Abstract and Applied Analysis

Stability Analysis Including Monostability and Multistability in Dynamical System and Applications

Guest Editors: Zhenkun Huang, Anke Meyer-Baese, Chunhua Feng, P. Balasubramaniam, Kelin Li, and Eva Kaslik

Stability Analysis Including Monostability and Multistability in Dynamical System and Applications

Abstract and Applied Analysis

Stability Analysis Including Monostability and Multistability in Dynamical System and Applications

Guest Editors: Zhenkun Huang, Anke Meyer-Baese,
Chunhua Feng, P. Balasubramaniam, Kelin Li, and Eva Kaslik

Editorial Board

Contents

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 695653, 2 pages
http://dx.doi.org/10.1155/2014/695653

Editorial

Stability Analysis Including Monostability and Multistability in Dynamical System and Applications

Zhenkun Huang,[1] Anke Meyer-Baese,[2] Chunhua Feng,[3] P. Balasubramaniam,[4] Kelin Li,[5] and Eva Kaslik[6]

[1] *School of Science, Jimei University, Xiamen 361021, China*
[2] *Department of Electrical Engineering, Florida State University, 2525 Pottsdamer Street, Tallahassee, FL 32310-6046, USA*
[3] *Department of Mathematics, Guangxi Normal University, Guangxi 541004, China*
[4] *Department of Mathematics, Gandhigram Rural Institute-Deemed University, Gandhigram, Tamil Nadu 624 302, India*
[5] *Department of Mathematics, Sichuan University of Science & Engineering, Sichuan 643000, China*
[6] *Department of Mathematics and Computer Science, West University of Timisoara, Boulevard Corneliu Coposu, 43300223 Timisoara, Romania*

Correspondence should be addressed to Zhenkun Huang; hzk974226@jmu.edu.cn

Received 3 December 2013; Accepted 3 December 2013; Published 27 January 2014

Stability theory has been significantly developed and extended to dynamical system which will provide a deep insight into a more comprehensive understanding of the dynamic structure of complex dynamical systems and therefore is of great significance for its application. The overall aim of this special issue is to open a discussion among researchers actively working in the field of stability theory and relative applications. The issue covers a wide variety of problems for neural networks and applications such as image processing and pedestrian detection, difference equations, dynamic equations on time scales, biological mathematics including multigroup and reaction-diffusion epidemic model, and soft computing. In the following, we briefly review each of the papers by highlighting the significance of the key contributions.

In "*Periodic oscillation analysis for a coupled FHN network model with delays*" by Y. Lin, the author provides new results of periodic oscillatory behavior of three coupled FHN neurons model by Chafee's criterion of limit cycle. In "*Dynamical behaviors of the stochastic hopfield neural networks with mixed time delays*" by L. Wan et al., the authors of this paper investigate dynamical behaviors of the stochastic Hopfield neural networks with mixed time delays. By employing the theory of stochastic functional differential equations and linear matrix inequality (LMI) approach, some novel criteria on asymptotic stability, ultimate boundedness, and weak attractor are derived.

Two papers in our special issue are devoted to discuss applications of neural networks. In "*Analysis of feature fusion based on HIK SVM and its application for pedestrian detection*" by S.-Z. Su and S.-Y. Chen, the author adopt support vector machine (SVM) with the histogram intersection kernel (HIK) as a classifier to detect pedestrians in low-resolution visual images and at night time. In "*Recursive neural networks based on PSO for image parsing*" by G.-R. Cai and S.-L. Chen, the authors give an image parsing algorithm based on Particle Swarm Optimization (PSO) and Recursive Neural Networks (RNNs).

A theoretical article titled "*$L\omega$-compactness in $L\omega$-spaces*" presents some important properties of $L\omega$-compactness. The authors, S.-L. Chen and J.-L. Huang, reveal the Alexander subbase lemma and the Tychonoff product theorem with respect to $L\omega$-compactness.

Six papers are concerned about dynamical analysis of difference equations or dynamic equations on time scales. In "*Oscillation for higher order dynamic equations on time scales*" by T. Sun et al., the authors present sufficient conditions to ensure every solution of higher order dynamic equations on time scales to be oscillatory or tend to zero. In "*h-Stability for differential systems relative to initial time difference*" by P. Wang and X. Liu, the authors discuss h-stability for differential systems with initial time difference and stability criteria are formulated by using variation of parameter techniques. In "*Periodic solutions of second-order difference problem with potential indefinite in sign*" by H. Bin, the author obtains some new results concerning the existence of nontrivial periodic solution of second-order difference problem with potential indefinite in sign by using Morse theory. In "*Dynamics of a family of nonlinear delay difference equations*" by Q. He et al., the authors give sufficient conditions guaranteeing the globally asymptotical stability of a unique positive equilibrium of nonlinear delay difference equations. In "*Subharmonics with minimal periods for convex discrete hamiltonian systems*" by H. Bin, by using variational methods and dual functional, the author considers the subharmonics with minimal periods for convex discrete Hamiltonian systems. In "*Leader-following consensus stability of discrete-time linear multiagent systems with observer-based protocols*" by B. Xu et al., the authors obtain two types of distributed observer-based consensus protocols to solve the leader-following consensus problem of discrete-time multiagent systems on a directed communication topology.

There are two new results about epidemic models in this special issue.

In "*Stability analysis of a multigroup epidemic model with general exposed distribution and nonlinear incidence rates*" by L. Zhang et al., the authors adopt Lyapunov functionals and a graph-theoretical technique to derive sufficient conditions ensuring the global dynamics. In "*Traveling wave solutions in a reaction-diffusion epidemic model*" by S. Wang et al., the authors investigate a unique and strictly monotonic traveling wave solutions in a reaction-diffusion epidemic model through monotone iteration of a pair of classical upper and lower solutions.

Acknowledgments

The guest editors of this special issue would like to express their thanks to the authors who have submitted papers for consideration and the referees of the submitted papers.

Zhenkun Huang
Anke Meyer-Baese
Chunhua Feng
P. Balasubramaniam
Kelin Li
Eva Kaslik

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 276972, 6 pages
http://dx.doi.org/10.1155/2013/276972

Research Article

Periodic Oscillation Analysis for a Coupled FHN Network Model with Delays

Yuanhua Lin

Department of Mathematics, Hechi University, Yizhou, Guangxi 546300, China

Correspondence should be addressed to Yuanhua Lin; lyh4773@163.com

Received 16 January 2013; Accepted 23 September 2013

Academic Editor: Chunhua Feng

The existence of periodic oscillation for a coupled FHN neural system with delays is investigated. Some criteria to determine the oscillations are given. Simple and practical criteria for selecting the parameters in this network are provided. Some examples are also presented to illustrate the result.

1. Introduction

Recently, several researchers have studied the dynamics of coupled FHN neural systems [1–5]. Wang et al. have investigated the following model [6]:

$$\begin{aligned} v_1'(t) &= -v_1^3(t) + av_1(t) - w_1(t) + c_1 \tanh(v_2(t-\tau)), \\ w_1'(t) &= v_1(t) - b_1 w_1(t), \\ v_2'(t) &= -v_2^3(t) + av_2(t) - w_2(t) + c_2 \tanh(v_1(t-\tau)), \\ w_2'(t) &= v_2(t) - b_2 w_2(t). \end{aligned} \tag{1}$$

The effects of time delay on bifurcation and synchronization in the two synaptically coupled FHN neurons have been investigated. The authors found that time delay can control the occurrence of bifurcation in the coupled FHN neural model and synchronization is sometimes related to bifurcation transition. Fan and Hong introduced second time delay in model (1) as follows [7]:

$$\begin{aligned} x_1'(t) &= -x_1^3(t) + ax_1(t) - x_2(t) + c_1 \tanh(x_3(t-\tau_1)), \\ x_2'(t) &= x_1(t) - b_1 x_2(t), \\ x_3'(t) &= -x_3^3(t) + ax_3(t) - x_4(t) + c_2 \tanh(x_1(t-\tau_2)), \\ x_4'(t) &= x_3(t) - b_2 x_4(t). \end{aligned} \tag{2}$$

Let $\tau = \tau_1 + \tau_2$ be a parameter. The authors have shown that there is a critical value of the parameter; the steady state of model (2) is stable when the parameter is less than the critical value and unstable when the parameter is greater than the critical value. Thus, the zero equilibrium loses its stability when the parameter passes through the critical value, and a Hopf bifurcation occurs and oscillations induced by the Hopf bifurcation appeared. Zhen and Xu generated models (1) and (2) to a three coupled FHN neurons network with time delay as follows [8]:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + cu_1^2 + du_1 - u_2 + \alpha u_1^2 \\ &\quad + \beta\left[f(u_3(t-\tau)) + f(u_5(t-\tau))\right], \\ u_2' &= \varepsilon(u_1 - bu_2), \\ u_3' &= -\frac{1}{3}u_3^3 + cu_3^2 + du_3 - u_4 + \alpha u_3^2 \\ &\quad + \beta\left[f(u_1(t-\tau)) + f(u_5(t-\tau))\right], \\ u_4' &= \varepsilon(u_3 - bu_4), \end{aligned}$$

$$\begin{aligned} u_5' &= -\frac{1}{3}u_5^3 + cu_5^2 + du_5 - u_6 + \alpha u_5^2 \\ &\quad + \beta\left[f\left(u_1\left(t-\tau\right)\right) + f\left(u_3\left(t-\tau\right)\right)\right], \\ u_6' &= \varepsilon\left(u_5 - bu_6\right), \end{aligned} \tag{3}$$

where α, β represent the synaptic strength of self-connection and neighborhood interaction, respectively, and $f(x)$ is a sufficiently smooth sigmoid amplification function such as $\tanh(x)$ and $\arctan(x)$. The method of Lyapunov functional is used to obtain the synchronization conditions of the neural system. Noting that, for each neuron of model (3), the synaptic strength of self-connection and neighborhoodinteraction are the same under the same restrictive condition, the dynamics of (3) are completely characterized by the following system:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + (c+\alpha)u_1^2 + du_1 - u_2 \\ &\quad + 2\beta f\left(u_1\left(t-\tau\right)\right), \\ u_2' &= \varepsilon\left(u_1 - bu_2\right), \end{aligned} \tag{4}$$

where $[u_1, u_2]^T$ is a completely synchronous solution of system (4). The Bautin bifurcation of synchronous solution for this neural system (4) in which α, β are regarded as the bifurcating parameters is investigated. However, generally speaking, the synaptic strength of self-connection, neighborhoodinteraction for each neuron, and the time delays are different. Therefore, in this paper, we will discuss the following model:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + c_1u_1^2 + d_1u_1 - u_2 + \alpha_1u_1^2 \\ &\quad + \beta_1\left[f\left(u_3\left(t-\tau_3\right)\right) + f\left(u_5\left(t-\tau_5\right)\right)\right], \\ u_2' &= \varepsilon_1\left(u_1 - b_1u_2\right), \\ u_3' &= -\frac{1}{3}u_3^3 + c_2u_3^2 + d_2u_3 - u_4 + \alpha_2u_3^2 \\ &\quad + \beta_2\left[f\left(u_1\left(t-\tau_1\right)\right) + f\left(u_5\left(t-\tau_5\right)\right)\right], \\ u_4' &= \varepsilon_2\left(u_3 - b_2u_4\right), \\ u_5' &= -\frac{1}{3}u_5^3 + c_3u_5^2 + d_3u_5 - u_6 + \alpha_3u_5^2 \\ &\quad + \beta_3\left[f\left(u_1\left(t-\tau_1\right)\right) + f\left(u_3\left(t-\tau_3\right)\right)\right], \\ u_6' &= \varepsilon_3\left(u_5 - b_3u_6\right), \end{aligned} \tag{5}$$

where $b_i, c_i, d_i, \alpha_i, \beta_i, \varepsilon_i$ $(i = 1, 2, 3)$ are constants. $\tau_j > 0$ $(j = 1, 3, 5)$ represent the time delays in signal transmission. System (5) can be rewritten as follows:

$$\begin{aligned} u_1' &= \left[d_1 + \left(\alpha_1 + c_1\right)u_1 - \frac{1}{3}u_1^2\right]u_1 - u_2 \\ &\quad + \beta_1\left[f\left(u_3\left(t-\tau_3\right)\right) + f\left(u_5\left(t-\tau_5\right)\right)\right], \\ u_2' &= \varepsilon_1u_1 - \varepsilon_1b_1u_2, \\ u_3' &= \left[d_2 + \left(\alpha_2 + c_2\right)u_3 - \frac{1}{3}u_3^2\right]u_3 - u_4 \\ &\quad + \beta_2\left[f\left(u_1\left(t-\tau_1\right)\right) + f\left(u_5\left(t-\tau_5\right)\right)\right], \\ u_4' &= \varepsilon_2u_3 - \varepsilon_2b_2u_4, \\ u_5' &= \left[d_3 + \left(\alpha_3 + c_3\right)u_5 - \frac{1}{3}u_5^2\right]u_5 - u_6 \\ &\quad + \beta_3\left[f\left(u_1\left(t-\tau_1\right)\right) + f\left(u_3\left(t-\tau_3\right)\right)\right], \\ u_6' &= \varepsilon_3u_5 - \varepsilon_3b_3u_6. \end{aligned} \tag{6}$$

It is known that if all solutions of system (6) are bounded and there exists a unique unstable equilibrium point of system (6), then this particular instability will force system (6) to generate a limit cycle, namely, a periodic oscillation [9]. We will provide some restrictive conditions which are easy to check to ensure the existence of periodic oscillation. It was pointed out that bifurcating method to determine the periodic solution of system (6) is very difficult.

In the following, we first assume that $f(u_i(t-\tau_i))$ $(i = 1, 3, 5)$ are continuous bounded monotone increasing functions, satisfying

$$\lim_{u_i \to 0} \frac{f\left(u_i(t)\right)}{u_i(t)} = \gamma_i\,(>0), \quad i = 1, 3, 5; \; f(0) = 0. \tag{7}$$

For example, activation functions $\tanh(u_i(t))$, $\arctan(u_i(t))$, and $(1/2)(|u_i(t)+1| - |u_i(t)-1|)$ satisfy condition (7). From assumption (7), the linearization of system (6) about the zero point leads to the following:

$$\begin{aligned} u_1' &= d_1u_1 - u_2 \\ &\quad + \beta_1\left[\gamma_3u_3\left(t-\tau_3\right) + \gamma_5u_5\left(t-\tau_5\right)\right], \\ u_2' &= \varepsilon_1u_1 - \varepsilon_1b_1u_2, \\ u_3' &= d_2u_3 - u_4 \\ &\quad + \beta_2\left[\gamma_1u_1\left(t-\tau_1\right) + \gamma_5u_5\left(t-\tau_5\right)\right], \\ u_4' &= \varepsilon_2u_3 - \varepsilon_2b_2u_4, \\ u_5' &= d_3u_5 - u_6 \\ &\quad + \beta_3\left[\gamma_1u_1\left(t-\tau_1\right) + \gamma_3u_3\left(t-\tau_3\right)\right], \\ u_6' &= \varepsilon_3u_5 - \varepsilon_3b_3u_6. \end{aligned} \tag{8}$$

The matrix form of system (8) is as follows:

$$U'(t) = AU(t) + BU(t-\tau), \tag{9}$$

where $U(t) = (u_1(t), u_2(t), \ldots, u_6(t))^T$, $U(t-\tau) = (u_1(t-\tau_1), 0, u_3(t-\tau_3), 0, u_5(t-\tau_5), 0)^T$,

$$A=\begin{pmatrix} d_1 & -1 & 0 & 0 & 0 & 0\\ \varepsilon_1 & -\varepsilon_1 b_1 & 0 & 0 & 0 & 0\\ 0 & 0 & d_2 & -1 & 0 & 0\\ 0 & 0 & \varepsilon_2 & -\varepsilon_2 b_2 & 0 & 0\\ 0 & 0 & 0 & 0 & d_3 & -1\\ 0 & 0 & 0 & 0 & \varepsilon_3 & -\varepsilon_3 b_3 \end{pmatrix},$$
$$B=\begin{pmatrix} 0 & 0 & \beta_1\gamma_3 & 0 & \beta_1\gamma_5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \beta_2\gamma_1 & 0 & 0 & 0 & \beta_2\gamma_5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \beta_3\gamma_1 & 0 & \beta_3\gamma_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

2. Preliminaries

Lemma 1. *Suppose that $b_i > 0$, $0 < \varepsilon_i \ll 1$, $d_i < 0$, $(\alpha_i + c_i)^2 + (4/3)d_i < 0$ $(i = 1,2,3)$; then each solution of system* (6) *is bounded.*

Proof. Note that the activation functions are bounded continuous nonlinear functions. Therefore, there exist $N_j > 0$ such that $|f(u_j(t-\tau_j))| \le N_j$ $(j = 1,3,5)$. Since $d_i < 0$, $(\alpha_i + c_i)^2 + (4/3)d_i < 0$ $(i = 1,2,3)$, this implies that there exist constants $k_i > 0$ such that for any values u_i we have $d_i+(\alpha_i + c_i)u_i - (1/3)u_i^2 \le -k_i < 0$ $(i = 1,2,3)$. From (6) we get

$$\begin{aligned}
\frac{d|u_1(t)|}{dt} &\le -k_1|u_1| + |u_2| + |\beta_1|(N_3+N_5),\\
\frac{d|u_2(t)|}{dt} &\le \varepsilon_1|u_1| - \varepsilon_1 b_1|u_2|,\\
\frac{d|u_3(t)|}{dt} &\le -k_2|u_3| + |u_4| + |\beta_2|(N_1+N_5),\\
\frac{d|u_4(t)|}{dt} &\le \varepsilon_2|u_3| - \varepsilon_2 b_2|u_4|,\\
\frac{d|u_5(t)|}{dt} &\le -k_3|u_5| + |u_6| + |\beta_3|(N_1+N_3),\\
\frac{d|u_6(t)|}{dt} &\le \varepsilon_3|u_5| - \varepsilon_3 b_3|u_6|.
\end{aligned} \tag{11}$$

Noting that system (11) is the first-order linear system of equations with constant coefficients, the eigenvalues of system (11) are $\lambda_{i1,i2} = (-(k_i + \varepsilon_i b_i) \pm \sqrt{(k_i+\varepsilon_i b_i)^2 - 4\varepsilon_i(k_i b_i + 1)})/2$ $(i = 1,2,3)$. Since $k_i > 0$, $\varepsilon_i > 0$, $b_i > 0$ $(i = 1,2,3)$, $\lambda_{i1,i2} < 0$ if $(k_i+\varepsilon_i b_i)^2 - 4\varepsilon_i(k_i b_i+1) > 0$ or $\lambda_{i1,i2}$ are complex numbers with $\mathrm{Re}\,\lambda_{i1,i2} < 0$ if $(k_i+\varepsilon_i b_i)^2 - 4\varepsilon_i(k_i b_i + 1) < 0$ $(i = 1,2,3)$. This implies that all solutions of system (11), as well as the system, (6) are bounded according to the theory of the first-order linear system of equations with constant coefficients.

According to [10], there is the same oscillatory behavior for systems (8) and (6). So, in order to investigate the periodic oscillatory behavior of system (6), we only need to deal with system (8). □

Lemma 2. *Suppose that matrix C (= $A + B$) is a nonsingular matrix. Then, system* (9) *has a unique equilibrium point.*

Proof. An equilibrium point $u^* = [u_1^*, u_2^*, \ldots, u_6^*]^T$ is the solution of the following algebraic equation:

$$AU^* + BU^* = (A+B)U^* = 0. \tag{12}$$

Assume that U^* and V^* are two equilibrium points of system (9); then we have

$$(A+B)(U^* - V^*) = C(U^* - V^*) = 0. \tag{13}$$

Since C is a nonsingular matrix, implying that $U^* - V^* = 0$ and $U^* = V^*$ system (9) has a unique equilibrium point. Obviously, this equilibrium point is exactly the zero point. □

3. Periodic Oscillation

Theorem 3. *Suppose that $b_i > 0$, $0 < \varepsilon_i \ll 1$, $d_i < 0$, $(\alpha_i + c_i)^2 + (4/3)d_i < 0$ $(i = 1,2,3)$, and C is a nonsingular matrix. Let $\tau_* = \min\{\tau_1,\tau_3,\tau_5\}$, $\tau^* = \max\{\tau_1,\tau_3,\tau_5\}$, $\rho_1,\rho_2,\ldots,\rho_6$, and $\omega_1,\omega_2,\ldots,\omega_6$ denote the eigenvalues of matrices A and B, respectively. Assume that there is at least one $\rho_i > 0$, $i \in (1,2,\ldots,6)$, and the following inequalities hold:*

$$|\omega_i|\,\tau_* e^{-\rho_i\tau_*} > 1, \qquad |\omega_i|\,\tau^* e^{-\rho_i\tau^*} > 1. \tag{14}$$

Then, the trivial solution of system (8) *is unstable, implying that there is a periodic oscillatory solution of system* (6).

Proof. From the assumptions, we know that system (8) has a unique equilibrium point and all solutions are bounded. We will prove that the unique equilibrium point is unstable. We first discuss the case that $\tau_1 = \tau_3 = \tau_5 = \tau_*$ in system (8). The characteristic equation of system (8) is as follows:

$$\det\left(\lambda I - A - Be^{-\lambda\tau_*}\right) = 0. \tag{15}$$

Equation (15) is equal to

$$\prod_{k=1}^{6}\left(\lambda-\rho_k-\omega_k e^{-\lambda\tau_*}\right)=0. \tag{16}$$

Therefore, we are led to an investigation of the nature of the roots for the following equations:

$$\lambda-\rho_k-\omega_k e^{-\lambda\tau_*}=0, \quad k=1,2,\ldots,6. \tag{17}$$

For some $\rho_i>0$, consider equation

$$\lambda-\rho_i-\omega_i e^{-\lambda\tau_*}=0. \tag{18}$$

If $\lambda<0$ is a solution of (18), then $|\lambda|=-\lambda$; from (18) we have

$$|\lambda|\geq|\omega_i|\,e^{|\lambda|\tau_*}-\rho_i. \tag{19}$$

Using the formula $ex\leq e^{x}(x>0)$, leads to the fact that

$$\begin{aligned}1\geq\ &\frac{|\omega_i|\,e^{|\lambda|\tau_*}}{|\lambda|+\rho_i}=\frac{|\omega_i|\,\tau_* e^{-\rho_i\tau_*}\cdot e^{(|\lambda|+\rho_i)\tau_*}}{\left(|\lambda|+\rho_i\right)\tau_*}\\ &\geq|\omega_i|\,\tau_* e^{-\rho_i\tau_*}.\end{aligned} \tag{20}$$

Equation (20) contradicts the first inequality of assumption (14). Then, we discuss the case that $\tau_1=\tau_3=\tau_5=\tau^*$ in system (8). Similarly, if $\lambda<0$ is a solution of the equation $\lambda-\rho_i-\omega_i e^{-\lambda\tau^*}=0$, we also have a contradiction with the second inequality of assumption (14). Since $\tau_*\leq\tau_i\leq\tau^*$ $(i=1,3,5)$, one can conclude that there exists a positive real part of the eigenvalue of system (8) for any τ_i $(i=1,3,5)$ under the assumptions. This means that the trivial solution of system (8) is unstable, implying that there is a periodic oscillatory solution of system (6) based on Chafee's criterion. □

Theorem 4. *Suppose that $b_i>0$, $0<\varepsilon_i\ll 1$, $d_i<0$, $(\alpha_i+c_i)^2+(4/3)d_i<0$ $(i=1,2,3)$, and C is a nonsingular matrix. Let $\rho_k=\rho_{k1}+i\rho_{k2}$ (ρ_{k2} may equal zero) and $\omega_k=\omega_{k1}+i\omega_{k2}$ (ω_{k2} may equal zero) $(k=1,2,\ldots,6)$ denote the eigenvalues of matrices A and B, respectively. If, for some ρ_i, $|\rho_{i1}|<\omega_{i1}$ as $\rho_{i1}<0$, then the trivial solution of system (8) is unstable, implying that system (6) has a periodic oscillatory solution.*

Proof. The assumptions guarantee that system (8) has a unique equilibrium point and all solutions are bounded. In this case, we first consider $\tau_1=\tau_3=\tau_5=\tau_*$ in system (8). Then, for some ρ_i, let $\lambda=\lambda_1+i\lambda_2$; from (18) we have

$$\begin{aligned}&\lambda_1-\rho_{i1}-\omega_{i1}e^{-\lambda_1\tau_*}\cos\left(\lambda_2\tau_*\right)=0,\\ &\lambda_2-\rho_{i2}+\omega_{i2}e^{-\lambda_1\tau_*}\sin\left(\lambda_2\tau_*\right)=0.\end{aligned} \tag{21}$$

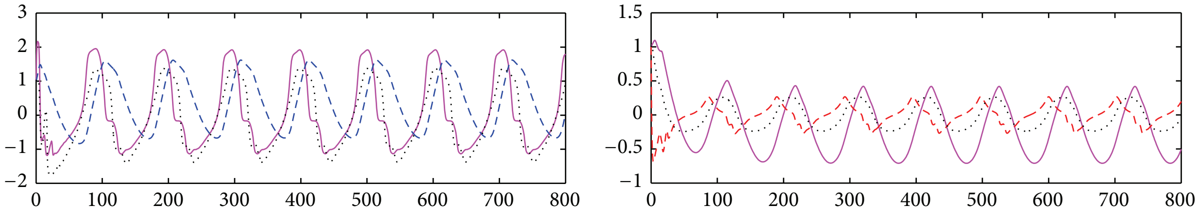
We will show that $\lambda_1>0$ and there is an eigenvalue which has positive real part of system (18). Let $f(\lambda_1)=\lambda_1-\rho_{i1}-\omega_{i1}e^{-\lambda_1\tau_*}\cos(\lambda_2\tau_*)$; then $f(\lambda_1)$ is a continuous function of λ_1. If $\rho_{i1}>0$, then select suitable delay τ_* such that $\omega_{i1}\cos(\lambda_2\tau_*)>-\rho_{i1}$. Therefore, $f(0)=-\rho_{i1}-\omega_{i1}\cos(\lambda_2\tau_*)<0$. Noting that $e^{-\lambda_1\tau_*}\to 0$ as $\lambda_1\to+\infty$, obviously, there exists a suitably large $\lambda_1(>0)$ such that $f(\lambda_1)=\lambda_1-\rho_{i1}-\omega_{i1}e^{-\lambda_1\tau_*}\cos(\lambda_2\tau_*)>0$. By the continuity of $f(\lambda_1)$, there exists a positive $\lambda_1{}^*\in(0,\lambda_1)$ such that $f(\lambda_1{}^*)=0$. If $\rho_{i1}<0$, since $|\rho_{i1}|<\omega_{i1}$ $(\omega_{i1}\neq 0)$, then there exists a suitable delay τ_* and a positive $\overline{\lambda}_1$ such that $\omega_{i1}\cos(\lambda_2\tau_*)<-\rho_{i1}$ and $\overline{\lambda}_1-\omega_{i1}e^{-\overline{\lambda}_1\tau_*}\cos(\lambda_2\tau_*)<0$ both hold. Then, $f(0)=-\rho_{i1}-\omega_{i1}\cos(\lambda_2\tau_*)>0$ and $f(\overline{\lambda}_1)=\overline{\lambda}_1-\omega_{i1}e^{-\overline{\lambda}_1\tau_*}\cos(\lambda_2\tau_*)<0$. Again, from the continuity of $f(\lambda_1)$, there exists a positive $\lambda_1{}^{**}\in(0,\overline{\lambda}_1)$ such that $f(\lambda_1{}^{**})=0$. Thus, there is an eigenvalue of system (18) that has positive real part. Implying that the trivial solution of system (8) is unstable. Thus, the trivial solution of system (6) is also unstable. Based on the theory of delay differential equation, the oscillatory behavior of the solution will maintain as time delay increasing. Therefore, for any $\tau_i\geq\tau_*$ $(i=1,3,5)$, system (8), as well as system, (6) generates a periodic oscillatory solution. We select a suitable delay τ_* such that system (6) has a periodic oscillatory solution. This oscillation is said to be induced by time delay. □

4. Simulation Result

The parameter values are selected as $\alpha_1=-1.5$, $\alpha_2=-1.5$, $\alpha_3=-1.2$; $b_1=0.16$, $b_2=0.25$, $b_3=0.12$; $c_1=1.3$, $c_2=1.302$, $c_3=1.305$; $d_1=-0.705$, $d_2=-0.706$, $d_3=-0.707$; $\beta_1=1.5$, $\beta_2=1.5$, $\beta_3=0.15$; $\varepsilon_1=0.05$, $\varepsilon_2=0.025$, $\varepsilon_3=0.085$, respectively. It is easy to check that the conditions of Lemmas 1 and 2 hold. The activation functions are selected as $\arctan(u)$ and $\tanh(u)$, respectively. In this case, $\gamma_1=\gamma_3=\gamma_5=1$, and eigenvalues of matrices A and B are $\rho_1=-0.6238$, $\rho_2=-0.0892$, $\rho_3=-0.6682$, $\rho_4=-0.0440$, $\rho_5=-0.5493$, and $\rho_6=-0.1679$, and $\omega_1=1.7562$, $\omega_2=-1.5000$, $\omega_3=-0.2562$, $\omega_4=0$, and $\omega_5=0$, $\omega_6=0$, respectively. Since $|\rho_1|=0.6238<\omega_1$, there is a periodic oscillatory solution based on Theorem 4. Both in Figures 1 and 2, the time delays are selected as $\tau_1=10$, $\tau_2=8$, and $\tau_3=4$. Then, we change delays as $\tau_1=1$, $\tau_2=2$, $\tau_3=3$; activation function is kept as $\tanh(u)$; periodic oscillatory solution also occurred (Figure 3). In Figure 4, the parameter values are selected as $\alpha_1=-0.95$, $\alpha_2=-1.2$, $\alpha_3=-1.25$; $b_1=0.18$, $b_2=0.2$, $b_3=0.16$; $c_1=1.4$, $c_2=1.42$, $c_3=1.45$; $d_1=-0.7$, $d_2=-0.72$, $d_3=-0.75$; $\beta_1=1.25$, $\beta_2=1.2$, $\beta_3=1.15$; $\varepsilon_1=0.05$, $\varepsilon_2=0.045$, and $\varepsilon_3=0.065$, respectively. The activation function is $\tanh(u)$. The eigenvalues of matrices A and B are $\rho_1=-0.6179$, $\rho_2=-0.0911$, $\rho_3=-0.6498$, $\rho_4=-0.0792$, $\rho_5=-0.6481$, $\rho_6=-0.1123$ and $\omega_1=2.3654$, $\omega_2=-1.2154$, $\omega_3=-1.1500$, $\omega_4=0$, $\omega_5=0$, and $\omega_6=0$, respectively. We see that periodic oscillatory solution appeared.

5. Conclusion

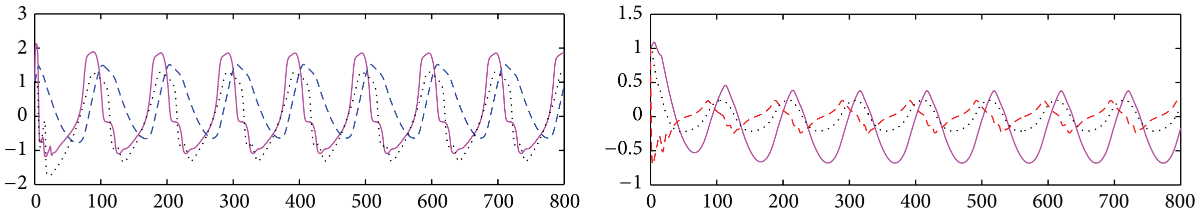
This paper discusses a three coupled FHN neurons model in which the synaptic strength of self-connection, neighborhoodinteraction for each neuron, and the time delays are



(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$

(b) Solid line: $u_4(t)$, dashed line: $u_5(t)$, dotted line: $u_6(t)$

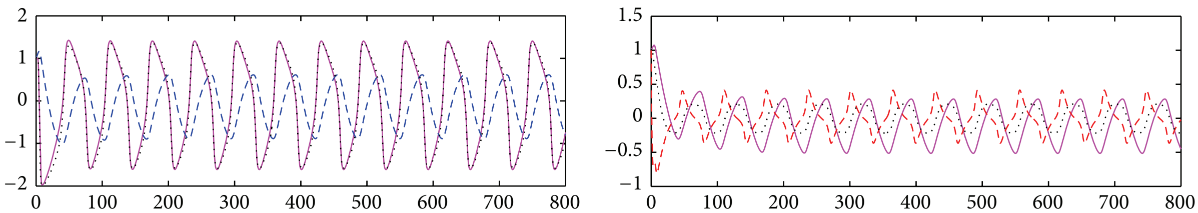
FIGURE 1: Periodic oscillatory behavior, activation function: arctan(u), and delays: (10, 8, 4).



(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$

(b) Solid line: $u_4(t)$, dashed line: $u_5(t)$, dotted line: $u_6(t)$

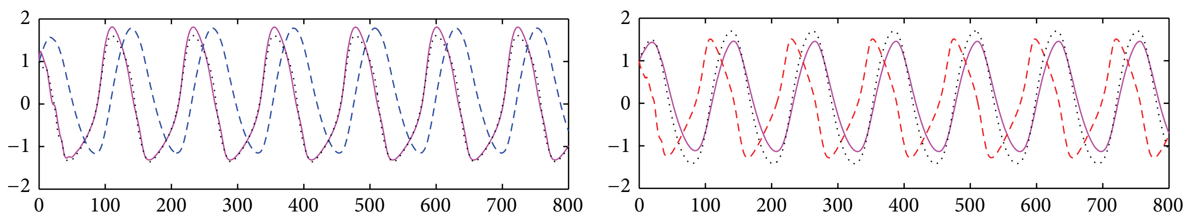
FIGURE 2: Periodic oscillatory behavior, activation function: tanh(u), and delays: (10, 8, 4).



(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$

(b) Solid line: $u_4(t)$, dashed line: $u_5(t)$, dotted line: $u_6(t)$

FIGURE 3: Periodic oscillatory behavior, activation function: tanh(u), and delays: (1, 2, 3).



(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$

(b) Solid line: $u_4(t)$, dashed line: $u_5(t)$, dotted line: $u_6(t)$

FIGURE 4: Periodic oscillatory behavior, activation function: tanh(u), and delays: (9, 10, 12).

different. Two theorems are provided to determine the periodic oscillatory behavior of the solutions based on Chafee's criterion of limit cycle. Computer simulation suggested that those theorems only are sufficient conditions.

Acknowledgments

This research was supported by NNSF of China (11361010) and Scientific Research Foundation of the Education Department of Guangxi Province (201010LX463).

References

[1] Q. Wang, Q. Lu, G. Chen, Z. Feng, and L. Duan, "Bifurcation and synchronization of synaptically coupled FHN models with time delay," *Chaos, Solitons & Fractals*, vol. 49, pp. 918–925, 2009.

[2] D. Fan and L. Hong, "Hopf bifurcation analysis in a synaptically coupled FHN neuron model with delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 7, pp. 1873–1886, 2010.

[3] U. Tetsushi, M. Hisayo, K. Takuji, and K. Hiroshi, "Bifurcation and chaos in coupled BVP oscillators," *International Journal of Bifurcation and Chaos*, vol. 4, no. 14, pp. 1305–1324, 2004.

[4] U. Tetsushi and K. Hiroshi, "Bifurcation in asymmetrically coupled BVP oscillators," *International Journal of Bifurcation and Chaos*, vol. 5, no. 13, pp. 1319–1327, 2003.

[5] T. Kunichika, A. Kazuyuki, and K. Hiroshi, "Bifurcations in synaptically coupled BVP neurons," *International Journal of Bifurcation and Chaos*, vol. 4, no. 11, pp. 1053–1064, 2001.

[6] Q. Wang, Q. Lu, G. Chen, Z. Feng, and L. Duan, "Bifurcation and synchronization of synaptically coupled FHN models with time delay," *Chaos, Solitons & Fractals*, vol. 39, pp. 918–925, 2009.

[7] D. Fan and L. Hong, "Hopf bifurcation analysis in a synaptically coupled FHN neuron model with delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 7, pp. 1873–1886, 2010.

[8] B. Zhen and J. Xu, "Bautin bifurcation analysis for synchronous solution of a coupled FHN neural system with delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 2, pp. 442–458, 2010.

[9] N. Chafee, "A bifurcation problem for a functional differential equation of finitely retarded type," *Journal of Mathematical Analysis and Applications*, vol. 35, pp. 312–348, 1971.

[10] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford, UK, 1991.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 357971, 10 pages
http://dx.doi.org/10.1155/2013/357971

Research Article

Leader-Following Consensus Stability of Discrete-Time Linear Multiagent Systems with Observer-Based Protocols

Bingbing Xu,[1] Lixin Gao,[1] Yan Zhang,[1] and Xiaole Xu[2]

[1] *Institute of Intelligent Systems and Decision, Wenzhou University, Zhejiang 325035, China*
[2] *Wenzhou Vocational College of Science & Technology, Zhejiang 325006, China*

Correspondence should be addressed to Lixin Gao; gao-lixin@163.com

Received 16 February 2013; Accepted 26 August 2013

Academic Editor: Pagavathi Balasubramaniam

We consider the leader-following consensus problem of discrete-time multiagent systems on a directed communication topology. Two types of distributed observer-based consensus protocols are considered to solve such a problem. The observers involved in the proposed protocols include full-order observer and reduced-order observer, which are used to reconstruct the state variables. Two algorithms are provided to construct the consensus protocols, which are based on the modified discrete-time algebraic Riccati equation and Sylvester equation. In light of graph and matrix theory, some consensus conditions are established. Finally, a numerical example is provided to illustrate the obtained result.

1. Introduction

In recent decades, the cooperate and control problem of distributed dynamic systems has been a challenging research field, owing to its widespread applications in many areas such as swarm of animals [1], collective motion of particles [2], schooling for underwater vehicles [3, 4], neural networks [5, 6], and distributed sensor networks [7].

The consensus problem, as one fundamental problem for coordinated control of multiagent systems, has gained significant attention from different research communities. Consensus problem considers how to design an information interaction protocol between agents and requires all agents to converge to a common value [8, 9]. Based on matrix theory, algebraic graph theory, and control theory, many researchers have acquired abundant results in studying consensus problem of multiagent systems. In [10], the authors proposed a general framework for consensus problem in fixed and switching networks and gave solution to the case with communication time delays. Olfati-Saber et al. established a general model for consensus problems of the multiagent systems and introduced Lyapunov method to reveal the contract with the connectivity of the graph theory and the stability of the system in [11]. Sometimes, it is better to consider a tracking consensus problem by adding a leader which can make all agents reach a command trajectory with any initial condition [12]. The leader-following consensus problem has been addressed in many references [13–17].

Many proposed distributed consensus protocols need to know neighbors' state information, but it may be difficult to measure this information. To make the system achieve consensus, it often contains an observer in the control protocol, which is used to estimate those unmeasurable state variables. The distributed observer-based control laws were proposed to solve first-order and second-order multiagent consensus problems in [12, 17]. To estimate the general active leader's unmeasurable state variables, [18] proposed a distributed algorithm for first-order agent, and [19] extended the results of [18] to the time-delay case. The distributed observer-based consensus protocols were addressed to solve multiagent consensus with general linear or linearized agent dynamics in [17, 20–24]. In [25], the author proposed an observer-type consensus protocol to the consensus problem for a class of fractional-order uncertain multiagent systems with general linear dynamics. In [26], the authors proposed distributed reduced-order observer-based protocols to solve consensus problem, which were generalized to solve leader-following consensus problem under switching topology by [27].

The observer-based consensus protocol can be viewed as a special case of the dynamic compensation method, which has been investigated by [28–30].

Discrete-time dynamic systems are commonly involved in the neural network, sampled control, signal filters, and state estimators. The discrete-time neural network was studied by [31–33]. The sampled-data discrete-time coordination of multiagent systems was investigated in [16, 34, 35]. The first-order discrete-time consensus has been investigated by [8, 9, 36–38]. In [39], the authors discussed discrete-time second-order consensus protocols for dynamics with nonuniform and time-varying communication delays under dynamically switching topology. The distributed H_∞ consensus problem was studied in [30] to solve multiagent consensus problem with discrete-time high-dimensional linear coupling dynamics subjected to external disturbances. The distributed state-feedback protocols for linear discrete-time multiagent were proposed in [40, 41]. The distributed observer-based protocol was proposed to solve leader-following consensus problem with linear discrete-time dynamics in [23, 42, 43].

Motivated by the above works, we focus our research on a group of agents with discrete-time high-dimensional linear coupling dynamics and directed interaction topology. We propose distributed observer-based protocols for leader-following multiagent systems. The full-order observer and reduced-order observer are adopted to reconstruct the state variables. Contrary to [23] and [40], the gain matrix design approach used in this paper is based on the modified discrete-time algebraic Riccati equations (MDARE) but not the normal discrete-time algebraic Riccati equations. The proposed design method must be feasible if spectral radius of system matrix is not greater than 1. Of course, the proposed design method can be used to construct the consensus protocols provided by [23] and [40]. Further, the separation principle is shown to be valid, from which we can establish consensus condition for closed-loop multiagent systems.

This paper is organized as follows. Section 2 presents the related notations and the problem formulated with graph theory. In Section 3, the distributed state feedback design is considered. In Sections 4 and 5, the distributed full-order and reduced-order observer-based consensus protocols are proposed, respectively, which are the main results of this paper. Section 6 presents a simulation example to illustrate our established results. Finally, the conclusion is given in Section 7.

2. Preliminaries and Problem Formulation

2.1. Notations and Graph Theory. $\mathrm{Re}(\xi)$ denotes the real part of $\xi \in C$. Let $R^{m\times n}$ and $C^{m\times n}$ be the set of $m \times n$ real matrices and complex matrices, respectively. $\mathbf{1}_n \in R^n$ is the column vector with all components equal to one. Let I be the identity matrix with compatible dimension. For a given matrix A, a_{ij} represents its element of ith row and jth column, A^T denotes its transpose, and A^H denotes its conjugate transpose. A matrix is said to be Schur-stable if all its eigenvalues are inside unit circle. $\rho(A)$ represents the spectral radius of matrix A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent its maximum and minimum eigenvalues of symmetric matrix A, respectively. For symmetric matrices A and B, $A > B$ means that $A - B$ is positive definite, that is, $A - B > 0$. $\otimes$ denotes Kronecker product, which satisfies $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$.

We describe the interaction relationship among n agents by a simple weighted diagraph $\mathcal{G} = \{\mathcal{V}, \varepsilon, W\}$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is the set of vertices and $\varepsilon \subset \mathcal{V}\times\mathcal{V}$ is the set of edges. If $(v_i, v_j) \in \varepsilon$, the vertex v_j is called a neighbor of vertex v_i, and the index set of neighbors of vertex v_i is denoted by $\mathcal{N}_i = \{j \mid (v_i, v_j) \in \varepsilon\}$. $W = [w_{ij}]_{n\times n}$ represents weighted adjacency matrix associated with graph $\mathcal{G}$, where $w_{ij} > 0$ if $(v_i, v_j) \in \varepsilon$ and $w_{ij} = 0$ otherwise. The degree matrix $D = \mathrm{diag}\{d_1, d_2, \dots, d_n\}$ of digraph $\mathcal{G}$ is a diagonal matrix with diagonal elements $d_i = \sum_{j=1}^{n} w_{ij}$. Then, the Laplacian matrix of $\mathcal{G}$ is defined as $L = D - W$. v_i is called globally reachable node if there exists at least a directed path from every other node to node v_i in digraph $\mathcal{G}$. A directed graph $\mathcal{G}$ has a globally reachable node if and only if there exists a directed spanning tree in $\mathcal{G}$ (see [9]).

For a multiagent system with leader (labeled as 0), the interaction topology is expressed by graph $\widehat{\mathcal{G}}$, which contains graph $\mathcal{G}$ and vertex v_0 and edges from other vertices to vertex v_0. Let g_i, $i = 1, 2, \dots, n$, be weight of (v_i, v_0). $g_i > 0$ if (v_i, v_0) is an edge of graph $\widehat{G}$ and $g_i = 0$ otherwise. Let $G_d = \mathrm{diag}\{g_1, g_2, \dots, g_n\}$. The matrix $L + G_d$ has the following property.

Lemma 1 (see [13]). *Matrix $L+G_d$ is positive stable if and only if graph $\widehat{\mathcal{G}}$ has a directed spanning tree with root v_0.*

2.2. Problem Formulation. Consider the multiagent system which is composed of n identical following agents and a leader. Each following agent has dynamics modeled by the discrete-time linear system:

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k), \\ y_i(k) &= Cx_i(k), \end{aligned} \tag{1}$$

where $x_i(k) \in R^m$, $u_i(k) \in R^p$, and $y_i(k) \in R^q$ are, respectively, the state variable, control input, and measured output of agent i.

The dynamics of the leader is given as

$$\begin{aligned} x_0(k+1) &= Ax_0(k), \\ y_0(k) &= Cx_0(k), \end{aligned} \tag{2}$$

where $x_0(k)$ is the state and $y_0(k)$ is the measured output of the leader. The leaderless consensus problem for multiagent system has been investigated by [26, 28, 44], which require the system matrix A to be Schur-stable. There is not such requirement to A in this paper. A, B, C are constant matrices with compatible dimensions. It is assumed that (A, B, and C) is stabilizable and detectable.

The $x_0(k)$ is often called as "consensus reference state" and assumed to be available only to a subgroup of the followers. The main objective of leader-following consensus problem is to design distributed consensus protocol to make multiagent system achieve consensus.

Definition 2. The leader-following multiagent system is said to achieve consensus if the state variables of all following agents satisfy $\lim_{k\to\infty}(x_i(k) - x_0(k)) = 0$, $i = 1, 2, \ldots, n$ for any initial state. One says that the protocol $u_i(k)$ can solve the leader-following consensus problem if the closed-loop system achieves consensus.

2.3. Preliminary Results. In this subsection, we introduce some preliminary results which will be used to establish our main results. Consider the following MDARE:

$$A^T PA - P - \delta A^T PB\left(I + B^T PB\right)^{-1} B^T PA + Q = 0, \quad (3)$$

where Q is any given positive definite matrix. Since Q is positive definite, $(A, Q^{1/2})$ must be detectable. The solvability of the MDARE is addressed by the following lemma.

Lemma 3 (see [45, 46]). *If $(A, Q^{1/2})$ is detectable, (A, B) is stabilizable, then there exists a $\delta_c \in [0, 1)$ such that the modified discrete time algebraic Riccati equation* (3) *has a unique positive-definite solution P for any $\delta_c < \delta \le 1$. Furthermore, $P = \lim_{k\to\infty} P_k$ for any initial condition $P_0 \ge 0$, where P_k satisfies*

$$P_{k+1} = A^T P_k A - \delta A^T P_k B\left(I + B^T P_k B\right)^{-1} B^T P_k A + Q. \quad (4)$$

Remark 4. The MDARE (3) is reduced, respectively, to the well-known discrete-time Riccati equation (DARE) and Stain equation as $\delta = 1$ and $\delta = 0$. The Stain equation has a unique positive-definite solution if A is Schur-stable. It is well known that DARE has a unique positive-definite solution if (A, B) is stablizable. If the involved matrix A is not Schur-stable, it is easy to see that $0 < \delta_c \le 1$. More details for issue δ_c can be referenced to [45]. Moreover, if the matrix A has no eigenvalues with magnitude larger than 1 and (A, C) is detectable, MDARE (3) has a unique positive-definite solution P for any δ satisfying $0 < \delta \le 1$.

Lemma 5. *For a given δ satisfying $\delta_c < \delta \le 1$, let P be the unique positive-definite solution of the MDARE* (3)*. Choose a feedback matrix $K = (I + B^T PB)^{-1} B^T PA$. Then, $A - sBK$ is Schur-stable for any $s \in \overline{C}(1, \sqrt{1-\delta})$.*

Proof. From the MDARE (3), we have

$$\begin{aligned}
&(A - sBK)^* P (A - sBK) - P \\
&\quad = A^T PA - (s + s^*) A^T PB\left(I + B^T PB\right)^{-1} B^T PA \\
&\qquad + ss^* K^T B^T PBK - P \\
&\quad = A^T PA - P - (s + s^* - ss^*) A^T PB\left(I + B^T PB\right)^{-1} \\
&\qquad \times B^T PA - |s|^2 K^T K \\
&\quad = A^T PA - P - \left(1 - |s-1|^2\right) A^T PB\left(I + B^T PB\right)^{-1} \\
&\qquad \times B^T PA - |s|^2 K^T K \\
&\quad \le A^T PA - P - \delta A^T PB\left(I + B^T PB\right)^{-1} B^T PA \\
&\quad \le -Q < 0.
\end{aligned} \quad (5)$$

Thus, we know that if $|s - 1| \le \sqrt{1-\delta}$, $A - sBK$ is Schur-stable. □

3. Distributed State Feedback Design

In this section, we investigate the multiagent consensus via state variable feedback control, which has been addressed by [23]. Here, we also use the control protocol proposed by [23] and provide a new design approach to construct the feedback gain matrix.

The neighborhood disagreement error of agent i is defined as

$$\xi_i(k) = \sum_{j\in\mathcal{N}_i} w_{ij}\left(x_j(k) - x_i(k)\right) + g_i\left(x_0(k) - x_i(k)\right). \quad (6)$$

Consider the following distributed state feedback protocol for agent i:

$$u_i(k) = c_1\left(1 + d_i + g_i\right)^{-1} K\xi_i(k), \quad (7)$$

where $d_i = \sum_{j\in\mathcal{N}_i} w_{ij}$, c_1 is the coupling strength and K is a feedback gain matrix, which will be determined later.

Denote $e_i(k) = x_i(k) - x_0(k)$ and $e(k) = [e_1^T(k), e_2^T(k), \ldots, e_n^T(k)]^T$. Then, we can derive that the close loop system has the global tracking error dynamics as follows [23]

$$e(k+1) = \left[I_n \otimes A - c_1\Gamma \otimes (BK)\right] e(k), \quad (8)$$

where $\Gamma = (I + D + G_d)^{-1}(L + G_d)$.

Definition 6 (see [23]). A covering circle $\overline{C}(c_0, r_0)$ related to matrix Γ is a closed circle in the complex plane centered at $c_0 \in R$ and $\lambda_i \in \overline{C}(c_0, r_0)$ for all $i = 1, 2, \ldots, n$.

Then, we provide a new design technique to construct feedback gain matrix K, which is presented in the following theorem.

Theorem 7. *For multiagent system* (1) *and* (2)*, assume that the interconnection topology $\widehat{\mathcal{G}}$ has a directed spanning tree with root v_0. If there exists a covering circle $\overline{C}(c_0, r_0)$ such that*

$$0 < \frac{r_0}{c_0} < \sqrt{1-\delta_c}, \quad (9)$$

then there must exist fitted c_1 and K such that the global tracking error dynamics (8) *is asymptotically stable. Furthermore, by taking δ which satisfies*

$$\frac{r_0}{c_0} \le \sqrt{1-\delta} < \sqrt{1-\delta_c} \quad (10)$$

and solving the MDARE (3) *to get the unique positive-definite solution P, the feedback matrix K and the coupling strength* c_1 *can be chosen as*

$$K = \left(I + B^T PB\right)^{-1} B^T PA,$$
$$c_1 = \frac{1}{c_0}. \tag{11}$$

Proof. From (10), we know $\delta > \delta_c$, which means that the MDARE (3) has a unique positive-definite solution P. Any λ_i satisfies $|\lambda_i - c_0| \leq r_0$. Thus, $|c_1\lambda_i - 1| \leq r_0/c_0 < \sqrt{1-\delta}$. According to Lemma 5, all $A - c_1\lambda_i BK$, $i = 1, 2, \ldots, n$ are Schur-stable.

Let U be a Schur transformation matrix of Γ such that

$$U^T \Gamma U = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \tag{12}$$

Then, we have

$$(U \times I)^T \left[I_n \otimes A - c_1\Gamma \otimes (BK)\right](U \times I)$$
$$= \begin{bmatrix} A - c_1\lambda_1 BK & * & \cdots & * \\ 0 & A - c_1\lambda_2 BK_c & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A - c_1\lambda_n BK_c \end{bmatrix}. \tag{13}$$

Certainly, $U \otimes I$ is also a unitary matrix. Matrix $[I_n \otimes A - c_1\Gamma \otimes (BK)]$ is Schur-stable if and only if all $A - c_1\lambda_i BK$, $i = 1, 2, \ldots, n$ are Schur-stable. Now, the proof is completed. □

Remark 8. From condition (9), it is required that $0 < c_0 < r_0$, which means that the covering circle should be located in the open right half plane. Moreover, the small enough r_0/c_0 will guarantee that the MDARE (3) is solvable, which is the key point in the proposed design approach. The weight parameter in the feedback law (7) need not take $c_1(1 + d_i + g_i)^{-1}$, which can be selected as $c_1(d_i + g_i)^{-1}$, c_1, and so on as long as there exists a covering circle for the related matrix $c_1\Gamma$ that satisfies the condition (9).

Next, we will discuss the covering circle of the matrix $c_1\Gamma$. Based on Gershgorin disk theorem [47], all the eigenvalues of $(I + D + G_d)^{-1}(I + W)$ are located in the union of n discs:

$$\bigcup_{j=1}^{n} \left\{ s \in C : \left| s - \frac{1}{1 + d_i + g_i} \right| \leq \frac{d_i}{1 + d_i + g_i} \right\}. \tag{14}$$

It is easy to see that this union is included in a unit circle $\{s : |s| \leq 1\}$ and the circular boundaries of the union of n discs have only one intersection with the circle at $s = 1$. If the interconnection topology $\widehat{\mathscr{G}}$ has a directed spanning tree with root v_0, we know that $L+G_d$ is nonsingular, and then, Γ is nonsingular too. Noting that $(I+D+G_d)^{-1}(I+W) = I-\Gamma$, then we know that all eigenvalues of matrix $(I + D + G_d)^{-1}(I + W)$ are not equal to 1. Thus, all eigenvalues of matrix Γ can be covered by circle $\overline{C}(1, r_0)$ with $r_0 < 1$. On the other hand, it is necessary to assume that the interconnection topology $\widehat{\mathscr{G}}$ has a directed spanning tree with root v_0. Otherwise, there exists at least one agent which cannot get the leader's information directly and indirectly. Certainly, if A is not Schur-stable, those agents cannot track the leader with some initial values. From this point, the assumption that the interconnection topology $\widehat{\mathscr{G}}$ has a directed spanning tree with root v_0 is necessary.

An interesting special case is that matrix A has no eigenvalues with magnitude larger than 1, that is, $\rho(A) \leq 1$. The well-known second-order discrete-time multiagent system

$$x_i(k+1) = x_i(k) + v_i(k),$$
$$v_i(k+1) = v_i(k) + u_i(k), \tag{15}$$

has been addressed in many references [34, 38]. The system matrix A of second-order discrete-time multiagent system is $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, which has no eigenvalues with magnitude larger than 1.

According to Theorem 7, we present the following corollary for this special case.

Corollary 9. *For multiagent system* (1) *and* (2) *with* $\rho(A) \leq 1$, *assume that the interconnection topology* $\widehat{\mathscr{G}}$ *has a directed spanning tree with root* v_0. *Take* $\delta = 1 - \max_i\{|\lambda_i - 1|^2\}$, *and solve the MDARE* (3) *to get the unique positive-definite solution P. Choose* $K = (I + B^T PB)^{-1} B^T PA$ *and* $C_1 = 1$. *Then, the distributed feedback control* (7) *guarantees that all following agents can track leader.*

Proof. According to Remark 4, we know $\delta_c = 0$ if $\rho(A) \leq 1$. Select $\delta = 1 - \max_i\{|\lambda_i - 1|^2\}$. From above analysis, we know that $\delta > 0$ and $C(1, \delta)$ are a covering circle. Thus, the MDARE (3) is solvable. According to Theorem 7, we can obtain the corollary directly. □

4. Consensus Protocol Design with Full-Order Observer

In many applications, each agent only accesses the neighbor's output variable. To solve leader-following consensus problem, we propose a new observer-based consensus protocol for agent i, which consists of a distributed estimation law and a feedback control law.

(i) Local estimation law for agent i:

$$z_i(k+1) = Fz_i(k) + Gy_i(k) + TBu_i(k),$$
$$\widehat{x}_i(k) = T^{-1}z_i(k), \tag{16}$$

where $z_i(k) \in R^m$ is the protocol state, $\widehat{x}(k)$ is the constructed variable to estimate $x_i(k)$, and $F \in R^{m\times m}$, $G \in R^{m\times q}$, and $T \in R^{m\times m}$ are the designed parameter matrices.

(ii) Neighbor-based feedback control law for agent i:

$$u_i(k) = c_1(1 + d_i + g_i)^{-1} K\eta_i(k), \tag{17}$$

where the neighborhood disagreement observer error $\eta_i(k)$ of agent i is denoted as

$$\eta_i(k) = \sum_{j\in\mathcal{N}_i} w_{ij}\left(\hat{x}_j(k) - \hat{x}_i(k)\right) + g_i\left(x_0(k) - \hat{x}_i(k)\right), \tag{18}$$

and K is a given feedback gain matrix.

Next, an algorithm is provided to select the parameter matrices used in estimation law (16).

Algorithm 10. Given that (A, C) is observable. The parameter matrices F, G, and T used in estimation law (16) can be constructed as follows.

(1) Select a Schur-stable $m \times m$ matrix F with a set of desired eigenvalues that contain no eigenvalues in common with those of A.

(2) Select $G \in R^{m\times q}$ randomly such that (F, G) is controllable.

(3) Solve Sylvester equation

$$TA - FT = GC \tag{19}$$

to get a nonsingular solution T. If T is singular, select another G until T is nonsingular.

Denote $\hat{e}_i(k) = z_i(k) - Tx_i(k)$ and $\hat{e}(k) = [\hat{e}_1^T(k), \hat{e}_2^T(k), \ldots, \hat{e}_n^T(k)]^T$. Then, after manipulations and combining (1) and (16), we can obtain

$$\begin{aligned}\hat{e}_i(k+1) &= z_i(k+1) - Tx_i(k+1)\\ &= Fz(k) + Gy_i(k) + TBu_i(k) - TAx_i(k) - TBU_i(k)\\ &= F\hat{e}_i(k) + (FT + GC - TA)x_i(k)\\ &= F\hat{e}_i(k).\end{aligned} \tag{20}$$

For tracking error $e_i(k) = x_i(k) - x_0(k)$, we have

$$\begin{aligned}e_i(k+1) &= Ae_i(k+1) + c_1(1 + d_i + g_i)^{-1} K\eta_i(k)\\ &= Ae_i(k+1) + c_1(1 + d_i + g_i)^{-1} K\xi_i(k)\\ &\quad + c_1(1 + d_i + g_i)^{-1} K\left[\eta_i(k) - \xi_i(k)\right]\\ &= Ae_i(k+1) - c_1(1 + d_i + g_i)^{-1}\\ &\quad\times\left[K\sum_{j\in\mathcal{N}_i} w_{ij}\left(e_i(k) - e_j(k)\right) + g_i e_i\right]\\ &\quad + c_1(1 + d_i + g_i)^{-1} KT^{-1}\\ &\quad\times\left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(\hat{e}_i(k) - \hat{e}_j(k)\right) + g_i\hat{e}_i\right].\end{aligned} \tag{21}$$

From (20) and (21), the error dynamics of closed-loop system will be expressed as

$$\begin{bmatrix} e(k+1)\\ \hat{e}(k+1)\end{bmatrix} = \begin{bmatrix} I_n\otimes A - c_1\Gamma\otimes(BK) & c_1\Gamma\otimes\left(BKT^{-1}\right)\\ 0 & I\otimes F\end{bmatrix} \times\begin{bmatrix} e(k)\\ \hat{e}(k)\end{bmatrix}. \tag{22}$$

Obviously, the error dynamics system (22) is Schur-stable if and only if $I_n\otimes A - c_1\Gamma\otimes(BK)$ and $I\otimes F$ are Schur-stable. Similar to Theorem 7, we present the following theorem directly, and the proof is omitted.

Theorem 11. *For multiagent system* (1) *and* (2), *assume that the interconnection topology* $\widehat{\mathcal{G}}$ *has a directed spanning tree with root* v_0. *If there exists a covering circle* $\overline{C}(c_0, r_0)$ *such that*

$$0 < \frac{r_0}{c_0} < \sqrt{1 - \delta_c}, \tag{23}$$

then the distributed observer-based protocols (16) *and* (17) *can solve the discrete-time leader-following consensus problem. Furthermore, the parameter matrices* F, G, *and* T *used in observer* (16) *are constructed by* Algorithm 10. *By taking* δ *satisfied*

$$\frac{r_0}{c_0} \le \sqrt{1 - \delta} < \sqrt{1 - \delta_c} \tag{24}$$

and solving the MDARE (3) *to get the unique positive-definite solution* P, *the feedback matrix* K *and the coupling strength* c_1 *can be chosen as*

$$\begin{aligned} K &= \left(I + B^T PB\right)^{-1} B^T PA,\\ c_1 &= \frac{1}{c_0}.\end{aligned} \tag{25}$$

Remark 12. Of course, when system matrix A satisfies $\rho(A) \le 1$, we can also establish similar corollaries as Corollary 9 in this section and the next section. In [23], three different observer/controller architectures are proposed for dynamic output feedback regulator design. Besides design feedback matrix K, another key technique is to choose an observer gain matrix L which makes $I_n \otimes A - c_1\Gamma\otimes(LC)$ Schur-stable. By using duality property, solve the following MDARE:

$$APA^T - P - \delta APC^T\left(I + CPC^T\right)^{-1} CPA^T + Q = 0 \tag{26}$$

to get the unique positive definite solution P. Then, the observer gain matrix L is chosen as $L = APC^T(I + CPC^T)^{-1}$. Thus, the proposed design method in this paper can also be applied to construct the protocols proposed by [23]. In this paper, we propose two new observer/controller architectures, which will replenish cooperative observer and regulator theory. Contrary to [23], our proposed approach must be feasible if system matrix A satisfies $\rho(A) \leq 1$.

5. Consensus Protocol Design with Reduced-Order Observer

In this section, we assume that C has full row rank, that is, $\mathrm{Rank}(C) = q$. The following reduced-order observer-based consensus protocol, which consists of a reduced-order estimation law and a feedback control law, is proposed for agent i.

(i) Local reduced-order estimation law for agent i:

$$v_i(k+1) = Fv_i(k) + Gy_i(k) + TBu_i(k), \tag{27}$$

where $v_i(k) \in R^{m-q}$ is the protocol state, $F \in R^{m-q \times m-q}$, and $G \in R^{m-q \times q}$ and $T \in R^{m-q \times m}$ are parameter matrices.

(ii) Neighbor-based feedback control law for agent i:

$$u_i(k) = c_1(1 + d_i + g_i)^{-1} K\zeta_i(k), \tag{28}$$

where the disagreement error $\zeta_i(k)$ of agent i is given as

$$\zeta_i(k) = Q_1\left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(y_j(k) - y_i(k)\right) + g_i\left(y_0(k) - y_i(k)\right)\right] + Q_2\left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(v_j(k) - v_i(k)\right) + g_i\left(Tx_0(k) - v_i(k)\right)\right], \tag{29}$$

and K is a gain matrix.

Similarly, an algorithm is presented to design the same parameter matrices used in the protocols (27) and (28).

Algorithm 13. Given that (A, C) is observable. The parameter matrices F, G, T, Q_1, and Q_2 can be constructed as follows.

(1) Select a Schur matrix $F \in R^{(m-q)\times(m-q)}$ with a set of desired eigenvalues that contain no eigenvalues in common with those of A.

(2) Select $G \in R^{(m-q)\times q}$ randomly such that (F, G) is controllable.

(3) Solve Sylvester equation

$$TA - FT = GC \tag{30}$$

to get the unique solution T, which satisfies that $\left[\begin{smallmatrix} C \\ T \end{smallmatrix}\right]$ is nonsingular. If $\left[\begin{smallmatrix} C \\ T \end{smallmatrix}\right]$ is singular, go back to step (2) to select another G until $\left[\begin{smallmatrix} C \\ T \end{smallmatrix}\right]$ is nonsingular.

(4) Compute matrices $Q_1 \in R^{m\times q}$ and $Q_2 \in R^{m\times(m-q)}$ by $[Q_1\ Q_2] = \left[\begin{smallmatrix} C \\ T \end{smallmatrix}\right]^{-1}$.

Now, we present the result related to reduced-order observer.

Theorem 14. *For multiagent system* (1) *and* (2)*, assume that the interconnection topology $\widehat{\mathcal{G}}$ has a directed spanning tree with root v_0. If there exists a covering circle $\overline{C}(c_0, r_0)$ such that*

$$0 < \frac{r_0}{c_0} < \sqrt{1 - \delta_c}, \tag{31}$$

then the distributed observer-based protocols (16) *and* (17) *can solve the discrete-time leader-following consensus problem. Furthermore, the parameter matrices F, G, T, Q_1, and Q_2 used in protocols* (27) *and* (28) *are constructed by Algorithm 13. By taking δ which satisfies*

$$\frac{r_0}{c_0} \leq \sqrt{1 - \delta} < \sqrt{1 - \delta_c} \tag{32}$$

and solving the MDARE (3) *to get the unique positive-definite solution P, the feedback matrix K and the coupling strength c_1 can be chosen as*

$$K = \left(I + B^T PB\right)^{-1} B^T PA, \qquad c_1 = \frac{1}{c_0}. \tag{33}$$

Proof. To analyze convergence, denote $e_i(k) = x_i(k) - x_0(k)$ and $\varepsilon_i = v_i(k) - Tx_0(k)$. Then, the dynamics of $e_i(k)$ and $\varepsilon_i(k)$ satisfy

$$\begin{aligned}
e_i(k+1) &= Ae_i(k) - c_1(1 + d_i + g_i)^{-1} BKQ_1C \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(e_i(k) - e_j(k)\right) + g_i e_i(k)\right] \\
&\quad - c_1(1 + d_i + g_i)^{-1} KQ_2 \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(\varepsilon_i(k) - \varepsilon_j(k)\right) + g_i \varepsilon_i(k)\right], \\
\varepsilon_i(k+1) &= v_i(k+1) - Tx_0(k+1) \\
&= Fv_i(k) + GCx_i(k) - TAx_0(k) \\
&\quad - c_1(1 + d_i + g_i)^{-1} TBKQ_1C \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i(t)} w_{ij}\left(x_i - x_j\right) + g_i\left(x_i - x_0\right)\right] \\
&\quad - c_1(1 + d_i + g_i)^{-1} TBKQ_2 \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(v_i - v_j\right) + g_i\left(v_i - Tx_0\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= F\varepsilon_i(k) + GCe_i(k) - c_1(1+d_i+g_i)^{-1}TBKQ_1C \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(e_i(k)-e_j(k)\right) + g_ie_i(k)\right] \\
&\quad - c_1(1+d_i+g_i)^{-1}TBKQ_2 \\
&\quad \times \left[\sum_{j\in\mathcal{N}_i} w_{ij}\left(\varepsilon_i(k)-\varepsilon_j(k)\right) + g_i\varepsilon_i(k)\right].
\end{aligned} \tag{34}$$

Let $e = (e_1^T, e_2^T, \ldots, e_n^T)^T$ and $\varepsilon = (\varepsilon_1^T, \varepsilon_2^T, \ldots, \varepsilon_n^T)^T$. From (34), the closed-loop error dynamics can be represented as

$$\begin{aligned}
&\begin{bmatrix} e(k+1) \\ \varepsilon(k+1) \end{bmatrix} \\
&= \begin{bmatrix} I_n\otimes A - c_1\Gamma\otimes BKQ_1C & -c_1\Gamma\otimes BKQ_2 \\ I_N\otimes GC - c_1\Gamma\otimes TBKQ_1C & I_n\otimes F - c_1\Gamma\otimes TBKQ_2 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e(k) \\ \varepsilon(k) \end{bmatrix} \triangleq H \begin{bmatrix} e(k) \\ \varepsilon(k) \end{bmatrix}.
\end{aligned} \tag{35}$$

It is easy to see that the leader-following multiagent system achieves consensus if the closed-loop error dynamics system (35) is Schur-stable.

Let $\overline{T} = \left[\begin{smallmatrix} I_n\otimes I_m & 0 \\ -I_n\otimes T & I_n\otimes I_{m-q} \end{smallmatrix}\right]$, which is nonsingular, and $\overline{T}^{-1} = \left[\begin{smallmatrix} I_n\otimes I_m & 0 \\ I_n\otimes T & I_n\otimes I_{m-q} \end{smallmatrix}\right]$. By step (2) of Algorithm 13, we have

$$\overline{H} \triangleq \overline{T}H\overline{T}^{-1} = \begin{bmatrix} I_n\otimes A - c_1\Gamma\otimes(BK) & -c_1\Gamma\otimes(BKQ_2) \\ 0 & I_n\otimes F \end{bmatrix}. \tag{36}$$

The matrix $\overline{H}$ is block upper triangular matrix with diagonal block matrix entries $I_n\otimes A - c_1\Gamma\otimes(BK)$ and F. Because F is Schur-stable, the matrix H is Schur-stable if and only if $A - c_1\Gamma\otimes(BK)$ is Schur-stable. The rest of the proof is omitted, because it is very similar to the proof of Theorem 7. □

6. Simulation Example

In this section, we give an example to illustrate the effectiveness of the obtained result. The multiagent system consists of four agents and one leader, that is, $n = 4$. The following agents and leader are, respectively, modeled by the linear dynamics (1) and (2) with system matrices

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0.2 & 1.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \qquad C = [0, 1, 1]. \tag{37}$$

The matrices L and G of the interaction graph $\widehat{\mathcal{G}}$ are given by

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \qquad G_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

By some simple computations, it is proper to take $c_0 = 0.5768$, $r_0 = 0.5001$. Therefore, take $c_1 = 1.7337$. By solving MDRAE (3) with $\delta = 0.2482$, the unique positive definite solution is

$$P = \begin{bmatrix} 2.7685 & 2.0965 & -8.6525 \\ 2.0965 & 17.0036 & -7.1766 \\ -8.6525 & -7.1766 & 59.3567 \end{bmatrix}. \tag{39}$$

Then, the gain matrix can be chosen as

$$K = \left(I + B^TPB\right)^{-1}B^TPA = [-0.0499, -0.0593, 0.2445]. \tag{40}$$

The multiagent system adopts the consensus protocols (16) and (17) with randomly initial state. The matrices F, G, and T are designed as follows:

$$\begin{gathered}
F = \begin{bmatrix} -0.1 & 0 & 0 \\ 1 & -0.2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \\
T = \begin{bmatrix} 0 & 1 & -0.9 \\ 0.4041 & 0.3342 & 0.4041 \\ 0.6306 & 4.3243 & -0.9459 \end{bmatrix}.
\end{gathered} \tag{41}$$

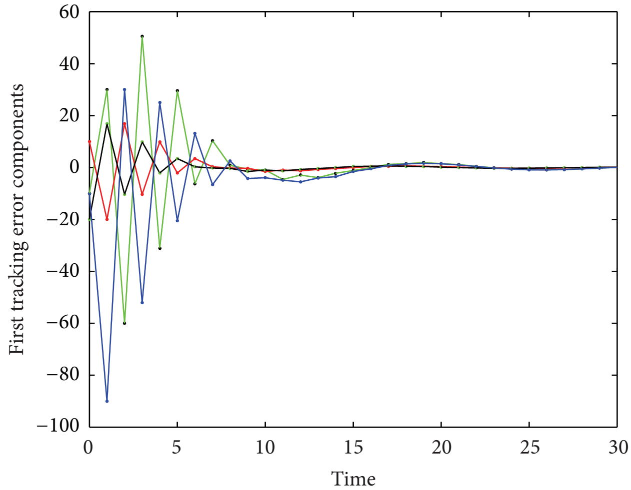
The state tracking errors showed in Figure 1, which show all following agents can track the leader. As for the reduced-order observer case, the matrices F, G, T, Q_1, and Q_2 used in the protocols (27) and (28) can be constructed by Algorithm 13 as follows:

$$\begin{gathered}
F = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad G = [4, 7]^T, \\
T = \begin{bmatrix} 1.1842 & 3.2895 & 0.5921 \\ 0.8535 & 5.6999 & 0.1009 \end{bmatrix}, \\
Q_1 = [-0.7031, 0.0892, 0.9168]^T, \\
Q_2 = \begin{bmatrix} 1.2936 & -0.6232 \\ -0.1972 & 0.2736 \\ 0.1972 & -0.2736 \end{bmatrix}.
\end{gathered} \tag{42}$$

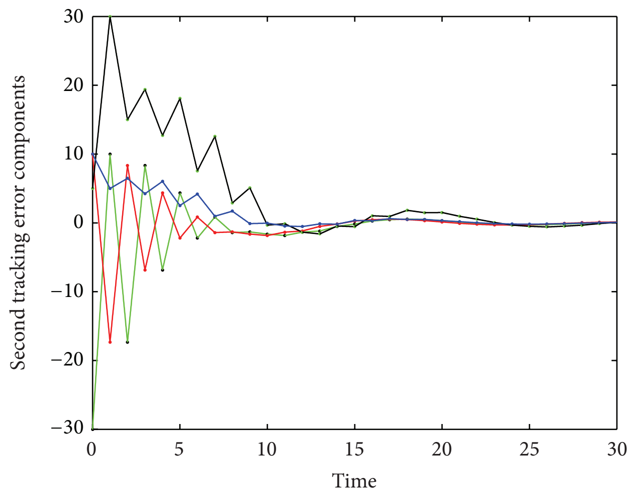
With consensus protocols (27) and (28), the state tracking errors showed in Figure 2, which also show all following agents, can track the leader.

