Abstract and Applied Analysis

Recent Contributions to Fixed Point Theory and Its Applications

Guest Editors: Mohamed A. Khamsi, Hichem Ben-El-Mechaiekh, and Bernd Schroeder

Recent Contributions to Fixed Point Theory and Its Applications

Abstract and Applied Analysis

Recent Contributions to Fixed Point Theory and Its Applications

Guest Editors: Mohamed A. Khamsi, Hichem Ben-El-Mechaiekh, and Bernd Schroeder

Editorial Board

Contents

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 329815, 1 page
http://dx.doi.org/10.1155/2014/329815

Editorial

Recent Contributions to Fixed Point Theory and Its Applications

Mohamed A. Khamsi,[1,2] Hichem Ben-El-Mechaiekh,[3] and Bernd Schroeder[4]

[1] *Department of Mathematical Sciences, UTEP, El Paso, TX 79968, USA*
[2] *Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia*
[3] *Department of Mathematics, Brock University, St. Catharines, ON, Canada L2S 3A1*
[4] *Program of Mathematics and Statistics, Louisiana Tech University, Ruston, LA 71272, USA*

Correspondence should be addressed to Mohamed A. Khamsi; mohamed@utep.edu

Received 26 February 2014; Accepted 26 February 2014; Published 22 April 2014

The flourishing field of fixed point theory started in the early days of topology with seminal contributions by Poincare, Lefschetz-Hopf, and Leray-Schauder at the turn of the 19th and early 20th centuries. The theory vigorously developed into a dense and multifaceted body of principles, results, and methods from topology and analysis to algebra and geometry as well as discrete and computational mathematics. This interdisciplinary theory *par excellence* provides insight and powerful tools for the solvability aspects of central problems in many areas of current interest in mathematics where topological considerations play a crucial role. Indeed, existence for linear and nonlinear problems is commonly translated into fixed point problems; for example, the existence of solutions to elliptic partial differential equations, the existence of closed periodic orbits in dynamical systems, and more recently the existence of answer sets in logic programming.

The classical fixed point theorems of Banach and Brouwer marked the development of the two most prominent and complementary facets of the theory, namely, *the metric fixed point* theory and *the topological fixed point theory*. The metric theory encompasses results and methods that involve properties of an essentially isometric nature. It originates with the concept of Picard successive approximations for establishing existence and uniqueness of solutions to nonlinear initial value problems of the 1st order and goes back as far as Cauchy, Liouville, Lipschitz, Peano, Fredholm, and most particularly, Emile Picard. However, the Polish mathematician Stefan Banach is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. Metric fixed point theory for important classes of mapping gained respectability and prominence to become a vast field of specialization partly and not only because many results have constructive proofs, but also because it sheds a revealing light on the geometry of normed spaces, not to mention its many applications in industrial fields such as image processing engineering, physics, computer science, economics, and telecommunications.

A particular interest in fixed points for set-valued operators developed towards the mid-20th century with the celebrated extensions of the Brouwer and Lefschetz theorems by Kakutani and Eilenberg-Montgomery, respectively. The Banach contraction principle was later on extended to multivalued contractions by Nadler. The fixed point theory for multivalued maps found numerous applications in control theory, convex and nonsmooth optimization, differential inclusions, and economics. The theory is also used prominently in denotational semantics (e.g., to give meaning to recursive programs). In fact, it is still too early to truly estimate the importance and impact of set-valued fixed point theorems in mathematics in general as the theory is still growing and finding renewed outlets.

This special issue adds to the development of fixed point theory by focusing on most recent contributions. It includes works on nonexpansive mappings in Banach and metric spaces, multivalued mappings in Banach and metric spaces, monotone mappings in ordered spaces, multivalued mappings in ordered spaces, and applications to such nonmetric spaces as modular spaces, as well as applications to logic programming and directed graphs.

Mohamed A. Khamsi
Hichem Ben-El-Mechaiekh
Bernd Schroeder

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 820482, 8 pages
http://dx.doi.org/10.1155/2014/820482

Research Article

A Suzuki Type Coupled Fixed Point Theorem for Generalized Multivalued Mapping

Pushpendra Semwal and Ramesh Chandra Dimri

Department of Mathematics, HNB Garhwal University, Srinagar Garhwal, Pauri Garhwal, Uttarakhand 246174, India

Correspondence should be addressed to Pushpendra Semwal; psrsdm@gmail.com

Received 4 November 2013; Accepted 24 January 2014; Published 15 April 2014

Academic Editor: Mohamed Amine Khamsi

We obtain a new Suzuki type coupled fixed point theorem for a multivalued mapping T from $X \times X$ into $\mathrm{CB}(X)$, satisfying a generalized contraction condition in a complete metric space. Our result unifies and generalizes various known comparable results in the literature. We also give an application to certain functional equations arising in dynamic programming.

1. Introduction and Preliminaries

In 2008, Suzuki [1] introduced a new type of mappings which generalize the well-known Banach contraction principle [2], and, further, Kikkawa and Suzuki [3] proved a Kannan [4] version of mappings.

Theorem 1 (see [3]). *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-map and let $\phi : [0, 1) \to (1/2, 1]$ be defined by*

$$\phi(r) = \begin{cases} 1 & if\ 0 \le r \le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r} & if\ \dfrac{1}{\sqrt{2}} \le r \le 1. \end{cases} \tag{1}$$

Let $\alpha \in [0, 1/2)$ and $r = \alpha/(1-\alpha) \in [0, 1)$. Suppose that

$$\begin{gathered} \phi(r)\, d(x, Tx) \le d(x, y) \text{ implies} \\ d(Tx, Ty) \le \alpha\left(d(x, Tx) + d(y, Ty)\right) \end{gathered} \tag{2}$$

for all $x, y \in X$. Then, T has a unique fixed point z, and $\lim_n T^n x = z$ holds for every $x \in X$.

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ the family of all nonempty, closed bounded subsets of X. Let H be a Hausdorff metric; that is,

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\} \tag{3}$$

for $A, B \in \mathrm{CB}(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

Nadler [5] proved multivalued extension of the Banach contraction principle as follows.

Let (X, d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists $r \in [0, 1)$ such that

$$H(Tx, Ty) \le r d(x, y) \tag{4}$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$. Many fixed point theorems have been proved by various authors as a generalization of Nadler's theorem [6–9]. One of the general fixed point theorems for a generalized multivalued mapping appears in [10].

Theorem 2 (see [11]). *Let (X, d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists a function ϕ from $[0,1)$ into $(0,1]$ defined by*

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \dfrac{1}{2}, \\ 1 - r & \text{if } \dfrac{1}{2} \leq r < 1, \end{cases} \tag{5}$$

such that

$$\phi(r)\, d(x, Tx) \leq d(x, y) \text{ implies} \tag{6}$$

$$H(Tx, Ty) \leq r \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \tag{7}$$

for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.

Bhaskar and Lakshmikantham [12] introduced the concept of coupled fixed point for a mapping F from $X \times X$ to X and established some coupled fixed point theorems in partially ordered sets. As an application, they studied the existence and uniqueness of solution for a periodic boundary value problem associated with a first order ordinary differential equation.

Definition 3. Let (X, d) be a metric space and $F : X \times X \to 2^X$. An element $(x, y) \in X \times X$ is called a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

The aim of this paper is to obtain coupled fixed point for a multivalued mapping $T : X \times X \to \mathrm{CB}(X)$ which satisfies the generalized contraction condition in complete metric spaces. Our results unify, extend, and generalize various known comparable results in the literature.

2. Main Results

Now, we shall prove our main result.

Firstly, we define a nonincreasing function ϕ from [0,1) into [0,1) by

$$\phi(r) = \begin{cases} r & \text{if } 0 \leq r < \dfrac{1}{2}, \\ 1 - r & \text{if } \dfrac{1}{2} \leq r < 1. \end{cases} \tag{8}$$

Theorem 4. *Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $\mathrm{CB}(X)$. Assume that there exists $r \in [0,1)$ such that*

$$\phi(r)\left[d(x, T(x, y)) + d(u, T(u, v))\right] \leq d(x, u) + d(y, v) \tag{9}$$

implies

$$\begin{aligned} &H(T(x, y), T(u, v)) \\ &\quad \leq \frac{r}{2} \max\Bigg\{ d(x, u) + d(y, v), d(x, T(x, y)) \\ &\qquad + d(y, T(y, x)), d(u, T(u, v)) + d(v, T(v, u)), \\ &\qquad \frac{d(x, T(u, v)) + d(y, T(v, u))}{2}, \\ &\qquad \frac{d(u, T(x, y)) + d(v, T(y, x))}{2} \Bigg\} \end{aligned} \tag{10}$$

for all $x, y, u, v \in X$. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

Proof. Let r_1 be a real number such that $0 \leq r < r_1 < 1$, and $x_1, y_1 \in X$ such that $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$. Since $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$, then

$$\begin{aligned} d(x_2, T(x_2, y_2)) &\leq H(T(x_1, y_1), T(x_2, y_2)), \\ d(y_2, T(y_2, x_2)) &\leq H(T(y_1, x_1), T(y_2, x_2)), \end{aligned} \tag{11}$$

and, as $\phi(r) < 1$,

$$\begin{aligned} &\phi(r)\left(d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2))\right) \\ &\quad \leq d(x_2, T(x_2, y_2)) + d(y_2, T(y_2, x_2)) \\ &\quad \leq d(x_2, x_3) + d(y_2, y_3). \end{aligned} \tag{12}$$

Thus, from assumption (10), we have

$$\begin{aligned} &d(x_2, T(x_2, y_2)) \\ &\quad \leq H(T(x_1, y_1), T(x_2, y_2)) \\ &\quad \leq \frac{r}{2} \max\Bigg\{ d(x_1, x_2) + d(y_1, y_2), d(x_1, T(x_1, y_1)) \\ &\qquad + d(y_1, T(y_1, x_1)), d(x_2, T(x, y_2)) \\ &\qquad + d(y_2, T(y_2, x_2)), \\ &\qquad \frac{d(x_1, T(x_2, y_2)) + d(y_1, T(y_2, x_2))}{2}, \\ &\qquad \frac{d(x_2, T(x_1, y_1)) + d(y_2, T(y_1, x_1))}{2} \Bigg\}, \end{aligned} \tag{13}$$

$$\begin{aligned} d(x_2,x_3) &\le \frac{r}{2}\max\Big\{d(x_1,x_2)+d(y_1,y_2),d(x_1,x_2) \\ &\quad +d(y_1,y_2),d(x_2,x_3)+d(y_2,y_3), \\ &\quad \frac{d(x_1,x_3)+d(y_1,y_3)}{2}, \\ &\quad \frac{d(x_2,x_2)+d(y_2,y_2)}{2}\Big\}, \end{aligned} \tag{14}$$

$$\begin{aligned} d(y_2,T(y_2,x_2)) &\le H(T(y_1,x_1),T(y_2,x_2)) \\ &\le \frac{r}{2}\max\Big\{d(x_1,x_2)+d(y_1,y_2),d(x_1,T(x_1,y_1)) \\ &\quad +d(y_1,T(y_1,x_1)),d(x_2,T(x,y_2)) \\ &\quad +d(y_2,T(y_2,x_2)), \\ &\quad \frac{d(x_1,T(x_2,y_2))+d(y_1,T(y_2,x_2))}{2}, \\ &\quad \frac{d(x_2,T(x_1,y_1))+d(y_2,T(y_1,x_1))}{2}\Big\}, \end{aligned} \tag{15}$$

$$\begin{aligned} d(y_2,y_3) &\le \frac{r}{2}\max\Big\{d(x_1,x_2)+d(y_1,y_2),d(x_1,x_2) \\ &\quad +d(y_1,y_2),d(x_2,x_3)+d(y_2,y_3), \\ &\quad \frac{d(x_1,x_3)+d(y_1,y_3)}{2}, \\ &\quad \frac{d(x_2,x_2)+d(y_2,y_2)}{2}\Big\}. \end{aligned} \tag{16}$$

Adding (14) and (16), we have

$$\begin{aligned} d(x_2,x_3)+d(y_2,y_3) &\le r\max\Big\{d(x_1,x_2)+d(y_1,y_2),d(x_2,x_3)+d(y_2,y_3), \\ &\quad \frac{d(x_1,x_3)+d(y_1,y_3)}{2}\Big\}. \end{aligned} \tag{17}$$

If $\max\{d(x_1,x_2)+d(y_1,y_2),d(x_2,x_3)+d(y_2,y_3),(d(x_1,x_3)+d(y_1,y_3))/2\}=d(x_2,x_3)+d(y_2,y_3)$, then we have $d(x_2,x_3)+d(y_2,y_3)\le r(d(x_2,x_3)+d(y_2,y_3))$ as $r<1$; a contradiction. Therefore,

$$\begin{aligned} d(x_2,x_3)+d(y_2,y_3) &\le r\max\Big\{d(x_1,x_2)+d(y_1,y_2),\frac{d(x_1,x_3)+d(y_1,y_3)}{2}\Big\}. \end{aligned} \tag{18}$$

Again, if $\max\{d(x_1,x_2)+d(y_1,y_2),(d(x_1,x_3)+d(y_1,y_3))/2\}=d(x_1,x_2)+d(y_1,y_2)$, we get

$$d(x_2,x_3)+d(y_2,y_3)\le r(d(x_1,x_2)+d(y_1,y_2)) \tag{19}$$

and if $\max\{d(x_1,x_2)+d(y_1,y_2),(d(x_1,x_3)+d(y_1,y_3))/2\}=(d(x_1,x_3)+d(y_1,y_3))/2$, we get

$$d(x_2,x_3)+d(y_2,y_3)\le r\frac{d(x_1,x_3)+d(y_1,y_3)}{2}. \tag{20}$$

Using triangle inequality, we obtain

$$\begin{aligned} d(x_2,x_3)+d(y_2,y_3) &\le r\frac{d(x_1,x_2)+d(x_2,x_3)+d(y_1,y_2)+d(y_2,y_3)}{2}. \end{aligned} \tag{21}$$

This implies

$$d(x_2,x_3)+d(y_2,y_3)\le\left(\frac{r}{2-r}\right)d(x_1,x_2)+d(y_1,y_2). \tag{22}$$

Hence, there exist $x_3,y_3\in X$ with $x_3\in T(x_2,y_2)$ and $y_3\in T(y_2,x_2)$ such that

$$d(x_2,x_3)+d(y_2,y_3)\le r(d(x_1,x_2)+d(y_1,y_2)), \tag{23}$$

$$d(x_2,x_3)+d(y_2,y_3)\le\left(\frac{r}{2-r}\right)d(x_1,x_2)+d(y_1,y_2). \tag{24}$$

Thus, we construct such sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1}\in T(x_n,y_n)$, $y_{n+1}\in T(y_n,x_n)$, and

$$\begin{aligned} d(x_n,x_{n+1})+d(y_n,y_{n+1}) &\le r(d(x_{n-1},x_n)+d(y_{n-1},y_n)), \\ d(x_n,x_{n+1})+d(y_n,y_{n+1}) &\le\left(\frac{r}{2-r}\right)d(x_{n-1},x_n)+d(y_{n-1},y_n). \end{aligned} \tag{25}$$

Then, we have

$$\begin{aligned} \sum_{n=1}^{\infty}(d(x_n,x_{n+1})+d(y_n,y_{n+1})) &\le\sum_{n=1}^{\infty}r^n(d(x_1,x_2)+d(y_1,y_2))<\infty, \\ \sum_{n=1}^{\infty}(d(x_n,x_{n+1})+d(y_n,y_{n+1})) &\le\sum_{n=1}^{\infty}\left(\frac{r}{2-r}\right)^n(d(x_1,x_2)+d(y_1,y_2))<\infty. \end{aligned} \tag{26}$$

Hence, we conclude that in both cases the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, there are some points z and z' in X such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} y_n = z'$.

Now, we shall show that

$$d(z, T(x,y)) + d(z', T(y,x)) \leq r \max\{d(z,x) + d(z', y), d(x, T(x,y)) + d(y, T(y,x))\} \tag{27}$$

for all $x \in X/\{z\}$ and $y \in X/\{z'\}$. Since $x_n \to x$ and $y_n \to y$, there exists $n_0 \in N$ such that $d(z, x_n) + d(z', y_n) \leq (1/3)(d(z,x) + d(z', y))$ for all $n \geq n_0$. Therefore

$$\begin{aligned}\phi(r)\, d(x_n, T(x_n,y_n)) + d(y_n, T(y_n,x_n)) &\leq r(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)) \\ &\leq d(x_n, z) + d(z, x_{n+1}) + d(y_n, z') + d(z', y_{n+1}) \\ &\leq d(x_n, z) + d(y_n, z') + d(z, x_{n+1}) + d(z', y_{n+1}) \\ &\leq \frac{2}{3}(d(x,z) + d(y,z')).\end{aligned} \tag{28}$$

Thus,

$$\phi(r)\, d(x_n, T(x_n,y_n)) + d(y_n, T(y_n,x_n)) \leq \frac{2}{3}(d(x,z) + d(y,z')). \tag{29}$$

Since

$$\begin{aligned}\frac{2}{3}(d(x,z) + d(y,z')) &= (d(x,z) + d(y,z')) - \frac{1}{3}(d(x,z) + d(y,z')) \\ &\leq (d(x,z) + d(y,z')) - (d(x_n,z) + d(y_n,z')) \\ &\leq d(x_n, x) + d(y_n, y),\end{aligned} \tag{30}$$

from (29) we have

$$\phi(r)\, d(x_n, T(x_n,y_n)) + d(y_n, T(y_n,x_n)) \leq d(x_n, x) + d(y_n, y). \tag{31}$$

Then, from (10) we have

$$\begin{aligned}H(T(x_n,y_n), T(x,y)) \leq \frac{r}{2}\max\Big\{& d(x_n,x) + d(y_n,y), d(x_n, T(x_n,y_n)) \\ &+ d(y_n, T(y_n,x_n)), d(x, T(x,y)) \\ &+ d(y, T(y,x)), \\ &\frac{d(x_n, T(x,y)) + d(y_n, T(y,x))}{2}, \\ &\frac{d(x, T(x_n,y_n)) + d(y, T(y_n,x_n))}{2}\Big\},\end{aligned} \tag{32}$$

$$\begin{aligned}H(T(y_n,x_n), T(y,x)) \leq \frac{r}{2}\max\Big\{& d(x_n,x) + d(y_n,y), d(x_n, T(x_n,y_n)) \\ &+ d(y_n, T(y_n,x_n)), d(x, T(x,y)) \\ &+ d(y, T(y,x)), \\ &\frac{d(x_n, T(x,y)) + d(y_n, T(y,x))}{2}, \\ &\frac{d(x, T(x_n,y_n)) + d(y, T(y_n,x_n))}{2}\Big\}.\end{aligned} \tag{33}$$

By adding, (32) and (33), we get

$$\begin{aligned}d(x_{n+1}, T(x,y)) + d(y_{n+1}, T(y,x)) \leq r\max\Big\{& d(x_n,x) + d(y_n,y), d(x_n, x_{n+1}) \\ &+ d(y_n, y_{n+1}), d(x, T(x,y)) \\ &+ d(y, T(y,x)), \\ &\frac{d(x_n, T(x,y)) + d(y_n, T(y,x))}{2}, \\ &\frac{d(x, x_{n+1}) + d(y, y_{n+1})}{2}\Big\}.\end{aligned} \tag{34}$$

Letting n tends to ∞, we obtain

$$\begin{aligned}d(z, T(x,y)) + d(z', T(y,x)) \leq r\max\Big\{& d(z,x) + d(z',y), d(x, T(x,y)) \\ &+ d(y, T(y,x)), \\ &\frac{d(z, T(x,y)) + d(z', T(y,x))}{2}\Big\}.\end{aligned} \tag{35}$$

Hence, we have:

$$d(z,T(x,y))+d(z',T(y,x)) \leq r\max\{d(z,x)+d(z',y),d(x,T(x,y)) +d(y,T(y,x))\}. \tag{36}$$

Now, we shall prove that $z\in T(z,z')$ and $z'\in T(z',z)$.

Case 1. First, we consider the case $0\leq r<1/2$. Suppose, on contrary, that $z\notin T(z,z')$ and $z'\notin (z',z)$. Let $a\in T(z,z')$ and $b\in T(z',z)$ be such that $2r\{d(a,z)+d(b,z')\}<d(z,T(z,z'))+d(z',T(z',z))$. Since $a\in T(z,z')$ and $b\in T(z',z)$ imply $a\neq z$ and $b\neq z'$, from (10), we have

$$d(z,T(a,b))+d(z',T(b,a)) \leq r\max\{d(z,a)+d(z',b),d(a,T(a,b))+d(b,T(b,a))\}. \tag{37}$$

On the other hand, since $\phi(r)d(z,T(z,z'))+d(z',T(z',z))\leq d(z,T(z,z'))+d(z',T(z',z))<d(z,a)+d(z',b)$, from (10), we get

$$\begin{aligned} H(T(z,z'),T(a,b)) \leq \frac{r}{2}\max\Big\{&d(z,a)+d(z',b),d(z,T(z,z')) \\ &+d(z',T(z',z)),d(a,T(a,b)) \\ &+d(b,T(b,a)), \\ &\frac{d(z,T(a,b))+d(z',T(b,a))}{2}, \\ &\frac{d(a,T(z,z'))+d(b,T(z',z))}{2}\Big\}, \end{aligned} \tag{38}$$

$$\begin{aligned} H(T(z',z),T(b,a)) \leq \frac{r}{2}\max\Big\{&d(z,a)+d(z',b),d(z,T(z,z')) \\ &+d(z',T(z',z)),d(a,T(a,b)) \\ &+d(b,T(b,a)), \\ &\frac{d(z,T(a,b))+d(z',T(b,a))}{2}, \\ &\frac{d(a,T(z,z'))+d(b,T(z',z))}{2}\Big\}. \end{aligned} \tag{39}$$

By adding, (38) and (39), we have

$$\begin{aligned} d(a,T(a,b))+d(b,T(b,a)) \leq r\max\Big\{&d(z,a)+d(z',b),d(a,T(a,b)) \\ &+d(b,T(b,a)), \\ &\frac{d(z,T(a,b))+d(z',T(b,a))}{2}\Big\}. \end{aligned} \tag{40}$$

This implies

$$d(a,T(a,b))+d(b,T(b,a)) \leq r\{d(z,a)+d(z',b)\}<d(z,a)+d(z',b). \tag{41}$$

And, from (40), we have

$$d(z,T(a,b))+d(z',T(b,a))\leq r\{d(z,a)+d(z',b)\}. \tag{42}$$

Therefore, we obtain

$$\begin{aligned} d(z,T(z,z'))+d(z',T(z',z)) &\leq d(z,T(a,b))+H(T(a,b),T(z,z')) \\ &\quad+d(z',T(b,a))+H(T(b,a),T(z',z)) \\ &\leq 2r(d(z,a)+d(z',b)) \\ &< d(z,T(z,z'))+d(z',T(z',z)). \end{aligned} \tag{43}$$

This is a contradiction. As a result, we have $z\in T(z,z')$ and $z'\in T(z',z)$.

