{N}} \in \ell_\infty$ be an increase with $r_0 > 1$. Suppose that Λ is a nonempty, ϕ convex, ϕ closed, and ϕ bounded subset of $(\ell(r))_\phi$, where $\phi(v) = [[\sum_{p=0}^{\infty}|v_p|^{r_p}]^{1/k}$, for all $v \in \ell(r)$. Assume that $W : \Lambda \longrightarrow \Lambda$ is a Kannan ϕ nonexpansive mapping. Then, W has a fixed point.*

Example 43. Let $W : \Lambda \longrightarrow \Lambda$ with

$$W(v) = \begin{cases} \frac{v}{4}, & \phi(v) \in [0,1), \\ \frac{v}{5}, & \phi(v) \in [1,\infty), \end{cases} \tag{86}$$

where $\Lambda = \{v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi : v_0 = v_1 = 0\}$ and $\phi(v) = \sqrt{\sum_{a\in\mathcal{N}} |v_a|^{2a+3/a+2}}$, for all $v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$. From Example 33, the map W is Kannan ϕ contraction mapping. So, it is Kannan ϕ nonexpansive mapping. Clearly, Λ is a nonempty, ϕ convex, ϕ closed, and ϕ bounded subset of $(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$. By Corollary 42, the map W has one fixed point $(v=\theta)$ in Λ.

7. Kannan Pre-Quasicontraction on Prequasi Ideal

We study the presence of a fixed point of Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach operator ideal constructed by $(\ell(r))_\phi$ and s-numbers.

Theorem 44 [9]. *Pick up Z and M to be Banach spaces, and $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is an increasing with $r_0 > 1$, and then, $(S_{(\ell(r))_\phi}, \Phi)$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$ is a pre-quasi-Banach operator ideal.*

Theorem 45. *If Z and M are Banach spaces and $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$, then, $(S_{(\ell(r))_\phi}, \Phi)$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$ is a pre-quasiclosed operator ideal.*

Proof. By Theorem 17, the space $(\ell(r))_\phi$ is a premodular (sss). So, from Theorem 12, we have $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$ which is a pre-quasinorm on $S_{(\ell(r))_\phi}$. Suppose that $W_q \in S_{(\ell(r))_\phi}(Z, M)$, for all $q \in \mathcal{N}$ and $\lim_{q\longrightarrow\infty} \Phi(W_q - W) = 0$. Therefore, there is $\varsigma > 0$ and as $\mathscr{L}(Z, M) \supseteq S_{(\ell(r))_\phi}(Z, M)$; one has

$$\begin{aligned} \Phi(W_q - W) &= \phi\left(\left(s_a(W_q - W)\right)_{a=0}^{\infty}\right) \\ &\geq \phi\left(s_0(W_q - W)(W_q - W), 0, 0, 0, \cdots\right) \\ &= \phi\left(\|W_q - W\|, 0, 0, 0, \cdots\right) \geq \varsigma \|W_q - W\|. \end{aligned} \tag{87}$$

Then, $(W_q)_{q\in\mathcal{N}}$ is convergent in $\mathscr{L}(Z, M)$. i.e., $\lim_{q\longrightarrow\infty} \|W_q - W\| = 0$, and since $(s_a(W_q - W))_{a=0}^{\infty} \in (\ell(r))_\phi$, for all $q \in \mathcal{N}$ and $(\ell(r))_\phi$ is a premodular (sss). Hence, one can see

$$\begin{aligned} \Phi(W) &= \phi\left((s_a(W))_{a=0}^{\infty}\right) = \phi\left(\left(s_a(W - W_q + W_q)\right)_{a=0}^{\infty}\right) \\ &\leq \phi\left(\left(s_{a/2}(W - W_q)\right)_{a=0}^{\infty}\right) + \phi\left(\left(s_{a/2}(W_q)\right)_{a=0}^{\infty}\right) \\ &\leq \phi\left(\left(\|W_q - W\|\right)_{a=0}^{\infty}\right) + (2)^{1/K} \phi\left(\left(s_a(W_q)\right)_{a=0}^{\infty}\right) < \varepsilon. \end{aligned} \tag{88}$$

We obtain $(s_a(W))_{a=0}^{\infty} \in (\ell(r))_\phi$; hence, $W \in S_{(\ell(r))_\phi}(Z, M)$. □

Definition 46. A pre-quasinorm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$, verifies the Fatou property if for all sequence $\{W_a\}_{a\in\mathcal{N}} \subseteq S_{\mathfrak{A}_\phi}(Z, M)$ with $\lim_{a\longrightarrow\infty} \Phi(W_a - W) = 0$ and every $V \in S_{\mathfrak{A}_\phi}(Z, M)$, then

$$\Phi(V - W) \leq \sup_a \inf_{i\geq a} \Phi(V - W_i). \tag{89}$$

Theorem 47. *The pre-quasinorm $\Phi(W) = [\sum_{a=0}^{\infty} |s_a(W)|^{r_a}]^{1/K}$, for each $W \in S_{(\ell(r))_\phi}(Z, M)$, does not verify the Fatou property, if $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$.*

Proof. Suppose that the conditions are verified and $\{W_p\}_{p\in\mathcal{N}} \subseteq S_{(\ell(r))_\phi}(Z, M)$ with $\lim_{p\longrightarrow\infty} \Phi(W_p - W) = 0$. As the space $S_{(\ell(r))_\phi}$ is a pre-quasiclosed ideal; hence, $W \in S_{(\ell(r))_\phi}(Z, M)$. Then, for all $V \in S_{(\ell(r))_\phi}(Z, M)$, one has

$$\begin{aligned} \Phi(V - W) &= \left[\sum_{a=0}^{\infty} |s_a(V - W)|^{r_a}\right]^{1/K} \\ &\leq \left[\sum_{a=0}^{\infty} |s_{a/2}(V - W)|^{r_a}\right]^{1/K} \\ &\quad + \left[\sum_{a=0}^{\infty} |s_{a/2}(W_i - W)|^{r_a}\right]^{1/K} \\ &\leq (2)^{1/K} \sup_p \inf_{i\geq p} \left[\sum_{a=0}^{\infty} |s_a(V - W)|^{r_a}\right]^{1/K}. \end{aligned} \tag{90}$$

Therefore, Φ does not verify the Fatou property. □

Now, we explain the definition of Kannan Φ-Lipschitzian mapping in the pre-quasioperator ideal.

Definition 48. For the pre-quasinorm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$. An operator $G : S_{\mathfrak{A}_\phi}(Z, M) \longrightarrow S_{\mathfrak{A}_\phi}(Z, M)$ is named a Kannan Φ -Lipschitzian, if there is $\xi \geq 0$ such that

$$\Phi(GW - GA) \leq \xi(\Phi(GW - W) + \Phi(GA - A)), \tag{91}$$

for every $W, A \in S_{\mathfrak{A}_\phi}(Z, M)$. An operator G is named

(1) Kannan Φ contraction, if $\xi \in [0, 1/2)$

(2) Kannan Φ nonexpansive, if $\xi = 1/2$

Definition 49. For the pre-quasi norm Φ on the ideal $S_{\mathfrak{A}_\phi}$, where $\Phi(W) = \phi((s_a(W))_{a=0}^{\infty})$, $G : S_{\mathfrak{A}_\phi}(Z, M) \longrightarrow S_{\mathfrak{A}_\phi}(Z, M)$, and $B \in S_{\mathfrak{A}_\phi}(Z, M)$. The operator G is named Φ sequentially continuous at B, if and only if, when $\lim_{p\longrightarrow\infty}\Phi(W_p - B) = 0$, then $\lim_{p\longrightarrow\infty}\Phi(GW_p - GB) = 0$.

Theorem 50. *Let $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ be an increase with $r_0 > 1$ and $G : S_{(\ell(r))_\phi}(Z, M) \longrightarrow S_{(\ell(r))_\phi}(Z, M)$, where $\Phi(W) = [\sum_{a=0}^{\infty}|s_a(W)|^{r_a}]^{1/K}$, for all $W \in S_{(\ell(r))_\phi}(Z, M)$. The point $A \in S_{(\ell(r))_\phi}(Z, M)$ is the unique fixed point of G, if the next settings are verified:*

(a) G is Kannan Φ contraction mapping

(b) G is Φ sequentially continuous at a point $A \in S_{(\ell(r))_\phi}(Z, M)$,

(c) We have $B \in S_{(\ell(r))_\phi}(Z, M)$ such that the sequence of iterates $\{G^p B\}$ has a subsequence $\{G^{p_i} B\}$ which converges to A

Proof. Suppose that the settings are satisfied. If A is not a fixed point of G, then, $GA \neq A$. From conditions (b) and (c), one has

$$\begin{aligned} &\lim_{p_i\longrightarrow\infty} \Phi(G^{p_i}B - A) = 0,\\ &\lim_{p_i\longrightarrow\infty} \Phi\left(G^{p_i+1}B - GA\right) = 0. \end{aligned} \tag{92}$$

As G is Kannan Φ contraction mapping, we have

$$\begin{aligned} 0 < \Phi(GA - A) &= \Phi\left(\left(GA - G^{p_i+1}B\right) + (G^{p_i}B - A) + \left(G^{p_i+1}B - G^{p_i}B\right)\right)\\ &\le (2)^{1/K}\Phi\left(G^{p_i+1}B - GA\right) + (2)^{1/K}\Phi(G^{p_i}B - A)\\ &\quad + (2)^{1/K}\xi\left(\frac{\xi}{1-\xi}\right)^{p_i-1}\Phi(GB - B). \end{aligned} \tag{93}$$

Since $p_i \longrightarrow \infty$, one has a contradiction. Hence, A is a fixed point of G. To prove that the fixed point A is unique, assume that we have two different fixed points $A, D \in S_{(\ell(r))_\phi}(Z, M)$ of G. Therefore, we have

$$\Phi(A - D) \le \Phi(GA - GD) \le \xi(\Phi(GA - A) + \Phi(GD - D)) = 0. \tag{94}$$

So, $A = D$. □

Example 51. Let Z and M be Banach spaces, $G : S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}(Z, M) \longrightarrow S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}(Z, M)$, where $\Phi(W) = \sqrt{\sum_{a=0}^{\infty}|s_z(W)|^{2a+3/a+2}}$, for every $W \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}(Z, M)$ and

$$G(W) = \begin{cases} \dfrac{W}{6}, & \Phi(W) \in [0, 1),\\ \dfrac{W}{7}, & \Phi(W) \in [1, \infty). \end{cases} \tag{95}$$

Since for all $W_1, W_2 \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in [0, 1)$, we have

$$\begin{aligned} \Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{6} - \frac{W_2}{6}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Phi\left(\frac{5W_1}{6}\right) + \Phi\left(\frac{5W_2}{6}\right)\right)\\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)). \end{aligned} \tag{96}$$

For all $W_1, W_2 \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1), \Phi(W_2) \in [1, \infty)$, we have

$$\begin{aligned} \Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{7} - \frac{W_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Phi\left(\frac{6W_1}{7}\right) + \Phi\left(\frac{6W_2}{7}\right)\right)\\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)). \end{aligned} \tag{97}$$

For all $W_1, W_2 \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W_1) \in [0, 1)$ and $\Phi(W_2) \in [1, \infty)$, we have

$$\begin{aligned} \Phi(GW_1 - GW_2) &= \Phi\left(\frac{W_1}{6} - \frac{W_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\Phi\left(\frac{5W_1}{6}\right) + \frac{\sqrt{2}}{\sqrt[4]{216}}\Phi\left(\frac{6W_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}(\Phi(GW_1 - W_1) + \Phi(GW_2 - W_2)). \end{aligned} \tag{98}$$

Therefore, the map W is Kannan Φ contraction mapping and $G^p(W) = \begin{cases} W/6^p, & \Phi(W) \in [0, 1),\\ W/7^p, & \Phi(W) \in [1, \infty). \end{cases}$

It is clear that G is Φ sequentially continuous at the zero operator $\Theta \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ and $\{G^p W\}$ has a subsequence $\{G^{p_i} W\}$ which converges to Θ. By Theorem 50, the zero operator $\Theta \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ is the only fixed point of G.

Let $\{W^{(n)}\} \subseteq S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ be such that $\lim_{n\longrightarrow\infty}\Phi(W^{(n)} - W^{(0)}) = 0$, where $W^{(0)} \in S_{(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi}$ with $\Phi(W^{(0)}) = 1$. Since the pre-quasinorm Φ is continuous, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\Phi\left(GW^{(n)} - GW^{(0)}\right) &= \lim_{n\longrightarrow\infty}\Phi\left(\frac{W^{(0)}}{6} - \frac{W^{(0)}}{7}\right)\\ &= \Phi\left(\frac{W^{(0)}}{42}\right) > 0.\end{aligned} \tag{99}$$

Hence, G is not Φ sequentially continuous at $W^{(0)}$. So, the map G is not continuous at $W^{(0)}$.

8. Application to the Existence of Solutions of Summable Equations

Summable equations like (100) studied by Salimi et al. [38], Agarwal et al. [39], and Hussain et al. [40]. In this section, we search for a solution to (100) in $(\ell(r))_\phi$, where $(r_a)_{a\in\mathcal{N}} \in \ell_\infty$ is increasing with $r_0 > 1$ and $\phi(v) = [\sum_{j=0}^{\infty}|v_j|^{r_j}]^{1/K}$, for all $v \in \ell(r)$. Consider the summable equations

$$v_a = p_a + \sum_{m=0}^{\infty} A(a,m) f(m, v_m), \tag{100}$$

and let $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by

$$W(v_a)_{a\in\mathcal{N}} = \left(p_a + \sum_{m=0}^{\infty} A(a,m) f(m, v_m)\right)_{a\in\mathcal{N}}. \tag{101}$$

Theorem 52. *The summable equation (100) has a solution in* $(\ell(r))_\phi$; *if* $A : \mathcal{N}^2 \longrightarrow \mathfrak{R}, f : \mathcal{N}\times\mathfrak{R} \longrightarrow \mathfrak{R}, p : \mathcal{N} \longrightarrow \mathfrak{R}$, *and for all* $a \in \mathcal{N}$, *suppose that*

$$\begin{aligned}&\left|\sum_{m\in\mathcal{N}} A(a,m)(f(m,v_m) - f(m,t_m))\right|^{r_a}\\ &\quad\le \frac{1}{2^K}\left[\left|p_a - v_a + \sum_{m=0}^{\infty} A(a,m) f(m,v_m)\right|^{r_a}\right.\\ &\qquad\left. + \left|p_a - t_a + \sum_{m=0}^{\infty} A(a,m) f(m,t_m)\right|^{r_a}\right].\end{aligned} \tag{102}$$

Proof. Let the conditions be verified. Consider the mapping $W : (\ell(r))_\phi \longrightarrow (\ell(r))_\phi$ defined by (101). We have

$$\begin{aligned}\phi(Wv - Wt) &= \left[\sum_{a\in\mathcal{N}}|Wv_a - Wt_a|^{r_a}\right]^{1/K}\\ &= \left[\sum_{a\in\mathcal{N}}\left|\sum_{m\in\mathcal{N}} A(a,m)[f(m,v_m) - f(m,t_m)]\right|^{r_a}\right]^{1/K}\\ &\le \frac{1}{2}\left(\left[\sum_{a\in\mathcal{N}}\left|p_a - v_a + \sum_{m=0}^{\infty} A(a,m) f(m,v_m)\right|^{r_a}\right]^{1/K}\right.\\ &\quad\left. + \left[\sum_{a\in\mathcal{N}}\left|p_a - t_a + \sum_{m=0}^{\infty} A(a,m) f(m,t_m)\right|^{r_a}\right]^{1/K}\right)\\ &= \frac{1}{2}(\phi(Wv - v) + \phi(Wt - t)).\end{aligned} \tag{103}$$

Then, from Theorem 41, we have a solution for equation (100) in $(\ell(r))_\phi$. □

Example 53. Given the sequence space $(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$, where $\phi(v) = \sqrt{\sum_{a\in\mathcal{N}}|v_a|^{2a+3/a+2}}$, for all $v \in \ell((2a+3/a+2)_{a=0}^{\infty})$. Consider the summable equations

$$v_a = e^{-(3a+6)} + \sum_{m=0}^{\infty}(-1)^{a+m}\left(\frac{v_a}{a^2+m^2+1}\right)^q, \tag{104}$$

with $a \ge 2$ and $q > 2$ and let $W : \Lambda \longrightarrow \Lambda$, where $\Lambda = \{v \in (\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi : v_0 = v_1 = 0\}$, defined by

$$W(v_a)_{a\ge2} = \left(e^{-(3a+6)} + \sum_{m=0}^{\infty}(-1)^{a+m}\left(\frac{v_a}{a^2+m^2+1}\right)^q\right)_{a\ge2}. \tag{105}$$

Clearly, Λ is a nonempty, ϕ convex, ϕ closed, and ϕ bounded subset of $(\ell((2a+3/a+2)_{a=0}^{\infty}))_\phi$. It is easy to see that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty}(-1)^a\left(\frac{v_a}{a^2+m^2+1}\right)^q((-1)^m - (-1)^m)\right|^{2a+3/a+2}\\ &\quad\le \frac{1}{\sqrt{2}}\left[\left|e^{-(3a+6)} - v_a + \sum_{m=0}^{\infty}(-1)^{a+m}\left(\frac{v_a}{a^2+m^2+1}\right)^q\right|^{2a+3/a+2}\right.\\ &\qquad\left. + \left|e^{-(3a+6)} - t_a + \sum_{m=0}^{\infty}(-1)^{a+m}\left(\frac{t_a}{a^2+m^2+1}\right)^q\right|^{2a+3/a+2}\right].\end{aligned} \tag{106}$$

By Theorem 52, the summable equation (104) has a solution in Λ.

9. Conclusion

We have introduced the concept of the pre-quasinormed space, which is more general than the quasinormed space. We investigate the sufficient conditions on Nakano (sss) such as its variable exponent in $(1,\infty)$ with the known pre-quasinorm to form pre-quasi-Banach and closed (sss), the concept of a fixed point of Kannan pre-quasi-norm contraction mapping in the pre-quasi-Banach (sss), which supports the property (R), and the pre-quasinormal structure property. The existence of a fixed point of Kannan pre-quasinorm nonexpansive mapping in the pre-quasi-Banach (sss) has been examined. Also, the existence of a fixed point of Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach operator ideal formed by Nakano (sss) and s -numbers has been investigated. Finally, we have explained some examples to show that the results obtained can solve a problem. The strength of this approach is that the existence results are established under flexible conditions provided by controlling the power of the Nakano sequence space. The novelty lies in the fact that our main results have improved some well-known theorems before, which concerned the variable exponent in the aforementioned space.

Data Availability

No data were used.

Disclosure

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Fixed Point of Generalized Weak Contraction in *b*-Metric Spaces

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In this manuscript, a class of generalized (ψ, α, β)-weak contraction is introduced and some fixed point theorems in the framework of b-metric space are proved. The result presented in this paper generalizes some of the earlier results in the existing literature. Further, some examples and an application are provided to illustrate our main result.

1. Introduction and Preliminaries

Fixed point theory plays a vital role in the development of nonlinear functional analysis. It has been used in various branches of engineering and sciences. Banach contraction principle is one of the most important results in fixed point theory introduced by great Polish mathematician Stefan Banach [1]. The concept of b-metric space or metric-type space was first introduced by Czerwik [2]. He provided a property which is weaker than the triangular inequality. The basic idea of b-metric was commenced by Bourbaki [3] and Bakhtin [4]. Later on, Khamsi and Hussain [5] reintroduced such spaces under the name of metric-type spaces for some results of fixed and common fixed points in the setting of b-metric spaces. Since then, several authors proved fixed point results of single valued and multivalued operators in b-metric space and its different type generalizations, we refer [6–22]. Every one of these applications captivated us to present the idea of b-metric space.

Definition 1. (see [23]). Let X be a nonempty set. A function $d : X \times X \longrightarrow 0, \infty$ is said to be a b -metric if it satisfies the following conditions:

$$\begin{aligned} &(1)\ 0 \leq d(x, y) \text{ and } d(x, y) = 0 \text{ if and only if } x = y \\ &(2)\ d(x, y) = d(y, x) \\ &(3)\ d(x, z) \leq s[d(x, y) + d(y, z)] \text{ for some } s \geq 1, \end{aligned} \tag{1}$$

for all $x, y, z \in X$. The pair (X, d) is called a b-metric with coefficient s.

Here, we observe that every metric space is a b-metric with $s = 1$. Conditions (1) and (2) of Definition 1 are similar to metric space but it is important how to use (3) effectively. An example is given to expound the concept of a third condition.

Example 1. Let $X = \mathbb{R}$. We define a mapping $d : X \times X \longrightarrow \mathbb{R}^+$ such that

$$d(x, y) = (x - y)^2, \forall x, y \in X. \tag{2}$$

The first two conditions of Definition 1 are clearly shown. The solution of third condition is as follows:

$$\begin{aligned} d(x, z) = (x - z)^2 = x^2 + z^2 - 2xz \leq x^2 + z^2 - 2xz \\ + x^2 + z^2 + 4y^2 - 4xy - 2. \end{aligned} \tag{3}$$

Since

$$x^2 + z^2 + 4y^2 > 2x(2y + z), x^2 + z^2 + 4y^2 - 2x(2y + z) > 0, \tag{4}$$

we have

$$\begin{aligned} d(x,z) &\le x^2+z^2-2xz+x^2+z^2+4y^2-4xy-2 \\ &= 2x^2+4y^2+2z^2-4xy-4xz \\ &= 2\left[x^2+2y^2+z^2-2xy-2xz\right] \\ &= 2\left[x^2+y^2-2xy+y^2+z^2-2xz\right] \\ &= 2\left[(x-y)^2+(y-z)^2\right] \\ &= 2[d(x,y)+d(y,z)]. \end{aligned} \tag{5}$$

Then, we obtain

$$d(x,z) \le 2[d(x,y)+d(y,z)]. \tag{6}$$

So the value of coefficient is $s=2$.

In this section, the concept of generalized (ψ,α,β)-weak contraction for metric space is provided with some basic notions and results. In 1997, Alber and Guerre-Delabriere [24] suggested a generalization of Banach contraction mapping by introducing the concept of ϕ-weak contraction in Hilbert space. In 2008, Dutta and Choudhury [25] gave a generalization of weakly contractive mapping by defining (ψ,ϕ)-weak contraction in complete metric spaces.

Definition 2. (see [26]). Let Ψ denote the class of function $\psi:[0,\infty)\longrightarrow[0,\infty)$ which satisfies the following conditions:

(1) ψ is continuous and nondecreasing

$$\psi(t)=0 \Longleftrightarrow t=0. \tag{7}$$

Definition 3. (see [26]). A self-map P is said to be a weakly contractive map if there exists a function $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ such that ϕ is continuous, nondecreasing, and $\phi(t)=0 \Longleftrightarrow t=0$ and satisfying

$$d(Px,Py) \le d(x,y)-\phi(d(x,y)), \forall x,y\in X. \tag{8}$$

Theorem 4. *(see [26]). Let(X,d) be a complete metric space and P be a weakly contractive self-map on X. Then, P has a unique fixed point in X.*

Definition 5. (see [25]). A self-map P is said to be (ψ,ϕ)-weak contraction, if for each $x,y\in X$,

$$\psi(d(Px,Py)) \le \psi(d(x,y))-\phi(d(x,y)), \tag{9}$$

where $\psi,\phi:[0,\infty]\longrightarrow[0,\infty]$ are both continuous and monotone nondecreasing functions with $\psi(t)=0=\phi(t) \Longleftrightarrow t=0$.

Theorem 6. *(see [25]). Let (X,d) be a complete metric space and a self-map P be a (ψ,ϕ)-weak contraction. Then, P has a unique fixed point.*

Definition 7. (see [27]) Two self-maps P and Q are said to be generalized ϕ -weakly contractive map if there exists a function $\phi:[0,+\infty]\longrightarrow[0,+\infty]$ such that ϕ is continuous, nondecreasing, and $\phi(t)=0 \Longleftrightarrow t=0$ and satisfying

$$d(Px,Qy) \le M(x,y)-\phi(M(x,y)), \tag{10}$$

where

$$M(x,y)=\max\left\{d(x,y),d(x,Px),d(y,Qy),\frac{[d(y,Px)+d(x,Qy)]}{2}\right\},\forall x,y\in X. \tag{11}$$

Theorem 8. *(see [27]). Let (X,d) be a complete metric space and T and R are generalized ϕ -weakly contractive self-maps on X. Then, P and Q have a unique common fixed point in X.*

Definition 9. (see [28]). Two self-maps P and Q are said to be generalized (ϕ,ψ)-weakly contractive maps if they satisfy

$$\psi(Px,Qy) \le M(x,y)-\phi(M(x,y)), \tag{12}$$

$\forall x,y\in X$, where $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ such that ψ is continuous, nondecreasing, and $\psi(t)=0 \Longleftrightarrow t=0$, $\phi:[0,+\infty]\longrightarrow[0,+\infty)$ such that ϕ is a lower semicontinuous function, $\phi(t)=0 \Longleftrightarrow t=0$ and

$$M(x,y)=\max\left\{d(x,y),d(x,Px),d(y,Qy),\frac{[d(y,Px)+d(x,Qy)]}{2}\right\},\forall x,y\in X. \tag{13}$$

Theorem 10. *(see [28]). Let (X,d) be a complete metric space and P and Q are generalized (ϕ,ψ) -weakly contractive self-maps on X. Then, P and Q have a unique common fixed point in X.*

Definition 11. (see [29]). Let $(X,\preccurlyeq)$ be a partially ordered set. Two mappings $P,Q:X\longrightarrow X$ are said to be weakly increasing if $Px\preccurlyeq QTx$ and $Qx\preccurlyeq PQx$ for all $x\in X$.

Remark 12. (see [29]). Note that two weakly increasing mappings need not be nondecreasing.

Definition 13. (see [30]). Let (X,d) be a metric space and $P,Q:X\longrightarrow X$ are given two self-mappings on X. The pair (P,Q) is said to be compatible if $\lim_{n\longrightarrow\infty} d(PQx_n,QPx_n)=0$, whenever x_n is a sequence in X such that

$$\lim_{n\longrightarrow\infty} Px_n=\lim_{n\longrightarrow\infty} Qx_n=t, \text{for some } t\in X. \tag{14}$$

Definition 14. (see [1]). Let $(X,\preccurlyeq)$ be a partially ordered set and let (X,d) be a b-metric space with coefficient $s\ge1$. Three maps T, R, and S are said to be a generalized (ψ,α,β)-weak contraction if for each $x,y\in X$,

$$\psi(d(Px,Qy)) \le \alpha((d(Sx,Sy))\beta(d(Sx,Sy)), \tag{15}$$

where $\psi\in\Psi$ and $\beta:[0,\infty)\longrightarrow[0,\infty)$ is a continuous function with condition

$$0<\beta(t)<\psi(t),\forall t>0. \tag{16}$$

Theorem 15. *(see [1]). Let $(X,\preccurlyeq)$ be a partially ordered set and assume that there exists a metric function d in X such that (X,d) is a complete metric space. Let $P,Q,S:X\longrightarrow X$ are generalized (ψ,α,β) -weak contraction mappings satisfying the following properties:*

$$PX\subseteq SX, \tag{17}$$

(1) P, Q, and S are continuous

(2) The pairs (P,S) and (Q,S) are compatible

(3) P and Q weakly increasing with respect to S

(4) Sx and Sy are comparable

(5) $\forall(x,y)\in X\times X$, there exists $u\in X$ such that $Px\preccurlyeq Pu$ and $Py\preccurlyeq Pu$

Then, P, Q, and S have a unique common fixed point $z\in X$.

The next section includes the concept of generalized (ψ,α,β)-weak contraction for b-metric space and theorem related to it.

2. Generalized (ψ,α,β)-Weak Contractions

Definition 16. Let $(X,\preccurlyeq)$ be a partially ordered set and let (X,d) be a b-metric space with coefficient $s\geq 1$. Three maps P, Q, and S are said to be a generalized (ψ,α,β)-weak contraction if for each $x,y\in X$ and $b\in(0,1)$.

$$b^s\psi(d(Px,Qy))\leq\alpha((d(Sx,Sy))\beta(d(Sx,Sy)),\forall x\geq y, \tag{18}$$

where $\alpha\in F,\psi\in\Psi$ and $\beta:[0,\infty)\longrightarrow[0,\infty)$ is a continuous function with the condition

$$0<\beta(t)<\psi(t),\forall t>0. \tag{19}$$

Theorem 17. *Let $(X,\preccurlyeq)$ be a partially ordered set and assume that there exist a b -metric function d in X such that (X,d) is complete b-metric space. Let $T,R,S:X\longrightarrow X$ are generalized (ψ,α,β) -weak contraction mappings satisfying the following properties:*

$$PX\subseteq SX, \tag{20}$$

(1) P, Q, and S are continuous

(2) The pair (P,S) and (Q,S) are compatible

(3) P and Q weakly increasing with respect to S

(4) Sx and Sy are comparable

(5) $\forall(x,y)\in X\times X$, there exist $u\in X$ such that $Px\preccurlyeq Pu$ and $Py\preccurlyeq Pu$

Then, P, Q, and S have a unique common fixed point $z\in X$.

Proof. The proof is done by using the concept of Banach contraction principle in which a Cauchy sequence is taken in complete b-metric space. Every Cauchy sequence is convergent in a complete metric space, and converging point of that sequence is proved to be a fixed point of contraction. ☐

Let us assume that $x_0\in X$ be an arbitrary point in X. By property (1), there exist $x_1,x_2\in X$ such that $Px_0=Sx_1$ and $Qx_1=Sx_2$. Continuing this process, sequences $\{x_n\}$ and $\{y_n\}$ can be constructed in X, defined as

$$Sx_{2n+1}=Px_{2n}=y_{2n},Sx_{2n+1}=Qx_{2n+1}=y_{2n+1},\forall n\in\mathbb{N}. \tag{21}$$

By using property (4), we obtain

$$Sx_1=Px_0\preccurlyeq Qx_1=Sx_2. \tag{22}$$

Similarly,

$$Sx_2=Px_1\preccurlyeq Qx_2=Sx_3. \tag{23}$$

Continuing this process, we get

$$Sx_1\preccurlyeq Sx_2\preccurlyeq Sx_3\preccurlyeq Sx_4\preccurlyeq\cdots\preccurlyeq Sx_{2n+1}\preccurlyeq Sx_{2n+2}\preccurlyeq\cdots. \tag{24}$$

Thus,

$$y_0\preccurlyeq y_1\preccurlyeq y_2\preccurlyeq\cdots\preccurlyeq y_{2n}\preccurlyeq y_{2n+1}\preccurlyeq\cdots. \tag{25}$$

According to our first supposition, if there exists $n\in\mathbb{N}$ such that $y_{2n-1}=y_{2n}$, then from (18),

$$\begin{aligned}b^s\psi(d(y_{2n},y_{2n+1}))&=b^s\psi(d(Px_{2n},Qx_{2n+1}))\\&\leq b^s\alpha(d(Sx_{2n},Sx_{2n+1}))\beta(d(Sx_{2n},Sx_{2n+1}))\\&=b^s\alpha(d(y_{2n-1},y_{2n}))\beta(d(y_{2n-1},y_{2n}))\\&=0,\end{aligned} \tag{26}$$

which implies that $y_{2n+1}=y_{2n}$. Consequently, $y_m=y_{2n-1}$ for any $m\geq 2n$. Hence, for every $m\geq 2n$, we have $Sx_m=Sx_{2n}$ which implies that $\{Sx_n\}$ is a b-Cauchy sequence.

According to our second supposition, $y_n\neq y_{n+1}$ for any integer n. Let $z_n=d(y_n,y_{n+1})$. Now, we have to show that $z_n\longrightarrow 0$ as $n\longrightarrow\infty$. Since Sx_{2n} and Sx_{2n+1} are comparable, then from (18), we have

$$\begin{aligned}b^s\psi(d(y_{2n+2},y_{2n+1}))&=b^s\psi(d(Sx_{2n+3},Sx_{2n+2}))\\&=b^s\psi(d(Px_{2n+2},Qx_{2n+1}))\\&\leq b^s\alpha(d(Sx_{2n+2},Sx_{2n+1}))\beta(d(Sx_{2n+2},Sx_{2n+1}))\\&=b^s\alpha(d(y_{2n+1},y_{2n}))\beta(d(y_{2n+1},y_{2n})).\end{aligned} \tag{27}$$

By property (2) of ψ and the fact that $\alpha \in F$, we get

$$d(y_{2n+2}, y_{2n+1}) \le d(y_{2n+1}, y_{2n}). \quad (28)$$

Similarly, we have

$$d(y_{2n+1}, y_{2n}) \le d(y_{2n}, y_{2n-1}). \quad (29)$$

By combining (28) and (29), we obtain

$$d(y_{2n+2}, y_{2n+1}) \le d(y_{2n+1}, y_{2n}) \le d(y_{2n}, y_{2n-1}). \quad (30)$$

This shows that the sequence $\{z_n\}$ is monotonically decreasing. So there exists $r \ge 0$ such that

$$\lim_{n\longrightarrow\infty} z_n = d(y_n, y_{n+1}) = r. \quad (31)$$

Suppose $r \ge 0$. Then,

$$b^s\psi(d(y_{2n+2}, y_{2n+1})) \le b^s\alpha(d(y_{2n+1}, y_{2n}))\beta(d(y_{2n+1}, y_{2n})). \quad (32)$$

Taking the limit as $n \longrightarrow \infty$, we get $\psi(r) \le \alpha(r) \le \beta(r)$. Since $\alpha \in F$, by using (19), we have $\psi(r) < \alpha(r) < \beta(r)$ but this is a contradiction. Then, $r = 0$. Hence,

$$\lim_{n\longrightarrow\infty} z_n = d(y_n, y_{n+1}) = 0. \quad (33)$$

Next, we have to show that $\{Sx_n\}$ is a b-Cauchy sequence. We prove this by contradiction. Now, we suppose that $\{Sx_{2n}\}$ is not a b-Cauchy sequence. Then, for any $\varepsilon > 0$, there exist two subsequences of positive integers m_k and n_k such that $n_k > m_k$ for all positive integers k,

$$d(Sx_{2m_k}, Sx_{2n_k}) > \varepsilon, \quad (34)$$

$$d(Sx_{2m_k}, Sx_{2n_{k-2}}) \le \varepsilon. \quad (35)$$

From (34) and (35) and by using triangle inequality, we get

$$\begin{aligned}\varepsilon \le d(Sx_{2m_k}, Sx_{2n_{k-2}}) &\le d(Sx_{2m_k}, Sx_{2n_{k-2}}) + d(Sx_{2n_{k-2}}, Sx_{2n_{k-1}}) \\ &+ d(Sx_{2n_{k-1}}, Sx_{2n_k}).\end{aligned} \quad (36)$$

Let $k \longrightarrow \infty$ in the above inequality and by using (33), we obtain

$$\lim_{k\longrightarrow\infty} d(Sx_{2m_k}, Sx_{2n_k}) = \varepsilon. \quad (37)$$

Again by using triangle inequality, we have

$$d(Sx_{2n_k}, Sx_{2m_{k-1}}) \le d(Sx_{2n_k}, Sx_{2m_k}) + d(Sx_{2m_k}, Sx_{2m_{k-1}}). \quad (38)$$

By taking limit as $k \longrightarrow \infty$ in above inequality and using (33)–(35), we get

$$\lim_{k\longrightarrow\infty} d(Sx_{2n_k}, Sx_{2m_{k-1}}) = \varepsilon. \quad (39)$$

Moreover, we obtain

$$\begin{aligned}d(Sx_{2n_k}, Sx_{2m_k}) &\le d(Sx_{2n_k}, Sx_{2n_{k+1}}) + d(Sx_{2n_{k+1}}, Sx_{2m_k}) \\ &= d(Sx_{2n_k}, Sx_{2n_{k+1}}) + d(Px_{2n_k}, Qx_{2m_{k-1}}).\end{aligned} \quad (40)$$

Using inequalities (33)–(37) and letting $\lim_{k\longrightarrow\infty}$, we have

$$\varepsilon \le \lim_{k\longrightarrow\infty} d(Px_{2n_k}, Qx_{2m_{k-1}}). \quad (41)$$

However, $\psi \in \Psi$, therefore

$$\psi(\varepsilon) \le \lim_{k\longrightarrow\infty} \psi(d(Px_{2n_k}, Qx_{2m_{k-1}})). \quad (42)$$

From (18), we have

$$b^s\psi(d(Px_{2n_k}, Qx_{2m_{k-1}})) \le b^s\alpha(d(Sx_{2n_k}, Sx_{2m_{k-1}}))\beta(d(Sx_{2n_k}, Sx_{2m_{k-1}})). \quad (43)$$

Taking limit as $k \longrightarrow \infty$ in the above inequality and using that fact that $\alpha \in F$, we have

$$\lim_{k\longrightarrow\infty} \psi(d(Px_{2n_k}, Qx_{2m_{k-1}})) < \beta(\varepsilon). \quad (44)$$

From (43) and (44), and using (19), we obtain

$$\psi(\varepsilon) \le \lim_{k\longrightarrow\infty} \psi(d(Px_{2n_k}, Qx_{2m_{k-1}})) < \beta(\varepsilon) < \psi(\varepsilon). \quad (45)$$

This is a contradiction. Therefore, $\{x_{2n}\}$ is a b-Cauchy sequence, and hence, $\{x_n\}$ is a b-Cauchy sequence for all $n \in \mathbb{N}$. Hence, there exists $u \in X$ such that

$$\lim_{n\longrightarrow\infty} Sx_n = u. \quad (46)$$

Next, claim that u is a coincidence point of P, Q, and S. From (46) and the continuity of S, we get

$$\lim_{n\longrightarrow\infty} S(Sx_n) = Su. \quad (47)$$

From triangle inequality, we have

$$\begin{aligned}d(Su, Pu) &\le d(Su, S(Sx_{2n+1})) + d(S(Tx_{2n})), P(Sx_{2n})) \\ &+ d(P(Sx_{2n}), Tu),\end{aligned} \quad (48)$$

From (46) and (21), we have

$$Sx_{2n} \longrightarrow u, Px_{2n} \longrightarrow u. \quad (49)$$

Since pair (P, S) are compatible, then

$$d(S(Px_{2n}), P(Sx_{2n})) \longrightarrow 0. \tag{50}$$

Using the continuity of P and (58), we get

$$d(P(Sx_{2n}), Pu) \longrightarrow 0, \tag{51}$$

letting $k \longrightarrow \infty$ in (48) and using (47)–(50) together with (51), we find

$$d(Su, Pu) \leq 0, \tag{52}$$

which means that $Su = Pu$. Similarly, from triangle inequality, we have

$$\begin{aligned} d(Su, Qu) \leq d(Su, S(Sx_{2n+2})) + d(S(Qx_{2n+1})), Q(Sx_{2n+1})) \\ + d(Q(Sx_{2n+1}), Qu). \end{aligned} \tag{53}$$

In a similar way, we obtain $d(Su, Qu) \leq 0$ which means that $Su = Qu$. Thus, we find that $Su = Qu = Pu$, that is, u is a coincidence point of P, Q, and S.

Now, we use the property (6) to show that u is a common fixed point of P, Q, and S. For this, we prove that P, Q, and S have a common fixed point. To prove this, we show that if p and q are coincidence points of P, Q, and S, i.e.,

$$\begin{aligned} Sp = Pp = Qp, \\ Sq = Pq = Qq. \end{aligned} \tag{54}$$

Then,

$$Sp = Sq. \tag{55}$$

From our assumption mentioned in property (6), there exists $u_0 \in X$ such that

$$Tp \preccurlyeq Tu_0, Tq \preccurlyeq Tu_0. \tag{56}$$

Now, we can define a sequence $\{Su_n\}$ as follows:

$$Su_{2n+1} = Pu_{2n}, Su_{2n+2} = Qu_{2n+1}, \forall n \in \mathbb{N}. \tag{57}$$

Again, we have

$$Pp = Sp \preccurlyeq Su_n, Pq = Sq \preccurlyeq Su_{2n}, \forall n \in \mathbb{N}. \tag{58}$$

Now, putting $x = u_n$ and $y = p$ in (18), we get

$$\begin{aligned} b^s\psi(d(Su_{2n+1}, Sp)) &= b^s\psi(d(Pu_{2n}, Qp)) \\ &\leq b^s\alpha(d(Su_{2n}, Sp))\beta(d(Su_{2n}, Sp)). \end{aligned} \tag{59}$$

Since $\alpha \in F$, we have the next inequality:

$$b^s\psi(d(Su_{2n+1}, Sp)) \leq b^s\beta(d(Su_{2n}, Sp)). \tag{60}$$

Similarly, again writing $y = u_{2n}$ and $x = p$ in (18), we find

$$b^s\psi(d(Su_{2n+2}, Sp)) \leq b^s\beta(d(Su_{2n+1}, Sp)). \tag{61}$$

By combining (60) and (61), we obtain for all $n \in \mathbb{N}$,

$$b^s\psi(d(Su_{2n+1}, Sp)) \leq b^s\beta(d(Su_{2n}, Sp)). \tag{62}$$

Consequently, by using property of $\psi(t)$ and $\beta(t)$, we get

$$d(Su_{2n+1}, Sp) \leq d(Su_{2n}, Sp). \tag{63}$$

Therefore, there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(Su_n, Sp) = r. \tag{64}$$

Let $r > 0$, then from (18), we obtain

$$b^s\psi(d(Su_{2n+1}, Sp)) \leq b^s\alpha((d(Su_{2n}, Sp))\beta(d(Su_{2n}, Sp)), \tag{65}$$

on taking limit as $n \longrightarrow \infty$ and using (19), we get

$$\psi(r) < \beta(r) < \psi(r). \tag{66}$$

This is a contradiction. Thus, $r = 0$ and from (64), we obtain

$$\lim_{n \longrightarrow \infty} d(Su_n, Sp) = 0. \tag{67}$$

In the same pattern, we can show that

$$\lim_{n \longrightarrow \infty} d(Su_n, Sq) = 0. \tag{68}$$

Now, using the fact that the limit is unique and by using (55)–(68), we can write

$$\begin{aligned} \lim_{n \longrightarrow \infty} d(S(Pu_{2n}, P(Su_{2n}))) = 0, \lim_{n \longrightarrow \infty} d(S(Qu_{2n+1}, Q(Su_{2n} + 1)) = 0, \\ \lim_{n \longrightarrow \infty} Pu_{2n} = Sp = Sq, \\ \lim_{n \longrightarrow \infty} Qu_{2n+1} = Sp = Sq. \end{aligned} \tag{69}$$

Since the pair P, S and Q, S are compatible, we have

$$\lim_{n \longrightarrow \infty} d(S(Pu_{2n}, P(Su_{2n}))) = 0, \lim_{n \longrightarrow \infty} d(S(Qu_{2n+1}, Q(Su_{2n} + 1)) = 0. \tag{70}$$

Let us take $z = Sp$. Consider

$$d(Sz, Pz) \leq d(Sz, S(Pu_{2n})) + d(S(Tu_{2n}), P(Su_{2n})) + d(P(Su_{2n}), Pz). \tag{71}$$

Letting $n \longrightarrow \infty$ and using the continuity of P, we get the above inequality as

$$d(Sz, Pz) \leq 0. \tag{72}$$

That is, $Sz = Pz$ and z is coincidence point of P and S. Similarly, $d(Sz, Qz) \leq 0$. That is, $Rz = Sz$ and z is the coincidence point of Q and S. Hence, from (55), we have

$$z = Sp = Sz = Pz = Qz. \tag{73}$$

This proves that z is a common fixed point of P, Q, and S. Now, we show that z is unique common fixed point. We will show this by contradiction. Assume that z is not unique, therefore, there exists another fixed point v as

$$v = Sp = Sv = Pv = Qv. \tag{74}$$

By using (55), we have $Sv = Sz$. Hence, we get

$$v = Sv = Sz = z, \tag{75}$$

but this is a contradiction to our assumption, and hence, a common fixed point is unique.

An example is given to support the main result because there is limited examples in the literature.

Example 2. Let $X = \mathbb{R}$ is the set of real numbers and a b -metric $d : X \times X \longrightarrow 0, \infty$ is defined as

$$d(x, y) = (x - y)^2, \forall x, y \in X, \tag{76}$$

with coefficient $s = 2$ and $b = 0.01$. Consider three self-maps $P, Q, S : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$Px = 2x, Qx = 5 - 2x, Sx = \frac{x}{3}. \tag{77}$$

Define maps $\psi, \beta : [0,\infty) \longrightarrow [0,\infty)$ and $\alpha : [0,\infty) \longrightarrow [0, 1]$ as

$$\begin{aligned} \psi(t) &= \frac{t}{2},. \\ \beta &= \frac{t}{3}, \\ \alpha(t) &= \frac{7}{3}. \end{aligned} \tag{78}$$

After substituting values in (18),

$$b^s \psi(d(Px, Qy)) \leq \alpha((d(Sx, Sy))\beta(d(Sx, Sy)), \tag{79}$$

we have

$$\begin{aligned} &0.01^2 \left(\psi(2x - 5 + 2y)^2\right) \leq \alpha\left(d\left(\frac{x}{3}, \frac{y}{3}\right)\right)\beta\left(d\left(\frac{x}{3}, \frac{y}{3}\right)\right), \\ &0.01^2 \left(\frac{(2x - 5 + 2y)^2}{2}\right) \leq \frac{7}{3}\left(\frac{(x - y)^2)}{27}\right). \end{aligned} \tag{80}$$

Now, for $(x, y) = (0, 1)$, we get

$$0.00045 \leq 0.08, \tag{81}$$

For $(x, y) = (1, 2)$, we have

$$0.00005 \leq 0.08, \tag{82}$$

Then, clearly, three maps P, Q, and S are generalized (ψ, α, β)-weak contraction for all values of $x, y \in \mathbb{R}$.

3. An Application to Fredholm Integral Equations

In this section, applying Theorem 17, we give an existence theorem for common solutions of Fredholm integral equations where the upper limits of equations are taken to be the coefficient of b-metric space $s \geq 1$. Here, we consider the following integral equations:

$$\int_0^s P_1(t, l, v(l))dl + \varkappa(t), \tag{83}$$

$$\int_0^s P_2(t, l, v(l))dl + \varkappa(t), \tag{84}$$

for all $t \in [0, s]$ where $s \geq 1$. Let us consider the space $X = C[I]$ $(I = [0, s])$ is a set of continuous functions defined on $[0, s]$. Obviously, the space $C[I]$ with the metric $d(p, q) = \max_{t\in I}|x(t) - y(t)|$ for all $x, y \in X$ is a complete metric space.

Here, b-metric space is defined on the partially ordered set. So, X can be prepared with partial order $\preccurlyeq$ given by

$$\text{for all } x, y \in X, x \preccurlyeq y : \iff x(t) \preccurlyeq y(t), \text{for all } t \in I. \tag{85}$$

Theorem 18. *Suppose the following hypotheses hold:*

(1) $P_1, P_2 : I \times I \times R \longrightarrow R$ and $\lambda : R \longrightarrow R$ are continuous

(2) The following inequalities hold:

$$\begin{aligned} P_1(t, l, u(l)) &\leq P_2\left(t, l, \int_0^s P_1(l, r, u(r))dr + \varkappa(l)\right), \\ P_2(t, l, u(l)) &\leq P_1\left(t, l, \int_0^s P_2(l, r, u(r))dr + \varkappa(l)\right) \end{aligned} \tag{86}$$

(3) There exists a continuous function $\xi : I \times I \longrightarrow R_+$ such that

$$b^s|P_1(t, l, x) - P_2(t, l, y)| \leq \xi(t, l)\sqrt{\frac{\log\left[(x - y)^2 + s\right]}{x - y}}, \tag{87}$$

for all $t, l \in I$ *and* $x, y \in X$ *such that* $x \preccurlyeq y$.

$$\max_{t\in I}\xi(t,i)\int_0^s \xi(t,i)^2 dl \le \frac{1}{s} \tag{88}$$

Then, the integral equations (83) and (84) have the unique solution $v^* \in C(I)$.

Proof. Let us define $P, Q : C(I) \longrightarrow C(I)$ by

$$\begin{aligned} P_x(t) &= \int_0^s T_1(t,l,v(l))dl + \varkappa(t), \\ Q_x(t) &= \int_0^s T_2(t,l,v(l))dl + \varkappa(t), \end{aligned} \tag{89}$$

where $t \in I$ and $p \in C(I)$. Here, P and Q are considered to be weakly increasing function according to the requirement of our result.

Now, for all $p, q \in C(I)$ such that $p \preccurlyeq q$, we have

$$\begin{aligned} b^s\left|Px(t) - Q_y(t)\right| &= \int_0^s |P_1(t,l,v(l)) - P_2(t,l,v(l))|dl \\ &\le \int_0^s \xi(t,l)\sqrt{\frac{\log\left[(x(l)-y(l))^2+s\right]}{x(l)-y(l)}}\,dl, \end{aligned} \tag{90}$$

where $b \in (0,1)$.

Using the Cauchy-Schwarz inequality in the R.H.S., we get

$$\begin{aligned} &\int_0^s \xi(t,l)\sqrt{\frac{\log\left[(x(l)-y(l))^2+s\right]}{x(l)-y(l)}}\,dl \\ &\le \left(\sqrt{\int_0^s \xi^2(t,l)dl}\right)\left(\sqrt{\int_0^s \frac{\log\left[(x(l)-y(l))^2+s\right]}{x(l)-y(l)}\,dl}\right). \end{aligned} \tag{91}$$

By using hypotheses (4), we have

$$\begin{aligned} &\int_0^s \xi(t,l)\sqrt{\frac{\log\left[(x(l)-y(l))^2+s\right]}{x(l)-y(l)}}\,dl \\ &\le \sqrt{\frac{1}{s}}\sqrt{\int_0^s \frac{\log\left[(x(l)-y(l))^2+s\right]}{x(l)-y(l)}\,dl} \\ &\le \left(\sqrt{\frac{1}{s}}\right)\left(\sqrt{\frac{\log\left[(d(x,y))^2+s\right]}{d(x,y)}}\right)(\sqrt{s}) \\ &\le \sqrt{\frac{\log\left[(d(x,y))^2+s\right]}{d(x,y)}}. \end{aligned} \tag{92}$$

This implies the following:

$$\begin{aligned} b^s\left|P_x(t) - Q_y(t)\right| &= b^s d\left(P_x, P_y\right) \\ &\le \sqrt{\frac{\log\left[(d(x,y))^2+s\right]}{d(x,y)}}, b^s d\left(P_x, P_y\right)^2 \\ &\le \frac{\log\left[(d(x,y))^2+s\right]}{d(x,y)} \\ &\le \left(\sqrt{\frac{\log\left[(d(x,y))^2+s\right]}{d(x,y)}}\right) \\ &\quad\cdot \left(\sqrt{\log\left[(d(x,y))^2+s\right]}\right), \end{aligned} \tag{93}$$

Suppose we choose the values of α, β, ψ as $\sqrt{(\log[t^2+s]/t)}$, $\sqrt{\log[t^2+s}$, and t^2, respectively. Therefore, from inequality (93),

$$b^s\psi(d(Px,Qy)) \le \alpha((d(Sx,Sy))\beta(d(Sx,Sy)), \forall x \ge y \tag{94}$$

Since all the hypotheses of Theorem 17 are satisfied so there exists a unique common fixed point $v^* \in C(I)$ of P and Q, that is, the solution of equations (83) and (84).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors equally conceived the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Research Article

Common Fixed Point Results for Generalized $(g-\alpha_{sp}, \psi, \varphi)$ Contractive Mappings with Applications

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In this paper, we introduce a new class of $g-\alpha_{sp}-$ admissible mappings and prove some common fixed point theorems involving this new class of mappings which satisfy generalized contractive conditions in the framework of $b-$ metric spaces. We also provide two examples to show the applicability and validity of our results. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

1. Introduction

The Banach contraction principle [1] is one of the essential pillars of the theory of metric fixed points. Many authors have obtained generalizations, extensions, and applications of their findings by investigating the Banach contraction principle in many directions. One of the most popular and interesting topics among them is the study of new classes of spaces and their fundamental properties.

Czerwik [2] introduced the concept of $b-$metric space and proved some fixed point theorems of contractive mappings in $b-$metric space. Subsequently, some authors have studied on the fixed point theorems of a various new type of contractive conditions in $b-$metric space. Aydi et al. in [3] proved common fixed point results for mappings satisfying a weak $\phi-$contraction in $b-$metric spaces. Following the results of Berinde [4], Pacurar [5] obtained the existence and uniqueness of fixed point of $\phi-$contractions, and Zada et al. [6] got fixed point results satisfying contractive conditions of rational type. In 2019, Hussain et al. studied the existence and uniqueness of a periodic common fixed point for pairs of mappings via rational type contraction in [7]. After that, authors obtained fixed point theorems for $L-$cyclic $(\alpha, \beta)_s-$contractions and cyclic $(\alpha, \beta)-(\psi, \phi)_s-$rational type contractions and discussed the existence of a unique solution to nonlinear fractional differential equations in [8, 9], respectively. Also using rational type contractive conditions, Hussain et al. [10] got the existence and uniqueness of common $n-$tupled fixed point for a pair of mappings. Using a contractive condition defined by means of a comparison function, [11] established results regarding the common fixed points of two mappings. In 2014, Abbas et al. obtained the results on common fixed points of four mappings in $b-$metric space in [12].

To generalize the concept of $b-$metric spaces, Hussain and Shah in [13] introduced the notion of a cone $b-$metric space, which means that it is a generalization of $b-$metric spaces and cone metric spaces; they considered topological properties of cone $b-$metric spaces and obtained some results on KKM mappings in the setting of cone $b-$metric spaces. In [14], some fixed point results for weakly contractive mappings in ordered partial metric space were obtained. Recently, Samet et al. [15] introduced the concept of $\alpha-$admissible and $\alpha-\psi-$contractive mappings and presented fixed point theorems for them. In [16], Jamal et al. used $(\psi, \phi)-$weak contraction to generalize coincidence point results which are established in the context of partially ordered $b-$metric spaces. In [17, 18], Zoto et al. studied generalized α_{sp} contractive mappings and $(\alpha-\psi, \phi)-$contractions in $b-$metric–like space. Recently, in [16], Jamal et al. used (ψ, ϕ) – weak contraction to generalize coincidence point results which are established in the context of partially ordered b – metric spaces. Abu-Donia et al. [19] proved the uniqueness and existence of the fixed points for five mappings from a complete intuitionistic fuzzy 3 – metric space into itself under weak compatible of type (α) and asymptotically regular. For recent development on fixed point theory, we refer to [20–26].

Motivated and inspired by Theorems 27 and 29 in [17], Theorem 3.13 in [18], and Theorem 2.1 in [20], in this paper, our purpose is to introduce the concept of $g-\alpha_{s^p}$ – admissible mappings and obtain a few common fixed point results involving generalized contractive conditions in the framework of b – metric space. Furthermore, we provide examples that elaborated the useability of our results. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

2. Preliminaries

First of all, we introduce some definitions as follows:

Definition 1 (see [2]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \longrightarrow [0,+\infty)$ is said to be a b – metric if and only if, for all x, y, z ∈ X, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq s(d(x, z) + d(y, z))$

Generally, we call (X, d) a b – metric space with parameter $s \geq 1$.

Remark 2. We should note that a b – metric space with $s = 1$ is a metric space. We can find several examples of b – metric spaces which are not metric spaces. (see [24]).

Example 3 (see [20]). Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number

Then, $d(x, y)$ is a b – metric space with $s = 2^{p-1}$.

Definition 4 (see [12]). Let (X, d) be a b – metric space with parameter $s \geq 1$. Then, a sequence $\{x_n\}$ in X is said to be:

(i) b – convergent if and only if there exists $x \in X$ such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow +\infty$

(ii) a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ when $n, m \longrightarrow +\infty$

In general, a b – metric space is called complete if and only if each Cauchy sequence in this space is b – convergent.

Definition 5 (see [21]). Let f and g be two self-mappings on a nonempty set X. If w = fx = gx, for some x ∈ X, then x is said to be the coincidence point of f and g, where w is called the point of coincidence of f and g. Let $C(f, g)$ denote the set of all coincidence points of f and g.

Definition 6 (see [21]). Let f and g be two self-mappings defined on a nonempty set X. Then, f and g are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

We need the following lemma to obtain our main results:

Lemma 7 (see [20]). *Let (X, d) be a b – metric space with parameter $s \geq 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are b – convergent to x and y, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\longrightarrow+\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow+\infty} d(x_n, y_n) \leq s^2 d(x, y). \tag{1}$$

In particular, if $x = y$, then we have $\lim_{n\longrightarrow+\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\longrightarrow+\infty} d(x_n, z) \leq \limsup_{n\longrightarrow+\infty} d(x_n, z) \leq s d(x, z). \tag{2}$$

3. Main Results

In this section, we will show the existence and uniqueness of common fixed point for generalized $(g-\alpha_{s^p}, \psi, \varphi)$ contractive mappings in complete b – metric space. Meanwhile, we give two examples to support our results.

Definition 8. Let (X, d) be a b – metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be given mappings and $p \geq 1$ be an arbitrary constant. The mapping $f : X \longrightarrow X$ is said to be $g-\alpha_{s^p}$ – admissible if, for all $x, y \in X, \alpha(gx, gy) \geq s^p$ implies $\alpha(fx, fy) \geq s^p$.

Remark 9.

(i) Note that, for $g = I$, the definition reduces to an α_{s^p} -admissible mapping in a b – metric space

(ii) For $s = 1$, the definition reduces to the definition of an α – admissible mapping in a metric space

Let (X, d) be a complete b – metric space with parameter $s \geq 1$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be a function. Then,

(H_{s^p}) If $\{x_n\}$ is a sequence in X such that $gx_n \longrightarrow gx$ as $n \longrightarrow +\infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha(gx_{n_k}, gx) \geq s^p$ for all $k \in N$

(U_{s^p}) For all $u, v \in C(f, g)$, we have the condition of $\alpha(gu, gv) \geq s^p$ or $\alpha(gv, gu) \geq s^p$.

We know that contraction-type mappings are extended in several directions. Since Samet introduced the concept of α – admissible mappings and $\alpha-\psi$ – contractive mapping, some papers have been published to study a series of generalizations. Afterwards, these classes of mappings are used under generalized weakly contractive conditions.

We shall consider the contractive conditions in this section are constructed via auxiliary functions defined with the families Ψ, Φ, respectively:

$$\begin{aligned}\Psi &= \{\psi : [0,+\infty) \longrightarrow [0,+\infty) \text{is an increasing and continuous function}\},\\ \Phi &= \{\varphi : [0,+\infty) \longrightarrow [0,+\infty) \text{is an increasing and continuous function and } \varphi(t) = 0 \text{ iff } t = 0\}.\end{aligned} \quad (3)$$

Now, we introduce the notion of rational $(g - \alpha_{s^p}, \psi, \varphi)$ contraction in the setting of b – metric spaces.

Definition 10. Let (X, d) be a b – metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be two self-mappings. Assume that $\alpha : X \times X \longrightarrow [0,+\infty)$ and $p \geq 1$ is a constant. A mapping f is called a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping, if there exist $\psi \in \Psi, \varphi \in \Phi$ such that

$$\psi(\alpha(gx, gy)d(fx, fy)) \leq \psi(N(x, y)) - \varphi(M(x, y)), \quad (4)$$

for all $x, y \in X$ with $\alpha(x, y) \geq s^p$, where

$$\begin{aligned}N(x, y) = \max \Big\{& d(gx, gy), d(fx, gx), d(fy, gy),\\ &\cdot \frac{d(gx, fy) + d(fx, gy)}{4s}, \frac{d(fx, gx)d(fy, gy)}{1 + d(fx, fy)},\\ &\cdot \frac{d(fy, gy)[1 + d(fx, gx)]}{1 + d(gx, gy)}\Big\},\\ M(x, y) = \max \Big\{& d(fx, gy), d(gx, gy), d(fx, gx), d(fy, gy),\\ &\cdot \frac{d(fx, gx)[1 + d(gx, gy)]}{1 + d(fx, gy)}, \frac{d(fx, gx)[1 + d(fx, gx)]}{1 + d(fx, gy)},\\ &\cdot \frac{d(fx, gx)[1 + d(fy, gy)]}{1 + d(fx, gy)}\Big\}.\end{aligned} \quad (5)$$

Example 11. Let $X = [0,+\infty)$ and $d(x, y) = (x - y)^2$. Define mappings $f, g : X \longrightarrow X$ by

$$fx = \begin{cases} \dfrac{(x + x^2)}{8}, & x \in [0, 1] \\ 2x, & x > 1 \end{cases}, \text{and } gx = \begin{cases} \dfrac{7(x + x^2)}{8}, & x \in [0, 1] \\ \dfrac{7x}{4}, & x > 1 \end{cases}. \quad (6)$$

Define mappings $\alpha : g(X) \times g(X) \longrightarrow [0,+\infty)$ by

$$\alpha(x, y) = \begin{cases} s^2, & x, y \in \left[0, \dfrac{7}{4}\right] \\ 0, & \text{otherwise} \end{cases}, \quad (7)$$

and $\psi, \varphi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\psi(t) = t/2, \varphi(t) = 64t/585$.

It is clear that $f(X) \subset g(X)$. For $x, y \in X$ such that $\alpha(gx, gy) \geq s^2$, we can know that $gx, gy \in [0, 7/4]$ and this implies that $x, y \in [0, 1]$. By definitions, we obtain $fx, fy \in [0, 7/4]$ and $\alpha(fx, fy) \geq s^2$. That is, f is a $g - \alpha_{s^2}$ – admissible mapping. For all $x, y \in [0, 1]$, we have

$$\begin{aligned}&\psi(\alpha(gx, gy)d(fx, fy))\\ &= \frac{1}{2} \cdot 4 \cdot d(fx, fy)\\ &= 2 \cdot \frac{1}{64}\left((x + x^2) - (y + y^2)\right)^2\\ &\leq \frac{1}{32}\max\left\{(x + x^2)^2, (y + y^2)^2\right\},\\ &\psi(N(x, y))\\ &\geq \psi(\max\{d(fx, gx), d(fy, gy)\})\\ &= \frac{9}{32}\max\left\{(x + x^2)^2 (y + y^2)^2\right\},\\ &\varphi(M(x, y))\\ &= \varphi\Bigg(\max\Bigg\{\left(\frac{(x + x^2)}{8} - \frac{7(y + y^2)}{8}\right)^2, \left(\frac{7(x + x^2)}{8} - \frac{7(y + y^2)}{8}\right)^2,\\ &\left(\frac{(x + x^2)}{8} - \frac{7(x + x^2)}{8}\right)^2, \left(\frac{(y + y^2)}{8} - \frac{7(y + y^2)}{8}\right)^2,\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2\left[1 + (7(x + x^2)/8 - 7(y + y^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2},\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2\left[1 + ((x + x^2)/8 - 7(x + x^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2},\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2\left[1 + ((y + y^2)/8 - 7(y + y^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2}\Bigg\}\Bigg)\\ &\leq \varphi\Bigg(\max\Bigg\{\frac{49}{64}\max\left\{(x + x^2)^2, (y + y^2)^2\right\},\\ &\frac{49}{64}\max\left\{(x + x^2)^2, (y + y^2)^2\right\}, \frac{9}{16}(x + x^2)^2, \frac{9}{16}(y + y^2)^2,\\ &\frac{9}{16}(x + x^2)^2\left[1 + \frac{49}{64}\max\left\{(x + x^2)^2, (y + y^2)^2\right\}\right],\\ &\frac{9}{16}(x + x^2)^2\left[1 + \frac{9}{16}(x + x^2)^2\right], \frac{9}{16}(x + x^2)^2\left[1 + \frac{9}{16}(y + y^2)^2\right]\Bigg\}\Bigg)\end{aligned}$$

$$\leq \frac{64}{585} \cdot \frac{9}{16} \cdot \frac{65}{16} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} = \frac{1}{4} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\}. \quad (8)$$

According to the above inequalities, we get that

$$\begin{aligned} \psi(\alpha(gx, gy)d(fx, fy)) &\leq \frac{1}{32} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &= \frac{9}{32} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &\quad - \frac{1}{4} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &\leq \psi(N(x, y)) - \varphi(M(x, y)). \end{aligned} \quad (9)$$

It follows that f is a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping.

Theorem 12. *Let (X, d) be a complete $b -$ metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on X such that $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of X, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. If the following conditions are satisfied:*

(i) f is a $g - \alpha_{s^p} -$ admissible mapping

(ii) f is a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping

(iii) there is $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$

(iv) properties (H_{s^p}) and (U_{s^p}) are satisfied

(v) α has a transitive property type s^p, that is, for $x, y, z \in X$

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \quad (10)$$

Then, f and g have a unique point of coincidence in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof. According to condition (3), there exists an $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq s^p$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in N$. If $y_n = y_{n+1}$ for some $n \in N$, then we have $y_n = y_{n+1} = fx_{n+1} = gx_{n+1}$ and it is easy to see that f and g have a point of coincidence. Without loss of generality, assume that $y_n \neq y_{n+1}$ for all $n \in N$. By the condition (1), we get

$$\begin{aligned} \alpha(gx_0, gx_1) &= \alpha(gx_0, fx_0) \geq s^p, \\ \alpha(gx_1, gx_2) &= \alpha(fx_0, fx_1) \geq s^p, \\ \alpha(gx_2, gx_3) &= \alpha(fx_1, fx_2) \geq s^p. \end{aligned} \quad (11)$$

Therefore, by induction, we obtain $\alpha(gx_n, gx_{n+1}) = \alpha(y_{n-1}, y_n) \geq s^p$ for all $n \in N$. It follows from (4) that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(s^p d(y_n, y_{n+1})) \\ &\leq \psi(\alpha(gx_n, gx_{n+1}) d(fx_n, fx_{n+1})) \\ &\leq \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})), \end{aligned} \quad (12)$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \right. \\ &\quad \cdot \frac{d(y_{n-1}, y_{n+1}) + d(y_n, y_n)}{4s}, \frac{d(y_{n-1}, y_n) d(y_{n+1}, y_n)}{1 + d(y_n, y_{n+1})}, \\ &\quad \left. \cdot \frac{d(y_{n+1}, y_n)[1 + d(y_n, y_{n-1})]}{1 + d(y_{n-1}, y_n)} \right\} \\ &\leq \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}), \right. \\ &\quad \left. \cdot \frac{s[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]}{4s} \right\} \\ &= \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}) \right\}, \end{aligned} \quad (13)$$

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{ d(y_n, y_n), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \right. \\ &\quad \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n-1}, y_n)]}{1 + d(y_n, y_n)}, \frac{d(y_n, y_{n-1})[1 + d(y_n, y_{n-1})]}{1 + d(y_n, y_n)}, \\ &\quad \left. \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n+1}, y_n)]}{1 + d(y_n, y_n)} \right\}. \end{aligned} \quad (14)$$

If we assume that, for some $n \in N$,

$$d(y_n, y_{n+1}) \geq d(y_n, y_{n-1}) > 0, \quad (15)$$

then from inequalities (13) and (14), we have

$$N(x_n, x_{n+1}) \leq d(y_n, y_{n+1}), \quad (16)$$

$$M(x_n, x_{n+1}) \geq \max\{d(y_n, y_{n+1}), d(y_n, y_{n-1})\} = d(y_n, y_{n+1}). \quad (17)$$

Using (12), (16), and (17), one can obtain that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})), \end{aligned} \quad (18)$$

which gives $\varphi(d(y_n, y_{n+1})) \leq 0$ and then $y_n = y_{n+1}$, a contradiction. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$, that is, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence and so there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = r. \quad (19)$$

By virtue of (13) and (14) again, we have

$$N(x_n, x_{n+1}) \le d(y_n, y_{n-1}),\quad M(x_n, x_{n+1}) \ge d(y_n, y_{n-1}). \tag{20}$$

It follows that

$$\psi(d(y_n, y_{n+1})) \le \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \le \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})). \tag{21}$$

Now suppose that $r > 0$, then taking the limit as $n \longrightarrow +\infty$ in above inequality, we have $\psi(r) \le \psi(s^p r) \le \psi(r) - \varphi(r)$, which gives a contradiction. Hence,

$$\lim_{n\longrightarrow+\infty} d(y_n, y_{n+1}) = 0. \tag{22}$$

Next, we aim to prove that $\{y_n\}$ is a Cauchy sequence. Suppose on the contrary that, $\lim_{n,m\longrightarrow+\infty} d(y_n, y_m) \ne 0$, then there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ satisfying n_k is the smallest index for which $n_k > m_k > k$,

$$\varepsilon \le d\left(y_{m_k}, y_{n_k}\right), \tag{23}$$

$$d\left(y_{m_k}, y_{n_k-1}\right) < \varepsilon. \tag{24}$$

In view of the triangle inequality, we have

$$\varepsilon \le d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{n_k-1}\right) + sd\left(y_{n_k-1}, y_{n_k}\right) < s\varepsilon + sd\left(y_{n_k}, y_{n_k-1}\right). \tag{25}$$

Taking the upper limit as $k \longrightarrow +\infty$ in the above inequality and using (22), we have

$$\varepsilon \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k}\right) \le s\varepsilon. \tag{26}$$

Also,

$$d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{n_k-1}\right) + sd\left(y_{n_k-1}, y_{n_k}\right), \tag{27}$$

$$d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{m_k-1}\right) + sd\left(y_{m_k-1}, y_{n_k}\right), \tag{28}$$

$$d\left(y_{m_k-1}, y_{n_k}\right) \le sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k}\right). \tag{29}$$

From (24) and (27), we obtain

$$\frac{\varepsilon}{s} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k-1}\right) \le \varepsilon. \tag{30}$$

Using (28) and (29), we have

$$\frac{\varepsilon}{s} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k}\right) \le s^2\varepsilon. \tag{31}$$

Similarly,

$$d\left(y_{m_k-1}, y_{n_k-1}\right) \le sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k-1}\right),$$
$$d\left(y_{m_k}, y_{n_k}\right) \le sd\left(y_{m_k}, y_{m_k-1}\right) + s^2 d\left(y_{m_k-1}, y_{n_k-1}\right) + s^2 d\left(y_{n_k-1}, y_{n_k}\right), \tag{32}$$

so there is

$$\frac{\varepsilon}{s^2} \le \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) \le s\varepsilon. \tag{33}$$

In view of the definition of $N(x, y)$, one can deduce that

$$N\left(x_{m_k}, x_{n_k}\right) = \max\left\{d\left(y_{m_k-1}, y_{n_k-1}\right), d\left(y_{m_k}, y_{m_k-1}\right), d\left(y_{n_k}, y_{n_k-1}\right), \frac{d\left(y_{m_k-1}, y_{n_k}\right) + d\left(y_{m_k}, y_{n_k-1}\right)}{4s}, \frac{d\left(y_{m_k-1}, y_{m_k}\right) d\left(y_{n_k}, y_{n_k-1}\right)}{1 + d\left(y_{m_k}, y_{n_k}\right)}, \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{m_k}, y_{m_k-1}\right)\right]}{1 + d\left(y_{m_k-1}, y_{n_k-1}\right)}\right\}, \tag{34}$$

which yields that

$$\limsup_{k\longrightarrow+\infty} N\left(x_{m_k}, x_{n_k}\right) \le \max\left\{\varepsilon s, 0, 0, \frac{\varepsilon s^2 + \varepsilon}{4s}, 0, 0\right\} = \varepsilon s. \tag{35}$$

Similarly, we obtain

$$M\left(x_{m_k}, x_{n_k}\right) = \max\left\{d\left(y_{m_k}, y_{n_k-1}\right), d\left(y_{m_k-1}, y_{n_k-1}\right), d\left(y_{m_k}, y_{m_k-1}\right), d\left(y_{n_k}, y_{n_k-1}\right), \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k-1}, y_{n_{k-1}}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k}, y_{m_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}\right\}. \tag{36}$$

So there is

$$\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) \geq \max\left\{\frac{\varepsilon}{s}, \frac{\varepsilon}{s^2}, 0, 0, 0, 0, 0\right\} \geq \frac{\varepsilon}{s^2},$$
$$\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) \leq \max\{\varepsilon, s\varepsilon, 0, 0, 0, 0, 0\} = s\varepsilon, \tag{37}$$

that is,

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) \leq s\varepsilon. \tag{38}$$

Using the transitive property type s^p of α, we get

$$\alpha\left(x_{m_k}, x_{n_k}\right) \geq s^p. \tag{39}$$

Applying (4) with $x = x_{n_k}$ and $y = x_{m_k}$,we get

$$\begin{aligned}\psi\left(d\left(y_{m_k}, y_{n_k}\right)\right) &\leq \psi\left(s^p d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(\alpha\left(g x_{m_k}, g x_{n_k}\right) d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(N\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{40}$$

By (35) and (38), we have

$$\begin{aligned}\psi(s\varepsilon) \leq \psi(s^p\varepsilon) &\leq \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(\limsup_{k\longrightarrow+\infty} N\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right)\\ &\leq \psi(s\varepsilon) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right),\end{aligned} \tag{41}$$

which implies that

$$\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) = 0, \tag{42}$$

a contradiction to (38). Therefore, $\{y_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists a $u \in X$ such that

$$\lim_{n\longrightarrow+\infty} d(y_n, u) = \lim_{n\longrightarrow+\infty} d(f x_n, u) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, u) = 0. \tag{43}$$

Since $g(X)$ is closed, we have $u \in g(X)$. It follows that one can choose a $z \in X$ such that $u = gz$, and we can write (43) as

$$\lim_{n\longrightarrow+\infty} d(y_n, gz) = \lim_{n\longrightarrow+\infty} d(f x_n, gz) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, gz) = 0. \tag{44}$$

The property (H_{s^p}) yields that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that $\alpha(y_{n_k-1}, gz) \geq s^p$ for all $k \in N$. If $fz \neq gz$, applying contractive condition (4) with $x = x_{n_k}$ and $y = z$, we have

$$\begin{aligned}\psi\left(d\left(y_{n_k}, fz\right)\right) = \psi\left(d\left(f x_{n_k}, fz\right)\right) &\leq \psi\left(s^p d\left(f x_{n_k}, fz\right)\right)\\ &\leq \psi\left(\alpha\left(g x_{n_k}, gz\right) d\left(f x_{n_k}, fz\right)\right)\\ &\leq \psi\left(N\left(x_{n_k}, z\right)\right) - \varphi\left(M\left(x_{n_k}, z\right)\right),\end{aligned} \tag{45}$$

where

$$\begin{aligned}N\left(x_{n_k}, z\right) = \max\Bigg\{& d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz),\\ & \cdot \frac{d\left(y_{n_k-1}, fz\right) + d\left(y_{n_k}, gz\right)}{4s},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right) d(fz, gz)}{1 + d\left(y_{n_k}, fz\right)},\\ & \cdot \frac{d(fz, gz)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k-1}, gz\right)}\Bigg\},\\ M\left(x_{n_k}, z\right) = \max\Bigg\{& d\left(y_{n_k}, gz\right), d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz),\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k-1}, gz\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)[1 + d(fz, gz)]}{1 + d\left(y_{n_k}, gz\right)}\Bigg\}.\end{aligned} \tag{46}$$

It is easy to show that

$$\begin{aligned}\limsup_{k\longrightarrow+\infty} N\left(x_{n_k}, z\right) &\leq \max\left\{0, 0, d(gz, fz), \frac{sd(fz, gz)}{4s}, 0, d(gz, fz)\right\}\\ &= d(gz, fz),\\ \liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right) &= \max\{0, 0, 0, d(fz, gz), 0, 0, 0\} = d(fz, gz).\end{aligned} \tag{47}$$

Taking the upper limit as $k \longrightarrow +\infty$ in (45), we have

$$\begin{aligned}\psi(d(gz, fz)) &\leq \psi\left(s^{p-1} d(gz, fz)\right) = \psi\left(s^p \frac{1}{s} d(gz, fz)\right)\\ &\leq \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(f x_{n_k}, fz\right)\right) \leq \psi\left(\limsup_{k\longrightarrow+\infty} N\left(x_{n_k}, z\right)\right)\\ &\quad - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right)\right) \leq \psi(d(gz, fz)) - \varphi(d(gz, fz)),\end{aligned} \tag{48}$$

which implies that

$$d(fz, gz) = 0. \tag{49}$$

That is, $fz = gz$. Therefore, $u = fz = gz$ is a point of coincidence for f and g. By using contractive condition (4) and the property (U_{s^p}), one can conclude that the point of coincidence is unique. Assume on the contrary that, there exist $z, z' \in C(f, g)$ and $z \neq z'$. According to the property of (U_{s^p}), without loss of generality, we assume that

$$\alpha\left(gz, gz'\right) \geq s^p. \tag{50}$$

Applying (4) with $x = z$ and $y = z'$, we obtain that

$$d\left(fz, fz'\right) = 0, \tag{51}$$

that is, $fz = fz'$. By the weak compatibility of f and g, it is easy to show that z is a unique common fixed point. This completes the proof.

Remark 13. It is obvious that the mappings defined in Example 11 satisfy the conditions of Theorem 12, so f and g have a unique common fixed point 0.

In Theorem 12, put $\psi(t) = t$, one can get the following result.

Corollary 14. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on X with $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of X, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. If the following conditions are satisfied:*

(i) f is a $g - \alpha_{s^p} -$ admissible mapping

(ii) there is function $\varphi \in \Phi$ such that

$$\alpha(gx, gy)d(fx, fy) \leq N(x, y) - \varphi(M(x, y)), \tag{52}$$

where $N(x, y), M(x, y)$ are same as Theorem 12,

(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$

(iv) properties (H_{s^p}) and (U_{s^p}) are satisfied

(v) α has a transitive property type s^p, that is, for $x, y, z \in X$

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \tag{53}$$

Then, f and g have a unique point of coincidence in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Theorem 15. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be given self-mappings on X with $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of X, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. Suppose that the following conditions are satisfied:*

(i) f is a $g - \alpha_{s^p} -$ admissible mapping

(ii) there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$

$$\psi(\alpha(gx, gy)d(fx, fy)) \leq \psi(L(x, y)) - \varphi(M(x, y)), \tag{54}$$

where $M(x, y)$ is same as Theorem 12 and

$$L(x, y) = \max\left\{d(fx, gy), d(fx, gx), d(fy, gy), \cdot \frac{d(gx, gy) + d(fx, gy)}{2s}\right\}. \tag{55}$$

(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$

(iv) properties (H_{s^p}) and (U_{s^p}) are satisfied

(v) α has a transitive property type s^p, that is, for $x, y, z \in X$

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p, \tag{56}$$

then f and g have a unique point of coincidence in X. Moreover, f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof. It is the same as the proof of Theorem 12, we also define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = g x_{n+1}$ for $n \in N$ and suppose that $y_n \neq y_{n+1}$ for each $n \in N$, so one can get that

$$\alpha(y_{n-1}, y_n) = \alpha(gx_n, gx_{n+1}) \geq s^p. \tag{57}$$

It follows from (54) that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(s^p d(y_n, y_{n+1})) \\ &\leq \psi(\alpha(gx_n, gx_{n+1})d(fx_n, fx_{n+1})) \\ &\leq \psi(L(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})), \end{aligned} \tag{58}$$

where

$$\begin{aligned} L(x_n, x_{n+1}) &= \max\left\{d(y_n, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \cdot \frac{d(y_{n-1}, y_n) + d(y_n, y_n)}{2s}\right\} \\ &\leq \max\left\{d(y_{n+1}, y_n), d(y_n, y_{n-1})\right\}, \end{aligned} \tag{59}$$

$$M(x_n, x_{n+1}) = \max \left\{ d(y_n, y_n), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n-1}, y_n)]}{1 + d(y_n, y_n)}, \cdot \frac{d(y_n, y_{n-1})[1 + d(y_n, y_{n-1})]}{1 + d(y_n, y_n)}, \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n+1}, y_n)]}{1 + d(y_n, y_n)} \right\}. \tag{60}$$

If we assume that, for some $n \in N$

$$d(y_n, y_{n+1}) \geq d(y_{n-1}, y_n) > 0, \tag{61}$$

then according to inequalities (59) and (60), we obtain

$$\begin{aligned} L(x_n, x_{n+1}) &\leq d(y_{n+1}, y_n), \\ M(x_n, x_{n+1}) &\geq d(y_{n+1}, y_n). \end{aligned} \tag{62}$$

In view of (58), we get

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(L(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})), \end{aligned} \tag{63}$$

which implies that $d(y_n, y_{n+1}) = 0$, a contradiction to $d(y_n, y_{n+1}) > 0$. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$. Hence, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence. Consequently, the limit of the sequence is a nonnegative number, say $r \geq 0$. That is, $\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = r$.

By (59) and (60), we have

$$\begin{aligned} L(x_n, x_{n+1}) &\leq d(y_n, y_{n-1}), \\ M(x_n, x_{n+1}) &\geq d(y_n, y_{n-1}). \end{aligned} \tag{64}$$

So,

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(L(\mathrm{x}_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})). \end{aligned} \tag{65}$$

If $r > 0$, then letting $n \longrightarrow +\infty$ in above inequality, we obtain that $\psi(r) = \psi(r) - \varphi(r)$, which implies that $r = 0$, i.e.,

$$\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = 0. \tag{66}$$

Now, we prove that $\{y_n\}$ is a Cauchy sequence. If not, as the proof of Theorem 12, there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ so that n_k is the smallest index for which $n_k > m_k > k$, and the following inequalities hold:

$$\varepsilon \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k}, y_{n_k}\right) \leq s\varepsilon, \tag{67}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k}, y_{n_k - 1}\right) \leq \varepsilon, \tag{68}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k - 1}, y_{n_k}\right) \leq s^2\varepsilon, \tag{69}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k - 1}, y_{n_k - 1}\right) \leq s\varepsilon. \tag{70}$$

In view of the definitions of $L(x, y)$ and $M(x, y)$, we have

$$\begin{aligned} L\left(x_{m_k}, x_{n_k}\right) &= \max \left\{ d\left(y_{m_k}, y_{n_k-1}\right), d\left(y_{m_k-1}, y_{m_k}\right), d\left(y_{n_k}, y_{n_k-1}\right), \frac{d\left(y_{m_k-1}, y_{n_k-1}\right) + d\left(y_{m_k}, y_{n_k-1}\right)}{2s} \right\}. \\ M\left(x_{m_k}, x_{n_k}\right) &= \max \left\{ d\left(y_{m_k}, y_{n_k-1}\right), d\left(y_{m_k-1}, y_{n_k-1}\right), d\left(y_{m_k-1}, y_{m_k}\right), d\left(y_{n_k}, y_{n_k-1}\right), \right. \\ &\quad \cdot \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k-1}, y_{n_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k}, y_{m_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \\ &\quad \left. \cdot \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)} \right\}. \end{aligned} \tag{71}$$

Letting $k \longrightarrow +\infty$ and using (67)–(70), one can obtain

$$\limsup_{k\longrightarrow+\infty} L\left(x_{m_k}, x_{n_k}\right) \le \max\left\{\varepsilon, 0, 0, \frac{s\varepsilon + s}{2s}\right\} = \varepsilon. \tag{72}$$

Similarly, we get that

$$\begin{aligned}\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) &\le \max\{\varepsilon, s\varepsilon, 0, 0, 0, 0, 0\} = s\varepsilon,\\ \liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) &\ge \max\left\{\frac{\varepsilon}{s}, \frac{\varepsilon}{s^2}, 0, 0, 0, 0, 0\right\} = \frac{\varepsilon}{s^2}.\end{aligned} \tag{73}$$

That is,

$$\frac{\varepsilon}{s^2} \le \liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) \le s\varepsilon. \tag{74}$$

Using the transitive property type s^p of α, we have

$$\alpha\left(x_{m_k}, x_{n_k}\right) \ge s^p. \tag{75}$$

Taking $x = x_{n_k}$ and $y = x_{m_k}$ in (54), one can deduce that

$$\begin{aligned}\psi\left(d\left(y_{m_k}, y_{n_k}\right)\right) &\le \psi\left(s^p d\left(fx_{m_k}, fx_{n_k}\right)\right)\\ &\le \psi\left(\alpha\left(gx_{m_k}, gx_{n_k}\right) d\left(fx_{m_k}, fx_{n_k}\right)\right)\\ &\le \psi\left(L\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{76}$$

Therefore,

$$\begin{aligned}\psi(\varepsilon) \le \psi(s^p\varepsilon) &\le \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(fx_{m_k}, fx_{n_k}\right)\right)\\ &\le \psi\left(\limsup_{k\longrightarrow+\infty} L\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right)\\ &\le \psi(\varepsilon) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{77}$$

It follows that $\liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) = 0$, and which gives a contradiction to (74). Hence,

$$\lim_{n,m\longrightarrow+\infty} d(y_n, y_m) = 0. \tag{78}$$

From the completeness of X and the closure of $g(X)$, there exists $u \in X$ such that

$$\lim_{n\longrightarrow+\infty} d(y_n, u) = \lim_{n\longrightarrow+\infty} d(fx_n, u) = \lim_{n\longrightarrow+\infty} d(gx_{n+1}, u) = 0. \tag{79}$$

It follows that one can choose a $z \in X$ such that $u = gz$, and write the above equality as

$$\lim_{n\longrightarrow+\infty} d(y_n, gz) = \lim_{n\longrightarrow+\infty} d(fx_n, gz) = \lim_{n\longrightarrow+\infty} d(gx_{n+1}, gz) = 0. \tag{80}$$

In view of the property (H_{s^p}), one can get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ with $\alpha(y_{n_k-1}, gz) \ge s^p$ for all $k \in N$. If $fz \ne gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (54), we have

$$\begin{aligned}\psi\left(d\left(y_{n_k}, fz\right)\right) = \psi\left(d\left(fx_{n_k}, fz\right)\right) &\le \psi\left(s^p d\left(fx_{n_k}, fz\right)\right)\\ &\le \psi\left(\alpha\left(gx_{n_k}, gz\right) d\left(fx_{n_k}, fz\right)\right)\\ &\le \psi\left(L\left(x_{n_k}, z\right)\right) - \varphi\left(M\left(x_{n_k}, z\right)\right),\end{aligned} \tag{81}$$

where

$$\begin{aligned}L\left(x_{n_k}, z\right) &= \max\left\{d\left(y_{n_k}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz), \frac{d\left(y_{n_k-1}, gz\right) + d\left(y_{n_k}, gz\right)}{2s}\right\}.\\ M\left(x_{n_k}, z\right) &= \max\left\{d\left(y_{n_k}, gz\right), d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz), \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k-1}, gz\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\right.\\ &\quad\left. \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k}, gz\right)}, \frac{d\left(y_{n_k}, y_{n_k-1}\right)[1 + d(fz, gz)]}{1 + d\left(y_{n_k}, gz\right)}\right\}.\end{aligned} \tag{82}$$

Consequently,

$$\limsup_{k\longrightarrow+\infty} L\left(x_{n_k}, z\right) \le \max\{0, 0, d(fz, gz), 0\} = d(fz, gz).$$
$$\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right) = d(fz, gz). \tag{83}$$

Taking the upper limit as $k \longrightarrow +\infty$ in (81), we get

$$\begin{aligned}\psi(d(gz, fz)) &\le \psi\left(s^{p-1}d(gz, fz)\right) = \psi\left(s^p \frac{1}{s} d(gz, fz)\right)\\ &\le \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(fx_{n_k}, fz\right)\right) \le \psi\left(\limsup_{k\longrightarrow+\infty} L\left(x_{n_k}, z\right)\right)\\ &\quad - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right)\right) \le \psi(d(gz, fz))\\ &\quad - \varphi(d(gz, fz)).\end{aligned} \tag{84}$$

It follows that $d(fz, gz) = 0$. That is, $u = fz = gz$ is a point of coincidence for f and g. Using the same technique in the proof of Theorem 12, one can complete the proof.

Example 16. Let $X = [0,+\infty)$ and $d(x, y) = (x - y)^2$. Define mappings $f, g : X \longrightarrow X$ by

$$fx = \begin{cases} \dfrac{x}{64}, & x \in [0, 1] \\ e^x - e + \dfrac{1}{2}, & x > 1 \end{cases}, \text{ and } gx = \begin{cases} \dfrac{x}{2}, & x \in [0, 1] \\ e^{2x} - e^2 + \dfrac{1}{2}, & x > 1 \end{cases}. \tag{85}$$

Define mappings $\alpha : g(X) \times g(X) \longrightarrow [0,+\infty)$ by

$$\alpha(x, y) = \begin{cases} s^2, & x, y \in \left[0, \dfrac{1}{2}\right], \\ 0, & \text{otherwise} \end{cases} \tag{86}$$

and $\psi, \varphi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\psi(t) = t, \varphi(t) = 3828t/4805$.

It is clear that $f(X) \subset g(X)$ and $g(X)$ is closed. For $x, y \in X$ such that $\alpha(gx, gy) \ge s^2$, we can know that $gx, gy \in [0, 1/2]$ and which implies that $x, y \in [0, 1]$. It follows that $fx, fy \in [0, 1/2]$ and $\alpha(fx, fy) \ge s^2$, that is, f is a $g - \alpha_{s^p}$ – admissible mapping.

For $x, y \in [0, 1]$, we have

$$\begin{aligned}\psi(\alpha(gx, gy)d(fx, fy)) &= 4 \cdot \left(\frac{x}{64} - \frac{y}{64}\right)^2 \le \frac{4}{64^2} \max\left\{x^2, y^2\right\},\\ \psi(L(x, y)) &\ge \psi(\max\{d(fx, gx), d(fy, gy)\}) = \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\},\\ \varphi(M(x, y)) &= \varphi\left(\max\left\{\left(\frac{x}{64} - \frac{y}{2}\right)^2, \left(\frac{x}{2} - \frac{y}{2}\right)^2, \left(\frac{x}{64} - \frac{x}{2}\right)^2, \left(\frac{y}{64} - \frac{y}{2}\right)^2, \frac{(x/64 - x/2)^2\left[1 + (x/2 - y/2)^2\right]}{1 + (x/64 - y/2)^2},\right.\right.\\ &\quad \left.\left. \cdot \frac{(x/64 - x/2)^2\left[1 + (x/2 - x/64)^2\right]}{1 + (x/64 - y/2)^2}, \frac{(x/64 - x/2)^2\left[1 + (y/2 - y/64)^2\right]}{1 + (x/64 - y/2)^2}\right\}\right)\\ &\le \varphi\left(\max\left\{\frac{1}{4}\max\left\{x^2, y^2\right\}, \frac{1}{4}\max\left\{x^2, y^2\right\}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\},\right.\right.\\ &\quad \left.\left. \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5057}{4096}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5057}{4096}\right\}\right)\\ &= \frac{3828}{4805} \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4}.\end{aligned} \tag{87}$$

Obviously, we conclude

$$\begin{aligned}\psi(\alpha(gx, gy)d(fx, fy)) &\le \frac{4}{64^2} \max\left\{x^2, y^2\right\} = \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} - \frac{3828}{4805} \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4}\\ &\le \psi(L(x, y)) - \varphi(M(x, y)).\end{aligned} \tag{88}$$

It follows that all conditions of Theorem 15 are satisfied. It is obvious that 0 is the unique common fixed point of f and g.

Remark 17. Taking $S = T = I_x$ in Theorem 2.1 of [12], Roshan et al. give that the existence of common fixed point for mappings f, g such that

$$d(fx, gy) \leq \frac{q}{s^4} \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\}, \tag{89}$$

where $q \in (0, 1)$ is a constant. Suppose all hypotheses in Example 16 are true. For $x = 0, y \in (0, 1/2)$, it is easy to calculate that

$$\begin{aligned} d(fx, gy) &= \frac{y^2}{4} > \frac{y^2}{16} \geq \frac{q}{16} \max \left\{ y^2, 0, \frac{y^2}{4}, \frac{y^2}{2 \cdot 64^2} + \frac{y^2}{2} \right\} \\ &= \frac{q}{s^4} \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\}, \end{aligned} \tag{90}$$

which implies that Theorem 2.1 of [12] cannot be applied to testify the existence of common fixed points of the mappings f and g in X.

If $\psi(t) = t$ and $\varphi(t) = t$ in Theorem 15, we get the following result immediately:

Corollary 18. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on X such that $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of X, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. If the following conditions are satisfied:*

(i) f is a $g - \alpha_{s^p}$ – admissible mapping,

(ii) for $x, y \in X$

$$\alpha(gx, gy) d(fx, fy) \leq L(x, y) - M(x, y), \tag{91}$$

where $L(x, y), M(x, y)$ are same as Theorem 15,

(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$

(iv) properties (H_{s^p}) and (U_{s^p}) are satisfied

(v) α has a transitive property type s^p, that is, for $x, y, z \in X$

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \tag{92}$$

Then, f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Let $g = I$, $\psi(t) = t$, and $\varphi(t) = Lt$ ($L > 0$ is a constant), we obtain that

Theorem 19. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and $f : X \longrightarrow X$ be a given self-mapping on X. Let $\alpha : X \times X \longrightarrow [0,+\infty)$ be a given mapping. If the following conditions are satisfied:*

(i) f is a α_{s^p} – admissible mapping

(ii) for $x, y \in X$

$$\alpha(x, y) d(fx, fy) \leq (1 - L) K^*(x, y), \tag{93}$$

where

$$K^*(x, y) = \max \{d(x, y), d(fx, x), d(fy, y), d(fx, y)\}, L \in (0, 1), \tag{94}$$

(iii) there exists $x_0 \in X$ with $\alpha(x_0, fx_0) \geq s^p$

(iv) properties (H_{s^p}) and (U_{s^p}) are satisfied when $g = I$

(v) α has a transitive property type s^p, that is, for $x, y, z \in X$

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \tag{95}$$

Then, f has a unique fixed point.

Proof. The proof of Theorem 19 is similar to that of Theorem 15, we omit it.

Remark 20. Since a $b-$metric space is a metric space when $s = 1$, so our results can be viewed as the generalization and the extension of comparable results.

4. Application

In this section, we will use Theorem 19 to show that there is a solution to the integral equation:

$$x(t) = \int_0^T G(t, r, x(r)) dr. \tag{96}$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$. The standard metric given by

$$\rho(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)| \text{for all } x, y \in X. \tag{97}$$

Now for $p \geq 1$, we define

$$d(x, y) = (\rho(x, y))^p = \sup_{t \in [0,T]} |x(t) - y(t)|^p \text{ for all } x, y \in X. \tag{98}$$

It is obvious that (X, d) is a complete $b-$metric space with $s = 2^{p-1}$.

Consider the mapping $f : X \longrightarrow X$ defined by

$$fx(t) = \int_0^T G(t, r, x(r))dr, \tag{99}$$

and let $\xi : R \times R \longrightarrow R$ be a given function.

Theorem 21. *Consider equation (96) and suppose that*

(i) $G : [0, T] \times [0, T] \times R \longrightarrow R^+$ *is continuous*

(ii) there exists $x_0 \in X$ *such that* $\xi(x_0(t), fx_0(t)) \geq 0$ *for all* $t \in [0, T]$

(iii) for all $t \in [0, T]$ *and* $x, y \in X$, $\xi(x(t), y(t)) \geq 0$ *implies* $\xi(fx(t), fy(t)) \geq 0$

(iv) properties (H_{s^p}) *and* (U_{s^p}) *are satisfied when* $g = I$

(v) there exists a continuous function $\gamma : [0, T] \times [0, T] \longrightarrow R^+$ *such that*

$$\sup_{t\in[0,T]} \int_0^T \gamma(t, r)dr \leq 1, \tag{100}$$

(vi) there exists a constant $L \in (0, 1)$ *such that for* $(t, r) \in [0, T] \times [0, T]$

$$|G(t, r, x(r)) - G(t, r, y(r))| \leq \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)|x(r) - y(r)|. \tag{101}$$

Then, the integral equation (96) has a unique solution $x \in X$.

Proof. Define $\alpha : X \times X \longrightarrow [0,+\infty)$ by

$$\alpha(x, y) = \begin{cases} s^p, & \text{if } \xi(x(t), y(t)) \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{102}$$

It is easy to prove that f is α_{s^p}-admissible. For $x, y \in X$, by virtue of assumptions (1)–(6), we have

$$\begin{aligned} s^p d(fx(t), fy(t)) &= s^p \sup_{t\in[0,T]} |fx(t) - fy(t)|^p \\ &= s^p \sup_{t\in[0,T]} \left| \int_0^T G(t, r, x(r))dr - \int_0^T G(t, r, y(r))dr \right|^p \\ &\leq s^p \sup_{t\in[0,T]} \left(\int_0^T |G(t, r, x(r)) - G(t, r, y(r))|dr \right)^p \\ &\leq s^p \sup_{t\in[0,T]} \left(\int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)|x(r) - y(r)|dr \right)^p \\ &\leq s^p \sup_{t\in[0,T]} \left(\int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)dr \right)^p \sup_{t\in[0,T]} |x(t) - y(t)|^p \\ &\leq (1 - L)K^*(x(t), y(t)), \end{aligned} \tag{103}$$

which implies that

$$\alpha(x(t), y(t))d(fx(t), fy(t)) \leq (1 - L)K^*(x(t), y(t)). \tag{104}$$

Therefore, all the conditions of Theorem 19 hold. As a result, the mapping f has a unique fixed point $x \in X$, which is a solution of the integral equation (96).

5. Conclusions

In this manuscript, we introduced a new class of $g - \alpha_{s^p} -$ admissible mappings and obtained common fixed point theorems for generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mappings in the framework of $b -$metric space. Further, we provided examples that elaborated the useability of our results. As an application of our result, we obtained a solution to an integral equation. The obtained results will be helpful for the variational iteration method, so we are going to study this topic in future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Periodic and Fixed Points for Caristi-Type G-Contractions in Extended b-Gauge Spaces

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In this paper, we introduce extended b-gauge spaces and the extended family of generalized extended pseudo-b-distances. Moreover, we define the sequential completeness and construct the Caristi-type G-contractions in the framework of extended b-gauge spaces. Furthermore, we develop periodic and fixed point results in this new setting endowed with a graph. The obtained results of this paper not only generalize but also unify and improve the existing results in the corresponding literature.

1. Introduction and Preliminaries

The famous Caristi fixed point theorem [1] states that a self-mapping T on a complete metric space (U, p) possesses a fixed point w in U if

$$p(u, Tu) \leq f(u) - f(Tu), \tag{1}$$

for all $u \in U$, where $f : U \longrightarrow [0,\infty)$ is a lower semicontinuous function.

Indeed, Caristi [1] observed these results when he searched for alternative proof of the outstanding fixed point theorem of Banach. It is known also Caristi-Kirk fixed point theorem [2]. In fact, Caristi's theorem is equivalent to metric completeness [3]. For some other contributions to this topic, we refer to [4–10].

In view of extending the concept of Banach contraction, Banach G-contraction was introduced by Jachymaski [11] in complete metric space accompanied with the graph G where the set of vertex matches with the metric space (see also [12–22]).

The notion of metric space has been refined and extended in several distinct directions, by many authors [23–25]. Among all, the notion of gauge space was initiated by Dugundji [26] as a generalization of a metric space. In 1973, Reilly [27] studied quasi-gauge spaces and proved that it generalizes topological spaces, quasi uniform spaces, and quasi metric spaces. This notion was extended as b-gauge spaces by Ali et al. [28] in 2015. For further facts on gauge spaces, we recommend the reader to [29–36].

In 2013, Wlodarczyk and Plebaniak [37] have given the notion of left (right) $\mathcal{J}$-families of generalized pseudo distances in quasi-gauge spaces that generalizes the abovementioned distances and provides powerful and useful tools for finding solutions to various problems of nonlinear analysis.

This paper is aimed at introducing extended b-gauge spaces $(U, Q_{\varphi;\Omega})$ and the extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo-b-distances generated by $(U, Q_{\varphi;\Omega})$. Moreover, by using extended $\mathcal{J}_{\varphi;\Omega}$-family, we define the extended $\mathcal{J}_{\varphi;\Omega}$-sequential completeness and construct the Caristi-type G-contractions $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$. Furthermore, we investigate periodic and fixed point results for these mappings in the new setting endowed with a graph, which generalizes and improves the existing results in the literature of fixed point theory.

In what follows, we recollect some essential concepts and basic results which shall be used in the sequel. For a nonempty set U, we use the notation 2^U to denote the set of all nonempty subsets of the space U. If $T : U \longrightarrow 2^U$ is a multivalued map, then the sets of all fixed points are denoted by Fix (T), that is, Fix $(T) = \{u \in U : u \in T(u)\}$. In addition, the set of all periodic points of T is denoted by Per (T), that is, Per $(T) = \{u \in U : u \in T^{[k]}(u) \text{for some } k \text{ in } N\}$, where $T^{[k]} = T \circ T \circ T \circ \cdots .. \circ T$ (k-times). A dynamic process of the system (U, T) starting at $w^0 \in U$ is a sequence $\{w^m : m \in \{0\} \cup N\}$ defined by $\forall_{m \in N}\{w^m \in T(w^{m-1})\}$.

One of the most interesting extension of a metric is the notion of b-metric [38, 39].

Definition 1. Let U be a nonempty set and $s \geq 1$. A map $q : U \times U \longrightarrow [0,\infty)$ is b-metric, if it satisfies the following properties:

(a) $q(e, f) = 0 \Leftrightarrow e = f$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq s\{q(e, f) + q(f, g)\}$

for all $e, f, g \in U$. Here, the pair (U, q, s) is called b-metric space.

Indeed, b-metric is one of the most interesting and original generalizations of the notion metric. As it is seen obviously, in the case of $s = 1$, the notions b-metric and standard metric coincide. On the other hand, despite the standard metric, b-metric is not continuous despite metric. Further, an open (closed) ball is not an open (closed) set. For more details on b-metric and interesting examples, we refer to, e.g., [40–49].

In 2017, Kamran et al. [50] refined the notion of b-metric under the name "extended b-metric."

Definition 2. Suppose U be a nonvoid set and let $\varphi : U \times U \longrightarrow [1,\infty)$. A map $q : U \times U \longrightarrow [0,\infty)$ is said to be an extended b -metric, if it satisfies the following properties:

(a) $q(e, f) = 0 \Leftrightarrow e = f$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$, for all $e, f, g \in U$

For given extended b-metric q on U, a pair (U, q) is called extended b-metric space.

Definition 3. Let U be a nonvoid set. The map $q : U \times U \longrightarrow [0,\infty)$ is called to be pseudo metric, if it satisfies the following properties:

(a) $q(e, e) = 0$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq q(e, f) + q(f, g)$, for all $e, f, g \in U$

The pair (U, q) is said to be pseudo metric space.