7. Conclusions

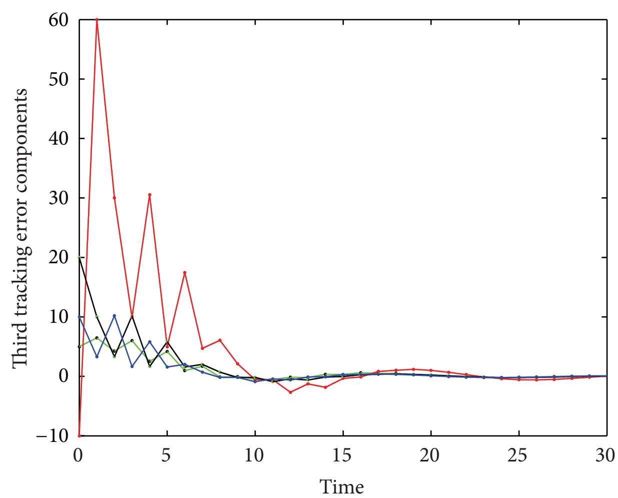
This paper solves a leader-following consensus problem of discrete-time multiagent system with distributed controllers and observers. We provide a general framework for designing



(a)

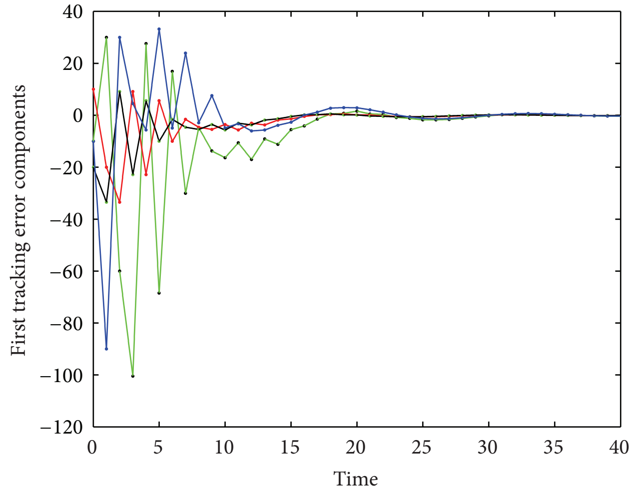


(b)

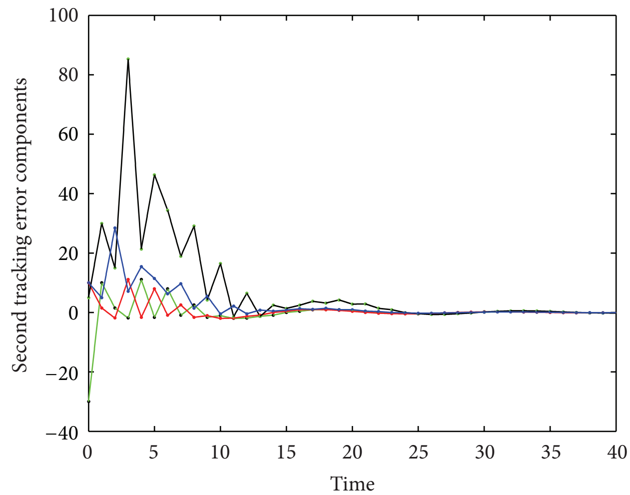


(c)

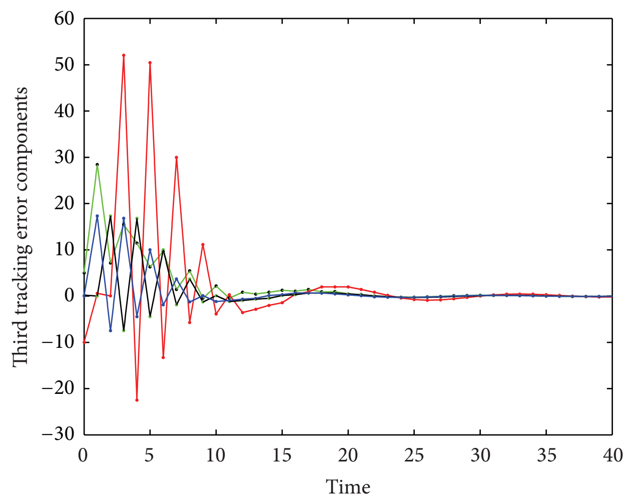
FIGURE 1: Error trajectories of three state components with full-order observer.



(a)



(b)



(c)

FIGURE 2: Error trajectories of three state components with reduced-order observer.

distributed consensus protocols by applying full state feedback information and measured output feedback information. Furthermore, we propose a reduced-order observer-based protocol to solve the leader-following consensus problem. The interconnection topology is modeled by graph, whose connectivity is a key factor to guarantee that the multiagent achieves consensus. The consensus problem is transformed into the stability problem of error dynamical system, which also preserves the property of the separation principle. The gain matrices can be designed by solving the MDARE and the Sylvester equation. Presented results could be generalized to switching and jumping interaction topology in future work.

Acknowledgments

This work was supported by the Zhejiang Provincial Natural Science Function of China under Grant no. LY13F030048 and the National Natural Science Function of China under Grant nos. 61074123, 61174063.

References

[1] J. Krause, G. D. Ruxton, and S. Krause, "Swarm intelligence in animals and humans," *Trends in Ecology & Evolution*, vol. 25, no. 1, pp. 28–34, 2010.

[2] T. Vicsek, A. Czirk, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[3] S. Kalantar and U. R. Zimmer, "Distributed shape control of homogeneous swarms of autonomous underwater vehicles," *Autonomous Robots*, vol. 22, no. 1, pp. 37–53, 2007.

[4] S. Yoon and C. Qiao, "Cooperative search and survey using Autonomous Underwater Vehicles (AUVs)," *IEEE Transactions on Parallel and Distributed Systems*, vol. 22, no. 3, pp. 364–379, 2011.

[5] Z. Huang, Q. Song, and C. Feng, "Multistability in networks with self-excitation and high-order synaptic connectivity," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 8, pp. 2144–2155, 2010.

[6] E. Kaslik and S. Sivasundaram, "Multiple periodic solutions in impulsive hybrid neural networks with delays," *Applied Mathematics and Computation*, vol. 217, no. 10, pp. 4890–4899, 2011.

[7] W. Yu, G. Chen, Z. Wang, and W. Yang, "Distributed consensus filtering in sensor networks," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 39, no. 6, pp. 1568–1577, 2009.

[8] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[9] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[10] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[11] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[12] Y. Hong, J. Hu, and L. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology," *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2006.

[13] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A*, vol. 374, no. 2, pp. 853–863, 2007.

[14] Y. Hong, G. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.

[15] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems & Control Letters*, vol. 59, no. 3-4, pp. 209–217, 2010.

[16] Z.-J. Tang, T.-Z. Huang, J.-L. Shao, and J.-P. Hu, "Leader-following consensus for multi-agent systems via sampled-data control," *IET Control Theory & Applications*, vol. 5, no. 14, pp. 1658–1665, 2011.

[17] L. Gao, X. Zhu, and W. Chen, "Leader-following consensus problem with an accelerated motion leader," *International Journal of Control, Automation, and Systems*, vol. 10, no. 5, pp. 931–939, 2012.

[18] Y. Hong and X. Wang, "Multi-agent tracking of a high-dimensional active leader with switching topology," *Journal of Systems Science and Complexity*, vol. 22, no. 4, pp. 722–731, 2009.

[19] L. Gao, Y. Tang, W. Chen, and H. Zhang, "Consensus seeking in multi-agent systems with an active leader and communication delays," *Kybernetika*, vol. 47, no. 5, pp. 773–789, 2011.

[20] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 1, pp. 213–224, 2010.

[21] Z. Li, Z. Duan, and G. Chen, "Dynamic consensus of linear multi-agent systems," *IET Control Theory and Applications*, vol. 5, no. 1, pp. 19–28, 2011.

[22] H. Zhang, F. L. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1948–1952, 2011.

[23] K. Hengster-Movric and F. Lewis, "Cooperative observers and regulators for discrete-time multiagent systems," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 14, pp. 1545–1562, 2013.

[24] L. Gao, X. Zhu, W. Chen, and H. Zhang, "Leader-following consensus of linear multiagent systems with state observer under switching topologies," *Mathematical Problems in Engineering*, vol. 2013, Article ID 873140, 12 pages, 2013.

[25] H. Li, "Observer-type consensus protocol for a class of fractional-order uncertain multiagent systems," *Abstract and Applied Analysis*, vol. 2012, Article ID 672346, 18 pages, 2012.

[26] Z. Li, X. Liu, P. Lin, and W. Ren, "Consensus of linear multi-agent systems with reduced-order observer-based protocols," *Systems & Control Letters*, vol. 60, no. 7, pp. 510–516, 2011.

[27] Y. Zhang, L. Gao, and C. Tong, "On distributed reduced-order observer-based protocol for linear multiagent consensus under switching topology," *Abstract and Applied Analysis*, vol. 2013, Article ID 793276, 13 pages, 2013.

[28] J. H. Seo, H. Shim, and J. Back, "Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach," *Automatica*, vol. 45, no. 11, pp. 2659–2664, 2009.

[29] Y. Liu and Y. Jia, "H_∞ consensus control of multi-agent systems with switching topology: a dynamic output feedback protocol," *International Journal of Control*, vol. 83, no. 3, pp. 527–537, 2010.

[30] L. Gao, C. Tong, and L. Wang, "H^∞ dynamic output feedback consensus control for discrete-time multi-agent systems with switching topology," *The Arabian Journal for Science and Engineering*, 2013.

[31] Z. Huang, X. Wang, and C. Feng, "Multiperiodicity of periodically oscillated discrete-time neural networks with transient excitatory self-connections and sigmoidal nonlinearities," *IEEE Transactions on Neural Networks*, vol. 21, no. 10, pp. 1643–1655, 2010.

[32] Z. Huang, X. Wang, and M. Sannay, "Self-excitation of neurons leads to multiperiodicity of discrete-time neural networks with distributed delays," *Science China*, vol. 54, no. 2, pp. 305–317, 2011.

[33] E. Kaslik and S. Sivasundaram, "Impulsive hybrid discrete-time Hopfield neural networks with delays and multistability analysis," *Neural Networks*, vol. 24, no. 4, pp. 370–377, 2011.

[34] Y. Cao and W. Ren, "Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction," *International Journal of Control*, vol. 83, no. 3, pp. 506–515, 2010.

[35] K. You and L. Xie, "Network topology and communication data rate for consensusability of discrete-time multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2262–2275, 2011.

[36] F. Xiao and L. Wang, "Dynamic behavior of discrete-time multiagent systems with general communication structures," *Physica A*, vol. 370, no. 2, pp. 364–380, 2006.

[37] F. Xiao and L. Wang, "Consensus protocols for discrete-time multi-agent systems with time-varying delays," *Automatica*, vol. 44, no. 10, pp. 2577–2582, 2008.

[38] P. Lin and Y. Jia, "Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies," *Automatica*, vol. 45, no. 9, pp. 2154–2158, 2009.

[39] J. Qin and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2417–2422, 2012.

[40] Y. Su and J. Huang, "Two consensus problems for discrete-time multi-agent systems with switching network topology," *Automatica*, vol. 48, no. 9, pp. 1988–1997, 2012.

[41] K. Hengster-Movric, K. You, F. L. Lewis, and L. Xie, "Synchronization of discrete-time multi-agent systems on graphs using Riccati design," *Automatica*, vol. 49, no. 2, pp. 414–423, 2013.

[42] K. You and L. Xie, "Coordination of discrete-time multi-agent systems via relative output feedback," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 13, pp. 1587–1605, 2011.

[43] X. Xu, S. Chen, W. Huang, and L. Gao, "Leader-following consensus of discrete-time multi-agent systems with observer-based protocols," *Neurocomputing*, vol. 118, no. 22, pp. 334–341, 2013.

[44] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.

[45] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[46] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[47] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, NY, USA, 1985.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 268721, 8 pages
http://dx.doi.org/10.1155/2013/268721

Research Article

Oscillation for Higher Order Dynamic Equations on Time Scales

Taixiang Sun,[1] Qiuli He,[2] Hongjian Xi,[3] and Weiyong Yu[1]

[1] *College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China*
[2] *College of Electrical Engineering, Guangxi University, Nanning, Guangxi 530004, China*
[3] *Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China*

Correspondence should be addressed to Taixiang Sun; stx1963@163.com

Received 15 February 2013; Accepted 26 August 2013

Academic Editor: Zhenkun Huang

We investigate the oscillation of the following higher order dynamic equation: $\{a_n(t)[(a_{n-1}(t)(\cdots(a_1(t)x^{\Delta}(t))^{\Delta}\cdots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + p(t)x^{\beta}(t) = 0$, on some time scale $\mathbf{T}$, where $n \geq 2$, $a_k(t)$ $(1 \leq k \leq n)$ and $p(t)$ are positive rd-continuous functions on $\mathbf{T}$ and α, β are the quotient of odd positive integers. We give sufficient conditions under which every solution of this equation is either oscillatory or tends to zero.

1. Introduction

In this paper, we investigate the oscillation of the following higher order dynamic equation:

$$\left\{a_n(t)\left[\left(a_{n-1}(t)\left(\cdots\left(a_1(t)x^{\Delta}(t)\right)^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta}\right]^{\alpha}\right\}^{\Delta} + p(t)x^{\beta}(t) = 0, \tag{E}$$

on some time scale $\mathbf{T}$, where $n \geq 2$, $a_k(t)$ $(1 \leq k \leq n)$ and $p(t)$ are positive rd-continuous functions on $\mathbf{T}$ and α, β are the quotient of odd positive integers. Write

$$S_k(t, x(t)) = \begin{cases} x(t), & \text{if } k = 0, \\ a_k(t) S_{k-1}^{\Delta}(t, x(t)), & \text{if } 1 \leq k \leq n-1, \\ a_n(t)\left[S_{n-1}^{\Delta}(t, x(t))\right]^{\alpha}, & \text{if } k = n, \end{cases} \tag{1}$$

then (E) reduces to the following equation:

$$S_n^{\Delta}(t, x(t)) + p(t)x^{\beta}(t) = 0. \tag{2}$$

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup \mathbf{T} = \infty$ and $t_0 \in \mathbf{T}$ is a constant. For any $a \in \mathbf{T}$, we define the time scale interval $[a, \infty)_{\mathbf{T}} = \{t \in \mathbf{T} : t \geq a\}$. By a solution of (2), we mean a nontrivial real-valued function $x(t) \in C_{\mathrm{rd}}^1[T_x, \infty)$, $T_x \geq t_0$, which has the property that $S_k(t, x(t)) \in C_{\mathrm{rd}}^1[T_x, \infty)$ for $0 \leq k \leq n$ and satisfies (2) on $[T_x, \infty)$, where C_{rd}^1 is the space of differentiable functions whose derivative is rd-continuous. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory.

The theory of time scale, which has recently received a lot of attention, was introduced by Hilger's landmark paper [1], a rapidly expanding body of the literature that has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an nonempty closed subset of the real numbers, and the cases when this time scale is equal to the real numbers or to the integers represent the classical theories of differential or of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). The new theory of the so-called "dynamic equations" not only unifies the theories of differential equations and difference equations, but also extends these classical cases to cases "in between," for example, to the so-called q-difference equations when $\mathbf{T} = \{1, q, q^2, \ldots, q^k, \ldots\}$, which has important applications in quantum theory (see [3]). In this work, knowledge and understanding of time scales and time scale notation are assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [2, 4]. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the papers [5–20].

Recently, Erbe et al. in [21–23] considered the third-order dynamic equations

$$\left(a(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\Delta}+p(t)f(x(t))=0,$$

$$x^{\Delta\Delta\Delta}(t)+p(t)x(t)=0, \tag{3}$$

$$\left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta}+f(t,x(t))=0,$$

respectively, and established some sufficient conditions for oscillation.

Hassan [24] studied the third-order dynamic equations

$$\left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta}+f(t,x(\tau(t)))=0 \tag{4}$$

and obtained some oscillation criteria, which improved and extended the results that have been established in [21–23].

2. Main Results

In this section, we investigate the oscillation of (2). To do this, we need the following lemmas.

Lemma 1 (see [25]). *Assume that*

$$\int_{t_0}^{\infty}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s=\int_{t_0}^{\infty}\frac{\Delta s}{a_i(s)}=\infty \quad \forall 1\le i\le n-1, \tag{5}$$

and $1\le m\le n$. *Then,*

(1) $\liminf_{t\to\infty}S_m(t,x(t))>0$ *implies* $\lim_{t\to\infty}S_i(t,x(t))=\infty$ *for* $0\le i\le m-1$;

(2) $\limsup_{t\to\infty}S_m(t,x(t))<0$ *implies* $\lim_{t\to\infty}S_i(t,x(t))=-\infty$ *for* $0\le i\le m-1$.

Lemma 2 (see [25]). *Assume that* (5) *holds. If* $S_n^{\Delta}(t,x(t))<0$ *and* $x(t)>0$ *for* $t\ge t_0$, *then there exists an integer* $0\le m\le n$ *with* $m+n$ *even such that*

(1) $(-1)^{m+i}S_i(t,x(t))>0$ *for* $t\ge t_0$ *and* $m\le i\le n$;

(2) *if* $m>1$, *then there exists* $T\ge t_0$ *such that* $S_i(t,x(t))>0$ *for* $1\le i\le m-1$ *and* $t\ge T$.

Remark 3. Let $a_n(t)=\cdots=a_1(t)=1$, and let $\mathbf{T}$ be the set of integers. Then, Lemmas 1 and 2 are Lemma 1.8.10 and Theorem 1.8.11 of [26], respectively.

Lemma 4. *Assume that* (5) *holds. Furthermore, suppose that*

$$\int_{t_0}^{\infty}\frac{1}{a_{n-1}(u)}\left\{\int_u^{\infty}\left[\frac{1}{a_n(s)}\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u=\infty. \tag{6}$$

If $x(t)$ *is an eventually positive solution of* (2), *then there exists* $T\ge t_0$ *sufficiently large such that*

(1) $S_n^{\Delta}(t,x(t))<0$ *for* $t\ge T$;

(2) *either* $S_i(t,x(t))>0$ *for* $t\ge T$ *and* $0\le i\le n$ *or* $\lim_{t\to\infty}x(t)=0$.

Proof. Pick $t_1\ge t_0$ so that $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$. It follows from (2) that

$$S_n^{\Delta}(t,x(t))=-p(t)x^{\beta}(t)<0 \quad \text{for } t\ge t_1. \tag{7}$$

By Lemma 2, we see that there exists an integer $0\le m\le n$ with $m+n$ even such that $(-1)^{m+i}S_i(t,x(t))>0$ for $t\ge t_1$ and $m\le i\le n$, and $x(t)$ is eventually monotone.

We claim that $\lim_{t\to\infty}x(t)\ne 0$ implies $m=n$. If not, then $S_{n-1}(t,x(t))<0(t\ge t_1)$ and $S_{n-2}(t,x(t))>0$ $(t\ge t_1)$, and there exist $t_2\ge t_1$ and a constant $c>0$ such that $x(t)\ge c$ on $[t_2,\infty)_{\mathbf{T}}$. Integrating (2) from t into ∞, we get that for $t\ge t_2$

$$-a_n(t)\left[S_{n-1}^{\Delta}(t,x(t))\right]^{\alpha}=-S_n(t,x(t))\le -c^{\beta}\int_t^{\infty}p(v)\Delta v. \tag{8}$$

Thus,

$$S_{n-1}(t,x(t))\le -c^{\beta/\alpha}\times\int_t^{\infty}\left[\frac{1}{a_n(s)}\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s \quad \text{for } t\ge t_2. \tag{9}$$

Again, integrating the above inequality from t_2 into t, we obtain that for $t\ge t_2$

$$\begin{aligned}S_{n-2}(t,x(t)) &\le S_{n-2}(t_2,x(t_2))\\ &\quad -c^{\beta/\alpha}\int_{t_2}^{t}\frac{1}{a_{n-1}(u)}\left\{\int_u^{\infty}\left[\frac{1}{a_n(s)}\times\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u.\end{aligned} \tag{10}$$

It follows from (6) that $\lim_{t\to\infty}S_{n-2}(t,x(t))=-\infty$, which is a contradiction to $S_{n-2}(t,x(t))>0$ $(t\ge t_1)$. The proof is completed. □

Lemma 5. *Assume that* $x(t)$ *is an eventually positive solution of* (2) *such that* $S_n^{\Delta}(t,x(t))<0$ *for* $t\ge T\ge t_0$ *and* $S_i(t,x(t))>0$ *for* $t\ge T$ *and* $0\le i\le n$. *Then,*

$$S_i(t,x(t))\ge S_n^{1/\alpha}(t,x(t))B_{i+1}(t,T) \quad \text{for } 0\le i\le n-1,\quad t\ge T, \tag{11}$$

and there exist $T_1>T$ *and a constant* $c>0$ *such that*

$$x(t)\le cB_1(t,T) \quad \text{for } t\ge T_1, \tag{12}$$

where

$$B_i(t,T) = \begin{cases} \int_T^t \left[\frac{1}{a_n(s)}\right]^{1/\alpha} \Delta s, & \text{if } i = n, \\ \int_T^t \frac{B_{i+1}(s,T)}{a_i(s)} \Delta s, & \text{if } 1 \le i \le n-1. \end{cases} \tag{13}$$

Proof. Since $S_n^\Delta(t,x(t)) < 0$ $(t \ge T)$, it follows that $S_n(t,x(t))$ is strictly decreasing on $[T,\infty)_{\mathbf{T}}$. Then, for $t \ge T$,

$$\begin{aligned} S_{n-1}(t,x(t)) &\ge S_{n-1}(t,x(t)) - S_{n-1}(T,x(T)) \\ &= \int_T^t \left[\frac{S_n(s,x(s))}{a_n(s)}\right]^{1/\alpha} \Delta s \\ &\ge S_n^{1/\alpha}(t,x(t))\, B_n(t,T) \\ S_{n-2}(t,x(t)) &\ge S_{n-2}(t,x(t)) - S_{n-2}(T,x(T)) \\ &= \int_T^t \frac{S_{n-1}(s,x(s))}{a_{n-1}(s)} \Delta s \\ &\ge S_n^{1/\alpha}(t,x(t))\, B_{n-1}(t,T) \\ &\vdots \\ S_1(t,x(t)) &\ge S_1(t,x(t)) - S_1(T,x(T)) \\ &= \int_T^t \frac{S_2(s,x(s))}{a_2(s)} \Delta s \ge S_n^{1/\alpha}(t,x(t))\, B_2(t,T) \\ S_0(t,x(t)) &\ge x(t) - x(T) \\ &= \int_T^t \frac{S_1(s,x(s))}{a_1(s)} \Delta s \ge S_n^{1/\alpha}(t,x(t))\, B_1(t,T). \end{aligned} \tag{14}$$

On the other hand, we have that for $t \ge T$,

$$\begin{aligned} S_{n-1}(t,x(t)) &= \int_T^t \left[\frac{S_n(s,x(s))}{a_n(s)}\right]^{1/\alpha} \Delta s + S_{n-1}(T,x(T)) \\ &\le S_{n-1}(T,x(T)) + S_n^{1/\alpha}(T,x(T))\, B_n(t,T). \end{aligned} \tag{15}$$

Thus, there exist $T_1 > T$ and a constant $b_1 > 0$ such that

$$S_{n-1}(t,x(t)) \le b_1 B_n(t,T) \quad \text{for } t \ge T_1. \tag{16}$$

Again,

$$\begin{aligned} S_{n-2}(t,x(t)) &= S_{n-2}(T_1,x(T_1)) + \int_{T_1}^t \frac{S_{n-1}(s,x(s))}{a_{n-1}(s)} \Delta s \\ &\le S_{n-2}(T_1,x(T_1)) + b_1 \int_T^t \frac{B_n(s,T)}{a_{n-1}(s)} \Delta s. \end{aligned} \tag{17}$$

Thus, there exists a constant $b_2 > 0$ such that

$$\begin{aligned} S_{n-2}(t,x(t)) &\le b_2 \int_T^t \frac{B_n(s,T)}{a_{n-1}(s)} \Delta s \\ &= b_2 B_{n-1}(t,T) \quad \text{for } t \ge T_1. \end{aligned} \tag{18}$$

Again,

$$\begin{aligned} S_{n-3}(t,x(t)) &= S_{n-3}(T_1,x(T_1)) + \int_{T_1}^t \frac{S_{n-2}(s,x(s))}{a_{n-2}(s)} \Delta s \\ &\le S_{n-3}(T_1,x(T_1)) + b_2 \int_T^t \frac{B_{n-1}(s,T)}{a_{n-2}(s)} \Delta s. \end{aligned} \tag{19}$$

Thus, there exists a constant $b_3 > 0$ such that

$$\begin{aligned} S_{n-3}(t,x(t)) &\le b_3 \int_T^t \frac{B_{n-1}(s,T)}{a_{n-2}(s)} \Delta s \\ &= b_3 B_{n-2}(t,T) \quad \text{for } t \ge T_1. \end{aligned} \tag{20}$$

The rest of the proof is by induction. The proof is completed. □

Lemma 6 (see [2]). *Let $f : \mathbf{R} \to \mathbf{R}$ be continuously differentiable and suppose that $g : \mathbf{T} \to \mathbf{R}$ is delta differentiable. Then, $f \circ g$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = g^\Delta(t) \int_0^1 f'(hg(t) + (1-h)g^\sigma(t))\, dh. \tag{21}$$

Lemma 7 (see [27]). *If A, B are nonnegative and $\lambda > 1$, then*

$$A^\lambda - \lambda A B^{\lambda-1} + (\lambda - 1) B^\lambda \ge 0. \tag{22}$$

Now, we state and prove our main results.

Theorem 8. *Suppose that (5) and (6) hold. If there exists a positive nondecreasing delta differentiable function θ such that for all sufficiently large $T \in [t_0,\infty)_{\mathbf{T}}$ and for any positive constants c_1, c_2, there is a $T_1 > T$ such that*

$$\limsup_{t\to\infty} \int_{T_1}^t \left[\theta(s) p(s) - \frac{\theta^\Delta(s)}{B_1^\alpha(s,T)} \delta_1(t,T,c_1,c_2)\right] \Delta s = \infty, \tag{23}$$

where

$$\delta_1(t,T,c_1,c_2) = \begin{cases} c_1, & \text{if } \alpha < \beta, \\ 1, & \text{if } \alpha = \beta, \\ c_2 B_1^{\alpha-\beta}(t,T), & \text{if } \alpha > \beta, \end{cases} \tag{24}$$

and $B_1(t,T)$ is as in Lemma 5. Then, every solution of (2) is either oscillatory or tends to zero.

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbf{T}}$. Then, without loss of generality, there is a $t_1 \ge t_0$, sufficiently large, such that $x(t) > 0$ for $t \ge t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \ge t_1$ such that

(i) $S_n^\Delta(t,x(t)) < 0$ for $t \ge t_2$;

(ii) either $S_i(t,x(t)) > 0$ for $t \ge t_2$ and $0 \le i \le n$ or $\lim_{t\to\infty} x(t) = 0$.

Let $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Consider

$$w(t) = \theta(t) \frac{S_n(t, x(t))}{x^\beta(t)} \quad \text{for } t \geq t_2. \tag{25}$$

It follows from Lemma 6 that

$$\left(x^\beta\right)^\Delta(t) = \beta x^\Delta(t) \int_0^1 \left(hx(t) + (1-h)x(t)^\sigma\right)^{\beta-1} dh > 0 \quad \text{for } t \geq t_2. \tag{26}$$

Then,

$$\begin{aligned} w^\Delta &= \left[\frac{\theta}{x^\beta}\right]^\Delta S_n^\sigma(\cdot, x) + \frac{\theta}{x^\beta} S_n^\Delta(\cdot, x) \\ &= \left[\frac{\theta^\Delta}{(x^\beta)^\sigma} - \frac{\theta\left(x^\beta\right)^\Delta}{x^\beta(x^\beta)^\sigma}\right] S_n^\sigma(\cdot, x) - \theta p \\ &\leq \frac{\theta^\Delta}{x^\beta} S_n(\cdot, x) - \theta p. \end{aligned} \tag{27}$$

From (11) and (27), we get

$$w^\Delta(t) \leq \frac{\theta^\Delta(t)}{B_1^\alpha(t, t_2)} x^{\alpha-\beta}(t) - \theta(t)p(t) \quad \text{for } t \geq t_2. \tag{28}$$

Now, we consider the following three cases.

Case 1. If $\alpha = \beta$, then

$$x^{\alpha-\beta}(t) = 1 \quad \text{for } t \geq t_2. \tag{29}$$

Case 2. If $\alpha > \beta$, then it follows from (12) that there exist $t_3 > t_2$ and a constant $c_2 > 0$ such that

$$x^{\alpha-\beta}(t) \leq c_2 B_1^{\alpha-\beta}(t, t_2) \quad \text{for } t \geq t_3. \tag{30}$$

Case 3. If $\alpha < \beta$, then

$$x(t) \geq x(t_2) \quad \text{for } t \geq t_2. \tag{31}$$

Thus,

$$x^{\alpha-\beta}(t) \leq c_1 = x^{\alpha-\beta}(t_2) \quad \text{for } t \geq t_2. \tag{32}$$

From (27)–(32), we obtain

$$w^\Delta(t) \leq \frac{\theta^\Delta(t)}{B_1^\alpha(t, t_2)} \delta_1(t, t_2, c_1, c_2) - \theta(t)p(t) \quad \text{for } t \geq t_3. \tag{33}$$

Integrating the above inequality from t_3 into t, we have

$$\int_{t_3}^t \left[\theta(s)p(s) - \frac{\theta^\Delta(s)}{B_1^\alpha(s, t_2)} \delta_1(s, t_2, c_1, c_2)\right] \Delta s \leq w(t_3) < \infty, \tag{34}$$

which gives a contradiction to (23). The proof is completed. □

Theorem 9. *Suppose that (5) and (6) hold. If there exists a positive nondecreasing delta differentiable function θ such that for all sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ and for any positive constants c_1, c_2, there is a $T_1 > T$ such that*

$$\limsup_{t\to\infty} \int_{T_1}^t \left[\theta(s)p(s) - \frac{(\alpha/\beta)^\alpha \left(\theta^\Delta(s)\right)^{\alpha+1} a_1^\alpha(s)}{(\alpha+1)^{\alpha+1}\left(B_2(s, T)\theta(s)\delta_2(s, T, c_1, c_2)\right)^\alpha}\right] \Delta s = \infty, \tag{35}$$

where

$$\delta_2(t, T, c_1, c_2) = \begin{cases} c_1 & \text{if } \alpha < \beta, \\ 1, & \text{if } \alpha = \beta, \\ c_2 B_1^{(\beta/\alpha)-1}(\sigma(t), T), & \text{if } \alpha > \beta, \end{cases} \tag{36}$$

and $B_1(t, T), B_2(t, T)$ are as in Lemma 5. Then, every solution of (2) is either oscillatory or tends to zero.

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t) > 0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that

(i) $S_n^\Delta(t, x(t)) < 0$ for $t \geq t_2$;

(ii) either $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t\to\infty} x(t) = 0$.

Let $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Note that

$$\begin{aligned} \left(x^\beta\right)^\Delta &= \beta x^\Delta \int_0^1 \left(hx + (1-h)x^\sigma\right)^{\beta-1} dh \\ &= \beta x^\Delta \int_0^1 \frac{\left(hx + (1-h)x^\sigma\right)^\beta}{hx + (1-h)x^\sigma} dh \\ &\geq \beta x^\Delta \frac{x^\beta}{x^\sigma}. \end{aligned} \tag{37}$$

From (11), we have

$$\begin{aligned} \frac{\left(x^\beta\right)^\Delta}{x^\beta} &\geq \beta \frac{x^\Delta}{x^\sigma} \geq \beta \frac{S_n^{1/\alpha}(\cdot, x) B_2(\cdot, t_2)}{a_1 x^\sigma} \\ &\geq \beta \frac{\left(S_n^\sigma(\cdot, x)\right)^{1/\alpha} B_2(\cdot, t_2)}{a_1 x^\sigma} \\ &= \beta \frac{\left(w^\sigma\right)^{1/\alpha}}{a_1(\theta^\sigma)^{1/\alpha}} \left(x^\sigma\right)^{(\beta/\alpha)-1} B_2(\cdot, t_2). \end{aligned} \tag{38}$$

Then it follows from (27) that for $t \geq t_2$,

$$\begin{aligned} w^\Delta &= \left[\frac{\theta}{x^\beta}\right]^\Delta S_n^\sigma(\cdot,x) + \frac{\theta}{x^\beta} S_n^\Delta(\cdot,x) \\ &= \left[\frac{\theta^\Delta}{(x^\beta)^\sigma} - \frac{\theta (x^\beta)^\Delta}{x^\beta (x^\beta)^\sigma}\right] S_n^\sigma(\cdot,x) - \theta p \\ &\leq \theta^\Delta \frac{w^\sigma}{\theta^\sigma} - \beta \frac{B_2(\cdot,t_2)\theta}{a_1} \frac{(w^\sigma)^{1+(1/\alpha)}}{(\theta^\sigma)^{1+(1/\alpha)}} (x^\sigma)^{(\beta/\alpha)-1} - \theta p. \end{aligned} \tag{39}$$

Now, we consider the following three cases.

Case 1. If $\alpha = \beta$, then

$$(x^\sigma)^{(\beta/\alpha)-1}(t) = 1 \quad \text{for } t \geq t_2. \tag{40}$$

Case 2. If $\alpha > \beta$, then it follows from (12) that there exist $t_3 > t_2$ and a constant c such that

$$x(t) \leq c B_1(t,t_2) \quad \text{for } t \geq t_3. \tag{41}$$

Thus,

$$(x^\sigma)^{(\beta/\alpha)-1}(t) \geq c_2 B_1^{(\beta/\alpha)-1}(\sigma(t),t_2), \tag{42}$$

with $c_2 = c^{(\beta/\alpha)-1}$.

Case 3. If $\alpha < \beta$, then

$$x(t) \geq x(t_2) \quad \text{for } t \geq t_2. \tag{43}$$

Thus,

$$(x^\sigma)^{(\beta/\alpha)-1}(t) \geq c_1 = x^{(\beta/\alpha)-1}(t_2). \tag{44}$$

From (39)–(44), we obtain that for $t \geq t_3$,

$$\begin{aligned} w^\Delta &\leq \frac{w^\sigma}{\theta^\sigma}\theta^\Delta - \frac{\beta B_2(\cdot,t_2)\theta\delta_2(\cdot,t_2,c_1,c_2)}{a_1} \frac{(w^\sigma)^{1+(1/\alpha)}}{(\theta^\sigma)^{1+(1/\alpha)}} - \theta p \\ &= -\frac{\beta B_2(\cdot,t_2)\theta\delta_2(\cdot,t_2,c_1,c_2)}{a_1} \\ &\quad \times \left\{\frac{(w^\sigma)^{1+(1/\alpha)}}{(\theta^\sigma)^{1+(1/\alpha)}} - \frac{w^\sigma}{\theta^\sigma} \frac{a_1\theta^\Delta}{\beta B_2(\cdot,t_2)\theta\delta_2(\cdot,t_2,c_1,c_2)}\right\} - \theta p. \end{aligned} \tag{45}$$

Let

$$A = \frac{w^\sigma}{\theta^\sigma}, \qquad B = \left[\frac{\alpha a_1 \theta^\Delta}{(\alpha+1)\beta B_2(\cdot,t_2)\theta\delta_2(\cdot,t_2,c_1,c_2)}\right]^\alpha, \tag{46}$$

with $\lambda = 1 + 1/\alpha$. By Lemma 7, we have

$$w^\Delta \leq \frac{(\alpha/\beta)^\alpha (\theta^\Delta)^{\alpha+1} a_1^\alpha}{(\alpha+1)^{\alpha+1}(B_2(\cdot,t_2)\theta\delta_2(\cdot,t_2,c_1,c_2))^\alpha} - \theta p. \tag{47}$$

Integrating the above inequality from t_3 into t, it follows that

$$\begin{aligned} &\int_{t_3}^{t} \Bigg[\theta(s)p(s) \\ &\quad - \frac{(\alpha/\beta)^\alpha (\theta^\Delta(s))^{\alpha+1} a_1^\alpha(s)}{(\alpha+1)^{\alpha+1}(B_2(s,t_2)\theta(s)\delta_2(s,t_2,c_1,c_2))^\alpha}\Bigg]\Delta s \\ &\leq w(t_3) < \infty, \end{aligned} \tag{48}$$

which gives a contradiction to (35). The proof is completed. □

Remark 10. The trick used in the proofs of Theorems 8 and 9 is from [16].

Theorem 11. *Suppose that* (5) *and* (6) *hold. If for all sufficiently large* $T \in [t_0,\infty)_{\mathbf{T}}$,

$$\int_T^\infty p(u)\left[\int_T^u \frac{\Delta s}{a_1(s)}\right]^\beta \Delta u = \infty, \tag{49}$$

then every solution of (2) *is either oscillatory or tends to zero.*

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbf{T}}$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t) > 0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that

(i) $S_n^\Delta(t,x(t)) < 0$ for $t \geq t_2$;

(ii) either $S_i(t,x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t\to\infty} x(t) = 0$.

Let $S_i(t,x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Then, for $t \geq t_2$,

$$\begin{aligned} x(t) &= x(t_2) + \int_{t_2}^{t} \frac{S_1(s,x(s))}{a_1(s)}\Delta s \\ &\geq S_1(t_2,x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1(s)}. \end{aligned} \tag{50}$$

It follows from (2) that

$$-S_n^\Delta(t,x(t)) \geq p(t)\left[S_1(t_2,x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1(s)}\right]^\beta. \tag{51}$$

Integrating the above inequality from t_2 into ∞, we have

$$S_n(t_2,x(t_2)) \geq S_1^\beta(t_2,x(t_2)) \int_{t_2}^{\infty} p(u)\left[\int_{t_2}^{u} \frac{\Delta s}{a_1(s)}\right]^\beta \Delta u, \tag{52}$$

which gives a contradiction to (49). The proof is completed. □

Theorem 12. *Suppose that* (5) *and* (6) *hold. If for all sufficiently large* $T \in [t_0,\infty)_{\mathbf{T}}$,

$$\limsup_{t\to\infty} B_1^\alpha(t,T)\delta_3(t,T,c_1,c_2)\int_t^\infty p(s)\Delta s > 1, \tag{53}$$

where

$$\delta_3(t,T,c_1,c_2) = \begin{cases} c_1, c_1 \text{ is any positive constant,} \\ \qquad\qquad \text{if } \alpha < \beta, \\ 1, \qquad\quad \text{if } \alpha = \beta, \\ c_2 B_1^{\beta-\alpha}(t,T), c_2 \text{ is any positive constant,} \\ \qquad\qquad \text{if } \alpha > \beta, \end{cases} \tag{54}$$

and $B_1(t,T)$ *is as in Lemma 5, then every solution of* (2) *is either oscillatory or tends to zero.*

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t) > 0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that

(i) $S_n^{\Delta}(t,x(t)) < 0$ for $t \geq t_2$;

(ii) either $S_i(t,x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t\to\infty} x(t) = 0$.

Let $S_i(t,x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Then, it follows from (2) and (11) that for $t \geq t_2$,

$$\int_t^{\infty} p(s)x^{\beta}(s)\Delta s \leq S_n(t,x(t)) \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^{\alpha}. \tag{55}$$

Using the fact that $x(t)$ is strictly increasing on $[t_2,\infty)_{\mathbf{T}}$, we obtain

$$x^{\beta}(t)\int_t^{\infty} p(s)\Delta s \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^{\alpha}. \tag{56}$$

Thus,

$$B_1^{\alpha}(t,t_2)\,x^{\beta-\alpha}(t)\int_t^{\infty} p(s)\Delta s \leq 1. \tag{57}$$

Now, we consider the following three cases.

Case 1. If $\alpha = \beta$, then

$$x^{\beta-\alpha}(t) = 1 \quad \text{for } t \geq t_2. \tag{58}$$

Case 2. If $\alpha > \beta$, then it follows from (12) that there exist $t_3 > t_2$ and a constant c such that

$$x(t) \leq cB_1(t,t_2) \quad \text{for } t \geq t_3. \tag{59}$$

Thus,

$$x^{\beta-\alpha}(t) \geq c_2 B_1^{\beta-\alpha}(t,t_2), \tag{60}$$

with $c_2 = c^{\beta-\alpha}$.

Case 3. If $\alpha < \beta$, then

$$x(t) \geq x(t_2) \quad \text{for } t \geq t_2. \tag{61}$$

Thus,

$$x^{\beta-\alpha}(t) \geq c_1 = x^{\beta-\alpha}(t_2). \tag{62}$$

From (57)–(62), we obtain that for $t \geq t_3$,

$$B_1^{\alpha}(t,t_2)\,\delta_3(t,t_2,c_1,c_2)\int_t^{\infty} p(s)\Delta s \leq 1, \tag{63}$$

which gives a contradiction to (53). The proof is completed. □

3. Examples

In this section, we give some examples to illustrate our main results.

Example 1. Consider the following higher order dynamic equation:

$$S_n^{\Delta}(t,x(t)) + t^{\gamma}x^{\beta}(t) = 0, \tag{64}$$

on an arbitrary time scale $\mathbf{T}$ with $\sup \mathbf{T} = \infty$, where $n \geq 2$, α, β and $S_k(t,x(t))$ $(0 \leq k \leq n)$ are as in (2) with $a_n(t) = t^{\alpha-1}, a_{n-1}(t) = \cdots = a_1(t) = t$, and $\gamma > -1$. Then, every solution of (64) is either oscillatory or tends to zero.

Proof. Note that

$$\begin{aligned} \int_{t_0}^{\infty}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s &= \int_{t_0}^{\infty}\left[\frac{1}{s^{\alpha-1}}\right]^{1/\alpha}\Delta s = \infty, \\ \int_{t_0}^{\infty}\frac{\Delta s}{a_i(s)} &= \int_{t_0}^{\infty}\frac{\Delta s}{s} = \infty \quad \text{for } 1 \leq i \leq n-1, \\ \int_{t_0}^{\infty} p(s)\Delta s &= \int_{t_0}^{\infty} s^{\gamma}\Delta s = \infty, \end{aligned} \tag{65}$$

by Example 5.60 in [4]. Pick $t_1 > t_0$ such that

$$\int_{t_0}^{t_1}\frac{1}{a_{n-1}(u)}\left\{\int_u^{t_1}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s\right\}\Delta u > 0. \tag{66}$$

Then,

$$\begin{aligned} &\int_{t_0}^{\infty}\frac{1}{a_{n-1}(u)}\left\{\int_u^{\infty}\left[\frac{1}{a_n(s)}\int_s^{\infty} p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u \\ &\quad \geq \left[\int_{t_1}^{\infty} p(v)\Delta v\right]^{1/\alpha} \\ &\quad \times \int_{t_0}^{t_1}\frac{1}{a_{n-1}(u)}\left(\int_u^{t_1}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s\right)\Delta u = \infty. \end{aligned} \tag{67}$$

Let $T \in [t_0, \infty)_{\mathbf{T}}$, sufficiently large, and $u_1 > T$ such that $\int_T^{u_1} (1/a_1(s))\Delta s > 1$, then

$$\begin{aligned}\int_T^\infty p(u)\left[\int_T^u \frac{1}{a_1(s)}\Delta s\right]^\beta \Delta u &\ge \int_{u_1}^\infty p(u)\left[\int_T^u \frac{1}{a_1(s)}\Delta s\right]^\beta \Delta u \\ &\ge \int_{u_1}^\infty p(u)\Delta u = \infty.\end{aligned} \quad (68)$$

Thus, conditions (5), (6), and (49) are satisfied. By Theorem 11, every solution of (64) is either oscillatory or tends to zero. □

Example 2. Consider the following higher order dynamic equation:

$$S_n^\Delta(t, x(t)) + \frac{1}{t^{1+\gamma}} x^\beta(t) = 0, \quad (69)$$

on an arbitrary time scale $\mathbf{T}$ with $\sup \mathbf{T} = \infty$, where $n \ge 2$, $S_k(t, x(t))$ $(0 \le k \le n)$ are as in (2) with $a_n(t) = 1, a_{n-1}(t) = t^{1/\alpha}, a_{n-2}(t) = \cdots = a_1(t) = t$, $0 < \gamma < \min\{1, \beta\}$, and α, β are the quotient of odd positive integers with $\alpha \ge 1$. Then, every solution of (69) is either oscillatory or tends to zero.

Proof. Note that

$$\begin{aligned}\int_{t_0}^\infty \left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s &= \int_{t_0}^\infty \Delta s = \infty, \\ \int_{t_0}^\infty \frac{1}{a_{n-1}(s)}\Delta s &= \int_{t_0}^\infty \frac{1}{s^{1/\alpha}}\Delta s = \infty, \\ \int_{t_0}^\infty \frac{1}{a_i(s)}\Delta s = \int_{t_0}^\infty &\frac{1}{s}\Delta s = \infty \quad \text{for } 1 \le i \le n-2.\end{aligned} \quad (70)$$

Pick $t_1 > t_0$ such that $\int_{t_0}^{t_1}(\Delta u / u^{1/\alpha}) > 0$, then

$$\begin{aligned}&\int_{t_0}^\infty \frac{1}{a_{n-1}(u)}\left\{\int_u^\infty \left[\frac{1}{a_n(s)}\int_s^\infty p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u \\ &= \int_{t_0}^\infty \frac{1}{u^{1/\alpha}}\left\{\int_u^\infty \left[\int_s^\infty \frac{1}{v^{\gamma+1}}\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u \\ &\ge \frac{1}{\gamma}\int_{t_0}^\infty \frac{1}{u^{1/\alpha}}\left\{\int_u^\infty \left[\int_s^\infty \frac{(v^\gamma)^\Delta}{v^\gamma (v^\gamma)^\sigma}\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u \\ &= \frac{1}{\gamma}\int_{t_0}^\infty \frac{1}{u^{1/\alpha}}\left[\int_u^\infty \left(\frac{1}{s^\gamma}\right)^{1/\alpha}\Delta s\right]\Delta u \\ &\ge \frac{1}{\gamma}\int_{t_0}^{t_1} \frac{1}{u^{1/\alpha}}\left[\int_{t_1}^\infty \left(\frac{1}{s^\gamma}\right)^{1/\alpha}\Delta s\right]\Delta u \\ &= \frac{1}{\gamma}\left[\int_{t_1}^\infty \left(\frac{1}{s^\gamma}\right)^{1/\alpha}\Delta s\right]\int_{t_0}^{t_1}\frac{\Delta u}{u^{1/\alpha}} \\ &= \infty.\end{aligned} \quad (71)$$

Let $M = \max\{c_1, 1, c_2\}$ with c_1, c_2 being positive constants, $\rho = \min\{\alpha, \beta\}$, and $\gamma < \tau < \min\{1, \beta\}$. Pick $T_1 > T > 0$ such that

$$\frac{1}{t^\gamma} \ge \frac{2}{t^\tau} \ge \frac{2M}{\left[(1/2)^{n+(1/\alpha)}\left(t - 2^{n-1}T\right)\right]^\rho} \quad \text{for } t \ge T_1. \quad (72)$$

Let $\theta(t) = t$, then

$$\begin{aligned}&B_1(t, T) \\ &= \int_T^t \frac{1}{a_1(u_1)} \times \left[\int_T^{u_1} \frac{1}{a_2(u_2)} \times \left[\cdots\left[\int_T^{u_{n-2}} \frac{1}{a_{n-1}(u_{n-1})}\left[\int_T^{u_{n-1}} \Delta u_n\right]^{1/\alpha} \times \Delta u_{n-1}\right]\cdots\right]\Delta u_2\right]\Delta u_1 \\ &= \int_{2^{n-1}T}^t \frac{1}{u_1} \times \left[\int_{2^{n-2}T}^{u_1} \frac{1}{u_2}\left[\cdots\left[\int_{2T}^{u_{n-2}}\left[\frac{1}{u_{n-1}}\int_T^{u_{n-1}}\Delta u_n\right]^{1/\alpha} \times \Delta u_{n-1}\right]\cdots\right]\Delta u_2\right]\Delta u_1 \\ &\ge \left(\frac{1}{2}\right)^{n+(1/\alpha)}\left(t - 2^{n-1}T\right), \\ &\int_{T_1}^t \left[\theta(s)p(s) - \frac{\theta^\Delta(s)}{B_1^\alpha(s, T)}\delta_1(s, T)\right]\Delta s \\ &= \int_{T_1}^t \left[\frac{1}{s^\gamma} - \frac{1}{B_1^\alpha(s, T)}\delta_1(s, T, c_1, c_2)\right]\Delta s \\ &\ge \int_{T_1}^t \left[\frac{2}{t^\tau} - \frac{M}{\left[(1/2)^{n+(1/\alpha)}\left(t - 2^{n-1}T\right)\right]^\rho}\right]\Delta s \\ &\ge \int_{T_1}^t \frac{1}{t^\tau}\Delta s.\end{aligned} \quad (73)$$

Thus,

$$\limsup_{t\to\infty} \int_{T_1}^t \left[\theta(s)p(s) - \frac{\theta^\Delta(s)}{B_1^\alpha(s, T)}\delta_1(s, T, c_1, c_2)\right]\Delta s = \infty. \quad (74)$$

So conditions (5), (6), and (23) are satisfied. Then, by Theorem 8, every solution of (69) is either oscillatory or tends to zero. □

Acknowledgments

This project is supported by NNSF of China (11261005 and 51267001), NSF of Guangxi (2011GXNSFA018135 and 2012GXNSFDA276040), and SF of ED of Guangxi (2013ZD061).

References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, 2002.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[5] M. Bohner and S. H. Saker, "Oscillation of second order nonlinear dynamic equations on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 34, no. 4, pp. 1239–1254, 2004.

[6] L. Erbe, "Oscillation results for second-order linear equations on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1061–1071, 2002.

[7] T. S. Hassan, "Oscillation criteria for half-linear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 176–185, 2008.

[8] R. P. Agarwal, M. Bohner, and S. H. Saker, "Oscillation of second order delay dynamic equations," *The Canadian Applied Mathematics Quarterly*, vol. 13, no. 1, pp. 1–17, 2005.

[9] M. Bohner, B. Karpuz, and Ö. Öcalan, "Iterated oscillation criteria for delay dynamic equations of first order," *Advances in Difference Equations*, vol. 2008, Article ID 458687, 12 pages, 2008.

[10] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 505–522, 2007.

[11] Z. Han, B. Shi, and S. Sun, "Oscillation criteria for second-order delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2007, Article ID 070730, 16 pages, 2007.

[12] Z. Han, S. Sun, and B. Shi, "Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 847–858, 2007.

[13] T. Sun, H. Xi, X. Peng, and W. Yu, "Nonoscillatory solutions for higher-order neutral dynamic equations on time scales," *Abstract and Applied Analysis*, vol. 2010, Article ID 428963, 16 pages, 2010.

[14] T. Sun, H. Xi, and X. Peng, "Asymptotic behavior of solutions of higher-order dynamic equations on time scales," *Advances in Difference Equations*, vol. 2011, Article ID 237219, 14 pages, 2011.

[15] T. Sun, H. Xi, and W. Yu, "Asymptotic behaviors of higher order nonlinear dynamic equations on time scales," *Journal of Applied Mathematics and Computing*, vol. 37, no. 1-2, pp. 177–192, 2011.

[16] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, and W. Sae-jie, "On the oscillation of certain second order nonlinear dynamic equations," *Mathematical and Computer Modelling*, vol. 50, no. 1-2, pp. 273–286, 2009.

[17] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, "Oscillation theorems for fourth-order halflinear delay dynamic equations with damping," *Mediterranean Journal of Mathematics*, 2013.

[18] C. Zhang, T. Li, R. P. Agarwal, and M. Bohner, "Oscillation results for fourth-order nonlinear dynamic equations," *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2058–2065, 2012.

[19] Y. Qi and J. Yu, "Oscillation criteria for fourth-order nonlinear delay dynamic equations," *Electronic Journal of Differential Equations*, vol. 2013, Article ID 740568, 11 pages, 2013.

[20] S. R. Grace, S. Sun, and Y. Wang, "On the oscillation of fourth order strongly superlinear and strongly sublinear dynamic equations," *Journal of Applied Mathematics and Computing*, 2013.

[21] L. Erbe, A. Peterson, and S. H. Saker, "Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales," *Journal of Computational and Applied Mathematics*, vol. 181, no. 1, pp. 92–102, 2005.

[22] L. Erbe, A. Peterson, and S. H. Saker, "Hille and Nehari type criteria for third-order dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 112–131, 2007.

[23] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation and asymptotic behavior of a third-order nonlinear dynamic equation," *The Canadian Applied Mathematics Quarterly*, vol. 14, no. 2, pp. 129–147, 2006.

[24] T. S. Hassan, "Oscillation of third order nonlinear delay dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1573–1586, 2009.

[25] T. Sun, W. Yu, and H. Xi, "Oscillatory behavior and comparison for higher order nonlinear dynamic equations on time scales," *Journal of Applied Mathematics & Informatics*, vol. 30, no. 1-2, pp. 289–304, 2012.

[26] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods and Applications*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[27] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1988.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 354287, 11 pages
http://dx.doi.org/10.1155/2013/354287

Research Article

Stability Analysis of a Multigroup Epidemic Model with General Exposed Distribution and Nonlinear Incidence Rates

Ling Zhang,[1] Jingmei Pang,[2] and Jinliang Wang[2]

[1] *School of Science, Department of Fundamental Mathematics, Jiamusi University, Jiamusi 154007, China*
[2] *School of Mathematical Science, Heilongjiang University, Harbin 150080, China*

Correspondence should be addressed to Jinliang Wang; jinliangwang@hit.edu.cn

Received 26 January 2013; Revised 16 June 2013; Accepted 22 July 2013

Academic Editor: Pagavathi Balasubramaniam

We investigate a class of multigroup epidemic models with general exposed distribution and nonlinear incidence rates. For a simpler case that assumes an identical natural death rate for all groups, and with a gamma distribution for exposed distribution is considered. Some sufficient conditions are obtained to ensure that the global dynamics are completely determined by the basic production number R_0. The proofs of the main results exploit the method of constructing Lyapunov functionals and a graph-theoretical technique in estimating the derivatives of Lyapunov functionals.

1. Introduction

Multigroup epidemic models have been used in the literature to describe the transmission dynamics of many different infectious diseases such as mumps, measles, gonorrhea, HIV/AIDS and vector borne diseases such as Malaria [1]. In the models, heterogeneous host population can be divided into several homogeneous groups according to modes of transmission, contact patterns, or geographic distributions, so that within-group and intergroup interactions can be modeled separately. It is well known that global dynamics of multigroup models with higher dimensions, especially the global stability of the endemic equilibrium, are a very challenging problem. Guo et al. [2] proposed a graph-theoretic approach to the method of global Lyapunov functions and used it to resolve the open problem on the uniqueness and global stability of the endemic equilibrium of a multigroup SIR model with varying subpopulation sizes. Subsequently, a series of studies on the global stability of multigroup epidemic models were produced in the literature (see e.g., [2–5]).

In the present paper, a more general multigroup epidemic model is proposed and studied to describe the disease spread in a heterogeneous host population with general exposed distribution and nonlinear incidence rate. The host population is divided into m distinct groups ($m \geq 1$). For $1 \leq i \leq m$, the ith group is further partitioned into four disjoint classes: the susceptible individuals, exposed individuals, infectious individuals, and recovered individuals, whose numbers of individuals at time t are denoted by $S_i(t)$, $E_i(t)$, $I_i(t)$, and $R_i(t)$, respectively. Susceptible individuals infected with the disease but not yet infective are in the exposed (latent) class.

It is pointed in [6] that a fixed latent period can be considered as an approximation of the mean latent period, and this would be appropriate for those diseases whose latent periods vary only relatively slightly. For example, poliomyelitis has a latent period of 1–3 days (comparing to its much longer infectious period of 14–20 days). However disease such as tuberculosis, including bovine tuberculosis (a disease spread from animal to animal mainly by direct contact), may take months to develop to the infectious stage and also can relapse. Since the time it takes from the moment of new infection to the moment of becoming infectious may differ from disease to disease, even for the same disease, it differs from individual to individual, and it is indeed a random variable. It is thus of interest from both mathematical and biological viewpoints to investigate whether sustained oscillations are the result of general exposed distribution.

Following the method of [6], we also assume that the disease does not cause deaths during the latent period, taking the natural death rate into consideration. Let $P(t)$ denote the probability that an exposed individual remains in the time t after entering the exposed class. For $1 \le i, j \le m$, $\beta_{ij} \ge 0$ denotes the coefficient of transmission between compartments S_i and I_j. It is assumed that m-square matrix $(\beta_{ij})_{1\le i,j\le m}$ is irreducible [7]. So the proportion of exposed individuals can be expressed by the integral

$$E_i(t) = \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} P_j(t-u)\,du, \quad (1)$$

where the sum takes into account cross-infections from all groups. Integrals in (1) are in the Riemann-Stieltjes sense. $P_j(t)$ satisfies the following reasonable properties:

(*A*) $P_j : [0,\infty) \to [0,1]$ is nonincreasing, piecewise continuous with possibly finitely many jumps and satisfies $P_j(0^+) = 1$, and $\lim_{t\to\infty} P_j(t) = 0$ with $\int_0^\infty P_j(t)dt$ is positive and finite.

Differentiating (1) gives

$$\begin{aligned} E_i'(t) &= \sum_{j=1}^{m} \beta_{ij} f_{ij}(S_i(t), I_j(t)) \\ &\quad + \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\quad \times P_j'(t-u)\,du - \delta_i E_i(t). \end{aligned} \quad (2)$$

The first term on the right hand side in (2) is the rate at which new infected individuals come into the exposed class, and the last term explains the natural deaths. The second term accounts for the rate at which the individuals move to the infectious class (noting that $P_j'(t-u) \le 0$ due to the aforementioned property) from the exposed class; hence

$$\begin{aligned} I_i'(t) = &- \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\times P_j'(t-u)\,du - (\delta_i + \varepsilon_i + \gamma_i) I_i(t). \end{aligned} \quad (3)$$

Let $h_j(t) = -P_j'(t)$ be the probability density function for the time (a random variable) it takes for an infected individual in the ith group to become infectious. Then (4) becomes

$$\begin{aligned} I_i'(t) = &\sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\times h_j(t-u)\,du - (\delta_i + \varepsilon_i + \gamma_i) I_i(t). \end{aligned} \quad (4)$$

Within the ith group, $\varphi_i(S_i)$ denotes the growth rate of S_i, which includes both the production and the natural death of susceptible individuals. Therefore, under the assumptions, the model to be studied takes the following differential and integral equations form:

$$\begin{aligned} S_i'(t) &= \varphi_i(S_i(t)) - \sum_{j=1}^{m} \beta_{ij} f_{ij}(S_i(t), I_j(t)), \\ E_i'(t) &= \sum_{j=1}^{m} \beta_{ij} f_{ij}(S_i(t), I_j(t)) \\ &\quad - \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - \delta_i E_i(t), \\ I_i'(t) &= \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - (\delta_i + \varepsilon_i + \gamma_i) I_i(t), \\ R_i'(t) &= \gamma_i I_i(t) - \delta_i R_i(t). \end{aligned} \quad (5)$$

Since the variables E_i and R_i do not appear in the first and third equations of model (5), $E_i(t)$ and $R_i(t)$, $i = 1, \ldots, m$, can be decoupled from the $S_i(t)$ and $I_i(t)$ equations; we only need to consider the subsystem of (5) consisting of only the S_i and I_i equations:

$$\begin{aligned} S_i'(t) &= \varphi_i(S_i(t)) - \sum_{j=1}^{m} \beta_{ij} f_{ij}(S_i(t), I_j(t)), \\ I_i'(t) &= \sum_{j=1}^{m} \beta_{ij} \int_0^t f_{ij}(S_i(u), I_j(u)) e^{-\delta_j(t-u)} \\ &\quad \times h_j(t-u)\,du - (\delta_i + \varepsilon_i + \gamma_i) I_i(t), \end{aligned} \quad (6)$$

where δ_i denotes the natural death rates of I_i compartments in the ith group, ε_i is the death rate caused by disease in the ith group, and γ_i is the rate of recovery of infectious individuals in the ith group. In what follows we investigate the global stability of system (5).

When $m = 1$, $P(t) = e^{\epsilon t}$, and with bilinear incidence rate, system (5) will reduce to the standard SEIR ordinary differential equation (ODE) model studied in [8, 9], and if we further assume that $P(t)$ is a step function, system (5) becomes the SEIR model with a discrete delay studied in [10]. Recently, a model of this type, but including the possibility of disease relapse, has been proposed in [11, 12] to investigate the transmission of herpes, and its global dynamics have been completely investigated in [5, 13].

To express the main idea and the approaches more clearly, we consider a simpler case in which all groups share the same natural death rate: $\delta_j = \delta$ for $j = 1, 2, \ldots, m$. Further, we assume that the functions $h_j(u)$ are disease specific only, implying that $h_j(u) = h(u)$ for $j = 1, 2, \ldots, m$. We choose the gamma distribution:

$$h(u) = h_{n,b}(u) = \frac{u^{n-1}}{(n-1)!b^n} e^{-u/b}, \tag{7}$$

where $b > 0$ is a real number and $n > 1$ is an integer, which is widely used and can approximate several frequently used distributions. For example, when $b \to 0^+$, $h_{n,b}(s)$ will approach the Dirac delta function, and when $n = 1$, $h_{n,b}(s)$ is an exponentially decaying function.

The main object of this paper is to carry out the well-known "linear chain trick" to system (6), transfer system into an equivalent ordinary differential equations system, and establish its global dynamics. We derive the basic reproductive number R_0 and show that R_0 completely determines the global dynamics of system (6). More specifically, if $R_0 \le 1$, the disease-free equilibrium is globally asymptotically stable and the disease dies out; if $R_0 > 1$, a unique endemic equilibrium exists and is globally asymptotically stable, and the disease persists at the endemic equilibrium. The global stability of P^* rules out any possibility for Hopf bifurcations and the existence of sustained oscillations. We should point out here that this work is motivated by Yuan and Zou [11, 12, 14]. In the proof we demonstrate that the graph-theoretic approach developed in [2, 3] can be successfully applied to construct suitable Lyapunov functionals and thus prove the global stability of the endemic equilibrium for model (6) with general exposed distribution and nonlinear incidence rate. Our work is also based on a recent work by Sun and Shi [15], which resolved the dynamics of multigroup SEIR epidemic models with nonlinear incidence of infection and nonlinear removal functions between compartments.

In Section 2, we first give the model, preliminaries and the basic reproduction number R_0. The global stability of the corresponding equilibria for $R_0 \le 1$ and $R_0 > 1$ is shown, respectively, in Section 3—the key results of this paper. And in Section 4, some numerical simulations are shown to illustrate the effectiveness of the proposed result.

2. Preliminaries

We make the following basic assumptions for the intrinsic growth rate of susceptible individuals in the ith group $\varphi_i(S_i)$ and the transmission functions $f_{ij}(S_i, I_j)$.

(A_1) φ_i are C^1 non-increasing functions on $[0, \infty)$ with $\varphi_i(0) > 0$, and there is a unique positive solution $\xi = S_i^0$ for the equation $\varphi_i(\xi) = 0$. $\varphi_i(S) > 0$ for $0 \le S < S_i^0$, and $\varphi_i(S) < 0$ for $S > S_i^0$; that is

$$\begin{gathered}\left[\varphi_i(S_i) - \varphi_i(S_i^0)\right](S_i - S_i^0) < 0, \\ \text{for } S_i \ne S_i^0, \ i = 1, 2, \ldots, m.\end{gathered} \tag{8}$$

(A_2) $f_{ij}(S_i, I_j) \le c_{ij}(S_i) I_j$ for all $I_j > 0$.

(A_3) $c_{ij}(S_i) \le c_{ij}(S_i^0)$, $0 < S_i < S_i^0$, $i, j = 1, \ldots, m$.

Following the technique and method in [14], define

$$\hat{b} \equiv \frac{b}{1 + \delta b}, \tag{9}$$

which can absorb the exponential term $e^{-\delta u}$ into the delay kernel. The second equation in (6) can be rewritten as

$$\begin{aligned} I_i'(t) = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} \int_0^t f_{ij}(S_i, I_j) h_{n,\hat{b}}(t-u)\, du \\ & - (\delta + \varepsilon_i + \gamma_i) I_i. \end{aligned} \tag{10}$$

For $l = 1, \ldots, n$, let

$$\begin{gathered} y_{i,l}(t) = \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_0^t f_{ij}(S_i, I_j) h_{l,\hat{b}}(t-u)\, du, \\ i = 1, 2, \ldots, m. \end{gathered} \tag{11}$$

Thus, for $l \in \{2, \ldots, n\}$, we obtain

$$\begin{aligned} \dot{y}_{i,l} = {} & h_{l,\hat{b}}(0) \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} f_{ij}(S_i, I_j) \\ & + \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{(l-1)(t-u)^{l-2}}{(l-1)!\hat{b}^l} e^{-(t-u)/\hat{b}} f_{ij}(S_i, I_j)\, du \\ & - \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{(t-u)^{l-1}}{(l-1)!\hat{b}^{l+1}} e^{-(t-u)/\hat{b}} f_{ij}(S_i, I_j)\, du \\ = {} & \frac{\left[y_{i,l-1} - y_{i,l}\right]}{\hat{b}}. \end{aligned} \tag{12}$$

For $l = 1$, we have

$$\begin{gathered} y_{i,1} = \sum_{j=1}^m \frac{\beta_{ij}\hat{b}}{(1+\delta b)^n} \int_{-\infty}^t \frac{e^{-(t-u)/\hat{b}}}{\hat{b}} f_{ij}(S_i, I_j)\, du, \\ i = 1, \ldots, m. \end{gathered} \tag{13}$$

It follows that

$$\begin{aligned} \dot{y}_{i,1} = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} f_{ij}(S_i, I_j) \\ & - \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} \int_{-\infty}^t \frac{e^{-(t-u)/\hat{b}}}{\hat{b}} f_{ij}(S_i, I_j)\, du \\ = {} & \sum_{j=1}^m \frac{\beta_{ij}}{(1+\delta b)^n} f_{ij}(S_i, I_j) - \frac{1}{\hat{b}} y_{i,1}, \quad i = 1, \ldots, m. \end{aligned} \tag{14}$$

Thus the integro-differential system (6) is equivalent to the ordinary differential equations

$$\begin{aligned}
S_i'(t) &= \varphi_i(S_i(t)) - \sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i(t), I_j(t)),\\
y_{i,1}'(t) &= \frac{1}{(1+\delta b)^n}\sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i(t), I_j(t)) - \frac{1}{\hat{b}}y_{i,1}(t),\\
y_{i,2}'(t) &= \frac{1}{\hat{b}}(y_{i,1}(t) - y_{i,2}(t)), \quad i = 1, 2, \dots, m,\\
&\vdots\\
y_{i,n}'(t) &= \frac{1}{\hat{b}}(y_{i,n-1}(t) - y_{i,n}(t)),\\
I_i'(t) &= \frac{1}{\hat{b}}y_{i,n}(t) - (\delta + \varepsilon_i + \gamma_i)I_i(t).
\end{aligned} \tag{15}$$

For initial condition

$$\begin{aligned}
&(S_1(0), y_{1,1}(0), \dots, y_{1,n}(0), I_1(0),\\
&\quad S_2(0), y_{2,1}(0), \dots, y_{1,n}(0), I_2(0), \dots,\\
&\quad S_m(0), y_{m,1}(0), \dots, y_{m,n}(0), I_m(0)) \in \mathrm{R}^{m(n+2)},
\end{aligned} \tag{16}$$

the existence, uniqueness, and continuity of the solution $(S_i, y_{i,1}, y_{i,2}, \dots, y_{i,n}, I_i)$ of system (15) follow from the standard theory of Volterra integro-differential equation [16]. It can also be verified that every solution of (15) with nonnegative initial condition remains nonnegative.

It follows from (A_1) and the first equation of (15) that $\limsup_{t\to\infty}S_i(t) \le S_i^0$ for all $i = 1, 2, \dots, m$. Let N_{φ_i} be the maximum of the function φ_i on R_+ and let q be a positive real number such that $q > \hat{b}N_{\varphi_i}$. Denote by Υ_i the ith tube for system (15); that is,

$$\Upsilon_i = (S_i, y_{i,1}, y_{i,2}, \dots, y_{i,n}, I_i). \tag{17}$$

It follows from a similar argument to that in [14] that we can show that the set D_ϵ defined by

$$\begin{aligned}
\Gamma_\epsilon = \Big\{&(S_i, y_{i,1}, y_{i,2}, \dots, y_{i,n}, I_i) \in \mathrm{R}_+^{m(n+2)} \mid\\
&S_i \le S_i^0 + \epsilon, S_i + (1+\delta b)^n y_{i,1} \le q + S_i^0,\\
&y_{i,l} \le \frac{q + S_i^0 + l\epsilon}{(1+\delta b)^n},\\
&I_i \le \frac{q + S_i^0 + (n+1)\epsilon}{\hat{b}(1+\delta b)^n(\delta + \epsilon_i + \gamma_i)},\\
&i = 1, 2, \dots, m,\ l = 2, 3, \dots, n\Big\}
\end{aligned} \tag{18}$$

is a forward invariant compact absorbing set for system (15) for $\epsilon > 0$ and that the set Γ_0 (i.e., when $\epsilon = 0$) is a forward invariant compact set.