Case 2. Now, we consider the case $1/2\leq r<1$. We shall first prove that

$$\begin{aligned} H(T(x,y),T(z,z'))+H(T(y,x),T(z',z)) \leq r\max\Big\{&d(x,z)+d(y,z'),d(x,T(x,y)) \\ &+d(y,T(y,x)),d(z,T(z,z')) \\ &+d(z',T(z',z)), \\ &\frac{d(x,T(z,z'))+d(y,T(z',z))}{2}, \\ &\frac{d(z,T(x,y))+d(z',T(y,x))}{2}\Big\} \end{aligned} \tag{44}$$

for all $x,y\in X$. If $x=z$ and $y=z'$, then the previous equation (44) obviously holds. Hence, let us assume $x\neq z$

and $y \neq z'$. Then, for every $n \in N$, there exist sequences $t_n \in T(x, y)$ and $t_n' \in T(y, x)$ such that

$$\begin{aligned} d(z, t_n) + d(z', t_n') &\leq r\{d(z, T(x, y)) + d(z', T(y, x))\} \\ &\quad + \frac{1}{n} d(z, x) + d(z', y). \end{aligned} \tag{45}$$

Then for all $n \in N$, we have

$$\begin{aligned} &d(x, T(x, y)) + d(y, T(y, x)) \\ &\quad \leq d(x, t_n) + d(y, t_n') \\ &\quad \leq d(x, z) + d(z, t_n) + d(y, z') + d(z', t_n') \\ &\quad \leq d(x, z) + d(y, z') + d(z, T(x, y)) + d(z', T(y, x)) \\ &\qquad + \frac{1}{n}(d(x, z) + d(y, z')) \\ &\quad \leq d(x, z) + d(y, z') \\ &\qquad + r\max\{d(x, z) + d(y, z'), d(x, T(x, y)) \\ &\qquad\qquad + d(y, T(y, x))\} + \frac{1}{n}(d(x, z) + d(y, z')). \end{aligned} \tag{46}$$

If $d(x,z) + d(y,z') \geq d(x,T(x,y)) + d(y,T(y,x))$, then we get

$$\begin{aligned} &d(x, T(x, y)) + d(y, T(y, x)) \\ &\quad \leq \left(1 + r + \frac{1}{n}\right)(d(x, z) + d(y, z')). \end{aligned} \tag{47}$$

Letting $n \to \infty$, we have

$$d(x, T(x, y)) + d(y, T(y, x)) \leq (1 + r)(d(x, z) + d(y, z')). \tag{48}$$

Thus

$$\begin{aligned} &\phi(r)(d(x, T(x, y)) + d(y, T(y, x))) \\ &\quad \leq (1 - r)(d(x, T(x, y)) + d(y, T(y, x))) \\ &\quad \leq \frac{1}{1 + r}(d(x, T(x, y)) + d(y, T(y, x))) \\ &\quad \leq (d(x, z) + d(y, z')), \end{aligned} \tag{49}$$

and, from (10), we obtain (44). If $d(x,z)+d(y,z') < d(x,T(x,y)) + d(y,T(y,x))$, then

$$\begin{aligned} &d(x, T(x, y)) + d(y, T(y, x)) \\ &\quad \leq (d(x, z) + d(y, z')) \\ &\qquad + r\{d(x, T(x, y)) + d(y, T(y, x))\} \\ &\qquad + \frac{1}{n}(d(x, z) + d(y, z')). \end{aligned} \tag{50}$$

And, therefore

$$\begin{aligned} &(1 - r)(d(x, T(x, y)) + d(y, T(y, x))) \\ &\quad \leq \left(1 + \frac{1}{n}\right)(d(x, z) + d(y, z')). \end{aligned} \tag{51}$$

Letting $n \to \infty$, we have

$$\begin{aligned} &\phi(r)(d(x, T(x, y)) + d(y, T(y, x))) \\ &\quad \leq (d(x, z) + d(y, z')), \end{aligned} \tag{52}$$

and, thus from condition (56), we obtain (44).

Finally, from (44), we obtain

$$\begin{aligned} &d(z, T(z, z')) + d(z', T(z', z)) \\ &\quad = \lim_{n\to\infty}(d(x_{n+1}, T(z, z')) + d(y_{n+1}, T(z', z))) \\ &\quad \leq \lim_{n\to\infty} r\max\Big\{d(x_n, z) + d(y_n, z'), d(x_n, x_{n+1}) \\ &\qquad\qquad + d(y_n, y_{n+1}), d(z, T(z, z')) \\ &\qquad\qquad + d(z', T(z', z)), \\ &\qquad\qquad \frac{d(x_n, T(z, z')) + d(z', T(z', z))}{2}, \\ &\qquad\qquad \frac{d(z, x_{n+1}) + d(z', y_{n+1})}{2}\Big\} \\ &\quad \leq r\max\{d(z, z) + d(z', z'), d(z, T(z, z')) \\ &\qquad\qquad + d(z', T(z', z))\} \\ &\quad = rd(z, T(z, z')) + d(z', T(z', z)). \end{aligned} \tag{53}$$

Hence, as $r < 1$, we obtain

$$d(z, T(z, z')) + d(z', T(z', z)) = 0. \tag{54}$$

This implies that $z \in T(z, z')$ and $z' \in T(z', z)$. □

Corollary 5. *Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $\mathrm{CB}(X)$. Assume that there exists $r \in [0, 1)$ such that $\phi(r)[d(x, T(x, y)) + d(u, T(u, v))] \leq (d(x, u) + d(y, v))$ implies*

$$\begin{aligned} &H(T(x, y), T(u, v)) \\ &\quad \leq \frac{r}{2}\max\{d(x, u) + d(y, v), d(x, T(x, y)) \\ &\qquad\qquad + d(y, T(y, x)), \\ &\qquad\qquad d(u, T(u, v)) + d(v, T(v, u))\} \end{aligned} \tag{55}$$

for all $x, y, u, v \in X$, where the function ϕ is defined as in Theorem 4. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

Corollary 6. *Let (X, d) be a complete metric space and let T be a mapping from $X \times X$ into $\mathrm{CB}(X)$. Assume that there exists $r \in [0, 1)$ such that $\phi(r)[d(x, T(x, y)) + d(u, T(u, v))] \leq (d(x, u) + d(y, v))$ implies*

$$\begin{aligned} &H(T(x,y), T(u,v)) \\ &\quad \leq \frac{r}{2} \max \Big\{ d(x,u) + d(y,v), d(x, T(x,y)) \\ &\qquad + d(y, T(y,x)), d(u, T(u,v)) + d(v, T(v,u)), \\ &\qquad \frac{d(u, T(x,y)) + d(v, T(y,x))}{2} \Big\} \end{aligned} \tag{56}$$

for all $x, y, u, v \in X$, where the function ϕ is defined as in Theorem 4. Then, there exist $z, z' \in X$ such that $z \in T(z, z')$ and $z' \in T(z', z)$.

3. An Application

The existence and uniqueness of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using various fixed point theorems. In this paper, we shall prove the existence and uniqueness of a solution for a class of functional equations using Corollary 6.

In this section, we assume that U and V are Banach spaces. $W \subset U$, $D \subset V$, and $\mathfrak{R}$ is a field of real numbers. Let $B(W)$ denote the set of all the real valued functions on W. It is known that $B(W)$ endowed with the metric

$$d_B(h,k) = \sup_{x \in W} |h(x) - k(x)|, \quad h, k \in B(W), \tag{57}$$

is a complete metric space. According to Bellman and Lee, the basic form of the functional equation of dynamic programming is given as

$$p(x) = \sup_{y} H(x, y, p(\tau(x,y))), \tag{58}$$

where x and y represent the state and decision vectors, respectively, $\tau : W \times D \to W$ represents the transformation of the process, and $p(x)$ represents the optimal return function with initial state x. In this section, we study the existence and uniqueness of a solution of the following functional equation:

$$p(x) = \sup_{y} [g(x,y) + G(x, y, p(\tau(x,y)))], \tag{59}$$

where $g : W \times D \to \mathfrak{R}$ and $G : W \times D \times \mathfrak{R} \to \mathfrak{R}$ are bounded functions. In this section, we study the existence and uniqueness of a solution of (59) in the following new form. Let a function ϕ be defined as in Theorem 4 and let the mapping T be defined by

$$\begin{aligned} &T(h,k)(x) \\ &\quad = \sup_{y \in D} [g(x,y) + G(x, y, h(\tau(x,y)), k(\tau(x,y)))], \\ &\qquad x \in W. \end{aligned} \tag{60}$$

Theorem 7. *Suppose that there exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k, h', k' \in B(W)$, and $t, s \in W$, the inequality*

$$\begin{aligned} &\phi(r) \left[d_B(h, T(h(t), k(s))) + d_B\left(h', T\left(h'(s), k'(t)\right)\right) \right] \\ &\quad \leq d\left(h(t), h'(s)\right) + d\left(k(s), k'(t)\right) \end{aligned} \tag{61}$$

implies

$$\begin{aligned} &\Big| G(x, y, h(\tau(x,y)), k(\tau(x,y))) \\ &\quad - G\left(x, y, h'(\tau(x,y)), k'(\tau(x,y))\right) \Big| \\ &\quad \leq r M\left(h(t), k(s), h'(s), k'(t)\right), \end{aligned} \tag{62}$$

where

$$\begin{aligned} &M\left(h(t), k(s), h'(s), k'(t)\right) \\ &\quad = \max \Big\{ \left|h(t) - h'(s)\right| + \left|k(s) - k'(t)\right|, \\ &\qquad |h(t) - T(h(t), k(s))| + |k(s) - T(k(s), h(t))|, \\ &\qquad \left|h(s)' - T\left(h'(s), k'(t)\right)\right| \\ &\qquad + \left|k'(t) - T\left(k'(t), h'(s)\right)\right|, \\ &\qquad \frac{\left|h'(s), T(h(t), k(s))\right| + \left|k'(t) - T(k(s), h(t))\right|}{2} \Big\}. \end{aligned} \tag{63}$$

Then, the functional equation (59) has a unique bounded solution in $B(W)$.

Proof. Note that T is a map from $B(W) \times B(W)$ onto $B(W)$ and that $(B(W), d_B)$ is a complete metric space, where d_B is the metric defined in (57). Let λ be an arbitrary positive real number, and $h, h', k, k' \in B(W)$.For arbitrary $x \in W$ and $y \in D$ so that

$$T(h,k)(x) < [g(x,y) + G(x, y, h(\tau_1), k(\tau_2))] + \lambda, \tag{64}$$

$$T\left(h', k'\right)(x) < \left[g(x,y) + G\left(x, y, h'(\tau_2), k'(\tau_1)\right)\right] + \lambda. \tag{65}$$

From the definition of mapping T, we have

$$T(h,k)(x) \geq [g(x,y) + G(x,y,h(\tau_2),k(\tau_1))], \quad (66)$$

$$T(h',k')(x) \geq [g(x,y) + G(x,y,h'(\tau_1),k'(\tau_2))]. \quad (67)$$

If inequality (61) holds, then from (64) and (67), we get

$$\begin{aligned} T(h,k)(x) - T(h',k')(x) &< G(x,y,h(\tau_1),k(\tau_2)) - G(x,y,h'(\tau_1),k'(\tau_2)) + \lambda \\ &\leq \left|G(x,y,h(\tau_1),k(\tau_2)) - G(x,y,h'(\tau_1),k'(\tau_2))\right| + \lambda \\ &\leq rM(h(t),k(s),h'(s),k'(t)) + \lambda. \end{aligned} \quad (68)$$

Similarly, (65) and (66) implies that

$$\begin{aligned} T(h',k')(x) - T(h,k)(x) \\ \leq rM(h(t),k(s),h'(s),k'(t)) + \lambda. \end{aligned} \quad (69)$$

Hence, from (68) and (69), we have

$$\begin{aligned} \left|T(h,k)(x) - T(h',k')(x)\right| \\ \leq rM(h(t),k(s),h'(s),k'(t)) + \lambda. \end{aligned} \quad (70)$$

Since inequality (70) is true for any $x \in W$ and arbitrary $\lambda > 0$, then (61) implies

$$\begin{aligned} &d_B(T(h(t),k(s)),T(h'(s),k'(t))) \\ &\leq r\max\Bigg\{ \left|h(t) - h'(s)\right| + \left|k(s) - k'(t)\right|, \\ &\qquad |h(t) - T(h(t),k(s))| \\ &\qquad + |k(s) - T(k(s),h(t))|, \\ &\qquad \left|h(s)' - T(h'(s),k'(t))\right| \\ &\qquad + \left|k'(t) - T(k'(t),h'(s))\right|, \\ &\qquad \frac{\left|h'(s),T(h(t),k(s))\right| + \left|k'(t) - T(k(s),h(t))\right|}{2} \Bigg\}. \end{aligned} \quad (71)$$

Therefore, all the conditions of Corollary 6 are met for the mapping T, and hence the functional equation (59) has a unique bounded solution. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[2] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[3] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 649749, 8 pages, 2008.

[4] R. Kannan, "Some results on fixed points. II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[5] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[6] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989.

[7] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 655–666, 1995.

[8] P. V. Semenov, "On fixed points of multivalued contractions," *Functional Analysis and Its Applications*, vol. 36, no. 2, pp. 159–161, 2002.

[9] L. Ćirić, "Multi-valued nonlinear contraction mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2716–2723, 2009.

[10] L. B. Ćirić, "Fixed points for generalized multi-valued contractions," *Matematički Vesnik*, vol. 9, no. 24, pp. 265–272, 1972.

[11] D. Doric and R. Lazović, "Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications," *Fixed Point Theory and Applications*, vol. 2011, article 40, 2011.

[12] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 580508, 8 pages
http://dx.doi.org/10.1155/2014/580508

Research Article

Expansive Mappings and Their Applications in Modular Space

A. Azizi, R. Moradi, and A. Razani

Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34149-16818, Iran

Correspondence should be addressed to A. Razani; razani@ipm.ir

Received 19 September 2013; Revised 30 January 2014; Accepted 31 January 2014; Published 14 April 2014

Academic Editor: Mohamed Amine Khamsi

Some fixed point theorems for ρ-expansive mappings in modular spaces are presented. As an application, two nonlinear integral equations are considered and the existence of their solutions is proved.

1. Introduction

Let (X, d) be a metric space and B a subset of X. A mapping $T : B \to X$ is said to be expansive with a constant $k > 1$ such that

$$d(Tx, Ty) \geq kd(x, y) \quad \forall x, y \in B. \tag{1}$$

Xiang and Yuan [1] state a Krasnosel'skii-type fixed point theorem as follows.

Theorem 1 (see [1]). *Let $(X, \|\cdot\|)$ be a Banach space and $K \subset X$ a nonempty, closed, and convex subset. Suppose that T and S map K into X such that*

(I) *S is continuous; $S(K)$ resides in a compact subset of X;*

(II) *T is an expansive mapping;*

(III) *$z \in S(K)$ implies that $T(K) + z \supset K$, where $T(K) + z = \{y + z \mid y \in T(K)\}$.*

Then there exists a point $x^* \in K$ with $Sx^* + Tx^* = x^*$.

For other related results, see also [2, 3].

In this paper, we study some fixed point theorems for $S+T$, where T is ρ-expansive and $S(B)$ resides in a compact subset of X_ρ, where B is a closed, convex, and nonempty subset of X_ρ and $T, S : B \to X_\rho$. Our results improve the classical version of Krasnosel'skii fixed point theorems in modular spaces.

Finally, as an application, we study the existence of a solution of some nonlinear integral equations in modular function spaces.

In order to do this, first, we recall the definition of modular space (see [4–6]).

Definition 2. Let X be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C})$. Then we have the following.

(a) A functional $\rho : X \to [0, \infty]$ is called modular if

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$;

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$.

If (iii) is replaced by

(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$, for all $x, y \in X$, then the modular ρ is called a convex modular.

(b) A modular ρ defines a corresponding modular space, that is, the space X_ρ given by

$$X_\rho = \{x \in X \mid \rho(\alpha x) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0\}. \tag{2}$$

(c) If ρ is convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0;\ \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}. \tag{3}$$

Remark 3. Note that ρ is an increasing function. Suppose that $0 < a < b$; then property (iii), with $y = 0$, shows that $\rho(ax) = \rho((a/b)(bx)) \leq \rho(bx)$.

Definition 4. Let X_ρ be a modular space. Then we have the following.

(a) A sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ is said to be

(i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$;

(ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.

(c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\in\mathbb{N}} \subset B$ and $x_n \to x$ then $x \in B$.

(d) A subset $B \subset X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$, for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ-diameter of B.

(e) ρ has the Fatou property if

$$\rho(x - y) \leq \liminf \rho(x_n - y_n), \tag{4}$$

whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

(f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

2. Expansive Mapping in Modular Space

In 2005, Hajji and Hanebaly [7] presented a modular version of Krasnosel'skii fixed point theorem, for a ρ-contraction and a ρ-completely continuous mapping.

Using the same argument as in [1], we state the modular version of Krasnosel'skii fixed point theorem for $S + T$, where T is a ρ-expansive mapping and the image of B under S ; that is, $S(B)$ resides in a compact subset of X_ρ, where B is a subset of X_ρ.

Due to this, we recall the following definitions and theorems.

Definition 5. Let X_ρ be a modular space and B a nonempty subset of X_ρ. The mapping $T : B \to X_\rho$ is called ρ-expansive mapping, if there exist constants $c, k, l \in \mathbb{R}^+$ such that $c > l$, $k > 1$ and

$$\rho(l(Tx - Ty)) \geq k\rho(c(x - y)), \tag{5}$$

for all $x, y \in B$.

Example 6. Let $X_\rho = B = \mathbb{R}^+$ and consider $T : B \to B$ with $Tx = x^n + 4x + 5$ for $x \in B$ and $n \in \mathbb{N}$. Then for all $x, y \in B$, we have

$$\begin{aligned} |Tx - Ty| &= |x^n - y^n + 4(x - y)| \\ &= \left|(x - y)\left(x^{n-1} + yx^{n-2} + \cdots + y^{n-1}\right) + 4(x - y)\right| \\ &\geq 4|x - y|. \end{aligned} \tag{6}$$

Therefore T is an expansive mapping with constant $k = 4$.

Theorem 7 (Schauder's fixed point theorem, page 825; see [1, 8]). *Let $(X, \|\cdot\|)$ be a Banach space and $K \subset X$ is a nonempty, closed, and convex subset. Suppose that the mapping $S : K \to K$ is continuous and $S(K)$ resides in a compact subset of X. Then S has at least one fixed point in K.*

We need the following theorem from [6, 9].

Theorem 8 (see [6, 9]). *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. $T : B \to B$ is a mapping such that there exist $c, k, l \in \mathbb{R}^+$ such that $c > l$, $0 < k < 1$ and for all $x, y \in B$ one has*

$$\rho(c(Tx - Ty)) \leq k\rho(l(x - y)). \tag{7}$$

Then there exists a unique fixed point $z \in B$ such that $Tz = z$.

Theorem 9. *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. $T : B \to X_\rho$ is a ρ-expansive mapping satisfying inequality (5) and $B \subset T(B)$. Then there exists a unique fixed point $z \in B$ such that $Tz = z$.*

Proof. We show that operator T is a bijection from B to $T(B)$. Let x_1 and x_2 be in B such that $Tx_1 = Tx_2$; by inequality (5), we have $x_1 = x_2$; also since $B \subset T(B)$ it follows that the inverse of $T : B \to T(B)$ exists. For all $x, y \in T(B)$,

$$\rho(c(fx - fy)) \leq \frac{1}{k}\rho(l(x - y)), \tag{8}$$

where $f = T^{-1}$. We consider $f = T^{-1}|_B : B \to B$, where $T^{-1}|_B$ denotes the restriction of the mapping T^{-1} to the set B. Since $B \subset T(B)$, then f is a ρ-contraction. Also since B is a ρ-closed subset of X_ρ, then, by Theorem 8, there exists a $z \in B$ such that $fz = z$. Also z is a fixed point of T.

For uniqueness, let z and w be two arbitrary fixed points of T; then

$$\begin{aligned} \rho(c(z - w)) \geq \rho(l(z - w)) &= \rho(l(Tz - Tw)) \\ &\geq k\rho(c(z - w)); \end{aligned} \tag{9}$$

hence $(k - 1)\rho(c(z - w)) \leq 0$ and $z = w$. □

We need the following lemma for the main result.

Lemma 10. *Suppose that all conditions of Theorem 9 are fulfilled. Then the inverse of $f := I - T : B \to (I - T)(B)$ exists and*

$$\rho\left(c\left(f^{-1}x - f^{-1}y\right)\right) \leq \frac{1}{k-1}\rho\left(l'(x - y)\right), \tag{10}$$

for all $x, y \in f(B)$, where $l' = \alpha l$ and α is conjugate of c/l; that is, $(l/c) + (1/\alpha) = 1$ and $c > 2l$.

Proof. For all $x, y \in B$,

$$\begin{aligned} \rho(l(Tx - Ty)) &= \rho(l((x - fx) - (y - fy))) \\ &\leq \rho(c(x - y)) + \rho(\alpha l(fx - fy)); \end{aligned} \tag{11}$$

$$k\rho(c(x - y)) - \rho(c(x - y)) \leq \rho(\alpha l(fx - fy)),$$

then

$$(k-1)\,\rho\left(c\left(x-y\right)\right) \le \rho\left(l'\left(fx-fy\right)\right). \quad (12)$$

Now, we show that f is an injective operator. Let $x, y \in B$ and $fx = fy$; then by inequality (12), $(k-1)\rho(c(x-y)) \le 0$ and $x = y$. Therefore f is an injective operator from B into $f(B)$, and the inverse of $f : B \to f(B)$ exists. Also for all $x, y \in f(B)$, we have $f^{-1}x, f^{-1}y \in B$. Then for all $x, y \in f(B)$, by inequality (12) we get

$$\rho\left(c\left(f^{-1}x - f^{-1}y\right)\right) \le \frac{1}{k-1}\rho\left(l'\left(x-y\right)\right). \quad (13)$$

□

Theorem 11. *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. Suppose that*

(I) *$S : B \to X_\rho$ is a ρ-continuous mapping and $S(B)$ resides in a ρ-compact subset of X_ρ;*

(II) *$T : B \to X_\rho$ is a ρ-expansive mapping satisfying inequality* (5) *such that $c > 2l$;*

(III) *$x \in S(B)$ implies that $B \subset x + T(B)$, where $T(B) + x = \{y + x \mid y \in T(B)\}$.*

There exists a point $z \in B$ such that $Sz + Tz = z$.