In 2015, Ali et al. [28] have defined gauge spaces in the setting of b_s-pseudo metrics called b-gauge spaces. In order to introduce extended b-gauge spaces, we start here with the introduction of the notion of extended pseudo-b-metric.

Definition 4. Let U be a nonempty set and $\varphi : U \times U \longrightarrow [1, \infty)$. A map $q : U \times U \longrightarrow [0,\infty)$ is an extended pseudo-b -metric, if it satisfies the following properties:

(a) $q(e, e) = 0$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$, for all $e, f, g \in U$

The pair (U, q) is called extended pseudo-b-metric space.

Example 1. Let $U = [0, 1]$. Define $q : U \times U \longrightarrow [0,\infty)$ and $\varphi : U \times U \longrightarrow [1,\infty)$ for all $e, f \in U$ as follows:

$$\begin{aligned} q(e, f) &= (e - f)^2, \\ \varphi(e, f) &= e + f + 2, \end{aligned} \tag{2}$$

for all $e, f, g \in U$. Then, q is an extended pseudo-b-metric on U. Indeed, $q(e, e) = 0$ and $q(e, f) = q(f, e)$. Further, $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$ holds.

Example 2. Let $U = \{e, f, g\}$ and $\varphi : U \times U \longrightarrow [1,\infty)$ with $\varphi(e, f) = |e| + |f| + 2$. Define $q : U \times U \longrightarrow [0,\infty)$ as follows:

$$\begin{aligned} q(e, e) &= 0, \\ q(e, f) &= q(f, e) = 1, \\ q(f, g) &= q(g, f) = \frac{1}{2}, \\ \text{and } q(e, g) &= q(g, e) = 2, \end{aligned} \tag{3}$$

for all $e, f, g \in U$. Further, $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$ holds. In conclusion, q is an extended pseudo-b-metric on U. Notice that $2 = q(e, g) > 3/2 = q(e, f) + q(f, g)$; thus, q is not a pseudo metric on U. This example shows that extended pseudo-b-metric is more general than pseudo metric.

Definition 5. Each family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended pseudo-b-metrics $q_\beta : U \times U \longrightarrow [0,\infty)$, $\beta \in \Omega$, is called an extended b-gauge on U.

Definition 6. Let the family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be an extended b-gauge on U. The topology $\mathcal{T}(Q_{\varphi;\Omega})$ whose subbase is defined by the family $\mathcal{B}(Q_{\varphi;\Omega}) = \{B(e, \varepsilon_\beta): e \in U, \varepsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e, \varepsilon_\beta) = \{f \in U : q_\beta(e, f) < \varepsilon_\beta\}$ is called the topology induced by $Q_{\varphi;\Omega}$ on U.

Definition 7. Suppose $(U, \mathcal{T})$ be a topological space and $Q_{\varphi;\Omega}$ be an extended b-gauge on U such that $\mathcal{T} = \mathcal{T}(Q_{\varphi;\Omega})$. Then, the topological space is called to be an extended b-gauge space, which is denoted by $(U, Q_{\varphi;\Omega})$.

Remark 8. (a) Each gauge space is b_s-gauge space (for s =1), and each b-gauge space is an extended b-gauge space (for $\varphi_\beta(u, v) = s$, for each $\beta \in \Omega$, where $s \geq 1$). Therefore, we can term extended b-gauge spaces as the largest general spaces.

(b) We observe that if $\varphi_\beta(u,v)=s$, for each $\beta\in\Omega$, where $s\geq 1$, the above definitions turn down to the corresponding definitions in b-gauge spaces, and if $\varphi_\beta(u,v)=1$ for each $\beta\in\Omega$, the above definitions turn down to the corresponding definitions in gauge spaces.

We now introduce the notion of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U (extended $\mathcal{J}_{\varphi;\Omega}$-families is the generalization of extended quasi-b-gauges).

Definition 9. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. The family $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ where $J_\beta:U\times U\longrightarrow[0,\infty)$, $\beta\in\Omega$ is said to be the extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo- b-distances on U (for short, extended $\mathcal{J}_{\varphi;\Omega}$-family on U) if there exists $\varphi=\{\varphi_\beta\}_{\beta\in\Omega}$, where $\varphi_\beta:U\times U\longrightarrow[1,\infty)$ such that for each $\beta\in\Omega$ and for all $u,v,w\in U$, the following hold:

(J1) $J_\beta(u,w)\leq\varphi_\beta(u,w)\{J_\beta(u,v)+J_\beta(v,w)\}$

(J2) For each sequences $(u_m:m\in N)$ and $(v_m:m\in N)$ in U fulfilling

$$\lim_{m\longrightarrow\infty}\sup_{n>m}J_\beta(u_m,u_n)=0, \tag{4}$$

$$\lim_{m\longrightarrow\infty}J_\beta(v_m,u_m)=0, \tag{5}$$

the following holds:

$$\lim_{m\longrightarrow\infty}q_\beta(v_m,u_m)=0. \tag{6}$$

We denote $J_{(U,Q_{\varphi;\Omega})}=\{\mathcal{J}_{\varphi;\Omega}:\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}\}$.

We mention here some trivial properties of extended $\mathcal{J}_{\varphi;\Omega}$-families in the following remark.

Remark 10. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Then, for each $\beta\in\Omega$ and for all $u,v,w\in U$, the following hold:

(a) $Q_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$

(b) Let $\mathcal{J}_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$. If $J_\beta(v,v)=0$ and $J_\beta(u,v)=J_\beta(v,u)$, then J_β is an extended pseudo-b-metric

(c) There exist examples of $\mathcal{J}_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$ which show that the maps J_β are not an extended pseudo-b-metrics (see following Example 3)

Example 3. Suppose $U=[0,1]\subset R$. Let $Q_{\varphi;\Omega}=\{q\}$ be the family of pseudo-b-metric where $q:U\times U\longrightarrow[0,\infty)$ be defined as in Example 1.

Let the set $F=[1/8,1]\subset U$. Let $d\in(0,\infty)^\Omega$ satisfies $\{\delta(F)<d\}$, where $\{\delta(F)=\sup\{q(e,f)\colon e,f\in F\}\}$. Let $J:U\times U\longrightarrow[0,\infty)$ and $\varphi:U\times U\longrightarrow[1,\infty)$ for all $e,f\in U$ define as follows:

$$J(e,f)=\begin{cases} q(e,f) & \text{if } F\cap\{e,f\}=\{e,f\},\\ d=4 & \text{if } F\cap\{e,f\}\neq\{e,f\},\end{cases} \tag{7}$$

and $\varphi(e,f)=e+f+2$. Then, $\mathcal{J}_{\varphi;\Omega}=\{J\}\in J_{(U,Q)}$.

We observe that $J(e,g)\leq\varphi(e,g)\{J(e,f)+J(f,g)\}$ for all $e,f,g\in U$; thus, condition $(\mathcal{J}_1)$ holds. Indeed, condition $(\mathcal{J}_1)$ will not hold in case if there exists some $e,f,g\in U$ such that $J(e,g)=d$, $J(e,f)=q(e,f)$, $J(f,g)=q(f,g)$, and $\varphi(e,g)\{q(e,f)+q(f,g)\}\leq d$. However, then this implies the existence of $h\in\{e,g\}$ with $h\notin F$, and on other hand, $e,f,g\in F$, which is impossible.

Now suppose that (4) and (5) are satisfied by the sequences $(u_m:m\in N)$ and $(v_m:m\in N)$ in U. Then, (5) implies

$$\forall_{0<\varepsilon<d}\exists_{m_1\in N}\forall_{m\geq m_1}\{J(v_m,u_m)<\varepsilon\}. \tag{8}$$

By (8) and (7), we have

$$\begin{gathered}\forall_{m\geq m_1}\{F\cap\{v_m,u_m\}=\{v_m,u_m\}\},\\ \forall_{0<\varepsilon<d}\exists_{m_1\in N}\forall_{m\geq m_1}\{q(v_m,u_m)=J(v_m,u_m)<\varepsilon\}.\end{gathered} \tag{9}$$

Thus, (6) is satisfied by the sequences $(u_m:m\in N)$ and $(v_m:m\in N)$. Therefore, $\mathcal{J}_{\varphi;\Omega}$ is an extended $\mathcal{J}_{\varphi;\Omega}$-family on U.

Now, using extended $\mathcal{J}_{\varphi;\Omega}$-family on U, we establish the following concepts of extended $\mathcal{J}_{\varphi;\Omega}$-completeness in the extended b-gauge space $(U,Q_{\varphi;\Omega})$.

Definition 11. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. A sequence $(u_m:m\in N)$ is extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U if, for all $\beta\in\Omega$, $\lim_{m\longrightarrow\infty}\sup_{n>m}J_\beta(u_m,u_n)=0$.

Definition 12. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. The sequence $(u_m:m\in N)$ is called to be extended $\mathcal{J}_{\varphi;\Omega}$-convergent to $u\in U$ if $\lim_{m\longrightarrow\infty}^{\mathcal{J}_{\varphi;\Omega}}u_m=u$, where

$$\lim_{m\longrightarrow\infty}^{\mathcal{J}_{\varphi;\Omega}}u_m=u\Leftrightarrow\lim_{m\longrightarrow\infty}J_\beta(u,u_m)=0=\lim_{m\longrightarrow\infty}J_\beta(u_m,u). \tag{10}$$

Definition 13. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. If $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)}\neq\varnothing$, where $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)}=\{u\in U:\lim_{m\longrightarrow\infty}^{\mathcal{J}_{\varphi;\Omega}}u_m=u\}$. Then, the sequence $(u_m:m\in N)$ in U is extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

Definition 14. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U.

The space $(U, Q_{\varphi;\Omega})$ is called $\mathcal{J}_{\varphi;\Omega}$-sequentially complete extended b-gauge space, if every extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy in U is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

Remark 15. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space.

(a) For each subsequence $(v_m : m \in N)$ of $(u_m : m \in N)$, where $(u_m : m \in N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, we have $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)} \subset S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in N)}$

(b) We observe that if $\varphi_\beta(u, v) = s$ for all $\beta \in \Omega$, where $s \geq 1$ and $\mathcal{J}_{\varphi;\Omega} = Q_{\varphi;\Omega}$, the above definitions of completeness reduce to the corresponding definitions in b-gauge spaces (see [28])

Definition 16. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. The map $T^{[k]} : U \longrightarrow U$, where $k \in N$ is called to be an extended $Q_{\varphi;\Omega}$-closed map on U if for each sequence $(x_m : m \in N)$ in $T^{[k]}(U)$, which is extended $Q_{\varphi;\Omega}$-converging in U, i.e., $S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)} \neq \varnothing$ and its subsequences $(v_m : m \in N)$ and $(u_m : m \in N)$ satisfy $\forall_{m\in N}\{v_m \in T^{[k]}(u_m)\}$ has the property that there exists $w \in S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)}$ such that $w \in T^{[k]}(w)$.

Definition 17. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. A set $Y \in 2^U$ is a $\mathcal{J}_{\varphi;\Omega}$-closed in U if $Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$, where $cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$, is the $\mathcal{J}_{\varphi;\Omega}$-closure in U, which indicates the set of all $x \in U$ for which there exists a sequence $(x_m : m \in N)$ in Y which $\mathcal{J}_{\varphi;\Omega}$-converges to x.

Define $Cl^{\mathcal{J}_{\varphi;\Omega}}(U) = \{Y \in 2^U : Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)\}$. Thus, $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ denotes the class of all nonempty $\mathcal{J}_{\varphi;\Omega}$-closed subsets of U.

Definition 18. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U, and let, for each $\beta \in \Omega$, $u \in U$ and for all $V \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$,

$$J_\beta(u, V) = \inf \{J_\beta(u, z): z \in V\}. \tag{11}$$

Define on $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ the distance $D_\beta^{\mathcal{J}_{\varphi;\Omega}}$ of Hausdorff type, where $D_\beta^{\mathcal{J}_{\varphi;\Omega}} : Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \times Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \longrightarrow [0,\infty)$, $\beta \in \Omega$ as follows:

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(U, V) = \begin{cases} \max \left\{ \sup_{u\in U} J_\beta(u, V), \sup_{v\in V} J_\beta(v, U) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise} \end{cases} \tag{12}$$

for each $\beta \in \Omega$ and for all $U, V \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$.

In this paper, Ω is a directed set and $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space enriched with the graph $G = (V, E)$ where the set of vertices coincides with set U and the set of edges E contains $\{(v, v): v \in V\}$. Also, G is such that no two edges are parallel.

2. Periodic and Fixed Point Theorems

Our main results for multivalued mappings are now given below.

Theorem 19. *Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \longrightarrow [0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\phi_\beta : U \longrightarrow [0,\infty)$, $\beta \in \Omega$ be a lower semicontinuous function such that for each $u \in U$ and $v \in Tu$ where $(u, v) \in E$, we have, for all $\beta \in \Omega$,*

$$J_\beta(v, Tv) \leq \phi_\beta(u) - \phi_\beta(v). \tag{13}$$

Assume, moreover, that the following conditions hold:

(i) There exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$

(ii) For each $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$, there exists $v \in Tu$ such that

$$J_\beta(u, v) \leq r_\beta J_\beta(u, Tu), \tag{14}$$

for all $\beta \in \Omega$. Then, the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup N)$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $\forall_{z^0\in U}\{S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing\}$

(II) Furthermore, assume that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(a_1) $Fix(T^{[k]}) \neq \varnothing$

(a_2) $\forall_{z^0\in U}\exists_{z\in Fix(T^{[k]})}\{z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\}$

Proof. (I) We first show that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.

By assumption (i), there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now using (13), we can write, for each $\beta \in \Omega$,

$$J_\beta(z^1, Tz^1) \leq \phi_\beta(z^0) - \phi_\beta(z^1). \tag{15}$$

Now by using assumption (ii) and (15), we have $r_\beta > 1$ for each $\beta \in \Omega$ and $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \le r_\beta J_\beta(z^1, Tz^1) \le r_\beta \left\{\phi_\beta(z^0) - \phi_\beta(z^1)\right\}. \quad (16)$$

As T is edge preserving, we can write $(z^1, z^2) \in E$. Proceeding in the same manner, we have a sequence $\{z^m : m \in \{0\} \cup N\}$ such that $(z^m, z^{m+1}) \in E$ and for each $m \in N$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^{m+1}) \le r_\beta J_\beta(z^m, Tz^m) \le r_\beta \left\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\right\}. \quad (17)$$

This implies that the sequence $\{\phi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta \ge 0$ such that $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$ as $m \longrightarrow \infty$. Now for $m, p \in N$ and each $\beta \in \Omega$, we have

$$\begin{aligned} J_\beta(z^m, z^{m+p}) &\le \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) \\ &\quad + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) \\ &\quad + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta \\ &\quad \cdot (z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\ &\quad + \cdots + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\quad \cdot (z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\ &\le \varphi_\beta(z^m, z^{m+p}) r_\beta \left\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\right\} \\ &\quad + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) r_\beta \\ &\quad \cdot \left\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\right\} \\ &\quad + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta \\ &\quad \cdot (z^{m+2}, z^{m+p}) r_\beta \left\{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\right\} \\ &\quad + \cdots + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\quad \cdot (z^{m+p-1}, z^{m+p}) r_\beta \left\{\phi_\beta(z^{m+p-2}) - \phi_\beta(z^{m+p-1})\right\}. \end{aligned} \quad (18)$$

Letting $m \longrightarrow \infty$, we have $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$. This implies that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta \in \Omega$ and for each $z^0 \in U$,

$$\forall_{\varepsilon>0} \exists_{k\in N} \forall_{n,m\in N; n>m\ge k} \left\{J_\beta(z^m, z^n) < \varepsilon\right\}. \quad (19)$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\} \cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$, we have, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N} \forall_{m\in N; m\ge k} \left\{J_\beta(z, z^m) < \varepsilon\right\}. \quad (20)$$

Thus, from (19) and (20), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N}$, defining $(u_m = z^m : m \in \{0\} \cup N)$ and $(v_m = z : m \in \{0\} \cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N} \forall_{m\in N; m\ge k} \left\{q_\beta(z, z^m) < \varepsilon\right\}. \quad (21)$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{for } m \in \{0\} \cup N, \quad (22)$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered} (z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing. \end{gathered} \quad (23)$$

Also, its subsequences

$$\begin{gathered} \left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U), \end{gathered} \quad (24)$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m) \quad (25)$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m\in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m\in N)}. \quad (26)$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}} \left\{z \in T^{[k]}(z)\right\}. \quad (27)$$

Thus, (a_1) holds. The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$. Hence, the theorem is proved.□

Theorem 20. *Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \longrightarrow [0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\phi_\beta : U \longrightarrow [0,\infty)$, $\beta \in \Omega$ be a lower semicontinuous function such that for each $u \in$*

U and $v \in Tu$ where $(u, v) \in E$, we have, for each $\beta \in \Omega$,

$$J_\beta(u, v) \le \phi_\beta(u) - \phi_\beta(v). \tag{28}$$

Assume, moreover, that the following condition holds:

(i) There exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$

Then, the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup N)$ is extended $Q_{\varphi;\Omega}$ -convergent sequence in U; thus, $\forall_{z^0 \in U}\{S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)} \neq \emptyset\}$

(II) Furthermore, assume that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(b_1) $Fix(T^{[k]}) \neq \emptyset$

(b_2) $\forall_{z^0 \in U}\exists_{z \in Fix(T^{[k]})}\{z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)}\}$

Proof. (I) We first show that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U. By assumption (i), there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now using (28), we can write, for each $\beta \in \Omega$,

$$J_\beta(z^0, z^1) \le \phi_\beta(z^0) - \phi_\beta(z^1). \tag{29}$$

As T is edge preserving, we can write $(z^1, z^2) \in E$. Proceeding in the same manner, we have a sequence $\{z^m : m \in \{0\} \cup N\}$ such that $(z^m, z^{m+1}) \in E$ and for each $m \in N$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^{m+1}) \le \phi_\beta(z^m) - \phi_\beta(z^{m+1}). \tag{30}$$

This implies that the sequence $\{\phi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta \ge 0$ such that $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$ as $m \longrightarrow \infty$. Now for $m, p \in N$ and each $\beta \in \Omega$, we have

$$\begin{aligned} J_\beta(z^m, z^{m+p}) \le{}& \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\ &+\cdots+\varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\cdot (z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\ \le{}& \varphi_\beta(z^m, z^{m+p})\left\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\left\{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) \\ &\cdot \left\{\phi_\beta(z^{m+2}) - \phi_\beta(z^{m+3})\right\} \\ &+\cdots+\varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta(z^{m+p-1}, z^{m+p}) \\ &\cdot \left\{\phi_\beta(z^{m+p-1}) - \phi_\beta(z^{m+p})\right\}. \end{aligned} \tag{31}$$

Letting $m \longrightarrow \infty$, we have $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$. This implies that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta \in \Omega$ and for each $z^0 \in U$,

$$\forall_{\varepsilon>0}\exists_{k \in N}\forall_{n,m \in N; n>m \ge k}\{J_\beta(z^m, z^n) < \varepsilon\}. \tag{32}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\} \cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)}$, we have, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k \in N}\forall_{m \in N; m \ge k}\{J_\beta(z, z^m) < \varepsilon\}. \tag{33}$$

Thus, from (32) and (33), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup N}$, defining $(u_m = z^m : m \in \{0\} \cup N)$ and $(v_m = z : m \in \{0\} \cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k \in N}\forall_{m \in N; m \ge k}\left\{q_\beta(z, z^m) < \varepsilon\right\}. \tag{34}$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)} \neq \emptyset$.

(II) To prove (b_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)} \neq \emptyset$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{for } m \in \{0\} \cup N, \tag{35}$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered} (z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)} \neq \emptyset. \end{gathered} \tag{36}$$

Also, its subsequences

$$\begin{gathered} \left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U), \end{gathered} \tag{37}$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m), \tag{38}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m : m \in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m \in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m : m \in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m \in N)}. \tag{39}$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)}=S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}}\left\{z\in T^{[k]}(z)\right\}. \tag{40}$$

Thus, (b_1) holds. The assertion (b_2) follows from (b_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\emptyset$. Hence, the theorem is proved. □

Theorem 21. *Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$, where $J_\beta:U\times U\longrightarrow[0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U,Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T:U\longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\psi_\beta:U\longrightarrow[0,\infty)$, $\beta\in\Omega$ be a upper semicontinuous function such that for each $u\in U$ and $v\in Tu$ where $(u,v)\in E$, we have, for each $\beta\in\Omega$,*

$$J_\beta(v,Tv)\le\psi_\beta(u)-\psi_\beta(v). \tag{41}$$

Assume, moreover, that the following conditions hold:

(i) There exist $z^0\in U$ and $z^1\in Tz^0$ such that $(z^0,z^1)\in E$

(ii) For each $\{r_\beta:r_\beta>1\}_{\beta\in\Omega}$ and $x\in U$, there exists $y\in Tx$ such that for each $\beta\in\Omega$,

$$J_\beta(x,y)\le r_\beta J_\beta(x,Tx). \tag{42}$$

Then, the following statements hold:

(I) For any $z^0\in U$, $(z^m:m\in\{0\}\cup N)$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $\forall_{z^0\in U}\{S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\emptyset\}$

(II) Furthermore, assume that $T^{[k]}$ for some $k\in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(c_1) $Fix(T^{[k]})\neq\emptyset$

(c_2) $\forall_{z^0\in U}\exists_{z\in Fix(T^{[k]})}\{z\in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\}$

Proof. (I) We first show that $(z^m:m\in\{0\}\cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U. By assumption (i), there exist $z^0\in U$ and $z^1\in Tz^0$ such that $(z^0,z^1)\in E$. Now using (41), we can write, for each $\beta\in\Omega$,

$$J_\beta(z^1,Tz^1)\le\psi_\beta(z^0)-\psi_\beta(z^1). \tag{43}$$

Now by using assumption (ii) and (43), we have $r_\beta>1$ for each $\beta\in\Omega$ and $z^2\in Tz^1$ such that

$$J_\beta(z^1,z^2)\le r_\beta J_\beta(z^1,Tz^1)\le r_\beta\left\{\psi_\beta(z^0)-\psi_\beta(z^1)\right\}. \tag{44}$$

As T is edge preserving, we can write $(z^1,z^2)\in E$. Proceeding as above, we have a sequence $\{z^m:m\in\{0\}\cup N\}$ such that $(z^m,z^{m+1})\in E$, and for each $m\in N$ and for all $\beta\in\Omega$, we have

$$J_\beta(z^m,z^{m+1})\le r_\beta J_\beta(z^m,Tz^m)\le r_\beta\left\{\psi_\beta(z^{m-1})-\psi_\beta(z^m)\right\}. \tag{45}$$

This implies that the sequence $\{\psi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta\ge0$ such that $\{\psi_\beta(z^m)\}\longrightarrow l_\beta$ as $m\longrightarrow\infty$. Now for $m,p\in N$ and each $\beta\in\Omega$, we have

$$\begin{aligned}J_\beta(z^m,z^{m+p})&\le\varphi_\beta(z^m,z^{m+p})J_\beta(z^m,z^{m+1})+\varphi_\beta(z^m,z^{m+p})\varphi_\beta\\&\quad\cdot(z^{m+1},z^{m+p})J_\beta(z^{m+1},z^{m+2})\\&\quad+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\varphi_\beta\\&\quad\cdot(z^{m+2},z^{m+p})J_\beta(z^{m+2},z^{m+3})\\&\quad+\cdots+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\cdots\varphi_\beta\\&\quad\cdot(z^{m+p-1},z^{m+p})J_\beta(z^{m+p-1},z^{m+p})\\&\le\varphi_\beta(z^m,z^{m+p})r_\beta\left\{\psi_\beta(z^{m-1})-\psi_\beta(z^m)\right\}\\&\quad+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})r_\beta\\&\quad\cdot\left\{\psi_\beta(z^m)-\psi_\beta(z^{m+1})\right\}+\varphi_\beta(z^m,z^{m+p})\varphi_\beta\\&\quad\cdot(z^{m+1},z^{m+p})\varphi_\beta(z^{m+2},z^{m+p})r_\beta\\&\quad\cdot\left\{\psi_\beta(z^{m+1})-\psi_\beta(z^{m+2})\right\}\\&\quad+\cdots+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\cdots\varphi_\beta\\&\quad\cdot(z^{m+p-1},z^{m+p})r_\beta\left\{\psi_\beta(z^{m+p-2})-\psi_\beta(z^{m+p-1})\right\}.\end{aligned} \tag{46}$$

Letting $m\longrightarrow\infty$, we have $\{\psi_\beta(z^m)\}\longrightarrow l_\beta$. This implies that $(z^m:m\in\{0\}\cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta\in\Omega$ and for each $z^0\in U$,

$$\forall_{\varepsilon>0}\exists_{k\in N}\forall_{n,m\in N;n>m\ge k}\{J_\beta(z^m,z^n)<\varepsilon\}. \tag{47}$$

Now, since $(U,Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m:m\in\{0\}\cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z\in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}$, we have, for all $\beta\in\Omega$ and for each $\varepsilon>0$,

$$\exists_{k\in N}\forall_{m\in N;m\ge k}\{J_\beta(z,z^m)<\varepsilon\}. \tag{48}$$

Thus, from (47) and (48), fixing $z\in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N}$, defining $(u_m=z^m:m\in\{0\}\cup N)$ and $(v_m=z:m\in\{0\}\cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta\in\Omega$ and for each $\varepsilon>0$,

$$\exists_{k\in N}\forall_{m\in N;m\ge k}\left\{q_\beta(z,z^m)<\varepsilon\right\}. \tag{49}$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\emptyset$.

(II) To prove (c_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{ for } m \in \{0\} \cup N, \quad (50)$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered} (z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing. \end{gathered} \quad (51)$$

Also, its subsequences

$$\begin{gathered} \left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U), \end{gathered} \quad (52)$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m), \quad (53)$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m:m\in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m:m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)}. \quad (54)$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}} \left\{z \in T^{[k]}(z)\right\}. \quad (55)$$

Thus, (c_1) holds. The assertion (c_2) follows from (c_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing$. Hence, the theorem is proved. □

Remark 22. (a) The fixed point results concerning Caristi-type contractions in gauge space in [51] require the completeness of the space (U, d). Therefore, our main theorems for Caristi-type G-contractions in the extended b-gauge space are a new generalization of the results in [51] in which assumptions are weaker and assertions are stronger.

(b) Our results for Caristi-type G-contractions in extended b-gauge space tell about periodic points as well, hence improve the results in [51]

(c) We observe that by taking $\forall_{\beta\in\Omega}\{\varphi_\beta(u, v) = s \geq 1\}$ in this paper, we obtain the results in b-gauge space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Some Qualitative Analyses of Neutral Functional Delay Differential Equation with Generalized Caputo Operator

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In this paper, a new class of a neutral functional delay differential equation involving the generalized ψ-Caputo derivative is investigated on a partially ordered Banach space. The existence and uniqueness results to the given boundary value problem are established with the help of the Dhage's technique and Banach contraction principle. Also, we prove other existence criteria by means of the topological degree method. Finally, Ulam-Hyers type stability and its generalized version are studied. Two illustrative examples are presented to demonstrate the validity of our obtained results.

1. Introduction

Fractional calculus has demonstrated high visibility and capability in the applications of various topics linked to physics, signal processing, mechanics, electromagnetics, economics, biology, and many more [1–3]. Even recently, fractional differential equations (FDEqs) have acquired particular attention because of their numerous applications in the fractional modeling [4–12]. FDEqs involving hybrid nonlinearity have been gained much attention during the past few years. This class of equations arises from various mathematical and physical phenomena such as three-layer beam, electromagnetic waves, curved beam's deflection with a constant or varying cross-section, and gravity-driven [13–19].

Almeida [20] introduced a new fractional derivative, named the ψ-fractional order derivative (FOD), with respect to another function, which extended the classical fractional derivative. Therefore, the generalizations of existing results in fractional calculus and FBVPs have been established by several mathematicians [21–25].

The qualitative analysis of FDEqs such as the solution's existence and uniqueness is the most popular problems that many researchers focus on. Various fixed point theorems are considered as the most effective tools for dealing with such problems. In this work, we follow some results presented by Ragusa et al. [26, 27] concerning the qualitative properties of some suitable FDEqs.

In the last decade, a new technique was developed by Dhage [28], named Dhage iteration principle, for investigating the numerical solutions' existence and approximation of integral and FDEqs by constructing a sequence of successive approximations with initial lower or upper solution. Dhage [29–32] provided a generalized form of hybrid fixed point theorem in the context of a metric space having the partial order without applying any geometric condition. In Dhage's research study, with the help of the measure of noncompactnessan, an algorithm for studying the solutions' existence of a certain nonlinear functional integral equation was investigated under weaker conditions. The advantages of the applied method were studied by Dhage to compare with the standard approaches that involve Banach, Schauder's, and Krasnoselskii's fixed point theorems. As a result, the iteration method due to Dhage has recently became an important tool for investigating the solution's existence and approximate results of nonlinear hybrid FDEqs that have various scientific applications such as air motion, electricity, fluid dynamics, process control with nonlinear structures, and electromagnetism. In addition, this method can be extended to other functional differential equations (FuDEqs) classes. On the other side, in recent years, the topological degree method has been considered as one of the main tools for studying the existence results to different fractional differential equations and inclusions. This method will be used in our research study to derive desired results in relation to the solutions of the proposed problem. For more details, see [33–38].

The FuDEqs' stability was first proposed by Ulam [39] and then by Hyers [40]. Later on, this type of stability and its generalization were called of the Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) type, respectively. Investigating the UH and GUH stability has been given a special attention in studying all FuDEqs kinds and FDEqs in particular [41–44].

Motivated by the novel developments in ψ-fractional calculus, the solution's existence, uniqueness, and UH stability of the proposed neutral functional differential equation (NFuDEq) is investigated in this research work. The NFuDEq is expressed as:

$$\begin{cases} {}^{c}D_{a^+}^{\nu;\psi}[\varpi(\tau) - \mathbb{F}(\tau, \varpi_\tau)] = \mathbb{H}(\tau, \varpi_\tau), \tau \in \mathrm{J} := [a, b], \\ \varpi(\tau) = \phi(\tau), \quad \tau \in [a - \delta, a], \end{cases} \tag{1}$$

where the ψ-Caputo FOD, denoted by ${}^{c}\mathbb{D}_{a^+}^{\nu;\psi}$, of order $\nu \in (0, 1)$, given $\mathbb{F}, \mathbb{H} : \mathrm{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that $\mathbb{F}(a, \varpi_a) = 0$, and $\phi : [a - \delta, a] \longrightarrow \mathbb{R}$ is a continuous function with $\phi(a) = 0$. For any function u defined on $[a - \delta, a]$ and any $\tau \in \mathrm{J}$, it is given by

$$u_\tau(\rho) = u(\tau + \rho), \rho \in [-\delta, 0]. \tag{2}$$

The main aim of this research work is to apply an iteration principle due to Dhage to ensure the solutions' existence along with approximation of (1) under weaker partial continuity and partial compactness type conditions.

This article is constructed as follows: some important definitions and lemmas which are needed for our results are provided in Section 2. The solutions' existence and approximation of (1) are proven in Section 3 via the Dhage iteration principle. In Section 4, a theorem, based on the coincidence degree theory for condensing maps, is established on the solutions' existence of the proposed NFuDEq (1). In Section 5, the solution's uniqueness for the NFuDEq (1) is proven by the Banach contraction principle of solutions. Moreover, we investigate the UH and GUH stability for the NFuDEq (1). Some illustrative examples for supposed problem are provided at the end to validate our theoretical results.

2. Fundamental Preliminaries

Some important definitions, theorems, and lemmas concerning advanced fractional calculus and nonlinear analysis are stated in this section which are needed for our approach in the next parts.

Consider the space of all continuous real-valued functions $\mathscr{C} = C(\mathrm{J}, \mathbb{R})$ endowed with the norm

$$\|\varpi\|_{\mathscr{C}} = \sup_{\tau \in \mathrm{J}} |\varpi(\tau)|. \tag{3}$$

Also, $\mathscr{C}_\delta = C([-\delta, 0], \mathbb{R})$ is endowed with norm

$$\|\phi\|_{\mathscr{C}_\delta} = \sup_{\tau \in -\delta, 0]} |\phi(\tau)|, \text{and } \|\varpi_\tau\|_{\mathscr{C}_\delta} = \sup_{\rho \in -\delta, 0]} |\varpi(\tau + \rho)|. \tag{4}$$

Consider the Banach space $\mathscr{C}_b = C([a - \delta, b], \mathbb{R})$ defined on $[a - \delta, b]$ with the norm

$$\|\varpi\|_{\mathscr{C}_b} = \|\varpi\|_{\mathscr{C}_\delta} + \|\varpi\|_{\mathscr{C}} = \sup_{\tau \in a - \delta, b]} |\varpi(\tau)|. \tag{5}$$

The order relation $\preccurlyeq$ is defined as follows:

$$[\varpi \preccurlyeq \omega \Leftrightarrow \varpi(t) \leq \omega(t)] \forall t \in [a - \delta, b], \tag{6}$$

which gives a partial ordering in $\mathscr{C}_b$.

From the research study in [29], let us now state some necessary definitions and preliminary results for our research work. Assume that $\bar{\mathbb{X}} = (\mathbb{X}, \preccurlyeq, \|.\|)$ displays a real partial order on $\mathbb{X}$. If for ϖ, ω in $\mathbb{X}$, either $\varpi \preccurlyeq \omega$ or $\omega \preccurlyeq \varpi$, then ϖ and ω are termed as comparable elements, and also when all members of $\varnothing \neq \mathscr{C} \subset \mathbb{X}$ are comparable, then $\mathscr{C}$ is named either totally ordered or a chain. If there exists a nondecreasing (resp., nonincreasing) sequence $(\varpi_n)_{n \in N}$ and $\varpi *$ in $\mathbb{X}$ such that $\varpi_n \longrightarrow \varpi *$ as $n \longrightarrow \infty$, then $\mathbb{X}$ is regular ($\varpi_n \preccurlyeq \varpi *$ (resp. $\varpi_n \pm \varpi *$)) for all $n \in \mathbb{N}$. By assuming this fact that there are lower and upper bounds in $\mathbb{X}$ for every both members of $\mathbb{X}$, in that case, the partially ordered Banach space $\mathbb{X}$ is named regular and lattice.

Definition 1 (see [29]). An operator: $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ is termed as nondecreasing or isotone if $\mathcal{Q}$ maintains the order relation $\preccurlyeq$, i.e., when $\varpi \preccurlyeq \omega$, it means that $\mathcal{Q}\varpi \preccurlyeq \mathcal{Q}\omega$ for all $\varpi, \omega \in \mathbb{X}$.

Definition 2 (see [29]). A mapping $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ has the compactness specification if $\mathcal{Q}(\mathbb{X})$ is a set in $\mathbb{X}$ with the relative compactness. In addition, $\mathcal{Q}$ is totally bounded if $\mathcal{Q}(S)$ has the relative compactness property in $\mathbb{X}$, where $S \subseteq \mathbb{X}$ is an arbitrary bounded set.

Every operator having the continuity and total boundedness properties will be completely continuous.

Definition 3 (see [29]). $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ has the partial continuity property at $a \in \mathbb{X}$, if for each $\varepsilon > 0$, $\delta > 0$ exists so that

$\|\mathcal{Q}\varpi - \mathcal{Q}a\| < \varepsilon$ whenever $\|\varpi - a\| < \delta$ and ϖ and a are comparable. Assuming $\mathcal{Q}$ as an operator with the partial continuity on $\mathbb{X}$, it is well-known that $\mathcal{Q}$ is continuous on each chain $\mathscr{C} \subset \mathbb{X}$. Furthermore, if $\mathcal{Q}(\mathscr{C})$ is bounded for every $\mathscr{C} \subseteq \mathbb{X}$, then $\mathcal{Q}$ is partially bounded. In addition, $\mathcal{Q}$ is uniformly partially bounded if all existing chains $\mathcal{Q}(\mathscr{C}) \subseteq \mathbb{X}$ involve the boundedness by a bound uniquely.

Definition 4 (see [29]). $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ has the partial compactness if $\mathcal{Q}(\mathscr{C}) \subset \mathbb{X}$ has the relative compactness with respect to all chains $\mathscr{C} \subseteq \mathbb{X}$. It has the partial total boundedness property if for each bounded and totally ordered set $\mathscr{C}$ contained in $\mathbb{X}$, $\mathcal{Q}(\mathscr{C}) \subset \mathbb{X}$ possesses the relative compactness.

Every operator with the partial continuity and the partial total boundedness is named as partially completely continuous on the underlying space.

Remark 5. Assume that $\mathcal{Q}$ is a nondecreasing selfmap on $\mathbb{X}$ and $\mathscr{C}$ is an arbitrary chain in it. In this case, $\mathcal{Q}$ possesses the partial compactness or the partial boundedness specifications whenever $\mathcal{Q}(\mathscr{C})$ is relatively compact or bounded in $\mathbb{X}$.

Definition 6 (see [28]). Regard d and $\preccurlyeq$ as a metric and an order relation on $\mathbb{X}$. We say that d and $\preccurlyeq$ are compatible if $\{\varpi_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ is monotone, and if a subsequence $\{\varpi_{n_k}\}_{k\in N}$ of $\{\varpi_n\}_{n\in\mathbb{N}}$ tends to ϖ^*, then $\{\varpi_n\}_{n\in\mathbb{N}}$ tends to ϖ^*. Similar definition can be applied on a partially order norm space. A subset S of $\mathbb{X}$ is named Janhavi if the order relation $\preccurlyeq$ and the metric d (or the norm $\|\cdot\|$) are compatible in it. Particularly, if $S = \mathbb{X}$, then we say that $\mathbb{X}$ is Janhavi metric (or Janhavi Banach space).

Definition 7 (see [29]). An operator $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ is $\mathfrak{D}$-Lipschitz if there exists an upper semicontinuous nondecreasing function: $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\Psi(0) = 0$ such that

$$\|\mathcal{Q}\varpi - \mathcal{Q}\omega\| \preccurlyeq \Psi(\|\varpi - \omega\|), \tag{7}$$

for all $\varpi, \omega \in \mathbb{X}$.

Definition 8 (see [29]). The same above operator $\mathcal{Q}$ is termed as partially nonlinear $\mathfrak{D}$-Lipschitz whenever a $\mathfrak{D}$-function $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ exists provide

$$\|\mathcal{Q}\varpi - \mathcal{Q}\omega\| \preccurlyeq \Psi(\|\varpi - \omega\|), \tag{8}$$

$\forall \varpi, \omega \in \mathbb{X}$. In addition, when $\mathcal{Q}$ is nonlinear $\mathfrak{D}$-Lipschitz subject to $\Psi(\tau) < \tau$ for $\tau > 0$, in that case, $\mathcal{Q}$ is nonlinear $\mathfrak{D}$-contraction.

Let us at present introduce a novel procedure, named Dhage iterative method, which is very useful for obtaining a scheme for the approximation of solutions to problems with nonlinearity.

Theorem 9 (see [29]). *Let $(\mathbb{X}, \preccurlyeq, \|\cdot\|)$ be a complete regular normed linear algebra via the partial order so that $\preccurlyeq$ and $\|\cdot\|$ are compatible. Consider two nondecreasing operators $\mathscr{K}$, $\mathscr{H} : \mathbb{X} \longrightarrow \mathbb{X}$ such that*

(a) $\mathscr{K}$ is partially nonlinear $\mathfrak{D}$-Lipschtiz and partially bounded with $\mathfrak{D}$-function $\psi_{\mathscr{K}}$

(b) $\mathscr{H}$ has the partial continuity and the compactness

(c) $\exists$ an element $\varpi_0 \in \mathbb{X}$ such that $\varpi_0 \preccurlyeq \mathscr{K}\varpi_0 + \mathscr{H}\varpi_0$ or $\varpi_0 \pm \mathscr{K}\varpi_0 + \mathscr{H}\varpi_0$

Then, $\mathscr{K}\varpi + \mathscr{H}\varpi = \varpi$ possesses a solution $\varpi*$ in $\mathbb{X}$, and the sequence of the successive iterations $\{\varpi_n\}_{n=0}^{\infty}$, expressed as $\varpi_{n+1} = \mathscr{K}\varpi_n + \mathscr{H}\varpi_n$, approaches to $\varpi*$ monotonically.

Theorem 10 (see [30]). *Let $\mathscr{H} : \mathbb{X} \longrightarrow \mathbb{X}$ be a nondecreasing and partially nonlinear $\mathfrak{D}$ -contraction. Assume that $\varpi_0 \in \mathbb{X}$ exists with $\varpi_0 \preccurlyeq \mathscr{H}\varpi_0$ or $\varpi_0 \pm \mathscr{H}\varpi_0$. If $\mathbb{X}$ is regular or $\mathscr{H}$ is continuous, then a fixed point $\varpi*$ is found, and the sequence of successive iterations $\{\mathscr{H}^n\varpi_0\}$ tends to $\varpi*$ monotonically. In addition, $\varpi*$ is unique if each of both members of $\mathbb{X}$ possesses a lower and an upper bound.*

Remark 11 (see [31]). Let every set contained in $\mathbb{X}$ with the partial compactness includes the compatibility specification with respect to $\preccurlyeq$ and $\|\cdot\|$. Then, every compact chain of $\mathbb{X}$ is Janhavi. This implication can be simply applied to establish the existence property of solutions in our research work.

Remark 12. The regularity property of $\mathbb{X}$ in Theorem 9 can be replaced with another strong continuity condition of the operators $\mathscr{K}$ and $\mathscr{H}$ on $\mathbb{X}$ where Dhage in [28] proved this result.

Remark 13 (see [30]).

(1) In a partially normed linear space, every compact operator has the partial compactness, and all partially compact operators has the partial total boundedness, while the converse is not valid

(2) Each completely continuous operator has the partial complete continuity, and each partially completely continuous operator has the continuity and the partial total boundedness, while the converse is not valid

In such a situation, the hypotheses regarding to the partial continuity and the partial compactness of an operator in Theorem 9 can be replaced by the continuity and compactness of that operator.

We state here the results below given by [45–47].

Definition 14. The mapping $\kappa : \mathfrak{M}_{\mathscr{C}} \longrightarrow [0,\infty)$ is named Kuratowski measure of non-compactness (KMNC) if

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with DIAM } (B) \le \varepsilon\}, \tag{9}$$

where $\mathfrak{M}_{\mathscr{C}}$ represents a class of all bounded mappings in $\mathscr{C}$.

Proposition 15. *The following are fulfilled for KMNC:*

(1) $A \subset E \Rightarrow \kappa(A) \leq \kappa(E)$

(2) $\kappa(A) = \kappa(\bar{A}) = \kappa(conv(A))$, *where* $\bar{A}$ *and* $conv(A)$ *represent the closure and the convex hull of A, respectively*

(3) $\kappa(A+E) \leq \kappa(A) + \kappa(E)$ *and* $\kappa(cA) = |c|\kappa(A), c \in \mathbb{R}$

Definition 16. Assume that $\mathscr{K} : A \longrightarrow \mathscr{C}$ be a continuous bounded mapping and $A \subset \mathscr{C}$. The operator $\mathscr{K}$ is said to be κ-Lipschitz if we can find a constant $\ell \geq 0$ satisfying the following condition:

$$\kappa(\mathscr{K}(B)) \leq \ell\kappa(B), \text{for every } B \subset A. \tag{10}$$

Moreover, $\mathscr{K}$ is called strict κ-contraction subject to $\ell < 1$.

Definition 17. $\mathscr{K}$ is called κ-condensing when

$$\kappa(\mathscr{K}(B)) < \kappa(B), \tag{11}$$

for every bounded and nonprecompact subset B of A. So,

$$\kappa(\mathscr{K}(B)) \geq \kappa(B), \text{which implies } \kappa(B) = 0. \tag{12}$$

Further, we have $\mathscr{K} : A \longrightarrow \mathscr{C}$ is Lipschitz if we can find $\ell > 0$ such that

$$\|\mathscr{K}(u) - \mathscr{K}(v)\| \leq \ell\|u - v\|, \text{forall } u, v \in A, \tag{13}$$

if $\ell < 1$, $\mathscr{K}$ is said to be strict contraction.

The following three interesting results are based on [48]:

Proposition 18. *Let* $\mathscr{K}, \mathscr{H} : A \longrightarrow \mathscr{C}$ *be* κ*-Lipschitz with constants* ℓ_1 *and* ℓ_2*. Then,* $\mathscr{K} + \mathscr{H} : A \longrightarrow \mathscr{C}$ *is* κ*-Lipschitz with* $\ell_1 + \ell_2$.

Proposition 19. *Every compact mapping* $\mathscr{K} : A \longrightarrow \mathscr{C}$ *is* κ*-Lipschitz with* $\ell = 0$.

Proposition 20. *Every Lipschitz mapping* $\mathscr{K} : A \longrightarrow \mathscr{C}$ *with* ℓ *is* κ*-Lipschitz with* ℓ.

Isaia [48] used the topological degree theory to introduce the following interesting results:

Theorem 21. *Let* $\mathscr{F} : A \longrightarrow \mathscr{C}$ *be* κ*-condensing and*

$$\Theta = \{u \in \mathscr{C} : \exists \xi \in [0,1] \; s.t. x = \xi \mathscr{F} u\}. \tag{14}$$

If $\Theta \subset \mathscr{C}$ *is bounded, i.e.,* $r > 0$ *exists subject to* $\Theta \subset B_r(0)$*; then, the degree*

$$\deg\,(I - \xi\mathscr{F}, B_r(0), 0) = 1, \text{for all } \xi \in [0,1]. \tag{15}$$

As a result, it is found a fixed-point for $\mathscr{F}$ and all possible fixed-points of $\mathscr{F}$ are contained in $B_r(0)$.

Let $\psi \in \mathscr{C}^1 = C^1(J, \mathbb{R})$ be an increasing differentiable function such that $\psi'(\tau) \neq 0, \forall \tau \in J$. Now, we start by defining ψ-FODs as follows:

Definition 22 (see [2]). The ψ-Riemann-Liouville fractional integral of order $\alpha > 0$ for an integrable function $\varpi : J \longrightarrow \mathbb{R}$ is given by

$$\mathbb{I}_{a^+}^{\alpha;\psi}\varpi(\tau) = \frac{1}{\Gamma(\alpha)}\int_a^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1}\varpi(s)ds, \tag{16}$$

where the Gamma function is denoted by Γ.

Definition 23 (see [2]). Let $n-1 < \alpha < n (n \in \mathbb{N})$, $\varpi : J \longrightarrow \mathbb{R}$ be an integrable function, and $\psi \in C^n(J, \mathbb{R})$. Then, the ψ-Riemann–Liouville FOD of a function ϖ of order α is expressed as:

$$\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(\tau) = \left(\frac{D_t}{\psi'(\tau)}\right)^n \mathbb{I}_{a^+}^{n-\alpha;\psi}\varpi(\tau), \tag{17}$$

where $n = [\alpha] + 1$ and $D_t = d/dt$.

Definition 24 (see [20]). For $n-1 < \alpha < n (n \in \mathbb{N})$ and ϖ, $\psi \in C^n(J, \mathbb{R})$, the ψ-Caputo FOD of a function ϖ of order α is given by

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(\tau) = \mathbb{I}_{a^+}^{n-\alpha;\psi}\varpi_\psi^{[n]}(\tau), \tag{18}$$

where $\varpi_\psi^{[n]}(\tau) = (D_t/\psi'(\tau))^n \varpi(\tau)$.

From the above definition, we can express ψ-Caputo FOD by the following formula:

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(\tau) = \begin{cases} \displaystyle\int_a^\tau \frac{\psi'(s)(\psi(\tau) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\varpi_\psi^{[n]}(s)ds, & \text{if } \alpha \notin \mathbb{N}, \\ \varpi_\psi^{[n]}(\tau), & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{19}$$

Also, the ψ–Caputo FOD of order α of ϖ is defined as

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(\tau) = \mathbb{D}_{a^+}^{\alpha;\psi}\left[\varpi(\tau) - \sum_{k=0}^{n-1}\frac{\varpi_\psi^{[k]}(a)}{k!}(\psi(\tau) - \psi(a))^k\right]. \tag{20}$$

For more details, see ([20], Theorem 3).

Lemma 25 (see [2]). *For* $\alpha, \beta > 0$*, and* $\varpi \in C(J, \mathbb{R})$*, we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\beta;\psi}\varpi(\tau) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi}\varpi(\tau), a.e. \tau \in J. \tag{21}$$

Lemma 26 (see [22]). *Assume that $\alpha > 0$. If $\varpi \in C(J, \mathbb{R})$, then*

$$ {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\alpha;\psi} \varpi(\tau) = \varpi(\tau), \tau \in J, \tag{22} $$

and if $\varpi \in C^{n-1}(J, \mathbb{R})$, then

$$ \mathbb{I}_{a^+}^{\alpha;\psi} {}^{c}\mathbb{D}_{a^+}^{\alpha;\psi} \varpi(\tau) = \varpi(\tau) - \sum_{k=0}^{n-1} \frac{\varpi_{\psi}^{[k]}(a)}{k!} [\psi(\tau) - \psi(a)]^k, \tau \in J. \tag{23} $$

It is easily to deduce that

$$ {}^{c}\mathbb{D}_{a^+}^{k;\psi} \mathbb{I}_{a^+}^{\alpha,\psi} = \mathbb{I}_{a^+}^{\alpha-k,\psi}. \tag{24} $$

Lemma 27 (see [2, 20]). *Let $\tau > a$, $\alpha \geq 0$, $\varsigma > 0$ and let $\chi(\tau) = \psi(\tau) - \psi(a)$. Then:*

(i) $\mathbb{I}_{a^+}^{\alpha;\psi} (\chi(\tau))^{\varsigma-1} = \Gamma(\varsigma)/\Gamma(\varsigma+\alpha)(\chi(\tau))^{\varsigma+\alpha-1}$

(ii) ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi} (\chi(\tau))^{\beta-1} = \Gamma(\varsigma)/\Gamma(\varsigma-\alpha)(\chi(\tau))^{\varsigma-\alpha-1}$

(iii) ${}^{c}\mathbb{D}_{a^+}^{\alpha;\psi} (\chi(\tau))^{k} = 0$*, for all* $k \in \{0, \cdots, n-1\}$, $n \in \mathbb{N}$

3. Existence and Approximation Results via Dhage's Technique

The solutions' existence and approximation of problem (1) are studied in this section.

Lemma 28. *Assume that $(\mathscr{C}_b, \preccurlyeq, \|.\|)$ is a partially ordered Banach space with the norm $\|.\|$, and the order relation $\preccurlyeq$ defined by (5) and (6), respectively. Then, every partially compact subset of $\mathscr{C}_b$ is Janhavi.*

Proof (see [31]). Let us now discuss exactly the problem (1). □

Definition 29. A function $\varpi \in \mathscr{C}_b$ is a lower solution for the NFuDEq (1) if:

(1) $\varpi_\tau \in \mathscr{C}_\delta$, $\forall \tau \in J$

(2) the function $\tau \mapsto [\varpi(\tau) - \mathbb{F}(\tau, \varpi_\tau)]$ is continuously differentiable on J and settles

$$ \begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi} [\varpi(\tau) - \mathbb{F}(\tau, \varpi_\tau)] \preccurlyeq \mathbb{H}(\tau, \varpi_\tau), \tau \in J, \\ \varpi(\tau) \preccurlyeq \phi(\tau). \end{cases} \tag{25} $$

Similarly, a differentiable function $\omega \in \mathscr{C}_b$ is named an upper solution of the NFuDEq (1) if the above inequality is satisfied with reverse sign.

To demonstrate the solutions' existence to (1), we state this lemma:

Lemma 30. *Assume that $0 < \nu < 1$, $\phi(a) = 0$, and $g, h : J \longrightarrow \mathbb{R}$ are continuous with $h(a) = 0$. The linear problem*

$$ {}^{c}\mathbb{D}_{a^+}^{\nu;\psi} [\varpi(\tau) - h(\tau)] = g(\tau), \quad \tau \in J; \quad \varpi(\tau) = \phi(\tau), \quad \tau \in [a-\delta, a], \tag{26} $$

has a unique solution $\varpi(\tau)$ defined by:

$$ \begin{cases} h(\tau) + \mathbb{I}_{a^+}^{\nu,\psi} g(\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \tag{27} $$

For the proof of Lemma 30, it is useful to refer to [2, 23, 41, 49].

With the help of the following hypothesis, we can investigate our results:

(H1) The functions $\mathbb{F}(\tau, \varpi)$ and $\mathbb{H}(\tau, \varpi)$ are monotone nondecreasing with respect to ϖ for any $\tau \in$ J.

(H2) $\exists$ a $\mathfrak{D}$-function Ψ that satisfies $\Psi(R) < R$ for $R > 0$ such that

$$ 0 \leq [\mathbb{F}(\tau, \varpi) - \mathbb{F}(\tau, \omega)] \leq \Psi(\varpi - \omega) \forall \tau \in \text{J and } \varpi, \omega \in \mathbb{R} \text{ with } \varpi \geq \omega. \tag{28} $$

(H3) $\exists M > 0$ such that $|\mathbb{H}(\tau, \varpi)| \leq M, \forall \tau \in$ J, and $\varpi \in \mathbb{R}$.

(H4) $\exists L > 0$ such that $|\mathbb{F}(\tau, \varpi)| \leq L$, $\forall \tau \in$ J, and $\varpi \in \mathbb{R}$.

(H5) The FBVP (1) possesses a lower solution $x \in \mathscr{C}_b$.

(H6) $\exists$ a positive constant $L_{\mathbb{H}}$ such that

$$ |\mathbb{H}(\tau, \varpi) - \mathbb{H}(\tau, \omega)| \leq L_{\mathbb{H}} |\varpi - \omega|, \forall \tau \in \text{J and } \varpi, \omega \in \mathbb{R} \text{ with } \varpi \geq \omega. \tag{29} $$

Theorem 31. *If the hypotheses (H1)-(H5) are fulfilled, then the NFuDEq (1) includes a solution ϖ^* formulated on $[a-\delta, b]$, and $\{\varpi_n\}$ containing the successive approximations expressed as:*

$$ \varpi_0 = x, \varpi_{n+1}(\tau) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi} \mathbb{H}(\tau, \varpi_{n,\tau}) + \mathbb{F}(\tau, \varpi_{n,\tau}), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a], \end{cases} \tag{30} $$

where $\varpi_{n,\tau}(\rho) = \varpi_n(\tau+\rho)$, $\rho \in [-\delta, 0]$, tends to ϖ^* monotonically.

Proof. Take $\mathbb{X} = \mathscr{C}_b = C([a-\delta, b], \mathbb{R})$. Then, using Lemma 28, each compact chain $\mathscr{C} \subset \mathscr{C}_b$ admits the compatibility property in $\|\cdot\|$ and $\leq$ such that $\mathscr{C}$ is Janhavi in $\mathscr{C}_b$. On the other side, $\mathscr{K}$ and $\mathscr{H}$ can be defined on $\mathscr{C}_b$ as follows:

$$ \mathscr{K}(\varpi)(\tau) := \begin{cases} \mathbb{F}(\tau, \varpi_\tau), & \text{if } \tau \in \text{J}, \\ 0, & \text{if } \tau \in [a-\delta, a], \end{cases} \tag{31} $$

$$ \mathscr{H}(\varpi)(\tau) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi} \mathbb{H}(\tau, \varpi_\tau), & \text{if } \tau \in \text{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \tag{32} $$

According to the structure of integral, it is obvious that $\mathscr{K}, \mathscr{H} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ are well-defined. In addition, the studied problem (1) can be reformulated by:

$$\mathscr{K}(\varpi)(\tau) + \mathscr{H}(\varpi)(\tau) = \varpi(\tau), \tau \in [a-\delta, b]. \tag{33}$$

To investigate the solutions' existence to this operator equation, we can sufficiently show that the operators $\mathscr{K}$ and $\mathscr{H}$ satisfy all items of Theorem 9. We follow our argument split into five steps. □

Step I. $\mathscr{K}$ and $\mathscr{H}$ are nondecreasing on $\mathscr{C}_b$.

For $\varpi, \omega \in \mathscr{C}_b$ with $\varpi_\tau \geq \omega_\tau$, using (H1), we get

$$\mathscr{K}(\varpi)(\tau) = \begin{cases} \mathbb{F}(\tau, \varpi_\tau), & \text{if } \tau \in \mathrm{J}, \\ 0, & \text{if } \tau \in [a-\delta, a] \end{cases} \geq \begin{cases} \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in \mathrm{J}, \\ 0, & \text{if } \tau \in [a-\delta, a] \end{cases} = \mathscr{K}(\omega)(\tau), \tag{34}$$

for all $\tau \in [a-\delta, b]$. It means that $\mathscr{K} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ is a nondecreasing operator.

Similarly, we obtain

$$\mathscr{H}(\varpi)(\tau) = \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi} \mathbb{H}(\tau, \varpi_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a] \end{cases} \geq \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi} \mathbb{H}(\tau, \omega_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a] \end{cases} = \mathscr{H}(\omega)(\tau), \tag{35}$$

for all $\tau \in [a-\delta, b]$. Thus, it is concluded that $\mathscr{H} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ is a nondecreasing operator.

Step II. $\mathscr{K}$ is a nonlinear $\mathfrak{D}$-contraction on $\mathscr{C}_b$.

For $\varpi, \omega \in \mathscr{C}_b$ with $\varpi \geq \omega$ and by (H2), we get that

$$\begin{aligned} |\mathscr{K}\varpi(\tau) - \mathscr{K}\omega(\tau)| &\leq |\mathbb{F}(\tau, \varpi_\tau) - \mathbb{F}(\tau, \omega_\tau)| \\ &\leq \Psi\left(\|\varpi_\tau - \omega_\tau\|_{C_\delta}\right) \\ &\leq \Psi(\|\varpi - \omega\|), \end{aligned} \tag{36}$$

$\forall \tau \in [a-\delta, b]$. By taking the supremum over τ, we get

$$\|\mathscr{K}\varpi - \mathscr{K}\omega\| \leq \Psi\left(\|\varpi - \omega\|_{\mathscr{C}_b}\right), \tag{37}$$

$\forall \varpi, \omega \in \mathscr{C}_b, \varpi \geq \omega$, where $r > \Psi(r)$ for $r > 0$. Therefore, according to Definition 8, our result is derived.

Step III. $\mathscr{H}$ is partially continuous on $\mathscr{C}_b$.

Regard $\{\varpi_n\}_{n\in\mathbb{N}}$ in a chain $\mathscr{C} \ni \varpi_n \longrightarrow \varpi$ as $n \longrightarrow \infty$. Then, $\varpi_{n,s} \longrightarrow \varpi_s$ for any $s \in \mathrm{J}$ letting $n \longrightarrow \infty$. The continuity of $\mathbb{H}$ yields

$$\begin{aligned} \lim_{n\longrightarrow\infty} (\mathscr{H}\varpi_n)(\tau) &= \begin{cases} \dfrac{1}{\Gamma(\nu)} \lim_{n\longrightarrow\infty} \displaystyle\int_a^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\nu-1} \mathbb{H}(s, \varpi_{n,s}) ds, & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in a-\delta, a] \end{cases} \\ &= \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_a^b \psi'(s)(\psi(\tau) - \psi(s))^{\nu-1} \lim_{n\longrightarrow\infty} \mathbb{H}(s, \varpi_{n,s}) ds, & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a] \end{cases} \\ &= \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi} \mathbb{H}(\tau, \varpi_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a] \end{cases} \\ &= \mathscr{H}\varpi(\tau), \end{aligned} \tag{38}$$

$\forall \tau \in [a-\delta, b]$. Hence, $\mathscr{H}\varpi_n$ converges to $\mathscr{H}\varpi$ pointwise on $[a-\delta, b]$.

In the following two cases, we prove that $\{\mathscr{H}\varpi_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in $\mathscr{C}_b$.

Case A. Take $\tau_1, \tau_2 \in \mathrm{J}$, with $\tau_1 < \tau_2$. Then,

$$\begin{aligned}
&|\mathscr{H}(\varpi_n)(\tau_2) - \mathscr{H}(\varpi_n)(\tau_1)| \\
&\quad \le \left|\mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau_2, \varpi_{n,\tau_2}) - \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau_1, \varpi_{n,\tau_1})\right| \\
&\quad \le \frac{1}{\Gamma(\nu)}\int_a^{\tau_1}\psi'(s)\big|(\psi(\tau_1)-\psi(s))^{\nu-1} - (\psi(\tau_2) \\
&\qquad - \psi(s))^{\nu-1}\big|\left|\mathbb{H}(s, \varpi_{n,s})\right|ds \\
&\qquad + \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\nu-1}\left|\mathbb{H}(s, \varpi_{n,s})\right|ds \\
&\quad \le \frac{L}{\Gamma(\nu+1)}((\psi(\tau_2)-\psi(a))^{\nu} - (\psi(\tau_1)-\psi(a))^{\nu} \\
&\qquad + 2(\psi(\tau_2)-\psi(\tau_1))^{\nu}),
\end{aligned} \tag{39}$$

which tends to zero as $\tau_1 \longrightarrow \tau_2$.

Case B. For $\tau_1, \tau_2 \in a-\delta, a]$. Then,

$$|\mathscr{H}(\varpi_n)(\tau_2) - \mathscr{H}(\varpi_n)(\tau_1)| = |\phi(\tau_2) - \phi(\tau_1)| \longrightarrow 0, as\, \tau_1 \longrightarrow \tau_2. \tag{40}$$

Clearly, if $\tau_1 \in [a-\delta, a]$ and $\tau_2 \in \mathrm{J}$ such that $\tau_1 \longrightarrow \tau_2$ has only one possibility that they are close to a at which $\mathscr{H}(\varpi_n)$ is close to zero.

Thus,

$$|\mathscr{H}(\varpi_n)(\tau_2) - \mathscr{H}(\varpi_n)(\tau_1)| \longrightarrow 0, as\, \tau_1 \longrightarrow \tau_2, \tag{41}$$

uniformly $\forall n \ge 1$. This proves that $\{\mathscr{H}\varpi_n\}$ is equi–continuous on $[a-\delta, b]$. Thus, the pointwise convergence of $\{\mathscr{H}\varpi_n\}$ on $[a-\delta, b]$ implies the uniform convergence, so $\mathscr{H}\varpi_n$ converges to $\mathscr{H}\varpi$ uniformly on $[a-\delta, b]$. Consequently, the selfmap $\mathscr{H}$ possesses the partial continuity on $\mathscr{C}_b$.

Step IV. $\mathscr{H}$ has the partial compactness property on $\mathscr{C}_b$.

Regard the chain $\mathscr{C}$ in $\mathscr{C}_b$ and $\omega \in \mathscr{H}(\mathscr{C})$. Then $\exists \varpi \in \mathscr{C} \ni \omega = \mathscr{H}\varpi$. Using hypothesis (H3), if $\tau \in [a-\delta, a]$, we have

$$|\omega(\tau)| = |(\mathscr{H}\varpi)(\tau)| \le |\phi(\tau)| \le \|\phi\|_{\mathscr{C}_\delta} \le \|\phi\|_{\mathscr{C}_b}. \tag{42}$$

Otherwise, if $\tau \in \mathrm{J}$, then

$$|\mathscr{H}(\varpi)(\tau)| \le \mathbb{I}_{a^+}^{\nu,\psi}|\mathbb{H}(\tau, \varpi_\tau)| \le M\mathbb{I}_{a^+}^{\nu,\psi}(1)(\tau) \le \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}M. \tag{43}$$

Hence,

$$\|\mathscr{H}(\varpi)\|_{\mathscr{C}_b} \le \|\phi\|_{\mathscr{C}_b} + \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}M \coloneqq R, \tag{44}$$

$\forall \tau \in [a-\delta, b]$. Thus, we obtain $\|\omega\| \le \|\mathscr{H}\varpi\| \le R$ for any $\omega \in \mathscr{H}(\mathscr{C})$. Thus, $\mathscr{H}(\mathscr{C})$ is a uniformly bounded subset of $\mathscr{C}_b$.

Let us now prove that $\mathscr{H}(\mathscr{C})$ is an equi–continuous set in $\mathscr{C}_b$. Let $\tau_1, \tau_2 \in J$, with $\tau_1 < \tau_2$. Then, according to Step III arguments, it is concluded that

$$|\omega(\tau_2) - \omega(\tau_1)| = |\mathscr{H}\varpi(\tau_2) - \mathscr{H}\varpi(\tau_1)| \longrightarrow 0 \text{ as } \tau_1 \longrightarrow \tau_2, \tag{45}$$

uniformly for any $\omega \in \mathscr{H}(C)$ which illustrates the equi–continuity of $\mathscr{H}(\mathscr{C})$ in $\mathscr{C}_b$. So, $\mathscr{H}(\mathscr{C})$ is compact in reference to Arzelà-Ascoli criterion. As a result, the selfmap $\mathscr{H} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ admits the partial compactness property on $\mathscr{C}_b$.

Step V. ϖ satisfies $\varpi \le \mathscr{K}\varpi + \mathscr{H}\varpi$.

By (H5), W is a lower solution of the NFuDEq (1) defined on $[a-\delta, b]$. Then, according to the lower solution definition, we get

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\nu;\psi}[W(\tau) - \mathbb{F}(\tau, W_\tau)] \le \mathbb{H}(\tau, W_\tau), \tau \in \mathrm{J} = [a, b], \\ W(\tau) \le \phi(\tau), \tau \in [a-\delta, a]. \end{cases} \tag{46}$$

Let us integrate the above inequality from a to τ, we obtain

$$\begin{aligned} W(\tau) &\le \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau, W_\tau) + \mathbb{F}(\tau, W_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \\ &= \mathscr{K}W(\tau) + \mathscr{H}W(\tau), \end{aligned} \tag{47}$$

$\forall \tau \in [a-\delta, b]$. Thus, $W \le \mathscr{K}W + \mathscr{H}W$. Obviously, both operators $\mathscr{K}$ and $\mathscr{H}$ satisfy all of the items of Theorem 9; therefore, the operator equation $\mathscr{K}\varpi + \mathscr{H}\varpi = \varpi$ has a solution ϖ^* defined on $[a-\delta, b]$. Furthermore, the sequence $\{\varpi_n\}_{n=0}^{\infty}$ of successive approximations defined by (30) tends to ϖ^* monotonically. So, our proof is ended.

Remark 32. Above theorem's conclusion also remains true if the hypothesis (H5) is replaced with (H7) such that the NFuDEq (1) has an upper solution: $y \in \mathscr{C}_b$.

Similarly, its proof under this replaced condition can be shown by the observation of the same arguments with some modifications.

Theorem 33. *Let (H1), (H5), and (H6) be valid. Then, the problem (1) has a unique solution $\varpi*$ defined on $[a-\delta, b]$ provided that $\Omega(R) < R, R > 0$, where*

$$\Omega(R) = \frac{L_{\mathbb{H}}(\psi(b)-\psi(a))^{\nu} R}{\Gamma(\nu+1)} + \Psi(R). \tag{48}$$

Moreover, the sequence $\{\varpi_n\}$ of successive approximations defined by (30) converges monotonically to $\varpi*$.

Proof. First, the operator: $\mathfrak{G} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ is defined by

$$\mathfrak{G}(\varpi)(\tau) := \begin{cases} \mathbb{I}_{a^+}^{\nu,\psi}\mathbb{H}(\tau, \varpi_\tau) + \mathbb{F}(\tau, \varpi_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in a-\delta, a], \end{cases} \tag{49}$$

for $\tau \in [a-\delta, b]$,. To prove this theorem, we establish the satisfaction of all items of Theorem 10 for $\mathfrak{G}$ in $\mathscr{C}_b$. We know that $\mathfrak{G}$ is nondecreasing and continuous.

The details are similar as in the proof of Theorem 31, so we omit them. Therefore, it is needed to be verified that $\mathfrak{G}$ is a partially $\mathfrak{D}$-contraction on $\mathscr{C}_b$. To arrive at such an aim, by taking $\varpi, \omega \in \mathscr{C}_b$ such that $\varpi \geq \omega$, if $\tau \in [a-\delta, a]$, then it is obvious that

$$|\mathfrak{G}(\varpi)(\tau) - \mathfrak{G}(\varpi)(\tau)| = 0. \tag{50}$$

Otherwise, let $\tau \in J$, it follows from (H1) and (H6), that

$$\begin{aligned} &|\mathfrak{G}(\varpi)(\tau) - \mathfrak{G}(\bar{\varpi})(\tau)| \\ &\quad \leq \mathbb{I}_{a^+}^{\nu,\psi}|\mathbb{H}(\tau, \varpi_\tau) - \mathbb{H}(\tau, \bar{\varpi}_\tau)| + |\mathbb{F}(\tau, \varpi_\tau) - \mathbb{F}(\tau, \bar{\varpi}_\tau)| \\ &\quad \leq L_{\mathbb{H}}\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta}\mathbb{I}_{a^+}^{\nu,\psi}(1)(\tau) + \Psi\left(\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta}\right) \\ &\quad \leq \frac{L_{\mathbb{H}}(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta} + \Psi\left(\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta}\right) \\ &\quad \leq \Omega\left(\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta}\right), \end{aligned} \tag{51}$$

for all $\tau \in J$, where $\Omega(R) < R, R > 0$. Let us now take the supremum over τ, we get

$$\|\mathfrak{G}\varpi - \mathfrak{G}\bar{\varpi}\| \leq \Omega\left(\|\varpi - \bar{\varpi}\|_{\mathscr{C}_b}\right), \tag{52}$$

for all $\varpi, \bar{\varpi} \in \mathscr{C}_b$, with $\varpi \geq \bar{\varpi}$. As a result, $\mathfrak{G}$ is a partially nonlinear $\mathfrak{D}$-contraction on $\mathscr{C}_b$. In addition, by using Theorem 31, it is proven that the given function x in (H5) satisfies the operator inequality $x \leq \mathfrak{G}x$ on $[a-\delta, b]$. Therefore, from Theorem 10, it is found a solution ϖ^* uniquely for the NFuDEq (1), and $\{\varpi_n\}$ defined by (30) tends to ϖ^* monotonically. □

4. Existence Result via Topological Degree Theory

The existence problem of the NFuDEq (1) is investigated in this section based on the Topological Degree Theory due to Isaia [48]. Let us first introduce the following hypothesis for convenience:

(M1) The functions $\mathbb{F}$ and $\mathbb{H}$ satisfy the following growth conditions for constants $M_i, N_i > 0$, $i = 1, 2$, $p \in (0, 1)$:

$$\begin{aligned} |\mathbb{F}(t, \varpi)| &\leq M_1|\varpi|^p + N_1, \\ |\mathbb{H}(t, \varpi)| &\leq M_2|\varpi|^p + N_2, \end{aligned} \tag{53}$$

for each $t \in \mathrm{J}$ and each $\varpi \in \mathbb{R}$.

(M2) For each $\tau \in \mathrm{J}$, and for each, $\varpi, \omega \in \mathbb{R}$, $\exists$ constants $L_{\mathbb{F}}, L_{\mathbb{H}} > 0$, provided

$$\begin{aligned} |\mathbb{F}(\tau, \varpi) - \mathbb{F}(\tau, \bar{\varpi})| &\leq L_{\mathbb{F}}|\varpi - \bar{\varpi}|, \\ |\mathbb{H}(\tau, \varpi) - \mathbb{H}(\tau, \bar{\varpi})| &\leq L_{\mathbb{H}}|\varpi - \bar{\varpi}|. \end{aligned} \tag{54}$$

In view of Lemma 30, we consider two operators $\mathscr{K}$, $\mathscr{H} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ given by (31) and (32), respectively. Then, we write the integral equation (27) as an operator equation:

$$\mathscr{F}\varpi(t) = \mathscr{K}\varpi(t) + \mathscr{H}\varpi(t), t \in [a-\delta, b]. \tag{55}$$

The continuity of $\mathbb{F}$ and $\mathbb{H}$ shows that the operator $\mathscr{F}$ is well-defined, and its fixed points are the same solutions of the existing equation (27) in Lemma 30.

Lemma 34. *If (M1) and (M2) hold, then the operator $\mathscr{K}$ is Lipschitz with constant $L_{\mathbb{F}}$ and satisfies*

$$\|\mathscr{K}\varpi\| \leq \left(M_1\|\varpi\|^p + N_1\right), \text{for every } \varpi \in \mathscr{C}_b. \tag{56}$$

Proof. Let $\varpi, \bar{\varpi} \in \mathscr{C}_b$, then we get

$$|\mathscr{K}\varpi(\tau) - \mathscr{K}\bar{\varpi}(\tau)| = |\mathbb{F}(\tau, \varpi) - \mathbb{F}(\tau, \bar{\varpi})| \leq L_{\mathbb{F}}\|\varpi - \bar{\varpi}\|, \tag{57}$$

$\forall \tau \in \mathrm{J}$. Let us take the supremum over τ, so we get

$$\|\mathscr{K}\varpi - \mathscr{K}\bar{\varpi}\| \leq L_{\mathbb{F}}\|\varpi - \bar{\varpi}\|. \tag{58}$$

Hence, $\mathscr{K} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ is a Lipschitzian on $\mathscr{C}_b$ with Lipschitz constant $L_{\mathbb{F}}$. From Proposition 20, $\mathscr{K}$ is κ–Lipschitz with constant $L_{\mathbb{F}}$. In addition, we get

$$|\mathscr{K}\varpi(t)| \leq \left(M_1\|\varpi\|^p + N_1\right), \tag{59}$$

for every $\varpi \in \mathscr{C}_b$. This finishes the proof. □

Lemma 35. *If (M1) holds, then $\mathscr{H}$ is continuous and satisfies the growth condition*

$$\|\mathscr{H}\varpi\| \leq \bar{M}\|\varpi\|^p + \bar{N}, \text{for every } \varpi \in \mathscr{C}_b. \tag{60}$$

Proof. Choose a bounded subset $\mathbb{D}_r = \{\varpi \in \mathscr{C}_b : \|\varpi\| \le r\} \subset \mathscr{C}_b$ and consider a sequence $\{\varpi_n\} \in \mathbb{D}_r$ via $\varpi_n \longrightarrow \varpi$ by letting $n \longrightarrow \infty$ in $\mathbb{D}_r$. We shall prove that $\|\mathscr{H}\varpi_n - \mathscr{H}\varpi\| \longrightarrow 0$, letting $n \longrightarrow \infty$. From the continuity of $\mathbb{H}$, it follows that $\mathbb{H}(s, \varpi_n) \longrightarrow \mathbb{H}(s, \varpi)$, as $n \longrightarrow \infty$. In view of (M1), we get $|\mathscr{H}\varpi_n(\tau)| \le \lambda(\tau)$, where

$$\lambda(\tau) = \begin{cases} |\phi(\tau)|, & \text{if } \tau \in [a-\delta, a], \\ \dfrac{(\psi(\tau) - \psi(a))^\nu}{\Gamma(\nu+1)} (M_2 r^p + N_2), & \text{if } \tau \in J, \end{cases} \tag{61}$$

which is Lebesgue's integrable bounded function. The Labesgue dominated convergence theorem ensures that $\|\mathscr{H}\varpi_n - \mathscr{H}\varpi\| \longrightarrow 0$, letting $n \longrightarrow +\infty$, which confirms the continuity of $\mathscr{H}$.

Next, it is easy as above to deduce that

$$|\mathscr{H}\varpi(\tau)| \le \begin{cases} \|\phi\|_{\mathscr{C}_b}, & \text{if } \tau \in [a-\delta, a], \\ \left(\dfrac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu+1)}\right) \left(M_2 \|\varpi\|^p + N_2\right), & \text{if } \tau \in J. \end{cases} \tag{62}$$

Therefore,

$$\begin{aligned} \|\mathscr{H}\varpi\| &\le \frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu+1)} \left(M_2 \|\varpi\|^p + N_2\right) + \|\phi\|_{\mathscr{C}_b} \\ &= \bar{M}\|\varpi\|^p + \bar{N}, \end{aligned} \tag{63}$$

where $\bar{M} = (\psi(b) - \psi(a))^\nu / \Gamma(\nu+1) M_2$ and $\bar{N} = (\psi(b) - \psi(a))^\nu / \Gamma(\nu+1) N_2 + \|\phi\|_{\mathscr{C}_b}$. This completes the proof. □

Lemma 36. *If (M1) holds, then the operator $\mathscr{H} : \mathscr{C}_b \longrightarrow \mathscr{C}_b$ is compact. As a result, $\mathscr{H}$ is κ-Lipschitz with zero constant.*

Proof. Take a bounded set $\bar{\Omega} \subset \mathbb{D}_r$. We need to establish the relative compactness of $\mathscr{H}(\bar{\Omega})$ in $\mathscr{C}_b$. For $\varpi \in \bar{\Omega}$, with the help of the estimate (63), we can obtain

$$\|\mathscr{H}\varpi\| \le \bar{M} r^p + \bar{N}, \tag{64}$$

which shows that $\mathscr{H}(\Omega)$ is uniformly bounded.

Now, we prove the equi–continuity of $\mathscr{H}$. For $\tau \in \mathrm{J}$, we can estimate the derivative operator using (24) as follows:

$$\begin{aligned} \left|(\mathscr{H}\varpi)'(\tau)\right| &\le \mathbb{I}_{a^+}^{\nu-1,\psi} |\mathbb{H}(\tau, \varpi(\tau))| \\ &\le \left(M_2 \|\varpi\|^p + N_2\right) \mathbb{I}_{a^+}^{\nu-1,\psi}(1)(\tau) \\ &\le \frac{(\psi(b) - \psi(a))^{\nu-1}}{\Gamma(\nu)} \left(M \|\varpi\|^p + N\right) \coloneqq \ell. \end{aligned} \tag{65}$$

Hence, for each $\tau_1, \tau_2 \in \mathrm{J}$ with $a < \tau_1 < \tau_2 < b$, we get

$$|(\mathscr{H}\varpi)(\tau_2) - (\mathscr{H}_2\varpi)(\tau_1)| \le \int_{\tau_1}^{\tau_2} \left|(\mathscr{H}\varpi)'(s)\right| ds \le \ell(\tau_2 - \tau_1), \tag{66}$$

which tends to zero independently of ϖ as $(\tau_2 - \tau_1) \longrightarrow 0$. So, $\mathscr{H}$ is equi–continuous. The equi–continuity for the case $\tau_1, \tau_2 \in [a-\delta, a]$ is obvious. From the foregoing arguments along with Arzela-Ascoli theorem, we deduce that $\mathscr{H}$ is compact on $\mathbb{D}_r$. Thus, from Proposition 19, $\mathscr{H}$ is κ–Lipschitz with zero constant. This completes our proof. □

Theorem 37. *If (M1) and (M2) hold, then the NFuDEq (1) has at least one solution $\varpi \in \mathscr{C}_b$ provided that $L_{\mathbb{F}} < 1$, and the set of the solutions is bounded in $\mathscr{C}_b$.*

Proof. Assume that $\mathscr{K}, \mathscr{H}, \mathscr{F}$ are the operators defined by (31), (32) and (55), respectively, which all of them are bounded and continuous, and also, by Lemma 34, $\mathscr{K}$ is κ–Lipschitz with $L_{\mathbb{F}}$ and by Lemma 36, $\mathscr{H}$ is κ–Lipschitz via constant 0. Thus, by Proposition 18, $\mathscr{F}$ is κ–Lipschitz with $L_{\mathbb{F}}$. Hence, $\mathscr{F}$ is strict κ–contraction with $L_{\mathbb{F}} > 0$. Since $L_{\mathbb{F}} < 1$, $\mathscr{F}$ is κ-condensing.

Now, let us consider the following set:

$$\Theta = \{\varpi \in \mathscr{C}_b : \text{there exists } \varsigma \in [0,1] \text{ such that } x = \varsigma\mathscr{F}\varpi\}. \tag{67}$$

We will show that the set Θ is bounded. For $\varpi \in \Theta$, we have $\varpi = \varsigma\mathscr{F}\varpi = \varsigma(\mathscr{K}(\varpi) + \mathscr{H}(\varpi))$, which implies that

$$\begin{aligned} \|\varpi\| &\le \varsigma(\|\mathscr{K}u\| + \|\mathscr{H}\varpi\|) \le \varsigma\left[M_1\|\varpi\|^p + N_1 + \bar{M}\|\varpi\|^p + \bar{N}\right] \\ &= \varsigma\left[\left(M_1 + \bar{M}\right)\|\varpi\|^p + \left(N_1 + \bar{N}\right)\right] = \mathscr{M}\|\varpi\|^p + \mathscr{N}, \end{aligned} \tag{68}$$

where $\mathscr{M} = (M + \bar{M})$ and $\mathscr{N} = (N + \bar{N})$. If Θ is unbounded in $\mathscr{C}_b$, in that case, we divide the obtained inequality by $a \coloneqq \|\varpi\|$ and supposing $a \longrightarrow \infty$, we get

$$1 \le \lim_{a \longrightarrow \infty} \frac{\mathscr{M} a^p + \mathscr{N}}{a} = 0, \tag{69}$$

which is impossible, and Θ is bounded. Accordingly, it is found a fixed point for $\mathscr{F}$ which is interpreted as the solution of the NFuDEq (1). This finishes the proof. □

Remark 38. If (M1) is represented for $p = 1$, then Theorem 37 is true so that $\mathscr{M} < 1$.

5. Uniqueness Result and UH Stability

The uniqueness of the solution for the NFuDEq (1) will be investigated below by using the standard Banach fixed point theorem. Moreover, The UH stability of the NFuDEq (1) will be also checked.

Theorem 39. *Suppose that assumption (M2) holds. Assume that*

$$\Delta := \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{H}} + L_{\mathbb{F}}\right) < 1. \tag{70}$$

Then, $\exists$ a unique solution for (1) on the interval $[a-\delta, b]$.

Proof. Define the set

$$U := \left\{\varpi \in \mathscr{C}_b : \varpi|_{[a-\delta,a]} \in \mathscr{C}_\delta, \varpi|_{\mathrm{J}} \in \mathscr{C}; {}^c\mathbb{D}^{\nu}_{a^+}\varpi \in \mathscr{C}\right\}, \tag{71}$$

and the operator $\mathscr{G} : U \longrightarrow U$:

$$\mathscr{G}(\varpi)(\tau) := \begin{cases} \mathbb{I}^{\nu,\psi}_{a^+}\mathbb{H}(\tau,\varpi_\tau) + \mathbb{F}(\tau,\varpi_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \tag{72}$$

Notice that $\mathscr{G}$ is well defined. Indeed, for $\varpi \in U$, $\tau \mapsto \mathscr{G}(\varpi)(\tau)$ is continuous, for any $\tau \in a-\delta, b]$. In addition, $\forall \tau \in \mathrm{J}$, ${}^c\mathbb{D}^{a,\psi}_{a^+}[\mathscr{G}(\varpi)(\tau) - \mathbb{F}(\tau,\varpi_\tau)] = \mathbb{H}(\tau,\varpi_\tau)$ exists, and it is continuous too due to the continuity of $\mathbb{H}$ and Lemma 26.

Now, we need to show that $\mathscr{G}$ is a contraction. If $\varpi, \bar{\varpi} \in U$ and $\tau \in a-\delta, a]$, then, $|\mathscr{G}(\varpi)(\tau) - \mathscr{G}(\bar{\varpi})(\tau)|$ equals to zero. On the contrary, for $\tau \in \mathrm{J}$, by (M2), it is derived that

$$\begin{aligned} &|\mathscr{G}(\varpi)(\tau) - \mathscr{G}(\bar{\varpi})(\tau)| \\ &\quad \le \mathbb{I}^{\nu,\psi}_{a^+}|\mathbb{H}(\tau,\varpi_\tau) - \mathbb{H}(\tau,\bar{\varpi}_\tau)| + |\mathbb{F}(\tau,\varpi_\tau) - \mathbb{F}(\tau,\bar{\varpi}_\tau)| \\ &\quad \le L_{\mathbb{H}}\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta}\mathbb{I}^{\nu,\psi}_{a^+}(1)(\tau) + L_{\mathbb{F}}\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta} \\ &\quad \le \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{H}} + L_{\mathbb{F}}\right)\|\varpi_\tau - \bar{\varpi}_\tau\|_{\mathscr{C}_\delta} \\ &\quad \le \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{H}} + L_{\mathbb{F}}\right)\|\varpi - \bar{\varpi}\|_{\mathscr{C}_b}, \end{aligned} \tag{73}$$

which implies

$$\|\mathscr{G}(\varpi) - \mathscr{G}(\bar{\varpi})\|_{\mathscr{C}_b} \le \Delta\|\varpi - \bar{\varpi}\|_{\mathscr{C}_b}. \tag{74}$$

Since $\Delta < 1$, the operator $\mathscr{G}$ is a contraction. Hence, Banach fixed point theorem shows that $\mathscr{G}$ admits a unique fixed point. This finishes the proof. ☐

Here, we discuss the UH and GUH stability types of (1).

Definition 40. The NFuDEq (1) is UH stable when $\exists c \in \mathbb{R}^+$ so that $\forall \varepsilon \in \mathbb{R}^+$ and $\forall \bar{\varpi} \in \mathscr{C}_b$ satisfying

$$\begin{cases} \left|{}^c\mathbb{D}^{\nu;\psi}_{a^+}[\bar{\varpi}(\tau) - \mathbb{F}(\tau,\bar{\varpi}_\tau)] - \mathbb{H}(\tau,\bar{\varpi}_\tau)\right| \le \varepsilon, \tau \in \mathrm{J}, \\ |\bar{\varpi}(\tau) - \phi(\tau)| \le \varepsilon, \tau \in [a-\delta, a], \end{cases} \tag{75}$$

exactly one solution $\varpi \in \mathscr{C}_b$ of (1) exists with

$$\|\bar{\varpi} - \varpi\| \le c\varepsilon. \tag{76}$$

Definition 41. The NFuDEq (1) is GUH stable if $\exists$ $\sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\sigma(0) = 0$ so that $\forall \varepsilon \in \mathbb{R}^+$ and $\forall \bar{\varpi} \in \mathscr{C}_b$ satisfying (75), one and exactly one solution $\varpi \in \mathscr{C}_b$ of (1) exists with

$$\|\varpi - \bar{\varpi}\| \le \sigma(\varepsilon). \tag{77}$$

Remark 42. A function $\bar{\varpi} \in \mathscr{C}_b$ is a solution of the inequality (75) *iff* $\exists$ a function $\eta \in \mathscr{C}$ such that

(1) $|\eta(\tau)| \le \varepsilon, \tau \in \mathrm{J}$

(2) ${}^c\mathbb{D}^{\nu;\psi}_{a^+}[\bar{\varpi}(\tau) - \mathbb{F}(\tau,\bar{\varpi}_\tau)] = \mathbb{H}(\tau,\bar{\varpi}_\tau) + \eta(\tau), \tau \in \mathrm{J}$

Theorem 43. *Suppose that (M2) and (70) hold. In this case, the solution of (1) is UH and GUH stable.*

Proof. Assume that each of these two members $\varepsilon \in \mathbb{R}^+$ and $\bar{\varpi} \in \mathscr{C}_b$ satisfy (75). Then, $\exists \eta \in \mathscr{C}$ such that $|\eta(\tau)| \le \varepsilon, \tau \in J$, and

$$\begin{cases} {}^c\mathbb{D}^{\nu;\psi}_{a^+}[\bar{\varpi}(\tau) - \mathbb{F}(\tau,\bar{\varpi}_\tau)] = \mathbb{H}(\tau,\bar{\varpi}_\tau) + \eta(\tau), \tau \in \mathrm{J}, \\ \bar{\varpi}(\tau) = \phi(\tau), \tau \in [a-\delta, a]. \end{cases} \tag{78}$$

Using Lemma 30, the NFuDEq (78) has a solution given as

$$\bar{\varpi}(t) := \begin{cases} \mathbb{I}^{\nu,\psi}_{a^+}[\mathbb{H}(\tau,\bar{\varpi}_\tau) + \eta(t)] + \mathbb{F}(\tau,\bar{\varpi}_\tau), & \text{if } \tau \in \mathrm{J}, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \tag{79}$$

☐

Theorem 44. *Ensures the existence of a unique solution $\varpi \in \mathscr{C}_b$ of the NFuDEq (1) which satisfies*

$$\varpi(t) := \begin{cases} \mathbb{I}^{\nu,\psi}_{a^+}\mathbb{H}(\tau,\varpi_\tau) + \mathbb{F}(\tau,\varpi_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a-\delta, a]. \end{cases} \tag{80}$$

Therefore, for any $\tau \in J$, we get:

$$\begin{aligned} &|\bar{\varpi}(\tau) - \varpi(\tau)| \\ &\quad \le \mathbb{I}^{\nu,\psi}_{a^+}|\mathbb{H}(\tau,\bar{\varpi}_\tau) - \mathbb{H}(\tau,\varpi_\tau)| + \mathbb{I}^{\nu,\psi}_{a^+}|\eta(\tau)| \\ &\qquad + |\mathbb{F}(\tau,\bar{\varpi}_\tau) - \mathbb{F}(\tau,\varpi_\tau)| \\ &\quad \le \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{H}}\|\bar{\varpi}_\tau - \varpi_\tau\|_{\mathscr{C}_\delta} + \frac{(\psi(\tau)-\psi(a))^{\nu}}{\Gamma(\nu+1)}\varepsilon \\ &\qquad + L_{\mathbb{F}}\|\bar{\varpi}_\tau - \varpi_\tau\|_{\mathscr{C}_\delta} \\ &\quad \le \left(\frac{(\psi(b)-\psi(a))^{\nu}}{\Gamma(\nu+1)}L_{\mathbb{H}} + L_{\mathbb{F}}\right)\|\bar{\varpi} - \varpi\|_{\mathscr{C}_b} + \kappa\varepsilon, \end{aligned} \tag{81}$$

where $\kappa = (\psi(b) - \psi(a))^{\nu}/\Gamma(\nu + 1)$. *Therefore, we have proved that*

$$\|\bar{\omega} - \omega\|_{\mathscr{C}_b} \le \Delta \|\bar{\omega} - \omega\|_{\mathscr{C}_b} + \kappa\varepsilon. \tag{82}$$

By the condition in Theorem 39, one can deduce that

$$\|\bar{\omega} - \omega\|_{\mathscr{C}_b} \le \frac{\kappa}{1 - \Delta}\varepsilon. \tag{83}$$

For $c = \kappa/1 - \Delta > 0$, *we infer that the solution of (1) is UH stable. In a similar manner, it is shown the existence of* $\sigma \in \mathscr{C}(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ *by* $\sigma(\varepsilon) = \kappa/1 - \Delta\varepsilon$ *with* $\sigma(0) = 0$. *Hence, the solution of (1) is GUH stable.*

6. Examples

Two illustrative examples are provided in this section to apply and validate our obtained results.

Example 45. Let us consider the NFuDEq according to (1) such that

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{3/4;\ln(\tau)}[\omega(\tau) - \mathbb{F}(\tau, \omega_\tau)] = \mathbb{H}(\tau, \omega_\tau), \tau \in \mathrm{J} := [1, e], \\ \omega(\tau) = 0, \quad \tau \in [1 - \delta, 1], \end{cases} \tag{84}$$

where
$\psi(\tau) = \ln(\tau)$, $a = 1$, $b = e$, $\nu = 3/4$,
and $\mathbb{F}(\tau, u_\tau)$, $\mathbb{H}(\tau, u_\tau)$ are given as

$$\mathbb{F}(\tau, u) = \frac{\ln(\tau)\cos u}{\sqrt{100 + \ln(\tau)}}, \mathbb{H}(\tau, u) = \frac{1}{(t+1)^2}\left(u + \sqrt{1 + u^2}\right). \tag{85}$$

To explain Theorem 39, let us take $\mathbb{H}(\tau, u)$ and $\mathbb{F}(\tau, u)$ given by (85) and $\mathbb{F}(1, u_1) = 0$. Clearly, the condition (M2) holds with $L_{\mathbb{F}} = 1/10$ and $L_{\mathbb{H}} = 1/4$. In addition, $\Delta \approx 0.3720 < 1$. Hence, all hypotheses of Theorem 39 are satisfied. Therefore, it is found exactly one solution for the NFuDEq (84) on $[1, e]$.

Example 46. Consider the NFuDEq as follows:

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{1/2;\tau}[z(\tau) - \mathbb{F}(\tau, z_\tau)] = \mathbb{H}(\tau, z_\tau), \tau \in \mathrm{J} := [0, 1], \\ z(0) = 0. \end{cases} \tag{86}$$

Notice that

$$\psi(\tau) = \tau, \phi(\tau) = 0, a = 0, b = 1, \nu = \frac{1}{2}. \tag{87}$$

To illustrate Theorem 37, let us take

$$\begin{aligned} \mathbb{H}(\tau, u_\tau) &= \frac{1}{e^{(-\pi\tau)} + 9}\left(\frac{|u(\tau)|}{1 + |u(\tau)|}\right) + \tau, \\ \mathbb{F}(\tau, \nu_\tau) &= \frac{\sqrt{3 + \tau^2}\,|\,\nu\,|}{10} + \left(1 + \tau^2\right). \end{aligned} \tag{88}$$

It is obvious that

$$\begin{aligned} |\mathbb{H}(\tau, u) - \mathbb{H}(\tau, \nu)| &\le \frac{1}{10}\|u - \nu\|_{\mathscr{C}_b}, \\ |\mathbb{F}(\tau, u) - \mathbb{F}(\tau, \nu)| &\le \frac{1}{5}\|u - \nu\|_{\mathscr{C}_b}. \end{aligned} \tag{89}$$

Hence, (M2) also holds with $L_{\mathbb{H}} = 1/10$, $L_{\mathbb{F}} = 1/5$. Further, from the above-given data, we can easily calculate

$$\left[\frac{(\psi(b) - \psi(a))^{\nu}}{\Gamma(\nu + 1)}\right] L_{\mathbb{H}} = 0.1128. \tag{90}$$

On the contrary, $\forall \tau \in \mathrm{J}$, $u \in \mathbb{R}$, we get

$$|\mathbb{H}(\tau, u)| \le 1 + \frac{1}{10}|u|, |\mathbb{F}(\tau, u)| \le 2 + \frac{1}{5}|u|. \tag{91}$$

Hence, (M1) holds with $M_1 = 1/10$, $M_2 = 1/5$, $p = N_1 = 1$, and $N_2 = 2$. In view of Theorem 37,

$$\Theta = \{u \in \mathscr{C}_b : \text{there exists } \varsigma \in [0, 1] \text{ such that } u = \varsigma\mathscr{K}u\} \tag{92}$$

is the solution set, then

$$\|u\| \le \xi\left(\|\mathscr{K}u\|_{\mathscr{C}_b} + \|\mathscr{H}u\|_{\mathscr{C}_b}\right) \le \mathscr{M}\|u\|_{\mathscr{C}_b} + \mathscr{N}. \tag{93}$$

Using the Matlab program, we have

$$\|u\|_{\mathscr{C}_b} \le \frac{\mathscr{N}}{1 - \mathscr{M}} = 4.8298. \tag{94}$$

By Theorem 37, the NFuDEq (86) with the data (87) and (88) has at least a solution.

7. Conclusion

In this paper, we considered and studied a fractional neutral functional delay differential equation involving a ψ-Caputo fractional derivative on a partially ordered Banach space. To do this, we proved the existence results with the help of the Dhage approximation technique, and then by topological degree method for condensing maps. We established the uniqueness result by the well-known Banach contraction principle. The different kinds of Hyers-Ulam stability were checked in the sequel. Finally, we supported the validity of our findings by providing two examples. This study can be extended to more general structures by using generalized fractional operators with singular or nonsingular kernels due to their high accuracy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Analytical Properties of the Generalized Heat Matrix Polynomials Associated with Fractional Calculus

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In this paper, we introduce a matrix version of the generalized heat polynomials. Some analytic properties of the generalized heat matrix polynomials are obtained including generating matrix functions, finite sums, and Laplace integral transforms. In addition, further properties are investigated using fractional calculus operators.

1. Overture

In the past few decades, matrix versions of the orthogonal polynomials have attracted a lot of research interest due to their close relations and various applications in many areas of mathematics, statistics, physics, and engendering, for example, see [1–11].

The recent advances of fractional order calculus (FOC) are dominated by its multidisciplinary applications. Moreover, special functions of fractional order calculus have many applications in various areas of mathematical analysis, probability theory, control systems, and engineering (see, for example, [12–15]).

Moreover, the development of fractional calculus associated with special matrix functions and polynomials has been investigated by many researchers, for example, the recent works [16–22].

Among these classical polynomials are the heat polynomials (also designated as Temperature polynomials) that are polynomial solutions of the heat equation and also are particularly useful in solving the Cauchy problem (see [23–26]). Special functions, such as the confluent hypergeometric function, integral error functions, and Laguerre polynomials, have a close link with the generalized heat polynomials introduced [27–29]. Further, the generalized heat polynomials are mainly used to construct an approximate solution of a given problem as a linear combination of the polynomials. Such solution satisfies the governing equation and other equations (cf., e.g., [30–35]).

In our investigation here, we define a generalized heat matrix polynomial $\mathbb{HP}_m(T; \xi, \nu)$. We then establish certain generating matrix functions, finite sum formulas, Laplace transforms, and fractional calculus operators for these polynomials in Sections 3, 4, 5, and 6, respectively. Further, some interesting special cases and concluding remarks of our main results are pointed out in Section 7.

2. Preliminaries

In this section, we give some basic definitions and terminologies; for more details, we can be referred to [36, 37].

Here and through the work, let $\mathbb{C}^{d\times d}$ be the vector space of all the square matrices of order $d \in \mathbb{N}$, ($\mathbb{N}$ is the set of all positive integers) whose entries are in the set of complex numbers $\mathbb{C}$. For a $E \in \mathbb{C}^{d\times d}$, let $\sigma(E)$ be the set of all eigenvalues of E which is called the spectrum of E. We have

$$\mu(E) := \max\{\Re(\xi) : \xi \in \sigma(E)\} \text{ and } \tilde{\mu}(E) := \min\{\Re(z) : z \in \sigma(E)\}, \quad (1)$$

which imply $\tilde{\mu}(E) = -\mu(-E)$. Here, $\mu(E)$ is called the spectral abscissa of E, and the matrix E is said to be positive stable if $\tilde{\mu}(E) > 0$. Further, let I and 0 denote the identity and zero matrices corresponding to a square matrix of any order, respectively.

If E is a positive stable matrix in $\mathbb{C}^{d\times d}$, then the gamma matrix function $\Gamma(E)$ is well defined as follows (cf., e.g., [11, 38, 39]):

$$\Gamma(E) = \int_0^\infty e^{-z} z^{E-I}\, dz, z^{E-I} := \exp\left((E-I)\ln z\right). \quad (2)$$

Moreover, if E is a matrix in $\mathbb{C}^{d\times d}$ which gratifies

$$E + nI \text{ is invertible for every integer } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad (3)$$

then $\Gamma(E)$ is invertible, its inverse coincides with $\Gamma^{-1}(E)$. Under condition (3), we can write the following Pochhammer matrix symbol

$$(E)_n = E(E+I)\cdots(E+(n-1)I) = \Gamma(E+nI)\,\Gamma^{-1}(E)(n \in \mathbb{N}). \quad (4)$$

Let $r, k \in \mathbb{N}_0$. Also let $(S)_r$ and $(Q)_k$ be arrays of r commutative matrices $S_1, S_2, \cdots, S_r$ and k commutative matrices $Q_1, Q_2, \cdots, Q_k$ in $\mathbb{C}^{d\times d}$, respectively, such that $Q_k + nI$ are invertible for $1 \le d \le k$ and all $n \in \mathbb{N}_0$. Then, the generalized hypergeometric matrix function ${}_r\mathbf{F}_k\left((S)_r ; (Q)_k ; z\right)(z \in \mathbb{C})$ can be defined by (see, e.g., [11, 39])

$${}_r\mathbf{F}_k\left((S)_r ; (Q)_k ; z\right) = \sum_{s=0}^{\infty} \prod_{j=1}^{r} \left(S_j\right)_s \prod_{i=1}^{k} \left[\left(Q_i\right)_s\right]^{-1} \frac{z^s}{s!}. \quad (5)$$

In particular, the hypergeometric matrix function ${}_2\mathbf{F}_1(S, P ; C ; z) \equiv \mathbf{F}(S, P ; C ; z)$ is defined by

$$\mathbf{F}(R, P ; C ; z) = \sum_{s=0}^{\infty} (R)_s (P)_s (C)_s^{-1} \frac{z^s}{s!}, \quad (6)$$

for matrices R, P, C in $\mathbb{C}^{d\times d}$ such that $C + nI$ is invertible for all $n \in \mathbb{N}_0$. Also, note that for $r = 1, k = 0$ in (9), we have the Binomial type matrix function ${}_1F_0(R;-;z)$ as follows:

$${}_1\mathbf{F}_0(R;-;z) = (1-z)^{-R} = I + Rz + \frac{R(R+I)z^2}{2!} + \cdots + \frac{(R)_n z^n}{n!} + \cdots, |z| < 1. \quad (7)$$

Let E be a positive stable invertible matrix in $\mathbb{C}^{d\times d}$. Then, the n^{th} Laguerre matrix polynomial is defined in the form (see, e.g., [11, 40])

$$\begin{aligned} \mathbf{L}_n^E(z) &= \sum_{m=0}^{n} \frac{(-1)^m z^m}{m!(n-m)!} (S+I)_n (E+I)_m^{-1} \\ &= \frac{(E+I)_n}{n!}\, {}_1\mathbf{F}_1(-nI ; E+I ; z), n \in N_0. \end{aligned} \quad (8)$$

The Laplace transform of $f(\xi)$ is defined by [7].

$$\mathscr{L}\{f(\xi) ; \lambda\} = \int_0^\infty e^{-\lambda\xi} f(\xi) d\xi, \Re(\lambda) > 0, \quad (9)$$

provided that the improper integral exists.

Lemma 1. *(see [7]). Let S be a positive stable invertible matrix in $\mathbb{C}^{d\times d}$. Then, we have*

$$\mathscr{L}\left\{\xi^S ; \lambda\right\} = \lambda^{-(S+I)}\Gamma(S+I), \text{Re}(\lambda) > 0. \quad (10)$$

3. Generalized Heat Matrix Polynomial and Generating Matrix Functions

A generalized heat matrix polynomial is defined in (11) below; then, a family of generating matrix functions are proposed, see Theorem 4 and Theorem 8 of this section.