Under the assumption (A_1), we know that system (15) always has the disease-free equilibrium

$$\begin{aligned}
P_0 = &(S_1^0, 0, \dots, 0, I_1^0, S_2^0, 0, \dots, 0, I_2^0, \dots, S_m^0, 0, \dots, 0, I_m^0)\\
&\in \mathrm{R}^{m(n+2)}.
\end{aligned} \tag{19}$$

An equilibrium P^* of (6) has the form $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_m^*, I_m^*) \in \mathrm{R}^{2m}$ with $S_i^* > 0$, $I_i^* > 0$, $i = 1, \dots, m$. Translating to the equivalent system (15), P^* is corresponding to

$$\begin{aligned}
\overline{P}^* = &(S_1^*, y_{1,1}^*, \dots, y_{1,n}^*, I_1^*, S_2^*, y_{2,1}^*, \dots, y_{2,n}^*, I_2^*, \dots,\\
&S_m^*, y_{m,1}^*, \dots, y_{m,n}^*, I_m^*) \in \mathrm{R}^{m(n+2)}.
\end{aligned} \tag{20}$$

$\overline{P}^*$ in the interior of Γ_0 is called an endemic equilibrium, and it satisfies the following equilibrium equations:

$$\begin{aligned}
0 &= \varphi_i(S_i^*) - \sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i^*, I_j^*),\\
0 &= \frac{1}{(1+\delta b)^n}\sum_{j=1}^{m}\beta_{ij}f_{ij}(S_i^*, I_j^*) - \frac{1}{\hat{b}}y_{i,1}^*,\\
0 &= \frac{1}{\hat{b}}(y_{i,1}^* - y_{i,2}^*),\\
&\vdots\\
0 &= \frac{1}{\hat{b}}(y_{i,n-1}^* - y_{i,n}^*),\\
0 &= \frac{1}{\hat{b}}y_{i,n}^* - (\delta + \varepsilon_i + \gamma_i)I_i^*.
\end{aligned} \tag{21}$$

The basic reproduction number R_0 is defined as the expected number of secondary cases produced by single infectious individual during its entire period of infectiousness in a completely susceptible population. For system (15), we can calculate it as the spectral radius of a matrix called the next generation matrix. Let

$$\mathscr{F} = \begin{pmatrix}
\frac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n} & \cdots & \frac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n}\\
\vdots & \ddots & \vdots\\
\frac{c_{m1}(S_m^0)\beta_{m1}}{(1+\delta b)^n} & \cdots & \frac{c_{mm}(S_m^0)\beta_{mm}}{(1+\delta b)^n}
\end{pmatrix}, \tag{22}$$

$$\begin{aligned}
\mathscr{V} &= \operatorname{diag}(\delta + \varepsilon_i + \gamma_i)\\
&= \begin{pmatrix}
\delta + \varepsilon_1 + \gamma_1 & 0 & \cdots & 0\\
0 & \delta + \varepsilon_2 + \gamma_2 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \delta + \varepsilon_m + \gamma_m
\end{pmatrix}.
\end{aligned}$$

Then the next generation matrix is

$$\mathscr{F}\mathscr{V}^{-1} = \begin{pmatrix} \frac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \\ \vdots & \ddots & \vdots \\ \frac{c_{m1}(S_1^0)\beta_{m1}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{mm}(S_1^0)\beta_{mm}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \end{pmatrix}, \tag{23}$$

and hence, the basic reproduction number R_0 is

$$R_0 = \rho\left(\mathscr{F}\mathscr{V}^{-1}\right) = \max\left\{|\lambda| ; \lambda \in \sigma\left(\mathscr{F}\mathscr{V}^{-1}\right)\right\}, \tag{24}$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ denote the spectral radius and the set of eigenvalues of a matrix, respectively. Since it can be verified that system (15) satisfies conditions (A_1)–(A_5) of Theorem 2 of [17], we have the following proposition.

Lemma 1. *For system* (15), *the disease-free equilibrium P_0 is locally asymptotically stable if $R_0 < 1$, while it is unstable if $R_0 > 1$.*

Following the method of [2], one defines a matrix

$$M^0 = \mathscr{V}^{-1}\mathscr{F} = \begin{pmatrix} \frac{c_{11}(S_1^0)\beta_{11}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} & \cdots & \frac{c_{1m}(S_1^0)\beta_{1m}}{(1+\delta b)^n(\delta+\varepsilon_1+\gamma_1)} \\ \vdots & \ddots & \vdots \\ \frac{c_{m1}(S_1^0)\beta_{m1}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} & \cdots & \frac{c_{mm}(S_1^0)\beta_{mm}}{(1+\delta b)^n(\delta+\varepsilon_m+\gamma_m)} \end{pmatrix}, \tag{25}$$

whose spectral radius has a similar threshold property to that of R_0, since both of the nonnegative matrices $\mathscr{F}\mathscr{V}^{-1}$ and M^0 are irreducible, and hence from the Perron-Frobenius theorem [7] that their spectral radii are given by each of their simple eigenvalues. Thus, we obtain $R_0 = \rho(\mathscr{F}\mathscr{V}^{-1}) = \rho(\mathscr{V}^{-1}\mathscr{F}) = \rho(M^0)$. Then the following lemma immediately follows.

Lemma 2. *$\rho(M^0) \le 1$ if and only if $R_0 \le 1$.*

3. Main Results

The following main theorems are summarized in terms of system (15).

Theorem 3. *Assume that the functions φ_i and f_{ij} satisfy assumptions (A_1)–(A_3), and the matrix $B = (\beta_{ij})_{m\times m}$ is irreducible and R_0 is defined by* (24).

(i) *If $R_0 \le 1$, then P_0 is the unique equilibrium of system* (15), *and P_0 is globally asymptotically stable in Γ_0.*

(ii) *If $R_0 > 1$, then P_0 is unstable, and system* (15) *is uniformly persistent in Γ_0.*

Proof. Let us define $M(S) = (\beta_{ij}c_{ij}(S_i)/(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i))_{m\times m}$, where $S = (S_1, S_2, \dots, S_m)^T$. Note that $M(S_0) = M^0$. Since $B = (\beta_{ij})_{m\times m}$ is irreducible, the matrix M^0 is also irreducible.

First we claim that there does not exist any endemic equilibrium $\overline{P}^*$ in Ω. Suppose that $S \ne S_0$. Then we have $0 < M(S) < M^0$. Since nonnegative matrix $M(S) + M^0$ is irreducible, it follows from the Perron-Frobenius theorem (see Corollary 2.1.5 of [7]) that $\rho(M(S)) < \rho(M^0) \le 1$. This implies that $M(S)I = I$ has only the trivial solution $I = 0$, where $I = (I_1, \dots, I_m)^T$. Hence the claim is true. Next we claim that the disease-free equilibrium P_0 is globally asymptotically stable in Γ_0. From the Perron-Frobenius (see Theorem 2.1.4 of [7]), the nonnegative irreducible matrix M^0 has a strictly positive left eigenvector $(\omega_1, \omega_2, \dots, \omega_m)$ associated with the eigenvalue $\rho(M^0)$ such that

$$(\omega_1, \omega_2, \dots, \omega_m)\,\rho\left(M^0\right) = (\omega_1, \omega_2, \dots, \omega_m)\,M^0. \tag{26}$$

Using this positive eigenvector, we construct the following Lyapunov function:

$$V_{\mathrm{DFE}} = \sum_{i=1}^{m} \frac{\omega_i}{\delta+\varepsilon_i+\gamma_i}\left(\sum_{j=1}^{n} y_{i,j} + I_i\right). \tag{27}$$

Computing the derivative of V_{DFE} along the solutions of (15) in Γ_0, we get

$$\begin{aligned} V'_{\mathrm{DFE}} &= \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} f_{ij}\left(S_i, I_j\right) - \omega_i I_i\right] \\ &\le \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}c_{ij}\left(S_i\right)}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} I_j - \omega_i I_i\right] \\ &\le \sum_{i=1}^{m}\left[\sum_{j=1}^{m} \frac{\omega_i\beta_{ij}c_{ij}\left(S_i^0\right)}{(1+\delta b)^n(\delta+\varepsilon_i+\gamma_i)} I_j - \omega_i I_i\right] \\ &= (\omega_1, \dots, \omega_m)\left[M^0 I - I\right] \\ &= \left[\rho\left(M^0\right) - 1\right](\omega_1, \dots, \omega_m)\,I. \end{aligned} \tag{28}$$

Thus, under the assumption $R_0 = \rho(M^0) < 1$, $V'_{\mathrm{DFE}} \le 0$, and $V'_{\mathrm{DFE}} = 0$ if and only if $I = 0$ and $S = S^0 = (S_1^0, S_2^0, \dots, S_m^0)$. Suppose that $\rho(M^0) = 1$. Then it follows from the previous that $V'_{\mathrm{DFE}} = 0$ implies

$$(\omega_1, \dots, \omega_m)\,M^0 I = (\omega_1, \dots, \omega_m)\,I. \tag{29}$$

Hence, if $S \ne S_0$, then $(\omega_1, \dots, \omega_m)M(S) < (\omega_1, \dots, \omega_m)M^0 = \rho(M^0)(\omega_1, \dots, \omega_m) = (\omega_1, \dots, \omega_m)$ and thus $I = 0$ is the only solution of (29). Summarizing the statements, we see that $V'_{\mathrm{DFE}} = 0$ if and only if $I = 0$ or $S = S_0$, which implies that the compact invariant subset of the set where $V'_{\mathrm{DFE}} = 0$ is only the singleton P_0. Thus, by LaSalle's invariance principle

[18], it follows that the disease-free equilibrium E^0 is globally asymptotically stable in Γ_0.

If $R_0 = \rho(M^0) > 1$, then

$$\begin{aligned}(\omega_1, \omega_2, \ldots, \omega_m) M^0 - (\omega_1, \omega_2, \ldots, \omega_m) \\ = \left[\rho\left(M^0\right) - 1\right](\omega_1, \omega_2, \ldots, \omega_m) > 0,\end{aligned} \tag{30}$$

and then, by continuity, we can obtain

$$V'_{\mathrm{DFE}} = (\omega_1, \ldots, \omega_m)\left[M^0 I - I\right] > 0, \tag{31}$$

in a neighborhood of P_0 in Γ_0; then P_0 is unstable.

Assume $R_0 = \rho(M^0) > 1$. By the uniform persistence result from [19] and a similar argument as in the proof of [2], the instability of P_0 implies the uniform persistence of (15). This together with the dissipativity of (15) resulted from the forward invariant and compact property of Γ_0 stated previously, implies which that (15) has an equilibrium in Γ_0, denoted by $\overline{P}^*$ (see, e.g., Theorem D.3 in [20]). □

In what follows we prove that the endemic equilibrium $\overline{P}^*$ of system (15) is globally asymptotically stable when $R_0 > 1$.

Throughout the paper, we denote

$$H(z) = z - 1 - \ln z. \tag{32}$$

Then $H(z) \geq 0$ for $z > 0$ and has global minimum at $z = 1$.

For convenience of notations, set $\overline{\beta}_{ij} = \beta_{ij} f_{ij}(S_i^*, I_j^*)$, $1 \leq i, j \leq m$, and

$$\overline{B} = \begin{bmatrix} \sum_{l\neq 1}\overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{ml} \\ -\overline{\beta}_{12} & \sum_{l\neq 2}\overline{\beta}_{2l} & \cdots & -\overline{\beta}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1m} & -\overline{\beta}_{2m} & \cdots & \sum_{l\neq m}\overline{\beta}_{ml} \end{bmatrix}. \tag{33}$$

Then, $\overline{B}$ is also irreducible. One knows that the solution space of the linear system

$$\overline{B}v = 0 \tag{34}$$

has dimension 1 and

$$(v_1, v_2, \ldots v_m) = (C_{11}, \ldots, C_{mm}) \tag{35}$$

gives a base of this space, where $C_{kk} > 0$, $k = 1, 2, \ldots, m$, is the cofactor of the kth diagonal entry of $\overline{B}$. To get the global stability of $\overline{P}^*$, the following assumptions in [15] are proposed:

(A_4) : $[\varphi_i(S_i) - \varphi_i(S_i^*)](S_i - S_i^*) < 0$ for $S_i \neq S_i^*$, $S_i \in [0, S_i^0]$,

(A_5) : For $S_i \neq S_i^*$, $[\varphi_i(S_i) - \varphi_i(S_i^*)] \cdot [f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)] < 0$.

(A_6) : For $S_i, I_j > 0$,

$$\begin{aligned}&\left(f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right) - f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)\right) \\ &\quad\cdot\left(\frac{f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right)}{I_j} - \frac{f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)}{I_j^*}\right) \leq 0.\end{aligned} \tag{36}$$

A difficult mathematical question for system (15) is that of whether the endemic equilibrium $\overline{P}^*$ is unique when $R_0 > 1$ and whether $\overline{P}^*$ is globally asymptotically stable when it is unique. Our main global stability result is given.

Theorem 4. *Consider system* (15). *Assume that* (A_4)–(A_6) *hold and the matrix* $B = (\beta_{ij})_{m\times m}$ *is irreducible. If* $R_0 > 1$, *then there is a unique endemic equilibrium* $\overline{P}^*$ *for system* (15), *and* $\overline{P}^*$ *is globally asymptotically stable in* Γ_0.

Proof. We show that $\overline{P}^*$ is globally asymptotically stable in Γ_0, which implies that there exists a unique endemic equilibrium.

Consider a Lyapunov function as

$$\begin{aligned}V_{\mathrm{EE}} = {} & S_i - f_{ii}\left(S_i^*, I_i^*\right)\int_{S_i^*}^{S_i} \frac{d\xi}{f_{ii}\left(\xi, I_i^*\right)} \\ & + (1+\delta b)^n\left[\sum_{j=1}^{n}\left(y_{i,j} - y_{i,j}^* - y_{i,j}^* \ln\frac{y_{i,j}}{y_{i,j}^*}\right)\right. \\ & \left. + I_i - I_i^* - I_i^* \ln\frac{I_i}{I_i^*}\right].\end{aligned} \tag{37}$$

This function has a linear part V_{EE} expressed by

$$L_{\mathrm{EE}} = S_i + (1+\delta b)^n\left[\sum_{j=1}^{n}\left(y_{i,j} - y_{i,j}^*\right) + I_i - I_i^*\right]. \tag{38}$$

First, calculating the derivatives of L_{EE}, we obtain

$$L'_{\mathrm{EE}} = \varphi_i\left(S_i\right) - (1+\delta b)^n\left(\delta + \varepsilon_i + \gamma_i\right) I_i. \tag{39}$$

Calculating the time derivative of V_{EE} along the solutions of system (15) and using equilibrium equation (21), we have

$$\begin{aligned}V'_{\mathrm{EE}} &= L'_{\mathrm{EE}} - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)}\dot{S}_i + (1+\delta b)^n\left[\sum_{j=1}^{n}\frac{y_{i,j}^*}{y_{i,j}}\dot{y}_{i,j} + \frac{I_i^*}{I_i}\dot{I}_i\right] \\ &= \varphi_i\left(S_i\right) - (1+\delta b)^n\left(\delta + \varepsilon_i + \gamma_i\right) I_i \\ &\quad - \left\{\left(\varphi_i\left(S_i\right) - \sum_{j=1}^{m}\beta_{ij} f_{ij}\left(S_i, I_j\right)\right)\frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)}\right.\end{aligned}$$

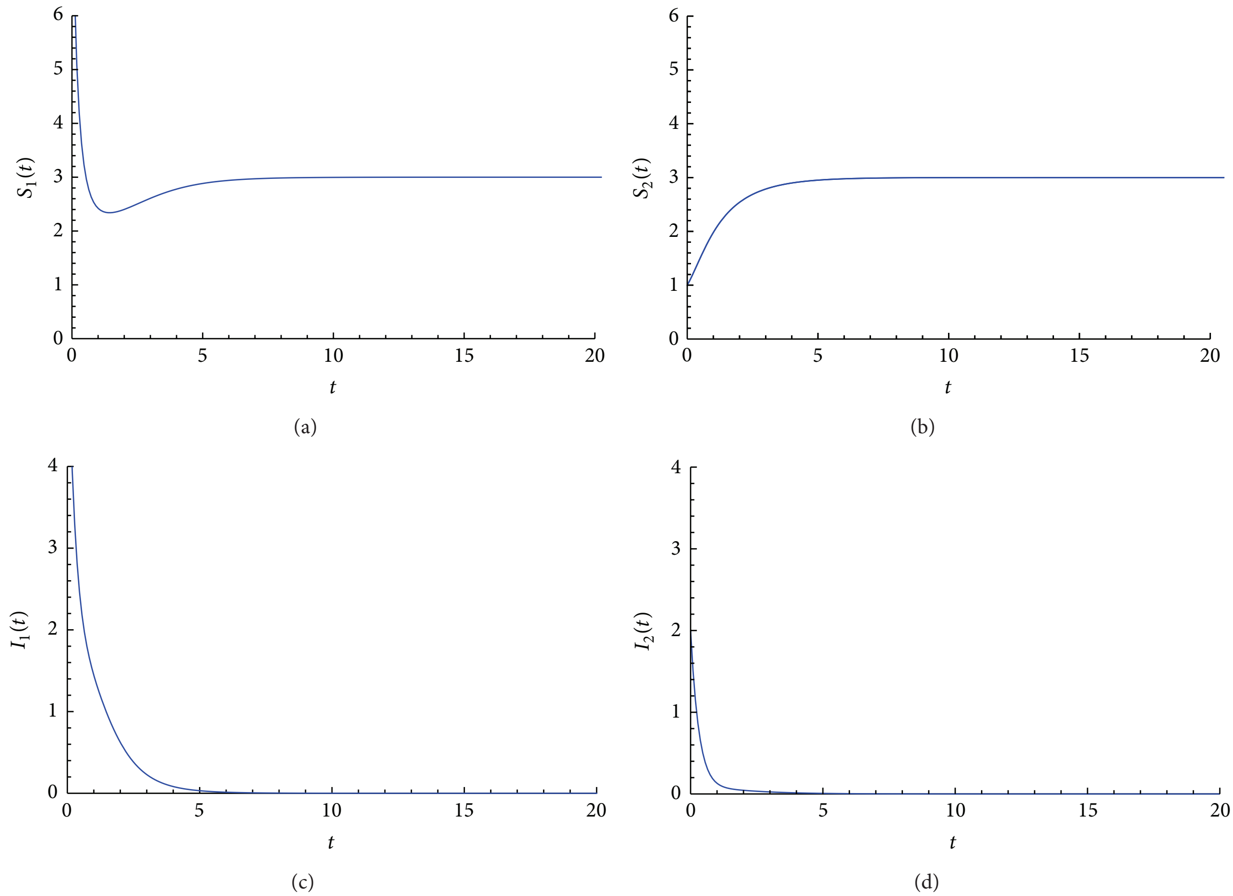


FIGURE 1: Trajectories of $S_1(t), I_1(t), S_2(t)$, and $I_2(t)$ for $R_0 = 0.051 < 1$, and $P_0 = (3, 0, 0, 0, 3, 0, 0, 0)$ is globally stable. $S_1(t)$, $S_2(t)$, $I_1(t)$, and $I_2(t)$ versus t are illustrated by (a), (b), (c), and (d). Initial values are $S_1(0) = 9, S_2(0) = 1, y_{1,1}(0) = 2, y_{1,2}(0) = 2, y_{2,1}(0) = 0, y_{2,2}(0) = 0, I_1(0) = 6$, and $I_2(0) = 2$.

$$
\begin{aligned}
&+ (1+\delta b)^n \left[\sum_{j=1}^{m} \frac{\beta_{ij} f_{ij}\left(S_i, I_j\right) y^*_{i,1}}{(1+\delta b)^n y_{i,1}} - \frac{y^*_{i,1}}{\widehat{b}} + \frac{1}{\widehat{b}} \sum_{k=2}^{n} y^*_{i,k} \left(\frac{y_{i,k-1}}{y_{i,k}} - 1 \right) + \frac{y_{i,n} I_i^*}{\widehat{b} I_i} - \left(\delta + \varepsilon_i + \gamma_i\right) I_i^* \right] \Bigg\} \\
&= \varphi_i\left(S_i\right) \left(1 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} \right) - \sum_{j=1}^{m} \beta_{ij} f_{ij}\left(S_i^*, I_j^*\right) \frac{y^*_{i,1} f_{ij}\left(S_i, I_j\right)}{y_{i,1} f_{ij}\left(S_i^*, I_j^*\right)} - \frac{(1+\delta b)^n}{\widehat{b}} \sum_{k=2}^{n} \frac{y^*_{i,k} y_{i,k-1}}{y_{i,k}} \\
&\quad + \frac{(1+\delta b)^n}{\widehat{b}} n y_i^* - \frac{(1+\delta b)^n}{\widehat{b}} y^*_{i,n} \frac{y_{i,n} I_i^*}{y^*_{i,n} I_i} + (1+\delta b)^n \left(\delta + \varepsilon_i + \gamma_i\right) I_i^* \\
&\quad + \sum_{j=1}^{m} \beta_{ij} f_{ij}\left(S_i^*, I_j^*\right) \frac{f_{ij}\left(S_i, I_j\right) f_{ii}\left(S_i^*, I_i^*\right)}{f_{ij}\left(S_i^*, I_j^*\right) f_{ii}\left(S_i, I_i^*\right)} - (1+\delta b)^n \left(\delta + \varepsilon_i + \gamma_i\right) I_i \\
&= \left(\varphi_i\left(S_i\right) - \varphi_i\left(S_i^*\right)\right) \left(1 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} \right) \\
&\quad + \sum_{j=1}^{m} \overline{\beta}_{ij} \left\{ n + 2 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} - \sum_{k=2}^{n} \frac{y^*_{i,k} y_{i,k-1}}{y_{i,k} y^*_{i,k-1}} - \frac{y_{i,n} I_i^*}{y^*_{i,n} I_i} - \frac{I_i}{I_i^*} - \frac{f_{ij}\left(S_i, I_j\right) y^*_{i,1}}{f_{ij}\left(S_i^*, I_j^*\right) y_{i,1}} + \frac{f_{ij}\left(S_i, I_j\right) f_{ii}\left(S_i^*, I_i^*\right)}{f_{ij}\left(S_i^*, I_j^*\right) f_{ii}\left(S_i, I_i^*\right)} \right\}.
\end{aligned}
\tag{40}
$$

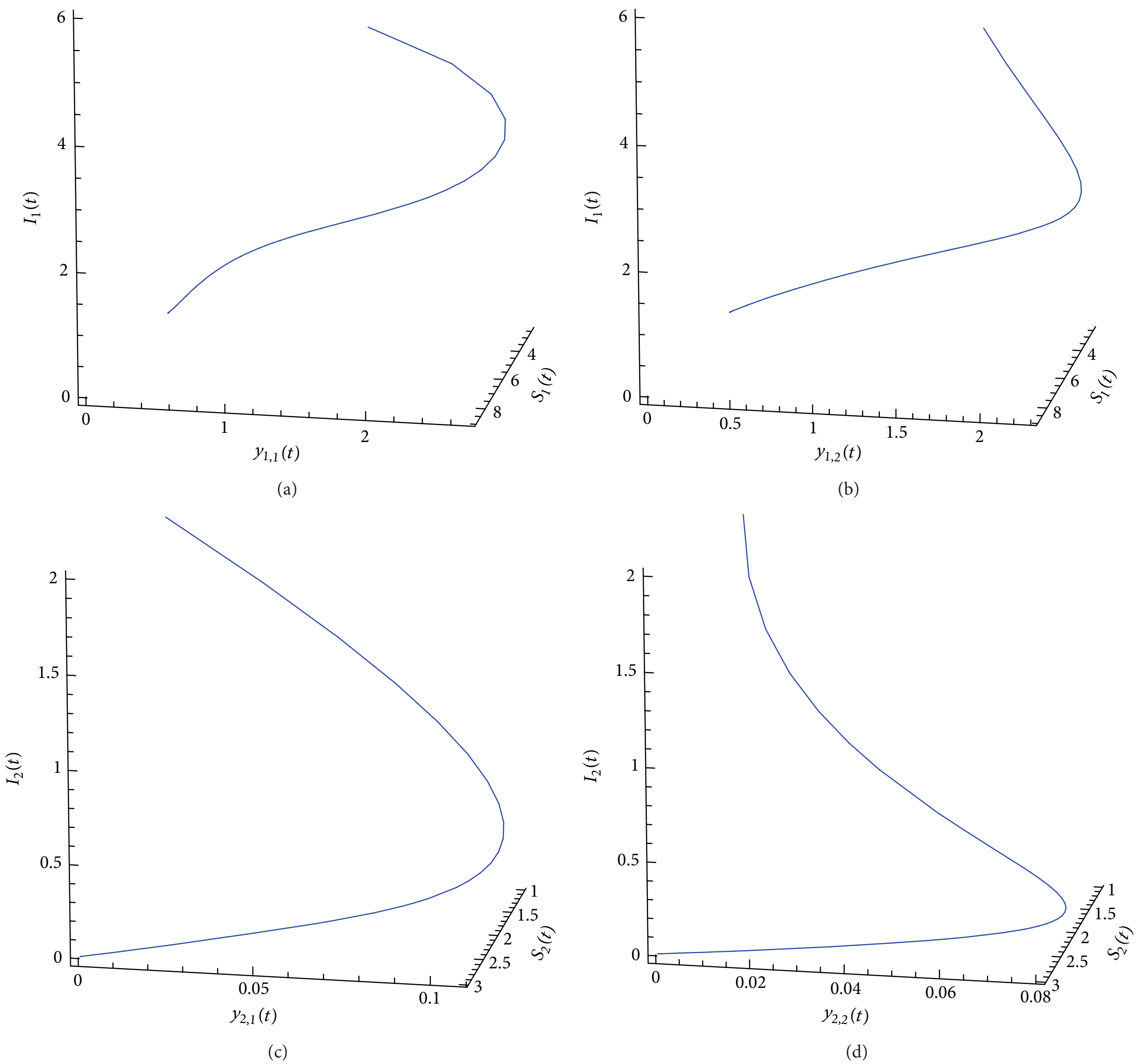


FIGURE 2: Numerical simulation of (45) with $R_0 = 0.051 < 1$; hence $P_0 = (3,0,0,0,3,0,0,0)$ is globally stable. Graphs (a) and (b) illustrate that $S_1(\mathrm{t})$, $y_{1,1}(t)$, $y_{1,2}(t)$ and $I_1(t)$ will eventually towards to steady state. Graphs (c) and (d) illustrate that $S_2(t)$, $y_{2,1}(t)$, $y_{2,2}(t)$, and $I_2(t)$ will eventually towards to steady state. Initial values are $S_1(0) = 9, S_2(0) = 1, y_{1,1}(0) = 2, y_{1,2}(0) = 2, y_{2,1}(0) = 0, y_{2,2}(0) = 0, I_1(0) = 6$, and $I_2(0) = 2$.

It follows from the assumptions (A_4)-(A_5) that V'_{EE} can be estimated by

$$\begin{aligned}
V'_{\mathrm{EE}} \le \sum_{i,j=1}^{m} \overline{\beta}_{ij} \Bigg\{ & G_i(I_i) - G_j(I_j) + H\left(\frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\right) \\
& + H\left(\frac{f_{ij}(S_i, I_j)\, y_{i,1}^*}{f_{ij}(S_i^*, I_j^*)\, y_{i,1}}\right) \\
& + \sum_{k=2}^{n} H\left(\frac{y_{i,k}^* y_{i,k-1}}{y_{i,k} y_{i,k-1}^*}\right) \\
& + H\left(\frac{y_{i,n} I_i^*}{y_{i,n}^* I_i}\right) \\
& + H\left(\frac{I_j f_{ii}(S_i, I_i^*)\, f_{ij}(S_i^*, I_j^*)}{I_j^* f_{ii}(S_i^*, I_i^*)\, f_{ij}(S_i, I_j)}\right)
\end{aligned}$$

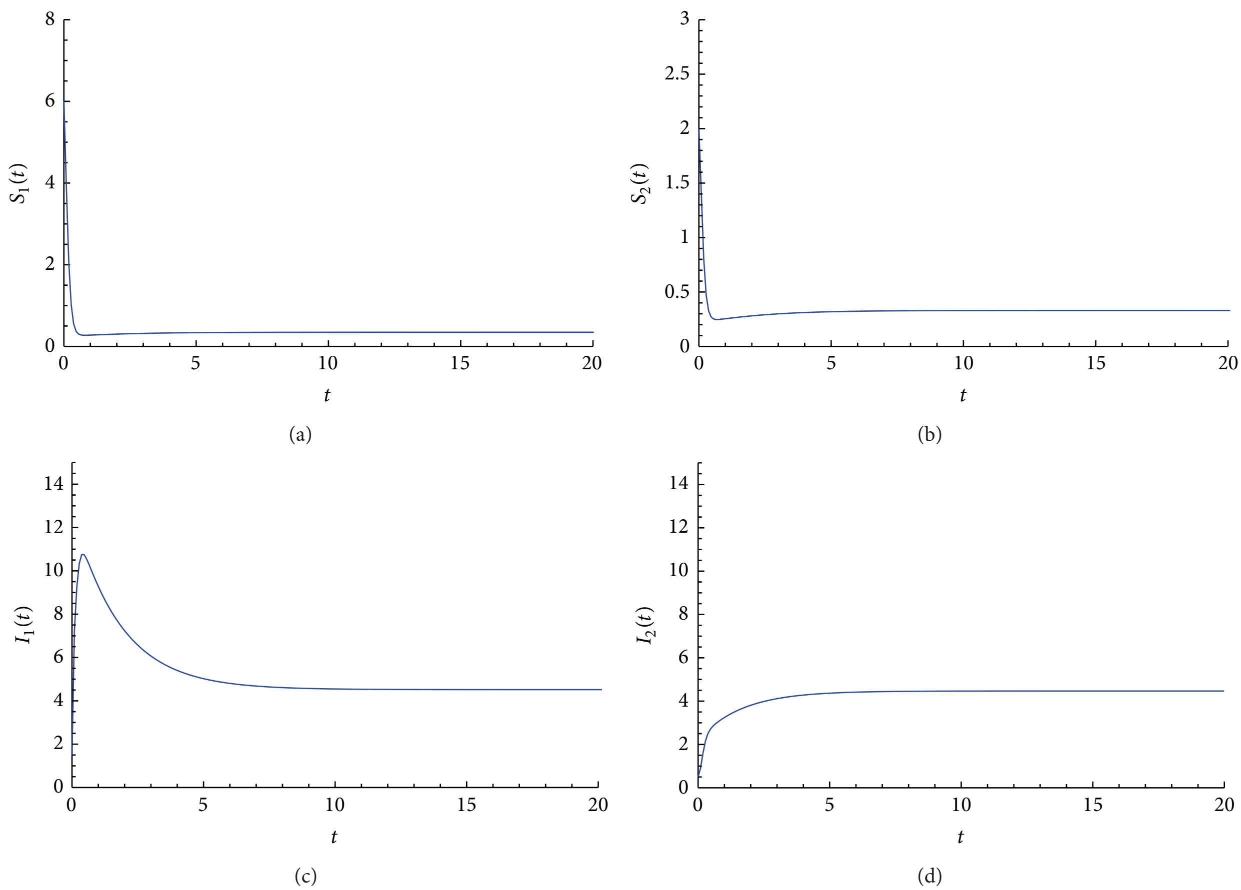


FIGURE 3: Trajectories of $S_1(t), I_1(t), S_2(t)$, and $I_2(t)$ for $R_0 = 1.67355 > 1$, and $\overline{P}^* = (0.347644, 0.0760948, 0.0760948, 4.51674, 0.330353, 0.0765909, 0.0765909, 4.4678)$ is globally stable. $S_1(t)$, $S_2(t)$, $I_1(t)$, and $I_2(t)$ versus t are illustrated by (a), (b), (c), and (d). Initial values are $S_1(0) = 6, S_2(0) = 2, y_{1,1}(0) = 3, y_{1,2}(0) = 3, y_{2,1}(0) = 0.1, y_{2,2}(0) = 0.1, I_1(0) = 1.5$, and $I_2(0) = 0.5$.

$$+\left[\frac{f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right)}{f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)} - 1\right] \cdot \left[1 - \frac{I_j f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right)}{I_j^* f_{ii}\left(S_i^*, I_i^*\right) f_{ij}\left(S_i, I_j\right)}\right]\Bigg\}. \tag{41}$$

From the assumption (A_6) and (32), we know that

$$V'_{\mathrm{EE}} \leq \sum_{i,j=1}^{m} \overline{\beta}_{ij} \left\{G_i\left(I_i\right) - G_j\left(I_j\right)\right\}, \tag{42}$$

where $G_i(I_i) = -I_i/I_i^* + \ln(I_i/I_i^*)$.

Obviously, the equalities in (41) and (42) hold if and only if

$$\frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)} = 1,$$

$$\left(1 - \frac{f_{ii}\left(S_i^*, I_i^*\right)}{f_{ii}\left(S_i, I_i^*\right)}\right)\left[\varphi_i\left(S_i\right) - \varphi_i\left(S_i^*\right)\right] = 0,$$

$$\left[\frac{f_{ij}\left(S_i, I_j\right) f_{ii}\left(S_i^*, I_i^*\right)}{f_{ij}\left(S_i^*, I_j^*\right) f_{ii}\left(S_i, I_i^*\right)} - 1\right] \times \left[1 - \frac{f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i^*, I_j^*\right) I_j}{f_{ii}\left(S_i, I_i^*\right) f_{ij}\left(S_i, I_j\right) I_j^*}\right] = 0. \tag{43}$$

That is, $S_i = S_i^*$, $I_i = I_i^*$, $i = 1, 2, \ldots, m$. We can show that V_{EE} and $\overline{\beta}_{ij}$ satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [21]. Therefore, the function

$$L = \sum_{i=1}^{n} \nu_i V_{\mathrm{EE}} \tag{44}$$

is a Lyapunov function for system (15); namely, $L'|_{(15)} \leq 0$ for $\overline{P}^* \in \Gamma_0$. One can only show that the largest invariant subset, where $L'|_{(15)} = 0$, is the singleton $\{\overline{P}^*\}$ by the same

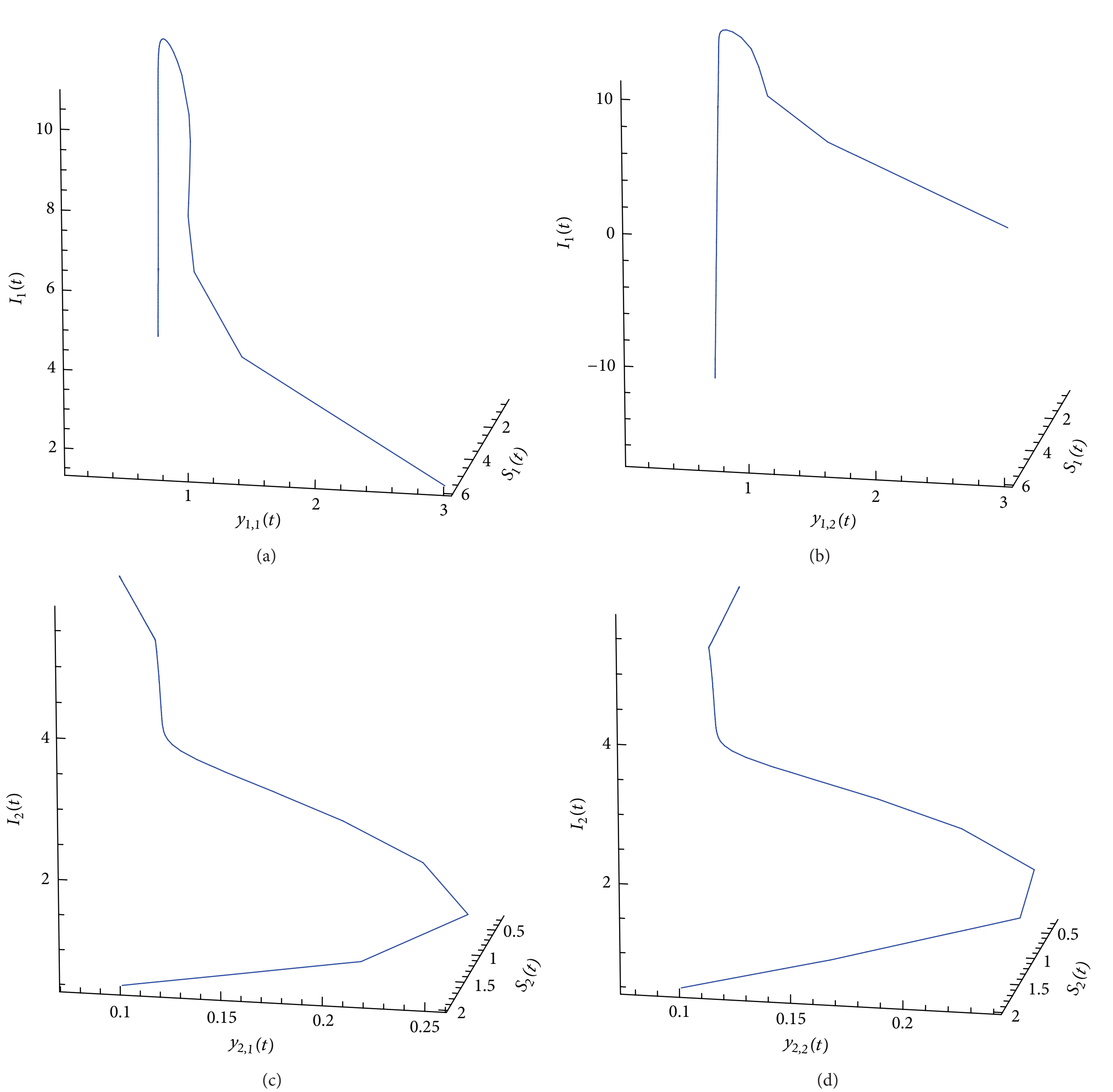


FIGURE 4: Numerical simulation of (45) with $R_0 = 1.67355 > 1$; hence $P^* = (0.347644, 0.0760948, 0.0760948, 4.51674, 0.330353, 0.0765909, 0.0765909, 4.4678)$ is globally stable. Graphs (a) and (b) illustrate that $S_1(t)$, $y_{1,1}(t)$, $y_{1,2}(t)$, and $I_1(t)$ will eventually towards to steady state. Graphs (c) and (d) illustrate that $S_2(t)$, $y_{2,1}(t)$, $y_{2,2}(t)$, and $I_2(t)$ will eventually towards to steady state. Initial values are $S_1(0) = 6, S_2(0) = 2$, $y_{1,1}(0) = 3$, $y_{1,2}(0) = 3$, $y_{2,1}(0) = 0.1$, $y_{2,2}(0) = 0.1$, $I_1(0) = 1.5$, and $I_2(0) = 0.5$.

argument as in [2–5, 13, 21]. By LaSalle's invariance principle, $\overline{P}^*$ is globally asymptotically stable in Γ_0. This completes the proof of Theorem 4. □

Remark 5. We show a complete proof for global asymptotic stability of unique endemic equilibrium of system (15). In the case of $f_{ij}(S_i, I_j) = S_i I_j$, system (15) will reduce to the system studied in [14, 22]. Here Theorem 4 extends related results in [14, 22] to a result to a more general case allowing a nonlinear incidence rate. Our result also cover the related results of single group model in [13] for the case of $f(S, I) = f(S)I$.

4. Numerical Example

Consider the system (15) when $m = 2$, $n = 2$, $\varphi_i(S_i(t)) = 3 - S_i$, and $f_{ij}(S_i, I_j) = S_i I_j$, $i, j = 1, 2$. One then has a two-group model as follows:

$$S_1'(t) = 3 - S_1 - [\beta_{11} S_1(t) I_1(t) + \beta_{12} S_1(t) I_2(t)],$$

$$y_{1,1}'(t) = \frac{1}{(1+\delta b)^n} [\beta_{11} S_1(t) I_1(t) + \beta_{12} S_1(t) I_2(t)] - \frac{1}{b} y_{1,1}(t),$$

$$y_{1,2}'(t) = \frac{1}{\bar{b}}\left(y_{1,1}(t) - y_{1,2}(t)\right),$$

$$I_1'(t) = \frac{1}{\bar{b}} y_{1,2}(t) - \left(\delta + \varepsilon_1 + \gamma_1\right) I_1(t),$$

$$S_2'(t) = 3 - S_2 - \left[\beta_{21} S_2(t) I_1(t) + \beta_{22} S_2(t) I_2(t)\right],$$

$$y_{2,1}'(t) = \frac{1}{(1+\delta b)^n}\left[\beta_{21} S_2(t) I_1(t) + \beta_{22} S_2(t) I_2(t)\right] - \frac{1}{\bar{b}} y_{2,1}(t),$$

$$y_{2,2}'(t) = \frac{1}{\bar{b}}\left(y_{2,1}(t) - y_{2,2}(t)\right),$$

$$I_2'(t) = \frac{1}{\bar{b}} y_{2,2}(t) - \left(\delta + \varepsilon_2 + \gamma_2\right) I_2(t). \tag{45}$$

If we choose parameters as $\beta_{11} = 5/24$, $\beta_{12} = 1$, $\beta_{21} = 1/36$, $\beta_{22} = 1/2$, $\delta = 0.8$, $\varepsilon_1 = 2$, $\varepsilon_2 = 2$, $\gamma_1 = 1/4$, and $\gamma_2 = 1/4$, we can compute $R_0 = 0.051 < 1$, and hence $P_0 = (3, 0, 0, 0, 3, 0, 0, 0)$ is the unique equilibrium of system (45) and it is globally stable from Theorem 4 (see Figures 1 and 2).

On the other hand, if β_{ij} are chosen as $\beta_{11} = 0.7$, $\beta_{12} = 1$, $\beta_{21} = 0.8$, $\beta_{22} = 1$, $\delta = 0.5$, $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.03$, $\gamma_1 = 0.05$, and $\gamma_2 = 0, 05$, we can compute $R_0 = 1.67355 > 1$, and hence $\overline{P}^* = (0.347644, 0.0760948, 0.0760948, 4.51674, 0.330353, 0.0765909, 0.0765909, 4.4678)$ is the unique equilibrium of system (45) and it is globally stable from Theorem 4 (see Figures 3 and 4).

Acknowledgments

The authors wish to thank the reviewers for their valuable comments and suggestions that led to truly significant improvement of the paper. J. Wang is supported by the Science and Technology Research Project of the Department of Education of Heilongjiang Province (no. 12531495).

References

[1] H. R. Thieme, *Mathematics in Population Biology*, Princeton University Press, Princeton, NJ, USA, 2003.

[2] H. Guo, M. Y. Li, and Z. Shuai, "Global stability of the endemic equilibrium of multigroup SIR epidemic models," *Canadian Applied Mathematics Quarterly*, vol. 14, no. 3, pp. 259–284, 2006.

[3] H. Guo, M. Y. Li, and Z. Shuai, "A graph-theoretic approach to the method of global Lyapunov functions," *Proceedings of the American Mathematical Society*, vol. 136, no. 8, pp. 2793–2802, 2008.

[4] H. Shu, D. Fan, and J. Wei, "Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 4, pp. 1581–1592, 2012.

[5] J. Wang, Y. Takeuchi, and S. Liu, "A multi-group SVEIR epidemic model with distributed delay and vaccination," *International Journal of Biomathematics*, vol. 5, no. 3, Article ID 1260001, 18 pages, 2012.

[6] P. van den Driessche, L. Wang, and X. Zou, "Modeling diseases with latency and relapse," *Mathematical Biosciences and Engineering*, vol. 4, no. 2, pp. 205–219, 2007.

[7] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1979.

[8] A. Korobeinikov and P. K. Maini, "A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence," *Mathematical Biosciences and Engineering*, vol. 1, no. 1, pp. 57–60, 2004.

[9] M. Y. Li and J. S. Muldowney, "Global stability for the SEIR model in epidemiology," *Mathematical Biosciences*, vol. 125, no. 2, pp. 155–164, 1995.

[10] G. Huang and Y. Takeuchi, "Global analysis on delay epidemiological dynamic models with nonlinear incidence," *Journal of Mathematical Biology*, vol. 63, no. 1, pp. 125–139, 2011.

[11] P. van den Driessche and X. Zou, "Modeling relapse in infectious diseases," *Mathematical Biosciences*, vol. 207, no. 1, pp. 89–103, 2007.

[12] P. van den Driessche, L. Wang, and X. Zou, "Impact of group mixing on disease dynamics," *Mathematical Biosciences*, vol. 228, no. 1, pp. 71–77, 2010.

[13] S. Liu, S. Wang, and L. Wang, "Global dynamics of delay epidemic models with nonlinear incidence rate and relapse," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 119–127, 2011.

[14] Z. Yuan and X. Zou, "Global threshold property in an epidemic model for disease with latency spreading in a heterogeneous host population," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3479–3490, 2010.

[15] R. Sun and J. Shi, "Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 280–286, 2011.

[16] R. K. Miller, *Nonlinear Volterra Integral Equations*, W. A. Benjamin, New York, NY, USA, 1971.

[17] P. van den Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, no. 1-2, pp. 29–48, 2002.

[18] J. P. Lasalle, *The Stability of Dynamical Systems*, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1976.

[19] H. I. Freedman, M. X. Tang, and S. G. Ruan, "Uniform persistence and flows near a closed positively invariant set," *Journal of Dynamics and Differential Equations*, vol. 6, no. 4, pp. 583–600, 1994.

[20] H. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, UK, 1995.

[21] M. Y. Li and Z. S. Shuai, "Global-stability problem for coupled systems of differential equations on networks," *Journal of Differential Equations*, vol. 248, no. 1, pp. 1–20, 2010.

[22] Z. Shuai and P. van den Driessche, "Impact of heterogeneity on the dynamics of an SEIR epidemic model," *Mathematical Biosciences and Engineering*, vol. 9, no. 2, pp. 393–411, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 929178, 8 pages
http://dx.doi.org/10.1155/2013/929178

Research Article

The $L\omega$-Compactness in $L\omega$-Spaces

Shui-Li Chen and Jin-Lan Huang

Department of Mathematics, School of Science, Jimei University, Xiamen, Fujian 361021, China

Correspondence should be addressed to Shui-Li Chen; chenshuili2013@gmail.com

Received 15 March 2013; Accepted 5 April 2013

Academic Editor: Zhenkun Huang

The concepts of $\alpha\omega$-remote neighborhood family, $\gamma\omega$-cover, and $L\omega$-compactness are defined in $L\omega$-spaces. The characterizations of $L\omega$-compactness are systematically discussed. Some important properties of $L\omega$-compactness such as ω-closed heredity, arbitrarily multiplicative property, and preserving invariance under ω-continuous mappings are obtained. Finally, the Alexander ω-subbase lemma and the Tychonoff product theorem with respect to $L\omega$-compactness are given.

1. Introduction

Compactness is one of the most important notions in general topology, fuzzy topology, and L-topology. Many research workers have presented various kinds of compactness [1–19] by means of introducing various operators, such as closure operator, θ-closure operator, δ-closure operator, R-closure operator, S-closure operator, SR-closure operator, and PS-closure operator; because the above operators are all order preserving. That is, they satisfy the following conditions: (i) if $A, B \in L^X$ and $A \leqslant B$, then $\omega(A) \leqslant \omega(B)$; (ii) for any $A \in L^X$, $A \leqslant \omega(A)$, where $\omega : L^X \to L^X$ can take and of the above operators, L^X is the family of all L-sets defined on X and with value in L, L is a fuzzy lattice, and 1_X is the greatest L-set of L^X. We introduced a kind of generalized fuzzy space called $L\omega$-space in [20] in order to unify various elementary concepts in L-topological spaces. In the present paper, we will propose and study a generalized compactness which will be called $L\omega$-compactness in $L\omega$-spaces. The $L\omega$-compactness is a unified form of N-compactness [16, 19], near N-compactness [5], almost N-compactness [6], S-compactness [13], SR-compactness [1], PS-compactness [2], δ-compactness [9], θ-compactness [18], and so forth.

2. Preliminaries

Throughout this paper, L denotes a fuzzy lattice, that is, a completely distributive lattice with order-reserving involution $'$, 0 and 1 denote the least and greatest elements of L, respectively, and M denotes the set that consisting of all nonzero $\vee$-irreducible elements of L. Let X be a nonempty crisp set, L^X the set of all L-fuzzy sets (briefly, L-sets) on X, and $M^*(L^X) = \{x_\alpha : \alpha \in M,\ x \in X\}$ the set of all nonzero $\vee$-irreducible elements (i.e., so-called molecules [17] or points for short) of L^X. The least and the greatest elements of L^X will be denoted by 0_X and 1_X, respectively. For any $\alpha \in M$, $\beta(\alpha)$ is called the greatest minimal set of α [12], and $\beta^*(\alpha) = \beta(\alpha) \cap M$ is said to be the standard minimal set of α [17].

Definition 1 (Chen and Cheng [20]). Let X be a nonempty crisp set.

(i) An operator ω: $L^X \to L^X$ is said to be an ω-operator if (1) for all $A, B \in L^X$ and $A \leqslant B$, $\omega(A) \leqslant \omega(B)$; (2) for all $A \in L^X$, $A \leqslant \omega(A)$.

(ii) An L-set $A \in L^X$ is called an ω-set if $\omega(A) = A$.

(iii) Put $\Omega = \{A \in L^X \mid \omega(A) = A\}$, and call the pair (L^X, ω) an $L\omega$-space.

Definition 2 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $x_\alpha \in M^*(L^X)$. If there exists a $Q \in \Omega$ such that $x_\alpha \nleq Q$ and $P \leqslant Q$, then call P an ω-remote neighborhood (briefly, ωR-neighborhood) of x_α. The collection of all ωR-neighborhoods of x_α is denoted by $\omega\eta(x_\alpha)$. If $A \nleq P$ for each $P \in \omega\eta(x_\alpha)$, then x_α is said to be an ω-adherence point of A and the union of all ω-adherence points of A is called the ω-closure of A and denoted by $\omega\,\mathrm{cl}(A)$. If $A = \omega\,\mathrm{cl}(A)$, then call A an ω-closed set and call A' an ω-open set. If P is an ω-closed set and $x_\alpha \not\leq P$, then P is said to be an ω-closed remote neighborhood (briefly, ωCR-neighborhood) of x_α and the collection of all ωCR-neighborhoods of x_α is denoted by $\omega\eta^-(x_\alpha)$. Note that $\omega C(L^X)$ and $\omega O(L^X)$ are the family of all ω-closed sets and all ω-open sets in L^X, respectively.

Definition 3 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $\omega\,\mathrm{int}(A) = \vee\{B \in L^X \mid B \leqslant A$ and B is an ω-open set in $L^X\}$. We call $\omega\,\mathrm{int}(A)$ the ω-interior of A. Obviously, A is ω-open if and only if $A = \omega\,\mathrm{int}(A)$.

Definition 4 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let N be a molecular net in L^X, and let $x_\alpha \in M^*(L^X)$. If N is eventually not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-limit point of N (or N ω-converges to x_α). If N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-cluster point of N (or N ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of N is written by ω-lim N (ω-*adN*).

Definition 5 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let I be an ideal in L^X, and let $x_\alpha \in M^*(L^X)$. If $\omega\eta^-(x_\alpha) \subseteq I$, then x_α is called an ω-limit point of I (or $I\omega$-converges to x_α). If $P \vee B \neq 1_X$ for each $P \in \omega\eta^-(x_\alpha)$ and each $B \in I$, then x_α is called an ω-cluster point of I (or I ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of I is denoted by ω-lim I (ω-*adI*).

Definition 6 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $x_\alpha \in M^*(L^X)$, and $\beta, \gamma \in \omega O(L^X)$. Then,

(i) β is said to be an ω-base in (L^X, Ω) if for each $G \in \omega O(L^X)$, there exists a subfamily φ of β such that $G = \vee_{B\in\varphi} B$;

(ii) γ is said to be an ω-subbase in (L^X, Ω) if the collection consisting of all intersections of any finite elements in γ is an ω-base in (L^X, Ω).

Definition 7 (Chen and Cheng [20]). Assume (L^X, Ω_i) to be an $L\omega_i$-space ($i = 1, 2$) and $f : (L^X, \Omega_1) \to (L^Y, \Omega_2)$ an L-valued Zadeh's type function [17]. If $f^{\leftarrow}(B) \in \omega_1 O(L^X)$ for each $B \in \omega_2 O(L^Y)$, then call $f(\omega_1, \omega_2)$-continuous.

3. $L\omega$-Compact Set and Its Characteristics

In this section, we will introduce the concepts of $\alpha\omega$-remote neighborhood family and $\gamma\omega$-cover in an $L\omega$-space first, propose the notion of $L\omega$-compactness by making use of $\alpha\omega$-remote neighborhood family next, and then discuss the characteristics of $L\omega$-compactness.

Definition 8. Suppose (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\alpha \in M$, and $\Phi \subseteq \omega C(L^X)$. If there exists a $P \in \Phi$ such that $P \in \omega\eta^-(x_\alpha)$ for each molecule x_α in A, then Φ is called an $\alpha\omega$-remote neighborhood family (briefly, $\alpha\omega$-RF) of A, in symbol $\wedge\Phi < A(\alpha\omega)$. If there exists a nonzero $\vee$-irreducible element $\lambda \in \beta^*(\alpha)$ with $\wedge\Phi < A(\lambda\omega)$, then Φ is said to be an $(\alpha\omega)^-$-RF, in symbol $\wedge\Phi \ll A(\alpha\omega)$.

Definition 9. Assume (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\gamma' \in M$, and $\Gamma \subseteq \omega O(L^X)$. If there is a $B \in \Gamma$ such that $B(x) \not\leq \gamma$ for each $x \in \tau_{\gamma'}(A) = \{x \in X \mid A(x) \geqslant \gamma'\}$, then Γ is known as a $\gamma\omega$-cover. If there exists a prime element $t \in \alpha^*(\gamma)$ such that Γ is a $t\omega$-cover of A, then Γ is said to be a $(\gamma\omega)^+$-cover of A, where $\alpha^*(\gamma)$ is the standard maximal set of γ [17].

Definition 10. Assume (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ such that Ψ is an $(\alpha\omega)^-$-RF, where $\alpha \in M$, then call A an $\alpha L\omega$-compact set. If A is an $\alpha L\omega$-compact set for any $\alpha \in M$, then call A an $L\omega$-compact set. Specially, when 1_X is $\alpha L\omega$-compact, we call (L^X, Ω) an $\alpha L\omega$-compact space, and if (L^X, Ω) is $\alpha L\omega$-compact for each $\alpha \in M$, we say that (L^X, Ω) is an $L\omega$-compact space.

Obviously, when ω is the L-closure operator on L^X, the $L\omega$-compactness is just the N-compactness in [19], and while ω takes the θ-closure operator (resp., δ-closure operator, R-closure operator, S-closure operator, PS-closure operator, and SR-closure operator) on L^X, the $L\omega$-compactness is just the θ-compactness (resp., δ-compactness, near N-compactness, S-compactness, PS-compactness, and SR-compactness). Therefore, the $L\omega$-compactness is of the universal significance.

Example 11. Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If the support $\sigma_0(A) = \{x \in X \mid A(x) > 0\}$ of A is a finite set, then A is an $L\omega$-compact set.

Proof. Assume that $\sigma_0(A) = \{x_1, x_2, \ldots, x_n\}$ and Φ is an $\alpha\omega$-RF of A. For each $i \in \{1, 2, \ldots, n\}$ we choose an ω-closed set $P_i \in \Phi$ with $\alpha \not\leq P_i(x_i)$. Being $\alpha = \sup \beta^*(\alpha)$, there is a $\lambda_i \in \beta^*(\alpha)$ such that $\lambda \not\leq P_i(x_i)$. Since $\beta^*(\alpha)$ is an upper directed set, there is a $\lambda \in \beta^*(\alpha)$ with $\lambda \geqslant \lambda_i$ for each $i \in \{1, 2, \ldots, n\}$, and thus $\lambda_i \not\leq P_i(x_i)$. Therefore Φ has a finite subfamily $\Psi = \{P_1, P_2, \ldots, P_n\}$ which is an $(\alpha\omega)^-$-RF of A. By Definition 10, A is an $L\omega$-compact set. □

Now we give some characteristics of $L\omega$-compactness as follows.

Theorem 12. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if the following conditions hold:*

(1) *for each $\alpha \in M$, every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ with $\wedge\Psi < A(\alpha\omega)$;*

(2) *for each $\alpha \in M$, if $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ is also an $(\alpha\omega)^-$-RF of A.*

Proof. Necessity. Assume that A is $L\omega$-compact and Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. According to Definition 10, Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$ and so it certainly holds that $\wedge\Psi < A(\alpha\omega)$. Thus (1) is satisfied. If $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ has a finite Ψ with $\wedge\Psi \ll A(\alpha\omega)$ by the

$L\omega$-compactness of A. Obviously, $\Psi = \Phi$, and hence Φ is an $(\alpha\omega)^-$-RF of A. Therefore (2) holds.

Sufficiency. Suppose that conditions (1) and (2) are satisfied, and Φ is an $\alpha\omega$-RF of A ($\alpha \in M$). By (1), there is a finite subfamily Ψ of Φ such that Ψ is an $\alpha\omega$-RF of A. Let $P = \wedge\Psi$. Then $\{P\}$ is an $\alpha\omega$-RF of A. According to (2), $\{P\}$ is also an $\alpha\omega$-RF of A; that is, there exists a $\lambda \in \beta^*(\alpha)$ with $\lambda \nleqslant P(x) = \wedge\{Q(x) \mid Q \in \Psi\}$ for each molecule $x_\lambda \leqslant A$. Since Ψ is finite, we can choose an ω-closed set $Q \in \Psi$ with $\lambda \nleqslant Q(x)$; that is, $Q \in \omega\eta^-(x_\lambda)$. This shows that Ψ is an $(\alpha\omega)^-$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 13. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if for each $\gamma' \in M$, every $\gamma\omega$-cover Γ of A has a finite subfamily Ξ such that Ξ is a $(\gamma\omega)^+$-cover of A.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and Γ is any $\gamma\omega$-cover of A ($\gamma' \in M$). Put $\Phi = \Gamma'$. Then $\Phi \subseteq \omega C(L^X)$, and there is an ω-closed set $B' \in \Phi$ with $B(x) \nleqslant \gamma$ for each $x \in \tau_{\gamma'}(A)$; that is, $\gamma' \nleqslant B'(x)$; equivalently, $B' \in \omega\eta^-(x_{\gamma'})$. This implies that Φ is a $\gamma'\omega$-RF of A. Thus Φ has a finite subfamily Ψ which is a $(\gamma'\omega)^-$-RF of A; that is, there exists $t' \in \beta^*(\gamma')$ such that for each $x \in \tau_{\gamma'}(A)$ we can take an ω-open set $B \in \Psi'$ with $t' \nleqslant B'(x)$. In other words, there are $t \in \alpha^*(\gamma)$ and $B \in \Psi' = \Xi$ with $B(x) \nleqslant t$ for each $x \in \tau_{\gamma'}(A)$. This means that Ξ is a finite subfamily of Γ and a $(\gamma\omega)^+$-cover of A.

Sufficiency. Assume that every $\gamma\omega$-cover of A has a finite subfamily which is a $(\gamma\omega)^+$-cover of A ($\gamma' \in M$). If Φ is an $\alpha\omega$-RF of A ($\alpha \in M$), then $\Gamma = \Phi'$ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Hence Γ has a finite subfamily Ξ which is a $(\gamma\omega)^+$-cover of A by the hypothesis. Write $\Psi = \Xi'$. One can easily see that Ψ is a finite subfamily of Φ and is an $(\alpha\omega)^-$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 14. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$ and each $\Phi \subseteq \omega C(L^X)$ having α-finite intersection property for A (i.e., for each finite subfamily Ψ of Φ and each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$), there exists a molecule $x_\alpha \leqslant A$ with $x_\alpha \leqslant \wedge\Phi$.*

Proof. Necessity. Grant that A is an $L\omega$-compact set, $\Phi \subseteq \omega C(L^X)$, and Φ has α-finite intersection property for A ($\alpha \in M$). If $x_\alpha \nleqslant \wedge\Phi$ for each $x_\alpha \leqslant A$, then Φ is an $\alpha\omega$-RF of A by the hypothesis of Φ. Hence Φ has a finite subfamily Ψ which is an $(\alpha\omega)^-$-RF of A; that is, there is a $\lambda \in \beta^*(\alpha)$ satisfying $x_\lambda \nleqslant \wedge\Psi$ for each $x_\lambda \leqslant A$; in other words, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \ngeqslant \lambda$. It contradicts the fact that Φ has α-finite intersection property for A. Hence the necessity is proved.

Sufficiency. Assume that the condition holds and that Φ is an $\alpha\omega$-RF of A. If for any finite subfamily Ψ of Φ, Ψ is not an $(\alpha\omega)^-$-RF of A, then for each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$; that is, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \geqslant \lambda$. This shows that Φ has α-finite intersection property for A. By the assumption we have $x_\alpha \leqslant A$ satisfying $x_\alpha \leqslant \wedge\Psi$. It contradicts that Φ is an $\alpha\omega$-RF of A. Therefore Φ has a finite subfamily Ψ which is an $(\alpha\omega)^-$-RF of A, and hence A is $L\omega$-compact. □

Theorem 15. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-net in A has an ω-cluster point in A with height α.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and that $N = \{N(n) \mid n \in D\}$ is an α-net [16] in A. If N does not have any ω-cluster point in A with height α, then there exists a $P[x] \in \omega\eta^-(x_\alpha)$ such that N is eventually in $P[X]$ for each $x_\alpha \leqslant A$; that is, there is a $n(x) \in D$ with $N(n) \leqslant P[x]$ whenever $n \geqslant n(x)$. Write $\Phi = \{P[x] \mid x_\alpha \leqslant A\}$. Obviously, Φ is $\alpha\omega$-RF of A. By the $L\omega$-compactness of A, Φ has a finite subfamily $\Psi = \{P[x_i] \mid i = 1, 2, \dots, m\}$ which is an $(\alpha\omega)^-$-RF of A; that is, there is an $i \in \{1, 2, \dots, m\}$ with $y_r \nleqslant P[X_i]$ for some $r \in \beta^*(\alpha)$ and each $y_r \leqslant A$. Take $P = \wedge_{i=1}^m P[x_i]$. Then $y_r \nleqslant P$ for each $y_r \leqslant A$. Since D is a directed set, there is an $n_0 \in D$, such that $n_0 \geqslant n(x_i)$ and $N(n) \leqslant P[x_i]$ ($i = 1, 2, \dots, m$) whenever $n \geqslant n_o$, and so $N(n) \leqslant P$. This shows that for each $y_r \leqslant A$, $\vee(N(n)) \ngeqslant r$ as long as $n \geqslant n_o$. It contradicts the fact that N is an α-net. Therefore N has at least an ω-cluster point in A with height α.

Sufficiency. Assume that every α-net in A has at least an ω-cluster point with hight α for each $\alpha \in M$, Φ is an $\alpha\omega$-RF of A, and $2^{(\Phi)}$ is the set of all finite subfamilies of Φ. If for each $r \in \beta^*(\alpha)$ and each $\Psi \in 2^{(\Phi)}$, Ψ is not an $r\omega$-RF of A; that is, $x_r \leqslant \wedge\Psi$ for each $x_r \leqslant A$, and hence there exists a molecule $N(r, \Psi) \leqslant A$ satisfying $N(r, \Psi) \leqslant \wedge\Psi$. In $\beta^*(\alpha) \times 2^{(\Phi)}$, we define the relation as follows: $(r_1, \Psi_1) \geqslant (r_2, \Psi_2)$ if and only if $r_1 \geqslant r_2$ and $\Psi_1 \supseteq \Psi_2$, then $\beta^*(\alpha) \times 2^{(\Phi)}$ is a directed set with the relation "$\geqslant$". Let $N = \{N(r, \Psi) \mid (r, \Psi) \in \beta^*(\alpha) \times 2^{(\Phi)}\}$. One can easily see that N is an α-net in A. We assert that N does not have any ω-cluster point in A with hight α. In fact, for each $x_\alpha \leqslant A$, we can choose an ω-closed set $P \in \Phi$ with $P \in \omega\eta^-(x_\alpha)$ by the definition of Φ. Taking $r_1 \in \beta^*(\alpha)$ and $\Psi \in 2^{(\Phi)}$, we have $P \in \Psi$ according to $(r, \Psi) \geqslant (r_1, \{P\})$, and hence $N(r, \Psi) \leqslant \wedge\Psi \subseteq P$. This implies that N is eventually in P, and thus x_α is not an ω-cluster point of N. It is in contradiction with the hypothesis of sufficiency. Consequently, A is $L\omega$-compact. □

Definition 16. Let (L^X, Ω) be an $L\omega$-space, let $\mathscr{F}$ be an α-filter in L^X; that is, $\vee_{x\in X}(F \wedge A)(x) \geqslant \alpha$ for each $F \in \mathscr{F}$ and $x_\alpha \in M^*(L^X)$. If $F \nleqslant P$ and for each $P \in \omega\eta^-(x_\alpha)$ and each $F \in \mathscr{F}$, then x_α is called an ω-cluster point of $\mathscr{F}$.

Theorem 17. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-filter containing A as an element has an ω-cluster point in A with hight α.*

Proof. Necessity. Grant that A is an $L\omega$-compact set and that $\mathscr{F}$ is an α-filter containing A as an element. Then $F \wedge A \in \mathscr{F}$

for each $F \in \mathscr{F}$ and $\vee_{x \in X}(F \wedge A)(x) \geqslant \alpha$, and thus there exists a molecule $N(F, r) \leqslant A$ with hight r for each $r \in \beta^*(\alpha)$. Define $N = \{N(F, r) \leqslant F \wedge A \mid (F, r) \in \mathscr{F} \times \beta^*(\alpha)\}$ and define a relation in $\mathscr{F} \times \beta^*(\alpha)$ as follows:

$$(F_1, r_1) \geqslant (F_2, r_2) \quad \text{iff } F_1 \leqslant F_2,\ r_1 \geqslant r_2. \tag{1}$$

Evidently, $\mathscr{F} \times \beta^*(\alpha)$ is a directed set with the relation "$\geqslant$", and then N is an α-net in A. By the $L\omega$-compactness of A and Theorem 15, N has an ω-cluster point in A with hight α, say x_α. We assert that x_α is also an ω-cluster point of $\mathscr{F}$. In reality, N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$; that is, for each $F \in \mathscr{F}$ there exist $F_1 \in \mathscr{F}$ with $F_1 \leqslant F$ and some $r \in \beta^*(\alpha)$ satisfying $N(F_1, r) \nleqslant P$. Hence we have $F \nleqslant P$ by virtue of the fact that $N(F_1, r) \leqslant F_1 \leqslant F$. This means that x_α is an ω-cluster point of $\mathscr{F}$. Therefore the necessity is proved.

Sufficiency. Suppose that every α-filter containing A as an element has an ω-cluster point in A with hight α for each $\alpha \in M$ and that Φ is an $\alpha\omega$-RF of A. If for each $\Psi \in 2^{(\Phi)}$, Ψ is not an $(\alpha\omega)^-$-RF of A, then there exists a molecule $x_r \leqslant A$ and $x_r \leqslant \wedge\Psi$ for each $r \in \beta^*(\alpha)$. Put $\mathscr{F} = \{F \in L^X \mid \exists \Psi \in 2^{(\Phi)}$ with $(\wedge\Psi) \wedge A \leqslant F\}$. One can easily verify that $\mathscr{F}$ is an α-filter containing A as an element, and hence $\mathscr{F}$ has an ω-cluster point in A with hight α by the supposition, say x_α. In accordance with Definition 16, we have $F \nleqslant P$ for each $P \in \omega\eta^-(x_\alpha)$ and each $F \in \mathscr{F}$, specially, $\wedge\Psi \nleqslant P$. Since Φ is an $\alpha\omega$-RF of A, there exists an ω-closed set $Q \in \Phi$ with $Q \in \omega\eta^-(x_\alpha)$ for each $x_\alpha \leqslant A$. Obviously, $\{Q\} \in 2^{(\Phi)}$, so $Q \nleqslant Q$, and this is impossible. Hence there must be a $\Psi \in 2^{(\Phi)}$ which is an $(\alpha\omega)^-$-RF of A. This shows that A is $L\omega$-compact. □

Definition 18. Let I be an ideal in L^X. If $\vee_{x \in X} B'(x) \geqslant \alpha$ for each $B \in I$, then I is called an α-ideal ($\alpha \in M$).

Theorem 19. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if every α-ideal I whose A is not in I has an ω-cluster point in A with hight α for each $\alpha \in M$.*

Proof. Necessity. Assume that A is an $L\omega$-compact set, I is an α-ideal whose A is not in I, and $N(I) = \{N(I)((b, B)) = b \leqslant A \mid (b, B) \in D(I)\}$ where $D(I) = \{(b, B) \mid b \in M^*(L^X), B \in I$ and $b \nleqslant B\}$. Then $N(I)$ is an α-net in A. Hence $N(I)$ has an ω-cluster point in A with hight α by Theorem 15, say x_α. Obviously, x_α is also an ω-cluster point of I. Consequently, the necessity is proved.

Sufficiency. Grant that every α-ideal whose A is not in it has an ω-cluster point in A with hight α for each $\alpha \in M$ and $\mathscr{F}$ is an α-filter containing A as an element. Let $I = \{F' \in L^X \mid F \in \mathscr{F}\}$. Evidently, I is an α-ideal whose A is not in I. Now we will prove that $\mathscr{F}$ has an ω-cluster point in A with hight α. Actually, by the hypothesis we know that I has an ω-cluster point in A with hight α, say x_α; that is, $F' \vee P \neq 1_X$; equivalently, $F \nleqslant P$, for each $F \in \mathscr{F}$ and each $P \in \omega\eta^-(x_\alpha)$. Therefore x_α is an ω-cluster point of $\mathscr{F}$ in line with Definition 16, and hence A is an $L\omega$-compact set by Theorem 17. This implies that the sufficiency holds. □

4. Some Important Properties of $L\omega$-Compactness

In this section, we still further deliberate the properties of $L\omega$-compactness in an $L\omega$-space.

Theorem 20. *Let (L^X, Ω) be an $L\omega$-space and $A, B \in L^X$. If A is $L\omega$-compact and B is ω-closed, then $A \wedge B$ is $L\omega$-compact.*

Proof. Assume that N is an α-net in $A \wedge B$ ($\alpha \in M$). Then N is also an α-net in A. Since A is ω-compact, N has an ω-cluster point in A with hight α, say x_α. We assert that $x_\alpha \leqslant B$. Actually, since N is an α-net in B and N ω-accumulates x_α, N has an α-subnet T which ω-converges to x_α and so $x_\alpha \leqslant \omega\,\mathrm{cl}(B) = B$. Hence $x_\alpha \leqslant A \wedge B$, and thus $A \wedge B$ is $L\omega$-compact in accordance with Theorem 15. □

This theorem shows that the $L\omega$-compactness is hereditary with respect to ω-closed sets.