Proof. Let $w \in S(B)$ and $T_w = T + w$. Consider the mapping $T_w : B \to X_\rho$; then by Theorem 9, the equation $Tx + w = x$ has a unique solution $x = \eta(w)$. Now, we show that η is a ρ-contraction. For $w_1, w_2 \in S(B)$, $T(\eta(w_1)) + w_1 = \eta(w_1)$ and $T(\eta(w_2)) + w_2 = \eta(w_2)$. Applying the same technique in Lemma 10,

$$(k-1)\,\rho\left(c\left(\eta\left(w_1\right) - \eta\left(w_2\right)\right)\right) \le \rho\left(l'\left(w_1 - w_2\right)\right), \quad (14)$$

where $l' = \alpha l$. Then

$$\rho\left(c\left(\eta\left(w_1\right) - \eta\left(w_2\right)\right)\right) \le \frac{1}{k-1}\rho\left(l'\left(w_1 - w_2\right)\right). \quad (15)$$

Therefore, mapping $\eta : S(B) \to B$ is a ρ-contraction and hence is a ρ-continuous mapping. By condition (I), $\eta S : B \to B$ is also ρ-continuous mapping and, by Δ_2-condition, ηS is $\|\cdot\|_\rho$-continuous mapping. Also $\eta S(B)$ resides in a $\|\cdot\|_\rho$-compact subset of X_ρ. Then using Theorem 7, there exists a $z \in B$ such that $z = \eta(S(z))$ which implies that $Tz + Sz = z$. □

The following theorem is another version of Theorem 11.

Theorem 12. *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. Suppose that*

(I) *$S : B \to X_\rho$ is a ρ-continuous mapping and $S(B)$ resides in a ρ-compact subset of X_ρ;*

(II) *$T : B \to X_\rho$ or $T : X_\rho \to X_\rho$ is a ρ-expansive mapping satisfying inequality* (5) *such that $c > 2l$;*

(III) *$S(B) \subset (I - T)(X_\rho)$ and $[x = Tx + Sy, y \in B$ implies that $x \in B]$ or $S(B) \subset (I - T)(B)$.*

Then there exists a point $z \in B$ such that $Sz + Tz = z$.

Proof. By condition (III), for each $w \in B$, there exists $x \in X_\rho$ such that $x - Tx = Sw$. If $S(B) \subset (I - T)(B)$, then $x \in B$; if $S(B) \subset (I - T)(X_\rho)$, then by Lemma 10 and condition (III), $x = (I - T)^{-1}Sw \in B$. Now $(I - T)^{-1}$ is a ρ-continuous and so $(I - T)^{-1}S$ is a ρ-continuous mapping of B into B. Since $S(B)$ resides in a ρ-compact subset of X_ρ, so $(I-T)^{-1}S(B)$ resides in a ρ-compact subset of the closed set B. By using Theorem 7, there exists a fixed point $z \in B$ such that $z = (I - T)^{-1}Sz$. □

Using the same argument as in [2], we can state a new version of Theorem 11, where S is ρ-sequentially continuous.

Definition 13. Let X_ρ be a modular space and B a subset of X_ρ. A mapping $T : B \to X_\rho$ is said to be

(1) ρ-sequentially continuous on the set B if for every sequence $\{x_n\} \subset B$ and $x \in B$ such that $\rho(x_n - x) \to 0$, then $\rho(Tx_n - Tx) \to 0$;

(2) ρ-closed if for every sequence $\{x_n\} \subset B$ such that $\rho(x_n - x) \to 0$ and $\rho(Tx_n - y) \to 0$, then $Tx = y$.

Definition 14. Let X_ρ be a modular space and B, C two subsets of X_ρ. Suppose that $T : B \to X_\rho$ and $S : C \to X_\rho$ are two mappings. Define

$$F = \left\{x \in B : x = Tx + Sy \text{ for some } y \in C\right\}. \quad (16)$$

Theorem 15. *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. Suppose that*

(I) *$S : B \to X_\rho$ is ρ-sequentially continuous;*

(II) *$T : B \to X_\rho$ is a ρ-expansive mapping satisfying inequality* (5) *such that $c > 2l$;*

(III) *$x \in S(B)$ implies that $B \subset x + T(B)$, where $T(B) + x = \{y + x \mid y \in T(B)\}$;*

(IV) *T is ρ-closed in F and F is relatively ρ-compact.*

Then there exists a point $z \in B$ such that $Sz + Tz = z$.

Proof. Let $w \in B$, and $T_{Sw} = T + Sw$. One considers the mapping $T_{Sw} : B \to X_\rho$; by Theorem 9, the equation

$$Tx + Sw = x \quad (17)$$

has a unique solution $x = \eta(Sw) \in B$.

Now, we show that $\eta S = (I - T)^{-1}$ exists. For any $w_1, w_2 \in B$ and by the same technique of Lemma 10, we have

$$\rho\left(c\left(\eta\left(Sw_1\right) - \eta\left(Sw_2\right)\right)\right) \le \frac{1}{k-1}\rho\left(l'\left(w_1 - w_2\right)\right), \quad (18)$$

where $l' = \alpha l$. This implies that $\eta S = (I - T)^{-1}$ exists and for all $w \in B$, $\eta Sw = (I - T)^{-1}Sw$ and $\eta S(B) \subset F$.

We show that ηS is ρ-sequentially continuous in B. Let $\{x_n\}$ be a sequence in B and $x \in B$ such that $\rho(x_n - x) \to 0$. Since $\eta S(x_n) \in F$ and F is relatively ρ-compact, then there exists $z \in B$ such that $\rho(\eta Sx_n - z) \to 0$. On the other hand, by condition (I), $\rho(Sx_n - Sx) \to 0$. Thus by (17), we get

$$T\left(\eta Sx_n\right) + Sx_n = \eta Sx_n; \quad (19)$$

then

$$\rho\left(\frac{T(\eta Sx_n)-(z-Sx)}{2}\right)=\rho\left(\frac{(\eta Sx_n-Sx_n)-(z-Sx)}{2}\right)$$
$$\leq \rho(\eta Sx_n - z)+\rho(Sx_n - Sx); \tag{20}$$

therefore when $n \to \infty$, condition (IV) implies that $Tz = z - Sx$; that is, $z = \eta Sx$ and

$$\rho(\eta Sx_n - \eta Sx) \longrightarrow 0; \tag{21}$$

then ηS is ρ-sequentially continuous in F. By Δ_2-condition, ηS is $\|\cdot\|_\rho$-sequentially continuous. Let $H = \overline{\mathrm{co}}^{\|\cdot\|_\rho} F$, where $\overline{\mathrm{co}}^{\|\cdot\|_\rho}$ denotes the closure of the convex hull in the sense of $\|\cdot\|_\rho$. Then $H \subset B$ and is a compact set. Therefore ηS is $\|\cdot\|_\rho$-sequentially continuous from H into H. Then using Theorem 7, ηS has a fixed point $z \in H$ such that $\eta Sz = z$. From (17), we have

$$T(\eta Sz)+Sz=\eta Sz; \tag{22}$$

that is, $Tz + Sz = z$. □

The following theorem is another version of Theorem 15.

Theorem 16. *Let X_ρ be a ρ-complete modular space. Assume that ρ is a convex modular satisfying the Δ_2-condition and B is a nonempty, ρ-closed, and convex subset of X_ρ. Suppose that*

(I) *$S : B \to X_\rho$ is ρ-sequentially continuous;*

(II) *$T : B \to X_\rho$ is a ρ-expansive mapping satisfying inequality (5), such that $c > 2l$;*

(III) *$S(B) \subset (I-T)(X_\rho)$ and $[x = Tx + Sy, y \in B]$ implies that $x \in B$ (or $S(B) \subset (I-T)(B)$).*

(IV) *T is ρ-closed in F and F is relatively ρ-compact.*

Then there exists a point $z \in B$ such that $Sz + Tz = z$.

Proof. By (III) for each $w \in B$, there exists $x \in X_\rho$ such that $x - Tx = Sw$ and $x = (I-T)^{-1}Sw \in B$. By the same technique of Theorem 15, $(I-T)^{-1}S : B \to B$ is ρ-sequentially continuous and there exists a $z \in B$ such that $z = (I-T)^{-1}Sz$. □

3. Integral Equation for ρ-Expansive Mapping in Modular Function Spaces

In this section, we study the following integral equation:

$$x(t)=\phi(t,x(t))+\int_0^t \psi(t,s,x(s))\,ds, \quad x\in C(I,L^\varphi), \tag{23}$$

where L^φ is the Musielak-Orlicz space and $I = [0, b] \subset \mathbb{R}$. $C(I, L^\varphi)$ denote the space of all ρ-continuous functions from I to L^φ with the modular $\sigma(x) = \sup_{t\in I}\rho(x(t))$. Also $C(I, L^\varphi)$ is a real vector space. If ρ is a convex modular, then σ is a convex modular. Also, if ρ satisfies the Fatou property and Δ_2-condition, then σ satisfies the Fatou property and Δ_2-condition (see [9]).

To study the integral equation (23), we consider the following hypotheses.

(1) $\phi : I \times L^\varphi \to L^\varphi$ is a ρ-expansive mapping; that is, there exist constants $c, k, l \in \mathbb{R}^+$ such that $c > 2l, k \geq 2$ and for all $x, y \in L^\varphi$

$$\rho(l(\phi(t,x)-\phi(t,y))) \geq k\rho(c(x-y)) \tag{24}$$

and ϕ is onto. Also for $t \in I$, $\phi(t,\cdot) : L^\varphi \to L^\varphi$ is ρ-continuous.

(2) ψ is a function from $I \times I \times L^\varphi$ into L^φ such that $\psi(t,s,\cdot) : x \to \psi(t,s,x)$ is ρ-continuous on L^φ for almost all $t, s \in I$ and $\psi(t,\cdot,x) : s \to \psi(t,s,x)$ is measurable function on I for each $x \in L^\varphi$ and for almost all $t \in I$. Also, there are nondecreasing continuous functions $\beta, \gamma : I \to \mathbb{R}^+$ such that

$$\lim_{t\to\infty}\beta(t)\int_0^t \gamma(s)\,ds = 0,$$
$$\rho(c(\psi(t,s,x))) \leq \beta(t)\gamma(s), \tag{25}$$

for all $t, s \in I$, $s \leq t$ and $x \in L^\varphi$.

(3) There exists measurable function $\eta : I \times I \times I \to \mathbb{R}^+$ such that

$$\rho(\psi(t,s,x)-\psi(r,s,x)) \leq \eta(t,r,s), \tag{26}$$

for all $t, r, s \in I$ and $x \in L^\varphi$; also $\lim_{t\to r}\int_0^b \eta(t,r,s)ds = 0$.

(4) $\rho(\psi(t,s,x)-\psi(t,s,y)) \leq \rho(x-y)$ for all $t, s \in I$ and $x, y \in L^\varphi$.

Remark 17 (see [7]). We consider L^φ, the Musielak-Orlicz space. Since ρ is convex and satisfies the Δ_2-condition, then

$$\|x_n - x\|_\rho \longrightarrow 0 \Longleftrightarrow \rho(x_n - x) \longrightarrow 0, \tag{27}$$

as $n \to \infty$ on L^φ. This implies that the topologies generated by $\|\cdot\|_\rho$ and ρ are equivalent.

Theorem 18. *Suppose that the conditions (1)–(4) are satisfied. Further assume that L^φ satisfies the Δ_2-condition. Also $\omega(t) = \beta(t)\int_0^t \gamma(s)ds$ and $\omega(0) = 0$; also $\sup\{\rho(c(\phi(t,v))), t \in I, v \in L^\varphi\} \leq \omega(t)$. Then integral equation (23) has at least one solution $x \in C(I, L^\varphi)$.*

Proof. Suppose that

$$Tx(t)=\phi(t,x(t)),$$
$$Sx(t)=\int_0^t \psi(t,s,x(s))\,ds. \tag{28}$$

Conditions (1) and (2) imply that T and S are well defined on $C(I, L^\varphi)$. Define the set $B = \{x \in C(I, L^\varphi); \rho(c(x(t))) \leq$

$\omega(t)$ *for all* $t \in I\}$. Then B is a nonempty, ρ-bounded, ρ-closed, and convex subset of $C(I, L^{\varphi})$. Equation (23) is equivalent to the fixed point problem $x = Tx + Sx$. By Theorem 12, we find the fixed point for $T + S$ in B. Due to this, we prove that S satisfies the condition (I) of Theorem 12. For $x \in B$, we show that $Sx \in B$. Indeed,

$$\begin{aligned}\rho(c(Sx(t))) &= \rho\left(c\left(\int_0^t \psi(t,s,x(s))\,ds\right)\right)\\ &\le \int_0^t \rho(c(\psi(t,s,x(s))))\,ds\\ &\le \int_0^t \beta(t)\gamma(s)\,ds\\ &= \omega(t);\end{aligned} \tag{29}$$

then $Sx \in B$. Since $S(B) \subset B$ and B is ρ-bounded, $S(B)$ is σ-bounded and by Δ_2-condition $\|\cdot\|_{\sigma}$-bounded.

We show that $S(B)$ is ρ-equicontinuous. For all $t, r \in I$ and $x \in L^{\varphi}$ such that $t < r$,

$$Sx(t) - Sx(r) = \int_0^t \psi(t,s,x(s))\,ds - \int_0^r \psi(r,s,x(s))\,ds; \tag{30}$$

then by condition (3),

$$\rho(Sx(t) - Sx(r)) \le \int_0^b \eta(t,r,s)\,ds; \tag{31}$$

since $\lim_{t\to r}\int_0^b \eta(t,r,s)ds = 0$, then $S(B)$ is ρ-equicontinuous. By using the Arzela-Ascoli theorem, we obtain that S is a σ-compact mapping. Next, we show that S is σ-continuous. Suppose that $\varepsilon > 0$ is given; we find a $\delta > 0$ such that $\sigma(x-y) < \delta$, for some $x, y \in B$. Note that

$$Sx(t) - Sy(t) = \int_0^t \psi(t,s,x(s))\,ds - \int_0^t \psi(t,s,y(s))\,ds; \tag{32}$$

also

$$\rho(Sx(t) - Sy(t)) \le \int_0^t \rho(x(s) - y(s))\,ds \le \int_0^t \sigma(x-y)\,ds; \tag{33}$$

then

$$\sigma(Sx - Sy) \le \int_0^b \sigma(x-y)\,ds \le \varepsilon; \tag{34}$$

therefore S is σ-continuous.

Since ϕ is ρ-continuous, it shows that T transforms $C(I, L^{\varphi})$ into itself. In view of supremum ρ and condition (1), it is easy to see that T is σ-expansive with constant $k \ge 2$. For $x, y \in B$,

$$\begin{aligned}&\rho(l(Tx(t) - Ty(t)))\\ &\quad\le \rho(c(x(t) - y(t)))\\ &\qquad + \rho(\alpha l((I-T)x(t) - (I-T)y(t)));\end{aligned} \tag{35}$$

then

$$\begin{aligned}\rho(\alpha l((I-T)x(t) - (I-T)y(t)))&\\ \ge (k-1)\rho(c(x(t) - y(t))),&\end{aligned} \tag{36}$$

where α is conjugate of c/l. Let $r = \alpha l$; since $k \ge 2$, then

$$\rho(r(I-T)x(t)) \ge (k-1)\rho(c(x(t))) \ge \rho(c(x(t))). \tag{37}$$

Now, assume that $x = Tx + Sy$ for some $y \in B$. Since $c > 2l$, then $r < c$, and

$$\begin{aligned}\rho(c(x(t))) &\le \rho(r(I-T)x(t)) = \rho(r(Sy(t)))\\ &\le \rho(c(Sy(t))) \le \omega(t),\end{aligned} \tag{38}$$

which shows that $x \in B$. Now, define a map T_z as follows:

$$T_z : C(I, L^{\varphi}) \longrightarrow C(I, L^{\varphi}), \tag{39}$$

for each $z \in C(I, L^{\varphi})$; by

$$T_z x(t) = Tx(t) + z(t), \tag{40}$$

for all $x, y \in C(I, L^{\varphi})$,

$$\begin{aligned}\rho(l(T_z x(t) - T_z y(t))) &= \rho(l(Tx(t) - Ty(t)))\\ &\ge k\rho(c(x(t) - y(t)));\end{aligned} \tag{41}$$

therefore

$$\sigma(l(T_z x - T_z y)) \ge k\sigma(c(x-y)); \tag{42}$$

then T_z is σ-expansive with constant $k \ge 2$ and T_z is onto. By Theorem 9, there exists $w \in C(I, L^{\varphi})$ such that $T_z w = w$; that is, $(I-T)w = z$. Hence $S(B) \subset (I-T)(L^{\varphi})$ and condition (III) of Theorem 12 holds. Therefore by Theorem 12, $S+T$ has a fixed point $z \in B$ with $Tz + Sz = z$; that is, z is a solution to (23). □

Now, we consider another integral equation.

Let L^{φ} be the Musielak-Orlicz space and $I = [0, b] \subset \mathbb{R}$. Suppose that ρ is convex and satisfies the Δ_2-condition. Since topologies generated by $\|\cdot\|_{\rho}$ and ρ are equivalent, then we consider Banach space $(L^{\varphi}, \|\cdot\|_{\rho})$ and $C(I, L^{\varphi})$ denote the space of all $\|\cdot\|_{\rho}$-continuous functions from I to L^{φ} with the modular $\|x\|_{\sigma} = \sup_{t\in I}\|x(t)\|_{\rho}$; also $C(I, L^{\varphi})$ is a real vector space. Consider the nonlinear integral equation

$$\begin{aligned}x(t) = \phi(t, x(t))&\\ + \lambda(t, x(t))\int_0^t \omega(t,s)\psi(s, x(s))\,ds,&\\ x \in C(I, L^{\varphi}),&\end{aligned} \tag{43}$$

where

(1) $\phi : I \times L^{\varphi} \to L^{\varphi}$ is a $\|\cdot\|_{\rho}$-expansive mapping; that is, there exists constant $l \ge 2$ such that

$$\|\phi(t,x) - \phi(t,y)\|_{\rho} \ge l\|x - y\|_{\rho}, \tag{44}$$

for all $x, y \in L^{\varphi}$ and ϕ is onto; also for $t \in I$, $\phi(t,\cdot): L^{\varphi} \to L^{\varphi}$ is $\|\cdot\|_{\rho}$-continuous;

(2) ψ is function from $I \times L^{\varphi}$ into L^{φ} such that $\psi(t,\cdot): L^{\varphi} \to L^{\varphi}$ is a $\|\cdot\|_{\rho}$-continuous and $t \to \psi(t,x)$ is measurable for every $x \in L^{\varphi}$. Also, there exist functions $\beta \in L^{1}(I)$ and a nondecreasing continuous function $\gamma : [0,\infty) \to (0,\infty)$ such that

$$\left\|\psi(t,x)\right\|_{\rho} \leq \beta(t)\,\gamma\left(\|x\|_{\rho}\right), \tag{45}$$

for all $t \in I$ and $x \in L^{\varphi}$. Also for $t \in I$, $x \to \psi(t,x)$ is nondecreasing on L^{φ};

(3) λ is function from $I \times L^{\varphi}$ into L^{φ} such that $\lambda(t,\cdot): L^{\varphi} \to L^{\varphi}$ is $\|\cdot\|_{\rho}$-continuous and there exists a $a \geq 0$ such that

$$\left\|\lambda(t,x)-\lambda(t,y)\right\|_{\rho} \leq a\left\|x-y\right\|_{\rho}, \tag{46}$$

for all $t \in I$ and $x \in L^{\varphi}$; also for $x \in L^{\varphi}$, $t \to \lambda(t,x)$ is nondecreasing on I and for $t \in I$, $x \to \lambda(t,x)$ is nondecreasing on L^{φ};

(4) ω is function from $I\times I$ into $\mathbb{R}^{+}$. For each $t \in I$, $\omega(t,s)$ is measurable on $[0,t]$. Also $\overline{\omega(t)} = \operatorname{esssup}|\omega(t,s)|$ is bounded on $[0,b]$ and $r = \sup|\overline{\omega(t)}|$. The map $\omega(\cdot,s): t \to \omega(t,s)$ is continuous from I to $L^{\infty}(I)$. Also for $s \in I$, $t \to \omega(t,s)$ is nondecreasing on I.

Theorem 19. *Suppose that the conditions (1)–(4) are satisfied and there exists a constant $k \geq 0$ such that for all $t \in I$,*

$$\int_0^t \beta(s)\,ds < \frac{k}{(ak+h)\,rb}\int_0^t \frac{1}{\gamma(k)}ds, \tag{47}$$

where $h := \sup\{\|\lambda(t,x)\|_{\rho}, t \in I, x \in L^{\varphi}\}$ and also $\sup\{\|\phi(t,x)\|_{\rho}, t \in I, x \in L^{\varphi}\} \leq k$. Then integral equation (43) *has at least one solution $x \in C(I,L^{\varphi})$.*

Proof. Define

$$B = \left\{x \in C\left(I,L^{\varphi}\right); \|x(t)\|_{\rho} \leq k \ \forall t \in I\right\}; \tag{48}$$

then B is a nonempty, $\|\cdot\|_{\rho}$-bounded, $\|\cdot\|_{\rho}$-closed, and convex subset of $C(I,L^{\varphi})$. Consider

$$\begin{gathered} Tx(t) = \phi(t,x(t)), \\ Sx(t) = \lambda(t,x(t))\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds. \end{gathered} \tag{49}$$

It is easy that by the hypothesis T and S are well defined on $C(I,L^{\varphi})$.