Definition 2. Let T be a positive stable matrix in the complex space $C^{d\times d}$ satisfying the spectral condition (3). Then, we define a generalized heat matrix polynomial of degree $m \in \mathbb{N}_0$ in the following explicit form:

$$\begin{aligned} \mathbb{HP}_m(T ; \xi, \nu) &= \sum_{s=0}^{m} 2^{2s} \binom{m}{s} \Gamma\left(T + \left(m + \frac{1}{2}\right)I\right)\Gamma^{-1} \\ &\quad \cdot \left(T + \left(m - s + \frac{1}{2}\right)I\right) \xi^{2m-2s} \nu^s \\ &= m!(4\nu)^m\, \mathbf{L}_m^{T-\frac{1}{2}I}\left(\frac{-\xi^2}{4\nu}\right) ; \nu > 0, \end{aligned} \quad (11)$$

where $\mathbf{L}_m^T(\xi)$ is the Laguerre matrix polynomial in (8).

Remark 3. Note that

$$\mathbb{HP}_m(T ; \xi, \nu) = (4\nu)^m \left(T + \frac{1}{2}I\right)_m {}_1\mathbf{F}_1\left(-mI, T + \frac{1}{2}I ; \frac{-\xi^2}{4\nu}\right), \quad (12)$$

and that for the scalar case $d = 1$, taking $T = a$ and $a > 0$, the m^{th} polynomial $\mathbb{HP}_m(a ; \xi, \nu)$ coincides with the classical scalar generalized heat polynomial, see [24, 26, 33]. Further, the ordinary heat polynomial defined in [23], when $T = 0$; $\mathbb{HP}_m(0 ; \xi, \nu) = \upsilon_{2m}(\xi, \nu)$.

Theorem 4. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, $|\xi^2 t| < |1 - 4\nu t|$ and T and R be positive stable matrices in $C^{d\times d}$ such that $T + nI$ is* *invertible for all $n \in \mathbb{N}_0$. A generating matrix function of $\mathbb{HP}_m(T;\xi,\nu)$ is*

$$\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}(R)_m \mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!} = (1-4\nu t)^{-R}{}_1\mathbf{F}_1\left(R, T+\frac{1}{2}I;\frac{\xi^2 t}{1-4\nu t}\right). \quad (13)$$

Proof. For convenience, suppose that the left-hand side of (13) is denoted by **J**. According to the series expression of (11) and (7) to **J**, we find that

$$\begin{aligned}\mathbf{J} &= \sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}(R)_m \mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\frac{(R)_m\left[(T+1/2I)_{m-s}\right]^{-1}}{s!(m-k)!}(\xi)^{2m-2s}(4\nu)^s t^m\\ &= \sum_{m=0}^{\infty}(R)_m\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\frac{\left(\xi^2 t\right)^m}{m!}\cdot\sum_{s=0}^{\infty}(R+mI)_s\frac{(\xi\nu t)^s}{s!}\\ &= (1-4\nu t)^{-R}\sum_{m=0}^{\infty}(R)_m\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\frac{\left(\xi^2 t/1-4\nu t\right)^m}{m!}.\end{aligned} \quad (14)$$

Upon using the relation (6), the last equality evidently leads us to the required result.

Corollary 5. *For $\mathbb{HP}_m(T;\xi,\nu)$, the following generating matrix function holds true*

$$\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!} = \exp\,(4\nu t)_0\mathbf{F}_1\left(-,T+\frac{1}{2}I;\xi^2 t\right). \quad (15)$$

Remark 6. The Bessel matrix function $J_R(z)$, for a positive stable matrix $R \in C^{d\times d}$, is expressible in terms of hypergeometric matrix function as follows (see, e.g., [11, 41])

$$J_R(w) = \left(\frac{w}{2}\right)^R \Gamma^{-1}(R+I)_0\mathbf{F}_1\left(-;R+I,\frac{-w^2}{4}\right). \quad (16)$$

Thus, by applying the relation (16) to (15) in Corollary 5, we can deduce the following Corollary.

Corollary 7. *For $\mathbb{HP}_m(T;\xi,\nu)$, the following holds true*

$$\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!} = \Gamma\left(\left(T+\frac{1}{2}I\right)\right)\cdot\left(i\xi\sqrt{t}\right)^{-\left(T-\frac{1}{2}I\right)}\exp\,(4\nu t)\,J_{T-\frac{1}{2}I}\left(2i\xi\sqrt{t}\right). \quad (17)$$

Theorem 8. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $m, l \in \mathbb{N}_0$ and T be a positive stable matrix in $C^{d\times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following relation holds true*

$$\sum_{m=0}^{\infty}\mathbb{HP}_{m+l}(T;\xi,\nu)\frac{t^m}{m!} = (1-4\nu t)^{-\left(T+\left(l+\frac{1}{2}\right)I\right)}\exp\cdot\left(\frac{\xi^2 t}{1-4\nu t}\right)\mathbb{HP}_{m+l}\left(T;\frac{\xi}{\sqrt{1-4\nu t}},\nu\right). \quad (18)$$

Proof. Follows by induction or by the successive application of Theorem 4 when $R = (T + 1/2I)$. The details are omitted.

Corollary 9. *For $l = 0$ in Theorem 8, the following holds true*

$$\sum_{m=0}^{\infty}\mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!} = (1-4\nu t)^{-\left(T+\frac{1}{2}I\right)}\exp\left(\frac{\xi^2 t}{1-4\nu t}\right). \quad (19)$$

Remark 10. The special cases of (18) and (19) when $d = 1$ are seen to yield the classical generating functions of the generalized Heat polynomials (see [24, 33]).

4. Finite Sums

Here, various finite sums of $\mathbb{HP}_m(T;\xi,\nu)$ can be obtained in the following results.

Theorem 11. *Let $\xi, z \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, and T be a positive stable matrix in $C^{d\times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. Then, we have*

$$\mathbb{HP}_m(T;\xi z,\nu) = \sum_{s=0}^{m}\binom{n}{k}\left(T+\frac{1}{2}I\right)_m\left[\left(T+\frac{1}{2}I\right)_s\right]^{-1}\cdot\left\{4\nu\left(1-z^2\right)\right\}^{m-s}z^{2s}\,\mathbb{HP}_s(T;\xi z,\nu). \quad (20)$$

Proof. From (15) and the following fact

$$e^{4\nu t}{}_0\mathbf{F}_1\left(;T+\frac{1}{2}I;\xi^2 z^2 t\right) = e^{4\nu t\left(1-z^2\right)+4\nu z^2 t}{}_0\mathbf{F}_1\left(;T+\frac{1}{2}I;\xi^2 z^2 t\right). \quad (21)$$

We thus find that

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\mathbb{HP}_m(T;\xi,\nu)\left[\left(T+\frac{1}{2}I\right)_s\right]^{-1}\frac{\left\{4\nu\left(1-z^2\right)\right\}^m\left(z^2 t\right)^s}{m!s!}\\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\frac{\left\{4\nu\left(1-z^2\right)\right\}^{m-s}\left(z^2 t\right)^s}{(m-s)!s!}\left[\left(T+\frac{1}{2}I\right)_s\right]^{-1}\mathbb{HP}_m(T;\xi,\nu).\end{aligned} \quad (22)$$

Comparing the coefficient of t^m on both side, we thus get the required a finite sum formula (20).

Theorem 12. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, also let T and R be positive stable matrices in $C^{d\times d}$ such that $T + nI$ and $R + nI$ are invertible for all $n \in \mathbb{N}_0$. Then, we have*

$$\mathbb{HP}_m(T;\xi,\nu) = \left(T+\frac{1}{2}I\right)_m \left[(R)_m\right]^{-1} \sum_{s=0}^{m} \binom{n}{s} \left(T-R+\frac{1}{2}I\right)_s \cdot \left[\left(T+\frac{1}{2}I\right)_s\right]^{-1} \times \mathbb{HP}_s(T;i\xi,\nu)\mathbb{HP}_{m-s} \cdot (2R-T-I;\xi,\nu). \tag{23}$$

Proof. By using the series (11) and Theorem 4 with applying to Kummer's matrix formula (see [5]), we observe that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}(R)_m \mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!} \\
&= (1-4\nu t)^{-R}\exp\left(\frac{\xi^2 t}{1-4\nu t}\right)\mathbf{F}_1\left(T-R+\frac{1}{2}I, T+\frac{1}{2}I;\frac{-\xi^2 t}{1-4\nu t}\right)_1 \\
&= (1-4\nu t)^{-(2R-T-I)+\frac{1}{2}I}\exp\left(\frac{\xi^2 t}{1-4\nu t}\right)(1-4\nu t)^{-\left(T-R+\frac{1}{2}I\right)}{}_1\mathbf{F}_1 \\
&\quad\cdot\left(T-R+\frac{1}{2}I, T+\frac{1}{2}I;\frac{-\xi^2 t}{1-4\nu t}\right) \\
&= \sum_{m=0}^{\infty}\frac{\mathbb{HP}_m(2R-T-I;\xi,\nu)}{m!}t^m\sum_{s=0}^{\infty}\left(T-R+\frac{1}{2}I\right)_s \\
&\quad\cdot\left[\left(T+\frac{1}{2}I\right)_s\right]^{-1}\frac{\mathbb{HP}_m(T;i\xi,\nu)}{m!}t^s.
\end{aligned} \tag{24}$$

Equating the coefficient of t^m on both sides, we thus arrive at the desired result (23).

5. Laplace Transforms

Here, Laplace integral transforms of the generalized heat matrix polynomials are derived as follows.

Theorem 13. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d\times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold*

$$\mathscr{L}\left\{u^A \mathbb{HP}_m\left(T;\xi\sqrt{u},\nu\right):\lambda\right\} = (4\nu)^m\left(T+\frac{1}{2}I\right)_m \lambda^{-(A+I)}\Gamma \cdot (A+I)_2\mathbf{F}_1\left(-mI, A+I;T+\frac{1}{2}I;\frac{-\xi^2}{4\nu\lambda}\right). \tag{25}$$

Proof. Making a particular use of (9) with (11), (6) and applying to Lemma 1, yields our desired result (25) in Theorem 13. The details are omitted.

A similar procedure yields the following Laplace integral transforms. So we prefer to omit the proofs.

Theorem 14. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d\times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold*

$$\begin{aligned}
&\mathscr{L}\left\{u^A \mathbb{HP}_m(T;\xi u,\nu):\lambda\right\} \\
&= (4\nu)^m\left(T+\frac{1}{2}I\right)_m \lambda^{-(A+I)}\Gamma(A+I)_3\mathbf{F}_1 \\
&\quad\cdot\left(-mI,\frac{1}{2}(A+I),\frac{1}{2}(A+2I);T+\frac{1}{2}I;\frac{-\xi^2}{\nu\lambda^2}\right).
\end{aligned} \tag{26}$$

Theorem 15. *Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d\times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold*

$$\begin{aligned}
&\mathscr{L}\left\{u^A \mathbb{HP}_m(T;u,\nu):\lambda\right\} \\
&= (4\nu)^m\left(T+\frac{1}{2}I\right)_m \lambda^{-(A+I)}\Gamma(A+I)_3\mathbf{F}_1 \\
&\quad\cdot\left(-mI,\frac{1}{2}(A+I),\frac{1}{2}(A+2I);T+\frac{1}{2}I;\frac{-1}{\nu\lambda^2}\right).
\end{aligned} \tag{27}$$

The above Theorems lead to the following special cases.

Corollary 16. *For T is a positive stable matrix in $C^{d\times d}$, and $\Re(\lambda) > 0$, then we have the following Laplace transforms*

$$\begin{aligned}
&\mathscr{L}\left\{u^{T-\frac{1}{2}I}\mathbb{HP}_m\left(T;\xi\sqrt{u},\nu\right):\lambda\right\} \\
&= (4\nu)^m\lambda^{-\left(A+\frac{1}{2}\right)}\Gamma\left(A+\left(m+\frac{1}{2}\right)I\right)\left(1+\frac{\xi^2}{4\nu\lambda}\right)^m, \\
&\mathscr{L}\left\{u^{2T-I}\mathbb{HP}_m(T;\xi u,\nu):\lambda\right\} \\
&= (4\nu)^m\left(T+\frac{1}{2}I\right)_m\lambda^{-2T}\Gamma(2T)_2\mathbf{F}_0\left(-mI,T;-;\frac{-\xi^2}{\nu\lambda^2}\right), \\
&\mathscr{L}\left\{u^{2T-I}\mathbb{HP}_m(T;u,\nu):\lambda\right\} \\
&= (4\nu)^m\left(T+\frac{1}{2}I\right)_m\lambda^{-(2T)}\Gamma(2T)_3\mathbf{F}_1\left(-mI,T;-;\frac{-1}{\nu\lambda^2}\right).
\end{aligned} \tag{28}$$

6. Fractional Calculus Approach

Here, we consider the Riemann. Liouville fractional integral and derivative operators $\mathbf{I}^\gamma$ and D_z^γ of order $\gamma \in C, \Re(\gamma) > 0$, respectively (see, for details, [19])

$$(\mathbf{I}^\gamma f(\tau))(z) = \frac{1}{\Gamma(\gamma)}\int_0^z f(\tau)(z-\tau)^{\gamma-1}d\tau, \tag{29}$$

where $f(\tau)$ is a function of τ and some square matrices so that this integral converges.

$$D_z^{\gamma}\{f(z)\}=\begin{cases}\dfrac{1}{\Gamma(-\gamma)}\displaystyle\int_0^z (z-\tau)^{-\gamma-1}f(\tau)d\tau, & (\Re(\gamma)<0),\\ \dfrac{d^n}{dz^n}(D_z^{\gamma-n}\{f(z)\}), & (n-1\le \Re(\gamma)<n\,(n\in N)),\end{cases} \tag{30}$$

where

$$D_z^{\gamma}\left\{z^{\beta}\right\}=\frac{\Gamma(\beta+I)}{\Gamma(\beta-(\gamma-1)I)}z^{\beta-\gamma}\,\Re(\beta)>-1. \tag{31}$$

It is noted in passing that (29) and (30) are applied in recent works, for example, (see [16–22]).

Theorem 17. *For the generalized heat matrix polynomials of degree $m\in\mathbb{N}_0$, the following fractional integral operator holds true:*

$$\begin{aligned}\mathbf{I}^{\gamma}\left\{\xi^{T-\frac{1}{2}I}\mathbb{HP}_m\left(T;\sqrt{\xi},\nu\right)\right\}&=\Gamma\left(T+\left(m+\frac{1}{2}\right)I\right)\Gamma^{-1}\\&\cdot\left(T+\left(\gamma+m+\frac{1}{2}\right)I\right)\\&\times\xi^{T+\left(\gamma-\frac{1}{2}\right)I}\mathbb{HP}_m\left(T+\gamma I;\sqrt{\xi},\nu\right).\end{aligned} \tag{32}$$

Proof. From (29) and (11), we have

$$\begin{aligned}\mathbf{I}^{\gamma}\left\{\xi^{T-\frac{1}{2}I}\,\mathbb{HP}_m\left(T;\sqrt{\xi},\nu\right)\right\}&=\frac{1}{\Gamma(\gamma)}\int_0^{\xi}(\xi-\tau)^{\gamma-1}\tau^{T-\frac{1}{2}I}\,\mathbb{HP}_m\\&\cdot\left(T;\sqrt{\tau},\nu\right)d\tau.\end{aligned} \tag{33}$$

Setting $\tau=\xi\eta$ and after a simplification, we get

$$\begin{aligned}&\mathbf{I}^{\gamma}\left\{\xi^{T-\frac{1}{2}I}\,\mathbb{HP}_m\left(T;\sqrt{\xi},\nu\right)\right\}\\&=\frac{\xi^{T+(\gamma-1/2)I}}{\Gamma(\gamma)}\int_0^1\eta^{T-\frac{1}{2}I}(1-\eta)^{\gamma-1}\,\mathbb{HP}_m\left(T;\sqrt{\xi\eta},\nu\right)d\eta\\&=\xi^{T+\left(\gamma-\frac{1}{2}\right)I}(4\nu)^m\left(T+\frac{1}{2}I\right)_m\sum_{s=0}^{m}(-mI)_s\left[\left(T+\frac{1}{2}I\right)_s\right]^{-1}\Gamma\\&\cdot\left(T+\left(s+\frac{1}{2}\right)I\right)\times\Gamma^{-1}\left(T+\left(s+\gamma+\frac{1}{2}\right)I\right)\frac{\left(-\left(\sqrt{\xi}\right)^2/4\nu\right)^s}{s!},\end{aligned} \tag{34}$$

whose last summation, in view of (11), is easily seen to arrive at the expression in (32). This completes the proof of Theorem 17.

Theorem 18. *For the generalized heat matrix polynomials $\mathbb{HP}_m(T;\xi,\nu)$, we have the following formula*

$$\begin{aligned}&\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\frac{(\beta)_m}{(\gamma)_m}\mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!}{}_1\mathbf{F}_1(\gamma-\beta;\gamma+m;4\nu t)\\&=\exp(4\nu t)\,{}_1\mathbf{F}_1\left(\beta I;T+\frac{1}{2}I;\xi^2t\right),\Re(\beta)>0,\Re(\gamma)>0.\end{aligned} \tag{35}$$

Proof. According to (15), we find that

$$\begin{aligned}&\exp(-4\nu t)\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m(T;\xi,\nu)\frac{t^m}{m!}\\&={}_0\mathbf{F}_1\left(-;T+\frac{1}{2}I;\xi^2t\right).\end{aligned} \tag{36}$$

Assume that the left-hand side of (35) be denoted by J. Multiply the left-hand side (35) by $t^{\beta-1}$ and applying the relation (31), we get

$$\begin{aligned}\mathrm{J}&=\frac{\Gamma(\beta)}{\Gamma(\gamma)}t^{\gamma-1}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m\\&\cdot(T;\xi,\nu)(4\nu)^s\frac{(-1)^s(\beta)_{m+s}t^{m+s}}{(\gamma)_{m+s}s!m!}\\&=\frac{\Gamma(\beta)}{\Gamma(\gamma)}t^{\gamma-1}\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m(T;\xi,\nu)\frac{(\beta)_mt^m}{(\gamma)_m m!}{}_1\mathbf{F}_1\\&\cdot\left(\beta+m;\gamma+\frac{1}{2};-4\nu t\right).\end{aligned} \tag{37}$$

Invoking Kummer's matrix transformation [5] leads to

$$\begin{aligned}\mathrm{J}&=\frac{\Gamma(\beta)}{\Gamma(\gamma)}t^{\gamma-1}\exp(-4\nu t)\sum_{m=0}^{\infty}\left[\left(T+\frac{1}{2}I\right)_m\right]^{-1}\mathbb{HP}_m\\&\cdot(T;\xi,\nu)\frac{(\beta)_mt^m}{(\gamma)_m m!}{}_1\mathbf{F}_1(\gamma-\beta;\gamma+m;4\nu t).\end{aligned} \tag{38}$$

It is easy to multiply right-hand side of (35) by $t^{\beta-1}$ and applying the fractional differentiation operator $D_t^{\beta-\gamma}$ from (31), whereupon this completes the establishment of the Theorem 18.

7. Concluding Remarks

In [23], Rosenbloom and Widder investigated expansions of solutions $u(x,t)$ of the heat equation $u_{xx}=u_t$ in series of polynomial solutions $v_n(x,t)$. Further, Haimo and Markett [33, 34] discussed the generalized heat equation

$$\Delta_x u(x,t)=\frac{\partial}{\partial t}u(x,t), \tag{39}$$

where $\Delta_x h(x)=h''(x)+(2\nu/x)h'(x),\nu$ is a fixed positive number and they introduced the generalized heat polynomial solution of (39) as

$$\mathbb{P}_{n,\nu}(x,t)=\sum_{j=0}^{n}2^{2j}\binom{n}{j}\frac{\Gamma(\nu+1/2+n)}{\Gamma(\nu+1/2+n-j)}x^{2n-2j}t^{j}. \quad (40)$$

Note that $\mathbb{P}_{n,0}(x,t)=v_{2n}(x,t)$ is the ordinary heat polynomials of even order. Also, $\mathbb{P}_{n,0}(x,-1)=H_{2n}(x/2)$ is the Hermite polynomials of even order.

Recently, many studies and extensions of the well-known special matrix polynomials have been in a focus of increasing attention leading to new and interesting problems. In this perspective, we defined the generalized heat matrix polynomials. Also, we have given some of their main properties, namely, the generating matrix functions, finite sum formulas, Laplace integral transforms, and fractional calculus operators. Further, This study is assumed to be a generalization of the scalar cases [27, 33, 34] to the matrix setting. In addition, this approach allows to derive several new integral and differential representations that can be used in theoretical, applicable aspects like the boundary value problems and the numerical algorithms. Additional research and application on this topic is now under preparation and will be presented in forthcoming works.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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Research Article

An Implicit Relation Approach in Metric Spaces under w-Distance and Application to Fractional Differential Equation

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The purpose of this work is to introduce a new class of implicit relation and implicit type contractive condition in metric spaces under w-distance functional. Further, we derive fixed point results under a new class of contractive condition followed by three suitable examples. Next, we discuss results about weak well-posed property, weak limit shadowing property, and generalized w-Ulam-Hyers stability of the mappings of a given type. Finally, we obtain sufficient conditions for the existence of solutions for fractional differential equations as an application of the main result.

1. Introduction and Preliminaries

In 1996, Kada et al. [1] introduced the concept of a w-distance on a metric space and proved a generalized Caristi fixed point theorem, Ekeland's ε-variational principle, and the nonconvex minimization theorem according to Mizoguchi and Takahashi [2].

Definition 1 (see [1]). Let (Ξ, d) be a metric space. A function $\omega : \Xi \times \Xi \longrightarrow [0,\infty)$ is called a w-distance on Ξ if it satisfies the following properties:

$(W1)$ $\omega(\vartheta, \mu) \le \omega(\vartheta, \nu) + \omega(\nu, \mu)$ for any $\vartheta, \nu, \mu \in \Xi$

$(W2)$ ω is lower semicontinuous in its second variable, i.e., if $\vartheta \in \Xi$ and $\nu_n \longrightarrow \nu \in \Xi$, then $\omega(\vartheta, \nu) \le \liminf_{n\longrightarrow\infty} \omega(\vartheta, \nu_n)$

$(W3)$ For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\omega(\mu, \vartheta) \le \delta$ and $\omega(\mu, \nu) \le \delta$ imply $d(\vartheta, \nu) \le \varepsilon$

The following examples show that a w-distance is not necessarily a metric.

Example 1.

(1) Let (Ξ, d) be a metric space and $c > 0$. Define $w_1(\vartheta, \nu) = cd(\vartheta, \nu)$ and $w_2(\vartheta, \nu) = c$ for all $\vartheta, \nu \in \Xi$. Then, w_1 and w_2 satisfy $(W1)$–$(W3)$. Obviously, w_2 is not a metric since $w_2(\vartheta, \vartheta) = c \neq 0$ for any $\vartheta \in \Xi$

(2) Consider $\mathbb{R}$ as a metric space with the usual metric. Define

$$w_3(\vartheta, \nu) = |\nu| \text{ for every } (\vartheta, \nu) \in \mathbb{R} \times \mathbb{R} \tag{1}$$

Then, w_3 is a w-distance on $\mathbb{R}$ which is not a metric (since it is not symmetric). Note that $(\vartheta_n) = (1, 1, 1, \cdots)$ is a convergent sequence in $\mathbb{R}$ but $w_3(\vartheta_n, \vartheta_m) = 1$ for all $m, n \in \mathbb{N}$.

To prove the main theorem, we need the following lemma, proved by Kada et al. [1].

Lemma 2. *Let (Ξ, d) be a metric space and let ω be a w-distance on Ξ. Suppose that $\{\vartheta_n\}$ and $\{\nu_n\}$ are sequences in Ξ, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, and let $\vartheta, \nu, \mu \in \Xi$. Then, the following assertions hold.*

(i) If $\omega(\vartheta_n, \nu) \le \alpha_n$ and $\omega(\vartheta_n, \mu) \le \beta_n$ for all $n \in \mathbb{N}$, then $\nu = \mu$. In particular, if $\omega(\vartheta, \nu) = \omega(\vartheta, \mu) = 0$, then $\nu = \mu$

(ii) If $\omega(\vartheta_n, \nu_n) \le \alpha_n$ and $\omega(\vartheta_n, \nu) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{\nu_n\}$ converges to ν

(iii) If $\omega(\vartheta_n, \vartheta_m) \le \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{\vartheta_n\}$ is a Cauchy sequence

(iv) If $\omega(\nu, \vartheta_n) \le \alpha_n$ for all $n \in \mathbb{N}$, then $\{\vartheta_n\}$ is a Cauchy sequence

Lemma 3 (see [1, 3]). *Let ω be a w-distance on a metric space (Ξ, d) and $\{\vartheta_n\}$ be a sequence in Ξ such that for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $\omega(\vartheta_n, \vartheta_m) < \varepsilon$, i.e., $\lim_{m,n?\infty} \omega(\vartheta_n, \vartheta_m) = 0$. Then, $\{\vartheta_n\}$ is a Cauchy sequence.*

Recall that the set $\mathcal{O}(\theta_0 ; \mathfrak{T}) = \{\mathfrak{T}^n \theta_0 : n = 0, 1, 2, \cdots\}$ is called the orbit of the self-map $\mathfrak{T}$ at the point $\theta_0 \in \Xi$.

Definition 4. Let (Ξ, d) be metric spaces and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a mapping. Then,

(1) $\mathrm{Fix}(\mathfrak{T}) = \{u \in \Xi \mid u = \mathfrak{T}u\}$ (the set of fixed points of $\mathfrak{T}$)

(2) a mapping $\mathfrak{T}$ is called a Picard operator if there exists $u \in \Xi$ such that $\mathrm{Fix}(\mathfrak{T}) = \{u\}$ and $\{\mathfrak{T}^n x\}$ converges to u, for all $x \in \Xi$

(3) [4] a metric space Ξ is said to be $\mathfrak{T}$-orbitally complete if every Cauchy sequence contained in $\mathcal{O}(x ; \mathfrak{T})$ (for some x in Ξ) converges in Ξ

(4) a mapping $\mathfrak{T}$ is said to be orbitally $\mathcal{U}$-continuous if, for some $\mathcal{U} \subset \Xi \times \Xi$, the following condition holds: for any $x \in \Xi$ and a strictly increasing sequence $\{n_i\}$ of positive integers

$$\lim_{i \longrightarrow \infty} \mathfrak{T}^{n_i} x = z \in \Xi, \tag{2}$$

and $(\mathfrak{T}^{n_i} x, z) \in \mathcal{U}$ for any $i \in \mathbb{N}$ imply that

$$\lim_{i \longrightarrow \infty} \mathfrak{T}^{n_i+1} x = \mathfrak{T}z \tag{3}$$

(5) $\mathfrak{T}$ is called orbitally continuous if, for any $x \in \Xi$ and a strictly increasing sequence $\{n_i\}$ of positive integers, $\mathfrak{T}^{n_i} x \longrightarrow z \in \Xi$ as $i \longrightarrow \infty$ implies that $\mathfrak{T}^{n_i+1} x \longrightarrow \mathfrak{T}z$ as $i \longrightarrow \infty$

In [5], Samet et al. defined the notion of α-admissible mapping which was further sharpened by Karapinar et al. [6] and extended in [7].

Definition 5. For a set $\Xi \neq \emptyset$, let $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ and $\mathfrak{T} : \Xi \longrightarrow \Xi$ be two mappings. Then, $\mathfrak{T}$ is said to be

(i) [5] α-admissible if

$$\nu, \vartheta \in \Xi \text{ with } \alpha(\nu, \vartheta) \ge 1 \Rightarrow \alpha(\mathfrak{T}\nu, \mathfrak{T}\vartheta) \ge 1 \tag{4}$$

(ii) [6] triangular α-admissible if $\mathfrak{T}$ is α-admissible and

$$\alpha(\nu, \vartheta) \ge 1 \text{ and } \alpha(\vartheta, \mu) \ge 1 \text{ imply } \alpha(\nu, \mu) \ge 1 \tag{5}$$

Lemma 6 (see [6]). *Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a triangular α-admissible mapping. Assume that there exists $\nu_0 \in \Xi$ such that $\alpha(\nu_0, \mathfrak{T}\nu_0) \ge 1$. Define a sequence $\{\nu_n\}$ by $\nu_{n+1} = \mathfrak{T}\nu_n$ for $n \in \mathbb{N}^*$. Then, $\alpha(\nu_n, \nu_m) \ge 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Similarly, we can state and prove the following lemma.

Lemma 7. *Let $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a triangular α-admissible mapping. Assume that there exists $\nu_0 \in \Xi$ such that $\alpha(\mathfrak{T}\nu_0, \nu_0) \ge 1$. Define a sequence $\{\nu_n\}$ by $\nu_{n+1} = \mathfrak{T}\nu_n$ for $n \in \mathbb{N}^*$. Then, $\alpha(\nu_n, \nu_m) \ge 1$ for all $m, n \in \mathbb{N}$ with $n > m$.*

To the best of our knowledge, there is no fixed point result in the literature which has been derived by implicit type contractive relation in a metric space under w-distance. Also, w-distance is not necessarily a metric (examples are given above). Motivated by this fact, there is a need for introducing such type of contractive conditions. With this in mind, in Section 2, we introduce the notion of a new implicit relation and ω-implicit contractive mapping in the respective structure. Then, we establish unique fixed point results under aforesaid implicit contractive condition for α -admissible and orbitally continuous mappings on orbitally complete spaces. We demonstrate the results by three illustrative examples. We note that the symmetry condition and full completeness of the underlying space are not required. In Section 4, some new results on weak well-posed property, weak limit shadowing property, and generalized w-Ulam-Hyers stability of mappings of the mentioned type are discussed. In Section 4, a sufficient condition for the existence of solutions for fractional differential equations as an application of the main result is given.

2. Implicit Relation for w-Distance on Metric Spaces

In this section, we introduce a modified version of implicit relation and examples discussed in [8, 9].

Let Ψ be the set of all continuous functions $\psi : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ satisfying the following conditions:

(ψ_1) ψ is nonincreasing in the fifth and sixth variables

(ψ_2) There exists $\hbar \in [0, 1)$ such that for all $\zeta, \xi, \mu \geq 0$, $k \geq 1$.

(ψ_{2a}) $\psi(k\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \leq 0$ implies that $k\zeta \leq \hbar\xi$

(ψ_{2b}) $\psi(k\zeta, \xi, \xi, \zeta, \mu, \zeta + \xi) \leq 0$ implies that $k\zeta \leq \hbar\xi$

(ψ_3) $\psi(k\zeta, \zeta, 0, 0, \zeta, \zeta) > 0$ and $\psi(k\zeta, 0, 0, \zeta, \zeta, 0) > 0$ for $\zeta > 0$ and $k \geq 1$

Example 2. Let $\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1^2 - a\ell_2^2 - b(\ell_3^2 + \ell_4^2/\ell_5 + \ell_6 + 1)$, $0 < a, b < 1$ and $a + 2b < 1$.

(ψ_{2a}) For $\zeta, \xi, \mu \geq 0$, we have

$$\psi(k\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = k^2\zeta^2 - a\xi^2 - b\frac{\zeta^2 + \xi^2}{1 + \zeta + \xi + \mu} \leq 0, \quad (6)$$

which implies that $k^2\zeta^2 \leq a\xi^2 + b(\zeta^2 + \xi^2)$. Then, $k^2\zeta^2 \leq (a + b/1 - b)\xi^2$. Hence, $k\zeta \leq \hbar\xi$, $\hbar = k\sqrt{a + b/k^2 - b} < 1$ for $k \geq 1$

(ψ_{2b}) Similarly as (ψ_{2a}), if $\psi(k\zeta, \xi, \xi, \zeta, \mu, \zeta + \xi) \leq 0$, then $k\zeta \leq \hbar\xi$ for $k \geq 1$

(ψ_3) For all $\zeta > 0$ and for $k \geq 1$, $\psi(k\zeta, \zeta, 0, 0, \zeta, \zeta) = (k - a)\zeta^2 > 0$

Example 3. Let $\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1^2 - a\ell_2^2 - b(\ell_3^2 + \ell_4^2/\ell_5^2 + \ell_6^2 + 1)$, $0 < a, b < 1$ and $a + 2b < 1$.

Similar to Example 2.

Example 4. Let $\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1 - a\ell_2 - b\ell_3 - c(\ell_4\ell_5/\ell_5 + \ell_6 + 1)$, $0 < a, b, c < 1$ and $a + b + c < 1$.

(ψ_{2a}) For $\zeta, \xi, \mu \geq 0$, we have

$$\psi(k\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = k\zeta - a\xi - b\xi - c\frac{\zeta(\zeta + \xi)}{1 + \zeta + \xi + \mu} \leq 0, \quad (7)$$

which implies that $k\zeta - (a + b)\xi \leq c\zeta$, that is, $k\zeta \leq \hbar\xi$, $\hbar = k(a + b)/k - c < 1$ for $k \geq 1$

(ψ_{2b}) Similarly as (ψ_{2a}), if $\psi(k\zeta, \xi, \zeta, \xi, 0, \mu) \leq 0$, then $k\zeta \leq \hbar\xi$

(ψ_3) $\psi(\hbar\zeta, \zeta, 0, 0, \zeta, \zeta) = (k - a)\zeta > 0$ for all $\zeta > 0$ and for $k \geq 1$

Example 5. Let $\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1^3 - a(\ell_3^2\ell_4^2/\ell_2 + \ell_5 + \ell_6 + 1)$, $0 \leq a < 1$.

(ψ_{2a}) For $\zeta, \xi, \mu \geq 0$ and $\psi(\hbar\zeta, \xi, \xi, \zeta, \mu, 0) = k^3\zeta^3 - a(\zeta^2\xi^2/\xi + \mu + 1) \leq 0$.

Then, $k^3\zeta \leq a(\xi^2/(\xi + \mu + 1)) < a\xi$, that is, $k\zeta \leq (a/k^2)\xi$. Hence, $k\zeta \leq \hbar\xi$, $\hbar = a/k^2 < 1$ for $k \geq 1$.

(ψ_{2b}) Similarly, if $\psi(k\zeta, \xi, \zeta, \xi, 0, \mu) \leq 0$, then $k\zeta \leq h\xi$

(ψ_3) $\psi(k\zeta, \zeta, 0, 0, \zeta, \zeta) = k^3\zeta^3 > 0$ for all $\zeta > 0$ for $k \geq 1$

Example 6. Let $\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1 - a\ell_2 - b(\ell_3\ell_6/(\ell_5 + \ell_6 + 1)) - c\ell_4$, $0 < a, b, c < 1$ and $a + b + c < 1$.

(ψ_{2a}) For $\zeta, \xi, \mu \geq 0$, we have

$$\psi(k\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = k\zeta - a\xi - b\frac{\zeta\xi}{1 + \zeta + \xi + \mu} - c\zeta \leq 0, \quad (8)$$

which implies that $k\zeta - a\xi - b\zeta \leq c\zeta$, that is, $k\zeta \leq \hbar\xi$, $\hbar = ka/(k - b - c) < 1$ for $k \geq 1$

(ψ_{2b}) Similarly as (ψ_{2a}), if $\psi(k\zeta, \xi, \zeta, \xi, 0, \mu) \leq 0$, then $k\zeta \leq \hbar\xi$

(ψ_3) $\psi(k\zeta, \zeta, 0, 0, \zeta, \zeta) = (k - a)\zeta > 0$ for all $\zeta > 0$ for $k \geq 1$

Now, we define ω-implicit contractive mapping in the metric space under w-distance using the above introduced implicit relation.

Definition 8. Let (Ξ, d) be a metric space with w-distance ω, $\mathfrak{T} : \Xi \longrightarrow \Xi$ be a given mapping, and $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be a functional. We say that $\mathfrak{T}$ is an ω-implicit contractive mapping, if there exists a function $\psi \in \Psi$ such that

$$\psi(\alpha(\vartheta, \nu)\omega(\mathfrak{T}\vartheta, \mathfrak{T}\nu), \omega(\vartheta, \nu), \omega(\vartheta, \mathfrak{T}\vartheta), \omega(\nu, \mathfrak{T}\nu), \omega(\vartheta, \mathfrak{T}\nu), \omega(\mathfrak{T}\vartheta, \nu)) \leq 0, \quad (9)$$

for all $(\vartheta, \nu) \in \Xi^2$.

If (9) is satisfied for $\vartheta, \nu \in \bar{\mathcal{O}}(\theta_0; \mathfrak{T})$ (for some $\theta_0 \in \Xi$), we say that $\mathfrak{T}$ is an orbitally ω-implicit contractive mapping (at θ_0).

Now, we are equipped to state and prove our first main result as follows.

Theorem 9. *Let (Ξ, d) be a metric space with w-distance ω and $\mathfrak{T} : \Xi \longrightarrow \Xi$. Suppose that the following conditions hold:*

(i) There exists $\theta_0 \in \Xi$ such that $\alpha(\theta_0, \mathfrak{T}\theta_0) \geq 1$ and $\alpha(\mathfrak{T}\theta_0, \theta_0) \geq 1$

(ii) $\mathfrak{T}$ is a triangular α-admissible mapping

(iii) $\mathfrak{T}$ is an orbitally ω-implicit contractive mapping

(iv) (Ξ, ω) is $\mathfrak{T}$-orbitally complete at θ_0

(v) $\mathfrak{T}$ is orbitally continuous

Then, there exists a point $\vartheta^* \in Fix(\mathfrak{T})$. In addition, $\omega(\vartheta^*, \vartheta^*) = 0$ provided $\alpha(\vartheta^*, \vartheta^*) \geq 1$ holds.

Proof. Let $\theta_0 \in \Xi$ be the point described in (i). Define a sequence $\{\theta_n\}$ by $\theta_{n+1} = \mathfrak{T}\theta_n$ for $n \geq 1$. If $\theta_{n(0)} = \theta_{n(0)+1}$ for some $n_0 \geq 1$, then obviously $\mathrm{Fix}(\mathfrak{T}) \neq \emptyset$. Hence, we suppose that $\theta_n \neq \theta_{n+1}$ for all $n \geq 1$. First, we show that

$$\lim_{n?\infty} \omega(\mathfrak{T}^n\theta_0, \mathfrak{T}^{n+1}\theta_0) = 0. \quad (10)$$

Using (ii) and Lemma 6, we have $\alpha(\theta_n, \theta_{n+1}) \geq 1$ for all $n \geq 1$. Then, for all $n \geq 1$, using (9) for $\vartheta = \theta_{n-1}$, $\nu = \theta_n$,

$$\psi\big(\alpha(\theta_{n-1},\theta_n)\omega\big(\mathfrak{I}^n\theta_0,\mathfrak{I}^{n+1}\theta_0\big),\omega\big(\mathfrak{I}^{n-1}\theta_0,\mathfrak{I}^n\theta_0\big),\\ \cdot\,\omega\big(\mathfrak{I}^{n-1}\theta_0,\mathfrak{I}^n\theta_0\big),\omega\big(\mathfrak{I}^n\theta_0,\mathfrak{I}^{n+1}\theta_0\big),\omega\big(\mathfrak{I}^{n-1}\theta_0,\mathfrak{I}^{n+1}\theta_0\big),\\ \cdot\,\omega\big(\mathfrak{I}^n\theta_0,\mathfrak{I}^n\theta_0\big)\big)\le 0. \tag{11}$$

Denoting $\rho_n=\omega(\mathfrak{I}^n\theta_0,\mathfrak{I}^{n+1}\theta_0)$ for all $n\in\mathbb{N}^*$ and applying (ψ_1) in the fifth variable, we have

$$\psi(\alpha(\theta_{n-1},\theta_n)\rho_n,\rho_{n-1},\rho_{n-1},\rho_n,\rho_{n-1}+\rho_n,\omega(\mathfrak{I}^n\theta_0,\mathfrak{I}^n\theta_0))\le 0. \tag{12}$$

It follows from (ψ_{2a}) that there is $\hbar\in[0,1)$ such that

$$\rho_n\le\alpha(\theta_n,\theta_{n+1})\rho_n\le\hbar\rho_{n-1}, \tag{13}$$

and so,

$$\rho_n\le\hbar\rho_{n-1}\,; \tag{14}$$

that is, the sequence $\{\rho_n\}$ is a nonincresing sequence of real numbers. Therefore, there exists ζ such that

$$\lim_{n\longrightarrow\infty}\rho_n=\lim_{n\longrightarrow\infty}\omega\big(\mathfrak{I}^n\theta_0,\mathfrak{I}^{n+1}\theta_0\big)=\zeta,\quad \forall n\in\mathbb{N}. \tag{15}$$

Applying the limit in (12), by the continuity of ψ, we get

$$\zeta\le\lim_{n?\infty}\alpha(\theta_n,\theta_{n+1})\rho_n\le h\zeta, \tag{16}$$

a contradiction, and therefore, $\zeta=0$.

For $\alpha(\mathfrak{I}\theta_0,\theta_0)\ge 1$, using condition (ii) and Lemma 7, we get $\alpha(\theta_{n+1},\theta_n)\ge 1$ for all $n\ge 1$. Using similar arguments as above, we can prove

$$\lim_{n?\infty}\omega\big(\mathfrak{I}^{n+1}\theta_0,\mathfrak{I}^n\theta_0\big)=0. \tag{17}$$

Next, we show that $\{\mathfrak{I}^n\theta_0\}$ is a Cauchy sequence in $\mathcal{O}(\theta_0\,;\mathfrak{I})$. For this, we show that

$$\lim_{m,n\longrightarrow\infty}\omega(\mathfrak{I}^n\theta_0,\mathfrak{I}^m\theta_0)=0. \tag{18}$$

On the contrary, suppose that condition (18) does not hold. Then, we can find a $\delta>0$ and increasing sequences $\{m_k\}_{k=1}^\infty,\{n_k\}_{k=1}^\infty$ of positive integers with $m_k>n_k$ such that

$$\omega(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0)\ge\delta,\quad \text{for all } k\in\{1,2,3,\cdots\}. \tag{19}$$

By (10), there exists a $k_0\in\mathbb{N}$, such that $n_k>k_0$ implies that

$$\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big)<\delta. \tag{20}$$

In view of the two last inequalities, we observe that $m_k\ne n_{k+1}$. We may assume that m_k is the minimal index such that (19) holds, so that

$$\omega(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^r\theta_0)<\delta,\quad \text{for } r\in\{n_{k+1},n_{k+2},\cdots,m_k-1\}. \tag{21}$$

Now, making use of (19), we get

$$0<\delta\le\omega(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0)\le\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k-1}\theta_0\big)+\omega\big(\mathfrak{I}^{m_k-1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)\\ <\delta+\omega\big(\mathfrak{I}^{m_k-1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big). \tag{22}$$

Thus,

$$\lim_{k?\infty}\omega(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0)=\delta. \tag{23}$$

Using the triangle inequality, we have

$$\omega(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0)\le\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)+\omega\big(\mathfrak{I}^{m_k+1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)\\ \le\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big)+\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)\\ +\omega\big(\mathfrak{I}^{m_k+1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big). \tag{24}$$

Taking the limit on both sides and making use of (10), (17), and (23), we obtain

$$\lim_{k?\infty}\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)\ge\delta. \tag{25}$$

Again, using the triangle inequality, we have

$$\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)\le\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)+\omega\big(\mathfrak{I}^{m_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)\\ \le\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{n_k}\theta_0\big)+\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)\\ +\omega\big(\mathfrak{I}^{m_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big). \tag{26}$$

Taking the limit on both sides and making use of (10), (17), and (23), we obtain

$$\lim_{k?\infty}\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)\le\delta. \tag{27}$$

Combining (25) and (27), we have

$$\lim_{k?\infty}\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)=\delta. \tag{28}$$

From Lemma 6, we have $\alpha(\theta_{n_k},\theta_{m_k})\ge 1$. Therefore, on applying condition (9), we get

$$\psi\big(\alpha\big(\theta_{n_k},\theta_{m_k}\big)\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0\big),\\ \cdot\,\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{m_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\\ \cdot\,\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)\big)\le 0. \tag{29}$$

Now applying (ψ_1) in the fifth and sixth variables, we have

$$\psi\Big(\alpha\Big(\theta_{n(k)},\theta_{m(k)}\Big)\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{m_k}\theta_0\big),$$
$$\cdot\,\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{m_k}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big)$$
$$+\,\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big),\omega\big(\mathfrak{I}^{n_k+1}\theta_0,\mathfrak{I}^{m_k+1}\theta_0\big)$$
$$+\,\omega\big(\mathfrak{I}^{m_k+1}\theta_0,\mathfrak{I}^{m_k}\theta_0\big)\Big)\leq 0. \tag{30}$$

Applying the limit and using continuity of ψ, we get

$$\psi\left(\delta\lim_{k?\infty}\alpha\big(\theta_{n_k},\theta_{m_k}\big),\delta,0,0,\delta,\delta\right)\leq 0, \tag{31}$$

a contradiction to (ψ_3). Hence, $\{\mathfrak{I}^n\theta_0\}$ must be a Cauchy sequence in $\mathcal{O}(\theta_0\,;\mathfrak{I})$.

Since Ξ is $\mathfrak{I}$-orbitally complete, there exists a point $\vartheta^* \in \Xi$ such that $\lim_{n\longrightarrow\infty}\mathfrak{I}^n\theta_0=\vartheta^*$. We shall show that ϑ^* is a fixed point of $\mathfrak{I}$.

Using the orbital continuity of $\mathfrak{I}$ (due to condition (v)), we have $\lim_{n\longrightarrow\infty}\mathfrak{I}\mathfrak{I}^n\theta_0=\mathfrak{I}\vartheta^*$. Owing to the uniqueness of the limit, we obtain $\mathfrak{I}\vartheta^*=\vartheta^*$.

Finally, assume that $\omega(\vartheta^*,\vartheta^*)>0$. Then, by (9) for $\alpha(\vartheta^*,\vartheta^*)\geq 1$, we have

$$\psi(\alpha(\vartheta^*,\vartheta^*)\omega(\mathfrak{I}\vartheta^*,\mathfrak{I}\vartheta^*),\omega(\vartheta^*,\vartheta^*),\omega(\vartheta^*,\mathfrak{I}\vartheta^*),$$
$$\cdot\,\omega(\mathfrak{I}\vartheta^*,\vartheta^*),\omega(\mathfrak{I}\vartheta^*,\vartheta^*),\omega(\vartheta^*,\mathfrak{I}\vartheta^*))\leq 0 \tag{32}$$

or

$$\psi(\alpha(\vartheta^*,\vartheta^*)\omega(\vartheta^*,\vartheta^*),\omega(\vartheta^*,\vartheta^*),\omega(\vartheta^*,\vartheta^*),\omega(\vartheta^*,\vartheta^*),$$
$$\cdot\,\omega(\vartheta^*,\vartheta^*),\omega(\vartheta^*,\vartheta^*))\leq 0. \tag{33}$$

It follows from (ψ_{2b}) (for $\zeta=\xi=\mu=\vartheta^*$) that there is $\hbar\in[0,1)$ such that

$$\alpha(\vartheta^*,\vartheta^*)\omega(\vartheta^*,\vartheta^*)\leq\hbar\,\omega(\vartheta^*,\vartheta^*), \tag{34}$$

a contradiction. Therefore, $\omega(\vartheta^*,\vartheta^*)=0$.

Next, we have the following result.

Theorem 10. *The conclusion of Theorem 9 remains true if condition (v) is replaced by the following one:*

(ν') For every $\nu\in\Xi$ with $\nu\neq\mathfrak{I}\nu$, $\inf\{\omega(\vartheta,\nu)+\omega(\vartheta,\mathfrak{I}\vartheta)\mid\vartheta\in\Xi\}>0$

Proof. Following the proof of Theorem 9, we observe that the sequence $\{\mathfrak{I}^n\theta_0\}$ is a Cauchy sequence, and so, there exists a point ϑ^* in Ξ such that $\lim_{n\longrightarrow\infty}\mathfrak{I}^n\theta_0=\vartheta^*$. Since $\lim_{m,n\longrightarrow\infty}\omega(\mathfrak{I}^n\theta_0,\mathfrak{I}^m\theta_0)=0$, for each $\varepsilon>0$, there exists an $N_\varepsilon\in\mathbb{N}$ such that $n>N_\varepsilon$ implies $\omega(\mathfrak{I}^{N_\varepsilon}\theta_0,\mathfrak{I}^n\theta_0)<\varepsilon$. Since $\lim_{n\longrightarrow\infty}\mathfrak{I}^n\theta_0=\vartheta^*$ and $\omega(\vartheta,\cdot)$ is lower semicontinuous,

$$\omega\big(\mathfrak{I}^{N_\varepsilon}\theta_0,\vartheta^*\big)\leq\liminf_{n?\infty}\omega\big(\mathfrak{I}^{N_\varepsilon}\theta_0,\mathfrak{I}^n\theta_0\big)<\varepsilon. \tag{35}$$

Therefore, $\omega(\mathfrak{I}^{N_\varepsilon}\theta_0,\vartheta^*)\leq\varepsilon$. Set $\varepsilon=1/k$, $N_\varepsilon=n_k$ so that

$$\lim_{k\longrightarrow\infty}\omega(\mathfrak{I}^{n_k}\theta_0,\vartheta^*)=0. \tag{36}$$

Assume that $\mathfrak{I}\vartheta^*\neq\vartheta^*$. Then, by the hypothesis (ν'), we have

$$0<\inf\{\omega(\vartheta,\vartheta^*)+\omega(\vartheta,\mathfrak{I}\vartheta)\mid\vartheta\in\Xi\}$$
$$\leq\inf\big\{\omega(\mathfrak{I}^{n_k}\theta_0,\vartheta^*)+\omega\big(\mathfrak{I}^{n_k}\theta_0,\mathfrak{I}^{n_k+1}\theta_0\big)\mid n\in\mathbb{N}\big\}\longrightarrow 0, \tag{37}$$

which contradicts our assumption. Therefore, $\mathfrak{I}\vartheta^*=\vartheta^*$.

The last conclusion is derived as in the proof of Theorem 9.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 9 and 10.

Theorem 11. *In addition to the hypotheses of Theorem 9 (or Theorem 10), if for all fixed points θ_* and η_* such that $\theta_*\neq\eta_*\in\Xi$, $\alpha(\theta_*,\eta_*)\geq 1$, then $\mathfrak{I}$ has a unique fixed point.*

Proof. Suppose that θ_* and η_* are two fixed points of $\mathfrak{I}$ such that $\theta_*\neq\eta_*$. Then, using (9) for $\alpha(\theta_*,\eta_*)\geq 1$,

$$\psi(\alpha(\theta_*,\eta_*)\omega(\mathfrak{I}\theta_*,\mathfrak{I}\eta_*),\omega(\theta_*,\eta_*),\omega(\theta_*,\mathfrak{I}\theta_*),$$
$$\cdot\,\omega(\eta_*,\mathfrak{I}\eta_*),\omega(\theta_*,\mathfrak{I}\eta_*),\omega(\mathfrak{I}\theta_*,\eta_*))\leq 0, \tag{38}$$

i.e.,

$$\psi(\alpha(\theta_*,\eta_*)\omega(\theta_*,\eta_*),\omega(\theta_*,\eta_*),0,0,\omega(\theta_*,\eta_*),\omega(\theta_*,\eta_*))\leq 0, \tag{39}$$

a contradiction to (ψ_3), and thus, $\omega(\theta_*,\eta_*)=0$. Also, we have $\omega(\theta_*,\theta_*)=0$. So, by using Lemma 2, we infer that $\theta_*=\eta_*$, i.e., the fixed point of $\mathfrak{I}$ is unique.

By choosing $\psi\in\Psi$ from Examples 2–6, we have the following consequences.

Corollary 12. *Let all the conditions of Theorems 9 and 10 be satisfied, except that the assumption of orbitally ω-implicit contractive mapping for $\psi\in\Psi$ is replaced by either of the form*

$$(i)\ [\alpha(\nu,\vartheta)\omega(\mathfrak{I}\nu,\mathfrak{I}\vartheta)]^2\leq a[\omega(\nu,\vartheta)]^2+b\frac{[\omega(\nu,\mathfrak{I}\nu)]^2+[\omega(\vartheta,\mathfrak{I}\vartheta)]^2}{1+\omega(\nu,\mathfrak{I}\vartheta)+\omega(\vartheta,\mathfrak{I}\nu)}, \tag{40}$$

where $0<a,b<1$, $a+2b<1$, $\alpha(\nu,\vartheta)\sqrt{a+b/\alpha(\nu,\vartheta)^2-b}<1$, or

$$(ii)\ [\alpha(\nu,\vartheta)\omega(\mathfrak{I}\nu,\mathfrak{I}\vartheta)]^2\leq a[\omega(\nu,\vartheta)]^2+b\frac{[\omega(\nu,\mathfrak{I}\nu)]^2+[\omega(\vartheta,\mathfrak{I}\vartheta)]^2}{1+[\omega(\nu,\mathfrak{I}\vartheta)]^2+[\omega(\vartheta,\mathfrak{I}\nu)]^2}, \tag{41}$$

where $0 < a, b < 1$*,* $a + 2b < 1$*,* $\alpha(\nu, \vartheta)\sqrt{a + b/\alpha(\nu, \vartheta)^2 - b} < 1$*, or*

$$(iii)\ \alpha(\nu, \vartheta)\omega(\Im\nu, \Im\vartheta) \le a\omega(\nu, \vartheta) + b\omega(\nu, \Im\nu) + c\frac{\omega(\vartheta, \Im\vartheta)\omega(\nu, \Im\vartheta)}{1 + \omega(\nu, \Im\vartheta) + \omega(\vartheta, \Im\nu)}, \quad (42)$$

where $0 < a, b, c < 1$*,* $a + b + c < 1$*,* $\alpha(\nu, \vartheta)(a + b)/\alpha(\nu, \vartheta) - c < 1$*, or*

$$(iv)\ \alpha(\nu, \vartheta)\omega(\Im\nu, \Im\vartheta) \le a\omega(\nu, \vartheta) + b\frac{\omega(\nu, \Im\nu)\omega(\nu, \Im\vartheta)}{1 + \omega(\nu, \Im\vartheta) + \omega(\vartheta, \Im\nu)} + c\,\omega(\vartheta, \Im\vartheta), \quad (43)$$

where $0 < a, b, c < 1$*,* $a + b + c < 1$*,* $\alpha(\nu, \vartheta)(a + b)/\alpha(\nu, \vartheta) - c < 1$*.*

Then, $\mathrm{Fix}(\Im)$ is a singleton.

3. Illustrations

Example 7. Consider the set $\Xi = \{0\} \cup \{1/8^n : n \in \mathbb{N}\}$ with the usual metric d. Define a w-distance $\omega : \Xi \times \Xi \longrightarrow [0,+\infty)$ by $\omega(\vartheta, \nu) = \nu$ for all $\vartheta, \nu \in \Xi$.

Consider the self-mapping $\Im$ on Ξ given by $\Im\vartheta = \vartheta/8$. Take $\theta_0 = 1/8$. It is simple to show that

$$\mathcal{O}(\theta_0 ; \Im) \subset \left\{\frac{1}{8^k} \mid k \in \mathbb{N} \cup \{0\}\right\}, \quad \mathcal{O}(\overline{\theta_0} ; \Im) = \mathcal{O}(\theta_0 ; \Im) \cup \{0\}, \quad (44)$$

and that (Ξ, ω) is $\Im$-orbitally complete at θ_0.

Define functional $\alpha : \Xi \times \Xi \longrightarrow [0,+\infty)$ as follows:

$$\alpha(\vartheta, \nu) = \begin{cases} 2, & \text{if } \vartheta, \nu \in \mathcal{O}(\theta_0 ; \Im), \\ 0, & \text{otherwise.} \end{cases} \quad (45)$$

At $\theta_0 = 1/8$ in $\mathcal{O}(\theta_0 ; \Im)$, $\alpha(\theta_0, \Im\theta_0) \ge 1$ and $\alpha(\Im\theta_0, \theta_0) \ge 1$. Also, $\Im$ is a triangular α-admissible mapping in $\mathcal{O}(\theta_0 ; \Im)$.

Considering Example 4, we can define $\psi \in \Psi$ as

$$\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1 - a\ell_2 - b\ell_3 - c\frac{\ell_4\ell_5}{1 + \ell_5 + \ell_6}. \quad (46)$$

Here, $k = \alpha(\nu, \vartheta) \ge 1$. One can easily check that $\hbar = k(a + b)/k - b < 1$ for $k \ge 1$, $0 < a, b, c < 1$ so that $a + b + c < 1$ and that ψ belongs to the set Ψ. We will show that $\Im$ is an orbitally ω-implicit contractive mapping.

Take $\vartheta, \nu \in \mathcal{O}(\overline{\theta_0} ; \Im)$, and so, $0 \le \vartheta, \nu \le 1/8$. Consider two cases.

Case 1. If $\vartheta = 0$ and $\nu = 1/8^n$, $n \in \mathbb{N}$, then (9) reduces to

$$\frac{1}{8^{n+1}} \le a \cdot \frac{1}{8^n} + c \cdot \frac{1}{8^{2(n+1)} + 8^{n+1}} \quad (47)$$

and is fulfilled for $a = 1/2$, $b = 1/5 = c$. If $\nu = 0$ and $\vartheta = 1/8^n$, $n \in \mathbb{N}$ or $\vartheta = 0$, then (9) holds trivially.

Case 2. Let $\vartheta, \nu \in \{1/8^n \mid n \in \mathbb{N}\}$. Then, (9) reduces to

$$2 \cdot \frac{\nu}{8} \le a \cdot \nu + b \cdot \frac{\vartheta}{8} + c \cdot \frac{\nu^2}{8(8 + \nu + \vartheta)} \quad (48)$$

and is fulfilled for $a = 1/2$, $b = 1/5 = c$.

Thus, $\Im$ is orbitally ω-implicit contractive mapping. Therefore, all the conditions of Theorem 9 are satisfied, and $\vartheta^* = 0$ is the unique fixed point of $\Im$ in $\mathcal{O}(\overline{\theta_0} ; \Im)$.

Example 8. Consider the set $\Xi = [0,+\infty)$ with the usual metric d. Define a w-distance $\omega : \Xi \times \Xi \longrightarrow [0,+\infty)$ *by* $\omega(\vartheta, \nu) = \nu$ for all $\vartheta, \nu \in \Xi$.

Consider the self-mapping $\Im$ on Ξ given by

$$\Im(\vartheta) = \begin{cases} \dfrac{1}{n+1}, & \text{if } \vartheta = \dfrac{1}{n}, \\ 0, & \text{if } \vartheta = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (49)$$

Take $\theta_0 = 1$. It is simple to show that

$$\mathcal{O}(\theta_0 ; \Im) \subset \left\{\frac{1}{n+1} \mid n \in \mathbb{N} \cup \{0\}\right\}, \quad \mathcal{O}(\overline{\theta_0} ; \Im) = \mathcal{O}(\theta_0 ; \Im) \cup \{0\}, \quad (50)$$

and that (Ξ, ω) is $\Im$-orbitally complete at θ_0.

Define a function $\alpha : \Xi \times \Xi \longrightarrow [0,\infty)$ as follows:

$$\alpha(\vartheta, \nu) = \begin{cases} 1, & \text{if } \vartheta, \nu \in \mathcal{O}(\theta_0 ; \Im), \\ 0, & \text{otherwise.} \end{cases} \quad (51)$$

At $\theta_0 = 1$ in $\mathcal{O}(\theta_0 ; \Im)$, $\alpha(\theta_0, \Im\theta_0) \ge 1$ and $\alpha(\Im\theta_0, \theta_0) \ge 1$. Also, $\Im$ is a triangular α-admissible mapping in $\mathcal{O}(\theta_0 ; \Im)$.

Considering Example 3, we can define $\psi \in \Psi$ as

$$\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1^2 - a\ell_2^2 - b\frac{\ell_3^2 + \ell_4^2}{\ell_5^2 + \ell_6^2 + 1}. \quad (52)$$

Here, $k = \alpha(\nu, \vartheta) \ge 1$. One can easily check that $\hbar = k\sqrt{a + b/k^2 - b} < 1$ for $k \ge 1$, $0 < a, b < 1$ so that $a + 2b < 1$ and that ψ belongs to the set Ψ. We will show that $\Im$ is an orbitally ω-implicit contractive mapping.

Take $\vartheta, \nu \in \mathcal{O}(\overline{\theta_0} ; \Im)$, and so, $0 \le \vartheta, \nu \le 1$. Consider two cases.

Case 1. If $\vartheta = 0$ and $\nu = 1/n$, $n \in \mathbb{N}$, then (9) reduces to

$$\left[\alpha\left(0, \frac{1}{m}\right)\omega\left(\mathfrak{T}(0), \mathfrak{T}\left(\frac{1}{n}\right)\right)\right]^2 \leq a \cdot \left[\omega\left(0, \frac{1}{n}\right)\right]^2 + b \cdot \frac{[\omega(0, \mathfrak{T}(0))]^2 + [\omega(1/n, \mathfrak{T}(1/n))]^2}{1 + [\omega(0, \mathfrak{T}(1/n))]^2 + [\omega(1/n, \mathfrak{T}(0))]^2}, \quad (53)$$

that is,

$$\frac{1}{(n+1)^2} \leq a \cdot \frac{1}{n^2} + b \cdot \frac{1/(n+1)^2}{1 + 1/(n+1)^2} = \frac{a}{n^2} + \frac{b}{(n+1)^2 + 1} \quad (54)$$

and is fulfilled, for $0 < a, b < 1$ so that $a + 2b < 1$. If $\nu = 0$ and $\vartheta = 1/n$, $n \in \mathbb{N}$, or $\vartheta = 0$, then (9) holds trivially.

Case 2. Let $\vartheta, \nu \in \{1/(n+1) \mid n \in \mathbb{N} \cup \{0\}\}$. Then, inequality (9) has the form

$$\left[\alpha\left(\frac{1}{n}, \frac{1}{m}\right)\omega\left(\mathfrak{T}\left(\frac{1}{n}\right), \mathfrak{T}\left(\frac{1}{m}\right)\right)\right]^2 \leq a \cdot \left[\omega\left(\frac{1}{n}, \frac{1}{m}\right)\right]^2 + b \cdot \frac{[\omega(1/n, \mathfrak{T}(1/n))]^2 + [\omega(1/m, \mathfrak{T}(1/m))]^2}{1 + [\omega(1/n, \mathfrak{T}(1/m))]^2 + [\omega(1/m, \mathfrak{T}(1/n))]^2}, \quad (55)$$

that is,

$$\left[\omega\left(\frac{1}{n+1}, \frac{1}{m+1}\right)\right]^2 \leq a \cdot \left[\omega\left(\frac{1}{n}, \frac{1}{m}\right)\right]^2 + b \cdot \frac{[\omega(1/n, 1/n+1)]^2 + [\omega(1/m, 1/m+1)]^2}{1 + [\omega(1/n, 1/m+1)]^2 + [\omega(1/m, 1/n+1)]^2}, \quad (56)$$

that is,

$$\frac{1}{(m+1)^2} \leq a\frac{1}{m^2} + b \cdot \frac{1/(n+1)^2 + 1/(m+1)^2}{1 + 1/(m+1)^2 + 1/(n+1)^2} = \frac{a}{m^2} + b \cdot \frac{(n+1)^2 + (m+1)^2}{(n+1)^2 + (m+1)^2 + 1} \quad (57)$$

and is fulfilled, for $0 < a, b < 1$ so that $a + 2b < 1$.

Thus, $\mathfrak{T}$ is an orbitally ω-implicit contractive mapping.

If $\nu > 0$, we have $\nu \neq \mathfrak{T}\nu$ so that

$$\inf\{\omega(\vartheta, \nu) + \omega(\vartheta, \mathfrak{T}\vartheta) : \vartheta \in \Xi\} = \inf\{\nu + \mathfrak{T}\vartheta : \vartheta \in \Xi\} > 0. \quad (58)$$

Thus, all the conditions of Theorem 10 are satisfied and $\vartheta^* = 0$ is the unique fixed point of $\mathfrak{T}$ in $\mathcal{O}(\overline{\theta_0}; \mathfrak{T})$.

Example 9. Consider the set $\Xi = [0, 1]$ with the usual metric d. Define a w-distance $\omega : \Xi \times \Xi \longrightarrow [0,+\infty)$ by $\omega(\vartheta, \nu) = 2|\vartheta - \nu|$ for all $\vartheta, \nu \in \Xi$.

Consider the self-mapping $\mathfrak{T}$ on Ξ given by

$$\mathfrak{T}(\vartheta) = \begin{cases} \dfrac{1}{2^{n+1}}, & \text{if } \vartheta = \dfrac{1}{2^n}, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases} \quad (59)$$

Take $\theta_0 = 1/2$. It is simple to show that

$$\mathcal{O}(\theta_0; \mathfrak{T}) \subset \left\{\frac{1}{2^n} \mid n \in \mathbb{N}\right\}, \quad \mathcal{O}(\overline{\theta_0}; \mathfrak{T}) = \mathcal{O}(\theta_0; \mathfrak{T}) \cup \{0\}, \quad (60)$$

and that (Ξ, ω) is $\mathfrak{T}$-orbitally complete at θ_0.

Define functional $\alpha : \Xi \times \Xi \longrightarrow [0,+\infty)$ as follows:

$$\alpha(\vartheta, \nu) = \begin{cases} 1, & \text{if } \vartheta, \nu \in \mathcal{O}(\theta_0; \mathfrak{T}), \\ 0, & \text{otherwise.} \end{cases} \quad (61)$$

At $\theta_0 = 1/2$ in $\mathcal{O}(\theta_0; \mathfrak{T})$, $\alpha(\theta_0, \mathfrak{T}\theta_0) \geq 1$ and $\alpha(\mathfrak{T}\theta_0, \theta_0) \geq 1$. Also, $\mathfrak{T}$ is a triangular α-admissible mapping in $\mathcal{O}(\theta_0; \mathfrak{T})$.

Considering Example 6, we can define $\psi \in \Psi$ as

$$\psi(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = \ell_1 - a\ell_2 - b\frac{\ell_3\ell_6}{\ell_5 + \ell_6 + 1} - c\ell_4, \quad (62)$$

$0 < a, b, c < 1$ and $a + b + c < 1$.

Take $\vartheta, \nu \in \mathcal{O}(\overline{\theta_0}; \mathfrak{T})$, and so, $0 \leq \vartheta, \nu \leq 1/2$. Consider two cases.

Case 1. If $\vartheta = 0$ and $\nu = 1/2^n$ (or $\nu = 0$ and $\vartheta = 1/2^n$), $n \in \mathbb{N}$, then (9) reduces to

$$\left|1 - \frac{1}{2^n}\right| \leq a \cdot \frac{1}{2^{n-1}} + b \cdot \frac{1/2^n}{1 + 1/2^n + 1/2^{n-1}} + c \cdot \left|1 - \frac{1}{2^n}\right| \quad (63)$$

and is fulfilled, for $0 < a, b, c < 1$, $a + b + c < 1$. If $\nu = 0$ and $\vartheta = 0$, then (9) holds trivially.

Case 2. Let $\vartheta, \nu \in \{1/2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider $\vartheta = 1/2^n$ and $\nu = 1/2^m (m > n)$. Then, the inequality (9) has the form

$$2\left|\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}\right| \leq 2a\left|\frac{1}{2^n} - \frac{1}{2^m}\right| + b\frac{2\left|1/2^n - 1/2^{n+1}\right| \cdot 2\left|1/2^n - 1/2^{m+1}\right|}{1 + 2\left|1/2^n - 1/2^{m+1}\right| + 2\left|1/2^m - 1/2^{n+1}\right|} + 2c\left|\frac{1}{2^m} - \frac{1}{2^{m+1}}\right|, \quad (64)$$

which is equivalent to

$$\frac{|2^n - 2^m|}{2^{m+n}}| \le a\frac{|2^n - 2^m|}{2^{m+n-1}} + \frac{4b|2^{m+1} - 2^n|}{2^{m+n} + |2^{m+1} - 2^n| + |2^{n+1} - 2^m|} + \frac{c}{2^m}, \tag{65}$$

which is obviously true for $0 < a, b, c < 1$ such that $a + b + c < 1$.

It can be noted that for $0 \ne \Im 0$, we have

$$\begin{aligned} &\inf\left\{\omega\left(\frac{1}{2^n}, 0\right) + \omega\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right): n \in \mathbb{N}\right\} \\ &= \inf\left\{\frac{2}{2^n} + \frac{2}{2^{n+1}} : n \in \mathbb{N}\right\} = 0. \end{aligned} \tag{66}$$

Thus, all the conditions of Theorem 10 are satisfied except $(v^{'})$. Clearly, $\Im$ has no fixed points in $\mathcal{O}(\bar{\theta_0}; \Im)$.

4. Weak Well-Posedness, Weak Limit Shadowing, and Generalized w-Ulam-Hyers Stability

The notion of well-posedness of a fixed point problem (fpp) has evoked much interest of several mathematicians, for example, Popa [10, 11]. In the paper [12], the authors defined a weak well-posed (wwp) property in metric space. In the following, we extend this notion to a w-distance in metric space.

Definition 13. Let (Ξ, d) be a metric space and ω be a w-distance in Ξ. Let $\Im : \Xi \longrightarrow \Xi$ be a mapping having a unique fixed point ϑ^* such that $\alpha(\vartheta^*, \vartheta^*) \ge 1$. The fpp of $\Im$ is said to be wwp with respect to ω if for any sequence $\{\theta_n\}$ in Ξ with $\lim_{n\longrightarrow\infty}\omega(\theta_n, \Im(\theta_n)) = 0$ and $\lim_{n,m\longrightarrow\infty}\omega(\Im(\theta_n), \Im(\theta_m)) = 0$, one has $\lim_{n?\infty}\omega(\theta_n, \vartheta^*) = 0$.

To guarantee the wwp of a mapping $\Im$, we add the following additional condition for functions $\psi \in \Psi$ and call the respective set Ψ':

(ψ_4) for all $\zeta, \xi > 0$, $k \ge 1$, $\psi(k\zeta, \xi, 0, 0, \xi, \zeta) \le 0$ implies that there exists $\hbar \in [0, 1)$ such that $k\zeta \le \hbar\xi$

Examples 2–4 and 6 satisfy the condition (ψ_4).

Theorem 14. *Let (Ξ, d) be a metric space and ω be a w-distance on Ξ. Suppose that all the hypotheses of Theorem 9 hold for $\psi \in \Psi'$. Then, the ffp for $\Im$ is wwp.*

Proof. Let $\{\theta_n\}$ be a sequence in Ξ such that $\lim_{n\longrightarrow\infty}\omega(\theta_n, \Im(\theta_n)) = 0$ and $\lim_{m,n\longrightarrow\infty}\omega(\Im\theta_m, \Im\theta_n) = 0$, for $n > m$. We obtain that

$$\omega(\theta_n, \vartheta^*) \le \omega(\theta_n, \Im\theta_m) + \omega(\Im\theta_m, \Im\theta_n) + \omega(\Im\theta_n, \vartheta^*). \tag{67}$$

Taking the limit as $n?\infty$, we get

$$\lim_{n\longrightarrow\infty} \omega(\theta_n, \vartheta^*) \le \lim_{n\longrightarrow\infty} \{\omega(\theta_n, \Im\theta_m) + \omega(\Im\theta_n, \vartheta^*)\}. \tag{68}$$

WLOG, we can assume that there exists a distinct subsequence $\{\Im\theta_{n_k}\}$ of $\{\Im\theta_n\}$. Otherwise, there exists $\theta_0 \in \Xi$ and $n_1 \in N$ such that $\Im\theta_n = \theta_0$ for $n \ge n_1$. Since $\lim_{n\longrightarrow\infty}\omega(\theta_n, \Im\theta_n) = 0$, we get $\lim_{n\longrightarrow\infty}\omega(\theta_n, \theta_0) = 0$. If $\theta_0 \ne \vartheta^*$, then $\theta_0 \ne \Im\theta_0$ due to uniqueness of the fixed point of $\Im$. For $n \ge n_1$, we obtain $\theta_0 = \Im\theta_n \ne \Im\theta_0$. From (9), we have

$$\begin{aligned} &\psi(\alpha(\theta_n, \theta_0)\omega(\Im\theta_n, \Im\theta_0), \omega(\theta_n, \theta_0), \omega(\theta_n, \Im\theta_n), \omega(\theta_0, \Im\theta_0), \\ &\cdot \omega(\theta_n, \Im\theta_0), \omega(\Im\theta_n, \theta_0)) \le 0, \end{aligned} \tag{69}$$

i.e.,

$$\psi\begin{pmatrix} \alpha(\theta_n, \theta_0)\omega(\theta_0, \Im\theta_0), \omega(\theta_n, \theta_0), \omega(\theta_n, \theta_0), \\ \omega(\theta_0, \Im\theta_0), \omega(\theta_n, \Im\theta_0), \omega(\theta_0, \theta_0) \end{pmatrix} \le 0, \tag{70}$$

i.e.,

$$\begin{aligned} &\psi(\alpha(\theta_n, \theta_0)\omega(\theta_0, \Im\theta_0), \omega(\theta_n, \theta_0), \omega(\theta_n, \theta_0), \omega(\theta_0, \Im\theta_0), \\ &\cdot \omega(\theta_n, \theta_0) + \omega(\theta_0, \Im\theta_0), 0) \le 0. \end{aligned} \tag{71}$$

It follows from (ψ_{2a}) that there is $\hbar \in [0, 1)$ such that

$$\omega(\theta_0, \Im\theta_0) \le \alpha(\theta_n, \theta_0)\omega(\theta_0, \Im\theta_0) \le \hbar\,\omega(\theta_n, \theta_0), \tag{72}$$

which on applying $n\longrightarrow\infty$ gives $\omega(\theta_0, \Im\theta_0) = 0$. Also, we have $\omega(\theta_0, \theta_0) = 0$. So, by Lemma 2, we get $\theta_0 = \Im\theta_0$, a contradiction. Hence, there exist $m, q, n > n_0 (m > q > n)$ such that $\Im\theta_m \ne \Im\theta_q \ne \Im\theta_n \ne \theta_n$. Then,

$$\omega(\theta_n, \Im\theta_m) \le \omega(\theta_n, \Im\theta_n) + \omega(\Im\theta_n, \Im\theta_q) + \omega(\Im\theta_q, \Im\theta_m), \tag{73}$$

which ?0 as $n?\infty$. On replacing the value in (68), we get

$$\lim_{n\longrightarrow\infty} \omega(\theta_n, \vartheta^*) \le \lim_{n\longrightarrow\infty}\omega(\Im\theta_n, \vartheta^*). \tag{74}$$

From (9), we have

$$\begin{aligned} &\psi(\alpha(\theta_n, \vartheta^*)\omega(\Im\theta_n, \Im\vartheta^*), \omega(\theta_n, \vartheta^*), \omega(\theta_n, \Im\theta_n), \omega(\vartheta^*, \Im\vartheta^*), \\ &\cdot \omega(\theta_n, \Im\vartheta^*), \omega(\Im\theta_n, \vartheta^*)) \le 0, \end{aligned} \tag{75}$$

i.e.,

$$\begin{aligned} &\psi(\alpha(\theta_n, \vartheta^*)\omega(\Im\theta_n, \vartheta^*), \omega(\theta_n, \vartheta^*), \omega(\theta_n, \Im\theta_n), \omega(\vartheta^*, \vartheta^*), \\ &\cdot \omega(\theta_n, \vartheta^*), \omega(\Im\theta_n, \vartheta^*)) \le 0, \end{aligned} \tag{76}$$

which on applying limit as $n?\infty$ gives

$$\psi\Big(\lim_{n\longrightarrow\infty}\alpha(\theta_n,\vartheta^*)\lim_{n\longrightarrow\infty}\omega(\Im\theta_n,\vartheta^*),\lim_{n\longrightarrow\infty}\omega(\theta_n,\vartheta^*),0,0, \cdot\lim_{n\longrightarrow\infty}\omega(\theta_n,\vartheta^*),\lim_{n\longrightarrow\infty}\omega(\Im\theta_n,\vartheta^*)\Big)\leq 0. \tag{77}$$

It follows from (ψ_4) that there is $\hbar\in[0,1)$ such that

$$\lim_{n?\longrightarrow\infty}\omega(\Im\theta_n,\vartheta^*)<\hbar\lim_{n\longrightarrow\infty}\omega(\theta_n,\vartheta^*). \tag{78}$$

Combining (74) and (78), we get $\lim_{n\longrightarrow\infty}\omega(\theta_n,\vartheta^*)=0$ as $\hbar<1$.

The limit shadowing property of fixed point problems has been discussed in the papers [13, 14]. We define weak limit shadowing property (wlsp) in metric spaces under w -distance.

Definition 15. Let (Ξ,d) be a complete metric space and ω be a w-distance in Ξ. Let $\Im:\Xi\longrightarrow\Xi$ be a mapping. The fpp of $\Im$ is said to have wlsp in Ξ if assuming that $\{\theta_n\}$ in Ξ satisfies $\omega(\theta_n,\Im\theta_n)\longrightarrow 0$ as $n\longrightarrow\infty$ and $\omega(\Im\theta_n,\Im\theta_m)\longrightarrow 0$, it follows that there exists $x\in\Xi$ such that $\omega(\theta_n,\Im^n\vartheta^*)\longrightarrow 0$ as $n\longrightarrow\infty$ $(\mathrm{Fix}(\Im)=\{\vartheta^*\})$.

Theorem 16. *In addition to the hypotheses of Theorem 9 (or Theorem 10) if $\{\theta_n\}$ in Ξ is such that $\lim_{n\longrightarrow\infty}\omega(\theta_n,\Im\theta_n)=0$, $\lim_{n,m\longrightarrow\infty}\omega(\Im\theta_n,\Im\theta_m)=0$ and $\vartheta^*\in Fix(\Im)$, then $\Im$ has the wlsp.*

Proof. Since ϑ^* is a fixed point of $\Im$, we have $\omega(\vartheta^*,\Im\vartheta^*)=0$ and let $\{\theta_n\}$ in Ξ such that $\lim_{n\longrightarrow\infty}\omega(\theta_n,\Im\theta_n)=0$, $\lim_{n,m\longrightarrow\infty}\omega(\Im\theta_n,\Im\theta_m)=0$, then by virtue of Theorem 9, we have $\lim_{n\longrightarrow\infty}\omega(\theta_n,\vartheta^*)=0$, and therefore, we can write $\lim_{n\longrightarrow\infty}\omega(\theta_n,\Im^n\vartheta^*)=0$.

Next, we define generalized-ω-Ulam-Hyers stability (G-ω-UHS) of fixed point problem (fpp) in metric spaces under w-distance.

Definition 17. Let (Ξ,d) be a metric space and ω be a w -distance on Ξ. Let $\Im:\Xi\longrightarrow\Xi$ be a mapping. The the fixed point equation (FPE)

$$\vartheta=\Im\vartheta,\vartheta\in\Xi \tag{79}$$

is said to be G-ω-UHS in the setting of metric spaces under w -distance, if there exists an increasing function $\phi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, continuous at 0, with $\phi(0)=0$, such that for each $\varepsilon>0$ and an ε-solution $\upsilon\in\Xi$, that is,

$$\omega(\upsilon,\Im\upsilon)\leq\varepsilon, \tag{80}$$

there exists a solution $\vartheta^*\in\Xi$ of (79) such that

$$\omega(\upsilon,\vartheta^*)\leq\phi(\varepsilon). \tag{81}$$

If $\phi(\xi)=b\xi$ for all $\xi\in\mathbb{R}_+$, where $b>0$, then FPE (79) is said to be ω-UHS in the framework of metric spaces under w-distance.

Remark 18. If $p=d$, then Definition 17 reduces to the notion of GUHS in metric spaces. Also, if $\phi(\xi)=b\xi$ for all $\xi\in\mathbb{R}_+$, where $b>0$, then it reduces to the notion of UHS in metric spaces. Finally, if $\omega(\vartheta,\upsilon)=|\vartheta-\upsilon|$, then it reduces to the classical UHS.

Theorem 19. *Let (Ξ,d) be a metric space and ω be a w-distance on Ξ. Suppose that all the hypotheses of Theorem 9 hold, using the contraction condition in the form (42). Then, the FPE (79) is G-ω-UHS.*

Proof. Following Theorem 9, we have $\Im\rho^*=\rho^*$; that is, $\rho^*\in\Xi$ is a solution of the FPE (79) with $\omega(\rho^*,\Im\rho^*)=0$. Let $\varepsilon>0$ and $\sigma^*\in\Xi$ be an ε-solution of (79), that is,

$$\omega(\sigma^*,\Im\sigma^*)\leq\varepsilon. \tag{82}$$

Since $\omega(\rho^*,\Im\rho^*)=\omega(\rho^*,\rho^*)=0\leq\varepsilon$, ρ^* and σ^* are ε -solutions. Now,

$$\omega(\rho^*,\sigma^*)\leq\omega(\rho^*,\Im\rho^*)+\omega(\Im\rho^*,\Im\sigma^*)+\omega(\Im\sigma^*,\sigma^*)\leq\omega(\Im\rho^*,\Im\sigma^*)+\varepsilon. \tag{83}$$

From the contractive condition (42) for $\Im$, we get

$$\begin{aligned}\omega(\Im\rho^*,\Im\sigma^*)&\leq a\omega(\rho^*,\sigma^*)+b\omega(\rho^*,\Im\rho^*)\\&\quad+c\frac{\omega(\sigma^*,\Im\sigma^*)\omega(\rho^*,\Im\sigma^*)}{1+\omega(\rho^*,\Im\sigma^*)+\omega(\sigma^*,\Im\rho^*)}\\&\leq a\omega(\rho^*,\sigma^*)+c\varepsilon\frac{\omega(\rho^*,\Im\sigma^*)}{1+\omega(\rho^*,\Im\sigma^*)+\omega(\sigma^*,\Im\rho^*)}\\&\leq a\omega(\rho^*,\sigma^*)+c\varepsilon.\end{aligned} \tag{84}$$

Therefore, from (83), we obtain

$$\omega(\rho^*,\sigma^*)\leq a\omega(\rho^*,\sigma^*)+c\varepsilon+\varepsilon=a\omega(\rho^*,\sigma^*)+(1+c)\varepsilon, \tag{85}$$

which implies that

$$\omega(\rho^*,\sigma^*)(1-a)\leq(1+c)\varepsilon, \tag{86}$$

i.e.,

$$\omega(\rho^*,\sigma^*)\leq\frac{1+c}{(1-a)}\varepsilon=\phi(\varepsilon)\text{ as }\frac{1+c}{(1-a)}>0. \tag{87}$$

Thus, the inequality (81) holds, and therefore, the FPE (79) is G-ω-UHS in the metric spaces under w-distance.

5. Application

Fractional differential/integral equations (FDE/DIE) have been extensively studied as an application of fixed point theory. In fact, to get the unique solution of FDE, one has to apply the Banach fixed point theorem or its variants. There are different types of FDEs in the literature, but the FDEs in the Caputo sense are the easiest to apply. The main advantage of Caputo derivative is that the derivative of the constant function is 0 while most of the other fractional derivatives do not have such an important property. This property helps in initial value problems to apply fixed point theorems. In paper [15], the existence of solutions for some Atangana-Baleanu fractional differential equations in the Caputo sense have been discussed. Some other FDE-related work can be seen in [16, 17].

The Caputo derivative of fractional order β is defined as

$$ {}^{c}\mathscr{D}^{\beta}(p(\rho)) = \frac{1}{\Gamma(n-\beta)}\int_0^{\rho}(\rho-\sigma)^{n-\beta-1}p^{(n)}(\sigma)\,ds\ (n-1<\beta<n, n=[\beta]+1), \tag{88} $$

where $p : [0,\infty)?\mathbb{R}$ is a continuous function, $[\beta]$ denotes the integer part of the positive real number β, and Γ is the gamma function.

In this section, we discuss the existence of solutions of following nonlinear fractional differential equation (FDE) [18] as an application of Theorem 9.

Consider the nonlinear FDE

$$ {}^{c}\mathscr{D}^{\beta}(\vartheta(\rho)) = \hbar(\rho, \vartheta(\rho))\ (0<\rho<1, 1<\beta\le 2), \tag{89} $$

with the integral boundary conditions

$$ \vartheta(0) = 0, \vartheta(1) = \int_0^{\eta}\vartheta(\sigma)ds(0<\eta<1), \tag{90} $$

where $J=[0,1]$, $\vartheta \in C(J,\mathbb{R})$, and $\hbar : J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function.

Let $\Xi = C(J,\mathbb{R})$ be endowed with the usual distance and w-distance $\omega : \Xi\times\Xi\longrightarrow[0,+\infty)$ by

$$ \omega(\vartheta,\nu) = \max_{\rho\in J}[|\vartheta(\rho)| + |\nu(\rho)|]. \tag{91} $$

Theorem 20. *Let $\mathfrak{T} : \Xi\longrightarrow\Xi$ be the operator defined by*

$$ \begin{aligned} \mathfrak{T}\vartheta(\rho) = {} & \frac{1}{\Gamma(\beta)}\int_0^{\rho}(\rho-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))d\sigma - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot\int_0^1(1-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))\,d\sigma + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot\int_0^{\eta}\left(\int_0^{\sigma}(\sigma-\varsigma)^{\beta-1}\hbar(z,\vartheta(z))\,d\varsigma\right)d\sigma, \end{aligned} \tag{92} $$

for $\vartheta\in\Xi$, $\rho\in J$. Also, let $\zeta : \mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be a given function. Assume that

(F1) $\hbar : J\times\mathbb{R}?\mathbb{R}$ is a continuous function, nondecreasing in the second variable

(F2) there exists $\vartheta_0\in\Xi$ such that $\zeta(\vartheta_0(\rho), \mathfrak{T}\vartheta_0(\rho))\ge 0$, $\zeta(\mathfrak{T}\vartheta_0(\rho), \vartheta_0(\rho))\ge 0$ for all $\rho\in J$

(F3) for $\vartheta,\nu,\upsilon\in\Xi$, $\zeta(\vartheta(\rho),\nu(\rho))\ge 0$ and $\zeta(\vartheta(\rho),\upsilon(\rho))\ge 0$ for all $\rho\in J$ implies that $\zeta(\vartheta(\rho),\upsilon(\rho))\ge 0$ for all $\rho\in J$

(F4) there exist $0<a,b,c<1$ with $a+b+c<1$ such that for $\vartheta,\nu\in\Xi$ and $\rho\in J$ we have

$$ \zeta(\vartheta,\nu)(|\hbar(\rho,\vartheta(\rho))| + |\hbar(\rho,\nu(\rho))|) \le \lambda\times\Theta(\vartheta,\nu)(\rho), \tag{93} $$

where

$$ \Theta(\vartheta,\nu)(\rho) = +c\cdot\frac{a\cdot(|\vartheta(\rho)|+|\nu(\rho)|) + b\cdot(|\vartheta(\rho)|+|\mathfrak{T}\vartheta(\rho)|)(|\vartheta(\rho)|+|\mathfrak{T}\vartheta(\rho)|)(|\vartheta(\rho)|+|\mathfrak{T}\nu(\rho)|)}{1+\max_{\rho\in J}(|\vartheta(\rho)|+|\mathfrak{T}\nu(\rho)|) + \max_{\rho\in J}(|\nu(\rho)|+|\mathfrak{T}\vartheta(\rho)|)}, \tag{94} $$

and $\lambda = \Gamma(\beta+1)/5$. Then, the problem (89)–(90) has at least one solution $\vartheta^*\in\Xi$.

Proof. Define a function $\alpha : \Xi^2\longrightarrow[0,\infty)$ by

$$ \alpha(\vartheta,\nu) = \begin{cases} 1, & \text{if for } \zeta(\vartheta(\rho),\nu(\rho))\ge 0, \text{for all } \rho\in J, \\ \gamma, & \text{otherwise,} \end{cases} \tag{95} $$

where $\gamma\in(0,1)$. It is easy to check that the assumption (F2)-(F3) implies the condition (i)-(ii) of Theorem 9, respectively.

Let $\vartheta,\nu\in\Xi$, then for each $\rho\in J$, by the definition (92) of operator $\mathfrak{T}$, we have

$$ \begin{aligned} & \zeta(\vartheta,\nu)(|\mathfrak{T}\vartheta(\rho)| + |\mathfrak{T}\nu(\rho)|) = \zeta(\vartheta,\nu) \\ & \quad\cdot\left(\left|\frac{1}{\Gamma(\beta)}\int_0^{\rho}(\rho-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))d\sigma - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\right.\right. \\ & \quad\left.\cdot\int_0^1(1-\sigma)^{\beta-1}\hbar(\sigma,\vartheta(\sigma))d\sigma + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^{\eta}\left(\int_0^{\sigma}(\sigma-\varsigma)^{\beta-1}\hbar(\varsigma,\vartheta(z))d\varsigma\right)d\sigma\right| \\ & \quad + \left|\frac{1}{\Gamma(\beta)}\int_0^{\rho}(\rho-\sigma)^{\beta-1}\hbar(\sigma,\nu(\sigma))d\sigma + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\right. \\ & \quad\left.\left.\cdot\int_0^1(1-\sigma)^{\beta-1}\hbar(\sigma,\nu(\sigma))d\sigma - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^{\eta}\left(\int_0^{\sigma}(\sigma-\varsigma)^{\beta-1}\hbar(\varsigma,\nu(\varsigma))\,d\varsigma\right)d\sigma\right|\right) \\ & \le\zeta(\vartheta,\nu)\left(\frac{1}{\Gamma(\beta)}\left\{\int_0^{\rho}(\rho-\sigma)^{\beta-1}(|\hbar(\sigma,\vartheta(\sigma))| + |\hbar(\sigma,\nu(\sigma))|)\,d\sigma\right\}\right. \\ & \quad + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\left\{\int_0^1(1-\sigma)^{\beta-1}(|\hbar(s,\vartheta(\sigma))| + |\hbar(\sigma,\nu(\sigma))|)\,d\sigma\right\} \\ & \quad\left. + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\left\{\int_0^{\eta}\left(\int_0^{\sigma}(\sigma-\varsigma)^{\beta-1}(|\hbar(\varsigma,\vartheta(\varsigma))| + |\hbar(\varsigma,\nu(\varsigma))|)\,d\varsigma\right)d\sigma\right\}\right), \end{aligned} \tag{96} $$

that is,

$$\begin{aligned}
&\zeta(\vartheta,\nu)(|\mathfrak{T}\vartheta(\rho)|+|\mathfrak{T}\nu(\rho)|)\\
&\quad\le \frac{1}{\Gamma(\beta)}\int_0^\rho(\rho-\sigma)^{\beta-1}\lambda\times\Theta(\vartheta,\nu)(\rho)d\sigma\\
&\qquad+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^1(1-\sigma)^{\beta-1}\lambda\times\Theta(\vartheta,\nu)(\rho)d\sigma\\
&\qquad+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^\sigma(\sigma-\varsigma)^{\beta-1}\lambda\times\Theta(\vartheta,\nu)(\rho)d\varsigma\, d\sigma\\
&\quad\le\lambda\times\Theta_{\mathfrak{T}}(\vartheta,\nu)\times\max_{\rho\in J}\left(\frac{1}{\Gamma(\beta)}\int_0^\rho(\rho-\sigma)^{\beta-1}\,d\sigma+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\right.\\
&\qquad\left.\cdot\int_0^1(1-\sigma)^{\beta-1}d\sigma+\frac{2\rho}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^\sigma(\sigma-\varsigma)^{\beta-1}\,d\varsigma\, d\sigma\right).
\end{aligned} \tag{97}$$

After easy calculations, we get

$$\zeta(\vartheta,\nu)(|\mathfrak{T}\vartheta(\rho)|+|\mathfrak{T}\nu(\rho)|)\le\Theta_{\mathfrak{T}}(\vartheta,\nu). \tag{98}$$

This implies that

$$\alpha(\vartheta,\nu)\omega(\mathfrak{T}\vartheta,\mathfrak{T}\nu)=\alpha(\vartheta,\nu)\max_{t\in I}(|(\mathfrak{T}\vartheta)(\rho)|+|(\mathfrak{T}\nu)(\rho)|)\le\Theta_{\mathfrak{T}}(\vartheta,\nu) \tag{99}$$

for all $\vartheta,\nu\in\Xi$, where

$$\Theta_{\mathfrak{T}}(\vartheta,\nu)=a\cdot\omega(\vartheta,\nu)+b\cdot\omega(\vartheta,\mathfrak{T}\vartheta)+c\cdot\frac{\omega(\vartheta,\mathfrak{T}\vartheta)\omega(\vartheta,\mathfrak{T}\nu)}{1+\omega(\vartheta,\mathfrak{T}\nu)+\omega(\nu,\mathfrak{T}\vartheta)}. \tag{100}$$

If we consider

$$\psi(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6)=\ell_1-a\ell_2-b\ell_3-c\frac{\ell_4\ell_5}{1+\ell_5+\ell_6}, \tag{101}$$

where $0<a,b,c<1$ so that $a+b+c<1$, then $\psi\in\Psi$ and the condition (iii) of Theorem 9 is satisfied. Therefore, all the requirements of Theorem 9 are fulfilled, and we conclude that there is a fixed point $\vartheta^*\in\Xi$ of the operator $\mathfrak{T}$. It is well known (see, e.g., [18], Theorem 3.1) that in this case ϑ^* is also a solution of the integral equation (92) and the FDE (89) with condition (90).

6. Further Work

On the line of our work, the following two types of FDE can also be discussed:

$$\mathscr{D}^\beta(\vartheta(\rho))+\hbar(\rho,\vartheta(\rho))=0\ (0\le\rho\le1,1<\beta), \tag{102}$$

with the two-point boundary conditions

$$\vartheta(0)=0,\vartheta(1)=0, \tag{103}$$

where $\hbar: I=[0,1]\times\mathbb{R}?\mathbb{R}$ is a continuous function.

$$\mathscr{D}^\alpha(\vartheta(\rho))+\mathscr{D}^\beta(\vartheta(t))=\hbar(\rho,\vartheta(\rho))\ (0\le t\le1,0<\beta<\alpha<1), \tag{104}$$

with the two-point boundary conditions

$$\vartheta(0)=0,\vartheta(1)=0, \tag{105}$$

where $\hbar: I=[0,1]\times\mathbb{R}?\mathbb{R}$ is a continuous function.

7. Conclusion

In this work, a new class of implicit relation and implicit type contractive condition under aforesaid implicit relation in the metric spaces under w-distance functional have been introduced. Next, some basic fixed point results under respective contractive conditions followed by three suitable examples have been discussed. Further, we have discussed weak well-posed property, weak limit shadowing property, and generalized w-Ulam-Hyers stability in the underlying spaces. Finally, sufficient conditions for the existence of solutions for the fractional differential equation as an application of the established result have been discussed.

Data Availability

This clause is not applicable to this paper.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Fuzzy Triple Controlled Metric Spaces and Related Fixed Point Results

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In this study, we introduce fuzzy triple controlled metric space that generalizes certain fuzzy metric spaces, like fuzzy rectangular metric space, fuzzy rectangular b-metric space, fuzzy b-metric space, and extended fuzzy b-metric space. We use f, g, h, three noncomparable functions as follows: $\mathfrak{m}_q(\mu, \eta, t+s+w) \geq \mathfrak{m}_q(\mu, \nu, t/f(\mu, \nu)) * \mathfrak{m}_q(\nu, \xi, s/g(\nu, \xi)) * \mathfrak{m}_q(\xi, \eta, w/h(\xi, \eta))$. We prove Banach fixed point theorem in the settings of fuzzy triple controlled metric space that generalizes Banach fixed point theorem for aforementioned spaces. An example is presented to support our main results. We also apply our technique to the uniqueness for the solution of an integral equation.

1. Introduction and Preliminaries

Banach contraction mapping principle (BCMP) [1] has many applications in various scientific fields ([2–6]). BCMP was proved in 1922 and has been investigated by many researchers in different ways ([7–11]). In 1965, Zadeh [12] defined a fuzzy set that generalizes the definition of a crisp set by associating all elements with membership values between the interval $[0, 1]$. Since then, the fuzzy set theory has been used extensively in mathematics ([13–18]) and many other areas ([19–21]). The definition of fuzzy metric space was given by Kramosil and Michálek [22] in 1975. The fuzzy version of BCMP was proved by Grabiec [23]. He also extended the Edelstein theorem to fuzzy metric spaces. George and Veeramani [24] introduced the Hausdorff topology in fuzzy metric space and modified the definition of fuzzy metric space given in [22]. In 1989, Bakhtin [25] introduced the notion of a b-metric space that generalizes the definition of classical metric space. Branciari [26] introduced the rectangular metric space and proved some fixed point results. Roshan et al. [27] introduced the notion of b-rectangular metric space that generalizes the definition of a rectangular metric space.

In [28], Nădăban utilized fuzzy sets in b-metric spaces and introduced the notion of a fuzzy b-metric space. He also discussed the topological properties of this new space. Kamran et al. [29] further generalized the definition of [25] by introducing the idea of extended b-metric space while in [30], Mehmood et al. applied the fuzzy sets to the definition in [29] by introducing the notion of an extended fuzzy b-metric space and proved BCMP on this space. In [31], authors defined fuzzy version of rectangular b-metric space while in [32], Asim et al. introduced the concept of extended rectangular b-metric space and proved related fixed point theorem. Recently, Saleem et al. [33] introduced the notion of fuzzy double controlled metric space and proved BCMP on such space. Abdeljawad et al. [34] modified the definition of controlled metric type space defined in [35] by giving the idea of a double controlled metric type space.

Definition 1 ([36]). Let $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a binary operation, then $*$ is said to be continuous triangular norm (in short, continuous t-norm), if for all $x_1, x_2, x_3, x_4 \in [0, 1]$, the following conditions are satisfied:

$(*1)$ $*(x_1, x_2) = *(x_2, x_1)$;

$(*2) * (x_1, *(x_2, x_3)) = *(*(x_1, x_2), x_3)$;
$(*3) *$ is continuous;
$(*4) * (x, 1) = x$ for every $x \in [0, 1]$;
$(*5) * (x_1, x_2) \leq *(x_3, x_4)$ whenever $x_1 \leq x_3, x_2 \leq x_4$.

Definition 2 ([28]). Let F be a nonempty set, $K \geq 1$ a real number, $*$ a continuous t-norm, and $\mathfrak{m}$ be a fuzzy set on $F \times F \times [0,\infty)$. Then, $\mathfrak{m}$ is said to be a fuzzy b-metric on F, if for all $\mu, \nu, \eta \in F$, $\mathfrak{m}$ satisfies the following:
$(b\mathfrak{m}1)$ $\mathfrak{m}(\mu, \nu, 0) = 0$ for $t = 0$;
$(b\mathfrak{m}2)$ $\mathfrak{m}(\mu, \nu, t) = 1$ for all $t > 0$, iff $\mu = \nu$;
$(b\mathfrak{m}3)$ $\mathfrak{m}(\mu, \nu, t) = \mathfrak{m}(\nu, \mu, t)$;
$(b\mathfrak{m}4)$ $\mathfrak{m}(\mu, \eta, K(t + s)) \geq \mathfrak{m}(\mu, \nu, t) * \mathfrak{m}(\nu, \eta, s)$ for all $s, t > 0$;
$(b\mathfrak{m}5)$ $\mathfrak{m}(\mu, \nu, \cdot)$: $(0,\infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{t\longrightarrow\infty} \mathfrak{m}(\mu, \nu, t) = 1$.

The quadruple $(F, \mathfrak{m}, *, K)$ is called fuzzy b-metric space.

Definition 3 ([30]). Let F be a nonempty set, $f : F \times F \longrightarrow [1, \infty)$, $*$ a continuous t-norm, and $\mathfrak{m}_f$ is a fuzzy set on $F \times F \times (0,\infty)$. Then, $\mathfrak{m}_f$ is extended fuzzy b-metric if for all $\mu, \nu, \eta \in F$, the following conditions are satisfied:
$(\mathfrak{m}_f 1) \mathfrak{m}_f(\mu, \nu, 0) = 0$ for $t = 0$;
$(\mathfrak{m}_f 2) \mathfrak{m}_f(\mu, \nu, t) = 1$ for all $t > 0$, iff $\mu = \nu$;
$(\mathfrak{m}_f 3) \mathfrak{m}_f(\mu, \nu, t) = \mathfrak{m}_f(\nu, \mu, t)$;
$(\mathfrak{m}_f 4) \mathfrak{m}_f(\mu, \eta, f(\mu, \eta)(t + s)) \geq \mathfrak{m}_f(\mu, \nu, t) * \mathfrak{m}_f(\nu, \eta, s)$ for all $s, t > 0$;
$(\mathfrak{m}_f 5) \mathfrak{m}_f(\mu, \nu, \cdot)$: $(0,\infty) \longrightarrow [0, 1]$ is left continuous.

Then, $(F, \mathfrak{m}_f, *, f)$ is known as an extended fuzzy b-metric space.

Definition 4 ([37]). Let F be a nonempty set, $*$ a continuous t-norm, and $\mathfrak{m}_r$ is a fuzzy set on $F \times F \times (0,\infty)$. Then, $\mathfrak{m}_r$ is called a fuzzy rectangular metric if for any $\mu, \nu \in F$ and all distinct points $u, \eta \in F \setminus \{\mu, \nu\}$, the following conditions are satisfied:
$(\mathfrak{m}_r 1) \mathfrak{m}_r(\mu, \nu, 0) = 0$ for $t = 0$;
$(\mathfrak{m}_r 2) \mathfrak{m}_r(\mu, \nu, t) = 1$ for all $t > 0$ iff $\mu = \nu$;
$(\mathfrak{m}_r 3) \mathfrak{m}_r(\mu, \nu, t) = \mathfrak{m}_r(\nu, \mu, t)$;
$(\mathfrak{m}_r 4)$
$\mathfrak{m}_r(\mu, \nu, t + s + w) \geq \mathfrak{m}_r(\mu, \xi, t) * \mathfrak{m}_r(\xi, \eta, s) * \mathfrak{m}_r(\eta, \nu, w)$ for all $t, s, w > 0$;
$(\mathfrak{m}_r 5) \mathfrak{m}_r(\mu, \nu, \cdot)$: $(0,\infty) \longrightarrow [0, 1]$ is left continuous, and $\lim_{t\longrightarrow\infty} \mathfrak{m}_r(\mu, \nu, t) = 1$.
Then, $(F, \mathfrak{m}_r, *)$ is called a fuzzy rectangular metric space.