Theorem 21. *Let A and B be both $L\omega$-compact sets in (L^X, Ω). Then $A \vee B$ is also an $L\omega$-compact set in (L^X, Ω).*

Proof. Suppose that Φ is an $\alpha\omega$-RF of $A \vee B$ ($\alpha \in M$). Then Φ is an $\alpha\omega$-RF of both A and B. Owing to the $L\omega$-compactness of A, there are $\lambda_1 \in \beta^*(\alpha)$ and $\Psi_1 \in 2^{(\Phi)}$ with $\wedge\Psi_1 < A(\lambda_1\omega)$. Similarly, there exist $\lambda_2 \in \beta^*(\alpha)$ and $\Psi_2 \in 2^{(\Phi)}$ satisfying $\wedge\Psi_2 < A(\lambda_2\omega)$. Take $\lambda = \lambda_1 \wedge \lambda_2$ and $\Psi = \Psi_1 \cup \Psi_2$; then $\lambda \in \beta^*(\alpha)$, $\Psi \in 2^{(\Phi)}$, and $\wedge\Psi < A(\lambda\omega)$; that is, Ψ is an $(\alpha\omega)^-$-RF of $A \vee B$. Consequently, $A \vee B$ is $L\omega$-compact. □

This theorem indicates that the $L\omega$-compactness is finitely additive.

Theorem 22. *Let $L = [0, 1]$, (L^X, Ω) be an $L\omega$-space and let $A \in L^X$ be an $L\omega$-compact set. Then there exists a crisp point $x \in X$ such that $A(x) = \sup\{A(t) \mid t \in X\}$.*

Proof. Let $\alpha = \sup\{A(t) \mid t \in X\}$; then $\alpha \in [0, 1]$. If $\alpha = 0$, then $A = 0_X$ and hence $A(x) = \sup\{A(t) \mid t \in X\}$ holds for each $x \in X$. If $\alpha > 0$, and D is the set of all natural numbers, then we choose $x^n \in X$ with $A(x^n) > \alpha - (1/n)$ and $N = \{x^n_{A(x^n)} \mid n \in D\}$. Obviously, N is an α-net in A, and N has an ω-cluster point x_α in A by virtue of the $L\omega$-compactness of A. Hence $A(x) \geqslant \alpha$ by $x_\alpha \leqslant A$. On the other hand, $A(x) \leqslant \alpha$ by the definition of α. Therefore $A(x) = \alpha = \sup\{A(t) \mid t \in X\}$. □

This theorem implies that an $L\omega$-compact set can reach the maximum at some point in X as a function.

Theorem 23. *Let (L^X, Ω_1) and (L^Y, Ω_2) be an $L\omega_1$-space and an $L\omega_2$-space, respectively, and let $f : L^X \to L^Y$ be an (ω_1, ω_2)-continuous L-valued Zadeh's type function. If A is an $L\omega_1$-compact set in (L^X, Ω_1), then $f^{\to}(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2).*

Proof. Assume that Φ is an $\alpha\omega_2$-RF of $f^{\to}(A)$ and $y_\alpha \in M^*(L^Y)$ with $y_\alpha \leqslant f^{\to}(A)(\alpha \in M)$. According to the definition of f, there is a molecule $x_\alpha \in M^*(L^X)$ such that $x_\alpha \leqslant A$ and $f^{\rightarrow}(x_\alpha) = y_\alpha$. Thus there is an ω-closed set $Q \in \Phi$ with $f^{\rightarrow}(x_\alpha) \nleqslant Q$; that is, $x_\alpha \nleqslant f^{\leftarrow}(Q)$. Since f is (ω_1, ω_2)-continuous, $f^{\leftarrow}(Q)$ is ω-closed in (L^X, ω_1), and hence $f^{\leftarrow}(Q) \in \omega_1\eta^-(x_\alpha)$. This means that $f^{\leftarrow}(\Phi) = \{f^{\leftarrow}(Q) \mid Q \in \Phi\}$ is an $\alpha\omega_1$-RF of A. Therefore Φ has a finite subfamily $\Psi = \{Q_1, Q_2, \ldots, Q_n\}$ such that $f^{\leftarrow}(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. We assert that Ψ is an $(\alpha\omega_2)^-$-RF of $f^{\rightarrow}(A)$. In reality, there exists a $\lambda \in \beta^*(\alpha)$ with $\wedge f^{\leftarrow}(\Psi) < A(\lambda\omega_1)$ by virtue of the fact that $f^{\leftarrow}(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. Since for each $y_\lambda \leqslant f^{\rightarrow}(A)$ there exists a $x_\lambda \leqslant A$ satisfying $f^{\rightarrow}(x_\lambda) = y_\lambda$, and there exists a $Q \in \Psi$ with $f^{\leftarrow}(Q) \in \omega_1\eta^-(x_\lambda)$, that is, $x_\lambda \nleqslant f^{\leftarrow}(Q)$. Hence $y_\lambda = f^{\rightarrow}(x_\lambda) \nleqslant Q$ by Lemma 3.1 in [19], and so Ψ is an $(\alpha\omega_2)^-$-RF of $f^{\rightarrow}(A)$. Consequently, $f^{\rightarrow}(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2). □

This theorem means that the $L\omega$-compactness is topological variant under (ω_1, ω_2)-continuous L-valued Zadeh's type functions.

Definition 24. Let (X, Ω) be a crisp ω-space, and let $\mathscr{P}(X)$ be the set of all subsets of X, that is, all crisp sets on X and $A \in L^X$, where $\omega : \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is a crisp ω-operator which satisfies the following conditions: (1)$\omega(U) \subseteq \omega(V)$ for each $U, V \in \mathscr{P}(X)$ and $U \subseteq V$; (2)$U \subseteq \omega(U)$ for each $U \in \mathscr{P}(X)$.

(i) If $\xi_\alpha(A) = \{x \in X \mid A(x) \leqslant \alpha\} \in \omega C(X)$, where $\omega C(X)$ denotes the set of all crisp ω-closed sets on X and $\alpha \in M$, then A is said to be an L-valued lower semicontinuous function on X.

(ii) Let $\Delta_L(\Omega)$ be the set of all L-valued lower semicontinuous functions on X, and call the pair $(L^X, \Delta_L(\Omega))$ the $L\omega$-space topologically generated by (X, Ω).

Theorem 25. *Let (X, Ω) be a crisp ω-space and let $(L^X, \Delta_L(\Omega))$ be the $L\omega$-space topologically generated by (X, Ω). Then $A \in L^X$ is $L\omega$-compact if and only if $\tau_\alpha(A) = \{x \in X \mid A(x) \geqslant \alpha\}$ is ω-compact for each $\alpha \in M$.*

Proof. Necessity. Provided that $A \in L^X$ is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ and Φ is an ω-open cover of $\tau_\alpha(A)(\alpha \in M)$, let $\Gamma = \{\chi_G \mid G \in \Phi\}$ and $\gamma = \alpha'$, where χ_G is the characteristic function of G. We assert that Γ is a $\gamma\omega$-cover of A. In fact, for each $x \in \tau_{\gamma'}(A)$, there is an ω-open set $G \in \Phi$ with $x \in G$; that is, $\chi_G(x) = 1$. Hence $\chi_G(x) \nleqslant \gamma$ by virtue of the fact that γ is a prime element in L with $\gamma \neq 1$. Thus Φ has a finite subfamily $\{G_1, G_2, \ldots, G_m\}$ such that $\mu = \{\chi_{G_i} \mid i = 1, 2, \ldots, m\} \in 2^{(\Gamma)}$ which is a $(\gamma\omega)^+$-cover of A in line with Theorem 13; that is, there is an $i \in \{1, 2, \ldots, m\}$ such that $\chi_{G_i} \in \mu$ with $\chi_{G_i}(x) \nleqslant \lambda$ for some $\lambda \in \alpha^*(\gamma)$ and each $x \in \tau_\alpha(A)$, and so $x \in G_i$. This implies that $\tau_\alpha(A) \subseteq \cup_{i=1}^m G_i$. Hence $\tau_\alpha(A)$ is an ω-compact set in (X, Ω).

Sufficiency. Grant that $\tau_\alpha(A)$ is an ω-compact set in (X, Ω) for each $\alpha \in M$ and that Γ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Then there is an ω-open set $B_x \in \Gamma$ with $B_x(x) \nleqslant \gamma$ for each $x \in \tau_\alpha(A)$, and hence there exists a prime element $t(x) \in \alpha^*(\gamma)$ satisfying $B_x(x) \nleqslant t(x)$. Put $l_{t(x)}(B_x) = \{y \in X \mid B_x(y) \nleqslant t(x)\}$ and $\Phi = \{l_{t(x)}(B_x) \mid x \in \tau_\alpha(A)\}$; then Φ is an ω-open cover of $\tau_\alpha(A)$ according to $x \in l_{t(x)}(B_x)$ and $B_x \in \Delta_L(\Omega)$. Because of the ω-compactness of $\tau_\alpha(A)$, Φ has a finite subfamily $\Psi = \{l_{t(x_i)}(B_{x_i}) \mid i = 1, 2, \ldots, m\}$ which is an ω-open cover of $\tau_\alpha(A)$; that is, there exists an $i \in \{1, 2, \ldots, m\}$ with $x \in l_{t(x_i)}(B_{x_i})$; in other words, $B_{x_i}(x) \nleqslant t(x_i)$ for each $x \in \tau_\alpha(A)$. Take $t = \wedge_{i=1}^m t(x_i)$; evidently, $t \in \alpha^*(\gamma)$ and $B_{x_i}(x) \nleqslant t$. Hence $\mu = \{B_{x_i} \mid i = 1, 2, \ldots, m\}$ is a $(\gamma\omega)^+$-cover of A, and thus A is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ by Theorem 13. □

This theorem indicates that the $L\omega$-compactness is a good extension in the sense of R. Lowen.

Theorem 26. *Let (L^X, Ω) be a stratified ωT_2 and $A \in L^X$. If A is $L\omega$-compact, then A is ω-closed.*

Proof. We only prove that $x_\alpha \leqslant A$ for each $x_\alpha \in M^*(L^X)$ with $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$ by the definition of ω-operator. Actually, if $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$, then there exists a molecular net $N = \{x_{t(n)}^{(n)} \in M^*(L^X) \mid n \in D\}$ in A which ω-converges to x_α in accordance with Theorem 2 in [11]. Write $\lambda = \wedge_{m\in D}\vee_{n\geqslant m} t(n)$; we assert that $\lambda \geqslant \alpha$. In fact, if $\lambda \ngeqslant \alpha$, then there is a $m \in D$ with $\vee_{n\geqslant m} t(n) \ngeqslant \alpha$, and let $d = \vee_{n\geqslant m} t(n)$. Since (L^X, Ω) is stratified, the constant L-set $[d]$ on X is ω-closed and $x_\alpha \nleqslant [d]$, that is, $[d] \in \omega\eta^-(x_\alpha)$. Obviously, N is eventually in $[d]$, and it contradicts the fact that N ω-converges to x_α. Hence $\lambda \geqslant \alpha$; that is, $\vee_{n\geqslant m} t(n) \geqslant \alpha$ for each $m \in D$. For each $r \in \beta^*(\alpha)$ and each $m \in D$ we choose $n(r, m) \in D$ such that $n(r, m) \geqslant m$ and $t(n(r, m)) \geqslant r$, and define the relation "$\geqslant$" in $\beta^*(\alpha) \times D$ as follows:

$$(r_1, m_1) \geqslant (r_2, m_2) \quad \text{iff } r_1 \geqslant r_2,\ m_1 \geqslant m_2. \tag{2}$$

Then $\beta^*(\alpha) \times D$ is a directed set with the relation. Write $S = \{x_{t(n(r,m))}^{n(r,m)} \mid (r, m) \in \beta^*(\alpha) \times D\}$; then $S = N \circ R$, where $R : \beta^*(\alpha) \times D \rightarrow D$ is defined as $R(n(r, m)) = n(r, m)$. Evidently S is a subnet of N and ω-converges to x_α, and S is an α-net in A. Being the $L\omega$-compactness of A, S has an ω-cluster point in A with hight α, say z_α. Since (L^X, Ω) is an ωT_2 space, S ω-converges to x_α and ω-accumulates to z_α, $z = x$ by Theorem 2.7 in [11], and hence $x_\alpha = z_\alpha \leqslant A$. This implies that $\omega\,\mathrm{cl}(A) \leqslant A$; that is, A is an ω-closed set. □

The following example shows that the stratified condition in Theorem 26 can not be omitted.

Example 27. Let $X = \{x\}$ be a single set, $L = [0, 1]$, and let $\omega : L^X \rightarrow L^X$ be the fuzzy closure operator. Define $\omega O(L^X) = \{0_X, x_{1/3}, 1_X\}$, where $A : x \rightarrow [0, 1]$ is defined as $A(x) = x_\alpha, \alpha \in [0, 1]$ for $x \in X$. Obviously, (L^X, Ω) is both an $L\omega$-compact space and an N-compact space. According to Example 11 we know that $A = x_{1/3}$ is an $L\omega$-compact set in (L^X, Ω), but A is not ω-closed.

The following theorems imply that the $L\omega$-compactness can strengthen ω-seperation properties.

Theorem 28. *If (L^X, Ω) is both ωT_2 and $L\omega$-compact $L\omega$-space, then (L^X, Ω) is an ω-regular space [11].*

Proof. Let $G \in L^X$ be an ω-closed pseudocrisp set and let x_λ be a molecule which x is not in supp G. By Definition 7.1 in [19], there is an $\alpha \in M$ such that $G(x) > 0$ implies $G(x) \geqslant \alpha$. For each $y_\alpha \in M^*(L^X)$, there are $P_y \in \omega\eta^-(x_\lambda)$ and $Q_y \in \omega\eta^-(y_\alpha)$ satisfying $P_y \vee P_y = 1_X$ by virtue of $x \neq y$ and the ωT_2 separation of (L^X, Ω). Put $\Phi = \{Q_y \mid y_\alpha \leqslant G\}$; then Φ is an $\alpha\omega$-RF of G. Since (L^X, Ω) is an $L\omega$-compact space, G is an $L\omega$-compact set in accordance with Theorem 20, and thus Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \dots, Q_{y_n}\}$ which is an $(\alpha\omega)^-$-RF of G; that is, there is an $r \in \beta^*(\alpha)$ such that for each molecule $z_r \leqslant G$ we have $i \leqslant n$ with $z_r \nleqslant Q_{y_i}$. Let $Q = \wedge_{i=1}^n Q_{y_i}$; then $z_r \nleqslant Q$; that is, $r \nleqslant Q(z)$ for each $z_r \leqslant G$. Since $G(z) > 0$ implies that $G(z) \geqslant \alpha \geqslant r$, $G(z) \nleqslant Q(z)$ for each $z \in \operatorname{supp} G$, and hence $Q \in \omega\eta^-(G)$. Write $P = \vee_{i=1}^n Py_i$; then $P \in \omega\eta^-(x_\lambda)$ and

$$P \vee Q = (\vee_{i=1}^n Py_i) \vee (\wedge_{i=1}^n Qy_i) \geqslant \vee_{i=1}^n (Py_i \vee Qy_i) = 1. \quad (3)$$

Consequently, (L^X, Ω) is an ω-regular space. □

Theorem 29. *Let (L^X, Ω) be an $L\omega$-compact ωT_2 space. Then (L^X, Ω) is an ω-normal space [11].*

Proof. Let both G, H be ω-closed pseudocrisp sets in (L^X, Ω) with $(\operatorname{supp} G) \cap (\operatorname{supp} H) = \phi$. Then there are $\lambda, \mu \in M$ such that $G(x) > 0$ if and only if $G(x) \geqslant \lambda$, and $H(x) > 0$ if and only if $H(x) \geqslant \mu$. According to the proof of Theorem 28, for each molecule $y_\mu \leqslant G$, there is an ω-closed set $P_y \in \omega\eta^-(G)$ satisfying $\lambda \nleqslant P_y(z)$ for each $z \in \operatorname{supp} G$, and there is a $Q_y \in \omega\eta^-(y_\mu)$ such that $P_y \vee Q_y = 1$. One can easily see that $\Phi = \{Q_y \mid y_\mu \leqslant B\}$ is a $\mu\omega$-RF of H. In line with Theorem 20 we know that H is an $L\omega$-compact set, and so Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \dots, Q_{y_n}\}$ such that Ψ is a $(\mu\omega)^-$-RF of H. Put $P = \vee_{i=1}^n Py_i$; $Q = \wedge_{i=1}^n Q_{y_i}$; then $P \in \omega\eta^-(G)$, $Q \in \omega\eta^-(H)$ and $P \vee Q = 1$. Therefore (L^X, Ω) is an ω-normal space. □

5. The Tychonoff Product Theorem

In this section, we will first extend Alexandar's subbase Lemma in general topology and give the Alexandar's ω-subbase lemma and next prove that the Tychonoff product theorem holds in $L\omega$-spaces.

Theorem 30 (Alexandar ω-subbase lemma). *Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and let γ be an ω-subbase [20] in L^X. If for each $\alpha\omega$-RF Φ of A where $\Phi \subseteq \gamma' \subseteq \omega C(L^X)$, there is a finite subfamily Ψ of Φ with $\wedge\Psi \ll A(\alpha\omega)(\alpha \in M)$, then A is $L\omega$-compact.*

Proof. Suppose that Φ is an arbitrary $\alpha\omega$-RF of A. We will prove that Φ has a finite subfamily Ψ which is an $(\alpha\omega)^+$-RF of A. In fact, if for each $\Psi \in 2^{(\Phi)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold, then $H = \{\Delta \mid \Phi \subseteq \Delta \subseteq \omega C(L^X)$, for all $\Psi \in 2^{(\Delta)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold$\} \neq \emptyset$, and H is a partial-ordered set with respect to the upper bound and hence there exists a maximal element Δ_0 in H by Zorn's Lemma. We assert that Δ_0 satisfies the following conditions:

(1) $\wedge\Delta_0 < A(\alpha \geqslant \omega)$;

(2) if $P \in \Delta_0$, then $Q \in \Delta_0$ for each $Q \in \omega C(L^X)$ with $Q \geqslant P$;

(3) if $P, Q \in \omega C(L^X)$ and $P \vee Q \in \Delta_0$, then $P \in \Delta_0$ or $Q \in \Delta_0$.

Actually, since $\wedge\Phi < A(\alpha\omega)$ and $\Phi \subseteq \Delta_0$, condition (1) holds. If $P \in \Delta_0, Q \in \omega C(L^X), Q \geqslant P$, and Q is not in Δ_0, then $\Delta^* = \Delta_0 \cup \{Q\} \in H$ and $\Delta_0 \subset \Delta^*$. It contradicts the fact that Δ_0 is the maximal element in H thus condition (2) holds. Let $P, Q \in \omega C(L^X)$. If P and Q are both not in Δ_0, then $\Delta_0 \cup \{P\}$ and $\Delta_0 \cup \{Q\}$ are both not in H by the maximality of Δ_0, and thus there are $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$ such that $\wedge(\Psi_1 \cup \{P\}) \ll A(\alpha\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) \ll A(\alpha\omega)$ according to the definition of H; that is, there are $s, t \in \beta^*(\alpha)$ with $\wedge(\Psi_1 \cup \{P\}) < A(s\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) < A(t\omega)$. Since $\beta^*(\alpha)$ is upper directed, we can choose $r \in \beta^*(\alpha)$ with $r \geqslant s \vee t$. Now we prove $\wedge\{\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}\} < A(r\omega)$. In reality, if $\Psi_2 \cup \Psi_2$ does not have any ωR-neighborhood of x_r for each $x_r \leqslant A$, then $\Psi_2 \cup \Psi$ does not have any ωR-neighborhood of x_s and x_t, respectively, and hence $P \in \omega\eta^-(x_s)$ and $Q \in \omega\eta^-(x_t)$. Particularly, $P, Q \in \omega\eta^-(x_r)$ and so $P \vee Q \in \omega\eta^-(x_r)$. This shows that $\wedge(\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}) < A(r\omega)$. Therefore $P \vee Q$ is not in Δ_0 by virtue of the definition of Δ_0 and $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$. So, condition (3) holds.

From (2) and (3) we have the following result:

(4) If $R \in \Delta_0$, $P_i \in \omega C(L^X)$ $(i = 1, 2, \dots, n)$ and $R \leqslant \vee_{i=1}^n P_i$, then there is an $i \in \{1, 2, \dots, n\}$ satisfying $P_i \in \Delta_0$.

Consider now $\gamma' \cap \Delta_0$. If $\gamma' \cap \Delta_0$ is an $\alpha\omega$-RF of A, then there is a finite subfamily δ of $\gamma' \cap \Delta_0$ which is an $(\alpha\omega)^-$-RF of A. Evidently, $\delta \in 2^{(\Delta_0)}$; it is in contradiction with $\Delta_0 \in H$. Hence $\gamma' \cap \Delta_0$ is not an $\alpha\omega$-RF of A; that is, there is a molecule x_α in A meeting $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0)$. We now verify that $x_\alpha \leqslant \wedge\Delta_0$. In fact, if there is $Q \in \Delta_0$ with $x_\alpha \nleqslant Q$, then by Definition 5 in [17] we can take a finite subfamily $\{P_{ij} \mid j \in J_i, i \in I\}$ of γ' satisfying $Q = \wedge_{i\in I} \vee_{j \in J_i} P_{ij}$, where J_i is a finite set for each $i \in I$. Because of $x_\alpha \nleqslant Q$, we can choose $i \in I$ with $x_\alpha \nleqslant \vee_{j\in J_i} P_{ij}$. Since $Q \leqslant \vee_{j\in J_i} P_{ij}$, there is a $j \in J_i$ such that $P_{ij} \in \Delta_0$ by (4). Hence $P_{ij} \in \gamma' \cap \Delta_0$ and $x_\alpha \nleqslant P_{ij}$; it contradicts the fact that $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0) \leqslant P_{ij}$; thus $x_\alpha \leqslant \wedge\Delta_0$. However, this is in contradiction with (1) again. This implies that Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 31. *Let $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ be a collection of $L\omega$-spaces and let (L^X, Ω) be the product space of them. If A_t is an $L\omega$-compact set in (L^{X_t}, Ω_t) for each $t \in T$, then the product $A = \Pi_{t\in T} A_t$ of all $L\omega$-compact sets $A_t (t \in T)$ is an $L\omega$-compact set in (L^X, Ω).*

Proof. Assume that Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. By Theorem 30 we can grant that every ω-closed set in Φ is of the form $\rho_t^{\leftarrow}(B_t)$ where $B_t \in \omega C(L^{X_t})$ and $\rho_t : L^X \to L^{X_t}$ is a protection because $\{\rho_t^{\leftarrow}(U_t) \mid U_t \in \omega O(L^X), t \in T\}$ is an ω-subbase in (L^X, Ω) [20]. Now we consider the following two cases.

(i) If there exists a $t_0 \in T$ such that no molecule with hight α is contained in A_{t_0}, then by the $L\omega$-compactness of A_{t_0}, there is an $r \in \beta^*(\alpha)$ such that no molecule with hight r is contained in A_{t_0}. In reality, if there exists a molecule with hight r in A_{t_0} for each $r \in \beta^*(\alpha)$, say $N(r)$, then $N = \{N(r) \mid r \in \beta^*(\alpha)\}$ is an α-net in A_{t_0} by the directivity of $\beta^*(\alpha)$. Since A_{t_0} is $L\omega$-compact, N has an ω-cluster point in A_{t_0} with hight α according to Theorem 15. It is in contradiction with the hypothesis of A_{t_0}. Thus it can be seen that there exists an $r \in \beta^*(\alpha)$ with $A_{t_0}(x^{t_0}) \ngeq r$ for each $x^{t_0} \in X_{t_0}$. Hence for each $x \in X$, we have

$$\begin{aligned} A(x) &= \left(\Pi_{t\in T} A_t\right)(x) \\ &= \wedge_{t\in T} A_t\left(\rho_t(x)\right) \leqslant A_{t_0}\left(\rho_{t_0}(x)\right) = A_{t_0}\left(x^{t_0}\right), \end{aligned} \tag{4}$$

and hence $A(x) \ngeq r$ for each $x \in X$; that is, no molecule with hight r is contained in A. This shows that for each $\Psi \in 2^{(\Phi)}$, Ψ is an $(\alpha\omega)^-$-RF of A.

(ii) Suppose that for each $t \in T$, A_t contains a molecule with hight α, say x_α^t. Since $\Phi \subseteq \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \omega C(L^X), t \in T\}$, we can take $R \subseteq T$ such that $\Phi = \cup_{t\in R}\Phi_t$, where $\Phi_t = \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \mathscr{B}_t \subseteq \omega C(L^X)\}$. Now we prove that there must be $s \in R$ with $\wedge\mathscr{B}_s < A(\alpha\omega)$. In fact, if there is a crisp point $y^t \in X_t$ such that $y^t \leqslant A_t \wedge (\wedge\mathscr{B}_t)$ for each $t \in R$, then we choose a crisp point z in X as follows: if $t \in R$, $z^t = y^t$; if t is not in R, $z^t = x^t$. Taking any ω-closed set $\rho_t^{\leftarrow}(B_t)$ in Φ, where $t \in R$ and $B_t \in \mathscr{B}_t$, we have

$$\rho_t^{\leftarrow}(B_t)(z) = B_t\left(z^t\right) = B_t\left(y^t\right) \geqslant \left(A_t \wedge (\wedge\mathscr{B}_t)\right)\left(y^t\right) \geqslant \alpha, \tag{5}$$

that is, $z_\alpha \leqslant \rho_t^{\leftarrow}(B_t)$, and hence $z_\alpha \leqslant \wedge\Phi$ by the arbitrariness of $\rho_t^{\leftarrow}(B_t) \in \Phi$. On the other hand,

$$A(z) = \wedge_{t\in R} A_t\left(z^t\right) = \left(\wedge_{t\in R} A_t\left(y^t\right)\right) \wedge \left(\wedge_{t\in R} A_t\left(x^t\right)\right) \geqslant \alpha. \tag{6}$$

This implies that z_α is a molecule in A; it contradicts the fact that Φ is an $\alpha\omega$-RF of A. Consequently, there is $s \in R$ with $\wedge\mathscr{B}_s < A_s(\alpha)$; thus there is a finite subfamily Γ_s of $\mathscr{B}_s$ with $\Gamma_s < A_s(r\omega)$ for some $r \in \beta^*(\alpha)$. Put $\Psi = \{\rho_s^{\leftarrow}(B_s) \mid B_s \in \Gamma_s\}$; then $\Psi \in 2^{(\Phi)}$. We assert that $\wedge\Psi < A(r\omega)$. Actually, for any molecule e_r in A with hight r we have $A_s(e^s) \geqslant A(e) \geqslant r$; that is, e_r^s is a molecule in A_s, where $e = \{e^t\}_{t\in T} \in X$. Hence there exists an ω-closed set $B_s \in \Gamma_s$ meeting $B_s \in \omega\eta^-(e_r^s)$ by virtue of the fact that Γ_s is an $r\omega$-RF of A_s; thus $\rho_s^{\leftarrow}(B_s)(e) = B_s(e^s) \ngeq r$; that is, $\rho_s^{\leftarrow}(B_s) \in \omega\eta^-(e_r)$. This shows that Ψ is an $r\omega$-RF of A. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 32 (Tychonoff product theorem). *Let (L^X, Ω) be the product space of a collection of $L\omega$-spaces $\{(L^{X_t}, \Omega_t) \mid t \in T\}$. Then (L^X, Ω) is $L\omega$-compact if and only if for each $t \in T$, (L^{X_t}, Ω_t) is $L\omega$-compact.*

Proof. Necessity. Assume that (L^X, Ω) is an $L\omega$-compact space. Since $\rho_t : (L^X, \Omega) \to (L^{X_t}, \Omega_t)$ is an ω-continuous L-valued Zadeh's type function for each $t \in T$, (L^{X_t}, Ω_t) is an $L\omega$-compact space by Theorem 23. Therefore the necessity holds.

Sufficiency. It follows from Theorem 31. □

The following example shows that the inverse theorem of Theorem 31 does not hold.

Example 33. Let $E = \{e_1, e_2, \ldots\}$ be a countably infinite set, $X_t = E$ for each $t \in T = \{1, 2, \ldots\}$, $L = [0, 1]$, $\Omega_t = [0, 1]^E$ and let ω be a fuzzy closure operator. Then (L^{X_t}, Ω_t) is a discrete $L\omega$-space for each $t \in T$. Define $A_t \in L^{X_t}$ $(t \in T)$ as follows:

if $j = 1$, $A_t(e_j) = 1$; if $j \geqslant 2$, $A_t(e_j) = 1/t$.

Suppose that (L^X, Ω) is the product space of $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ and $A = \Pi_{t\in T} A_t$. Now we prove that A is an $L\omega$-compact set in (L^X, Ω), but A_t is not an ω-compact set in (L^{X_t}, Ω_t) for each $t \in T$. In reality, for each $x = (x_1, x_2, \ldots) \in X$ we put $x_t = e_{j(t)}^t$, where x_t is a crisp point e_j in X_t; then from the definitions of A_t and fuzzy product set A we know

$$\begin{aligned} A(x) &= \left(\Pi_{t\in T} A_t\right)(x) = \wedge_{t\in T} A_t\left(x_t\right) = \wedge_{t\in T} A_t\left(e_{j(t)}^t\right) \\ &= \begin{cases} 0, & \text{if there are infinite elements } t \\ & \text{such that } j(t) \geqslant 2. \\ \dfrac{1}{t_R}, & \text{if there is a } t_R \in T \text{ such that } j(t_R) \geqslant 2 \\ & \text{and } j(t) = 1 \text{ whenever } t > t_R. \end{cases} \end{aligned} \tag{7}$$

Thus it can be seen that $A \neq 0_X$ and if $A(x) \geqslant 1/t_R$, then the coordinates $x_t = e_{j(t)}^t = e_1$ of x whenever $t > t_R$. Obviously, points in X satisfying the condition are only finite. Let $\alpha \in M$, that is, $\alpha > 0$, and let Φ be an $\alpha\omega$-RF of A. Choose $t_R \in T$ with $1/t_R < \alpha$. Since there are only finite molecule in A with hight α, denote the finite crisp points as $x^1, x^2, \ldots, x^n$. If $(x^i)_\alpha \leqslant A$ for each $i \in \{1, 2, \ldots, n\}$, then there is $P_i \in \Phi$ with $P_i(x^i) < \alpha$. Put $s = \max\{P_i(x^i) \mid P_i(x^i) < \alpha, i \leqslant n\}$; then $s < \alpha$. Taking $s_1 \in (s, \alpha)$ and $r = \max(s_1, 1/t_R)$, we know that A has at most n molecules with hight r, say $(x^i)_r$ $(i \leqslant n)$. By the definition of Φ, there is a $P_i \in \Phi$ such that $P_i \in \omega\eta^-((x^i)_r)$ for each $(x^i)_r$ in A. Denote $\Psi = \{P_i \in \Phi \mid P_i \in \omega\eta^-((x^i)_r), i \leqslant n\}$; then $\Psi \in 2^{(\Phi)}$ and Ψ is an $r\omega$-RF of A. This implies that Ψ is an $(\alpha\omega)^-$-RF of A by $r \in \beta^*(\alpha)$. Hence A is $L\omega$-compact in (L^X, Ω). On the other hand, take $D = T$ and $N = \{N(m) \mid m \in D\}$ where $N(m) = (e_m)_{1/t}$ for each $m \in D$ and each $t \in T$; then N is a $(1/t)$-net in A_t. Since (L^{X_t}, Ω_t) is discrete, N does not have any ω-cluster point in A_t with hight $1/t$. Therefore A_t is not $L\omega$-compact in (L^{X_t}, Ω_t) for each $t \in T$ according to Theorem 15.

Acknowledgments

The project was supported by the Natural Science Foundation of Fujian Province of China (no. 2011J01013) and the

Programs of Science and Technology Department of Xiamen City (nos. 3502Z20110010 and 3502Z20123022).

References

[1] S.-Z. Bai, "The SR-compactness in L-fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 87, no. 2, pp. 219–225, 1997.

[2] S. Z. Bai, "L-fuzzy PS-compactness," *International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems*, vol. 10, no. 2, pp. 201–209, 2002.

[3] C. L. Chang, "Fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182–190, 1968.

[4] S. L. Chen, "Almost F-compactness in L-fuzzy topological spaces," *Northeastern Mathematical Journal*, vol. 7, no. 4, pp. 428–432, 1991.

[5] S. L. Chen, "The near N-compactness in L-fuzzy topological spaces," *Journal of Mathematics*, vol. 16, no. 1, pp. 67–71, 1996.

[6] S. L. Chen, "L-fuzzy almost nice compactness," *Journal of Fuzzy Mathematics*, vol. 7, no. 1, pp. 91–100, 1999.

[7] T. E. Gantner, R. C. Steinlage, and R. H. Warren, "Compactness in fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 62, no. 3, pp. 547–562, 1978.

[8] A. Di Concilio and G. Gerla, "Almost compactness in fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 13, no. 2, pp. 187–192, 1984.

[9] I. M. Hanafy, "δ-compactness in fuzzy topological spaces," *Journal of Fuzzy Mathematics*, vol. 5, no. 2, pp. 277–284, 1997.

[10] C. M. Hu, "$\mathscr{C}$-structure of FTS. IV. $\mathscr{C}$-compactness," *Journal of Fuzzy Mathematics*, vol. 3, no. 1, pp. 109–123, 1995.

[11] Z. X. Huang and S. L. Chen, "ω-separation axioms in $L\omega$-spaces," *Journal of Mathematics*, vol. 25, no. 1, pp. 383–388, 2005.

[12] B. Hutton, "Uniformities on fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 58, no. 3, pp. 559–571, 1977.

[13] S. R. T. Kudri, "Semicompactness and S^*-closedness in L-fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 109, no. 2, pp. 223–231, 2000.

[14] R. Lowen, "Fuzzy topological spaces and fuzzy compactness," *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 621–633, 1976.

[15] F.-G. Shi, "A new motion of fuzzy compactness in L-topological spaces," *Information Sciences*, vol. 173, no. 1-3, pp. 35–48, 2005.

[16] G. J. Wang, "A new fuzzy compactness defined by fuzzy nets," *Journal of Mathematical Analysis and Applications*, vol. 94, no. 1, pp. 1–23, 1983.

[17] G. J. Wang, *Theory of* L*-Fuzzy Topological Spaces*, Shaanxi Normal University Press, 1988.

[18] J. X. Yang, "L-fuzzy θN-compact spaces," *Fuzzy Systems and Mathematics*, vol. 14, no. 2, pp. 30–37, 2000.

[19] D. S. Zhao, "The N-compactness in L-fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 128, no. 1, pp. 64–79, 1987.

[20] S. L. Chen and J. S. Cheng, "On $L\omega$-spaces," Tsinghua University Press and Springer, vol. 1, pp. 257–264.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 827482, 4 pages
http://dx.doi.org/10.1155/2013/827482

Research Article

h-Stability for Differential Systems Relative to Initial Time Difference

Peiguang Wang[1] **and Xiaowei Liu**[2]

[1] *College of Electronic and Information Engineering, Hebei University, Baoding 071002, China*
[2] *College of Mathematics and Computer Science, Hebei University, Baoding 071002, China*

Correspondence should be addressed to Peiguang Wang; pgwang@mail.hbu.edu.cn

Received 22 March 2013; Accepted 22 April 2013

Academic Editor: Zhenkun Huang

This paper investigates the relationship between an unperturbed differential system and a perturbed differential system that have initial time difference. Notions of h-stability for differential systems with initial time difference are introduced, and stability criteria are formulated by using variation of parameter techniques.

1. Introduction

It is well known that, in applications, asymptotic stability is more important than stability, because the desirable feature is to know the size of the region of asymptotic stability. However, when we study the asymptotic stability, it is not easy to deal with nonexponential types of stability. Pinto [1] introduced the notion of h-stability with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential stability and uniform Lipschitz stability) under some perturbations and developed the study of exponential stability to a variety of reasonable systems called h-systems. Since then, Choi and Ryu [2], Choi et al. [3], and Choi and Koo [4] investigated h-stability problem for the nonlinear differential systems respectively, and Choi et al. [5, 6] characterized the h-stability in variation for nonlinear difference systems via n_∞-similarity and Lyapunov functions and obtained some relative results. For the detailed results of h-stability of impulsive dynamic systems on time scale and others systems can be found in [7–10].

At present, the investigation of differential systems with initial time difference has attracted a lot of attention. This is mainly because of the fact that when considering initial value problems, it is impossible not to make errors in the starting time in dealing with real world phenomena, that is, the solutions of the unperturbed differential system may start at some initial time and the solutions of the perturbed systems may start at a different initial time. When we consider such a change of initial time for each solution, we need to deal with the problem of comparing between any two solutions which start at different times. At present, there are two methods to discuss the stability problem with initial time difference: one is the differential inequalities and comparison principle, and the other is the method of variation of parameters. For the pioneering works in this area we can refer to the papers [11, 12]. After that, there are many stability results for various of differential and difference systems; see [13–20]. However, the above results were obtained by using comparison principle and differential inequalities; there are few stability criteria by using the method of variation of parameters; see [21–24].

In this paper, we attempt to extend the notion of h-stability to differential systems with initial time difference, namely, initial time difference h-stability (ITDhS) and then establish some stability criteria for such differential systems by using the method of variation of parameters. The remainder of this paper is organized in the following manner. Some preliminaries are presented in Section 2. The notions of h-stability for differential systems with initial time difference are given in this section. In Section 3, several stability criteria are established. Finally, an example is added to illustrate the result obtained.

2. Preliminaries

Let $R^+ = [0, +\infty)$ and R^n denotes the n-dimensional Euclidean space with appropriate norm $\|\cdot\|$.

Consider the differential systems:

$$x' = f(t, x), \quad x(t_0) = x_0, \quad t \geq t_0, \; t_0 \in R^+, \tag{1}$$

$$x' = f(t, x), \quad x(\tau_0) = y_0, \quad t \geq \tau_0, \; \tau_0 \in R^+ \tag{2}$$

and the perturbed differential system of (2):

$$y' = F(t, y), \quad y(\tau_0) = y_0, \quad t \geq \tau_0, \; \tau_0 \in R^+, \tag{3}$$

where $f, F \in C[R^+ \times R^n, R^n]$ are locally Lipschitzian and f has continuous partial derivatives $\partial f/\partial x$ on $R^+ \times R^n$. The above assumptions imply the existence and uniqueness of solutions through (t_0, x_0) and (τ_0, y_0). A special case of (3) is where $F(t, y) = f(t, y) + R(t, y)$, $R(t, y)$ is the perturbation term. Let $\eta = \tau_0 - t_0 > 0$. Furthermore, suppose that $x(t, t_0, x_0)$ is the given solution with respect to which we shall study stability criteria.

Let us begin by defining the following notions.

Definition 1. The solution $x(t, \tau_0, y_0)$ of the system (2) through (τ_0, y_0) is said to be initial time difference h-stability (ITDhS) with respect to the solution $x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (1), if and only if there exist $c \geq 1$ and a positive bounded continuous function h defined on R^+ such that

$$\begin{aligned} &\|x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq c\left[\|y_0 - x_0\| + |\tau_0 - t_0|\right] h(t)\, h^{-1}(\tau_0) \end{aligned} \tag{4}$$

for $t \geq \tau_0$ and $h^{-1}(t) = 1/h(t)$.

Similarly, we can define initial time difference h-stability (ITDhS) with respect to the solution $y(t, \tau_0, y_0)$ of the system (3) through (τ_0, y_0).

We are now in a position to give the Alekseev's formula, which is an important tool in the subsequent discussion.

Lemma 2 (see [25]). *If $x(t, \tau_0, y_0)$ is the solution of (2) and exists for $t \geq \tau_0$, any solution $y(t, \tau_0, y_0)$ of (3), with $y(\tau_0) = y_0$, satisfies the integral equation:*

$$\begin{aligned} y(t, \tau_0, y_0) &= x(t, \tau_0, y_0) \\ &\quad + \int_{\tau_0}^{t} \Phi(t, s, y(s, \tau_0, y_0))\, R(s, y(s, \tau_0, y_0))\, ds \end{aligned} \tag{5}$$

for $t \geq \tau_0$, where $\Phi(t, \tau_0, y_0) = \partial x(t, \tau_0, y_0)/\partial y_0$.

The following lemma will also be needed in our investigations.

Lemma 3 (see [25]). *Assume that $x(t, t_0, x_0)$ is the solution of (1) through (t_0, x_0), which exists for $t \geq t_0$, and then*

$$x(t, t_0, x_0) = \left[\int_0^1 \Phi(t, t_0, s x_0)\, ds\right] x_0, \tag{6}$$

where $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$.

3. Stability Criteria

We shall present, in this section, the stability criteria for differential systems with initial time difference.

Theorem 4. *Let $x(t, \tau_0, y_0)$ and $x(t - \eta, t_0, x_0)$ be the solutions of (2) and (1) through (τ_0, y_0) and (t_0, x_0), respectively, $t \geq \tau_0$. Assume that*

(i) *$v(t, \tau_0, v_0) = x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)$, in which $v_0 = y_0 - x_0$;*

(ii) *there exists a positive bounded continuously differentiable function $h(t)$ on R^+ such that*

$$\begin{aligned} &\liminf_{\delta \to 0^-} \frac{\left\|v(t, \tau_0, v_0) + \left(\tilde{f}(t, v, \eta)\right)\delta\right\| - \|v(t, \tau_0, v_0)\|}{\delta} \\ &\quad \leq h'(t)\, h^{-1}(t)\, \|v(t, \tau_0, v_0)\|, \end{aligned} \tag{7}$$

where $\tilde{f}(t, v, \eta) = f(t, x(t - \eta, t_0, x_0) + v(t, \tau_0, v_0)) - f(t, x(t - \eta, t_0, x_0))$;

(iii) *f is locally Lipschitzian in time such that*

$$\begin{aligned} &\|f(t, x(t - \eta, t_0, x_0)) - f(t - \eta, x(t - \eta, t_0, x_0))\| \\ &\quad \leq L_1(t)\, \frac{|\eta|}{L_2(\tau_0)}, \end{aligned} \tag{8}$$

where $L_2(\tau_0) = \int_{\tau_0}^{+\infty} h^{-1}(s) h(\tau_0) L_1(s) ds$, $L_1(s) \in C[R^+, R^+]$.

Then the solution $x(t, \tau_0, y_0)$ of the system (2) is ITDhS with respect to the solution $x(t - \eta, t_0, x_0)$.

Proof. Define $z(t) = \|v(t, \tau_0, v_0)\|$ for $t \geq \tau_0$, and then $z(\tau_0) = \|y_0 - x_0\|$. Also,

$$\begin{aligned} v'(t, \tau_0, v_0) &= x'(t, \tau_0, y_0) - x'(t - \eta, t_0, x_0) \\ &= f(t, v(t, \tau_0, v_0) + x(t - \eta, t_0, x_0)) \\ &\quad - f(t - \eta, x(t - \eta, t_0, x_0)). \end{aligned} \tag{9}$$

Using a Taylor approximation for $v(t, \tau_0, v_0)$ and the conditions (i) and (ii), we arrive at

$$\begin{aligned} &D\_z(t) \\ &= \liminf_{\delta \to 0} \frac{\left\|v(t, \tau_0, v_0) + v'(t, \tau_0, v_0)\,\delta\right\| - \|v(t, \tau_0, v_0)\|}{\delta} \\ &\leq h'(t)\, h^{-1}(t)\, z(t) + L_1(t)\, \frac{|\eta|}{L_2(\tau_0)}. \end{aligned} \tag{10}$$

And then, from (10), we have

$$\begin{aligned} &z(t) \leq h(t)\, h^{-1}(\tau_0) \\ &\quad \times \left(z(\tau_0) + \frac{|\eta|}{L_2(\tau_0)} \int_{\tau_0}^{+\infty} h^{-1}(s)\, h(\tau_0)\, L_1(s)\, ds\right). \end{aligned} \tag{11}$$

Moreover, using the condition (iii), we obtain

$$z(t) \leq h(t) h^{-1}(\tau_0)(\|y_0 - x_0\| + |\eta|). \tag{12}$$

Then from (12), we get

$$\begin{aligned} &\|x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq [\|y_0 - x_0\| + |\tau_0 - t_0|] h(t) h^{-1}(\tau_0). \end{aligned} \tag{13}$$

So by Definition 1 with $c = 1$, the solution $x(t, \tau_0, y_0)$ of (2) is ITDhS with respect to the solution $x(t - \eta, t_0, x_0)$. This completes the proof. □

Remark 5. Set $h(t) = e^{-\beta_0 t}$, and then we can obtain Theorem 3.4 in [8].

Theorem 6. *Let $y(t, \tau_0, y_0)$ be the solution of (3) through (τ_0, y_0). Assume that*

(i) *the solution $x(t, \tau_0, y_0)$ of (2) is ITDhS with respect to the solution $x(t - \eta, t_0, x_0)$ for $t \geq \tau_0$, where $x(t, t_0, x_0)$ is any solution of (1);*

(ii) *there exist $c \geq 1$, $\alpha > 0$ and a positive bounded continuous function h defined on R^+ such that*

$$\begin{aligned} &\|\Phi(t, s, y(s))\| \leq c h(t) h^{-1}(s), \\ &\|R(s, y(s))\| \leq r(s) \|y(s)\|, \end{aligned} \tag{14}$$

provided that $y(s, \tau_0, y_0) \leq \alpha$, $r(s) \in C(R^+, R^+)$ and $\int_{\tau_0}^{+\infty} r(s) ds < +\infty$.

Then the solution $y(t, \tau_0, y_0)$ of (3) is ITDhS with respect to the solution $x(t - \eta, t_0, x_0)$.

Proof. Define $v(t, \tau_0, v_0) = x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)$ and $z(t) = \|v(t, \tau_0, v_0)\|$, and then $z(\tau_0) = \|y_0 - x_0\|$. The condition (i) yields

$$\begin{aligned} &\|x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq c[\|y_0 - x_0\| + |\tau_0 - t_0|] h(t) h^{-1}(\tau_0). \end{aligned} \tag{15}$$

By Lemma 2, it follows that

$$\begin{aligned} &y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0) \\ &\quad = v(t, \tau_0, v_0) + \int_{\tau_0}^{t} \Phi(t, s, y(s)) R(s, y(s)) ds. \end{aligned} \tag{16}$$

Now taking the norms of both sides and using the triangle inequality, we have

$$\begin{aligned} &\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq z(t) + \int_{\tau_0}^{t} \|\Phi(t, s, y(s))\| \|R(s, y(s))\| ds. \end{aligned} \tag{17}$$

From (15), we obtain

$$\begin{aligned} &\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq c[\|y_0 - x_0\| + |\tau_0 - t_0|] h(t) h^{-1}(\tau_0) \\ &\qquad + \int_{\tau_0}^{t} \|\Phi(t, s, y(s))\| \|R(s, y(s))\| ds. \end{aligned} \tag{18}$$

Setting $M^*(t) = \|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\|$ and using the triangle inequality, we have

$$\begin{aligned} M^*(t) &\leq c[\|y_0 - x_0\| + |\tau_0 - t_0|] h(t) h^{-1}(\tau_0) \\ &\quad + \int_{\tau_0}^{t} c h(t) h^{-1}(s) r(s) M^*(s) ds \\ &\quad + \int_{\tau_0}^{t} c h(t) h^{-1}(s) r(s) \|x(s - \eta, t_0, x_0)\| ds. \end{aligned} \tag{19}$$

By using Lemma 3 and the condition (ii), we obtain

$$\|x(t - \eta, t_0, x_0)\| \leq \alpha c h(t) h^{-1}(\tau_0), \quad \text{for } \|x_0\| \leq \alpha. \tag{20}$$

Hence,

$$\begin{aligned} M^*(t) &\leq c\{\|y_0 - x_0\| + |\tau_0 - t_0|\} h(t) h^{-1}(\tau_0) \\ &\quad + \int_{\tau_0}^{t} c h(t) h^{-1}(s) r(s) M^*(s) ds \\ &\quad + c^2 \alpha h(t) h^{-1}(\tau_0) \int_{\tau_0}^{t} r(s) ds. \end{aligned} \tag{21}$$

Then we have

$$\begin{aligned} N^*(t) &\leq c\{\|y_0 - x_0\| + |\tau_0 - t_0|\} \\ &\quad + \int_{\tau_0}^{t} r^*(s) N^*(s) ds + c^2 \alpha N_1(\tau_0), \end{aligned} \tag{22}$$

where $r^*(t) = c r(t)$, $N^*(t) = h^{-1}(t) h(\tau_0) M^*(t)$, and $\int_{\tau_0}^{+\infty} r(s) ds = N_1(\tau_0)$.

By Gronwall's inequality, one gets

$$\begin{aligned} M^*(t) \leq &\left\{c[\|y_0 - x_0\| + |\tau_0 - t_0|] + c^2 \alpha N_1(\tau_0)\right\} \\ &\times h(t) h^{-1}(\tau_0) e^{c N_1(\tau_0)}. \end{aligned} \tag{23}$$

Moreover, set

$$\begin{aligned} &c_1\{\|y_0 - x_0\| + |\tau_0 - t_0|\} \\ &\quad = \left\{c[\|y_0 - x_0\| + |\tau_0 - t_0|] + c^2 \alpha N_1(\tau_0)\right\} e^{c N_1(\tau_0)} \end{aligned} \tag{24}$$

and $c_1 \geq 1$, we get

$$\begin{aligned} &\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \\ &\quad \leq c_1\{\|y_0 - x_0\| + |\tau_0 - t_0|\} h(t) h^{-1}(\tau_0). \end{aligned} \tag{25}$$

From Definition 1, it follows that the solution of (3) is ITDhS with respect to the solution $x(t - \eta, t_0, x_0)$. This completes the proof. □

4. Example

Now, we shall illustrate Theorem 6 by a simple example. Consider the differential systems

$$x' = -x, \quad x(t_0) = x_0, \quad t \geq t_0, \quad t_0 \in R^+, \tag{26}$$

$$x' = -x, \quad x(\tau_0) = y_0, \quad t \geq \tau_0, \ \tau_0 \in R^+, \tag{27}$$

and the perturbed differential system of (27):

$$y' = -y + \frac{1}{t^2}y, \quad y(\tau_0) = y_0, \quad t \geq \tau_0, \ \tau_0 \in R^+. \tag{28}$$

Define $z(t) = x(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$; by direct calculation, we have the solution of (27) given by $x(t,\tau_0,y_0) = y_0e^{-t+\tau_0}$, which exists for all $t \geq \tau_0$, and $\Phi(t,\tau_0,y_0) = \partial x(t,\tau_0,y_0)/\partial y_0 = e^{-t+\tau_0}$, $\Phi(\tau_0,\tau_0,y_0) = I$. $\|z(t)\| \leq (\|y_0 - x_0\| + |\tau_0 - t_0|)e^{-t+\tau_0}$, and then the solution of system (27) is ITDhS with respect to $x(t-\eta,\tau_0,x_0)$.

Now, let us begin to consider the perturbation term $F(t,y) = (1/t^2)y$ of (28), and we have $\|(1/t^2)y\| \leq (1/t^2)\|y\|$, where $\int_{\tau_0}^{+\infty}(1/t^2)dt < +\infty$. Then by Theorem 6, we can conclude that the solution $y(t,\tau_0,y_0)$ of (28) is ITDhS with respect to the solution $x(t-\eta,t_0,x_0)$.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China (11271106) and the Natural Science Foundation of Hebei Province of China (A2013201232).

References

[1] M. Pinto, "Perturbations of asymptotically stable differential systems," *Analysis*, vol. 4, no. 1-2, pp. 161–175, 1984.

[2] S. K. Choi and H. S. Ryu, "h-stability in differential systems," *Bulletin of the Institute of Mathematics*, vol. 21, no. 3, pp. 245–262, 1993.

[3] S. K. Choi, N. J. Koo, and H. S. Ryu, "h-stability of differential systems via t_∞-similarity," *Bulletin of the Korean Mathematical Society*, vol. 34, no. 3, pp. 371–383, 1997.

[4] S. K. Choi and N. J. Koo, "h-stability for nonlinear perturbed systems," *Annals of Differential Equations*, vol. 11, no. 1, pp. 1–9, 1995.

[5] S. K. Choi and N. J. Koo, "Variationally stable difference systems by n_∞-similarity," *Journal of Mathematical Analysis and Applications*, vol. 249, no. 2, pp. 553–568, 2000.

[6] S. K. Choi, N. J. Koo, and Y. H. Goo, "Variationally stable difference systems," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 587–605, 2001.

[7] S. K. Choi, Y. H. Goo, and N. Koo, "h-stability of dynamic equations on time scales with nonregressivity," *Abstract and Applied Analysis*, vol. 2008, Article ID 632473, 13 pages, 2008.

[8] D. Kong and L. Q. Zhang, "h-stability of impulsive dynamic systems on time scales," *Science Technology and Engineering*, vol. 11, no. 32, pp. 7870–7872, 2011.

[9] P. G. Wang, S. J. Li, and Z. R. Zhan, "h-stability of differential systems," *Mathematics in Practice and Theory*, vol. 38, no. 10, pp. 187–190, 2008.

[10] Z. M. Wu and L. L. Yin, "h-stability for a class of perturbed difference systems," *Journal of Mathematics and System Science*, vol. 32, no. 9, pp. 1138–1144, 2012.

[11] V. Lakshmikantham and A. S. Vatsala, "Differential inequalities with initial time difference and applications," *Journal of Inequalities and Applications*, vol. 3, no. 3, pp. 233–244, 1999.

[12] V. Lakshmikantham, S. Leela, and J. V. Devi, "Another approach to the theory of differential inequalities relative to changes in the initial times," *Journal of Inequalities and Applications*, vol. 4, no. 2, pp. 163–174, 1999.

[13] T. Jankowski, "Systems of differential inequalities with initial time difference," *Ukrainian Mathematical Journal*, vol. 56, no. 1, pp. 139–145, 2004.

[14] T. Jankowski, "Integro-differential inequalities with initial time difference and applications," *Acta Mathematica Hungarica*, vol. 100, no. 4, pp. 329–342, 2003.

[15] C. Yakar, "Boundedness criteria with initial time difference in terms of two measures," *Dynamics of Continuous, Discrete & Impulsive Systems A*, vol. 14, no. Advances in Dynamical Systems, suppl. S2, pp. 270–274, 2007.

[16] C. Yakar and M. D. Shaw, "Practical stability in terms of two measures with initial time difference," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 12, pp. 781–785, 2009.

[17] C. Yakar, "Fractional differential equations in terms of comparison results and Lyapunov stability with initial time difference," *Abstract and Applied Analysis*, vol. 2010, Article ID 762857, 16 pages, 2010.

[18] W. Chen, Z. Han, S. Sun, and T. Li, "Practical φ_0-stability for impulsive dynamic systems with time scales and initial time difference," *Journal of Applied Mathematics & Informatics*, vol. 29, no. 3-4, pp. 891–900, 2011.

[19] J. Y. Bao, P. G. Wang, and C. X. Gao, "Stability criteria for differential equations with initial time difference," *Acta Mathematicae Applicatae Sinica*, vol. 35, no. 4, pp. 608–616, 2012.

[20] A. Li, E. Feng, and S. Li, "Stability and boundedness criteria for nonlinear differential systems relative to initial time difference and applications," *Nonlinear Analysis*, vol. 10, no. 2, pp. 1073–1080, 2009.

[21] M. D. Shaw and C. Yakar, "Generalized variation of parameters with initial time difference and a comparison result in term Lyapunov-like functions," *International Journal of Non-Linear Differential Equations-Theory Methods and Applications*, vol. 5, pp. 86–108, 1999.

[22] M. D. Shaw and C. Yakar, "Stability criteria and slowly growing motions with initial time difference," *Problems of Nonlinear Analysis in Engineering Systems*, vol. 1, pp. 50–66, 2000.

[23] C. Yarak and M. D. Shaw, "Initial time difference stability in terms of two measures and a variational comparison result," *Dynamics of Continuous, Discrete & Impulsive Systems A*, vol. 15, no. 3, pp. 417–425, 2008.

[24] C. Yakar and S. G. Deo, "Variation of parameters formulae with initial time difference for linear integrodifferential equations," *Applicable Analysis*, vol. 85, no. 4, pp. 333–343, 2006.

[25] V. Lakshmikantham and S. G. Deo, *Method of Variation of Parameters for Dynamic Systems*, vol. 1, Gordon and Breach Science, London, UK, 1998.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 456530, 4 pages
http://dx.doi.org/10.1155/2013/456530

Research Article

Dynamics of a Family of Nonlinear Delay Difference Equations

Qiuli He,[1] Taixiang Sun,[2] and Hongjian Xi[3]

[1] *College of Electrical Engineering, Guangxi University, Nanning, Guangxi 530004, China*
[2] *College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China*
[3] *Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China*

Correspondence should be addressed to Taixiang Sun; stx1963@163.com

Received 16 March 2013; Accepted 18 April 2013

Academic Editor: Zhenkun Huang

We study the global asymptotic stability of the following difference equation: $x_{n+1} = f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t})$, $n = 0, 1, \ldots$, where $0 \le k_1 < k_2 < \cdots < k_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive, and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$. We give sufficient conditions under which the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable.

1. Introduction

In this note, we consider a nonlinear difference equation and deal with the question of whether the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable. Recently, there has been much interest in studying the global attractivity, the boundedness character, and the periodic nature of nonlinear difference equations; for example, see [1–22].

Amleh et al. [1] studied the characteristics of the difference equation:

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}. \tag{E1}$$

They confirmed a conjecture in [13] and showed that the unique positive equilibrium $\overline{x} = p + 1$ of (E1) is globally asymptotically stable provided $p > 1$.

Fan et al. [8] investigated the following difference equation:

$$x_{n+1} = f\left(x_n, x_{n-k}\right). \tag{E2}$$

They showed that the length of finite semicycle of (E2) is less than or equal to k and gave sufficient conditions under which every positive solution of (E2) converges to the unique positive equilibrium.

Kulenović et al. [11] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{E3}$$

where the initial values $x_{-1}, x_0 \in R_+ \equiv (0, +\infty)$ and p_n is the period-two sequence

$$p_n = \begin{cases} \alpha, & \text{if } n \text{ is even}, \\ \beta, & \text{if } n \text{ is odd}, \end{cases} \quad \text{with } \alpha, \beta \in R_+. \tag{1}$$

Sun and Xi [20] studied the more general equation

$$x_{n+1} = f\left(x_{n-s}, x_{n-t}\right), \quad n = 0, 1, 2, \ldots, \tag{E4}$$

where $s, t \in \{0, 1, 2, \ldots\}$ with $s < t$, the initial values $x_{-t}, x_{-t+1}, \ldots, x_0 \in R_+$ and gave sufficient conditions under which every positive solution of (E4) converges to the unique positive equilibrium.

In this paper, we study the global asymptotic stability of the following difference equation:

$$x_{n+1} = f\left(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t}\right), \quad n = 0, 1, \ldots, \tag{2}$$

where $0 \leq k_1 < k_2 < \cdots < k_s$ and $0 \leq m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$ and $a = \inf_{(u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_t)\in E^{s+t}} \times f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \in E$ satisfies the following conditions:

(H_1) $f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t)$ is decreasing in u_i for any $i \in \{1, 2, \ldots, s\}$ and increasing in v_j for any $j \in \{1, 2, \ldots, t\}$.

(H_2) Equation (2) has the unique positive equilibrium, denoted by $\overline{x}$.

(H_3) The function $f(a, a, \ldots, a; x, x, \ldots, x)$ has only fixed point in the interval $(a, +\infty)$, denoted by A.

(H_4) For any $y \in E$, $f(y, \ldots, y; x, \ldots, x)/x$ is nonincreasing in $x \in (0, +\infty)$.

(H_5) If $(x, y) \in E \times E$ is a solution of the system

$$\begin{aligned} y &= f(x, \ldots, x; y, \ldots, y), \\ x &= f(y, \ldots, y; x, \ldots, x), \end{aligned} \tag{3}$$

then $x = y$.

2. Main Result

Theorem 1. *Assume that (H_1)–(H_5) hold. Then the unique positive equilibrium $\overline{x}$ of (2) is globally asymptotically stable.*

Proof. Let $l = \max\{m_t, k_s\}$. Since

$$\begin{aligned} a = {} & \inf_{(u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_t)\in E^{s+t}} f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \\ & \in E, \end{aligned} \tag{4}$$

we have

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}) > f(\overline{x} + 1, \overline{x}, \ldots, \overline{x}) \geq a. \tag{5}$$

Claim 1. $f(A, \ldots, A; a, \ldots, a) < \overline{x} < A$.

Proof of Claim 1. Assume on the contrary that $\overline{x} \geq A$. Then it follows from (H_1), (H_3), and (H_4) that

$$\begin{aligned} A &= f(a, \ldots, a; A, \ldots, A) > f(\overline{x}, \ldots, \overline{x}; A, \ldots, A) \\ &= \frac{f(\overline{x}, \ldots, \overline{x}; A, \ldots, A)}{A} A \geq \frac{f(\overline{x}, \ldots, \overline{x})}{\overline{x}} A \\ &= A. \end{aligned} \tag{6}$$

This is a contradiction. Therefore $\overline{x} < A$. Obviously

$$f(A, \ldots, A; a, \ldots, a) < f(\overline{x}, \ldots, \overline{x}; \overline{x}, \ldots, \overline{x}) = \overline{x}. \tag{7}$$

Claim 1 is proven.

Claim 2. For any $M \geq A$, $J = [a, M]$ is an invariable interval of (2).

Proof of Claim 2. For any $x_0, x_{-1}, \ldots, x_{-l} \in J$, we have from (H_4) that

$$\begin{aligned} a &\leq x_1 \\ &= f(x_{-k_1}, x_{-k_2}, \ldots, x_{-k_s}; x_{-m_1}, x_{-m_2}, \ldots, x_{-m_t}) \\ &\leq \frac{f(a, \ldots, a; M, \ldots, M)}{M} M \leq \frac{f(a, \ldots, a; A, \ldots, A)}{A} M \\ &= M. \end{aligned} \tag{8}$$

By induction, we may show that $x_n \in J$ for any $n \geq 1$. Claim 2 is proven.

Let $m_0 = a$, $M_0 = M \geq A$ and for any $i \geq 0$,

$$\begin{aligned} m_{i+1} &= f(M_i, \ldots, M_i; m_i, \ldots, m_i), \\ M_{i+1} &= f(m_i, \ldots, m_i; M_i, \ldots, M_i). \end{aligned} \tag{9}$$

Claim 3. For any $n \geq 0$, we have

$$\begin{gathered} m_n \leq m_{n+1} < \overline{x} < M_{n+1} \leq M_n, \\ \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = \overline{x}. \end{gathered} \tag{10}$$

Proof of Claim 3. From Claim 2, we obtain

$$\begin{aligned} m_0 \leq m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1 \leq M_0, \\ m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &\leq f(M_1, \ldots, M_1; m_1, \ldots, m_1) = m_2 \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_1, \ldots, m_1; M_1, \ldots, M_1) = M_2 \\ &\leq f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1. \end{aligned} \tag{11}$$

By induction, we have that for $n \geq 0$,

$$m_n \leq m_{n+1} < \overline{x} < M_{n+1} \leq M_n. \tag{12}$$

Set

$$\beta = \lim_{n\to\infty} m_n \quad \textit{and} \quad \alpha = \lim_{n\to\infty} M_n. \tag{13}$$

Then

$$\beta = f(\alpha, \dots, \alpha; \beta, \dots, \beta), \quad \alpha = f(\beta, \dots, \beta; \alpha, \dots, \alpha). \tag{14}$$

This with (H_2) and (H_5) implies $\alpha = \beta = \overline{x}$. Claim 3 is proven.

Claim 4. The equilibrium $\overline{x}$ of (2) is locally stable.

Proof of Claim 4. Let $M = A$ and m_n, M_n be the same as Claim 3. For any $\varepsilon > 0$ with $0 < \varepsilon < \min\{A - \overline{x}, \overline{x} - a\}$, there exists $n > 0$ such that

$$\overline{x} - \varepsilon < m_n < \overline{x} < M_n < \overline{x} + \varepsilon. \tag{15}$$

Set $0 < \delta = \min\{\overline{x} - m_n, M_n - \overline{x}\}$. Then for any $x_0, x_{-1}, \dots, x_{-l} \in (\overline{x} - \delta, \overline{x} + \delta)$, we have

$$\begin{aligned} x_1 &= f\left(x_{-k_1}, \dots, x_{-k_s}; x_{-m_1}, \dots, x_{-m_t}\right) \\ &\le f(m_n, \dots, m_n; M_n, \dots, M_n) \\ &= M_{n+1} \le M_n, \\ x_1 &= f\left(x_{-k_1}, \dots, x_{-k_s}; x_{-m_1}, \dots, x_{-m_t}\right) \\ &\ge f(M_n, \dots, M_n; m_n, \dots, m_n) \\ &= m_{n+1} \ge m_n. \end{aligned} \tag{16}$$

In similar fashion,we can show that for any $k \ge 1$,

$$x_k \in [m_n, M_n] \subset (\overline{x} - \varepsilon, \overline{x} + \varepsilon). \tag{17}$$

Claim 4 is proven.

Claim 5. $\overline{x}$ is the global attractor of (2).

Proof of Claim 5. Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (2), and let $M = \max\{x_1, \dots, x_{l+1}, A\}$ and m_n, M_n be the same as Claim 3. From Claim 2, we have $x_n \in [m_0, M_0] = [a, M]$ for any $n \ge 1$. Moreover, we have

$$\begin{aligned} & x_{l+2} \\ &\quad = f\left(x_{l+1-k_1}, \dots, x_{l+1-k_s}; x_{l+1-m_1}, \dots, x_{l+1-m_t}\right) \\ &\quad \le f(m_0, \dots, m_0; M_0, \dots, M_0) = M_1, \\ & x_{l+2} \\ &\quad = f\left(x_{l+1-k_1}, \dots, x_{l+1-k_s}; x_{l+1-m_1}, \dots, x_{l+1-m_t}\right) \\ &\quad \ge f(M_0, \dots, M_0; m_0, \dots, m_0) = m_1. \end{aligned} \tag{18}$$

In similar fashion, we may show $x_n \in [m_1, M_1]$ for any $n \ge l + 2$. By induction, we obtain

$$x_n \in [m_k, M_k] \quad \text{for } n \ge k(l+1) + 1. \tag{19}$$

It follows from Claim 3 that $\lim_{n\to\infty} x_n = \overline{x}$. Claim 5 is proven.

From Claims 4 and 5, Theorem 1 follows. □

3. Applications

In this section, we will give two applications of Theorem 1.

Example 2. Consider equation

$$x_{n+1} = p + \frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}} + \sqrt{\frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}}}, \quad n = 0, 1, \dots, \tag{20}$$

where $0 \le n_1 < n_2 < \cdots < n_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{n_1, n_2, \dots, n_s\} \bigcap \{m_1, m_2, \dots, m_t\} = \emptyset$, $p > 0, a_i > 0$ for any $i \in \{1, 2, \dots, t\}$ and $b_k > 0$ for any $k \in \{1, 2, \dots, s\}$, and the initial conditions $x_{-l}, \dots, x_0 \in (0, \infty)$ with $l = \max\{m_t, n_s\}$. Write $A = \sum_{i=1}^{t} a_i$ and $B = \sum_{k=1}^{s} b_k$. If $pB > A$, then the unique positive equilibrium $\overline{x}$ of (20) is globally asymptotically stable.

Proof. Let $E = (0, +\infty)$. It is easy to verify that (H_1), (H_2), and (H_4) hold for (20). Note that $a = \inf_{(u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_t) \in E^{s+t}} f(u_1, u_2, \dots, u_s; v_1, v_2, \dots, v_t) = p$. Then

$$x = f(a, a, \dots, a; x, x, \dots, x) = p + \frac{Ax}{Bp} + \sqrt{\frac{Ax}{Bp}} \tag{21}$$

has only solution

$$x = \sqrt{\left[\sqrt{pAB} + \sqrt{pAB + 4p^2 B(Bp - A)}\right] / 2(Bp - a)} \tag{22}$$

in the interval $(p, +\infty)$, which implies that (H_3) holds for (20). In addition, let

$$x = p + \frac{xA}{yB} + \sqrt{\frac{xA}{yB}}, \quad y = p + \frac{yA}{xB} + \sqrt{\frac{yA}{xB}}, \tag{23}$$

then

$$\frac{x}{y} = \frac{p + xA/yB + \sqrt{xA/yB}}{p + yA/xB + \sqrt{yA/xB}}. \tag{24}$$

Therefore $x/y = 1$, which implies that (23) has unique solution

$$x = y = \overline{x} = p + \frac{A}{B} + \sqrt{A/B}. \tag{25}$$

Thus (H_5) holds for (20). It follows from Theorem 1 that the equilibrium $\overline{x} = p + A/B + \sqrt{A/B}$ of (20) is globally asymptotically stable. □

Example 3. Consider equation

$$x_{n+1} = \frac{q + \sum_{i=1}^{t} a_i x_{n-m_i}}{p + \sum_{k=1}^{s} b_k x_{n-n_k}}, \quad n = 0, 1, \ldots, \tag{26}$$

where $0 \le n_1 < n_2 < \cdots < n_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{n_1, n_2, \ldots, n_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, $p > 0$, $q > 0$, $a_i > 0$ for any $1 \le i \le t$ and $b_j > 0$ for any $1 \le j \le s$, and the initial conditions $x_{-l}, \ldots, x_0 \in (0, \infty)$ with $l = \max\{m_t, n_s\}$. Write $A = \sum_{i=1}^{t} a_i$ and $B = \sum_{k=1}^{s} b_k$. If $p > A$, then the unique positive equilibrium $\overline{x}$ of (26) is globally asymptotically stable.

Proof. Let $E = [0, +\infty)$. It is easy to verify that (H_1)–(H_4) hold for (26). In addition, the following equation

$$\begin{aligned} x &= \frac{q + xA}{p + yB}, \\ y &= \frac{q + yA}{p + xB} \end{aligned} \tag{27}$$

has unique solution

$$x = y = \overline{x} = \frac{A - p + \sqrt{(p - A)^2 + 4Bq}}{2B}, \tag{28}$$

which implies that (H_5) holds for (26). It follows from Theorem 1 that the equilibrium $\overline{x} = (A - p + \sqrt{(p - A)^2 + 4Bq})/2B$ of (26) is globally asymptotically stable. □

Acknowledgments

This project is supported by NNSF of China (11261005, 51267001) and NSF of Guangxi (2011GXNSFA018135, 2012GXNSFDA276040).

References

[1] A. M. Amleh, D. A. Georgiou, E. A. Grove, and G. Ladas, "On the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n$," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 790–798, 1999.

[2] K. S. Berenhaut and R. T. Guy, "Periodicity and boundedness for the integer solutions to a minimum-delay difference equation," *Journal of Difference Equations and Applications*, vol. 16, no. 8, pp. 895–916, 2010.

[3] K. S. Berenhaut, R. T. Guy, and C. L. Barrett, "Global asymptotic stability for minimum-delay difference equations," *Journal of Difference Equations and Applications*, vol. 17, no. 11, pp. 1581–1590, 2011.

[4] K. S. Berenhaut and A. H. Jones, "Asymptotic behaviour of solutions to difference equations involving ratios of elementary symmetric polynomials," *Journal of Difference Equations and Applications*, vol. 18, no. 6, pp. 963–972, 2012.

[5] E. Camouzis and G. Ladas, "When does local asymptotic stability imply global attractivity in rational equations?" *Journal of Difference Equations and Applications*, vol. 12, no. 8, pp. 863–885, 2006.

[6] R. Devault, V. L. Kocic, and D. Stutson, "Global behavior of solutions of the nonlinear difference equation $x_{n+1} = p_n + x_{n-1}/x_n$," *Journal of Difference Equations and Applications*, vol. 11, no. 8, pp. 707–719, 2005.