For $x \in B$, we show that $Sx \in B$. Consider

$$\begin{aligned} &\|Sx(t)\|_{\rho} \\ &= \left\|\lambda(t,x(t))\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds\right\|_{\rho} \\ &= \left\|(\lambda(t,x(t))-\lambda(t,0)+\lambda(t,0))\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds\right\|_{\rho} \\ &\leq \left(a\|x(t)\|_{\rho}+h\right) r\int_0^t \beta(s)\,\gamma\left(\|x(s)\|_{\rho}\right)ds \\ &\leq (ak+h)\,r\int_0^t \beta(s)\,\gamma(k)\,ds \\ &\leq (ak+h)\,r\int_0^b \frac{k\gamma(k)}{(ak+h)\,rb\gamma(k)}ds \\ &\leq k. \end{aligned} \tag{50}$$

Let $x \in B$ and assume that $t > \tau \in I$ such that $|t-\tau| < \delta$, for a given positive constant δ. We have

$$\begin{aligned} &\|Sx(t)-Sx(\tau)\|_{\rho} \\ &\quad= \left\|\lambda(t,x(t))\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds\right. \\ &\qquad\left.-\lambda(\tau,x(\tau))\int_0^{\tau} \omega(\tau,s)\,\psi(s,x(s))\,ds\right\|_{\rho} \\ &\quad= \left\|\lambda(t,x(t))\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds\right. \\ &\qquad\pm\lambda(t,x(t))\int_0^t \omega(\tau,s)\,\psi(s,x(s))\,ds \\ &\qquad\pm\lambda(\tau,x(\tau))\int_0^t \omega(\tau,s)\,\psi(s,x(s))\,ds \\ &\qquad\left.-\lambda(\tau,x(\tau))\int_0^{\tau} \omega(\tau,s)\,\psi(s,x(s))\,ds\right\|_{\rho} \\ &\quad\leq \left\|\lambda(t,x(t))\left(\int_0^t \omega(t,s)\,\psi(s,x(s))\,ds\right.\right. \\ &\qquad\qquad\left.\left.-\int_0^t \omega(\tau,s)\,\psi(s,x(s))\,ds\right)\right\|_{\rho} \\ &\qquad+\left\|(\lambda(\tau,x(\tau))-\lambda(\tau,x(\tau)))\right. \\ &\qquad\left.\times\int_0^t \omega(\tau,s)\,\psi(s,x(s))\,ds\right\|_{\rho} \\ &\qquad+\left\|\lambda(\tau,x(\tau))\int_{\tau}^t \omega(\tau,s)\,\psi(s,x(s))\,ds\right\|_{\rho}; \end{aligned} \tag{51}$$

since

$$\begin{aligned}
&\left\| \lambda (t, x(t)) \left(\int_0^t \omega(t,s)\psi(s,x(s))\,ds - \int_0^t \omega(\tau,s)\psi(s,x(s))\,ds \right) \right\|_\rho \\
&= \left\| \lambda (t, x(t)) \left(\int_0^t (\omega(t,s) - \omega(\tau,s))\psi(s,x(s))\,ds \right) \right\|_\rho \\
&\le \left\| (\lambda(\tau, x(\tau)) - \lambda(\tau,0) + \lambda(\tau,0)) \times \left(\int_0^t (\omega(t,s) - \omega(\tau,s))\psi(s,x(s))\,ds \right) \right\|_\rho \\
&\le (ak+h)\, |\omega(t,0) - \omega(\tau,0)|_{L_\infty} \int_0^t \beta(s)\gamma(k)\,ds \\
&\le \frac{k}{r} |\omega(t,0) - \omega(\tau,0)|_{L_\infty}, \\
&\left\| (\lambda(t,x(t)) - \lambda(\tau,x(\tau))) \int_0^t \omega(\tau,s)\psi(s,x(s))\,ds \right\|_\rho \\
&\le \left\| (\lambda(t,x(t)) - \lambda(\tau,x(\tau)))\, r \int_0^t \beta(s)\gamma(k)\,ds \right\|_\rho \\
&\le \frac{k}{ak+h} \left(\|\lambda(t,x(t)) - \lambda(t,x(\tau))\|_\rho + \|\lambda(\tau,x(\tau)) - \lambda(t,x(\tau))\|_\rho \right) \\
&\le \frac{k}{ak+h} \left(a\|x(t) - x(\tau)\|_\rho + h \right), \\
&\left\| \lambda(\tau,x(\tau)) \int_\tau^t \omega(\tau,s)\psi(s,x(s))\,ds \right\|_\rho \\
&= \left\| (\lambda(\tau,x(\tau)) - \lambda(\tau,0) + \lambda(\tau,0)) \times \int_\tau^t \omega(\tau,s)\psi(s,x(s))\,ds \right\|_\rho \\
&\le (ak+h)\, r \int_\tau^t \beta(s)\gamma(k)\,ds \\
&\le \frac{k}{b} |t-\tau|,
\end{aligned} \tag{52}$$

then $S(B)$ is $\|\cdot\|_\rho$-equicontinuous. By using the Arzela-Ascoli Theorem, we obtain that S is a $\|\cdot\|_\rho$-compact mapping.

We show that S is $\|\cdot\|_\rho$-continuous. Suppose that $\varepsilon > 0$ is given. We find a $\delta > 0$ such that $\|x - y\|_\sigma < \delta$. We have

$$\begin{aligned}
&\|Sx(t) - Sy(t)\|_\rho \\
&= \left\| \lambda(t,x(t)) \int_0^t \omega(t,s)\psi(s,x(s))\,ds - \lambda(t,y(t)) \int_0^t \omega(t,s)\psi(s,y(s))\,ds \right\|_\rho \\
&\le \left\| (\lambda(t,x(t)) - \lambda(t,y(t))) \int_0^t \omega(t,s)\psi(s,x(s))\,ds \right\|_\rho + \left\| \lambda(t,y(t)) \int_0^t (\psi(s,x(s)) - \psi(s,y(s)))\,ds \right\|_\rho \\
&\le \frac{ka}{ak+h} \|x(t) - y(t)\|_\rho + (ak+h)\, r \int_0^t \|x(s) - y(s)\|_\rho ds \\
&\le \frac{ka}{ak+h} \|x - y\|_\sigma + (ak+h)\, rb \|x - y\|_\sigma \\
&\le \varepsilon.
\end{aligned} \tag{53}$$

Since ϕ is $\|\cdot\|_\rho$-continuous, it shows that T transforms $C(I, L^\varphi)$ into itself. In view of supremum $\|\cdot\|_\rho$ and condition (1), it is easy to see that T is $\|\cdot\|_\sigma$-expansive with constant $l \ge 2$.

For $x, y \in B$,

$$\begin{aligned}
&\|Tx(t) - Ty(t)\|_\rho \\
&\quad \le \|x(t) - y(t)\|_\rho + \|(I-T)x(t) - (I-T)y(t)\|_\rho;
\end{aligned} \tag{54}$$

then

$$\|(I-T)x(t) - (I-T)y(t)\|_\rho \ge (l-1)\|x(t) - y(t)\|_\rho; \tag{55}$$

since $l \ge 2$, then

$$\|(I-T)x(t)\|_\rho \ge (l-1)\|x(t)\|_\rho \ge \|x(t)\|_\rho. \tag{56}$$

Now, assume that $x = Tx + Sy$ for some $y \in B$. Then

$$\|x(t)\|_\rho \le \|(I-T)x(t)\|_\rho = \|Sy(t)\|_\rho \le k, \tag{57}$$

which shows that $x \in B$. Now for each $z \in C(I, L^\varphi)$ we define a map T_z as follows:

$$T_z : C(I, L^\varphi) \longrightarrow C(I, L^\varphi); \tag{58}$$

by

$$T_z x(t) = Tx(t) + z(t); \tag{59}$$

for all $x, y \in C(I, L^\varphi)$,

$$\|T_z x(t) - T_z y(t)\|_\rho = \|Tx(t) - Ty(t)\|_\rho \ge l\|x(t) - y(t)\|_\rho; \tag{60}$$

therefore

$$\|T_z x - T_z y\|_\sigma \ge l\|x - y\|_\sigma; \tag{61}$$

then T_z is $\|\cdot\|_\sigma$-expansive with constant $l \ge 2$ and T_z is onto. By Theorem 9, there exists $w \in C(I, L^\varphi)$ such that $T_z w = w$; that is, $(I-T)w = z$. Hence $S(B) \subset (I-T)(L^\varphi)$. Therefore by Theorem 12, $S + T$ has a fixed point $z \in B$ with $Tz + Sz = z$; that is, z is a solution of (43). □

Finally, some examples are presented to guarantee Theorems 18 and 19.

Example 20. Consider the following integral equation:

$$x(t) = \frac{9x(t)}{1+t^2} + \int_0^t \arctan\left(\frac{5t(1+s)\sqrt{x(s)}}{(1+t)^3\left(1+\sqrt{x(s)}\right)}\right) ds, \tag{62}$$

where $L^{\varphi} = \mathbb{R}^+$, $I = [0,1]$.

For $x, y \in \mathbb{R}^+$ and $t \in I$, we have

$$\left|\phi(t,x) - \phi(t,y)\right| = \left|\frac{9x}{1+t^2} - \frac{9y}{1+t^2}\right| \geq \frac{9}{2}\left|x-y\right|. \tag{63}$$

Therefore by Theorem 18, the integral equation (62) has at least one solution.

Example 21. Consider the following integral equation:

$$x(t) = \frac{9x(t)}{1+t^2} + \frac{1}{8}\arcsin x(t)\int_0^t \frac{t}{t+s}x(s)\,ds, \tag{64}$$

where $\phi(t,x) = (9x/(1+t^2))$, $\lambda(t,x) = (1/8)\arcsin x$, $\omega(t,s) = t/(t+s)$, and $\psi(t,x) = x$. Also $L^{\varphi} = \mathbb{R}^+$, $I = [0,1]$. Therefore by Theorem 19, the integral equation (64) has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. Xiang and R. Yuan, "A class of expansive-type Krasnosel'skii fixed point theorems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3229–3239, 2009.

[2] T. Xiang and R. Yuan, "Krasnoselskii-type fixed point theorems under weak topology settings and applications," *Electronic Journal of Differential Equations*, vol. 2010, no. 35, pp. 1–15, 2010.

[3] T. Xiang, "Notes on expansive mappings and a partial answer to Nirenberg's problem," *Electronic Journal of Differential Equations*, vol. 2013, no. 2, pp. 1–16, 2013.

[4] M. A. Khamsi, "Nonlinear semigroups in modular function spaces," *Mathematica Japonica*, vol. 37, no. 2, pp. 291–299, 1992.

[5] M. A. Khamsi, W. M. Kozlowski, and S. Reich, "Fixed point theory in modular function spaces," *Nonlinear Analysis*, vol. 14, pp. 935–953, 1999.

[6] A. Razani and R. Moradi, "Common fixed point theorems of integral type in modular spaces," *Bulletin of the Iranian Mathematical Society*, vol. 35, no. 2, pp. 11–24, 2009.

[7] A. Hajji and E. Hanebaly, "Fixed point theorem and its application to perturbed integral equations in modular function spaces," *Electronic Journal of Differential Equations*, vol. 2005, no. 105, pp. 1–11, 2005.

[8] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*, vol. 9 of *Mathematical Analysis and Applications*, Chapman & Hall/CRC, 2006.

[9] A. Ait Taleb and E. Hanebaly, "A fixed point theorem and its application to integral equations in modular function spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 2, pp. 419–426, 2000.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 959612, 4 pages
http://dx.doi.org/10.1155/2014/959612

Research Article

Noncompact Equilibrium Points for Set-Valued Maps

Souhail Chebbi and Bessem Samet

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Souhail Chebbi; schebbi@ksu.edu.sa

Received 23 August 2013; Accepted 17 January 2014; Published 25 March 2014

Academic Editor: Hichem Ben-El-Mechaiekh

We prove a generalized result on the existence of equilibria for a monotone set-valued map defined on noncompact domain and take its values in an order of topological vector space. As consequence, we give a new variational inequality.

1. Introduction

In the literature, the notion of an equilibrium point (or equilibrium problem) has been firstly introduced by Karamardian in [1] and Allen in [2]. By using the well-known KKM principle, they proved that for a real valued function f defined on a product of two sets X and Y, there exists an element $\overline{x}$ of X, which will be called an *equilibrium point*, satisfying for all $y \in X$:

$$f(\overline{x}, y) \geq 0. \tag{1}$$

The classical hypothesis used to prove this type of equilibrium result concerns the convexity and compactness of the domain X, the monotonicity, the convexity, and the continuity of f and all extensions of this result obtained in the literature are about these hypotheses. In a recent work (see [3]), this result was extended to the noncompact case by using a coercivity type condition on a bifunction f. In this context the function f is supposed to take its values in a topological vector space endowed with an order defined by a cone C in the same way that has been used by [4–7]. Note that the result on the existence of equilibrium points proved in [3] was obtained via a result on the existence of what we called weak equilibrium points, that is, a point $\overline{x} \in X$, satisfying the following condition:

$$f(\overline{x}, y) \notin -\operatorname{int} C, \quad \forall y \in X, \tag{2}$$

where $\operatorname{int} C$ denotes the interior of the cone C in Y.

In this paper, we investigate the extension of equilibrium points to set-valued maps F in the same context. Generally, we have many choices to formulate the notion of equilibrium point. In fact, if P is a closed convex cone of a topological vector space Y with nonempty interior, $(P \neq Y)$, and $F : X \times X \to Y$ is a set-valued map, then the equilibrium point for a set-valued map can be extended in several possible ways (see [8, 9]) as follows: $F(x, y) \subseteq P$; $F(x, y) \cap -\operatorname{int} P = \emptyset$; $F(x, y) \nsubseteq -\operatorname{int} P$; $F(x, y) \cap P \neq \emptyset$. In this paper, we select the one that will be more adapted technically to our arguments. We will put a "moving" order on Y by a cone and the notions of convexity and continuity are naturally extended in our setting. We will use the pseudomonotonicity condition on F borrowed from [10]. As an application, we prove a variational inequality. The results obtained in this paper generalize the corresponding one in [9, 10].

2. Preliminaries

We extend the notions of convexity, monotonicity, and continuity given previously to set-valued maps. If X and Y are two sets, a set-valued map $F : X \to 2^Y$, where 2^X denotes the family of all subsets of X, is a map that is assigned to each $x \in X$, a subset $F(x) \subseteq Y$. Note that for the notation of set-valued maps, we will simply write $F : X \to Y$ instead of $F : X \to 2^Y$.

We firstly need to define an order on the codomain of set-valued maps as it has done for single valued maps. If X is a subset of some real topological vector space E, let Y be another real topological vector space, and let $C \subseteq Y$ be a closed convex cone (not necessarily pointed) with nonempty interior and $C \neq Y$. Then C defines an ordering "$\geq$" on Y by means of

$$y \geq 0 \Longleftrightarrow y \in C, \quad y > 0 \Longleftrightarrow y \in \operatorname{int} C. \tag{3}$$

We extend this notation to arbitrary subset $S \subseteq Y$ by setting

$$\begin{gathered} S \geq 0 \Longleftrightarrow S \subseteq C, \quad S > 0 \Longleftrightarrow S \subseteq \operatorname{int} C, \\ S \leq 0 \Longleftrightarrow S \subseteq -C, \quad S < 0 \Longleftrightarrow S \subseteq -\operatorname{int} C. \end{gathered} \tag{4}$$

By using this order, we naturally extend the notion of convexity for set-valued maps as follows.

Definition 1. Given a set-valued map $F : X \rightarrow Y$ defined on a vector space X with values in a vector space Y endowed with an order defined by a convex cone $C \subseteq Y$, we say that F is convex with respect to C if for all $x, y \in X$ and $\alpha \in [0, 1]$:

$$F(\alpha x + (1-\alpha) y) \leq \alpha F(x) + (1-\alpha) F(y), \tag{5}$$

which means that

$$F(\alpha x + (1-\alpha) y) \subseteq \alpha F(x) + (1-\alpha) F(y) - C. \tag{6}$$

Note that in particular, if $X = Y = \mathbb{R}$ and $C = \mathbb{R}^{+}$, we obtain the standard definition of convex set-valued maps.

As in the case of single valued maps, we can find many kinds of monotonicity for set-valued maps in the literature. We will use the notion of pseudomonotonicity defined in [10] which in turn extends the corresponding one defined in [7] for single valued maps.

Definition 2. Let E and Y be two real topological vector spaces, let $X \subset E$ be a nonempty closed and convex set, and let $C : X \rightarrow Y$ be a set-valued map such that for every $x \in X, C(x)$ is a closed and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. Consider a set-valued map $F : X \times X \rightarrow Y$. F is said to be pseudomonotone if, for any given $x, y \in X$,

$$F(x, y) \nsubseteq -\operatorname{int} C(x) \Longrightarrow F(y, x) \subseteq -C(y). \tag{7}$$

We recall the classical notions of continuity for set-valued maps as follows.

Definition 3. Given a set-valued map $F : X \rightarrow Y$ defined on a vector space X with values in a vector space Y. Then

(1) F is said to be lower semicontinuous (l.s.c) at $x_0 \in X$ if, for every open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U \subseteq X$ of x with $F(x) \cap V \neq \emptyset$ for all $x \in U$. F is said to be l.s.c. on X if F is l.s.c. at every $x \in X$.

(2) F is said to be upper semicontinuous (u.s.c) at $x_0 \in X$ if, for every open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighborhood set $U \subseteq X$ of x with $F(x) \subseteq V$ for all $x \in U$. F is said to be u.s.c. on X if F is u.s.c. at every $x \in X$.

(3) A set-valued map which is both lower and upper semicontinuous is called continuous.

In this paper, we will use the definition of coercing family borrowed from [11].

Definition 4. Consider a subset X of a topological vector space and a topological space Y. A family $\{(C_i, K_i)\}_{i \in I}$ of pair of sets is said to be coercing for a set-valued map $F : X \rightarrow Y$ if and only if

(i) for each $i \in I$, C_i is contained in a compact convex subset of X and K_i is a compact subset of Y;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subset K_i$.

Remark 5. Definition 1 can be reformulated by using the "dual" set-valued map $F^{*} : Y \rightarrow X$ defined for all $y \in Y$ by $F^{*}(y) = X \setminus F^{-1}(y)$. Indeed, a family $\{(C_i, K_i)\}_{i \in I}$ is coercing for F if and only if it satisfies conditions (i), (ii) of Definition 4, and the following one:

$$\forall i \in I, \quad \exists k \in I, \quad \forall y \in Y \setminus K_i, \quad F^{*}(y) \cap C_k \neq \emptyset. \tag{8}$$

Note that in the case where the family is reduced to one element, condition (iii) of Definition 4 and in the sense of Remark 5 appeared first in this generality (with two sets K and C) in [12] and generalized condition of Karamardian [1] and Allen [2]. Condition (iii) is also an extension of the coercivity condition given by Fan [13]. For other examples of set-valued maps admitting a coercing family that is not necessarily reduced to one element, see [11].

The following generalization of KKM principle obtained in [11] will be used in the proof of the main result of this paper.

Proposition 6. *Let E be a Hausdorff topological vector space, Y a convex subset of E, X a nonempty subset of Y, and $F : X \rightarrow Y$ a KKM map with compactly closed values in Y (i.e., for all $x \in X$, $F(x) \cap C$ is closed for every compact set C of Y). If F admits a coercing family, then $\bigcap_{x \in X} F(x) \neq \phi$.*

3. The Main Result

As it is mentioned in the introduction, at an abstract level all possible extension of equilibria can be handled equally well. But there are great practical differences if we try to replace the resulting abstract conditions by simpler, verifiable hypotheses like convexity or semicontinuity. This is even more so if we admit a "moving" ordering cone $P(x)$ (see [10]). For these reasons we choose to consider here the following generalized equilibrium problem.

Definition 7. Let X be a nonempty convex subset of some real topological vector space E, Y a real topological vector space, and $P : X \rightarrow Y$ a set-valued map such that for any $x \in X$, $P(x)$ is a closed convex cone with $\operatorname{int} P(x) \neq \emptyset$ and $P(x) \neq Y$. Let $F : X \times X \rightarrow Y$ be a set-valued map. The generalized equilibrium problem is to find $\bar{x} \in X$ such that

$$F(\bar{x}, y) \nsubseteq -\operatorname{int} P(\bar{x}) \quad \forall y \in X; \tag{9}$$

in this case, $\bar{x}$ is said to be an *equilibrium point*.

Theorem 8. *Let E and Y be real topological vector spaces (not necessarily Hausdorff). Let a nonempty, convex set $X \subseteq E$ and three set-valued mappings $F : X \times X \to Y$, $C : X \to Y$, and $D : X \to Y$ be given. Suppose that the following conditions are satisfied.*

(1) *For all $x, y \in X$, $F(x,y) \nsubseteq C(x)$ implies $F(y,x) \subseteq D(y)$ (pseudomonotonicity).*

(2) *For all $y \in X$, $\{x \in X : F(y,x) \subseteq D(y)\}$ is closed in X.*

(3) *For all $x \in X$, $\{y \in X : F(x,y) \subseteq C(x)\}$ is convex.*

(4) *For all $x \in X$, $F(x,x) \nsubseteq C(x)$.*

(5) *There exists a family $\{(C_i, K_i)\}_{i\in I}$ satisfying conditions (i) and (ii) of Definition 4 and the following one: for each $i \in I$, there exists $k \in I$ such that*

$$\{x \in X : F(y,x) \subseteq D(y), \forall y \in C_k\} \subseteq K_i. \tag{10}$$

Then there exists $\overline{x} \in X$ such that $F(y,\overline{x}) \subseteq D(y)$ for all $y \in X$.

Proof. Let us consider a set-valued map $S : X \to Y$ defined for every $y \in X$ by

$$S(y) := \{x \in X : F(y,x) \subseteq D(y)\}. \tag{11}$$

Then we can see firstly that S is a KKM map; that is, for every finite subset $\{y_1, \ldots, y_n\}$ of X there holds

$$\operatorname{co}\{y_1, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} S(y_i). \tag{12}$$

In fact, let $z \in \operatorname{co}\{y_1, \ldots, y_n\}$ and assume by contradiction that $z \notin \bigcup_{i=1}^{n} S(y_i)$; it means that $z = \sum_{i\in I} \lambda_i y_i$ with $\lambda_i \geq 0$, $\sum_{i\in I} \lambda_i = 1$ and $z \notin S(y_i)$ for all i. Then $F(y_i, z) \nsubseteq D(y_i)$ for all i, hence from condition (1) $F(z, y_i) \subseteq C(z)$ for all i. It follows from condition (3) that $F(z, \sum_{i\in I} \lambda_i y_i) \subseteq C(z)$, and then $F(z,z) \subseteq C(z)$, which contradicts condition (4); thus S is a KKM map.

It is also clear from condition (2) that, for all $y \in X$, $S(y)$ is closed.