Definition 5 ([31]). Let F be a nonempty set, $b \geq 1$ a real number, $*$ a continuous t-norm, and $\mathfrak{m}_{rb}$ be a fuzzy set on $F \times F \times (0,\infty)$. Then, $\mathfrak{m}_{rb}$ is called fuzzy rectangular b-metric if for any $\mu, \nu \in F$ and all distinct points $\xi, \eta \in F \setminus \{\mu, \nu\}$, the following conditions are satisfied:
$(\mathfrak{m}_{rb} 1) \mathfrak{m}_{rb}(\mu, \nu, 0) = 0$;
$(\mathfrak{m}_{rb} 2) \mathfrak{m}_{rb}(\mu, \nu, t) = 1$ for all $t > 0$ iff $\mu = \nu$;
$(\mathfrak{m}_{rb} 3) \mathfrak{m}_{rb}(\mu, \nu, t) = \mathfrak{m}_{rb}(\nu, \mu, t)$;
$(\mathfrak{m}_{rb} 4)$

$\mathfrak{m}_{rb}(\mu, \nu, b(t + s + w)) \geq \mathfrak{m}_{rb}(\mu, \xi, t) * \mathfrak{m}_{rb}(\xi, \eta, s) * \mathfrak{m}_{rb}(\eta, \nu, w)$ for all $t, s, w > 0$;
$(\mathfrak{m}_{rb} 5) \mathfrak{m}_{rb}(\mu, \nu, \cdot)$: $(0,\infty) \longrightarrow [0, 1]$ is left continuous, and $\lim_{t\longrightarrow\infty} \mathfrak{m}_{rb}(\mu, \nu, t) = 1$.
Then, $(F, \mathfrak{m}_{rb}, *)$ is known as fuzzy rectangular b-metric space.

Definition 6 ([34]). Let $f, g : F \times F \longrightarrow [1, \infty)$ be two non-comparable functions, if for all $\mu, \nu, \eta \in F$, $q : F \times F \longrightarrow [0, \infty)$ satisfies the following:
$(q_{fg} 1) q(\mu, \nu) = 0$ iff $\mu = \nu$;
$(q_{fg} 2) q(\mu, \nu) = q(\nu, \mu)$;
$(q_{fg} 3) q(\mu, \nu) \leq f(\mu, \eta) q(\mu, \eta) + g(\eta, \nu) q(\eta, \nu)$.
Then, (F, q) is known as double controlled metric type space.

2. Main Results

We first give the definition of a fuzzy triple controlled metric space as follows.

Definition 7. Let $f, g, h : F \times F \longrightarrow [1,\infty)$ be three noncomparable functions, $*$ a continuous t-norm, and $\mathfrak{m}_q$ is a fuzzy set on $F \times F \times (0,\infty)$. Then, $\mathfrak{m}_q$ is called fuzzy triple controlled metric if for any $\mu, \nu \in F$ and all distinct $\xi, \eta \in F \setminus \{\mu, \nu\}$, the following conditions are satisfied:
$(\mathfrak{m}_q 1) \mathfrak{m}_q(\mu, \nu, t) > 0$;
$(\mathfrak{m}_q 2) \mathfrak{m}_q(\mu, \nu, t) = 1$ for all $t > 0$ iff $\mu = \nu$;
$(\mathfrak{m}_q 3) \mathfrak{m}_q(\mu, \nu, t) = \mathfrak{m}_q(\nu, \mu, t)$;
$(\mathfrak{m}_q 4)$

$\mathfrak{m}_q(\mu, \nu, t + s + w) \geq \mathfrak{m}_q(\mu, \xi, t/f(\mu, \xi) * \mathfrak{m}_q(\xi, \eta, s/g(\xi, \eta)) * \mathfrak{m}_q(\eta, \nu, w/h(\eta, \nu))$, for all $t, s, w > 0$;
$(\mathfrak{m}_q 5) \mathfrak{m}_q(\mu, \nu, \cdot)$: $(0,\infty) \longrightarrow [0, 1]$ is continuous.

Then, $(F, \mathfrak{m}_q, *)$ is called a fuzzy triple controlled metric space.

Remark 8.

(i) Taking $f(\mu, \xi) = g(\xi, \eta) = h(\eta, \nu) = 1$ in $(\mathfrak{m}_q 4)$, then we get the definition of fuzzy rectangular metric space [37].

(ii) Taking $f(\mu, \xi) = g(\xi, \eta) = h(\eta, \nu) = b \geq 1$ in $(\mathfrak{m}_q 4)$, then our definition reduces to the definition of fuzzy rectangular b-metric space [31].

(iii) Taking $\eta = \nu$ and $s + w = t'$ in $(\mathfrak{m}_q 4)$, then our definition reduces to the definition of fuzzy double controlled metric space defined in [33].

(iv) Taking $\eta = \nu$, $f(\mu, \xi) = g(\xi, \eta)$ and $s + w = t'$ in ($\mathfrak{m}_q$4), then our definition reduces to the definition of extended fuzzy b-metric space defined in [30].

(v) Taking $\eta = \nu$, $f(\mu, \xi) = g(\xi, \eta) = b \geq 1$ and $s + w = t'$ in ($\mathfrak{m}_q$4), then our definition reduces to the definition of fuzzy b-metric space defined in [28].

The following example justifies Definition 7.

Example 1. Consider $F = \{1, 2, 3, 4\}$ and let $f, g, h : F \times F \longrightarrow [1,\infty)$ be three noncomparable functions defined as $f(\mu, \nu) = 1 + \mu + \nu$, $g(\mu, \nu) = \mu^2 + \nu + 1$ and $h(\mu, \nu) = \mu^2 + \nu^2 - 1$. Then, $\mathfrak{m}_q : F \times F \times (0,\infty) \longrightarrow [0, 1]$ defined by

$$\mathfrak{m}_q(\mu, \nu, t) = \frac{\min\{\mu, \nu\} + t}{\max\{\mu, \nu\} + t}, \quad (1)$$

with product t-norm, that is, $t_1 * t_2 = t_1 t_2$, $(F, \mathfrak{m}_q, *)$ is a fuzzy triple controlled metric space.

Now,

$$\begin{aligned} f(1,2) &= f(2,1) = 4, f(1,3) = f(3,1) = 5, f(1,4) = f(4,1) \\ &= 6, f(2,3) = f(3,2) = 6, f(2,4) = f(4,2) = 7, f(3,4) \\ &= f(4,3) = 8, f(1,1) = 3, f(2,2) = 5, f(3,3) \\ &= 7, f(4,4) = 9, \end{aligned}$$

$$\begin{aligned} g(1,1) &= 3, g(1,2) = 4, g(1,3) = 5, g(1,4) = 6, g(2,1) \\ &= 6, g(2,2) = 7, g(2,3) = 8, g(2,4) = 9, g(3,1) \\ &= 11, g(3,2) = 12, g(3,3) = 13, g(3,4) = 14, g(4,1) \\ &= 18, g(4,2) = 19, g(4,3) = 20, g(4,4) = 21, \end{aligned}$$

$$\begin{aligned} h(1,2) &= h(2,1) = 4, h(1,3) = h(3,1) = 9, h(1,4) = h(4,1) \\ &= 16, h(2,3) = h(3,2) = 12, h(2,4) = h(4,2) \\ &= 19, h(3,4) = h(4,3) = 24, h(1,1) = 1, h(2,2) \\ &= 7, h(3,3) = 17, h(4,4) = 31. \end{aligned} \quad (2)$$

Axioms ($\mathfrak{m}_q$1) to ($\mathfrak{m}_q$3) and ($\mathfrak{m}_q$5) can easily be verified, and we check only ($\mathfrak{m}_q$4).

Let $\mu = 1, \eta = 4$, then either $\nu = 2$ and $\xi = 3$ or $\nu = 3$ and $\xi = 2$. We prove for $\nu = 2$ and $\xi = 3$. The proof for $\nu = 3$ and $\xi = 2$ is similar.

$$\mathfrak{m}_q(1, 4, t + s + w) = \frac{\min\{1, 4\} + t + s + w}{\max\{1, 4\} + t + s + w} = \frac{1 + t + s + w}{4 + t + s + w}. \quad (3)$$

Now,

$$\begin{aligned} \mathfrak{m}_q\left(1, 2, \frac{t}{f(1,2)}\right) &= \frac{\min\{1,2\} + t/f(1,2)}{\max\{1,2\} + t/f(1,2)} = \frac{1 + t/4}{2 + t/4} = \frac{4 + t}{8 + t}, \\ \mathfrak{m}_q\left(2, 3, \frac{s}{g(2,3)}\right) &= \frac{\min\{2,3\} + s/g(2,3)}{\max\{2,3\} + s/g(2,3)} = \frac{2 + s/8}{3 + s/8} = \frac{16 + s}{24 + s}, \\ \mathfrak{m}_q\left(3, 4, \frac{w}{h(3,4)}\right) &= \frac{\min\{3,4\} + w/h(3,4)}{\max\{3,4\} + w/h(3,4)} = \frac{3 + w/24}{4 + w/24} = \frac{72 + w}{92 + w}. \end{aligned} \quad (4)$$

Clearly,

$$\mathfrak{m}_q(1, 4, t + s + w) \geq \mathfrak{m}_q\left(1, 2, \frac{t}{f(1,2)}\right) * \mathfrak{m}_q\left(2, 3, \frac{s}{g(2,3)}\right) * \mathfrak{m}_q \cdot \left(3, 4, \frac{w}{h(3,4)}\right). \quad (5)$$

Working like same steps, remaining cases can easily be proved. Hence, $(F, \mathfrak{m}_q, *)$ is a fuzzy triple controlled metric space.

Remark 9. (i) In Example 1, $(F, \mathfrak{m}_q, *)$ is not a fuzzy triple controlled metric space, if we use minimum t-norm, i.e., $t_1 * t_2 = \min\{t_1, t_2\}$ instead of product t-norm.

Next, we define the convergence of a sequence as well as Cauchy sequence in the context of fuzzy triple controlled metric space.

Definition 10. Consider a fuzzy triple controlled metric space $(F, \mathfrak{m}_q, *)$. Then, a sequence $\{\mu_n\}$ in F is said to be

(1) convergent, if for all $t > 0$, there exists μ in F such that

$$\lim_{n \longrightarrow \infty} \mathfrak{m}_q(\mu_n, \mu, t) = 1, \quad (6)$$

(2) Cauchy iff for all $t > 0$ and for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathfrak{m}_q(\mu_n, \mu_m, t) \geq 1 - \varepsilon, \text{for all } m, n \geq n_0. \quad (7)$$

$(F, \mathfrak{m}_q, *)$ is called complete fuzzy triple controlled metric space, if every Cauchy sequence in F converges to some point μ in F.

Definition 11. Let $(F, \mathfrak{m}_q, *)$ be a fuzzy triple controlled metric space. Then, an open ball $B(x, r, t)$, with center x, radius r, $r \in]0, 1[$, and $t > 0$, is given by

$$B(x, r, t) = \{y \in F : m_q(x, y, t) > 1 - r\}, \quad (8)$$

and the corresponding topology is defined as

$$\tau_{\mathfrak{m}_q} = \{S \subset F : B(x, r, t) \subset S\}. \quad (9)$$

Next example shows that a fuzzy triple controlled metric space is not Hausdorff.

Example 2. Consider the fuzzy triple controlled metric space as given in Example 1. Then, the open ball centered at 1, radius 0.3, and $t = 5$ is given by

$$B(1{,}0.3{,}5) = \{y \in F : \mathfrak{m}_q(1, y, 5) > 0.7\}. \tag{10}$$

Now,

$$\begin{aligned}\mathfrak{m}_q(1, 2, 5) &= \frac{1+5}{2+5} = \frac{6}{7} = 0.8571,\\ \mathfrak{m}_q(1, 3, 5) &= \frac{1+5}{3+5} = \frac{6}{8} = 0.75,\\ \mathfrak{m}_q(1, 4, 5) &= \frac{1+5}{4+5} = \frac{6}{9} = 0.6666.\end{aligned} \tag{11}$$

Thus, $B(1{,}0.3{,}5) = \{2, 3\}$. Now, consider the open ball $B(2{,}0.2{,}7)$ with centered at 2, radius $r = 0.2$, and $t = 7$. Then,

$$\begin{aligned}B(2{,}0.2{,}7) &= \{y \in F : \mathfrak{m}_q(2, y, 7) > 0.8\},\\ \mathfrak{m}_q(2, 1, 7) &= \frac{1+7}{2+7} = \frac{8}{9} = 0.8888,\\ \mathfrak{m}_q(2, 3, 7) &= \frac{2+7}{3+7} = \frac{9}{10} = 0.9,\\ \mathfrak{m}_q(2, 4, 7) &= \frac{2+7}{4+7} = \frac{9}{11} = 0.8181.\end{aligned} \tag{12}$$

Thus, $B(2{,}0.2{,}7) = \{1, 3, 4\}$ and so $B(1{,}0.3{,}5) \cap B(2{,}0.2{,}7) = \{2, 3\} \cap \{1, 3, 4\} = \{3\} \neq \varnothing$. Hence, fuzzy triple controlled metric space $(F, \mathfrak{m}_q, *)$ is not Hausdorff.

Theorem 12. *Let $f, g, h : F \times F \longrightarrow [1, 1/k)$ be three noncomparable functions ($k \in (0, 1)$) and $(F, \mathfrak{m}_q, *)$ be a complete fuzzy triple controlled metric space such that*

$$\lim_{t \longrightarrow \infty} \mathfrak{m}_q(\mu, \nu, t) = 1. \tag{13}$$

Let $T : F \longrightarrow F$ be a given mapping such that

$$\mathfrak{m}_q(T\mu, T\nu, kt) \geq \mathfrak{m}_q(\mu, \nu, t), \text{for all } \mu, \nu \in F. \tag{14}$$

Then, T has a unique fixed point.

Proof. Choose a_0 an arbitrary point in F. If $Ta_0 = a_0$, then we are done. If $Ta_0 \neq a_0$ then $Ta_0 = a_1 \in F$. Continuing in this way, we have

$$\begin{aligned}Ta_1 &= T(Ta_0) = T^2 a_0 = a_2,\\ Ta_2 &= T(Ta_1) = T^2 a_1 = T^2(Ta_0) = T^3 a_0 = a_3,\end{aligned} \tag{15}$$

and so on,

$$Ta_n = T(Ta_{n-1}) = T^2 a_{n-1} = \cdots = T^{n+1} a_0 = a_{n+1}. \tag{16}$$

So, we have iterative sequence $a_n = Ta_{n-1} = T^n a_0$. Applying (14) successively, we get

$$\begin{aligned}\mathfrak{m}_q(a_n, a_{n+1}, t) &= \mathfrak{m}_q(Ta_{n-1}, Ta_n, t) \geq \mathfrak{m}_q\left(a_{n-1}, a_n, \frac{t}{k}\right)\\ &= \mathfrak{m}_q\left(Ta_{n-2}, Ta_{n-1}, \frac{t}{k}\right) \geq \mathfrak{m}_q\left(a_{n-2}, a_{n-1}, \frac{t}{k^2}\right)\\ &\geq \mathfrak{m}_q\left(a_{n-3}, a_{n-2}, \frac{t}{k^3}\right) \geq \cdots \geq \mathfrak{m}_q\left(a_0, a_1, \frac{t}{k^n}\right).\end{aligned} \tag{17}$$

Now, writing $t = t/3 + t/3 + t/3$ and for any sequence $\{a_n\}$, using rectangular property, we have the following cases:

Case 1. When $p = 2m + 1$, i.e., p is odd, then

$$\begin{aligned}\mathfrak{m}_q(a_n, a_{n+2m+1}, t) &\geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q \cdot \left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)\\
&* \mathfrak{m}_q\left(a_{n+2}, a_{n+2m+1}, \frac{t/3}{h(a_{n+2}, a_{n+2m+1})}\right)\\
&\geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q\left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)\\
&* \mathfrak{m}_q\left(a_{n+2}, a_{n+3}, \frac{t/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+3}, a_{n+4}, \frac{t/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+4}, a_{n+2m+1}, \frac{t/3^2}{h(a_{n+4}, a_{n+2m+1})h(a_{n+2}, a_{n+2m+1})}\right)\\
&\geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q\left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)\\
&* \mathfrak{m}_q\left(a_{n+2}, a_{n+3}, \frac{t/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+3}, a_{n+4}, \frac{t/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+4}, a_{n+5}, \frac{t/3^3}{f(a_{n+4}, a_{n+5})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+5}, a_{n+6}, \frac{t/3^3}{g(a_{n+5}, a_{n+6})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+6}, a_{n+7}, \frac{t/3^4}{f(a_{n+6}, a_{n+7})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})h(a_{n+6}, a_{n+2m+1})}\right)\\
&* \mathfrak{m}_q\left(a_{n+7}, a_{n+8}, \frac{t/3^4}{g(a_{n+7}, a_{n+8})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})h(a_{n+6}, a_{n+2m+1})}\right) \vdots\\
&* \mathfrak{m}_q\left(a_{n+2m-2}, a_{n+2m-1}, \frac{t/3^m}{f(a_{n+2m-2}, a_{n+2m-1})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})}\right)\end{aligned}$$

$$* \mathfrak{m}_q\left(a_{n+2m-1}, a_{n+2m}, \frac{t/3^m}{g(a_{n+2m-1}, a_{n+2m})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2m}, a_{n+2m+1}, \frac{t/3^m}{h(a_{n+2m}, a_{n+2m+1})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})}\right). \tag{18}$$

Applying (17) on right hand side, we deduce

$$\mathfrak{m}_q(a_n, a_{n+2m+1}, t) \geq \mathfrak{m}_q\left(a_0, a_1, \frac{t/3}{f(a_n, a_{n+1})k^n}\right) * \mathfrak{m}_q \cdot \left(a_0, a_1, \frac{t/3}{g(a_{n+1}, a_{n+2})k^{n+1}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m+1})k^{n+2}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m+1})k^{n+3}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^3}{f(a_{n+4}, a_{n+5})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})k^{n+4}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^3}{g(a_{n+5}, a_{n+6})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})k^{n+5}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^4}{f(a_{n+6}, a_{n+7})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})h(a_{n+6}, a_{n+2m+1})k^{n+6}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^4}{g(a_{n+7}, a_{n+8})h(a_{n+2}, a_{n+2m+1})h(a_{n+4}, a_{n+2m+1})h(a_{n+6}, a_{n+2m+1})k^{n+7}}\right) \vdots$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^m}{f(a_{n+2m-2}, a_{n+2m-1})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})k^{n+2m-2}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^m}{g(a_{n+2m-1}, a_{n+2m})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})k^{2m-1}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^m}{h(a_{n+2m}, a_{n+2m+1})h(a_{n+2m-2}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m+1})k^{2m}}\right). \tag{19}$$

Case 2. When $p = 2m$, i.e., p is even, then

$$\mathfrak{m}_q(a_n, a_{n+2m}, t) \geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q \cdot \left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2}, a_{n+2m}, \frac{t/3}{h(a_{n+2}, a_{n+2m})}\right)$$
$$\geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q\left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2}, a_{n+3}, \frac{t/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+3}, a_{n+4}, \frac{t/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+4}, a_{n+2m}, \frac{t/3^2}{h(a_{n+4}, a_{n+2m})h(a_{n+2}, a_{n+2m})}\right)$$
$$\geq \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{f(a_n, a_{n+1})}\right) * \mathfrak{m}_q\left(a_{n+1}, a_{n+2}, \frac{t/3}{g(a_{n+1}, a_{n+2})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2}, a_{n+3}, \frac{t/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+3}, a_{n+4}, \frac{t/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+4}, a_{n+5}, \frac{t/3^3}{f(a_{n+4}, a_{n+5})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+5}, a_{n+6}, \frac{t/3^3}{g(a_{n+5}, a_{n+6})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+6}, a_{n+7}, \frac{t/3^4}{f(a_{n+6}, a_{n+7})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m})h(a_{n+6}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+7}, a_{n+8}, \frac{t/3^4}{g(a_{n+7}, a_{n+8})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m})h(a_{n+6}, a_{n+2m})}\right) \vdots$$
$$* \mathfrak{m}_q\left(a_{n+2m-4}, a_{n+2m-3}, \frac{t/3^{m-1}}{f(a_{n+2m-4}, a_{n+2m-3})h(a_{n+2m-4}, a_{n+2m}) \cdots h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2m-3}, a_{n+2m-2}, \frac{t/3^{m-1}}{g(a_{n+2m-3}, a_{n+2m-2})h(a_{n+2m-4}, a_{n+2m}) \cdots h(a_{n+2}, a_{n+2m})}\right)$$
$$* \mathfrak{m}_q\left(a_{n+2m-2}, a_{n+2m}, \frac{t/3^{m-1}}{h(a_{n+2m-2}, a_{n+2m})h(a_{n+2m-4}, a_{n+2m+1}) \cdots h(a_{n+2}, a_{n+2m})}\right). \tag{20}$$

Now applying (17), we deduce

$$\mathfrak{m}_q(a_n, a_{n+2m+1}, t) \geq \mathfrak{m}_q\left(a_0, a_1, \frac{t/3}{f(a_n, a_{n+1})k^n}\right) * \mathfrak{m}_q \cdot \left(a_0, a_1, \frac{t/3}{g(a_{n+1}, a_{n+2})k^{n+1}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3/3^2}{f(a_{n+2}, a_{n+3})h(a_{n+2}, a_{n+2m})k^{n+2}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3/3^2}{g(a_{n+3}, a_{n+4})h(a_{n+2}, a_{n+2m})k^{n+3}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^3}{f(a_{n+4}, a_{n+5})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m+1})k^{n+4}}\right)$$
$$* \mathfrak{m}_q\left(a_0, a_1, \frac{t/3^3}{g(a_{n+5}, a_{n+6})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m+1})k^{n+5}}\right)$$

$$*\mathfrak{m}_q\left(a_0, a_1, \frac{t/3^4}{f(a_{n+6}, a_{n+7})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m+1})h(a_{n+6}, a_{n+2m})k^{n+6}}\right)$$
$$*\mathfrak{m}_q\left(a_0, a_1, \frac{t/3^4}{g(a_{n+7}, a_{n+8})h(a_{n+2}, a_{n+2m})h(a_{n+4}, a_{n+2m})h(a_{n+6}, a_{n+2m})k^{n+7}}\right)\vdots$$
$$*\mathfrak{m}_q\left(a_0, a_1, \frac{t/3^{m-1}}{f(a_{n+2m-3}, a_{n+2m-2})h(a_{n+2m-4}, a_{n+2m})\cdots h(a_{n+2}, a_{n+2m})k^{n+2m-4}}\right)$$
$$*\mathfrak{m}_q\left(a_0, a_1, \frac{t/3^{m-1}}{g(a_{n+2m-1}, a_{n+2m})h(a_{n+2m-2}, a_{n+2m+1})\cdots h(a_{n+2}, a_{n+2m+1})k^{n+2m-3}}\right)$$
$$*\mathfrak{m}_q\left(a_0, a_1, \frac{t/3^{m-1}}{h(a_{n+2m-2}, a_{n+2m})h(a_{n+2m-4}, a_{n+2m})\cdots h(a_{n+2}, a_{n+2m})k^{n+2m-2}}\right). \tag{21}$$

Using (13) for each case, we obtain

$$\lim_{n\longrightarrow\infty}\mathfrak{m}_q\left(a_n, a_{n+p}, t\right) = 1 * 1*\cdots*1 = 1, \tag{22}$$

which shows that $\{a_n\}$ is a Cauchy sequence in F and converges to some $a \in F$ (as F is complete), i.e.,

$$\lim_{n\longrightarrow\infty}\mathfrak{m}_q(a_n, a, t) = 1. \tag{23}$$

Now, we show that a is the fixed point of T. From (14),

$$\begin{aligned}\mathfrak{m}_q(a, Ta, t) &\geq \mathfrak{m}_q\left(a, a_n, \frac{t/3}{f(a, a_n)}\right) * \mathfrak{m}_q\left(a_n, a_{n+1}, \frac{t/3}{g(a_n, a_{n+1})}\right) * \mathfrak{m}_q\\ &\cdot\left(a_{n+1}, Ta, \frac{t/3}{h(a_{n+1}, Ta)}\right) \geq \mathfrak{m}_q\left(a, a_n, \frac{t/3}{f(a, a_n)}\right) * \mathfrak{m}_q\\ &\cdot\left(Ta_{n-1}, Ta_n, \frac{t/3}{g(a_n, a_{n+1})}\right) * \mathfrak{m}_q\left(Ta_n, Ta, \frac{t/3}{h(a_{n+1}, Ta)}\right)\\ &\geq \mathfrak{m}_q\left(a, a_n, \frac{t/3}{f(a, a_n)}\right) * \mathfrak{m}_q\left(a_{n-1}, a_n, \frac{t/3}{g(a_n, a_{n+1})k}\right) * \mathfrak{m}_q\\ &\cdot\left(a_n, a, \frac{t/3}{h(a_{n+1}, Ta)k}\right) \longrightarrow 1 * 1 * 1 = 1 \text{ as } n \longrightarrow \infty,\end{aligned} \tag{24}$$

which shows that $Ta = a$, i.e., a is a fixed point of T. To prove a is unique, we assume a' is also a fixed point of T, i.e., $Ta' = a'$. Then,

$$\mathfrak{m}_q\left(a, a', t\right) = \mathfrak{m}_q\left(Ta, Ta', t\right) \geq \mathfrak{m}_q\left(a, a', \frac{t}{k}\right), \tag{25}$$

which implies that $a = a'$, so the fixed point of T is unique.

Example 3. Let $F = [0, 1]$ and $\mathfrak{m}_q : F \times F \times (0,\infty) \longrightarrow [0, 1]$ be defined as $\mathfrak{m}_q(\mu, \nu, t) = e^{-|\mu-\nu|/t}$ for all $t > 0$. Let $f, g, h : F \times F \longrightarrow [1, 1/k)$ be defined by $f(\mu, \nu) = \mu + \nu + 1$, $g(\nu, \xi) = \nu^2 + \xi + 1$ and $h(\xi, \eta) = \xi^2 + \eta^2 + 1$. Then, $(F, \mathfrak{m}_q,*)$ is a complete fuzzy triple controlled metric space. Let $T : F \longrightarrow F$ be given by $T\mu = 1 - \mu/4$. Then,

$$\begin{aligned}\mathfrak{m}_q(T\mu, T\nu, kt) &= \mathfrak{m}_q\left(1 - \frac{\mu}{4}, 1 - \frac{\nu}{4}, kt\right) = e^{-|1-\mu/4-1+\nu/4|/kt}\\ &= e^{-|\mu/4-\nu/4|/kt} \geq e^{-|\mu-\nu|/4kt} \geq e^{-|\mu-\nu|/t} = \mathfrak{m}_q(\mu, \nu, t),\end{aligned} \tag{26}$$

for all $\mu, \nu \in F$, where $k \in [13/50, 1)$. Since all the conditions of Theorem 12 are satisfied, therefore T has a unique fixed point which is $\mu = 4/5$.

Theorem 12 generalizes Theorem 2.1 of [31] as follows.

Corollary 13. *Let $(F, \mathfrak{m}_q,*)$ be a complete fuzzy rectangular b -metric space with $b \geq 1$ such that*

$$\lim_{t\longrightarrow\infty}\mathfrak{m}_q(\mu, \nu, t) = 1. \tag{27}$$

Let $T : F \longrightarrow F$ be a mapping such that

$$\mathfrak{m}_q(T\mu, T\nu, kt) \geq \mathfrak{m}_q(\mu, \nu, t), \text{for all } \mu, \nu \in F. \tag{28}$$

Then, T has a unique fixed point.

Remark 14. If we choose $f(\mu, \nu) = g(\mu, \nu) = h(\mu, \nu) = 1$, then our main result reduces to the Banach contraction principle for the fuzzy rectangular metric space defined in [37].

3. Application

Consider the integral equation

$$\mu(\xi) = g(\xi) + \int_0^\xi G(\xi, \gamma, \mu(\gamma))d\gamma, \tag{29}$$

where $\xi \in I = [0, 1]$.

Define $f(\mu, \nu) = 3(\mu + \nu + 1)$, $g(\nu, \xi) = 3(\mu^2 + \nu^2 + 1)$, and $h(\xi, \eta) = 3(\mu^2 + \nu^2 - 1)$. Also, for all $t > 0$, $\mu, \nu \in C(I, \mathbb{R})$,

$$\mathfrak{m}_q(\mu, \nu, t) = e^{-\sup_{\xi\in I}|\mu(\xi)-\nu(\xi)|^2/t}. \tag{30}$$

Then, $(C(I, \mathbb{R}), \mathfrak{m}_q)$ is a complete fuzzy triple controlled metric space with product t-norm, where $C(I, \mathbb{R})$ is the space of all real valued continuous functions defined on I.

Theorem 15. *Consider an integral operator defined on $C(I, \mathbb{R})$ as follows:*

$$T\mu(\xi) = g(\xi) + \int_0^\xi G(\xi, \gamma, \mu(\gamma))d\gamma, g \in C(I, \mathbb{R}), \tag{31}$$

where G satisfies the following condition.

There exists $f : I \times I \longrightarrow [0, \infty)$ such that for all $\xi, \gamma \in I$, $f(\xi, \gamma) \in L^1(I, \mathbb{R})$, and for all $\mu, \nu \in C(I, \mathbb{R})$, we have

$$\left|G(\xi, \gamma, \mu(\gamma)) - G(\xi, \gamma, \nu(\gamma))\right|^2 \leq f^2(\xi, \gamma)\left|\mu(\gamma) - \nu(\gamma)\right|^2, \tag{32}$$

where

$$0 < \sup_{\xi\in I}\int_0^\xi f^2(\xi,\gamma)d\gamma \leq k < 1. \tag{33}$$

Then, the integral equation (29) has a unique solution.

Proof. Let $\mu, \nu \in C(I, \mathbb{R})$. Note that

$$\begin{aligned}
\mathfrak{m}_q(T\mu, T\nu, kt) &= e^{-\sup_{\xi\in I}|T\mu(\xi)-T\nu(\xi)|^2/kt}\\
&= e^{-\sup_{\xi\in I}\left|\int_0^\xi (G(\xi,\gamma,\mu(\gamma))-G(\xi,\gamma,\nu(\gamma)))d\gamma\right|^2/kt}\\
&\geq e^{-\sup_{\xi\in I}\int_0^\xi |G(\xi,\gamma,\mu(\gamma))-G(\xi,\gamma,\nu(\gamma))|^2 d\gamma/kt}\\
&\geq e^{-\sup_{\xi\in I}\int_0^\xi f^2(\xi,\gamma)|\mu(\gamma)-\nu(\gamma)|^2 d\gamma/kt}\\
&\geq e^{-|\mu(\gamma)-\nu(\gamma)|^2 \sup_{\xi\in I}\int_0^\xi f^2(\xi,\gamma)d\gamma/kt}\\
&\geq e^{-k|\mu(\gamma)-\nu(\gamma)|^2/kt} = e^{-|\mu(\gamma)-\nu(\gamma)|^2/t}\\
&\geq e^{-\sup_{\gamma\in I}|\mu(\gamma)-\nu(\gamma)|^2/t} = \mathfrak{m}_q(\mu,\nu,t).
\end{aligned} \tag{34}$$

Thus, $\mathfrak{m}_q(T\mu, T\nu, kt) \geq \mathfrak{m}_q(\mu,\nu,t)$ for all $\mu, \nu \in C(I, \mathbb{R})$ and consequently, all the conditions of Theorem 12 are satisfied. Therefore, the integral equation (29) has a unique solution.

4. Conclusion

In this article, the concept of fuzzy triple controlled metric space is given which generalizes fuzzy rectangular b-metric space, rectangular fuzzy metric space. We have established Banach fixed point theorem in this space. An example is also presented that illustrates our main result. Our newly defined results can be used for further investigation in many existing results in the literature.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare to have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Rational Fuzzy Cone Contractions on Fuzzy Cone Metric Spaces with an Application to Fredholm Integral Equations

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This paper is aimed at proving some common fixed point theorems for mappings involving generalized rational-type fuzzy cone-contraction conditions in fuzzy cone metric spaces. Some illustrative examples are presented to support our work. Moreover, as an application, we ensure the existence of a common solution of the Fredholm integral equations: $\mu(\tau) = \int_0^\tau \Gamma(\tau, \nu, \mu(\nu))d\nu$ and $\nu(\tau) = \int_0^\tau \Gamma(\tau, \nu, \nu(\nu))d\nu$, for all $\mu \in U$, $\nu \in [0, \eta]$, and $0 < \eta \in \mathbb{R}$, where $U = C([0, \eta], \mathbb{R})$ is the space of all $\mathbb{R}$-valued continuous functions on the interval $[0, \eta]$ and $\Gamma : [0, \eta] \times [0, \eta] \times \mathbb{R} \longrightarrow \mathbb{R}$.

1. Introduction

In 1922, Banach [1] proved a "Banach contraction principle," which is stated as follows: "A self-mapping on a complete metric space verifying the contraction condition has a unique fixed point." This principle plays a very important role in the fixed point theory. A number of researches have generalized it in many directions for single-valued and multivalued mappings in the context of metric spaces. Some of the findings can be found in [2–13] and the references therein. Currently, the fixed point theory is one of the most interested research areas in the field of mathematics. In the last decades, it has been investigated in many fields, such as game theory, graph theory, economics, computer sciences, and engineering.

The theory of fuzzy sets was introduced by Zadeh [14], while the concept of a fuzzy metric space (FM space) was given by Kramosil and Michalek [15]. After that, the stronger form of the metric fuzziness was presented by George and Veeramani in [16]. Later on, in [17], Gregori and Sapena proved some contractive-type fixed point results in complete FM spaces. Some more fixed point results in FM spaces can be found in [18–27] and the references therein.

Initially, in 2007, the concept of a cone metric space was reintroduced by Huang and Zhang [28]. They proved some nonlinear contractive-type fixed point results in cone metric spaces. After the publication of this article, a number of researchers have contributed their ideas in cone metric spaces. Some of such works can be found in [29–34] and the references therein.

In 2015, the basic concept of a fuzzy cone metric space (FCM space) was given by Öner et al. [35]. They presented some key attributes and a "fuzzy cone Banach contraction theorem" in FCM spaces. Later, Rehman and Li [36] extended and improved a "fuzzy cone Banach contraction theorem" and proved some generalized fixed point theorems in FCM spaces. Some more properties and related fixed point results can be found in [37–47].

The aim of this research work is to establish some rational-type fuzzy cone-contraction results in FCM spaces. We use the concept of [36, 39] and prove some common fixed theorems under generalized rational-type fuzzy cone-contraction conditions in FCM spaces. Some illustrative examples are presented. In the last section, we give an application of two Fredholm integral equations (FIEs).

2. Preliminaries

Definition 1 [47]. An operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if

(i) $*$ is commutative, associative, and continuous

(ii) $1 * \eta_1 = \eta_1$ and $\eta_1 * \eta_2 \leq \eta_3 * \eta_4$, whenever $\eta_1 \leq \eta_3$ and $\eta_2 \leq \eta_4$, for all $\eta_1, \eta_2, \eta_3, \eta_4 \in [0,1]$

The basic t-norms: the minimum, the product, and the Lukasiewicz continuous t-norms are defined by [47]

$$\begin{aligned} \eta_1 * \eta_2 = \min\{\eta_1, \eta_2\}\, \eta_1 * \eta_2 = \eta_1\eta_2, \\ \eta_1 * \eta_2 = \max\{\eta_1 + \eta_2 - 1, 0\}. \end{aligned} \quad (1)$$

Definition 2 [35]. A 3-tuple $(U, M_r, *)$ is said to be a FCM space if P is a cone of E, U is an arbitrary set, $*$ is a continuous t-norm, and M_r is a fuzzy set on $U^2 \times \text{int}(P)$ satisfying the following conditions:

(1) $\forall_{\nu_1,\nu_2 \in U}$; $M_r(\nu_1, \nu_2, t) > 0$ and $M_r(\nu_1, \nu_2, t) = 1 \Leftrightarrow \nu_1 = \nu_2$

(2) $\forall_{\nu_1,\nu_2 \in U}$; $M_r(\nu_1, \nu_2, t) = M_r(\nu_2, \nu_1, t)$

(3) $\forall_{\nu_1,\nu_2,\nu_3 \in U}$; $M_r(\nu_1, \nu_2, t) * M_r(\nu_2, \nu_3, s) \leq M_r(\nu_1, \nu_3, t+s)$

(4) $\forall_{\nu_1,\nu_2 \in U}$; $M_r(\nu_1, \nu_2, .)$: $\text{int}(P) \longrightarrow [0,1]$ is continuous

for all $t, s \in$ sin int (P).

Definition 3 [35]. Let $(U, M_r, *)$ be a FCM space and $\nu_1 \in U$ and (ν_j) be a sequence in U.

(i) (ν_j) converges to ν_1 if for $c \in (0,1)$ and $t \gg \theta$ there is $j_1 \in N$ such that $M_r(\nu_j, \nu_1, t) > 1 - c$, for $j \geq j_1$. We may write this $\lim_{j \longrightarrow \infty} \nu_j = \nu_1$ or $\nu_j \longrightarrow \nu_1$ as $j \longrightarrow \infty$

(ii) (ν_j) is Cauchy if for $c \in (0,1)$ and $t \gg \theta$ there is $j_1 \in N$ such that $M_r(\nu_j, \nu_k, t) > 1 - c$, for $j, k \geq j_1$

(iii) $(U, M_r, *)$ is complete if every Cauchy sequence is convergent in U

(iv) (ν_j) is fuzzy cone contractive if there is $a \in (0,1)$ so that

$$\frac{1}{M_r(\nu_j, \nu_{j+1}, t)} - 1 \leq a\left(\frac{1}{M_r(\nu_{j-1}, \nu_j, t)} - 1\right), \quad \text{for } t \gg \theta, j \geq 1. \quad (2)$$

Lemma 4 [35]. *Let $(U, M_r, *)$ be a FCM space and let (ν_j) be sequence in U converging to a point $\nu_1 \in U$ iff $M_r(\nu_j, \nu_1, t) \longrightarrow 1$ as $j \longrightarrow \infty$ for each $t \gg \theta$.*

Definition 5 [36]. Let $(U, M_r, *)$ be a FCM space. The fuzzy cone metric M_r is triangular if

$$\frac{1}{M_r(\nu_1, \nu_3, t)} - 1 \leq \left(\frac{1}{M_r(\nu_1, \nu_2, t)} - 1\right) + \left(\frac{1}{M_r(\nu_2, \nu_3, t)} - 1\right), \quad \forall \nu_1, \nu_2, \nu_3 \in U, t \gg \theta. \quad (3)$$

Definition 6 [35]. Let $(U, M_r, *)$ be a FCM space and $\ell : U \longrightarrow U$. Then, ℓ is said to be fuzzy cone contractive if there is $a \in (0,1)$ such that

$$\frac{1}{M_r(\ell\nu_1, \ell\nu_2, t)} - 1 \leq a\left(\frac{1}{M_r(\nu_1, \nu_2, t)} - 1\right), \quad \forall \nu_1, \nu_2 \in U, t \gg \theta. \quad (4)$$

A "fuzzy cone Banach contraction theorem" [35] is stated as follows: "Let $(U, M_r, *)$ be a complete FCM space in which fuzzy cone contractive sequences are Cauchy and $\ell : U \longrightarrow U$ be a fuzzy cone contractive mapping. Then, ℓ has a unique fixed point."

In this paper, we present some rational-type fuzzy cone-contraction theorems in FCM spaces by using the concept of [36, 39]. Namely, we prove some common fixed theorems under generalized rational-type fuzzy cone-contraction conditions in FCM spaces without the assumption that the fuzzy cone contractive sequences are Cauchy. We use "the triangular property of the fuzzy cone metric." We also present some illustrative examples to support our work. In the last section, an application of Fredholm integral equations is provided.

3. Main Results

In this section, we prove some common fixed point theorems via generalized rational-type fuzzy cone-contraction conditions in FCM spaces.

Theorem 7. *Let $(U, M_r, *)$ be a complete FCM space in which M_r is triangular. Let $\ell, \hbar : U \longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned} &\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 \\ &\leq a\left(\frac{1}{M_r(\mu, \nu, t)} - 1\right) \\ &+ b\left(\frac{M_r(\mu, \nu, t)}{M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1\right) \\ &+ c\left(\frac{M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t)}{M_r(\mu, \nu, t) * M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1\right) \\ &+ d\left(\frac{1}{M_r(\mu, \ell\mu, t)} - 1 + \frac{1}{M_r(\nu, \hbar\nu, t)} - 1\right), \end{aligned} \quad (5)$$

for all $\mu, \nu \in U$, $t \gg \theta$, $a \in (0, 1)$, and $b, c, d \geq 0$ with $a + b + c + 2d < 1$. Then, ℓ and $\hbar$ have a common fixed point in U.

Proof. Fix $\mu_0 \in U$ and construct a sequence of points in U such that

$$\begin{gathered} \mu_{2j+1} = \ell \mu_{2j}, \\ \mu_{2j+2} = \hbar \mu_{2j+1}, \\ j \geq 0. \end{gathered} \tag{6}$$

Then, by (5), for $t \gg \theta$,

$$\begin{aligned}
&\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1 \\
&\quad = \frac{1}{M_r\left(\ell\mu_{2j}, \hbar\mu_{2j+1}, t\right)} - 1 \\
&\quad \leq a\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1\right) \\
&\qquad + b\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)}{M_r\left(\mu_{2j}, \hbar\mu_{2j+1}, 2t\right) * M_r\left(\mu_{2j+1}, \ell\mu_{2j}, 2t\right)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j}, \ell\mu_{2j}, t\right) * M_r\left(\mu_{2j+1}, \hbar\mu_{2j+1}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r\left(\mu_{2j}, \hbar\mu_{2j+1}, 2t\right) * M_r\left(\mu_{2j+1}, \ell\mu_{2j}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j}, \ell\mu_{2j}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \hbar\mu_{2j+1}, t\right)} - 1\right) \\
&\quad = a\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1\right) \\
&\qquad + b\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+2}, 2t\right) * M_r(x_{2j+1}, x_{2j+1}, 2t)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r\left(\mu_{2j}, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+1}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) \\
&\quad = a\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1\right) + b\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+2}, 2t\right)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+2}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right).
\end{aligned} \tag{7}$$

By Definition 2 (3), $M_r(\mu_{2j}, \mu_{2j+2}, 2t) \geq M_r(\mu_{2j}, \mu_{2j+1}, t) * M_r(\mu_{2j+1}, \mu_{2j+2}, t)$, for $t \gg \theta$. One writes

$$\begin{aligned}
&\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1 \\
&\quad \leq a\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1\right) \\
&\qquad + b\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right).
\end{aligned} \tag{8}$$

After simplification, we get that

$$\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1 \leq \gamma\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1\right), \quad \text{for } t \gg \theta, \tag{9}$$

where $\gamma = (a + c + d)/(1 - b - d) < 1$ since $(a + b + c + 2d) < 1$. Similarly,

$$\begin{aligned}
&\frac{1}{M_r\left(\mu_{2j+2}, \mu_{2j+3}, t\right)} - 1 \\
&\quad = \frac{1}{M_r\left(\ell\mu_{2j+2}, \hbar\mu_{2j+1}, t\right)} - 1 \\
&\quad \leq a\left(\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) \\
&\qquad + b\left(\frac{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j+2}, \hbar\mu_{2j+1}, 2t\right) * M\left(\mu_{2j+1}, \ell\mu_{2j+2}, 2t\right)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j+2}, \ell\mu_{2j+2}, t\right) * M_r\left(\mu_{2j+1}, \hbar\mu_{2j+1}, t\right)}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right) * M_r\left(\mu_{2j+2}, \hbar\mu_{2j+1}, 2t\right) * M_r\left(\mu_{2j+1}, \ell\mu_{2j+2}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j+2}, \ell\mu_{2j+2}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \hbar\mu_{2j+1}, t\right)} - 1\right) \\
&\quad = a\left(\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) + b\left(\frac{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j+2}, \mu_{2j+2}, 2t\right) * M_r(x_{2j+1}, x_{2j+3}, 2t)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j+2}, \mu_{2j+3}, t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right) * M_r\left(\mu_{2j+2}, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \mu_{2j+3}, 2t\right)} - 1\right) \\
&\qquad + d\left(\frac{1}{M_r\left(\mu_{2j+2}, \mu_{2j+3}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) \\
&\quad = a\left(\frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) + b\left(\frac{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\mu_{2j+1}, \mu_{2j+3}, 2t\right)} - 1\right) \\
&\qquad + c\left(\frac{M_r\left(\mu_{2j+2}, \mu_{2j+3}, t\right)}{M_r\left(\mu_{2j+1}, \mu_{2j+3}, 2t\right)} - 1\right) + d\left(\frac{1}{M_r\left(\mu_{2j+2}, \mu_{2j+3}, t\right)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right).
\end{aligned} \tag{10}$$

Again, by Definition 2 (3), $M_r(\mu_{2j+1}, \mu_{2j+3}, 2t) \geq M_r(\mu_{2j+1}, \mu_{2j+2}, t) * M_r(\mu_{2j+2}, \mu_{2j+3}, t)$, for $t \gg \theta$. We have

$$\begin{aligned}
&\frac{1}{M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}-1\\
&\leq a\left(\frac{1}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}-1\right)\\
&+b\left(\frac{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)*M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}-1\right)\\
&+c\left(\frac{M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)*M_r\left(\mu_{2j+2},\mu_{2j+3},2t\right)}-1\right)\\
&+d\left(\frac{1}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}-1+\frac{1}{M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}-1\right).
\end{aligned} \tag{11}$$

After simplification, we have

$$\frac{1}{M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}-1\leq\gamma\left(\frac{1}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}-1\right),\quad \text{for } t\gg\theta, \tag{12}$$

where the value of γ is the same as in (9). Now, from (9) and (12) and by induction, we have

$$\begin{aligned}
&\frac{1}{M_r\left(\mu_{2j+2},\mu_{2j+3},t\right)}-1\\
&\leq\gamma\left(\frac{1}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}-1\right)\\
&\leq\gamma^2\left(\frac{1}{M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)}-1\right)\leq\cdots\\
&\leq\gamma^{2j+2}\left(\frac{1}{M_r(\mu_0,\mu_1,t)}-1\right)\longrightarrow 0,\quad \text{as } j\longrightarrow\infty,
\end{aligned} \tag{13}$$

which shows that (μ_j) is a fuzzy cone-contractive sequence in U, and we get that

$$\lim_{j\longrightarrow\infty}M_r\left(\mu_{2j+1},\mu_{2j+2},t\right)=1,\quad \text{for } t\gg\theta. \tag{14}$$

Note that M_r is triangular; then, for all $k>j\geq j_0$,

$$\begin{aligned}
\frac{1}{M_r\left(\mu_j,\mu_k,t\right)}-1\leq&\left(\frac{1}{M_r\left(\mu_j,\mu_{j+1},t\right)}-1\right)\\
&+\left(\frac{1}{M_r\left(\mu_{j+1},\mu_{j+2},t\right)}-1\right)+\cdots\\
&+\left(\frac{1}{M_r(\mu_{k-1},\mu_k,t)}-1\right)\\
&\leq\left(\gamma^j+\gamma^{j+1}+\cdots+\gamma^{k-1}\right)\left(\frac{1}{M_r(\mu_0,\mu_1,t)}-1\right)\\
&\leq\frac{\gamma^j}{1-\gamma}\left(\frac{1}{M_r(\mu_0,\mu_1,t)}-1\right)\longrightarrow 0,\quad \text{as } j\longrightarrow\infty,
\end{aligned} \tag{15}$$

which yields that (μ_j) is a Cauchy sequence in U. Since $(U,M_r,*)$ is complete, there is $\nu_1\in U$ such that

$$\lim_{j\longrightarrow\infty}M_r\left(\mu_{2j+1},\nu_1,t\right)=1,\quad \text{for } t\gg\theta. \tag{16}$$

Now, we prove that $\hbar\nu_1=\nu_1$. Since M_r is triangular,

$$\begin{aligned}
\frac{1}{M_r(\nu_1,\hbar\nu_1,t)}-1\leq&\left(\frac{1}{M_r\left(\nu_1,\mu_{2j+1},t\right)}-1\right)\\
&+\left(\frac{1}{M_r\left(\mu_{2j+1},\hbar\nu_1,t\right)}-1\right),\quad \text{for } t\gg\theta.
\end{aligned} \tag{17}$$

By (5), (14), and (16), for $t\gg\theta$,

$$\begin{aligned}
\frac{1}{M_r\left(\mu_{2j+1},\hbar\nu_1,t\right)}-1=&\frac{1}{M_r\left(\ell\mu_{2j},\hbar\nu_1,t\right)}-1\leq a\left(\frac{1}{M_r\left(\mu_{2j},\nu_1,t\right)}-1\right)+b\left(\frac{M_r\left(\mu_{2j},\nu_1,t\right)}{M_r\left(\mu_{2j},\hbar\nu_1,2t\right)*M_r\left(\nu_1,\ell\mu_{2j},2t\right)}-1\right)\\
&+c\left(\frac{M_r\left(\mu_{2j},\ell\mu_{2j},t\right)*M_r(\nu_1,\hbar\nu_1,t)}{M_r\left(\mu_{2j},\nu,t\right)*M_r\left(\mu_{2j},\hbar\nu_1,2t\right)*M_r\left(\nu_1,\ell\mu_{2j},2t\right)}-1\right)+d\left(\frac{1}{M_r\left(\mu_{2j},\ell\mu_{2j},t\right)}-1+\frac{1}{M_r(\nu_1,\hbar\nu_1,t)}-1\right)\\
=&a\left(\frac{1}{M_r\left(\mu_{2j},\nu_1,t\right)}-1\right)+b\left(\frac{M_r\left(\mu_{2j},\nu_1,t\right)}{M_r\left(\mu_{2j},\hbar\nu_1,2t\right)*M_r\left(\nu_1,\mu_{2j+1},2t\right)}-1\right)\\
&+c\left(\frac{M_r\left(\mu_{2j},\mu_{2j+1},t\right)*M_r(\nu_1,\hbar\nu_1,t)}{M_r\left(\mu_{2j},\nu_1,t\right)*M_r\left(\mu_{2j},\hbar\nu_1,2t\right)*M_r\left(\nu_1,\mu_{2j+1},2t\right)}-1\right)+d\left(\frac{1}{M_r\left(\mu_{2j},\mu_{2j+1},t\right)}-1+\frac{1}{M_r(\nu_1,\hbar\nu_1,t)}-1\right).
\end{aligned} \tag{18}$$

Again, by Definition 2 (3), $M_r(\mu_{2j}, \hbar v_1, 2t) \geq M_r(\mu_{2j}, v_1, t) * M_r(v_1, \hbar v_1, t)$, for $t \gg \theta$. It follows that

$$\begin{aligned}\frac{1}{M_r\left(\mu_{2j+1}, \hbar v_1, t\right)} - 1 \leq{}& a\left(\frac{1}{M_r\left(\mu_{2j}, v_1, t\right)} - 1\right) + b\left(\frac{M_r\left(\mu_{2j}, v_1, t\right)}{M_r\left(\mu_{2j}, v_1, t\right) * M_r(v_1, \hbar v_1, t) * M_r\left(v_1, \mu_{2j+1}, 2t\right)} - 1\right) \\ &+ c\left(\frac{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right) * M_r(v_1, \hbar v_1, t)}{M_r\left(\mu_{2j}, v_1, t\right) * M_r\left(\mu_{2j}, v_1, t\right) * M_r(v_1, \hbar v_1, t) * M_r\left(v_1, \mu_{2j+1}, 2t\right)} - 1\right) \\ &+ d\left(\frac{1}{M_r\left(\mu_{2j}, \mu_{2j+1}, t\right)} - 1 + \frac{1}{M_r(v_1, \hbar v_1, t)} - 1\right) \longrightarrow (b+d)\left(\frac{1}{M_r(v_1, \hbar v_1, t)} - 1\right), \quad \text{as } j \longrightarrow \infty.\end{aligned} \tag{19}$$

Then,

$$\begin{aligned}\limsup_{j\longrightarrow\infty}\left(\frac{1}{M_r\left(\mu_{2j+1}, \hbar v_1, t\right)} - 1\right) \\ \leq (b+d)\left(\frac{1}{M_r(v_1, \hbar v_1, t)} - 1\right), \quad \text{for } t \gg \theta.\end{aligned} \tag{20}$$

This together with (17) and (16) implies

$$\frac{1}{M_r(v_1, \hbar v_1, t)} - 1 \leq (b+d)\left(\frac{1}{M_r(v_1, \hbar v_1, t)} - 1\right), \quad \text{for } t \gg \theta. \tag{21}$$

Note that $(b+d) < 1$ because $a + b + c + 2d < 1$. Then, $M_r(v_1, \hbar v_1, t) = 1$, that is, $\hbar v_1 = v_1$. Similarly, we can show that $\ell v_1 = v_1$ because M_r is triangular. Therefore,

$$\begin{aligned}\frac{1}{M_r(v_1, \ell v_1, t)} - 1 \leq{}& \left(\frac{1}{M_r\left(v_1, \mu_{2j+2}, t\right)} - 1\right) \\ &+ \left(\frac{1}{M_r\left(\mu_{2j+2}, \ell v_1, t\right)} - 1\right), \quad \text{for } t \gg \theta.\end{aligned} \tag{22}$$

Now, again by (5), (14), and (16), one writes for $t \gg \theta$

$$\begin{aligned}\frac{1}{M_r\left(\mu_{2j+2}, \ell v_1, t\right)} - 1 ={}& \frac{1}{M_r\left(\ell v_1, \hbar \mu_{2j+1}, t\right)} - 1 \leq a\left(\frac{1}{M_r\left(v_1, \mu_{2j+1}, t\right)} - 1\right) + b\left(\frac{M_r\left(v_1, \mu_{2j+1}, t\right)}{M_r\left(v_1, \hbar \mu_{2j+1}, 2t\right) * M_r\left(\mu_{2j+1}, \ell v_1, 2t\right)} - 1\right) \\ &+ c\left(\frac{M_r(v_1, \ell v_1, t) * M_r\left(\mu_{2j+1}, \hbar \mu_{2j+1}, t\right)}{M_r\left(v_1, \mu_{2j+1}, t\right) * M_r\left(v_1, \hbar \mu_{2j+1}, 2t\right) * M_r\left(\mu_{2j+1}, \ell v_1, 2t\right)} - 1\right) + d\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \hbar \mu_{2j+1}, t\right)} - 1\right) \\ ={}& a\left(\frac{1}{M_r\left(v_1, \mu_{2j+1}, t\right)} - 1\right) + b\left(\frac{M_r\left(v_1, \mu_{2j+1}, t\right)}{M_r\left(v_1, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \ell v_1, 2t\right)} - 1\right) \\ &+ c\left(\frac{M_r(v_1, \ell v_1, t) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(v_1, \mu_{2j+1}, t\right) * M_r\left(v_1, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \ell v_1, 2t\right)} - 1\right) + d\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right).\end{aligned} \tag{23}$$

Again, by Definition 2 (3), $M_r(\mu_{2j+1}, \ell\nu_1, 2t) \geq M_r(\mu_{2j+1}, \nu_1, t) * M_r(\nu_1, \ell\nu_1, t)$, for $t \gg \theta$. It follows that

$$\begin{aligned}\frac{1}{M_r\left(\mu_{2j+1}, \ell\nu_1, t\right)} - 1 &\leq a\left(\frac{1}{M_r\left(\nu_1, \mu_{2j+1}, t\right)} - 1\right) + b\left(\frac{M_r\left(\nu_1, \mu_{2j+1}, t\right)}{M_r\left(\nu_1, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \nu_1, t\right) * M_r(\nu_1, \ell\nu_1, t)} - 1\right) \\ &+ c\left(\frac{M_r(\nu_1, \ell\nu_1, t) * M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)}{M_r\left(\nu_1, \mu_{2j+1}, t\right) * M_r\left(\nu_1, \mu_{2j+2}, 2t\right) * M_r\left(\mu_{2j+1}, \nu_1, t\right) * M_r(\nu_1, \ell\nu_1, t)} - 1\right) \\ &+ d\left(\frac{1}{M_r(\nu_1, \ell\nu_1, t)} - 1 + \frac{1}{M_r\left(\mu_{2j+1}, \mu_{2j+2}, t\right)} - 1\right) \longrightarrow (b+d)\left(\frac{1}{M_r(\nu_1, \ell\nu_1, t)} - 1\right), \quad \text{as } j \longrightarrow \infty.\end{aligned} \tag{24}$$

Then,

$$\begin{aligned}&\limsup_{j\longrightarrow\infty}\left(\frac{1}{M_r\left(\mu_{2j+2}, \ell\nu_1, t\right)} - 1\right) \\ &\quad \leq (b+d)\left(\frac{1}{M_r(\nu_1, \ell\nu_1, t)} - 1\right), \quad \text{for } t \gg \theta.\end{aligned} \tag{25}$$

This together with (22) and (16) implies

$$\frac{1}{M_r(\nu_1, \ell\nu_1, t)} - 1 \leq (b+d)\left(\frac{1}{M_r(\nu_1, \ell\nu_1, t)} - 1\right), \quad \text{for } t \gg \theta. \tag{26}$$

Note that $(b+d) < 1$ since $a + b + c + 2d < 1$. Then, $M_r(\nu_1, \ell\nu_1, t) = 1$, that is, $\ell\nu_1 = \nu_1$.

Hence, ν_1 is a common fixed point of ℓ and $\hbar$.

Example 1. Let $U = [0,\infty)$, $*$ be a continuous t-norm and $M_r : U^2 \times (0,\infty) \longrightarrow [0, 1]$ be written as

$$M_r(\mu, \nu, t) = \frac{t}{t + |\mu - \nu|}, \quad \forall \mu, \nu \in U, t \gg \theta. \tag{27}$$

Then, easily one can verify that M_r is triangular and $(U, M_r, *)$ is a complete FCM space. Now, we define $\ell, \hbar : U \longrightarrow U$ by

$$\ell(\mu) = \hbar(\mu) = \begin{cases} \dfrac{3\mu}{8}, & \text{if } \mu \in [0, 1), \\ \dfrac{4\mu}{5} + \dfrac{7}{5}, & \text{if } \mu \in [1,\infty). \end{cases} \tag{28}$$

Then, for $t \gg \theta$, we have

$$\frac{1}{M_r(\ell(\mu), \hbar(\nu), t)} - 1 = \left|\frac{\ell(\mu) - \hbar(\nu)}{t}\right| = \frac{3}{8}\left(\frac{1}{M_r(\mu, \nu, t)} - 1\right). \tag{29}$$

Hence, the pair of self-mapping $(\ell, \hbar)$ is a fuzzy cone-contraction. Now, from Definition 2 (3), $M_r(\mu, \hbar\nu, 2t) \geq M_r(\mu, \nu, t) * M_r(\nu, \hbar\nu, t)$ and $M_r(\nu, \ell\mu, 2t) \geq M_r(\nu, \mu, t) * M_r(\mu, \ell\mu, t)$, for $t \gg \theta$. We get the following:

$$\begin{aligned}&\left(\frac{M_r(\mu, \nu, t)}{M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1\right) \\ &\leq \frac{1}{M_r(\mu, \nu, t) * M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t)} - 1 \\ &= \frac{1}{(t/(t+|\mu-\nu|))(t/(t+|\mu-\ell\mu|))(t/(t+|\nu-\hbar\nu|))} - 1 \\ &= \frac{(t+|\mu-\nu|)(t+|\mu-\ell\mu|)(t+|\nu-\hbar\nu|)}{t^3} - 1 \\ &= \frac{(t+|\mu-\nu|)[(5t/8)(\mu+\nu) + (25/64)\mu\nu] + t^2|\mu-\nu|}{t^3} \\ &= \frac{5(t+|\mu-\nu|)[8t(\mu+\nu) + 5\mu\nu]}{64t^3} + \frac{|\mu-\nu|}{t},\end{aligned}$$

$$\begin{aligned}&\left(\frac{M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t)}{M_r(\mu, \nu, t) * M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1\right) \\ &\leq \frac{1}{(M_r(\mu, \nu, t))^3} - 1 = \frac{1}{(t/(t+|\mu-\nu|))^3} - 1 \\ &= \frac{(t+|\mu-\nu|)^3}{t^3} - 1 = \frac{(|\mu-\nu|)^3 + 3t|\mu-\nu|(t+|\mu-\nu|)}{t^3},\end{aligned}$$

$$\left(\frac{1}{M_r(\mu,\ell\mu,t)}-1+\frac{1}{M_r(\nu,\hbar\nu,t)}-1\right) = \frac{1}{t/(t+|\mu-\ell\mu|)}-1+\frac{1}{t/(t+|\nu-\hbar\nu|)}-1=\frac{5(\mu+\nu)}{8t}. \quad (30)$$

Hence, from the above, we conclude that all the conditions of Theorem 7 are satisfied with $a=3/8$, $b=c=1/6$, and $d=1/8$. The mappings ℓ and $\hbar$ have a common fixed point, i.e., $\ell(7)=\hbar(7)=7\in[0,\infty)$.

Putting $b=0$ in Theorem 7, we get the following corollary.

Corollary 8. *Let $(U,M_r,*)$ be a complete FCM space in which M_r is triangular. Let $\ell,\hbar:U\longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}\frac{1}{M_r(\ell\mu,\hbar\nu,t)}-1 &\le a\left(\frac{1}{M_r(\mu,\nu,t)}-1\right)\\ &+c\left(\frac{M_r(\mu,\ell\mu,t)*M_r(\nu,\hbar\nu,t)}{M_r(\mu,\nu,t)*M_r(\mu,\hbar\nu,2t)*M_r(\nu,\ell\mu,2t)}-1\right)\\ &+d\left(\frac{1}{M_r(\mu,\ell\mu,t)}-1+\frac{1}{M_r(\nu,\hbar\nu,t)}-1\right),\end{aligned} \quad (31)$$

for all $\mu,\nu\in U$, $t\gg\theta$, $a\in(0,1)$, and $c,d\ge0$ with $(a+c+2d)<1$. Then, ℓ and $\hbar$ have a common fixed point in U.

In the following corollary, we prove that the mappings ℓ and $\hbar$ have a unique common fixed point in U by using the constant $c=0$ in Theorem 7.

Corollary 9. *Let $(U,M_r,*)$ be a complete FCM space in which M_r is triangular. Let $\ell,\hbar:U\longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}\frac{1}{M_r(\ell\mu,\hbar\nu,t)}-1 &\le a\left(\frac{1}{M_r(\mu,\nu,t)}-1\right)\\ &+b\left(\frac{M_r(\mu,\nu,t)}{M_r(\mu,\hbar\nu,2t)*M_r(\nu,\ell\mu,2t)}-1\right)\\ &+d\left(\frac{1}{M_r(\mu,\ell\mu,t)}-1+\frac{1}{M_r(\nu,\hbar\nu,t)}-1\right),\end{aligned} \quad (32)$$

for all $\mu,\nu\in U$, $t\gg\theta$, $a\in(0,1)$, and $b,d\ge0$ with $a+b+2d<1$. Hence, ℓ and $\hbar$ have a unique common fixed point in U.

Proof. It follows from the proof of Theorem 7 that v_1 is a common fixed point of ℓ and $\hbar$ in U. For uniqueness, let u_1 be another common fixed point of ℓ and $\hbar$ in U such that $\ell u_1=\hbar u_1=u_1$ and $\ell v_1=\hbar v_1=v_1$. Then, by view of (32),

$$\begin{aligned}\frac{1}{M_r(u_1,v_1,t)}-1 &= \frac{1}{M_r(\ell u_1,\hbar v_1,t)}-1\\ &\le a\left(\frac{1}{M_r(u_1,v_1,t)}-1\right)\\ &+b\left(\frac{M_r(u_1,v_1,t)}{M_r(u_1,\hbar v_1,2t)*M_r(v_1,\ell u_1,2t)}-1\right)\\ &+d\left(\frac{1}{M_r(\mu_1,\ell\mu_1,t)}-1+\frac{1}{M_r(v_1,\hbar v_1,t)}-1\right).\end{aligned} \quad (33)$$

By Definition 2 (3),

$$\begin{aligned}M_r(v_1,\ell u_1,2t) &\ge M_r(v_1,u_1,t)*M_r(u_1,\ell u_1,t)\\ &=M_r(v_1,u_1,t)*1=M_r(v_1,u_1,t), \quad \text{for } t\gg\theta,\end{aligned}$$

$$\begin{aligned}M_r(u_1,\hbar v_1,2t) &\ge M_r(u_1,v_1,t)*M_r(v_1,\hbar v_1,t)\\ &=M_r(u_1,v_1,t)*1=M_r(u_1,v_1,t), \quad \text{for } t\gg\theta.\end{aligned} \quad (34)$$

It follows that

$$\begin{aligned}\frac{1}{M_r(u_1,v_1,t)}-1 &\le a\left(\frac{1}{M_r(u_1,v_1,t)}-1\right)\\ &+b\left(\frac{M_r(u_1,v_1,t)}{M_r(u_1,v_1,t)*M_r(v_1,u_1,t)}-1\right)\\ &+d\left(\frac{1}{M_r(\mu_1,\mu_1,t)}-1+\frac{1}{M_r(v_1,v_1,t)}-1\right)\\ &=(a+b)\left(\frac{1}{M_r(u_1,v_1,t)}-1\right)\\ &=(a+b)\left(\frac{1}{M_r(\ell u_1,\hbar v_1,t)}-1\right)\\ &\le(a+b)^2\left(\frac{1}{M_r(u_1,v_1,t)}-1\right)\le\cdots\\ &\le(a+b)^j\left(\frac{1}{M_r(u_1,v_1,t)}-1\right)\longrightarrow0, \quad \text{as } j\longrightarrow\infty.\end{aligned} \quad (35)$$

Since $a+b<1$, one writes $M_r(u_1,v_1,t)=1$, i.e., $u_1=v_1$ for $t\gg\theta$.

Corollary 10. *Let $(U,M_r,*)$ be a complete FCM space in which M_r is triangular. Let $\ell,\hbar:U\longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}\frac{1}{M_r(\ell\mu,\hbar\nu,t)}-1 &\le a\left(\frac{1}{M_r(\mu,\nu,t)}-1\right)\\ &+d\left(\frac{1}{M_r(\mu,\ell\mu,t)}-1+\frac{1}{M_r(\nu,\hbar\nu,t)}-1\right),\end{aligned} \quad (36)$$

for all $\mu,\nu\in U$, $t\gg\theta$, $a\in(0,1)$, and $d\ge0$ with $a+2d<1$. Then, ℓ and $\hbar$ have a unique common fixed point in U.

Example 2. As in Example 1, let $M_r : U^2 \times (0,\infty) \longrightarrow [0,1]$ be defined by

$$M_r(\mu,\nu,t) = \frac{t}{t + |(\mu-\nu)/2|}, \quad \forall \mu,\nu \in U, t \gg \theta. \tag{37}$$

Then, easily one can verify that M_r is triangular and $(U, M_r, *)$ is a complete FCM space. Now, we define self-mappings $\ell, \hbar : U \longrightarrow U$ by

$$l(\mu) = \begin{cases} \frac{2\mu}{3} + \frac{1}{3}, & \mu \in [0,1], \\ \frac{4\mu}{5} + \frac{8}{5}, & \mu \in (1,\infty), \end{cases} \\ \hbar(\nu) = \begin{cases} \frac{2\nu}{3} + \frac{1}{3}, & \nu \in [0,1], \\ \frac{5\nu}{6} + \frac{4}{3}, & \nu \in (1,\infty). \end{cases} \tag{38}$$

Then, from (36), for $t \gg \theta$, we have

$$\begin{aligned}\left(\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1\right) &= \left|\frac{(\mu-\nu)}{3t}\right| \le \frac{2}{3}\left|\frac{(\mu-\nu)}{2t}\right| + \left|\frac{(\mu+\nu-2)}{42t}\right| \\ &\le \frac{2}{3}\left|\frac{(\mu-\nu)}{2t}\right| + \frac{1}{7}\left|\frac{(\mu+\nu-2)}{6t}\right| \\ &= a\left(\frac{1}{M_r(\mu,\nu,t)} - 1\right) \\ &\quad + d\left(\frac{1}{M_r(\mu,\ell\mu,t)} - 1 + \frac{1}{M_r(\nu,\hbar\nu,t)} - 1\right).\end{aligned} \tag{39}$$

Hence, all the conditions of Corollary 10 are satisfied with $a = 2/3$ and $d = 1/7$. The mappings ℓ and $\hbar$ have a common fixed point, i.e., $\ell(8) = \hbar(8) = 8 \in [0,\infty)$.

Theorem 11. *Let $(U, M_r, *)$ be a complete FCM space in which M_r is triangular. Let $\ell, \hbar : U \longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}&\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 \\ &\quad \le a\left(\frac{1}{M_r(\mu,\nu,t)} - 1\right) \\ &\qquad + b\left(\frac{M_r(\mu,\nu,t)}{M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1\right) \\ &\qquad + c\left(\frac{M_r(\mu,\ell\mu,t) * M_r(\nu,\hbar\nu,t)}{M_r(\mu,\nu,t) * M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1\right) \\ &\qquad + d\left(\frac{1}{M_r(\nu,\ell\mu,t)} - 1 + \frac{1}{M_r(\mu,\hbar\nu,t)} - 1\right),\end{aligned} \tag{40}$$

for all $\mu,\nu \in U$, $t \gg \theta$, $a \in (0,1)$, and $b,c,d \ge 0$ with $a + b + c + 2d < 1$. Then, ℓ and $\hbar$ have a common fixed point in U.

Proof. The proof is similar as the proof of Theorem 7.

Corollary 12. *Let $(U, M_r, *)$ be a complete FCM space in which M_r is triangular. Let $\ell, \hbar : U \longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}&\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 \\ &\quad \le a\left(\frac{1}{M_r(\mu,\nu,t)} - 1\right) \\ &\qquad + c\left(\frac{M_r(\mu,\ell\mu,t) * M_r(\nu,\hbar\nu,t)}{M_r(\mu,\nu,t) * M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1\right) \\ &\qquad + d\left(\frac{1}{M_r(\nu,\ell\mu,t)} - 1 + \frac{1}{M_r(\mu,\hbar\nu,t)} - 1\right),\end{aligned} \tag{41}$$

for all $\mu,\nu \in U$, $t \gg \theta$, $a \in (0,1)$, and $c,d \ge 0$ with $(a + c + 2d) < 1$. Then, ℓ and $\hbar$ have a common fixed point in U.

Corollary 13. *Let $(U, M_r, *)$ be a complete FCM space in which M_r is triangular. Let $\ell, \hbar : U \longrightarrow U$ be a pair of self-mappings so that*

$$\begin{aligned}\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 &\le a\left(\frac{1}{M_r(\mu,\nu,t)} - 1\right) \\ &\quad + b\left(\frac{M_r(\mu,\nu,t)}{M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1\right) \\ &\quad + d\left(\frac{1}{M_r(\nu,\ell\mu,t)} - 1 + \frac{1}{M_r(\mu,\hbar\nu,t)} - 1\right),\end{aligned} \tag{42}$$

$\forall \mu,\nu \in U$, $t \gg \theta$, $a \in (0,1)$, and $b,d \ge 0$ with $(a + b + 2d) < 1$. Then, ℓ and $\hbar$ have a unique common fixed point in U.

Proof. It is as the proof of Theorem 7. Let ν_1 be a common fixed point of ℓ and $\hbar$ in U. Let u_1 be another common fixed point of ℓ and $\hbar$ in U such that $\ell u_1 = \hbar u_1 = u_1$ and $\ell \nu_1 = \hbar \nu_1 = \nu_1$. Then, by view of (42),

$$\begin{aligned}\frac{1}{M_r(u_1,\nu_1,t)} - 1 &= \frac{1}{M_r(\ell u_1,\hbar\nu_1,t)} - 1 \\ &\le a\left(\frac{1}{M_r(u_1,\nu_1,t)} - 1\right) \\ &\quad + b\left(\frac{M_r(u_1,\nu_1,t)}{M_r(u_1,\hbar\nu_1,2t) * M_r(\nu_1,\ell u_1,2t)} - 1\right) \\ &\quad + d\left(\frac{1}{M_r(\nu_1,\ell u_1,t)} - 1 + \frac{1}{M_r(u_1,\hbar\nu_1,t)} - 1\right).\end{aligned} \tag{43}$$

By Definition 2 (3),

$$\begin{aligned}M_r(\nu_1,\ell u_1,2t) &\ge M_r(\nu_1,u_1,t) * M_r(u_1,\ell u_1,t) \\ &= M_r(\nu_1,u_1,t) * 1 = M_r(\nu_1,u_1,t), \quad \text{for } t \gg \theta,\end{aligned}$$

$$M_r(u_1, \hbar v_1, 2t) \geq M_r(u_1, v_1, t) * M_r(v_1, \hbar v_1, t) = M_r(u_1, v_1, t) * 1 = M_r(u_1, v_1, t), \quad \text{for } t \gg \theta. \tag{44}$$

It follows that

$$\begin{aligned}\frac{1}{M_r(u_1, v_1, t)} - 1 &\leq a\left(\frac{1}{M_r(u_1, v_1, t)} - 1\right) + b\left(\frac{M_r(u_1, v_1, t)}{M_r(u_1, v_1, t) * M_r(v_1, u_1, t)} - 1\right) \\ &\quad + d\left(\frac{1}{M_r(v_1, u_1, t)} - 1 + \frac{1}{M_r(u_1, v_1, t)} - 1\right) \\ &= (a + b + 2d)\left(\frac{1}{M_r(u_1, v_1, t)} - 1\right), \quad \text{for } t \gg \theta.\end{aligned} \tag{45}$$

Since $0 < (a + b + 2d) < 1$, $M_r(u_1, v_1, t) = 1$, i.e., $u_1 = v_1$.

Corollary 14. *Let $(U, M_r, *)$ be a complete FCM space in which M_r is triangular. Let $\ell, \hbar : U \longrightarrow U$ be a pair of self-mappings so that*

$$\frac{1}{M_r(\ell\mu, \hbar v, t)} - 1 \leq a\left(\frac{1}{M_r(\mu, v, t)} - 1\right) + d\left(\frac{1}{M_r(v, \ell\mu, t)} - 1 + \frac{1}{M_r(\mu, \hbar v, t)} - 1\right), \tag{46}$$

for all $\mu, v \in U$, $t \gg \theta$, $a \in (0, 1)$, and $d \geq 0$ with $a + 2d < 1$. Then, ℓ and $\hbar$ have a unique common fixed point in U.

Example 3. Let $U = [0, 1]$. As in Example 2, we define self-mappings $\ell, \hbar : U \longrightarrow U$ by

$$\begin{aligned}\ell(\mu) &= \begin{cases} \dfrac{2\mu}{5} + \dfrac{1}{7}, & \mu \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{3\mu}{4} + \dfrac{3}{16}, & \mu \in \left(\dfrac{1}{2}, 1\right], \end{cases} \\ \hbar(v) &= \begin{cases} \dfrac{2v}{5} + \dfrac{1}{7}, & v \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{2v}{3} + \dfrac{1}{4}, & v \in \left(\dfrac{1}{2}, 1\right]. \end{cases}\end{aligned} \tag{47}$$

Now, from (46), for $t \gg \theta$, we have

$$\begin{aligned}\frac{1}{M_r(\ell\mu, \hbar v, t)} - 1 &= \left|\frac{\ell\mu - \hbar v}{2t}\right| = \left|\frac{\mu - v}{5t}\right| \leq \frac{2}{5}\left|\frac{\mu - v}{2t}\right| + \frac{2}{7}\left|\frac{21(\mu + v) - 10}{70t}\right| \\ &\leq \frac{2}{5}\left|\frac{\mu - v}{2t}\right| + \frac{2}{7}\left(\left|\frac{v - (2\mu/5) - (1/7)}{2t}\right| + \left|\frac{\mu - (2v/5) - (1/7)}{2t}\right|\right) \\ &= a\left(\frac{1}{M_r(v, \mu, t)} - 1\right) + d\left(\frac{1}{M_r(v, \ell\mu, t)} - 1 + \frac{1}{M_r(\mu, \hbar v, t)} - 1\right).\end{aligned} \tag{48}$$

Hence, all the conditions of Corollary 14 are satisfied with $a = 2/5$ and $d = 2/7$. The mappings ℓ and $\hbar$ have a common fixed point, i.e., $\ell(3/4) = \hbar(3/4) = 3/4 \in [0, 1]$.

4. Application

In this section, we present an application on Fredholm integral equations. Let $U = C([0, \eta], \mathbb{R})$ be the space of all $\mathbb{R}$-valued continuous functions on the interval $[0, \eta]$, where $0 < \eta \in \mathbb{R}$. The Fredholm integral equations are

$$\begin{aligned}\mu(\tau) &= \int_0^\eta K_1(\tau, v, \mu(v))dv, \\ v(\tau) &= \int_0^\eta K_2(\tau, v, v(v))dv, \\ &\forall \mu, v \in U,\end{aligned} \tag{49}$$

where $\tau \in [0, \eta]$ and $K_1, K_2 : [0, \eta] \times [0, \eta] \times \mathbb{R} \longrightarrow \mathbb{R}$. The induced metric $d : U^2 \longrightarrow \mathbb{R}$ be defined as

$$d(\mu, v) = \sup_{\tau \in [0,\eta]} |\mu(\tau) - v(\tau)| = \|\mu - v\|, \quad \text{where } \mu, v \in C([0, \eta], \mathbb{R}) = U. \tag{50}$$

The binary operation $*$ is defined by $\alpha * \lambda = \alpha\lambda$ for all $\alpha, \lambda \in [0, \eta]$. A standard fuzzy metric $M_r : U^2 \times (0,\infty) \longrightarrow [0, 1]$ is given as

$$M_r(\mu, v, t) = \frac{t}{t + d(\mu, v)}, \quad \text{for } t > 0, \forall \mu, v \in U. \tag{51}$$

Then, easily one can verify that M_r is triangular and $(U, M_r, *)$ is a complete FCM space.

Theorem 15. *The two FIEs are*

$$\mu(\tau) = \int_0^\eta K_1(\tau, \nu, x(\nu))d\nu,$$
$$\nu(\tau) = \int_0^\eta K_2(\tau, \nu, \nu(\nu))d\nu, \quad (52)$$

where $\tau \in [0, 1]$ and $\mu, \nu \in U$. Assume that $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are such that $A_\mu, B_\nu \in \mathbf{E}$ for every $\mu, \nu \in \mathbf{E}$, where

$$A_\mu(\tau) = \int_0^\eta K_1(\tau, \nu, \mu(\nu))d\nu,$$
$$B_\nu(\tau) = \int_0^\eta K_2(\tau, \nu, \nu(\nu))d\nu. \quad (53)$$

If there exists $\beta \in (0, 1)$ such that for all $\mu, \nu \in U$,

$$\left\|A_\mu - B_\nu\right\| \le \beta N(\ell, \hbar, \mu, \nu), \quad (54)$$

where

$$N(\ell, \hbar, \mu, \nu) = \max\left\{\begin{array}{c} \|\mu-\nu\|, \left\|\mu - A_\mu\right\| + \|\nu - B_\nu\|, \dfrac{1}{t^2}\left(3t\|\mu-\nu\|^2 + \|\mu-\nu\|^3\right), \\ \dfrac{1}{t^2}\left(t + \|\mu-\nu\|\right)\left(t\left\|\mu - A_\mu\right\| + t\|\nu - B_\nu\| + \left\|\mu - A_\mu\right\| \cdot \|\nu - B_\nu\|\right) \end{array}\right\}. \quad (55)$$

Then, the two FIEs defined in (49) have a common solution in U.

Proof. Define the mappings $\ell, \hbar : \mathbf{E} \longrightarrow \mathbf{E}$ by

$$\ell(\mu) = A_\mu,$$
$$\hbar(\nu) = B_\nu. \quad (56)$$

The FIEs in (49) have a common solution if and only if ℓ and $\hbar$ have a common fixed point in U. Now, we have to show that Theorem 7 is applied to the integral operators ℓ and $\hbar$. Then, for all $\mu, \nu \in U$, we have the following four cases.

(a) If $N(\ell, \hbar, \mu, \nu) = \|\mu - \nu\|$ in (55), then from (51) and (54), we have

$$\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 = \frac{d(\ell\mu, \hbar\nu)}{t} \le \beta\frac{N(\ell, \hbar, \mu, \nu)}{t} = \beta\frac{\|\mu-\nu\|}{t} = \beta\left(\frac{1}{M_r(\mu, \nu, t)} - 1\right). \quad (57)$$

This implies that

$$\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 \le \beta\left(\frac{1}{M_r(\mu, \nu, t)} - 1\right), \quad \text{for } t \gg \theta, \quad (58)$$

for all $\mu, \nu \in U$ such that $\ell\mu \neq \hbar\nu$. It is obvious that the inequality (58) holds if $\ell\mu = \hbar\nu$. Thus, the integral operators ℓ and $\hbar$ satisfy all the conditions of Theorem 7 with $\beta = a$ and $b = c = d = 0$ in (5). The integral operators ℓ and $\hbar$ have a common fixed point, i.e., (49) has a common solution in U.

(b) If $N(\ell, \hbar, \mu, \nu) = (1/t^2)(t + \|\mu - \nu\|)(t\|\mu - A_\mu\| + t\|\nu - B_\nu\| + \|\mu - A_\mu\| \cdot \|\nu - B_\nu\|)$ in (55), then from (51) and (54), we have

$$\begin{aligned}\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 &= \frac{d(\ell\mu, \hbar\nu)}{t} \\ &\le \beta\frac{N(\ell, \hbar, \mu, \nu)}{t} \\ &= \beta\frac{1}{t^3}\left(t + \|\mu-\nu\|\right)\left(t\left\|\mu - A_\mu\right\| \right. \\ &\quad \left. + t\|\nu - B_\nu\| + \left\|\mu - A_\mu\right\| \cdot \|\nu - B_\nu\|\right).\end{aligned} \quad (59)$$

It yields that

$$\begin{aligned}\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 &\le \beta\frac{1}{t^3}\left(t + \|\mu-\nu\|\right)\left(t\left\|\mu - A_\mu\right\| \right. \\ &\quad \left. + t\|\nu - B_\nu\| + \left\|\mu - A_\mu\right\| \cdot \|\nu - B_\nu\|\right),\end{aligned} \quad (60)$$

for all $\mu, \nu \in U$ and for $t \gg \theta$. Here, we simplify the term $((M_r(\mu, \nu, t))/(M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t))) - 1$ by using Definition 2 (3) and (51), for $t \gg \theta$, we have

$$\begin{aligned}
\frac{M_r(\mu,\nu,t)}{M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1 &\le \frac{M_r(\mu,\nu,t)}{M_r(\mu,\nu,t) * M_r(\nu,\hbar\nu,t) * M_r(\nu,\mu,t) * M_r(\mu,\ell\mu,t)} - 1 \\
&= \frac{1}{M_r(\mu,\nu,t) * M_r(\nu,\hbar\nu,t) * M_r(\mu,\ell\mu,t)} - 1 \\
&= \frac{(t+\|\mu-\nu\|)(t+\|\nu-\hbar\nu\|)(t+\|\mu-\ell\mu\|)}{t^3} - 1 \\
&= \frac{1}{t^3}\begin{pmatrix} t^2\|\mu-\nu\| + t^2(\|\nu-\hbar\nu\| + \|\mu-\ell\mu\|) + t\|\mu-\nu\|(\|\nu-\hbar\nu\| + \|\mu-\ell\mu\|) \\ t\|\mu-\ell\mu\|\cdot\|\nu-\hbar\nu\| + \|\mu-\nu\|\cdot\|\mu-\ell\mu\|\cdot\|\nu-\hbar\nu\| \end{pmatrix} \\
&= \frac{1}{t^3}\left(t^2\|\mu-\nu\| + (t+\|\mu-\nu\|)(t\|\nu-\hbar\nu\| + t\|\mu-\ell\mu\| + \|\mu-\ell\mu\|\cdot\|\nu-\hbar\nu\|)\right).
\end{aligned} \tag{61}$$

This implies that

$$\begin{aligned}
&\frac{M_r(\mu,\nu,t)}{M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1 \\
&\quad \le \frac{1}{t^3}\left(t^2\|\mu-\nu\| + (t+\|\mu-\nu\|)\right. \\
&\quad \left.\cdot(t\|\nu-\hbar\nu\| + t\|\mu-\ell\mu\| + \|\mu-\ell\mu\|\cdot\|\nu-\hbar\nu\|)\right),
\end{aligned} \tag{62}$$

for all $\mu, \nu \in U$ and for $t \gg \theta$. Now, from (60) and (62), we have

$$\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 \le \beta\left(\frac{M_r(\mu,\nu,t)}{M_r(\mu,\ell\mu,t) * M_r(\nu,\hbar\nu,2t)} - 1\right), \quad \text{for } t \gg \theta, \tag{63}$$

for all $\mu, \nu \in U$ such that $\ell\mu \ne \hbar\nu$. It is obvious that the inequality (63) holds if $\ell\mu = \hbar\nu$. Thus, the integral operators ℓ and $\hbar$ satisfy all the conditions of Theorem 7 with $\beta = b$ and $a = c = d = 0$ in (5). The integral operators ℓ and $\hbar$ have a common fixed point, i.e., (49) has a common solution in U.

(c) If $N(\ell, \hbar, \mu, \nu) = (1/t^2)(3t\|\mu-\nu\|^2 + \|\mu-\nu\|^3)$ in (55), then from (51) and (54), we have

$$\begin{aligned}
\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 = \frac{d(\ell\mu,\hbar\nu)}{t} &\le \beta\frac{N(\ell,\hbar,\mu,\nu)}{t} \\
&= \beta\frac{3t\|\mu-\nu\|^2 + \|\mu-\nu\|^3}{t^3}.
\end{aligned} \tag{64}$$

This implies

$$\frac{1}{M_r(\ell\mu,\hbar\nu,t)} - 1 \le \beta\frac{3t\|\mu-\nu\|^2 + \|\mu-\nu\|^3}{t^3}, \tag{65}$$

for all $\mu, \nu \in U$ and for $t \gg \theta$. Here, we simplify the term $((M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t))/(M_r(\mu, \nu, t) * M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t))) - 1$, by Definition 2 (3). For $t \gg \theta$, we have

$$\begin{aligned}
\frac{M_r(\mu,\ell\mu,t) * M_r(\nu,\hbar\nu,t)}{M_r(\mu,\nu,t) * M_r(\mu,\hbar\nu,2t) * M_r(\nu,\ell\mu,2t)} - 1 &\le \frac{M_r(\mu,\ell\mu,t) * M_r(\nu,\hbar\nu,t)}{M_r(\mu,\nu,t) * M_r(\mu,\nu,t) * M_r(\nu,\hbar\nu,t) * M_r(\nu,\mu,t) * M_r(\mu,\ell\mu,t)} - 1 \\
&= \frac{1}{M_r(\mu,\nu,t) * M_r(\mu,\nu,t) * M_r(\nu,\mu,t)} - 1.
\end{aligned} \tag{66}$$

In view of (51) and after routine calculation, we get

$$\frac{M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t)}{M_r(\mu, \nu, t) * M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1 \leq \frac{\left(3t^2\|\mu - \nu\| + 3t\|\mu - \nu\|^2 + \|\mu - \nu\|^3\right)}{t^2}, \tag{67}$$

for $t \gg \theta$. Now, from (65) and (67), we have

$$\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 \leq \beta\left(\frac{M_r(\mu, \ell\mu, t) * M_r(\nu, \hbar\nu, t)}{M_r(\mu, \nu, t) * M_r(\mu, \hbar\nu, 2t) * M_r(\nu, \ell\mu, 2t)} - 1\right), \quad \text{for } t \gg \theta, \tag{68}$$

for all $\mu, \nu \in U$ such that $\ell\mu \neq \hbar\nu$. It is obvious that the inequality (68) holds if $\ell\mu = \hbar\nu$. Thus, the integral operators ℓ and $\hbar$ satisfy all the conditions of Theorem 7 with $\beta = c$ and $a = b = d = 0$ in (5). The integral operators ℓ and $\hbar$ have a common fixed point, i.e., (49) has a common solution in U.

(d) If $N(\ell, \hbar, \mu, \nu) = \|\mu - A_\mu\| + \|\nu - B_\nu\|$ in (55), then from (51) and (54), we have

$$\begin{aligned}\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 &= \frac{d(\ell\mu, \hbar\nu)}{t} \leq \beta\frac{N(\ell, \hbar, \mu, \nu)}{t} \\ &= \beta\frac{\|\mu - A_\mu\| + \|\nu - B_\nu\|}{t} \\ &= \beta\left(\frac{1}{M_r(\mu, \ell\mu, t)} - 1 + \frac{1}{M_r(\nu, \hbar\nu, t)} - 1\right).\end{aligned} \tag{69}$$

This implies that

$$\frac{1}{M_r(\ell\mu, \hbar\nu, t)} - 1 \leq \beta\left(\frac{1}{M_r(\mu, \ell\mu, t)} - 1 + \frac{1}{M_r(\nu, \hbar\nu, t)} - 1\right), \quad \text{for } t \gg \theta, \tag{70}$$

for all $\mu, \nu \in U$ such that $\ell\mu \neq \hbar\nu$. It is obvious that the inequality (70) holds if $\ell\mu = \hbar\nu$. Thus, the integral operators ℓ and $\hbar$ satisfy all the conditions of Theorem 7 with $\beta = d$ and $a = b = c = 0$ in (5). The integral operators ℓ and $\hbar$ have a common fixed point, i.e., (49) has a common solution in U.

5. Conclusion

In this paper, we presented the concept of rational-type fuzzy cone contractions in FCM spaces and some common fixed point results under generalized rational-type fuzzy cone-contraction conditions in complete FCM spaces by using the "triangular property of fuzzy cone metric" as a basic tool. Moreover, we resolved some Fredholm integral equations as an application. So, one can use this concept to prove more rational-type fuzzy cone-contraction results in complete FCM spaces with different types of applications.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analysed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The authors have equally contributed to the final manuscript.

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Research Article

On Unique and Nonunique Fixed Points in Metric Spaces and Application to Chemical Sciences

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We introduce the notions of a generalized Θ-contraction, a generalized $\Theta_{\mathscr{E}}$-weak contraction, a $\Psi_{\mathscr{E}}$-weak JS-contraction, an integral-type $\Theta_{\mathscr{E}}$-weak contraction, and an integral-type $\Psi_{\mathscr{E}}$-weak JS-contraction to establish the fixed point, fixed ellipse, and fixed elliptic disc theorems. Further, we verify these by illustrative examples with geometric interpretations to demonstrate the authenticity of the postulates. The motivation of this work is the fact that the set of nonunique fixed points may include a geometric figure like a circle, an ellipse, a disc, or an elliptic disc. Towards the end, we provide an application of Θ-contraction to chemical sciences.

1. Introduction and Preliminaries

The study of the geometry of the set of nonunique fixed points of a map is a significant area of research. There are numerous examples of a map where the set of nonunique fixed points of the self-map includes some geometric shapes. For example, consider a self-map $\mathscr{M}$ on the metric space ($\mathscr{U}, d$) with the usual metric defined on the two-dimensional plane $\mathbb{R}^2$ as

$$\mathscr{M}(\mathfrak{u},\mathfrak{v}) = \begin{cases} (\mathfrak{u},\mathfrak{v}), & (\mathfrak{u},\mathfrak{v}) \in \mathfrak{u}^2+\mathfrak{v}^2=1, \\ (1,0), & \text{otherwise}. \end{cases} \quad (1)$$

Noticeably, the set of nonunique fixed points $\{(\cos n\theta, \sin n\theta): n \in \mathbb{Z}, \theta \in [0, 2\pi)\}$ includes the circle $\mathscr{C}((0,0),1)$ centered at $(0,0)$ having radius 1; that is, $\mathscr{C}((0,0),1)$ is a fixed circle of $\mathscr{M}$. It is significant to mention that there exist maps that map the circle $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$ to itself but do not fix all the points of the circle $\mathscr{C}(\mathfrak{u}_0,\mathfrak{r})$. For example, let $\mathscr{M}$ be a self-map on the two-dimensional plane $\mathbb{R}^2$ defined by

$$\mathscr{M}(\mathfrak{u},\mathfrak{v}) = \left(\frac{u}{\mathfrak{u}^2+\mathfrak{v}^2}, \frac{v}{\mathfrak{u}^2+\mathfrak{v}^2}\right), \mathfrak{u},\mathfrak{v} \in \mathbb{R}. \quad (2)$$

Then, $\mathscr{M}\mathscr{C}(0,1) = \mathscr{C}(0,1)$, but map $\mathscr{M}$ fixes only two points $(1,0)$ and $(-1,0)$ of the circle $\mathscr{C}(0,1)$. Noticeably, $\mathscr{M}$ does not fix all the points of $\mathscr{C}(0,1)$. For details on this work, one may refer to [1–23] and the references therein. A geometric figure (a circle, a disc, an ellipse, and so on) included in the set of nonunique fixed points is called a fixed figure (a fixed circle, a fixed disc, a fixed ellipse, and so on) of the self-map [15].

The aim of the present work is to introduce notions of a generalized Θ-contraction, a generalized $\Theta_{\mathscr{E}}$-weak contraction, a $\Psi_{\mathscr{E}}$-weak JS-contraction, a generalized integral-type $\Theta_{\mathscr{E}}$-weak contraction, and an integral-type $\Psi_{\mathscr{E}}$-weak JS-contraction to study the geometry of nonunique fixed points. In the sequel, we establish the fixed point, fixed ellipse, and fixed elliptic disc theorems. Further, we verify these by illustrative examples to demonstrate the authenticity of the postulates. Further, we provide an application of Ćirić-type Θ

-contraction to obtain an application to chemical sciences. To be specific, we solve a boundary value problem arising when a diffusing material is kept in an absorbing medium between parallel walls of specified concentrations.

Definition 1 [24]. A metric on a nonempty set $\mathscr{U}$ is a function $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ satisfying

(i) $d(\mathfrak{u}, \mathfrak{v}) = 0$ iff $\mathfrak{u} = \mathfrak{v}$

(ii) $d(\mathfrak{u}, \mathfrak{v}) = d(\mathfrak{v}, \mathfrak{u})$

(iii) $d(\mathfrak{u}, \mathfrak{v}) \leq d(\mathfrak{u}, \mathfrak{w}) + d(\mathfrak{w}, \mathfrak{v})$, $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathscr{U}$

Definition 2 [5]. An ellipse having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in a metric space $(\mathscr{U}, d)$ is defined as

$$\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \{\mathfrak{u} \in \mathscr{U} : d(\mathfrak{c}_1, \mathfrak{u}) + d(\mathfrak{c}_2, \mathfrak{u}) = \mathfrak{a}, \mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{U}, \mathfrak{a} \in [0,\infty)\}. \tag{3}$$

The midpoint $\mathscr{C}$ of a line $\mathfrak{c}_1\mathfrak{c}_2$ is known as a center of an ellipse. Here, the segment of length $\mathfrak{a}$ on line $\mathfrak{c}_1\mathfrak{c}_2$ is the major axis, the line perpendicular to it through the center is the minor axis, and $\mathfrak{a}/2$ is the length of a semimajor axis of an ellipse. The distance $f = (1/2)d(\mathfrak{c}_1, \mathfrak{c}_2)$ is the linear eccentricity, and the ratio of linear eccentricity and semimajor axis is the eccentricity; that is, $e = d(\mathfrak{c}_1, \mathfrak{c}_2)/\mathfrak{a}$. Visibly, the circles are the ellipses of vanishing eccentricity in which both the focal points are the same; that is, $d(\mathfrak{c}_1, \mathfrak{c}_2) = 0$. Actually, an ellipse is a compressed circle. Generally, eccentricity is the measure of the deviation of the curve from the circularity of the particular shape.

Example 1. Let $\mathscr{U} = \mathbb{R}$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}|$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$; then,

$$\begin{aligned}\mathscr{E}(5,10,8) &= \{\mathfrak{u} \in \mathscr{M} : d(5, \mathfrak{u}) + d(10, \mathfrak{u}) = 8\} \\ &= \{\mathfrak{u} \in \mathscr{U} : |5 - \mathfrak{u}| + |10 - \mathfrak{u}| = 8\} = \{3.5,11.5\}.\end{aligned} \tag{4}$$

That is, an ellipse centered at 7.5 having foci at 5 and 10 is $\{3.5,11.5\}$.

Definition 3 [25]. Let Ω symbolize the class of functions $\Theta : (0,\infty) \longrightarrow (1,\infty)$ such that the subsequent conditions hold:

(Θ_1): Θ is nondecreasing;

(Θ_2): for every sequence $\{\mathfrak{u}_n\} \subseteq (0,\infty)$, $\lim_{n\longrightarrow\infty}\Theta\mathfrak{u}_n = 1 \Leftrightarrow \lim_{n\longrightarrow\infty}\mathfrak{u}_n = 0^+$;

(Θ_3): there exist $\alpha \in (0, 1)$ and $\beta \in (0,\infty]$ such that $\lim_{\mathfrak{u}\longrightarrow 0^+}(\Theta(\mathfrak{u}) - 1)/\mathfrak{u}^\alpha = \beta$.

Definition 4 [5]. Let $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in a metric space $(\mathscr{U}, d)$. Then, $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is said to be a fixed ellipse of $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ if $\mathscr{M}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, $\mathfrak{a} \in [0,\infty)$.

2. Main Results

In this section, we are dealing with maps satisfying some novel contractions which fix one element of the space or more than one element of the space under suitable conditions and a set of nonunique fixed points, including some geometrical shapes, may be either an ellipse or an elliptic disc. First, we define a generalized Θ-contraction to establish a unique fixed point by giving a short and simple proof.

Definition 5. Let $\Theta : (0,\infty) \longrightarrow (1,\infty) \in \Omega$, and the map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ of a metric space $(\mathscr{U}, d)$ is said to be a generalized Θ-contraction with $\mathfrak{u} \neq \mathfrak{v}$ if

$$d(\mathscr{M}\mathfrak{u}, \mathscr{M}\mathfrak{v}) > 0 \Rightarrow \Theta(d(\mathscr{M}\mathfrak{u}, \mathscr{M}\mathfrak{v})) \leq [\Theta(\mathscr{L}(\mathfrak{u}, \mathfrak{v}))]^\alpha, \tag{5}$$

where $\mathscr{L}(\mathfrak{u}, \mathfrak{v}) = \max\{d(\mathfrak{u}, \mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + (1-\gamma)d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), (1-\gamma)d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + (1-\gamma)d(\mathfrak{v}, \mathscr{M}\mathfrak{u}), (1-\gamma)d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{u})\}$, $\gamma \in [0, 1)$, $\alpha \in (0, 1)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$.

Remark 6. In the above contraction, if $\gamma = 0$, then $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ is a Ćirić-type Θ-contraction.

Theorem 7. *Let $(\mathscr{U}, d)$ be a complete metric space and map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ be a continuous generalized Θ-contraction. Then, $\mathscr{M}$ has a unique fixed point. Also, the sequence of iterates $\{\mathscr{M}^n\mathfrak{u}\}$ converges to a fixed point of $\mathscr{M}$ in $\mathscr{U}$.*

Proof. Define a Picard sequence $\{\mathfrak{u}_n\} \subseteq \mathscr{U}$, $\mathfrak{u}_{n+1} = \mathscr{M}\mathfrak{u}_n$, $n \in \mathbb{N}_0$, with initial point $\mathfrak{u}_0 \in \mathscr{U}$. If for some $n \in \mathbb{N}$, $\mathscr{M}^n\mathfrak{u} = \mathscr{M}^{n+1}\mathfrak{u}$, then $\mathscr{M}^n\mathfrak{u}$ is a fixed point of $\mathscr{M}$ and the proof is complete. So, presume that for each n, $d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}) > 0$; then,

$$\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) \leq \Theta\left(\mathscr{L}\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right)\right), \tag{6}$$

where

$$\begin{aligned}\mathscr{L}\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) &= \max\left\{d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right), \gamma d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right)\right. \\ &\quad + (1-\gamma)d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right), (1-\gamma)d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) \\ &\quad + \gamma d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right), \gamma d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right) \\ &\quad + (1-\gamma)d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^n\mathfrak{u}), (1-\gamma)d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right) \\ &\quad \left.+ \gamma d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^n\mathfrak{u})\right\} \\ &= \max\left\{d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right), \gamma d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right)\right. \\ &\quad + (1-\gamma)d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right), (1-\gamma)d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) \\ &\quad + \gamma d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right), \gamma d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right), \\ &\quad \left.\cdot (1-\gamma)d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right\}.\end{aligned} \tag{7}$$

Case 1. If $d(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}) \leq d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})$, then

$$\mathscr{L}\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) = d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right). \tag{8}$$

That is, $\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})) \leq [\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}))]^\alpha$, $\alpha \in (0, 1)$, a contradiction.

Case 2. If $d(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}) \geq d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})$, then

$$\mathscr{L}\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) = d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right). \tag{9}$$

That is, $\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})) \leq [\Theta(d(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}))]^\alpha$.

Following a similar pattern,

$$\begin{aligned}\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) &\leq \left[\Theta\left(d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right)\right)\right]^\alpha \cdots \\ &\leq [\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))]^{\alpha^n} \longrightarrow 1, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{10}$$

Using (Θ_2), $\lim_{n\longrightarrow\infty} d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 0$.

Using (Θ_3), there exist $\beta \in (0,\infty)$ such that $\lim_{n\longrightarrow\infty}(\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})) - 1)/(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}))^\alpha = \beta$.

If $\beta \in (0,\infty)$, then for $\varepsilon_1 = \beta/4 > 0$, there exists $N_1 > 0$ such that

$$\begin{aligned}&\left|\frac{\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) - 1}{\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right)^\alpha} - \beta\right| < \varepsilon, n \geq N_1, \\ &\Rightarrow \frac{\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) - 1}{\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right)^\alpha} > \beta - \varepsilon_1 \\ &= \frac{3}{4}\beta > \varepsilon_1, n \geq N_1.\end{aligned} \tag{11}$$

That is, $(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}))^\alpha < (1/\varepsilon_1)(\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})) - 1), n \geq N_1$.

If $\beta = \infty$, then for any $\varepsilon_2 > 0$, there exists $N_2 > 0$ such that

$$\frac{\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) - 1}{\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right)^\alpha} > \varepsilon_2, n \geq N_2. \tag{12}$$

That is, $(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}))^\alpha < (1/\varepsilon_2)(\Theta(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u})) - 1), n > N_2$.