[7] H. El-Metwally, "Qualitative properties of some higher order difference equations," *Computers & Mathematics with Applications*, vol. 58, no. 4, pp. 686–692, 2009.

[8] Y. Fan, L. Wang, and W. Li, "Global behavior of a higher order nonlinear difference equation," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 1, pp. 113–126, 2004.

[9] A. Gelişken, C. Çinar, and A. S. Kurbanli, "On the asymptotic behavior and periodic nature of a difference equation with maximum," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 898–902, 2010.

[10] B. D. Iričanin, "Global stability of some classes of higher-order nonlinear difference equations," *Applied Mathematics and Computation*, vol. 216, no. 4, pp. 1325–1328, 2010.

[11] M. R. S. Kulenović, G. Ladas, and C. B. Overdeep, "On the dynamics of $x_{n+1} = p_n + x_{n-1}/x_n$ with a period-two coefficient," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 905–914, 2004.

[12] A. S. Kurbanlı, C. Çinar, and G. Yalçinkaya, "On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = x_{n-1}/(y_n x_{n-1}+1)$, $y_{n+1} = y_{n-1}/(x_n y_{n-1}+1)$," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 1261–1267, 2011.

[13] G. Ladas, "Open problems and conjecture," *Journal of Differential Equations and Applications*, vol. 5, pp. 317–321, 1995.

[14] G. Papaschinopoulos, M. A. Radin, and C. J. Schinas, "On the system of two difference equations of exponential form: $x_{n+1} = a+bx_{n-1}e^{-y_n}$, $y_{n+1} = c+dy_{n-1}e^{-x_n}$," *Mathematical and Computer Modelling*, vol. 54, no. 11-12, pp. 2969–2977, 2011.

[15] G. Papaschinopoulos and C. J. Schinas, "On the dynamics of two exponential type systems of difference equations," *Computers & Mathematics with Applications*, vol. 64, no. 7, pp. 2326–2334, 2012.

[16] G. Papaschinopoulos, C. J. Schinas, and G. Stefanidou, "On the nonautonomous difference equation $x_{n+1} = A_n + x_{n-1}^p/x_n^q$," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5573–5580, 2011.

[17] C. Qian, "Global attractivity of periodic solutions in a higher order difference equation," *Applied Mathematics Letters*, vol. 26, pp. 578–583, 2013.

[18] S. Stević, "Boundedness character of a class of difference equations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 70, no. 2, pp. 839–848, 2009.

[19] S. Stević, "Periodicity of a class of nonautonomous max-type difference equations," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9562–9566, 2011.

[20] T. Sun and H. Xi, "Global behavior of the nonlinear difference equation $x_{n+1} = f(x_{n-s}, x_{n-t})$," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 760–765, 2005.

[21] T. Sun, H. Xi, and Q. He, "On boundedness of the difference equation $x_{n+1} = p_n + x_{n-3s+1}/x_{n-s+1}$ with period-*k* coefficients," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5994–5997, 2011.

[22] N. Touafek and E. M. Elsayed, "On the solutions of systems of rational difference equations," *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 1987–1997, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 436062, 11 pages
http://dx.doi.org/10.1155/2013/436062

Research Article

Analysis of Feature Fusion Based on HIK SVM and Its Application for Pedestrian Detection

Song-Zhi Su[1,2] and Shu-Yuan Chen[3]

[1] *School of Information Science and Technology, Xiamen University, Xiamen, Fujian Province 361005, China*
[2] *Fujian Key Laboratory of the Brain-Like Intelligent Systems (Xiamen University), Xiamen, Fujian Province 361005, China*
[3] *Department of Computer Science and Engineering, Yuan Ze University, Taoyuan 320, Taiwan*

Correspondence should be addressed to Song-Zhi Su; ssz@xmu.edu.cn

Received 5 March 2013; Accepted 1 April 2013

Academic Editor: Zhenkun Huang

This work presents the fusion of integral channel features to improve the effectiveness and efficiency of pedestrian detection. The proposed method combines the histogram of oriented gradient (HOG) and local binary pattern (LBP) features by a concatenated fusion method. Although neural network (NN) is an efficient tool for classification, the time complexity is heavy. Hence, we choose support vector machine (SVM) with the histogram intersection kernel (HIK) as a classifier. On the other hand, although many datasets have been collected for pedestrian detection, few are designed to detect pedestrians in low-resolution visual images and at night time. This work collects two new pedestrian datasets—one for low-resolution visual images and one for near-infrared images—to evaluate detection performance on various image types and at different times. The proposed fusion method uses only images from the INRIA dataset for training but works on the two newly collected datasets, thereby avoiding the training overhead for cross-datasets. The experimental results verify that the proposed method has high detection accuracies even in the variations of image types and time slots.

1. Introduction

Pedestrian detection is an active area of research in the field of computer vision [1, 2] and a preliminary task in various applications, including intelligent video surveillance, automotive robotics, content-based image annotation/retrieval, and management of personal digital images. Large variations in appearance caused by articulated body motion, viewpoint, lighting conditions, occlusions, and cluttered backgrounds present serious challenges. Hence, pedestrian detection in still images is more difficult than that of faces [3].

Most pedestrian detection methods use a pretrained binary classifier to find pedestrians in still images by scanning the entire image. Such a method is called the "sliding window method" (or scanning window). The classifier is "fired" if the image features inside the local search subwindow satisfy certain criteria. At the core of the sliding window framework are image descriptors and classifiers that are based on these descriptors. According to features used for pedestrian detection, these methods can be divided into three groups: holistic-based methods, part-based methods, and patch-based methods.

Holistic-based methods use global features, such as edge templates, histogram of oriented gradient [4], and Haar-like wavelet [5]. One popular holistic-based method is the histogram of oriented gradient (HOG) method, which has near-perfect classification performance when applied to the original MIT pedestrian database and is widely used in other computer vision tasks, such as scene classification, object tracking, and pose estimation. Part-based methods model a pedestrian as a set of parts, which include legs, torso, arms, and head. Hypotheses concerning these parts are generated by learning local features such as edgelet [6] and orientation features. These parts are then assembled to form a final human model based on geometric constraints. Accurate pedestrian detection depends on accurate part detection and pedestrian representation by parts. Though this approach is effective for dealing with partially occluded pedestrian detection, part detection is a difficult task. One example of a patch-based method is implicit-shape-model- (ISM-) based object detection, developed by Leibe et al. [7], which combines both detection and segmentation in a probability framework and requires only a few training images. However, constructing a smart and discriminative codebook from various perspectives remains an open problem.

Numerous descriptors used in pedestrian detection have recently been proposed. Zhao and Thorpe [8] proposed a pedestrian method by a pair of moving camera through stereo-based segmentation and neural network-based recognition. Dalal and Triggs [4] developed a descriptor similar to scale invariant feature transform (SIFT) [9], which encodes HOG in the detection window. HOG has been subsequently extended to describe histograms that present information on motion. Felzenszwalb et al. [10] recently applied HOG to their deformable part models and obtained promising results in the PASCAL VOC Challenge. Zhu et al. [11] implemented a cascade of rejecters based on HOG descriptors to achieve near-real-time performance. Cascade models have also been successfully used with other types of pedestrian descriptors, such as edgelet features and the region of covariance (COV) [12].

In order to integrate various pedestrian descriptors, many works have proposed fusing multiple features to detect pedestrians. Wojek et al. [13] combined the oriented histogram of flow with HOG or Haar on an onboard camera setup and concluded that incorporating motion information considerably enhances detection performance. Y. T. Chen and C. S. Chen [14] proposed a method for detecting humans in a single image, based on a novel cascade structure with metastages. Their method includes both intensity-based rectangular features and gradient-based 1D features in the feature pool of the Real AdaBoost algorithm for weak classifier selection. Wang et al. [15] combined HOG and local binary pattern, trained by a linear SVM, to solve the partial occlusion problem.

However, multicue pedestrian detection methods have the following disadvantages for detecting pedestrians in still images. First, optical flow information cannot be extracted from a single image. Second, edgelet extraction or the COV feature is computation-intensive. Finally, the AdaBoost has too many parameters to tune, and the cascading test is time-consuming and sensitive to occlusion. Therefore, this work uses HOG and LBP features, which can be extracted efficiently by integral images. An SVM with a linear kernel or HIK [16] has the advantage of ease of training in the training stage and fast prediction in the test stage [17].

Although many datasets have been collected for pedestrian detection, few are designed to detect pedestrians in cross-dataset, which is still a hot topic in computer vision. Vazquez et al. [18] proposed an unsupervised domain adaptation of virtual and real worlds for pedestrian detection. Jain and Learned-Miller [19] proposed an online approach for quickly adapting a pretrained classifier to a new test dataset without retraining the classifier. In this work, we collect two new pedestrian datasets—one for low-resolution visual images and one for near infrared images—to evaluate detection performance on various image types and at different times. This work proposes cross-dataset pedestrian detection by fusing integral channel features, which use only images from the INRIA dataset for training but are effective on the two newly collected datasets, thereby avoiding the training overhead for cross-datasets.

The remainder of this paper is organized as follows. Section 2 offers a description of the proposed method, including the features, classifiers, and fusion. Section 3 presents and offers a discussion of the relevant experimental results. Finally, Section 4 draws a conclusion and presents suggestions and directions for future work.

2. Proposed Pedestrian Detection Method

Sliding window-based object detection algorithms for static images consist of two components: feature extraction and classifier training. Feature extraction encodes the visual appearance of a detected object using object descriptors. Classifier training trains a classifier to determine whether the current searching window contains a pedestrian. In this section, we discuss the features and classifiers.

2.1. Feature Extraction. Several methods for describing pedestrians have recently been proposed. This work uses HOG and LBP as pedestrian descriptors. All of these features can be extracted using integral histogram techniques, accelerating the computation process. They are complementary because they encode gradient and texture information, respectively.

HOG. The HOG proposed by Dalal and Triggs [4] has been widely used in the computer vision field, including object detection, recognition, and classification. HOG is similar to edge orientation histograms, shape context, and the SIFT descriptor, but it is computed on a dense grid of uniformly titled cells. Overlapping local contrast normalization in blocks is conducted to improve accuracy. HOG implementation involves dividing search windows into small-connected regions, called cells, for which the histogram of gradient directions is computed (Figure 1(a)). In this work, an HOG descriptor is implemented using the following parameters. Image derivatives in x and y directions are obtained by applying the masks $[-1\ \ 0\ \ 1]$ and $[-1\ \ 0\ \ 1]^T$, respectively. The gradient orientation is linearly voted into nine orientation bins in the range 0°–180°. A block size is 16×16; a cell size is 8×8; blocks overlap half of a cell in each direction; Gaussian is weighting with $\sigma = 4$ using an L2-norm for the feature vector in a block. The final vector consists of all normalized block histograms, yielding 3780 dimensions.

LBP. Various applications have applied the local binary pattern (LBP) extensively, which is highly effective in texture classification and face recognition because it is invariant to monotonic changes in the gray level [20]. Wang et al. [15] noted that HOG performs poorly when the background is cluttered with noisy edges and LBP is complementary when it exploits the uniform pattern concept (Figure 2). In this work, we adopt eight sample points and require bilinear interpolation to find the red points in Figure 2(a) with a radius of one and take the l_∞ distance as the distance to the central pixel. The number of 0/1 transitions is no more than

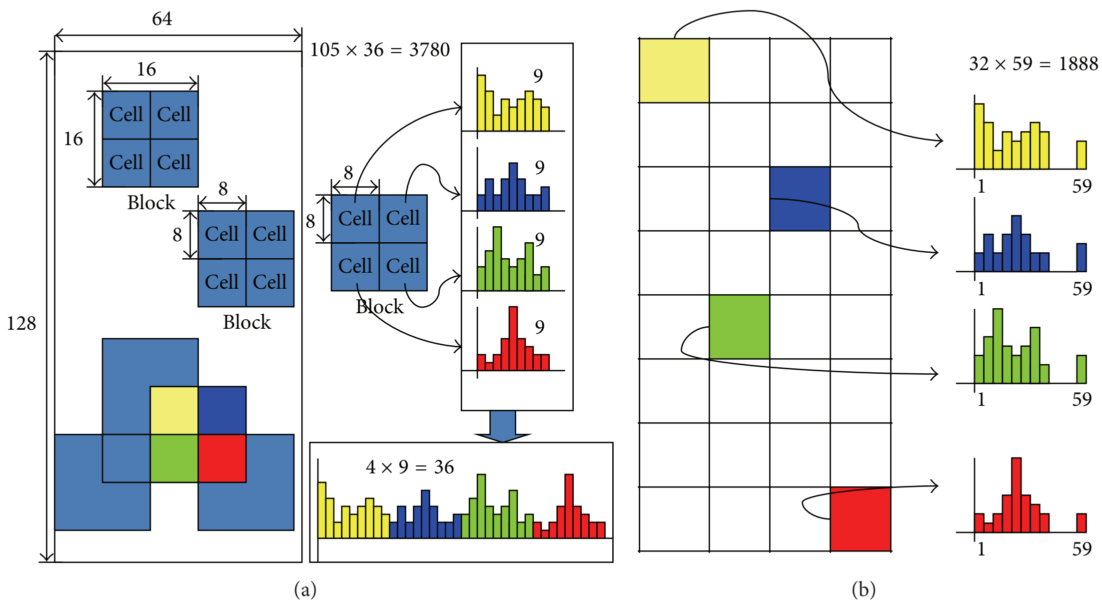


Figure 1: Feature extraction using HOG and LBP.

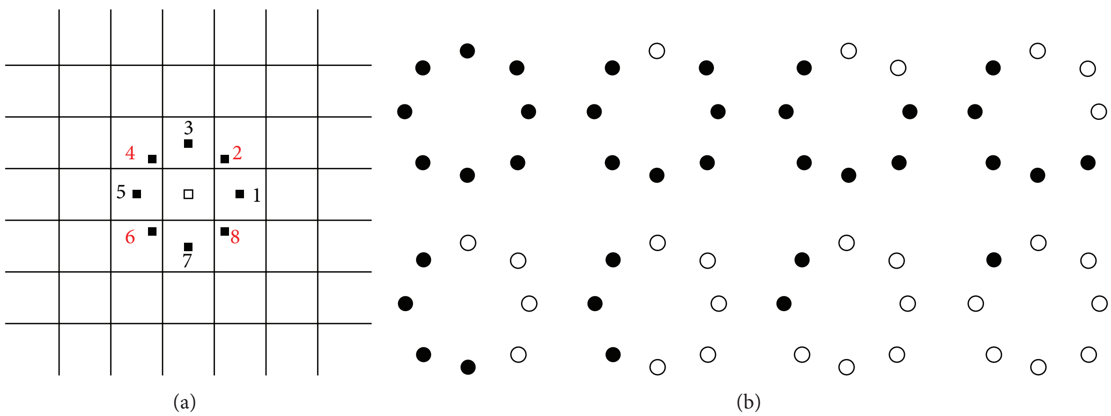


Figure 2: (a) Eight sample points of the local binary pattern; (b) examples of the uniform local binary pattern.

two. Different uniform patterns are counted into different bins, and all nonuniform patterns are voted into one bin. A cell includes 59 bins, and the L2-norm is adopted to normalize the histogram. We used the procedure by Wang et al. to extract the LBP feature and to directly establish pattern histograms in the cells (16 × 16, without overlap, as shown in Figure 1(b)). LBP histograms in the 32 cells are then concatenated into a feature vector with dimensions of 59 × 32 = 1888 to describe the texture in the current search window.

2.2. Classifier Training. Linear SVM and AdaBoost are widely used for detecting pedestrians. This work focuses on an SVM with different kernel functions because it is easy to train in the training stage and can make rapid predictions in the test stage. Linear SVMs learn the hyperplane that separates pedestrians from the background in the original feature space. Extended versions of SVM, such as RBF kernel SVMs, transform data to a high and potentially infinite number of dimensions. However, the extensions are seldom used in pedestrian detection because more dimensions lead to computational overload.

Maji et al. [16] recently approximated the histogram intersection kernel of SVM (HIKSVM) to accelerate prediction, and Wu [17] proposed a fast dual method for HIKSVM learning. Section 3 describes experiments conducted to compare the performance of a linear SVM with that of HIKSVM. The experimental results show that HIKSVM outperforms the linear SVM. A brief introduction of HIKSVM follows.

Swain and Ballard [21] first proposed the HIK, which is widely used as a measure of similarity between histograms. Researchers have proven that HIK is positive definite and can be used as a discriminative kernel function for SVMs. However, the HIK requires memory and computation time that is linearly proportional to the number of support vectors because it is nonlinear. Maji et al. presented HIKSVMs with a runtime complexity, that is, the logarithm of the number of

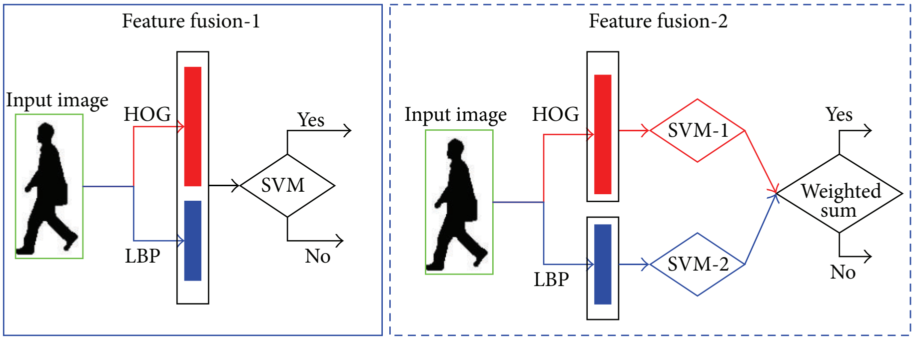


Figure 3: Examples of two feature fusion methods using the SVM as a classifier.

support vectors. Based on precomputing auxiliary tables, an approximate classifier can be constructed with runtime and space requirements that are independent of the number of support vectors.

For feature vectors $\mathbf{x}, \mathbf{y} \in R^n_+$, the HIK can be expressed as follows:

$$k_{\mathrm{HI}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \min(x(i), y(i)), \quad (1)$$

and the corresponding discriminative function for a new input vector $\mathbf{x}$ is

$$\begin{aligned} h(\mathbf{x}) &= \sum_{l=1}^{m} \alpha_l y_l k(x, x_l) + b \\ &= \sum_{l=1}^{m} \alpha_l y_l \left(\sum_{i=1}^{n} \min(x(i), x_l(i)) \right) + b. \end{aligned} \quad (2)$$

Maji et al. noticed that for intersection kernels, the summations in (2) can be reformed as follows:

$$\begin{aligned} h(\mathbf{x}) &= \sum_{i=1}^{n} \left(\sum_{l=1}^{m} \alpha_l y_l \min(x(i), x_l(i)) \right) + b \\ &= \sum_{i=1}^{n} h_i(x(i)) + b, \end{aligned} \quad (3)$$

where $h_i(s) = \sum_{l=1}^{m} \alpha_l y_l \min(s, x_l(i))$. Consider the functions $h_i(s)$ at fixed point i; $\overline{x}_l(i)$ represents the sorted values of $x_l(i)$ in increasing order with corresponding values of α and labels given by $\overline{\alpha}_l$ and $\overline{y}_l$. According to the HIK, let r be the largest integer, such that $\overline{x}_r(i) \le s$; therefore,

$$\begin{aligned} h_i(s) &= \sum_{l=1}^{m} \alpha_l y_l \min(s, x_l(i)) \\ &= \sum_{1 \le l \le r} \overline{\alpha}_l \overline{y}_l \overline{x}_l(i) + s \sum_{r < l \le m} \overline{\alpha}_l \overline{y}_l \\ &= A_i(r) + sB_i(r), \end{aligned} \quad (4)$$

where $A_i(r) = \sum_{1 \le l \le r} \overline{\alpha}_l \overline{y}_l \overline{x}_l(i)$, $B_i(r) = \sum_{r < l \le m} \overline{\alpha}_l \overline{y}_l$. Clearly, (4) is piecewise linear, and functions A_i and B_i are independent of the input data. Therefore, s can be precomputed by first finding the position of $s = x(i)$ in the sorted list by binary search, with a runtime complexity of $O(\log m)$. Although the runtime complexity of computing $h(x)$ is $O(n \log m)$, it necessitates to double the storage that is required by the standard implementation because the modified version must store $\overline{x}_l$ and $h_i(\overline{x}_l)$.

Maji et al. found that the support distributions in each dimension tend to be smooth and concentrated. Therefore, the $h(x)$ is relatively smooth and can be approximated by simpler functions, greatly reducing the required storage and accelerating the prediction. In this work, $h_i(s)$ is computed using a lookup table with a piecewise constant approximation.

2.3. Feature Fusion. The two main feature fusion methods (Figure 3) are concatenated fusion (FF1) and weighted sum (FF2). Concatenated fusion concatenates different feature descriptors and then feeds the concatenated results into the classifier. The weighted sum feeds different features into individual classifiers and then combines classification scores using a weighted sum.

This work fuses HOG and LBP features for detecting pedestrians because both can be implemented by integral histogram approaches, accelerating the subsequent prediction process, as described in Section 3. Let the output scores of the individual SVM classifiers using HOG and LBP features be f_{HOG} and f_{LBP}, respectively. For the FF2 fusion method, the final output score is then defined by the weighted sum

$$f = \alpha f_{\mathrm{HOG}} + (1 - \alpha) f_{\mathrm{LBP}}, \quad 0 < \alpha < 1. \quad (5)$$

The values of α to $\alpha \in \{\alpha \mid \alpha = 0.1K, K = 1, 2, \ldots, 9\}$ are herein. Section 3 verifies that FF1 performance is superior to FF2 for all of the values of α, and FF2 has the best performance when $\alpha = 0.5$. Hence, this work fuses HOG, LBP, and Haar using HIKSVM by the FF1 method because this method is highly accurate, as confirmed in Section 3.

3. Experimental Results

The accuracies achieved using various integral channel features, different kernels of support vector machines, and two feature fusion methods for detecting pedestrians are extensively compared. Random noise blocks are added to

(a) (b) (c)

Figure 4: Examples of pedestrian images: (a) INRIA; (b) XMU-VIS; (c) XMU-NIR.

Table 1: INRIA training and test sets, XMU-VIS test sets, and XMU-NIR datasets.

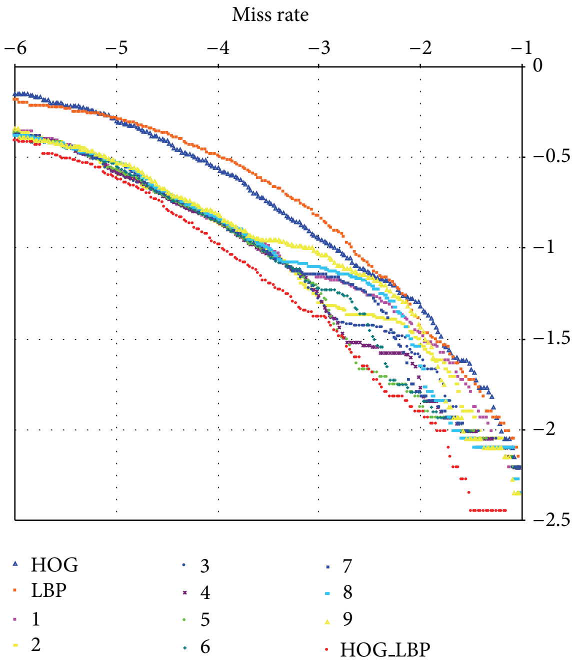
	Training				Test			
	Pedestrians		Nonpedestrians		Pedestrians		Nonpedestrians	
	#imgs	#win	#imgs	#win	#imgs	#win	#imgs	#win
INRIA	615	2416	1218	22111	288	1126	453	4484965
XMU-VIS	—	—	—	—	4207	10154	413	1834994
XMU-NIR	—	—	—	—	1057	2596	—	—

the pedestrian image to test the robustness achieved using various features and classifiers. Experimental results obtained using the INRIA person dataset and two newly collected Xiamen databases indicate that the combined HOG and LBP features by the concatenated-fusion method using the SVM with the HIK as a classifier yield the highest accuracy. The multiple feature combination outperforms single features, and the HIK consistently outperforms the linear SVM.

3.1. Dataset and Performance Evaluation Measures. This work evaluates the performance of pedestrian detection using three databases: the INRIA person database [4] and two new databases collected at Xiamen University, called XMU-VIS and XMU-NIR, respectively. The INRIA dataset contains human images taken from several viewing angles under various lighting conditions both indoors and outdoors. Figure 4(a) shows samples of the INRIA dataset. INRIA images fall into three groups, which are further divided into training and testing sets. The first group is composed of 615 full-size positive images containing 1208 pedestrian instances for training and 288 images containing 566 instances, for testing. The second group comprises scale-normalized crops of humans sized 64×128, including 2416 positive images for training and 1126 positive images for testing. The third group comprises full-size negative images including 1218 images for training and 453 images for testing.

This work used 2416 scale-normalized crops of human images as positive training samples and randomly sampled 22111 subimages from 1218 person-free training photographs as negative training samples. All of the training images are from the INRIA dataset, including the situations of test images from XMU-VIS or XMU-NIR datasets, to show cross-dataset human detection. For the INRIA dataset, the 1126 cropped images of pedestrians were used for testing. The negative test samples were obtained by scanning the 453 testing images in steps of eight pixels in the x- and y-directions using five scales (0.8, 0.9, 1.0, 1.1, and 1.2) of image size, yielding 4484965 negative cropping windows.

The XMU-VIS dataset was collected at various places around Xiamen University and at different time. The size of each pedestrian image in the XMU-VIS dataset is 640×480 smaller than that of INRIA in 720×576. The goal is to simulate images captured by onboard cameras in intelligent vehicles for detecting pedestrians in low resolution. The XMU-VIS test set is composed of 4207 pedestrian images with 10154 cropped images and 413 negative images with 1834994 cropped images. The XMU-NIR dataset was also collected at various locations around Xiamen University and at different times. The images captured by near-infrared sensors were sized 1280×720. The XMU-NIR dataset consists of 1057 pedestrian images, in which 2596 are pedestrians. Table 1 summarizes the three datasets.



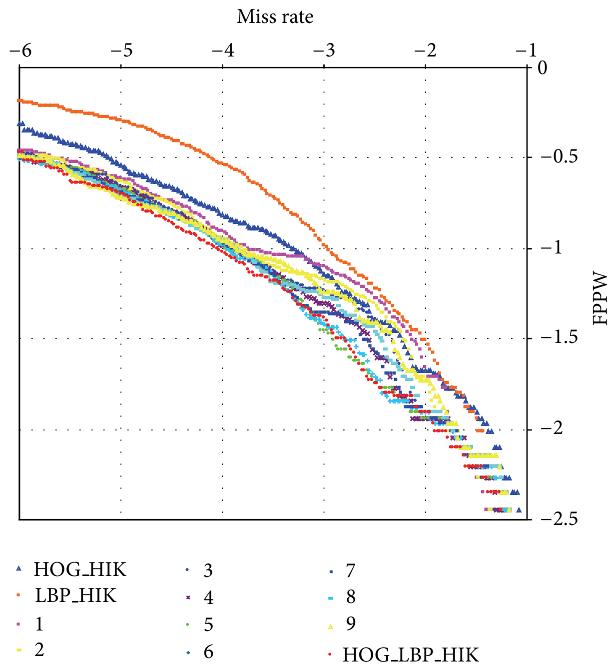


Figure 5: Comparison of accuracies achieved by combining HOG and LBP features through FF1 and FF2 methods on the INRIA dataset. The classifiers are linear SVM and HIKSVM in (a) and (b), respectively.

Feature fusion performance is measured by plotting the number of false positives per window (FPPW) versus the miss rate, as proposed by Dalal and Triggs [4] in Section 3.2. This only measures classification performance and excludes nonmaximum suppression and other postprocessing steps. The FPPW miss rate curves are plotted in log-log space. To avoid sampling bias, negative samples are selected as a fixed set and are not boosted by bootstrap learning.

Pedestrian detection performance over cross-datasets is measured by precision and recall curves, described in Section 3.3. This is a measure of both classification and location performance, including nonmaximum suppression and various postprocessing steps. Efficiency and robustness to occlusion of the proposed method are also discussed in Sections 3.4.

3.2. Performance Evaluation of Feature Fusion. As mentioned in Section 2.3, the two main feature fusion methods are FF1 and FF2. This experiment was conducted to compare the performances of FF1 and FF2. Both the linear SVM and HIKSVM are applied on the INRIA dataset. HIKSVM is approximated as 20 linear segments with a piecewise constant function. Experimental results show that the FF1 method outperforms FF2 for all of the values of α and FF2 has the best performance when $\alpha = 0.5$ (Figure 5). Therefore, FF1 is selected by default for feature fusion hereafter.

The experimental results show that combining HOG and LBP features through the FF1 method using the HIKSVM classifier yields the best performance. Figure 6 shows a comparison of the results obtained by applying combined features (single features or combining HOG and LBP features) and different SVMs (HIKSVM or linear SVM) on the INRIA and XMU-VIS datasets, respectively. Figure 6(a) shows that applying feature HOG to the INRIA dataset is better than applying feature LBP. In contrast, Figure 7(b) shows that applying feature LBP is better than applying HOG on the XMU-VIS dataset. The HIKSVM outperforms the linear SVM, regardless of the features used. Combining HOG and LBP features through the FF1 method with HIKSVM as a classifier yields the best performance, regardless of the INRIA or XMU-VIS datasets. Therefore, the proposed method fuses HOG and LBP features through the FF1 method and uses the HIKSVM as a classifier. The method is then applied to test images using the sliding window strategy to evaluate pedestrian detection performance over cross-datasets in Section 3.3.

3.3. Performance Evaluation of Pedestrian Detection over Cross-Datasets. As shown in [2], the per-window measure for pedestrian classification is flawed and fails to predict full image performance for pedestrian detection. Therefore, the proposed method is also evaluated on full images using the PASCAL criteria in this section. The details are described as follows. The proposed pedestrian detection, fusing HOG and LBP features through the FF1 method with the HIKSVM as a classifier, is used to find pedestrians in an image by scanning the entire image with a fixed size rectangle. A denoted window, labeled as a rectangle in Figure 7, presents the framework of the proposed HIKSVM-based pedestrian detection with sliding window scanning on full images, called sliding window scanning. Various sized windows are scanned to detect multiscale humans.

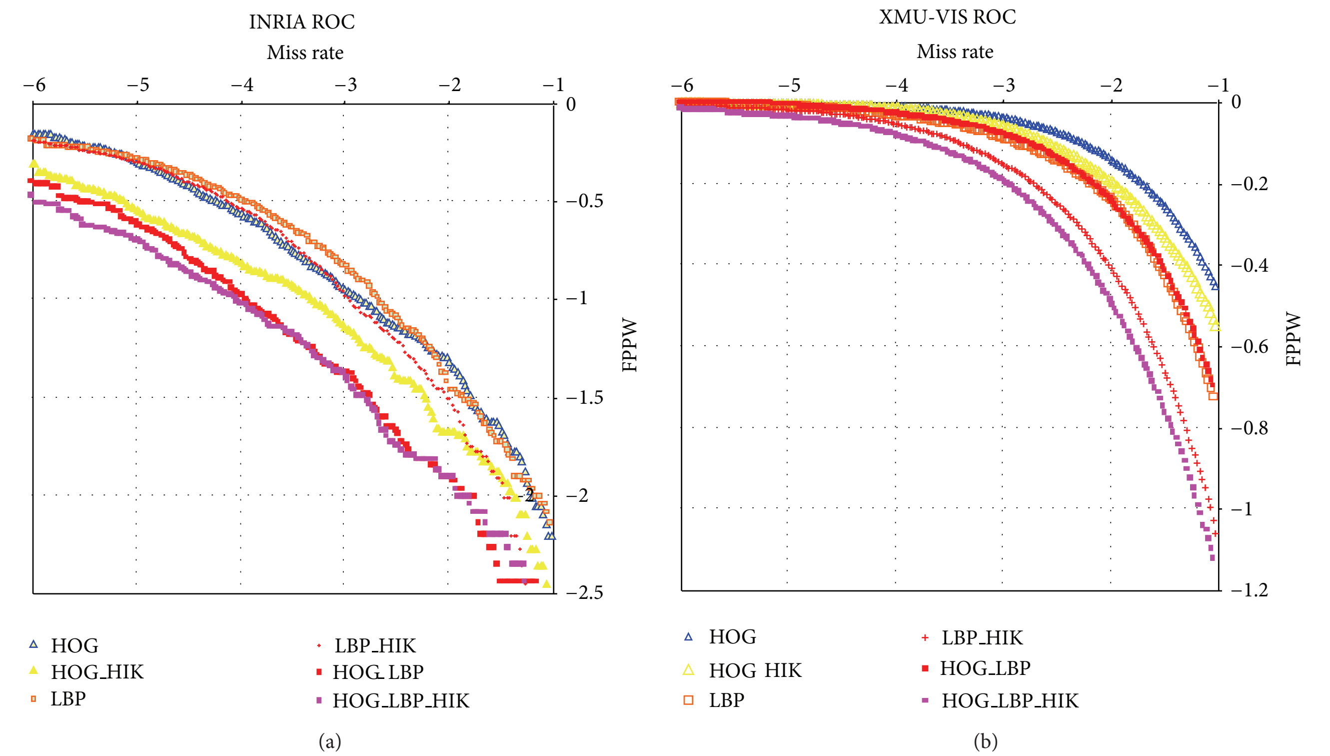


FIGURE 6: Comparing accuracies achieved by applying single features (HOG and LBP) and fusing features (HOG + LBP) using HIKSVM and FF1 to INRIA and XMU-VIS datasets.

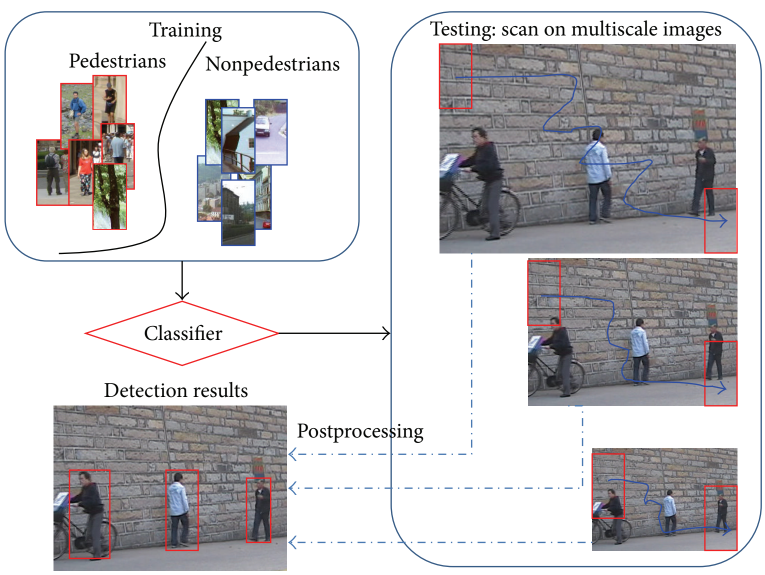
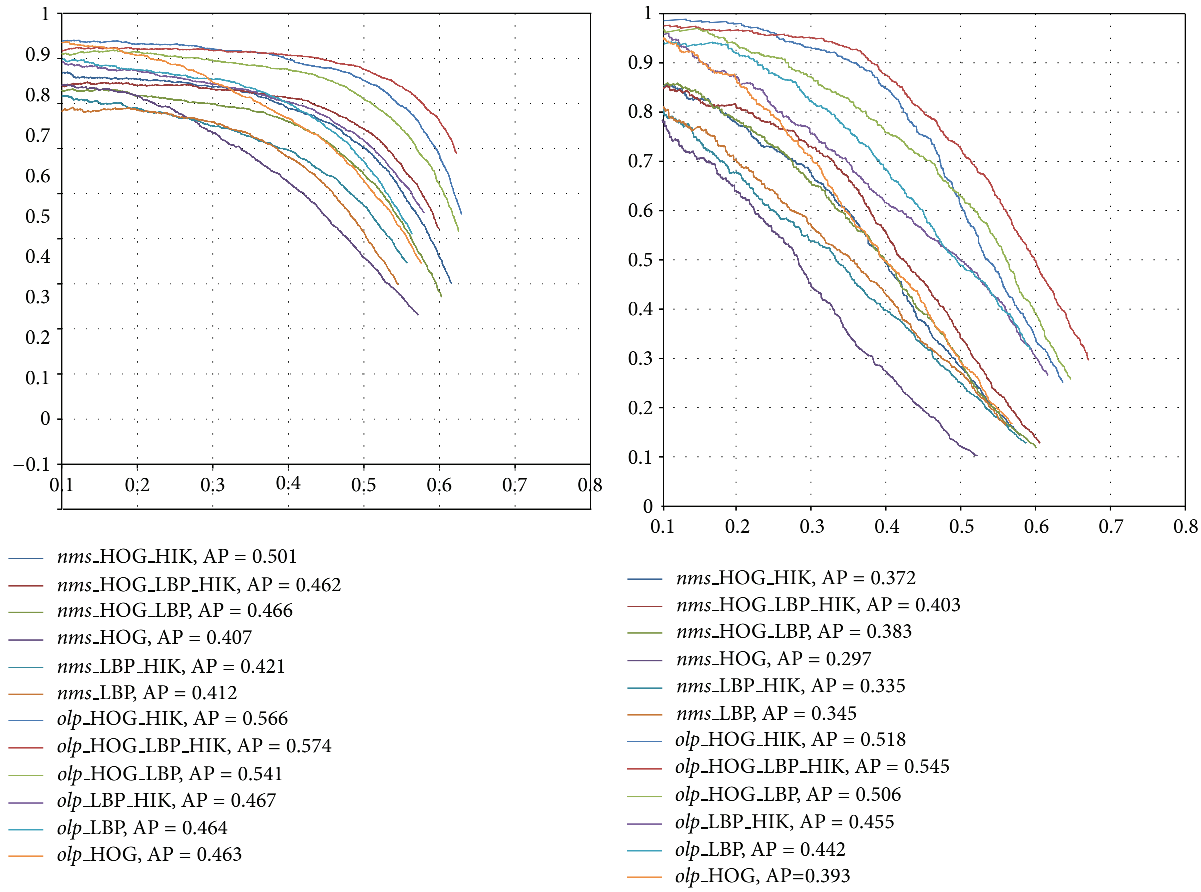


FIGURE 7: Framework of the proposed HIKSVM-based pedestrian detection with sliding window scanning.

The local block features of each window are fed to the HIKSVM-based pedestrian classifier to determine whether a human exists in the window. Windows determining whether a human exists are considered as candidate windows. After performing multiscale sliding window scanning, candidate windows of various sizes may overlap each other, specifically surrounding authentic humans. Overlapping windows should be postprocessed to locate humans with an accurate position. Two typical postprocessing methods, mean-shift location and window overlapping handling, denoted by nms and olp, respectively, are used and compared to determine the proper postprocessing methods. Experimental results show that the proposed pedestrian detection, fusing HOG and LBP features through the FF1 method with the HIKSVM classifier and window overlap postprocessing, is superior (Figure 8).



(a) (b)

FIGURE 8: Comparison of precision and recalls achieved by using single features (HOG and LBP) and fusing HOG and LBP through various postprocessing: (a) XMU-VIS; (b) XMU_NIR.

3.4. Robustness Evaluation to Partial Occlusion. An experiment was conducted to show that the proposed method by concatenating HOG and LBP features through the FF1 method with an HIKSVM classifier is typically robust to partial occlusion (Figures 9 and 10). The experiment was designed to randomly add one to five random blocks of size 16 × 16 to the 1126 test-cropped images of pedestrians in the INRIA dataset (Figure 11). Figure 11 shows that when random blocks are added to the test-cropped images, the number of missed pedestrians increases, regardless of the features and SVMs used. The number of missed pedestrians increases when more random blocks are added. Figure 12 shows that the number of missing pedestrians for HOG and LBP is lower than that when using a single feature, regardless of the SVM that is used. In this experiment, a test sample is considered to include a pedestrian when the SVM output score exceeds 0.5.

4. Conclusion

This work systematically compares integral channel features, fusion methods, and kernels of SVM. The experimental results show that fusing HOG and LBP features through concatenation with the HIKSVM classifier yields the best performance, even for cross-datasets. The comparison is conducted using the INRIA person dataset for training and two newly collected Xiamen databases, XMU-VIS and XMU-NIR, combined with INRIA for testing. The results are as follows. First, directly concatenating various features as the final feature for classification is better than the weighted fusion of individual classifier results. Second, combining HOG and LBP features outperforms using a single feature, regardless of whether HIKSVM or linear SVM is used. As to kernel mapping, there are also some non-linear kernels [22], such as RBF and Chi2 kernel, which have reported obtaining better performance than HIK. But non-linear kernels are time-consuming in testing state; so, in this paper, we only discuss the linear kernels for pedestrian detection. Third, HIKSVM consistently outperforms linear SVM, even when noise blocks are added that cause the occlusion problem. Fourth, for the postprocessing method, window-overlap-based postprocessing outperforms the mean-shift-based postprocessing. Finally, the proposed method is effective to detect pedestrian locations, even for cross-datasets collected in Xiamen University and captured by low-resolution visual sensors or near-infrared sensors. However, the method proposed in this work has certain limitations. Therefore, future works should extend

Figure 9: Detection results on XMU-VIS. From left to right and top to down, the classifiers are HOG_lin, LBP_lin, HOG_LBP_lin, HOG_HIK, LBP_HIK, and HOG_LBP_HIK.

Figure 10: Detection results on XMU-NIR. From left to right and top to down, the classifiers are HOG_lin, LBP_lin, HOG_LBP_lin, HOG_HIK, LBP_HIK, and HOG_LBP_HIK.

Figure 11: Examples of adding random blocks of size 16 × 16 to test-cropped images in the INRIA dataset.

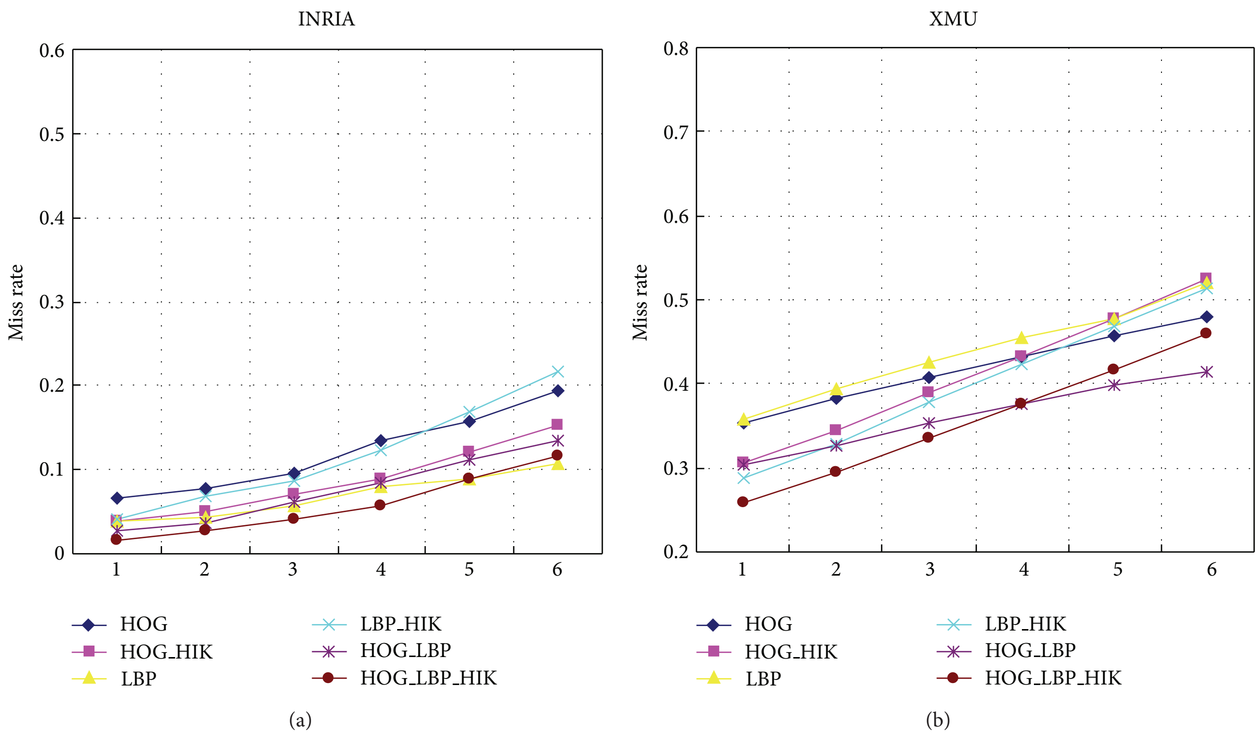


Figure 12: Comparison of missing rates using various combinations of features (single features or feature fusion) when random blocks were added to test-cropped images.

the proposed method to construct a practical pedestrian detection system for videos that integrates additional motion features and scene geometry information.

Acknowledgments

The authors wish to thank the Intel Open Source Computer Vision Library (OpenCV), fiksvm [16], and libSVM [23], for making their implementation convenient. This work was supported by the National Natural Science Foundation of China (no. 61202143) and the National Science Council of Taiwan (no. NSC 101-2221-E-155-060).

References

[1] D. Gerónimo, A. M. López, A. D. Sappa, and T. Graf, "Survey of pedestrian detection for advanced driver assistance systems," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 32, no. 7, pp. 1239–1258, 2010.

[2] P. Dollar, C. Wojek, B. Schiele, and P. Perona, "Pedestrian detection: an evaluation of the state of the art," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 34, no. 4, pp. 743–761, 2012.

[3] P. Viola and M. J. Jones, "Robust real-time face detection," *International Journal of Computer Vision*, vol. 57, no. 2, pp. 137–154, 2004.

[4] N. Dalal and B. Triggs, "Histograms of oriented gradients for human detection," in *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR '05)*, pp. 886–893, San Diego, Calif, USA, June 2005.

[5] A. Mohan, C. Papageorgiou, and T. Poggio, "Example-based object detection in images by components," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 23, no. 4, pp. 349–361, 2001.

[6] B. Wu and R. Nevatia, "Detection and tracking of multiple, partially occluded humans by Bayesian combination of edgelet based part detectors," *International Journal of Computer Vision*, vol. 75, no. 2, pp. 247–266, 2007.

[7] B. Leibe, A. Leonardis, and B. Schiele, "Robust object detection with interleaved categorization and segmentation," *International Journal of Computer Vision*, vol. 77, no. 1–3, pp. 259–289, 2008.

[8] L. Zhao and C. E. Thorpe, "Stereo- and neural network-based pedestrian detection," *IEEE Transactions on Intelligent Transportation Systems*, vol. 1, no. 3, pp. 148–154, 2000.

[9] D. G. Lowe, "Distinctive image features from scale-invariant keypoints," *International Journal of Computer Vision*, vol. 60, no. 2, pp. 91–110, 2004.

[10] P. F. Felzenszwalb, R. B. Girshick, D. McAllester, and D. Ramanan, "Object detection with discriminatively trained part-based models," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 32, no. 9, pp. 1627–1645, 2010.

[11] Q. Zhu, M. C. Yeh, K. T. Cheng, and S. Avidan, "Fast human detection using a cascade of histograms of oriented gradients," in *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR '06)*, vol. 2, pp. 1491–1498, June 2006.

[12] O. Tuzel, F. Porikli, and P. Meer, "Pedestrian detection via classification on Riemannian manifolds," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 30, no. 10, pp. 1713–1727, 2008.

[13] C. Wojek, S. Walk, and B. Schiele, "Multi-Cue onboard pedestrian detection," in *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition Workshops (CVPR '09)*, pp. 794–801, June 2009.

[14] Y. T. Chen and C. S. Chen, "Fast human detection using a novel boosted cascading structure with meta stages," *IEEE Transactions on Image Processing*, vol. 17, no. 8, pp. 1452–1464, 2008.

[15] X. Wang, X. Han, and S. Yan, "A HOG-LBP human detector with partial occlusion handling," in *Proceedings of the IEEE 12th International Conference on Computer Vision (ICCV '09)*, pp. 32–39, Kyoto, Japan, October 2009.

[16] S. Maji, A. C. Berg, and J. Maliks, "Classification using intersection kernel support vector machines is efficient," in *Proceedings of the 26th IEEE Conference on Computer Vision and Pattern Recognition (CVPR '08)*, pp. 1–8, Anchorage, Alaska, USA, June 2008.

[17] J. X. Wu, "Efficient HIK SVM learning for image classification," *IEEE Transactions on Image Processing*, vol. 21, no. 10, pp. 4442–4453, 2012.

[18] D. Vazquez, A. M. Lopez, and D. Ponsa, "Unsupervised domain adaptation of virtual and real worlds for pedestrian detection," in *Proceedings of the 21st International Conference on Pattern Recognition (ICPR '12)*, pp. 3492–3495, Tsukuba, Japan, 2012.

[19] V. Jain and E. Learned-Miller, "Online domain adaptation of a pre-trained cascade of classifiers," in *Proceedings of the Conference on Computer Vision and Pattern Recognition (CVPR '11)*, 2011.

[20] T. Ahonen, A. Hadid, and M. Pietikäinen, "Face description with local binary patterns: application to face recognition," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 28, no. 12, pp. 2037–2041, 2006.

[21] M. J. Swain and D. H. Ballard, "Color indexing," *International Journal of Computer Vision*, vol. 7, no. 1, pp. 11–32, 1991.

[22] A. Vedaldi and A. Zisserman, "Efficient additive kernels via explicit feature maps," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 34, no. 3, pp. 480–492, 2012.

[23] C. C. Chang and C. J. Lin, "LIBSVM: a library for support vector machines," 2001, http://www.csie.ntu.edu.tw/~cjlin/libsvm.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 216913, 13 pages
http://dx.doi.org/10.1155/2013/216913

Research Article

Traveling Wave Solutions in a Reaction-Diffusion Epidemic Model

Sheng Wang,[1] Wenbin Liu,[2] Zhengguang Guo,[1] and Weiming Wang[1]

[1] *College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China*
[2] *College of Physics and Electronic Information Engineering, Wenzhou University, Wenzhou 325035, China*

Correspondence should be addressed to Weiming Wang; weimingwang2003@163.com

Received 4 February 2013; Revised 17 March 2013; Accepted 17 March 2013

Academic Editor: Anke Meyer-Baese

We investigate the traveling wave solutions in a reaction-diffusion epidemic model. The existence of the wave solutions is derived through monotone iteration of a pair of classical upper and lower solutions. The traveling wave solutions are shown to be unique and strictly monotonic. Furthermore, we determine the critical minimal wave speed.

1. Introduction

Recently, great attention has been paid to the study of the traveling wave solutions in reaction-diffusion models [1–17]. In the sense of epidemiology, the traveling wave solutions describe the transition from a disease-free equilibrium to an endemic equilibrium; the existence and nonexistence of nontrivial traveling wave solutions indicate whether or not the disease can spread [11]. The results contribute to predicting the developing tendency of infectious diseases, to determining the key factors of the spread of infectious disease, and to seeking the optimum strategies of preventing and controlling the spread of the infectious diseases [18–21].

Some methods have been used to derive the existence of traveling wave solutions in reaction-diffusion models, and the monotone iteration method has been proved to be an effective one. Such a method reduces the existence of traveling wave solutions to that of an ordered pair of upper-lower solutions [6, 7, 9, 10, 14, 15].

In [22], Berezovsky and coworkers introduced a simple epidemic model through the incorporation of variable population, disease-induced mortality, and emigration into the classical model of Kermack and McKendrick [23]. The total population (N) is divided into two groups of susceptible (S) and infectious (I); that is to say, $N = S + I$. The model describing the relations between the state variables is

$$\begin{aligned} \frac{dS}{dt} &= rN\left(1-\frac{N}{K}\right)-\beta\frac{SI}{N}-(\mu+m)S, \\ \frac{dI}{dt} &= \beta\frac{SI}{N}-(\mu+d)I, \end{aligned} \tag{1}$$

where the reproduction of susceptible follows a logistic equation with the intrinsic growth rate r and the carrying capacity K, β denotes the contact transmission rate (the infection rate constant), μ is the natural mortality; d denotes the disease-induced mortality, and m is the per-capita emigration rate of uninfected.

For model (1), the epidemic threshold, the so-called basic reproduction number R_0, is then computed as $R_0 = \beta/(\mu + d)$. The disease will successfully invade when $R_0 > 1$ but will die out if $R_0 < 1$. $R_0 = 1$ is usually a threshold whether the disease goes to extinction or goes to an endemic. Large values of R_0 may indicate the possibility of a major epidemic [19]. In addition, the basic demographic reproductive number R_d is given by $R_d = r/(\mu + m)$. It can be shown that if $R_d > 1$ the population grows, while $R_d \le 1$ implies that the population does not survive [22].

For simplicity, rescaling model (1) by letting $S \to S/K$, $I \to I/K$, and $t \to t/(\mu + d)$ leads to the following model:

$$\begin{aligned} \frac{dS}{dt} &= \nu R_d (S+I)[1-(S+I)] - R_0 \frac{SI}{S+I} - \nu S, \\ \frac{dI}{dt} &= R_0 \frac{SI}{S+I} - I, \end{aligned} \tag{2}$$

where $\nu = (\mu+m)/(\mu+d)$ is defined by the ratio of the average life span of susceptibles to that of infectious.

For details, we refer the reader to [20, 22].

In this paper, we are interested in the existence of traveling wave solutions in the following reaction-diffusion epidemic model [20]:

$$\begin{aligned} \frac{\partial S}{\partial t} &= \nu R_d (S+I)[1-(S+I)] - R_0 \frac{SI}{S+I} - \nu S + d\frac{\partial^2 S}{\partial x^2}, \\ \frac{\partial I}{\partial t} &= R_0 \frac{SI}{S+I} - I + d\frac{\partial^2 I}{\partial x^2}, \\ S(x,0) &= S_0(x), \qquad I(x,0) = I_0(x), \end{aligned} \tag{3}$$

where ν, R_0, R_d are all positive constants, d is the diffusion coefficient, and $(x,t) \in R \times R^+$.

We are looking for the traveling wave solutions of model (3) with the following form:

$$S(x,t) = S(\xi), \qquad I(x,t) = I(\xi), \qquad \xi = x + ct, \tag{4}$$

satisfying the following boundary value conditions:

$$(S(-\infty), I(-\infty))^T = E_1, \qquad (S(+\infty), I(+\infty))^T = E_2, \tag{5}$$

where E_1, E_2 are the equilibrium points of model (3).

This paper is arranged as follows. In Section 2, we construct a pair of ordered upper-lower solutions of model (3) and establish the uniqueness and strict monotonicity of the traveling wave solutions.

2. Existence and Asymptotic Decay Rates

In this section, we will establish the existence of traveling wave solutions of model (3) by constructing a pair of ordered upper-lower solutions. The definition of the upper solution and the lower solution is standard. We assume that the inequality between two vectors throughout this paper is componentwise.

Setting

$$\widehat{S} = \frac{R_d - 1}{R_d} - S, \qquad \widehat{I} = I, \tag{6}$$

then model (3) can be written as

$$\begin{aligned} \frac{\partial \widehat{S}}{\partial t} &= -\nu R_d \left(\frac{R_d-1}{R_d} - \widehat{S} + \widehat{I}\right)\left[1 - \left(\frac{R_d-1}{R_d} - \widehat{S} + \widehat{I}\right)\right] \\ &\quad + R_0 \frac{\left((R_d-1)/R_d - \widehat{S}\right)\widehat{I}}{(R_d-1)/R_d - \widehat{S} + \widehat{I}} + \nu\left(\frac{R_d-1}{R_d} - \widehat{S}\right) + d\frac{\partial^2 \widehat{S}}{\partial x^2}, \\ \frac{\partial \widehat{I}}{\partial t} &= R_0 \frac{\left((R_d-1)/R_d - \widehat{S}\right)\widehat{I}}{(R_d-1)/R_d - \widehat{S} + \widehat{I}} - \widehat{I} + d\frac{\partial^2 \widehat{I}}{\partial x^2}, \\ \left(\widehat{S}, \widehat{I}\right)^T(-\infty) &= (0,0)^T, \qquad \left(\widehat{S}, \widehat{I}\right)^T(+\infty) = \left(\widehat{S}^*, \widehat{I}^*\right)^T. \end{aligned} \tag{7}$$

For model (3), the equilibria are $E_1 = ((R_d-1)/R_d, 0)$ and $E_2 = (S^*, I^*)$, where

$$S^* = \frac{\nu R_0 R_d - R_0 - \nu + 1}{\nu R_0^2 R_d}, \qquad I^* = (R_0 - 1) S^*, \tag{8}$$

and for model (7), the equilibria are $\widehat{E_1} = (0,0)$ and $\widehat{E_2} = (\widehat{S}^*, \widehat{I}^*)$, where

$$\begin{aligned} \widehat{S}^* &= \frac{(R_0-1)(\nu R_0 R_d - \nu R_0 - \nu + 1)}{\nu R_0^2 R_d}, \\ \widehat{I}^* &= \frac{(R_0-1)(\nu R_0 R_d - R_0 - \nu + 1)}{\nu R_0^2 R_d}. \end{aligned} \tag{9}$$

Obviously,

$$\begin{aligned} \widehat{I}^* - \widehat{S}^* &= \frac{(\nu-1)(R_0-1)}{\nu R_0 R_d}, \\ \widehat{S}^* &= \frac{R_d-1}{R_d} - S^*, \quad \widehat{I}^* = I^*. \end{aligned} \tag{10}$$

For simplicity, we define the following functions and constants:

$$\begin{aligned} \alpha_0 &= \frac{R_d-1}{R_d}, \qquad \phi(I) = \alpha_0 \widehat{I}^* + \left(\widehat{I}^* - \widehat{S}^*\right) I; \\ \beta_0 &= \alpha_0 \left(\widehat{I}^*\right)^2 \frac{R_0-1}{R_0}(R_0 - \nu + 1); \\ \gamma_0 &= 2\nu (R_d - 1)\left(\widehat{I}^* - \widehat{S}^*\right) - \left[\nu \widehat{I}^* + (R_0-1)\widehat{S}^*\right]; \\ \psi(I) &= \nu R_d \left(\widehat{I}^* - \widehat{S}^*\right)^2 I^2 + \gamma_0 \widehat{I}^* I + \beta_0; \\ \eta_0 &= -\frac{\gamma_0 \widehat{I}^*}{2\nu R_d \left(\widehat{I}^* - \widehat{S}^*\right)^2}; \\ \varphi(I) &= 1, \quad I > 0; \qquad \varphi(I) = -1, \quad I \le 0; \\ B &= \frac{\widehat{I}^*}{2}\left[1 + \varphi\left(\frac{\widehat{I}^*}{2} - \eta_0\right)\right]. \end{aligned} \tag{11}$$

And we will always assume the following hypotheses throughout the rest of this paper:

$[H1]$

$$R_0 > 1, \quad 1 < R_d < \frac{2R_0^2 + 2R_0 - 2}{3R_0^2 - 2R_0},$$
$$\max\left\{\frac{27R_0(R_d-1)^2}{R_d^3}, \frac{R_0-1}{R_0R_d-1}\right\} < \nu < \frac{-1}{R_0R_d - R_0 - 1}. \tag{12}$$

$[H2]$

$$\nu \geq \max\left\{\frac{R_0}{2-R_d}, \frac{R_0^3 - 2R_0^2 + 4R_0 - 2}{2R_0^2 + 2R_0 - 2 - (3R_0^2 - 2R_0)R_d}\right\},$$
$$\psi(B) \leq 0. \tag{13}$$

Then we can obtain the following.

Lemma 1. *If [H1] holds, then E_2 and $\widehat{E_2}$ are endemic points of model* (3) *and model* (7)*, respectively.*

Lemma 2. *For model* (7)*, if [H1] holds, then $\widehat{E_1}$ is unstable, and $\widehat{E_2}$ is stable.*

For the sake of convenience, let $x = \sqrt{d}\,\tilde{x}$. For simplicity, we still use the variables S, I, and x instead of $\widehat{S}$, $\widehat{I}$, and $\tilde{x}$, respectively, then model (7) could be rewritten as

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\nu R_d(\alpha_0 - S + I)[1 - (\alpha_0 - S + I)] \\ &\quad + R_0\frac{(\alpha_0 - S)I}{\alpha_0 - S + I} + \nu(\alpha_0 - S) + \frac{\partial^2 S}{\partial x^2}, \\ \frac{\partial I}{\partial t} &= R_0\frac{(\alpha_0 - S)I}{\alpha_0 - S + I} - I + \frac{\partial^2 I}{\partial x^2},\end{aligned} \tag{14}$$
$$(S,I)^T(-\infty) = (0,0)^T, \qquad (S,I)^T(+\infty) = \left(\widehat{S}^*, \widehat{I}^*\right)^T.$$

Following the definition of quasi-monotonicity [17], we can obtain the following results.

Lemma 3. *Model* (14) *is a quasi-monotone decreasing system in $(S,I) \in [\widehat{E_1}, \widehat{E_2}]$.*

Proof. Let

$$\begin{aligned}F_1(S,I) &= -\nu R_d(\alpha_0 - S + I)[1 - (\alpha_0 - S + I)] \\ &\quad + R_0\frac{(\alpha_0 - S)I}{\alpha_0 - S + I} + \nu(\alpha_0 - S), \\ F_2(S,I) &= R_0\frac{(\alpha_0 - S)I}{\alpha_0 - S + I} - I.\end{aligned} \tag{15}$$

From [17], we can know that the functions $F_1(S,I)$ and $F_2(S,I)$ are said to possess a quasi-monotone nonincreasing system, if the sign of $\partial F_1(S,I)/\partial I$ and $\partial F_2(S,I)/\partial S$ are both nonpositive.

Since

$$\begin{aligned}\frac{\partial F_2(S,I)}{\partial S} &= -R_0\left(\frac{I}{\alpha_0 - S + I}\right)^2 \leq 0, \\ \frac{\partial F_1(S,I)}{\partial I} &= R_0\left(\frac{\alpha_0 - S}{\alpha_0 - S + I}\right)^2 + 2\nu R_d(\alpha_0 - S + I) - \nu R_d, \\ \frac{\partial}{\partial S}\left(\frac{\partial F_1(S,I)}{\partial I}\right) &= -2\nu R_d - 2R_0\frac{(\alpha_0 - S)I}{(\alpha_0 - S + I)^3} \leq -2\nu R_d < 0.\end{aligned} \tag{16}$$

Then,

$$\frac{\partial F_1(S,I)}{\partial I} \leq R_0\left(\frac{\alpha_0}{\alpha_0 + I}\right)^2 + 2\nu R_d(\alpha_0 + I) - \nu R_d. \tag{17}$$

Let

$$G(z) = \frac{\alpha_0^2 R_0}{z^2} + 2\nu R_d z - \nu R_d, \quad z \in \left[\alpha_0, \alpha_0 + \widehat{I^*}\right], \tag{18}$$

then

$$G'(z) = 2\nu R_d - \frac{2\alpha_0^2 R_0}{z^3} = 0, \tag{19}$$

obviously, $z^* = \sqrt[3]{\alpha_0^2 R_0/\nu R_d}$ is the unique real root of $G'(z)$.

Since $\nu > 27R_0(R_d - 1)^2/R_d^3$, consider $\alpha_0 = (R_d - 1)/R_d$, then we can get

$$G(z^*) = \frac{(\alpha_0^2 R_0)^{2/3}\left[(27\alpha_0^2 R_0)^{1/3} - (\nu R_d)^{1/3}\right]}{(z^*)^2} < 0. \tag{20}$$

And

$$\lim_{z \to 0^+} G(z) = \lim_{z \to +\infty} G(z) = +\infty; \tag{21}$$

hence, $G(z)$ has two positive roots.

Since $\nu \geq R_0/(2 - R_d)$, thus $G(\alpha_0) = R_0 + \nu R_d - 2\nu \leq 0$.

According to conditions $[H1]$ and $[H2]$, we can get

$$\begin{aligned}G\left(\alpha_0 + \widehat{I^*}\right) &= \nu R_d - 2\nu + 2\nu R_d\widehat{I^*} + R_0\left(\frac{\alpha_0}{\alpha_0 + \widehat{I^*}}\right)^2 \\ &< \nu R_d - 2\nu + 2\nu R_d\widehat{I^*} + R_0 \\ &= \frac{\left(3R_0^2 R_d - 2R_0^2 - 2R_0R_d - 2R_0 + 2\right)\nu}{R_0^2} \\ &\quad + \frac{\left(R_0^3 - 2R_0^2 + 4R_0 - 2\right)}{R_0^2} \\ &\leq 0.\end{aligned} \tag{22}$$

Then, $G([\alpha_0, \alpha_0 + \widehat{I^*}]) \leq 0$. Hence, $\partial F_1(S,I)/\partial I \leq 0$.

That is to say, model (14) is a quasi-monotone system in $(S,I) \in [\widehat{E_1}, \widehat{E_2}]$. □

Since the traveling wave solution of model (14) has the following form

$$S(\xi) = S(x+ct), \quad I(\xi) = I(x+ct), \quad \xi = x+ct, \quad c>0; \tag{23}$$

substituting (23) into model (14), we can get the following model:

$$\begin{gathered} S'' - cS' - \nu R_d(\alpha_0 - S + I)[1-(\alpha_0 - S + I)] \\ + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} + \nu(\alpha_0 - S) = 0, \\ I'' - cI' + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} - I = 0, \end{gathered} \tag{24}$$

$$(S,I)^T(-\infty) = (0,0)^T, \qquad (S,I)^T(+\infty) = \left(\widehat{S}^*, \widehat{I}^*\right)^T.$$

Obviously, we can know the following.

Remark 4. Model (24) is also a quasi-monotone system in $(S,I) \in [\widehat{E_1}, \widehat{E_2}]$.

Now we establish the existence of traveling wave solutions of model (24) through monotone iteration of a pair of smooth upper and lower solutions. Following [17], we give the definitions of the upper and lower solutions of model (24) as follows, respectively.

Definition 5. A smooth function $(\overline{S}(\xi), \overline{I}(\xi))^T$ $(\xi \in \mathbb{R})$ is an upper solution of model (24) if its derivatives $(\overline{S}', \overline{I}')^T$ and $(\overline{S}'', \overline{I}'')$ are continuous on $\mathbb{R}$, and $(\overline{S}, \overline{I})^T$ satisfies

$$\begin{gathered} S'' - cS' - \nu R_d(\alpha_0 - S + I)[1-(\alpha_0 - S + I)] \\ + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} + \nu(\alpha_0 - S) \le 0, \\ I'' - cI' + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} - I \le 0, \end{gathered} \tag{25}$$

with the following boundary value conditions

$$\begin{pmatrix} S \\ I \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} S \\ I \end{pmatrix}(+\infty) \ge \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix}. \tag{26}$$

Definition 6. A smooth function $(\underline{S}(\xi), \underline{I}(\xi))^T$ $(\xi \in \mathbb{R})$ is a lower solution of model (24) if its derivatives $(\underline{S}', \underline{I}')^T$ and $(\underline{S}'', \underline{I}'')$ are continuous on $\mathbb{R}$, and $(\underline{S}, \underline{I})^T$ satisfies

$$\begin{gathered} S'' - cS' - \nu R_d(\alpha_0 - S + I)[1-(\alpha_0 - S + I)] \\ + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} + \nu(\alpha_0 - S) \ge 0, \\ I'' - cI' + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} - I \ge 0, \end{gathered} \tag{27}$$

with the following boundary value conditions

$$\begin{pmatrix} S \\ I \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} S \\ I \end{pmatrix}(+\infty) \le \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix}. \tag{28}$$

The construction of the smooth upper-lower solution pair is based on the solution of the following KPP equation:

$$\begin{gathered} w'' - cw' + f(w) = 0, \\ w(-\infty) = 0, \qquad w(+\infty) = b, \end{gathered} \tag{29}$$

where $f \in C^2([0,b])$ and $f > 0$ in the open interval $(0,b)$ with $f(0) = f(b) = 0$, $f'(0) = a_1 > 0$, and $f'(b) = -b_1 < 0$ [15]. First, let us recall the following result.

Lemma 7 (see [1, 15]). *Corresponding to every $c \ge 2\sqrt{a_1}$, model (29) has a unique (up to a translation of the origin) monotonically increasing traveling wave solution $w(\xi)$ for $\xi \in R$. The traveling wave solution w has the following asymptotic behaviors.*

(i) *For the wave solution with noncritical speed $c > 2\sqrt{a_1}$, one has*

$$\begin{aligned} w(\xi) &= a_\omega e^{((c-\sqrt{c^2-4a_1})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2-4a_1})/2)\xi}\right) \quad \text{as } \xi \longrightarrow -\infty, \\ w(\xi) &= b - b_\omega e^{((c-\sqrt{c^2+4b_1})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2+4b_1})/2)\xi}\right) \quad \text{as } \xi \longrightarrow +\infty, \end{aligned} \tag{30}$$

where a_ω and b_ω are positive constants.

(ii) *For the wave with critical speed $c = 2\sqrt{a_1}$, one has*

$$\begin{aligned} w(\xi) &= (a_c + d_c\xi)e^{\sqrt{a_1}\xi} + o\left(\xi e^{\sqrt{a_1}\xi}\right) \quad \text{as } \xi \longrightarrow -\infty, \\ w(\xi) &= b - b_c e^{(\sqrt{a_1} - \sqrt{a_1+b_1})\xi} \\ &\quad + o\left(e^{(\sqrt{a_1} - \sqrt{a_1+b_1})\xi}\right) \quad \text{as } \xi \longrightarrow +\infty, \end{aligned} \tag{31}$$

where the constant d_c is negative, b_c is positive, and $a_c \in R$.

For constructing the upper solution of the model (24), we start with the following model:

$$\begin{gathered} I'' - cI' + I\left(\widehat{I}^* - I\right)\frac{\alpha_0(R_0 - 1)}{\alpha_0 \widehat{I}^* + \left(\widehat{I}^* - \widehat{S}^*\right) I} = 0, \\ I(-\infty) = 0, \qquad I(+\infty) = \widehat{I}^*. \end{gathered} \tag{32}$$

Define $f(I) = I(\widehat{I}^* - I)(\alpha_0(R_0 - 1)/\phi(I))$, $I \in [0, \widehat{I}^*]$, one can verify that all of the following conditions are satisfied:

(i) $f(I) = I(\widehat{I}^* - I)(\alpha_0(R_0 - 1)/\phi(I)) \in C^2([0, \widehat{I}^*])$;

(ii) $f(I) > 0$, for all $I \in (0, \widehat{I}^*)$ and $f(0) = f(\widehat{I}^*) = 0$;

(iii) $f'(0) = R_0 - 1 > 0$, $f'(\widehat{I}^*) = -\alpha_0(R_0-1)^2/R_0\widehat{I}^* < 0$.

From Lemma 7, we know that, for each $c \ge 2\sqrt{R_0 - 1}$, equation (32) has a unique traveling wave solution $\overline{I}(\xi)$ (up to a translation of the origin), satisfying the given boundary value conditions (26).

Define

$$\begin{pmatrix} \overline{S}(\xi) \\ \overline{I}(\xi) \end{pmatrix} = \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}\overline{I}(\xi) \\ \overline{I}(\xi) \end{pmatrix}, \quad \xi \in R, \tag{33}$$

then we can get the following result.