In addition, we can verify that condition (5) implies that the family $\{(C_i, K)\}_{i\in I}$ satisfies the following condition: for all $i \in I$ there exists $k \in I$ with

$$\bigcap_{y\in C_k} S(y) \subset K_i. \tag{13}$$

We deduce that S satisfies all hypothesis of Proposition 6, so we have

$$\bigcap_{y\in X} S(y) \neq \emptyset. \tag{14}$$

Therefore there exists $\overline{x} \in X$ such that for any $y \in X$, $\overline{x} \in S(y)$. Hence

$$F(y,\overline{x}) \subseteq D(y), \quad \forall y \in X. \tag{15}$$

□

Theorem 9. *Let E, Y, X, F, C, and D satisfy the assumptions of Theorem 8 and the additional following conditions.*

(6) *For all $x, y \in X$ with $y \neq x$ and $u \in (x,y)$ if $F(u,x) \subseteq D(u)$ and $F(u,y) \subseteq C(u)$, then $F(u,v) \subseteq C(u)$ for all $v \in (x,y)$.*

(7) *For all $x, y \in X$ with $y \neq x$, $\{u \in [x,y] : F(u,y) \subseteq C(u)\}$ is open in $[x,y]$.*

Then there exists $\overline{x} \in X$ such that $F(\overline{x}, y) \nsubseteq C(\overline{x})$ for all $y \in X$.

Proof. By Theorem 8, there exists $\overline{x} \in X$ with $F(y,\overline{x}) \subseteq D(y)$ for all $y \in X$. Assume that $F(\overline{x}, y) \subseteq C(\overline{x})$ for some $y \in X$; then $y \neq \overline{x}$ by (6) and from (7) there exists $u \in (\overline{x}, y)$ such that $F(u,y) \subseteq C(u)$. Since $F(u,\overline{x}) \subseteq D(u)$, we deduce that $F(u,u) \subseteq C(u)$, but this contradicts (6) and the theorem is proved. □

The following result, which corresponds to Theorem 1 in [10], can be deduced from the two previous theorems.

Corollary 10. *Let F, C, D satisfy hypothesis (1–4) of Theorem 8, (6, 7) of Theorem 9 and the following condition.*

($5'$) *There exists a nonempty compact set $A \subseteq X$ and a compact convex set $B \subseteq X$ such that for every $x \in X \setminus A$ there exists $y \in B$ with $F(x,y) \subseteq C(x)$.*

Then there exists $\overline{x} \in A$ such that $F(\overline{x}, y) \nsubseteq C(x)$ for all $y \in X$.

Proof. By taking for all $i \in I$, $C_i = B$, which is convex compact set, and $K_i = A$, which is compact set, and by using hypothesis ($5'$), we can see that S admits a coercing family in the sense of Remark 5; that is, for all $x \in X \setminus A$, $S^*(x) \cap B \neq \emptyset$. Suppose, per absurdum, that there exists $x_0 \in X \setminus A$ with $S^*(x_0) \cap B = \emptyset$. Hence for all $y \in B$, $y \notin S^*(x_0)$. This means that for all $y \in B$, $y \in S^{-1}(x_0)$ and so $x_0 \in S(y)$. Therefore, there exists $x_0 \in X \setminus A$ such that for all $y \in B$, we have

$$F(y, x_0) \subseteq D(y). \tag{16}$$

Then by Theorem 9, we deduce that there exists $x_0 \in X \setminus A$ such that for all $y \in B$

$$F(x_0, y) \nsubseteq C(x_0), \tag{17}$$

but this contradicts hypothesis ($5'$). □

Corollary 11. *Let $F : X \times X \to Y$ be a set-valued map satisfy the following conditions.*

(1) *For all $x, y \in X$, $F(x,y) \nsubseteq -\operatorname{int} P(x)$ implies $F(y,x) \subseteq -P(y)$.*

(2) *For all $y \in X$, $F(y, \cdot)$ is lower semicontinuous.*

(3) *For all $x \in X$, $F(x, \cdot)$ is convex with respect to $P(x)$.*

(4) *The map $\operatorname{int} P(x)$ has open graph in $X \times Y$.*

(5) *For all $x, y \in X$, $F(\cdot, y)$ is upper semicontinuous and compact valued on $[x,y]$.*

(6) *For all $x \in X$, $F(x,x) \nsubseteq -\operatorname{int} P(x)$.*

(7) *There exists a family* $\{C_i, K_i\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition 4 coercing and the following one. For each* $i \in I$, *there exists* $k \in I$ *such that*

$$\{x \in X : F(y,x) \subseteq -P(y), \forall y \in C_k\} \subseteq K_i. \qquad (18)$$

Then there exists $\overline{x} \in X$ *such that* $F(\overline{x}, y) \nsubseteq -\operatorname{int} P(\overline{x})$ *for all* $y \in X$.

Proof. Following [10], if the map $F(y,\cdot)$ is lower semicontinuous and $D(y)$ is closed, then condition (7) of Theorem 9 is satisfied. Furthermore and also by [10], condition (7) of Theorem 9 is fulfilled, if for all $x \in X$, the map $F(\cdot, x)$ is upper semicontinuous along line segments $[x, y] \subseteq X$ with compact values, and the map $C(\cdot)$ has open graph in $X \times Y$. □

Now let $L : X \to Z$ denote the space of all continuous linear operators $X \to Z$. For $\phi \in L(X,Z)$, we write $\langle \phi, x\rangle$: $\phi(x)$ and for $\Phi \subseteq L(X,Z)$, we write $\langle \Phi, x\rangle := \{\langle \phi, x\rangle : \phi \in \Phi\}$. The following result is a variational inequality formulation of our main result.

Corollary 12. *Let a map* $\Phi : K \to L(X,Z)$ *be given such that for all* $x \in K$, $\Phi(x)$ *is nonempty. Suppose the following.*

(1) *For all* $x, y \in K$, $\langle \Phi(x), y-x\rangle \nsubseteq -\operatorname{int} P(x)$ *implies* $\langle \Phi(y), x-y\rangle \subseteq -P(y)$.

(2) *The map* $\operatorname{int} P(\cdot)$ *has open graph in* $K \times Z$.

(3) *For all* $x, y \in K$, $\langle \Phi(\cdot), y-x\rangle$ *is upper semicontinuous on* $[x, y]$ *and compact valued.*

(4) *There exists a family* $\{C_i, K_i\}_{i\in I}$ *satisfying conditions* (*i*) *and* (*ii*) *of Definition 4 and the following one: for each* $i \in I$, *there exists* $k \in I$ *such that*

$$\{x \in X : \langle \Phi(y), x-y\rangle \subseteq -P(y), \forall y \in C_k\} \subseteq K_i. \qquad (19)$$

Then there exists $\overline{x} \in X$ *such that* $\langle \Phi(\overline{x}), y-\overline{x}\rangle) \nsubseteq -\operatorname{int} P(\overline{x})$ *for all* $y \in K$.

Proof. Take $F(x,y) := \langle \Phi(x), y-x\rangle$, $C(x) := -\operatorname{int} P(x)$, and $D(x) := -P(x)$. Then conditions (1) and (5) of Theorem 9 are clearly satisfied. (2) holds since each member of $\Phi(y)$ is continuous and $D(y)$ is closed. (4) is satisfied since $F(x,x) = \{0\}$ and $P(x) \neq Z$. (3) and (6) hold since for all $\alpha \in [0,1]$:

$$F(x, \alpha y_1 + (1-\alpha) y_2) \subseteq \alpha F(x, y_1) + (1-\alpha) F(x, y_2). \qquad (20)$$

To verify hypothesis (7), we have to show that $R = \{u \in [x,y] : \langle \Phi(u), y-u\rangle\}$ is closed in $[x,y]$. Let $\{u_i\}$ be a net in R converging to $u \in [x,y]$; we may assume $u \neq y$, since $y \in R$, and we may assume $u_i \neq y$ for all i as well. Thus $y-u = \lambda(y-x)$ with $\lambda \neq 0$ and $y-u_i = \lambda_i(y-x)$ with $\lambda_i \neq 0$. For every i, there exists $w_i \in \langle \Phi(u_i), y-u_i\rangle$ with $w_i \notin -\operatorname{int} P(u_i)$; then $z_i = \lambda_i^{-1} w_i \in \langle \Phi(u_i), y-x\rangle$. We conclude as above that there is a subnet z_j converging to some $z \in \langle \Phi(u), y-x\rangle$. The corresponding w_j converges to $w = \lambda z \langle \Phi(u), y-u\rangle$, since $-\operatorname{int} P(\cdot)$ has open graph; we obtain $w \notin -\operatorname{int} P(u)$; hence $u \in R$. □

Note that Corollaries 11 and 12 extend, respectively, Corollaries 1 and 2 in [10] obtained in noncompact case since our coercivity condition is more general.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group Project (RGP-VPP 237).

References

[1] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, pp. 161–168, 1971.

[2] G. Allen, "Variational inequalities, complementarity problems, and duality theorems," *Journal of Mathematical Analysis and Applications*, vol. 58, no. 1, pp. 1–10, 1977.

[3] Z. Al-Rumaih, S. Chebbi, and H. K. Xu, "Non-compact equilibrium point and applications," *Journal of Applied Mathematics*, vol. 2012, Article ID 373462, 9 pages, 2012.

[4] M. Bianchi, N. Hadjisavvas, and S. Schaible, "Vector equilibrium problems with generalized monotone bifunctions," *Journal of Optimization Theory and Applications*, vol. 92, no. 3, pp. 527–542, 1997.

[5] M. Bianchi and S. Schaible, "Generalized monotone bifunctions and equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 90, no. 1, pp. 31–43, 1996.

[6] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[7] N. X. Tan and P. N. Tinh, "On the existence of equilibrium points of vector functions," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 141–156, 1998.

[8] G. Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem," *Journal of Optimization Theory and Applications*, vol. 74, no. 3, pp. 445–456, 1992.

[9] Q. H. Ansari, W. Oettli, and D. Schläger, "A generalization of vectorial equilibria," *Mathematical Methods of Operations Research*, vol. 46, no. 2, pp. 147–152, 1997.

[10] W. Oettli and D. Schläger, "Existence of equilibria for monotone multivalued mappings," *Mathematical Methods of Operations Research*, vol. 48, no. 2, pp. 219–228, 1998.

[11] H. Ben-El-Mechaiekh, S. Chebbi, and M. Florenzano, "A generalized KKMF principle," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 583–590, 2005.

[12] H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, "Points fixed et coincidences pour les applications multivoque (application de Ky fan)," *Comptes Rendus des Séances de l'Académie des Sciences I*, vol. 295, no. 3, pp. 257–259, 1982.

[13] K. Fan, "Some properties of convex sets related to fixed point theorems," *Mathematische Annalen*, vol. 266, no. 4, pp. 519–537, 1984.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 180698, 14 pages
http://dx.doi.org/10.1155/2014/180698

Research Article

A Unification of G-Metric, Partial Metric, and b-Metric Spaces

Nawab Hussain,[1] Jamal Rezaei Roshan,[2] Vahid Parvaneh,[3] and Abdul Latif[1]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Islamic Azad University, Qaemshahr Branch, Qaemshahr, Iran*
[3] *Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran*

Correspondence should be addressed to Vahid Parvaneh; vahid.parvaneh@kiau.ac.ir

Received 9 October 2013; Accepted 18 January 2014; Published 5 March 2014

Academic Editor: Mohamed Amine Khamsi

Using the concepts of G-metric, partial metric, and b-metric spaces, we define a new concept of generalized partial b-metric space. Topological and structural properties of the new space are investigated and certain fixed point theorems for contractive mappings in such spaces are obtained. Some examples are provided here to illustrate the usability of the obtained results.

1. Introduction and Mathematical Preliminaries

The concept of a b-metric space was introduced by Czerwik in [1, 2]. After that, several interesting results about the existence of fixed point for single-valued and multivalued operators in (ordered) b-metric spaces have been obtained (see, e.g., [3–13]).

Definition 1 (see [1]). Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [14].

Definition 2 (see [14]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all the three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Aghajani et al. in [15] introduced the class of generalized b-metric spaces (G_b-metric spaces) and then they presented some basic properties of G_b-metric spaces.

The following is their definition of G_b-metric spaces.

Definition 3 (see [15]). Let X be a nonempty set and $s \geq 1$ a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

On the other hand, Matthews [16] has introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, self-distance of an arbitrary point need not to be equal to zero.

Definition 4 (see [16]). A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(p_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \le p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

In this case, (X, p) is called a partial metric space.

For a survey of fixed point theory, its applications, and comparison of different contractive conditions and related results in both partial metric spaces and G-metric spaces we refer the reader to, for example, [17–27] and references mentioned therein.

Recently, Zand and Nezhad [28] have introduced a new generalized metric space (G_p-metric spaces) as a generalization of both partial metric spaces and G-metric spaces.

We will use the following definition of a G_p-metric space.

Definition 5 (see [29]). Let X be a nonempty set. Suppose that a mapping $G_p : X \times X \times X \to \mathbb{R}^+$ satisfies

(G_p1) $x = y = z$ if $G_p(x,y,z) = G_p(z,z,z) = G_p(y,y,y) = G_p(x,x,x)$;

(G_p2) $G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x, y, z \in X$ with $z \ne y$;

(G_p3) $G_p(x,y,z) = G_p(p\{x,y,z\})$, where p is any permutation of x, y, and z (symmetry in all the three variables);

(G_p4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G_p is called a G_p-metric and (X, G_p) is called a G_p-metric space.

As a generalization and unification of partial metric and b-metric spaces, Shukla [30] presented the concept of a partial b-metric space as follows.

Definition 6 (see [30]). A partial b-metric on a nonempty set X is a mapping $p_b : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(p_{b1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,

(p_{b2}) $p_b(x,x) \le p_b(x,y)$,

(p_{b3}) $p_b(x,y) = p_b(y,x)$,

(p_{b4}) $p_b(x,y) \le s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A partial b-metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $s \ge 1$ is called the coefficient of (X, p_b).

In a partial b-metric space (X, p_b), if $x, y \in X$ and $p_b(x,y) = 0$, then $x = y$, but the converse may not be true. It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of these facts needs not to be hold.

Example 7 (see [30]). Let $X = \mathbb{R}^+$, $q > 1$ a constant, and $p_b : X \times X \to \mathbb{R}^+$ defined by

$$p_b(x,y) = [\max\{x,y\}]^q + |x-y|^q \quad \forall x, y \in X. \tag{1}$$

Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Note that in a partial b-metric space the limit of a convergent sequence may not be unique (see [30, Example 2]).

In [31], Mustafa et al. introduced a modified version of ordered partial b-metric spaces in order to obtain that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 8 (see [31]). Let X be a (nonempty) set and $s \ge 1$ a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(p_{b1}) $x = y \Leftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$,

(p_{b2}) $p_b(x,x) \le p_b(x,y)$,

(p_{b3}) $p_b(x,y) = p_b(y,x)$,

($p_{b4'}$) $p_b(x,y) \le s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + ((1-s)/2)(p_b(x,x) + p_b(y,y))$.

The pair (X, p_b) is called a partial b-metric space.

Since $s \ge 1$, from ($p_{b4'}$), we have

$$\begin{aligned} p_b(x,y) &\le s(p_b(x,z) + p_b(z,y) - p_b(z,z)) \\ &\le s(p_b(x,z) + p_b(z,y)) - p_b(z,z). \end{aligned} \tag{2}$$

Hence, a partial b-metric in the sense of Definition 8 is also a partial b-metric in the sense of Definition 6.

The following example shows that a partial b-metric on X (in the sense of Definition 8) is neither a partial metric nor a b-metric on X.

Example 9 (see [31]). Let (X, d) be a metric space and $p_b(x,y) = d(x,y)^q + a$, where $q > 1$ and $a \ge 0$ are real numbers. Then p_b is a partial b-metric with $s = 2^{q-1}$.

Proposition 10 (see [31]). *Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y) \tag{3}$$

for all $x, y \in X$.

Hence, the importance of our definition of partial b-metric is that by using it we can define a dependent b-metric which we call the b-metric associated with p_b.

Now, we present some definitions and propositions in a partial b-metric space.

Definition 11 (see [31]). Let (X, p_b) be a partial b-metric space. Then, for an $x \in X$ and an $\epsilon > 0$, the p_b-ball with center x and radius $\epsilon > 0$ is

$$B_{p_b}(x,\epsilon) = \{y \in X \mid p_b(x,y) < p_b(x,x) + \epsilon\}. \quad (4)$$

Lemma 12 (see [31]). *Let (X, p_b) be a partial b-metric space. Then,*

(A) *if $p_b(x,y) = 0$, then $x = y$;*

(B) *if $x \neq y$, then $p_b(x,y) > 0$.*

Proposition 13 (see [31]). *Let (X, p_b) be a partial b-metric space, $x \in X$, and $\epsilon > 0$. If $y \in B_{p_b}(x,\epsilon)$ then there exists a $\delta > 0$ such that $B_{p_b}(y,\delta) \subseteq B_{p_b}(x,\varepsilon)$.*

Thus, from the above proposition the family of all open p_b-balls

$$\Delta = \left\{B_{p_b}(x,r) \mid x \in X, r > 0\right\} \quad (5)$$

is a base of a T_0-topology τ_{p_b} on X which we call the p_b-metric topology.

The topological space (X, p_b) is T_0 but need not be T_1.

The following lemma shows the relationship between the concepts of p_b-convergence, p_b-Cauchyness, and p_b-completeness in two spaces (X, p_b) and (X, d_{p_b}).

Lemma 14 (see [31]). *(1) A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial b-metric space (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).*

(2) A partial b-metric space (X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x,x_n) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x,x_n) = \lim_{n,m\to\infty} p_b(x_n,x_m) = p_b(x,x). \quad (6)$$

Now, we introduce the concept of generalized partial b-metric space, a G_{p_b}-metric space, as a generalization of both partial b-metric space and G-metric space.

Definition 15. Let X be a nonempty set. Suppose that the mapping $G_{p_b} : X \times X \times X \to \mathbb{R}^+$ satisfies the following conditions:

($G_{p_b}1$) $x = y = z$ if $G_{p_b}(x,y,z) = G_{p_b}(z,z,z) = G_{p_b}(y,y,y) = G_{p_b}(x,x,x)$;

($G_{p_b}2$) $G_{p_b}(x,x,x) \leq G_{p_b}(x,x,y) \leq G_{p_b}(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

($G_{p_b}3$) $G_{p_b}(x,y,z) = G_{p_b}(p\{x,y,z\})$, where p is any permutation of x, y, or z (symmetry in all three variables);

($G_{p_b}4$) $G_{p_b}(x,y,z) \leq s[G_{p_b}(x,a,a) + G_{p_b}(a,y,z) - G_{p_b}(a,a,a)] + ((1-s)/3)[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)]$ for all $x,y,z,a \in X$ (rectangle inequality).

Then G_{p_b} is called a G_{p_b}-metric and (X, G_{p_b}) is called a G_{p_b}-metric space.

Since $s \geq 1$, so from $G_{p_b}4$ we have the following inequality:

$$G_{p_b}(x,y,z) \leq s\left[G_{p_b}(x,a,a) + G_{p_b}(a,y,z) - G_{p_b}(a,a,a)\right]. \quad (7)$$

The G_{p_b}-metric space G_{p_b} is called symmetric if

$$G_{p_b}(x,x,y) = G_{p_b}(x,y,y) \quad (8)$$

holds for all $x, y \in X$. Otherwise, G_{p_b} is an asymmetric G_{p_b}-metric.

Now we present some examples of G_{p_b}-metric space.

Example 16. Let $X = [0,+\infty)$ and let $G_{p_b} : X^3 \to \mathbb{R}^+$ be given by $G_{p_b}(x,y,z) = [\max\{x,y,z\}]^p$, where $p > 1$. Obviously, (X, G_{p_b}) is a symmetric G_{p_b}-metric space which is not a G-metric space. Indeed, if $x = y = z > 0$, then $G_{p_b}(x,y,z) = x^p > 0$. It is easy to see that $G_{p_b}1$–$G_{p_b}3$ are satisfied. Now we show that $G_{p_b}4$ holds. For each $x,y,z,a \in X$, we have

$$\frac{x^p + y^p + z^p}{3} \leq [\max\{x,y,z\}]^p, \quad (9)$$

so

$$\begin{aligned}
&[\max\{x,y,z\}]^p + \frac{s-1}{3}(x^p + y^p + z^p) + sa^p \\
&\quad\leq [\max\{x,y,z\}]^p + (s-1)[\max\{x,y,z\}]^p + sa^p \\
&\quad= s[\max\{x,y,z\}]^p + sa^p \\
&\quad\leq s[\max\{x,a\}]^p + s[\max\{a,y,z\}]^p.
\end{aligned} \quad (10)$$

Thus,

$$\begin{aligned}
&[\max\{x,y,z\}]^p \\
&\quad\leq s\left([\max\{x,a\}]^p + [\max\{a,y,z\}]^p - a^p\right) \\
&\qquad+ \frac{1-s}{3}(x^p + y^p + z^p)
\end{aligned} \quad (11)$$

which implies the required inequality

$$\begin{aligned}
&G_{p_b}(x,y,z) \\
&\quad\leq s\left[G_{p_b}(x,a,a) + G_{p_b}(a,y,z) - G_{p_b}(a,a,a)\right] \\
&\qquad+ \frac{1-s}{3}\left[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)\right].
\end{aligned} \quad (12)$$

Example 17. Let $X = \{0, 1, 2, 3\}$. Let

$$\begin{aligned} A = \{&(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),\\ &(0,0,2),(3,0,0),(0,3,0),(0,0,3),(1,2,2),\\ &(2,1,2),(2,2,1),(2,3,3),(3,2,3),(3,3,2)\},\\ B = \{&(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),\\ &(2,2,0),(0,3,3),(3,0,3),(3,3,0),(2,1,1),\\ &(1,2,1),(1,1,2),(3,2,2),(2,3,2),(2,2,3)\}. \end{aligned} \tag{13}$$

Define $G_{p_b} : X^3 \to \mathbb{R}^+$ by

$$G_{p_b}(x,y,z) = \begin{cases} \dfrac{3}{2}, & \text{if } x = y = z \neq 2, \\ 0, & \text{if } x = y = z = 2, \\ 2, & \text{if } (x,y,z) \in A, \\ \dfrac{5}{2}, & \text{if } (x,y,z) \in B, \\ 3, & \text{otherwise.} \end{cases} \tag{14}$$

It is easy to see that (X, G_{p_b}) is an asymmetric G_{p_b}-metric space with coefficient $s \geq 6/5$.

Proposition 18. *Every G_{p_b}-metric space (X, G_{p_b}) defines a b-metric space $(X, d_{G_{p_b}})$, where*

$$\begin{aligned} d_{G_{p_b}}(x,y) = G_{p_b}(x,y,y) + G_{p_b}(y,x,x) \\ - G_{p_b}(x,x,x) - G_{p_b}(y,y,y), \end{aligned} \tag{15}$$

for all $x, y \in X$.