Thus, for all $\beta \in (0,\infty]$ and $\mu = \max\{1/\varepsilon_1, 1/\varepsilon_2\}$, there exists $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned}\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right)^\alpha &< \mu\left(\Theta\left(d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right)\right) - 1\right), n > N, \\ &\leq \mu[\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))] - 1 \longrightarrow 0, \text{ as } n \longrightarrow \infty(\text{using } (\Theta_2)).\end{aligned} \tag{13}$$

That is, $\lim_{n\longrightarrow\infty}(d(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}))^\alpha = 0$ implies that there exists $n \geq N$ such that

$$d\left(\mathscr{M}^n\mathfrak{u}, \mathscr{M}^{n+1}\mathfrak{u}\right) \leq \frac{1}{n^{1/\alpha}}, n \geq N. \tag{14}$$

If $n > m$,

$$\begin{aligned}d(\mathscr{M}^m\mathfrak{u}, \mathscr{M}^n\mathfrak{u}) &\leq d\left(\mathscr{M}^m\mathfrak{u}, \mathscr{M}^{m+1}\mathfrak{u}\right) + d\left(\mathscr{M}^{m+1}\mathfrak{u}, \mathscr{M}^{m+2}\mathfrak{u}\right) \\ &\quad + \cdots + d\left(\mathscr{M}^{n-1}\mathfrak{u}, \mathscr{M}^n\mathfrak{u}\right) \\ &\leq \frac{1}{m^{1/\alpha}} + \frac{1}{(m+1)^{1/\alpha}} + \cdots + \frac{1}{(n-1)^{1/\alpha}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/\alpha}}.\end{aligned} \tag{15}$$

Since $\alpha \in (0, 1)$, series $\Sigma_{i=n}^{\infty} 1/i^{1/\alpha}$ is convergent and $\lim_{n,m\longrightarrow\infty} d(\mathscr{M}^m\mathfrak{u}, \mathscr{M}^n\mathfrak{u})$ exists and is finite; that is, $\{\mathscr{M}^n\mathfrak{u}\}$ is a Cauchy sequence.

Since $\mathscr{U}$ is complete, $\{\mathscr{M}^n\mathfrak{u}\}$ converges to $\mathfrak{u}^* \in \mathscr{U}$. Since $\mathscr{M}$ is continuous, $\{\mathscr{M}^n\mathfrak{u}\} \longrightarrow \mathfrak{u}^* \Rightarrow \{\mathscr{M}^{n+1}\mathfrak{u}\} \longrightarrow \mathscr{M}\mathfrak{u}^*$. By definition of limit $\mathscr{M}\mathfrak{u}^* = \mathfrak{u}^*$, that is, $\mathfrak{u}^*$ is a fixed point of $\mathscr{M}$.

Let $\mathfrak{w}^*$ be another fixed point of $\mathscr{U}$. So $d(\mathscr{M}\mathfrak{u}^*, \mathscr{M}\mathfrak{w}^*) = d(\mathfrak{u}^*, \mathfrak{w}^*) > 0$. Now,

$$\Theta(d(\mathscr{M}\mathfrak{u}^*, \mathscr{M}\mathfrak{w}^*)) \leq [\Theta(\mathscr{L}(\mathfrak{u}^*, \mathfrak{w}^*))]^\alpha, \tag{16}$$

where

$$\begin{aligned}\mathscr{L}(\mathfrak{u}^*, \mathfrak{w}^*) &= \max\{d(\mathfrak{u}^*, \mathfrak{w}^*), \gamma d(\mathfrak{u}^*, \mathscr{M}\mathfrak{u}^*) + (1-\gamma)d(\mathfrak{w}^*, \mathscr{M}\mathfrak{w}^*), \\ &\quad \cdot (1-\gamma)d(\mathfrak{u}^*, \mathscr{M}\mathfrak{u}^*) + \gamma d(\mathfrak{w}^*, \mathscr{M}\mathfrak{w}^*), \gamma d(\mathfrak{u}^*, \mathscr{M}\mathfrak{w}^*) \\ &\quad + (1-\gamma)d(\mathfrak{w}^*, \mathscr{M}\mathfrak{u}^*), (1-\gamma)d(\mathfrak{u}^*, \mathscr{M}\mathfrak{w}^*) + \gamma d(\mathfrak{w}^*, \mathscr{M}\mathfrak{u}^*)\} \\ &= \max\{d(\mathfrak{u}^*, \mathfrak{w}^*), \gamma d(\mathfrak{u}^*, \mathfrak{u}^*) + (1-\gamma)d(\mathfrak{w}^*, \mathfrak{w}^*), (1-\gamma)d(\mathfrak{u}^*, \mathfrak{u}^*) \\ &\quad + \gamma d(\mathfrak{w}^*, \mathfrak{w}^*), \gamma d(\mathfrak{u}^*, \mathfrak{w}^*) + (1-\gamma)d(\mathfrak{w}^*, \mathfrak{u}^*), (1-\gamma)d(\mathfrak{u}^*, \mathfrak{w}^*) \\ &\quad + \gamma d(\mathfrak{w}^*, \mathfrak{u}^*)\} \\ &= \max\{d(\mathfrak{u}^*, \mathfrak{w}^*), \gamma d(\mathfrak{u}^*, \mathfrak{w}^*) + (1-\gamma)d(\mathfrak{w}^*, \mathfrak{u}^*), (1-\gamma)d(\mathfrak{u}^*, \mathfrak{w}^*) \\ &\quad + \gamma d(\mathfrak{w}^*, \mathfrak{u}^*)\} = d(\mathfrak{u}^*, \mathfrak{w}^*).\end{aligned} \tag{17}$$

That is, $\Theta(d(\mathscr{M}\mathfrak{u}^*, \mathscr{M}\mathfrak{w}^*)) \leq [\Theta(d(\mathfrak{u}^*, \mathfrak{w}^*))]^\alpha \leq \Theta(d(\mathfrak{u}^*, \mathfrak{w}^*))$.

That is, $\Theta(d(\mathfrak{u}^*, \mathfrak{w}^*)) \leq \Theta(d(\mathfrak{u}^*, \mathfrak{w}^*))$, a contradiction.

Hence, $\mathscr{M}$ has a unique fixed point in $\mathscr{U}$.

Theorem 8. *Let $(\mathscr{U}, d)$ be a complete metric space and map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ be a continuous Ćirić-type Θ-contraction. Then, $\mathscr{M}$ has a unique fixed point. Also, the sequence of iterates $\{\mathscr{M}^n\mathfrak{u}\}$ converges to a fixed point of $\mathscr{M}$ in $\mathscr{U}$.*

Proof. The proof follows the pattern of Theorem 7 on taking $\gamma = 0$.

The subsequent example appreciates that Theorem 8 gives assurance of the uniqueness of the fixed point.

Example 2. Let $\mathscr{U} = \{\mathfrak{u}_n = 2n - 1 : n \in \mathbb{N}\}$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}|$. Then, $(\mathscr{U}, d)$ is a complete metric space.

Let $\Theta(t) = e^{te^t} \in \Omega$, $\gamma = 0$.

Define a self-map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ as

$$\mathscr{M}\mathfrak{u} = \begin{cases} \mathfrak{u}_1, & \mathfrak{u} = \mathfrak{u}_1, \\ \mathfrak{u}_{n-1}, & \mathfrak{u} = \mathfrak{u}_n, n \geq 2. \end{cases} \tag{18}$$

Then,

$$\begin{aligned} \mathscr{L}(\mathfrak{u}_n, \mathfrak{u}_1) &= \max\{d(\mathfrak{u}_n, \mathfrak{u}_1), d(\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_n), d(\mathfrak{u}_1, \mathscr{M}\mathfrak{u}_1), d(\mathfrak{u}_1, \mathscr{M}\mathfrak{u}_n), d(\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_1)\} \\ &= \max\{d(2n-1, 1), d(2n-1, 2n-3), 0, d(1, 2n-3), d(2n-1, 1)\} \\ &= \max\{|2n-2|, |2|, |2n-4|\} = 2n-2, n \geq 2. \end{aligned} \tag{19}$$

Now,

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \frac{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_1)}{\mathscr{L}(\mathfrak{u}_n, \mathfrak{u}_1)} \\ &= \lim_{n\longrightarrow\infty} \frac{|\mathscr{M}\mathfrak{u}_n - \mathscr{M}\mathfrak{u}_1|}{\mathscr{L}(\mathfrak{u}_n, \mathfrak{u}_1)} \\ &= \lim_{n\longrightarrow\infty} \frac{|2n-3-1|}{2n-2} = 1, n \geq 2. \end{aligned} \tag{20}$$

Clearly, $\mathscr{M}$ is neither a Ćirić-type contraction [26] nor a Banach contraction [27].

Now, we claim that $\mathscr{M}$ satisfies Ćirić-type Θ-contraction; that is,

$$\begin{aligned} d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m) \neq 0 &\Rightarrow e^{\sqrt{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)e^{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)}}} \\ &\leq e^{\alpha\sqrt{d(\mathfrak{u}_n, \mathfrak{u}_m)e^{d(\mathfrak{u}_n, \mathfrak{u}_m)}}}, \alpha \in (0,1) \\ &\Rightarrow d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)e^{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)} \\ &\leq \alpha^2\left[d(\mathfrak{u}_n, \mathfrak{u}_m)e^{d(\mathfrak{u}_n, \mathfrak{u}_m)}\right], \alpha \in (0,1) \\ &\Rightarrow \frac{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)e^{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)}}{d(\mathfrak{u}_n, \mathfrak{u}_m)e^{d(\mathfrak{u}_n, \mathfrak{u}_m)}} \leq \alpha^2, \alpha \in (0,1). \end{aligned} \tag{21}$$

Case 1. When $n = 1$ and $m \geq 2$,

$$\begin{aligned} &\frac{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)e^{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)}}{d(\mathfrak{u}_n, \mathfrak{u}_m)e^{d(\mathfrak{u}_n, \mathfrak{u}_m)}} \\ &= \frac{|4-2m|e^{|4-2m|}}{|2-2m|e^{|2-2m|}} = \frac{(2m-4)e^{(2m-4)}}{(2m-2)e^{(2m-2)}} \leq e^{-2}. \end{aligned} \tag{22}$$

Case 2. When $n > m > 1$,

$$\begin{aligned} &\frac{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)e^{d(\mathscr{M}\mathfrak{u}_n, \mathscr{M}\mathfrak{u}_m)}}{d(\mathfrak{u}_n, \mathfrak{u}_m)e^{d(\mathfrak{u}_n, \mathfrak{u}_m)}} \\ &= \frac{|2n-2m-6|e^{|2n-2m-6|}}{|2n-2m-2|e^{|2n-2m-2|}} \\ &= \frac{(2n-2m-6)e^{(2n-2m-6)}}{(2n-2m-2)e^{(2n-2m-2)}} \leq e^{-4}. \end{aligned} \tag{23}$$

Thus, $\mathscr{M}$ is a Ćirić-type Θ-contraction with $\alpha = \max\{e^{-4}, e^{-2}\} = e^{-2}$ and has a unique fixed point 1. Further, $\lim_{n\longrightarrow\infty}\mathscr{M}^n\mathfrak{u}_1 = 1$.

Remark 9. Theorems 7 and 8 are improvements, extensions, and generalizations of Banach [27], Ćirić [26], Jleli and Samet [25], and references therein. Further, on taking $\Theta(t) = e^t$, $t > 0$, in these results, we obtain some novel results which are generalizations of existing results in the literature.

Next, following Joshi et al. [5], we define an elliptic disc and a fixed elliptic disc to study the geometry of nonunique fixed points in a metric space.

Definition 10. An elliptic disc having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in a metric space $(\mathscr{U}, d)$ is defined as $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \{\mathfrak{u} \in \mathscr{U} : d(\mathfrak{c}_1, \mathfrak{u}) + d(\mathfrak{c}_2, \mathfrak{u}) \leq \mathfrak{a}, \mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{M}, \mathfrak{a} \in [0,\infty)\}$.

Remark 11. For defining an ellipse or elliptic disc $\mathfrak{a} \geq d(\mathfrak{c}_1, \mathfrak{c}_2)$.

Example 3. Let $\mathscr{U} = \mathbb{R}^2$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = \sqrt{(\mathfrak{u}_1 - \mathfrak{v}_1)^2 + (\mathfrak{u}_2 - \mathfrak{v}_2)^2}$, $\mathfrak{u} \ll (\mathfrak{u}_1, \mathfrak{u}_2)$, $\mathfrak{v} \ll (\mathfrak{v}_1, \mathfrak{v}_2) \in \mathscr{U}$; then,

$$\begin{aligned} \mathscr{E}_{\mathscr{D}}((-5,0), (5,0), 12) &= \{\mathfrak{u} \in \mathscr{U} : d((-5,0), (\mathfrak{u}_1, \mathfrak{u}_2)) \\ &\quad + d((5,0), (\mathfrak{u}_1, \mathfrak{u}_2)) \leq 12\} \\ &= \left\{\mathfrak{u} \in \mathscr{U} : \sqrt{(\mathfrak{u}_1 + 5)^2 + \mathfrak{u}_2^2} \right. \\ &\quad \left. + \sqrt{(\mathfrak{u}_1 - 5)^2 + \mathfrak{u}_2^2} \leq 12\right\}, \end{aligned} \tag{24}$$

which is shown by the blue shaded portion in Figure 1.

Definition 12. Let $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an elliptic disc having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in a metric space $(\mathscr{U}, d)$. Then, $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is said to be a fixed elliptic disc of map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ if $\mathscr{M}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in \mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, $\mathfrak{a} \in [0,\infty)$.

We now introduce and exploit a generalized $\Theta_{\mathscr{E}}$-weak contraction to demonstrate that the set of nonunique fixed points of a map includes an ellipse or an elliptic disc.

Definition 13. Let $\Theta : (0,\infty) \longrightarrow (1,\infty)$ be an increasing function. A map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ of a metric space $(\mathscr{U}, d)$ is said to be a generalized $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{u} \neq \mathfrak{v}$, if

$$d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) > 0 \Rightarrow \Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) \leq [\Theta(\mathscr{L}(\mathfrak{u}, \mathfrak{v}))]^\alpha, \tag{25}$$

where $\mathscr{L}(\mathfrak{u}, \mathfrak{v}) = \max\{d(\mathfrak{u}, \mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + (1-\gamma)d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), (1-\gamma)d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + (1-\gamma)d(\mathfrak{v}, \mathscr{M}\mathfrak{u}), (1-\gamma)d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{u})\}$, $\gamma \in [0,1)$, $\alpha \in (0,1)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$.

Remark 14. In the above contraction, if $\gamma = 0$, then $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ is said to be a Ćirić-type $\Theta_{\mathscr{E}}$-weak contraction.

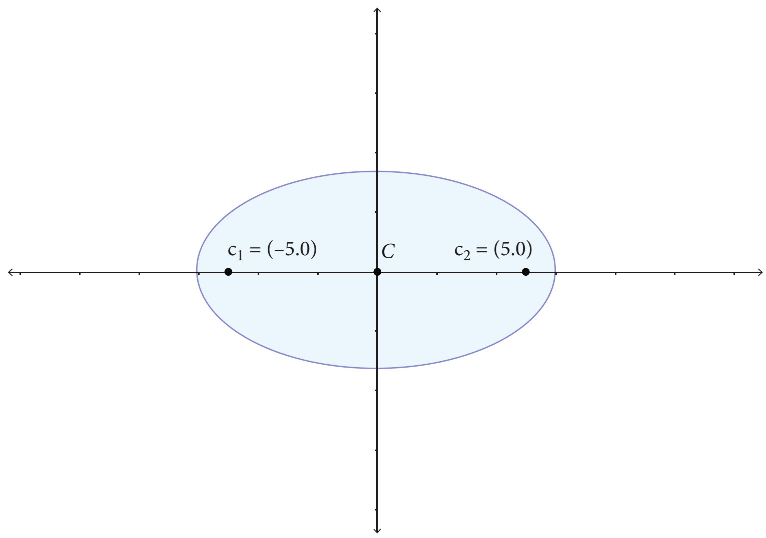

FIGURE 1: The elliptic disc $\mathscr{E}_{\mathscr{D}}((-5,0),(5,0),12)$ centered at $(0,0)$ having foci at $(-5,0)$ and $(5,0)$ is shown in this figure.

Theorem 15. *Let $\mathscr{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be an ellipse in a metric space $(\mathscr{U},d)$ and $\mathfrak{a}=(1/2)\{\inf d(\mathfrak{u},\mathscr{M}\mathfrak{u})\colon \mathfrak{u}\neq\mathscr{M}\mathfrak{u}\}$. If map $\mathscr{M}: \mathscr{U}\longrightarrow\mathscr{U}$ is a generalized $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{c}_1,\mathfrak{c}_2\in\mathscr{U}$ and $d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})+d(\mathfrak{c}_2,\mathscr{M}\mathfrak{u})=\mathfrak{a}$, $\mathfrak{u}\in\mathscr{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, then $\mathscr{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.*

Proof. Let $\mathfrak{u}\in\mathscr{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be any arbitrary point and $\mathscr{M}\mathfrak{u}\neq\mathfrak{u}$. From the definition of $\mathfrak{a}$, $d(\mathfrak{u},\mathscr{M}\mathfrak{u})\geq 2\mathfrak{a}$, suppose $\mathscr{M}\mathfrak{c}_1\neq\mathfrak{c}_1$ and $\mathscr{M}\mathfrak{c}_2\neq\mathfrak{c}_2$, so we have $d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1))>0$, $d(\mathfrak{c}_2,\mathscr{M}\mathfrak{c}_2))>0$, and

$$\begin{aligned}\Theta(d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)) &\leq [\Theta(\mathscr{L}(\mathfrak{c}_1,\mathfrak{c}_1))]^{\alpha}\\ &= [\Theta(\max\{d(\mathfrak{c}_1,\mathfrak{c}_1),\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1),(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\\ &\quad +\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1),(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\\ &\quad +\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\})]^{\alpha}\\ &= [\Theta(\max\{0,d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\})]^{\alpha} = [\Theta(d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1))]^{\alpha}\\ &< \Theta(d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)),\ \alpha\in(0,1),\ \text{a contradiction.}\end{aligned} \tag{26}$$

So $\mathscr{M}\mathfrak{c}_1=\mathfrak{c}_1$. Similarly, $\mathscr{M}\mathfrak{c}_2=\mathfrak{c}_2$.

Again, since $d(\mathfrak{u},\mathscr{M}\mathfrak{u})>0$,

$$\begin{aligned}\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u})) &\leq [\Theta(\mathscr{L}(\mathfrak{u},\mathfrak{c}_1))]^{\alpha}\\ &= [\Theta(\max\{d(\mathfrak{u},\mathfrak{c}_1),\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u})\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u})\\ &\quad +\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{c}_1)\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{c}_1)\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\})]^{\alpha}\\ &< [\Theta(\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\})]^{\alpha}\\ &< \Theta(\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)\\ &\quad +(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}),\ \alpha\in(0,1).\end{aligned} \tag{27}$$

Case 1. If $\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}=2\mathfrak{a}$, then $\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(2\mathfrak{a})$.

By definition of $\mathfrak{a}$ and Θ, $\Theta(2\mathfrak{a})\leq\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(2\mathfrak{a})$, a contradiction.

Case 2. If $\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}=\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u})$, then $\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}))$.

If $\gamma=0$, $\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(0)$, a contradiction.

If $\gamma\in(0,1)$, $\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))$, a contradiction.

Case 3. If $\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}=(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u})$, then $\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta((1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}))\leq\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))$, a contradiction.

Case 4. If $\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}=\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})$, then

$$\begin{aligned}\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u})) &< \Theta(\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}))\\ &< \Theta(\gamma\mathfrak{a}+(1-\gamma)\mathfrak{a})=\Theta(\mathfrak{a}).\end{aligned} \tag{28}$$

By definition of $\mathfrak{a}$ and Θ, $\Theta(2\mathfrak{a})\leq\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))<\Theta(\mathfrak{a})$, a contradiction.

Case 5. If $\max\{2\mathfrak{a},\gamma d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathscr{M}\mathfrak{u}),\gamma d(\mathfrak{u},\mathfrak{c}_1)+(1-\gamma)d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}),(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})\}=(1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u})$, then

$$\begin{aligned}\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u})) &< \Theta((1-\gamma)d(\mathfrak{u},\mathfrak{c}_1)+\gamma d(\mathfrak{c}_1,\mathscr{M}\mathfrak{u}))\\ &< \Theta(\gamma\mathfrak{a}+(1-\gamma)\mathfrak{a})=\Theta(\mathfrak{a}).\end{aligned} \tag{29}$$

By definition of $\mathfrak{a}$ and Θ, $\Theta(2\mathfrak{a}) \leq \Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) < \Theta(\mathfrak{a})$, a contradiction.

Similarly, we can prove for $\mathfrak{c}_2 \in \mathscr{U}$.

Hence, $\mathscr{M}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$; that is, $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.

Theorem 16. *If in the above theorem $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) \leq \mathfrak{a}$, then $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$.*

Proof. Now, to show $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$, it is sufficient to demonstrate that $\mathscr{M}$ fixes an ellipse $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{b})$ with $\mathfrak{b} \lhd \mathfrak{a}$. Since $\mathscr{M}$ is a generalized $\Theta_{\mathscr{E}}$-weak contraction, then proceeding as in Theorem 15, $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{b})$ is a fixed ellipse of $\mathscr{M}$ as $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) = \mathfrak{b} \leq \mathfrak{a}$; that is, $\mathscr{M}\mathfrak{u} = \mathfrak{u}, \forall \mathfrak{u} \in \mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{b})$. Hence, $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$.

Theorem 17. *Theorem 15 remains true if we substitute Ćirić-type $\Theta_{\mathscr{E}}$-weak contraction in place of generalized $\Theta_{\mathscr{E}}$-weak contraction.*

Proof. The proof follows the pattern of Theorem 15 on taking $\gamma = 0$.

Theorem 18. *Theorem 16 remains true if we substitute Ćirić-type $\Theta_{\mathscr{E}}$-weak contraction in place of generalized $\Theta_{\mathscr{E}}$-weak contraction.*

Proof. The proof follows the pattern of Theorem 16 on taking $\gamma = 0$.

The subsequent examples elucidate Theorems 17 and 18.

Example 4. Let $\mathscr{U} = [5,\infty)$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}|$.

Let $\Theta(t) = e^t$, $\mathfrak{c}_1 = -2$, $\mathfrak{c}_2 = 3$, $\mathfrak{a} = 6$, $\gamma = 0$, and $\alpha = 6/7$.

The ellipse

$$\begin{aligned}\mathscr{E}(-2, 3, 6) &= \{\mathfrak{u} \in \mathscr{U} : d(-2, \mathfrak{u}) + d(3, \mathfrak{u}) = 6\} \\ &= \{\mathfrak{u} \in \mathscr{U} : |-2 - \mathfrak{u}| + |3 - \mathfrak{u}| = 6\} = \{-2.5,3.5\}.\end{aligned} \tag{30}$$

Define a self-map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ as $\mathscr{M}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in [-5, 5] \\ \mathfrak{u} + 12, & \text{otherwise} \end{cases}$.

Since for $\mathfrak{u} \in [-5, 5]$, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 0$, and for $\mathfrak{u} \in (5,\infty)$, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 12 > 0$.

Case 1. For $\mathfrak{u} > 5$ and $\mathfrak{c}_1 = -2$,

$$\begin{aligned}\mathscr{L}(\mathfrak{u},-2) &= \max\{d(\mathfrak{u},-2), d(-2, \mathscr{M}(-2)), d(\mathfrak{u}, \mathscr{M}\mathfrak{u}), d(-2, \mathscr{M}\mathfrak{u}), d(\mathfrak{u}, \mathscr{M}(-2))\} \\ &= \max\{d(\mathfrak{u},-2), d(-2,-2), d(\mathfrak{u}, \mathscr{M}\mathfrak{u}), d(-2, \mathscr{M}\mathfrak{u}), d(\mathfrak{u},-2)\} \\ &= \max\{d(\mathfrak{u},-2), 0, 12, d(-2, \mathfrak{u} + 12)\} = \max\{|\mathfrak{u} + 2|, 12, |\mathfrak{u} + 14|\} \\ &= |\mathfrak{u} + 14| > 19,\end{aligned} \tag{31}$$

and $\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) = \Theta(12) = e^{12} < e^{(12/13)|\mathfrak{u}+8|} = e^{(\mathscr{L}(\mathfrak{u},-2))^{(12/19)}} = [\Theta(\mathscr{L}(\mathfrak{u},-2))]^{(12/19)}$.

Case 2. For $\mathfrak{u} > 5$ and $\mathfrak{c}_2 = 3$,

$$\begin{aligned}\mathscr{L}(\mathfrak{u}, 3) &= \max\{d(\mathfrak{u}, 3), d(3, \mathscr{M}3), d(\mathfrak{u}, \mathscr{M}\mathfrak{u}), d(3, \mathscr{M}\mathfrak{u}), d(\mathfrak{u}, \mathscr{M}3)\} \\ &= \max\{d(\mathfrak{u}, 3), d(3, 3), d(\mathfrak{u}, \mathscr{M}\mathfrak{u}), d(3, \mathscr{M}\mathfrak{u}), d(\mathfrak{u}, 3)\} \\ &= \max\{d(\mathfrak{u}, 3), 0, 12, d(3, \mathfrak{u} + 12)\} \\ &= \max\{|\mathfrak{u} - 3|, 12, |\mathfrak{u} + 9|\} = |\mathfrak{u} + 9| > 14,\end{aligned} \tag{32}$$

and $\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) = \Theta(12) = e^{12} < e^{(12/14)|\mathfrak{u}+9|} = e^{(\mathscr{L}(\mathfrak{u},3))^{(12/14)}} = [\Theta(\mathscr{L}(\mathfrak{u}, 3))]^{(12/14)}$.

That is, $\mathscr{M}$ is a Ćirić-type $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{c}_1 = -2$, $\mathfrak{c}_2 = 3$, and $\alpha = \max\{12/19, 12/14\} = 12/14$. Hence, $\mathscr{E}(-2, 3, 6) = \{-2.5,3.5\}$ is a fixed ellipse and $\mathscr{E}_{\mathscr{D}}(-2, 3, 6) = [-2.5,3.5]$ is a fixed elliptic disc of $\mathscr{M}$. One may verify that $d(-2, \mathfrak{u}) + d(3, \mathfrak{u}) \leq 6$, $\mathfrak{u} \in \mathscr{E}_{\mathscr{D}}(-2, 3, 6)$.

Example 5. Let $\mathscr{U} = \mathbb{R}^2$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = \sqrt{(\mathfrak{u}_1 - \mathfrak{v}_1)^2 + (\mathfrak{u}_2 - \mathfrak{v}_2)^2}$, where $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2)$ and $\mathfrak{v} = (\mathfrak{v}_1, \mathfrak{v}_2)$.

Let $\Theta(t) = 1 + t$, $\mathfrak{c}_1 = (3 + 2\sqrt{3},-1)$, $\mathfrak{c}_2 = (3 - 2\sqrt{3},-1)$, $\mathfrak{a} = 8$, $\gamma = 0$, and $\alpha = 6/7$.

The ellipse

$$\begin{aligned}\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, 8) &= \{\mathfrak{u} \in \mathscr{U} : d(\mathfrak{c}_1, \mathfrak{u}) + d(\mathfrak{c}_2, \mathfrak{u}) = 8\} \\ &= \left\{\mathfrak{u} \in \mathscr{U} : \sqrt{\left(\mathfrak{u}_1 - 3 - 2\sqrt{3}\right)^2 + (\mathfrak{u}_2 + 1)^2} + \sqrt{\left(\mathfrak{u}_1 - 3 + 2\sqrt{3}\right)^2 + (\mathfrak{u}_2 + 1)^2} = 8\right\} \\ &= \left\{\mathfrak{u} \in \mathscr{U} : \frac{(\mathfrak{u}_1 - 3)^2}{16} + \frac{(\mathfrak{u}_2 + 1)^2}{4} = 1\right\},\end{aligned} \tag{33}$$

which is shown by the blue line in Figure 2.

Further, the elliptic disc $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, 8) = \{\mathfrak{u} \in \mathscr{U} : ((\mathfrak{u}_1 - 3)^2/16) + ((\mathfrak{u}_2 + 1)^2/4) \leq 1\}$, which is shown as the blue shaded portion in Figure 2.

Define a self-map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ as $\mathscr{M}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in (3 + 6\cos\theta,-1 + 6\sin\theta) \\ \mathfrak{u} + (8\sqrt{2}, 8\sqrt{2}), & \text{otherwise} \end{cases}$.

Since for $\mathfrak{u} \in (3 + 6\cos\theta,-1 + 6\sin\theta)$, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 0$, and for $\mathfrak{u} \in \mathbb{R}^2 \setminus (3 + 6\cos\theta,-1 + 6\sin\theta)$, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 16 > 0$.

Case 1. For $\mathfrak{u} \in \mathbb{R}^2 \setminus (3+6\cos\theta, -1+6\sin\theta)$ and $\mathfrak{c}_1 = (3+2\sqrt{3}, -1)$,

$$\begin{aligned}
\mathscr{L}\left(\mathfrak{u}, \left(3+2\sqrt{3},-1\right)\right) &= \max\left\{d\left(\mathfrak{u}, \left(3+2\sqrt{3},-1\right)\right), d\right.\\
&\quad \cdot \left(\left(3+2\sqrt{3},-1\right), \mathscr{M}\left(3+2\sqrt{3},-1\right)\right), d\\
&\quad \cdot (\mathfrak{u}, \mathscr{M}\mathfrak{u}), d\left(\left(3+2\sqrt{3},-1\right), \mathscr{M}\mathfrak{u}\right), d\\
&\quad \left.\cdot \left(\mathfrak{u}, \mathscr{M}\left(3+2\sqrt{3},-1\right)\right)\right\}\\
&= \max\left\{d\left(\mathfrak{u}, \left(3+2\sqrt{3},-1\right)\right), d\right.\\
&\quad \cdot \left(\left(3+2\sqrt{3},-1\right), \left(3+2\sqrt{3},-1\right)\right), d\\
&\quad \cdot (\mathfrak{u}, \mathscr{M}\mathfrak{u}), d\left(\left(3+2\sqrt{3},-1\right), \mathscr{M}\mathfrak{u}\right), d\\
&\quad \left.\cdot \left(\mathfrak{u}, \left(3+2\sqrt{3},-1\right)\right)\right\}\\
&= \max\left\{d\left(\mathfrak{u}, \left(3+2\sqrt{3},-1\right)\right), 0, 16, d\right.\\
&\quad \left.\cdot \left(\left(3+2\sqrt{3},-1\right), \mathfrak{u} + \left(8\sqrt{2}, 8\sqrt{2}\right)\right)\right\}\\
&= \max\left\{\sqrt{\left(\mathfrak{u}_1 - 3 - 2\sqrt{3}\right)^2 + (\mathfrak{u}_2+1)^2}, 16,\right.\\
&\quad \left.\cdot \sqrt{\left(\mathfrak{u}_1 + 8\sqrt{2} - 3 - 2\sqrt{3}\right)^2 + \left(\mathfrak{u}_2 + 8\sqrt{2} + 1\right)^2}\right\}\\
&> 16,
\end{aligned} \tag{34}$$

and $\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) = \Theta(16) = e^{16} < e^{(16/17)\mathscr{L}(\mathfrak{u},(3+2\sqrt{3},-1))|} = [\Theta(\mathscr{L}(\mathfrak{u}, (3+2\sqrt{3}, -1)))]^{(16/17)}$.

Case 2. For $\mathfrak{u} \in \mathbb{R}^2 \setminus (3+6\cos\theta, -1+6\sin\theta)$ and $\mathfrak{c}_1 = (3-2\sqrt{3}, -1)$,

$$\begin{aligned}
\mathscr{L}\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right) &= \max\left\{d\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right), d\right.\\
&\quad \cdot \left(\left(3-2\sqrt{3},-1\right), \mathscr{M}\left(3-2\sqrt{3},-1\right)\right), d\\
&\quad \cdot (\mathfrak{u}, \mathscr{M}\mathfrak{u}), d\left(\left(3-2\sqrt{3},-1\right), \mathscr{M}\mathfrak{u}\right), d\\
&\quad \left.\cdot \left(\mathfrak{u}, \mathscr{M}\left(3-2\sqrt{3},-1\right)\right)\right\}\\
&= \max\left\{d\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right), d\right.\\
&\quad \cdot \left(\left(3-2\sqrt{3},-1\right), \left(3-2\sqrt{3},-1\right)\right), d(\mathfrak{u}, \mathscr{M}\mathfrak{u}), d\\
&\quad \left.\cdot \left(\left(3-2\sqrt{3},-1\right), \mathscr{M}\mathfrak{u}\right), d\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right)\right\}\\
&= \max\left\{d\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right), 0, 16, d\right.\\
&\quad \left.\cdot \left(\left(3-2\sqrt{3},-1\right), \mathfrak{u} + \left(8\sqrt{2}, 8\sqrt{2}\right)\right)\right\}\\
&= \max\left\{\sqrt{\left(\mathfrak{u}_1 - 3 + 2\sqrt{3}\right)^2 + (\mathfrak{u}_2+1)^2}, 16,\right.\\
&\quad \left.\cdot \sqrt{\left(\mathfrak{u}_1 + 8\sqrt{2} - 3 + 2\sqrt{3}\right)^2 + \left(\mathfrak{u}_2 + 8\sqrt{2} + 1\right)^2}\right\} > 21,
\end{aligned} \tag{35}$$

$$\begin{aligned}
\Theta(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) &= \Theta(16) = e^{16} < e^{(16/21)\mathscr{L}(\mathfrak{u},(3-2\sqrt{3},-1))}\\
&= \left[\Theta\left(\mathscr{L}\left(\mathfrak{u}, \left(3-2\sqrt{3},-1\right)\right)\right)\right]^{(16/21)}.
\end{aligned} \tag{36}$$

That is, $\mathscr{M}$ is a Ćirić-type $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{c}_1 = (3+2\sqrt{3}, -1)$, $\mathfrak{c}_2 = (3-2\sqrt{3}, -1)$, and $\alpha = \max\{16/17, 16/21\} = 16/17$. Hence, $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, 8)$ is a fixed ellipse and $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, 8)$ is a fixed elliptic disc of $\mathscr{M}$. One may verify that $d(\mathfrak{c}_1, \mathfrak{u}) + d(\mathfrak{c}_2, \mathfrak{u}) \leq 8$, $\mathfrak{u} \in \mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, 8)$.

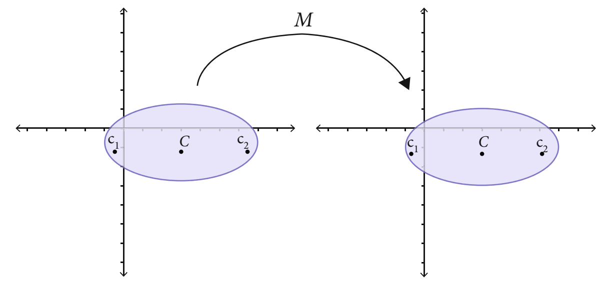

Figure 2: The blue line and its interior demonstrate the ellipse ($\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, 8)$) and elliptic disc ($\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, 8)$), respectively, which is fixed by map $\mathscr{M}$.

Now, we introduce $\Psi_{\mathscr{E}}$-weak JS-contraction to study the geometry of nonunique fixed points.

Definition 19. Let $\Psi : [0,\infty) \longrightarrow [1,\infty)$ be an increasing function with $\Psi(0) = 1$; then, a map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ of a metric space $(\mathscr{U}, d)$ is said to be a $\Psi_{\mathscr{E}}$-weak JS-contraction with $\mathfrak{u} \neq \mathfrak{v}$, if

$$\begin{aligned}
d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) > 0 \Rightarrow \Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))&\\
\leq [\Psi(d(\mathfrak{u}, \mathfrak{v}))]^a [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))]^b [\Theta(d(\mathfrak{v}, \mathscr{M}\mathfrak{v}))]^c&\\
\cdot [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{v}))]^e [\Psi(d(\mathfrak{v}, \mathscr{M}\mathfrak{u}))]^f,&
\end{aligned} \tag{37}$$

where a, b, c, e, and f are nonnegative and $a+b+c+e+f \in [0,1)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$.

Theorem 20. *Let $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse in a metric space $(\mathscr{U}, d)$ and $\mathfrak{a} = (1/2)\{\inf d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) : \mathfrak{u} \neq \mathscr{M}\mathfrak{u}\}$. If map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ is a $\Psi_{\mathscr{E}}$-weak JS-contraction with $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{U}$ and $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) = \mathfrak{a}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.*

Proof. Let $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be any arbitrary point and $\mathscr{M}\mathfrak{u} \neq \mathfrak{u}$. From the definition of $\mathfrak{a}$, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) \geq 2\mathfrak{a}$, since $\mathscr{M}$ is $\Psi_{\mathscr{E}}$-weak JS-contraction for $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{U}$, suppose $\mathscr{M}\mathfrak{c}_1 \neq \mathfrak{c}_1$ and $\mathscr{M}\mathfrak{c}_2 \neq \mathfrak{c}_2$, so we have $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1)) > 0$, $d(\mathfrak{c}_2, \mathscr{M}\mathfrak{c}_2)) > 0$, and

$$\begin{aligned}
\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1)) &\leq [\Psi(d(\mathfrak{c}_1, \mathfrak{c}_1))]^a [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^b [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^c\\
&\quad \cdot [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^e [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^f\\
&= [\Psi(0)]^a [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^{b+c+e+f}\\
&= [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^{1-a} < \Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1)), \text{ a contradiction.}
\end{aligned} \tag{38}$$

So $\mathscr{M}\mathfrak{c}_1 = \mathfrak{c}_1$. Similarly, $\mathscr{M}\mathfrak{c}_2 = \mathfrak{c}_2$.

Again, since $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) > 0$, so

$$\begin{aligned}\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) &\leq [\Psi(d(\mathfrak{u}, \mathfrak{c}_1))]^a [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))]^b [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^c \\ &\quad \cdot [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{c}_1))]^e [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}))]^f \\ &< [\Psi(\mathfrak{a})]^a [\Psi(2\mathfrak{a})]^b [\Psi(d(\mathfrak{c}_1, \mathfrak{c}_1))]^c [\Psi(\mathfrak{a})]^e [\Psi(\mathfrak{a})]^f \\ &< [\Theta(2\mathfrak{a})]^a [\Psi(2\mathfrak{a})]^b [\Theta(0)]^c [\Psi(2\mathfrak{a})]^e [\Psi(2\mathfrak{a})]^f \\ &\quad \cdot (\text{as } \Psi \text{ is increasing}) = [\Psi(2\mathfrak{a})]^{a+b+e+f} \\ &< [\Psi(2\mathfrak{a})]^{1-c} < \Psi(2\mathfrak{a}).\end{aligned} \tag{39}$$

Since $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) \geq 2\mathfrak{a}$ and Ψ is increasing, $\Psi(2\mathfrak{a}) \leq \Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) < \Psi(2\mathfrak{a})$, a contradiction.

Similarly, we can prove for $\mathfrak{c}_2 \in \mathscr{U}$.

Hence, $\mathscr{M}\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$; that is, $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.

Theorem 21. *If in Theorem 20, $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) \leq \mathfrak{a}$, then $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$.*

Proof. The proof follows the pattern of Theorem 16.

The subsequent example elucidates Theorems 20 and 21.

Example 6. Let $\mathscr{U} = \{-2, 0, (1/2)\ln(6/e), (1/2)\ln(15/e), (1/2)\ln(18/e), (1/2)\ln(21/e), (1/2)\ln(24/e), (1/2)\ln(27/e), (1/2)\ln(30/e), (1/2)\ln(6e), (1/2)\ln(9e), (1/2)\ln(12e), (1/2)\ln(15e), \ln 2, \ln 3, \ln 5\}$ and a metric $d : \mathscr{U} \times \mathscr{U} \longrightarrow \mathbb{R}^+$ be defined as $d(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}|$. Let $\Psi(t) = e^t$, $\mathfrak{c}_1 = \ln 3$, $\mathfrak{c}_2 = \ln 5$, $\mathfrak{a} = 1$, $\gamma = 0$, and $\alpha = 3/4$.

The ellipse

$$\begin{aligned}\mathscr{E}(\ln 3, \ln 5, 1) &= \{\mathfrak{u} \in \mathscr{U} : d(\ln 3, \mathfrak{u}) + d(\ln 5, \mathfrak{u}) = 1\} \\ &= \{\mathfrak{u} \in \mathscr{U} : |\ln 3 - \mathfrak{u}| + |\ln 5 - \mathfrak{u}| = 1\} \\ &= \left\{\frac{1}{2}\ln\left(\frac{15}{e}\right), \frac{1}{2}\ln(15e)\right\}.\end{aligned} \tag{40}$$

Define a self-map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ as

$$\mathscr{M}\mathfrak{u} = \begin{cases} 0, & \mathfrak{u} = -2, \\ -2, & \mathfrak{u} = 0, \\ \mathfrak{u}, & \text{otherwise.}\end{cases} \tag{41}$$

Then, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = \begin{cases} 2, & \mathfrak{u} \in \{-2, 0\} \\ 0, & \text{otherwise}\end{cases}$.

Then, $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) = 2 > 0$.

Case 1. For $\mathfrak{u} = \{-2, 0\}$ and $\mathfrak{c}_1 = \ln 3$,

$$\begin{aligned}&[\Psi(d(\mathfrak{u}, \ln 3))]^a [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))]^b [\Psi(d(\ln 3, \mathscr{M}\ln 3))]^c \\ &\quad \times [\Psi(d(\mathfrak{u}, \mathscr{M}\ln 3))]^e [\Psi(d(\ln 3, \mathscr{M}\mathfrak{u}))]^f \\ &= [\Psi(|\mathfrak{u} - \ln 3\,|)]^a [\Psi(|\mathfrak{u} - \mathscr{M}\mathfrak{u}\,|)]^b [\Psi(|\ln 3 - \mathscr{M}\ln 3\,|)]^c \\ &\quad \times [\Psi(|\mathfrak{u} - \mathscr{M}\ln 3\,|)]^e [\Psi(|\ln 3 - \mathscr{M}\mathfrak{u}\,|)]^f \\ &= [\Psi(|\mathfrak{u} - \ln 3\,|)]^a [\Psi(2)]^b [\Psi(|\ln 3 - \ln 3\,|)]^c [\Psi(|\mathfrak{u} - \ln 3\,|)]^e \\ &\quad \times [\Psi(|\ln 3 - \mathscr{M}\mathfrak{u}\,|)]^f \\ &= [\Psi(|\mathfrak{u} - \ln 3\,|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 - \mathscr{M}\mathfrak{u}\,|)]^f \\ &= [\Psi(|\mathfrak{u} - \ln 3\,|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 - \mathscr{M}\mathfrak{u}|)]^f,\end{aligned}$$

$$\begin{aligned}&[\Psi(|\mathfrak{u} - \ln 3|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 - \mathscr{M}\mathfrak{u}|)]^f \\ &= \begin{cases} [\Psi(\ln 3)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 3 + 2|)]^f, & \text{if } \mathfrak{u} = 0 \\ [\Psi(|2 + \ln 3|)]^{a+e} [\Psi(2)]^b [\Psi(\ln 3)]^f, & \text{if } \mathfrak{u} = -2 \end{cases} \\ &= \begin{cases} [\Psi(\ln 3)]^{a+e} [\Psi(2)]^b \left[\Psi\left(\ln\left(3e^2\right)\right)\right], & \text{if } \mathfrak{u} = 0 \\ \left[\Psi\left(\ln\left(3e^2\right)\right)\right]^{a+e} [\Psi(2)]^b [\Psi(\ln 3)]^f, & \text{if } \mathfrak{u} = -2 \end{cases} \\ &= \begin{cases} 3^{a+e} e^{2b} \left(3e^2\right)^f, & \text{if } \mathfrak{u} = 0 \\ \left(3e^2\right)^{a+e} e^{2b} 3^f, & \text{if } \mathfrak{u} = -2 \end{cases} > e^2 = \Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})),\end{aligned} \tag{42}$$

for $a = e = 1/4$, $b = 1/4$, $c = 0$, and $f = 1/3$, satisfying $a + b + c + e + f < 1$; that is,

$$\begin{aligned}\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) &< [\Psi(d(\mathfrak{c}_1, \mathfrak{c}_1))]^a [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^b [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^c \\ &\quad \cdot [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^e [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^f.\end{aligned} \tag{43}$$

Case 2. For $\mathfrak{u} \in \{-2, 0\}$ and $\mathfrak{c}_2 = \ln 5$,

$$\begin{aligned}&[\Psi(d(\mathfrak{u}, \ln 5))]^a [\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u}))]^b [\Psi(d(\ln 5, \mathscr{M}\ln 5))]^c \\ &\quad \times [\Psi(d(\mathfrak{u}, \mathscr{M}\ln 5))]^e [\Psi(d(\ln 5, \mathscr{M}\mathfrak{u}))]^f \\ &= [\Psi(|\mathfrak{u} - \ln 5|)]^a [\Psi(|\mathfrak{u} - \mathscr{M}\mathfrak{u}|)]^b [\Psi(|\ln 5 - \mathscr{M}\ln 5|)]^c \\ &\quad \times [\Psi(|\mathfrak{u} - \mathscr{M}\ln 5|)]^e [\Psi(|\ln 5 - \mathscr{M}\mathfrak{u}|)]^f \\ &= [\Psi(|\mathfrak{u} - \ln 5|)]^a [\Psi(2)]^b [\Psi(|\ln 5 - \ln 5|)]^c \\ &\quad \times [\Psi(|\mathfrak{u} - \ln 5|)]^e [\Psi(|\ln 5 - \mathscr{M}\mathfrak{u}|)]^f \\ &= [\Psi(|\mathfrak{u} - \ln 5|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 5 - \mathscr{M}\mathfrak{u}|)]^f,\end{aligned}$$

$$\begin{aligned}&[\Psi(|\mathfrak{u} - \ln 5|)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 5 - \mathscr{M}\mathfrak{u}|)]^f \\ &= \begin{cases} [\Psi(\ln 5)]^{a+e} [\Psi(2)]^b [\Psi(|\ln 5 + 2|)]^f, & \text{if } \mathfrak{u} = 0 \\ [\Psi(|2 + \ln 5|)]^{a+e} [\Psi(2)]^b [\Psi(\ln 5)]^f, & \text{if } \mathfrak{u} = -2 \end{cases} \\ &= \begin{cases} [\Psi(\ln 5)]^{a+e} [\Psi(2)]^b \left[\Psi\left(\ln\left(5e^2\right)\right)\right]^f, & \text{if } \mathfrak{u} = 0 \\ \left[\Psi\left(\ln\left(5e^2\right)\right)\right]^{a+e} [\Psi(2)]^b [\Psi(\ln 5)]^f, & \text{if } \mathfrak{u} = -2 \end{cases} \\ &= \begin{cases} 5^{a+e} e^{2b} \left(5e^2\right)^f, & \text{if } \mathfrak{u} = 0 \\ \left(5e^2\right)^{a+e} e^{2b} 5^f, & \text{if } \mathfrak{u} = 2 \end{cases} > e^2 = \Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})),\end{aligned} \tag{44}$$

for $a = e = 1/4$, $b = 1/4$, $c = 0$, and $f = 1/3$, satisfying $a + b + c + e + f < 1$; that is,

$$\begin{aligned}\Psi(d(\mathfrak{u}, \mathscr{M}\mathfrak{u})) &< [\Psi(d(\mathfrak{c}_1, \mathfrak{c}_1))]^a [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^b \\ &\cdot [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^c [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^e \\ &\cdot [\Psi(d(\mathfrak{c}_1, \mathscr{M}\mathfrak{c}_1))]^f .\end{aligned} \quad (45)$$

That is, $\mathscr{M}$ is a Ψ-weak JS-contraction with $\mathfrak{c}_1 = \ln 3$, $\mathfrak{c}_2 = \ln 5$, and $a = e = 1/4$, $b = 1/4$, $c = 0$, and $f = 1/3$. Hence, $\mathscr{E}(\ln 3, \ln 5, 1) = \{(1/2)\ln(15/e), (1/2)\ln(15e)\}$ is a fixed ellipse and $\mathscr{E}_{\mathscr{D}}(\ln 3, \ln 5, 1) = \mathscr{U} \setminus \{-2, 0\}$ is a fixed elliptic disc of $\mathscr{M}$. One may verify that $d(\ln 3, \mathfrak{u}) + d(\ln 5, \mathfrak{u}) \le 1$, $\mathfrak{u} \in \mathscr{E}_{\mathscr{D}}(\ln 3, \ln 5, 1)$. Noticeably, a fixed ellipse or a fixed elliptic disc is not essentially unique, as $\mathscr{E}(\ln 3, \ln 5, 1)$ is also a fixed ellipse and $\mathscr{E}_{\mathscr{D}}(\ln 3, \ln 5, 1)$ is also a fixed elliptic disc of $\mathscr{M}$.

Now, we prove the fixed ellipse and fixed elliptic disc conclusions of the integra-type. Let $\Phi : [0,\infty) \longrightarrow [0,\infty)$ be a locally integrable function so that, for each $t > 0$, $\int_0^1 \Phi(t)dt > 0$.

Definition 22. Let $\Theta : (0,\infty) \longrightarrow (1,\infty)$ be an increasing function. A map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ of a metric space $(\mathscr{U}, d)$ is said to be a generalized integral-type $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{u} \ne \mathfrak{v}$, if

$$d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) > 0 \Rightarrow \int_0^{\Theta(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))} \Phi(t)dt \le \int_0^{[\Theta(\mathscr{L}(\mathfrak{u},\mathfrak{v}))]^{\alpha}} \Phi(t)dt, \quad (46)$$

where $\mathscr{L}(\mathfrak{u}, \mathfrak{v}) = \max\{d(\mathfrak{u}, \mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + (1-\gamma) d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), (1-\gamma) d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{v}), \gamma d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + (1-\gamma) d(\mathfrak{v}, \mathscr{M}\mathfrak{u}), (1-\gamma) d(\mathfrak{u}, \mathscr{M}\mathfrak{v}) + \gamma d(\mathfrak{v}, \mathscr{M}\mathfrak{u})\}$, $\gamma \in [0, 1)$, $\alpha \in (0, 1)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$.

Theorem 23. *Let $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse in a metric space $(\mathscr{U}, d)$ and $\mathfrak{a} = (1/2)\{\inf d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) : \mathfrak{u} \ne \mathscr{M}\mathfrak{u}\}$. If map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ is a generalized integral-type $\Theta_{\mathscr{E}}$-weak contraction with $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{U}$ and $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) = \mathfrak{a}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.*

Proof. The proof follows the pattern of Theorem 15.

Theorem 24. *If in Theorem 23, $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) \le \mathfrak{a}$, then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$.*

Proof. The proof follows the pattern of Theorem 16.

Next, we introduce an integral-type $\Psi_{\mathscr{E}}$-weak JS-contraction.

Definition 25. Let $\Psi : [0,\infty) \longrightarrow [1,\infty)$ be an increasing function with $\Psi(0) = 1$; then, map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ of a metric space $(\mathscr{U}, d)$ is said to be an integral-type $\Psi_{\mathscr{E}}$-weak JS-contraction with $\mathfrak{u} \ne \mathfrak{v}$, if $d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) > 0$ implies that

$$\int_0^{\Psi(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))} \Phi(t)dt \le \int_0^{[\Psi(d(\mathfrak{u},\mathfrak{v}))]^{\mathbf{a}} [\Psi(d(\mathfrak{u},\mathscr{M}\mathfrak{u}))]^{\mathbf{b}} [\Theta(d(\mathfrak{v},\mathscr{M}\mathfrak{v}))]^{\mathbf{c}} [\Psi(d(\mathfrak{u},\mathscr{M}\mathfrak{v}))]^{\mathbf{e}} [\Psi(d(\mathfrak{v},\mathscr{M}\mathfrak{u}))]^{\mathbf{f}}} \Phi(t)dt, \quad (47)$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{e}$, and $\mathbf{f}$ are nonnegative and $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{e} + \mathbf{f} \in [0, 1)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{U}$.

Theorem 26. *Let $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse in a metric space $(\mathscr{U}, d)$ and $\mathfrak{a} = (1/2)\{\inf d(\mathfrak{u}, \mathscr{M}\mathfrak{u}) : \mathfrak{u} \ne \mathscr{M}\mathfrak{u}\}$. If map $\mathscr{M} : \mathscr{U} \longrightarrow \mathscr{U}$ is an integral-type $\Psi_{\mathscr{E}}$-weak JS-contraction with $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{U}$ and $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{v}) = \mathfrak{a}$, $\mathfrak{u} \in \mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathscr{M}$.*

Proof. The proof follows the pattern of Theorem 15.

Theorem 27. *If in Theorem 26, $d(\mathfrak{c}_1, \mathscr{M}\mathfrak{u}) + d(\mathfrak{c}_2, \mathscr{M}\mathfrak{u}) \le \mathfrak{a}$, then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathscr{M}$.*

Proof. The proof follows the pattern of Theorem 16.

Remark 28.

(1) It is interesting to mention here that if a self-map fixes an elliptic disc, then it also fixes an ellipse (see Examples 4, 5, and 6). However, the reverse may not be true

(2) The ellipses (elliptic discs) in metric spaces change their shapes on changing the center, lengths of a semiminor or semimajor axis, or metric under consideration. Also, it is not necessary that an ellipse or an elliptic disc defined in a metric space be the same as an ellipse or an elliptic disc in a Euclidean space. Noticeably, ellipses discussed in Examples 1, 4, and 6 and elliptic discs in Examples 4 and 6 are different from the ellipse and elliptic disc in a Euclidean space

(3) If there exists a self-map that maps the ellipse (elliptic disc) to itself, then that ellipse (elliptic disc) may not be a fixed ellipse (elliptic disc); that is, the self-map may not fix all the points of the ellipse (elliptic disc)

(4) All the elliptic discs inside a fixed elliptic disc are also fixed elliptic discs of a self-map. Further, a fixed ellipse is not essentially unique (see Example 6)

(5) If both the focuses coincide, then fixed ellipse results, as well as fixed elliptic disc results, reduce to analogous fixed circle and fixed disc results, respectively. Noticeably, if $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{u}_0$ (say), then $\mathscr{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathscr{C}(\mathfrak{u}_0, \mathfrak{a}/2)$ and $\mathscr{E}_{\mathscr{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathscr{D}(\mathfrak{u}_0, \mathfrak{a}/2)$, with center $\mathfrak{u}_0$ and radius $\mathfrak{a}/2$

(6) For work on the set of nonunique fixed points forming a fixed figure, one may refer to [1–23] and references therein

3. Applications

Inspired by the fact that the fixed point theory is applicable in many real-world problems, we solve a boundary value problem arising when a diffusing matter is kept in an absorbing medium between parallel walls of fixed concentrations γ and δ. The concentration $\mathfrak{u}(t)$ of the substance at time t is given by

$$-\frac{d^2\mathfrak{u}}{dt^2}+\mathfrak{f}(t)\mathfrak{u}=\xi(t),\ \mathfrak{u}(0)=\gamma,\ \mathfrak{u}(1)=\delta, \tag{48}$$

where $\mathfrak{f}(t)$ is the absorption coefficient and $\xi(t)$ is the source density.

The Green function associated with an initial value problem (48) is

$$\mathcal{G}(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le s\le 1.\end{cases} \tag{49}$$

Problem (48) is equivalent to

$$\mathfrak{u}(t)=\gamma+(\delta-\gamma)t+\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{u}(s))ds,\ s\in[0,1]. \tag{50}$$

Let $\mathcal{U}$ be a set of Riemann integrable functions on $[0,1]$; that is, $\mathcal{U}=R[0,1]$. Define $d:\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ by

$$d(\mathfrak{u},\mathfrak{v})=\|\mathfrak{u}-\mathfrak{v}\|_\infty,\ \mathfrak{u},\mathfrak{v}\in\mathcal{U}, \tag{51}$$

where $\|\mathfrak{u}\|_\infty=\sup_{t\in[0,1]}|\mathfrak{u}(t)|$. Clearly, $(\mathcal{U},d)$ is a complete metric space.

Theorem 29. *Consider the boundary value problem (48). Let $\mathcal{M}:\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}$ be a self-map in a complete metric space ($\mathcal{U},d$), satisfying*

$$\begin{aligned}\|\mathfrak{u}(t)-\mathfrak{v}(t)\|_\infty>0\Rightarrow\|\mathfrak{f}(t)\mathfrak{u}(t)-\mathfrak{f}(t)\mathfrak{v}(t)\|_\infty\\ \le e^{-\lambda}\|\mathfrak{u}(t)-\mathfrak{v}(t)\|_\infty,\ \lambda>0.\end{aligned} \tag{52}$$

Then, the boundary value problem (48) has a solution.

Proof. Define a map $\mathcal{M}:\mathcal{U}\longrightarrow\mathcal{U}$ by

$$\mathcal{M}\mathfrak{u}(t)=\gamma+(\delta-\gamma)t+\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{u}(s))ds,\ s\in[0,1]. \tag{53}$$

Clearly, a fixed point of map $\mathcal{M}$ is a solution to the problem (48).

Since $\|\mathfrak{u}(t)-\mathfrak{u}(t)\|_\infty>0$,

$$\begin{aligned}d(\mathcal{M}\mathfrak{u},\mathcal{M}\mathfrak{v})&=\left|\gamma+(\delta-\gamma)t+\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{u}(s))ds\right.\\&\quad\left.-\gamma-(\delta-\gamma)t-\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{v}(s))ds\right|\\&=\left|\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{u}(s))ds\right.\\&\quad\left.-\int_0^1\mathcal{G}(t,s)(\xi(s)-\mathfrak{f}(s)\mathfrak{v}(s))ds\right|\\&=\left|\int_0^1\mathcal{G}(t,s)(\mathfrak{f}(s)\mathfrak{u}(s)-\mathfrak{f}(s)\mathfrak{v}(s))ds\right|\\&<\|\mathfrak{f}(s)\mathfrak{u}(s)-\mathfrak{f}(s)\mathfrak{v}(s)\|_\infty\sup_{t\in[0,1]}\left|\int_0^1\mathcal{G}(t,s)ds\right|\\&<e^{-\lambda}\|\mathfrak{u}(s)-\mathfrak{v}(s)\|_\infty\sup_{t\in[0,1]}\left|\int_0^1\mathcal{G}(t,s)ds\right|\\&<\frac{1}{8}e^{-\lambda}\|\mathfrak{u}(s)-\mathfrak{v}(s)\|_\infty=\frac{1}{8}e^{-\lambda}d(\mathfrak{u},\mathfrak{v}).\end{aligned} \tag{54}$$

If $\Theta(t)=e^t$, $t\in(0,\infty)$, then

$$\begin{aligned}\Theta(d(\mathcal{M}\mathfrak{u},\mathcal{M}\mathfrak{v}))&=e^{d(\mathcal{M}\mathfrak{u},\mathcal{M}\mathfrak{v})}<e^{(1/8)e^{-\lambda}d(\mathfrak{u},\mathfrak{v})}=e^{d(\mathfrak{u},\mathfrak{v})^{\left((1/8)e^{-\lambda}\right)}}\\&=[\Theta(d(\mathfrak{u},\mathfrak{v}))]^\alpha\le[\Theta(\max\{d(\mathfrak{u},\mathfrak{v}),d(\mathfrak{v},\mathcal{M}\mathfrak{v}),\\&\quad d(\mathfrak{u},\mathcal{M}\mathfrak{u}),d(\mathfrak{v},\mathcal{M}\mathfrak{u}),d(\mathfrak{u},\mathcal{M}\mathfrak{v})]^\alpha,\end{aligned} \tag{55}$$

where $\alpha=(1/8)e^{-\lambda}$ and $\alpha\in(0,1)$. Therefore, all the conditions of Theorem 8 are verified. Hence, $\mathcal{M}$ has a unique fixed point, which is indeed a unique solution to a boundary value problem (48).

4. Conclusion

We have explored new directions as a fixed ellipse and a fixed elliptic disc to the geometry of the set of nonunique fixed points of a map on a metric space via novel contractions. Further, we have utilized a generalized Θ-contraction and Ćirić-type Θ-contraction to establish a unique fixed point. Also, we have verified established results by illustrative examples to demonstrate the authenticity of the postulates and substantiated utility of our results by solving a boundary value problem of chemical sciences. The study of a set of unique and nonunique fixed points in the current context would be an interesting area for future study.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this research.

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Research Article

Hyers-Ulam Stability of Functional Equation Deriving from Quadratic Mapping in Non-Archimedean (n, β)-Normed Spaces

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In this work, we have to introduce a generalized quadratic functional equation and derive its solution. The main objective of this work is to investigate the Hyers-Ulam stability of quadratic functional equation in non-Archimedean (n, β)-normed spaces.

1. Introduction

The theory of functional equation is one of the most interesting topics in the field of Mathematics. It deals with the search of functions which satisfies a given equation. A functional equation is like a regular algebraic equation; though instead of unknown elements in some set, we are interested in finding a function satisfying our equation.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. Aoki was the first author who has generalized the theorem of Hyers (see [3]).

Moreover, Gavruta [4], Rassias [5], and Bourgin [6] have considered the stability problem with unbounded Cauchy difference (see also [7]). On the other hand, Rassias [8–13] considered the Cauchy difference controlled by a product of different powers of norm. This stability phenomenon is called the Ulam-Gavruta-Rassias stability (see also [14]).

The Hyers-Ulam stability issue for the quadratic functional equation was cleared by Skof [15]. In [16], Czerwik demonstrated the Hyers-Ulam-Rassias stability of the quadratic functional equation. Afterward, Jung [17] has summed up the outcomes gotten by Skof and Czerwik.

The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [15]. In [16], Czerwik proved the Hyers-Ulam-Rassias stability of quadratic functional equation. Later, Jung [17] has generalized the results obtained by Skof and Czerwik.

The first work on the Hyers-Ulam stability of functional equations in complete non-Archimedean normed spaces (some particular cases were considered earlier; see [18] for details) is [19]. After it, a lot of articles (see, for instance, [20] and the references given there) on the stability of other equations in such spaces were published. In [21], the stability of the additive Cauchy equation in non-Archimedean fuzzy normed spaces under the strongest t-norm TM Rassias has been established.

In 1897, Hensel [22] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [23–26].

Initially, Liu [27] introduced the notions of (n, β)-normed space and non-Archimedean (n, β)-normed space. Then, they investigated Hyers-Ulam stability of the Cauchy functional equation and the Jensen functional equation in non-Archimedean (n, β)-normed spaces and that of the pexiderized Cauchy functional equation in (n, β)-normed spaces. During the most recent many years, a few stability issues of functional equations have been researched by various mathematicians (see [28–40]).

The objective of this work is to introduce that a generalized quadratic functional equation is

$$\sum_{1\le i<j\le l}\phi(v_i+v_j)+\sum_{1\le i<j\le l}\phi(v_i-v_j)=2(l-1)\sum_{1\le i\le l}\phi(v_i), \quad (1)$$

where $l \geq 2$ and examine the Hyers-Ulam stability of the above mentioned equation in non-Archimedean (n, β)-normed spaces.

2. Preliminaries

Now, we recall some notions and results which will be used.

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers and $i, t, p, n \in \mathbb{N}$, and let $n \geq 2$ be fixed.

Definition 1 (see [27]). Let E be a linear space over $\mathbb{R}$ with $\dim E \geq n$, $n \in \mathbb{N}$, and $0 < \beta \leq 1$, let a mapping $\|\cdot,\cdots,\cdot\|_\beta : E^n \longrightarrow \mathbb{R}$ satisfies the following conditions:

(i) $\|m_1,\cdots,m_n\|_\beta = 0$ if and only if $m_1, \cdots, m_n$ are linearly dependent

(ii) $\|m_1,\cdots,m_n\|_\beta$ is invariant under permutations of $m_1, \cdots, m_n$

(iii) $\|\lambda m_1,\cdots,m_n\|_\beta = |\lambda|^\beta \|m_1,\cdots,m_n\|_\beta$

(iv) $\|m_1,\cdots,m_{n-1}, u+v\|_\beta \leq \|m_1,\cdots,m_{n-1}, u\|_\beta + \|m_1,\cdots,m_{n-1}, v\|_\beta$

for all $m_1, \cdots, m_n \in E$, and $\lambda \in \mathbb{R}$.

Then, $\|\cdot, \cdots, \cdot\|_\beta$ is called as (n, β)-norm on E, and $(E, \|\cdot, \cdots, \cdot\|_\beta)$ is called a linear (n, β)-normed space or (n, β)-normed space.

We remark that the notion of a linear (n, β)-normed space is a summed up of a linear n-normed space $(\beta = 1)$ and of a β-normed space $(n = 1)$.

Definition 2 (see [27]). Let E be a real vector space with $\dim E \geq n$ over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$, where $n \in \mathbb{Z}^+$ and a constant β with $0 < \beta \leq 1$. A real-valued function $\|\bullet,\cdots,\cdot\|_\beta : E^n \longrightarrow \mathbb{R}$ is called an (n, β)-norm on E if the upcoming conditions true:

(a) $\|m_1,\cdots,m_n\|_\beta = 0$ if and only if $m_1, \cdots, m_n$ are linearly dependent

(b) $\|m_1,\cdots,m_n\|_\beta$ is invariant under permutations of $m_1, \cdots, m_n$

(c) $\|\lambda m_1,\cdots,m_n\|_\beta = |\lambda|^\beta \|m_1,\cdots,m_n\|_\beta$

(d) $\|m_0 + m_1,\cdots,m_n\|_\beta \leq \max\{\|m_0, m_2,\cdots,m_n\|_\beta, \|m_1, m_2\cdots,m_n\|_\beta\}$

for all $\lambda \in K$ and $m_0, m_1, \cdots, m_n \in E$. Then, $(E, \|\cdot, \cdots, \cdot\|_\beta)$ is called as non-Archimedean (n, β)-normed space.

Example 3. Let p be a prime number. For any nonzero rational number $x = (a/b)p^r$ such that a and b are coprime to the prime number p, define the p-adic absolute value $\|x\|_p := p^{-r}$. Then, $\|\cdot\|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $\|\cdot\|_p$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.

Note that if $p > 3$; then, $\|2^n\| = 1$ in for each integer n.

Remark 4 (see [27]). A sequence $\{v_m\}$ in a non-Archimedean (n, β)-normed space E is a Cauchy sequence if and only if $\{v_{m+1} - v_m\}$ converges to zero.

Lemma 5 (see [27]). *For a convergent sequence $\{v_p\}$ in a linear (n, β)-normed space E,*

$$\lim_{p\to\infty}\|v_p, w_1,\cdots,w_{n-1}\|_\beta = \left\|\lim_{p\longrightarrow\infty} v_p, w_1,\cdots,w_{n-1}\right\|_\beta, \quad (2)$$

for all $w_1, \cdots, w_{n-1} \in E$.

Lemma 6 (see [27]). *Let $(E, \|\cdot, \cdots, \cdot\|_\beta)$ be a linear (n, β)-normed space, $n \geq 2$, $0 < \beta \leq 1$. If $v_1 \in E$ and $\|v_1, w_1,\cdots,w_{n-1}\|_\beta = 0$ for all $w_1, \cdots, w_{n-1} \in E$, then $v_1 = 0$.*

3. General Solution

Here, the authors discussed the general solution of the equation (1). Consider E and F are real vector spaces.

Theorem 7. *If $\phi : E \longrightarrow F$ be a mapping satisfies the functional equation (1) for all $v_1, v_2, \cdots, v_l \in E$, then the mapping $\phi : E \longrightarrow F$ is quadratic, that is ϕ satisfies the equality*

$$\phi(v_1 + v_2) + \phi(v_1 - v_2) = 2\phi(v_1) + 2\phi(v_2), \quad (3)$$

for all $v_1, v_2 \in E$.

Proof. Suppose that the mapping $\phi : E \longrightarrow F$ satisfies the functional equation (1). Replacing $(v_1, v_2,\cdots,v_l)$ by $(0, 0,\cdots,0)$ in (1), we get $\phi(0) = 0$. Now, setting $(v_1, v_2,\cdots,v_l)$ by $(v, 0, \cdots,0)$ in (1), we get $\phi(-v) = \phi(v)$ for all $v \in E$. Therefore, the function ϕ is an even function. Substituting $(v_1, v_2,\cdots,v_l)$ by $(v, v, 0,\cdots,0)$ and $(v, v, v, 0,\cdots,0)$ in (1), we obtain $\phi(2v) = 2^2 \phi(v)$, $\phi(3v) = 3^2\phi(v)$, and so on, for every $v \in E$. In general, for any nonnegative integer $n > 0$, we get $\phi(nv) = n^2\phi(v)$ for all $v \in E$. Next, replacing $(v_1, v_2,\cdots,v_l)$ by $(v_1, v_2, 0,\cdots,0)$ in (1), we get our desired outcome.

4. Stability Results

Here, we consider $|2| \neq 1$ and examine the Hyers-Ulam stability of the functional equation (1).

Let us assume E and F are non-Archimedean β_1-normed space and complete non-Archimedean (n,β)-normed space, respectively, where $n \geq 2$, $0 < \beta, \beta_1 \leq 1$.

Define a mapping $\Delta\phi : E \longrightarrow F$ by

$$\Delta\phi(v_1, v_2, \cdots, v_l) = \sum_{1\leq i<j\leq l} \phi(v_i + v_j) + \sum_{1\leq i<j\leq l} \phi(v_i - v_j) - 2(l-1) \sum_{1\leq i\leq l} \phi(v_i), \tag{4}$$

for all $v_1, v_2, \cdots, v_l \in E$.

Theorem 8. *Let $\tau \in [0,\infty)$ and $s \in (0,\infty)$ with $s\beta_1 > \beta$, and let $\chi : \underbrace{F \times F \times \cdots \times F}_{n-1} \longrightarrow [0,\infty)$ be a function. Suppose that $\phi : E \longrightarrow F$ be a mapping satisfies*

$$\|\Delta\phi(v_1, v_2, \cdots, v_l), w_1, \cdots, w_{n-1}\|_\beta \leq \tau \sum_{1\leq j\leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), \tag{5}$$

for all $v_1, v_2, \cdots, v_l$ and $m_1, \cdots, m_{n-1} \in F$. Then, there exists a unique quadratic mapping $Q_2 : E \longrightarrow F$ satisfies

$$\|\phi(v) - Q_2(v), w_1, \cdots, w_{n-1}\|_\beta \leq 2\tau \left|2^{-2\beta}\right| \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), \tag{6}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$.

Proof. Replacing $(v_1, v_2, \cdots, v_l)$ by $(v, v, 0, \cdots, 0)$ in (5), we obtain

$$\left\|\phi(2v) - 2^2\phi(v), w_1, \cdots, w_{n-1}\right\|_\beta \leq 2\tau \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), \tag{7}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Dividing both sides by $|2^{2\beta}|$, we have

$$\left\|\frac{\phi(2v)}{2^2} - \phi(v), w_1, \cdots, w_{n-1}\right\|_\beta \leq 2\left|2^{-2\beta}\right| \tau \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), \tag{8}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Replacing v by $2^p v$ in (8), we attain

$$\begin{aligned} &\left\|\frac{\phi(2^{p+1}v)}{2^{2(p+1)}} - \frac{\phi(2^p v)}{2^{2p}}, m_1, \cdots, m_{n-1}\right\|_\beta \\ &\leq 2\left|2^{-2(p+1)\beta}\right| \tau \|2^p v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}) \\ &\leq 2\left|2^{-2\beta}\right| \left|2^{s\beta_1 - 2\beta}\right|^p \tau \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}). \end{aligned} \tag{9}$$

As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain that

$$\lim_{p\longrightarrow\infty} \left\|\frac{\phi(2^{p+1}v)}{2^{2(p+1)}} - \frac{\phi(2^p v)}{2^{2p}}, m_1, \cdots, m_{n-1}\right\|_\beta = 0, \tag{10}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. From Remark 4, we conclude that the sequence $\{\phi(2^p v)/2^{2p}\}$ is a Cauchy sequence in F. Since F is complete space, we can define $Q_2 : E \longrightarrow F$ by

$$Q_2(v) = \lim_{p\longrightarrow\infty} \frac{\phi(2^p v)}{2^{2p}}, \tag{11}$$

for all $v \in E$. Next, our aim is to prove that the function Q_2 is quadratic. From (5), (11) and Lemma 5 that

$$\begin{aligned} &\|\Delta Q_2(v_1, v_2, \cdots, v_l), m_1, \cdots, m_{n-1}\|_\beta \\ &= \lim_{p\longrightarrow\infty} \left|2^{-2p\beta}\right| \|\Delta\phi(2^p v_1, 2^p v_2, \cdots, 2^p v_l), m_1, \cdots, m_{n-1}\|_\beta \\ &\leq \lim_{p\longrightarrow\infty} \left|2^{-2p\beta}\right| \tau \sum_{1\leq j\leq l} \|2^p v_j\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}) \\ &\leq \lim_{p\longrightarrow\infty} \tau \left|2^{s\beta_1 - 2\beta}\right|^p \sum_{1\leq j\leq l} \|v_j\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), \end{aligned} \tag{12}$$

for all $v_1, v_2, \cdots, v_l \in E$ and $m_1, \cdots, m_{n-1} \in F$. As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain

$$\|\Delta Q_2(v_1, v_2, \cdots, v_l), m_1, \cdots, m_{n-1}\|_\beta = 0, \tag{13}$$

for all $v_1, v_2, \cdots, v_l \in E$, $m_1, \cdots, m_{n-1} \in F$. By Lemma 6, we have

$$\Delta Q_2(v_1, v_2, \cdots, v_l) = 0, \tag{14}$$

for all $v_1, v_2, \cdots, v_l \in E$. Therefore, the function Q_2 is quadratic. Switching v through $2v$ in (7) and divide by $|2^{2\beta}|$, we attain

$$\left\|\frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1, \cdots, m_{n-1}\right\|_\beta \leq 2\tau \left|2^{-4\beta}\right| \|2v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}). \tag{15}$$

Thus, by (7) and (15), we reach

$$\begin{aligned} &\left\|\phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1, \cdots, m_{n-1}\right\|_\beta \\ &\leq \max\left\{\left\|\frac{\phi(2v)}{2^2} - \phi(v), m_1, \cdots, m_{n-1}\right\|_\beta, \left\|\frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1, \cdots, m_{n-1}\right\|_\beta\right\} \\ &\leq \max\left\{2\tau\left|2^{-2\beta}\right| \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1}), 2\tau\left|2^{-4\beta}\right| \|v\|_{\beta_1}^s \chi(m_1, \cdots, m_{n-1})\right\}. \end{aligned} \tag{16}$$

As $s\beta_1 > \beta$ and $|2| \neq 1$, we obtain

$$\left\|\phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1,\cdots,m_{n-1}\right\|_\beta \leq \left|2^{-2\beta}\right| 2\tau\|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (17)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. By induction on p, we can conclude that

$$\left\|\phi(v) - \frac{\phi(2^p v)}{2^{2p}}, m_1,\cdots,m_{n-1}\right\|_\beta \leq 2\left|2^{-2\beta}\right| \tau\|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (18)$$

for all $p \in \mathbb{N}$, $v \in E$, and $m_1, \cdots, m_{n-1} \in F$. Replacing v by $2v$ in (18) and dividing both sides by $|2^{2\beta}|$, we have

$$\left\|\frac{\phi(2v)}{2^2} - \frac{\phi(2^{p+1} v)}{2^{2(p+1)}}, m_1,\cdots,m_{n-1}\right\|_\beta \leq 2\left|2^{-4\beta}\right| \tau\|2v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (19)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $p \in \mathbb{N}$. It follows from (7) and (19) that

$$\left\|\phi(v) - \frac{\phi(2^{p+1} v)}{2^{2(p+1)}}, m_1,\cdots,m_{n-1}\right\|_\beta \leq 2\left|2^{-2\beta}\right| \tau\|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (20)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in inequality (18), we can get (6). Next, we want to prove that the function Q_2 is unique. Let Q_2' be an another quadratic mapping which satisfies (6),

$$\begin{aligned}
&\|Q_2(v) - Q_2'(v), m_1,\cdots,m_{n-1}\|_\beta \\
&= \left|2^{-2p\beta}\right| \|Q_2(2^p v) - Q_2'(2^p v), m_1,\cdots,m_{n-1}\|_\beta \\
&\leq \left|2^{-2p\beta}\right| \max\left\{\|Q_2(2^p v) - \phi(2^p v), m_1,\cdots,m_{n-1}\|_\beta, \|\phi(2^p v) \right. \\
&\quad \left. - Q_2'(2^p v), m_1,\cdots,m_{n-1}\|_\beta\right\} \\
&\leq 2\left|2^{-2p\beta}\right| \left|2^{-2\beta}\right| \tau\|2^p v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}) \\
&\leq 2\tau\left|2^{s\beta_1 - 2\beta}\right|^p \left|2^{-2\beta}\right| \|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}).
\end{aligned} \quad (21)$$

Taking the limit as p tends to ∞ in the last inequality, we obtain that

$$\|Q_2(v) - Q_2'(v), m_1,\cdots,m_{n-1}\|_\beta = 0, \quad (22)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. By Lemma 6, we conclude that $Q_2(v) = Q_2'(v)$ for all $v \in E$. Hence, Q_2 is the unique quadratic mapping which satisfies (6).

Theorem 9. *Let $\tau \in [0,\infty)$, $s \in (0,\infty)$ with $s\beta_1 < \beta$. Let $\chi : \underbrace{F \times F \times \cdots \times F}_{n-1} \longrightarrow [0,\infty)$ be a function. Suppose that a mapping $\phi : E \longrightarrow F$ satisfies*

$$\|\Delta\phi(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta \leq \tau \sum_{1\leq j\leq l} \|v_j\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (23)$$

for all $v_1, v_2, \cdots, v_l$ and $m_1, \cdots, m_{n-1} \in F$. Then, there exists a unique quadratic mapping $Q_2 : E \longrightarrow F$ satisfies

$$\|\phi(v) - Q_2(v), m_1,\cdots,m_{n-1}\|_\beta \leq 2\tau\left|2^{-s\beta_1}\right| \|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (24)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$.

Proof. Switching $(v_1, v_2,\cdots,v_l)$ by $(v, v, 0,\cdots,0)$ in (23), we have

$$\|\phi(2v) - 2^2\phi(v), m_1,\cdots,m_{n-1}\|_\beta \leq 2\tau\|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (25)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Interchanging v by $v/2$ in (25), we obtain

$$\left\|\phi(v) - 2^2\phi\left(\frac{v}{2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \leq 2\tau\left|2^{-s\beta_1}\right| \|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}), \quad (26)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Replacing v by $v/2^p$ in (26), we reach

$$\begin{aligned}
&\left\|2^{2p}\phi\left(\frac{v}{2^p}\right) - 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\
&\quad \leq 2\left|2^{2p\beta}\right| \tau \left|2^{-(p+1)s\beta_1}\right| \|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}) \\
&\quad \leq 2\left|2^{-s\beta_1}\right| \left|2^{2\beta - s\beta_1}\right| p\tau\|v\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}),
\end{aligned} \quad (27)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. As $s\beta_1 < \beta$ and $|2| \neq 1$, we have

$$\lim_{p\longrightarrow\infty} \left\|2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right) - 2^{2p}\phi\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta = 0, \quad (28)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. From Remark 4, we conclude that the sequence $\{2^{2p}\phi(v/2^p)\}$ is a Cauchy sequence in F. As F is complete space. We can define $Q_2 : E \longrightarrow F$ by

$$Q_2(v) = \lim_{p\longrightarrow\infty} 2^{2p}\phi\left(\frac{v}{2^p}\right), \quad (29)$$

for all $v \in E$. Next, our aim is to prove that the function Q_2 is quadratic. From (23), (29) and Lemma 5 that

$$\begin{aligned}\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta &= \lim_{p\longrightarrow\infty}\left|2^{2p\beta}\right|\left\|\Delta\phi\left(\frac{v_1}{2^p}, \frac{v_2}{2^p},\cdots,\frac{v_l}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\le \lim_{p\longrightarrow\infty}\left|2^{2p\beta}\right|\tau\sum_{1\le j\le l}\left\|\frac{v_j}{2^p}\right\|_{\beta_1}^s \chi(m_1,\cdots,m_{n-1}) \\ &\le \lim_{p\longrightarrow\infty}\tau\left|2^{2\beta-s\beta_1}\right|^p\sum_{1\le j\le l}\|v_j\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{30}$$

for all $v_1, v_2, \cdots, v_l \in E$ and $m_1, \cdots, m_{n-1} \in F$. Since $s\beta_1 < \beta$ and $|2| \neq 1$, we have

$$\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta = 0, \tag{31}$$

for all $v_1, v_2, \cdots, v_l \in E$, $m_1, m_2, \cdots, m_{n-1} \in F$. By Lemma 6, we obtain

$$\Delta Q_2(v_1, v_2,\cdots,v_l) = 0, \tag{32}$$

for all $v_1, v_2, \cdots, v_l \in E$. Therefore, the function Q_2 is quadratic. Switching v through $v/2$ in (25) and multiplying by $|2^{2\beta}|$, we reach

$$\left\|2^4\phi\left(\frac{v}{2^2}\right) - 2^2\phi\left(\frac{v}{2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \le 2\tau\left|2^{2\beta}\right|\left\|\frac{v}{2^2}\right\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}). \tag{33}$$

Thus, by (25) and (33), we obtain

$$\begin{aligned}&\left\|\phi(v) - 2^4\phi\left(\frac{v}{2^2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\le \max\left\{\left\|2^2\phi\left(\frac{v}{2}\right) - \phi(v), m_1,\cdots,m_{n-1}\right\|_\beta, \left\|2^4\phi\left(\frac{v}{2^2}\right) - 2^2\phi\left(\frac{v}{2}\right), m_1,\cdots,m_{n-1}\right\|_\beta\right\} \\ &\le \max\left\{2\tau\left\|\frac{v}{2}\right\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}), 2\tau\left|2^{2\beta}\right|\left\|\frac{v}{2^2}\right\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1})\right\}.\end{aligned} \tag{34}$$

Since $s\beta_1 < \beta$ and $|2| \neq 1$, we attain

$$\left\|\phi(v) - 2^4\phi\left(\frac{v}{2^2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \le \left|2^{-s\beta_1}\right|2\tau\|v\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}), \tag{35}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. By induction on p, we can conclude that

$$\left\|\phi(v) - 2^{2p}\phi\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta \le 2\left|2^{-s\beta_1}\right|\tau\|v\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}), \tag{36}$$

for all $p \in \mathbb{N}$, $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Replacing v by $v/2$ in (36) and multiplying both sides by $|2^{2\beta}|$, we get.

$$\begin{aligned}&\left\|2^2\phi\left(\frac{v}{2}\right) - 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\le 2\left|2^{2\beta}\right|\left|2^{-2s\beta_1}\right|\tau\|v\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{37}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. It follows from (25) and (37) that

$$\left\|\phi(v) - 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right), m_1,\cdots,m_{n-1}\right\|_\beta \le 2\left|2^{-s\beta_1}\right|\tau\|v\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}), \tag{38}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in (36), we can get (24). Finally, we want to prove that the function Q_2 is unique. Consider an another quadratic mapping Q_2' satisfying (24),

$$\begin{aligned}&\|Q_2(v) - Q_2'(v), m_1,\cdots,m_{n-1}\|_\beta \\ &= \left|2^{2p\beta}\right|\left\|Q_2\left(\frac{v}{2^p}\right) - Q_2'\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\le \left|2^{2p\beta}\right|\max\left\{\left\|Q_2\left(\frac{v}{2^p}\right) - \phi\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta, \left\|\phi\left(\frac{v}{2^p}\right) - Q_2'\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta\right\} \\ &\le 2\left|2^{2p\beta}\right|\left|2^{-s\beta_1}\right|\tau\left\|\frac{v}{2^p}\right\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}) \\ &\le 2\tau\left|2^{2\beta-s\beta_1}\right|^p\left|2^{-s\beta_1}\right|\|v\|_{\beta_1}^s\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{39}$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Taking the limit as p tends to ∞, we obtain that

$$\|Q_2(v) - Q_2'(v), m_1,\cdots,m_{n-1}\|_\beta = 0, \tag{40}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. By Lemma 6, we conclude that $Q_2(v) = Q_2'(v)$ for all $v \in E$. So that the function Q_2 is the unique quadratic function. Hence, the proof is completed.

We obtain the following results of theorem with a generalized control function when the domain E is a vector space and codomain F be a complete non-Archimendean (n, β)-normed space, where $n \ge 2$ and $0 < \beta \le 1$.

Theorem 10. *Let $\psi : E^l \longrightarrow [0,\infty)$ be a function such that*

$$\lim_{p\longrightarrow\infty}\left|\frac{1}{2^{2p\beta}}\right|\psi(2^p v_1, 2^p v_2,\cdots,2^p v_l) = 0, \tag{41}$$

for all $v_1, v_2, \cdots, v_l \in E$, and let $\chi : \underbrace{F \times F \times \cdots \times F}_{n-1} \longrightarrow [0,\infty)$ be a function. Suppose $\phi : E \to F$ be a mapping which satisfies

$$\|\Delta\phi(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta \le \psi(v_1, v_2,\cdots,v_l)\chi(m_1,\cdots,m_{n-1}), \tag{42}$$

for all $v_1, v_2, \cdots, v_l \in E$ *and* $m_1, \cdots, m_{n-1} \in F$. *Then, there exists a quadratic mapping* $Q_2 : E \longrightarrow F$ *such that*

$$\|\phi(v) - Q_2(v), m_1,\cdots,m_{n-1}\|_\beta \le \tilde{\psi}(v)\chi(m_1,\cdots,m_{n-1}), \quad (43)$$

where

$$\tilde{\psi}(v) = \lim_{p\longrightarrow\infty} \max\left\{\left|2^{-2i\beta}\right|\psi\left(2^{i-1}v, 2^{i-1}v, 0,\cdots,0\right): 1 \le i \le p\right\}, \quad (44)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. *Moreover, if*

$$\lim_{t\longrightarrow\infty}\lim_{p\longrightarrow\infty} \max\left\{\left|2^{-2i\beta}\right|\psi\left(2^{i-1}v, 2^{i-1}v, 0,\cdots,0\right): 1 + t \le i \le p + t\right\} = 0, \quad (45)$$

for all $v \in E$. *Then, the unique mapping* Q_2 *is quadratic which satisfies the inequality (43).*

Proof. Setting $(v_1, v_2,\cdots,v_l)$ by $(v, v, 0,\cdots,0)$ in (42) and dividing both sides by $|2^{2\beta}|$, we have

$$\left\|\frac{\phi(2v)}{2^2} - \phi(v), m_1,\cdots,m_{n-1}\right\|_\beta \le \left|2^{-2\beta}\right|\psi(v, v, 0,\cdots,0)\chi(m_1,\cdots,m_{n-1}), \quad (46)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Replacing v by $2^i v$ in (46) and dividing both sides by $|2^{2i\beta}|$, we obtain

$$\left\|\frac{\phi\left(2^{i+1}v\right)}{2^{2(i+1)}} - \frac{\phi\left(2^i v\right)}{2^{2i}}, m_1,\cdots,m_{n-1}\right\|_\beta \le \left|2^{-2i\beta}\right|\left|2^{-2\beta}\right|\psi\left(2^i v, 2^i v, 0,\cdots,0\right)\chi(m_1,\cdots,m_{n-1}), \quad (47)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $i \in \mathbb{N}$. Taking the limit as i tends to ∞ and considering (41), we attain

$$\lim_{i\longrightarrow\infty}\left\|\frac{\phi\left(2^{i+1}v\right)}{2^{2(i+1)}} - \frac{\phi\left(2^i v\right)}{2^{2i}}, m_1,\cdots,m_{n-1}\right\|_\beta = 0, \quad (48)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Utilizing Remark 4, it is clear that the sequence $\{\phi(2^m v)/2^{2m}\}$ is a Cauchy sequence. As F is a complete space. We can define the mapping $Q_2 : E \longrightarrow F$ by

$$\lim_{i\longrightarrow\infty}\left\|\frac{\phi\left(2^{i+1}v\right)}{2^{2(i+1)}} - \frac{\phi\left(2^i v\right)}{2^{2i}}, m_1,\cdots,m_{n-1}\right\|_\beta = 0, \quad (49)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Next, we want to prove that the function Q_2 is quadratic. So,

$$\begin{aligned}&\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta \\ &\le \left|2^{-2p\beta}\right|\|\Delta Q_2(2^p v_1, 2^p v_2,\cdots,2^p v_l), m_1,\cdots,m_{n-1}\|_\beta \\ &\le \left|2^{-2p\beta}\right|\psi(2^p v, 2^p v, 0,\cdots,0)\chi(m_1,\cdots,m_{n-1}),\end{aligned} \quad (50)$$

for all $v_1, v_2, \cdots, v_l \in E$ and $m_1, \cdots, m_{n-1} \in F$. Taking the limit as $p \longrightarrow \infty$ and considering (41), we arrive

$$\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta = 0, \quad (51)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$. Using Lemma 6, we conclude that Q_2 is quadratic. Switching v through $2v$ in (46) and dividing by $|2^{2\beta}|$, we obtain

$$\left\|\frac{\phi(2^2 v)}{2^4} - \frac{\phi(2v)}{2^2}, m_1,\cdots,m_{n-1}\right\|_\beta \le \left|2^{-4\beta}\right|\psi(2v, 2v, 0,\cdots,0)\chi(m_1,\cdots,m_{n-1}), \quad (52)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Considering (46), we receive

$$\left\|\phi(v) - \frac{\phi(2^2 v)}{2^4}, m_1,\cdots,m_{n-1}\right\|_\beta \le \max\left\{\left|2^{-2\beta}\right|\psi(v, v, 0,\cdots,0), \left|2^{-4\beta}\right|\psi(2v, 2v, 0,\cdots,0)\right\}\chi(m_1,\cdots,m_{n-1}). \quad (53)$$

By induction on p, we reach

$$\left\|\phi(v) - \frac{\phi(2^p v)}{2^{2p}}, m_1,\cdots,m_{n-1}\right\|_\beta \le \max\left\{\frac{\psi\left(2^{t-1}v, 2^{t-1}v, 0,\cdots,0\right)}{\left|2^{2t\beta}\right|} : 1 \le t \le p\right\}\chi(m_1,\cdots,m_{n-1}), \quad (54)$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Replacing v by $2v$ in (54) and dividing both sides by $|2^{2\beta}|$, we have

$$\left\|\frac{\phi(2v)}{2^2} - \frac{\phi\left(2^{(m+1)}v\right)}{2^{2(m+1)}}, m_1,\cdots,m_{n-1}\right\|_\beta \le \max\left\{\frac{\psi(2^t v, 2^t v, 0,\cdots,0)}{\left|2^{2(t+1)\beta}\right|} : 1 \le t \le p\right\}\chi(m_1,\cdots,m_{n-1}), \quad (55)$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $p \in \mathbb{N}$, which together with

(46) implies

$$\begin{aligned}\left\|\phi(v)-\frac{\phi(2^{p+1}v)}{2^{2(p+1)}},m_1,\cdots,m_{n-1}\right\|_\beta &\le \max\left\{\frac{\psi(v,v,0,\cdots,0)}{|2^{2\beta}|},\frac{\psi(2^tv,2^tv,0,\cdots,0)}{|2^{2(t+1)\beta}|}:1\le t\le p\right\}\chi(m_1,\cdots,m_{n-1})\\ &\le \max\left\{\left|2^{-2(t+1)\beta}\right|\psi(2^tv,2^tv,0,\cdots,0):0\le t\le p\right\}\chi(m_1,\cdots,m_{n-1})\\ &\le \max\left\{\left|2^{-2t\beta}\right|\psi(2^{t-1}v,2^{t-1}v,0,\cdots,0):1\le t\le p+1\right\}\chi(m_1,\cdots,m_{n-1}),\end{aligned}\tag{56}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$, and $p\in\mathbb{N}$. Passing the limit as p tends to ∞ in (54), we get (43). Finally, we want to show that the quadratic function Q_2 is unique. Consider an another quadratic function Q_2' satisfying (43). Hence,

$$\begin{aligned}\lim_{p\to\infty}\left|2^{-2t\beta}\right|\tilde\psi(2^tv) &= \lim_{t\to\infty}\left|2^{-2t\beta}\right|\lim_{p\to\infty}\max\left\{\left|2^{-2i\beta}\right|\psi(2^{i+t-1}v,2^{i+t-1}v,0,\cdots,0):1\le i\le p\right\}\\ &\le \lim_{t\to\infty}\lim_{p\to\infty}\max\left\{\left|2^{-2i\beta}\right|\psi(2^{i-1}v,2^{i-1}v,0,\cdots,0):1+t\le i\le p+t\right\},\end{aligned}\tag{57}$$

for all $v\in E$ and $m_1,\cdots,m_{n-1}\in F$. It follows from (45) that

$$\begin{aligned}\|Q_2(v)-Q_2'(v),m_1,\cdots,m_{n-1}\|_\beta &\le \lim_{t\to\infty}\left|2^{-2t\beta}\right|\|Q_2(2^tv)-Q_2'(2^tv),m_1,\cdots,m_{n-1}\|_\beta\\ &\le \lim_{t\to\infty}\left|2^{-2t\beta}\right|\max\left\{\|Q_2(2^tv)-\phi(2^tv),m_1,\cdots,m_{n-1}\|_\beta,\|\phi(2^tv)-Q_2'(2^tv),m_1,\cdots,m_{n-1}\|_\beta\right\}\\ &\le \lim_{t\to\infty}\left|2^{-2t\beta}\right|\tilde\psi(2^tv)\chi(m_1,\cdots,m_{n-1})=0.\end{aligned}\tag{58}$$

From Lemma 6, we conclude that the quadratic function Q_2 is unique. This ends the proof of the theorem.

Theorem 11. *Let $\psi:E^l\to[0,\infty)$ be a mapping such that*

$$\lim_{p\to\infty}\left|2^{2p\beta}\right|\psi\left(\frac{v_1}{2^p},\frac{v_2}{2^p},\cdots,\frac{v_l}{2^p}\right)=0,\tag{59}$$

for all $v_1,v_2,\cdots,v_l\in E$, and let $\chi:\underbrace{F\times F\times\cdots\times F}_{n-1}\to[0,\infty)$ be a function. Suppose that $\phi:E\to F$ be a mapping satisfies

$$\|\Delta\phi(v_1,v_2,\cdots,v_l),m_1,\cdots,m_{n-1}\|_\beta\le\psi(v_1,v_2,\cdots,v_l)\chi(m_1,\cdots,m_{n-1}),\tag{60}$$

for all $v_1,v_2,\cdots,v_l\in E$ and $m_1,\cdots,m_{n-1}\in F$. Then, there exists a quadratic mapping $Q_2:E\to F$ such that

$$\|\phi(v)-Q_2(v),m_1,\cdots,m_{n-1}\|_\beta\le\tilde\psi(v)\chi(m_1,\cdots,m_{n-1}),\tag{61}$$

where

$$\tilde\psi(v)=\lim_{p\to\infty}\max\left\{\left|2^{2(i-1)\beta}\right|\psi(2^{-i}v,2^{-i}v,0,\cdots,0):1\le i\le p\right\},\tag{62}$$

for all $v\in E$ and $m_1,\cdots,m_{n-1}\in F$. Moreover, if

$$\lim_{t\to\infty}\lim_{p\to\infty}\max\left\{\left|2^{2(i-1)\beta}\right|\psi(2^{-i}v,2^{-i}v,0,\cdots,0):1+t\le i\le p+t\right\}=0,\tag{63}$$

for all $v\in E$, then the quadratic mapping Q_2 is unique which satisfies (61).

Proof. Switching $(v_1,v_2,\cdots,v_l)$ by $(v,v,0,\cdots,0)$ in (60), we have

$$\|\phi(2v)-2^2\phi(v),m_1,\cdots,m_{n-1}\|_\beta\le\psi(v,v,0,\cdots,0)\chi(m_1,\cdots,m_{n-1}),\tag{64}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$. Interchanging v by $v/2$ in (64), we obtain

$$\left\|\phi(v)-2^2\phi\left(\frac{v}{2}\right),m_1,\cdots,m_{n-1}\right\|_\beta\le\psi\left(\frac{v}{2},\frac{v}{2},0,\cdots,0\right)\chi(m_1,\cdots,m_{n-1}),\tag{65}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$. Replacing v by $v/2^{i-1}$ in (65) and multiplying both sides by $|2^{2(i-1)\beta}|$, we obtain

$$\begin{aligned}&\left\|2^{2(i)}\phi\left(\frac{v}{2^i}\right)-2^{2(i-1)}\phi\left(\frac{v}{2^{(i-1)}}\right),m_1,\cdots,m_{n-1}\right\|_\beta\\ &\quad\le\left|2^{2(i-1)\beta}\right|\psi\left(\frac{v}{2^i},\frac{v}{2^i},0,\cdots,0\right)\chi(m_1,\cdots,m_{n-1}),\end{aligned}\tag{66}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$, and $i\in\mathbb{N}$. Taking the limit as i tends to ∞ and considering (59), we attain

$$\lim_{i\to\infty}\left\|2^{2(i)}\phi\left(\frac{v}{2^i}\right)-2^{2(i-1)}\phi\left(\frac{v}{2^{(i-1)}}\right),m_1,\cdots,m_{n-1}\right\|_\beta=0,\tag{67}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$. From Remark 4, we can conclude that the sequence $\{2^{2m}\phi(v/2^m)\}$ is a Cauchy sequence. We know that F is a complete space; we define a mapping $Q_2:E\to F$ by

$$\lim_{i\to\infty}\left\|2^{2(i)}\phi\left(\frac{v}{2^i}\right)-2^{2(i-1)}\phi\left(\frac{v}{2^{(i-1)}}\right),m_1,\cdots,m_{n-1}\right\|_\beta=0,\tag{68}$$

for all $v\in E$, $m_1,\cdots,m_{n-1}\in F$. Next, we prove that Q_2 is quadratic. So,

$$\begin{aligned}\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta &\le \left|2^{2p\beta}\right|\left\|\Delta Q_2\left(\frac{v_1}{2^p}, \frac{v_2}{2^p},\cdots,\frac{v_l}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\le \left|2^{2p\beta}\right|\psi\left(\frac{v}{2^p}, \frac{v}{2^p}, 0,\cdots,0\right)\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{69}$$

for all $v_1, v_2, \cdots, v_l \in E$ and $m_1, \cdots, m_{n-1} \in F$. Taking the limit as $p \to \infty$ and considering (59), we arrive

$$\|\Delta Q_2(v_1, v_2,\cdots,v_l), m_1,\cdots,m_{n-1}\|_\beta = 0, \tag{70}$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. By Lemma 6, we know that Q_2 is quadratic. Replacing v by $v/2$ in (65) and multiplying both sides by $|2^{2\beta}|$, we obtain

$$\begin{aligned}&\left\|2^4\phi\left(\frac{v}{2^2}\right) - 2^2\phi\left(\frac{v}{2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\quad\le \left|2^{2\beta}\right|\psi\left(\frac{v}{2^2}, \frac{v}{2^2}, 0,\cdots,0\right)\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{71}$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Considering (65), we receive

$$\begin{aligned}&\left\|\phi(v) - 2^4\phi\left(\frac{v}{2^2}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\quad\le \max\left\{\psi\left(\frac{v}{2}, \frac{v}{2}, 0,\cdots,0\right), \left|2^{2\beta}\right|\psi\left(\frac{v}{2^2}, \frac{v}{2^2}, 0,\cdots,0\right)\right\}\chi(m_1,\cdots,m_{n-1}).\end{aligned} \tag{72}$$

By induction on p, we reach

$$\begin{aligned}&\left\|\phi(v) - 2^{2p}\phi\left(\frac{v}{2^p}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\quad\le \max\left\{\left|2^{2(t-1)\beta}\right|\psi(2^{-t}v, 2^{-t}v, 0,\cdots,0)\colon 1 \le t \le p\right\}\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{73}$$

for all $v \in E$ and $m_1, \cdots, m_{n-1} \in F$. Replacing v by $v/2$ in (73) and multiplying both sides by $|2^{2\beta}|$, we have

$$\begin{aligned}&\|2^2\phi\left(\frac{v}{2}\right) - 2^{2(m+1)}\phi\left(\frac{v}{2^{(m+1)}}\right), m_1, \cdots, m_{n-1}\|_\beta \\ &\quad\le \max\left\{\left|2^{2(t-1)\beta}\right|\psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0,\cdots,0\right)\colon 1 \le t \le p\right\}\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{74}$$

which together with (65) implies

$$\begin{aligned}&\|\phi(v) - 2^{2(p+1)}\phi\left(\frac{v}{2^{p+1}}\right), m_1, \cdots, m_{n-1}\|_\beta \\ &\quad\le \max\left\{\psi\left(\frac{v}{2}, \frac{v}{2}, 0,\cdots,0\right), \left|2^{2(t-1)\beta}\right|\psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0,\cdots,0\right)\colon 1 \le t \le p\right\}\chi(m_1,\cdots,m_{n-1}) \\ &\quad\le \max\left\{\left|2^{2(t-1)\beta}\right|\psi\left(\frac{v}{2^t}, \frac{v}{2^t}, 0,\cdots,0\right)\colon 1 \le t \le p+1\right\}\chi(m_1,\cdots,m_{n-1}),\end{aligned} \tag{75}$$

for all $v \in E$, $m_1, \cdots, m_{n-1} \in F$, and $p \in \mathbb{N}$. Passing the limit as p tends to ∞ in (73), we get (61). At the end, we show that the quadratic function Q_2 is unique. Consider an another quadratic mapping Q_2' which satisfies (61). Therefore,

$$\begin{aligned}\lim_{p\to\infty}\left|2^{2t\beta}\right|\tilde{\psi}\left(\frac{v}{2^t}\right) &= \lim_{t\to\infty}\left|2^{2t\beta}\right|\lim_{p\to\infty}\max\left\{\left|2^{2(i-1)\beta}\right|\psi\left(\frac{v}{2^{i+t}}, \frac{v}{2^{i+t}}, 0,\cdots,0\right)\colon 1 \le i \le p\right\} \\ &\le \lim_{t\to\infty}\lim_{p\to\infty}\max\left\{\left|2^{2(i-1)\beta}\right|\psi\left(\frac{v}{2^i}, \frac{v}{2^i}, 0,\cdots,0\right)\colon 1+t \le i \le p+t\right\},\end{aligned} \tag{76}$$

for all $v \in E$. It follows from (63) that

$$\begin{aligned}&\|Q_2(v) - Q_2'(v), m_1,\cdots,m_{n-1}\|_\beta \\ &\quad\le \lim_{t\to\infty}\left|2^{2t\beta}\right|\left\|Q_2\left(\frac{v}{2^t}\right) - Q_2'\left(\frac{v}{2^t}\right), m_1,\cdots,m_{n-1}\right\|_\beta \\ &\quad\le \lim_{t\to\infty}\left|2^{2(t)\beta}\right|\max\left\{\left\|Q_2\left(\frac{v}{2^t}\right) - \phi\left(\frac{v}{2^t}\right), m_1,\cdots,m_{n-1}\right\|_\beta, \|\phi(2^tv)\right. \\ &\qquad\left. - Q_2'(2^tv), m_1,\cdots,m_{n-1}\|_\beta\right\} \\ &\quad\le \lim_{t\to\infty}\left|2^{2(t-1)\beta}\right|\tilde{\psi}\left(\frac{v}{2^t}\right)\chi(m_1,\cdots,m_{n-1}) = 0.\end{aligned} \tag{77}$$

By Lemma 6, we conclude that the quadratic function Q_2 is unique. This ends the proof of the theorem.