Lemma 8. *For each $c \ge 2\sqrt{R_0 - 1}$, (33) is a smooth upper solution of model (24).*

Proof. On the boundary,

$$\begin{pmatrix} \overline{S} \\ \overline{I} \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \overline{S} \\ \overline{I} \end{pmatrix}(+\infty) \ge \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix}. \tag{34}$$

As for the I component, we have

$$\begin{aligned}
&\overline{I}'' - c\overline{I}' + R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} - \overline{I} \\
&= -\overline{I}(\widehat{I}^* - \overline{I})\frac{\alpha_0(R_0-1)}{\phi(\overline{I})} + R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} - \overline{I} \\
&= -\overline{I}(\widehat{I}^* - \overline{I})\frac{\alpha_0(R_0-1)}{\phi(\overline{I})} + R_0\frac{\alpha_0\widehat{I}^* - \widehat{S}^*\overline{I}}{\phi(\overline{I})}\overline{I} - \overline{I} \\
&= -\overline{I}(\widehat{I}^* - \overline{I})\frac{\alpha_0(R_0-1)}{\phi(\overline{I})} \\
&\quad + \overline{I}\frac{\alpha_0(R_0-1)\widehat{I}^* - (R_0\widehat{S}^* - \widehat{S}^* + \widehat{I}^*)\overline{I}}{\phi(\overline{I})} \\
&= -\overline{I}(\widehat{I}^* - \overline{I})\frac{\alpha_0(R_0-1)}{\phi(\overline{I})} \\
&\quad + \overline{I}\frac{\alpha_0(R_0-1)\widehat{I}^* - \alpha_0(R_0-1)\overline{I}}{\phi(\overline{I})} \\
&= 0.
\end{aligned} \tag{35}$$

As for the S component, since $\nu > 1$, then $\widehat{I}^* - \widehat{S}^* = (\nu - 1)(R_0 - 1)/\nu R_0 R_d > 0$. And

(i) if $\eta_0 < \widehat{I}^*/2$, then $\max_{\xi\in[0,\widehat{I}^*]}\psi(\overline{I}) = \psi(\widehat{I}^*) = \psi(B)$;

(ii) if $\eta_0 \ge \widehat{I}^*/2$, then $\max_{\xi\in[0,\widehat{I}^*]}\psi(\overline{I}) = \psi(0) = \psi(B)$.

Thus we can get:

$$\begin{aligned}
&\overline{S}'' - c\overline{S}' - \nu R_d(\alpha_0 - \overline{S} + \overline{I})[1 - (\alpha_0 - \overline{S} + \overline{I})] \\
&\quad + R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} + \nu(\alpha_0 - \overline{S}) \\
&= \frac{\widehat{S}^*}{\widehat{I}^*}(\overline{I}'' - c\overline{I}') - \nu R_d(\alpha_0 - \overline{S} + \overline{I})[1 - (\alpha_0 - \overline{S} + \overline{I})] \\
&\quad + R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} + \nu(\alpha_0 - \overline{S}) \\
&= \frac{\widehat{S}^*}{\widehat{I}^*}\left(-R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} + \overline{I}\right) - \nu R_d(\alpha_0 - \overline{S} + \overline{I}) \\
&\quad \times [1 - (\alpha_0 - \overline{S} + \overline{I})] + R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} + \nu(\alpha_0 - \overline{S}) \\
&= \left(1 - \frac{\widehat{S}^*}{\widehat{I}^*}\right)R_0\frac{(\alpha_0 - \overline{S})\overline{I}}{\alpha_0 - \overline{S} + \overline{I}} + \frac{\widehat{S}^*}{\widehat{I}^*}\overline{I} - \nu R_d(\alpha_0 - \overline{S} + \overline{I}) \\
&\quad \times [1 - (\alpha_0 - \overline{S} + \overline{I})] + \nu(\alpha_0 - \overline{S}) \\
&= \frac{\widehat{I}^* - \widehat{S}^*}{\widehat{I}^*}R_0\frac{\alpha_0\widehat{I}^* - \widehat{S}^*\overline{I}}{\phi(\overline{I})}\overline{I} + \frac{\widehat{S}^*}{\widehat{I}^*}\overline{I} - \nu R_d\frac{\phi(\overline{I})}{\widehat{I}^*}\left[1 - \frac{\phi(\overline{I})}{\widehat{I}^*}\right] \\
&\quad + \nu\frac{\alpha_0\widehat{I}^* - \widehat{S}^*\overline{I}}{\widehat{I}^*} \\
&= \frac{(\widehat{I}^* - \widehat{S}^*)\overline{I}\psi(\overline{I})}{(\widehat{I}^*)^2\phi(\overline{I})} \le 0.
\end{aligned} \tag{36}$$

Hence, $(\overline{S}, \overline{I})$ forms a smooth upper solution for model (24). ☐

For constructing the lower solution of the model (24), we start with the following model:

$$\begin{gathered}
I'' - cI' + I[\widehat{I}^* - (1+\varepsilon)I]\frac{\alpha_0(R_0-1)}{\alpha_0\widehat{I}^* + (\widehat{I}^* - \widehat{S}^*)I} = 0, \\
I(-\infty) = 0, \qquad I(+\infty) = \frac{\widehat{I}^*}{1+\varepsilon}.
\end{gathered} \tag{37}$$

Define $g(I) = I[\widehat{I}^* - (1+\varepsilon)I](\alpha_0(R_0-1)/(\alpha_0\widehat{I}^* + (\widehat{I}^* - \widehat{S}^*)I))$, $I \in [0, \widehat{I}^*/(1+\varepsilon)]$. One can easily verify that all of the following conditions hold:

(i) $g(I) = I[\widehat{I}^* - (1+\varepsilon)I](\alpha_0(R_0-1)/(\alpha_0\widehat{I}^* + (\widehat{I}^* - \widehat{S}^*)I)) \in C^2([0, \widehat{I}^*/(1+\varepsilon)])$;

(ii) $g(I) > 0$, for all $I \in (0, \widehat{I}^*/(1+\varepsilon))$ and $g(0) = g(\widehat{I}^*/(1+\varepsilon)) = 0$;

(iii) $g'(0) = R_0 - 1 > 0$, $g'(\widehat{I}^*/(1+\varepsilon)) = -(1+\varepsilon)\alpha_0(R_0 - 1)/(\varepsilon\alpha_0 + (R_0/(R_0-1))\widehat{I}^*) < 0$.

From Lemma 7, we know that, for each fixed $c \geq 2\sqrt{R_0 - 1}$, model (37) has a unique traveling wave solution $\underline{I}(\xi)$ (up to a translation of the origin), satisfying the given boundary value conditions (28).

Define

$$\begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix} = \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}\underline{I}(\xi) \\ \underline{I}(\xi) \end{pmatrix}, \quad \xi \in R, \tag{38}$$

then we have the following result:

Lemma 9. *For each fixed $c \geq 2\sqrt{R_0 - 1}$, (38) is a lower solution of model (24).*

Proof. On the boundary,

$$\begin{pmatrix} \underline{S} \\ \underline{I} \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \underline{S} \\ \underline{I} \end{pmatrix}(+\infty) = \begin{pmatrix} \dfrac{\widehat{S}^*}{1+\varepsilon} \\ \dfrac{\widehat{I}^*}{1+\varepsilon} \end{pmatrix} \leq \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix}. \tag{39}$$

As for the I component, we have

$$\begin{aligned} &\underline{I}'' - c\underline{I}' + R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} - \underline{I} \\ &\quad = -\underline{I}\left[\widehat{I}_* - (1+\varepsilon)\underline{I}\right]\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})} + R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} - \underline{I} \\ &\quad = -\underline{I}\left[\widehat{I}_* - (1+\varepsilon)\underline{I}\right]\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})} + \underline{I}\left(\widehat{I}^* - \underline{I}\right)\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})} \\ &\quad = \varepsilon(\underline{I})^2\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})} \geq 0. \end{aligned} \tag{40}$$

As for the S component, we have

$$\begin{aligned} &\underline{S}'' - c\underline{S}' - \nu R_d(\alpha_0 - \underline{S} + \underline{I})\left[1 - (\alpha_0 - \underline{S} + \underline{I})\right] \\ &\qquad + R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} + \nu(\alpha_0 - \underline{S}) \\ &\quad = \frac{\widehat{S}^*}{\widehat{I}^*}\left(\underline{I}'' - c\underline{I}'\right) - \nu R_d(\alpha_0 - \underline{S} + \underline{I})\left[1 - (\alpha_0 - \underline{S} + \underline{I})\right] \\ &\qquad + R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} + \nu(\alpha_0 - \underline{S}) \\ &\quad = \frac{\widehat{S}^*}{\widehat{I}^*}\left\{\left[\varepsilon(\underline{I})^2\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})}\right] - \left[R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} - \underline{I}\right]\right\} \\ &\qquad - \nu R_d(\alpha_0 - \underline{S} + \underline{I})\left[1 - (\alpha_0 - \underline{S} + \underline{I})\right] + R_0\frac{(\alpha_0 - \underline{S})\underline{I}}{\alpha_0 - \underline{S} + \underline{I}} \\ &\qquad + \nu(\alpha_0 - \underline{S}) \\ &\quad = \varepsilon\frac{\widehat{S}^*}{\widehat{I}^*}(\underline{I})^2\frac{\alpha_0(R_0 - 1)}{\phi(\underline{I})} + \frac{\left(\widehat{I}^* - \widehat{S}^*\right)\underline{I}\psi(\underline{I})}{\left(\widehat{I}^*\right)^2\phi(\underline{I})} \geq 0. \end{aligned} \tag{41}$$

Thus $(\underline{S}, \underline{I})$ forms a smooth lower solution for model (24). □

Next, we show that, by shifting the upper solution far enough to the left, then the upper-lower solution in Lemmas 8 and 9 are ordered.

Lemma 10. *Let $c \geq 2\sqrt{R_0 - 1}$, $(\overline{S}, \overline{I})^T$ and $(\underline{S}, \underline{I})^T$ be the upper solution and the lower solution defined in (33) and (38), then there exists a positive number r, such that $(\overline{S}, \overline{I})^T(\xi + r) \geq (\underline{S}, \underline{I})^T(\xi)$ for all $\xi \in R$.*

Proof. Our proof is only for $c > 2\sqrt{R_0 - 1}$, and the proof for the case of $c = 2\sqrt{R_0 - 1}$ is similar to it.

First, we derive the asymptotic behaviors of the upper solution and the lower solution at infinities.

According to Lemma 7, when $\xi \to -\infty$, we can obtain:

$$\begin{aligned} \begin{pmatrix} \overline{S} \\ \overline{I} \end{pmatrix}(\xi) &= \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}A_1 \\ A_1 \end{pmatrix} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right), \\ \begin{pmatrix} \underline{S} \\ \underline{I} \end{pmatrix}(\xi) &= \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}B_1 \\ B_1 \end{pmatrix} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right). \end{aligned} \tag{42}$$

And let $\sigma_0 = (1/2)(c - \sqrt{c^2 + 4(\alpha_0(R_0-1)^2/R_0\widehat{I}^*)} < 0$, $\delta_0 = (1/2)(c - \sqrt{c^2 + 4((1+\varepsilon)\alpha_0(R_0-1)/(\varepsilon\alpha_0 + (R_0/(R_0-1))\widehat{I}^*))} < 0$, when $\xi \to +\infty$, we can get

$$\begin{aligned} \begin{pmatrix} \overline{S} \\ \overline{I} \end{pmatrix}(\xi) &= \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}A_2 \\ A_2 \end{pmatrix} e^{\sigma_0\xi} + o\left(e^{\sigma_0\xi}\right), \\ \begin{pmatrix} \underline{S} \\ \underline{I} \end{pmatrix}(\xi) &= \frac{1}{1+\varepsilon}\begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \dfrac{\widehat{S}^*}{\widehat{I}^*}B_2 \\ B_2 \end{pmatrix} e^{\delta_0\xi} + o\left(e^{\delta_0\xi}\right), \end{aligned} \tag{43}$$

where, A_1, A_2, B_1, B_2 are all positive constants.

Since for any $\tilde{r} > 0$, $\overline{I}^{\tilde{r}}(\xi) \equiv \overline{I}(\xi + \tilde{r})$ is also a solution of model (32). Thus, $(\overline{S}^{\tilde{r}}, \overline{I}^{\tilde{r}})^{T}(\xi)$ is an upper solution of model (24). So, according to Lemma 7, when $\xi \to -\infty$, we can get:

$$\begin{pmatrix} \overline{S}^{\tilde{r}} \\ \overline{I}^{\tilde{r}} \end{pmatrix}(\xi) = \begin{pmatrix} \frac{\widehat{S}^*}{\widehat{I}^*} A_1 \\ A_1 \end{pmatrix} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\tilde{r}} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right). \tag{44}$$

Since $(c - \sqrt{c^2 - 4(R_0 - 1)})/2 > 0$, we can choose a large enough number $\tilde{r} \gg 0$, such that

$$\begin{pmatrix} \frac{\widehat{S}^*}{\widehat{I}^*} A_1 \\ A_1 \end{pmatrix} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\tilde{r}} > \begin{pmatrix} \frac{\widehat{S}^*}{\widehat{I}^*} B_1 \\ B_1 \end{pmatrix}, \tag{45}$$

hence, there exists a large number $N_1 \gg 1$, such that

$$\begin{pmatrix} \overline{S}^{\tilde{r}}(\xi) \\ \overline{I}^{\tilde{r}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix}, \quad \xi \in (-\infty, -N_1]. \tag{46}$$

By using a similar argument as above, there exists a large enough number $N_2 \gg 1$, such that

$$\begin{pmatrix} \overline{S}^{\tilde{r}}(\xi) \\ \overline{I}^{\tilde{r}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix}, \quad \xi \in [N_2, +\infty). \tag{47}$$

Second, we show that

$$\begin{pmatrix} \overline{S}^{\tilde{r}}(\xi) \\ \overline{I}^{\tilde{r}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix}, \quad \xi \in [-N_1, N_2]. \tag{48}$$

We deal with such two possible cases:

Case 1. If

$$\begin{pmatrix} \overline{S}^{\tilde{r}}(\xi) \\ \overline{I}^{\tilde{r}}(\xi) \end{pmatrix} > \begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix}, \quad \xi \in [-N_1, N_2], \tag{49}$$

then, the proof is completed.

Case 2. If there exists a point $\xi_0 \in (-N_1, N_2)$, such that

$$\begin{pmatrix} \overline{S}^{\tilde{r}}(\xi_0) \\ \overline{I}^{\tilde{r}}(\xi_0) \end{pmatrix} \leq \begin{pmatrix} \underline{S}(\xi_0) \\ \underline{I}(\xi_0) \end{pmatrix} \tag{50}$$

satisfying $\overline{S}^{\tilde{r}}(\xi_0) < \underline{S}(\xi_0)$ or $\overline{I}^{\tilde{r}}(\xi_0) < \underline{I}(\xi_0)$.

In this case, we use the Sliding Domain method [15].

Step 1. we shift $(\overline{S}^{\tilde{r}}, \overline{I}^{\tilde{r}})^T$ to the left by increasing the number $\tilde{r}$ until finding a new number $r_1 > \tilde{r}$ such that $(\overline{S}^{r_1}, \overline{I}^{r_1})^T > (\underline{S}, \underline{I})^T$ on the smaller interval $[-N_1, N_2 - (r_1 - \tilde{r})]$.

Step 2. we shift $(\overline{S}^{r_1}, \overline{I}^{r_1})^T$ back to the right by decreasing r_1 to a smaller number $\tilde{r} < r_2 < r_1$ such that one of the branches of the upper solution touches its counterpart of the lower solution at some point ξ_1 in the interval $(-N_1 + (r_1 - r_2), N_2 - (r_1 - \tilde{r}))$. On the endpoints of the interval $(-N_1 + (r_1 - r_2), N_2 - (r_1 - \tilde{r}))$, we still have $(\overline{S}^{r_2}, \overline{I}^{r_2})^T > (\underline{S}, \underline{I})^T$.

Let $\vec{W}(\xi) = (\overline{S}^{r_2}, \overline{I}^{r_2})^T - (\underline{S}, \underline{I})^T$ and $\vec{F} = (F_1, F_2)^T$, where

$$\begin{aligned} F_1 &= -\nu R_d (\alpha_0 - S + I)\left[1 - (\alpha_0 - S + I)\right] \\ &\quad + R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} + \nu (\alpha_0 - S), \\ F_2 &= R_0 \frac{(\alpha_0 - S) I}{\alpha_0 - S + I} - I. \end{aligned} \tag{51}$$

For $\xi \in (-N_1 + (r_1 - r_2), N_2 - (r_1 - \tilde{r}))$, we get that

$$\vec{W}'' - c\vec{W}' + \begin{pmatrix} \frac{\partial F_1}{\partial S}\left(\underline{S} + \zeta_1 \omega_1, \overline{I}^{r_2}\right) & \frac{\partial F_1}{\partial I}\left(\overline{S}^{r_2}, \underline{I} + \zeta_2 \omega_2\right) \\ \frac{\partial F_2}{\partial S}\left(\underline{S} + \zeta_3 \omega_1, \overline{I}^{r_2}\right) & \frac{\partial F_2}{\partial I}\left(\overline{S}^{r_2}, \underline{I} + \zeta_4 \omega_2\right) \end{pmatrix} \vec{W} = 0, \tag{52}$$

where $\zeta_i \in [0, 1]$, $i = 1, 2, 3, 4$. Since the above model is monotone and the cube $[(0, 0), (\widehat{S}^*, \widehat{I}^*)]$ is convex, thus we can deduce by Maximum Principle that $\vec{W} > 0$ for $\xi \in [-N_1 + (r_1 - r_2), N_2 - (r_1 - \tilde{r})]$. So ξ_1 does not exist and we can decrease r_2 further to $\tilde{r}$. It is calculated that the point ξ_0 does not exist either. The proof of this lemma is completed. □

To ease the burden of notations, we still use $(\overline{S}, \overline{I})^T$ to denote the shifted upper solution as given in Lemma 8. Let

$$\begin{aligned} D_{11} &= -\frac{R_0^2 + \nu R_0 R_d + \nu R_0 - 4R_0 - 2\nu + 3}{R_0}, \\ D_{12} &= \frac{\nu R_0 R_d - 2R_0 - 2\nu + 3}{R_0}, \\ D_{21} &= -\frac{(R_0 - 1)^2}{R_0}, \\ D_{22} &= -\frac{R_0 - 1}{R_0}, \\ \mu_1 &= \frac{-(D_{11} + D_{22}) + \sqrt{(D_{11} - D_{22})^2 + 4D_{12}D_{21}}}{2}, \\ \mu_2 &= \frac{-(D_{11} + D_{22}) - \sqrt{(D_{11} - D_{22})^2 + 4D_{12}D_{21}}}{2}. \end{aligned} \tag{53}$$

With such constructed ordered upper-lower solution pair, we can get the following.

Theorem 11. *For $c \geq 2\sqrt{R_0 - 1}$, model (24) has a unique (up to a translation of the origin) traveling wave solution. The traveling wave solution is strictly increasing and has the following asymptotic properties:*

(i) *if* $c > 2\sqrt{R_0 - 1}$, *when* $\xi \to -\infty$,

$$\begin{aligned}\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right).\end{aligned} \tag{54}$$

when $\xi \to +\infty$, *and if* $\mu_1 \neq \mu_2$, *then*

$$\begin{aligned}\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \overline{A_1} \\ \overline{A_2} \end{pmatrix} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right),\end{aligned} \tag{55}$$

while $\mu_1 = \mu_2$:

$$\begin{aligned}\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \overline{\overline{A_{11}}} + \overline{\overline{A_{12}}}\xi \\ \overline{\overline{A_{21}}} + \overline{\overline{A_{22}}}\xi \end{pmatrix} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right),\end{aligned} \tag{56}$$

where, $\mu = \min\{\mu_1, \mu_2\} > 0$, $\overline{\overline{A_{11}}}, \overline{\overline{A_{21}}} \in \mathbb{R}$, A_1, A_2, $\overline{A_1}$, $\overline{A_2}$, $\overline{\overline{A_{12}}}$ *and* $\overline{\overline{A_{22}}}$ *are all positive constants.*

(ii) *if* $c = 2\sqrt{R_0 - 1}$, *when* $\xi \to -\infty$,

$$\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} B_{11} + B_{12}\xi \\ B_{21} + B_{22}\xi \end{pmatrix} e^{\sqrt{R_0-1}\xi} + o\left(\xi e^{\sqrt{R_0-1}\xi}\right), \tag{57}$$

when $\xi \to +\infty$, *and if* $\mu_1 \neq \mu_2$, *then*

$$\begin{aligned}\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \overline{B_{11}} \\ \overline{B_{22}} \end{pmatrix} e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} \\ + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right),\end{aligned} \tag{58}$$

while $\mu_1 = \mu_2$,

$$\begin{aligned}\begin{pmatrix} S \\ I \end{pmatrix}(\xi) = \begin{pmatrix} \widehat{S}^* \\ \widehat{I}^* \end{pmatrix} - \begin{pmatrix} \overline{\overline{B_{11}}} + \overline{\overline{B_{12}}}\xi \\ \overline{\overline{B_{21}}} + \overline{\overline{B_{22}}}\xi \end{pmatrix} e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} \\ + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right),\end{aligned} \tag{59}$$

where $\mu = \min\{\mu_1, \mu_2\} > 0$, $B_{12}, B_{22} < 0$, $B_{11}, B_{21}, \overline{\overline{B_{11}}}, \overline{\overline{B_{21}}} \in \mathbb{R}$, *and* $\overline{B_{11}}$, $\overline{B_{22}}$, $\overline{\overline{B_{12}}}$, $\overline{\overline{B_{22}}}$ *are all positive constants.*

Proof. From Lemma 3 and Remark 4, we know that model (24) is a quasi-monotone nonincreasing system in $(S, I) \in [\widehat{E_1}, \widehat{E_2}]$, and by using the monotone iteration scheme given in [3, 13], we can obtain the existence of the solution $(S, I)^T$ to the first two equations in model (24) for every $c \geq 2\sqrt{R_0 - 1}$, which satisfies

$$\begin{pmatrix} \underline{S}(\xi) \\ \underline{I}(\xi) \end{pmatrix} \leq \begin{pmatrix} S(\xi) \\ I(\xi) \end{pmatrix} \leq \begin{pmatrix} \overline{S}(\xi) \\ \overline{I}(\xi) \end{pmatrix}. \tag{60}$$

According to the above inequality, we can get that, on the boundary, the solution tends to $(0,0)^T$ as $\xi \to -\infty$ and $(\widehat{S}^*, \widehat{I}^*)^T$ as $\xi \to +\infty$.

To derive the asymptotic decay rate of the traveling wave solutions as $\xi \to \pm\infty$, we just let $c > 2\sqrt{R_0 - 1}$ and

$$U(\xi) = (S(\xi), I(\xi))^T, \quad -\infty < \xi < +\infty \tag{61}$$

be the traveling wave solution of model (24) generated form the monotone iteration, since the case of (ii) $c = 2\sqrt{R_0 - 1}$ is similar to it.

We differentiate model (24) with respect to ξ, and note that $U'(\xi) = (\chi_1, \chi_2)^T(\xi)$ satisfies

$$\begin{aligned}\chi_1'' - c\chi_1' + C_{11}(S, I)\chi_1 + C_{12}(S, I)\chi_2 = 0, \\ \chi_2'' - c\chi_2' + C_{21}(S, I)\chi_1 + C_{22}(S, I)\chi_2 = 0,\end{aligned} \tag{62}$$

where

$$\begin{aligned}C_{11}(S, I) &= \nu R_d\left[1 - (\alpha_0 - S + I)\right] - \nu R_d(\alpha_0 - S + I) \\ &\quad - \frac{R_0 I}{\alpha_0 - S + I} + \frac{R_0(\alpha_0 - S)I}{(\alpha_0 - S + I)^2} - \nu, \\ C_{12}(S, I) &= -\nu R_d\left[1 - (\alpha_0 - S + I)\right] + \nu R_d(\alpha_0 - S + I) \\ &\quad + \frac{R_0(\alpha_0 - S)}{\alpha_0 - S + I} - \frac{R_0(\alpha_0 - S)I}{(\alpha_0 - S + I)^2}, \\ C_{21}(S, I) &= -\frac{R_0 I}{\alpha_0 - S + I} + \frac{R_0(\alpha_0 - S)I}{(\alpha_0 - S + I)^2}, \\ C_{22}(S, I) &= \frac{R_0(\alpha_0 - S)}{\alpha_0 - S + I} - \frac{R_0(\alpha_0 - S)I}{(\alpha_0 - S + I)^2} - 1.\end{aligned} \tag{63}$$

Now, we study the exponential decay rate of the traveling wave solution as $\xi \to -\infty$. The asymptotic model of model (62) as $\xi \to -\infty$ is

$$\begin{aligned}\lambda'' - c\lambda' + E_{11}\lambda + E_{12}\mu = 0, \\ \mu'' - c\mu' + E_{21}\lambda + E_{22}\mu = 0,\end{aligned} \tag{64}$$

where

$$\begin{aligned}E_{11} = -\nu(R_d - 1), \qquad E_{12} = \nu R_d + R_0 - 2\nu, \\ E_{21} = 0, \qquad E_{22} = R_0 - 1.\end{aligned} \tag{65}$$

The second equation of model (64) has two independent solutions with the following form:

$$\begin{aligned}\mu^{(1)}(\xi) = e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}, \\ \mu^{(2)}(\xi) = e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi}.\end{aligned} \tag{66}$$

Relating the second equation of model (62) with the second equation of model (64), we can deduce that $\chi_2(\xi)$ has the following property as $\xi \to -\infty$:

$$\begin{aligned} \chi_2(\xi) &= \alpha [1 + o(1)] e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ &\quad + \beta [1 + o(1)] e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi} \end{aligned} \tag{67}$$

for some constants α and β. Thus, we can obtain that

$$\begin{aligned} \chi_2(\xi) &= \alpha e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} + \beta e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi} \\ &\quad + \Upsilon_1(\xi) + \Upsilon_2(\xi), \end{aligned} \tag{68}$$

where

$$\begin{aligned} \lim_{\xi \to -\infty} \frac{\Upsilon_1(\xi)}{e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}} &= 0, \\ \lim_{\xi \to -\infty} \frac{\Upsilon_2(\xi)}{e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi}} &= 0. \end{aligned} \tag{69}$$

So we obtain that

$$\begin{aligned} &\lim_{\xi \to -\infty} \frac{\chi_2(\xi) - \alpha e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}}{e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}} \\ &= \lim_{\xi \to -\infty} \frac{\Upsilon_1(\xi) + \Upsilon_2(\xi) + \beta e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi}}{e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}} \\ &= \lim_{\xi \to -\infty} \frac{\Upsilon_1(\xi)}{e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}} + \beta \lim_{\xi \to -\infty} e^{\sqrt{c^2-4(R_0-1)}\xi} \\ &\quad + \lim_{\xi \to -\infty} \frac{\Upsilon_2(\xi)}{e^{((c+\sqrt{c^2-4(R_0-1)})/2)\xi}} \lim_{\xi \to -\infty} e^{\sqrt{c^2-4(R_0-1)}\xi} = 0. \end{aligned} \tag{70}$$

Thus, $\chi_2(\xi) = \alpha[1 + o(1)]e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}$.

Now, we consider the first equation of model (64). We rewrite it as

$$\lambda'' - c\lambda' - \nu(R_d - 1)\lambda = -(\nu R_d + R_0 - 2\nu)\mu. \tag{71}$$

One can verify that $(c - \sqrt{c^2 - 4(R_0 - 1)})/2$ is not a characteristic of

$$\lambda'' - c\lambda' - \nu(R_d - 1)\lambda = 0. \tag{72}$$

The above equation has two independent solutions of the following form:

$$\begin{aligned} \lambda^{(1)}(\xi) &= e^{((c-\sqrt{c^2+4\nu(R_d-1)})/2)\xi}, \\ \lambda^{(2)}(\xi) &= e^{((c+\sqrt{c^2+4\nu(R_d-1)})/2)\xi}. \end{aligned} \tag{73}$$

Thus, when $\xi \to -\infty$, $\chi_1(\xi)$ has the following property:

$$\begin{aligned} \chi_1(\xi) &= \overline{\alpha} [1 + o(1)] e^{((c+\sqrt{c^2+4\nu(R_d-1)})/2)\xi} \\ &\quad + \overline{\beta} [1 + o(1)] e^{((c-\sqrt{c^2+4\nu(R_d-1)})/2)\xi} \\ &\quad + \gamma [1 + o(1)] e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \end{aligned} \tag{74}$$

for some constants $\overline{\alpha}, \overline{\beta}$; $\gamma \neq 0$. Since $\chi_1(-\infty) = 0$, thus $\overline{\beta} = 0$. So, when $\xi \to -\infty$, we have the following formula:

$$\begin{pmatrix} \chi_1(\xi) \\ \chi_2(\xi) \end{pmatrix} = \begin{pmatrix} \gamma [1 + o(1)] e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ \alpha [1 + o(1)] e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \end{pmatrix}. \tag{75}$$

Then, we study the exponential decay rate of the traveling wave solution as $\xi \to +\infty$. The asymptotic model of model (62) as $\xi \to +\infty$ is

$$\begin{aligned} \psi_1'' - c\psi_1' + D_{11}\psi_1 + D_{12}\psi_2 &= 0, \\ \psi_2'' - c\psi_2' + D_{21}\psi_1 + D_{22}\psi_2 &= 0. \end{aligned} \tag{76}$$

By setting $(\psi_i)'(\xi) = \widetilde{\psi}_i$, $i = 1, 2$, we rewrite model (76) as a first order model of ordinary differential equation in the four components $(\psi_1, \widetilde{\psi}_1, \psi_2, \widetilde{\psi}_2)^T$:

$$\begin{aligned} \psi_1' &= \widetilde{\psi}_1, \\ \widetilde{\psi}_1' &= c\widetilde{\psi}_1 - D_{11}\psi_1 - D_{12}\psi_2, \\ \psi_2' &= \widetilde{\psi}_2, \\ \widetilde{\psi}_2' &= c\widetilde{\psi}_2 - D_{21}\psi_1 - D_{22}\psi_2. \end{aligned} \tag{77}$$

In the case of (i) $\mu_1 \neq \mu_2$, we can obtain that the solution of model (77) has the following form:

$$(\psi_1, \widetilde{\psi}_1, \psi_2, \widetilde{\psi}_2)^T = \sum_{i=1}^{4} c_i h_i e^{\lambda_i \xi}, \tag{78}$$

where

$$\begin{aligned} \lambda_1 &= \frac{c + \sqrt{c^2 + 4\mu_1}}{2}, \qquad \lambda_2 = \frac{c - \sqrt{c^2 + 4\mu_1}}{2}, \\ \lambda_3 &= \frac{c + \sqrt{c^2 + 4\mu_2}}{2}, \qquad \lambda_4 = \frac{c - \sqrt{c^2 + 4\mu_2}}{2}, \end{aligned} \tag{79}$$

and h_i $(i = 1, 2, 3, 4)$ are the eigenvectors of the constant matrix with λ_i $(i = 1, 2, 3, 4)$ as the corresponding eigenvalues, c_i $(i = 1, 2, 3, 4)$ are arbitrary constants. Since

$$\lim_{\xi \to +\infty} (\psi_1, \widetilde{\psi}_1, \psi_2, \widetilde{\psi}_2)^T = 0, \tag{80}$$

thus $(\psi_1, \widetilde{\psi}_1, \psi_2, \widetilde{\psi}_2)^T = c_2 h_2 e^{\lambda_2 \xi} + c_4 h_4 e^{\lambda_4 \xi}$, so when $\xi \to +\infty$, we can get that

$$\begin{aligned} \begin{pmatrix} \chi_1(\xi) \\ \chi_2(\xi) \end{pmatrix} &= \begin{pmatrix} \kappa_1 [\Lambda_1 + o(1)] e^{((c-\sqrt{c^2+4\mu_1})/2)\xi} \\ \kappa_1 [\Gamma_1 + o(1)] e^{((c-\sqrt{c^2+4\mu_1})/2)\xi} \end{pmatrix} \\ &\quad + \begin{pmatrix} \kappa_2 [\Lambda_2 + o(1)] e^{((c-\sqrt{c^2+4\mu_2})/2)\xi} \\ \kappa_2 [\Gamma_2 + o(1)] e^{((c-\sqrt{c^2+4\mu_2})/2)\xi} \end{pmatrix}. \end{aligned} \tag{81}$$

Furthermore, we can obtain that

$$\begin{aligned}\chi_1(\xi) &= \kappa_1\Lambda_1 e^{((c-\sqrt{c^2+4\mu_1})/2)\xi} + \kappa_2\Lambda_2 e^{((c-\sqrt{c^2+4\mu_2})/2)\xi} \\ &\quad + \Omega_{11}(\xi) + \Omega_{12}(\xi), \\ \chi_2(\xi) &= \kappa_1\Gamma_1 e^{((c-\sqrt{c^2+4\mu_1})/2)\xi} + \kappa_2\Gamma_2 e^{((c-\sqrt{c^2+4\mu_2})/2)\xi} \\ &\quad + \Omega_{21}(\xi) + \Omega_{22}(\xi),\end{aligned} \tag{82}$$

where

$$\begin{gathered}\lim_{\xi\to+\infty}\frac{\Omega_{11}(\xi)}{e^{((c-\sqrt{c^2+4\mu_1})/2)\xi}} = 0, \qquad \lim_{\xi\to+\infty}\frac{\Omega_{12}(\xi)}{e^{((c-\sqrt{c^2+4\mu_2})/2)\xi}} = 0, \\ \lim_{\xi\to+\infty}\frac{\Omega_{21}(\xi)}{e^{((c-\sqrt{c^2+4\mu_1})/2)\xi}} = 0, \qquad \lim_{\xi\to+\infty}\frac{\Omega_{22}(\xi)}{e^{((c-\sqrt{c^2+4\mu_2})/2)\xi}} = 0.\end{gathered} \tag{83}$$

$\kappa_1, \kappa_2, \Lambda_1, \Lambda_2, \Gamma_1$, and Γ_2 are all constants.

Let $\mu = \min\{\mu_1, \mu_2\}$, then

$$\begin{aligned}&\lim_{\xi\to+\infty}\frac{\chi_1(\xi)}{e^{((c-\sqrt{c^2+4\mu})/2)\xi}} \\ &= \kappa_1\Lambda_1\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_1})/2)\xi} \\ &\quad + \kappa_2\Lambda_2\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_2})/2)\xi} \\ &\quad + \lim_{\xi\to+\infty}\frac{\Omega_{11}(\xi)}{e^{((c-\sqrt{c^2+4\mu_1})/2)\xi}}\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_1})/2)\xi} \\ &\quad + \lim_{\xi\to+\infty}\frac{\Omega_{12}(\xi)}{e^{((c-\sqrt{c^2+4\mu_2})/2)\xi}}\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_2})/2)\xi} \\ &= \Delta_1(\kappa_1, \kappa_2, \Lambda_1, \Lambda_2), \\ &\lim_{\xi\to+\infty}\frac{\chi_2(\xi)}{e^{((c-\sqrt{c^2+4\mu})/2)\xi}} \\ &= \kappa_1\Gamma_1\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_1})/2)\xi} \\ &\quad + \kappa_2\Gamma_2\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_2})/2)\xi} \\ &\quad + \lim_{\xi\to+\infty}\frac{\Omega_{21}(\xi)}{e^{((c-\sqrt{c^2+4\mu_1})/2)\xi}}\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_1})/2)\xi} \\ &\quad + \lim_{\xi\to+\infty}\frac{\Omega_{22}(\xi)}{e^{((c-\sqrt{c^2+4\mu_2})/2)\xi}}\lim_{\xi\to+\infty} e^{((\sqrt{c^2+4\mu}-\sqrt{c^2+4\mu_2})/2)\xi} \\ &= \Delta_2(\kappa_1, \kappa_2, \Gamma_1, \Gamma_2),\end{aligned} \tag{84}$$

where

$$\begin{gathered}\Delta_1(\kappa_1, \kappa_2, \Lambda_1, \Lambda_2) = \begin{cases}\kappa_1\Lambda_1, & \mu_1 < \mu_2, \\ \kappa_2\Lambda_2, & \mu_1 > \mu_2,\end{cases} \\ \Delta_2(\kappa_1, \kappa_2, \Gamma_1, \Gamma_2) = \begin{cases}\kappa_1\Gamma_1, & \mu_1 < \mu_2, \\ \kappa_2\Gamma_2, & \mu_1 > \mu_2;\end{cases}\end{gathered} \tag{85}$$

thus, when $\xi \to +\infty$, we can get that

$$\begin{aligned}&\begin{pmatrix}\chi_1(\xi) \\ \chi_2(\xi)\end{pmatrix} \\ &= \begin{pmatrix}\Delta_1(\kappa_1, \kappa_2, \Lambda_1, \Lambda_2)(1 + o(1)) e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ \Delta_2(\kappa_1, \kappa_2, \Gamma_1, \Gamma_2)(1 + o(1)) e^{((c-\sqrt{c^2+4\mu})/2)\xi}\end{pmatrix}.\end{aligned} \tag{86}$$

In the case of (ii) $\mu_1 = \mu_2$, we can obtain that the solution of model (77) has the following form:

$$\begin{aligned}(\psi_1, \tilde{\psi}_1, \psi_2, \tilde{\psi}_2)^T &= (G_1 + G_2\xi) H_{1,2} e^{\overline{\lambda_1}\xi} \\ &\quad + (G_3 + G_4\xi) H_{3,4} e^{\overline{\lambda_3}\xi},\end{aligned} \tag{87}$$

where $H_{1,2}$ is the eigenvector of the constant matrix with $\overline{\lambda_1} = \overline{\lambda_2}$ as the corresponding eigenvalues, $H_{3,4}$ is the eigenvector of the constant matrix with $\overline{\lambda_3} = \overline{\lambda_4}$ as the corresponding eigenvalues, G_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

Since $\lim_{\xi\to+\infty}(\psi_1, \tilde{\psi}_1, \psi_2, \tilde{\psi}_2)^T = 0$, thus

$$(\psi_1, \tilde{\psi}_1, \psi_2, \tilde{\psi}_2)^T = (G_3 + G_4\xi) H_{3,4} e^{\overline{\lambda_3}\xi}. \tag{88}$$

So, when $\xi \to +\infty$, we can get that

$$\begin{pmatrix}\chi_1(\xi) \\ \chi_2(\xi)\end{pmatrix} = \begin{pmatrix}(G_{1,3} + G_{1,4}\xi) e^{\overline{\lambda_3}\xi} + o\left(e^{\overline{\lambda_3}\xi}\right) \\ (G_{2,3} + G_{2,4}\xi) e^{\overline{\lambda_4}\xi} + o\left(e^{\overline{\lambda_4}\xi}\right)\end{pmatrix}. \tag{89}$$

By comparing the upper solution and roughness of the exponential dichotomy [24], we obtain the asymptotic decay rate of the traveling wave solutions at $+\infty$ given in Theorem 11.

According to the monotone iteration process [3], the traveling wave solution $U(\xi)$ is increasing; thus $U'(\xi) \geq 0$ and $U'(\xi) = (\chi_1, \chi_2)^T(\xi)$ hold

$$\begin{aligned}\chi_1'' - c\chi_1' + \frac{\partial F_1}{\partial S}\chi_1 + \frac{\partial F_1}{\partial I}\chi_2 &= 0, \\ \chi_2'' - c\chi_2' + \frac{\partial F_2}{\partial S}\chi_1 + \frac{\partial F_2}{\partial I}\chi_2 &= 0,\end{aligned} \tag{90}$$

satisfying

$$(\chi_1, \chi_2)^T(\xi) \geq 0, \qquad (\chi_1, \chi_2)^T(\pm\infty) = 0. \tag{91}$$

The strong Maximum Principle implies that $(\chi_1, \chi_2)^T(\xi) > 0$. So the strict monotonicity of the traveling wave solutions is concluded.

Now, we use the Sliding domain method to prove the uniqueness of the traveling wave solution. Let $U_1(\xi) = (S_1, I_1)^T(\xi)$ and $U_2(\xi) = (S_2, I_2)^T(\xi)$ be the traveling wave solution of model (24), with $c > 2\sqrt{R_0 - 1}$. Thus, there are some positive numbers A_i, B_i $(i = 1, 2, 3, 4)$, such that for a big enough number $N \gg 1$, when $\xi < -N$, we have

$$\begin{gathered}\begin{pmatrix} S_1(\xi) \\ I_1(\xi) \end{pmatrix} = \begin{pmatrix} A_1 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ A_2 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right), \\ \begin{pmatrix} S_2(\xi) \\ I_2(\xi) \end{pmatrix} = \begin{pmatrix} A_3 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ A_4 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right),\end{gathered} \tag{92}$$

when $\xi > N$,

$$\begin{gathered}\begin{pmatrix} S_1(\xi) \\ I_1(\xi) \end{pmatrix} = \begin{pmatrix} \widehat{S}^* - B_1 e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ \widehat{I}^* - B_2 e^{((c-\sqrt{c^2+4\mu})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right), \\ \begin{pmatrix} S_2(\xi) \\ I_2(\xi) \end{pmatrix} = \begin{pmatrix} \widehat{S}^* - B_3 e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ \widehat{I}^* - B_4 e^{((c-\sqrt{c^2+4\mu})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right).\end{gathered} \tag{93}$$

Since the traveling wave solutions of model (24) are translation-invariant, then for any $\theta \in R$, $U_1^\theta(\xi) \equiv U_1(\xi + \theta) \equiv (S_1(\xi+\theta), I_1(\xi + \theta))^T$ is also a traveling wave solution of model (24). Thus, by using the same method as above, when $\xi < -N$, we can get

$$\begin{pmatrix} S_1(\xi+\theta) \\ I_1(\xi+\theta) \end{pmatrix} = \begin{pmatrix} A_1 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\theta} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ A_2 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\theta} e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right), \tag{94}$$

when $\xi > N$,

$$\begin{pmatrix} S_1(\xi+\theta) \\ I_1(\xi+\theta) \end{pmatrix} = \begin{pmatrix} \widehat{S}^* - B_1 e^{((c-\sqrt{c^2+4\mu})/2)\theta} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ \widehat{I}^* - B_2 e^{((c-\sqrt{c^2+4\mu})/2)\theta} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \end{pmatrix} + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right). \tag{95}$$

If θ is large enough, then we can obtain the following inequalities:

$$\begin{gathered} A_1 e^{((c+\sqrt{c^2-4(R_0-1)})/2)\theta} > A_3, \\ A_2 e^{((c+\sqrt{c^2-4(R_0-1)})/2)\theta} > A_4, \\ B_1 e^{((c-\sqrt{c^2+4\mu})/2)\theta} < B_3, \\ B_2 e^{((c-\sqrt{c^2+4\mu})/2)\theta} < B_4. \end{gathered} \tag{96}$$

Thus, if θ is large enough, then $U_1^\theta(\xi) > U_2(\xi)$, for all $\xi \in R \setminus [-N, N]$.

Now, we consider model (24) on the interval $[-N, N]$.

First, suppose that

$$W(\xi) \equiv U_1^\theta(\xi) - U_2(\xi) \geq 0, \quad \xi \in [-N, N], \tag{97}$$

then

$$\begin{gathered} W'' - cW' + \begin{pmatrix} \dfrac{\partial F_1}{\partial S}(S_2 + \zeta_1\omega_1, I_1) & \dfrac{\partial F_1}{\partial I}(S_1, I_2 + \zeta_2\omega_2) \\ \dfrac{\partial F_2}{\partial S}(S_2 + \zeta_3\omega_1, I_1) & \dfrac{\partial F_2}{\partial I}(S_1, I_2 + \zeta_4\omega_2) \end{pmatrix} W = 0, \\ W(-N) > 0, \qquad W(N) > 0, \end{gathered} \tag{98}$$

where, $\zeta_i \in (0, 1)$ $(i = 1, 2, 3, 4)$, $\xi \in (-N, N)$. Since the above model is monotone, by the Maximum Principle, we can deduce that $W(\xi) > 0$, $\xi \in [-N, N]$. Consequently, we get that $U_1^\theta(\xi) > U_2(\xi), \xi \in R$.

Second, we suppose that there exists a point $\xi_* \in (-N, N)$ such that

$$S_1^\theta(\xi_*) < S_2(\xi_*) \tag{99}$$

or

$$I_1^\theta(\xi_*) < I_2(\xi_*). \tag{100}$$

In this case, we increase θ, that is shifting U_1^θ to the left, so that $U_1^\theta(-N) > U_2(-N)$ and $U_1^\theta(N) > U_2(N)$. According to the monotonicity of U_1^θ and U_2, we can find a number $\overline{\theta} \in (0, 2N)$ such that $U_1^\theta(\xi + \overline{\theta}) > U_2(\xi)$, $\xi \in (-N, N)$. Shifting $U_1^\theta(\xi + \overline{\theta})$ back until one component of $U_1^\theta(\xi + \overline{\theta})$ touches its counterpart of $U_2(\xi)$ at some point $\overline{\overline{\xi}} \in (-N, N)$. Since $U_1^\theta(\xi + \overline{\theta})$ and $U_2(\xi)$ are strictly increasing, $\overline{\overline{\xi}} \in (-N, N)$, thus, we get that $U_1^\theta(\xi + \overline{\theta}) > U_2(\xi)$, $\xi = \pm N$. However, by the Maximum Principle for that component again, we find that components of U_1^θ and U_2 are identically equal for all $\xi \in [-N, N]$ for a larger number θ. This is a contradiction, thus $U_1^\theta(\xi) > U_2(\xi)$, $\xi \in R$. Here, θ is a new number which is chosen by the above mean.

Now, decrease the θ until one of the following happens.

Case (a). There is a $\overline{\theta} \geq 0$, such that $U_1^{\overline{\theta}} = U_2(\xi)$, $\xi \in R$. In this case, we have finished the proof.

Case (b). There are a $\overline{\theta}$ and a point $\xi_1 \in R$, such that one of the components of $U_1^{\overline{\theta}}$ and U_2 are equal. And $U_1^{\overline{\theta}} \geq U_2$, $\xi \in R$. On R for that component, according to the Maximum Principle, we find that $U_1^{\overline{\theta}}$ and U_2 must be identical on that component. We can return to Case (a).

Consequently, in either situation, their exists a number $\overline{\theta} \geq 0$ such that

$$U_1^{\overline{\theta}}(\xi) = U_2(\xi), \quad \xi \in (-\infty, +\infty). \tag{101}$$

This ends of the proof. □

By Theorem 11, we can get the following theorem:

Theorem 12. *For each $c \geq 2\sqrt{R_0 - 1}$, model* (3) *has a unique (up to a translation of the origin) traveling wave solution. The traveling wave solution is strictly increasing and has the following asymptotic properties:*

(i) $c > 2\sqrt{R_0 - 1}$: *when* $\xi \to -\infty$,

$$\begin{aligned} S(\xi) &= \frac{R_d - 1}{R_d} - A_1 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right), \\ I(\xi) &= A_2 e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi} + o\left(e^{((c-\sqrt{c^2-4(R_0-1)})/2)\xi}\right). \end{aligned} \tag{102}$$

when $\xi \to +\infty$, and if $\mu_1 \neq \mu_2$, then

$$\begin{aligned} S(\xi) &= S^* + \overline{A_1} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right), \\ I(\xi) &= I^* - \overline{A_2} e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right), \end{aligned} \tag{103}$$

if $\mu_1 = \mu_2$,

$$\begin{aligned} S(\xi) &= S^* + \left(\overline{\overline{A_{11}}} + \overline{\overline{A_{12}}}\xi\right) e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right), \\ I(\xi) &= I^* - \left(\overline{\overline{A_{21}}} + \overline{\overline{A_{22}}}\xi\right) e^{((c-\sqrt{c^2+4\mu})/2)\xi} \\ &\quad + o\left(e^{((c-\sqrt{c^2+4\mu})/2)\xi}\right), \end{aligned} \tag{104}$$

where $\mu = \min\{\mu_1, \mu_2\} > 0$, $\overline{\overline{A_{11}}}, \overline{\overline{A_{21}}} \in \mathbb{R}$ A_1, A_2, $\overline{A_1}$, $\overline{A_2}$, $\overline{\overline{A_{12}}}$ and $\overline{\overline{A_{22}}}$ are all positive constants.

(ii) $c = 2\sqrt{R_0 - 1}$: *when* $\xi \to -\infty$,

$$\begin{aligned} S(\xi) &= \frac{R_d - 1}{R_d} - (A_{11} + A_{12}\xi) e^{\sqrt{R_0-1}\xi} + o\left(\xi e^{\sqrt{R_0-1}\xi}\right), \\ I(\xi) &= (A_{11} + A_{12}\xi) e^{\sqrt{R_0-1}\xi} + o\left(\xi e^{\sqrt{R_0-1}\xi}\right). \end{aligned} \tag{105}$$

when $\xi \to +\infty$, and if $\mu_1 \neq \mu_2$, then

$$\begin{aligned} S(\xi) &= S^* + \overline{B_{11}} e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right), \\ I(\xi) &= I^* - \overline{B_{22}} e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right), \end{aligned} \tag{106}$$

if $\mu_1 = \mu_2$, then

$$\begin{aligned} S(\xi) &= S^* + \left(\overline{\overline{B_{11}}} + \overline{\overline{B_{12}}}\xi\right) e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} \\ &\quad + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right), \\ I(\xi) &= I^* - \left(\overline{\overline{B_{21}}} + \overline{\overline{B_{22}}}\xi\right) e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi} \\ &\quad + o\left(e^{(\sqrt{R_0-1}-\sqrt{R_0-1+\mu})\xi}\right), \end{aligned} \tag{107}$$

where $\mu = \min\{\mu_1, \mu_2\} > 0$, $B_{12}, B_{22} < 0$, $B_{11}, B_{21}, \overline{\overline{B_{11}}}, \overline{\overline{B_{21}}} \in \mathbb{R}$, $\overline{B_{11}}$, $\overline{B_{22}}$, $\overline{\overline{B_{12}}}$, $\overline{\overline{B_{22}}}$ are all positive constants.

Theorem 13. *There is no monotone traveling wave solution of model* (24) *for any $0 < c < 2\sqrt{R_0 - 1}$. In other words, there is no monotone traveling wave solution of model* (3) *for any $0 < c < 2\sqrt{R_0 - 1}$.*

Proof. Suppose there is a monotone traveling wave solution $\mathbb{L}(\xi) = (l_1(\xi), l_2(\xi))^T$ of model (24) with the wave speed c_0, where $c_0 \in (0, 2\sqrt{R_0 - 1})$.

The asymptotic model of $\mathbb{L}(\xi) = (l_1(\xi), l_2(\xi))^T$ as $\xi \to -\infty$ is

$$\begin{aligned} \overline{\lambda}'' - c_0\overline{\lambda}' - \nu(R_d - 1)\overline{\lambda} + (\nu R_d + R_0 - 2\nu)\overline{\mu} &= 0, \\ \overline{\mu}'' - c_0\overline{\mu}' + (R_0 - 1)\overline{\mu} &= 0. \end{aligned} \tag{108}$$

The second function of (108) has two characteristics as the following ones: $(c_0 + \sqrt{4(R_0 - 1) - c_0^2}i)/2$, $(c_0 - \sqrt{4(R_0 - 1) - c_0^2}i)/2$. Thus it has two independent solutions of the following form:

$$\begin{aligned} \overline{\mu_{11}} &= e^{(c_0/2)\xi} \cos\left(\frac{\sqrt{4(R_0 - 1) - c_0^2}}{2}\xi\right), \\ \overline{\mu_{22}} &= e^{(c_0/2)\xi} \sin\left(\frac{\sqrt{4(R_0 - 1) - c_0^2}}{2}\xi\right). \end{aligned} \tag{109}$$

Similar to the proof of Theorem 11, we can get that, when $\xi \to -\infty$, $\chi_2(\xi)$ can be described as the following equation:

$$\begin{aligned}\chi_2(\xi) &= K_1 e^{(c_0/2)\xi}\cos\left(\frac{\sqrt{4(R_0-1)-c_0^2}}{2}\xi\right)\\ &\quad + K_2 e^{(c_0/2)\xi}\sin\left(\frac{\sqrt{4(R_0-1)-c_0^2}}{2}\xi\right) + \text{h.o.t},\\ &= \sqrt{K_1^2+K_2^2}e^{(c_0/2)\xi}\sin\left(\frac{\sqrt{4(R_0-1)-c_0^2}}{2}\xi+\tau(\xi)\right)\\ &\quad + \text{h.o.t},\end{aligned} \tag{110}$$

where $\tan(\tau(\xi)) = K_1/K_2$, and h.o.t is the short notation for the higher order terms.

That is to say, $l_2(\xi)$ is oscillating. Thus, any solution of model (24) with $0 < c < 2\sqrt{R_0-1}$ is not strictly monotone. □

Theorems 12 and 13 indicate that $c = 2\sqrt{R_0-1}$ is the critical minimal wave speed.

Acknowledgments

The authors would like to thank the anonymous referee for the very helpful suggestions and comments which led to improvements of our original paper. This research was supported by the Natural Science Foundation of Zhejiang Province (LY12A01014, R1110261, and LQ12A01009), the National Science Foundation of China (61272018), and the National Basic Research Program of China (2012CB426510).

References

[1] D. H. Sattinger, "Weighted norms for the stability of traveling waves," *Journal of Differential Equations*, vol. 25, no. 1, pp. 130–144, 1977.

[2] Y. Hosono and B. Ilyas, "Traveling waves for a simple diffusive epidemic model," *Mathematical Models & Methods in Applied Sciences*, vol. 5, no. 7, pp. 935–966, 1995.

[3] J. Wu and X. Zou, "Traveling wave fronts of reaction-diffusion systems with delay," *Journal of Dynamics and Differential Equations*, vol. 13, no. 3, pp. 651–687, 2001.

[4] P. Weng and X. Q. Zhao, "Spreading speed and traveling waves for a multi-type SIS epidemic model," *Journal of Differential Equations*, vol. 229, no. 1, pp. 270–296, 2006.

[5] K. F. Zhang and X. Q. Zhao, "Spreading speed and travelling waves for a spatially discrete SIS epidemic model," *Nonlinearity*, vol. 21, no. 1, pp. 97–112, 2008.

[6] W. T. Li, G. Lin, and S. Ruan, "Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems," *Nonlinearity*, vol. 19, no. 6, pp. 1253–1273, 2006.

[7] S. Ma, "Traveling waves for non-local delayed diffusion equations via auxiliary equations," *Journal of Differential Equations*, vol. 237, no. 2, pp. 259–277, 2007.

[8] R. Liu, V. R. S. K. Duvvuri, and J. Wu, "Spread pattern formation of H5N1-avian influenza and its implications for control strategies," *Mathematical Modelling of Natural Phenomena*, vol. 3, no. 7, pp. 161–179, 2008.

[9] K. Li and X. Li, "Travelling wave solutions in diffusive and competition-cooperation systems with delays," *IMA Journal of Applied Mathematics*, vol. 74, no. 4, pp. 604–621, 2009.

[10] Z. C. Wang, W. T. Li, and S. Ruan, "Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay," *Journal of Differential Equations*, vol. 238, no. 1, pp. 153–200, 2007.

[11] Z. C. Wang and J. Wu, "Travelling waves of a diffusive Kermack-McKendrick epidemic model with non-local delayed transmission," *Proceedings of the Royal Society A*, vol. 466, no. 2113, pp. 237–261, 2010.

[12] Z. C. Wang, J. Wu, and R. Liu, "Traveling waves of the spread of avian influenza," *Proceedings of the American Mathematical Society*, vol. 140, no. 11, pp. 3931–3946, 2012.

[13] A. Boumenir and V. M. Nguyen, "Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations," *Journal of Differential Equations*, vol. 244, no. 7, pp. 1551–1570, 2008.

[14] X. Hou and A. W. Leung, "Traveling wave solutions for a competitive reaction-diffusion system and their asymptotics," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 5, pp. 2196–2213, 2008.

[15] X. Hou and W. Feng, "Traveling waves and their stability in a coupled reaction diffusion system," *Communications on Pure and Applied Analysis*, vol. 10, no. 1, pp. 141–160, 2011.

[16] Q. Gan, R. Xu, and P. Yang, "Travelling waves of a delayed SIRS epidemic model with spatial diffusion," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 52–68, 2011.

[17] Q. Ye, Z. Li, M. Wang, and Y. Wu, *An Introduction to Reaction-Diffusion Equation*, Science Press, Beijing, China, 2nd edition, 2011.

[18] H. W. Hethcote, "The mathematics of infectious diseases," *SIAM Review*, vol. 42, no. 4, pp. 599–653, 2000.

[19] Z. Ma, Y. Zhou, and J. Wu, *Modeling and Dynamics of Infectious Diseases*, vol. 11 of *Series in Contemporary Applied Mathematics CAM*, Higher Education Press, Beijing, China, 2009.

[20] W. Wang, Y. Cai, M. Wu, K. Wang, and Z. Li, "Complex dynamics of a reaction-diffusion epidemic model," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 5, pp. 2240–2258, 2012.

[21] Y. Cai, W. Liu, Y. Wang, and W. Wang, "Complex dynamics of a diffusive epidemic model with strong Allee effect," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 4, pp. 1907–1920, 2013.

[22] F. Berezovsky, G. Karev, B. Song, and C. Castillo-Chavez, "A simple epidemic model with surprising dynamics," *Mathematical Biosciences and Engineering*, vol. 2, no. 1, pp. 133–152, 2005.

[23] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society A*, vol. 115, no. 772, pp. 700–721, 1927.

[24] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Tata McGraw-Hill, New York, NY, USA, 1972.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 517036, 8 pages
http://dx.doi.org/10.1155/2013/517036

Research Article

Periodic Solutions of Second-Order Difference Problem with Potential Indefinite in Sign

Honghua Bin

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Honghua Bin; hhbin@jmu.edu.cn

Received 19 January 2013; Accepted 24 February 2013

Academic Editor: Zhenkun Huang

We investigate the periodic solutions of second-order difference problem with potential indefinite in sign. We consider the compactness condition of variational functional and local linking at 0 by introducing new number λ_*. By using Morse theory, we obtain some new results concerning the existence of nontrivial periodic solution.

1. Introduction

We consider the second-order discrete Hamiltonian systems

$$\Delta^2 x_{n-1} + W'(n, x_n) = 0, \qquad x_{n+T} = x_n, \tag{1}$$

where $T \geq 2$ is a given integer, $n \in \mathbb{Z}$, $x_n \in \mathbb{R}^N$, $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, W' stands for the gradient of W with respect to the second variable. $W \in C^2(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in the first variable and has the form $W(n,x) = (1/2)a|x|^2 + H(n,x)$, where $a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, $r = [T/2]$, $[\cdot]$ stands for the greatest-integer function. For integers $a \leq b$, the discrete interval $\{a, a+1, \ldots, b\}$ is denoted by $Z[a,b]$.

In this paper we consider that H is sign changing, that is,

$$\begin{aligned} H(n,x) &= b(n)\left(\frac{1}{s}|x|^s + \overline{G}_s(n,x)\right) \\ &\triangleq \frac{1}{s} b(n)|x|^s + G_s(n,x), \end{aligned} \tag{2}$$

$\Omega_+ = \{n \in Z[1,T] | b(n) > 0)\}$, $\Omega_- = \{n \in Z[1,T] | b(n) < 0)\}$ are two nonempty subsets of $Z[1,T]$, where $s > 1$, $b(\cdot)$ is a T-periodic real function, $G_s \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$, and $G_s(n,0) = 0$.

Consider the second-order Hamiltonian system

$$\begin{gathered} \ddot{x}(t) + W'(t,x) = 0, \qquad x(0) = x(T), \\ \dot{x}(0) = \dot{x}(T), \end{gathered} \tag{3}$$

where $W \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t, $W(t,x) = (1/2)(A(t)x, x) + H(t,x)$. Here $A(\cdot)$ is a continuous, T-periodic matrix-value function.

Systems (1) and (3) have been investigated by many authors using various methods, see [1–5]. The dynamical behavior of differential and difference equations was studied by using various methods, and many interesting results have obtained, see [6–10] and references therein. The critical point theory [11–14] is a useful tool to investigate differential equations. Morse theory [15–19] has also been used to solve the asymptotically linear problem. By minimax methods in critical point theory, Tang and Wu [4], Antonacci [20, 21] considered the problem (3) with potential indefinite in sign, where H is superquadratic at zero and infinity. By using Morse theory, Zou and Li [10] study the existence of T-periodic solution of (3), where H is asymptotically superquadratic and sign changing. Moroz [19] studies system (3) where H is asymptotically subquadratic and sign changing. Motivated by [5, 10, 19], we investigate periodic solutions for asymptotically superquadratic or subquadratic discrete system (1).

By expression of $H(n,x)$, system (1) possesses a trivial solution $x = 0$. Here we are interested in finding the nonzero T-periodic solution of (1), and we divide the problem into two cases: $s > 2$ and $1 < s < 2$. For $s = 2$, one can refer to [22].

Case 1 (asymptotically superquadratic case: $s > 2$). In this case, we replace p with s in (2). Letting $g_p(n,x) = G'_p(n,x)$, we rewrite (1) as

$$\Delta^2 x_{n-1} + a x_n + b(n)|x_n|^{p-2} x_n + g_p(n, x_n) = 0, \qquad x_{n+T} = x_n. \tag{4}$$

Furthermore, for all $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$, we assume that g_p satisfies

(A1) $g_p(n,x) = o(|x|)$ as $|x| \to \infty$ uniformly in n,

(A2) $g_p(n,x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly in n.

Case 2 (asymptotically subquadratic case: $1 < s < 2$). Here we replace q with s in (2). Letting $g_q(n,x) = G'_q(n,x)$, we rewrite (1) as

$$\Delta^2 x_{n-1} + a x_n + b(n)|x_n|^{q-2} x_n + g_q(n, x_n) = 0, \qquad x_{n+T} = x_n. \tag{5}$$

For all $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$, we assume that g_q satisfies

(B1) $g_q(n,x) = o(|x|^{q-1})$ as $|x| \to \infty$ uniformly in n,

(B2) $g_q(n,x) = o(|x|)$ as $|x| \to 0$ uniformly in n.

Before stating the main results, we introduce space $E_T = \{x = \{x_n\} \in S | x_{n+T} = x_n, n \in \mathbb{Z}\}$, where $S = \{x = \{x_n\} | x_n \in \mathbb{R}^N, n \in \mathbb{Z}\}$. For any $x, y \in S, a, b \in \mathbb{R}$, we define $ax + by = \{ax_n + by_n\}_{n\in\mathbb{Z}}$. Then S is a linear space. Let $\langle x, y\rangle_{E_T} = \sum_{n=1}^{T}(x_n, y_n)$, $\|x\|_{E_T} = (\sum_{n=1}^{T}|x_n|^2)^{1/2}$, for all $x, y \in E_T$, where $(\cdot,\cdot)$ and $|\cdot|$ are the usual inner product and norm in $\mathbb{R}^N$, respectively. Obviously, E_T is a Hilbert space with dimension NT and homeomorphism to $\mathbb{R}^{NT}$. For $r > 1$, let $\|x\|_r = (\sum_{n=1}^{T}|x_n|^r)^{1/r}$, $x \in E_T$. Moreover, for simplicity, we write $\langle x, y\rangle$ and $\|x\|$ instead of $\langle x, y\rangle_{E_T}$ and $\|x\|_{E_T}$, respectively.

Lemma 1. *There exist positive numbers a_1, a_2, such that $a_1 \| x\|_r \leq \| x \| \leq a_2 \| x\|_r$.*

Inspired by [10, 19], one introduces two numbers as follows:

$$\lambda_*(p) = \inf_{\|x\|=1}\left\{\|\Delta x\|^2 \mid \sum_{n=1}^{T} b(n)|x_n|^p = 0\right\}, \qquad \lambda_*(q) = \inf_{\|x\|=1}\left\{\|\Delta x\|^2 \mid \sum_{n=1}^{T} b(n)|x_n|^q = 0\right\}. \tag{6}$$

Theorem 2. *If $a < \lambda_*(p)$, then (4) has a nonzero T-periodic solution.*

Theorem 3. *If $a < \lambda_*(q)$, then (5) has a nonzero T-periodic solution.*

This paper is divided into four sections. Section 2 contains some preliminaries, and the proofs of Theorems 2 and 3 are given in Sections 3 and 4, respectively.

2. Preliminaries

2.1. Variational Functional and (PS) Condition. For seeking T-periodic solution of (1), we consider variational functional J_p associated with (4) as $J_p(x) = (1/2)\sum_{n=1}^{T}|\Delta x_n|^2 - (1/2)a\sum_{n=1}^{T}|x_n|^2 - 1/p\sum_{n=1}^{T} b(n)|x_n|^p - \sum_{n=1}^{T} G_p(n, x_n)$, that is

$$J_p(x) = \frac{1}{2}\|\Delta x\|^2 - \frac{1}{2}a\|x\|^2 - \frac{1}{p}\sum_{n=1}^{T} b(n)|x_n|^p - \sum_{n=1}^{T} G_p(n, x_n), \quad x \in E_T. \tag{7}$$

Moreover, T-periodic solution of (5) is associated with the critical point of functional

$$J_q(x) = \frac{1}{2}\|\Delta x\|^2 - \frac{1}{2}a\|x\|^2 - \frac{1}{q}\sum_{n=1}^{T} b(n)|x_n|^q - \sum_{n=1}^{T} G_q(n, x_n), \quad x \in E_T. \tag{8}$$

We say that a C^1-functional φ on Hilbert space X satisfies the Palais-Smale (PS) condition if every sequence $\{x^{(j)}\}$ in X, such that $\{\varphi(x^{(j)})\}$, is bounded and $\varphi'(x^{(j)}) \to 0$ as $j \to \infty$ contains a convergent subsequence.

Lemma 4. *Functional J_p satisfies (PS) condition if $a < \lambda_*(p)$.*

Proof. Let $\{x^{(j)}\} \subset E_T$ be the (PS) sequence for functional J_p, such that $J_p(x^{(j)})$ is bounded, and $J'_p(x^{(j)}) \to 0$ as $j \to \infty$. Hence, for any $\varepsilon > 0$, there exist $N_\varepsilon > 0$ and constant $c_1 > 0$, such that

$$\left|\left\langle J'_p(x^{(j)}), x^{(j)}\right\rangle\right| \leq \varepsilon\left\|x^{(j)}\right\| \quad \text{for } j \geq N_\varepsilon, \qquad \left|J_p(x^{(j)})\right| \leq c_1. \tag{9}$$

To prove that J_p satisfies (PS) condition, it suffices to show that $\|x^{(j)}\|$ is bounded in E_T. Suppose not that there exists a subsequence $\{x^{(j_k)}\}$, $\| x^{(j_k)} \| \to \infty$ as $k \to \infty$. For simplicity, we write as $\{x^{(j)}\}$ instead of $\{x^{(j_k)}\}$. Without loss of generality, we assume that there exists $k \in Z[1,T]$, such that

$$\left|x_n^{(j)}\right| \longrightarrow \infty \quad \text{as} \quad j \longrightarrow \infty \quad \text{for } n \in Z[1,k], \qquad x_n^{(j)} \text{ are bounded for } n \in Z[k+1,T]. \tag{10}$$

Therefore for all $n \in [1,T]$, by assumption (A1), there exists $c_2 > 0$ such that

$$\left|G_p(n, x_n^{(j)})\right| \leq \varepsilon\left|x_n^{(j)}\right|^2 + c_2, \qquad \left|g_p(n, x_n^{(j)})\right| \leq \varepsilon\left|x_n^{(j)}\right| + c_2 \tag{11}$$

for large j. By the previous argument, it follows that

$$\left|\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right)\right| \le \sum_{n=1}^{T}\left|g_p\left(n,x_n^{(j)}\right)\right| \left|x_n^{(j)}\right| \le \varepsilon\left\|x^{(j)}\right\|^2 + c_2T\left\|x^{(j)}\right\|. \tag{12}$$

By (7), we have

$$pJ_p\left(x^{(j)}\right) - \left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle = \left(\frac{p}{2}-1\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) - p\sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) + \sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right). \tag{13}$$

In terms of (9) and (11), for large j, it follows that

$$\left(\frac{p}{2}-1\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) \le pc_1 + \varepsilon\left\|x^{(j)}\right\| + (p+1)\varepsilon\left\|x^{(j)}\right\|^2 + pc_2T + c_2T\left\|x^{(j)}\right\|. \tag{14}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$. Dividing by $\|x^{(j)}\|^2$ in the previous formula, it follows that

$$\left\|\Delta y^{(j)}\right\|^2 \le a + \frac{2}{p-2}\left((p+1)\varepsilon + \frac{c_2T+\varepsilon}{\left\|x^{(j)}\right\|} + \frac{pc_2T+pc_1}{\left\|x^{(j)}\right\|^2}\right) \tag{15}$$

for large j. Therefore, by ε being chosen arbitrarily, there is a subsequence that converges to $y^0 \in E_T$ such that

$$\left\|\Delta y^0\right\|^2 \le a, \quad \left\|y^0\right\| = 1. \tag{16}$$

On the other hand, we have

$$J_p\left(x^{(j)}\right) - \frac{1}{2}\left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle = \left(\frac{1}{2}-\frac{1}{p}\right)\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p - \sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) + \frac{1}{2}\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right). \tag{17}$$

Then, by (9) and (11), for large j, we get

$$\begin{aligned}\left|\left(\frac{1}{2}-\frac{1}{p}\right)\sum_{n=1}^{T}b(n)\left|x_n^{(j)}\right|^p\right| &= \left|J_p\left(x^{(j)}\right) - \frac{1}{2}\left\langle J_p'\left(x^{(j)}\right),x^{(j)}\right\rangle + \sum_{n=1}^{T}G_p\left(n,x_n^{(j)}\right) - \frac{1}{2}\sum_{n=1}^{T}\left(g_p\left(n,x_n^{(j)}\right),x_n^{(j)}\right)\right| \\ &\le c_1 + \frac{\varepsilon}{2}\left\|x^{(j)}\right\| + \varepsilon\left\|x^{(j)}\right\|^2 + c_2T + \frac{1}{2}\left(\varepsilon\left\|x^{(j)}\right\|^2 + c_2T\left\|x^{(j)}\right\|\right).\end{aligned} \tag{18}$$

By dividing by $\|x^{(j)}\|^p$ in the previous formula, then by $p > 2$, we have $\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p \to 0$ as $j \to \infty$, that is, $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p = 0$. By the definition of $\lambda_*(p)$, see (6), we have $\|\Delta y^0\|^2 \ge \lambda_*(p)$. This contradicts with (16) and assumption $a < \lambda_*(p)$. The proof is completed. □

Lemma 5. *Functional J_q satisfies (PS) condition if $a < \lambda_*(q)$.*

The proof is similar to that of Lemma 4 and is omitted.