Proof. Let $x, y, z \in X$. Then we have

$$\begin{aligned} &d_{G_{p_b}}(x,y) \\ &= G_{p_b}(x,y,y) + G_{p_b}(y,x,x) - G_{p_b}(x,x,x) \\ &\quad - G_{p_b}(y,y,y) \\ &\leq s\left(G_{p_b}(x,z,z) + G_{p_b}(z,y,y) - G_{p_b}(z,z,z)\right) \\ &\quad + \left(\frac{1-s}{3}\right)\left(G_{p_b}(x,x,x) + 2G_{p_b}(y,y,y)\right) \\ &\quad + s\left(G_{p_b}(y,z,z) + G_{p_b}(z,x,x) - G_{p_b}(z,z,z)\right) \\ &\quad + \left(\frac{1-s}{3}\right)\left(2G_{p_b}(x,x,x) + G_{p_b}(y,y,y)\right) \\ &\quad - G_{p_b}(x,x,x) - G_{p_b}(y,y,y) \\ &= s\left[G_{p_b}(x,z,z) + G_{p_b}(z,x,x) - G_{p_b}(x,x,x)\right. \\ &\quad - G_{p_b}(z,z,z) + G_{p_b}(z,y,y) \\ &\quad \left. + G_{p_b}(y,z,z) - G_{p_b}(z,z,z) - G_{p_b}(y,y,y)\right] \\ &= s\left[d_{G_{p_b}}(x,z) + d_{G_{p_b}}(z,y)\right]. \end{aligned} \tag{16}$$

□

With straightforward calculations, we have the following proposition.

Proposition 19. *Let X be a G_{p_b}-metric space. Then for each $x, y, z, a \in X$ it follows that*

(1) $G_{p_b}(x,y,z) \leq sG_{p_b}(x,a,a) + s^2G_{p_b}(y,a,a) + s^2G_{p_b}(z,a,a) - (s+s^2)G_{p_b}(a,a,a)$;

(2) $G_{p_b}(x,y,z) \leq s[G_{p_b}(x,x,y) + G_{p_b}(x,x,z) - G_{p_b}(x,x,x)] + ((1-s)/3)[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)]$;

(3) $G_{p_b}(x,y,y) \leq 2sG_{p_b}(x,x,y) + ((1-4s)/3)G_{p_b}(x,x,x) + (2/3)(1-s)G_{p_b}(y,y,y)$;

(4) $G_{p_b}(x,y,z) \leq s[G_{p_b}(x,a,z) + G_{p_b}(a,y,z) - G_{p_b}(a,a,a)] + ((1-s)/3)[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)]$, $a \neq z$.

Lemma 20. *Let (X, G_{p_b}) be a G_{p_b}-metric space. Then,*

(A) *if $G_{p_b}(x,y,z) = 0$, then $x = y = z$;*

(B) *if $x \neq y$, then $G_{p_b}(x,y,y) > 0$.*

Proof. If $G_{p_b}(x,y,z) = 0$, then from $G_{p_b}2$ we have $G_{p_b}(x,x,x) = G_{p_b}(y,y,y) = G_{p_b}(z,z,z) = 0$, so from $G_{p_b}1$ we obtain (A). To prove (B), on the contrary, if $G_{p_b}(x,y,y) = 0$, then from (A) we have $x = y$, a contradiction. □

Definition 21. Let (X, G_{p_b}) be a G_{p_b}-metric space. Then for an $x \in X$ and an $\epsilon > 0$, the G_{p_b}-ball with center x and radius $\epsilon > 0$ is

$$B_{G_{p_b}}(x,\epsilon) = \left\{y \in X \mid G_{p_b}(x,x,y) < G_{p_b}(x,x,x) + \epsilon\right\}. \tag{17}$$

By motivation of Proposition 4 in [31], we provide the following proposition.

Proposition 22. *Let (X, G_{p_b}) be a G_{p_b}-metric space, $x \in X$, and $\epsilon > 0$. If $y \in B_{G_{p_b}}(x,\epsilon)$, then there exists a $\delta > 0$ such that $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\varepsilon)$.*

Proof. Let $y \in B_{G_{p_b}}(x,\epsilon)$; if $y = x$, then we choose $\delta = \epsilon$. Suppose that $y \neq x$; then, by Lemma 20, we have $G_{p_b}(x,x,y) \neq 0$. Now, we consider two cases.

Case 1. If $G_{p_b}(x,x,y) = G_{p_b}(x,x,x)$, then for $s = 1$ we choose $\delta = \epsilon$. If $s > 1$, then we consider the set

$$A = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2s^{n+1}(s-1)} < G_{p_b}(x,x,x)\right\}. \tag{18}$$

By Archmedean property, A is a nonempty set; then by the well ordering principle, A has a least element m. Since $m - 1 \notin A$, we have $G_{p_b}(x,x,x) \leq (\epsilon/(2s^m(s-1)))$ and we choose $\delta = \epsilon/2s^{m+1}$. Let $z \in B_{G_{p_b}}(y, \delta)$; by property $G_{p_b}4$ we have

$$\begin{aligned} G_{p_b}(x,x,z) &\le s\left(G_{p_b}(x,x,y) + G_{p_b}(y,y,z) - G_{p_b}(y,y,y)\right) \\ &\le s\left(G_{p_b}(x,x,x) + \delta\right) \\ &\le G_{p_b}(x,x,x) + \frac{\epsilon}{2s^m} + \frac{\epsilon}{2s^m} \\ &= G_{p_b}(x,x,x) + \frac{\epsilon}{s^m} \\ &< G_{p_b}(x,x,x) + \epsilon. \end{aligned} \tag{19}$$

Hence, $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$.

Case 2. If $G_{p_b}(x,x,y) \neq G_{p_b}(x,x,x)$, then, from property $G_{p_b}2$, we have $G_{p_b}(x,x,x) < G_{p_b}(x,x,y)$, and. for $s = 1$, we consider the set

$$B = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2^{n+3}} < G_{p_b}(x,x,y) - G_{p_b}(x,x,x)\right\}. \tag{20}$$

Similarly, by the well ordering principle there exists an element m such that $G_{p_b}(x,x,y) - G_{p_b}(x,x,x) \le \epsilon/2^{m+2}$, and we choose $\delta = \epsilon/2^{m+2}$. One can easily obtain that $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$.

For $s > 1$, we consider the set

$$C = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2s^{n+2}} < G_{p_b}(x,x,y) - \frac{1}{s}G_{p_b}(x,x,x)\right\} \tag{21}$$

and by the well ordering principle there exists an element m such that $G_{p_b}(x,x,y) - (1/s)G_{p_b}(x,x,x) \le \epsilon/2s^{m+1}$ and we choose $\delta = \epsilon/2s^{m+1}$. Let $z \in B_{G_{p_b}}(y,\delta)$. By property $G_{p_b}4$ we have

$$\begin{aligned} G_{p_b}(x,x,z) &\le s\left(G_{p_b}(x,x,y) + G_{p_b}(y,y,z) - G_{p_b}(y,y,y)\right) \\ &\le s\left(G_{p_b}(x,x,y) + \delta\right) \\ &\le G_{p_b}(x,x,x) + \frac{\epsilon}{2s^m} + \frac{\epsilon}{2s^m} \\ &= G_{p_b}(x,x,x) + \frac{\epsilon}{s^m} < G_{p_b}(x,x,x) + \epsilon. \end{aligned} \tag{22}$$

Hence, $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$. □

Thus, from the above proposition the family of all open G_{p_b}-balls

$$F = \left\{B_{G_{p_b}}(x,\epsilon) \mid x \in X, \epsilon > 0\right\} \tag{23}$$

is a base of a T_0-topology $\tau_{G_{p_b}}$ on X which we call the G_{p_b}-metric topology.

The topological space (X, G_{p_b}) is T_0, but need not be T_1.

Definition 23. Let (X, G_{p_b}) be a G_{p_b}-metric space. Let $\{x_n\}$ be a sequence in X.

(1) A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{n,m\to\infty} G_{p_b}(x, x_n, x_m) = G_{p_b}(x,x,x)$.

(2) $\{x_n\}$ is said to be a G_{p_b}-Cauchy sequence, if $\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, x_m)$ exists (and is finite).

(3) (X, G_{p_b}) is said to be G_{p_b}-complete if every G_{p_b}-Cauchy sequence in X is G_{p_b}-convergent to an $x \in X$.

Using the above definitions, one can easily prove the following proposition.

Proposition 24. *Let (X, G_{p_b}) be a G_{p_b}-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following statements are equivalent:*

(1) $\{x_n\}$ is G_{p_b}-convergent to x.

(2) $G_{p_b}(x_n, x_n, x) \to G_{p_b}(x,x,x)$, as $n \to \infty$.

(3) $G_{p_b}(x_n, x, x) \to G_{p_b}(x,x,x)$, as $n \to \infty$.

Based on Lemma 2.2 of [27], we prove the following essential lemma.

Lemma 25. *(1) A sequence $\{x_n\}$ is a G_{p_b}-Cauchy sequence in a G_{p_b}-metric space (X, G_{p_b}) if and only if it is a b-Cauchy sequence in the b-metric space $(X, d_{G_{p_b}})$.*

(2) A G_{p_b}-metric space (X, G_{p_b}) is G_{p_b}-complete if and only if the b-metric space $(X, d_{G_{p_b}})$ is b-complete. Moreover, $\lim_{n\to\infty} d_{G_{p_b}}(x, x_n) = 0$ if and only if

$$\begin{aligned} \lim_{n\to\infty} G_{p_b}(x, x_n, x_n) &= \lim_{n\to\infty} G_{p_b}(x_n, x, x) \\ &= \lim_{n,m\to\infty} G_{p_b}(x_n, x_n, x_m) = G_{p_b}(x,x,x). \end{aligned} \tag{24}$$

Proof. First, we show that every G_{p_b}-Cauchy sequence in (X, G_{p_b}) is a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Let $\{x_n\}$ be a G_{p_b}-Cauchy sequence in (X, G_{p_b}). Then, there exists $\alpha \in \mathbb{R}$ such that, for arbitrary $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with

$$\left|G_{p_b}(x_n, x_m, x_m) - \alpha\right| < \frac{\varepsilon}{4}, \tag{25}$$

for all $n, m \ge n_\varepsilon$. Hence,

$$\begin{aligned} &\left|d_{G_{p_b}}(x_n, x_m)\right| \\ &\quad = G_{p_b}(x_n, x_m, x_m) + G_{p_b}(x_m, x_n, x_n) \\ &\qquad - G_{p_b}(x_n, x_n, x_n) - G_{p_b}(x_m, x_m, x_m) \\ &\quad = \left|G_{p_b}(x_n, x_m, x_m) - \alpha + \alpha - G_{p_b}(x_n, x_n, x_n)\right. \\ &\qquad \left. + G_{p_b}(x_m, x_n, x_n) - \alpha + \alpha - G_{p_b}(x_m, x_m, x_m)\right| \\ &\quad \le \left|G_{p_b}(x_n, x_m, x_m) - \alpha\right| + \left|\alpha - G_{p_b}(x_n, x_n, x_n)\right| \\ &\qquad + \left|G_{p_b}(x_m, x_n, x_n) - \alpha\right| + \left|\alpha - G_{p_b}(x_m, x_m, x_m)\right| \\ &\quad < \varepsilon, \end{aligned} \tag{26}$$

for all $n, m \geq n_\varepsilon$. Hence, we conclude that $\{x_n\}$ is a b-Cauchy sequence in $(X, d_{G_{p_b}})$.

Next, we prove that b-completeness of $(X, d_{G_{p_b}})$ implies G_{p_b}-completeness of (X, G_{p_b}). Indeed, if $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}), then it is also a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Since the b-metric space $(X, d_{G_{p_b}})$ is b-complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} d_{G_{p_b}}(y, x_n) = 0$. Hence,

$$\lim_{n\to\infty} \left[G_{p_b}(x_n, y, y) - G_{p_b}(y, y, y) + G_{p_b}(y, x_n, x_n) - G_{p_b}(x_n, x_n, x_n)\right] = 0, \tag{27}$$

therefore; $\lim_{n\to\infty}[G_{p_b}(x_n, y, y) - G_{p_b}(y, y, y)] = 0$.

On the other hand,

$$\begin{aligned}
\lim_{n,m\to\infty} & G_{p_b}(x_n, x_m, y) \\
\leq & \lim_{n,m\to\infty} sG_{p_b}(x_n, y, y) + \lim_{n,m\to\infty} sG_{p_b}(x_m, y, y) \\
& - sG_{p_b}(y, y, y) \\
& + \frac{1-s}{3}\left[\lim_{n,m\to\infty} G_{p_b}(x_n, x_n, x_n) + \lim_{n,m\to\infty} G_{p_b}(x_m, x_m, x_m) + G_{p_b}(y, y, y)\right] \\
= & G_{p_b}(y, y, y).
\end{aligned} \tag{28}$$

Also, from $(G_{p_b}2)$,

$$G_{p_b}(y, y, y) \leq \lim_{n,m\to\infty} G_{p_b}(x_n, x_m, y). \tag{29}$$

Hence, we obtain that $\{x_n\}$ is a G_{p_b}-convergent sequence in (X, G_{p_b}).

Now, we prove that every b-Cauchy sequence $\{x_n\}$ in $(X, d_{G_{p_b}})$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}). Let $\varepsilon = 1/2$. Then, there exists $n_0 \in \mathbb{N}$ such that $d_{G_{p_b}}(x_n, x_m) < 1/2$ for all $n, m \geq n_0$. Since

$$G_{p_b}\left(x_n, x_{n_0}, x_{n_0}\right) - G_{p_b}\left(x_{n_0}, x_{n_0}, x_{n_0}\right) \leq d_{G_{p_b}}\left(x_n, x_{n_0}\right) < \frac{1}{2}, \tag{30}$$

then

$$\begin{aligned}
G_{p_b}(x_n, x_n, x_n) &\leq d_{G_{p_b}}\left(x_n, x_{n_0}\right) + G_{p_b}\left(x_n, x_{n_0}, x_{n_0}\right) \\
&< \frac{1}{2} + G_{p_b}\left(x_{n_0}, x_{n_0}, x_{n_0}\right).
\end{aligned} \tag{31}$$

Consequently, the sequence $\{G_{p_b}(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$ and so there exists $a \in \mathbb{R}$ such that a subsequence $\{G_{p_b}(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to a; that is,

$$\lim_{k\to\infty} G_{p_b}\left(x_{n_k}, x_{n_k}, x_{n_k}\right) = a. \tag{32}$$

Now, we prove that $\{G_{p_b}(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a b-Cauchy sequence in $(X, d_{G_{p_b}})$, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_{G_{p_b}}(x_n, x_m) < \varepsilon$, for all $n, m \geq n_\varepsilon$. Thus, for all $n, m \geq n_\varepsilon$,

$$\begin{aligned}
G_{p_b}&(x_n, x_n, x_n) - G_{p_b}(x_m, x_m, x_m) \\
&\leq G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x_m, x_m, x_m) \\
&\leq d_{G_{p_b}}(x_m, x_n) < \varepsilon.
\end{aligned} \tag{33}$$

Therefore, $\lim_{n\to\infty} G_{p_b}(x_n, x_n, x_n) = a$.

On the other hand,

$$\begin{aligned}
&\left|G_{p_b}(x_n, x_m, x_m) - a\right| \\
&\quad = \left|G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x_n, x_n, x_n) + G_{p_b}(x_n, x_n, x_n) - a\right| \\
&\quad \leq d_{G_{p_b}}(x_m, x_n) + \left|G_{p_b}(x_n, x_n, x_n) - a\right|,
\end{aligned} \tag{34}$$

for all $n, m \geq n_\varepsilon$. Hence, $\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, x_m) = a$, and consequently $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}).

Conversely, let $\{x_n\}$ be a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Then, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}) and so it is G_{p_b}-convergent to a point $x \in X$ with

$$\begin{aligned}
\lim_{n\to\infty} G_{p_b}(x, x_n, x_n) &= \lim_{n\to\infty} G_{p_b}(x_n, x, x) \\
&= \lim_{n,m\to\infty} G_{p_b}(x, x_m, x_n) = G_{p_b}(x, x, x).
\end{aligned} \tag{35}$$

Then, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}
G_{p_b}(x, x_n, x_n) - G_{p_b}(x, x, x) &< \frac{\varepsilon}{4}, \\
G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x) &< \frac{\varepsilon}{4}.
\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
&\left|d_{G_{p_b}}(x_n, x)\right| \\
&\quad = \left|G_{p_b}(x_n, x, x) - G_{p_b}(x_n, x_n, x_n) + G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\right| \\
&\quad \leq \left|G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x)\right| \\
&\qquad + \left|G_{p_b}(x, x, x) - G_{p_b}(x_n, x_n, x_n)\right| \\
&\qquad + \left|G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\right| < \varepsilon,
\end{aligned} \tag{37}$$

whenever $n \geq n_\varepsilon$. Therefore, $(X, d_{G_{p_b}})$ is b-complete.

Finally, let $\lim_{n\to\infty} d_{G_{p_b}}(x_n, x) = 0$. So

$$\begin{aligned}
&\lim_{n\to\infty}\left[G_{p_b}(x_n, x, x) - G_{p_b}(x_n, x_n, x_n)\right] \\
&\quad + \lim_{n\to\infty}\left[G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\right] = 0, \\
&\lim_{n\to\infty}\left[G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x)\right] \\
&\quad + \lim_{n\to\infty}\left[G_{p_b}(x_n, x_n, x) - G_{p_b}(x_n, x_n, x_n)\right] = 0.
\end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned}
&\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x, x, x) \\
&\quad \le s\left[\lim_{n\to\infty} G_{p_b}(x_n, x, x) + \lim_{m\to\infty} G_{p_b}(x, x_m, x_m)\right. \\
&\qquad \left. - G_{p_b}(x, x, x)\right] + \frac{1-s}{3} \\
&\qquad \times\left[\lim_{n\to\infty} G_{p_b}(x_n, x_n, x_n) + 2\lim_{m\to\infty} G_{p_b}(x_m, x_m, x_m)\right] \\
&\qquad - G_{p_b}(x, x, x) = 0.
\end{aligned} \tag{39}$$

□

Definition 26. Let (X, G_{p_b}) and (X', G'_{p_b}) be two generalized partial b-metric spaces and let $f : (X, G_{p_b}) \to (X', G'_{p_b})$ be a mapping. Then f is said to be G_{p_b}-continuous at a point $a \in X$ if, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $G_{p_b}(a, a, x) < \delta + G_{p_b}(a, a, a)$ imply that $G'_{p_b}(f(a), f(a), f(x)) < \varepsilon + G'_{p_b}(f(a), f(a), f(a))$. The mapping f is G_{p_b}-continuous on X if it is G_{p_b}-continuous at all $a \in X$. For simplicity, we say that f is continuous.

From the above definition, with straightforward calculations, we have the following proposition.

Proposition 27. *Let (X, G_{p_b}) and (X', G'_{p_b}) be two generalized partial b-metric spaces. Then a mapping $f : X \to X'$ is G_{p_b}-continuous at a point $x \in X$ if and only if it is G_{p_b} sequentially continuous at x; that is, whenever $\{x_n\}$ is G_{p_b}-convergent to x, $\{f(x_n)\}$ is G'_{p_b}-convergent to $f(x)$.*

Definition 28. A triple $(X, \preceq, G_{p_b})$ is called an ordered generalized partial b-metric space if $(X, \preceq)$ is a partially ordered set and G_{p_b} is a generalized partial b-metric on X.

We will need the following simple lemma of the G_{p_b}-convergent sequences in the proof of our main results.

Lemma 29. *Let (X, G_{p_b}) be a G_{p_b}-metric space and suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_{p_b}-convergent to x, y, and z, respectively. Then we have*

$$\begin{aligned}
&\frac{1}{s^3} G_{p_b}(x, y, z) - \frac{1}{s^2} G_{p_b}(x, x, x) \\
&\quad - \frac{1}{s} G_{p_b}(y, y, y) - G_{p_b}(z, z, z) \\
&\quad \le \liminf_{n\to\infty} G_{p_b}(x_n, y_n, z_n) \le \limsup_{n\to\infty} G_{p_b}(x_n, y_n, z_n) \\
&\quad \le s^3 G_{p_b}(x, y, z) + s G_{p_b}(x, x, x) + s^2 G_{p_b}(y, y, y) \\
&\qquad + s^3 G_{p_b}(z, z, z).
\end{aligned} \tag{40}$$

In particular, if $\{y_n\} = \{z_n\} = a$ are constant, then

$$\begin{aligned}
&\frac{1}{s} G_{p_b}(x, a, a) - G_{p_b}(x, x, x) \\
&\quad \le \liminf_{n\to\infty} G_{p_b}(x_n, a, a) \le \limsup_{n\to\infty} G_{p_b}(x_n, a, a) \\
&\quad \le s G_{p_b}(x, a, a) + s G_{p_b}(x, x, x).
\end{aligned} \tag{41}$$

Proof. Using the rectangle inequality, we obtain

$$\begin{aligned}
G_{p_b}(x, y, z) &\le s G_{p_b}(x, x_n, x_n) + s^2 G_{p_b}(y, y_n, y_n) \\
&\quad + s^3 G_{p_b}(z, z_n, z_n) + s^3 G_{p_b}(x_n, y_n, z_n), \\
G_{p_b}(x_n, y_n, z_n) &\le s G_{p_b}(x_n, x, x) + s^2 G_{p_b}(y_n, y, y) \\
&\quad + s^3 G_{p_b}(z_n, z, z) + s^3 G_{p_b}(x, y, z).
\end{aligned} \tag{42}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

If $\{y_n\} = \{z_n\} = a$, then

$$\begin{aligned}
G_{p_b}(x, a, a) &\le s G_{p_b}(x, x_n, x_n) + s G_{p_b}(x_n, a, a), \\
G_{p_b}(x_n, a, a) &\le s G_{p_b}(x_n, x, x) + s G_{p_b}(x, a, a).
\end{aligned} \tag{43}$$

□

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{44}$$

In order to generalize the Banach contraction principle, Geraghty proved the following result.

Theorem 30 (see [32]). *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx, fy) \le \beta(d(x, y))\, d(x, y) \tag{45}$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In [33], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces.