5. Conclusion

In this work, we introduced a new finite dimensional quadratic functional equation (1) and obtained its solution to show which satisfies the quadratic properties. Mainly, investigated Hyers-Ulam stability of considered domain E is a non-Archimedean β-normed space in Theorem 8 and Theorem 9 and examined by considered the domain E is a vector space in Theorem 10 and Theorem 11. The results obtained in this article may be further generalized to be in non-Archimedean quasi(n, β)-Banach spaces. This could be a potential future work.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version manuscript.

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Research Article

Terminal Value Problem for Implicit Katugampola Fractional Differential Equations in *b*-Metric Spaces

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This manuscript deals with a class of Katugampola implicit fractional differential equations in *b*-metric spaces. The results are based on the $\alpha-\varphi$-Geraghty type contraction and the fixed point theory. We express an illustrative example.

1. Introduction and Preliminaries

An interesting extension and unification of fractional derivatives of the type Caputo and the type Caputo-Hadamard is called Katugampola fractional derivative that has been introduced by Katugampola [1, 2]. Some fundamental properties of this operator are presented in [3, 4]. Several results of implicit fractional differential equations have been recently provided (see [4–14] and the references therein). A new class of mixed monotone operators with concavity and applications to fractional differential equations has been considered in [15]. In [16], the authors presented some existence and uniqueness results for a class of terminal value problem for differential equations with Hilfer-Katugampola fractional derivative.

On the other side, a novel extension of *b*-metric was suggested by Czerwik [17, 18]. Although the *b*-metric standard looks very similar to the metric definition, it has a quite different structure and properties. For example, in the *b*-metric topology framework, an open (closed) set is not open (closed). Additionally, the *b*-metric function is not continuous. These weaknesses make this new structure more interesting (see [19–28]).

Throughout the paper, any mentioned set is nonempty. We consider the following type of terminal value problems of Katugampola implicit differential equations of noninteger orders:

$$\begin{cases} \left({}^{\rho}D^{r}_{0^+}+\vartheta\right)(\tau)=\kappa\left(\tau,\vartheta(\tau),\left({}^{\rho}D^{r}_{0^+}+\vartheta\right)(\tau)\right), & \tau\in I:=[0,T],\\ \vartheta(T)=\vartheta_T\in\mathbb{R}, \end{cases} \tag{1}$$

with $T>0$ and the function $\kappa: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous. Here, ${}^{\rho}D^{r}_{0^+}$ is the Katugampola fractional derivative of order $r\in(0,1]$.

Set $C(I):=\{h\mid h \text{ real continuous functions on } I:=[0,T]\}$. Then, $C(I)$ forms a Banach space with the norm $\|\vartheta\|_\infty=\sup_{\tau\in I}|\vartheta(\tau)|$.

Set $L^1(I):=\{\vartheta: \mathrm{I}\to\mathbb{R}|\vartheta \text{ is measurable function and Lebesgue integrable}\}$. Then, $L^1(I)$ becomes a Banach space with the norm $\|\vartheta\|_{L^1}=\int_0^T|\vartheta(\tau)|dt$.

Set $C_{r,\rho}(I)=\{\vartheta:(0,T]\to\mathbb{R}|\tau^{\rho(1-r)}\vartheta(\tau)\in C(I)\}$. Then, it forms a Banach space $\|\vartheta\|_C:=\sup_{\tau\in I}\|\tau^{\rho(1-r)}\vartheta(\tau)\|$. Here, $C_{r,\rho}(I)$ is called the weighted space of continuous functions.

Definition 1 (Katugampola fractional integral) [1]. The Katugampola fractional integrals of order $r>0$ and $\rho>0$ of a function $y\in X_c^p(I)$ are defined by

$$ {}^{\rho}T_{0^+}^{r}y(\tau)\,\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t\frac{s^{\rho-1}y(s)}{(\tau^\rho-s^\rho)^{1-r}}ds,\quad \tau\in I. \tag{2}$$

Definition 2 (Katugampola fractional derivatives) [1, 2]. The generalized fractional derivatives of order $r>0$ and $\rho>0$ corresponding to the Katugampola fractional integrals (2) defined for any $\tau\in I$ by

$$ {}^{\rho}D_{0^+}^{r}y(\tau)=\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\left({}^{\rho}T_{0^+}^{n-r}y\right)(\tau)=\frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\int_0^t\frac{s^{\rho-1}y(s)}{(\tau^\rho-s^\rho)^{r-n+1}}ds, \tag{3}$$

where $n=[r]+1$; if the integrals exist.

Remark 1 ([1, 2]). As a basic example, we quote for $r,\rho>0$ and $\theta>-\rho$,

$$ {}^{\rho}D_{0^+}^{r}\tau^\theta=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1-r+(\theta/\rho))}\tau^{\theta-r\rho}. \tag{4}$$

Giving in particular,

$$ {}^{\rho}D_{0^+}^{r}\tau^{\rho(r-i)}=0,\quad \text{for each } i=1,2,\cdots,n. \tag{5}$$

In fact, for $r,\rho>0$ and $\theta>-\rho$, we have

$$\begin{aligned} {}^{\rho}D_{0^+}^{r}\tau^\theta&=\frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(\tau^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho+\theta-1}(\tau^\rho-s^\rho)^{n-r-1}ds\\ &=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1+n-r+(\theta/\rho))}\left[n-r+\frac{\theta}{\rho}\right]\cdots\left[1-r+\frac{\theta}{\rho}\right]\tau^{\theta-r}\rho\\ &=\frac{\rho^{r-1}\Gamma(1+(\theta/\rho))}{\Gamma(1-r+(\theta/\rho))}\tau^{\theta-r}\rho. \end{aligned} \tag{6}$$

If we put $i=r-(\theta/\rho)$, we obtain from (6):

$$ {}^{\rho}D_{0^+}^{r}\tau^{\theta(r-i)}=\rho^{r-1}\frac{\Gamma(r-i+1)}{\Gamma(n-i+1)}(n-i)(n-i-1)\cdots(1-m)\tau^{-\rho i}. \tag{7}$$

So, ${}^{\rho}D_{0^+}^{r}\tau^{\rho(r-i)}=0,\forall r,\rho>0$.

Theorem 1 ([2]). *Let $r,\rho,c\in\mathbb{R}$, be such that $r,\rho>0$. Then, for any $\kappa,\omega\in X_c^p(I)$, where $1\le p\le\infty$, we have*

(1) Inverse property:

$$ {}^{\rho}D_{0^+}^{r}\,{}^{\rho}I_{0^+}^{r}\kappa(\tau)=\kappa(\tau),\quad \textit{for all } r\in(0,1]. \tag{8}$$

(2) Linearity property: for all $r\in(0,1)$, we have

$$\begin{cases} {}^{\rho}D_{0^+}^{r}(\kappa+\omega)(\tau)={}^{\rho}D_{0^+}^{r}\kappa(\tau)+{}^{\rho}D_{0^+}^{r}\omega(\tau).\\ {}^{\rho}I_{0^+}^{r}(\kappa+\omega)(\tau)={}^{\rho}I_{0^+}^{r}\kappa(\tau)+{}^{\rho}I_{0^+}^{r}\omega(\tau).\end{cases} \tag{9}$$

Lemma 1 ([2]). *Let $r,\rho>0$. If $\vartheta\in C(I)$; then the fractional differential equation ${}^{\rho}D_{0^+}^{r}+\vartheta(\tau)=0$, has a unique solution*

$$\vartheta(\tau)=C_1\tau^{\rho(r-1)}+C_2\tau^{\rho(r-2)}+\cdots+C_n\tau^{\rho(r-n)}, \tag{10}$$

where $C_i\in\mathbb{R}$ with $i=1,2,\cdots,n$.

Proof. Let $r,\rho>0$. from Remark 1, we have

$$ {}^{\rho}D_{0^+}^{r}\tau^{\rho(r-i)}=0,\quad \text{for each } i=1,2,\cdots,n. \tag{11}$$

Then, the fractional equation ${}^{\rho}D_{0^+}^{r}\vartheta(\tau)=0$ has a particular solution as follows:

$$\vartheta(\tau)=C_i\tau^{\rho(r-i)},\quad C_i\in\mathbb{R},\text{ for each } i=1,2,\cdots,n. \tag{12}$$

Thus, the general solution of ${}^{\rho}D_{0^+}^{r}\vartheta(\tau)=0$ is a sum of particular solutions (12), i.e.

$$\vartheta(\tau)=C_1\tau^{\rho(r-1)}+C_2\tau^{\rho(r-2)}+\cdots+C_n\tau^{\rho(r-n)},\,C_i\in\mathbb{R};\,(i=1,2,\cdots,n). \tag{13}$$

Lemma 2. *Let $r,\rho>0$. If $\vartheta,{}^{\rho}D_{0^+}^{r}\vartheta\in C(I)$ and $0<r\le1$, then*

$$ {}^{\rho}I_{0^+}^{r}\,{}^{\rho}D_{0^+}^{r}\vartheta(\tau)=\vartheta(\tau)+c\tau^{\rho(r-1)}, \tag{14}$$

for some constant $c\in\mathbb{R}$.

Proof. Let ${}^{\rho}D_{0^+}^{r}\vartheta\in C(I)$ be the fractional derivative (3) of order $0<r\le1$. If we apply the operator ${}^{\rho}D_{0^+}^{r}$ to ${}^{\rho}I_{0^+}^{r}\,{}^{\rho}D_{0^+}^{r}\vartheta(\tau)-\vartheta(\tau)$ and use the properties (8) and (9), we get

$$\begin{aligned} {}^{\rho}D_{0^+}^{r}\left[{}^{\rho}I_{0^+}^{r}\,{}^{\rho}D_{0^+}^{r}\vartheta(\tau)-\vartheta(\tau)\right]&={}^{\rho}D_{0^+}^{r}\,{}^{\rho}I_{0^+}^{r}\,{}^{\rho}D_{0^+}^{r}\vartheta(\tau)-{}^{\rho}D_{0^+}^{r}\vartheta(\tau)\\ &={}^{\rho}D_{0^+}^{r}\vartheta(\tau)-{}^{\rho}D_{0^+}^{r}\vartheta(\tau)=0. \end{aligned} \tag{15}$$

From the proof of Lemma 1, there exists $c\in\mathbb{R}$, such that

$$ {}^{\rho}I_{0^+}^{r}\,{}^{\rho}D_{0^+}^{r}\vartheta(\tau)-\vartheta(\tau)=c\tau^{\rho(r-1)}, \tag{16}$$

which implies (14).

Lemma 3. *Let $h \in L^1(I, \mathbb{R})$ and $0 < r \leq 1$ and $\rho > 0$. A function $\vartheta \in C(I)$ forms a solution for*

$$\begin{cases} ({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau), & \tau \in I, \\ \vartheta(T) = \vartheta_T, \end{cases} \tag{17}$$

if and only if ϑ fulfills

$$\vartheta(\tau) = (\vartheta_T - {}^{\rho}I^{r}_{0^+}z(T))\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{r-1}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}z(s)ds. \tag{18}$$

Proof. Let $r, \rho > 0$. and $0 < r \leq 1$. Suppose that ϑ satisfies (17). Employing the operator ${}^{\rho}I^{r}_{0^+}$ to the each side of the equation

$$({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau), \tag{19}$$

we find

$${}^{\rho}I^{r}_{0^+}{}^{\rho}D^{r}_{0^+}\vartheta(\tau) = {}^{\rho}I^{r}_{0^+}z(\tau). \tag{20}$$

From Lemma 2, we get

$$\vartheta(\tau) + c\tau^{\rho(r-1)} = {}^{\rho}I^{r}_{0^+}z(\tau), \tag{21}$$

for some $c \in \mathbb{R}$. If we use the terminal condition $\vartheta(T) = \vartheta_T$ in (21), we find

$$\vartheta(T) = \vartheta_T = {}^{\rho}I^{r}_{0^+}z(T) - cT^{\rho(r-1)}, \tag{22}$$

which shows

$$c = ({}^{\rho}I^{r}_{0^+}z(T) - \vartheta_T)T^{\rho(1-r)}. \tag{23}$$

Henceforth, we deduce (18).

Contrariwise, if ϑ achieves (18), then $({}^{\rho}D^{r}_{0^+}\vartheta)(\tau) = z(\tau)$; for $\tau \in I$ and $\vartheta(\tau) = \vartheta_T$.

Lemma 4. *Contemplate the problem (1), and set $g \in C(I)$, and $\omega(\tau) = \varkappa(\tau, \vartheta(\tau), \omega(\tau))$.*

We presume ϑ achieves

$$\vartheta(\tau) = (\vartheta_T - {}^{\rho}I^{r}_{0^+}\omega(T))\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{r-1}\omega(s)ds. \tag{24}$$

Then, ϑ forms a solution of (1).

Definition 3 [29, 30]. A function $d : S \times S \longrightarrow [0,\infty)$ is called b-metric if there is $c \geq 1$ and d fulfills

(i) $(bM1)$ $d(\nu, \vartheta) = 0$ if and only if $\nu = \vartheta$

(ii) $(bM2)$ $d(\nu, \mu) = d(\mu, \nu)$

(iii) $(bM3)$ $d(\mu, \vartheta) \leq c[d(\mu, \nu) + d(\nu, \vartheta)]$

for all $\mu, \nu, \vartheta \in S$. We say that the tripled (S, d, c) is b-metric space (in short, b.m.s.).

Example 1 [29, 30]. *Let $d : C(I) \times C(I) \longrightarrow [0,\infty)$ be described as*

$$d(\nu, \vartheta) = \left\|(\nu - \vartheta)^2\right\|_{\infty} \coloneqq \sup_{\tau \in I}\|\nu(\tau) - \vartheta(\tau)\|^2, \quad \text{for all } \nu, \vartheta EC(I). \tag{25}$$

Ergo, $(C(I), d, 2)$ is b-metric space.

Example 2 [29, 30]. Set $S = [0, 1]$ and $d : S \times S \longrightarrow [0,\infty)$ be designated by

$$d(\nu, \vartheta) = |\nu^r - \vartheta^r|, \quad \text{for all } \nu, \vartheta \in S. \tag{26}$$

Henceforth, (S, d, r) with $r \geq 2$ is b-metric space.

We set the following: $\{\phi : [0,\infty) \to [0,\infty) | \phi$ is continuous, increasing, $\phi(0) = 0$ and $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$ for $c > 1\}$.

For some $c \geq 1$, we set $\mathcal{F} \coloneqq \{\lambda : [0,\infty) \to [0, (1/c^2)) | \lambda$ is nondecreasing$\}$.

Definition 4 [29, 30]. A self-operator T, on a *b.m.s.* (S, d, c), is called a generalized $\alpha - \phi -$ Geraghty contraction whenever there exists $\alpha : S \times S \longrightarrow [0,\infty)$, and some $L \geq 0$ such that for

$$D(\nu, \vartheta) = \max\left\{d(\nu, \vartheta), d(\vartheta, T(\vartheta)), d(\nu, T(\nu)), \frac{d(\nu, T(\vartheta)) + d(\vartheta, T(\nu))}{2s}\right\}, \tag{27}$$

$$N(\nu, \vartheta) = \min\{d(\nu, \vartheta), d(\vartheta, T(\vartheta)), d(\nu, T(\nu))\}, \tag{28}$$

we have

$$\alpha(\mu, \nu)\varphi\left(c^3 d(T(\mu), T(\nu))\right) \leq \lambda(\varphi(D(\mu, \nu))(\varphi(D(\mu, \nu)) + L\psi(N(\mu, \nu), \tag{29}$$

for all $\mu, \nu, \vartheta \in S$, where $\lambda \in \mathcal{F}, \varphi, \psi \in \Phi$.

Remark 2. In the case when $L = 0$ in Definition 4 and the fact that

$$d(\mu, \nu) \leq D(\mu, \nu), \quad \text{for all } \mu, \nu \in S, \tag{30}$$

the inequality (29) becomes

$$\alpha(\mu, \nu)\varphi\left(c^3 d(T(\mu), T(\nu)\right) \leq \lambda(\varphi(d(\mu, \nu))\varphi(d(\mu, \nu))). \tag{31}$$

Definition 5 [29, 30]. Set $\alpha : S \times S \longrightarrow [0,\infty)$. An operator $T : S \longrightarrow S$, is $\alpha -$ admissible if

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(T(\mu), T(\nu)) \geq 1, \tag{32}$$

for all $\mu, \nu \in S$.

Definition 6 [29, 30]. Let (S, d, c) with $c \geq 1$ be a *b.m.s* and $\alpha : S \times S\mathbb{R}_+^*$.

We say that S is $\alpha-$ regular if for any sequence $\{v_n\}_{n\in\mathbb{N}}$ in S such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\alpha(v_n, v_{n+1}) \geq 1$ for each n; there exists a subsequence $\{v_n(\kappa)\}_{\kappa\in\mathbb{N}}$ of $\{v_n\}n$ with $\alpha(v_{n(\kappa)}, x) \geq 1$ for all k.

Theorem 2 [29, 30]. *We presume that a self-operator T over a complete b.m.s.*

(S, d, c) with $c \geq 1$ forms a generalized $\alpha-\varphi-$ Geraghty contraction. Furthermore,

(i) T is $\alpha-$ admissible with initial value $\alpha(\mu 0, T(\mu 0)) \geq 1$ for some $\mu 0 \in M$

(ii) either T is continuous or M is $\alpha-$ regular

Then T possesses a fixed point. Furthermore, if

(iii) for all fixed points μ, v of T, either $\alpha(\mu, v) \geq 1$ or $\alpha(v, \mu) \geq 1$, then the found fixed point is unique

This manuscript launches the study of Katugampola implicit fractional differential equations on b.m.s.

2. Main Results

Observe that $(C_{r,\rho}(I), d, 2)$ is a complete b.m.s. with $d : C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow [0,\infty)$ described as

$$d(v, \vartheta) = \left\|(v-\vartheta)^2\right\|C := \sup_{\tau\in I} \tau^{\rho(1-r)}|v(\tau)-\vartheta(\tau)|^2. \quad (33)$$

A function $\vartheta \in C_{r,\rho}(I)$ is called a solution of (1) if it archives

$$\vartheta(\tau) = \left(\vartheta_T - {}^{\rho}I_{0^+}^{r}\omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}\omega(s)ds, \quad (34)$$

with $\omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau)) \in C(I)$.

In the sequel, we shall need the following hypotheses:

(H_1) There exist $\varphi \in \Phi, p : C(I) \times C(I) \longrightarrow (0,\infty)$ and $q : I \longrightarrow (0, 1)$ so that for each $\vartheta, v, \vartheta_1, v_1 \in C_{r,\rho}(I)$, and $\tau \in I$

$$|\kappa(\tau, \vartheta, v) - \kappa(\tau, \vartheta_1, v_1)| \leq \tau^{\rho/2(1-r)}p(\vartheta, v)|\vartheta-\vartheta_1| + q(\tau)|v-v_1|, \quad (35)$$

with

$$\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{T} s^{\rho-1}\left(T^{\rho}-s^{\rho}\right)^{r-1}\frac{p(\vartheta, v)}{1-q*}ds\right\|_C^2 + \left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}\frac{p(\vartheta, v)}{1-q*}ds\right\|_C^2 \leq \varphi\left(\left\|(\vartheta-v)^2\right\|C\right) \quad (36)$$

(H_2) There are $\mu_0 \in C_{r,\rho}(\mathrm{I})$ and $\theta : C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow \mathbb{R}$, so that

$$\theta\left(\mu_0(\tau), \left(\vartheta_T - {}^{\rho}I_{0+}^{r}\omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}\omega(s)ds\right) \geq 0, \quad (37)$$

with $g \in C(I)$ and $\omega(\tau) = \kappa(\tau, \mu 0(\tau), \omega(\tau))$

(H_3) For any $\tau \in I$, and $\vartheta, v \in C_{r,\rho}(I)$, $\theta(\vartheta(\tau), v(\tau)) \geq 0$ implies

$$\theta\left(\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}\omega(s)ds, \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}\mathfrak{z}(s)ds\right) \geq 0, \quad (38)$$

with $\omega, \mathfrak{z} \in C(I)$ so that

$$\begin{cases} \mathfrak{z}(\tau) = \kappa(\tau, v(\tau), \mathfrak{z}(\tau)), \\ \omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau)). \end{cases} \quad (39)$$

(H_4) If $\vartheta_{nn\in N} \subset C(I)$ with $\vartheta_n \longrightarrow \vartheta$ and $\theta(\vartheta_n, \vartheta_{n+1}) \geq$, then

$$\theta(\vartheta_n, \vartheta) \geq 1. \quad (40)$$

Theorem 3. *We presume (H_1)–(H_4). Then, the problem (1) possesses at least a solution on I.*

Proof. Take the operator $N : C_r, \rho(I) \longrightarrow C_r, \rho(I)$ into account that is described as

$$(N\vartheta)(\tau) = \left(\vartheta T - {}^{\rho}I_{0+}^{r}\omega(T)\right)\left(\frac{\tau}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}s^{\rho}\right)^{r-1}\omega(s)ds, \quad (41)$$

where $\omega \in C(I)$, with $\omega(\tau) = \kappa(\tau, \vartheta(\tau), \omega(\tau))$.

On account of Lemma 4, we deduce that solutions of (1) are the fixed points of N.

Let $C_{r,\rho}(I) \times C_{r,\rho}(I) \longrightarrow (0,\infty)$ be the function defined by

$$\begin{cases} \alpha(\vartheta, v) = 1, & \text{if } \theta(\vartheta(\tau)v(\tau)) \geq 0, \tau \in I, \\ \alpha(\vartheta, v) = 0, & \text{otherwise}. \end{cases} \quad (42)$$

First, we demonstrate that N form a generalized $\alpha-\varphi$ -Geraghty operator. For any $\tau \in I$ and each $\vartheta, v \in C(I)$, we derive that

$$\begin{aligned} &\left|\tau^{\rho(1-r)}(N\vartheta)(\tau) - \tau^{\rho(1-r)}(Nv)(\tau)\right| \\ &\quad\leq \tau^{\rho(1-r)}\left|{}^{\rho}I_{0+}^{r}(g-h)(T)\right|\left(\frac{\tau}{T}\right)^{\rho(r-1)} \\ &\qquad+ \frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{r-1}|\omega(s)-\mathfrak{z}(s)|ds, \end{aligned} \quad (43)$$

where ω, $\mathfrak{z}\in C(I)$, with

$$\omega(\tau)=\kappa(\tau,\vartheta(\tau),\omega(\tau)), \quad (44)$$

$$\mathfrak{z}(\tau)=\kappa(\tau,\nu(\tau),\mathfrak{z}(\tau)). \quad (45)$$

From (H_1), we have

$$\begin{aligned}|\omega(\tau)-\mathfrak{z}(\tau)| &= |\kappa(\tau,\vartheta(\tau),\omega(\tau))-\kappa(\tau,\upsilon(\tau),\mathfrak{z}(\tau))| \\ &\le p(\vartheta,\upsilon)\tau^{\rho/2(1-r)}|\vartheta(\tau)-\upsilon(\tau)|+q(\tau)|\omega(\tau)-\mathfrak{z}(\tau)| \\ &\le p(\vartheta,\upsilon)\left(\tau^{\rho(1-r)}|\vartheta(\tau)-\upsilon(\tau)|^2\right)^{1/2}+q(\tau)|\omega(\tau)-\mathfrak{z}(\tau)|.\end{aligned} \quad (46)$$

Thus,

$$|\omega(\tau)-\mathfrak{z}(\tau)|\frac{p(\vartheta,\upsilon)}{1-q*}\left\|(\vartheta-\upsilon)^2\right\|_C^{1/2}, \quad (47)$$

where $q*=\sup_{\tau\in I}|q(\tau)|$.

Next, we have

$$\begin{aligned}\left|\tau^{\rho(1-r)}(N\vartheta)(\tau)-\tau^{\rho(1-r)}(N\upsilon)(\tau)\right| &\le \tau^{\rho(1-r)}|{}^{\rho}I_{0+}^{r}(g-h)(T)|\left(\frac{\tau}{T}\right)^{\rho(r-1)} \\ &+\frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{\tau}s^{\rho-1}(\tau^{\rho}-s^{\rho})^{r-1}\frac{p(\vartheta,\upsilon)}{1-q*}\left\|(\vartheta-\upsilon)^2\right\|_C^{1/2}ds \\ &\le \frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{T}s^{\rho-1}(T^{\rho}-s^{\rho})^{r-1}\frac{p(\vartheta,\upsilon)}{1-q*}\left\|(\vartheta-\upsilon)^2\right\|_C^{1/2}ds \\ &+\frac{\rho^{1-r}\tau^{\rho(1-r)}}{\Gamma(r)}\int_0^{\tau}s^{\rho-1}(\tau^{\rho}-s^{\rho})^{r-1}\frac{p(\vartheta,\upsilon)}{1-q*}\left\|(\vartheta-\upsilon)^2\right\|_C^{1/2}ds.\end{aligned} \quad (48)$$

Thus,

$$\begin{aligned}&\alpha(\vartheta,\upsilon)\left|\tau^{\rho(1-r)}(N\vartheta)(\tau)-\tau^{\rho(1-r)}(N\upsilon)(\tau)\right|^2 \\ &\le \left\|(\vartheta-\upsilon)^2\right\|C\alpha(\vartheta,\upsilon)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{T}s^{\rho-1}(T^{\rho}-s^{\rho})^{r-1}\frac{p(\vartheta,\upsilon)}{1-q*}ds\right\|_C^2 \\ &+\left\|(\vartheta-\upsilon)^2\right\|C\alpha(\vartheta,\upsilon)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^{T}s^{\rho-1}(\tau^{\rho}-s^{\rho})^{r-1}\frac{p(\vartheta,\upsilon)}{1-q*}ds\right\|_C^2 \\ &\le \left\|(\vartheta-\upsilon)^2\right\|C\phi\left(\left\|(\vartheta-\upsilon)^2\right\|C\right).\end{aligned} \quad (49)$$

Hence,

$$\alpha(\vartheta,\nu)\varphi\left(2^3d(N(\vartheta),N(\nu)\right)\le\lambda(\varphi(d(\vartheta,\nu))\varphi(d(\vartheta,\nu), \quad (50)$$

where $\lambda\in$ F, $\varphi\in\Phi$, with $\lambda(\tau)=1/8t$, and $\varphi(\tau)=\tau$.

So, N is generalized $\alpha-\varphi-$ Geraghty operator.

Let $\vartheta,\nu\in C_{r},_{\rho}(I)$ such that

$$\alpha(\vartheta,\nu)\ge 1. \quad (51)$$

Accordingly, for any $t\in I$, we find

$$\theta(\vartheta(\tau),\nu(\tau))\ge 0. \quad (52)$$

This implies from (H_3) that

$$\theta(Nu(\tau),N\nu(\tau))\ge 0, \quad (53)$$

which gives $\alpha(N(\vartheta),N(\nu))\ge 1$.

Ergo, N is a α-admissible.

Now, from (H_2), there exists $\mu_0\in C_{r},_{\rho}(I)$ such that

$$\alpha(\mu_0,N(\mu_0))\ge 1. \quad (54)$$

Finally, from (H_4), if $\mu_{nn\in}N\subset M$ with $\mu_n\longrightarrow\mu$ and $\alpha(\mu_n,\mu_n+1)\ge 1$, then,

$$\alpha(\mu_n,\mu)\ge 1. \quad (55)$$

Theorem 2 implies that fixed point ϑ of N forms a solution for (1).

3. An Example

The tripled $(C_{r},_{\rho}([0,1]),d,2)$ is a complete b.m.s. with $d: C_{r},_{\rho}([0,1])\times C_{r,\rho}([0,1])\longrightarrow[0,\infty)$ such that

$$d(\mu,\vartheta)=\left\|(\mu-\vartheta)^2\right\|C. \quad (56)$$

We take the following fractional differential problem into consideration

$$\begin{cases}({}^{\rho}D_{0+}^{r}\mu)(\tau)=\kappa(\tau,\mu(\tau),(\rho D_{0+}^{r}\mu)(\tau)), & \tau\in[0,1],\\ \mu(1)=2,\end{cases} \quad (57)$$

with

$$\kappa(\tau,\mu(\tau),\vartheta(\tau))=\frac{\tau\rho/2(1-r)(1+\sin(|\mu(\tau)|))}{4(1+|\mu(\tau)|)}+\frac{e^{-\tau}}{2(1+|\vartheta(\tau)|)};\tau\in[0,1]. \quad (58)$$

Let $\tau\in(0,1]$, and $\mu,\vartheta\in C_{r,\rho}([0,1])$. If $|\mu(\tau)|\le|\vartheta(\tau)|$, then

$$
\begin{aligned}
&|\kappa(\tau,\mu(\tau),\mu_1(\tau))-\kappa(\tau,\vartheta(\tau),\vartheta_1(\tau))| \\
&\quad = \tau^{\rho/2(1-r)}\left|\frac{1+\sin(|\mu(\tau)|)}{4(1+|\mu(\tau)|)}-\frac{1+\sin(|\vartheta(\tau)|)}{4(1+|\vartheta(\tau)|)}\right| \\
&\qquad + \left|\frac{e^{-\tau}}{2(1+|\mu_1(\tau)|)}-\frac{e^{-\tau}}{2(1+|\vartheta_1(\tau)|)}\right| \\
&\quad \le \frac{\tau^{\rho/2(1-r)}}{4}\left||\mu(\tau)|-|\vartheta(\tau)|\right| + \frac{\tau^{\rho/2(1-r)}}{4}|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad + \frac{\tau^{\rho/2(1-r)}}{4}\left||\mu(\tau)|\sin(|\vartheta(\tau)|)-|\vartheta(\tau)|\sin(|\mu(\tau)|)\right| \\
&\qquad + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau))| \le \frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad + \frac{\tau^{\rho/2(1-r)}}{4}|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad + \frac{\tau^{\rho/2(1-r)}}{4}\left||\vartheta(\tau)|\sin(|\vartheta(\tau)|)-|\vartheta(\tau)|\sin(|\mu(\tau)|)\right| \\
&\qquad + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau))| = \frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad + \frac{\tau^{\rho/2(1-r)}}{4}(1+|\upsilon(\tau)|)|\sin(|\mu(\tau)|)-\sin(|\vartheta(\tau)|)| \\
&\qquad + \frac{e^{-\tau}}{2}|u_1(\tau)-\vartheta_1(\tau))| \le \frac{\tau^{\rho/2(1-r)}}{4}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad + \frac{\tau^{\rho/2(1-r)}}{2}(1+|\vartheta(\tau)|)\times\left|\sin\left(\frac{\left||\mu(\tau)|-|\vartheta(\tau)|\right|}{2}\right)\right|\left|\cos\left(\frac{|\mu(\tau)|+|\vartheta(\tau)|}{2}\right)\right| \\
&\qquad + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)| \le \frac{\tau^{\rho/2(1-r)}}{4}(2+|\upsilon(\tau)|)|\mu(\tau)-\vartheta(\tau)| + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.
\end{aligned}
\tag{59}
$$

In the case when $|\vartheta(\tau)| \le |\mu(\tau)|$, we get

$$
|\kappa(\tau,\mu(\tau))-\kappa(\tau,\vartheta(\tau))| \le \frac{\tau^{p/2(1-r)}}{4}\Big(2+|\mu(\tau)||\mu(\tau)-\vartheta(\tau)| + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.
\tag{60}
$$

Hence,

$$
\begin{aligned}
&|\kappa(\tau,\mu(\tau))-\kappa(\tau,\vartheta(\tau))| \\
&\quad \le \frac{T^{p/2(1-r)}}{4}\min_{\tau\in I}\{2+|\mu(\tau)|, 2+|\vartheta(\tau)|\}|\mu(\tau)-\vartheta(\tau)| \\
&\qquad + \frac{e^{-\tau}}{2}|\mu_1(\tau)-\vartheta_1(\tau)|.
\end{aligned}
\tag{61}
$$

Thus, hypothesis (H_1) is achieved with

$$
p(\mu,\vartheta) = \frac{T^{\rho/2(1-r)}}{4}\min_{r\in I}\{2+|\mu(\tau)|, 2+|\vartheta(\tau)|\},
\tag{62}
$$

$$
q(\tau) = \frac{1}{2}e^{-\tau}.
\tag{63}
$$

Define the functions $\lambda(\tau)=(1/8)t$, $\phi(\tau)=\tau$, $\alpha : C_{r,\rho}([0,1])\times C_{r,\rho}([0,1]) \to \mathbb{R}_+^*$ with

$$
\begin{cases}
\alpha(\mu,\vartheta)=1, & \text{if } \delta(\mu(\tau),\vartheta(\tau)) \ge 0, \tau\in I, \\
\alpha(\mu,\vartheta)=0, & \text{else}
\end{cases}
\tag{64}
$$

and $\delta : C_{r,\rho}([0,1])\times C_{r,\rho}([0,1]) \longrightarrow R$ with $\delta(\mu,\vartheta)=k\mu-\vartheta$ k_C.

Hypothesis (H_2) is satisfied with $\mu_0(\tau)=\mu_0$. Also, (H_3) holds the definition of the function δ. So, Theorem 3 yields that problem (57) admits a solution.

Data Availability

No data is used. No data is available in this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Complex-Valued Triple Controlled Metric Spaces and Applications

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In this manuscript, we introduce the concept of complex-valued triple controlled metric spaces as an extension of rectangular metric type spaces. To validate our hypotheses and to show the usability of the Banach and Kannan fixed point results discussed herein, we present an application on Fredholm-type integral equations and an application on higher degree polynomial equations.

1. Introduction

Since the breakthrough of Banach [1] in 1922, where he was able to show that a contractive mapping on a complete metric space has a unique fixed point, the field of fixed point theory has become an important research focus in the field of mathematics; see [2–6]. Due to the fact that fixed point theory has many applications in many fields of science, many researchers have been working on generalizing his result by either generalizing the type of contractions [7–10] or by extending the metric space itself (b-metric spaces [11, 12], controlled metric spaces [13], double controlled metric spaces [14], etc.). On the other hand, Azam et al. [15] defined complex-valued metric spaces and gave common fixed point results. Rao et al. [16] introduced the complex-valued b-metric spaces in the year 2013. Going in the same direction, recently, Ullah et al. [17] presented complex-valued extended b-metric spaces to extend the idea of extended b-metric spaces.

In this manuscript, following the path of the work done in [18], we extend complex-valued rectangular extended b-metric spaces [19] to complex-valued triple controlled metric spaces. The layout of our manuscript is as follows. In the second section, we present some backgrounds along with the definition of complex-valued triple controlled metric spaces. In the third section, we prove some fixed point results in such spaces. In the fourth section, we present an application for our findings. In closing, we present two open questions.

2. Preliminaries

In what follows, owing to Azam et al. [15], we recall several notations and definitions which will be used in the sequel.

Let $\mathbb{C}$ be the set of all complex numbers and $s_1, s_2 \in \mathbb{C}$. The partial order on $\mathbb{C}$ is defined as $s_1 \preceq s_2$ if and only if Re $(s_1) \leq$ Re (s_2) and Im $(s_1) \leq$ Im (s_2). This implies that $s_1 \preceq s_2$ if one of the below conditions is fulfilled:

(i) $\mathrm{Re}\,(s_1) = \mathrm{Re}\,(s_2), \mathrm{Im}\,(s_1) < \mathrm{Im}\,(s_2)$

(ii) $\mathrm{Re}\,(s_1) < \mathrm{Re}\,(s_2), \mathrm{Im}\,(s_1) = \mathrm{Im}\,(s_2)$

(iii) $(\mathrm{Re}\,(s_1) < \mathrm{Re}\,(s_2), \mathrm{Im}\,(s_1) < \mathrm{Im}\,(s_2)$

(iv) $\mathrm{Re}\,(s_1) = \mathrm{Re}\,(s_2), \mathrm{Im}\,(s_1) = \mathrm{Im}\,(s_2)$

Following [15], the authors in [17] developed the notion of complex-valued extended b-metric spaces.

Definition 1 (see [17]). Let $\mathfrak{X}$ be a nonempty set and $\xi : \mathfrak{X} \times \mathfrak{X} \longrightarrow [1,\infty)$ be a function. Then, $L_e : \mathfrak{X}^2 \longrightarrow \mathbb{C}$ is known as a complex-valued extended b-metric space if the following are satisfied for all $s, \kappa, u \in \mathfrak{X}$:

(1) $0 \preceq L_e(s, \kappa)$ and $L_e(s, \kappa) = 0$ if and only if $s = \kappa$

(2) $L_e(s, \kappa) = L_e(\kappa, s)$

(3) $L_e(s, \kappa) \preceq \xi(s, \kappa)[L_e(s, u) + L_e(u, \kappa)]$

Then, the pair $(\mathfrak{X}, L_e)$ is known as a complex-valued extended b-metric space.

As an extension of complex-valued extended b-metric spaces, Ullah et al. in [19] introduced the concept of complex-valued rectangular extended b-metric spaces.

Definition 2 (see [19]). Let $\mathfrak{X}$ be a nonempty set and $\xi : \mathfrak{X}^2 \longrightarrow [1,\infty)$ and $L_r : \mathfrak{X}^2 \longrightarrow \mathbb{C}$. We say that $(\mathfrak{X}, L_r)$ is a complex-valued rectangular extended b-metric space if for all $a, b \in \mathfrak{X}$ each of which is different from $\kappa, \nu \in \mathfrak{X}$, we have

(1) $L_r(s, \kappa) = 0$ if and only if $s = \kappa$

(2) $L_r(s, \kappa) = L_r(\kappa, s)$

(3) $L_r(a, b) \preceq \xi(a, b)[L_r(a, \kappa) + L_r(\kappa, \nu) + L_r(\nu, b)]$

The authors in [20] have recently introduced the idea of triple controlled metric type spaces as follows.

Definition 3 (see [20]). Let $\mathfrak{X}$ be a nonempty set. Given three functions $\xi, \rho, \varsigma : \mathfrak{X}^2 \longrightarrow [1,\infty)$ and $L_T : \mathfrak{X}^2 \longrightarrow [0,\infty)$. We say that $(\mathfrak{X}, L_T)$ is a triple controlled metric type space if for all $a, b, \kappa, \nu \in \mathfrak{X}$, we have

(1) $L_T(s, \kappa) = 0$ if and only if $s = \kappa$

(2) $L_T(s, \kappa) = L_T(\kappa, s)$

(3) $L_T(a, b) \leq \xi(a, \kappa)L_T(a, \kappa) + \rho(\kappa, \nu)L_T(\kappa, \nu) + \varsigma(\nu, b)$ $L_T(\nu, b)$

Highly motivated by the abovementioned concepts, we now present the definition of complex-valued triple controlled metric spaces.

Definition 4. Let $\mathfrak{X}$ be a nonempty set. Given three functions $\xi, \rho, \varsigma : \mathfrak{X}^2 \longrightarrow [1,\infty)$ and $L_t : \mathfrak{X}^2 \longrightarrow \mathbb{C}$. We say that $(\mathfrak{X}, L_t)$ is a complex-valued triple controlled metric space if for all $a, b \in \mathfrak{X}$, each of which is different from $\kappa, \nu \in \mathfrak{X}$, we have

(1) $L_t(s, \kappa) = 0$ if and only if $s = \kappa$

(2) $L_t(s, \kappa) = L_t(\kappa, s)$

(3) $L_t(a, b) \preceq \xi(a, \kappa)L_t(a, \kappa) + \rho(\kappa, \nu)L_t(\kappa, \nu) + \varsigma(\nu, b)L_t(\nu, b)]$

Throughout the rest of this paper, we will denote a complex-valued triple controlled metric space by (CV-TCMS). Next, we present the topology of (CV-TCMSs).

Definition 5. Let $(\mathfrak{X}, L_t)$ be a (CV-TCMS).

(1) We say that a sequence $\{a_n\}$ is L_t-convergent to some $a \in \mathfrak{X}$ if $|L_t(a_n, a)| \longrightarrow 0$ as $n \longrightarrow \infty$

(2) We say that a sequence $\{a_n\}$ is L_t-Cauchy if and only if $\lim_{n,m\longrightarrow\infty} | L_t(a_n, a_m) | = 0$

(3) We say that $(\mathfrak{X}, L_t)$ is L_t-complete if for every L_t-Cauchy sequence is L_t-convergent

(4) Let $x \in \mathfrak{X}$. An open ball of center x and radius $\eta > 0$ in the (CV-TCMS) $(\mathfrak{X}, L_t)$ is $B_\xi(x, \eta) = \{b \in \mathfrak{X} \mid L_t(x, b) \preceq \eta\}$

Note that a CV rectangular metric space is a CV-TCMS. The converse is not true. Next, we present an example that confirms this statement.

Example 1. Let $\mathfrak{X} = \mathfrak{Y} \cup \mathfrak{Z}$ where $\mathfrak{Y} = \{(1/k) \mid k \in \mathbb{N}\}$ and $\mathfrak{Z}$ is the set of positive integers. We define $L_t : \mathfrak{X}^2 \longrightarrow \mathbb{C}$ by

$$L_t(a, b) = \begin{cases} 0, & \Leftrightarrow a = b, \\ 2i\beta, & \text{if} \quad a, b \in \mathfrak{Y}, \\ \dfrac{i\beta}{2}, & \text{otherwise}, \end{cases} \tag{1}$$

where $\beta > 0$. Now, define $\xi : \mathfrak{X}^2 \longrightarrow [1,\infty)$ by $\xi(a, b) = 4\beta$. Given $\rho : \mathfrak{X}^2 \longrightarrow [1,\infty)$ as $\rho(a, b) = 3\beta$ and $\varsigma : \mathfrak{X}^2 \longrightarrow [1,\infty)$ as $\varsigma(a, b) = \max\{a, b\} + 2\beta$.

Note that $(\mathfrak{X}, L_t)$ is a CV-TCMS. On the other hand, $(\mathfrak{X}, L_t)$ is not a CV rectangular metric space. Indeed,

$$L_t\left(\frac{1}{2}, \frac{1}{3}\right) = 2i\beta \succ L_t\left(\frac{1}{2}, 2\right) + L_t(2, 3) + L_t\left(3, \frac{1}{3}\right) = \frac{3i\beta}{2}. \tag{2}$$

In this paper, we prove the Banach and Kannan fixed point results in the setting of CV-TCMSs. Two related applications are also investigated.

3. Main Results

Theorem 1. *Let $(\mathfrak{X}, L_t)$ be a L_t-complete CV-TCMS. Let $\mathbb{T}: \mathfrak{X} \longrightarrow \mathfrak{X}$ satisfy $L_t(\mathbb{T}x, \mathbb{T}y) \preceq \delta L_t(x, y)$ where $0 < \delta < 1$. Assume that there exists $x_0 \in \mathfrak{X}$ such that the sequence $\{x_n\}$ defined by $x_n = \mathbb{T}^n x_0$ satisfies the following:*

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \xi(x_n, x_{n+1}) \le \frac{1}{\delta}, \\ &\lim_{n\longrightarrow\infty} \rho(x_n, x_{n+1}) \le \frac{1}{\delta}, \\ &\lim_{n\longrightarrow\infty} \varsigma(x_n, y) < \infty, \\ &\lim_{n\longrightarrow\infty} \xi(y, x_n) < \infty \text{for any } y \in \mathfrak{X}, \\ &\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \xi(x_n, x_{n+1})\varsigma(x_n, x_m) \le \frac{1}{\delta}, \\ &\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \rho(x_n, x_{n+1})\varsigma(x_n, x_m) \le \frac{1}{\delta}. \end{aligned} \tag{3}$$

Then, $\mathbb{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. First, we have $L_t(x_n, x_{n+1}) \preceq \delta L_t(x_{n-1}, x_n) \preceq \delta^2 L_t(x_{n-2}, x_{n-1}) \preceq \cdots \preceq \delta^n L_t(x_0, x_1)$. Then,

$$|L_t(x_n, x_{n+1})| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4}$$

Now, let $L_i = L_t(x_{n+i}, x_{n+i+1})$. We need to consider the following two cases.

Case 1. Let $x_n = x_m$ for some natural numbers n and m with $n \neq m$. Without loss of generality, take $m > n$. If $\mathbb{T}^{m-n}(x_n) = x_n$; then, by choosing $y = x_n$ and $p = m - n$, we get $\mathbb{T}^p y = y$, which implies that y is a periodic point of $\mathbb{T}$. Hence, $L_t(y, \mathbb{T}y) = L_t(\mathbb{T}^p y, \mathbb{T}^{p+1} y) \preceq \delta^p L_t(y, \mathbb{T}y)$. Since $\delta \in (0, 1)$, we get $|L_t(y, \mathbb{T}y)| = 0$, so $y = \mathbb{T}y$, that is, $\mathbb{T}$ has a fixed point.

From now on, we consider the following case.

Case 2. Assume that for all natural numbers $n \neq m$, we have $x_n = \mathbb{T}^n x_0 \neq \mathbb{T}^m x_0 = x_m$. Let $n < m$. To prove that $\{x_n\}$ is a L_t-Cauchy sequence, we need to consider the following two subcases.

Subcase 1. If $m = n + 2p + 1$ (where $p \ge 1$ is a fixed natural number), then by the rectangle inequality of the CV-TCMS, we have

$$\begin{aligned} |L_t(x_n, x_{n+2p+1})| &\le \xi(x_n, x_{n+1})|L_t(x_n, x_{n+1})| + \rho(x_{n+1}, x_{n+2}) \\ &\quad\cdot |L_t(x_{n+1}, x_{n+2})| + \varsigma(x_{n+2}, x_{n+2p+1})|L_t(x_{n+2}, x_{n+2p+1})|\,] \\ &\le \xi(x_n, x_{n+1})|L_t(x_n, x_{n+1})| + \rho(x_{n+1}, x_{n+2})|L_t(x_{n+1}, x_{n+2})| \\ &\quad + \varsigma(x_{n+2}, x_{n+2p+1})[\xi(x_{n+2}, x_{n+3})\,|L_t(x_{n+2}, x_{n+3})| \\ &\quad \cdot + \rho(x_{n+3}, x_{n+4})\,|L_t(x_{n+3}, x_{n+4})| + \varsigma(x_{n+4}, x_{n+2p+1}) \\ &\quad \cdot\, |L_t(x_{n+4}, x_{n+2p+1})|\,] \le \xi(x_n, x_{n+1})\delta^n|L_0| + \rho(x_{n+1}, x_{n+2})\delta^{n+1} \\ &\quad \cdot |L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\left[\xi(x_{n+2}, x_{n+3})\delta^{n+2}\,|L_0| + \rho(x_{n+3}, x_{n+4})\delta^{n+3}\right. \\ &\quad \cdot\, |L_0| + \varsigma(x_{n+4}, x_{n+2p+1})\,|L_t(x_{n+4}, x_{n+2p+1})|\,] \le \xi(x_n, x_{n+1}) \\ &\quad \cdot \delta^n|L_0| + \rho(x_{n+1}, x_{n+2})\delta^{n+1}|L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\xi(x_{n+2}, x_{n+3})\delta^{n+2} \\ &\quad \cdot |L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\rho(x_{n+3}, x_{n+4})\delta^{n+3}|L_0| \\ &\quad + \varsigma(x_{n+2}, x_{n+2p+1})\varsigma(x_{n+4}, x_{n+2p+1})[\xi(x_{n+4}, x_{n+5})\,|L_t(x_{n+4}, x_{n+5})| \\ &\quad + \rho(x_{n+5}, x_{n+6})\,|L_t(x_{n+5}, x_{n+6})| + \varsigma(x_{n+6}, x_{n+2p+1}) \\ &\quad \cdot\, |L_t(x_{n+6}, x_{n+2p+1})|\,] \le \xi(x_n, x_{n+1})\delta^n|L_0| + \rho(x_{n+1}, x_{n+2})\delta^{n+1}|L_0| \\ &\quad + \varsigma(x_{n+2}, x_{n+2p+1})\xi(x_{n+2}, x_{n+3})\delta^{n+2}|L_0| + \varsigma(x_{n+2}, x_{n+2p+1}) \\ &\quad \cdot \rho(x_{n+3}, x_{n+4})\delta^{n+3}|L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\varsigma(x_{n+4}, x_{n+2p+1}) \\ &\quad \cdot \xi(x_{n+4}, x_{n+5})\delta^{n+4}|L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\varsigma(x_{n+4}, x_{n+2p+1}) \\ &\quad \cdot \rho(x_{n+5}, x_{n+6})\delta^{n+5}|L_0| + \varsigma(x_{n+2}, x_{n+2p+1})\varsigma(x_{n+4}, x_{n+2p+1}) \\ &\quad \cdot \varsigma(x_{n+6}, x_{n+2p+1})|L_t(x_{n+6}, x_{n+2p+1})| \le \cdots = \xi(x_n, x_{n+1})\delta^n|L_0| \\ &\quad + \rho(x_{n+1}, x_{n+2})\delta^{n+1}|L_0| + \sum_{l=1}^{p}\prod_{i=1}^{l}\xi(x_{n+2l}, x_{n+2l+1}) \\ &\quad \cdot \varsigma(x_{n+2i}, x_{n+2p+1})\delta^{n+2l}|L_0 + \sum_{l=1}^{p}\prod_{i=1}^{l}\rho(x_{n+2l+1}, x_{n+2l+2}) \\ &\quad \cdot \varsigma(x_{n+2i}, x_{n+2p+1})\delta^{n+2l+1}|L_0|. \end{aligned} \tag{5}$$

Now, given that

$$\begin{aligned} &\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \xi(x_n, x_{n+1})\varsigma(x_n, x_m) \le \frac{1}{\delta}, \\ &\sup_{m\ge 1} \lim_{n\longrightarrow\infty} \rho(x_n, x_{n+1})\varsigma(x_n, x_m) \le \frac{1}{\delta}, \end{aligned} \tag{6}$$

we can easily deduce that

$$|L_t(x_n, x_{n+2p+1})| \le \left[\delta^{n-1} + \delta^{n-1} + \sum_{l=1}^{p}\delta^{n+l} + \sum_{l=1}^{p}\delta^{n+l+1}\right]|L_0|. \tag{7}$$

Since $\lim_{n\longrightarrow\infty}\delta^n = 0$, the last right-hand side goes to zero at the limit $n \longrightarrow \infty$ (for any integer $p \ge 1$). Therefore, $\{|L_t(x_n, x_{n+2p+1})|\}_n$ is convergent.

Subcase 2. Let $m = n + 2p$ (where $p \ge 1$ is a fixed integer). First, notice the following:

$$L_t(x_n, x_{n+2}) \preceq \delta L_t(x_{n-1}, x_{n+1}) \preceq \delta^2 L_t(x_{n-2}, x_n) \preceq \cdots \preceq \delta^n L_t(x_0, x_2), \tag{8}$$

which leads us to conclude that

$$|L_t(x_n, x_{n+2})| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{9}$$

Thus, by Subcase 1 and using the rectangular inequality of the complex-valued triple controlled metric, we have

$$\begin{aligned} \left|L_t\left(x_n, x_{n+2p}\right)\right| &\le \xi\left(x_n, x_{n+2p-3}\right)\left|L_t\left(x_n, x_{n+2p-3}\right)\right| \\ &+ \rho\left(x_{n+2p-3}, x_{n+2p-2}\right)\left|L_t\left(x_{n+2p-3}, x_{n+2p-2}\right)\right| \\ &+ \varsigma\left(x_{n+2p-2}, x_{n+2p}\right)\left|L_t\left(x_{n+2p-2}, x_{n+2p}\right)\right| \le \xi(x_n, x_{n+1})\delta^n|L_0| \\ &+ \rho(x_{n+1}, x_{n+2})\delta^{n+1}|L_0| + \sum_{l=1}^{p-1}\prod_{i=1}^{l}\xi(x_{n+2l}, x_{n+2l+1}) \\ &\cdot \varsigma\left(x_{n+2i}, x_{n+2p-3}\right)\delta^{n+2l}|L_0| + \sum_{l=1}^{p-1}\prod_{i=1}^{l}\rho(x_{n+2l+1}, x_{n+2l+2}) \\ &\cdot \varsigma\left(x_{n+2i}, x_{n+2p-3}\right)\delta^{n+2l+1}|L_0| + \varsigma\left(x_{n+2p-2}, x_{n+2p}\right)\delta^n|L_t(x_0, x_2)|. \end{aligned} \tag{10}$$

Now, similar to Subcase 1, one can easily deduce that $\{|L_t(x_n, x_{n+2p})|\}_n$ is a convergent sequence as $n \longrightarrow \infty$ (for any integer $p \ge 1$). Hence, by Subcases 1 and 2, we conclude that $\{x_n\}$ is a L_t-Cauchy sequence. Since $(\mathfrak{X}, L_t)$ is a L_t-complete CV-TCMS, there is $\nu \in \mathfrak{X}$ such that $\{x_n\} \longrightarrow \nu$ as $n \longrightarrow \infty$.

Now, if there exists $N \in \mathbb{N}$ such that $x_N = \nu$, then since we deal with Case 2, one writes $x_n = \mathbb{T}^n x_0 \ne \nu$ for all $n > N$. Also, $x_n = \mathbb{T}^n x_0 \ne \mathbb{T}\nu$ for all $n > N$. Next, assume that there exists $N \in \mathbb{N}$ with $x_N = \mathbb{T}^N x_0 = \mathbb{T}\nu$. Once again, we confirm that $x_n = \mathbb{T}^n x_0 \in \{\nu, \mathbb{T}\nu\}$ for all $n > N$. Thus, without loss of generality, we may assume $x_n \in \{\nu, \mathbb{T}\nu\}$ for all natural numbers n. We have

$$\begin{aligned} L_t(\nu, \mathbb{T}\nu) &\preceq \xi(\nu, x_n)L_t(\nu, x_n) + \rho(x_n, x_{n+1})L_t(x_n, x_{n+1}) \\ &+ \varsigma(x_{n+1}, \mathbb{T}\nu)L_t(x_{n+1}, \mathbb{T}\nu) \preceq \xi(\nu, x_n)L_t(\nu, x_n) \\ &+ \rho(x_n, x_{n+1})L_t(x_n, x_{n+1} + \varsigma(x_{n+1}, \mathbb{T}\nu)\delta L_t(x_n, \nu), \end{aligned} \tag{11}$$

which implies

$$\begin{aligned} |L_t(\nu, \mathbb{T}\nu)| &\le \xi(\nu, x_n)|L_t(\nu, x_n)| + \rho(x_n, x_{n+1})|L_t(x_n, x_{n+1})| \\ &+ \varsigma(x_{n+1}, \mathbb{T}\nu)\delta|L_t(x_n, \nu)|. \end{aligned} \tag{12}$$

Therefore, in view of the assumptions in the theorem, as $n \longrightarrow \infty$, we deduce that $|L_t(\nu, \mathbb{T}\nu)| = 0$ and that is $\mathbb{T}\nu = \nu$ as required.

In closing, assume there exist two fixed points of $\mathbb{T}$, say ν and μ where $\nu \ne \mu$. Thus,

$$L_t(\nu, \mu) = L_t(\mathbb{T}\nu, \mathbb{T}\mu) \preceq \delta L_t(\nu, \mu) \prec L_t(\nu, \mu), \tag{13}$$

which is a contradiction. Therefore, the fixed point of $\mathbb{T}$ is unique.

Theorem 2. *Let $(\mathfrak{X}, L_t)$ be a L_t-complete CV-TCMS and $\mathbb{T}$ be a self mapping on $\mathfrak{X}$ satisfying the following condition: for all $a, b \in \mathfrak{X}$, there exists $0 < \delta < 1/2$ such that*

$$L_t(\mathbb{T}a, \mathbb{T}b) \preceq \delta[L_t(a, \mathbb{T}a) + L_t(b, \mathbb{T}b)], \tag{14}$$

and there exists $x_0 \in \mathfrak{X}$ in order that the sequence $\{x_n\}$ defined by $x_n = \mathbb{T}^n x_0$ satisfies the following:

$$\begin{aligned} &\lim_{n\longrightarrow\infty}\xi(y, x_n) \le \frac{1}{\delta}, \\ &\lim_{n\longrightarrow\infty}\varsigma(x_n, y) < \frac{1}{\delta} \quad \text{for any } y \in \mathfrak{X}, \\ &\lim_{n\longrightarrow\infty}\rho(x_n, x_{n+1}) \le \frac{1}{\delta}. \end{aligned} \tag{15}$$

Then, $\mathbb{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. First of all, note that for all $n \ge 1$, we have

$$L_t(x_n, x_{n+1}) \preceq \delta[L_t(x_{n-1}, x_n) + L_t(x_n, x_{n+1})]. \tag{16}$$

Consequently,

$$L_t(x_n, x_{n+1}) \preceq \frac{\delta}{1-\delta}L_t(x_{n-1}, x_n). \tag{17}$$

Since $0 < \delta < 1/2$, one has $0 < (\delta/(1-\delta)) < 1$. Set $\mu = \delta/(1-\delta)$. One writes

$$\begin{aligned} |L_t(x_n, x_{n+1})| &\le \mu|L_t(x_{n-1}, x_n)| \le \mu^2|L_t(x_{n-2}, x_{n-1})| \\ &\le \cdots \le \mu^n|L_t(x_0, x_1)|. \end{aligned} \tag{18}$$

Therefore,

$$|L_t(x_n, x_{n+1})| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{19}$$

Also, for all $n, m \ge 1$, we have

$$L_t(x_n, x_m) \preceq \delta[L_t(x_{n-1}, x_n) + L_t(x_{m-1}, x_m)]. \tag{20}$$

By (19), we deduce that $|L_t(x_n, x_m)| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence, $\{x_n\}$ is a L_t-Cauchy sequence. Since $(\mathfrak{X}, L_t)$ is a L_t-complete CV-TCMS, the sequence $\{x_n\}$ converges to some $\nu \in \mathfrak{X}$.

By the argument of the proof of Theorem 1, assume that for all $n \ge 1$, we have $x_n \in \{\nu, \mathbb{T}\nu\}$. Thus,

$$\begin{aligned} L_t(\nu, \mathbb{T}\nu) &\preceq \xi(\nu, x_n)L_t(\nu, x_n) + \rho(x_n, x_{n+1})L_t(x_n, x_{n+1}) \\ &+ \varsigma(x_{n+1}, \mathbb{T}\nu)L_t(x_{n+1}, \mathbb{T}\nu) \preceq \xi(\nu, x_n)L_t(\nu, x_n) \\ &+ \rho(x_n, x_{n+1})L_t(x_n, x_{n+1}) + \varsigma(x_{n+1}, \mathbb{T}\nu) \\ &\cdot \delta[L_t(x_n, \mathbb{T}x_n) + L_t(\nu, \mathbb{T}\nu)]. \end{aligned} \tag{21}$$

As $n \longrightarrow \infty$, we obtain

$$\begin{aligned}|L_t(\nu, \mathbb{T}\nu)| &\le 0 + 0 + 0 + \limsup_{n\longrightarrow\infty} \varsigma(x_{n+1}, \mathbb{T}\nu)\delta |L_t(\nu, \mathbb{T}\nu)| \\ &< |L_t(\nu, \mathbb{T}\nu)|.\end{aligned} \tag{22}$$

At the limit $n \longrightarrow \infty$, we find that $|L_t(\nu, \mathbb{T}\nu)| = 0$ and that is $\mathbb{T}\nu = \nu$ as required. Now, assume that we have two fixed points of $\mathbb{T}$, say ν and s. Therefore,

$$|L_t(\nu, s)| = |L_t(\mathbb{T}\nu, \mathbb{T}s)| \le \delta(|L_t(\nu, \nu)| + |L_t(s, s)|) = 0. \tag{23}$$

Hence, $\nu = s$, as desired.

4. Applications

4.1. A Fredholm-Type Integral Equation. Consider the set $\mathfrak{X} = C([0, 1], \mathbb{R})$. Given the following Fredholm-type integral equation

$$a'(t) = \int_0^1 \mathbb{M}\left(t, s, a'(t)\right) ds, \quad \text{for } t, s \in [0, 1], \tag{24}$$

where $\mathbb{M}(t, s, a'(t))$ is a continuous function from $[0, 1]^2$ into $\mathbb{R}$. Now, define

$$\begin{aligned} L_t : \mathfrak{X} &\times \mathfrak{X} \longrightarrow \mathbb{C} \\ (a, b) &\mapsto i \sup_{t\in[0,1]} \left(\frac{|a(t)| + |b(t)|}{2}\right). \end{aligned} \tag{25}$$

Note that $(\mathfrak{X}, L_t)$ is a complete CV-TCMS, where

$$\xi(a, b) = 2, \rho(a, b) = 1 \text{ and } \varsigma(a, b) = 3. \tag{26}$$

Theorem 3. *Assume that for all $a, b \in \mathfrak{X}$*

(1) $|\mathbb{M}(t, s, a'(t))| + |\mathbb{M}(t, s, b(t))| \le \delta(|a'(t)| + |b(t)|)$, for some $\delta \in [0, 1/4)$

(2) $\mathbb{M}(t, s, \int_0^1 \mathbb{M}(t, s, a'(t))ds) < \mathbb{M}(t, s, a'(t))$ for all t, s

Then, the above integral equation has a unique solution.

Proof. Let $\mathbb{T} : \mathfrak{X} \longrightarrow \mathfrak{X}$ be defined by $\mathbb{T}a'(t) = \int_0^1 \mathbb{M}(t, s, a'(t))ds$. Then,

$$L_t\left(\mathbb{T}a', \mathbb{T}b\right) = i \sup_{t\in[0,1]} \left(\frac{|\mathbb{T}a'(t)| + |\mathbb{T}b(t)|}{2}\right). \tag{27}$$

Now, we have

$$\begin{aligned} L_t\left(\mathbb{T}a'(t), \mathbb{T}b(t)\right) &= i \frac{|\mathbb{T}a'(t)| + |\mathbb{T}b(t)|}{2} \\ &= i \frac{|\int_0^1 \mathbb{M}\left(t, s, a'(t)\right)ds| + |\int_0^1 \mathbb{M}(t, s, b(t))ds|}{2} \\ &\preceq i \frac{\int_0^1 |\mathbb{M}\left(t, s, a'(t)\right)| ds + \int_0^1 |\mathbb{M}(t, s, b(t))| ds}{2} \\ &= i \frac{\int_0^1 \left(|\mathbb{M}\left(t, s, a'(t)\right)| + |\mathbb{M}(t, s, b(t))|\right) ds}{2} \\ &\preceq i \frac{\int_0^1 \delta\left(|a'(t)| + |b(t)|\right) ds}{2} \\ &\preceq \delta L_t\left(a'(t), b(t)\right). \end{aligned} \tag{28}$$

Thus, $L_t(\mathbb{T}a', \mathbb{T}b) \preceq \delta L_t(a', b)$. Since $\delta \in [0, 1/4)$, one gets

$$\begin{aligned} \xi(a, b) &< \frac{1}{\delta}, \\ \rho(a, b) &< \frac{1}{\delta}, \\ \varsigma(a, b) &< \frac{1}{\delta}. \end{aligned} \tag{29}$$

Therefore, all the hypotheses of Theorem 1 are satisfied, and hence, equation (24) has a unique solution.

4.2. A Polynomial Equation of a Degree Greater or Equal to 3. The following is an application on higher degree polynomial equations.

Theorem 4. *For any natural number $\beta \ge 3$ and real $|\alpha| \le 1$, the following equation*

$$\alpha^\beta + 1 = \left(\beta^4 - 1\right)\alpha^{\beta+1} + \beta^4 \alpha \tag{30}$$

has a unique real solution.

Proof. It is not difficult to see that if $|\alpha| > 1$, equation (30) does not have a solution. So, let $\mathfrak{X} = [-1, 1]$ and for all $\alpha, r \in \mathfrak{X}$, let $L_t(\alpha, r) = |\alpha - r| + i|\alpha - r|$ and $\xi(u, v) = 3, \rho(u, v) = 4$ and $\varsigma(u, v) = \max\{u, v\} + 2$. Note that $(\mathfrak{X}, L_t)$ is a L_t-complete CV-TCMS. Now, let

$$\mathbb{T}\alpha = \frac{\alpha^\beta + 1}{\left(\beta^4 - 1\right)\alpha^\beta + \beta^4}. \tag{31}$$

Notice that, since $\beta \geq 2$, we can deduce that $\beta^4 \geq 6$. Thus,

$$\begin{aligned}
L_t(\mathbb{T}\alpha, \mathbb{T}r) &= \left| \frac{\alpha^\beta + 1}{(\beta^4 - 1)\alpha^\beta + \beta^4} - \frac{r^\beta + 1}{(\beta^4 - 1)r^\beta + \beta^4} \right| \\
&\quad + i\left| \frac{\alpha^\beta + 1}{(\beta^4 - 1)\alpha^\beta + \beta^4} - \frac{r^\beta + 1}{(\beta^4 - 1)r^\beta + \beta^4} \right| \\
&= \left| \frac{\alpha^\beta - r^\beta}{((\beta^4 - 1)\alpha^\beta + \beta^4)((\beta^4 - 1)r^\beta + \beta^4)} \right| \\
&\quad + i\left| \frac{\alpha^\beta - r^\beta}{((\beta^4 - 1)\alpha^\beta + \beta^4)((\beta^4 - 1)r^\beta + \beta^4)} \right| \\
&\preceq \frac{|\alpha - r|}{\beta^4} + i\frac{|\alpha - r|}{\beta^4} \preceq \frac{|\alpha - r|}{6} + i\frac{|\alpha - r|}{6} \\
&= \frac{1}{6} L_t(\alpha, r).
\end{aligned} \tag{32}$$

Hence,

$$L_t(\mathbb{T}\alpha, \mathbb{T}r) \preceq \delta L_t(\alpha, r), \quad \text{where}\, \delta = \frac{1}{6}. \tag{33}$$

Moreover, it is easy to see that for all $\alpha_0 \in \mathfrak{X}$, we have

$$\alpha_n = \mathbb{T}^n \alpha_0 \leq \frac{2}{\beta^4}. \tag{34}$$

Note that all the conditions of Theorem 1 are satisfied. Thus, $\mathbb{T}$ possesses a unique fixed point in $\mathfrak{X}$, and equation (30) has a unique real solution.

5. Conclusion

Finally, we would like to leave the following questions.

Question 1. Let $(\mathfrak{X}, L_t)$ be a CV-TCMS and $\mathbb{T} : \mathfrak{X} \longrightarrow \mathfrak{X}$. Given a function $\varsigma : \mathfrak{X}^2 \longrightarrow [1,\infty)$. Suppose there exists $\delta \in (0, 1)$ such that, for all $s, r \in \mathfrak{X}$,

$$L_t(\mathbb{T}s, \mathbb{T}r) \preceq \delta\varsigma(s, r) L_t(s, r). \tag{35}$$

Under what conditions does $\mathbb{T}$ have a unique fixed point in $\mathfrak{X}$?

Question 2. Let $(\mathfrak{X}, L_t)$ be a CV-TCMS, and $\mathbb{T} : \mathfrak{X} \longrightarrow \mathfrak{X}$. Given a function $\varsigma : \mathfrak{X}^2 \longrightarrow [1,\infty)$. Suppose there exists $\delta \in (0, 1/2)$ such that, for all $s, r \in \mathfrak{X}$,

$$L_t(\mathbb{T}s, \mathbb{T}r) \preceq \delta\varsigma(s, r)[L_t(s, \mathbb{T}s) + L_t(r, \mathbb{T}r)]. \tag{36}$$

Under what conditions does $\mathbb{T}$ have a unique fixed point in $\mathfrak{X}$?

Data Availability

Data sharing is not applicable to this article as no data set were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All the authors have equally contributed to the final manuscript.

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Research Article

Asymptotic Behavior of Solutions to Free Boundary Problem with Tresca Boundary Conditions

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In this paper, we study the asymptotic behavior of an incompressible Herschel-Bulkley fluid in a thin domain with Tresca boundary conditions. We study the limit when the ε tends to zero, we prove the convergence of the unknowns which are the velocity and the pressure of the fluid, and we obtain the limit problem and the specific Reynolds equation.

1. Introduction

In 1926, the model of Herschel-Bulkley fluid introduced is called a non-Newtonian fluid, whose flow properties differ in any way from those of any Newtonian fluids. There are many phenomena in nature and industry exhibiting the behavior of the Herschel-Bulkley fluid medium and has been used in various publications to describe the flow of metals, plastic solids, and some polymers. The literature concerning this topic is extensive (see e.g., [1–14]). Further, let us mention the works which is realized by many authors in this area, for example, (see [2, 4, 9, 10, 13–21]).

This paper is to discuss the asymptotic behavior of steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer with Tresca boundary conditions.

The paper is organized as follows. In Section 2, we introduce some notations, preliminaries, and the mechanical problem of the steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer. In Section 3, we investigate some estimates and convergence theorem. To this aim, we use the change of variable x_3/ε, to transform the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. Finally, the a priori estimate allows us to pass to the limit when ε tends to zero, and we prove the convergence results and limit problem with a specific weak form of the Reynolds equation and two-dimensional constitutive equation of the model flow.

2. Problem Statement and Variational Formulation

Let ω be fixed region in plan $x = (x_1, x_2) \in \mathbb{R}^2$. We assume that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x)$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < h_* < h(x) < h^*$ for all $(x, 0) \in \omega$ and Γ_L^ε the lateral surface. We denote by Ω^ε the domain of the flow:

$$\Omega^\varepsilon = \left\{ (x, x_3) \in \mathbb{R}^3 : (x, 0) \in \omega, 0 < x_3 < \varepsilon h(x) \right\}. \quad (1)$$

The boundary of Ω^ε is Γ^ε. We have $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \bar{\omega}$ where $\overline{\Gamma}_L^\varepsilon$ is the lateral boundary.

(i) The law of conservation of momentum is defined by

$$u^\varepsilon \nabla u^\varepsilon = \text{div}\,(\sigma^\varepsilon) + f^\varepsilon \text{ in } \Omega^\varepsilon, \tag{2}$$

where div $(\sigma^\varepsilon) = (\sigma^\varepsilon_{ij,j})$ and $f^\varepsilon = (f_i^\varepsilon)_{1\le i\le 3}$ denote the body forces.

(ii) The stress tensor σ^ε is decomposed as follows

$$\begin{cases} \sigma^\varepsilon_{ij} = \tilde{\sigma}^\varepsilon_{ij} - p^\varepsilon \delta_{ij}, \\ \tilde{\sigma}^\varepsilon = \alpha^\varepsilon \dfrac{D(u^\varepsilon)}{|D(u^\varepsilon)|} + \mu |D(u^\varepsilon)|^{r-2} D(u^\varepsilon) \text{ if } D(u^\varepsilon) \neq 0, \\ |\tilde{\sigma}^\varepsilon| \le \alpha^\varepsilon \text{ if } D(u^\varepsilon) = 0. \end{cases} \tag{3}$$

where $\alpha^\varepsilon \ge 0$ is the yield stress, $\mu > 0$ is the constant viscosity, u^ε is the velocity field, p^ε is the pressure, δ_{ij} is the Kronecker symbol, $1 < r \le 2$ and $D(u^\varepsilon) = 1/2(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. For any tensor $D = (d_{ij})$, the notation $|D|$ represents the matrix norm: $|D| = (\sum_{i,j} d_{ij} d_{ij})^{1/2}$.

(iii) The incompressibility equation

$$\text{div}\,(u^\varepsilon) = 0 \text{ in } \Omega^\varepsilon. \tag{4}$$

Our boundary conditions is described as

(iv) At the surface $\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon$, we assume that

$$u^\varepsilon = 0. \tag{5}$$

(v) On ω, there is a no-flux condition across ω so that

$$u^\varepsilon \times n = 0. \tag{6}$$

(vi) The tangential velocity on ω is unknown and satisfies Tresca boundary conditions:

$$\begin{cases} |\sigma^\varepsilon_\tau| < k^\varepsilon u^\varepsilon_\tau = 0 \\ |\sigma^\varepsilon_\tau| = k^\varepsilon \exists \lambda \ge 0, u^\varepsilon_\tau = -\lambda \sigma^\varepsilon_\tau \end{cases} \text{in } \omega. \tag{7}$$

Here, k^ε is the friction yield coefficient and $|.|$ is the Euclidean norm in $\mathbb{R}^2$; $n = (n_1, n_2, n_3)$ is the unitoutward normal to Γ_1^ε, and

$$\begin{aligned} u_n^\varepsilon &= u^\varepsilon.n = u_i^\varepsilon.n_i, \\ u_{\tau_i}^\varepsilon &= u_i^\varepsilon - u_n^\varepsilon n_i, \\ \sigma_n^\varepsilon &= (\sigma.n)n = \sigma^\varepsilon_{ij} n_i n_j, \\ \sigma^\varepsilon_{\tau_i} &= \sigma^\varepsilon_{ij} n_j - \sigma^\varepsilon_n n_i. \end{aligned} \tag{8}$$

In order to, we observe that

$$\begin{aligned} K^\varepsilon &= \left\{ \varphi \in W^{1,r}(\Omega^\varepsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \varphi.n = 0 \text{ on } \omega \right\}, \\ K^\varepsilon_{\text{div}} &= \{ \varphi \in K^\varepsilon : \text{div}\,(\varphi) = 0 \}, \\ L_0^{r'}(\Omega^\varepsilon) &= \left\{ q \in L^{r'}(\Omega^\varepsilon) \colon \int_{\Omega^\varepsilon} q\, dx dx_3 = 0 \right\}. \end{aligned} \tag{9}$$

A formal application of Green's formula, using (1)–(6), leads to the following weak formulation:

Find a velocity field $u^\varepsilon \in K^\varepsilon_{\text{div}}$ and $p^\varepsilon \in L_0^{r'}(\Omega^\varepsilon)$, $(1/r + 1/r' = 1)$ such that:

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) + B(u^\varepsilon, u^\varepsilon, \varphi - u^\varepsilon) - (p^\varepsilon, \text{div } \varphi) + j(\varphi) \\ - j(u^\varepsilon) \ge (f^\varepsilon, \varphi - u^\varepsilon), \end{aligned} \tag{10}$$

for all $\varphi \in K^\varepsilon$, where

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) &= \mu \int_{\Omega^\varepsilon} |D(u^\varepsilon)|^{r-2} D(u^\varepsilon) D(v^\varepsilon)\, dx dx_3, \\ B(u^\varepsilon, u^\varepsilon, v) &= \sum_{i=1}^3 \int_{\Omega^\varepsilon} u_i^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} v\, dx dx_3, \\ (p^\varepsilon, \text{div } \varphi) &= \int_{\Omega^\varepsilon} p^\varepsilon \text{ div } \varphi\, dx dx_3, \\ j(v) &= \int_\omega k^\varepsilon |v| dx + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(v)|\, dx dx_3, \\ (f^\varepsilon, v) &= \int_{\Omega^\varepsilon} f^\varepsilon v\, dx dx_3 = \sum_{i=1}^3 \int_{\Omega^\varepsilon} f_i^\varepsilon v_i\, dx dx_3. \end{aligned} \tag{11}$$

As in [6, 8], we can show that this variational problem has a unique solution.

Now, we state some the following results (see, [15]).

$$\|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \le C \|D(u^\varepsilon)\|_{L^r(\Omega^\varepsilon)} \text{ (Korn inequality)}, \tag{12}$$

$$\|u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \le \varepsilon h^* \left\| \frac{\partial u^\varepsilon}{\partial z} \right\|_{L^r(\Omega^\varepsilon)} \quad \text{for} \quad i = 1, 2 \left(\text{Poincare}' \text{ inequality} \right), \tag{13}$$

$$ab \le \frac{a^r}{r} + \frac{b^{r'}}{r'}, \forall (a, b) \in \mathbb{R}^2 \text{ (Young inequality)}. \tag{14}$$

3. Change of the Domain and Study of Convergence

Here, we apply the technique of scaling in Ω^ε on the coordinate x_3. With the variables $z = x_3/\varepsilon$, we get

$$\Omega = \left\{(x,z) \in \mathbb{R}^3 : (x,0) \in \omega, 0 < z < h(x\right\}. \quad (15)$$

Next, we denote by $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\omega}$ its boundary, then, we define the following functions in Ω:

$$\begin{aligned} &\widehat{u}_i^\varepsilon(x,z) = u_i^\varepsilon(x,x_3), i = 1,2, \\ &\widehat{u}_3^\varepsilon(x,z) = \varepsilon^{-1} u_3^\varepsilon(x,x_3), \\ &\widehat{p}^\varepsilon(x,z) = \varepsilon^r p^\varepsilon(x,x_3). \end{aligned} \quad (16)$$

Now, we assume that

$$\widehat{f}(x,z) = \varepsilon^r f^\varepsilon(x,x_3), \widehat{\alpha} = \varepsilon^{r-1}\alpha^\varepsilon, \widehat{k} = \varepsilon^{r-1}k^\varepsilon, \quad (17)$$

and we consider the sets

$$\begin{aligned} &K(\Omega) = \left\{\widehat{\varphi} \in \left(W^{1,r}(\Omega)\right)^3 : \widehat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L ; \widehat{\varphi}.n = 0 \text{ on } \omega\right\}, \\ &K_{\text{div}}(\Omega) = \{\widehat{\varphi} \in K(\Omega): \text{ div } \widehat{\varphi} = 0\}, \\ &V_z = \left\{\widehat{\varphi} \in (L^r(\Omega))^2 ; \frac{\partial \widehat{\varphi}_i}{\partial z} \in L^r(\Omega): \widehat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L\right\}, \\ &\tilde{V}_z = \left\{\widehat{\varphi} \in V_z : \widehat{\varphi} \text{ satisfy } \left(\mathrm{D}'\right)\right\}, \end{aligned} \quad (18)$$

where the condition (D') is given by

$$\left(\mathrm{D}'\right) \int_\omega \left(\widehat{\varphi}_1 \frac{\partial \theta}{\partial x_1} + \widehat{\varphi}_2 \frac{\partial \theta}{\partial x_2}\right) dxdz = 0, \quad \text{for all} \quad \widehat{\varphi} \in (L^r(\Omega))^2 \text{ and } \theta \in C_0^\infty(\Omega). \quad (19)$$

By injecting the new data, unknown factors in (10) and after multiplication by ε^{r-1}, we deduce that

$$\begin{aligned} &a_0(\widehat{u}^\varepsilon, \widehat{\varphi} - \widehat{u}^\varepsilon) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \widehat{\varphi}) - (\widehat{p}^\varepsilon, \text{div } (\widehat{\varphi} - \widehat{u}^\varepsilon)) \\ &\quad + j_0(\widehat{\varphi}) - j_0(\widehat{u}^\varepsilon) \geq \left(\widehat{f}, \widehat{\varphi} - \widehat{u}^\varepsilon\right), \forall \widehat{\varphi} \in K(\Omega), \end{aligned} \quad (20)$$

where

$$\begin{aligned} a_0(\widehat{u}^\varepsilon, \widehat{\varphi} - \widehat{u}^\varepsilon) &= \sum_{i,j=1}^2 \int_\Omega \left[\varepsilon^2 \mu \left|\tilde{D}(\widehat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right)\right)\right] \\ &\quad \cdot \frac{\partial(\widehat{\varphi}_i - \widehat{u}_i^\varepsilon)}{\partial x_j} dxdz + \sum_{i=1}^2 \int_\Omega \mu \left|\tilde{D}(\widehat{u}^\varepsilon)\right|^{r-2} \\ &\quad \left(\frac{1}{2}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right)\right) \frac{\partial(\widehat{\varphi}_i - \widehat{u}_i^\varepsilon)}{\partial z} dxdz \\ &\quad + \int_\Omega \left(\mu \left|\tilde{D}(\widehat{u}^\varepsilon)\right|^{r-2} \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\right) \frac{\partial(\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon)}{\partial z} dxdz \\ &\quad + \sum_{j=1}^2 \int_\Omega \varepsilon^2 \mu \left|\tilde{D}(\widehat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial z}\right)\right) \\ &\quad \cdot \frac{\partial(\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon)}{\partial x_j} dxdz, \end{aligned}$$

$$(\widehat{p}^\varepsilon, \text{div } (\widehat{\varphi} - \widehat{u}^\varepsilon)) = \int_{\Omega^\varepsilon} \widehat{p}^\varepsilon \text{ div } (\widehat{\varphi} - \widehat{u}^\varepsilon) dxdz,$$

$$\begin{aligned} B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \widehat{\varphi}) &= \sum_{i,j=1}^2 \int_\Omega \varepsilon^2 \widehat{u}_i^\varepsilon \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \widehat{\varphi} dxdz + \sum_{i=1}^2 \int_\Omega \varepsilon^4 \widehat{u}_i^\varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \widehat{\varphi} dxdz \\ &\quad + \sum_{j=1}^2 \int_\Omega \varepsilon^2 \widehat{u}_3^\varepsilon \frac{\partial \widehat{u}_j^\varepsilon}{\partial z} \widehat{\varphi}_j dxdz + \int_\Omega \varepsilon^4 \widehat{u}_3^\varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \widehat{\varphi}_3 dxdz, \end{aligned}$$

$$j_0(\widehat{\varphi}) = \widehat{\alpha} \int_\Omega \left|\tilde{D}(\widehat{\varphi})\right| dxdz + \int_\omega \widehat{k} |\widehat{\varphi}| dx,$$

$$\left(\widehat{f}^\varepsilon, \widehat{\varphi} - \widehat{u}^\varepsilon\right) = \sum_{j=1}^2 \int_\Omega \widehat{f}_i (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) dxdz + \int_\Omega \varepsilon \widehat{f}_3 (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) dxdz,$$

$$\begin{aligned} \left|\tilde{D}(\widehat{u}^\varepsilon)\right| &= \left(\frac{1}{4} \sum_{i,j=1}^2 \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right)^2 \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right)^2 + \varepsilon^2 \left(\frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\right)^2\right)^{1/2}. \end{aligned} \quad (21)$$

We now establish the estimates for the velocity field $\widehat{u}^\varepsilon$ and the pressure $\widehat{p}^\varepsilon$ in Ω.

Theorem 1. *Let $(\widehat{u}^\varepsilon, \widehat{p}^\varepsilon) \in K_{\text{div}}(\Omega) \times L_0^{r'}(\Omega)$ be the solution of variational problem (20), then there exists a constant $C > 0$ independent of ε such that:*

$$\begin{aligned} &\sum_{i,j=1}^2 \left\|\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^r(\Omega)}^r + \left\|\varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\right\|_{L^r(\Omega)}^r \\ &\quad + \sum_{i=1}^2 \left(\left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial z}\right\|_{L^r(\Omega)}^r + \left\|\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^r(\Omega)}^r\right) \leq C. \end{aligned} \quad (22)$$

Proof. Choosing $\varphi = 0$ as test function in inequality (10), we get

$$\begin{aligned} &a(u^\varepsilon, u^\varepsilon) + B(u^\varepsilon, u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dxdx_3 \\ &\quad + \int_\omega k^\varepsilon |u^\varepsilon| dx \leq (f^\varepsilon, u^\varepsilon), \end{aligned} \quad (23)$$

and because $B(u^\varepsilon, u^\varepsilon, u^\varepsilon) = 0$, we obtain

$$a(u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dx dx_3 + \int_\omega k^\varepsilon |u^\varepsilon| dx \le (f^\varepsilon, u^\varepsilon). \quad (24)$$

Using now (13) and (14) will yield after some algebra

$$(f^\varepsilon, u^\varepsilon) \le \varepsilon h^* \|\nabla u^\varepsilon\|_{L^r(\Omega)} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)} \le \frac{1}{2}\mu C_k \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} + \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|^{r'}_{L^{r'}(\Omega^\varepsilon)}. \quad (25)$$

From (24) and (25), we deduce that

$$a(u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dx dx_3 + \int_\omega k^\varepsilon |u^\varepsilon - s| dx \le \frac{1}{2}\mu C_k \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} + \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|^{r'}_{L^{r'}(\Omega^\varepsilon)}. \quad (26)$$

We multiply (26) by ε^{r-1}, we get

$$\varepsilon^{r-1} a(u^\varepsilon, u^\varepsilon) + \widehat{\alpha} \int_{\Omega^\varepsilon} \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz + \int_\omega \widehat{k} |\widehat{u}^\varepsilon| dx \le \frac{1}{2}\mu C_k \varepsilon^{r-1} \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} + \varepsilon^{r-1} \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|^{r'}_{L^{r'}(\Omega^\varepsilon)}. \quad (27)$$

Now, since $\varepsilon^{r'} \|f^\varepsilon\|^{r'}_{L^{r'}(\Omega^\varepsilon)} = \varepsilon^{1-r} \|\widehat{f}\|^{r'}_{L^{r'}(\Omega)}$, it follows that

$$\varepsilon^{r-1} a(u^\varepsilon, u^\varepsilon) + \widehat{\alpha} \int_{\Omega} \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz + \int_\omega \widehat{k} |\widehat{u}^\varepsilon| dx \le \frac{1}{2}\mu C_K \varepsilon^{r-1} \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} + \frac{(h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \left\|\widehat{f}\right\|^{r'}_{L^{r'}(\Omega)}. \quad (28)$$

According to Korn's inequality and (28), such that C_K independent of ε, we have

$$\frac{1}{2}\mu C_K \varepsilon^{r-1} \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} + \widehat{\alpha} \int_{\Omega} \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz + \int_\omega \widehat{k} |\widehat{u}^\varepsilon| dx \le \frac{(h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \left\|\widehat{f}\right\|^{r'}_{L^{r'}(\Omega)}. \quad (29)$$

Using (29), we deduce (22), with $C = (1/2\mu C_K)^{-1} (h^*)^{r'} / r'(1/2\mu r C_k)^{r'/r} \|\widehat{f}\|^{r'}_{L^{r'}(\Omega)}$, and

$$\varepsilon^{r-1} \|\nabla u^\varepsilon\|^r_{L^r(\Omega^\varepsilon)} = \|\nabla \widehat{u}^\varepsilon\|^r_{L^r(\Omega)} = \sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|^r_{L^r(\Omega)} + \left\| \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \right\|^r_{L^r(\Omega)} + \sum_{i=1}^{2} \left(\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|^r_{L^r(\Omega)} + \left\| \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right\|^r_{L^r(\Omega)} \right). \quad (30)$$

Theorem 2. *Under the conditions in Theorem (1), there exists a constant $C' > 0$ independent of ε such that*

$$\left\| \frac{\partial \widehat{p}^\varepsilon}{\partial x_i} \right\|_{W^{-1,r'}(\Omega)} \le C' \quad \text{for} \quad i = 1, 2, \quad (31)$$

$$\left\| \frac{\partial \widehat{p}^\varepsilon}{\partial z} \right\|_{W^{-1,r'}(\Omega)} \le \varepsilon C'. \quad (32)$$

Proof. To get the first estimate on the pressure in (31)–(32), we choose in (20), $\widehat{\varphi} = \widehat{u}^\varepsilon + \psi$, $\psi \in W_0^{1,r}(\Omega)^3$, to obtain

$$\begin{aligned} & a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) - (\widehat{p}^\varepsilon, \operatorname{div} \psi) + \widehat{\alpha} \int_\Omega \left|\tilde{D}(\widehat{u}^\varepsilon + \psi)\right| dx dz - \widehat{\alpha} \int_\Omega \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz \ge \left(\widehat{f}^\varepsilon, \psi\right), \\ & (\widehat{p}^\varepsilon, \operatorname{div} \psi) \le a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) + \widehat{\alpha} \int_\Omega \left|\tilde{D}(\widehat{u}^\varepsilon + \psi)\right| dx dz - \widehat{\alpha} \int_\Omega \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz - \left(\widehat{f}^\varepsilon, \psi\right). \end{aligned} \quad (33)$$

Keeping in mind that $|\tilde{D}(\widehat{u}^\varepsilon + \psi)| \le \sqrt{2}(|\tilde{D}(\widehat{u}^\varepsilon)| + |\tilde{D}(\psi)|)$, it follows that

$$(\widehat{p}^\varepsilon, \operatorname{div} \psi) \le a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) + \widehat{\alpha}\sqrt{2} \int_\Omega \left|\tilde{D}(\psi)\right| dx dz + \left(\sqrt{2} - 1\right) \widehat{\alpha} \int_\Omega \cdot \left|\tilde{D}(\widehat{u}^\varepsilon)\right| dx dz - \int_\Omega \widehat{f} \psi dx dz. \quad (34)$$

Using Hölder formula, we get

$$(\widehat{p}^\varepsilon, \operatorname{div} \psi) \le \mu \|D(\widehat{u}^\varepsilon)\|^{r/r'}_{L^r(\Omega)} \|\psi\|_{W^{1,r}(\Omega)^3} + \|\widehat{u}^\varepsilon\|^2_{W^{1,r}(\Omega)^3} \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha}\sqrt{2} |\Omega|^{1/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + \left(\sqrt{2} - 1\right) \widehat{\alpha} \left\|\tilde{D}(\widehat{u}^\varepsilon)\right\|_{L^r(\Omega)^3} + \left\|\widehat{f}\right\|_{L^{r'}(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}. \quad (35)$$

By similar arguments, we choose in (20) $\widehat{\varphi} = \widehat{u}^\varepsilon - \psi$ and $\psi \in W_0^{1,r}(\Omega)^3$ to obtain

$$-(\widehat{p}^{\varepsilon}, \operatorname{div} \psi) \leq \mu \|D(\widehat{u}^{\varepsilon})\|_{L^r(\Omega)}^{r/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + C\|\widehat{u}^{\varepsilon}\|_{W^{1,r}(\Omega)^3}^2 \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha}\sqrt{2}|\Omega|^{1/r'} \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \left(\sqrt{2}-1\right)\widehat{\alpha}\left\|\tilde{D}(\widehat{u}^{\varepsilon})\right\|_{L^r(\Omega)^3} + \left\|\widehat{f}\right\|_{L^{r'}(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}. \tag{36}$$

We combine now (35) and (36) to see that

$$|(\widehat{p}^{\varepsilon}, \operatorname{div} \psi)| \leq \mu \|D(\widehat{u}^{\varepsilon})\|_{L^r(\Omega)}^{r/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + \|\widehat{u}^{\varepsilon}\|_{W^{1,r}(\Omega)^3}^2 \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha}\sqrt{2}|\Omega|^{1/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + \left(\sqrt{2}-1\right)\widehat{\alpha}\left\|\tilde{D}(\widehat{u}^{\varepsilon})\right\|_{L^r(\Omega)^3} + \left\|\widehat{f}\right\|_{L^{r'}(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}. \tag{37}$$

Next, for $i = 1, 2$, we choose $\psi = (\psi_1, 0, 0)$ then $\psi = (0, \psi_2, 0)$ in the inequality (37) and using (22), we find

$$\left|\int_{\Omega} \frac{\partial \widehat{p}^{\varepsilon}}{\partial x_i} \psi dx dz\right| \leq \left(C_1 + \widehat{\alpha}\sqrt{2}|\Omega|^{1/r'} + \left\|\widehat{f}_i\right\|_{L^{r'}(\Omega)^3}\right) \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \left(\sqrt{2}-1\right)\widehat{\alpha} C, \tag{38}$$

where $|\Omega| = \operatorname{mes}(\Omega)$. Then, (31) holds for $i = 1, 2$.

To get (32), we take $\psi = (0, 0, \psi_3)$ in the inequality (37) to see that

$$\frac{1}{\varepsilon}\left|\int_{\Omega} \frac{\partial \widehat{p}^{\varepsilon}}{\partial z} \psi dx dz\right| \leq \left(C_1 + \widehat{\alpha}\sqrt{2}|\Omega|^{1/r'} + \left\|\widehat{f}_3\right\|_{L^{r'}(\Omega)}\right) \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \left(\sqrt{2}-1\right)\widehat{\alpha} C. \tag{39}$$

The question which naturally arises is to know what will be the asymptotic behavior of the fluid when the thickness of the thin film is very small. Mathematically, it is about knowing: do the speed field and the pressure admit a limit when ε tends towards zero and what is the limit problem who should check this limit?

The answer to the first question is given in Theorem (3). However, the answer to the second question will be dealt with in Theorems (4), (7), and (8).

Theorem 3. *Under the same assumptions as in Theorem (1) and Theorem (2), there exist $u^{\star} = (u_1^{\star}, u_2^{\star}) \in \tilde{V}_z$ and $p^{\star} \in L_0^{r'}(\Omega)$ such that:*

$$\widehat{u}_i^{\varepsilon} \rightharpoonup u_i^{\star}, i = 1, 2 \quad \text{weakly} \quad \text{in } \tilde{V}_z, \tag{40}$$

$$\varepsilon \frac{\partial \widehat{u}_i^{\varepsilon}}{\partial x_j} \rightharpoonup 0, i, j = 1, 2 \quad \text{weakly} \quad \text{in } L^r(\Omega), \tag{41}$$

$$\varepsilon \frac{\partial \widehat{u}_3^{\varepsilon}}{\partial z} \rightharpoonup 0, \quad \text{weakly} \quad \text{in } L^r(\Omega), \tag{42}$$

$$\varepsilon^2 \frac{\partial \widehat{u}_3^{\varepsilon}}{\partial x_i} \rightharpoonup 0, i = 1, 2 \quad \text{weakly} \quad \text{in } L^r(\Omega), \tag{43}$$

$$\varepsilon \widehat{u}_3^{\varepsilon} \rightharpoonup 0, \quad \text{weakly} \quad \text{in } L^r(\Omega), \tag{44}$$

$$\widehat{p}^{\varepsilon} \rightharpoonup p^{\star}, \quad \text{weakly in} \quad L^{r'}(\Omega), p^{\star} \quad \text{depend only of} \quad x. \tag{45}$$

Proof. By Theorem (1), there exists a constant C independent of ε such that

$$\left\|\frac{\partial \widehat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^r(\Omega)} \leq C, i = 1, 2, \tag{46}$$

and using Poincare's inequality, we deduce that

$$\|\widehat{u}_i^{\varepsilon}\|_{L^r(\Omega)} \leq h^* \left\|\frac{\partial \widehat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^r(\Omega)}, i = 1, 2. \tag{47}$$

that is to say, $\widehat{u}_i^{\varepsilon}$ is bounded in V_z, $i = 1, 2$, this implies the existence of $\widehat{u}_i^*$ in V_z such that $\widehat{u}_i^{\varepsilon}$converges to $\widehat{u}_i^*$ in $L^r(\Omega)$. The same, the inequality (22), we give

$$\varepsilon \left\|\frac{\partial \widehat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^r(\Omega)} \leq C, \tag{48}$$

so $\varepsilon \partial \widehat{u}_i^{\varepsilon} / \partial x_j$ converges to $\partial \widehat{u}_i^* / \partial x_j$ and as $\|\widehat{u}_i^{\varepsilon}\|_{L^r(\Omega)} \leq C$, then $\partial \widehat{u}_i^{\varepsilon} / \partial x_j$ converges weakly to $\partial \widehat{u}_i^* / \partial x_j$; which gives the converges weakly of $\partial \widehat{u}_i^{\varepsilon} / \partial x_j$ to 0 in $L^r(\Omega)$.

Well thanks to the inequality: $\varepsilon^2 \|\partial \widehat{u}_3^{\varepsilon} / \partial x_j\|_{L^r(\Omega)} \leq C$, we have the convergence $\varepsilon^2 \partial \widehat{u}_3^{\varepsilon} / \partial x_j \longrightarrow \partial \widehat{u}_3^* / \partial x_j$ and $\varepsilon \partial \widehat{u}_3^{\varepsilon} / \partial x_j \longrightarrow \partial \widehat{u}_3^* / \partial x_j$. This shows that $\partial \widehat{u}_3^{\varepsilon} / \partial x_j$ converges weakly to 0 in $L^r(\Omega)$. Finally, using (31) and (32), we get (45).

4. Study of the Limit Problem

In this section, we give both the equations satisfied by p^* and u^* in Ω and the inequalities for the trace of the velocity $u^*(x, 0)$ and the stress $\partial u^* / \partial z(x, 0)$ on ω.

Theorem 4. *With the same assumptions of Theorem (3) the solution* (u^*, p^*) *satisfying the following relations*

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{r-2/2}\frac{\partial(u_i^*)}{\partial z}\frac{\partial(\widehat{\varphi}_i-u_i^*)}{\partial z}dxdz - \int_{\Omega}p^*(x)\left(\frac{\partial\widehat{\varphi}_1}{\partial x_1}+\frac{\partial\widehat{\varphi}_2}{\partial x_2}\right)dxdz + \widehat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial\widehat{\varphi}}{\partial z}\right|-\left|\frac{\partial u^*}{\partial z}\right|\right)dxdz+\int_{\omega}\widehat{k}(|\widehat{\varphi}|-|u^*|)dx \geq \sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i(\widehat{\varphi}_i-u_i^*)dxdz, \forall\widehat{\varphi}\in W_{\Gamma_I\cup\Gamma_L}, \tag{49}$$

where

$$W_{\Gamma_I\cup\Gamma_L}=\left\{\widehat{\varphi}=(\widehat{\varphi}_1,\widehat{\varphi}_2)\in W^{1,r}(\Omega)^2, \widehat{\varphi}=0 \text{ on } \Gamma_1\cup\Gamma_L\right\}. \tag{50}$$

The proof of this theorem is based on the following lemma.

Lemma 5 (Minty). *Let E be a Banach spaces,* $T: E\longrightarrow E'$ *a monotone and hemicontinuous operator, and* $J: E\longrightarrow]-\infty, +\infty]$ *a proper and convex functional. Let* $u\in E$ *and* $f\in E'$. *Then, the followings assertions are equivalent:*

$$\langle Tu\,;v-u\rangle_{E'\times E}+J(v)-J(u)\geq\langle f\,;v-u\rangle_{E'\times E'}, \forall v\in E,$$
$$\langle Tv\,;v-u\rangle_{E'\times E}+J(v)-J(u)\geq\langle f\,;v-u\rangle_{E'\times E}, \forall v\in E. \tag{51}$$

Proof. By using *Minty*'s Lemma (5) and the fact that div $(\widehat{u}^\varepsilon) = 0$ in Ω, then (20) is equivalent to

$$a_0(\widehat{\varphi},\widehat{\varphi}-\widehat{u}^\varepsilon)+B_0(\widehat{\varphi},\widehat{\varphi},\widehat{\varphi}-\widehat{u}^\varepsilon) - \sum_{i=1}^{2}\left(\widehat{p}^\varepsilon,\frac{\partial\widehat{\varphi}_i}{\partial x_i}\right)-\left(\widehat{p}^\varepsilon,\frac{\partial\widehat{\varphi}_3}{\partial z}\right)+j_0(\widehat{\varphi})-j_0(\widehat{u}^\varepsilon) \geq\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i(\widehat{\varphi}_i-\widehat{u}_i^\varepsilon)dxdz+\int_{\Omega}\varepsilon\widehat{f}_3(\widehat{\varphi}_3-\widehat{u}_3^\varepsilon)dxdz. \tag{52}$$

Using Theorem (3) and the fact that j_0 is convex and lower semicontinuous, $(\liminf j_0(\widehat{u}^\varepsilon)\geq j_0(u^*))$, we obtain

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\widehat{\varphi}_i}{\partial z}\right)^2\right)^{r-2/2}\frac{\partial(\widehat{\varphi}_i)}{\partial z}\frac{\partial(\widehat{\varphi}_i-u_i^*)}{\partial z}dxdz - \int_{\Omega}p^*\left(\frac{\partial\widehat{\varphi}_1}{\partial x_1}+\frac{\partial\widehat{\varphi}_2}{\partial x_2}\right)dxdz-\int_{\Omega}p^*\frac{\partial\widehat{\varphi}_3}{\partial z}dxdz + j_0(\widehat{\varphi})-j_0(u^*)\geq\sum_{j=1}^{2}\int_{\Omega}\widehat{f}_i(\widehat{\varphi}_i-u_i^*)dxdz, \tag{53}$$

and as $\int_\Omega p^*\partial\widehat{\varphi}_3/\partial z dxdz=0$, because p^* independent of z, we deduce that

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\widehat{\varphi}_i}{\partial z}\right)^2\right)^{r-2/2}\frac{\partial(\widehat{\varphi}_i)}{\partial z}\frac{\partial(\widehat{\varphi}_i-u_i^*)}{\partial z}dxdz - \int_{\Omega}p^*\left(\frac{\partial\widehat{\varphi}_1}{\partial x_1}+\frac{\partial\widehat{\varphi}_2}{\partial x_2}\right)dxdz+j_0(\widehat{\varphi})-j_0(u^*) \geq\sum_{j=1}^{2}\int_{\Omega}\widehat{f}_i(\widehat{\varphi}_i-u_i^*)dxdz. \tag{54}$$

Using again *Minty*'s Lemma for the second time, thus, (54) is equivalent to (49).