2.2. Eigenvalue Problem. Consider eigenvalue problem:

$$-\Delta^2 x_{n-1} = \lambda x_n, \qquad x_{n+T} = x_n, \quad x_n \in \mathbb{R}^N, \tag{19}$$

that is, $x_{n+1}+(\lambda-2)x_n+x_{n-1} = 0$, $x_{n+T} = x_n$. By the periodicity, the difference system has complexity solution $x_n = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/T$, $k \in \mathbb{Z}$. Moreover, $\lambda = 2-e^{-i\theta}-e^{i\theta} = 2(1-\cos\theta) = 4\sin^2(k\pi/T)$. Let η_k denote the real eigenvector corresponding to the eigenvalues $\lambda_k = 4\sin^2(k\pi/T)$, where $k \in Z[0,r]$ and $r = [T/2]$. Since $a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, we can split space E_T as follows:

$$E_T = W^- \bigoplus W^0 \bigoplus W^+, \tag{20}$$

where

$$W^- = \operatorname{span}\{\eta_k \mid k \in Z[0,m-1]\}, \qquad W^0 = \operatorname{span}\{\eta_m\}, \qquad W^+ = \operatorname{span}\{\eta_k \mid k \in Z[m+1,r]\}. \tag{21}$$

By means of eigenvalue problem, we have $|\Delta x_n|^2 - a|x_n|^2 = (\Delta x_n, \Delta x_n) - a(x_n,x_n) = (-\Delta^2 x_{n-1}, x_n) - a(x_n,x_n) = (\lambda - a)(x_n,x_n) = (\lambda-a)|x_n|^2$. Let

$$\delta = \begin{cases} \min\left\{4\sin^2\dfrac{(m+1)\pi}{T} - 4\sin^2\dfrac{m\pi}{T},\ 4\sin^2\dfrac{m\pi}{T} - 4\sin^2\dfrac{(m-1)\pi}{T}\right\}, & m \in Z[1,r], \\ 4\sin^2\dfrac{\pi}{T}, & m = 0. \end{cases} \tag{22}$$

Then $\pm(\|\Delta x\|^2 - a\|x\|^2) \ge \delta\|x\|^2$ for $x \in W^{\pm}$.

On the other hand, associating to numbers $\lambda_*(p)$ and $\lambda_*(q)$ (see (6)), we set

$$\Lambda_*(p) = \sum_{n=1}^{T} b(n)\left|e_n\right|^p, \qquad \Lambda_*(q) = \sum_{n=1}^{T} b(n)\left|e_n\right|^q, \tag{23}$$

where $e_n = u \in \mathbb{R}^N$ $(n \in [1,T])$ is the real eigenvector corresponding to eigenvalue $\lambda_0 = 0$. $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in E_T$, where $\bullet^T$ denotes the transpose of a vector or a matrix. Moreover, letting $|u| = T^{-1/2}$, we have $\|e\| = 1$, $\|\Delta e\| = 0$. Therefore, by definition of $\lambda_*(p)$, if $\Lambda_*(p) = 0$ then $\lambda_*(p) = 0$.

However, by assumption $\lambda_*(p) > a = 4\sin^2(m\pi/T)$ for some $m \in Z[0,r]$, thus $\lambda_*(p) > 0$. That is to say the equality $\Lambda_*(p) = 0$ cannot hold. Therefore our discussion will be distinguished in two cases: $\Lambda_*(p) > 0$ and $\Lambda_*(p) < 0$.

2.3. Preliminaries. Let X be a Hilbert space, and let $\varphi \in C^1(X,\mathbb{R})$ be a functional satisfying the (PS) condition. Write $\mathrm{crit}(\varphi) = \{x \in X \mid \varphi'(x) = 0\}$ for the set of critical points of functional φ and $\varphi^c = \{x \in X \mid \varphi(x) \le c\}$ for the level set. Denote by $H_k(A,B)$ the kth singular relative homology group with integer coefficients. Let $x_0 \in \mathrm{crit}(\varphi)$ be an isolated critical point with value $c = \varphi(x_0)$, $c \in \mathbb{R}$, the group $C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\})$, and $k \in \mathbb{Z}$ is called the kth critical group of φ at x_0, where U is a closed neighbourhood of u. Due to the excision of homology [13], $C_k(\varphi, x_0)$ is dependent on U.

Suppose that $\varphi(\mathrm{crit}(\varphi))$ is strictly bounded from below by $a \in \mathbb{R}$, then the critical groups of φ at infinity are formally defined [11] as $C_k(\varphi, \infty) = H_k(X, \varphi^a)$, $k \in \mathbb{Z}$.

Proposition 6 (Proposition 2.3, [11]). *Assume that C^2-functional φ satisfying (PS) condition has a local linking at* 0 *with respect to $X = X_0^+ \bigoplus X_0^-$; that is, there exists $\rho > 0$ such that*

$$\varphi(x) \le \varphi(0) \quad \text{for } x \in X_0^- \text{ and } \|x\| \le \rho, \qquad \varphi(x) > \varphi(0) \quad \text{for } x \in X_0^+ \text{ and } 0 < \|x\| \le \rho. \tag{24}$$

Then $C_k(\varphi, 0) \ne 0$, $k = \dim X_0^-$.

By Propostion 6, one proves the following lemmas with respect to $E_T = X^+ \bigoplus X^-$.

Lemma 7. *If $a < \lambda_*(p)$, then $C_k(J_p, 0) \ne 0$, $k = \dim X^-$, where $X^- = W^- \bigoplus W^0$ as $\Lambda_*(p) > 0$, $X^- = W^-$ as $\Lambda_*(p) < 0$. $\Lambda_*(p)$ is defined by* (23).

Proof. We first consider the following.

Case 1 ($\Lambda_*(p) > 0$ and $X^+ = W^+$, $X^- = W^- \bigoplus W^0$). By $p > 2$, $|x|^p = o(|x|^2)$ as $|x| \to 0$, then there exists $\theta \in (0,1)$ suitably small, such that $|x|^p \le \delta/3(b/p + \varepsilon)|x|^2$ as $|x| < \theta$, where $\delta > 0$ see (22) and $b = \max\{|b(1)|, \dots, |b(T)|\} > 0$. By assumption (A2) and $G_p(n,0) = 0$, for any given $\varepsilon > 0$, there exists $\rho_n \in (0,\theta)$, such that $|G_p(n,x_n)| \le \varepsilon|x_n|^p$ as $|x_n| \le \rho_n$, $n \in Z[1,T]$. Thus

$$\frac{1}{p}\sum_{n=1}^{T} b(n)\left|x_n\right|^p + \sum_{n=1}^{T} G_p(n,x_n) \le \left(\frac{b}{p} + \varepsilon\right)\sum_{n=1}^{T}\left|x_n\right|^p \le \frac{1}{3}\delta\|x\|^2. \tag{25}$$

Let $\rho = \min\{\rho_1, \dots, \rho_T\}$. For $0 < \|x\| \le \rho < 1$, it follows that

$$J_p(x) \ge \frac{1}{2}\delta\|x\|^2 - \frac{1}{3}\delta\|x\|^2 > 0, \quad x \in W^+ = X^+. \tag{26}$$

We need to prove that $J_p(x) \le 0$ for $x \in X^- = W^- \bigoplus W^0$, $\|x\| \le \rho$. We first claim that

$$\sum_{n=1}^{T} b(n)\left|x_n\right|^p > 0, \quad \forall x \in W^- \bigoplus W^0,\ x \ne 0. \tag{27}$$

Indeed, by contradiction, assume that $\sum_{n=1}^{T} b(n)|x_n|^p \le 0$, for some $x \in W^- \bigoplus W^0$, $x \ne 0$. Since $\Lambda_*(p) = \sum_{n=1}^{T} b(n)|e_n|^p > 0$, where $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in W^- \bigoplus W^0$, and $(W^- \bigoplus W^0) \setminus \{0\}$ is arcwise connected, then there exists a $x^0 \in (W^- \bigoplus W^0) \setminus \{0\}$, such that $\sum_{n=1}^{T} b(n)|x_n^0|^p = 0$. Thus $\|\Delta x^0\|^2 \ge \lambda_*(p)\|x^0\|^2$ by the definition of $\lambda_*(p)$. On the other hand, by the definition of $W^- \bigoplus W^0$, we have $\|\Delta x^0\|^2 \le a\|x^0\|^2$. This is a contradiction with assumption $a < \lambda_*(p)$. So the claim (27) holds.

There exists $c_4 > 0$ by (27), such that $\sum_{n=1}^{T} b(n)|x_n|^p \ge c_4\|x\|_p^p$ for all $x \in W^- \bigoplus W^0 \setminus \{0\}$, where $\|x\|_p = (\sum_{n=1}^{T}|x_n|^p)^{1/p}$. For $x \in W^- \bigoplus W^0$, $\|x\| \le \rho$, ε sufficiently small, we have

$$J_p(x) \le -\frac{1}{p}\sum_{n=1}^{T} b(n)\left|x_n\right|^p - \sum_{n=1}^{T} G_p(n,x_n) \le -\frac{c_4}{p}\|x\|_p^p + \varepsilon\|x\|_p^p \le 0. \tag{28}$$

Since $J_p(0) = 0$ and J_p satisfies (PS) condition by Lemma 4, so by Proposition 6, we obtain that $C_k(J_p, 0) \ne 0$ for $k = \dim(W^- \bigoplus W^0)$.

Case 2 ($\Lambda_*(p) < 0$, $X^+ = W^+ \bigoplus W^0$, $X^- = W^-$). It is easy to see that $J_p(x) \le 0$ by $\|\Delta x\|^2 - a\|x\|^2 \le -\delta\|x\|^2$ and $p > 2$, where $x \in W^-$ and $\|x\| \le \rho$. We need to claim that $J_p(x) > 0$, for $x \in W^+ \bigoplus W^0$, $0 < \|x\| \le \rho$.

Suppose not that there exists a sequence $\{x^{(j)}\} \subset E_T$ such that

$$\left\{x^{(j)}\right\} \subset W^+ \bigoplus W^0 \setminus \{0\}, \quad 0 < \left\|x^{(j)}\right\| \le \rho, \qquad J_p\left(x^{(j)}\right) \le 0, \tag{29}$$

for large j. For $\| x^{(j)} \| \le \rho$, by Lemma 1, we get

$$\left|\sum_{n=1}^{T}\left[\frac{1}{p}b(n)\left|x_n^{(j)}\right|^p + G_p\left(n, x_n^{(j)}\right)\right]\right| \le \sum_{n=1}^{T}\left[\frac{b}{p}\left|x_n^{(j)}\right|^p + \varepsilon\left|x_n^{(j)}\right|^p\right] \le \left(\frac{b}{p}+\varepsilon\right)\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^p. \tag{30}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$. Then by (29) and the previous formula, we have

$$0 \ge \frac{J_p\left(x^{(j)}\right)}{\left\|x^{(j)}\right\|^2} \ge \frac{1}{2}\left(\left\|\Delta y^{(j)}\right\|^2 - a\right) - \left(\frac{b}{p}+\varepsilon\right)\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^{p-2}. \tag{31}$$

On the other hand, $\|\Delta y^{(j)}\|^2 \ge a$ by the definition of $W^+ \bigoplus W^0$. Hence by $p > 2$, there exists a subsequence converges to $y^0 \in E_T$, such that $\|\Delta y^0\|^2 = a$, that is $y^0 \in W^0$ and $\|y^0\| = 1$. Since $\|\Delta x^{(j)}\|^2 \ge a\|x^{(j)}\|^2$ for $\{x^{(j)}\} \subset W^+ \bigoplus W^0$, it follows from $J_p(x^{(j)}) \le 0$ that

$$0 \le \frac{1}{p}\sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p + \sum_{n=1}^{T} G_p\left(n, x_n^{(j)}\right) \le \frac{1}{p}\sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p + \varepsilon\left(\frac{1}{a_1}\right)^p\left\|x^{(j)}\right\|^p. \tag{32}$$

Dividing by $\|x^{(j)}\|^p$ in the previous inequality, then $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p \ge 0$.

Since e, $y^0 \in W^- \bigoplus W^0$, $\Lambda_*(p) = \sum_{n=1}^{T} b(n)|e_n|^p < 0$ and $(W^- \bigoplus W^0) \setminus \{0\}$ is arcwise connected, then there exists a $\overline{y} \in (W^- \bigoplus W^0) \setminus \{0\}$ such that $\sum_{n=1}^{T} b(n)|\overline{y}_n|^p = 0$. Thus $\| \Delta\overline{x}\|^2 \ge \lambda_*(p) \| \overline{x}\|^2$ by the definition of $\lambda_*(p)$. On the other hand, $\| \Delta\overline{x}\|^2 \le a \| \overline{x}\|^2$ by the definition of $W^- \bigoplus W^0$. This is a contradiction with assumption $a < \lambda_*(p)$. That is to say, the claim is valid.

By Proposition 6, we obtain $C_k(J_p, 0) \ne 0$, $k = \dim W^-$. The proof is completed. □

Lemma 8. *If $a < \lambda_*(q)$, then $C_k(J_q, \infty) \ne 0$ for $k = \dim X^-$, where $X^- = W^- \bigoplus W^0$ as $\Lambda_*(q) > 0$, $X^- = W^-$ as $\Lambda_*(q) < 0$.*

The proof is similar to that of Lemma 7 and is omitted.

3. Proof of Theorem 2

Lemma 9. *Let $a < \lambda_*(p)$. If there exists $K_1 > 0$ such that for any $K > K_1$, $J_p(x) \le -K$, then one has $\sum_{n=1}^{T} b(n)|x_n|^p > 0$, and $(d/dt)J_p(tx)|_{t=1} < 0$.*

Proof. We first claim that $\| x \|$ is sufficiently large, if x satisfies condition of Lemma 9. Suppose not there exists $M > 0$ such that $\| x \| \le M$. So there exists $\{x^{(j)}\} \subset E_T$, $x^0 \in E_T$, such that $x^{(j)} \to x^0$ as $j \to \infty$. Since for any $j > K_1$, we have $J_p(x^{(j)}) \le -j$, thus $J_p(x^0) = \lim_{j\to\infty}J_p(x^{(j)}) = -\infty$. It is a contradiction with $J_p(x^0) = c$.

If $\|x\|$ is large enough, then we can assume that $|x_n|$ is large enough for $n \in Z[1, k]$ and $|x_n|$ are bounded for $n \in Z[k+1, T]$. Therefore, by assumption (A1), for any given $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$\left|g_p(n, x_n)\right| \le \varepsilon|x_n| + \frac{M_1}{T}, \qquad \left|G_p(n, x_n)\right| \le \varepsilon|x_n|^2 + \frac{M_1}{T}, \quad \forall (n, x_n) \in Z[1, T] \times \mathbb{R}^N. \tag{33}$$

We claim that $\sum_{n=1}^{T} b(n)|x_n|^p > 0$. Suppose not that, for $j > K_1$, there exists $\{x^{(j)}\} \subset E_T$ such that

$$\sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p \le 0. \tag{34}$$

By $J_p(x^{(j)}) \le -j \le 0$, (33) and (34), we have

$$\frac{1}{2}\left\|\Delta x^{(j)}\right\|^2 \le \frac{a}{2}\left\|x^{(j)}\right\|^2 + \sum_{n=1}^{T} G_p\left(n, x_n^{(j)}\right) \le \frac{a}{2}\left\|x^{(j)}\right\|^2 + \varepsilon\left\|x^{(j)}\right\|^2 + M_1. \tag{35}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$ and divided by $\|x^{(j)}\|^2$ in the previous inequality. Since ε can be small enough, then there exists a subsequence that converges to $y^0 \in E_T$, such that $\|\Delta y^0\|^2 \le a$, $\|y^0\| = 1$. Moreover, by (33) and (34), we get

$$0 \ge \frac{1}{p}\sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p \ge j + \frac{1}{2}\left\|\Delta x^{(j)}\right\|^2 - \frac{a}{2}\left\|x^{(j)}\right\|^2 - \sum_{n=1}^{T} G_p\left(n, x_n^{(j)}\right) \ge -\left(\frac{a}{2}+\varepsilon\right)\left\|x^{(j)}\right\|^2 - M_1. \tag{36}$$

Since $p > 2$ and $\lim_{j\to\infty}\|x^{(j)}\| = \infty$, divided by $\|x^{(j)}\|^p$ in the previous inequality, we have $\sum_{n=1}^{T} b(n)|y_n^0|^p = \lim_{j\to\infty}\sum_{n=1}^{T} b(n)|y_n^{(j)}|^p = 0$, that is, $\| \Delta y^0 \| \ge \lambda_*(q)$, which deduce a contradiction. So the claim $\sum_{n=1}^{T} b(n)|x_n|^p > 0$ holds.

Next we prove that $(d/dt)J_p(tx)|_{t=1} < 0$ holds. By contradiction, there exists a sequence $\{x^{(j)}\} \subset E_T$ such that, for $j > K_1$,

$$\left.\frac{d}{dt}J_p\left(tx^{(j)}\right)\right|_{t=1} \ge 0. \tag{37}$$

Then, by (7), we get

$$\left.\frac{d}{dt}J_p\left(tx^{(j)}\right)\right|_{t=1} = \left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2 - \sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p - \sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right), \tag{38}$$

and by (37) and $J_p(x^{(j)}) \le -j < 0$, it follows that

$$\begin{aligned}\left(1-\frac{p}{2}\right)\left(\left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2\right) \\ -\sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right) + p\sum_{n=1}^{T}G_p\left(n, x_n^{(j)}\right) \\ = \frac{d}{dt}J_p\left(tx^{(j)}\right)\Big|_{t=1} - pJ_p\left(x^{(j)}\right) \ge 0.\end{aligned} \tag{39}$$

Set $y_n^{(j)} = x_n^{(j)}/\|x^{(j)}\|$ and divided by $\|x^{(j)}\|^2$ in the previous formula; since $p > 2$ and ε can be small enough, then there exists a subsequence converges to $y^0 \in E_T$ such that $\|\Delta y^0\|^2 \le a$, $\|y^0\| = 1$. Moreover, by (37) and the first claim, we get

$$\begin{aligned}0 < \sum_{n=1}^{T} b(n)\left|x_n^{(j)}\right|^p \le \left\|\Delta x^{(j)}\right\|^2 - a\left\|x^{(j)}\right\|^2 \\ -\sum_{n=1}^{T}\left(g_p\left(n, x_n^{(j)}\right), x_n^{(j)}\right).\end{aligned} \tag{40}$$

Divided by $\|x^{(j)}\|^p$ in the previous formula, and by $p > 2$, it follows that $\sum_{n=1}^{T} b(n)|y_n^0|^p = 0$. This is a contradiction with the definition of $\lambda_*(p)$ and condition $a < \lambda_*(p)$. So the second claim holds. The proof is completed. □

Based on Lemma 9, we introduce the following notations:

$$\begin{aligned}J_p^{-K} &= \left\{x \in E_T : J_p(x) \le -K\right\}, \\ E_p^+ &= \left\{x \in E_T : \sum_{n=1}^{T} b(n)\left|x_n\right|^p > 0\right\}, \\ E(\Omega_+) &= \left\{x \in E_T : x_n = 0 \text{ for } n \in Z[1,T] \setminus \Omega_+\right\} \setminus \{0\}.\end{aligned} \tag{41}$$

Clearly, $E(\Omega_+) \subset E_p^+$. And by Lemma 9, we have $J_p^{-K} \subset E_p^+$. In order to describe the $H_q(E_T, J_p^{-K})$, we need to show the following lemma.

Lemma 10. *If $a < \lambda_*(p)$, then there exists $K_1 > 0$, such that for any $K > K_1$, J_p^{-K} is a strong deformation retraction of E_p^+. Moreover, $E(\Omega_+)$ and E_p^+ are homotopy equivalent.*

Proof. Now we prove that J_p^{-K} is a strong deformation retraction of E_p^+.

By Lemma 9, we have $J_p^{-K} \subset E_p^+$. Let $x \in E_p^+$. By Lemma 9, there exists a unique $t_p = t_p(x) > 0$ such that $J_p(t_p x) = -K$. By applying Implicit Function Theorem, $t_p(x)$ is a continuous function in E_p^+. Let $T_p(x) = \max\{t_p(x), 1\}$ and define $f_p(s,x) = (1-s)x + sT_p(x)x$, then $f_p : [0,1] \times E_p^+ \to J_p^{-K}$ is a strong deformation retraction. Thus J_p^{-K} is a strong deformation retraction of E_p^+.

We next claim that $E(\Omega_+)$ is a strong deformation retraction of E_p^+. Clearly, in terms of the notations, we have $E(\Omega_+) \subset E_p^+$. Let $\xi_p : Z[1,T] \to \mathbb{R}$ be a function such that

$$\begin{aligned}\xi_p(n) = 1 \quad \text{if } n \in \Omega_+, \qquad \xi_p(n) = 0 \quad \text{if } n \in \Omega_-, \\ \xi_p(n) \in [0,1] \quad \text{if } n \in Z[1,T] \setminus (\Omega_+ \cup \Omega_-).\end{aligned} \tag{42}$$

Define

$$\zeta_p(s, x_n) = \begin{cases}(1-2s)x_n + 2s\xi_p(n)x_n & \text{if } 0 \le s \le \dfrac{1}{2}, \\ 2(1-s)\xi_p(n)x_n + 2\left(s-\dfrac{1}{2}\right)P\left(\xi_p(n)x_n\right) & \\ & \text{if } \dfrac{1}{2} \le s \le 1,\end{cases} \tag{43}$$

where $P : E_T \to E(\Omega_+)$ is a projection operator. Then $\zeta_p : [0,1] \times E_p^+ \to E(\Omega_+)$ is a deformation retraction. Indeed,

$$\begin{aligned}\zeta_p(0,x) = x, \quad \zeta_p(1,x) \in E(\Omega_+), \quad \text{for } x \in E_p^+, \\ \zeta_p(s,x) = x, \quad \text{for } x \in E(\Omega_+) \text{ and } s \in [0,1].\end{aligned} \tag{44}$$

For $x \in E_p^+$, if $s \in [0, 1/2]$, then

$$\begin{aligned}&\sum_{n=1}^{T} b(n)\left|\zeta_p(s,x_n)\right|^p \\ &\quad= \sum_{n\in\Omega_+} b(n)\left|x_n\right|^p + \sum_{n\in\Omega_-} b(n)(1-2s)^p\left|x_n\right|^p \\ &\quad\ge \sum_{n=1}^{T} b(n)\left|x_n\right|^p > 0,\end{aligned} \tag{45}$$

where $0 \le (1-2s)^p \le 1$, that is, $\zeta_p(s,x) \in E_p^+$. If $s \in (1/2, 1]$, it follows that

$$\begin{aligned}&\sum_{n=1}^{T} b(n)\left|\zeta_p(s,x_n)\right|^p \\ &\quad= \sum_{n\in\Omega_+} b(n)\left|2(1-s)\xi_p(n)x_n + 2\left(s-\frac{1}{2}\right)P\left(\xi_p(n)x_n\right)\right|^p \\ &\quad\ge 0.\end{aligned} \tag{46}$$

We claim that the equality of the previous formula cannot hold. Otherwise, $Px_n = -((1-s)/(s-(1/2)))x_n$, for $n \in \Omega_+$, which implies that $Px_n = 0$. Hence $x_n = 0$ in Ω_+, which contradicts with the fact $x \in E_p^+$. So $\sum_{n=1}^{T} b(n)|\zeta_p(s,x_n)|^p > 0$, that is, $\zeta_p(s,x) \in E_p^+$ as $s \in (1/2, 1]$. Therefore, ζ_p is a deformation retraction from E_p^+ onto $E(\Omega_+)$, and this completes the proof. □

Proof of Theorem 2. Since $E(\Omega_+)$ is well known to be contractile in itself, and by Lemma 10, it follows that J_p^{-K} is homotopically equivalent to $E(\Omega_+)$ for K large enough, then the Betti numbers (cf. [11, 13]) are

$$\begin{aligned}\beta_k &= \dim C_k\left(J_p,\infty\right) = \dim H_k\left(E_T, J_p^{-K}\right)\\ &= \dim H_k\left(E_T, E\left(\Omega_+\right)\right) = 0, \quad k \in Z\left[0, NT\right].\end{aligned} \tag{47}$$

Now we suppose that system (4) has only trivial solution; that is, J_p has only critical point $x = 0$, then we have the Morse-type numbers $M_k = \dim C_k(J_p, 0)$ for $k \in Z[0, NT]$ (cf. [13]). Moreover, by Lemma 7, $C_k(J_p, 0) \neq 0$ for $k = \dim W^-$ or $k = \dim(W^- \bigoplus W^0)$. Since J_p satisfies (PS) condition by Lemma 4, then using Morse Relation, we have the following.

$$0 = \sum_{k=0}^{NT}(-1)^k\beta_k = \sum_{k=0}^{NT}(-1)^k M_k \neq 0, \tag{48}$$

which is a contradiction. Therefore, J_p has at least one critical point $x^* \neq 0$ and system (4) has at least a nonzero T-periodic solution. □

4. Proof of Theorem 3

For convenience, we introduce the following notations:

$$\begin{aligned}J_q^c &= \left\{x \in E_T : J_q\left(x\right) \leq c\right\}, \quad c \in \mathbb{R},\\ E_q^+ &= \left\{x \in E_T : \sum_{n=1}^{T} b\left(n\right)\left|x_n\right|^q > 0\right\}.\end{aligned} \tag{49}$$

Clearly, $E_q^+ \cup \{0\}$ is star-shaped with respect to the origin and $E(\Omega_+) \subset E_q^+$, where $E(\Omega_+)$ is given in Section 3. Similarly with the proof of Lemmas 9 and 10, we have the following.

Lemma 11. *Let $a < \lambda_*(q)$. Then there exists $\rho > 0$ such that $(d/dt)J_q(tx)|_{t=1} > 0$ for any $x \in M_\rho = \{x \in B_\rho \cap E_q^+ : J_q(x) \geq 0\}$, where B_ρ stands for the closed ball in E_T of radius $\rho > 0$ with the center at zero.*

Lemma 12. *Let $a < \lambda_*(q)$. Then there exists $\rho > 0$ such that $(J_q^0 \cap B_\rho) \setminus \{0\}$ is a retract of $E_q^+ \cap B_\rho$, and $E(\Omega^+)$ is a strong deformation retraction of E_q^+.*

Proof of Theorem 3. We first prove that $J_q^0 \cap B_\rho$ is contractible in itself. In fact, it is sufficient to show that $J_q^0 \cap B_\rho$ is starshaped with respect to the origin; that is, $x \in J_q^0 \cap B_\rho$ implies that $tx \in J_q^0 \cap B_\rho$ for all $t \in [0, 1]$.

Assume, by a contradiction, that there exists $x_0 \in J_q^0 \cap B_\rho$ and $t_0 \in (0, 1)$, such that $J_q(t_0x_0) > 0$. It follows from Lemma 11 that $(d/dt)J_q(t_0x_0) > 0$. By the monotonicity arguments, this implies that

$$J_q\left(tx_0\right) > 0 \quad \forall t \in \left[t_0, 1\right]. \tag{50}$$

This contradicts the assumption $x_0 \in J_q^0$, which implies $J_q(x_0) \leq 0$.

On the other hand, since $E(\Omega_+)$ is contractible in itself, and $E_q^+ \cup \{0\}$ is starshaped with respect to the origin, then $E_q^+ \cap B_\rho$ is contractible in itself. The retract of the set which is contractible in itself is also contractible (cf. [19]); it follows that the set $(J_q^0 \cap B_\rho) \setminus \{0\}$ is contractible by Lemma 12.

Combining the previous argument, $J_q^0 \cap B_\rho$ and $(J_q^0 \cap B_\rho) \setminus \{0\}$ are contractible in themselves.

$$\begin{aligned}\dim C_k\left(J_q, 0\right) = \dim H_k\left(J_q^0 \cap B_\rho, \left(J_q^0 \cap B_\rho\right) \setminus \{0\}\right) = 0,\\ k \in Z\left[0, NT\right].\end{aligned} \tag{51}$$

By Lemma 8, $C_k(J_q, \infty) \neq 0$ for $k = \dim(W^- \bigoplus W^0)$ or $k = \dim W^-$. Therefore, by Morse Relation and the same methods in proof of Theorem 2, it follows that J_q has at least one critical point $x^* \neq 0$ and system (5) has at least a nonzero T-periodic solution. □

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187) and NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), and the Foundation of Education of Fujian Province (JA09152).

References

[1] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China A*, vol. 46, no. 4, pp. 506–515, 2003.

[2] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.

[3] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, Boston, Mass, USA, 1991.

[4] C.-L. Tang and X.-P. Wu, "Periodic solutions for second order Hamiltonian systems with a change sign potential," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 506–516, 2004.

[5] Y.-T. Xu and Z.-M. Guo, "Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 51, no. 7, pp. 1273–1283, 2002.

[6] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, vol. 16, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[7] H. Amann and E. Zehnder, "Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 7, no. 4, pp. 539–603, 1980.

[8] Z. Huang, C. Feng, and S. Mohamad, "Multistability analysis for a general class of delayed Cohen-Grossberg neural networks," *Information Sciences*, vol. 187, pp. 233–244, 2012.

[9] Z. Huang and Y. N. Raffoul, "Biperiodicity in neutral-type delayed difference neural networks," *Advances in Difference Equations*, vol. 2012, article 5, 2012.

[10] W. Zou and S. Li, "Infinitely many solutions for Hamiltonian systems," *Journal of Differential Equations*, vol. 186, no. 1, pp. 141–164, 2002.

[11] T. Bartsch and S. Li, "Critical point theory for asymptotically quadratic functionals and applications to problems with resonance," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 28, no. 3, pp. 419–441, 1997.

[12] K. C. Chang, *Critical Point Theory and Its Applications*, Science and Technical Press, Shanghai, China, 1980.

[13] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[14] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[15] V. Benci, "A new approach to the Morse-Conley theory and some applications," *Annali di Matematica Pura ed Applicata*, vol. 158, pp. 231–305, 1991.

[16] C. Conley, *Isolated Invariant Sets and the Morse Index*, vol. 38 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1978.

[17] K. C. Chang, "Solutions of asymptotically linear operator equations via Morse theory," *Communications on Pure and Applied Mathematics*, vol. 34, no. 5, pp. 693–712, 1981.

[18] K.-C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, Mass, USA, 1993.

[19] V. Moroz, "On the Morse critical groups for indefinite sublinear elliptic problems," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 52, no. 5, pp. 1441–1453, 2003.

[20] F. Antonacci, "Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign," *Unione Matematica Italiana. Bollettino B*, vol. 10, no. 2, pp. 303–324, 1996.

[21] F. Antonacci, "Existence of periodic solutions of Hamiltonian systems with potential indefinite in sign," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 29, no. 12, pp. 1353–1364, 1997.

[22] H.-H. Bin, J.-S. Yu, and Z.-M. Guo, "Nontrivial periodic solutions for asymptotically linear resonant difference problem," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 477–488, 2006.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 617618, 7 pages
http://dx.doi.org/10.1155/2013/617618

Research Article

Recursive Neural Networks Based on PSO for Image Parsing

Guo-Rong Cai and Shui-Li Chen

School of Sciences, Jimei University, Xiamen, China

Correspondence should be addressed to Shui-Li Chen; sgzx@jmu.edu.cn

Received 24 February 2013; Accepted 3 March 2013

Academic Editor: Zhenkun Huang

This paper presents an image parsing algorithm which is based on Particle Swarm Optimization (PSO) and Recursive Neural Networks (RNNs). State-of-the-art method such as traditional RNN-based parsing strategy uses L-BFGS over the complete data for learning the parameters. However, this could cause problems due to the nondifferentiable objective function. In order to solve this problem, the PSO algorithm has been employed to tune the weights of RNN for minimizing the objective. Experimental results obtained on the Stanford background dataset show that our PSO-based training algorithm outperforms traditional RNN, Pixel CRF, region-based energy, simultaneous MRF, and superpixel MRF.

1. Introduction

Image parsing is an important step towards understanding an image, which is to perform a full-scene labeling. The task of image parsing consists in labeling every pixel in the image with the category of the object it belongs to. After a perfect image parsing, every region and every object are delineated and tagged [1]. Image parsing is frequently used in a wide variety of tasks including parsing scene [2, 3], aerial image [4], and facade [5].

During the past decade, the image parsing technique has undergone rapid development. Some methods for this task such as [6] rely on a global descriptor which can do very well for classifying scenes into broad categories. However, these approaches fail to gain a deeper understanding of the objects in the scene. Many other methods rely on CRFs [7], MRFs [8], or other types of graphical models [9, 10] to ensure the consistency of the labeling and to account for context. Also, there are many approaches for image annotation and semantic segmentation of objects into regions [11]. Note that most of the graphical-based methods rely on a presegmentation into superpixels or other segment candidates and extract features and categories from individual segments and from various combinations of neighboring segments. The graphical model inference pulls out the most consistent set of segments which covers the image [1]. Recently, these ideas have been combined to provide more detailed scene understanding [12–15].

It is well known that many graphical methods are based on neural networks. The main reason is that neural networks have promising potential for tasks of classification, associative memory, parallel computation, and solving optimization problems [16]. In 2011, Socher et al. proposed a RNN-based parsing algorithm that aggregates segments in a greedy strategy using a trained scoring function [17]. It recursively merges pairs of segments into supersegments in a semantically and structurally coherent way. The main contribution of the approach is that the feature vector of the combination of two segments is computed from the feature vectors of the individual segments through a trainable function. Experimental results on Stanford background dataset revealed that RNN-based method outperforms state-of-the-art approaches in segmentation, annotation, and scene classification. That being said, it is worth noting that the objective function is nondifferentiable due to the hinge loss. This could cause problems since one of the principles of L-BFGS, which is employed as the training algorithm in RNN, is that the objective should be differentiable.

Since Particle Swarm Optimization (PSO) [18] has proven to be an efficient and powerful problem-solving strategy, we use a novel nonlinear PSO [19] to tune the weights of RNN. The main idea is to use particle swarm for searching good

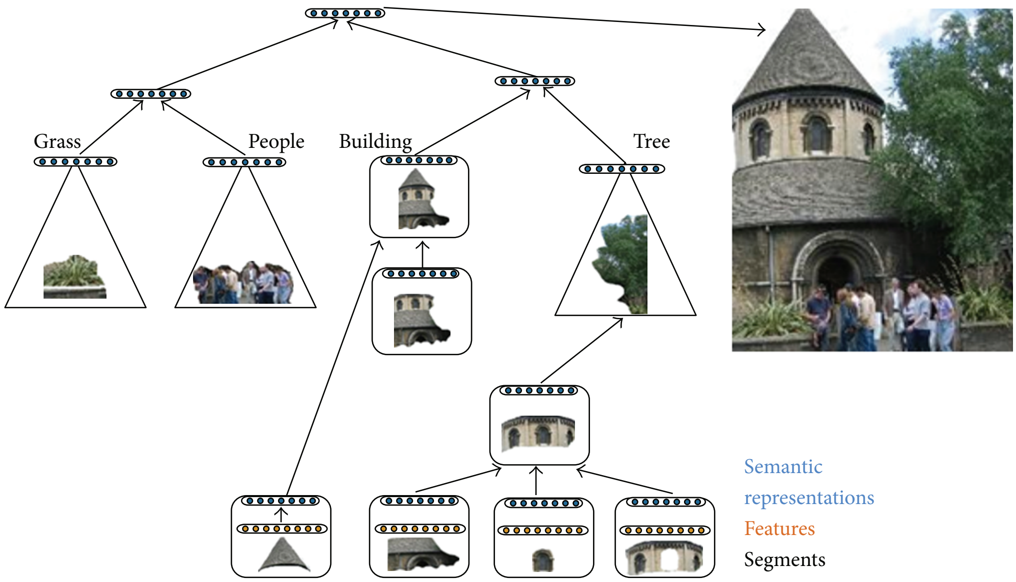


FIGURE 1: Hierarchical architecture of image parsing based on recursive neural network.

combination of weights to minimize the objective function. The experimental results show that the proposed algorithm has better performance than traditional RNN on Stanford background dataset.

The rest of the paper is organized as follows: Section 2 provides a brief description of the RNN-based image parsing algorithm. Section 3 describes how PSO and the proposed algorithm work. Section 4 presents the dataset and the experimental results. Section 5 draws conclusions.

2. Image Parsing Based on Recursive Neural Networks

The main idea behind recursive neural networks for image parsing lies in that images are oversegmented into small regions and each segment has a vision feature. These features are then mapped into a "semantic" space using a recursive neural network. Figure 1 outlines the approach for RNN-based image parsing method. Note that the RNN computes (i) a score that is higher when neighboring regions should be merged into a larger region, (ii) a new semantic feature representation for this larger region, and (iii) its class label. After regions with the same object label are merged, neighboring objects are merged to form the full scene image. These merging decisions implicitly define a tree structure in which each node has associated with the RNN outputs (i)–(iii), and higher nodes represent increasingly larger elements of the image. Details of the algorithm are given from Sections 2.1 to 2.3.

2.1. Input Representation of Scene Images. Firstly, an image x is oversegmented into superpixels (also called segments) using the algorithm from [20]. Secondly, for each segment, compute 119 features via [10]. These features include color and texture features, boosted pixel classifier scores (trained on the labeled training data), and appearance and shape features. Thirdly, a simple neural network layer has been used to map these features into the "semantic" n-dimensional space in which the RNN operates, given as follows.

Let F_i be the features described previously for each segment, where $i = 1, \ldots, N_{\text{segs}}$ and N_{segs} denotes the number of segments in an image. Then the representation is given as

$$a_i = f\left(W^{\text{sem}} F_i + b^{\text{sem}}\right), \tag{1}$$

where $W^{\text{sem}} \in R^{n\times 119}$ is the matrix of parameters we want to learn, b^{sem} is the bias, and f is applied element wise and can be any sigmoid-like function. In [17], the original sigmoid is function $f(x) = 1/(1 + e^{-x})$ (Figure 2).

2.2. Greedy Structure Predicting. Since there are more than exponentially many possible parse trees and no efficient dynamic programming algorithms for RNN setting, therefore, Socher recommended a greedy strategy. The algorithm finds the pairs of neighboring segments and adds their activations to a set of potential child node pairs. Then the network computes the potential parent representation for these possible child nodes:

$$p(i, j) = f\left(W\left[c_i; c_j\right] + b\right). \tag{2}$$

With this representation, a local score can be determined by using a simple inner product with a row vector $W^{\text{score}} \in R^{1\times n}$:

$$s(i, j) = W^{\text{score}} p(i, j). \tag{3}$$

As illustrated in Figure 3, the recursive neural network is different from the original RNN in that it predicts a score for being a correct merging decision. The process repeats until all pairs are merged and only one parent activation is left, as

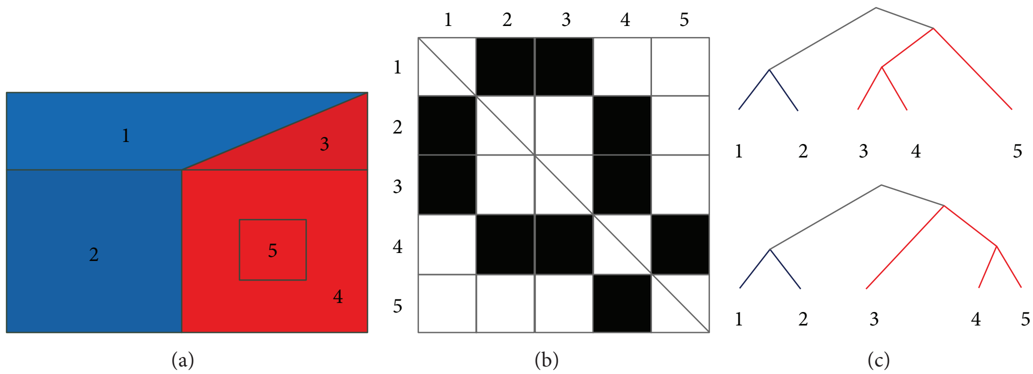


FIGURE 2: Illustration of the RNN training inputs: (a) a training image (red and blue are differently labeled regions). (b) An adjacency matrix of image segments. (c) A set of correct trees which is oblivious to the order in which segments with the same label are merged.

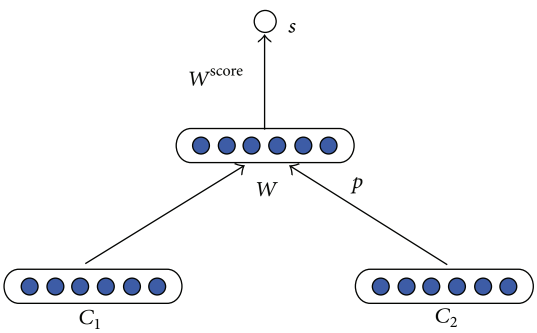


FIGURE 3: Recursive neural network which is replicated for each pair of input vectors.

shown in Figure 1. The final score that we need for structure prediction is simply the sum of all the local decisions:

$$s\left(\mathrm{RNN}\left(\theta, x_i, \hat{y}\right)\right) = \sum_{d \in N(\hat{y})} s_d, \tag{4}$$

where θ are all the parameters needed to compute a score s with an RNN, $\hat{y}$ is a parse for input x_i, and $N(\hat{y})$ is the set of nonterminal nodes.

2.3. Category Classifiers in the Tree. The main advantage of the algorithm is that each node of the tree built by the RNN has associated with it a distributed feature representation. To predict class labels, a simple softmax layer is added to each RNN parent node, as shown later:

$$\text{label}_p = \text{softmax}\left(W^{\text{label}} p\right). \tag{5}$$

When minimizing the cross-entropy error of this softmax layer, the error will backpropagate and influence the RNN parameters.

3. Nonlinear Particle Swarm Optimization for Training FNN

As for traditional RNN-based method, the objective J of (5) is not differentiable due to the hinge loss. For training RNN, Socher used L-BFGS over the complete training data to minimize the objective, where the iteration of the swarm relates to the update of the parameters of RNN. That being said, it is worth noting that the basic principle of L-BFGS is that the objective function should be differentiable. Since the objective function for RNN is nondifferentiable, L-BFGS could cause problems for computing the weights of RNN. To solve this problem, a novel nonlinear PSO (NPSO) has been used to tune the parameters of RNN.

3.1. Nonlinear Particle Swarm Optimization. As a population-based evolutionary algorithm, PSO is initialized with a population of candidate solutions. The activities of the population are guided by some behavior rules. For example, let $X_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{iD}(t))$ $(x_{id}(t) \in [-x_{d\max}, x_{d\max}])$ be the location of the ith particle in the tth generation, where $x_{d\max}$ is the boundary of the dth search space for a given problem and $d = 1, \ldots, D$. The location of the best fitness achieved so far by the ith particle is denoted as $p_i(t)$ and the index of the global best fitness by the whole population as $p_g(t)$. The velocity of ith particle is $V_i(t) = (v_{i1}(t), v_{i2}(t), \ldots, v_{iD}(t))$, where v_{id} is in $[-v_{d\max}, v_{d\max}]$ and $v_{d\max}$ is the maximal speed of dth dimension. The velocity and position update equations of the ith particle are given as follows:

$$\begin{aligned} v_{id}(t+1) &= w \cdot v_{id}(t) + c_1 r_1 \left(p_{id} - x_{id}(t)\right) \\ &\quad + c_2 r_2 \left(p_{gd} - x_{id}(t)\right), \\ x_{id}(t+1) &= x_{id}(t) + v_{id}(t+1), \end{aligned} \tag{6}$$

where $i = 1, \ldots, n$ and $d = 1, \ldots, D$. $w, c_1, c_2 \geq 0$. w is the inertia weight, c_1 and c_2 denote the acceleration coefficients, and r_1 and r_2 are random numbers, generated uniformly in the range $[0, 1]$.

Note that a suitable value for the inertia weight provides a balance between the global and local exploration abilities of the swarm. Based on the concept of decrease strategy, our nonlinear inertia weight strategy [19] chooses a lower value of w during the early iterations and maintains higher value

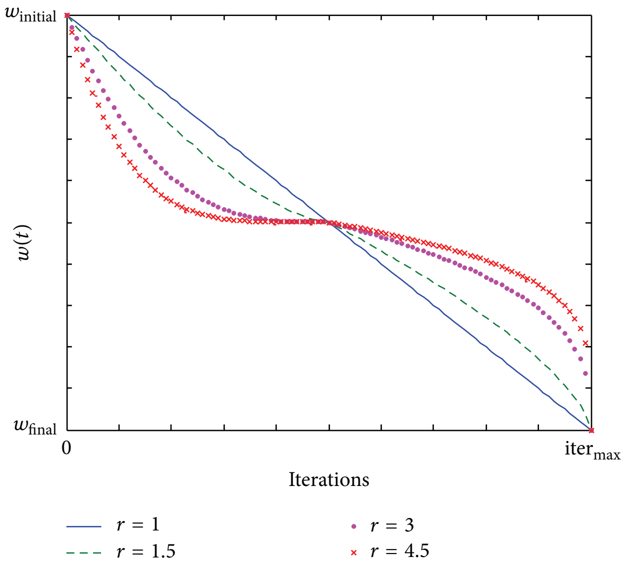


FIGURE 4: Nonlinear strategy of inertia weight.

of w than linear model [21]. This strategy enables particles to search the solution space more aggressively to look for "better areas", thus will avoid local optimum effectively.

The proposed update scheme of $w(t)$ is given as follows:

$$w(t) = \begin{cases} \left(1 - \dfrac{2t}{iter_{max}}\right)^{r} \dfrac{(w_{initial} + w_{final})}{2} \\ \quad + \dfrac{(w_{initial} - w_{final})}{2}, & t \leq \dfrac{iter_{max}}{2}, \\ \left(1 - \dfrac{2(t - (iter_{max}/2))}{iter_{max}}\right)^{1/r} \dfrac{(w_{initial} - w_{final})}{2} \\ \quad + w_{final}, & t > \dfrac{iter_{max}}{2}, \end{cases} \tag{7}$$

where $iter_{max}$ is the maximum number of iterations, t denotes the iteration generation, and $r > 1$ is the nonlinear modulation index.

Figure 4 illustrates the variations of nonlinear inertia weight for different values of r. Note that $r = 1$ is equal to the linear model. In [19], we showed that a choice of r within [2-3] is normally satisfactory.

3.2. Encoding Strategy and Fitness Evaluation. Let $\theta = (W^{sem}; W; W^{score}; W^{label})$ be the set of RNN parameters; then each particle can be the expressed as the combination of all parameters, as shown later:

W^{sem}	W	W^{score}	W^{label}

(8)

During the iteration, each particle relates to a combination of weights of neural networks. The goal is to minimize a fitness function, given as

$$J(\theta) = \frac{1}{N}\sum_{i=1}^{N} r_i(\theta) + \frac{\lambda}{2}\|\theta\|^2, \tag{9}$$

where $r_i(\theta) = s(\mathrm{RNN}(\theta, x_i, y^*)) + \Delta(x_i, l_i, y^*) - \max_{y_i \in Y(x_i, l_i)}(s(\mathrm{RNN}(\theta, x_i, y_i)))$ and y^* denote the parse tree generated by the greedy strategy according to parameter θ. Minimizing this objective means minimize the error between the parsing results, which is generated by the best particle and the labeled training images (ground truth).

3.3. Summary of PSO-Based Training Algorithm.

Input includes a set of labeled images, the size of the hidden layer n, the value of penalization term for incorrect parsing decisions κ, the regularization parameter λ, the population of particles m, the values of nonlinear parameter r and the number of iterations $iter_{max}$.

Output includes the set of model parameters $\theta = (W^{sem}, W, W^{score}$, and $W^{label})$, each with respect to weights of a recursive neural network.

(1) Randomly initialize m particles and randomize the positions and velocities for entire population. Record the global best location p_g of the population and the local best locations p_i of the ith particle according to (9), where $i = 1, 2, \ldots, m$.

(2) For each iteration, evaluate the fitness value of the ith particle through (9). If $(f(x_i)) < (f(p_i))$, set $p_i = x_i$ as the so far best position of the ith particle. If $(f(x_i)) < (f(p_g))$, set $p_g = x_i$ as the so far best position of the population.

(3) Calculate the inertia weight through (7). Update the position and velocity of particles according to (6).

(4) Repeat Step 2 and Step 3 until maximum number of generation.

(5) Compute the weights of RNN according to the best particle.

4. Experimental Results and Discussion

4.1. Description of the Experiments. In this section, PSO-based RNN method is compared with traditional RNN [17], pixel CRF [10], region-based energy [10], simultaneous MRF [8], and superpixel MRF [8], by using images from Stanford background dataset. All the experiments have been conducted on a computer with Intel sixteen-core processor 2.67 GHz processor and 32 GB RAM.

As for RNN, Socher recommends that the size of the hidden layer $n = 100$, the penalization term for incorrect parsing decisions $\kappa = 0.05$, and the regularization parameter $\lambda = 0.001$. As for the particle swarm optimization, we set

Figure 5: Typical results of multiclass image segmentation and pixel-wise labeling with PSO-based recursive neural networks.

the population of particles $m = 100$, the number of iterations $\text{iter}_{\max} = 500$, $c_1 = c_2 = 2$, $w_{\text{initial}} = 0.95$, $w_{\text{final}} = 0.4$, and $r = 2.5$.

4.2. Scene Annotation. The first experiment aims at evaluating the accuracy of scene annotation on the Stanford background dataset. Like [17], we run fivefold cross-validation and report pixel level accuracy in Table 1. Note that the traditional RNN model influences the leaf embeddings through backpropagation, while we use PSO to tune the weights of RNN. As for traditional RNN model, we label the superpixels by their most likely class based on the multinomial distribution from the softmax layer at the leaf nodes. One can see that in Table 1, our approach outperforms previous methods that report results on this data, which means that the PSO-based RNN constructs a promising strategy for scene annotation. Some typical parsing results are illustrated in Figure 5.

4.3. Scene Classification. As described in [17], the Stanford background dataset can be roughly categorized into three

Table 1: Accuracy of pixel accuracy of state-of-the-art methods on Stanford background dataset.

Method and semantic pixel accuracy in %
Pixel CRF, Gould et al. (2009) 74.3
Log. Regr. on superpixel features 75.9
Region-based energy, Gould et al. (2009) 76.4
Local labeling, Tighe and Lazebnik (2010) 76.9
Superpixel MRF, Tighe and Lazebnik (2010) 77.5
Simultaneous MRF, Tighe and Lazebnik (2010) 77.5
Traditional RNN, Socher and Fei-Fei (2011) 78.1
PSO-based RNN (our method) 78.3

scene types: city, countryside, and sea side. Therefore, like traditional RNN, we trained SVM that using the average over all nodes' activations in the tree as features. That means the entire parse tree and the learned feature representations of the RNN are taken into account. As a result, the accuracy has been promoted to 88.4%, which is better than traditional RNN (88.1%) and Gist descriptors (84%) [6]. If only the top node of the scene parse tree is considered, we will get 72%. The results reveal that it does lose some information that is captured by averaging all tree nodes.

5. Conclusions

In this paper, we have proposed an image parsing algorithm that is based on PSO and Recursive Neural Networks (RNNs). The algorithm is an incremental version of RNN. The basic idea is to solve the problem of nondifferentiable objective function of traditional training algorithm such as L-BFGS. Hence, PSO has been employed as an optimization tool to tune the weights of RNN. The experimental results reveal that the proposed algorithm has better performance than state-of-the-art methods on Stanford background dataset. That being said, the iteration of swarms dramatically increases the runtime of the training process. Our future work may focus on reducing the time complexity of the algorithm.

Acknowledgments

This work was supported by the National Nature Science Foundation of China (No. 61202143, No. 61103052, and No. 11101187); Major Project of Industry University of Fujian Province (2011H6020); Doctoral Program Foundation of Institutions of Higher Education of China (No. 20090121110032); Shenzhen Science and Technology Research Foundation (Nos. JC200903180630A, ZYB200907110169A); The Natural Science Foundation of Xiamen City, China (3502Z20123022).

References

[1] C. Farabet, C. Couprie, L. Najman, and Y. LeCun, "Learning hierarchical features for scene labeling," *IEEE Transactions on Pattern Analysis and Machine Learning*, no. 99, pp. 1–15, 2012.

[2] C. Liu, J. Yuen, and A. Torralba, "Nonparametric scene parsing via label transfer," *IEEE Transactions on Pattern Analysis and Machine Learning*, vol. 33, no. 12, pp. 2368–2382, 2011.

[3] Z. Tu and S. C. Zhu, "Parsing images into regions, curves, and curve groups," *International Journal of Computer Vision*, vol. 69, no. 2, pp. 223–249, 2006.

[4] J. Porway, Q. Wang, and S. C. Zhu, "A hierarchical and contextual model for aerial image parsing," *International Journal of Computer Vision*, vol. 88, no. 2, pp. 254–283, 2010.

[5] O. Teboul, L. Kokkinos, L. Simon, P. Koutsourakis, and N. Paragios, "Parsing facades with shape grammars and reinforcement learning," *IEEE Transactions on Pattern Analysis and Machine Learning*. In press.

[6] A. Oliva and A. Torralba, "Modeling the shape of the scene: a holistic representation of the spatial envelope," *International Journal of Computer Vision*, vol. 42, no. 3, pp. 145–175, 2001.

[7] S. Nowozin, P. V. Gehler, and C. H. Lampert, "On parameter learning in CRF-based approaches to object class image segmentation," *Lecture Notes in Computer Science*, vol. 6316, no. 6, pp. 98–111, 2010.

[8] J. Tighe and S. Lazebnik, "Superparsing: scalable nonparametric image parsing with superpixels," *Lecture Notes in Computer Science*, vol. 6315, no. 5, pp. 352–365, 2010.

[9] X. He and R. Zemel, "Learning hybrid models for image annotation with partially labeled data," in *Proceeding of Advances in Neural Information Processing Systems 21, Proceedings of the 22nd Annual Conference on Neural Information Processing Systems*, pp. 625–632, Vancouver, British Columbia, Canada, December 2008.

[10] S. Gould, R. Fulton, and D. Koller, "Decomposing a scene into geometric and semantically consistent regions," in *Proceedings of the 12th International Conference on Computer Vision (ICCV '09)*, pp. 1–8, Kyoto, Japan, October 2009.

[11] L. Zhu, Y. Chen, Y. Lin, C. Lin, and A. Yuille, "Recursive segmentation and recognition templates for image parsing," *IEEE Transactions on Pattern Analysis and Machine Learning*, vol. 34, no. 2, pp. 359–371, 2012.

[12] L. Zhu, Y. Chen, and A. Yuille, "Learning a hierarchical deformable template for rapid deformable object parsing," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 32, no. 6, pp. 1029–1043, 2010.

[13] F. Han and S. C. Zhu, "Bottom-up/top-down image parsing with attribute grammar," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 31, no. 1, pp. 59–73, 2009.

[14] L. Ladický, P. Sturgess, C. Russell et al., "Joint optimization for object class segmentation and dense stereo reconstruction," *International Journal of Computer Vision*, vol. 100, no. 2, pp. 122–133, 2012.

[15] R. Socher and L. Fei-Fei, "Connecting modalities: semi-supervised segmentation and annotation of images using unaligned text corpora," in *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR '10)*, pp. 966–973, San Francisco, Calif, USA, June 2010.

[16] Z. K. Huang, C. H. Feng, and S. Mohamad, "Multistability analysis for a general class of delayed Cohen-Grossberg neural networks," *Information Sciences*, vol. 187, pp. 233–244, 2012.

[17] R. Socher, C. C. Lin, A. Y. Ng, and C. D. Manning, "Parsing natural scenes and natural language with recursive neural networks," in *Proceedings of the International Conference on Machine Learning*, pp. 129–136, 2011.

[18] J. Kennedy and R. Eberhart, "Particle Swarm Optimization," in *Proceedings of the IEEE International Conference on Neural Networks*, pp. 1942–1948, Perth, Australia, December 1995.

[19] G. R. Cai, S. Z. Li, S. L. Chen, and Y. D. Wu, "A fuzzy neural network model of linguistic dynamic systems based on computing with words," *Journal of Donghua University*, vol. 27, no. 6, pp. 813–818, 2010.

[20] D. Comaniciu and P. Meer, "Mean shift: a robust approach toward feature space analysis," *IEEE Transactions on Pattern Analysis and Machine Learning*, vol. 24, no. 5, pp. 603–619, 2002.

[21] Y. Shi and R. Eberhart, "Parameter selection in particle swarm optimization," in *Proceeding of the 7th International Conference on Evolutionary Programming VII (EP '89)*, pp. 591–600, 1998.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 508247, 9 pages
http://dx.doi.org/10.1155/2013/508247

Research Article

Subharmonics with Minimal Periods for Convex Discrete Hamiltonian Systems

Honghua Bin

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Honghua Bin; hhbin@jmu.edu.cn

Received 19 January 2013; Accepted 24 February 2013

Academic Editor: Zhengkun Huang

We consider the subharmonics with minimal periods for convex discrete Hamiltonian systems. By using variational methods and dual functional, we obtain that the system has a pT-periodic solution for each positive integer p, and solution of system has minimal period pT as H subquadratic growth both at 0 and infinity.

1. Introduction

Consider Hamiltonian systems

$$J\dot{u}(t) + \nabla H(t, u(t)) = 0, \qquad u(0) = u(pT), \tag{1}$$

where $u(t) \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$, ∇H stands for the gradient of H with respect to the second variable, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the symplectic matrix with I_N the identity in $\mathbb{R}^N$. Moreover, H is T-periodic in the variable t, $p \in \mathbb{N} \setminus \{0\}$.

Classically, solutions for systems (1) are called subharmonics. The first result concerning the subharmonics problem traced back to Birkhoff and Lewis in 1933 (refer to [1]), in which there exists a sequence of subharmonics with arbitrarily large minimal period, using perturbation techniques. More results can be found in [1–5], where H is convex with subquadratic growth both at 0 and infinity. Using Z_p index theory and Clarke duality, Xu and Guo [1, 5] proved that the number of solutions for systems (1) with minimal period pT tends towards infinity as $p \to \infty$.

For periodic and subharmonic solutions for discrete Hamiltonian systems, Guo and Yu [6, 7] obtained some existence results by means of critical point theory, where they introduced the action functional

$$F(u) = -\frac{1}{2}\sum_{n=1}^{pT}(J\Delta Lu(n-1), u(n)) - \sum_{n=1}^{pT} H(n, Lu(n)). \tag{2}$$

Using Clarke duality, periodic solution of convex discrete Hamiltonian systems with forcing terms has been investigated in [8]. Clarke duality was introduced in 1978 by Clarke [9], and developed by Clarke, Ekeland, and others, see [10–12]. This approach is different from the direct method of variations; some scholars applied it to consider the periodic solutions, subharmonic solutions with prescribed minimal period of Hamiltonian systems; one can refer to [3, 5, 12–14] and references therein. The dynamical behavior of differential and difference equations was studied by using various methods; see [15–19]. We refer the reader to Agarwal [20] for a broad introduction to difference equations.

Motivated by the works of Mawhin and Willem [12] and Xu and Guo [21], we use variational methods and Clarke duality to consider the subharmonics with minimal periods for discrete Hamiltonian systems

$$J\Delta u(n) + \nabla H(n, Lu(n)) = 0, \quad u(n) = u(n+pT), \tag{3}$$

where $u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix}$, $Lu(n) = \begin{pmatrix} u_1(n+1) \\ u_2(n) \end{pmatrix}$, $u_i(n) \in \mathbb{R}^N$ $(i = 1, 2)$ with N a given positive integer, and $\Delta u(n) = u(n+1) - u(n)$ is the forward difference operator. $p, T \in \mathbb{N} \setminus \{0\}$. Moreover, hamiltonian function H satisfies the following assumption:

(A1) $H : \mathbb{Z} \times \mathbb{R}^{2N} \to \mathbb{R}$ is continuous differentiable on $\mathbb{R}^{2N}$, $H(n, \cdot)$ convex for each $n \in \mathbb{Z}$ and $H(n+T, u) = H(n, u)$ for each $u \in \mathbb{R}^{2N}$;

(A2) there exist constants $a_1 > 0$, $a_2 > 0$, $1 < \theta < 2$, such that

$$\frac{a_1}{\theta}|u|^\theta \le H(n,u) \le \frac{a_2}{\theta}|u|^\theta, \quad u \in \mathbb{R}^{2N}, \tag{4}$$

which implies H subquadratic growth both at 0 and infinity.

Theorem 1. *Assume (A1) holds. $H(n,u) \to +\infty$, $H(n,u)/|u|^2 \to 0$, as $|u| \to \infty$ uniformly in $n \in \mathbb{Z}$. Then there exists a pT-periodic solution u_p of (3), such that $\|u_p\|_\infty \triangleq \max_{n\in Z[1,pT]}\{|u_p(n)|\} \to \infty$, and the minimal period T_p of u_p tends to $+\infty$ as $p \to \infty$.*

Theorem 2. *Under the assumptions (A1) and (A2), if*

$$\frac{a_2}{a_1} \le \begin{cases} \left(\dfrac{1}{4}\sin\dfrac{\pi}{pT}\right)^{\theta/2}, & \text{when } pT \text{ is even}, \\ \left(\dfrac{1}{2}\sin\dfrac{\pi}{2pT}\right)^{\theta/2}, & \text{when } pT \text{ is odd} \end{cases} \tag{5}$$

for given integer $p > 1$, then the solution of (3) has minimal period pT.

2. Clarke Duality and Eigenvalue Problem

First we introduce a space E_{pT} with dimension $2NpT$ as follows:

$$\begin{aligned} E_{pT} = \{u = \{u(n)\} \in S \mid u(n+pT) \\ = u(n), p \in \mathbb{N}\setminus\{0\}, n \in \mathbb{Z}\}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} S = \Bigg\{u = \{u(n)\} \mid u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix} \in \mathbb{R}^{2N}, \\ u_j(n) \in \mathbb{R}^N, j = 1,2, n \in \mathbb{Z}\Bigg\}. \end{aligned} \tag{7}$$

Equipped with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ in E_{pT} as

$$\begin{aligned} \langle u,v\rangle &= \sum_{n=1}^{pT}(u(n),v(n)), \\ \|u\| &= \left(\sum_{n=1}^{pT}|u(n)|^2\right)^{1/2}, \quad \forall u,v \in E_{pT}, \end{aligned} \tag{8}$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual scalar product and corresponding norm in $\mathbb{R}^{2N}$, respectively. It is easy to see that $(E_{pT},\langle\cdot,\cdot\rangle)$ is a Hilbert space with $2NpT$ dimension, which can be identified with $\mathbb{R}^{2NpT}$. Then for any $u \in E_{pT}$, it can be written as $u = (u^T(1), u^T(2), \dots, u^T(pT))^T$, where $u(j) = \binom{u_1(j)}{u_2(j)} \in \mathbb{R}^{2N}$, $j \in Z[1,pT]$, the discrete interval $\{1,2,\dots,pT\}$ is denoted by $Z[1,pT]$, and $\cdot^T$ denotes the transpose of a vector or a matrix.

Denote the subspace $\overline{Y} = \{u \in E_{pT} \mid u(1) = u(2) = \cdots = u(pT) \in \mathbb{R}^{2N}\}$. Let Y be the direct orthogonal complement of E_{pT} to $\overline{Y}$. Thus E_{pT} can be split as $E_{pT} = Y \oplus \overline{Y}$, and for any $u \in E_{pT}$, it can be expressed in the form $u = \tilde{u} + \overline{u}$, where $\tilde{u} \in Y$, $\overline{u} \in \overline{Y}$.

Next we recall Clarke duality and some lemmas.

The Legendre transform (see [12]) $H^*(t,\cdot)$ of $H(t,\cdot)$ with respect to the second variable is defined by

$$H^*(t,v) = \sup_{u\in R^{2N}}\{(v,u) - H(t,u)\}, \tag{9}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{2N}$. It follows from (A1) and (A2) that

(B1) $H^*(n,\cdot)$ is continuous differentiable on $\mathbb{R}^{2N}$,

(B2) for $\tau = \theta/(\theta-1)$, $v \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, one has

$$\frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}|v|^\tau \le H^*(n,v) \le \frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}|v|^\tau. \tag{10}$$

Associated with variational functional (2), the dual action functional is defined by

$$\begin{aligned} \chi(v) = \sum_{n=1}^{pT}\frac{1}{2}(L(J\Delta v(n-1)), v(n)) \\ + \sum_{n=1}^{pT}H^*(n,\Delta v(n)), \quad v \in E_{pT}. \end{aligned} \tag{11}$$

Indeed, by (11), we have $\chi(v+\overline{u}) = \chi(v)$ for any $\overline{u} \in \overline{Y}$ and $v \in Y$. Therefore, χ can be restricted in subspace Y of E_{pT}. Moreover, in terms of Lemma 2.6 in [8] and the following lemma, the critical points of (11) correspond to the subharmonic solutions of (3).

Lemma 3 (see [8, Theorem 1]). *Assume that*

(H1) *$H(n,\cdot) \in C^1(\mathbb{R}^{2N},\mathbb{R})$, $H(n,\cdot)$ is convex in the second variable for $n \in \mathbb{Z}$,*

(H2) *there exists $\beta : \mathbb{Z} \to \mathbb{R}^{2N}$ such that for all $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$, $H(n,u) \ge (\beta(n),u)$, and $\beta(n+T) = \beta(n)$,*

(H3) *there exist $\alpha \in (0, 2\sin(\pi/pT))$ and $\gamma : \mathbb{Z} \to \mathbb{R}^+$, such that for any $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$, $H(n,u) \le (\alpha/2)|u|^2 + \gamma(n)$, and $\gamma(n+T) = \gamma(n)$,*

(H4) *for each $u \in \mathbb{R}^{2N}$, $\sum_{n=1}^{pT} H(n,u) \to +\infty$ as $|u| \to \infty$.*

Then system (3) has at least one periodic solution u, such that $v = -J[u - (1/pT)\sum_{n=1}^{pT} u(n)]$ minimizes the dual action $\chi(v) = \sum_{n=1}^{pT}(1/2)(LJ\Delta v(n-1), v(n)) + \sum_{n=1}^{pT} H^*(n,\Delta v(n))$.

The following lemmas will be useful in our proofs, where Lemma 4 can be proved by means of Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, and Lemma 5 is a Hölder inequality.

Lemma 4. *For any $k \in \mathbb{Z}$, $\sum_{n=1}^{pT}\sin((2k\pi/pT)n) = \sum_{n=1}^{pT}\cos((2k\pi/pT)n) = 0$.*

Lemma 5. *For any $u_j > 0$, $v_j > 0$, $k \in \mathbb{Z}$, one has $\sum_{j=1}^{k} u_j v_j \le (\sum_{j=1}^{k} u_j^p)^{1/p}(\sum_{j=1}^{k} v_j^q)^{1/q}$, where $p > 1$, $q > 1$ and $1/p + 1/q = 1$.*

Lemma 6 (see [12, proposition 2.2]). *Let $H : \mathbb{R}^m \to \mathbb{R}$ be of C^1 and convex functional, $-\beta \le H(u) \le \alpha q^{-1}|u|^q + \gamma$, where $u \in \mathbb{R}^m$, $\alpha > 0$, $q > 1$, $\beta \ge 0$, $\gamma \ge 0$. Then $\alpha^{-p/q} p^{-1} |\nabla H(u)|^p \le (\nabla H(u), u) + \beta + \gamma$, where $1/p + 1/q = 1$.*

In order to know the form of $u \in E_{pT}$, we consider eigenvalue problem

$$LJ\Delta u(n-1) = \lambda u(n), \qquad u(n+pT) = u(n), \tag{12}$$

where $u(n) = \binom{u_1(n)}{u_2(n)}$, $Lu(n-1) = \binom{u_1(n)}{u_2(n-1)} \in \mathbb{R}^{2N}$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. We can rewrite (12) as the following form:

$$\begin{gathered} u_1(n+1) = (1-\lambda^2) u_1(n) + \lambda u_2(n), \\ u_2(n+1) = -\lambda u_1(n) + u_2(n), \\ u_1(n+pT) = u_1(n), \qquad u_2(n+pT) = u_2(n). \end{gathered} \tag{13}$$

Denoting

$$M(\lambda) = \begin{pmatrix} (1-\lambda^2) I_N & \lambda I_N \\ -\lambda I_N & I_N \end{pmatrix}, \tag{14}$$

then problem (12) is equivalent to

$$u(n+1) = M(\lambda) u(n), \qquad u(n+pT) = u(n). \tag{15}$$

Letting $u(n) = \mu^n c$ be the solution of (15), for some $c \in \mathbb{C}^{2N}$, we have $\mu c = M(\lambda)c$ and $\mu^{pT} = 1$. Consider the polynomial $|M(\lambda) - \mu I_{2N}| = 0$ and condition $\mu^{pT} = 1$; it follows that

$$\begin{gathered} \mu = e^{2k\pi i/pT}, \qquad \lambda = 2\sin\frac{k\pi}{pT}, \\ k \in Z[-pT+1, pT-1]. \end{gathered} \tag{16}$$

In the following we denote by $\mu_k = e^{2k\pi i/pT}$, $\lambda_k = 2\sin(k\pi/pT)$, $k \in Z[-pT+1, pT-1]$, and $\rho \in \mathbb{R}^N$. By $(M(\lambda_k) - \mu_k I_{2N})c = 0$, it follows that

$$c_k = \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)}\rho \end{pmatrix}. \tag{17}$$

Thus

$$\begin{aligned} u_k(n) &= \mu_k^n c_k = e^{2k\pi n i/pT} \begin{pmatrix} \rho \\ ie^{(-k\pi i/pT)}\rho \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix} \\ &\quad + i \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{aligned} \tag{18}$$

Let

$$\begin{aligned} \xi_k(n) &= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right)\rho \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}, \\ \eta_k &= \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right)\rho \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}. \end{aligned} \tag{19}$$

Obviously, $\xi_k(n)$ and $\eta_k(n)$ satisfy (15). Moreover $LJ\Delta\xi_k(n-1) = \lambda_k \xi_k(n)$, $LJ\Delta\eta_k(n-1) = \lambda_k \eta_k(n)$, $\xi_{2pT+k}(n) = \xi_k(n)$, $\eta_{2pT+k}(n) = \eta_k(n)$, $\xi_{pT-k}(n) = \xi_k(n)$, $\eta_{pT-k}(n) = -\eta_k(n)$.

For $k \ne pT/2$, subspace Y_k is defined by

$$Y_k = \begin{cases} \operatorname{span}\{\xi_k(n), \eta_{k+(pT/2)}(n)\}, & k \in Z\left[-\frac{pT}{2}+1, \frac{pT}{2}-1\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is even}, \\ \operatorname{span}\{\xi_k(n), \eta_{k+((pT+1)/2)}(n)\}, & k \in Z\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \\ & n \in \mathbb{Z}, \text{ if } pT \text{ is odd}, \end{cases} \tag{20}$$

where $[\cdot]$ denotes the greatest-integer function and

$$\begin{gathered} Y_{pT/2} = \operatorname{span}\{\xi_{pT/2}(n), n \in \mathbb{Z}\}, \\ Y_{-pT/2} = \operatorname{span}\{\xi_{-pT/2}(n), n \in \mathbb{Z}\}. \end{gathered} \tag{21}$$

Therefore,

$$\begin{gathered} Y = \oplus Y_k, \quad k \in Z\left[-\frac{pT}{2}, \frac{pT}{2}\right] \setminus \{0\}, \text{ if } pT \text{ is even}, \\ Y = \oplus Y_k, \quad k \in Z\left[\left[-\frac{pT}{2}\right], \left[\frac{pT}{2}\right]\right] \setminus \{0\}, \text{ if } pT \text{ is odd}. \end{gathered} \tag{22}$$

Moreover, for any $u = \{u(n)\} \in E_{pT}$, we may express $u(n)$ as

$$\begin{aligned} u(n) &= \sum_{k=-pT+1}^{pT-1} \left[\begin{pmatrix} \cos\left(\frac{2k\pi}{pT} n\right) a_k \\ -\sin\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right) a_k \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \sin\left(\frac{2k\pi}{pT} n\right) b_k \\ \cos\left(\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\right) b_k \end{pmatrix} \right], \end{aligned} \tag{23}$$

where $a_k, b_k \in \mathbb{R}^N$.