As in [33], we will consider the class of functions $\mathscr{F}$, where $\beta \in \mathscr{F}$ if $\beta : [0, \infty) \to [0, 1/s)$ and has the property

$$\beta(t_n) \longrightarrow \frac{1}{s} \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

Theorem 31 (see [33]). *Let $s > 1$ and let (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$D(fx, fy) \leq \beta(D(x, y)) D(x, y) \tag{47}$$

for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$ and for each $x \in X\{f^n x\}$ converges to z in (X, D, s).

The aim of this paper is to present certain new fixed point theorems for hybrid rational Geraghty-type and ψ-contractive mappings in partially ordered G_{p_b}-metric spaces. Our results improve and generalize many comparable results in literature. Some examples are established to prove the generality of our results.

2. Main Results

Recall that $\mathcal{F}$ denotes the class of all functions $\beta : [0, \infty) \to [0, 1/s)$ satisfying the following condition:

$$\beta(t_n) \longrightarrow \frac{1}{s} \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{48}$$

Theorem 32. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \leq \beta(G_{p_b}(x, y, z)) M(x, y, z) \tag{49}$$

for all comparable elements $x, y, z \in X$, where

$$\begin{aligned} M(x, y, z) &= \max \left\{ G_{p_b}(x, y, z), \right. \\ &\left. \frac{G_{p_b}(x, x, fx) G_{p_b}(y, y, fy) G_{p_b}(z, z, fz)}{1 + [sG_{p_b}(fx, fy, fz)]^2} \right\}. \end{aligned} \tag{50}$$

If f is continuous, then f has a fixed point.

Proof. Put $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots. \tag{51}$$

Step 1. We will show that $\lim_{n \to \infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (49) we have

$$\begin{aligned} &sG_{p_b}(x_n, x_{n+1}, x_{n+2}) \\ &\quad = sG_{p_b}(fx_{n-1}, fx_n, fx_{n+1}) \\ &\quad \leq \beta(G_{p_b}(x_{n-1}, x_n, x_{n+1})) M(x_{n-1}, x_n, x_{n+1}) \\ &\quad \leq \frac{1}{s} G_{p_b}(x_{n-1}, x_n, x_{n+1}) \\ &\quad \leq G_{p_b}(x_{n-1}, x_n, x_{n+1}), \end{aligned} \tag{52}$$

because

$$\begin{aligned} &M(x_{n-1}, x_n, x_{n+1}) \\ &\quad = \max \left\{ G_{p_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\qquad (G_{p_b}(x_{n-1}, x_{n-1}, fx_{n-1}) G_{p_b}(x_n, x_n, fx_n) \\ &\qquad \times G_{p_b}(x_{n+1}, x_{n+1}, fx_{n+1})) \\ &\qquad \left. \times \left(1 + [sG_{p_b}(fx_{n-1}, fx_n, fx_{n+1})]^2\right)^{-1} \right\} \\ &\quad = \max \left\{ G_{p_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\qquad (G_{p_b}(x_{n-1}, x_{n-1}, x_n) G_{p_b}(x_n, x_n, x_{n+1}) \\ &\qquad \times G_{p_b}(x_{n+1}, x_{n+1}, x_{n+2})) \\ &\qquad \left. \times \left(1 + [sG_{p_b}(x_n, x_{n+1}, x_{n+2})]^2\right)^{-1} \right\} \\ &\quad = G_{p_b}(x_{n-1}, x_n, x_{n+1}). \end{aligned} \tag{53}$$

Therefore, $\{G_{p_b}(x_n, x_{n+1}, x_{n+2})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n \to \infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = r$. Letting $n \to \infty$ in (52) we have

$$sr \leq r. \tag{54}$$

Since $s > 1$, we deduce that $r = 0$; that is,

$$\lim_{n \to \infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = 0. \tag{55}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a G_{p_b}-Cauchy sequence. By rectangular inequality and (49), we have

$$\begin{aligned} &G_{p_b}(x_n, x_m, x_m) \\ &\quad \leq sG_{p_b}(x_n, x_{n+1}, x_{n+1}) + s^2 G_{p_b}(x_{n+1}, x_{m+1}, x_{m+1}) \\ &\qquad + s^2 G_{p_b}(x_{m+1}, x_m, x_m) \\ &\quad \leq sG_{p_b}(x_n, x_{n+1}, x_{n+1}) + s\beta(G_{p_b}(x_n, x_m, x_m)) \\ &\qquad \times M(x_n, x_m, x_m) + s^2 G_{p_b}(x_{m+1}, x_m, x_m). \end{aligned} \tag{56}$$

Letting $m, n \to \infty$ in the above inequality and applying (55) we have

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \leq s \lim_{n,m\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right) \lim_{n,m\to\infty} M(x_n, x_m, x_m). \quad (57)$$

Here,

$$\begin{aligned} G_{p_b}(x_n, x_m, x_m) &\leq M(x_n, x_m, x_m) \\ &= \max\left\{G_{p_b}(x_n, x_m, x_m), \frac{G_{p_b}(x_n, x_n, f(x_n))\left[G_{p_b}(x_m, x_m, f(x_m))\right]^2}{1 + \left[sG_{p_b}(fx_n, fx_m, fx_m)\right]^2}\right\} \\ &= \max\left\{G_{p_b}(x_n, x_m, x_m), \frac{G_{p_b}(x_n, x_n, x_{n+1})\left[G_{p_b}(x_m, x_m, x_{m+1})\right]^2}{1 + \left[sG_{p_b}(x_{n+1}, x_{m+1}, x_{m+1})\right]^2}\right\}. \end{aligned} \quad (58)$$

Letting $m, n \to \infty$ in the above inequality we get

$$\lim_{m,n\to\infty} M(x_n, x_m, x_m) = \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m). \quad (59)$$

Hence, from (57) and (59), we obtain

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \leq s \lim_{m,n\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right) \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \quad (60)$$

and so we get

$$\frac{1}{s} \leq \lim_{m,n\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right). \quad (61)$$

Since $\beta \in \mathcal{F}$ we deduce that

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) = 0. \quad (62)$$

Consequently, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in X. Thus, from Lemma 25, $\{x_n\}$ is a b-Cauchy sequence in the b-metric space $(X, d_{G_{p_b}})$. Since (X, G_{p_b}) is G_{p_b}-complete, then, from Lemma 25, $(X, d_{G_{p_b}})$ is a b-complete b-metric space. Therefore, the sequence $\{x_n\}$ b-converges to some $u \in X$; that is, $\lim_{n\to\infty} d_{G_{p_b}}(x_n, u) = 0$. Again, from Lemma 25 and (62),

$$\begin{aligned} \lim_{n\to\infty} G_{p_b}(u, x_n, x_n) &= \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \\ &= G_{p_b}(u, u, u) = 0. \end{aligned} \quad (63)$$

Step 3. Now, we show that u is a fixed point of f. Suppose to the contrary; that is, $fu \neq u$; then, from Lemma 20, we have $G_{p_b}(u, u, fu) > 0$.

Using the rectangular inequality, we get

$$G_{p_b}(u, u, fu) \leq sG_{p_b}(fu, fx_n, fx_n) + sG_{p_b}(fx_n, u, u). \quad (64)$$

Letting $n \to \infty$ and using the continuity of f and (63), we get

$$\begin{aligned} G_{p_b}(u, u, fu) &\leq s \lim_{n\to\infty} G_{p_b}(fu, fx_n, fx_n) \\ &\quad + s \lim_{n\to\infty} G_{p_b}(fx_n, u, u) = sG_{p_b}(fu, fu, fu). \end{aligned} \quad (65)$$

Note that, from (49), we have

$$sG_{p_b}(fu, fu, fu) \leq \beta\left(G_{p_b}(u, u, u)\right) M(u, u, u), \quad (66)$$

where by (65)

$$\begin{aligned} M(u, u, u) &= \max\left\{G_{p_b}(u, u, u), \frac{G_{p_b}(u, u, fu)\, G_{p_b}(u, u, fu)\, G_{p_b}(u, u, fu)}{1 + \left[sG_{p_b}(fu, fu, fu)\right]^2}\right\} \\ &\leq G_{p_b}(u, u, fu). \end{aligned} \quad (67)$$

Hence, as $\beta(t) \leq 1$ for all $t \in [0, \infty)$, we have $sG_{p_b}(fu, fu, fu) \leq G_{p_b}(u, u, fu)$. Thus, by (65) we obtain that $sG_{p_b}(fu, fu, fu) = G_{p_b}(u, u, fu)$. But then, using (66), we get that $G_{p_b}(u, u, fu) = sG_{p_b}(fu, fu, fu) \leq \beta(G_{p_b}(u, u, u))M(u, u, u) < G_{p_b}(u, u, fu)$, which is a contradiction. Hence, we have $fu = u$. Thus, u is a fixed point of f. □

Now we replace the continuity of f in Theorem 32 by the regularity of the space to get the required conclusion.

Theorem 33. *Under the same hypotheses of Theorem 32, instead of the continuity assumption of f, assume that, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Repeating the proof of Theorem 32, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 29 and (63)

$$\begin{aligned} s\left[\frac{1}{s}G_{p_b}(u, fu, fu) - G_{p_b}(u,u,u)\right] &\le s\limsup_{n\to\infty} G_{p_b}(x_{n+1}, fu, fu) \\ &\le \limsup_{n\to\infty}\left(\beta\left(G_{p_b}(x_n,u,u)\right)M(x_n,u,u)\right) \\ &\le \frac{1}{s}\limsup_{n\to\infty} M(x_n,u,u), \end{aligned} \tag{68}$$

where

$$\begin{aligned} &\lim_{n\to\infty} M(x_n,u,u) \\ &= \lim_{n\to\infty}\max\left\{G_{p_b}(x_n,u,u), \frac{\left[G_{p_b}(x_n,x_n,fx_n), G_{p_b}(u,u,fu)\right]^2}{1+\left[sG_{p_b}(fx_n,fu,fu)\right]^2}\right\} \\ &= \lim_{n\to\infty}\max\left\{G_{p_b}(x_n,x_n,u), \frac{\left[G_{p_b}(x_n,x_n,x_{n+1}), G_{p_b}(u,u,fu)\right]^2}{1+\left[sG_{p_b}(x_{n+1},fu,fu)\right]^2}\right\} \\ &= \max\left\{G_{p_b}(u,u,u), 0\right\} = 0 \quad (\text{see (55) and (63)}). \end{aligned} \tag{69}$$

Therefore, we deduce that $G_{p_b}(u, fu, fu) \le sG_{p_b}(u,u,u) = 0$. Hence, we have $u = fu$. □

If in the above theorems we assume $\beta(t) = r$, where $0 \le r \le 1/s$, we obtain the following corollary.

Corollary 34. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G_{p_b}-metric on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \le rM(x,y,z) \tag{70}$$

for all comparable elements $x, y, z \in X$, where $0 \le r < 1/s$ and

$$\begin{aligned} &M(x,y,z) \\ &= \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\, G_{p_b}(y,y,fy)\, G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\}. \end{aligned} \tag{71}$$

If f is continuous or for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 35. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G_{p_b}-metric space G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$\begin{aligned} &sG_{p_b}(fx, fy, fz) \\ &\quad\le aG_{p_b}(x,y,z) \\ &\qquad + b\frac{G_{p_b}(x,x,fx)\, G_{p_b}(y,y,fy)\, G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2} \end{aligned} \tag{72}$$

for all comparable elements $x, y, z \in X$, where $a, b \ge 0$ and $a + b \le 1/s$.

If f is continuous or for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Proof. Since

$$\begin{aligned} &aG_{p_b}(x,y,z) + b\frac{G_{p_b}(x,x,fx)\, G_{p_b}(y,y,fy)\, G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2} \\ &\quad\le (a+b) \\ &\quad\times \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\, G_{p_b}(y,y,fy)\, G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\}, \end{aligned} \tag{73}$$

taking $r = a + b$, all the conditions of Corollary 34 hold and hence f has a fixed point. □

Let Ψ be the family of all continuous and nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty}\psi^n(t) = 0 \tag{74}$$

for all $t > 0$. It is known that, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Theorem 36. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \le \psi(M(x,y,z)), \tag{75}$$

where

$$M(x,y,z) = \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\} \quad (76)$$

for all comparable elements $x,y,z \in X$. If f is continuous, then f has a fixed point.

Proof. Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots . \quad (77)$$

Putting $x_n = f^n(x_0)$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \quad (78)$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, for all $n \in \mathbb{N}$, we assume that $x_n \neq x_{n+1}$.

Step 1. We will prove that

$$\lim_{n\to\infty} G_{p_b}(x_n,x_{n+1},x_{n+2}) = 0. \quad (79)$$

Using condition (75), we obtain

$$\begin{aligned} G_{p_b}(x_n,x_{n+1},x_{n+2}) &\leq sG_{p_b}(x_n,x_{n+1},x_{n+2}) \\ &= sG_{p_b}(fx_{n-1},fx_n,fx_{n+1}) \\ &\leq \psi(M(x_{n-1},x_n,x_{n+1})). \end{aligned} \quad (80)$$

Here,

$$\begin{aligned} &M(x_{n-1},x_n,x_{n+1}) \\ &= \max\Big\{G_{p_b}(x_{n-1},x_n,x_{n+1}), \\ &\qquad \times\left(G_{p_b}(x_{n-1},x_{n-1},fx_{n-1})\,G_{p_b}(x_n,x_n,fx_n)\right. \\ &\qquad \left.\times G_{p_b}(x_{n+1},x_{n+1},fx_{n+1})\right) \\ &\qquad \times\left(1+\left[G_{p_b}(fx_{n-1},fx_n,fx_{n+1})\right]\right)^{-1}\Big\} \\ &= G_{p_b}(x_{n-1},x_n,x_{n+1}). \end{aligned} \quad (81)$$

Hence,

$$\begin{aligned} G_{p_b}(x_n,x_{n+1},x_{n+2}) &\leq sG_{p_b}(x_n,x_{n+1},x_{n+2}) \\ &\leq \psi\left(G_{p_b}(x_{n-1},x_n,x_{n+1})\right). \end{aligned} \quad (82)$$

By induction, we get that

$$\begin{aligned} G_{p_b}(x_{n+2},x_{n+1},x_n) &\leq \psi\left(G_{p_b}(x_{n+1},x_n,x_{n-1})\right) \\ &\leq \psi^2\left(G_{p_b}(x_n,x_{n-1},x_{n-2})\right) \leq \cdots \\ &\leq \psi^n\left(G_{p_b}(x_2,x_1,x_0)\right). \end{aligned} \quad (83)$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} G_{p_b}(x_n,x_{n+1},x_{n+2}) = 0. \quad (84)$$

Step 2. We will prove that $\{x_n\}$ is a G_{p_b}-Cauchy sequence.

Suppose to the contrary that $\{x_n\}$ is not a G_{p_b}-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad G_{p_b}\left(x_{m_i},x_{n_i},x_{n_i}\right) \geq \varepsilon. \quad (85)$$

This means that

$$G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) < \varepsilon. \quad (86)$$

From (85) and using the rectangle inequality, we get

$$\begin{aligned} \varepsilon \leq G_{p_b}\left(x_{m_i},x_{n_i},x_{n_i}\right) &\leq sG_{p_b}\left(x_{m_i},x_{m_i+1},x_{m_i+1}\right) \\ &\quad + sG_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right). \end{aligned} \quad (87)$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} G_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right). \quad (88)$$

From the definition of $M(x,y)$ we have

$$\begin{aligned} &M\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) \\ &= \max\Big\{G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right), \\ &\qquad \Big(G_{p_b}\left(x_{m_i},x_{m_i},fx_{m_i}\right) \\ &\qquad \times\left[G_{p_b}\left(x_{n_i-1},x_{n_i-1},fx_{n_i-1}\right)\right]^2\Big) \\ &\qquad \times\left(1+\left[sG_{p_b}\left(fx_{m_i},fx_{n_i-1},fx_{n_i-1}\right)\right]^2\right)^{-1}\Big\} \\ &= \max\Big\{G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right), \\ &\qquad \Big(G_{p_b}\left(x_{m_i},x_{m_i},x_{m_i+1}\right) \\ &\qquad \times\left[G_{p_b}\left(x_{n_i-1},x_{n_i-1},x_{n_i}\right)\right]^2\Big) \\ &\qquad \times\left(1+\left[sG_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right)\right]^2\right)^{-1}\Big\} \end{aligned} \quad (89)$$

and if $i \to \infty$, by (84) and (86), we have

$$\limsup_{i\to\infty} M\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) \leq \varepsilon. \quad (90)$$

Now, from (75) we have

$$sG_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right)=sG_{p_b}\left(fx_{m_i},fx_{n_i-1},fx_{n_i-1}\right)\leq\psi\left(M\left(x_{m_i},x_{n_i-1},x_{n_{i-1}}\right)\right).\tag{91}$$

Again, if $i\to\infty$ by (88) we obtain

$$\varepsilon=s\left(\frac{\varepsilon}{s}\right)\leq\left(s\limsup_{i\to\infty}G_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right)\right)\leq\psi(\varepsilon)<\varepsilon,\tag{92}$$

which is a contradiction. Consequently, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in X. Thus, from Lemma 25 we have proved that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space $(X,d_{G_{p_b}})$. Since (X,G_{p_b}) is G_{p_b}-complete, then, from Lemma 25, $(X,d_{G_{p_b}})$ is a b-complete b-metric space. Therefore, the sequence $\{x_n\}$ b-converges to some $u\in X$; that is, $\lim_{n\to\infty}d_{G_{p_b}}(x_n,u)=0$. Again, from Lemma 25 and (62),

$$\lim_{n\to\infty}G_{p_b}(u,x_n,x_n)=\lim_{m,n\to\infty}G_{p_b}(x_n,x_m,x_m)=G_{p_b}(u,u,u)=0.\tag{93}$$

Step 3. Now we show that u is a fixed point of f. Suppose to the contrary, that $fu\neq u$; then, from Lemma 20, we have $G_{p_b}(u,u,fu)>0$.

Using the rectangle inequality, we get

$$G_{p_b}(u,u,fu)\leq sG_{p_b}(fu,fx_n,fx_n)+sG_{p_b}(fx_n,u,u).\tag{94}$$

Letting $n\to\infty$ and using the continuity of f, we get

$$G_{p_b}(u,u,fu)\leq sG_{p_b}(fu,fu,fu).\tag{95}$$

Note that, from (75), we have

$$sG_{p_b}(fu,fu,fu)\leq\psi(M(u,u,u)),\tag{96}$$

where

$$M(u,u,u)=\max\left\{G_{p_b}(u,u,u),\frac{G_{p_b}(u,u,fu)G_{p_b}(u,u,fu)G_{p_b}(u,u,fu)}{1+\left[sG_{p_b}(fu,fu,fu)\right]^2}\right\}\leq G_{p_b}(u,u,fu).\tag{97}$$

Hence, as ψ is nondecreasing, we have $sG_{p_b}(fu,fu,fu)\leq G_{p_b}(u,u,fu)$. Thus, by (95) we obtain that

$$G_{p_b}(u,u,fu)=sG_{p_b}(fu,fu,fu).\tag{98}$$

Equation (96) yields that $G_{p_b}(u,u,fu)\leq\psi(M(u,u,u))\leq\psi(G_{p_b}(u,u,fu))$. This is impossible, according to our assumptions on ψ. Hence, we have $fu=u$. Thus, u is a fixed point of f. □

Theorem 37. *Under the hypotheses of Theorem 36, instead of the continuity assumption of f, assume that, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to u\in X$, one has $x_n\preceq u$ for all $n\in\mathbb{N}$. Then f has a fixed point.*

Proof. Following the proof of Theorem 36, we construct an increasing sequence $\{x_n\}$ in X such that $x_n\to u\in X$. Using the given assumption on X we have $x_n\preceq u$. Now, we show that $u=fu$. By (75) we have

$$sG_{p_b}(fu,fu,x_n)=sG_{p_b}(fu,fu,fx_{n-1})\leq\psi(M(u,u,x_{n-1})),\tag{99}$$

where

$$M(u,u,x_{n-1})=\max\left\{G_{p_b}(u,u,x_{n-1}),\frac{\left[G_{p_b}(u,u,fu)\right]^2G_{p_b}(x_{n-1},x_{n-1},fx_{n-1})}{1+\left[sG_{p_b}(fu,fu,fx_{n-1})\right]^2}\right\}.\tag{100}$$

Letting $n\to\infty$ in the above, from (93), we get

$$\lim_{n\to\infty}M(u,u,x_{n-1})=0.\tag{101}$$

Again, taking the upper limit as $n\to\infty$ in (99) and using Lemma 29 and (101) we get

$$s\left[\frac{1}{s}G_{p_b}(u,fu,fu)-G_{p_b}(u,u,u)\right]\leq s\limsup_{n\to\infty}G_{p_b}(x_n,fu,fu)\leq\limsup_{n\to\infty}\psi(M(u,u,x_{n-1}))=0.\tag{102}$$

So we get $G_{p_b}(u,fu,fu)=0$. That is, $fu=u$. □

Corollary 38. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X,G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f:X\to X$ be an increasing mapping with $x_0\preceq f(x_0)$ for some $x_0\in X$. Suppose that*

$$sG_{p_b}(fx,fy,fz)\leq kM(x,y,z),\tag{103}$$

where $0\leq k<1/s$ and

$$M(x,y,z)=\max\left\{G_{p_b}(x,y,z),\frac{G_{p_b}(x,x,fx)G_{p_b}(y,y,fy)G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\}\tag{104}$$

for all comparable elements $x, y, z \in X$. If f is continuous or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$, we have $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

We conclude this section by presenting some examples that illustrate our results.

Example 39. Let $X = [0,1]$ be equipped with the usual order and G_{p_b}-metric function G_{p_b} given by $G_{p_b}(x,y,z) = [\max\{x,y,z\}]^2 = \max\{x^2,y^2,z^2\}$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = (1/4)x(e^{-x^2})^{1/2}$ and the function $\beta \in \mathcal{F}$ given by $\beta(t) = (1/2)e^{-t}$, $t > 0$, and $\beta(0) \in [0, 1/2)$. It is easy to see that f is an increasing function on X and $0 \preceq f(0) = 0$. We show that f is G_{p_b}-continuous on X. By Proposition 27 it is sufficient to show that f is G_{p_b}sequentially continuous on X. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G_{p_b}(x_n,x,x) = G_{p_b}(x,x,x)$, so we have $\lim_{n\to\infty}\max\{x_n^2,x^2\} = x^2$, equally $\max\{\lim_{n\to\infty}x_n^2, x^2\} = x^2$, and hence $\lim_{n\to\infty}x_n^2 = \alpha \le x^2$. On the other hand we have

$$
\begin{aligned}
&\lim_{n\to\infty} G_{p_b}(fx_n, fx, fx)\\
&\quad = \lim_{n\to\infty}\max\left\{(fx_n)^2, (fx)^2\right\}\\
&\quad = \lim_{n\to\infty}\max\left\{\frac{1}{16}x_n^2e^{-x_n^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = \max\left\{\frac{1}{16}\lim_{n\to\infty}x_n^2e^{-x_n^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = \max\left\{\frac{1}{16}\alpha e^{-\alpha}, \frac{1}{16}x^2e^{-x^2}\right\} = \frac{1}{16}x^2e^{-x^2}\\
&\quad = \max\left\{\frac{1}{16}x^2e^{-x^2}, \frac{1}{16}x^2e^{-x^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = G_{p_b}(fx, fx, fx).
\end{aligned}
\tag{105}
$$

So f is G_{p_b}sequentially continuous on X.