Theorem 6. *The variational inequality (49) is equivalent the following system*

$$\mu\int_{\Omega}\left(\frac{1}{2}\right)^{r/2}\left|\frac{\partial u^*}{\partial z}\right|^r dxdz+\widehat{\alpha}\int_{\Omega}\left|\frac{\partial u^*}{\partial z}\right|dxdz+\int_{\omega}\widehat{k}|u^*|dx = \int_{\Omega}\widehat{f}u^*dxdz,$$

$$\mu\int_{\Omega}\left(\frac{1}{2}\right)^{r/2}\left|\frac{\partial u^*}{\partial z}\right|^{r-2}\frac{\partial u^*}{\partial z}\frac{\partial\widehat{\psi}}{\partial z}dxdz+\widehat{\alpha}\int_{\Omega}\left|\frac{\partial\widehat{\psi}}{\partial z}\right|dxdz + \int_{\omega}\widehat{k}|\widehat{\psi}|dx\geq\int_{\Omega}\widehat{f}\widehat{\psi}dxdz, \forall\widehat{\psi}\in\Sigma(K), \tag{55}$$

where

$$\Sigma(K)=\left\{\widehat{\psi}=(\widehat{\psi}_1,\widehat{\psi}_2)\in H^1(\Omega)^2:\widehat{\psi}\quad\text{satisfy}\quad(D')\right\}. \tag{56}$$

Theorem 7. *Let us set*

$$\sigma^*=\widetilde{\sigma}^*-\nabla p^* \text{ and } \widetilde{\sigma}^*=\left(\frac{1}{2}\right)^{r/2}\mu\left|\frac{\partial u^*}{\partial z}\right|^{r-2}\frac{\partial u^*}{\partial z}+\widehat{\alpha}\pi, \tag{57}$$

then

$$-\frac{\partial}{\partial z}\left[\frac{1}{2}\mu\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{r-2/2}\frac{\partial u^*}{\partial z}+\frac{\sqrt{2}}{2}\widehat{\alpha}\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}\right]=\widehat{f}-\nabla p^*, \tag{58}$$

in $W^{-1,r'}(\Omega)^2$, *where* $\pi\in L^\infty(\Omega)^2$ *and* $\|\pi\|_{\infty,\Omega}\leq 1$.

Proof. For the proof of this theorem, we follow the same steps as in [13] (Theorem (9)).

Theorem 8. *Under the assumptions of preceding theorems, u^* and p^* satisfy the following inequality*

$$\begin{aligned}\int_\omega \Bigg[& \frac{h^3}{12}\nabla p^* + \tilde{F} + \mu\int_0^h\int_0^y A^*(x,\zeta)\frac{\partial u^*(x,\xi)}{\partial\xi}d\xi dy \\ & + \widehat{\alpha}\frac{\sqrt{2}}{2}\int_0^h\int_0^y\left|\frac{\partial u^*}{\partial z}\right|(x,\xi)d\xi dy \\ & - \frac{h\mu}{2}\int_0^h A^*(x,\zeta)\frac{\partial u^*(x,\xi)}{\partial\xi}d\xi \\ & - \frac{\widehat{\alpha}\sqrt{2}h}{4}\int_0^h\left|\frac{\partial u^*}{\partial z}\right|(x,\xi)d\xi\Bigg].\nabla\varphi(x)dx = 0,\end{aligned} \tag{59}$$

for all $\varphi \in W^{1,r}(\omega)$, where

$$\begin{aligned}\tilde{F}(x) &= \int_0^h F(x,y)dy - \frac{h}{2}F(x,h),\ F(x,y) = \int_0^h\int_0^\xi \widehat{f}(x,t)dtd\xi, \\ A^*(x,\xi) &= \frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u^*}{\partial z}(x,\xi)\right)^2\right)^{r-2/2}.\end{aligned} \tag{60}$$

Proof. The proof can be found in [13].

The uniqueness of the limit velocity and pression are given by the following theorem.

Theorem 9. *The solution (u^*, p^*) in $V_z \times L_0^{r'}(\omega)$ of inequality (49) is unique.*

Proof. Let $(u^{*,1},p^{*,1})$ and $(u^{*,2},p^{*,2})$ be two solutions of (49); taking $\varphi = u^{*,2}$ and $\varphi = u^{*,1}$, respectively, as test function in (59), we get

$$\begin{aligned}\mu\sum_{i=1}^{2}\int_\Omega\left(\frac{1}{2}\right)^{r/2}\left(\sum_{i=1}^{2}\left(\frac{\partial u_i^{*,1}}{\partial z}\right)^2\right)^{r-2/2}\frac{\partial u_i^{*,1}}{\partial z}\frac{\partial}{\partial z}\left(u_i^{*,1}-u_i^{*,2}\right)dxdz \\ \le \mu\sum_{i=1}^{2}\int_\Omega\left(\frac{1}{2}\right)^{r/2}\left(\sum_{i=1}^{2}\left(\frac{\partial u_i^{*,2}}{\partial z}\right)^2\right)^{r-2/2} \\ \cdot\frac{\partial u_i^{*,2}}{\partial z}\frac{\partial}{\partial z}\left(u_i^{*,1}-u_i^{*,2}\right)dxdz.\end{aligned} \tag{61}$$

Keeping in mind that for every $x, y \in \mathbb{R}^n$

$$\left(|x|^{r-2}x - |y|^{r-2}y, x-y\right) \ge (r-1)(|x|+|y|)^{r-2}|x-y|^2, \forall 1 < r \le 2, \tag{62}$$

we obtain

$$\int_\Omega\left[\left|\frac{\partial u^{*,1}}{\partial z}\right|+\left|\frac{\partial u^{*,2}}{\partial z}\right|\right]^{r-2}\left|\frac{\partial u^{*,1}}{\partial z}-\frac{\partial u^{*,2}}{\partial z}\right|^2 dxdz = 0, \tag{63}$$

where $|\partial u^{*,j}/\partial z| = \left(\sum_{i=1}^{2}\left(\partial u_i^{*,j}/\partial z\right)^2\right)^{1/2}$, $j = 1, 2$.

Using *Hölder*'s inequality, we deduce

$$\begin{aligned}\int_\Omega & \left[\frac{\partial}{\partial z}\left(u^{*,1}-u^{*,2}\right)\right]^r dxdz \\ & \le C\left(\int_\Omega\left[\left|\frac{\partial u^{*,1}}{\partial z}\right|+\left|\frac{\partial u^{*,2}}{\partial z}\right|\right]^{r-2}\left|\frac{\partial u^{*,1}}{\partial z}-\frac{\partial u^{*,2}}{\partial z}\right|^2 dxdz\right)^{r/2} \\ & \times\left(\int_\Omega\left[\left|\frac{\partial u^{*,1}}{\partial z}\right|+\left|\frac{\partial u^{*,2}}{\partial z}\right|\right]^r dxdz\right)^{2-r/2},\end{aligned} \tag{64}$$

from (63) and (64), we deduce that $\|u^{*,1}-u^{*,2}\|_{V_z} = 0$.

Finally, to prove the uniqueness of the pressure, we use equation (59) with the two pressures $p^{*,1}$ and $p^{*,2}$, we find

$$\int_\omega \frac{h^3}{12}\nabla\left(p^{*,1}-p^{*,2}\right)\nabla\varphi dx = 0. \tag{65}$$

Taking $\varphi = p^{*,1} - p^{*,2}$ and using *Poincaré* inequality, we obtain $\|p^{*,1}-p^{*,2}\|_{L^{r'}(\omega)} = 0$. Then, $p^{*,1} = p^{*,2}$.

5. Conclusion

In this work, the asymptotic behavior of an incompressible *Herschel-Bulkley* fluid in a thin domain with *Tresca* boundary conditions is considered, where we prove the convergence of the unknowns which are the velocity and the pressure of the fluid when the ε tends to zero. In addition, the limit problem and the specific *Reynolds* equation are studied. The aim of our next study is to complement and improve our current results, which is to weaken the hypotheses of fixed point theory by using the following concepts: weak contractual applications, applications that verify some characteristics, normal global operating system, and closed graph applications. We will state and give conclusions on the fixed point theory using the concepts mentioned in recent references. On the other hand, we will study the uniform convergent behavior of a series of designations, of Banach space towards itself, with fixed points, or nearly fixed points in order to show some results in the fixed point theory applied on the problem studied in this paper. With the help of these results, we will introduce some applications and provide some examples and some notes regarding weak contraction mappings. In addition, we will mention and give some results of the fixed point theory of weak contraction mappings by using the studied algorithm in ([15, 22–35]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

A New Result of Stability for Thermoelastic-Bresse System of Second Sound Related with Forcing, Delay, and Past History Terms

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We consider a one-dimensional linear thermoelastic Bresse system with delay term, forcing, and infinity history acting on the shear angle displacement. Under an appropriate assumption between the weight of the delay and the weight of the damping, we prove the well-posedness of the problem using the semigroup method, where an asymptotic stability result of global solution is obtained.

1. Introduction

In this work, we considered with the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \int_0^\infty g(s)\psi_{xx}(x,t-s)ds + \gamma\theta_x + f(\psi) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \end{cases} \tag{1}$$

$(x, t) \in (0, 1) \times (0,\infty)$, with initial-boundary conditions

$$\begin{aligned} \varphi(0,t) = \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t) = w_x(0,t) \\ = w(1,t) = \theta(0,t) = q(1,t) = 0, t \geq 0, \end{aligned} \tag{2}$$

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), x \in (0,1), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), x \in (0,1), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in (0,1), \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau), \end{cases} \tag{3}$$

with $\tau > 0$ is a time delay and μ_1 and μ_2 are positive real numbers. The function θ is the temperature difference, q is the heat flux, and $\rho_1, \rho_2, \rho_3, k, l, k_0, b, \gamma, \kappa, \alpha, \beta$ are positive constants. We use the energy method and assume that the relaxation function g satisfies the following hypotheses:

(G1) $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^1 function such that

$$g(0) > 0, b - \int_0^\infty g(s)ds = b - g_0 = L > 0. \tag{4}$$

(G2) Let ζ be a positive constant with

$$g'(t) \leq -\zeta g(t), \forall t \geq 0, \tag{5}$$

and we suppose that the forcing term $f(\psi(x,t))$ satisfies some hypotheses.

(A1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\left|f(\psi^2) - f(\psi^1)\right| \le k_0\left(\left|\psi^1\right|^\theta - \left|\psi^2\right|^\theta\right)\left|\psi^1 - \psi^2\right| \tag{6}$$

for all $\psi^1, \psi^2 \in \mathbb{R}$,

where $k_0 > 0, \theta > 0$.

(A2)

$$0 \le \widehat{f}(\psi) \le f(\psi)\psi \text{ for all } \psi \in \mathbb{R}, \tag{7}$$

with

$$\widehat{f}(z) = \int_0^z f(s)ds. \tag{8}$$

Depending on some of the following parameters, we consider

$$\begin{aligned} \tilde{\eta} &= \left(1 - \frac{\alpha k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\gamma^2\alpha}{b}, \\ k &= k_0. \end{aligned} \tag{9}$$

It is well known that, in the single wave equation, if $\mu_2 = 0$, that is, in the absence of a delay, the energy of system exponentially decays (see, e.g., [1–22]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system becomes unstable.

Bresse system is a mathematical model that describes the vibration of a planar, linear shearable curved beam. The model was first derived by Bresse [23], and it consists of three coupled wave equations given by

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - IQ + F_3, \end{cases} \tag{10}$$

where

$$\begin{aligned} N &= k_0(w_x - l\varphi), \\ Q &= k(\varphi_x + lw + \psi), \\ M &= b\psi_x. \end{aligned} \tag{11}$$

We use N,Q, and M to denote the axial force, the shear force, and the bending moment. By w,φ, and ψ, we are denoting the longitudinal, vertical, and shear angle displacements. Here, $\rho_1 = \rho A$,$\rho_2 = \rho I$, $b = EI$, $k_0 = EA$, $k = k' GA$, and $l = R^{-1}$ (see, e.g., [23]).

The Bresse system (10) is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l = 0$. The reader may refer to, for example, [24–34].

System (10) is an undamped system, and its associated energy remains constant when the time t evolves. To stabilize system (10), many damping terms have been considered by several authors (see, e.g., [1, 35–40]).

In the succeeding text, we will present some works, which studied the stability of the dissipatif Bresse system. The paper [41] was concerned with asymptotic stability of a Bresse system with two frictional dissipations.

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) = -\gamma_1\varphi_t, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) = -\gamma_2\psi_t, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0. \end{cases} \tag{12}$$

Under the condition of equal speeds of wave propagation, the authors proved that the system is exponentially stable. Otherwise, they show that Bresse system is not exponentially stable. Then, they proved that the solution decays polynomially to zero with optimal decay rate, depending on the regularity of initial data.

There are several works dedicated to the mathematical analysis of the Bresse system. They are mainly concerned with decay rates of solutions of the linear system. This is done by adding suitable damping effects that can be of thermal, viscous, or viscoelastic nature (see for instance [42–44]), among others.

Concerning thermoelastic Bresse system, [37] considered

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + l\gamma\theta_1 = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + lw + \psi) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \gamma\theta_{1x} = 0, \\ \rho_3\theta_t - \theta_{xx} + \gamma\psi_{tx} = 0, \\ \rho_3\theta_{1x} - \theta_{1xx} + \gamma(w_{tx} - l\varphi_t) = 0, \end{cases} \tag{13}$$

together with initial and specific boundary conditions and proved an exponential and only polynomial-type decay stabilities results.

2. Preliminaries and Well-Posedness

Firstly, we assume the following hypothesis:

$$|\mu_2| < \mu_1. \tag{14}$$

Using semigroup theory, we will prove that systems (1)–(3) are well posed by introducing the following new variable [17].

$$z(x,\rho,t) = \varphi_t(x, t - \tau\rho), x \in (0,1), \rho \in (0,1), t > 0. \tag{15}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } (0,1) \times (0,1) \times (0,\infty). \tag{16}$$

Further, let

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s), s \ge 0. \tag{17}$$

For this reason, we observe that

$$\eta_t^t(x,s) + \eta_s^t(x,s) = \psi_t(x,t). \tag{18}$$

Therefore, problem (1) takes the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - lk_0(w_x - l\varphi) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \\ \rho_2\psi_{tt} - L\psi_{xx} + k(\varphi_x + lw + \psi) + \int_0^\infty g(s)\eta_{xx}^t(x,s)ds + \gamma\theta_x + f(\psi(x,t)) = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \\ \eta_t^t(x,s) + \eta_s^t(x,s) = \psi_t(x,t). \end{cases} \tag{19}$$

The following are with the boundary conditions:

$$\begin{aligned} \varphi(0,t) &= \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t) = w_x(0,t) \\ &= w(1,t) = \theta(0,t) = q(1,t) = 0, \; t \ge 0. \end{aligned} \tag{20}$$

The initial conditions are as follows:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), x \in (0,1), \\ \psi_t(x,0) = \psi_1(x), w(x,0) = w_0(x), \\ w_t(x,0) = w_1(x), x \in (0,1), \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), x \in (0,1), \\ \varphi_t(x,-t) = f_0(x,t) \text{ in } (0,1)\times(0,\tau), \\ z(x,1,t) = f(x,t-\tau) \text{ in } (0,1)\times(0,\tau), \\ \eta^t(x,0) = 0, \forall t \ge 0, \\ \eta^t(0,s) = \eta^t(1,s) = 0 \forall s \ge 0, \\ \eta^0(x,s) = \eta_0(s) = 0 \forall s \ge 0. \end{cases} \tag{21}$$

Let ξ be positive constants such that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \tag{22}$$

where τ is a real number with $0 < \tau$ and μ_1, μ_2 are a positive constants, and the initial data are $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, f, \theta_0, q_0, \eta_0)$.

If we set

$$U = \left(\varphi, \varphi_t, z, \psi, \psi_t, w, w_t, \theta, q, \eta^t\right)^T, \tag{23}$$

then

$$U' = \left(\varphi_t, \varphi_{tt}, z_t, \psi_t, \psi_{tt}, w_t, w_{tt}, \theta_t, q_t, \eta_t^t\right)^T. \tag{24}$$

Therefore, problems (19)–(21) can be written as

$$\begin{cases} U'(t) = AU(t) + F, \\ U(0) = (\varphi_0, \varphi_1, f_1(.-,\tau), \psi_0, \psi_1, w_0, w_1, \theta_0, q_0, \eta_0), \end{cases} \tag{25}$$

where the operator A is defined by

$$A\begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \\ \phi \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \dfrac{k_0 l}{\rho_1}(w_x - l\varphi) - \dfrac{\mu_1}{\rho_1}u - \dfrac{\mu_2}{\rho_1}z(.,1) \\ -\left(\dfrac{1}{\tau}\right)z_\rho \\ v \\ \dfrac{L}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + lw + \psi) + \dfrac{1}{\rho_2}\int_0^\infty g(s)\phi_{xx}(s)ds - \dfrac{\gamma}{\rho_2}\theta_x \\ \varpi \\ \dfrac{k_0}{\rho_1}(w_x - l\varphi)_x - \dfrac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\gamma}{\rho_3}v_x \\ -\dfrac{\beta}{\alpha}q - \dfrac{k}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}, \tag{26}$$

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{1}{\rho_2}f(\psi) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{27}$$

We consider the following spaces:

$$\begin{aligned} H_*^1(0,1) &= \left\{h \in H^1(0,1) \colon h(0) = 0\right\}, \\ \tilde{H}_*^1(0,1) &= \left\{h \in H^1(0,1) \colon h(1) = 0\right\}, \\ H_*^2(0,1) &= H^2(0,1) \cap H_*^1(0,1), \\ \tilde{H}_*^2(0,1) &= H^2(0,1) \cap \tilde{H}_*^1(0,1), \\ \mathscr{H} &= H_*^1(0,1) \times L^2(0,1) \times L^2\left((0,1), H_0^1(0,1)\right) \\ &\quad \times \tilde{H}_*^1(0,1) \times L^2(0,1) \times \tilde{H}_*^1(0,1) \times L^2(0,1) \\ &\quad \times \tilde{H}_*^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1) \\ &\quad \times L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right), \end{aligned} \tag{28}$$

where $L_g^2(\mathbb{R}^+, H_0^1(0,1))$ denotes the Hilbert space of H_0^1 –valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(V_1, V_2)_{L_g^2(\mathbb{R}^+, H_0^1(\Omega))} = \int_0^1\int_0^1 g(s)V_{1x}(s)V_{2x}(s)dsdx. \quad (29)$$

We will show under the assumption (22) that A generates a C_0 semigroup on $\mathscr{H}$.

Now, we consider the vectors

$$\begin{aligned} U &= (\varphi, u, z, \psi, v, w, \varpi, \theta, q, \phi)^T, \\ \bar{U} &= \left(\bar{\varphi}, \bar{u}, \bar{z}, \bar{\psi}, \bar{v}, \bar{w}, \bar{\varpi}, \bar{\theta}, \bar{q}, \bar{\phi}\right)^T, \end{aligned} \quad (30)$$

and we define the inner product

$$\begin{aligned} \langle U, \bar{U}\rangle_{\mathscr{H}} = {} & k\int_0^1 (\varphi_x + \psi + lw)(\bar{\varphi}_x + \bar{\psi} + l\bar{w})dx + \rho_2\int_0^1 v\bar{v}dx \\ & + \rho_1\int_0^1 \varpi\bar{\varpi}dx + k_0\int_0^1 (w_x - l\varphi)(\bar{w}_x - l\bar{\varphi})dx \\ & + l\int_0^1 \psi_x\bar{\psi}_x dx + \rho_1\int_0^1 u\bar{u}dx + \xi\int_0^1\int_0^1 z\bar{z}d\rho dx \\ & + \rho_3\int_0^1 \theta\bar{\theta}dx + \alpha\int_0^1 q\bar{q}dx \\ & + \int_0^1\int_0^\infty g(s)\phi_x(s)\bar{\phi}_x(s)dxds, \end{aligned} \quad (31)$$

where the domain of A is defined by

$$D(A) = \left\{ \begin{array}{c} U \in \mathscr{H}/\varphi \in H_*^2(0,1)\,;\, \psi, w \in \tilde{H}_*^2(0,1), u, \theta \in H_*^1(0,1)\,; \\ v, \varpi, q \in \tilde{H}_*^1(0,1), u = z(.,0), z_\rho \in L^2\left((0,1)\,;\,L^2(0,1)\right) \\ , \varphi_x(1) = 0, w_x(0) = \psi_x(0) = 0, \\ \phi_s \in L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right), \phi(x,0) = 0, \end{array} \right\}. \quad (32)$$

Important properties of the above metrics are stated in the following lemmas. Although most of these results are followed straightforwardly from the known results, they are crucial for what follows. So for the convenience of the reader, we give their proofs here.

Lemma 1. *The operator A is dissipative and satisfies, for any $U \in D(A)$,*

$$\begin{aligned} \langle AU, U\rangle_{\mathscr{H}} = {} & -\beta\int_0^1 q^2 dx + \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\right)\int_0^1 u^2 dx \\ & + \left(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)\int_0^1 z^2(x,1)dx \\ & + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\phi_x(x,s)|^2 dsdx \le 0. \end{aligned} \quad (33)$$

Proof. For any $U \in D(A)$, using the inner product,

$$\langle AU, U\rangle_{\mathscr{H}} = \left\langle \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(.,1) \\ -\left(\frac{1}{\tau}\right)z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + lw + \psi) + \frac{1}{\rho_2}\int_0^\infty g(s)\phi_{xx}(s)ds - \frac{\gamma}{\rho_2}\theta_x \\ -\varpi \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\frac{1}{\rho_3}q_x - \frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q - \frac{1}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \\ \phi \end{pmatrix} \right\rangle_{\mathscr{H}}. \quad (34)$$

Then,

$$\begin{aligned}\langle AU,U\rangle_{\mathscr{H}} = {} & k\int_0^1 (u_x+v+l\varpi)(\varphi_x+lw+\psi)dx + k_0\int_0^1(\varpi_x - lu)\\ & \cdot (w_x - l\varphi)dx + k\int_0^1(\varphi_x+lw+\psi)udx\\ & + k_0 l\int_0^1 (w_x-l\varphi)udx - \mu_1\int_0^1 u^2dx\\ & - \mu_2\int_0^1 z(x,1)udx + L\int_0^1 \psi_{xx}vdx\\ & - k\int_0^1(\varphi_x+lw+\psi)vdx - \gamma\int_0^1\theta_x vdx\\ & + k_0\int_0^1 (w_x-l\varphi)\varpi dx - kl\int_0^1(\varphi_x+lw+\psi)\varpi dx\\ & + L\int_0^1 v_x\psi_x dx + \int_0^1\int_0^\infty g(s)\phi_x(s)(-\phi_s+v)dxds\\ & - \int_0^1 q_x\theta dx - \gamma\int_0^1 u_x\theta dx - \beta\int_0^1 q^2dx - \int_0^1\theta_x q dx\\ & - \xi\int_0^1\int_0^1 zz_\rho d\rho dx.\end{aligned} \tag{35}$$

By the fact that

$$\begin{aligned}& -\beta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \mu_2\int_0^1 z(x,1)udx + \int_0^1\int_0^\infty g(s)\phi_x(s)\\ & \quad \cdot(-\phi_s+v)dxds - \frac{\xi}{\tau}\int_0^1\int_0^1 z(x,\rho)z_\rho(x,\rho)d\rho dx\\ & = -\beta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \mu_2\int_0^1 z(x,1)udx + \int_0^1\int_0^\infty g(s)\phi_x(s)\\ & \quad \cdot(-\phi_s+v)dxds - \frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho)d\rho dx\\ & = -\beta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \mu_2\int_0^1 z(x,1)udx + \int_0^1\int_0^\infty g(s)\phi_x(s)\\ & \quad \cdot(-\phi_s+v)dxds - \frac{\xi}{2\tau}\int_0^1 \left\{z^2(x,1)-z^2(x,0)\right\}dx\\ & = -\beta\int_0^1 q^2dx - \mu_1\int_0^1 u^2dx - \mu_2\int_0^1 z(x,1)udx + \int_0^1\int_0^\infty g(s)\phi_x(s)\\ & \quad \cdot(-\phi_s+v)dxds - \frac{\xi}{2\tau}\int_0^1 z^2(x,1)dx + \frac{\xi}{2\tau}\int_0^1 u^2dx,\end{aligned} \tag{36}$$

and using Young's inequality, we find

$$\begin{aligned}\langle AU,U\rangle_H \le {} & -\beta\int_0^1 q^2dx + \left(-\mu_1+\frac{\mu_2}{2}+\frac{\xi}{2\tau}\right)\int_0^1 u^2dx\\ & + \left(\frac{\mu_2}{2}-\frac{\xi}{2\tau}\right)\int_0^1 z^2(x,1)dx\\ & + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\phi_x(x,s)|^2dsdx.\end{aligned} \tag{37}$$

Keeping in mind condition (22), the desired result yields.

Lemma 2. *The operator $I-A$ is surjective.*

Proof. We need to show that for all $\mathscr{F}=(f_1,f_2,f_3,f_4,f_5,f_6,f_7f_8,f_9,f_{10})^T\in\mathscr{H}$, there exists $U\in D(A)$ such that

$$U - AU = \mathscr{F}, \tag{38}$$

that is,

$$\begin{cases} -u+\varphi = f_1\in H_*^1(0,1),\\ -k(\varphi_x+lw+\psi)_x - k_0 l(w_x-l\varphi)+\rho_1 u+\mu_1 u+\mu_2 z(.,1) = \rho_1 f_2\in L^2(0,1),\\ z+\tau^{-1}z_\rho = \tau f_3\in L^2\left((0,1),H^1(0,1)\right),\\ -v+\psi = f_4\in \tilde{H}_*^1(0,1),\\ -L\psi_{xx}+k(\varphi_x+lw+\psi)+\rho_2 v - \int_0^\infty g(s)\phi_{xx}(s)ds+\gamma\theta_x = \rho_2 f_5\in L^2(0,1),\\ -\varpi+w = f_6\in\tilde{H}_*^1(0,1),\\ -k_0(w_x-l\varphi)_x + kl(\varphi_x+lw+\psi)+\rho_1\varpi = \rho_1 f_7\in L^2(0,1),\\ q_x+\gamma v_x+\rho_3\theta = \rho_3 f_8\in L^2(0,1),\\ (\beta+\alpha)q+\theta_x = \alpha f_9\in L^2(0,1),\\ \phi+\phi_s - v = f_{10}\in L^2(0,1). \end{cases} \tag{39}$$

From (39), we define

$$\theta = \frac{\alpha}{k}\int_0^x f_9(y)dy - \frac{\alpha}{k}(\beta+\alpha)\int_0^x q(y)dy, \tag{40}$$

so $\theta(0,t)=0$.

Inserting $u=\varphi-f_1$, $v=\psi-f_4$, $\varpi=w-f_6$ and (39) into (40), we get

$$\begin{cases} -k(\varphi_x+lw+\psi)_x - k_0 l(w_x-l\varphi)+\rho_1\varphi+\mu_1 u+\mu_2 z(.,1) = h_1\in L^2(0.1),\\ -L\psi_{xx}+k(\varphi_x+lw+\psi)+\rho_2\psi - \int_0^\infty g(s)\phi_{xx}(s)ds - \gamma(\beta+\alpha)q = h_2\in L^2(0.1),\\ -k_0(w_x-l\varphi)_x+kl(\varphi_x+lw+\psi)+\rho_1 w = h_3\in L^2(0.1),\\ q_x+(\beta+\alpha)\int_0^x q(y)dy - \gamma\psi_x = h_4\in L^2(0.1),\\ z+\tau^{-1}z_\rho = h_5\in L^2(0.1),\\ \phi+\phi_s-v = h_6\in L^2(0.1), \end{cases} \tag{41}$$

where

$$\begin{cases} h_1 = \rho_1(f_1+f_2),\\ h_2 = \rho_2(f_4+f_5) - \dfrac{\alpha}{k}\gamma f_9,\\ h_3 = \rho_1(f_6+f_7),\\ h_4 = \gamma f_{4x} + \rho_3\left(f_8 - \dfrac{\alpha}{k}\displaystyle\int_0^x f_9(y)dy\right),\\ h_5 = z+\tau^{-1}z_\rho,\\ h_6 = \phi+\phi_s - v. \end{cases} \tag{42}$$

Furthermore, by (39), we can find as $z(x,0)=u(x)$ for $x\in(0,1)$. Following the same last approach, we obtain by using equation for z in (39)

$$z(x,\rho)=u(x)e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho f_3(x,s)e^{\tau\rho s}ds. \tag{43}$$

From (39), we obtain

$$z(x,\rho)=\varphi(x)e^{-\tau\rho}-f_1e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho f_3(x,s)e^{\tau\rho s}ds. \tag{44}$$

Then,

$$z(x,1)=\varphi(x)e^{-\tau}+z_0(x), \tag{45}$$

such that

$$z_0(x)=-f_1e^{-\tau}+\tau e^{-\tau}\int_0^\rho f_3(x,s)e^{\tau s}ds. \tag{46}$$

We note that the last equation in (41) with $\phi(x,0)=0$ has a unique solution

$$\begin{aligned}\phi(x,s)&=\left(\int_0^x e^y(f_{10}(x,y)+v(x)dy)e^{-s}\right)\\&=\left(\int_0^x e^y(f_{10}(x,y)+\psi(x)-f_4(x)dy)e^{-s}\right).\end{aligned} \tag{47}$$

In order to solve (42), we consider

$$a((\varphi,\psi,w,q),(\tilde\varphi,\tilde\psi,\tilde w,\tilde q))=L(\tilde\varphi,\tilde\psi,\tilde w,\tilde q), \tag{48}$$

where

$$a:\left[H_*^1(0,1)\times\tilde H_*^1(0,1)\times\tilde H_*^1(0,1)\times L^2(0,1)\right]^2\longrightarrow\mathbb{R} \tag{49}$$

is the bilinear form given by

$$\begin{aligned}&a((\varphi,\psi,w,q),(\tilde\varphi,\tilde\psi,\tilde w,\tilde q))\\&\quad=k\int_0^1(\varphi_x+lw+\psi)(\tilde\varphi_x+l\tilde w+\tilde\psi)dx+(\beta+\alpha)\int_0^1 q\tilde q dx\\&\qquad+b\int_0^1\psi_x\tilde\psi_x dx+\rho_2\int_0^1\psi\tilde\psi dx-\gamma(\beta+\alpha)\int_0^1 q\tilde\psi dx\\&\qquad+\rho_1\int_0^1\psi\tilde\psi dx+\gamma(\beta+\alpha)\int_0^1\psi\tilde q dx+\rho_1\int_0^1 w\tilde w dx\\&\qquad+k_0\int_0^1(w_x-l\varphi)(\tilde w_x-l\tilde\varphi)dx+\int_0^1\varphi\tilde\varphi(\mu_1+\mu_2e^{-\tau})dx\\&\qquad+\rho_3(\beta+\alpha)\int_0^1\left(\int_0^x q(y)dy\int_0^x\tilde q(y)dy\right)dx.\\&L:\left[H_*^1(0,1)\times\tilde H_*^1(0,1)\times\tilde H_*^1(0,1)\times L^2(0,1)\right]\longrightarrow\mathbb{R}\end{aligned} \tag{50}$$

is the linear form defined by

$$\begin{aligned}L(\tilde\varphi,\tilde\psi,\tilde w,\tilde q)&=\int_0^1 h_1\tilde\varphi dx+\int_0^1 h_2\tilde\psi dx+\int_0^1 h_3\tilde w dx\\&\quad+(\alpha+\beta)\int_0^1 h_4\int_0^x\tilde q(y)dydx+\int_0^1(\mu_1f_1\mu_2z_0)\tilde\varphi dx.\end{aligned} \tag{51}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\tilde\varphi,\tilde\psi,\tilde w,\tilde q)\in H_*^1(0,1)\times\tilde H_*^1(0,1)\times\tilde H_*^1(0,1)\times L^2(0,1)$, problem (48) admits a unique solution $(\varphi,\psi,w,q)\in H_*^1(0,1)\times\tilde H_*^1(0,1)\times\tilde H_*^1(0,1)\times L^2(0,1)$. Since $D(A)$ is dense in $\mathscr{H}$ consequently, using Lemmas 1 and 2, we conclude that A is a maximal monotone operator. Hence, by Hille-Yosida theorem (see [45]), we have the following well-posedness result such that (25) is satisfied.

Theorem 3. *Let $U_0\in\mathscr{H}$, then there exists a unique weak solution $U\in C(\mathbb{R}^+,\mathscr{H})$ of problems (1)–(3). Moreover, if $U_0\in D(A)$, then $U\in C(\mathbb{R}^+,D(A))\cap C^1(\mathbb{R}^+,\mathscr{H})$.*

Lemma 4. *The operator F defined in (26) is locally Lipschitz in $\mathscr{H}$.*

Proof. Let $U=(\varphi,u,z,\psi,v,w,\varpi,\theta,q,\phi)^T$, $\bar U=(\bar\varphi,\bar u,\bar z,\bar\psi,\bar v,\bar w,\bar\varpi,\bar\theta,\bar q,\bar\phi)^T$, then we have

$$\left\|F(U)-F(\bar U)\right\|_{\mathscr{H}}\le\|f(\psi)-f(\bar\psi)\|_{L^2}. \tag{52}$$

By using (6), Holder's and Poincaré's inequalities, we can obtain

$$\|f(\psi)-f(\bar\psi)\|_{L^2}\le\left(\|\psi\|_{2\theta}^\theta+\|\bar\psi\|_{2\theta}^\theta\right)\|\psi-\bar\psi\|\le c_1\|\psi-\bar\psi\|, \tag{53}$$

which gives us

$$\left\|F(U)-F(\bar U)\right\|_{\mathscr{H}}\le c_1\|\psi-\bar\psi\|_{\mathscr{H}}. \tag{54}$$

Then, the operator F is locally Lipschitz in $\mathscr{H}$. The proof is hence complete.

3. Exponential Stability

Here, we present our stability result for the energy of the solution of systems (1)–(3), by using the multiplier technique. So we define the energy of our system by

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + b\psi_x^2 + \rho_3\theta^2 + \alpha q^2 + k(\varphi_x + \psi + lw)^2 + k_0(w_x - l\varphi)^2\right] dx + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds\,dx + \int_0^1 \widehat{f}(\psi(t))\,dx. \quad (55)$$

The proof of the stability for our system is based on the following lemmas:

Lemma 5. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)-(21). Then, the energy functional, defined by (55), satisfies*

$$E'(t) \le -\beta\int_0^1 q^2 dx - C\int_0^1 \psi_t^2 dx - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \cdot \|\varphi_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|z(x,1,t)\|_2^2 + \frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 ds\,dx, \quad (56)$$

such that $C > 0$.

Proof. Multiplying $(1.1)_1$, $(1.1)_2$, $(1.1)_3$, $(1.1)_4$, and $(1.1)_5$ by φ_t,ψ_t,w_t,θ, and q, respectively, and after simplification, we have (56).

With the fact

$$\frac{d}{dt}\widehat{f}(\psi) = f(\psi)\psi, \quad (57)$$

it gives us (56).

Lemma 6. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). We have*

$$F_1(t) \coloneqq \alpha\rho_3\int_0^1 \theta\int_0^x q(y)\,dy\,dx \quad (58)$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_1'(t) \le -\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_1\int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 q^2 dx. \quad (59)$$

Proof. Taking the derivative of F_1, using the fourth and fifth equations in (1) and performing integration by parts, we get

$$F_1'(t) = -\rho_3 k\int_0^1 \theta^2 dx - \alpha k\int_0^1 q^2 dx - \alpha\gamma\int_0^1 \psi_{tx}\int_0^1 q(y)\,dy\,dx - \beta\rho_3\int_0^1 \theta\int_0^x q(y)\,dy\,dx. \quad (60)$$

According to Cauchy–Schwarz and Young's inequalities with $\varepsilon_1 > 0$, we get (59).

Lemma 7. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). We have*

$$F_2(t) \coloneqq -\frac{\rho_2\rho_3}{\gamma}\int_0^1 \theta\int_0^x \psi_t(y)\,dy\,dx \quad (61)$$

satisfies, for any $\varepsilon_1, \varepsilon_2, \delta_1 > 0$, the estimate

$$F_2'(t) \le -\frac{\rho_2}{\gamma}\int_0^1 \psi_t^2 dx + \varepsilon_2\int_0^1 (\varphi_x + \psi + lw)^2 dx + \left(\varepsilon_3 + \frac{\rho_3}{\gamma}\left(\frac{\varepsilon_2}{b^2\lambda_2} + \frac{b^2}{2\varepsilon_2\lambda_2}\right)\right)\int_0^1 \psi_x^2 dx + c\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds\,dx. + (\delta_1 + C_1)\int_0^1 \theta_x^2 dx. \quad (62)$$

Proof. For differentiation of F_2, using equations in (1) and integration by parts, we obtain

$$F_2'(t) = -\rho_2\int_0^1 \psi_t^2 dx - \frac{\rho_2 k}{\gamma}\int_0^1 q\psi_t\,dx + \rho_3\int_0^1 \theta^2 dx - \frac{b\rho_3}{\gamma}\int_0^1 \theta\psi_x\,dx + \frac{k\rho_3}{\gamma}\int_0^1 (\varphi_x + \psi + lw)\int_0^x \theta(y)\,dy\,dx + \frac{\rho_3}{\gamma}\int_0^1\int_0^\infty g(s)\eta_x^t(x,s)\,ds\int_0^x \theta_x(y)\,dy\,dx + \frac{\rho_3}{\gamma}\int_0^1 \theta\int_0^x f(\psi)\,dy\,dx. \quad (63)$$

Estimate (62) follows by using Cauchy–Schwarz, Young's, and Poincaré's inequalities that

$$\int_0^1 |f(\psi)\theta|\,dx \le \int_0^1 |\psi|^\theta|\psi||\theta|\,dx \le \|\psi\|_{2(\theta+1)}^\theta\|\psi\|_{2(\theta+1)}\|\theta\| \le C_1\int_0^1 \theta^2 dx. \quad (64)$$

Lemma 8. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional*

$$F_3(t) \coloneqq -\rho_1\int_0^1 (\varphi\varphi_t + ww_t)\,dx \quad (65)$$

satisfies the estimate

$$F_3'(t) \le -\left(\rho_1 - \frac{1}{4\varepsilon_4}\right)\int_0^1 \varphi_t^2 dx + c\int_0^1 \psi_x^2 dx + k_0\int_0^1 (w_x - l\varphi)^2 dx + c\int_0^1 (\varphi_x + \psi + lw)^2 dx - \rho_1\int_0^1 w_t^2 dx + (\varepsilon_5\mu_2 + \mu_1\varepsilon_4)\int_0^1 \varphi^2 dx + \frac{\mu_2}{4\varepsilon_5}\int_0^1 z^2(x,1,t)dx. \quad (66)$$

Proof. Using (1)–(3) gives

$$F_3'(t) = -\rho_1\int_0^1 \varphi_t^2 dx + k\int_0^1 (\varphi_x + \psi + lw)^2 dx - k\int_0^1 (\varphi_x + \psi + lw)\psi dx - \rho_1\int_0^1 w_t^2 dx + k_0\int_0^1 (w_x - l\varphi)^2 dx + \mu_1\int_0^1 \varphi\varphi_t dx + \mu_2\int_0^1 \varphi z(x,1,t)dx. \quad (67)$$

Using Young's and Poincaré's inequalities, estimate (66) is established.

Lemma 9. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional*

$$F_4(t) := \rho_2\int_0^1 \psi\psi_t dx \quad (68)$$

satisfies for any $\delta_2 > 0$ the estimate

$$F_4'(t) \le \left(\frac{b}{2} + \delta_2 + C_2\right)\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \frac{k^2}{b}\int_0^1 (\varphi_x + \psi + lw)^2 dx + c\int_0^1 \theta^2 dx + \frac{g_0}{4\delta_2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx. \quad (69)$$

Proof. Taking the derivative of F_4 and using the second equation in (1), it follows that

$$F_4'(t) = -b\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \gamma\int_0^1 \psi_x\theta dx - k\int_0^1 (\varphi_x + \psi + lw)dx + \int_0^1 \psi_x(x)\int_0^\infty g(s)\eta_x^t(x,s)ds dx. - \int_0^1 \psi f(\psi)dx, \quad (70)$$

$$\int_0^1 |f(\psi)\psi| dx \le \int_0^1 |\psi|^\theta |\psi||\psi| dx \le \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \le C_2\int_0^1 \psi_x^2 dx. \quad (71)$$

Young's and Poincaré's inequalities for (70) yield (69).

Lemma 10. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional*

$$F_5(t) := -\rho_1\int_0^1 \varphi_t(w_x - l\varphi)dx - \rho_1\int_0^1 w_t(\varphi_x + \psi + lw)dx \quad (72)$$

satisfies the estimate

$$F_5'(t) \le -\left(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\right)\int_0^1 (w_x - l\varphi)^2 dx - \frac{l\rho_1}{2}\int_0^1 w_t^2 dx + (l\rho_1 + \varepsilon_6\mu_1)\int_0^1 \varphi_t^2 dx + c\int_0^1 \psi_t^2 dx + lk\int_0^1 (\varphi_x + \psi + lw)^2 dx + \varepsilon_7\mu_2\int_0^1 z^2(x,1,t)dx. \quad (73)$$

Proof. For differentiation of F_5, using $(1.1)_1$ and $(1.1)_3$, we arrive at

$$F_5'(t) = -lk_0\int_0^1 (w_x - l\varphi)^2 dx - l\rho_1\int_0^1 w_t^2 dx + l\rho_1\int_0^1 \varphi_t^2 dx + lk\int_0^1 (\varphi_x + \psi + lw)^2 dx - \rho_1\int_0^1 \psi_t w_t dx + \mu_1\int_0^1 \varphi_t(w_x - l\varphi)dx + \mu_2\int_0^1 z(x,1,t)(w_x - l\varphi)dx. \quad (74)$$

Young's inequality for (74) yields (73).

Lemma 11. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)-(21) and let $k = k_0$. Then, the functional*

$$F_6(t) := -\rho_1\int_0^1 (w_x - l\varphi)\int_0^x w_t(y)dy dx - \rho_1\int_0^1 \varphi_t\int_0^x (\varphi_x + \psi + lw)dy dx \quad (75)$$

satisfies the estimate

$$F_6'(t) \le -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx - k_0\int_0^1 (w_x - l\varphi)^2 dx + \rho_1\int_0^1 w_t^2 dx + k\int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{\rho_1}{2}\int_0^1 \psi_t^2 dx. \quad (76)$$

Proof. A simple differentiation of F_6, using the first and third equations in (1), leads to

$$\begin{aligned} F_6'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - k_0\int_0^1 (w_x - l\varphi)^2 dx + \rho_1\int_0^1 w_t^2 dx \\ &- \rho_1\int_0^1 \varphi_t \int_0^x \psi_t(y) dy + k\int_0^1 (\varphi_x + \psi + lw)^2 dx \\ &+ l(k - k_0)\int_0^1 (w_x - l\varphi)\int_0^x (\varphi_x + \psi + lw) dy dx, \end{aligned} \quad (77)$$

and using Young's and Cauchy-Schwarz inequalities, with the fact that $k = k_0$, gives (76).

Lemma 12. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21) and let (9) holds, and we have*

$$\begin{aligned} F_7(t) &:= \rho_2\int_0^1 \psi_t(\varphi_x + \psi + lw) dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_x dx \\ &+ \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_t dx \\ &- \frac{b}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x + \psi + lw) dx \\ &- \frac{bl^2\rho_2}{k_0}\int_0^1 \psi\psi_t dx + \frac{bl\rho_1}{k_0}\int_0^1 \psi w_t dx \end{aligned} \quad (78)$$

satisfies, for any $\varepsilon_4, \varepsilon_5, \delta_3 > 0$, the estimate

$$\begin{aligned} F_7'(t) &\le -\left(\frac{k}{2} - \frac{b\eta}{\gamma\alpha\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right. \\ &+ \left.\frac{b}{4\varepsilon_3}\right)\int_0^1 (\varphi_x + \psi + lw)^2 dx + \varepsilon_8\int_0^1 w_t^2 dx \\ &+ \left(\frac{b^2 l^2}{k} + \frac{bl^2\rho_2\delta_3}{k_0} + b\varepsilon_3 + \frac{bl^2}{k_0}c_1\right. \\ &+ \left. 2\left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right) + c_2\right)\int_0^1 \psi_x^2 dx \\ &+ \varepsilon_9\int_0^1 (w_x - l\varphi)^2 dx + c\left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k}\right) \\ &\cdot \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_8}\right)\int_0^1 q^2 dx + c\left(1 + \frac{1}{\varepsilon_9}\right. \\ &+ \left.\frac{b\rho_3\mu_1}{\gamma\rho_1}\varepsilon_5\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \theta^2 dx + \left(\frac{b\eta}{\gamma\alpha}\varepsilon_{10}\right. \\ &+ \left.\gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1}\varepsilon_2\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \theta_x^2 dx \\ &+ \left(\frac{b\rho_1}{k4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \varphi_t^2 dx \\ &+ \frac{l\varepsilon_1}{b^2}\int_0^1 (w + \psi)^2 dx + \frac{\varepsilon}{2}\int_0^1 (\varphi_x + \psi)^2 \\ &+ \frac{g_0 bl^2\rho_2}{k_0 4\delta_3}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx. \end{aligned} \quad (79)$$

Proof. Taking the deviate of F_7, we obtain

$$\begin{aligned} F_7'(t) &= \rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi + lw) dx + \rho_2\int_0^1 \psi_t(\varphi_x + \psi + lw)_t dx \\ &+ \frac{b\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx - \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_{xt} dx \\ &+ \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \theta_t\varphi_t dx \\ &+ \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_{tt} dx \\ &- \frac{b}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 q_t(\varphi_x + \psi + lw) dx \\ &- \frac{b}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x + \psi + lw)_t dx \\ &- \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_t^2 dx - \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_{tt}\psi dx \\ &+ \frac{bl\rho_1}{k_0}\int_0^1 w_{tt}\psi dx + \frac{bl\rho_1}{k_0}\int_0^1 w_t\psi_t dx. \end{aligned} \quad (80)$$

From the RHS of (80) and the relations in (1)–(3), we arrive at

$$\begin{aligned} &\rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi + lw) dx \\ &= -k\int_0^1 (\varphi_x + \psi + lw)^2 dx - \gamma\int_0^1 \theta_x(\varphi_x + \psi + lw) dx \\ &- b\int_0^1 \psi_x(\varphi_x + \psi + lw)_x dx \\ &- \int_0^1\int_0^\infty g(s)\psi_{xx}(x, t-s) ds(\varphi_x + \psi + lw) dx \\ &- \int_0^1 f(\psi)(\varphi_x + \psi + lw) dx, \end{aligned} \quad (81)$$

$$\begin{aligned} \rho_1\int_0^1 \varphi_{tt}\psi_x dx &= k\int_0^1 \psi_x(\varphi_x + \psi + lw)_x dx \\ &+ k_0 l\int_0^1 (w_x - l\varphi) dx - \mu_1\int_0^1 \varphi_t\psi_x dx, \end{aligned} \quad (82)$$

$$\rho_3\int_0^1 \theta_t\varphi_t dx = k\int_0^1 q\varphi_{xt} dx + \gamma\int_0^1 \psi_t\varphi_{xt} dx, \quad (83)$$

$$\begin{aligned} \int_0^1 \theta\varphi_{tt} dx &= -\frac{k}{\rho_1}\int_0^1 \theta_x(\varphi_x + \psi + lw) dx \\ &+ \frac{lk_0}{\rho_1}\int_0^1 \theta(w_x - l\varphi) dx - \frac{\mu_1}{\rho_1}\int_0^1 \theta\varphi_t dx, \end{aligned} \quad (84)$$

$$\begin{aligned} -\int_0^1 q_t(\varphi_x + \psi + lw) dx &= \frac{\beta}{\alpha}\int_0^1 q(\varphi_x + \psi + lw) dx \\ &+ \frac{k}{\alpha}\int_0^1 \theta_x(\varphi_x + \psi + lw) dx, \end{aligned} \quad (85)$$

$$-\rho_2\int_0^1 \psi_{tt}\psi dx = b\int_0^1 \psi_x^2 dx + k\int_0^1 \psi(\varphi_x+\psi+lw)dx - \gamma\int_0^1 \theta\psi_x dx + \int_0^1\int_0^\infty g(s)\psi_{xx} ds\psi dx + \int_0^1 f(\psi)\psi dx, \quad (86)$$

$$\rho_1\int_0^1 w_{tt}\psi dx = -k_0\int_0^1 \psi_x(w_x - l\varphi)dx - kl\int_0^1 \psi(\varphi_x+\psi+lw)dx. \quad (87)$$

Invoking to (81)–(87) into (80), we arrive at

$$\begin{aligned} F_7'(t) &= -k\int_0^1 (\varphi_x+\psi+lw)^2 dx + \left(\rho_2 - \frac{bl^2\rho_2}{k_0}\right)\int_0^1 \psi_t^2 dx \\ &+ \left(l\rho_2 + \frac{bl\rho_1}{k_0}\right)\int_0^1 \psi_t w_t dx + \frac{b\eta}{\alpha\gamma}\int_0^1 \theta_x(\varphi_x+\psi+lw)dx \\ &- \frac{b}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q\psi_t dx - \frac{bl}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q w_t dx \\ &+ \frac{blk_0\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta(w_x - l\varphi)dx - \frac{\gamma bl^2}{k_0}\int_0^1 \theta\psi_x dx \\ &+ \frac{b\beta}{\alpha\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x+\psi+lw)dx + \frac{b^2l^2}{k_0}\int_0^1 \psi_x^2 dx \\ &- bl\int_0^1 \psi_x(w_x-l\varphi)dx - \gamma\int_0^1 \theta_x(\varphi_x+\psi+lw)dx \\ &- \frac{kb\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta_x(\varphi_x+\psi+lw)dx \\ &- b\int_0^1 \psi_x(\varphi_x+\psi+lw)_x dx + \frac{b\rho_1}{k}\int_0^1 \psi_t\varphi_{xt} dx \\ &- \frac{b\rho_3\mu_1}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_t dx - \int_0^1 f(\psi)(\varphi_x+\psi+lw)dx \\ &+ \frac{bl^2}{k_0}\int_0^1 f(\psi)\psi dx + \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_x(x)\int_0^\infty g(s)\eta_x^t(x,s).dsdx. \end{aligned} \quad (88)$$

We thus have

$$\begin{aligned} \int_0^1 |\varphi_x f(\psi)|dx &\le \|\varphi_x\|\|\psi\|_{2(\theta+1)}^{\theta}\|\psi\|_{2(\theta+1)} \\ &\le \frac{\varepsilon}{2b^2}\int_0^1 \varphi_x^2 dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{\varepsilon}{b^2}\int_0^1 \psi^2 dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x+\psi)^2 dx + \left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2 dx. \end{aligned} \quad (89)$$

Estimate (79) follows thanks to Young's inequality and the fact that $k = k_0$.

Lemma 13. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, the energy functional*

$$F_8(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx. \quad (90)$$

Then, we have the following estimate, for any $\varepsilon_{11} > 0$,

$$F_8'(t) \le \left(-K + \varepsilon_{11}\left(\frac{K}{2} + \frac{\mu_2 c}{2}\right)\right)\int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_{11}}\int_0^1 \psi_x^2 dx + \frac{\mu_2}{2\varepsilon_{11}}\int_0^1 z^2(x,1,t)dx + \rho_1\int_0^1 \varphi_t^2 dx, \quad (91)$$

where $c = 1/\pi^2$ is the Poincaré constant.

Proof. Taking the derivative of (90) with respect to t, we have

$$F_8'(t) = \rho_1\int_0^1 \varphi_{tt}\varphi dx + \rho_1\int_0^1 \varphi_t^2 dx + \mu_1\int_0^1 \varphi_t\varphi dx. \quad (92)$$

Then, by using the first equation in (1), we find

$$F_8'(\mathrm{t}) = k\int_0^1 (\varphi_x+\psi+lw)_x\varphi dx - \mu_2\int_0^1 \varphi z(x,1,t)dx + \rho_1\int_0^1 \varphi_t^2 dx. \quad (93)$$

Consequently, we arrive at

$$F_8'(t) = -k\int_0^1 (\varphi_x+\psi+lw)\varphi_x dx - \mu_2\int_0^1 \varphi z(x,1,t)dx + \rho_1\int_0^1 \varphi_t^2 dx. \quad (94)$$

Applying Young's inequality and Poincaré's inequality, we find (90).

Lemma 14. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (19)–(21). Then, we define the functional*

$$F_9(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho dx. \quad (95)$$

Then, the following result holds.

$$F_9'(t) \le -F_9(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)dx + \frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)dx, \quad (96)$$

where c is a positive constant.

Proof. Taking the deviate of (95) with respect to t and using the equation (16), we get

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx\right) &= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)d\rho dx \\ &= -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx \\ &\quad -\frac{1}{2\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx.\end{aligned} \tag{97}$$

Making use of the estimate above, implies that there exists a positive constant c_1 such that (96) holds.

Theorem 15. *Assume that $\eta = 0$ and $k = k_0$. Then, $(\varphi, \psi, w, \theta, q, z, \eta^t)$ the solution of (19)–(21) satisfies*

$$E(t) \le c_0 e^{-c_1 t}, t \ge 0, \tag{98}$$

where the positive constant c_0 is directly depending on initial data and the uniform constant c_1 is depending only on the coefficients of the system. For $N, N_i > 0$,

$$\mathscr{L}(t) := NE(t) + \sum_{i=1}^{i=9} N_i F_i(t), \tag{99}$$

Then, from (56), (59), (62), (66), (69), (73), (76), (79), (91), and (96), we have

$$\begin{aligned}\mathscr{L}'(t) \le{}& \left[-\beta N + c_1\left(1+\frac{1}{\varepsilon_1}\right) + cN_2 + c\left(1+\frac{1}{\varepsilon_8}\right)N_7\right]\int_0^1 q^2dx \\ &- N\left[\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2}\right]\|\varphi_t\|_2^2 - N\left[\frac{\xi}{2} - \frac{|\mu_2|}{2}\right]\|z(x,1,t)\|_2^2 \\ &- \left[\frac{N_1\rho_3}{2} - N_2\left(C_1 + 1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) - cN_4\right. \\ &\left. - c\left(1 + \frac{1}{\varepsilon_9} + \frac{b\rho_3\mu_1}{\gamma\rho_1 c}\varepsilon_5\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)N_7\right]\int_0^1\theta^2dx \\ &+ \left[\varepsilon_1 N_1 - CN - N_2\frac{\rho_2}{\gamma} + \rho_2 N_4 + cN_5 + \frac{\rho_1}{2}N_6\right. \\ &\left. + c\left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k}\right)N_7 + \rho_1 N_8 + \frac{1}{2\tau}N_9\right]\int_0^1\psi_t^2dx \\ &+ \left[\varepsilon_2 N_2 + cN_3 + \frac{k^2}{b}N_4 + lkN_5 + kN_6\right. \\ &- \left(\frac{k}{2} - \frac{b\tilde{\eta}}{\alpha\gamma\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right. \\ &\left.\left. + \frac{b}{4\varepsilon_3}\right)N_7\right]\int_0^1(\varphi_x + lw + \psi)^2dx \\ &+ \left[\left(\varepsilon_3 + \frac{\rho_3}{\gamma}\left(\frac{\varepsilon_2}{b^2\lambda_2} + \frac{b^2}{2\varepsilon_2\lambda_2}\right)\right)N_2 + cN_3\right. \\ &+ \left(\delta_2 + \frac{b}{2} + C_2\right)N_4 + \left(\frac{bl^2\rho_2\delta_3}{k_0} + \frac{b^2l^2}{k} + b\varepsilon_3\right. \\ &\left. + \frac{bl^2}{k_0}c_1 + 2\left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right) + c_2\right)N_7 \\ &\left. + \left(\frac{k}{2\varepsilon_{11}} - k + \varepsilon_{11}\frac{k}{2} + \varepsilon_{11}\frac{\mu_2 c}{2}\right)N_8\right]\int_0^1\psi_x^2dx \\ &+ \left[-\rho_1 N_3 - \frac{l\rho_1}{2}N_5 + \rho_1 N_6 + \varepsilon_8 N_7\right]\int_0^1 w_t^2dx \\ &+ \left[k_0 N_3 - \left(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\right)N_5 - k_0 N_6 + \varepsilon_9 N_7\right] \\ &\cdot \int_0^1(w_x - l\varphi)^2dx + \left[\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8 - \frac{c_1}{2\tau}N_9\right] \\ &\cdot \int_0^1 z^2(x,1,t)dx + [\varepsilon_5\mu_2 + \varepsilon_4\mu_1]N_3\int_0^1\varphi^2dx \\ &+ \left[-\left(1 - \frac{1}{4\varepsilon_4}\right)N_3 + (l\rho_1 + \varepsilon_6\mu_1)N_5 - \frac{\rho_1}{2}N_6\right. \\ &\left. + \rho_1 N_8 + \left(\frac{b\rho_1}{k4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)N_7\right]\int_0^1\varphi_t^2dx \\ &+ \left[\left(-k + \varepsilon_{11}\left(\frac{k}{2} + \frac{\mu_2 c}{2}\right)\right)N_8\right]\int_0^1\varphi_x^2dx \\ &+ \left[\left(\frac{b\tilde{\eta}\varepsilon_{10}}{\alpha\gamma} + \gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1}\varepsilon_2\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)N_7\right. \\ &\left. + N_2\delta_1\right]\int_0^1\theta_x^2dx - N_9F_9(t) + \left(\frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2}\right. \\ &\left. + \frac{N_7 g_0 bl^2\rho_2}{k_0 4\delta_3} - N\frac{\zeta}{2}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned} \tag{100}$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_i i = 1, \ldots 10$ small enough such that

$$\varepsilon_1 \le \frac{N_2(\rho_2/\gamma) + \rho_2 N_4 + cN_5 + (\rho_1/2)N_6}{N_1}. \tag{101}$$

Moreover, we pick N_9 large enough so that

$$\begin{aligned}&\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8 - \frac{c_1}{2\tau}N_9 \le 0, \\ &N_9 \ge \frac{(\mu_2/4\varepsilon_5)N_3 + \varepsilon_7\mu_2 N_5 + (\mu_2/2\varepsilon_{11})N_8}{c_1/2\tau},\end{aligned} \tag{102}$$

and we take ε_{11} small enough such that

$$\varepsilon_{11} \le \frac{k}{(k/2 + \mu_2 c/2)N_8}. \tag{103}$$

Next, choosing N_5 large enough such that

$$\frac{N_5\rho_3\kappa}{4} \geq N_4\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4}(b+2\kappa)\right). \tag{104}$$

After that, we can choose N large enough such that

$$\begin{gathered} N \geq \frac{c_1(1+1/\varepsilon_1) + cN_2 + c(1+1/\varepsilon_8)N_7}{\beta}, \\ \frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2} + \frac{g_0 b l^2 \rho_2}{k_0 4\delta_3} - N\frac{\zeta}{2} \leq 0. \end{gathered} \tag{105}$$

Thus, the relation (100) becomes

$$\begin{aligned} \frac{d}{dt}\mathscr{L}(t) \leq & -\eta_1\int_0^1 \left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + lw + \psi)^2 \right. \\ & \left. + \theta^2 + q^2\right)dx - \eta_1\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx, \end{aligned} \tag{106}$$

which leads by (55) that there exists also η_2, such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\eta_2 E(t), \forall t \geq 0. \tag{107}$$

Lemma 16. *For N large enough, there exist two positive constants β_1 and β_2 depending on $N_i, i = 1, \cdots, 9$ and $\varepsilon_i, i = 1, \cdots, 11$ such that*

$$\beta_1 E(t) \leq \mathscr{L}(t) \leq \beta_2 E(t), \forall t \geq 0. \tag{108}$$

Proof. We consider the functional

$$H(t) = \sum_{i=1}^{i=9} N_i F_i(t) \tag{109}$$

and show that

$$|H(t)| \leq CE(t), C > 0. \tag{110}$$

From (58), (61), (65), (68), (72), (75), (78), (90), and (95), we obtain

$$\begin{aligned} |H(t)| \leq & N_1\left|\alpha\rho_3\int_0^1\theta\int_0^x q(y)dydx\right| \\ & + N_2\left|-\frac{\rho_2\rho_3}{\gamma}\int_0^1\theta dx\int_0^x \psi_t(y)dydx\right| \\ & + N_3\left|\rho_1\int_0^1(\varphi\varphi_t + ww_t)dx\right| \\ & + N_4\left|\rho_2\int_0^1\int_0^x \psi\psi_t(t,x)dx\right| \\ & + N_5\left|-\rho_1\int_0^1\varphi_t(w_x - l\varphi)dx - \rho_1\int_0^1 w_t(\varphi_x + \psi + lw)dx\right| \\ & + N_6\left|-\rho_1\int_0^1(w_x - l\varphi)\int_0^x w_t(y)dydx\right. \\ & \left. -\rho_1\int_0^1\varphi_t\int_0^x(\varphi_x + \psi + lw)dydx\right| \\ & + N_7\left|\rho_2\int_0^1(\varphi_x + \psi + lw)dx + \frac{b\rho_1}{k}\int_0^1\varphi_t\psi_x dx\right| \\ & + N_8\left|\int_0^1\rho_1\varphi\varphi_t dx + \frac{\mu_1}{2}\int_0^1\varphi^2 dx\right| \\ & + N_9\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx. \end{aligned} \tag{111}$$

By using, the trivial relation

$$\int_0^1(\varphi + lw)^2 dx \leq 2c\int_0^1(\varphi_x + lw + \psi)^2 dx + 2c\int_0^1\psi_x^2 dx, \tag{112}$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| \leq & \alpha_1\int_0^1\varphi_t^2 dx + \alpha_2\int_0^1\psi_t^2 dx + \alpha_3\int_0^1 w_t^2 \\ & + \alpha_4\int_0^1\psi_x^2 + \alpha_5\int_0^1\theta^2 dx + \alpha_6\int_0^1 q^2 dx \\ & + \alpha_7\int_0^1\left((\varphi_x + lw + \psi)^2 + (w_x - l\varphi)^2\right)dx \\ & + \int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx, \end{aligned} \tag{113}$$

where $\alpha_1, \cdots, \alpha_6$ are the positive constants as follows:

$$\begin{cases} \alpha_1 := \frac{1}{2}(N_3\rho_1 + N_8\rho_1), \\ \alpha_2 := \frac{1}{2}\left(N_4\rho_2 + N_2\frac{\rho_2\rho_3}{\gamma}\right), \\ \alpha_3 = \frac{1}{2}(N_3\rho_1 + N_6\rho_1), \\ \alpha_4 := \frac{b\rho_1}{2k}, \\ \alpha_5 := \frac{1}{2}\left(N_1\rho_3 + \frac{\rho_2\rho_3}{\gamma}\right), \\ \alpha_6 := \frac{1}{2}(N_1\rho_3 + N_5\tau_0\rho_3), \\ \alpha_7 := \frac{1}{2}(N_7\rho_2 + 3\rho_1). \end{cases} \tag{114}$$

From (113), we have

$$|H(t)| \leq \widehat{C}E(t), \tag{115}$$

for

$$\widehat{C} = \frac{\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}}{\min\{\rho_1, \rho_2, \rho_3, k, b, \kappa, \gamma, \delta, \tau_0\}}. \tag{116}$$

Therefore, we get

$$|\mathscr{L}(t) - NE(t)| \leq \widehat{C}E(t). \tag{117}$$

Then, we can choose N large enough so that $\beta_1 = N - \widehat{C} > 0$. Then, (108) holds true for $\beta_2 = N + \widehat{C} > 0$, and this concludes the proof of the Lemma.

Combining now (107) and (108), we conclude that there exists some $\Lambda > 0$ such that

$$\frac{d}{dt}\mathscr{L}(t) \leq -\Lambda\mathscr{L}(t), \forall t \geq 0. \tag{118}$$

Integration of (118) yields

$$\mathscr{L}(t) \leq \mathscr{L}(0)e^{-\Lambda t}, \forall t \geq 0. \tag{119}$$

Finally, using (108) and (119), so (98) is satisfied, we thus immediately reach to Theorem 15.

4. Conclusion and Perspective

In this current study, a one-dimensional linear thermoelastic Bresse system with delay term, forcing, and infinity history acting on the shear angle displacement is considered. According to an appropriate assumption between the weight of the delay and the weight of the damping, the well-posedness of the problem using the semigroup method is proved, where an asymptotic stability result of global solution is obtained. In next article, we will generalize this result to convex bounded domain with a holomorphic map, and let x and y be two distinct fixed points for our problem. We will suppose there is at least one complex geodesics passing through two distinct variables. We will see that this method of proof cannot be generalized to the case of a bounded domain of a complex Banach space. Also, in the last part of the next article, we will study the fixed points of the analytical automorphisms of the open unit-ball B of a complex Banach space. More precisely, we will assume that B is homogeneous and we will show that, if the right hand side is an analytical automorphism of B, there exists a complex geodesic which we will specify formed of fixed points of the right hand. We will see that the set of fixed points of the right hand can be much larger by using the studied algorithm in ([46–51]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Solvability for a New Class of Moore-Gibson-Thompson Equation with Viscoelastic Memory, Source Terms, and Integral Condition

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This paper deals with the existence and uniqueness of solutions of a new class of Moore-Gibson-Thompson equation with respect to the nonlocal mixed boundary value problem, source term, and nonnegative memory kernel. Galerkin's method was the main used tool for proving our result. This work is a generalization of recent homogenous work.

1. Introduction

In this contribution, we are interested to study the existence and uniqueness of solutions of the following problem

$$\begin{cases} \mathscr{L}u(x,t) = au_{ttt} + \beta u_{tt} - c^2\Delta u - b\Delta u_t - \displaystyle\int_0^t h(t-\sigma)\Delta u(\sigma)d\sigma = F(x,t), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), u_{tt}(x,0) = u_2(x), \\ \dfrac{\partial u}{\partial \eta} = \displaystyle\int_0^t\int_\Omega (\xi,\tau)d\xi d\tau, x \in \partial\Omega. \end{cases} \quad (1)$$

Here, a and β are physical parameters, and c is the speed of sound. The convolution term $\int_0^t h(t-s)\Delta u(s)ds$ reflects the memory effect of materials due to vicoelasticity, F is a given function, and h is the relaxation function satisfying

(H1) $h \in C^1(\mathbb{R}+,\mathbb{R}+)$ is a nonincreasing function satisfying

$$h(0) > 0, 1 - h_0 = l > 0, \quad (2)$$

where $h_0 = G(\infty) = \int_0^\infty h(s)ds > 0$, $G(t) = \int_0^t h(s)ds$, and $h'' > 0$.

(H2) $\exists\zeta > 0$ satisfying

$$h'(t) \le -\zeta h(t), t \ge 0. \quad (3)$$

(H3)

$$\beta - a > 0 \quad (4)$$

The phenomena resulting from sound waves (diffraction, interference, reflection) in terms of modeling are very important. As the existence of the third derivative is very important, especially in the field of thermodynamics (EIT), the study of these models is considered the beginning of an in-depth understanding of both convergent and good behavior. From the results extracted, the equation of MGT resulted in nonlinear acoustics, for much depth, see ([1–7]) and especially [8] where equation of MGT appeared for the first time. Also, nonlinear problems of great importance can be considered [9], where Galerkin's method was applied in solving them, for more depth ([2, 3, 10–13]). Recently, in [14], the authors studied the equation of MGT with memory. Likewise, in [1], the authors used Galerkin's method to demonstrate the ability to solve a mixed problem of MGT equation in the absence of viscous elasticity and memory. Based on work [9] and the works we mentioned earlier, we want to prove the existence and uniqueness of a weak solution to the problem (1).

We divide this paper into the following: in the second part, we put some definitions and appropriate spaces. Then, we apply Galerkin's method to prove the existence, and in the fourth part, we demonstrate the uniqueness.

2. Preliminaries

We will define the spaces: $V(Q_T)$ and $W(Q_T)$ by

$$\begin{aligned} V(Q_T) &= \left\{u \in W_2^1(Q_T):\ u_t \in W_2^1(Q_T), u,\nabla u \in L_h^2(Q_T)\right\},\\ W(Q_T) &= \{u \in V(Q_T):\ u(x,T)=0\},\\ L_h^2(Q_T) &= \left\{u \in V(Q_T):\ \int_0^T h \circ u(t)dt<\infty\right\}, \end{aligned} \tag{5}$$

where

$$h \circ u(t) = \int_\Omega \int_0^t h(t-\sigma)(u(t)-u(\sigma))^2 d\sigma dx. \tag{6}$$

Consider the equation

$$\begin{aligned} & a(u_{ttt}, v)_{L^2(Q_T)} + \beta(u_{tt}, v)_{L^2(Q_T)} - c^2(\Delta u, v)_{L^2(Q_T)} \\ & \quad - b(\Delta u_t, v)_{L^2(Q_T)} - (\Delta w, v)_{L^2(Q_T)} = (F, v)_{L^2(Q_T)}, \end{aligned} \tag{7}$$

where

$$w(x,t) = \int_0^t h(t-\sigma)u(x,\sigma)d\sigma, \tag{8}$$

and $(.,.)_{L2(QT)}$ stands for the inner product in $L^2(Q_T)$, u is supposed to be a solution of (1) and $v \in W(Q_T)$. Evaluation of the inner product in [9] gives

$$\begin{aligned} & -a(u_{tt}, v_t)_{L^2(Q_T)} - \beta(u_t, v_t)_{L^2(Q_T)} + c^2(\nabla u,\nabla v)_{L^2(Q_T)} \\ & \quad + b(\nabla u_t,\nabla v)_{L^2(Q_T)} + (\nabla w,\nabla v)_{L^2(Q_T)} \\ & = (F, v)_{L^2(Q_T)} + c^2\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega v(\xi,\tau)d\xi d\tau\right)ds_x dt \\ & \quad + b\int_0^T\int_{\partial\Omega} v\int_\Omega u(\xi,t)d\xi ds_x dt - b\int_0^T\int_{\partial\Omega} v\int_\Omega u_0(\xi)d\xi ds_x dt \\ & \quad + a(u_2(x), v(x,0))_{L^2(\Omega)} + \beta(u_1(x), v(x,0))_{L^2(\Omega)} \\ & \quad + \int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega w(\xi,\tau)d\xi d\tau\right)ds_x dt. \end{aligned} \tag{9}$$

We give two useful inequalities:

(i) Gronwall inequality. Let the nonnegative integrable functions $\varphi(t), \phi(t)$ on the interval I with the nondecreasing function $h(t)$. If $\forall t \in I$, we have

$$\phi(t) \le \varphi(t) + c\int_0^t \phi(s)ds, \tag{10}$$

where $c>0$, hence,

$$\phi(\mathrm{t}) \le \varphi(\mathrm{t}) \exp(\mathrm{ct}). \tag{11}$$

(ii) Trace inequality (see [15]). If $\Phi \in W_1^2(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, then for any $\varepsilon>0$,

$$\|\Phi\|^2_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\Phi\|^2_{L^2(\Omega)} + l(\varepsilon)\|\Phi\|^2_{L^2(\Omega)}, \tag{12}$$

where $l(\varepsilon)>0$.

Definition 1. We call a generalized solution to the problem (1) for each function $u \in V(Q_T)$ that fulfills the equation (9) for each $v \in W(Q_T)$.

3. Solvability of the Problem

In this section, we use Galerkin's method to prove the existence of a generalized solution of our problem.

Theorem 2. *If $u_0 \in W_2^1(\Omega)$, $u_1 \in W_2^1(\Omega)$, $u_2 \in L^2(\Omega)$, and $F \in L^2(Q_T)$, then there is at least one generalized solution in $V(Q_T)$ to problem (1).*

Proof. Let $\{Z_k(x)\}_{k\ge1}$ be a fundamental system in $W_2^1(\Omega)$, such that $(Z_k, Z_l)\Omega = \delta_{k,l}$.

First, we will give an approximate solution of the problem (1) in the form

$$u^N(x,t)=\sum_{k=1}^{N}C_k(t)Z_k(x), \tag{13}$$

where $C_k(t)$ are constants given by the conditions, for $k=1,\cdots,N$,

$$(\mathscr{L}u(x,t),Z_l(x))_{L^2(\Omega)}=(F(x,t),Z_l(x))_{L^2(\Omega)} \tag{14}$$

and can be determined from the relations

$$\begin{aligned}
&a\left(u_{ttt}^N,Z_l(x)\right)_{L^2(\Omega)}+\beta\left(u_{tt}^N,Z_l(x)\right)_{L^2(\Omega)}+c^2\left(\nabla u^N,\nabla Z_l(x)\right)_{L^2(\Omega)}\\
&\quad+b\left(\nabla u_t^N,\nabla Z_l(x)\right)_{L^2(\Omega)}+\left(\nabla w^N,\nabla Z_l(x)\right)_{L^2(\Omega)}\\
&=(F(x,t),Z_l(x))_{L^2(\Omega)}+c^2\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau\right)ds_x\\
&\quad+b\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau\right)ds_x\\
&\quad+\int_{\partial\Omega}Z_l(x)\left(\int_0^t\int_\Omega w^N(\xi,\tau)d\xi d\tau\right)ds_x,
\end{aligned} \tag{15}$$

substitution of (13) into (15), and we find for $l=1,\cdots,N$.

$$\begin{aligned}
&\int_\Omega\sum_{k=1}^{N}\Big\{aC_k'''(t)Z_k(x)Z_l(x)\beta C_k'''(t)Z_k(x)Z_l(x)\\
&\quad+c^2C_k(t)\nabla Z_k(x).\nabla Z_l(x)+bC_k'(t)\nabla Z_k(x).\nabla Z_l(x)\\
&\quad+\left(\int_0^t h(t-\sigma)C_k(\sigma)d\sigma\right)\nabla Z_k(x).\nabla Z_l(x)\Big\}dx\\
&=(F(x,t),Z_l(x))_{L^2(\Omega)}+c^2\sum_{k=1}^{N}\int_0^t C_k(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\
&\quad+b\sum_{k=1}^{N}\int_0^t C_k'(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\
&\quad+\sum_{k=1}^{N}\int_0^t\left(\int_0^\tau h(\tau-\sigma)C_k(\sigma)d\sigma\right)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau
\end{aligned} \tag{16}$$

From (15) it follows that

$$\begin{aligned}
&\sum_{k=1}^{N}\Big\{aC_k'''(t)(Z_k(x),Z_l(x))_{L^2(\Omega)}+\beta C_k''(t)(Z_k(x),Z_l(x))_{L^2(\Omega)}\\
&\quad+c^2C_k(t)(\nabla Z_k(x),\nabla Z_l(x))_{L^2(\Omega)}\\
&\quad+bC_k'(t)(\nabla Z_k(x),\nabla Z_l(x))_{L^2(\Omega)}\\
&\quad+\left(\int_0^t h(t-\sigma)C_k(\sigma)d\sigma\right)(\nabla Z_k(x),\nabla Z_l(x))_{L^2(\Omega)}\Big\}\\
&=(F(x,t),Z_l(x))_{L^2(\Omega)}+c^2\sum_{k=1}^{N}\int_0^t C_k(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\
&\quad+b\sum_{k=1}^{N}\int_0^t C_k'(\tau)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau\\
&\quad+\sum_{k=1}^{N}\int_0^t\left(\int_0^\tau h(\tau-\sigma)C_k(\sigma)d\sigma\right)\left(\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds_x\right)d\tau
\end{aligned} \tag{17}$$

Let

$$\begin{aligned}
&(Z_k,Z_l)_{L^2(\Omega)}=\delta_{kl}=\begin{cases}1, & k=l\\ 0, & k\neq l\end{cases},\\
&(\nabla Z_k,\nabla Z_l)_{L^2(\Omega)}=\gamma_{kl},\\
&\int_{\partial\Omega}Z_l(x)\int_\Omega Z_k(\xi)d\xi ds=\chi_{kl},\\
&(F(x,t),Z_l(x))_{L^2(\Omega)}=F_l(t).
\end{aligned} \tag{18}$$

Then, (17) can be written as

$$\begin{aligned}
&\sum_{k=1}^{N}\Big\{a\delta_{kl}C_k'''(t)+\beta\delta_{kl}C_k''(t)+c^2C_k(t)\gamma_{kl}\\
&\quad+\left(\int_0^t h(t-\sigma)C_k(\sigma)d\sigma\right)\gamma_{kl}+bC_k'(t)\gamma_{kl}\\
&\quad-\int_0^t\left[c^2C_k(\tau)\chi_{kl}-\left(\int_0^\tau h(\tau-\sigma)C_k(\sigma)d\sigma\right)\chi_{kl}\right]d\tau\Big\}=F_l(t).
\end{aligned} \tag{19}$$

By differentiating (two times) with respect to t, it gives

$$\begin{aligned}
&\sum_{k=1}^{N}\Big\{a\delta_{kl}C_k'''(t)+\beta\delta_{kl}C_k'''(t)+c^2C_k''(t)\gamma_{kl}+h(0)C_k'(t)\gamma_{kl}\\
&\quad+bC_k''(t)\gamma_{kl}-c^2C_k(t)\chi_{kl}-bC_k''(t)\chi_{kl}-h(0)C_k(t)\chi_{kl}\Big\}=F_l''(t).
\end{aligned} \tag{20}$$

$$\sum_{k=1}^{N}\left\{a\delta_{kl}C_k'''(0)+\beta\delta_{kl}C_k''\right\}. \tag{21}$$

We find a system of differential equations of fifth order with respect to t, constant coefficients, and the initial conditions (21). Hence, we obtain a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. Thus, $\forall n$, $\exists u^N(x)$ satisfying (15).

Now, we prove that u^N is sequence bounded. To do this, we multiply each equation of (15) by the appropriate $C_k'(t)$ summing over k from 1 to N. Hence, by integration the result equality with respect to t from 0 to τ, and $\tau\leq T$, it gives

$$\begin{aligned}
&a\left(u_{ttt}^N,u_t^N\right)_{L^2(Q_\tau)}+\beta\left(u_{ttt}^N,u_t^N\right)_{L^2(Q_\tau)}+c^2\left(\nabla u^N,\nabla u_t^N\right)_{L^2(Q_\tau)}\\
&\quad+b\left(\nabla u_t^N,\nabla u_t^N\right)_{L^2(Q_\tau)}+\left(\nabla w^N,\nabla u_t^N\right)_{L^2(Q_\tau)}\\
&=\left(F,u_t^N\right)_{L^2(Q_\tau)}+c^2\int_0^\tau\int_0^\tau u_t^N(x,t)\left(\int_0^t\int_\Omega(\xi,\eta)d\xi d\eta\right)ds_x dt\\
&\quad+b\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt\\
&\quad+\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt.
\end{aligned} \tag{22}$$

Evaluation of the terms on the LHS of (22) gives

$$a\left(u_{ttt}^N, u_t^N\right)_{L^2(Q_\tau)} = a\left(u_{\tau\tau}^N, u_\tau^N\right)_{L^2(\Omega)} - a\left(u_{tt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)} - a\int_0^\tau \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{23}$$

$$\beta\left(u_{tt}^N, u_t^N\right)_{L^2(Q_\tau)} = \frac{\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\beta}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{24}$$

$$c^2\left(\nabla u^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = \frac{c^2}{c}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{c^2}{c}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{25}$$

$$b\left(\nabla u_t^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = b\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{26}$$

$$\left(\nabla w^N, \nabla u_t^N\right)_{L^2(Q_\tau)} = \frac{1}{2} h\circ\nabla u^N(\tau) - \frac{1}{2} G(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{1}{2}\int_0^\tau h'^N(t)dt + \frac{1}{2}\int_0^\tau h(t)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 dt, \tag{27}$$

$$\begin{aligned} c^2\int_0^\tau\int_{\partial\Omega} u_t^N\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt &= c^2\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad - c^2\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x, \end{aligned} \tag{28}$$

So,

$$\begin{aligned} b\int_0^\tau\int_{\partial\Omega} u_t^N\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt &= b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad - b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x. \end{aligned} \tag{29}$$

Thus,

$$\begin{aligned} \int_0^\tau\int_{\partial\Omega} u_t^N\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt &= \int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad - \int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x. \end{aligned} \tag{30}$$

Taking into account the equalities (23)–(30) in (22), we end up with

$$\begin{aligned} & a\left(u_{\tau\tau}^N, u_\tau^N\right)_{L^2(\Omega)} + \frac{\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \frac{1}{2} h\circ\nabla u^N(\tau) - \frac{1}{2} G(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &= a\left(u_{tt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)} + \frac{\beta}{2}\left\|u_\tau^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{c^2}{2}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + a\int_0^\tau \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &\quad + b\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \frac{1}{2}\int_0^\tau h'^N(t)dt - \frac{1}{2}\int_0^\tau h(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &\quad + c^2\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x - c^2\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega (\xi,t)d\xi dt ds_x \\ &\quad + b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x - b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x \\ &\quad + \int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega (\xi,t)d\xi dt ds_x - \int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x + \left(F, u_t^N\right)_{L^2(Q_\tau)} \end{aligned} \tag{31}$$

Now, multiplying the equations of (15) by $C_k''(t)$, collect them from 1 to N and integrating the result with respect to t from 0 to τ, and $\tau \le T$, we find

$$\begin{aligned} & a\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + \beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + c^2\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} + b\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} + \left(\nabla w^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} \\ &= \left(F, u_{tt}^N\right)_{L^2(Q_\tau)} + c^2\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &\quad + b\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_\eta^N(\xi,\eta)d\xi d\eta\right)ds_x dt - \int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt \end{aligned} \tag{32}$$

With the same reasoning in (22), we find

$$a\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} = \frac{a}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{a}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{33}$$

$$\beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} = \beta\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{34}$$

$$\begin{aligned} c^2\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= c^2\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(Q_\tau)} \\ &\quad - c^2\left(\nabla u^N(x,0), \nabla u_\tau^N(x,0)\right)_{L^2(\Omega)} \\ &\quad - c^2\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \end{aligned} \tag{35}$$

$$\begin{aligned} b\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= \frac{b}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad - \frac{b}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \end{aligned} \tag{36}$$

$$\begin{aligned} &\left(\nabla w^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} \\ &= \frac{1}{2}\left\{h'^N(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - 2\left(\nabla w^N(\tau), \nabla u_\tau^N\right)_{L^2(\Omega)}\right\} \\ &\quad + \frac{1}{2}\int_0^\tau h''^N(t)dt - \frac{1}{2}\int_0^\tau h'(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \end{aligned} \tag{37}$$

$$\begin{aligned} &c^2\int_0^\tau\int_{\partial\Omega} u_{tt}^N\left(\int_0^\tau\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &= c^2\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi t ds_x \\ &\quad - c^2\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x, \end{aligned} \tag{38}$$

$$\begin{aligned} &b\int_0^\tau\int_{\partial\Omega} u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_\eta^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &= b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x \\ &\quad - b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x \\ &\quad - b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds. \end{aligned} \tag{39}$$

$$\begin{aligned} &\int_0^\tau\int_{\partial\Omega} u_{tt}^N\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\ &= \int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega w^N(\xi,t)d\xi dt ds_x \\ &\quad - \int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x. \end{aligned} \tag{40}$$

A substitution of equalities (33)–(40) in (22) gives

$$\begin{aligned} &\frac{a}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{b}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad + c^2\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)} \\ &\quad - \frac{1}{2}\left\{h'^N(\tau) + h(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right. \\ &\quad \left. - 2\left(\nabla w^N(\tau), \nabla u_\tau^N\right)_{L^2(\Omega)}\right\} \\ &= \frac{a}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 + c^2\left(\nabla u^N(x,0), \nabla u_t^N(x,0)\right)_{L^2(\Omega)} \\ &\quad + \frac{b}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 - \beta\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &\quad + c^2\int_0^\tau \left\|\nabla u_t(x,t)\right\|_{L^2(\Omega)}^2 - \frac{1}{2}\int_0^\tau h''^N(t)dt \\ &\quad + \frac{1}{2}\int_0^\tau h'(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &\quad + c^2\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad - c^2\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x \\ &\quad + b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x \\ &\quad - b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x \\ &\quad - b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)d\xi dt ds_x \\ &\quad \cdot \int_{\partial\Omega} u_\tau^N(x,t)\int_0^\tau\int_\Omega w^N(\xi,t)d\xi dt ds_x \\ &\quad - \int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x + \left(F, u_{tt}^N\right)_{L^2(Q_\tau)}. \end{aligned} \tag{41}$$

Multiplying (32) by λ and using (41), we get

$$\begin{aligned} &\lambda a\left(u_{\tau\tau}^N, u_\tau^N\right)_{L^2(\Omega)} + \frac{\lambda\beta}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\lambda c^2}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \frac{\lambda}{2} h\circ\nabla u^N(\tau) \\ &\quad - \frac{\lambda}{2}G(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \frac{a}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad + \frac{b}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 + c^2\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)} \\ &\quad - \frac{1}{2}\left\{h'^N(\tau) + h(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right. \\ &\quad \left. - 2\left(\nabla w^N(\tau), \nabla u_\tau^N\right)_{L^2(\Omega)}\right\} \\ &= \lambda\left(F, u_t^N\right)_{L^2(Q_\tau)} + \left(F, u_{tt}^N\right)_{L^2(Q_\tau)} \\ &\quad + \lambda a\left(u_{tt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)} \\ &\quad + (\lambda a - \beta)\int_0^\tau \left\|u_{tt}(x,t)\right\|_{L^2(\Omega)}^2 dt \end{aligned}$$

$$+\left(c^2-\lambda b\right)\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt+\frac{\lambda}{2}\int_0^\tau h'^N(t)dt$$
$$-\frac{\lambda}{2}\int_0^\tau h(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+\lambda c^2\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$+\frac{\lambda c^2}{2}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$-\lambda c^2\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$+\frac{\lambda\beta}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2$$
$$+\left(\lambda b-c^2\right)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$-\lambda b\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x$$
$$+\lambda\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega w^N(\xi,t)d\xi dt ds_x$$
$$-\lambda\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x$$
$$+\frac{a}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2+c^2\left(\nabla u^N(x,0),\nabla u_t^N(x,0)\right)_{L^2(\Omega)}$$
$$+\frac{b}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2-\frac{1}{2}\int_0^\tau h''^N(t)dt$$
$$+\frac{1}{2}\int_0^\tau h'(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+c^2\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$+b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(x,\tau)d\xi ds_x$$
$$-b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$-b\int_{\partial\Omega}\int_0^\tau u_t^N(\xi,t)d\xi dt ds_x\int_{\partial\Omega}$$
$$\cdot u_\tau^N(x,\tau)\int_0^\tau\int_\Omega w^N(\xi,t)d\xi dt ds_x$$
$$-\int_{\partial\Omega}\int_0^\tau u_\tau^N(x,t)\int_\Omega w^N(\xi,t)d\xi dt ds_x. \tag{42}$$

where $0<\lambda<1$.

With the help of Cauchy and the trace inequalities, we can estimate all the terms in the RHS of (42) that gives

$$c^2\lambda\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{c^2\lambda}{2_{\varepsilon_1}}\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \tag{43}$$
$$+\frac{c^2\lambda}{2}\varepsilon_1 T|\Omega||\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt,$$

$$-c^2\lambda\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{c^2\lambda}{2}\varepsilon\int_0^\tau\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt \tag{44}$$
$$+\frac{c^2\lambda}{2}\left(l(\varepsilon)+|\Omega||\partial\Omega|\right)\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt,$$

$$\left(b\lambda-c^2\right)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,\tau)d\xi dt ds_x$$
$$\le\frac{\left(b\lambda-c^2\right)}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt\right.$$
$$\left.+l(\varepsilon)\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt\right) \tag{45}$$
$$+\frac{\left(b\lambda-c^2\right)}{2}|\Omega||\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt,$$

$$-b\lambda\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x$$
$$\le\frac{b\lambda}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt+l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt\right)$$
$$+\frac{b\lambda}{2}|\Omega||\partial\Omega|T\left\|u^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{46}$$

$$c^2\int_{\partial\Omega} u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)d\xi dt ds_x$$
$$\le\frac{c^2}{2}\left(\frac{\varepsilon}{\varepsilon_2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+\frac{l(\varepsilon)}{\varepsilon_2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)$$
$$+\frac{c^2}{2}\varepsilon_2|\Omega||\partial\Omega|T\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{47}$$

$$+b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)d\xi ds_x$$
$$\le\frac{b}{2\varepsilon_3}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \tag{48}$$
$$+\frac{b}{2}\varepsilon_3|\Omega||\partial\Omega|\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2,$$

$$-b\int_{\partial\Omega} u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x$$
$$\le\frac{b}{2\varepsilon_4}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \tag{49}$$
$$+\frac{b}{2}\varepsilon_4|\Omega||\partial\Omega|\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2,$$

$$-b\int_{\partial\Omega} u_\tau^N(x,t)\int_\Omega u_\tau^N(\xi,\tau)d\xi dt ds_x$$
$$\le\frac{b}{2}\int_0^\tau\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 dt \tag{50}$$
$$+\frac{b}{2}\left(l(\varepsilon)|\Omega||\partial\Omega|\right)\int_0^\tau\left\|u_\tau^N(x,t)\right\|_{L^2(\Omega)}^2 dt,$$

$$-\frac{\lambda a}{2}\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}-\frac{\lambda a}{2}\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \le \lambda a\left(u^N_{\tau\tau}(x,\tau),u^N_{\tau}(x,\tau)\right)_{L^2(\Omega)}, \tag{51}$$

$$-\frac{c^2\varepsilon_7}{2}\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}-\frac{c^2}{2\varepsilon_7}\left\|\nabla u^N_{\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \le c^2\left(u^N(x,\tau),\nabla u^N_{\tau}(x,\tau)\right)_{L^2(\Omega)}, \tag{52}$$

$$\lambda a\left(u^N_{tt}(x,0),u^N_{\tau}(x,0)\right) \le \frac{\lambda a}{2}\left\|u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)}+\frac{\lambda a}{2}\left\|u^N_{\tau}(x,0)\right\|^2_{L^2(\Omega)}, \tag{53}$$

$$c^2\left(\nabla u^N(x,0),u^N_{\tau}(x,0)\right)_{L^2(\Omega)} \le \frac{c^2}{2}\left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)}+\frac{c^2}{2}\left\|\nabla u^N_{\tau}(x,0)\right\|^2_{L^2(\Omega)}. \tag{54}$$

$$-\frac{G(\tau)}{2}\left(\frac{1}{\varepsilon_8}+\frac{1}{\varepsilon_9}\right)\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}-\frac{\varepsilon_8}{2}h\circ\nabla u^N(\tau) - \frac{\varepsilon_9}{2}G(\tau)\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} \le 2\left(\nabla w^N(x,\tau),\nabla u^N_{\tau}(x,\tau)\right)_{L^2(\Omega)}, \tag{55}$$

$$\int_{\partial\Omega}u^N_{\tau}(x,\tau)\int_0^{\tau}\int_{\Omega}w^N(\xi,t)d\xi dt ds_x \le \left(\frac{1}{2\varepsilon_5}+\frac{1}{2\varepsilon_5'}\right)\left(\varepsilon\left\|\nabla u^N_{\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u^N_{\tau}(x,\tau)\right\|^2_{L^2(\Omega)}\right) + |\Omega||\partial\Omega|T\left(\frac{\varepsilon_5}{2}\int_0^{\tau}h\circ u^N(t)dt+\frac{\varepsilon_5'}{2}h_0\int_0^{\tau}\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}\right), \tag{56}$$

$$\lambda\int_{\partial\Omega}u^N(x,\tau)\int_0^{\tau}\int_{\Omega}w^N(\xi,t)d\xi dt ds_x \le \lambda\left(\frac{1}{2\varepsilon_6}+\frac{1}{2\varepsilon_6'}\right)\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) + \lambda|\Omega||\partial\Omega|T\left(\frac{\varepsilon_6}{2}\int_0^{\tau}h\circ u^N(t)dt+\frac{\varepsilon_6'}{2}h_0\int_0^{\tau}\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)}\right), \tag{57}$$

$$\int_{\partial\Omega}\int_0^{\tau}u^N_t(x,\tau)\int_{\Omega}w^N(\xi,t)d\xi dt ds_x \le \left(\varepsilon\int_0^{\tau}\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^{\tau}\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt\right) + \frac{1}{2}|\Omega||\partial\Omega|T\left(\int_0^{\tau}h\circ u^N(t)dt+h_0\int_0^{\tau}\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt,\right) \tag{58}$$

$$\lambda\int_{\partial\Omega}\int_0^{\tau}u^N(x,t)\int_{\Omega}u^N(\xi,t)d\xi dt ds_x \le \lambda\left(\varepsilon\int_0^{\tau}\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}dt+l(\varepsilon)\int_0^{\tau}\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt\right) + \frac{\lambda}{2}|\Omega||\partial\Omega|T\left(\int_0^{\tau}h\circ u^N(t)dt+h_0\int_0^{\tau}\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt\right), \tag{59}$$

$$\lambda\left(F,u^N_{\tau}\right)_{L^2(Q_{\tau})} \le \frac{\lambda}{2}\int_0^{\tau}\|F(x,t)\|^2_{L^2(\Omega)}dt+\frac{1}{2}\int_0^{\tau}\|u_{tt}(x,t)\|^2_{L^2(\Omega)}dt. \tag{60}$$

Combining inequalities (45)-(60) and equality (44) and make use of the following inequality

$$\begin{aligned}
&m_1\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)} \le m_1\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}+m_1\left\|u^N_{\tau}(x,t)\right\|^2_{L^2(\Omega)} + m_1\left\|u^N(x,0)\right\|^2_{L^2(\Omega)},\\
&m_2\left\|u^N_t(x,\tau)\right\|^2_{L^2(\Omega)} \le m_2\left\|u^N_t(x,t)\right\|^2_{L^2(\Omega)}+m_2\left\|u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)} + m_2\left\|u^N_t(x,0)\right\|^2_{L^2(\Omega)},\\
&m_3\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} \le m_3\left\|\nabla u^N(x,t)\right\|^2_{L^2(Q_{\tau})}+m_3\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_{\tau})} + m_3\left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)},\\
&h\circ u^N(\tau) \le \frac{1}{2}h\circ u^N(t)dt+\frac{h_0}{2}\left\|u^N_t(x,t)\right\|^2_{L^2(Q_{\tau})}-\int_0^{\tau}h'^N(t)dt,\\
&h\circ\nabla u^N(\tau) \le \frac{1}{2}\int_0^{\tau}h\circ\nabla u^N(t)dt+\frac{h_0}{2}\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_{\tau})}-\int_0^{\tau}h'^N(t)dt,
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
m_1 &:= \frac{c^2\lambda}{\varepsilon_1}l(\varepsilon)+\frac{b}{2}\varepsilon_3|\Omega||\partial\Omega|+\lambda\left(\frac{1}{2\varepsilon_6}+\frac{1}{2\varepsilon_6'}\right)l(\varepsilon),\\
m_2 &:= l(\varepsilon)\left(\frac{c^2}{2\varepsilon_2}+\frac{b}{2\varepsilon_2}+\frac{b}{2\varepsilon_4}+\left(\frac{1}{2\varepsilon_5}+\frac{1}{2\varepsilon_5'}\right)\right),\\
m_3 &:= \frac{c^2\varepsilon_7}{2}+h(0)+\frac{c^2\lambda}{2\varepsilon_1}+\frac{c^2\varepsilon_9}{2}+\lambda\left(\frac{1}{2\varepsilon_6}+\frac{1}{2\varepsilon_6'}\right)\varepsilon,
\end{aligned} \tag{62}$$

and we have

$$\begin{aligned}
&\frac{c^2\lambda}{2\varepsilon_1} l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}(\beta-a)\left\|u_t^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\lambda}{2}\left(c^2 - G(\tau)\right)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left\{\frac{b}{2} - \frac{c^2\varepsilon}{2\varepsilon_2} - \frac{b\varepsilon}{2\varepsilon_3} - \frac{b\varepsilon}{2\varepsilon_4} - \frac{c^2}{2\varepsilon_7} - \frac{\varepsilon}{2\varepsilon_5} \right. \\
&\quad \left. - \frac{\varepsilon}{2\varepsilon_5'} - \frac{G(\tau)}{2}\left(\frac{1}{\varepsilon_8} + \frac{1}{\varepsilon_9}\right)\right\}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{a}{2} - \frac{\lambda a}{2}\right)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{\lambda+1}{2}\right) h\diamond\nabla u^N(\tau) + h\circ u^N(\tau) \\
&\le \gamma_1\int_0^\tau \left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \gamma_2\int_0^\tau \left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
&\quad + \underbrace{\left\{\frac{c^2\lambda}{2}\varepsilon + \lambda\varepsilon + m_3\right\}}_{\alpha_1}\int_0^\tau \left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
&\quad + \gamma_3\int_0^\tau \left\|\nabla u_\tau^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \underbrace{\frac{(\lambda+3)\xi}{2}}_{\alpha_2}\int_0^\tau h\diamond\nabla u^N(\tau) \\
&\quad + \gamma_4\int_0^\tau h\circ u^N(\tau) + \underbrace{\left\{\frac{\lambda a + \beta + m_2}{2}\right\}}_{\alpha_3}\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
&\quad + \underbrace{\left\{\frac{b\lambda}{2}|\Omega||\partial\Omega|T + \frac{b}{2}\varepsilon_4|\Omega||\partial\Omega| + m_1\right\}}_{\alpha_4}\left\|u^N(x,0)\right\|_{L^2(\Omega)}^2 \\
&\quad + \underbrace{\left\{\frac{\lambda a}{2} + \frac{\lambda\beta}{2} + m_2\right\}}_{\alpha_5}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{\lambda+1}{2}\right)\int_0^\tau \left\|F(x,t)\right\|_{L^2(\Omega)}^2 dt \\
&\quad + \underbrace{\left\{\frac{\lambda}{2}c^2 + \frac{c^2}{2} + \frac{1}{2}h(0) + m_3\right\}}_{\alpha_6}\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 \\
&\quad + \underbrace{\left\{\frac{b}{2} + \frac{c^2}{2}\right\}}_{\alpha_7}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 \\
&\quad + \underbrace{\left\{\frac{\lambda a}{2} + \frac{a}{2}\right\}}_{\alpha_7}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\gamma_1 &:= \left\{\frac{c^2\lambda}{2}\varepsilon_1 T|\partial\Omega||\Omega| + \frac{c^2\lambda}{2}\left(l(\varepsilon) + |\partial\Omega||\Omega|\right) \right. \\
&\quad + \frac{(b\lambda + c^2)}{2}|\partial\Omega||\Omega| + \left(\frac{c^2}{2}\varepsilon_2 + \frac{c^2}{2}\varepsilon_5' h_0 + \frac{\lambda+1}{2}\right) \\
&\quad \left. \cdot |\partial\Omega||\Omega|T + \lambda l(\varepsilon) + m_1\right\}, \\
\gamma_2 &:= \left\{\frac{(b\lambda + c^2)}{2} l(\varepsilon) + \frac{b\lambda}{2} l(\varepsilon) + \frac{b}{2}\left(l(\varepsilon) + |\partial\Omega||\Omega|\right) + l(\varepsilon) \right. \\
&\quad \left. + \frac{h_0}{2\xi} + \frac{\lambda}{2} + m_2\right\},
\end{aligned} \tag{64}$$

$$\begin{aligned}
\gamma_3 &:= \left\{c^2 + \lambda b + \frac{(b\lambda + c^2)}{2}\varepsilon + \frac{b(\lambda+1)}{2}\varepsilon \right. \\
&\quad \left. + \lambda\varepsilon + \frac{h_0\lambda}{2} + \frac{h_0}{2\xi} + m_3\right\},
\end{aligned}$$

$$\gamma_4 := \left\{\left(\frac{\varepsilon_5}{2} + \frac{\lambda+1}{2}\right)|\partial\Omega||\Omega|T + \frac{3\xi}{2}\right\}. \tag{65}$$

Choosing $\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_5', \varepsilon_7, \varepsilon_8$ and ε_9 sufficiently large

$$\begin{aligned}
\alpha_0 &= \frac{b}{2} - \frac{c^2\varepsilon}{2\varepsilon_2} - \frac{b\varepsilon}{2\varepsilon_3} - \frac{b\varepsilon}{2\varepsilon_4} - \frac{c^2}{2\varepsilon_7} - \frac{\varepsilon}{2\varepsilon_5} \\
&\quad - \frac{\varepsilon}{2\varepsilon_5'} - \frac{G(\tau)}{2}\left(\frac{1}{\varepsilon_8} + \frac{1}{\varepsilon_9}\right) > 0.
\end{aligned} \tag{66}$$

By using (2)-(4), the relation (64) reduces to

$$\begin{aligned}
&\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad + \left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + h\circ u^N(\tau) + h\diamond\nabla u^N(\tau) \\
&\le D\int_0^\tau \left\{\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 + \left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 \right. \\
&\quad + \left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2 + \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 + \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 \\
&\quad \left. + h\circ u^N(t) + h\diamond\nabla u^N(t) + \|F\|_{L^2(\Omega)}^2\right\} dt \\
&\quad + D\left(\left\|u^N(x,0)\right\|_{W_2^1(\Omega)}^2 + \left\|u_t^N(x,0)\right\|_{W_2^1(\Omega)}^2 \right. \\
&\quad \left. + \left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2\right), \\
&\le D\int_0^\tau \left\{\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 + \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 \right. \\
&\quad \left. + h\circ u^N(t) + h\diamond\nabla u^N(t) + \|F\|_{L^2(\Omega)}^2\right\} dt \\
&\quad + D\left(\left\|u^N(x,0)\right\|_{W_2^1(\Omega)}^2 + \left\|u_t^N(x,0)\right\|_{W_2^1(\Omega)}^2 \right. \\
&\quad \left. + \left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 + h\circ u^N(0) + h\diamond\nabla u^N(0)\right),
\end{aligned} \tag{67}$$

where

$$D := \frac{\max\{\alpha_i, i = 1 \cdots 8\}}{\min\{(c^2\lambda/2\varepsilon_1)l(\varepsilon), (\lambda/2)(\beta - a), (\lambda/2)(c^2 - G(\tau)), (a/2)(1-\lambda), (\lambda + 1/2), 1, \alpha_0\}}. \tag{68}$$

Using the inequality of Gronwall to (67) and integrating the result from 0 to τ that gives

$$\begin{aligned}&\|u^N(x,t)\|^2_{W_2^1(Q_\tau)} + \|u_t^N(x,t)\|^2_{W_2^1(Q_\tau)} \\ &\quad + \|u^N(x,t)\|^2_{h,W_2^1(Q_\tau)} + \|u_{tt}^N(x,t)\|^2_{L^2(Q_\tau)} \\ &\le De^{DT}\Big\{\|u_0(x)\|^2_{W_2^1(\Omega)} + \|u_1(x)\|^2_{W_2^1(\Omega)} \\ &\quad + \|u_2(x)\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(\Omega)}\Big\}.\end{aligned} \tag{69}$$

where

$$\begin{aligned}&\|u(x,t)\|^2_{h,W_2^1(Q_\tau)} := \int_0^\tau h \circ u(t)dt + \int_0^\tau h\circ\nabla u(t)dt, \\ &\|u(x,0)\|^2_{h,W_2^1(Q_\tau)} = 0.\end{aligned} \tag{70}$$

We deduce from (69) that

$$\begin{aligned}&\|u^N(x,t)\|^2_{w_2^1(Q_\tau)} + \|u_t^N(x,t)\|^2_{w_2^1(Q_\tau)} \\ &\quad + \|u^N(x,t)\|^2_{h,W_2^1(Q_\tau)} + \|u_{tt}^N(x,t)\|^2_{L^2(Q_\tau)} \le A.\end{aligned} \tag{71}$$

Hence, $\{u^N\}_{N\ge1}$ is sequence bounded in $V(Q_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(Q_T)$ to a limit function $u(x,t)$, and we have to show that $u(x,t)$ is a generalized solution of (1). Since $u^N(x,t) \longrightarrow u(x,t)$ in $L^2(Q_T)$ and $u^N(x,0) \longrightarrow \zeta(x)$ in $L^2(\Omega)$, then $u(x,0) = \zeta(x)$.