Since $(\Delta u(n), \Delta u(n)) = -(\Delta^2 u(n-1), u(n))$, we consider eigenvalue problem

$$-\Delta^2 u(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad u(n) \in \mathbb{R}^N, \tag{24}$$

where $\Delta^2 u(n-1) = \Delta u(n) - \Delta u(n-1) = u(n+1) - 2u(n) + u(n-1)$. The second order difference equation (24) has complexity solution $u(n) = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/pT$. Moreover, $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1-\cos\theta) = 4\sin^2(\theta/2)$; that is, $\lambda = 4\sin^2(k\pi/pT)$, $k \in Z[0, pT-1]$.

By the previous, it follows Lemma 7.

Lemma 7. *For any $u \in E_{pT}$, one has $-\lambda_{\max}\|u\|^2 \le \sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n)) \le \lambda_{\max}\|u\|^2$, and $0 \le \sum_{n=1}^{pT}|\Delta u(n)|^2 \le \lambda_{\max}^2\|u\|^2$, where*

$$\begin{aligned}\lambda_{\max} &= \max_{k\in[0,pT-1]}\left\{2\sin\frac{k\pi}{pT}\right\}\\ &= \begin{cases}2, & \text{if } pT \text{ is even},\\ 2\cos\dfrac{\pi}{2pT}, & \text{if } pT \text{ is odd}.\end{cases}\end{aligned} \tag{25}$$

Moreover, if $u \in Y$, then $4\sin^2(\pi/pT)\|u\|^2 \le \sum_{n=1}^{pT}|\Delta u(n)|^2 \le \lambda_{\max}^2\|u\|^2$.

3. Proofs of Main Results

Lemma 8. *Consider*

$$\begin{aligned}&\sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n))\\ &\quad\ge -\left(2\sin\frac{\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta u(n)|^2, \quad \forall u \in E_{pT}.\end{aligned} \tag{26}$$

Proof. Letting $\tilde{u}(n) = u(n) - (1/pT)\sum_{n=1}^{pT}u(n)$, then $\tilde{u} \in Y$. By Lemmas 5 and 7, we have

$$\begin{aligned}&\sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n))\\ &\quad= \sum_{n=1}^{pT}(LJ\Delta u(n-1), \tilde{u}(n))\\ &\quad\ge -\left(\sum_{n=1}^{pT}|LJ\Delta u(n-1)|^2\right)^{1/2}\\ &\qquad\cdot\left(\sum_{n=1}^{pT}|\tilde{u}(n)|^2\right)^{1/2}\\ &\quad\ge -\left(\sum_{n=1}^{pT}|\Delta u(n)|^2\right)^{1/2}\\ &\qquad\cdot\left(2\sin\frac{\pi}{pT}\right)^{-1}\left(\sum_{n=1}^{pT}|\Delta\tilde{u}(n)|^2\right)^{1/2}\\ &\quad= -\left(2\sin\frac{\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta u(n)|^2.\end{aligned} \tag{27}$$

□

Lemma 9. *If there exist $\alpha \in (0, \sin(\pi/pT))$, $\beta \ge 0$ and $\delta > 0$, such that*

$$\delta|u| - \beta \le H(n,u) \le \frac{\alpha}{2}|u|^2 + \gamma \tag{28}$$

for all $n \in [1, pT]$ and $u \in \mathbb{R}^{2N}$, then each solution of (3) satisfies the inequalities

$$\begin{aligned}\sum_{n=1}^{pT}|\Delta u(n)|^2 &\le \frac{2\alpha(\beta+\gamma)pT\sin(\pi/pT)}{\sin(\pi/pT)-\alpha},\\ \sum_{n=1}^{pT}|Lu(n)| &\le \frac{(\beta+\gamma)pT\sin(\pi/pT)}{\delta(\sin(\pi/pT)-\alpha)}.\end{aligned} \tag{29}$$

Proof. Let u be the solution of (3). By Lemma 6, we have

$$\begin{aligned}\frac{1}{2\alpha}|\nabla H(n, Lu(n))|^2 &\le (\nabla H(n, Lu(n)), Lu(n)) + \beta + \gamma\\ &= -(J\Delta u(n), Lu(n)) + \beta + \gamma.\end{aligned} \tag{30}$$

Obviously, $|J\Delta u(n)|^2 = (-\nabla H(n, Lu(n)), J\Delta u(n)) = |\nabla H(n, Lu(n))|^2$ by (3), and it follows that $(1/2\alpha)\sum_{n=1}^{pT}|J\Delta u(n)|^2 + \sum_{n=1}^{pT}(J\Delta u(n), Lu(n)) \le (\beta+\gamma)pT$; that is,

$$\begin{aligned}&\frac{1}{2\alpha}\sum_{n=1}^{pT}|\Delta u(n)|^2 + \sum_{n=1}^{pT}(LJ\Delta u(n-1), u(n))\\ &\quad\le (\beta+\gamma)pT.\end{aligned} \tag{31}$$

By means of Lemma 8, we have

$$\left[\frac{1}{2\alpha} - \left(2\sin\frac{\pi}{pT}\right)^{-1}\right]\sum_{n=1}^{pT}|\Delta u(n)|^2 \le (\beta+\gamma)pT, \tag{32}$$

which gives first conclusion.

Now, $H(n,0) \le \gamma$ in view of (28); therefore by convex and Lemma 8, we have

$$\begin{aligned}&\delta\sum_{n=1}^{pT}|Lu(n)| - \beta pT\\ &\quad\le \sum_{n=1}^{pT}H(n, Lu(n))\\ &\quad\le \sum_{n=1}^{pT}[H(n,0) + (\nabla H(n, Lu(n)), Lu(n))]\end{aligned}$$

$$\begin{aligned}
&\le \gamma pT - \sum_{n=1}^{pT} (J\Delta u(n), Lu(n)) \\
&= \gamma pT - \sum_{n=1}^{pT} (JL\Delta u(n-1), u(n)) \\
&\le \gamma pT + \left(2\sin\frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2 \\
&\le \gamma pT + \frac{\alpha(\beta+\gamma)pT}{\sin(\pi/pT)-\alpha},
\end{aligned} \tag{33}$$

which gives the second conclusion. The proof is completed. □

Proof of Theorem 1. Let $c_1 = \max_{n\in\mathbb{Z}}|H(n,0)|$. By assumption in Theorem 1, there exists $R > 0$, such that $H(n,u) \ge 1+c_1$, for $n \in \mathbb{Z}$ and $|u| \ge R$. Moreover, there exist $\alpha \in (0, 2\sin(\pi/pT))$, $\gamma > 0$ such that

$$H(n,u) \le \frac{\alpha}{2}|u|^2 + \gamma, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}. \tag{34}$$

Thus, by convex of H, for all $(n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}$ with $|u| \ge R$, we have

$$\begin{aligned}
1 + c_1 &\le H\left(n, \frac{R}{|u|}u\right) \\
&\le H(n,0) + \frac{R}{|u|}(H(n,u) - H(n,0)) \\
&\le \frac{R}{|u|}H(n,u) + c_1.
\end{aligned} \tag{35}$$

Therefore there exist $\beta > 0$ and $\delta > 0$, such that

$$H(n,u) \ge \delta|u| - \beta, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}. \tag{36}$$

Combining the previous argument, by Lemma 3, the system (3) has a pT-periodic solution u_p such that $v_p = -J[u_p - (1/pT)\sum_{n=1}^{pT} u_p(n)] \in Y$ minimizes the dual action

$$\begin{aligned}
\chi_p(v_p) = &\sum_{n=1}^{pT}\frac{1}{2}\left(LJ\Delta v_p(n-1), v_p(n)\right) \\
&+ \sum_{n=1}^{pT} H^*\left(n, \Delta v_p(n)\right) \quad \text{on } E_{pT}.
\end{aligned} \tag{37}$$

It follows that $\Delta u_p(n) = J\Delta v_p(n)$ and $Jv_p(n) = u_p(n) - (1/pT)\sum_{n=1}^{pT} u_p(n)$.

We next prove that $\|u_p\|_\infty \to \infty$ as $p \to \infty$.

Suppose not, there exist $c_2 > 0$ and a subsequence $\{p_k\}$ such that

$$p_k \to \infty, \qquad \|u_{p_k}\|_\infty \le c_2 \quad \text{as } k \longrightarrow \infty. \tag{38}$$

In terms of (3), it follows that $\|\Delta u_{p_k}\|_\infty \le c_3$ for some $c_3 > 0$, and $\|v_{p_k}\|_\infty \le 2c_2$, $\|\Delta v_{p_k}\|_\infty \le c_3$. Consequently, by $H^*(n,v) \ge -H(n,0) \ge -c_1$, we have

$$\begin{aligned}
c_{p_k} &= \chi_{p_k}(v_{p_k}) \\
&= \sum_{n=1}^{p_kT}\frac{1}{2}\left(LJ\Delta v_{p_k}(n-1), v_{p_k}(n)\right) \\
&\quad + \sum_{n=1}^{p_kT} H^*\left(n, \Delta v_{p_k}(n)\right) \\
&\ge -\frac{1}{2}\sum_{n=1}^{p_kT}\left|LJ\Delta v_{p_k}(n-1)\right|\left|v_{p_k}(n)\right| - c_1 p_k T \\
&\ge -\left(\sqrt{2}c_2c_3 + c_1\right)p_kT,
\end{aligned} \tag{39}$$

where $n \in Z[1, p_kT]$ and

$$\begin{aligned}
\left|LJ\Delta v_{p_k}(n-1)\right| &= \left(\left|\Delta v_{2,p_k}(n)\right|^2 + \left|\Delta v_{1,p_k}(n-1)\right|^2\right)^{1/2} \\
&\le \sqrt{2}\left\|\Delta v_{p_k}\right\|_\infty \le \sqrt{2}c_3.
\end{aligned} \tag{40}$$

By (36), if $|v| \le \delta$, we have $(v,u) - H(n,u) \le (v,u) - \delta|u| + \beta \le \beta$, and $H^*(n,v) \le \beta$. Letting $\rho \in \mathbb{R}^N$ and $|\rho| = 1$, in terms of (12), h_p associated with $\lambda_{-1} = -2\sin(\pi/pT)$ is given by

$$\begin{aligned}
h_p(n) = &\frac{\delta}{4\sin(\pi/pT)} \\
&\cdot \begin{pmatrix} \left(\cos\frac{2\pi}{pT}n - \sin\frac{2\pi}{pT}n\right)\rho \\ \left(\sin\frac{2\pi}{pT}\left(n-\frac{1}{2}\right) + \cos\frac{2\pi}{pT}\left(n-\frac{1}{2}\right)\right)\rho \end{pmatrix}
\end{aligned} \tag{41}$$

which belongs to E_{pT}, and

$$\begin{aligned}
&\left|\Delta h_p(n)\right|^2 \\
&= \left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 \\
&\quad \cdot \left|2\sin\frac{\pi}{pT}\begin{pmatrix} \left(-\sin\frac{2\pi}{pT}\left(n+\frac{1}{2}\right) - \cos\frac{2\pi}{pT}\left(n+\frac{1}{2}\right)\right)\rho \\ \left(\cos\frac{2\pi}{pT}n - \sin\frac{2\pi}{pT}n\right)\rho \end{pmatrix}\right|^2 \\
&= \frac{1}{4}\left[2 + \sin\frac{2\pi}{pT}(2n+1) - \sin\frac{2\pi}{pT}(2n)\right]\cdot|\rho|^2\delta^2 \\
&\le \delta^2.
\end{aligned} \tag{42}$$

Moreover, by Lemma 4 we have

$$\begin{aligned}
&\sum_{n=1}^{pT}\left|h_p(n)\right|^2 \\
&=\sum_{n=1}^{pT}\left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 \\
&\quad\cdot\left(2+\sin\frac{2\pi}{pT}(2n-1)-\sin\frac{2\pi}{pT}(2n)\right)|\rho|^2 \\
&=\left(\frac{\delta}{4\sin(\pi/pT)}\right)^2 2|\rho|^2 pT=\frac{\delta^2 pT}{8\sin^2(\pi/pT)}.
\end{aligned}\tag{43}$$

Thus $c_p = \chi_p(h_p) \le \sum_{n=1}^{pT}(1/2)(LJ\Delta h_p(n-1), h_p(n)) + \beta pT = \sum_{n=1}^{pT}(1/2)(-2\sin(\pi/pT))|h_p(n)|^2 + \beta pT = -\delta^2 pT/8\sin(\pi/pT) + \beta pT$. Combining (39), we have $8(\sqrt{2}c_2c_3 + c_1 + \beta_1)\sin(\pi/p_kT) \ge \delta^2$, which is impossible as k large. So the claim $\lim_{p\to\infty}\|u_p\|_\infty = \infty$ is valid.

It remains only to prove that the minimal period T_p of u_p tends to $+\infty$ as $p \to \infty$.

If not, there exists $T > 0$ and a sequence $\{p_k\}$ such that the minimal period T_{p_k} of u_{p_k} satisfies $1 \le T_{p_k} \le T$. By assumption in Theorem 1, there exists $\alpha \in (0, \sin(\pi/T))$ and $\gamma > 0$ such that

$$H(n,u) \le \frac{\alpha}{2}|u|^2 + \gamma, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^{2N}. \tag{44}$$

By (36) and Lemma 9 with pT replaced by T_{p_k}, we get

$$\begin{aligned}
\sum_{n=1}^{T_{p_k}}\left|\Delta u_{p_k}(n)\right|^2 &\le \frac{2\alpha(\beta+\gamma)T_{p_k}\sin\left(\pi/T_{p_k}\right)}{\sin\left(\pi/T_{p_k}\right)-\alpha} \\
&\le \frac{2\alpha(\beta+\gamma)T\sin(\pi/T)}{\sin(\pi/T)-\alpha},
\end{aligned}\tag{45}$$

$$\begin{aligned}
\sum_{n=1}^{T_{p_k}}\left|Lu_{p_k}(n)\right| &\le \frac{(\beta+\gamma)T_{p_k}\sin\left(\pi/T_{p_k}\right)}{\delta\left(\sin\left(\pi/T_{p_k}\right)-\alpha\right)} \\
&\le \frac{(\beta+\gamma)T_{p_k}\sin(\pi/T)}{\delta(\sin(\pi/T)-\alpha)}.
\end{aligned}\tag{46}$$

Write $u_{p_k} = \tilde{u}_{p_k} + \overline{u}_{p_k}$, where $\overline{u}_{p_k} = (1/T_{p_k})\sum_{n=1}^{T_{p_k}} u_{p_k}(n) = (1/T_{p_k})\sum_{n=1}^{T_{p_k}} Lu_{p_k}(n) \in \overline{Y}$. Inequality (46) implies that

$$\begin{aligned}
\left\|\overline{u}_{p_k}\right\|_\infty &\triangleq \max_{n\in Z[1,T_{p_k}]}\left\{\left|\overline{u}_{p_k}\right|\right\} \\
&\le \frac{1}{T_{p_k}}\sum_{n=1}^{T_{p_k}}\left|Lu_{p_k}(n)\right| \le \frac{(\beta+\gamma)\sin(\pi/T)}{\delta(\sin(\pi/T)-\alpha)}.
\end{aligned}\tag{47}$$

By Lemma 7 and (45), it follows that

$$\begin{aligned}
\left\|\tilde{u}_{p_k}\right\|^2 &= \sum_{n=1}^{T_{p_k}}\left|\tilde{u}_{p_k}(n)\right|^2 \\
&\le \left(2\sin\frac{\pi}{T_{p_k}}\right)^{-1}\sum_{n=1}^{T_{p_k}}\left|\Delta u_{p_k}(n)\right|^2 \\
&\le (2\sin(\pi/T))^{-1}\frac{2\alpha(\beta+\gamma)T\sin(\pi/T)}{\sin(\pi/T)-\alpha} \\
&\le \frac{\alpha(\beta+\gamma)T}{\sin(\pi/T)-\alpha},
\end{aligned}\tag{48}$$

which implies that $\{\|\tilde{u}_{p_k}\|_\infty\}$ is bounded, therefore $\{\|u_{p_k}\|_\infty\}$ is bounded; a contradiction with the second claim $\lim_{p\to\infty}\|u_p\|_\infty = \infty$. This completes the proof. □

Proof of Theorem 2. Under the assumptions (A1) and (A2), all conditions in Theorem 1 are satisfied. Then, for each integer $p > 1$, there exists a pT-periodic solution u of (3) such that $v = -J[u - (1/pT)\sum_{n=1}^{pT} u(n)] \in Y$ minimizes the dual action

$$\begin{aligned}
\chi(v) &= \sum_{n=1}^{pT}\frac{1}{2}(LJ\Delta v(n-1), v(n)) \\
&\quad+\sum_{n=1}^{pT}H^*(n,\Delta v(n)) \quad \text{on } E_{pT}.
\end{aligned}\tag{49}$$

If the critical point v of dual action functional χ has minimal period $pT/l \in \mathbb{N}\setminus\{0\}$, where $l \in \mathbb{N}\setminus\{0\}$, then by Lemma 7 with pT replaced by pT/l, we have the following estimate:

$$4\sin^2\frac{l\pi}{pT}\sum_{n=1}^{pT}|v(n)|^2 \le \sum_{n=1}^{pT}|\Delta v(n)|^2. \tag{50}$$

By Lemma 5 and the previous inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{pT}(LJ\Delta v(n-1), v(n)) \\
&\ge -\left(\sum_{n=1}^{pT}|LJ\Delta v(n-1)|^2\right)^{1/2} \\
&\quad\cdot\left(\sum_{n=1}^{pT}|v(n)|^2\right)^{1/2} \\
&\ge -\left(\sum_{n=1}^{pT}|\Delta v(n)|^2\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\cdot\left(2\sin\frac{l\pi}{pT}\right)^{-1}\left(\sum_{n=1}^{pT}|\Delta v(n)|^2\right)^{1/2}\\
&=-\left(2\sin\frac{l\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta v(n)|^2\\
&\geq-\left(2\sin\frac{l\pi}{pT}\right)^{-1}(pT)^{(1-2/\tau)}\left(\sum_{n=1}^{pT}|\Delta v(n)|^\tau\right)^{2/\tau},
\end{aligned}\tag{51}$$

where $\tau=\theta/(\theta-1)>2$ for $1<\theta<2$. It follows from assumption (B2) that

$$H^*(n,\Delta v(n))\geq\frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}|\Delta v(n)|^\tau,\tag{52}$$

thus

$$\chi(v)\geq-\left(2\sin\frac{l\pi}{pT}\right)^{-1}(pT)^{(\tau-2)/\tau}\left(\sum_{n=1}^{pT}|\Delta v(n)|^\tau\right)^{2/\tau}+\frac{1}{\tau}\left(\frac{1}{a_2}\right)^{\tau-1}\sum_{n=1}^{pT}|\Delta v(n)|^\tau\tag{53}$$

$$\geq\frac{(1/\tau-1/2)\,pT\left(a_2^2\right)^{(\tau-1)/(\tau-2)}}{\left(\sin(l\pi/pT)\right)^{\tau/(\tau-2)}}.\tag{54}$$

One can obtain the previous inequality by minimizing in (53) with respect to $\left(\sum_{n=1}^{pT}|\Delta v(n)|^\tau\right)^{1/\tau}$, and the minimum is attained at $(pT)^{1/\tau}(a_2)^{(\tau-1)/(\tau-2)}/(\sin(l\pi/pT))^{1/(\tau-2)}$.

On the other hand, let

$$v(n)=\frac{1}{\sqrt{pT}}\begin{pmatrix}\cos\dfrac{2k\pi}{pT}n\cdot a_k\\ -\sin\dfrac{2k\pi}{pT}\left(n-\dfrac{1}{2}\right)\cdot a_k\end{pmatrix},\tag{55}$$

where $a_k\in\mathbb{R}^N$, $k\in Z[[-pT/2],[pT/2]]\setminus\{0\}$. Then $v\in Y_k$, and

$$\Delta v(n)=-2\sin\frac{k\pi}{pT}\frac{1}{\sqrt{pT}}\begin{pmatrix}\sin\dfrac{2k\pi}{pT}\left(n+\dfrac{1}{2}\right)\cdot a_k\\ \cos\dfrac{2k\pi}{pT}n\cdot a_k\end{pmatrix}.\tag{56}$$

Taking $a_k=(d,0,\ldots,0)^T\in\mathbb{R}^N$, where $d\in\mathbb{R}$, by Lemma 4, it follows that

$$\begin{aligned}
&\sum_{n=1}^{pT}(LJ\Delta v(n-1),v(n))\\
&=\sum_{n=1}^{pT}\left[-\Delta v_2(n)v_1(v)+\Delta v_1(n-1)v_2(n)\right]\\
&=\sum_{n=1}^{pT}\frac{1}{pT}\cdot 2\sin\frac{k\pi}{pT}\\
&\quad\cdot\left(\cos^2\frac{2k\pi}{pT}n\cdot|d|^2+\sin^2\frac{2k\pi}{pT}\left(n-\frac{1}{2}\right)\cdot|d|^2\right)\\
&=\lambda_k\cdot|d|^2,
\end{aligned}\tag{57}$$

where $\lambda_k=2\sin(k\pi/pT)$ and

$$\begin{aligned}
&\sum_{n=1}^{pT}|\Delta v(n)|^\tau\\
&=\sum_{n=1}^{pT}|\lambda_k|^\tau(pT)^{-\tau/2}\\
&\quad\cdot\left(\sin^2\frac{2k\pi}{pT}\left(n+\frac{1}{2}\right)+\cos^2\frac{2k\pi}{pT}n\right)^{\tau/2}|d|^\tau\\
&\leq\lambda_{\max}^\tau\cdot(pT)^{1-(\tau/2)}\cdot 2^{\tau/2}|d|^\tau.
\end{aligned}\tag{58}$$

Therefore, taking $k=-[pT/2]$, by eigenvalue problem (24) and (B2), it follows that

$$\begin{aligned}
\chi(v)&=\frac{1}{2}\sum_{n=1}^{pT}(LJ\Delta v(n-1),v(n))\\
&\quad+\sum_{n=1}^{pT}H^*(n,\Delta v(n))\\
&\leq-\frac{1}{2}\lambda_{\max}\cdot|d|^2\\
&\quad+\frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}\sum_{n=1}^{pT}|\Delta v(n)|^\tau\\
&\leq-\frac{1}{2}\lambda_{\max}\cdot|d|^2+\frac{1}{\tau}\left(\frac{1}{a_1}\right)^{\tau-1}\lambda_{\max}^\tau\\
&\quad\cdot(pT)^{1-(\tau/2)}\cdot 2^{\tau/2}|d|^\tau.
\end{aligned}\tag{59}$$

Let $f(\rho)$ equal the right-hand side of (59) where $\rho=|d|$. It is easy to see that the absolute minimum of f is attained at $\rho_{\min}=(a_1)^{(\tau-1)/(\tau-2)}(pT)^{1/2}/[\lambda_{\max}^{(\tau-1)/(\tau-2)}\cdot 2^{\tau/2(\tau-2)}]$ and given by $f_{\min} = (1/\tau - 1/2)pT(a_1^2)^{(\tau-1)/(\tau-2)}/(2\lambda_{\max})^{\tau/(\tau-2)}$. Hence, by (19), let

$$\xi(n) = \xi_{-[pT/2]}(n) = \begin{pmatrix} \cos\frac{2k\pi}{pT} n \cdot \rho \\ -\sin\frac{2k\pi}{pT}\left(n - \frac{1}{2}\right) \cdot \rho \end{pmatrix}, \tag{60}$$

where $\rho \in \mathbb{R}^N$, $k = -[pT/2]$.

If pT is even, then $\xi(n) = (1,1)^T \cdot (-1)^n \rho$. Set

$$Y_{\rho_{\min}} = \left\{ v \in Y_{-[pT/2]} : v(n) = \xi(n), \; \rho = (d, 0, \ldots, 0)^T \in \mathbb{R}^N, d \in \mathbb{R} \right\}. \tag{61}$$

For $v \in Y_{\rho_{\min}}$, we have

$$\chi(v) \le f_{\min}. \tag{62}$$

Combining (54), (59), and (62), we have

$$\frac{(1/\tau - 1/2)\, pT\left(a_2^2\right)^{(\tau-1)/(\tau-2)}}{\left(\sin(l\pi/pT)\right)^{\tau/(\tau-2)}} \le \frac{(1/\tau - 1/2)\, pT\left(a_1^2\right)^{(\tau-1)/(\tau-2)}}{\left(2\lambda_{\max}\right)^{\tau/(\tau-2)}}. \tag{63}$$

By $\tau > 2$, and $\theta = \tau/(\tau-1)$, it follows that

$$\frac{\sin(l\pi/pT)}{(2\lambda_{\max}) \le (a_2/a_1)^{2/\theta}}. \tag{64}$$

For integer $p > 1$, $T \ge 1$, $l \in \mathbb{N} \setminus \{0\}$, $pT/l \in \mathbb{N} \setminus \{0\}$, we have $0 < l\pi/pT \le \pi$, $0 < \pi/pT \le \pi/2$.

If pT is even, then $\lambda_{\max} = 2$. By assumption $a_2/a_1 \le ((1/4)\sin(\pi/pT))^{\theta/2}$ we have $\sin(l\pi/pT) \le \sin(\pi/pT)$, which implies that $l = 1$ or $l = pT - 1$. If $pT > 2$, then $pT/l = pT/(pT-1) \notin \mathbb{N}$. So we have $l = 1$.

If pT is odd, then $\lambda_{\max} = 2\cos(\pi/2pT)$. By assumption $a_2/a_1 \le ((1/2)\sin(\pi/2pT))^{\theta/2}$, we have $\sin(l\pi/pT) \le \sin(\pi/pT)$, so $l = 1$. This completes the proof. □

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187), NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), and the Foundation of Education of Fujian Province (JA09152).

References

[1] Y.-T. Xu, "Subharmonic solutions for convex nonautonomous Hamiltonian systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 28, no. 8, pp. 1359–1371, 1997.

[2] J. Q. Liu and Z. Q. Wang, "Remarks on subharmonics with minimal periods of Hamiltonian systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 20, no. 7, pp. 803–821, 1993.

[3] R. Michalek and G. Tarantello, "Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems," *Journal of Differential Equations*, vol. 72, no. 1, pp. 28–55, 1988.

[4] P. H. Rabinowitz, "Minimax methods on critical point theory with applications to differentiable equations," in *Proceedings of the CBMS*, vol. 65, American Mathematical Society, Providence, RI, USA, 1986.

[5] Y.-T. Xu and Z.-M. Guo, "Applications of a Z_p index theory to periodic solutions for a class of functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 189–205, 2001.

[6] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China A*, vol. 46, no. 4, pp. 506–515, 2003.

[7] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 55, no. 7-8, pp. 969–983, 2003.

[8] J. Yu, H. Bin, and Z. Guo, "Periodic solutions for discrete convex Hamiltonian systems via Clarke duality," *Discrete and Continuous Dynamical Systems A*, vol. 15, no. 3, pp. 939–950, 2006.

[9] F. H. Clarke, "A classical variational principle for periodic Hamiltonian trajectories," *Proceedings of the American Mathematical Society*, vol. 76, no. 1, pp. 186–188, 1979.

[10] F. H. Clarke, "Periodic solutions of Hamilton's equations and local minima of the dual action," *Transactions of the American Mathematical Society*, vol. 287, no. 1, pp. 239–251, 1985.

[11] F. H. Clarke and I. Ekeland, "Nonlinear oscillations and boundary value problems for Hamiltonian systems," *Archive for Rational Mechanics and Analysis*, vol. 78, no. 4, pp. 315–333, 1982.

[12] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[13] A. Ambrosetti and G. Mancini, "Solutions of minimal period for a class of convex Hamiltonian systems," *Mathematische Annalen*, vol. 255, no. 3, pp. 405–421, 1981.

[14] G. Cordaro, "Existence and location of periodic solutions to convex and non coercive Hamiltonian systems," *Discrete and Continuous Dynamical Systems A*, vol. 12, no. 5, pp. 983–996, 2005.

[15] F. M. Atici and A. Cabada, "Existence and uniqueness results for discrete second-order periodic boundary value problems," *Computers & Mathematics with Applications*, vol. 45, no. 6-9, pp. 1417–1427, 2003.

[16] F. M. Atici and G. S. Guseinov, "Positive periodic solutions for nonlinear difference equations with periodic coefficients," *Journal of Mathematical Analysis and Applications*, vol. 232, no. 1, pp. 166–182, 1999.

[17] Z. Huang, C. Feng, and S. Mohamad, "Multistability analysis for a general class of delayed Cohen-Grossberg neural networks," *Information Sciences*, vol. 187, pp. 233–244, 2012.

[18] Z. Huang and Y. N. Raffoul, "Biperiodicity in neutral-type delayed difference neural networks," *Advances in Difference Equations*, vol. 2012, article 5, 2012.

[19] D. Jiang and R. P. Agarwal, "Existence of positive periodic solutions for a class of difference equations with several deviating

arguments," *Computers & Mathematics with Applications*, vol. 45, no. 6-9, pp. 1303–1309, 2003.

[20] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, vol. 228, Marcel Dekker Inc., New York, NY, USA, 2nd edition, 2000.

[21] Y. T. Xu and Z. M. Guo, "Applications of a geometrical index theory to functional differential equations," *Acta Mathematica Sinica*, vol. 44, no. 6, pp. 1027–1036, 2001.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 384981, 8 pages
http://dx.doi.org/10.1155/2013/384981

Research Article

Dynamical Behaviors of the Stochastic Hopfield Neural Networks with Mixed Time Delays

Li Wan,[1] Qinghua Zhou,[2] Zhigang Zhou,[1] and Pei Wang[3]

[1] *School of Mathematics and Computer Science, Wuhan Textile University, Wuhan 430073, China*
[2] *Department of Mathematics, Zhaoqing University, Zhaoqing 526061, China*
[3] *School of Mathematics and Information Sciences, Henan University, Kaifeng 475004, China*

Correspondence should be addressed to Qinghua Zhou; zqhmath@yahoo.com.cn

Received 24 December 2012; Accepted 8 February 2013

Academic Editor: Kelin Li

This paper investigates dynamical behaviors of the stochastic Hopfield neural networks with mixed time delays. The mixed time delays under consideration comprise both the discrete time-varying delays and the distributed time-delays. By employing the theory of stochastic functional differential equations and linear matrix inequality (LMI) approach, some novel criteria on asymptotic stability, ultimate boundedness, and weak attractor are derived. Finally, a numerical example is given to illustrate the correctness and effectiveness of our theoretical results.

1. Introduction

The well-known Hopfield neural networks were firstly introduced by Hopfield [1, 2] in early 1980s. Since then, both the mathematical analysis and practical applications of Hopfield neural networks have gained considerable research attention. The Hopfield neural networks have already been successfully applied in many different areas such as combinatorial optimization, knowledge acquisition, and pattern recognition, see, for example, [3–5]. In both the biological and artificial neural networks, the interactions between neurons are generally asynchronous, which give rise to the inevitable signal transmission delays. Also, in electronic implementation of analog neural networks, time delay is usually time-varying due to the finite switching speed of amplifiers. Note that continuously distributed delays have gained particular attention, since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths.

Recently, it has been well recognized that stochastic disturbances are ubiquitous and inevitable in various systems, ranging from electronic implementations to biochemical systems, which are mainly caused by thermal noise and environmental fluctuations as well as different orders of ongoing events in the overall systems [6, 7]. Therefore, considerable attentions have been paid to investigate the dynamics of stochastic neural networks, and many results on stochastic neural networks with delays have been reported in the literature, see, for example, [8–30] and references therein. Among which, some sufficient criteria on the stability of uncertain stochastic neural networks were derived in [8–10]. Almost sure exponential stability of stochastic neural networks was discussed in [11–15]. In [16–22], mean square exponential stability and pth moment exponential stability of stochastic neural networks were investigated; Some sufficient criteria on the exponential stability for impulsive stochastic neural networks were established in [23–26]. In [27], the stability of discrete-time stochastic neural networks was analyzed, while exponential stability of stochastic neural networks with Markovian jump parameters is investigated in [28–30]. These references mainly considered the stability of equilibrium point of stochastic neural networks. What do we study when the equilibrium point does not exist?

Except for stability property, boundedness and attractor are also foundational concepts of dynamical systems. They play an important role in investigating the uniqueness of equilibrium, global asymptotic stability, global exponential stability, the existence of periodic solution, its control and synchronization [31, 32], and so on. Recently, ultimate boundedness and attractor of several classes of neural networks with time delays have been reported. Some sufficient criteria were derived in [33, 34], but these results hold only under constant delays. Following, in [35], the globally robust ultimate boundedness of integrodifferential neural networks with uncertainties and varying delays was studied. After that, some sufficient criteria on the ultimate boundedness of neural networks with both varying and unbounded delays were derived in [36], but the concerned systems are deterministic ones. In [37, 38], a series of criteria on the boundedness, global exponential stability, and the existence of periodic solution for nonautonomous recurrent neural networks were established. In [39–41], the ultimate boundedness and attractor of the stochastic Hopfield neural networks with time-varying delays were discussed. To the best of our knowledge, for stochastic neural networks with mixed time delays, there are few published results on the ultimate boundedness and weak attractor. Therefore, the arising questions about the ultimate boundedness, weak attractor, and asymptotic stability of the stochastic Hopfield neural networks with mixed time delays are important yet meaningful.

The left of the paper is organized as follows and some preliminaries are in Section 2, Section 3 presents our main results, a numerical example and conclusions will be in Sections 4 and 5, respectively.

2. Preliminaries

Consider the following stochastic Hopfield neural networks with mixed time delays:

$$\begin{aligned} dx(t) = \Big[& -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) \\ & + D\int_{t-\tau(t)}^{t} g(x(s))\,ds + J\Big]dt \\ & + [\sigma_1 x(t) + \sigma_2 x(t-\tau(t))]\,dw(t), \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)^T$ is the state vector associated with the neurons, $C = \operatorname{diag}\{c_1, \ldots, c_n\}$, $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when being disconnected from the network and the external stochastic perturbation; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ represent the connection weight matrix; $J = (J_1, \ldots, J_n)^T$, J_i denotes the external bias on the ith unit; f_j and g_j denote activation functions, $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^T$, $g(x(t)) = (g_1(x_1(t)), \ldots, g_n(x_n(t)))^T$; $\sigma_1, \sigma_2 \in R^{n\times n}$ are the diffusion coefficient matrices; $w(t)$ is one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a natural filtration $\{\mathscr{F}_t\}_{t\ge 0}$ generated by $\{w(s) : 0 \le s \le t\}$; there exists a positive constant τ such that the transmission delay $\tau(t)$ satisfies

$$0 \le \tau(t) \le \tau. \tag{2}$$

The initial conditions are given in the following form:

$$x(s) = \xi(s), \quad -\tau \le s \le 0,\ j = 1, \ldots, n, \tag{3}$$

where $\xi(s) = (\xi_1(s), \ldots, \xi_n(s))^T$ is $C([-\tau, 0]; R^n)$-valued function, $\mathscr{F}_0$-measurable R^n-valued random variable satisfying $\|\xi\|_\tau^2 = \sup_{-\tau\le s\le 0} E\|\xi(s)\|^2 < \infty$, $\|\cdot\|$ is the Euclidean norm, and $C([-\tau, 0]; R^n)$ is the space of all continuous R^n-valued functions defined on $[-\tau, 0]$.

Let $F(x_t, t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + D\int_{t-\tau(t)}^{t} g(x(s))ds + J$, $G(x_t, t) = \sigma_1 x(t) + \sigma_2 x(t-\tau(t))$, where

$$x_t = \{x(t+\theta) : -\tau \le \theta \le 0, t \ge 0\} = \varphi(\theta). \tag{4}$$

Then system (1) can be written by

$$dx(t) = F(x_t, t)\,dt + G(x_t, t)\,dw(t). \tag{5}$$

Throughout this paper, the following assumption will be considered.

(A1) There exist constants l_i^-, l_i^+, m_i^- and m_i^+ such that

$$\begin{aligned} l_i^- &\le \frac{f_i(x) - f_i(y)}{x - y} \le l_i^+, \\ m_i^- &\le \frac{g_i(x) - g_i(y)}{x - y} \le m_i^+, \end{aligned} \qquad \forall x, y \in R. \tag{6}$$

Remark 1. It follows from [42] that under the assumption (A1), system (1) has a global solution on $t \ge 0$. Moreover, under assumption 1, it is not difficult to prove that $F(x_t, t)$ and $G(x_t, t)$ satisfy the local Lipschitz condition in [43].

Remark 2. We note that assumption (A1) is less conservative than that in [8, 9, 39], since the constants l_i^-, l_i^+, m_i^- and m_i^+ are allowed to be positive, negative numbers, or zeros.

The notation $A > 0$ (resp., $A \ge 0$) means that matrix A is symmetric positive definite (resp., positive semidefinite). A^T denotes the transpose of the matrix A. $\lambda_{\min}(A)$ represents the minimum eigenvalue of matrix A. Denote by $C(R^n \times [-\tau, \infty); R^+)$ the family of continuous functions from $R^n \times [-\tau, \infty)$ to $R^+ = [0, \infty)$. Let $C^{2,1}(R^n \times [-\tau, \infty); R^+)$ be the family of all continuous nonnegative functions $V(x, t)$ defined on $R^n \times [-\tau, \infty)$ such that they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(R^n \times [-\tau, \infty); R^+)$, we define the functional $\mathbb{L}V : C([-\tau, 0]; R^n) \times R^+ \to R$ by

$$\begin{aligned} \mathbb{L}V(\varphi, t) = {} & V_t(\varphi(0), t) + V_x(\varphi(0), t)F(\varphi, t) \\ & + \frac{1}{2}\operatorname{trace}\left[G^T(\varphi, t)V_{xx}(\varphi(0), t)G(\varphi, t)\right], \end{aligned} \tag{7}$$

where $V_x(x, t) = (V_{x_1}(x, t), \ldots, V_{x_n}(x, t))$ and $V_{xx}(x, t) = (V_{x_i x_j}(x, t))_{n\times n}$.

The following lemmas will be used in establishing our main results.

Lemma 3 (see [44]). *For any positive definite matrix $P > 0$, scalar $\gamma > 0$, vector function $f : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, and the following inequality holds:*

$$\left(\int_0^\gamma f(s)\,ds\right)^T P \left(\int_0^\gamma f(s)\,ds\right) \le \gamma \int_0^\gamma f^T(s) P f(s)\,ds. \tag{8}$$

Lemma 4 (see [43]). *Suppose that system* (5) *satisfies the local Lipschitz condition and the following assumptions hold.*

(A2) *There are two functions* $V \in C^{2,1}(R^n \times [-\tau, \infty); R^+)$ *and* $U \in C(R^n \times [-\tau, \infty); R^+)$ *and two probability measures* $\mu(\cdot)$ *and* $\overline{\mu}(\cdot)$ *on* $[-\tau, 0]$ *such that*

$$\lim_{\|x\| \to \infty} \inf_{0 \le t \le \infty} V(x,t) = \infty, \tag{9}$$

while for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R^+$,

$$\begin{aligned} \mathbb{L}V(\varphi, t) \le{}& \alpha_1 - \alpha_2 V(\varphi(0), t) \\ &+ \alpha_3 \int_{-\tau}^{0} V(\varphi(\theta), t+\theta)\, d\overline{\mu}(\theta) \\ &- U(\varphi(0), t) \\ &+ \alpha \int_{-\tau}^{0} U(\varphi(\theta), t+\theta)\, d\mu(\theta), \end{aligned} \tag{10}$$

where $\alpha_1 \ge 0$, $\alpha_2 > \alpha_3 \ge 0$ *and* $\alpha \in (0,1)$.

(A3) *If there is a pair of positive constants* c *and* p *such that*

$$c\|x\|^p \le V(x,t), \quad \forall (x,t) \in R^n \times [-\tau, \infty). \tag{11}$$

Then the unique global solution $x(t)$ *to system* (5) *obeys*

$$\limsup_{t \to \infty} E\|x(t)\|^p \le \frac{\alpha_1}{c\varepsilon}, \tag{12}$$

where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ *while* $\varepsilon_2 = -\ln(\alpha)/\tau$ *and* $\varepsilon_1 > 0$ *is the unique root to the following equation:*

$$\alpha_2 = \varepsilon_1 + \alpha_3 e^{\varepsilon_1 \tau}. \tag{13}$$

If, furthermore, $\alpha_1 = 0$, *then*

$$\begin{aligned} &\limsup_{t \to \infty} \frac{1}{t} \ln\left(E\|x(t)\|^p\right) \le -\varepsilon, \\ &\limsup_{t \to \infty} \frac{1}{t} \ln\left(\|x(t)\|\right) \le -\frac{\varepsilon}{p} \quad \textit{almost surely}. \end{aligned} \tag{14}$$

3. Main Results

Theorem 5. *Suppose that there exist some matrices* $P > 0$, $U_i = \operatorname{diag}\{u_{i1}, \ldots, u_{in}\} \ge 0$ $(i = 1, 2, 3)$ *and positive constants* γ_1, γ_2, λ *such that* $\lambda^{-1}\tau\gamma_2^{-1} \in (0,1)$ *and*

$$\Sigma = \begin{pmatrix} \Delta_1 & 0 & PA + L_2U_1 & PB & U_3M_2 \\ * & \Delta_2 & 0 & L_2U_2 & 0 \\ * & * & \Delta_3 & 0 & 0 \\ * & * & * & \Delta_4 & 0 \\ * & * & * & * & \Delta_5 \end{pmatrix} < 0, \tag{15}$$

where $\Delta_1 = (\gamma_1 + 2\lambda)P + 2\sigma_1^T P \sigma_1 - PC - CP + U_1(\lambda I - 2L_1) + U_3(\lambda I - 2M_1)$, $\Delta_2 = 2\sigma_2^T P \sigma_2 + (\lambda I - 2L_1)U_2$, $\Delta_3 = 2(\lambda - 1)U_1$, $\Delta_4 = 2(\lambda - 1)U_2$, $\Delta_5 = 2(\lambda - 1)U_3 + \gamma_2 D^T P D$, $L_1 = \operatorname{diag}\{l_1^- l_1^+, \ldots, l_n^- l_n^+\}$, $L_2 = \operatorname{diag}\{l_1^- + l_1^+, \ldots, l_n^- + l_n^+\}$, $M_1 = \operatorname{diag}\{m_1^- m_1^+, \ldots, m_n^- m_n^+\}$, $M_2 = \operatorname{diag}\{m_1^- + m_1^+, \ldots, m_n^- + m_n^+\}$, $*$ *means the symmetric terms.*

Then, the following results hold.

(i) *System* (1) *is stochastically ultimately bounded; that is, for any* $\delta \in (0,1)$, *there exists a positive constant* $C = C(\delta)$ *such that the solution* $x(t)$ *of system* (1) *satisfies*

$$\limsup_{t \to \infty} P\{\|x(t)\| \le C\} \ge 1 - \delta. \tag{16}$$

(ii) *If* $\alpha_1 = 0$, *where* $\alpha_1 = \max(\gamma_3, 0)$, $\varepsilon > 0$ *is the same as defined in Lemma 4,*

$$\begin{aligned} \gamma_3 ={}& \lambda^{-1} J^T P J \\ &+ \sum_{i=1}^{n} (u_{1i} + u_{2i}) \\ &\quad \times \Big\{ -2f_i^2(0) + 2\lambda^{-1} f_i^2(0) \\ &\qquad + \lambda^{-1}(l_i^+ + l_i^-)^2 f_i^2(0) \Big\} \\ &+ \sum_{i=1}^{n} u_{3i} \Big\{ -2g_i^2(0) + 2\lambda^{-1} g_i^2(0) \\ &\qquad + \lambda^{-1}(m_i^+ + m_i^-)^2 g_i^2(0) \Big\}, \end{aligned} \tag{17}$$

then

$$\begin{aligned} &\limsup_{t \to \infty} \frac{1}{t} \ln\left(E\|x(t)\|^2\right) \le -\varepsilon, \\ &\limsup_{t \to \infty} \frac{1}{t} \ln\left(\|x(t)\|\right) \le -\frac{\varepsilon}{2} \quad \textit{almost surely}. \end{aligned} \tag{18}$$

Proof. Let the Lyapunov function $V(x,t) = x^T(t) P x(t)$. Applying Itô's formula in [42] to $V(t)$ along with system (1), one may obtain the following:

$$\begin{aligned} dV(x,t) ={}& 2x^T(t) P [\sigma_1 x(t) + \sigma_2 x(t - \tau(t))]\, dw(t) \\ &+ \mathbb{L}V(\varphi, t)\, dt, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathbb{L}V(\varphi, t) ={}& 2x^T(t) P \Big[-Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \\ &+ D \int_{t-\tau(t)}^{t} g(x(s))\, ds + J \Big] \\ &+ [\sigma_1 x(t) + \sigma_2 x(t - \tau(t))]^T \\ &\times P [\sigma_1 x(t) + \sigma_2 x(t - \tau(t))] \\ \le{}& 2x^T(t) P [-Cx(t) + Af(x(t)) \\ &+ Bf(x(t - \tau(t)))] \end{aligned}$$

$$
\begin{aligned}
&+2\lambda x^T(t)Px(t) \\
&+\lambda^{-1}J^TPJ+\lambda^{-1}\left(\int_{t-\tau(t)}^{t}Dg(x(s))\,ds\right)^T \\
&\times P\left(\int_{t-\tau(t)}^{t}Dg(x(s))\,ds\right) \\
&+2\left[x^T(t)\sigma_1^TP\sigma_1x(t)\right. \\
&\qquad\left.+x^T(t-\tau(t))\sigma_2^TP\sigma_2x(t-\tau(t))\right].
\end{aligned}
\tag{20}
$$

From Lemma 3, it follows that

$$
\begin{aligned}
&\lambda^{-1}\left(\int_{t-\tau(t)}^{t}Dg(x(s))\,ds\right)^TP\left(\int_{t-\tau(t)}^{t}Dg(x(s))\,ds\right) \\
&\quad\le\lambda^{-1}\left(\int_{t-\tau}^{t}Dg(x(s))\,ds\right)^TP\left(\int_{t-\tau}^{t}Dg(x(s))\,ds\right) \\
&\quad\le\lambda^{-1}\tau\int_{t-\tau}^{t}g^T(x(s))D^TPDg(x(s))\,ds \\
&\quad=\lambda^{-1}\tau\int_{-\tau}^{0}g^T(x(t+\theta))D^TPDg(x(t+\theta))\,d\theta \\
&\quad=\lambda^{-1}\tau\int_{-\tau}^{0}g^T(\varphi(\theta))D^TPDg(\varphi(\theta))\,d\theta.
\end{aligned}
\tag{21}
$$

From (A1), it follows that for $i=1,\dots,n$,

$$
\begin{aligned}
0\le&-2\sum_{i=1}^{n}u_{1i}\left[f_i(x_i(t))-f_i(0)-l_i^+x_i(t)\right] \\
&\times\left[f_i(x_i(t))-f_i(0)-l_i^-x_i(t)\right] \\
=&-2\sum_{i=1}^{n}u_{1i}\left\{f_i^2(x_i(t))-(l_i^++l_i^-)x_i(t)f_i(x_i(t))\right. \\
&\qquad+l_i^+l_i^-x_i^2(t)+f_i^2(0)-2f_i(0)f_i(x_i(t)) \\
&\qquad\left.+(l_i^++l_i^-)x_i(t)f_i(0)\right\} \\
=&-2\sum_{i=1}^{n}u_{1i}\left\{f_i^2(x_i(t))-(l_i^++l_i^-)x_i(t)f_i(x_i(t))\right. \\
&\qquad\left.+l_i^+l_i^-x_i^2(t)\right\} \\
&+\sum_{i=1}^{n}u_{1i}\left[-2f_i^2(0)+4f_i(0)f_i(x_i(t))\right. \\
&\qquad\left.-2(l_i^++l_i^-)x_i(t)f_i(0)\right] \\
\le&-2\sum_{i=1}^{n}u_{1i}\left\{f_i^2(x_i(t))-(l_i^++l_i^-)x_i(t)f_i(x_i(t))\right. \\
&\qquad\left.+l_i^+l_i^-x_i^2(t)\right\} \\
&+\sum_{i=1}^{n}u_{1i}\left\{-2f_i^2(0)+2\left[\lambda f_i^2(x_i(t))+\lambda^{-1}f_i^2(0)\right]\right. \\
&\qquad\left.+\left[\lambda x_i^2(t)+\lambda^{-1}(l_i^++l_i^-)^2f_i^2(0)\right]\right\} \\
=&\,2(\lambda-1)f^T(x(t))U_1f(x(t))+2f^T(x(t))U_1L_2x(t) \\
&+x^T(t)U_1(\lambda I-2L_1)x(t) \\
&+\sum_{i=1}^{n}u_{1i}\left\{-2f_i^2(0)+2\lambda^{-1}f_i^2(0)+\lambda^{-1}(l_i^++l_i^-)^2f_i^2(0)\right\}.
\end{aligned}
\tag{22}
$$

Similarly, one derives that

$$
\begin{aligned}
0\le&-2\sum_{i=1}^{n}u_{2i}\left[f_i(x_i(t-\tau(t)))-f_i(0)-l_i^+x_i(t-\tau(t))\right] \\
&\times\left[f_i(x_i(t-\tau(t)))-f_i(0)-l_i^-x_i(t-\tau(t))\right] \\
\le&\,2(\lambda-1)f^T(x(t-\tau(t)))U_2f(x(t-\tau(t))) \\
&+2f^T(x(t-\tau(t)))U_2L_2x(t-\tau(t)) \\
&+x^T(t-\tau(t))U_2(\lambda I-2L_1)x(t-\tau(t)) \\
&+\sum_{i=1}^{n}u_{2i}\left\{-2f_i^2(0)+2\lambda^{-1}f_i^2(0)+\lambda^{-1}(l_i^++l_i^-)^2f_i^2(0)\right\}, \\
0\le&-2\sum_{i=1}^{n}u_{3i}\left[g_i(x_i(t))-g_i(0)-m_i^+x_i(t)\right] \\
&\times\left[g_i(x_i(t))-g_i(0)-m_i^-x_i(t)\right] \\
\le&\,2(\lambda-1)g^T(x(t))U_3g(x(t))+2g^T(x(t))U_3M_2x(t) \\
&+x^T(t)U_3(\lambda I-2M_1)x(t) \\
&+\sum_{i=1}^{n}u_{3i}\left\{-2g_i^2(0)+2\lambda^{-1}g_i^2(0)+\lambda^{-1}(m_i^++m_i^-)^2g_i^2(0)\right\}.
\end{aligned}
\tag{23}
$$

Further from (20)–(23), one derives

$$
\begin{aligned}
\mathbb{L}V(\varphi,t)\le&\,2x^T(t)P\left[-Cx(t)+Af(x(t))+Bf(x(t-\tau(t)))\right] \\
&+2\lambda x^T(t)Px(t) \\
&+\lambda^{-1}\tau\int_{-\tau}^{0}g^T(\varphi(\theta))D^TPDg(\varphi(\theta))\,d\theta
\end{aligned}
$$

$$\begin{aligned}
&+2\left[x^{T}(t)\sigma_1^T P\sigma_1 x(t)\right.\\
&\qquad\left.+x^{T}(t-\tau(t))\sigma_2^T P\sigma_2 x(t-\tau(t))\right]\\
&+2(\lambda-1)f^{T}(x(t))U_1 f(x(t))\\
&+2f^{T}(x(t))U_1 L_2 x(t)\\
&+x^{T}(t)U_1(\lambda I-2L_1)x(t)\\
&+2(\lambda-1)f^{T}(x(t-\tau(t)))U_2 f(x(t-\tau(t)))\\
&+2f^{T}(x(t-\tau(t)))U_2 L_2 x(t-\tau(t))\\
&+x^{T}(t-\tau(t))U_2(\lambda I-2L_1)x(t-\tau(t))\\
&+2(\lambda-1)g^{T}(x(t))U_3 g(x(t))\\
&+2g^{T}(x(t))U_3 M_2 x(t)\\
&+x^{T}(t)U_3(\lambda I-2M_1)x(t)\\
&+\gamma_3+\gamma_1 x^{T}(t)Px(t)-\gamma_1 V(\varphi(0),t)\\
&+\gamma_2 g^{T}(x(t))D^{T}PDg(x(t))\\
&-\gamma_2 g^{T}(\varphi(0))D^{T}PDg(\varphi(0))\\
\le{}&\eta^{T}(t)\Sigma\eta(t)-\gamma_1 V(\varphi(0),t)\\
&-\gamma_2 g^{T}(\varphi(0))D^{T}PDg(\varphi(0))+\gamma_3\\
&+\lambda^{-1}\tau\int_{-\tau}^{0}g^{T}(\varphi(\theta))D^{T}PDg(\varphi(\theta))\,d\theta\\
\le{}&\gamma_3-\gamma_1 V(\varphi(0),t)-U(\varphi(0),t)\\
&+\lambda^{-1}\tau\gamma_2^{-1}\int_{-\tau}^{0}U(\varphi(\theta),t+\theta)\,d\theta,
\end{aligned}\tag{24}$$

where $\eta(t)=(x^T(t),x^T(t-\tau(t)),f^T(x(t)),f^T(x(t-\tau(t))),g^T(x(t)))^T$, $\varphi(0)=x(t)$, $U(x,t)=\gamma_2 g^T(x)D^TPDg(x)$,

$$\begin{aligned}
\gamma_3={}&\lambda^{-1}J^TPJ\\
&+\sum_{i=1}^{n}(u_{1i}+u_{2i})\\
&\times\left\{-2f_i^2(0)+2\lambda^{-1}f_i^2(0)+\lambda^{-1}(l_i^{+}+l_i^{-})^2f_i^2(0)\right\}\\
&+\sum_{i=1}^{n}u_{3i}\left\{-2g_i^2(0)+2\lambda^{-1}g_i^2(0)\right.\\
&\qquad\left.+\lambda^{-1}(m_i^{+}+m_i^{-})^2g_i^2(0)\right\}.
\end{aligned}\tag{25}$$

Let $\alpha_1=\max(\gamma_3,0)$, $\alpha_2=\gamma_1$, $\alpha_3=0$, $\alpha=\lambda^{-1}\tau\gamma_2^{-1}$, $c=\lambda_{\min}(P)$. Then it follows from Lemma 4 that

$$\limsup_{t\to\infty}E\|x(t)\|^2\le\frac{\alpha_1}{\lambda_{\min}(P)\varepsilon}\le\frac{\gamma_4}{\lambda_{\min}(P)\varepsilon},\tag{26}$$

where $\varepsilon>0$ is the same as defined in Lemma 4,

$$\begin{aligned}
\gamma_4={}&\lambda^{-1}J^TPJ\\
&+\sum_{i=1}^{n}(u_{1i}+u_{2i})\left\{2\lambda^{-1}f_i^2(0)+\lambda^{-1}(l_i^{+}+l_i^{-})^2f_i^2(0)\right\}\\
&+\sum_{i=1}^{n}u_{3i}\left\{2\lambda^{-1}g_i^2(0)+\lambda^{-1}(m_i^{+}+m_i^{-})^2g_i^2(0)\right\}.
\end{aligned}\tag{27}$$

Therefore, for any $\delta>0$, it follows from Chebyshev's inequality that

$$\begin{aligned}
\limsup_{t\to\infty}P&\left\{\|x(t)\|>\sqrt{\frac{\gamma_4}{\delta\lambda_{\min}(P)\varepsilon}}\right\}\\
&\le\limsup_{t\to\infty}\frac{E\|x(t)\|^2}{\gamma_4/\delta\lambda_{\min}(P)\varepsilon}\\
&\le\delta.
\end{aligned}\tag{28}$$

If, furthermore, $\alpha_1=0$, then it follows from Lemma 4 that (ii) holds. □

Theorem 5 shows that there exists $t_0>0$ such that for any $t\ge t_0$, $P\{\|x(t)\|\le C\}\ge 1-\delta$. Let B_C denote by

$$B_C=\{x\in R^n\mid\|x(t)\|\le C,t\ge t_0\}.\tag{29}$$

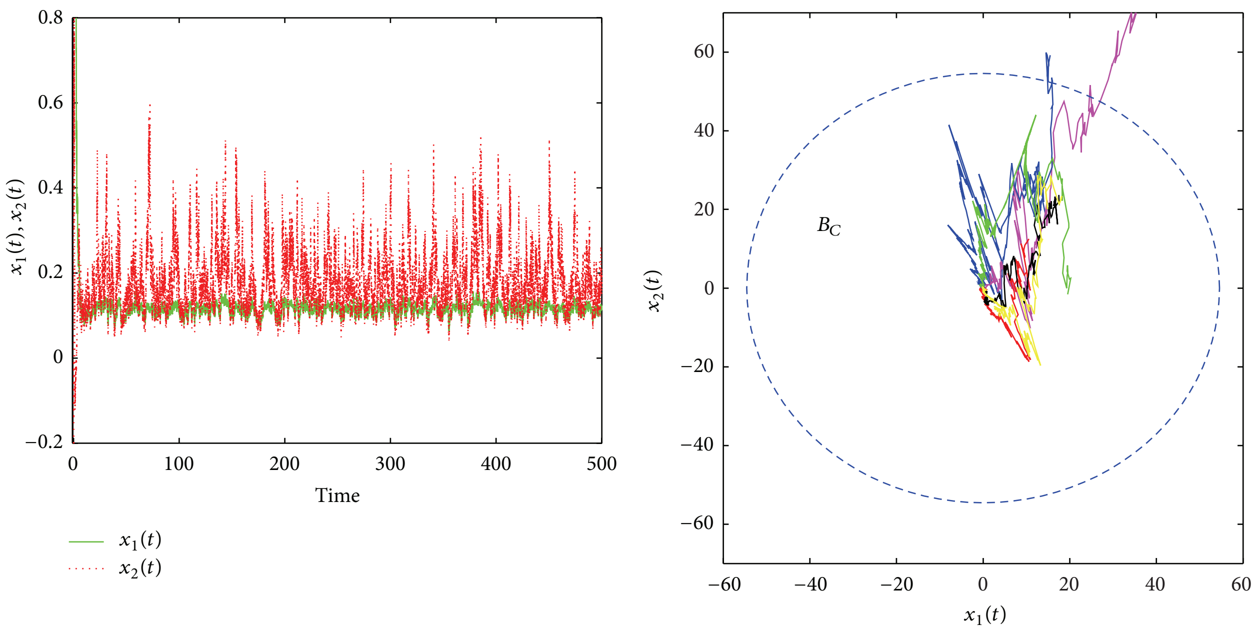
Clearly, B_C is closed, bounded, and invariant. Moreover,

$$\limsup_{t\to\infty}\inf_{y\in B_C}\|x(t)-y\|=0,\tag{30}$$

with no less than probability $1-\delta$, which means that B_C attracts the solutions infinitely many times with no less than probability $1-\delta$, so we may say that B_C is a weak attractor for the solutions.

Theorem 6. *Suppose that all conditions of Theorem 5 hold, then there exists a weak attractor B_C for the solutions of system* (1).

Remark 7. Compared with [39–41], assumption (A1) is less conservative than that in [39] and system (1) includes mixed time delays, which is more complex than that in [39–41]. In addition, Lemma 4 is firstly used to investigate the dynamical behaviors of stochastic neural networks with mixed time delays. The bound for $\mathbb{L}V$ may be in a much weaker form. Our results do not only deal with the asymptotic moment estimation but also the path wise (almost sure) estimation.



(a) (b)

Figure 1: Time trajectories (a) as well as the set B_C and several typical phase portraits (b) for the system in Example 8 (color online). Where initial values for $t < 0$ are chosen as $x(t) = (50, 80)$. For (b), only phase portraits for $t \geq 0$ are shown.

4. Numerical Example

In this section, a numerical example is presented to demonstrate the validity and effectiveness of our theoretical results.

Example 8. Consider the following stochastic Hopfield neural networks with mixed time delays:

$$\begin{aligned} dx(t) = \Big[& -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) \\ & + D\int_{t-\tau(t)}^{t} g(x(s))\,ds + J \Big] dt \\ & + [\sigma_1 x(t) + \sigma_2 x(t-\tau(t))]\,dw(t), \end{aligned} \tag{31}$$

where $J = (0.01, 0.01)^T$, $\tau(t) = 0.2|\sin t|$,

$$A = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.5 & -1 \\ -1.4 & 0.8 \end{pmatrix},$$
$$C = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.3 & -0.1 \\ 0.1 & 0.4 \end{pmatrix}, \tag{32}$$
$$\sigma_1 = \begin{pmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0.1 & -0.2 \\ -0.2 & 0.3 \end{pmatrix},$$

and $f(x) = g(x) = \tanh(x)$, $w(t)$ is one-dimensional Brownian motion. Then $L_1 = M_1 = 0$, $L_2 = M_2 = \operatorname{diag}\{1, 1\}$. By using the Matlab LMI Control Toolbox [45], for $\gamma_1 = \gamma_2 = \lambda = 0.5$, $\tau = 0.2$, based on Theorem 5, such system is stochastically ultimately bounded when P, U_1, U_2 and U_3 are chosen as follows:

$$P = \begin{pmatrix} 0.3278 & 0.2744 \\ 0.2744 & 0.3567 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 0.0668 & 0 \\ 0 & 0.2255 \end{pmatrix},$$
$$U_2 = \begin{pmatrix} 0.1263 & 0 \\ 0 & 0.1218 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 0.0901 & 0 \\ 0 & 0.1031 \end{pmatrix}. \tag{33}$$

For $\delta = 0.01$, $\gamma_1 = \gamma_2 = \lambda = 0.5$, $\tau = 0.2$, we obtain $\varepsilon = 0.5$ and constant $C = C(\delta) = \sqrt{J^T PJ/\lambda_{\min}(P)\delta\lambda\varepsilon} = \sqrt{0.1141/0.0675} = 1.3001$. Then $B_C = \{x \in R^2 \mid \|x(t)\| \leq 1.3001, t \geq t_0\}$, $P(x \in B_C) \geq 0.99$. For the system in Example 8 (color online), Figure 1(a) shows time trajectories, and Figure 1(b) shows the set B_C and several typical phase portraits, where initial value for $t < 0$ is chosen as $x(t) = (50, 80)$. In Figure 1(b), only phase portraits for $t \geq 0$ are shown. From Figure 1, one can easily find that these trajectories are almost all attracted by the set B_C.

5. Conclusion

In this paper, by using the theory of stochastic functional differential equations and linear matrix inequality, new results and sufficient criteria on the asymptotic stability, ultimate boundedness, and attractor of stochastic Hopfield neural networks with mixed time delays are established. A numerical example is also presented to demonstrate the correctness of our theoretical results.

Acknowledgments

The authors thank the Editor and the Reviewers for their detailed comments and valuable suggestions. This work was supported by the National Natural Science Foundation of China (no. 11271295, 10926128, and 11047114), Science and Technology Research Projects of Hubei Provincial Department of Education (no. Q20111607, Q20111611, D20131602, and D20131604), and Young Talent Cultivation Projects of Guangdong (LYM09134).

References

[1] J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 79, no. 8, pp. 2554–2558, 1982.

[2] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 81, no. 10, part 1, pp. 3088–3092, 1984.

[3] W. J. Li and T. Lee, "Hopfield neural networks for affine invariant matching," *IEEE Transactions on Neural Networks*, vol. 12, no. 6, pp. 1400–1410, 2001.

[4] Y. Liu, Z. Wang, and X. Liu, "Global exponential stability of generalized recurrent neural networks with discrete and distributed delays," *Neural Networks*, vol. 19, no. 5, pp. 667–675, 2006.

[5] S. S. Young, P. D. Scott, and N. M. Nasrabadi, "Object recognition using multilayer Hopfield neural network," *IEEE Transactions on Image Processing*, vol. 6, no. 3, pp. 357–372, 1997.

[6] M. Kærn, T. C. Elston, W. J. Blake, and J. J. Collins, "Stochasticity in gene expression: from theories to phenotypes," *Nature Reviews Genetics*, vol. 6, no. 6, pp. 451–464, 2005.

[7] K. Sriram, S. Soliman, and F. Fages, "Dynamics of the interlocked positive feedback loops explaining the robust epigenetic switching in *Candida albicans*," *Journal of Theoretical Biology*, vol. 258, no. 1, pp. 71–88, 2009.

[8] H. Huang and G. Feng, "Delay-dependent stability for uncertain stochastic neural networks with time-varying delay," *Physica A*, vol. 381, no. 1-2, pp. 93–103, 2007.

[9] W. H. Chen and X. M. Lu, "Mean square exponential stability of uncertain stochastic delayed neural networks," *Physics Letters A*, vol. 372, no. 7, pp. 1061–1069, 2008.

[10] Q. T. Gan and R. Xu, "Global robust exponential stability of uncertain neutral high-order stochastic hopfield neural networks with time-varying delays," *Neural Processing Letters*, vol. 32, no. 1, pp. 83–96, 2010.

[11] H. Y. Zhao, N. Ding, and L. Chen, "Almost sure exponential stability of stochastic fuzzy cellular neural networks with delays," *Chaos, Solitons and Fractals*, vol. 40, no. 4, pp. 1653–1659, 2009.

[12] H. Y. Zhao and N. Ding, "Dynamic analysis of stochastic bidirectional associative memory neural networks with delays," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1692–1702, 2007.

[13] C. Huang and J. D. Cao, "Almost sure exponential stability of stochastic cellular neural networks with unbounded distributed delays," *Neurocomputing*, vol. 72, no. 13–15, pp. 3352–3356, 2009.

[14] C. Huang, P. Chen, Y. He, L. Huang, and W. Tan, "Almost sure exponential stability of delayed Hopfield neural networks," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 701–705, 2008.

[15] L. P. Chen, Y. Chai, and R. C. Wu, "Covergence of stochastic fuzzy Cohen-Grossberg neural networks with reaction-diffusion and distributed delays," *Expert Systems with Applications*, vol. 39, no. 5, pp. 5767–5773, 2012.

[16] C. Huang and J. D. Cao, "On pth moment exponential stability of stochastic Cohen-Grossberg neural networks with time-varying delays," *Neurocomputing*, vol. 73, no. 4-6, pp. 986–990, 2010.

[17] C. Huang, Y. He, and H. Wang, "Mean square exponential stability of stochastic recurrent neural networks with time-varying delays," *Computers & Mathematics with Applications*, vol. 56, no. 7, pp. 1773–1778, 2008.

[18] R. Rakkiyappan and P. Balasubramaniam, "Delay-dependent asymptotic stability for stochastic delayed recurrent neural networks with time varying delays," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 526–533, 2008.

[19] L. Ma and F. P. Da, "Mean-square exponential stability of stochastic Hopfield neural networks with time-varying discrete and distributed delays," *Physics Letters A*, vol. 373, no. 25, pp. 2154–2161, 2009.

[20] Y. Sun and J. D. Cao, "pth moment exponential stability of stochastic recurrent neural networks with time-varying delays," *Nonlinear Analysis: Real World Applications*, vol. 8, no. 4, pp. 1171–1185, 2007.

[21] Q. Song and Z. Wang, "Stability analysis of impulsive stochastic Cohen-Grossberg neural networks with mixed time delays," *Physica A*, vol. 387, no. 13, pp. 3314–3326, 2008.

[22] Z. Wang, J. Fang, and X. Liu, "Global stability of stochastic high-order neural networks with discrete and distributed delays," *Chaos, Solitons and Fractals*, vol. 36, no. 2, pp. 388–396, 2008.

[23] X. D. Li and X. L. Fu, "Global asymptotic stability of stochastic Cohen-Grossberg-type BAM neural networks with mixed delays: an LMI approach," *Journal of Computational and Applied Mathematics*, vol. 235, no. 12, pp. 3385–3394, 2011.

[24] X. D. Li, "Existence and global exponential stability of periodic solution for delayed neural networks with impulsive and stochastic effects," *Neurocomputing*, vol. 73, no. 4-6, pp. 749–758, 2010.

[25] X. L. Fu and X. D. Li, "LMI conditions for stability of impulsive stochastic Cohen-Grossberg neural networks with mixed delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 435–454, 2011.

[26] C. H. Wang, Y. G. Kao, and G. W. Yang, "Exponential stability of impulsive stochastic fuzzy reactiondiffusion Cohen-Grossberg neural networks with mixed delays," *Neurocomputing*, vol. 89, pp. 55–63, 2012.

[27] Y. Ou, H. Y. Liu, Y. L. Si, and Z. G. Feng, "Stability analysis of discrete-time stochastic neural networks with time-varying delays," *Neurocomputing*, vol. 73, no. 4–6, pp. 740–748, 2010.

[28] Q. Zhu and J. Cao, "Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 41, no. 2, pp. 341–353, 2011.

[29] Q. Zhu, C. Huang, and X. Yang, "Exponential stability for stochastic jumping BAM neural networks with time-varying and distributed delays," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 1, pp. 52–77, 2011.

[30] J. K. Tian, Y. M. Li, J. Z. Zhao, and S. M. Zhong, "Delay-dependent stochastic stability criteria for Markovian jumping neural networks with mode-dependent time-varying delays and partially known transition rates," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5769–5781, 2012.

[31] P. Wang, D. Li, and Q. Hu, "Bounds of the hyper-chaotic Lorenz-Stenflo system," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2514–2520, 2010.

[32] P. Wang, D. Li, X. Wu, J. Lü, and X. Yu, "Ultimate bound estimation of a class of high dimensional quadratic autonomous dynamical systems," *International Journal of Bifurcation and Chaos*, vol. 21, no. 9, article 2679, 2011.

[33] X. Liao and J. Wang, "Global dissipativity of continuous-time recurrent neural networks with time delay," *Physical Review E*, vol. 68, no. 1, Article ID 016118, 7 pages, 2003.

[34] S. Arik, "On the global dissipativity of dynamical neural networks with time delays," *Physics Letters A*, vol. 326, no. 1-2, pp. 126–132, 2004.

[35] X. Y. Lou and B. T. Cui, "Global robust dissipativity for integro-differential systems modeling neural networks with delays," *Chaos, Solitons and Fractals*, vol. 36, no. 2, pp. 469–478, 2008.

[36] Q. Song and Z. Zhao, "Global dissipativity of neural networks with both variable and unbounded delays," *Chaos, Solitons and Fractals*, vol. 25, no. 2, pp. 393–401, 2005.

[37] H. Jiang and Z. Teng, "Global eponential stability of cellular neural networks with time-varying coefficients and delays," *Neural Networks*, vol. 17, no. 10, pp. 1415–1425, 2004.

[38] H. Jiang and Z. Teng, "Boundedness, periodic solutions and global stability for cellular neural networks with variable coefficients and infinite delays," *Neurocomputing*, vol. 72, no. 10–12, pp. 2455–2463, 2009.

[39] L. Wan and Q. H. Zhou, "Attractor and ultimate boundedness for stochastic cellular neural networks with delays," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 5, pp. 2561–2566, 2011.

[40] L. Wan, Q. H. Zhou, and P. Wang, "Ultimate boundedness of stochastic Hopfield neural networks with time-varying delays," *Neurocomputing*, vol. 74, no. 17, pp. 2967–2971, 2011.

[41] L. Wan, Q. H. Zhou, P. Wang, and J. Li, "Ultimate boundedness and an attractor for stochastic Hopfield neural networks with time-varying delays," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 953–958, 2012.

[42] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Limited, Chichester, UK, 1997.

[43] Q. Luo, X. R. Mao, and Y. Shen, "Generalised theory on asymptotic stability and boundedness of stochastic functional differential equations," *Automatica*, vol. 47, no. 9, pp. 2075–2081, 2011.

[44] K. Gu, "An integral inequality in the stability problem of time-delay systems," in *Proceedings of the 39th IEEE Confernce on Decision and Control*, pp. 2805–2810, Sydney, Australia, December 2000.

[45] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox User'S Guide*, MathWorks, 1995.