Now to prove that (15) holds, we multiply each of the relations (15) by a function $p_l(t) \in W_2^1(0,T)$, $p_l(T) = 0$. Hence, collect them the obtained equalities ranging from $l = 1$ to $l = N$ and integrating the result over t on $(0,T)$. If we let $\eta^N = \sum_{k=1}^N p_l(t)Z_k(x)$, then we have

$$\begin{aligned}&a(u_{tt}^N, \eta_t^N)_{L^2(Q_T)} + \beta(u_t^N, \eta_t^N)_{L^2(Q_T)} - c^2(\nabla u^N, \nabla\eta^N)_{L^2(Q_T)} \\ &\quad - b(\nabla u_t^N, \nabla\eta^N)_{L^2(Q_T)} - (\nabla w^N, \nabla\eta^N)_{L^2(Q_T)} \\ &= a(u_{tt}^N(x,0), \eta^N(0))_{L^2(\Omega)} + \beta(u_t^N(x,0), \eta^N(0))_{L^2(\Omega)} \\ &\quad + c^2\int_{\partial\Omega}\int_0^T \eta^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\tau)d\xi d\tau\right)dtds_x \\ &\quad + b\int_{\partial\Omega}\int_0^T \eta^N(x,t)\int_\Omega u^N(\xi,\tau)d\xi dtds_x \\ &\quad - b\int_{\partial\Omega}\int_0^T \eta^N(x,t)\int_\Omega u^N(\xi,0)d\xi dtds_x \\ &\quad + \int_{\partial\Omega}\int_0^T \eta^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\tau)d\xi d\tau\right)dtds_x \\ &\quad + (F, \eta_t^N)_{L^2(Q_T)},\end{aligned} \tag{72}$$

for all η^N of the form $\sum_{k=1}^N p_l(t)Z_k(x)$ and $\alpha > 0$.

Since

$$\begin{aligned}&\int_0^t\int_\Omega\left(\left(u^N(\xi,\tau) - u(\xi,\tau)\right)d\xi d\tau\right) \le \sqrt{T|\Omega|}\left\|u^N - u\right\|_{L^2(Q_T)}, \\ &\left\|u^N - u\right\|_{L^2(Q_T)} \to 0 \text{ as } N \to \infty.\end{aligned} \tag{73}$$

Thus, the limit function u satisfies (15) for every $\eta^N = \sum_{k=1}^N p_l(t)Z_k(x)$.

We define the totality of all functions of the form $\eta^N = \sum_{k=1}^N p_l(t)Z_k(x)$ by $\mathbb{Q}_N$, with $p_l(t) \in W_2^1(0,T)$, $p_l(T) = 0$.

But $\cup_{l=1}^N \mathbb{Q}_N$ is dense in $W(Q_T)$, hence the relation (15) holds $\forall u \in W(Q_T)$. Then, we have shown that the limit function $u(x,t)$ is a generalized solution of problem (1) in $V(Q_T)$.

4. Uniqueness of the Problem

Theorem 3. *The problem (1) cannot have more than one generalized solution in $V(Q_T)$.*

Proof. Suppose that $\exists_{u_1,u_2} \in V(Q_T)$ two different generalized solutions for the problem (1). Hence, the difference $U = u_1 - u_2$ solves

$$\begin{cases} aU_{ttt} + \beta U_{tt} - c^2\Delta U - b\Delta U_t - \int_0^t h(t-s)\Delta U(s)ds = 0, \\ U(x,0) = U_t(x,0) = U_{tt}(x,0) = 0, \\ \dfrac{\partial U}{\partial\eta} = \int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau, x \in \partial\Omega, \end{cases} \tag{74}$$

and (9) gives

$$\begin{aligned} a(U_{tt}, v_t)_{L^2(Q_T)} &+ \beta(U_t, v_t)_{L^2(Q_T)} + c^2(\nabla U, \nabla v)_{L^2(Q_T)} \\ &+ b(\nabla U_t, \nabla v)_{L^2(Q_T)} + (\nabla W_t, \nabla v)_{L^2(Q_T)} \\ &= -c^2\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau\right)ds_x dt \\ &- b\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,\tau)d\xi ds_x dt \\ &- \int_0^T\int_{\partial\Omega} v_t\left(\int_\Omega W(\xi,\tau)d\xi\right)ds_x dt. \end{aligned} \tag{75}$$

where

$$W(x,t) = \int_0^t h(t-\sigma)U(x,\sigma)d\sigma. \tag{76}$$

Let the function

$$v(x,t) = \begin{cases} \int_t^\tau U(x,s)ds & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{77}$$

It is obvious that $v \in W(Q_T)$ and $v_t(x,t) = -U(x,t)$ for all $t \in [0,\tau]$. By integration by parts in the LHS of (75) that yields

$$\begin{aligned} -a(U_{tt}, v_t)_{L^2(Q_T)} &= a(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)} \\ &- a\int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)}dt, \end{aligned} \tag{78}$$

$$-\beta(U_t, v_t)_{L^2(Q_T)} = \frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}, \tag{79}$$

$$c^2(\nabla U, \nabla v)_{L^2(Q_T)} = \frac{c^2}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}, \tag{80}$$

$$\begin{aligned} (\nabla W, \nabla v)_{L^2(Q_T)} &\le h_0\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}dt \\ &+ \frac{h_0}{2}\int_0^\tau \|\nabla U(x,t)\|^2_{L^2(\Omega)}dt \\ &+ \frac{h_0}{2}\int_0^\tau h_0\nabla U(t)dt, \end{aligned} \tag{81}$$

$$b(\nabla U_t, \nabla v)_{L^2(Q_T)} = b\int_0^\tau \|\nabla v_t(x,t)\|^2_{L^2(\Omega)}dt. \tag{82}$$

Plugging (78)–(82) into (75), we obtain

$$\begin{aligned} a(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)} &+ \frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)} \\ &= a\int_0^\tau \|U_\tau(x,t)\|^2_{L^2(\Omega)}dt - b\int_0^\tau \|v_t(x,t)\|^2_{L^2(\Omega)}dt h_0 \\ &\cdot \int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}dt + \frac{h_0}{2}\int_0^\tau \|\nabla U(x,t)\|^2_{L^2(\Omega)}dt \\ &+ \frac{h_0}{2}h_0\nabla U(t)dt - \int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega W(\xi,\tau)d\xi d\tau\right)ds_x dt \\ &- c^2\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau\right)ds_x dt \\ &- b\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,\tau)d\xi ds_x dt. \end{aligned} \tag{83}$$

Now since

$$v^2(x,t) = \left(\int_t^\tau U(x,s)ds\right)^2 \le \tau\int_0^\tau U^2(x,s)ds, \tag{84}$$

then

$$\|v\|^2_{L^2(Q_T)} \le \tau^2\|U\|^2_{L^2(Q_T)} \le T^2\|U\|^2_{L^2(Q_T)}. \tag{85}$$

Applying the inequality of the trace, the RHS of (83) gives

$$\begin{aligned} c^2\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)d\xi d\tau\right)ds_x dt \\ \le \frac{c^2}{2}T^2\{l(\varepsilon) + |\Omega||\partial\Omega|\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt \\ + \frac{c^2}{2}\varepsilon\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}dt, \end{aligned} \tag{86}$$

$$\begin{aligned} b\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,t)d\xi ds_x dt \\ \le \frac{b}{2}\left\{T^2 l(\varepsilon) + |\Omega||\partial\Omega|\right\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt \\ + \frac{b}{2}\varepsilon\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}dt, \end{aligned} \tag{87}$$

$$\begin{aligned} \int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega W(\xi,\tau)d\xi d\tau\right)ds_x dt \\ \le \frac{1}{2}T^2\{l(\varepsilon) + |\Omega||\partial\Omega|\}\int_0^\tau h\circ U(t)dt \\ + \frac{1}{2}T^2 h\circ\{l(\varepsilon) + |\Omega||\partial\Omega|\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt \\ + \varepsilon\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{88}$$

Combining the relations (86)–(83) and (87)–(88), we get

$$\begin{aligned}
&a(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)}+\frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}\\
&\quad+\frac{c^2}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}\\
&\le\left\{\frac{c^2}{2}T^2(l(\varepsilon)+|\Omega||\partial\Omega|)+\frac{(b+h_0)}{2}\left(T^2 l(\varepsilon)+|\Omega||\partial\Omega|\right)\right\}\\
&\quad\cdot\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt+\left(\left(\frac{c^2+b}{2}+1\right)\varepsilon+h(0)\right)\\
&\quad\cdot\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt+a\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt\\
&\quad+\frac{h_0}{2}\int_0^\tau\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt+\frac{h_0}{2}\int_0^\tau h\circ\nabla U(t)dt\\
&\quad+\frac{1}{2}T^2(l(\varepsilon)+|\Omega||\partial\Omega|)\int_0^\tau h\circ U(t)dt.
\end{aligned}\tag{89}$$

Next, multiplying (74) by U_{tt} and integrating the result over $Q_\tau=\Omega\times(0,\tau)$, we find

$$\begin{aligned}
&a(U_{ttt},U_{tt})_{L^2(Q_T)}+\beta(U_{tt},U_{tt})_{L^2(Q_T)}-c^2(\Delta U,U_{tt})_{L^2(Q_T)}\\
&\quad-b(\Delta U_t,U_{tt})_{L^2(Q_T)}-(\Delta W,U_{tt})_{L^2(Q_T)}=0.
\end{aligned}\tag{90}$$

An integration by parts in (91) yields

$$a(U_{ttt},U_{tt})_{L^2(Q_T)}=\frac{a}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)},\tag{91}$$

$$\beta(U_{tt},U_{tt})_{L^2(Q_T)}=\beta\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt,\tag{92}$$

$$\begin{aligned}
-c^2(\Delta U,U_{tt})_{L^2(Q_T)}&=c^2(\nabla U_\tau(x,\tau),\nabla U(x,\tau))_{L^2(\Omega)}\\
&\quad-c^2\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}dt\\
&\quad-c^2\int_{\partial\Omega}U_\tau(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x\\
&\quad+c^2\int_{\partial\Omega}\int_0^\tau U_t(x,t)\int_\Omega U(\xi,t)d\xi dt ds_x,
\end{aligned}\tag{93}$$

$$\begin{aligned}
-b(\Delta U_t,U_{tt})_{L^2(Q_T)}&=\frac{b}{2}\|\nabla U_t(x,\tau)\|^2_{L^2(\Omega)}\\
&\quad-b\int_{\partial\Omega}U_t(x,\tau)\int_\Omega U(\xi,\tau)d\xi ds_x\\
&\quad+b\int_0^\tau\int_{\partial\Omega}U_t(x,t)\int_\Omega U_t(\xi,t)d\xi ds_x dt,
\end{aligned}\tag{94}$$

$$\begin{aligned}
(\nabla W,\nabla U_{tt})_{L^2(Q_T)}&=-\frac{1}{2}h'\circ\nabla U(\tau)-\frac{1}{2}h(\tau)\|\nabla U(x,\tau)\|^2_{L^2(Q_T)}\\
&\quad+2(\nabla W(\tau),\nabla U_\tau)_{L^2(\Omega)}\\
&\quad-\frac{1}{2}\int_0^\tau h'(t)\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt\\
&\quad+\frac{1}{2}\int_0^\tau h''\circ\nabla U(t)dt\\
&\quad+\int_{\partial\Omega}U_\tau(x,\tau)\int_0^\tau\int_\Omega W(\xi,\sigma)d\xi d\sigma ds_x\\
&\quad-\int_0^\tau\int_{\partial\Omega}U_t(x,t)\int_\Omega W(\xi,t)d\xi ds_x dt.
\end{aligned}\tag{95}$$

Substitution (91)–(95) into (90), we get the equality

$$\begin{aligned}
&\frac{a}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}+c^2(\nabla U_\tau(x,\tau),\nabla U(x,\tau))_{L^2(\Omega)}\\
&\quad+\frac{b}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}h'\circ\nabla U(\tau)\\
&\quad-\frac{1}{2}h(\tau)\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}+2(\nabla W(\tau),\nabla U_\tau)_{L^2(\Omega)}\\
&=-\beta\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt+c^2\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}\\
&\quad+c^2\int_{\partial\Omega}U_\tau(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,t)d\xi dt\right)ds_x\\
&\quad-c^2\int_{\partial\Omega}\int_0^\tau U_t(x,t)\int_\Omega U(\xi,t)d\xi dt ds_x\\
&\quad+b\int_{\partial\Omega}U_\tau(x,\tau)\int_\Omega U(\xi,\tau)d\xi ds_x\\
&\quad-b\int_0^\tau\int_{\partial\Omega}U_t(x,t)\int_\Omega W(\xi,t)d\xi ds_x dt\\
&\quad+\int_{\partial\Omega}U_\tau(x,\tau)\int_0^\tau\int_\Omega W(\xi,\sigma)d\xi d\sigma ds_x\\
&\quad-\int_0^\tau\int_{\partial\Omega}U_t(x,t)\int_\Omega W(\xi,t)d\xi ds_x dt\\
&\quad+\frac{1}{2}\int_0^\tau h'(t)\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt\\
&\quad-\frac{1}{2}\int_0^\tau h''\circ\nabla U(t)dt.
\end{aligned}\tag{96}$$

The RHS of (96) can be bounded as follows

$$\begin{aligned}
&c^2\int_{\partial\Omega}U_\tau(x,\tau)\left(\int_0^T\int_\Omega U(\xi,t)d\xi dt\right)ds_x\\
&\quad\le\frac{c^2}{2_{\varepsilon_1'}}\left(\varepsilon\|\nabla U_\tau(\xi,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\right)\\
&\quad+\frac{c^2}{2\,\varepsilon_1'}T|\partial\Omega||\Omega|\int_0^T\|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}\tag{97}$$

$$
\begin{aligned}
&-c^2\int_{\partial\Omega}\int_0^\tau U_t(x,t)\int_\Omega U(\xi,\tau)d\xi dt ds_x \\
&\quad\le \frac{c^2}{2}\int_0^T\left\{\varepsilon\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_t(x,t)\|^2_{L^2(\Omega)}\right\}dt \\
&\qquad+\frac{c^2}{2}|\Omega||\partial\Omega|\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
&b\int_{\partial\Omega}U_\tau(x,\tau)\|\|\int_\Omega U(\xi,\tau)d\xi ds_x \\
&\quad\le \frac{b}{2\varepsilon_2'}\left(\varepsilon\|\nabla U_\tau x(\xi,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\right) \\
&\qquad+\frac{b}{2}\varepsilon_2' T|\Omega||\partial\Omega|\|U(x,\tau)\|^2_{L^2(\Omega)},
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
&-b\int_0^\tau\int_{\partial\Omega}U_t(x,t)\int_\Omega U(\xi,t)d\xi ds_x dt \\
&\quad\le \frac{b}{2}\{l(\varepsilon)+T|\Omega||\partial\Omega|\}\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&\qquad+\frac{b}{2}\varepsilon\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
&\int_{\partial\Omega}U_\tau(x,\tau)\left(\int_0^\tau\int_\Omega W(\xi,\tau)d\xi dt\right)ds_x \\
&\quad\le\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)\left(\varepsilon\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\right) \\
&\qquad+\frac{1}{2}\varepsilon_3 Th_0|\partial\Omega||\Omega|\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&\qquad+\frac{1}{2}\varepsilon_4 T|\partial\Omega||\Omega|\int_0^\tau h\circ U(t)dt,
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
&-\int_{\partial\Omega}\int_0^\tau U_t(x,t)\int_\Omega W(\xi,t)d\xi dt ds_x \\
&\quad\le\int_0^\tau\left\{\varepsilon\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_t(x,t)\|^2_{L^2(\Omega)}\right\}dt \\
&\qquad+\frac{1}{2}h_0|\Omega||\partial\Omega|\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&\qquad+\frac{1}{2}|\Omega||\partial\Omega|\int_0^\tau h\circ U(t)dt,
\end{aligned}
\tag{102}
$$

So, combining inequalities (97)–(102), we obtain

$$
\begin{aligned}
&\frac{a}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}h'\circ\nabla U(\tau)-\frac{1}{2}h(\tau)\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} \\
&\quad+c^2(\nabla U_\tau(x,\tau),\nabla U(x,\tau))_{L^2(\Omega)} \\
&\quad+2(\nabla W(x,\tau),\nabla U_\tau(x,\tau))_{L^2(\Omega)} \\
&\quad+\left\{\frac{b}{2}-\frac{c^2}{2\varepsilon_1'}\varepsilon-\frac{b}{2\varepsilon_2'}\varepsilon-\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)\varepsilon\right\}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&\quad-\left\{\frac{c^2 l(\varepsilon)}{2\varepsilon_1'}+\frac{bl(\varepsilon)}{2\varepsilon_2'}+\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)l(\varepsilon)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\right. \\
&\quad-\frac{b}{2}\varepsilon_2' T|\Omega||\partial\Omega|\|U(x,\tau)\|^2_{L^2(\Omega)} \\
&\le\beta\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt+\frac{1}{2}T|\Omega|\left(\varepsilon_3+1\right)\int_0^\tau h\circ U(t)dt \\
&\quad+\frac{1}{2}\int_0^\tau h''\circ U(t)dt-\frac{1}{2}\int_0^\tau h'(t)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}dt \\
&\quad+\left\{c^2+\frac{c^2}{2}\varepsilon+\frac{b}{2}\varepsilon\right\}\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad+\left\{\frac{c^2}{2}\varepsilon_1' T|\Omega||\partial\Omega|+\frac{1}{2}\left(c^2+h_0+Th_0\varepsilon_4\right)|\Omega||\partial\Omega|\right\} \\
&\quad\cdot\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad+\left(\frac{b}{2}\{l(\varepsilon)+T|\Omega||\partial\Omega|\}+\frac{1}{2}\left(c^2+1\right)l(\varepsilon)\right) \\
&\quad\cdot\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt.
\end{aligned}
\tag{103}
$$

Adding side to side (89) and (103) that gives

$$
\begin{aligned}
&a(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)} \\
&\quad+\left\{\frac{\beta}{2}-\frac{b}{2}\varepsilon_2' T|\Omega||\partial\Omega|\right\}\|U(x,\tau)\|^2_{L^2(\Omega)} \\
&\quad\cdot\frac{c^2}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}+\frac{a}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&\quad+c^2(\nabla U_\tau(x,\tau),\nabla U(x,\tau))_{L^2(\Omega)} \\
&\quad+\left\{\frac{b}{2}-\frac{c^2}{2\varepsilon_1'}\varepsilon-\frac{b}{2\varepsilon_2'}\varepsilon-\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)\varepsilon\right\} \\
&\quad\cdot\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}-\left\{\frac{c^2}{2\varepsilon_1'}+\frac{b}{2\varepsilon_2'}+\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)l(\varepsilon)\right\} \\
&\quad\cdot l(\varepsilon)\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}h'\circ\nabla U(\tau) \\
&\quad+2(\nabla W(x,\tau),\nabla U_\tau(x,\tau))_{L^2(\Omega)} \\
&\le\gamma_4\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt+\left\{a+\frac{b}{2}(l(\varepsilon)+T|\Omega||\partial\Omega|)\right. \\
&\quad\left.+\left(\frac{c^2}{2}+1\right)l(\varepsilon)\right\}\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad+\left(\left(\frac{c^2+b}{2}+1\right)\varepsilon+h_0\right)\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt \\
&\quad+\left\{c^2+\frac{c^2}{2}\varepsilon+\frac{b}{2}\varepsilon+\frac{h_0}{4}\right\}\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}dt^2 \\
&\quad\cdot\left(\frac{1}{2}|\Omega||\partial\Omega|\left(\varepsilon_3 T+1+T^2\right)+\frac{1}{2}T^2 l(\varepsilon)\right)\int_0^\tau h\circ U(t)dt \\
&\quad-\frac{1}{2}\int_0^\tau h''\circ\nabla U(t)dt+\frac{h_0}{2}\int_0^\tau h\circ\nabla U(t)dt \\
&\quad+\left(\frac{h(0)}{4}+\frac{h_0}{2}\right)\int_0^\tau\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}
\tag{104}
$$

where

$$\gamma_4 := \left\{ \frac{c^2}{2} T^2 (l(\varepsilon) + |\Omega||\partial\Omega|) + \frac{b}{2} \left(T^2 l(\varepsilon) + |\Omega||\partial\Omega| \right) + \frac{1}{2} T^2 (l(\varepsilon) + |\Omega||\partial\Omega| h_0) + \frac{c^2}{2\varepsilon_1} \varepsilon_1' T |\Omega||\partial\Omega| + \frac{c^2}{2} |\Omega||\partial\Omega| + \frac{1}{2} h_0 |\Omega||\partial\Omega| \left(\varepsilon_4 T^2 + 1 \right) \right\}. \quad (105)$$

Now, the last term on the RHS of (104), we give the function θ (x,t) by

$$\theta(x,t) := \int_0^t U(x,s) ds. \quad (106)$$

Hence, we use (77), and we get

$$\begin{aligned} &\mathrm{v}(\mathrm{x},\mathrm{t}) = \theta(\mathrm{x},\tau) - \theta(\mathrm{x},\mathrm{t}), \nabla \mathrm{v}(\mathrm{x},0) = \nabla\theta(\mathrm{x},\tau), \\ &\|\nabla v\|^2_{L^2(Q_\tau)} = \|\nabla\theta(x,\tau) - \nabla\theta(x,t)\|^2_{L^2(\Omega)} \\ &\quad \le 2\left(\tau \|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla\theta(x,t)\|^2_{L^2(Q_\tau)} \right). \end{aligned} \quad (107)$$

And using the inequalities

$$\begin{aligned} &m_1' \|U(x,\tau)\|^2_{L^2(\Omega)} \le m_1' \|U(x,t)\|^2_{L^2(Q_\tau)} + m_1 \|U_t(x,t)\|^2_{L^2(Q_\tau)}, \\ &m_2' \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \le m_2' \|U_t(x,t)\|^2_{L^2(Q_\tau)} + m_2 \|U_{tt}(x,t)\|^2_{L^2(Q_\tau)}, \\ &m_3' \|\nabla U(x,\tau)\|^2_{L^2(\Omega)} \le m_3' \|\nabla U(x,t)\|^2_{L^2(Q_\tau)} + m_3 \|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)}, \\ &m_4' h \circ U(\tau) \le m_4' \left(1 + \frac{\zeta}{2} \right) \int_0^\tau h' \circ U(t) dt + m_4 \|\nabla U_t(x,\tau)\|^2_{L^2(Q_\tau)}, \\ &-\frac{a}{2} \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \frac{a}{2} \|U(x,\tau)\|^2_{L^2(\Omega)} \le a(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)}, \\ &-\frac{c^2}{2\varepsilon_3'} \|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \frac{c^2}{2} \varepsilon_3' \|\nabla U(x,\tau)\|^2_{L^2(\Omega)} \le c^2 (\nabla U_\tau(x,\tau), \nabla U(x,\tau))_{L^2(\Omega)}, \\ &-\left(\frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_8} \right) \|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} - h_0 \varepsilon_8 \|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \varepsilon_9 h_0 \nabla U(\tau) \le 2(\nabla W(x,\tau), \nabla U_\tau(x,\tau))_{L^2(\Omega)}. \end{aligned} \quad (108)$$

Let

$$\begin{cases} m_1' := \dfrac{a}{2} + \dfrac{b}{2} \varepsilon_2' T |\Omega||\partial\Omega|, \\ m_2' := 1 + \dfrac{a}{2} - \dfrac{c^2}{2\varepsilon_3'} - \left\{ \dfrac{c^2}{2\varepsilon_1'} + \dfrac{b}{2\varepsilon_2'} + \dfrac{1}{2\varepsilon_3} + \dfrac{1}{2\varepsilon_4} \right\} l(\varepsilon), \end{cases}$$
$$m_3' := \frac{c^2}{2} \varepsilon_3' + h_0 \varepsilon_8, \quad (109)$$

and we choose $\varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_8$ and ε_9 sufficiently large

$$\frac{c^2}{2\varepsilon_1'} \varepsilon + \frac{b}{2\varepsilon_2'} \varepsilon + \frac{c^2}{2\varepsilon_3'} + \left(\frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_9} \right) \varepsilon < \frac{b}{2}. \quad (110)$$

As τ is arbitrary, and we get

$$A := \frac{c^2}{2} - 2\tau\varepsilon \left(\left(c^2 + b \right)/2 + 1 + h_0 \right) > 0. \quad (111)$$

Thus, inequality (104) takes the form

$$\begin{aligned} &\frac{\beta}{2} \|U(x,\tau)\|^2_{L^2(\Omega)} + \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla U(x,\tau)\|^2_{L^2(\Omega)} \\ &\quad + \left\{ \underbrace{\frac{b}{2} - \frac{c^2}{2\varepsilon_1'} \varepsilon - \frac{b}{2\varepsilon_2'} \varepsilon - \frac{c^2}{2\varepsilon_3'} + \frac{1}{4\varepsilon_9} + \frac{h_0}{4\varepsilon_8}}_{\gamma_6} \right\} \cdot \|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2} h' \circ \nabla U(\tau) - h' \circ U(\tau) \\ &\quad + \frac{a}{2} \|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + A \|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} \\ &\le \left(\gamma_4 + m_1' \right) \int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)} dt + \gamma_5 \int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)} dt \\ &\quad + \left(m_3' + \frac{h_0}{4} \right) \int_0^\tau \|\nabla U(x,t)\|^2_{L^2(\Omega)} dt \\ &\quad + \left(\beta + m_2' \right) \int_0^\tau \|U_{tt}(x,t)\|^2_{L^2(\Omega)} dt \\ &\quad + \left\{ \underbrace{c^2 + \frac{c^2}{2} \varepsilon + \frac{b}{2} \varepsilon + \frac{h_0}{4} + \frac{h(0)}{2} + m_3'}_{\gamma_7} \right\} \cdot \int_0^\tau \|\nabla U_t(x,t)\|^2_{L^2(\Omega)} dt + \underbrace{2\varepsilon \left(\frac{(c^2 + b)}{2} + 1 + h_0 \right)}_{\gamma_8} \\ &\quad \cdot \int_0^\tau \|\nabla\theta(x,t)\|^2_{L^2(\Omega)} dt - \frac{1}{2} \int_0^\tau h'' \circ \nabla U(t) dt \\ &\quad - \left(\frac{\varepsilon_9}{\zeta} + 1 \right) \int_0^\tau h' \circ \nabla U(t) dt \end{aligned}$$

$$-\underbrace{\frac{1}{\zeta}\left(\frac{1}{2}|\Omega||\partial\Omega|\left(\varepsilon_3 T+1+T^2\right)+\frac{1}{2}T^2 l(\varepsilon)+1\right)}_{\gamma_9} \cdot\int_0^\tau h'\circ U(t)dt, \quad (112)$$

where

$$\gamma_5 := a+\frac{b}{2}(l(\varepsilon)+T|\Omega||\partial\Omega|)+\left(\frac{c^2}{2}+1\right)l(\varepsilon)+\frac{h(0)}{2}+m_1'+m_2'. \quad (113)$$

We get

$$\begin{aligned}&\|U(x,\tau)\|^2_{L^2(\Omega)}+\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}+\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}\\&\quad-h'\circ U(\tau)+\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}+\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\\&\quad+\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)}-h'\circ\nabla U(\tau)\\&\le D'\int_0^\tau\left\{\|U(x,t)\|^2_{L^2(\Omega)}+\|U_t(x,t)\|^2_{L^2(\Omega)}\right.\\&\quad+\|\nabla U(x,t)\|^2_{L^2(\Omega)}-h'\circ U(t)+\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}\\&\quad\left.+\|U_{tt}(x,t)\|^2_{L^2(\Omega)}+\|\nabla\theta(x,t)\|^2_{L^2(\Omega)}-h'\circ\nabla U(t)\right\}dt,\end{aligned} \quad (114)$$

where

$$D' := \frac{\max\left\{(\beta/2),1,\left(\eta^2/2\right),\gamma_6,(a/2),A\right\}}{\min\left\{\left(\gamma_4+m_1'\right),\gamma_5,\left(m_3'+(h_0/4)\right),\gamma_7,\left(\beta+m_2'\right),((\varepsilon_9/\zeta)+1),\gamma_8,\gamma_9\right\}}. \quad (115)$$

Hence, applying Gronwall's lemma to (114) gives

$$\begin{aligned}&\|U(x,\tau)\|^2_{L^2(\Omega)}+\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}+\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}\\&\quad-h'\circ U(\tau)+\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}+\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}\\&\quad+\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)}-h'\circ\nabla U(\tau)\le 0,\end{aligned} \quad (116)$$

for all $\tau\in[0,(c^2/4\varepsilon(((c^2+b)/2)+1+h_0))]$.

For the intervals, we use the same method

$$\tau\in\left[\frac{(m-1)c^2}{4\varepsilon(((c^2+b)/2)+1+h_0)},\frac{mc^2}{4\varepsilon(((c^2+b)/2)+1+h_0)}\right] \quad (117)$$

to cover the whole interval $[0,T]$ and thus proving that $U(x,\tau)=0$, $\forall\tau$ in $[0,T]$. Hence, the uniqueness is proved.

5. Conclusion

The objective of this work is the study of solvability of the Moore-Gibson-Thompson equation with viscoelastic memory term and integral condition by using the Galerkin method. The MGT equation is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. Recent developments in numerical schemes for solving MGT have placed immense interest in nonlinear dispersive wave models. In the next work, we will try to use the same method with Boussinesque and Hall-MHD equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see [6, 15–24], for example, [10, 11, 25, 26]) by using some famous algorithms (see [27–29]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

New Estimates of Solution to Coupled System of Damped Wave Equations with Logarithmic External Forces

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In the paper, we consider new stability results of solution to class of coupled damped wave equations with logarithmic sources in $\mathbb{R}^n$. We prove a new scenario of stability estimates by introducing a suitable Lyapunov functional combined with some estimates.

1. Introduction

In the present paper, we consider an initial boundary value problem with damping terms and logarithmic sources, for $x \in \mathbb{R}^n$, $t > 0$

$$\begin{cases} \partial_{tt} v_1 + b v_2 = \phi(x)\Delta\left(v_1 - \int_0^t \varpi_1(t-p)v_1(p)dp\right) + k v_1 \ln|v_1|, \\ \partial_{tt} v_2 + b v_1 = \phi(x)\Delta\left(v_2 - \int_0^t \varpi_2(t-p)v_2(p)dp\right) + k v_2 \ln|v_2|, \\ v_1(x,0) = v_{10}(x), v_2(x,0) = v_{20}(x), \\ \partial_t v_1(x,0) = v_{11}(x), \partial_t v_2(x,0) = v_{21}(x), \end{cases} \tag{1}$$

where $b > 0$, $n \geq 3$, and k is a small positive real number. The density function $\rho(x) > 0$, for all $x \in \mathbb{R}^n$, where $(\phi(x))^{-1} = 1/\phi(x) \equiv \rho(x)$, under homogeneous Drichlet boundary conditions.

A related initial boundary value problem was considered by Han in [1]:

$$\begin{cases} u_{tt} + u_t - \Delta u + u + |u|^2 u = u \ln|u|^2, & x \in \Omega, t \in [0,T), \\ u(x,0) = u_0(x) u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in [0,T), \end{cases} \tag{2}$$

and the global existence of weak solutions was proved, for all $(u_0, u_1) \in H_0^1 \times L^2$ in $\mathbb{R}^3$. The weak and strong damping terms in logarithmic wave equation

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln|u|, & x \in \Omega, t \in (0,\infty), \\ u(x,0) = u_0(x) u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,\infty) \end{cases} \tag{3}$$

were introduced by Lian and Xu [2]. The global existence, asymptotic behavior, and blowup at three different initial energy levels (subcritical energy $E(0) < d$, critical initial energy $E(0) = d$, and the arbitrary high initial energy $E(0) > 0 (\omega = 0)$) were proved. In [3], Al-Gharabli established explicit and general energy decay results for the problem

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u ds = k u \ln|u|, & x \in \Omega, t \in (0,\infty), \\ u(x,0) = u_0(x) u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \dfrac{\partial u}{\partial \upsilon} = 0, & x \in \partial\Omega, t \in (0,\infty). \end{cases} \tag{4}$$

When the density $\phi(x) \neq 1$, Papadopoulos and Stavrakakis [4] considered the following semilinear hyperbolic initial value problem:

$$u_{tt} + \phi(x)\Delta u + \delta u_t + \lambda f(u) = \eta(x), (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. \quad (5)$$

The authors proved local existence of solutions and established the existence of a global attractor in the energy space $\mathscr{D}^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$, where $(\phi(x))^{-1} := g(x)$. Miyasita and Zennir [5] proved the global existence of the following viscoelastic wave equation:

$$\begin{cases} u_{tt} + au_t - \phi(x)\left(\Delta u + \omega \Delta u_t - \int_0^t g(t-s)\Delta u(s)ds\right) = u|u|^{p-1}, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (6)$$

The novelty of our work lies primarily in the use of a new condition between the weights of damping the external forces, where we outline the effects of the damping term with less conditions on the viscoelastic terms. We also propose logarithmic nonlinearities in sources and used classical arguments to estimate them. These nonlinearities make the problem very interesting in the application point of view. In order to compensate for the lack of classical Poincaré's inequality in $\mathbb{R}^n$, we use the weighted function to use generalized Poincaré's one. The main contribution of this paper is introduced in Theorem 8, where we obtain decay estimates with positive initial energy under a general assumption on the kernel. The rest of the paper is outline as follows. In Section 2, we give some preliminaries and our main results. In Section 3, we will prove the general decay of energy to the problem.

2. Preliminaries and Main Results

We state some assumptions and definitions that will be useful in this paper. With respect to the relaxation functions ϖ_1, ϖ_2, we assume for $i = 1, 2$.

(H1) $\varpi_1, \varpi_2 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy for any $t \geq 0$,

$$\varpi_i(0) > 0, \int_0^\infty \varpi_i(p)dp = l_{i0} < \infty, 1 - \int_0^t \varpi_i(p)dp = l_i > 0 \quad (7)$$

(H2) There exist nonincreasing differentiable functions $\zeta_1, \zeta_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy

$$\zeta_i(t) > 0, \varpi_{i'}(t) \leq -\zeta_i(t)\varpi_i(t) \text{ for } t \geq 0 \quad (8)$$

(H3) The function $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = 2n/2n - qn + 2q$

Definition 1 (see [4]). We define the function spaces of our problem and their norms as follows:

$$\mathscr{H} = \left\{ v \in L^{2n/(n-2)}(\mathbb{R}^n) | \nabla v \in \left(L^2(\mathbb{R}^n)\right)^n \right\}. \quad (9)$$

Let the function spaces $\mathscr{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|v\|_{\mathscr{H}} = (v, v)^{1/2}_{\mathscr{H}}$ for the inner product:

$$(v, w)_{\mathscr{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx, \quad (10)$$

and $L^2_\rho(\mathbb{R}^n)$ be defined with the norm $\|v\|_{L^2_\rho} = (v, v)^{1/2}_{L^2_\rho}$ for

$$(v, w)_{L^2_\rho} = \int_{\mathbb{R}^n} \rho v w \, dx. \quad (11)$$

For general $q \in [1, +\infty)$, $L^q_\rho(\mathbb{R}^n)$ is the weighted L^q space under a weighted norm

$$\|v\|_{L^q_\rho} = \left(\int_{\mathbb{R}^n} \rho |v|^q \, dx \right)^{1/q}. \quad (12)$$

To distinguish the usual L^q space from the weighted one, we denote the standard L^q norm by

$$\|v\|_q = \left(\int_{\mathbb{R}^n} |v|^q \, dx \right)^{1/q}. \quad (13)$$

We denote an eigenpair $\{(\lambda_j, w_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times \mathscr{H}$ by

$$-\phi(x)\Delta w_j = \lambda_j w_j \, \mathrm{x} \in \mathbb{R}^\mathrm{n} \quad (14)$$

for any $j \in \mathbb{N}$. Then, according to [4],

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \uparrow +\infty \quad (15)$$

holds and $\{w_j\}$ is a complete orthonormal system in $\mathscr{H}$.

Now, we introduce Sobolev embedding and generalized Poincaré's inequalities.

Lemma 2. *Let ρ satisfy (H3). Then, there are positive constants $C_S > 0$ and $C_P > 0$ that depend only on n and ρ such* *that*

$$\|v\|_{2n/(n-2)} \le C_S \|v\|_{\mathscr{H}}, \quad \|v\|_{L^2_\rho} \le C_P \|v\|_{\mathscr{H}} \tag{16}$$

for $v \in \mathscr{H}$.

Lemma 3 (see Lemma 2.2 in [6]). *Let* ρ *satisfy (H3). Then, we have*

$$\|v\|_{L^q_\rho} \le C_q \|v\|_{\mathscr{H}}, \quad C_q = C_S \|\rho\|_s^{1/q} \tag{17}$$

for $v \in \mathscr{H}$, *where* $s = 2n/(2n - qn + 2q)$ *for* $1 \le q \le 2n/(n-2)$.

The energy functional associated to problem (1) is given by

$$\begin{aligned}\mathscr{E}(t) &= \frac{1}{2}\sum_{i=1}^{2} \|\partial_t v_i(t)\|^2_{L^2_\rho} + \frac{1}{2}\sum_{i=1}^{2} \left(1 - \int_0^t \varpi_i(p)dp\right)\|\nabla v_i(t)\|^2 \\ &\quad + b\|v_1(t)v_2(t)\|^2 + \frac{1}{2}\sum_{i=1}^{2} (\varpi_i \circ \nabla v_i)(t) \\ &\quad - \frac{k}{2}\sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln|v_i| dx + \frac{k}{4}\sum_{i=1}^{2} \|v_i\|^2_{L^2_\rho} \\ &\ge \frac{1}{2}\sum_{i=1}^{2} \|\partial_t v_i(t)\|^2_{L^2_\rho} + \frac{1}{2}\sum_{i=1}^{2} \left(1 - \int_0^t \varpi_i(p)dp\right)\|\nabla v_i(t)\|^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{2} (\varpi_i \circ \nabla v_i)(t) - \frac{k}{2}\sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln|v_i| dx \\ &\quad + \frac{k}{4}\sum_{i=1}^{2} \|v_i\|^2_{L^2_\rho},\end{aligned} \tag{18}$$

where

$$(\varpi \circ v) = \int_0^t \varpi(t-p)\|v(t) - v(p)\|_2^2 dp. \tag{19}$$

With direct differentiation of (18), using (1), we obtain

$$\partial_t \mathscr{E}(t) = -\frac{1}{2}\sum_{i=1}^{2} \left(\varpi_i(t)\|v_i\|^2_{\mathscr{H}} - \left(\varpi_i' \circ \nabla v_i\right)\right) \le 0, \tag{20}$$

which let our system dissipative.

Lemma 4 (see [7]) (logarithmic Sobolev inequality). *Lets u be any function in* $H_0^1(\Omega)$ *and* $a > 0$ *be any number. Then,*

$$\int_\Omega v^2 \ln|v| dx \le \frac{1}{2}\|v\|_2^2 \ln\|v\|_2^2 + \frac{a^2}{2\pi}\|\nabla v\|_2^2 - (1 + \ln a)\|v\|_2^2. \tag{21}$$

Lemma 5 (see [8]) (logarithmic Gronwall inequality). *Let* $c > 0$, $\gamma \in L^1(0, T; \mathbb{R}^+)$, *and assume that the function* $\omega : [0, T] \longrightarrow [1, \infty)$ *satisfies*

$$\omega(t) \le c\left(1 + \int_0^t \gamma(p)\omega(p)\ln\omega(p)dp\right), \quad 0 \le t \le T, \tag{22}$$

then

$$\omega(t) \le c \exp\left(c\int_0^t \gamma(p)dp\right), \quad 0 \le t \le T. \tag{23}$$

We define the following functionals

$$\begin{aligned}J(t) &= \frac{1}{2}\sum_{i=1}^{2} \left(1 - \int_0^t \varpi_i(p)dp\right)\|\nabla v_i(t)\|^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{2} (\varpi_i \circ \nabla v_i)(t) - \frac{k}{2}\sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln|v_i| dx \\ &\quad + \frac{k}{4}\sum_{i=1}^{2} \|v_i\|^2_{L^2_\rho},\end{aligned}$$

$$\begin{aligned}I(t) &= \frac{1}{2}\sum_{i=1}^{2} \left(1 - \int_0^t \varpi_i(p)dp\right)\|\nabla v_i(t)\|^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{2} (\varpi_i \circ \nabla v_i)(t) - \frac{k}{2}\sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln|v_i| dx.\end{aligned} \tag{24}$$

Then, we introduce

$$W = \{(v_1, v_2): v_1, v_2 \in \mathscr{H} : I(t) > 0, J(t) < d\} \cup \{0\}. \quad \sum_{i=1}^{2} \|v_i\|^2 < 4d \text{ for all } t \in [0, T). \tag{25}$$

Lemma 6. *Let* $(v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathscr{H} \times L^2_\rho(\mathbb{R}^n)$ *such that* $0 < \mathscr{E}(0) < d$ *and* $I(t_0) > 0$. *Then, we have*

$$(v_1, v_2) \in W, \tag{26}$$

Theorem 7 (see [5]). *Let* $(v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathscr{H} \times L^2_\rho(\mathbb{R}^n)$. *Under the assumptions (H1)–(H3). Then, problem (1) has a global weak solution u in the space*

$$(v_1, v_2) \in \left(C([0, +\infty); \mathscr{H}) \cap C^1\left([0, +\infty); L^2_\rho(\mathbb{R}^n)\right)\right)^2. \tag{27}$$

Then, the main result in this paper is the general decay of energy to problem (1) which is given in the following theorem.

Theorem 8. *Assume the assumptions (H1)–(H3) hold and* $0 < \mathscr{E}(0) < d$. *Let* (v_1, v_2) *be the weak solution of problem (1) with the initial data* $(v_{10}, v_{11}), (v_{20}, v_{21}) \in \mathscr{H}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$. *Then, there exist constant* $\beta > 0$ *such that the energy* $\mathscr{E}(t)$ *defined by (18) satisfies for all* $t > 0$,

$$\mathscr{E}(t) \le \beta \left(1 + \int_{t_0}^{t} \xi^{\varepsilon_0 + 1}(p) dp \right)^{-1/\varepsilon_0}, \quad \varepsilon_0 \in (0, 1). \tag{28}$$

3. Asymptotic Behavior for $\mathscr{E}(0) < d$

The following technical lemmas are useful to prove the general decay of energy to problem (1).

Lemma 9. *Under the assumptions in Theorem 8, then the functional* $\Phi(t)$ *defined by*

$$\Phi(t) = \int_{\mathbb{R}^n} \rho(x)(v_1(t)\partial_t v_1 + v_2(t)\partial_t v_2(t)) dx dx \tag{29}$$

satisfies for any $t \ge 0$,

$$\begin{aligned}\Phi'(t) \le & \sum_{i=1}^{2} \|\partial_t v_i(t)\|^2_{L^2_\rho} - \frac{1}{2}\sum_{i=1}^{2} l_i \|\nabla v_i(t)\|^2 \\ & + \sum_{i=1}^{2} \frac{1 - l_i}{4\varepsilon} (\varpi_i \circ \nabla v_i)(t) \\ & + k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx.\end{aligned} \tag{30}$$

Proof. We differentiate $\Phi(t)$, using (1), we can get

$$\begin{aligned}\Phi'(t) = & \sum_{i=1}^{2} \|\partial_t v_i\|^2_{L^2_\rho} - \sum_{i=1}^{2} \|\nabla v_i\|^2 \\ & + \sum_{i=1}^{2} \int_{\mathbb{R}^n} \nabla v_i(t) \cdot \int_0^t \varpi_i(t-p) \nabla v_i(p) dp dx \\ & - 2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 dx + k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx.\end{aligned} \tag{31}$$

It follows from Young and Poincaré's inequality that for any $\varepsilon > 0$,

$$\begin{aligned}& \int_{\mathbb{R}^n} \nabla v_i(t) \cdot \int_0^t \varpi_i(t-p) \nabla v_i(p) dp dx \\ & = \int_{\mathbb{R}^n} \nabla v_i(t) \cdot \int_0^t \varpi_i(t-p)(\nabla v_i(p) - \nabla v_i(t)) dp dx \\ & \quad + \int_0^t \varpi_i(p) dp \|\nabla v_i(t)\|_2^2 \\ & \le (1 - l_i)\|\nabla v_i\|^2 + \varepsilon \|\nabla v_i\|_2^2 + \frac{1}{4\varepsilon} \int_{\mathbb{R}^n} \\ & \quad \cdot \left(\int_0^t \varpi_i(t-p)(\nabla v_i(p) - \nabla v(t)) dp \right)^2 dx \\ & \le (1 - l_i + \varepsilon)\|\nabla v_i\|_2^2 + \frac{1 - l_i}{4\varepsilon} (\varpi_i \circ \nabla v_i)(t).\end{aligned} \tag{32}$$

Exploit Young and Poincaré's inequalities to estimate

$$2b \int_{\mathbb{R}^n} \rho(x) v_1 v_2 dx \le \varepsilon c_* \|\nabla v_i\|^2_{L^2_\rho} + \frac{c_*}{4\varepsilon} \|\nabla v_2\|^2_{L^2_\rho}. \tag{33}$$

Inserting (32)–(33) into (31) yields for any $\varepsilon > 0$,

$$\begin{aligned}\Phi'(t) \le & \sum_{i=1}^{2} \|\partial_t v_i(t)\|^2_{L^2_\rho} - \sum_{i=1}^{2} (l - \varepsilon - \varepsilon c_*) \|\nabla v_i(t)\|^2 \\ & + \sum_{i=1}^{2} \frac{1 - l_i}{4\varepsilon} (\varpi_i \circ \nabla v_i)(t) \\ & + k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx.\end{aligned} \tag{34}$$

Taking $\varepsilon > 0$ small enough in (34) such that

$$l_i - \varepsilon - \varepsilon c_* > \frac{l}{2}. \tag{35}$$

The proof is hence complete.

Lemma 10. *Under the assumptions in Theorem 8, then the functional* $\psi(t)$ *defined by*

$$\begin{aligned}\psi(t) = & -\int_{\mathbb{R}^n} \rho(x) \partial_t v_1(t) \int_0^t \varpi_1(t-p)(v_1(t) - v_1(p)) dp dx \\ & - \int_{\mathbb{R}^n} \rho(x) \partial_t v_2(t) \int_0^t \varpi_2(t-p)(v_2(t) - v_2(p)) dp dx\end{aligned} \tag{36}$$

satisfies for any $\delta > 0$,

$$\begin{aligned}\psi'(t) \le & \sum_{i=1}^{2} \delta\left[(1-l_i)^2 + 1 + bc_*\right]\|\nabla v_i(t)\|^2 \\ & - \sum_{i=1}^{2}\left[\left(\int_0^t \varpi_i(s)dp\right) - 2\delta\right]\|\partial_t v_i(t)\|_{L^2_\rho}^2 \\ & + C\sum_{i=1}^{2}\left(\int_0^t \varpi_i(s)dp\right)(\varpi_i\circ\nabla v_i)(t) \\ & - \frac{\varpi_i(0)c_*}{4\delta}\sum_{i=1}^{2}\left(\varpi_i'\circ\Delta v_i\right)(t) \\ & + c_{\varepsilon_0}\sum_{i=1}^{2}(\varpi_i\circ\nabla v_i)^{1/(1+\varepsilon_0)}.\end{aligned} \tag{37}$$

Proof. Taking the derivative of $\psi(t)$ and using (1), we conclude that

$$\begin{aligned}\psi'(t) = & \sum_{i=1}^{2}\int_{\mathbb{R}^n}\nabla v_i(t)\int_0^t \varpi_i(t-p)(\nabla v_i(t)-\nabla v_i(p))dpdx \\ & - \int_{\mathbb{R}^n}\left(\int_0^t \varpi_1(t-p)\nabla v_1(p)dp\right) \\ & \cdot\left(\int_0^t \varpi_1(t-p)(\nabla v_1(t)-\nabla v_1(p))dp\right)dx \\ & - \int_{\mathbb{R}^n}\left(\int_0^t \varpi_2(t-p)\nabla v_2(p)dp\right) \\ & \cdot\left(\int_0^t \varpi_2(t-p)(\nabla v_2(t)-\nabla v_2(p))dp\right)dx \\ & + b\int_{\mathbb{R}^n}\rho(x)v_2\int_0^t \varpi_1(t-p)(v_1(t)-v_1(p))dpdx \\ & + b\int_{\mathbb{R}^n}\rho(x)v_1\int_0^t \varpi_2(t-p)(v_2(t)-v_2(p))dpdx \\ & - k\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)v_i\ln|v_i|\int_0^t \varpi_i(t-p)(v_i(t)-v_i(p))dpdx \\ & - \sum_{i=1}^{2}\int_0^t \varpi_i(p)dp\|\partial_t v_i\|_{L^2_\rho}^2 \\ & - \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\partial_t v_i\int_0^t \varpi_i'(t-p)(v_i(t)-v_i(p))dpdx.\end{aligned} \tag{38}$$

We then use Young and Poincaré's inequalities; we can get for any $\delta > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla v_i(t)\int_0^t \varpi_i(t-p)(\nabla v_i(t)-\nabla v_i(p))dpdx \\ &\quad\le \delta\|\nabla v_i\|^2 + \frac{1}{4\delta}\left(\int_0^t \varpi_i(p)dp\right)(\varpi_i\circ\nabla v_i)(t).\end{aligned} \tag{39}$$

The second and third terms can be treated as

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\int_0^t \varpi_i(t-p)\nabla v_i(p)dp\right)\left(\int_0^t \varpi_i(t-p)(\nabla v_i(t)-\nabla v_i(p))dp\right)dx \\ &\quad\le \delta(1-l_i)^2\|\nabla v_i\|^2 + \left(1+\frac{1}{4\delta}\right)\left(\int_0^t \varpi_i(p)dp\right)(\varpi_i\circ\nabla v_i)(t).\end{aligned} \tag{40}$$

The fourth and fifth terms will be estimated by

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(x)v_2\int_0^t \varpi_1(t-p)(v_1(t)-v_1(p))dpdx \\ &\quad\le \delta c_*\|\nabla v_2\|^2 + \frac{c_*}{4\delta}\left(\int_0^t \varpi_1(p)dp\right)(\varpi_1\circ\nabla v_1)(t), \\ &\int_{\mathbb{R}^n}\rho(x)v_1\int_0^t \varpi_2(t-p)(v_2(t)-v_2(p))dpdx \\ &\quad\le \delta c_*\|\nabla v_1\|^2 + \frac{c_*}{4\delta}\left(\int_0^t \varpi_2(p)dp\right)(\varpi_2\circ\nabla v_2)(t),\end{aligned} \tag{41}$$

respectively.

For the last term, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(x)\partial_t v_i\int_0^t \varpi_i'(t-p)(v_i(t)-v_i(p))dpdx \\ &\quad\le \delta\|\partial_t v_i\|_{L^2_\rho}^2 + \frac{c_*}{4\delta}\left(\int_0^t -\varpi_i'(p)dp\right)\left(\varpi_i'\circ\nabla v_i\right)(t) \\ &\quad\le \delta\|\partial_t v_i\|_{L^2_\rho}^2 - \frac{\varpi_i(0)c_*}{4\delta}\left(\varpi_i'\circ\nabla v_i\right)(t).\end{aligned} \tag{42}$$

Let $\varepsilon_0 \in (0,1)$ and $g(s) = s^{\varepsilon_0}(|\ln s| - s)$. Notice that g is continous on $(0,\infty)$, its limit at 0 is 0, and its limit at ∞ is $-\infty$. Then, g has a maximum m_{ε_0} on $[0,\infty)$, so the following inequality holds

$$s|\ln s| \le s^2 + m_{\varepsilon_0}s^{1-\varepsilon_0}, \quad \text{for all } s > 0. \tag{43}$$

Using the Cauchy-Schwartz's inequality and applying (43), yields, for any $\delta > 0$,

$$\begin{aligned}&k\int_{\mathbb{R}^n}\rho(x)v_i\ln|v_i|\int_0^t \varpi_i(t-p)(v_i(t)-v_i(p))dpdx \\ &\quad\le k\int_{\mathbb{R}^n}\rho(x)\left(v_i^2 + m_{\varepsilon_0}|v_i|^{1-\varepsilon_0}\right) \\ &\qquad\cdot\left|\int_0^t \varpi_i(t-p)(v_i(t)-v_i(p))dpdx\right| \\ &\quad\le c\int_{\mathbb{R}^n}\rho(x)v_i^2\left|\int_0^t \varpi_i(t-p)(v_i(t)-v_i(p))dpdx\right| \\ &\qquad+ \delta\|v_i\|_{L^2_\rho}^2 + \int_{\mathbb{R}^n}\left|\int_0^t \varpi_i(t-p)(v_i(t)-v_i(p))dpdx\right|^{2/(1+\varepsilon_0)} \\ &\quad\le \delta c_*\|\nabla v_i\|_2^2 + \frac{1}{4\delta}(\varpi_i\circ\nabla v_i)(t) + c_{\varepsilon_0}(\varpi_i\circ\nabla v_i)^{1/(1+\varepsilon_0)}.\end{aligned} \tag{44}$$

Combining (39)–(44) with (39) gives us (37) with

$$C = \frac{bc_* + 2}{4\delta} + 2\delta. \quad (45)$$

Therefore, the proof is complete.

Now, we define a Lyapunov functional $\mathscr{L}(t)$ by

$$\mathscr{L}(t) = M\mathscr{E}(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \psi(t), \quad (46)$$

where M, ε_1, and ε_2 are positive constants which will be taken later.

It is easy to see that $\mathscr{L}(t)$ and $\mathscr{E}(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 such that

$$\beta_1 \mathscr{E}(t) \leq \mathscr{L}(t) \leq \beta_2 \mathscr{E}(t). \quad (47)$$

Remark 11 (see [3]). Since ζ_i is nonincreasing, we have

$$\zeta_i(t)(\varpi_i \circ \nabla v_i))^{1/(1+\varepsilon_0)} \leq C\left(-\mathscr{E}'(t)\right)^{1/(1+\varepsilon_0)}. \quad (48)$$

Proof of Theorem 8. For any fixed $t_0 > 0$, we have for any $t \geq t_0$,

$$\int_0^t \varpi_i(p)dp \geq \int_0^{t_0} \varpi_i(p)dp \coloneqq \varpi_{i0}. \quad (49)$$

It follows from (37), (30), and (20) that

$$\begin{aligned}
\mathscr{L}'(t) &= M\mathscr{E}'(t) + \varepsilon_1 \Phi'(t) + \psi'(t) \\
&\leq -\sum_{i=1}^{2} (\varpi_{i0} - 2\delta - \varepsilon_1) \|\partial_t v_i(t)\|_{L^2_\rho}^2 \\
&\quad - \sum_{i=1}^{2} \left[\frac{l_i}{2}\varepsilon_1 - \delta\left((1-l_i)^2 + 1 + bc_*\right)\right] \|\nabla v_i(t)\|_2^2 \\
&\quad + \sum_{i=1}^{2} [C_1\varepsilon_1 + Cl_i](\varpi_i \circ \nabla v_i)(t) - \frac{M}{2}\sum_{i=1}^{2} \varpi_i(t)\|v_i(t)\|^2 \\
&\quad + \varepsilon_1 k \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) v_i^2 \ln |v_i| dx + \varepsilon_1 c_{\varepsilon_0} \sum_{i=1}^{2} (\varpi_i \circ \nabla v_i))^{1/(1+\varepsilon_0)} \\
&\quad + C_3 \sum_{i=1}^{2} \left(\varpi_i' \circ \nabla v_i\right)(t).
\end{aligned} \quad (50)$$

Using the logarithmic Sobolev inequality, we have

$$\begin{aligned}
\mathscr{L}'(t) &\leq -\sum_{i=1}^{2} (\varpi_{i0} - 2\delta - \varepsilon_1) \|\partial_t v_t(i)\|_{L^2_\rho}^2 \\
&\quad + C_3 \sum_{i=1}^{2} \left(\varpi_i' \circ \nabla v_i\right)(t) \\
&\quad - \sum_{i=1}^{2} \left[\frac{l_i}{2}\varepsilon_1 - \delta\left((1-l_i)^2 + 1 + bc_* - \varepsilon_1 k \frac{\alpha^2}{2\pi}\right)\right] \\
&\quad \|\nabla v_i(t)\|_2^2 + \sum_{i=1}^{2} [C_1\varepsilon_1 + Cl_i](\varpi_i \circ \nabla v_i)(t) \\
&\quad - \frac{M}{2}\sum_{i=1}^{2} \varpi_i(t)\|v_i(t)\|^2 + \varepsilon_1 k \frac{1}{2}\sum_{i=1}^{2} \|v_i\|_2^2 \ln \|v_i\|_2^2 \\
&\quad - \varepsilon_1 k(1 + \ln \alpha) \sum_{i=1}^{2} \|v_i\|_2^2 + \varepsilon_1 c_{\varepsilon_0} \sum_{i=1}^{2} (\varpi_i \circ \nabla v_i))^{1/(1+\varepsilon_0)}.
\end{aligned} \quad (51)$$

Recalling (18) and $\mathscr{E}(t) \leq \mathscr{E}(0) < d$, we get

$$\ln \|v\|_2^2 < \ln \left(\frac{4}{k}\mathscr{E}(t)\right) < \ln \left(\frac{4}{k}\mathscr{E}(0)\right) < \ln \left(\frac{4}{k}d\right). \quad (52)$$

Now, we take $\varepsilon_1 > 0$ small enough so that

$$(\varpi_{i0} - 2\delta - \varepsilon_1) > 0. \quad (53)$$

For any fixed $\varepsilon_1 > 0$, we pick $\delta > 0$ so small that

$$\frac{l_i}{2}\varepsilon_1 - \delta\left((1-l_i)^2 + 1\right) > \frac{l_i}{4}\varepsilon_1. \quad (54)$$

On the other hand, we choose $M > 0$ large enough so that (47) holds, and further

$$C_3 = \frac{M}{2} - \frac{\varpi_i(0)}{4\delta} > 0. \quad (55)$$

We can conclude that there exist two positive constant m and C' such that

$$\mathscr{L}'(t) \leq -m\mathscr{E}(t) + C' \sum_{i=1}^{2} (\varpi_i \circ \nabla v_i)(t) + \varepsilon_1 c_{\varepsilon_0} \sum_{i=1}^{2} (\varpi_i \circ \nabla v_i))^{1/(1+\varepsilon_0)}. \quad (56)$$

Multiplying (56) by $\zeta(t) = \min\{\zeta_1, \zeta_2\}$ by (H2) and use the fact that

$$(\varpi_i \circ \nabla v_i)(t) \leq c(\varpi_i \circ \nabla v_i)^{1/(1+\varepsilon_0)}(t), \quad (57)$$

and (48), we get

$$\zeta(t)\mathscr{L}'(t) \leq -m\zeta(t)\mathscr{E}(t) + c\left(-\mathscr{E}'(t)\right)^{1/(1+\varepsilon_0)}. \quad (58)$$

Multiply (58) by $\zeta^{\varepsilon_0}(t)\mathscr{E}^{\varepsilon_0}(t)$ and recall that $\zeta'(t) \le 0$ to obtain

$$\zeta^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0}(t)\mathscr{L}'(t) \le -m\zeta^{\varepsilon_0}(t)\mathscr{E}^{\varepsilon_0+1}(t) + c(\zeta\mathscr{E})^{\varepsilon_0}(t)\left(-\mathscr{E}'(t)\right)^{1/(1+\varepsilon_0)}. \tag{59}$$

Using Young's inequality, for any $\delta > 0$,

$$\begin{aligned}\zeta^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0}(t)\mathscr{L}'(t) &\le -m\zeta^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0+1}(t) \\ &\quad + c\left(\delta\zeta^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0+1}(t) - c_\delta\mathscr{E}'(t)\right) \\ &\le -(m-\delta c)\zeta^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0+1}(t) - c\mathscr{E}'(t),\end{aligned} \tag{60}$$

which implies

$$\left(\zeta^{\varepsilon_0+1}\mathscr{E}^{\varepsilon_0}\mathscr{L} + c\mathscr{E}\right)(t) \le -(m-\delta c)v^{\varepsilon_0+1}(t)\mathscr{E}^{\varepsilon_0+1}(t). \tag{61}$$

It is clear that to get

$$\mathscr{L}_1(t) = \left(\zeta^{\varepsilon_0+1}\mathscr{E}^{\varepsilon_0}\mathscr{L} + c\mathscr{E}\right) \sim \mathscr{E}(t). \tag{62}$$

By using (61) and $\zeta'(t) \le 0$, we arrive at

$$\mathscr{L}_{1'}(t) = \left(\zeta^{\varepsilon_0+1}\mathscr{E}^{\varepsilon_0}\mathscr{L} + c\mathscr{E}\right) \le -m'\zeta^{\varepsilon_0}(t)\mathscr{E}^{\varepsilon_0+1}(t). \tag{63}$$

Integration over (t_0, t) leads to for some constant $m' > 0$ such that

$$\mathscr{L}_1(t) \le m'\left(1 + \int_{t_0}^{t}\zeta^{\varepsilon_0+1}(p)dp\right)^{-1/\varepsilon_0}. \tag{64}$$

The equivalence of $\mathscr{L}_1(t)$ and $\mathscr{E}$ completes Proof of Theorem 8.

Remark 12.

(1) We mention here that we have coupled our system without the classical way, i.e., our idea is not to couple equations in the logarithmic nonlinear terms

(2) Most contribution here is to obtain our nonexistence result with less conditions on the viscoelastic terms

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Research Article

Blow-Up of Certain Solutions to Nonlinear Wave Equations in the Kirchhoff-Type Equation with Variable Exponents and Positive Initial Energy

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This paper is concerned with the blow-up of certain solutions with positive initial energy to the following quasilinear wave equation: $u_{tt} - \mathscr{M}(\mathscr{N}u(t))\Delta_{p(\cdot)}u + g(u_t) = f(u)$. This work generalizes the blow-up result of solutions with negative initial energy.

1. Introduction

Let Ω be an open bounded Lipschitz domain in $\mathbb{R}^n (n \geq 1)$, $T > 0$, $Q_T = \Omega \times (0, T)$. We consider the following nonlinear hyperbolic equation:

$$\begin{cases} u_{tt} - \mathscr{M}(\mathscr{N}u(t))\Delta_{p(x)}u + g(u_t) = f(u), & (x, t) \in Q_T, \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \tag{1}$$

Here, $\partial\Omega$ is a Lipschitz continuous boundary. The initial conditions meet the following:

$$\begin{aligned} u_0 &\in W_0^{1,p(\cdot)}(\Omega), \\ u_1 &\in L^2(\Omega). \end{aligned} \tag{2}$$

The Kirchhoff function $\mathscr{M} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and has the standard form:

$$\mathscr{M}(\tau) = a + b\gamma\tau^{\gamma-1}, \quad a, b \geq 0, \gamma \geq 1, a + b > 0, \gamma > 1 \text{ if } b > 0. \tag{3}$$

The elliptic nonhomogeneous $p(x)$-Laplacian operator is defined by

$$\Delta_{p(x)}u = \nabla \cdot \left(|\nabla u|^{p(x)-2}\nabla u\right), \tag{4}$$

where $\nabla\cdot$ is the vectorial divergence and ∇ is the gradient of u. The functional

$$\mathscr{N}u(t) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx, \tag{5}$$

is the naturally associated $p(x)$-Dirichlet energy integral. The term with a variable exponent

$$f(u) = c(x, t)|u|^{q(x)-2}u, \tag{6}$$

plays the role of a source, and the dissipative term with a variable exponent

$$g(u_t) = d(x, t)|u_t|^{r(x)-2}u_t, \tag{7}$$

is a strong damping term.

The coefficients c and d are continuous in Q_T and satisfy

$$0 < c^- = \inf_{(x,t)\in Q_T} c(x,t) \le c(x,t) \le c^+ = \sup_{(x,t)\in Q_T} c(x,t) < \sigma < \infty, \tag{8}$$

$$0 < d^- = \inf_{(x,t)\in Q_T} d(x,t) \le d(x,t) \le d^+ = \sup_{(x,t)\in Q_T} d(x,t) < \infty, \tag{9}$$

where σ is a constant defined in (38). We assume that the Kirchhoff function $\mathcal{M}$, defined by (3), satisfies the following hypotheses:

(i) For $1 < \alpha \le \beta < \min\{n/p^+, np^-/p^+(n-p^-)\}$, there exist $m_2 \ge m_1 > 0$ such that

$$m_1 \tau^{\alpha-1} \le \mathcal{M}(\tau) \le m_2 \tau^{\beta-1}, \quad \tau \in \mathbb{R}^+. \tag{10}$$

(ii) For all $\tau \in \mathbb{R}^+$, it holds that

$$\int_0^\tau \mathcal{M}(s)ds = \widehat{\mathcal{M}}(\tau) \ge \mathcal{M}(\tau)\tau. \tag{11}$$

The exponents $p(\cdot)$, $q(\cdot)$, and $r(\cdot)$ are continuous and satisfy

$$\begin{aligned} 2 &\le \min\{p^-, r^-\} \le \{p(x), r(x)\} \le \max\{p^+, r^+\} < q^- \\ &\le q(x) \le q^+ < p^-\alpha \le p^-\beta \le p^+\alpha \le p^+\beta \le p_*(x), \end{aligned} \tag{12}$$

where the constants α and β are given in (10) and

$$\begin{aligned} p^- &= \text{ess}\inf_{x\in\Omega} p(x), \\ p^+ &= \text{ess}\sup_{x\in\Omega} p(x), \\ q^- &= \text{ess}\inf_{x\in\Omega} q(x), \\ q^+ &= \text{ess}\sup_{x\in\Omega} q(x), \\ r^- &= \text{ess}\inf_{x\in\Omega} r(x), \\ r^+ &= \text{ess}\sup_{x\in\Omega} r(x). \end{aligned} \tag{13}$$

Also, we can define $p_*(x)$ by

$$p_*(x) = \begin{cases} \dfrac{np(x)}{\text{ess}\sup_{x\in\Omega}(n - r(x))}, & \text{if } p^+ < n, \\ +\infty, & \text{if } p^+ \ge n. \end{cases} \tag{14}$$

We also assume that $p(\cdot)$, $q(\cdot)$, and $r(\cdot)$ satisfy the log-Hölder continuity condition

$$|\xi(x) - \xi(y)| \le -\frac{L}{\log|x-y|}, \quad \text{for a.e.,} x, y \in \Omega, |x-y| < \delta, \tag{15}$$

for $L > 0$, $0 < \delta < 1$.

Problem (1) models several physical and biological systems such as viscoelastic fluids, filtration processes through a porous medium, and fluids with viscosity dependent on temperature. In the intention of problem (1), we can see that it is linked to the following equation presented by Kirchhoff and Hensel [1] in 1883:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right| dx\right)\frac{\partial^2 u}{\partial x^2} + g(u_t) = f(u). \tag{16}$$

The parameters L, h, E, ρ, and P_0 represent, respectively, the length of the string, the area of the cross-section, Young's modulus of the material, the mass density, and the initial tension. This equation is an extension of the classic d'Alembert's wave equation by looking at the effects of changes in the length of the string during the vibrations. As for this problem, it has been studied. More precisely, for $g(u_t) = u_t$, the global existence and nonexistence results can be found in [2, 3], and for $g(u_t) = |u_t|^p u_t$, $p > 0$, the main results of existence and nonexistence are in the paper [4]. In recent years, hyperbolic problems with a constant exponent have been studied by many authors; we refer to interesting works [5–7]. However, only a little research has been done regarding hyperbolic problems with nonlinearities of the variable exponent type; some interesting works can be found in [8–13].

Recently, in [14], Piskin studied the following wave equation with variable exponent nonlinearities:

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{17}$$

The author proved, by using the modified energy functional method, the existence of solutions. We have also looked at the asymptotic behavior of the Kirchhoff wave equation problems. We can say that the investigation into the determination of the type, as well as the rate of decay, was the focus of attention of many researchers whose work was represented in [15, 16]. Motivated by previous studies, in this work, we consider problem (1), which is more interesting and applicable in the real approach of sciences, so a finite-time blow-up for certain solutions with positive and also negative initial energy has been proved. More precisely, our aim here is to find sufficient conditions on the variable exponents $p(\cdot)$, $q(\cdot)$, and $r(\cdot)$ and the initial data for which the blow-up occurs. This paper is organized as follows. After the introduction in the first section, we will give some preliminaries in Section 2. Then, in Section 3, we state the main results which will be proved in Sections 4 and 5.

2. Preliminaries

Regarding some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$, we refer to the book of Musielak [17] and the papers [18, 19]. Let

$$C_+(\bar{\Omega}) := \{h : h \in C(\bar{\Omega}), h(x) > 1, \quad \text{for all } x \in \bar{\Omega}\}. \quad (18)$$

For any $h \in C(\bar{\Omega})$, we write

$$h^+ = \sup_{x\in\Omega} h(x), \quad h^- = \inf_{x\in\Omega} h(x). \quad (19)$$

Then, for any $p(x) \in C^+(\bar{\Omega})$, we define the variable exponent Lebesgue space as follows:

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \varrho_{p(\cdot)}(\mu u) < +\infty, \quad \text{for some } \mu > 0\right\}, \quad (20)$$

where $\varrho_{p(\cdot)}$ is the $p(\cdot)$ modular of u, and it is defined by

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx. \quad (21)$$

It is equipped with the following so-called Luxemburg norm on this space defined by the formula

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}. \quad (22)$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$, and their continuous functions are dense if $p^+ < \infty$.

Lemma 1. *Suppose that $p(\cdot)$ satisfies (15); then,*

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega), \quad (23)$$

where $C > 0$ is a constant that depends only on p^-, p^+, and Ω.

Lemma 2. *If $p(\cdot) \in C(\bar{\Omega})$ and $q : \Omega \longrightarrow [1,\infty)$ are measurable functions such that*

$$\operatorname{ess\,inf}_{x\in\Omega}(p_*(x) - q(x)) > 0, \quad \text{with } p_*(x) = \begin{cases} \dfrac{np(x)}{\operatorname{ess\,sup}_{x\in\Omega}(n - p(x))}, & \text{if } q^- < n, \\ \infty, & \text{if } q^- \ge n, \end{cases} \quad (24)$$

then the embedding $W_0^{1,p(\cdot)}(\Omega)^\circ L^{q(\cdot)}(\Omega)$ is continuous and compact.

Lemma 3. *Let $1 < p^- \le p^+ < +\infty$. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, uniformly convex, and reflexive Banach spaces. The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad \forall x \in \Omega. \quad (25)$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\left|\int_\Omega u(x)v(x)\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p}{}^{'(\cdot)}. \quad (26)$$

Lemma 4. *If $p \ge 1$ is a measurable function on Ω and $u \in L^{p(\cdot)}(\Omega)$, then $\|u\|_{p(\cdot)} \le 1$ and $\rho_{p(\cdot)}(u) \le 1$ are equivalent. For $u \in L^{p(\cdot)}(\Omega)$, we have*

$$\|u\|_{p(\cdot)} \le 1 \text{ implies } \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}, \quad \|u\|_{p(\cdot)} > 1 \text{ implies } \rho_{p(\cdot)}(u) \ge \|u\|_{p(\cdot)}. \quad (27)$$

Lemma 5. *If $p(x) \in [1,\infty)$ is a measurable function on Ω, then*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \le \rho_{p(\cdot)}(u) \le \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\}, \quad (28)$$

for all $u \in L^{p(\cdot)}(\Omega)$.

3. Main Results

Now, we state without proof the following existence result.

Proposition 6. *Assume that (2) holds and the coefficients a, b, c, and d satisfy (3) and (9) and the exponents p, q, and r satisfy (12). Then, problem (1) has a unique weak solution such that*

$$\begin{cases} u \in L^\infty\left((0,T), W_0^{1,p(\cdot)}(\Omega)\right), \\ u_t \in L^\infty\left((0,T), L^2(\Omega)\right), \\ u_{tt} \in L^\infty\left((0,T), W_0^{-1,p'(\cdot)}(\Omega)\right), \end{cases} \quad (29)$$

where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$.

Remark 7. The proof can be established by employing the Galerkin method as in the work of Antontsev [8].

We first define the energy function. Let

$$\mathscr{E}(t) := \frac{1}{2}\|u_t\|_{L^2(\Omega)}^2 + \widehat{\mathscr{M}}(\mathscr{N}u(t)) - \Psi(t), \quad (30)$$

where

$$\widehat{\mathscr{M}}(\mathscr{N}u(t)) = a\mathscr{N}u(t) + b\gamma[\mathscr{N}u(t)]^{\gamma},$$
$$\Psi(t) = \int_\Omega \frac{c(x,t)}{q(x)}|u|^{q(x)}dx. \tag{31}$$

In order to investigate the properties of $\mathscr{E}(t)$, the following lemma is necessary.

Lemma 8. *Suppose that u is a solution of problem (1) that satisfies (29); then, we have*

$$\mathscr{E}_t(t) = -\int_\Omega \frac{c_t(x,t)}{q(x)}|u|^{q(x)}dx - \int_\Omega d(x,t)|u_t|^{r(x)}dx. \tag{32}$$

Proof. By using the energy function (30) and problem (1), we directly deduce (32).

We also introduce the following lemma.

Lemma 9. *Suppose that the conditions of Lemmas 1–5 hold. Then, there exists a constant C > 1, which is a generic constant that depends on Ω only, such that*

$$\varrho_{q(\cdot)}^{s/q^-}(u) \le C\left(\|\nabla u\|_{p(\cdot)}^{\alpha p^-} + \varrho_{q(\cdot)}(u)\right), \tag{33}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$ *and* $\alpha p^- \le s \le \alpha q^-$.

Proof. If $\varrho_{q(\cdot)}(u) > 1$, then

$$\varrho_{q(\cdot)}^{s/q^-}(u) \le \varrho_{q(\cdot)}^{\alpha}(u) \le C\left(\|\nabla u\|_{p(\cdot)}^{\alpha p^-} + \varrho_{q(\cdot)}(u)\right). \tag{34}$$

If $\varrho_{q(\cdot)}(u) \le 1$, then we deduce by Lemma 4 that $\|u\|_{q(\cdot)} \le 1$. Then, Lemmas 2 and 5 imply

$$\varrho_{q(\cdot)}^{s/q^-}(u) \le \varrho_{q(\cdot)}^{\alpha p^-/q^-}(u) \le \max\left\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\right\}^{\alpha p^-/q^-}$$
$$= \|u\|_{q(\cdot)}^{\alpha p^-} \le C\|\nabla u\|_{p(\cdot)}^{\alpha p^-}. \tag{35}$$

Let B be the best constant of the Sobolev embedding

$$W_0^{1,p(\cdot)^{\circ}}L^{q(\cdot)}. \tag{36}$$

We set

$$B_1 = \max\left\{1, B, \left(\frac{m_1}{c^+\alpha(p^+)^{\alpha-1}}\right)^{1/\alpha p^+}\right\}, \tag{37}$$

$$\sigma = \min\left\{1, \left(\frac{q^- m_1}{B_1^{q^-}\alpha(p^+)^{\alpha}}\right)\min\left\{\Gamma_1^{1-(q^-/\alpha p^+)}, \Gamma_1^{1-(q^+/\alpha p^+)}\right\}\right\}, \tag{38}$$

$$\Gamma_1 = \left(\frac{m_1(p^+)^{1-\alpha}}{c^+B_1^{q^-}}\right)^{\alpha p^+/(q^- - \alpha p^+)}, \tag{39}$$

$$\mathscr{E}_1 = \left[\frac{m_1(p^+)^{-\alpha}}{q^-\alpha}(q^- - p^+)\right]\Gamma_1. \tag{40}$$

Now, the main results of the blow-up for certain solutions with positive/negative initial energy are given by the following theorems.

Theorem 10. *Let the assumptions of Proposition 6 be satisfied, and assume that*

$$c^+ < \sigma, c_t(x,t) \ge \tilde{\sigma} \ge 0, \quad \forall (x,t) \in Q_T,$$
$$0 < \mathscr{E}(0) < \mathscr{E}_1, \quad \Gamma_1 < \Gamma_0 = \|\nabla u_0\|_{p(\cdot)}^{\alpha p^+} \le B_1^{-\alpha p^+}. \tag{41}$$

Then, the solutions of (1) blow up in finite time:

$$T^* \le \frac{1-\lambda}{\mu\lambda\mathscr{F}^{\lambda/(1-\lambda)}(0)}, \quad \lambda, \mu > 0. \tag{42}$$

Theorem 11. *Let the assumptions of Proposition 6 be satisfied, and assume that*

$$\mathscr{E}(0) < 0. \tag{43}$$

Then, the solution of (1) blows up in finite time (42).

4. Proof of Theorem 10

To prove Theorem 10, we need the following lemmas.

Lemma 12. *Let the assumptions of Theorem 10 hold; then, there exists σ > 0 such that for any* $c^+ < \sigma$, *there exist a constant* $\Gamma_2 > \Gamma_1$ *such that*

$$\|\nabla u(\cdot,t)\|_{p(\cdot)}^{\alpha p^+} \ge \Gamma_2, \quad \forall t \ge 0. \tag{44}$$

Proof. By using the hypothesis (10) and the function (30), we obtain

$$\begin{aligned}
\mathscr{E}(t) &\ge \widehat{\mathscr{M}}(t) - \Psi(t) \ge \frac{m_1}{\alpha(p^+)^{\alpha}}\rho_{p(\cdot)}^{\alpha}(\nabla u) - \frac{c^+}{q^-}\rho_{q(\cdot)}(u)\\
&\ge \frac{m_1}{\alpha(p^+)^{\alpha}}\min\left\{\|\nabla u\|_{p(\cdot)}^{\alpha p^-}, \|\nabla u\|_{p(\cdot)}^{\alpha p^+}\right\}\\
&\quad - \frac{c^+}{q^-}\max\left\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\right\}\\
&\ge \frac{m_1}{\alpha(p^+)^{\alpha}}\min\left\{\|\nabla u\|_{p(\cdot)}^{\alpha p^-}, \|\nabla u\|_{p(\cdot)}^{\alpha p^+}\right\}\\
&\quad - \frac{c^+}{q^-}\max\left\{\left(B_1\|\nabla u\|_{p(\cdot)}\right)^{q^-}, \left(B_1\|\nabla u\|_{p(\cdot)}\right)^{q^+}\right\}\\
&= \frac{m_1}{\alpha(p^+)^{\alpha}}\min\left\{\Gamma^{p^-/p^+}, \Gamma\right\}\\
&\quad - \frac{c^+}{q_-}\max\left\{\left(B_1^{\alpha p^+}\Gamma\right)^{q^-/\alpha p^+}, \left(B_1^{\alpha p^+}\Gamma\right)^{q^+/\alpha p^+}\right\}\\
&\coloneqq \psi(\Gamma), \quad \forall \Gamma \in \mathbb{R}^+,
\end{aligned} \tag{45}$$

where $\Gamma = \|\nabla u\|_{p(\cdot)}^{\alpha p^+}$. Let the function

$$\phi : [0, 1] \longrightarrow \mathbb{R}, \tag{46}$$

be defined by

$$\phi(\Gamma) = \frac{m_1}{\alpha (p^+)^\alpha} \Gamma - \frac{c^+}{q^-} \left(B_1^{\alpha p^+} \Gamma \right)^{q^- / \alpha p^+}. \tag{47}$$

Notice that $\phi(\Gamma) = \psi(\Gamma)$, for $0 < \Gamma \le B_1^{-\alpha p^+}$. It is easy to check that the function ϕ is increasing for $0 < \Gamma < \Gamma_1$ and decreasing for $\Gamma_1 < \Gamma \le \infty$. On the other hand, by (38), we deduce that, for any $c^+ < \sigma$, since $\mathscr{E}(0) < \mathscr{E}_1 = \phi(\Gamma_1)$, there exists a positive constant $\Gamma_2 \in (\Gamma_1, \infty)$ such that $\phi(\Gamma_2) = \mathscr{E}(0)$. Then, we have $\phi(\Gamma_0) = \psi(\Gamma_0) \le \mathscr{E}(0) = \phi(\Gamma_2)$. This implies that $\Gamma_0 \ge \Gamma_2$.

Now, we suppose on the contrary that $\|\nabla u(t_0)\|_{p(\cdot)}^{\alpha p^+} < \Gamma_2$ for some $t_0 > 0$. Then, there exists $t_1 > 0$ such that $\Gamma_1 < \|\nabla u(t_1)\|_{p(\cdot)}^{\alpha p^+}$. Using the monotonicity of $\phi(\Gamma)$, we have

$$\mathscr{E}(t_1) \ge \phi\left(\|\nabla u(t_1)\|_{p(\cdot)}^{\alpha p^+} \right) > \phi(\Gamma_2) = \mathscr{E}(0), \tag{48}$$

which contradicts $\mathscr{E}(t) < \mathscr{E}(0)$, for all $t \in (0, T)$.

Lemma 13. *Let the assumptions of Theorem 10 hold. Then, in light of Lemma 12, we have*

$$\rho_{q(\cdot)}(u) \ge \kappa, \quad \kappa > 0. \tag{49}$$

Proof. By using (30), we get

$$\begin{aligned} \frac{c^+}{q^-} \rho_{q(\cdot)}(u) &\ge \Psi(t) \ge -\mathscr{E}(0) + \frac{m_1}{\alpha (p^+)^\alpha} \rho_{p(\cdot)}^\alpha (\nabla u) + \frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 \\ &\ge \frac{m_1}{\alpha (p^+)^\alpha} \rho_{p(\cdot)}^\alpha (\nabla u) - \psi(\Gamma_2) \\ &\ge \frac{c^+}{q_-} \max \left\{ \left(B_1^{\alpha p^+} \Gamma_2 \right)^{q^- / \alpha p^+}, \left(B_1^{\alpha p^+} \Gamma_2 \right)^{q^+ / \alpha p^+} \right\} \\ &:= \kappa. \end{aligned} \tag{50}$$

Let

$$\mathscr{H}(t) = \mathscr{E}_1 - \mathscr{E}(t). \tag{51}$$

Lemma 14. *Let the assumptions of Theorem 10 be satisfied; then, we have*

$$0 < \mathscr{H}(0) \le \mathscr{H}(t) \le \frac{\sigma}{q^-} \rho_{q(\cdot)}(u). \tag{52}$$

Proof. Using (30), (32), and (51), we obtain

$$0 < \mathscr{H}(0) \le \mathscr{H}(t) \le \mathscr{E}_1 - \frac{1}{2} \left[\|u_t\|_{L^2(\Omega)}^2 + \widehat{\mathscr{M}}(\mathscr{N} u(t)) \right] + \Psi(t). \tag{53}$$

Then, the use of (10) gives

$$\begin{aligned} \mathscr{E}_1 - \left[\frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 - \widehat{\mathscr{M}}(\mathscr{N} u(t)) \right] &\le \mathscr{E}_1 - \int_0^{\mathscr{N} u(t)} \mathscr{M}(\tau) d\tau \\ &\le \mathscr{E}_1 - \frac{m_1}{\alpha (p^+)^\alpha} \min \left\{ \|\nabla u\|_{p(\cdot)}^{\alpha p^-}, \|\nabla u\|_{p(\cdot)}^{\alpha p^+} \right\} \\ &\le \mathscr{E}_1 - \frac{m_1}{\alpha (p^+)^\alpha} \min \left\{ \Gamma_2^{p^- / p^+}, \Gamma_2 \right\} \\ &\le \mathscr{E}_1 - \frac{m_1}{\alpha (p^+)^\alpha} \min \left\{ \Gamma_1^{p^- / p^+}, \Gamma_1 \right\} \\ &= \mathscr{E}_1 - \frac{m_1}{\alpha (p^+)^\alpha} \Gamma_1. \end{aligned} \tag{54}$$

Now, recalling $\mathscr{E}_1$ in (38), we have

$$\mathscr{E}_1 - \left[\frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 - \widehat{\mathscr{M}}(\mathscr{N} u(t)) \right] \le -\frac{m_1 (p^+)^{1-\alpha}}{q^- \alpha} \Gamma_1 < 0. \tag{55}$$

On the other hand, we use (9) to get

$$\Psi(t) \le \frac{c^+}{q^-} \rho_{q(\cdot)}(u) \le \frac{\sigma}{q^-} \rho_{q(\cdot)}(u). \tag{56}$$

Combining (55) with (56) gives (52).

Corollary 15. *Under the assumptions of Lemma 9, we have*

(i) $\|u\|_{q^-}^s \le C(\|\nabla u\|_{p(\cdot)}^{\alpha p^-} + \|u\|_{q^-}^{q^-})$

(ii) $\rho_{q(\cdot)}^{s/q^-} \le C(|\mathscr{H}(t)| + \|u_t\|_{L^2(\Omega)}^2 + \rho_{q(\cdot)}(u))$

(iii) $\|u\|_{q^-}^s \le C(|\mathscr{H}(t)| + \|u_t\|_{L^2(\Omega)}^2 + \|u\|_{q^-}^{q^-})$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\alpha p^- \le s \le \alpha q^-$.

Lemma 16. *Assume that (12) and (15) hold. Then, the solution of (1) satisfies*

$$\varrho_{q(\cdot)}(u) \ge C \|u\|_{q^-}^{q^-}, \tag{57}$$

for some $C > 0$.

Proof. Let

$$\begin{aligned} \Omega^+ &= \{x \in \Omega / |u| \ge 1\}, \\ \Omega^- &= \{x \in \Omega / |u| < 1\}. \end{aligned} \tag{58}$$

We have

$$\varrho_{q(\cdot)}(u) = \int_{\Omega^+} |u|^{q(x)} dx + \int_{\Omega^-} |u|^{q(x)} dx \geq \int_{\Omega^+} |u|^{q(x)} dx + c_1 \left(\int_{\Omega^-} |u|^{q(x)} dx \right)^{q^+/q^-}. \quad (59)$$

This implies

$$c_2 \left(\varrho_{q(\cdot)}(u) \right)^{q^-/q^+} + \varrho_{q(\cdot)}(u) \geq \|u\|_{q^-}^{q^-}. \quad (60)$$

Now, given (52), (60) leads to

$$\varrho_{q(\cdot)}(u) \geq \left[1 + c_2 \left(\frac{q^-}{\sigma} \right)^{(q^-/q^+)-1} \right]^{-1} \|u\|_{q^-}^{q^-}. \quad (61)$$

Thus, (57) follows.

Lemma 17. *Suppose that (12) holds, and u is a solution of (1). Then,*

$$\varrho_{r(\cdot)}(u) \leq C \left(\varrho_{q(\cdot)}^{r^-/q^-}(u) + \varrho_{q(\cdot)}^{r^+/q^-}(u) \right). \quad (62)$$

Proof.

$$\begin{aligned} \varrho_{r(\cdot)}(u) &\leq \int_{\Omega^-} |u|^{r^-} dx + \int_{\Omega^+} |u|^{r^+} dx \\ &\leq C \left[\left(\int_{\Omega^-} |u|^{q^-} dx \right)^{r^-/q^-} + \left(\int_{\Omega^+} |u|^{q^-} dx \right)^{r^+/q^-} \right] \\ &\leq C \left(\|u\|_{q^-}^{r^-} + \|u\|_{q^-}^{r^+} \right) \leq C \left(\varrho_{q(\cdot)}^{r^-/q^-}(u) + \varrho_{q(\cdot)}^{r^+/q^-}(u) \right). \end{aligned} \quad (63)$$

We set

$$\mathcal{F}(t) = \mathcal{H}^{1-\lambda}(t) + \varepsilon |u, u_t|_{L^2(\Omega)}, \quad (64)$$

for ε small, which will be specified later, and for

$$0 < \lambda \leq \min \left\{ \frac{\alpha q^- - r^+}{q^-(r^+ - 1)}, \frac{q^- - 2}{2q^-} \right\}. \quad (65)$$

Now, we are in a position to prove Theorem 10.

Proof. We differentiate (64) and use the equation in (1) to get

$$\begin{aligned} \mathcal{F}_t(t) &= (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 + |u, u_{tt}|_{L^2(\Omega)} \right) \\ &\geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) \\ &\quad + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 - \mathcal{M}(\mathcal{N}u(t))\varrho_{p(\cdot)}(\nabla u) + c^- \varrho_{q(\cdot)}(u) \right) \\ &\quad - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)} \geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) \\ &\quad + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 - \frac{m_2}{(p^-)^{\beta-1}} \varrho_{p(\cdot)}^{\beta}(\nabla u) + c^- \varrho_{q(\cdot)}(u) \right) \\ &\quad - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)} \geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) \\ &\quad + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 - \frac{m_2}{(p^-)^{\beta-1}} \varrho_{p(\cdot)}^{\alpha}(\nabla u) + c^- \varrho_{q(\cdot)}(u) \right) \\ &\quad - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)}. \end{aligned} \quad (66)$$

Adding and subtracting the term $\varepsilon(1-\eta)q^-\mathcal{H}(t)$, for $0 < \eta < 1$, from the right-hand side of (56), by using (49) and (10), we get

$$\begin{aligned} \mathcal{F}_t(t) &\geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) + \varepsilon(1-\eta)q^-\mathcal{H}(t) - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)} \\ &\quad + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 - \frac{m_2}{(p^-)^{\beta-1}} \varrho_{p(\cdot)}^{\alpha}(\nabla u) + c^- \varrho_{q(\cdot)}(u) \right) \\ &\quad + \varepsilon(1-\eta)q^- \left(-\mathcal{E}_1 + \frac{1}{2}\|u_t\|_{L^2(\Omega)}^2 + \widehat{\mathcal{M}}(\mathcal{N}u(t)) - \frac{c^-}{q^-} \varrho_{q(\cdot)}(u) \right) \\ &\geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) + \varepsilon(1-\eta)q^-\mathcal{H}(t) - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)} \\ &\quad + \varepsilon \left(\|u_t\|_{L^2(\Omega)}^2 - \frac{m_2}{(p^-)^{\beta-1}} \rho_{p(\cdot)}^{\alpha}(\nabla u) + c^- \varrho_{q(\cdot)}(u) \right) \\ &\quad + \varepsilon(1-\eta)q^- \left(\left(-\frac{\mathcal{E}_1 c^+}{q^-\kappa} - \frac{c^-}{q^-} \right) \varrho_{q(\cdot)}(u) + \frac{1}{2}\|u_t\|_{L^2(\Omega)}^2 \right) \\ &\quad + \varepsilon(1-\eta)q^- \left(\frac{m_1}{\alpha(p^+)^{\alpha}} \varrho_{p(\cdot)}^{\alpha}(\nabla u) \right). \end{aligned} \quad (67)$$

Then, for η small enough, we have

$$\begin{aligned} \mathcal{F}_t(t) &\geq (1-\lambda)\mathcal{H}^{-\lambda}(t)\mathcal{H}_t(t) - \varepsilon d^+ |g(u_t), u|_{L^2(\Omega)} \\ &\quad + \varepsilon\delta \left(\mathcal{H}(t) + \|u_t\|_{L^2(\Omega)}^2 + \varrho_{p(\cdot)}^{\alpha}(\nabla u) + \varrho_{q(\cdot)}(u) \right), \end{aligned} \quad (68)$$

where

$$\delta = \min \left\{ 1 - \eta q^-, 1 + \frac{(1-\eta)q^-}{2}, \frac{(1-\eta)q^- m_1}{\alpha(p^+)^{\alpha}} - \frac{m_2}{(p^-)^{\beta-1}}, \eta c^- - \frac{(1-\eta)\mathcal{E}_1 c^+}{\kappa} \right\}. \quad (69)$$

Recall Young's inequality

$$XY \le \frac{\delta_1 X^{l_1}}{l_1} + \frac{\delta_1^{-l_2} Y^{l_2}}{l_2}, \tag{70}$$

where $X, Y \ge 0, \delta_1 > 0, l_1, l_2 \in \mathbb{R}^+$, such that $(1/l_1) + (1/l_2) = 1$. Applying (70) to estimate the term $|g(u_t), u|_{L^2(\Omega)}$, we get

$$\int_\Omega |u_t|^{r(x)-1}|u| dx \le \frac{1}{r^-}\int_\Omega \delta_1^{r(x)}|u|^{r(x)} dx + \frac{r^+ - 1}{r^+}\int_\Omega \delta_1^{-r(x)/(r(x)-1)}|u_t|^{r(x)} dx, \tag{71}$$

where

$$\delta_1^{-r(x)/(r(x)-1)} = \xi \mathscr{H}^{-\lambda}(t), \tag{72}$$

where ξ is a large constant to be specified later.

Now, by using (32) and (49), we get

$$\varrho_{r(\cdot)}(u_t) \le \frac{\mathscr{H}_t(t)}{d^+} - \frac{\tilde{\sigma}}{q^+ d^+}\varrho_{q(\cdot)}(u) \le \frac{\mathscr{H}_t(t)}{d^+} - \frac{\tilde{\sigma}\kappa}{q^+ d^+} \le \frac{\mathscr{H}_t(t)}{d^+}. \tag{73}$$

Combining (73) and (71) yields

$$\begin{aligned} \int_\Omega |u_t|^{r(x)-1}|u| dx &\le \frac{1}{r^-}\int_\Omega \xi^{1-r(x)}|u|^{r(x)}\mathscr{H}^{\lambda(r(x)-1)}(t) d \\ &+ \frac{(r^+ - 1)\xi}{r^+ d^+}\mathscr{H}^{-\lambda}(t)\mathscr{H}_t(t). \end{aligned} \tag{74}$$

Substituting (74) in (68), we obtain

$$\begin{aligned} \mathscr{F}_t(t) \ge & \left[(1-\lambda) - \varepsilon\frac{(r^+ - 1)\xi}{r^+}\right]\mathscr{H}^{-\lambda}(t)\mathscr{H}_t(t) \\ &+ \varepsilon\delta\left(\mathscr{H}(t) + \|u_t\|^2_{L^2(\Omega)} + \varrho^{\alpha}_{p(\cdot)}(\nabla u) + \varrho_{q(\cdot)}(u)\right) \\ &- \varepsilon\frac{\xi^{1-r^-} d^+}{r^-} C_1 \mathscr{H}^{\lambda(r^+ - 1)}(t)\varrho_{r(\cdot)}(u). \end{aligned} \tag{75}$$

To estimate the last term in (75), we use (62) and (52) to get

$$\mathscr{H}^{\lambda(r^+ - 1)}(t)\varrho_{r(\cdot)}(u) \le C\left(\varrho_{q(\cdot)}^{(r^-/q^-)+\lambda(r^+ - 1)}(u) + \varrho_{q(\cdot)}^{(r^+/q^-)+\lambda(r^+ - 1)}(u)\right). \tag{76}$$

Then, we use (65) and Lemma 9, for

$$\begin{aligned} s &= r^+ + \lambda q^-(r^+ - 1) \le \alpha q^-, \\ s &= r^- + \lambda q^-(r^+ - 1) \le \alpha q^-, \end{aligned} \tag{77}$$

to deduce from (76) that

$$\mathscr{H}^{\lambda(r^+ - 1)}(t)\varrho_{r(\cdot)}(u) \le C\left(\|\nabla u\|^{\alpha p^-}_{p(\cdot)} + \varrho_{q(\cdot)}(u)\right). \tag{78}$$

By exploiting Lemmas 5 and 12, we get

$$\varrho^{\alpha}_{p(\cdot)}(\nabla u) \ge C\|\nabla u\|^{\alpha p^-}_{p(\cdot)}. \tag{79}$$

Combining (75), (78), and (79) leads to

$$\begin{aligned} \mathscr{F}_t(t) \ge & \left[(1-\lambda) - \varepsilon\frac{(r^+ - 1)\xi}{r^+}\right]\mathscr{H}^{-\lambda}(t)\mathscr{H}_t(t) + \varepsilon\left(\delta - \frac{\xi^{1-r^-} d^+}{r^-}C\right) \\ &\cdot \left[\mathscr{H}(t) + \|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^{\alpha p^-}_{p(\cdot)} + \varrho_{q(\cdot)}(u)\right]. \end{aligned} \tag{80}$$

Now, we pick ξ large enough and ε so small such that

$$\begin{aligned} \delta_2 &= \delta - \frac{\xi^{1-r^-} d^+}{r^-} C > 0, \\ \varepsilon &\le \frac{(1-\lambda) r^+}{\xi(r^+ - 1)}, \\ \mathscr{F}(0) &= \mathscr{H}^{1-\lambda}(0) + \varepsilon|u_0, u_1|_{L^2(\Omega)} > 0. \end{aligned} \tag{81}$$

Then, by using (57), (80) takes the form

$$\mathscr{F}_t(t) \ge \delta_2\varepsilon\left[\mathscr{H}(t) + \|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^{\alpha p^-}_{p(\cdot)} + \|u\|^{q^-}_{q^-}\right]. \tag{82}$$

Therefore, we get

$$\mathscr{F}(t) \ge \mathscr{F}(0) > 0, \quad \text{for all } t \ge 0. \tag{83}$$

On the other hand, the application of the Hölder inequality yields

$$\left||u, u_t|_{L^2(\Omega)}\right|^{1-\lambda} \le C\|u\|^{1/(1-\lambda)}_{q^-}\|u_t\|^{1/(1-\lambda)}_{L^2(\Omega)}, \tag{84}$$

and from (70), we get

$$\left||u, u_t|_{L^2(\Omega)}\right|^{1-\lambda} \le C\left[\|u\|^{\theta_1/(1-\lambda)}_{q^-} + \|u_t\|^{\theta_2/(1-\lambda)}_{L^2(\Omega)}\right], \tag{85}$$

for $(1/\theta_1) + (1/\theta_2) = 1$. Setting $\theta_1/(1-\lambda) = (2/(1-2\lambda)) \le q^-$, we get $\theta_2 = 2(1-\lambda)$ by virtue of (65). Therefore, (85) takes the form

$$\left||u, u_t|_{L^2(\Omega)}\right|^{1-\lambda} \le C\left[\|u\|^{s}_{q^-} + \|u_t\|^2_{L^2(\Omega)}\right], \tag{86}$$

where $s = 2/(1-2\lambda)$. By recalling Corollary 15, we get

$$\left||u, u_t|_{L^2(\Omega)}\right|^{1-\lambda} \le C\left(|\mathscr{H}(t)| + \|u_t\|^2_{L^2(\Omega)} + \|u\|^{q^-}_{q^-}\right). \tag{87}$$

Now, (87) and the following Minkowski's inequality

$$(X+Y)^i \le 2^{i-1}\left(X^i+Y^i\right), \tag{88}$$

will give

$$\begin{aligned}\mathscr{F}^{1/(1-\lambda)}(t) &= \left[\mathscr{H}^{1-\lambda}(t)+\varepsilon|u,u_t|_{L^2(\Omega)}\right]^{1/(1-\lambda)}\\ &\le 2^{\lambda/(1-\lambda)}\left(\mathscr{H}(t)+\varepsilon^{1/(1-\lambda)}\left||u,u_t|_{L^2(\Omega)}\right|^{1-\lambda}\right)\\ &\le C\left(|\mathscr{H}(t)|+\|u_t\|^2_{L^2(\Omega)}+\|u\|^{q^-}_{q^-}\right)\\ &\le C\left(\mathscr{H}(t)+\|u_t\|^2_{L^2(\Omega)}+\|\nabla u\|^{\alpha p^-}_{p(\cdot)}+\|u\|^{q^-}_{q^-}\right).\end{aligned} \tag{89}$$

By combining (82) and (89), we obtain

$$\mathscr{F}_t(t) \ge \mu\mathscr{F}(t)^{1/(1-\lambda)}(t), \tag{90}$$

where μ is a positive constant. A simple integration of (90) over $(0,t)$ yields

$$\mathscr{F}^{1/(1-\lambda)}(t) \ge \frac{1}{\mathscr{F}^{-\lambda/(1-\lambda)}(0)-(\mu\lambda t/(1-\lambda))}, \tag{91}$$

which implies that the solution blows up in finite time T^*, such that

$$T^* \le \frac{1-\lambda}{\mu\lambda\mathscr{F}^{\lambda/(1-\lambda)}(0)}. \tag{92}$$

This completes the proof of Theorem 10.

5. Proof of Theorem 11

We set

$$\mathscr{H}(t) := -\mathscr{E}(t). \tag{93}$$

To prove our main result, we first establish the following lemma.

Lemma 18. *Let u be the solution of (1). Then, there exists a constant $C>0$ such that*

$$\|\nabla u(t,x)\|_{p(\cdot)} \ge C, \quad \forall t \ge 0. \tag{94}$$

Proof. Suppose that, by contradiction, there exists a sequence t_k such that

$$\|\nabla u(t,x)\|_{p(\cdot)} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{95}$$

Then, by using Lemmas 2 and 5, we get

$$\lim_{k\to\infty} \mathscr{E}(t_k) \ge 0, \tag{96}$$

which contradicts the fact that $\mathscr{E}(t)<0, \forall t \ge 0$.

Using (93) and (94) and applying the same procedures used to prove Theorem 10 will give the proof of Theorem 11.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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