For all comparable elements $x, y, z \in X$ and the fact that $g(x) = x^2e^{-x^2}$ is an increasing function on X we have

$$
\begin{aligned}
sG_{p_b}(fx, fy, fz) &= 2\max\left\{\frac{1}{16}x^2e^{-x^2}, \frac{1}{16}y^2e^{-y^2}, \frac{1}{16}z^2e^{-z^2}\right\}\\
&= \frac{1}{8}\max\left\{x^2e^{-x^2}, y^2e^{-y^2}, z^2e^{-z^2}\right\}\\
&= \frac{1}{8}e^{-\max\{x^2,y^2,z^2\}}\max\left\{x^2,y^2,z^2\right\}\\
&\le \frac{1}{2}e^{-\max\{x^2,y^2,z^2\}}\max\left\{x^2,y^2,z^2\right\}\\
&= \beta\left(G_{p_b}(x,y,z)\right)G_{p_b}(x,y,z)\\
&\le \beta\left(G_{p_b}(x,y,z)\right)M(x,y,z).
\end{aligned}
\tag{106}
$$

Hence, f satisfies all the assumptions of Theorem 32 and thus it has a fixed point (which is $u = 0$).

Example 40. Let $X = [0,1]$ be equipped with the usual order and G_{p_b}-metric function G_{p_b} given by $G_{p_b}(x,y,z) = [\max\{x,y,z\}]^2 = \max\{x^2,y^2,z^2\}$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = (1/4)\sqrt{\ln(x^2+1)}$ and the function $\psi \in \Psi$ given by $\psi(t) = (1/8)t$, $t \ge 0$. It is easy to see that f is increasing function and $0 \preceq f(0) = 0$. Now we show that f is G_{p_b}-continuous function on X.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}G_{p_b}(x_n,x,x) = G_{p_b}(x,x,x)$, so we have $\lim_{n\to\infty}\max\{x_n^2,x^2\} = x^2$, equally $\max\{\lim_{n\to\infty}x_n^2,x^2\} = x^2$, and hence $\lim_{n\to\infty}x_n^2 = \alpha \le x^2$. On the other hand we have

$$
\begin{aligned}
&\lim_{n\to\infty} G_{p_b}(fx_n, fx, fx)\\
&\quad = \lim_{n\to\infty}\max\left\{(fx_n)^2, (fx)^2\right\}\\
&\quad = \lim_{n\to\infty}\max\left\{\frac{1}{16}\ln\left(x_n^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = \max\left\{\frac{1}{16}\ln\left(\lim_{n\to\infty}x_n^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = \max\left\{\frac{1}{16}\ln(\alpha+1), \frac{1}{16}\ln\left(x^2+1\right)\right\} = \frac{1}{16}\ln\left(x^2+1\right)\\
&\quad = \max\left\{\frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = G_{p_b}(fx, fx, fx).
\end{aligned}
\tag{107}
$$

So f is G_{p_b}-sequentially continuous on X.

For all comparable elements $x, y, z \in X$, we have

$$
\begin{aligned}
&sG_{p_b}(fx, fy, fz)\\
&\quad = 2\max\left\{\left(\frac{1}{4}\sqrt{\ln(x^2+1)}\right)^2, \left(\frac{1}{4}\sqrt{\ln(y^2+1)}\right)^2,\right.\\
&\qquad\qquad \left.\left(\frac{1}{4}\sqrt{\ln(z^2+1)}\right)^2\right\}\\
&\quad = 2\max\left\{\frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(y^2+1\right), \frac{1}{16}\ln\left(z^2+1\right)\right\}\\
&\quad = \frac{1}{8}\max\left\{\ln\left(x^2+1\right), \ln\left(y^2+1\right), \ln\left(z^2+1\right)\right\}\\
&\quad \le \frac{1}{8}\max\left\{x^2,y^2,z^2\right\} = \psi\left(G_{p_b}(x,y,z)\right)\\
&\quad \le \psi\left(M(x,y,z)\right).
\end{aligned}
\tag{108}
$$

Hence, f satisfies all the assumptions of Theorem 36 and thus it has a fixed point (which is $u = 0$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the first and fourth authors acknowledge with thanks DSR, KAU, for financial support.

References

[1] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[2] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, no. 2, pp. 263–276, 1998.

[3] A. Aghajani, M. Abbas, and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces," *Mathematica Slovaca*. In press.

[4] H. Aydi, M.-F. Bota, E. Karapinar, and S. Moradi, "A common fixed point for weak φ-contractions on b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[5] H. Aydi, M. F. Bota, E. Karapınar, and S. Mitrovic, "A fixed point theorem for set-valued quasicontractions in b-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 88, 2012.

[6] M. Boriceanu, "Fixed point theory for multivalued generalized contraction on a set with two b-metrics," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 54, no. 3, pp. 3–14, 2009.

[7] M. Boriceanu, "Strict fixed point theorems for multivalued operators in b-metric spaces," *International Journal of Modern Mathematics*, vol. 4, no. 3, pp. 285–301, 2009.

[8] N. Hussain, D. Dorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 2012.

[9] N. Hussain and M. H. Shah, "KKM mappings in cone b-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[10] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 9, pp. 3123–3129, 2010.

[11] M. A. Khamsi, "Remarks on cone metric spaces and fixed point theorems of contractive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 315398, 7 pages, 2010.

[12] M. Păcurar, "Sequences of almost contractions and fixed points in b-metric spaces," *Analele Universitatii de Vest din Timisoara*, vol. 48, no. 3, pp. 125–137, 2010.

[13] J. R. Roshan, N. Shobkolaei, S. Sedghi, and M. Abbas, "Common fixed point of four maps in b-metric spaces," *Hacettepe Journal of Mathematics and Statistics*. In press.

[14] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[15] A. Aghajani, M. Abbas, and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces," *Filomat*. In press.

[16] S. G. Matthews, "Partial metric topology," Annals of the New York Academy of Sciences, vol. 728, Proceedings of the 8th Summer Conference, Queens College, 1992, General Topology and its Applications, pp. 183–197, 1994.

[17] H. Aydi, "Common fixed point results for mappings satisfying (ψ,φ)-weak contractions in ordered partial metric space," *International Journal of Mathematics and Statistics*, vol. 12, no. 2, pp. 53–64, 2012.

[18] Z. Kadelburg, H. K. Nashine, and S. Radenović, "Fixed point results under various contractive conditions in partial metric spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales A*, vol. 107, no. 2, pp. 241–256.

[19] H. K. Nashine, Z. Kadelburg, and S. Radenović, "Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2355–2365, 2013.

[20] N. Shobkolaei, S. Sedghi, J. R. Roshan, and I. Altun, "Common fixed point of mappings satisfying almost generalized (S,T)-contractive condition in partially ordered partial metric spaces," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 443–452, 2012.

[21] M. Abbas, T. Nazir, and S. Radenović, "Some periodic point results in generalized metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 4094–4099, 2010.

[22] H. Alaeidizaji and V. Parvaneh, "Coupled fixed point results in complete partial metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 670410, 12 pages, 2012.

[23] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology and Its Applications*, vol. 157, no. 18, pp. 2778–2785, 2010.

[24] L. Ćirić, B. Samet, H. Aydi, and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2398–2406, 2011.

[25] M. Khandaqji, S. Al-Sharif, and M. Al-Khaleel, "Property P and some fixed point results on (ψ,φ)-weakly contractive G-metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 675094, 11 pages, 2012.

[26] Z. Mustafa and B. Sims, "Fixed point theorems for contractive mappings in complete G-metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[27] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste of Mathematics*, vol. 36, no. 1-2, pp. 17–26, 2004.

[28] M. R. A. Zand and A. D. Nezhad, "A generalization of partial metric spaces," *Journal of Contemporary Applied Mathematics*, vol. 24, pp. 86–93, 2011.

[29] V. Parvaneh, J. R. Roshan, and S. Radenović, "Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 130, 2013.

[30] S. Shukla, "Partial b-metric spaces and fixed point theorems," *Mediterranean Journal of Mathematics*, 2013.

[31] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Some common fixed point results in ordered partial b-metric spaces," *Journal of Inequalities and Applications*, vol. 2013, article 562, 2013.

[32] M. A. Geraghty, "On contractive mappings," *Proceedings of the American Mathematical Society*, vol. 40, pp. 604–608, 1973.

[33] D. Dukić, Z. Kadelburg, and S. Radenović, "Fixed points of Geraghty-type mappings in various generalized metric spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 561245, 13 pages, 2011.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 131482, 10 pages
http://dx.doi.org/10.1155/2014/131482

Research Article

Research on Adjoint Kernelled Quasidifferential

Si-Da Lin,[1] Fu-Min Xiao,[2] Zun-Quan Xia,[1] and Li-Ping Pang[1]

[1] *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*
[2] *Department of Hydrography & Cartography, PLA Dalian Naval Academy, Dalian 116018, China*

Correspondence should be addressed to Si-Da Lin; lsdsl@163.com

Received 10 October 2013; Accepted 19 January 2014; Published 5 March 2014

Academic Editor: Hichem Ben-El-Mechaiekh

The quasidifferential of a quasidifferentiable function in the sense of Demyanov and Rubinov is not uniquely defined. Xia proposed the notion of the kernelled quasidifferential, which is expected to be a representative for the equivalence class of quasidifferentials. Although the kernelled quasidifferential is known to have good algebraic properties and geometric structure, it is still not very convenient for calculating the kernelled quasidifferentials of $-f$ and $\min\{f_i \mid i \in$ a finite index set $I\}$, where f and f_i are kernelled quasidifferentiable functions. In this paper, the notion of adjoint kernelled quasidifferential, which is well-defined for $-f$ and $\min\{f_i \mid i \in I\}$, is employed as a representative of the equivalence class of quasidifferentials. Some algebraic properties of the adjoint kernelled quasidifferential are given and the existence of the adjoint kernelled quasidifferential is explored by means of the minimal quasidifferential and the Demyanov difference of convex sets. Under some condition, a formula of the adjoint kernelled quasidifferential is presented.

1. Introduction

Quasidifferential calculus, developed by Demyanov and Rubinov, plays an important role in nonsmooth analysis and optimization. The class of quasidifferentiable functions is fairly broad. It contains not only convex, concave, and differentiable functions but also convex-concave, D.C. (i.e., difference of two convex), maximum, and other functions. In addition, it even includes some functions which are not locally Lipschitz continuous. Quasidifferentiability can be employed to study a wide range of theoretical and practical issues in many fields, such as in mechanics, engineering, and economics nonsmooth analysis and fuzzy control theory (see, e.g., [1–13]).

A function f defined on an open set $\mathcal{O} \subset R^n$ is called quasidifferentiable (q.d.) at a point $x \in \mathcal{O}$, in the sense of Demyanov and Rubinov [5], if it is directionally differentiable at x and there exist two nonempty convex compact sets $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ such that the directional derivative can be represented in the form as

$$f'(x;d) = \max_{u\in\underline{\partial} f(x)} \langle u, d\rangle + \min_{v\in\overline{\partial} f(x)} \langle v, d\rangle, \quad \forall d \in R^n, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in R^n. The pair of sets $Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a quasidifferential of f at x and $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ are called a subdifferential and a superdifferential, respectively.

It is well known that the quasidifferential is not uniquely defined. Let Y_n be the set of all nonempty convex compact sets in R^n. Denote $A \pm B = \{a \pm b \mid a \in A, b \in B\}$ and $\lambda A = \{\lambda a \mid a \in A\}$, where $A, B \in Y_n$ and $\lambda \geq 0$. Suppose that $[U, V]$ is a quasidifferential of f; then, for any $A \in Y_n$, the pair of sets $[U+A, V-A]$ is still a quasidifferential of f. And the set $\mathcal{D} f(x)$ of quasidifferentials of f at x is so large that the whole space R^n could be covered by the union of subdifferentials or superdifferentials; that is,

$$R^n = \bigcup_{Df(x)\in\mathcal{D} f(x)} \underline{\partial} f(x) = \bigcup_{Df(x)\in\mathcal{D} f(x)} \overline{\partial} f(x). \tag{2}$$

The quasidifferential uniqueness is an essential problem in quasidifferential calculus, so it is necessary to find a way by which a quasidifferential, particularly a small quasidifferential in some sense, as a representative of the equivalence class of quasidifferentials, can be determined automatically. The problem was for the first time considered in a discussion at IIASA, by Demyanov and Xia in 1984 [4]. There were many reports and publications mentioning or dealing with this subject from different points of view (see, for instance, [9–26], etc.).

Pallaschke et al. [18] introduced the notion of the minimal quasidifferential and proved the existence of equivalent minimal quasidifferential. $[U, V] \in \mathscr{D} f(x)$ is called minimal, provided that $[U_1, V_1] \in \mathscr{D} f(x)$ satisfying $U_1 \subset U$ and $V_1 \subset V$ implies $U = U_1$ and $V = V_1$. Nevertheless, the minimal quasidifferential is not uniquely defined either. Indeed, any translation of a minimal quasidifferential is still a minimal quasidifferential; in other words, if $[A, B]$ is a minimal quasidifferential, then, for any singleton $\{c\}$, the pair of sets $[A + \{c\}, B - \{c\}]$ is still a minimal quasidifferential. For one-dimensional space, equivalent minimal pairs are uniquely determined up to translations, according to [8]. Grzybowski [15] and Scholtes [22] proved independently the fact that equivalent minimal quasidifferentials, in the two-dimensional case, are uniquely determined up to a translation. For the n-dimensional case ($n \geq 3$), Grzybowski [15] gave the first example of two equivalent minimal pairs in R^3 which are not related by translations, and, as in [19], Pallaschke and Unbański indicated that a continuum of equivalent pairs are not related by translation for different indices. Some sufficient conditions and both sufficient and necessary conditions for the minimality of pairs of compact convex sets were given and some reduction techniques for the reduction of pairs of compact convex sets via cutting hyperplanes or excision of compact convex subsets were proposed according to Pallaschke and Urbański [20, 21].

For the same purpose, Xia [24, 25] introduced the notion of the kernelled quasidifferential. It was proved that

$$\begin{aligned} S &= \bigcap_{Df(x) \in \mathscr{D} f(x)} \left(\underline{\partial} f(x) + \overline{\partial} f(x) \right), \\ \overline{S} &= \bigcap_{Df(x) \in \mathscr{D} f(x)} \left(\overline{\partial} f(x) - \overline{\partial} f(x) \right) \end{aligned} \tag{3}$$

are nonempty, according to Deng and Gao [14]. S and $\overline{S}$ (defined by (3)) are called sub- and super-kernel, respectively, and $[S, \overline{S}]$ is called a quasi-kernel of $\mathscr{D} f(x)$. The quasi-kernel is said to be a kernelled quasidifferential of f at x if and only if the quasi-kernel $[S, \overline{S}]$ is a quasidifferential, denoted by $D_k f(x) = [\underline{\partial}_k f(x), \overline{\partial}_k f(x)]$. If f has a kernelled quasidifferential at $x \in R^n$, then f is said to be a kernelled quasidifferentiable function at x. For the case of one-dimensional space, the existence of the kernelled quasidifferential was given by Gao [16]. In the two dimensional case, based on the translation of minimal quasidifferentials, it was proved that the kernelled quasidifferential exists for any q.d. function (see [17]). In the n-dimensional case ($n \geq 3$), whether the pair of sets given in (3) is a quasidifferential of f at x is still an open problem, some progress has been made in the last years. Zhang et al. [26] gave a sufficient condition for a quasi-kernel being a kernelled quasidifferential. In [11], Gao presented a condition in terms of Demyanov difference, called g-condition, in which the kernelled quasidifferential exists. The corresponding subclasses and augmented class of g-q.d. functions on R^n were defined and some more properties on this class were presented according to Song and Xia [23].

Although the kernelled quasidifferential is known to have good algebraic properties and geometric structure (see [25]), it is still not very convenient for calculating the kernelled quasidifferentials of $-f$ and $\min\{f_i \mid i \in$ a finite index set $I\}$, where f and f_i are kernelled quasidifferentiable functions. Hence, in this paper, the notion of adjoint kernelled quasidifferential, which is well-defined for $-f$ and $\min\{f_i \mid i \in I\}$, is employed as a representative of the equivalence class of quasidifferentials. Some algebraic properties of the adjoint kernelled quasidifferential are given and the existence of the adjoint kernelled quasidifferential is explored by means of the minimal quasidifferential and the Demyanov difference of convex sets. The rest of the paper is organized as follows. In Section 2, some preliminary definitions and results used in the paper are provided. In Section 3, definitions of adjoint kernelled quasidifferential will be introduced and some operations of adjoint kernelled quasidifferentiable functions are given. In Section 4, we prove that the adjoint kernelled quasidifferential exists in one- and two-dimensional cases and two sufficient conditions for the existence of the adjoint kernelled quasidifferential in R^n ($n \geq 3$) are given. In Section 5, under some condition, a formula of the adjoint kernelled quasidifferential is presented.

2. Preliminaries

The support function $\delta^*(\cdot \mid C)$ of a set $C \in Y_n$ is defined by

$$\delta^*(x \mid C) = \max_{v \in C} \langle v, x \rangle, \quad \forall x \in R^n. \tag{4}$$

It is well known (see, e.g., [6]) that the mapping $A \mapsto \delta^*(\cdot \mid A)$ called the Minkowski duality is one-to-one correspondence between Y_n and the set P_n of all finite sublinear functions is defined on R^n.

Proposition 1. *Let $A, B \in Y_n$; then*

$$A \subset B \Longleftrightarrow \delta^*(x \mid A) \leq \delta^*(x \mid B), \quad \forall x \in R^n. \tag{5}$$

It is true that $\delta^*(\cdot \mid C)$ is convex with

$$\partial \delta^*(x \mid C) = \left\{ u \in C \mid \langle u, x \rangle = \max_{v \in C} \langle v, x \rangle \right\}, \tag{6}$$

particularly, $\partial \delta^*(0 \mid C) = C$, where ∂ denotes the subdifferential in the sense of convex analysis [27].

For any $d \in R^n$ and $C \in Y_n$, we denote the max-face of C with respect to d by the formula

$$C(d) = \{ x \in C \mid \langle d, x \rangle = \delta^*(d \mid C) \}. \tag{7}$$

Obviously, the max-face $C(d)$ coincides with the subdifferential $\partial \delta^*(d \mid C)$. Denote by $N_C(x)$ the normal cone to C at $x \in C$; that is,

$$N_C(x) = \{ d \in R^n \mid \langle d, y - x \rangle \leq 0, \forall y \in C \}. \tag{8}$$

Proposition 2. *Let $C \in Y_n$, for $x \in C$; it holds*

$$x \in C(d) \Longleftrightarrow d \in N_C(x). \tag{9}$$

Proposition 3. *Let $C \in Y_n$ and $x \in C$. If $d_1, d_2 \in N_C(x)$, then*

$$\delta^*(d_1 + d_2 \mid C) = \delta^*(d_1 \mid C) + \delta^*(d_2 \mid C). \quad (10)$$

Let the function f defined on R^n be locally Lipschitz continuous and let D_f denote the set where ∇f exists. The Clarke subdifferential $\partial_{Cl} f(x)$ of f at x is defined as follows:

$$\partial_{Cl} f(x) = \text{co}\left\{\lim_{x_n \to x} \nabla f(x_n) \mid x_n \longrightarrow x, x_n \in D_f\right\}, \quad (11)$$

where "co" denotes the convex hull. In the convex case, the Clarke subdifferential coincides with the subdifferential in the sense of convex analysis [28].

A set $T \subset R^n$ is called of full measure (with respect to R^n), if $R^n \setminus T$ is a set of measure zero. Let $A \in Y_n$ and $T_A = D_{\delta^*(\cdot|A)}$ be the set of all points $x \in R^n$ such that $\nabla \delta^*(x \mid A)$ exists. The set T_A is of full measure in R^n. Let $A, B \in Y_n$ and T be a subset of $T_A \cap T_B$ of full measure; then the set

$$A \dot{-} B = \text{cl co}\{\nabla \delta^*(x \mid A) - \nabla \delta^*(x \mid B) \mid x \in T\} \quad (12)$$

is called Demyanov difference of A and B, where "cl" refers to the closure. This construction was applied implicitly by Demyanov for the study of connections between the Clarke subdifferential and the quasidifferential [3]. In general, the Demyanov difference is smaller than the Minkowski difference. It is true that

$$A \dot{-} B \subset A - B. \quad (13)$$

According to [6], the Demyanov difference can be rewritten as

$$A \dot{-} B = \partial_{Cl}(\delta^*(y \mid A) - \delta^*(y \mid B))|_{y=0}. \quad (14)$$

Define the algebraic operations of addition and multiplication by a real number in $Y_n^2 = Y_n \times Y_n$ and the equivalence relation $\sim$ as follows:

$$\begin{aligned}
(A, B) + (C, D) &= (A + C, B + D), \\
c(A, B) &= (cA, cB), \quad c \geq 0, \\
c(A, B) &= (cB, cA), \quad c < 0, \\
(A, B) \sim (C, D) &\Longleftrightarrow A - D = C - B,
\end{aligned} \quad (15)$$

where $c \in R$, (A, B), and $(C, D) \in Y_n^2$. It is easy to check that $\mathscr{D} f(x) \in Y_n^2/_\sim$.

Proposition 4. *If $[\underline{\partial}_1 f(x), \overline{\partial}_1 f(x)], [\underline{\partial}_2 f(x), \overline{\partial}_2 f(x)] \in \mathscr{D} f(x)$, then*

$$\underline{\partial}_1 f(x) \dot{-} (-\overline{\partial}_1 f(x)) = \underline{\partial}_2 f(x) \dot{-} (-\overline{\partial}_2 f(x)). \quad (16)$$

The main formulas of quasidifferential calculus will be stated as Proposition 5. Algebraic operations over quasidifferentials are performed as over elements of the space of compact sets (or what is the same, as over pairs of sets).

Proposition 5. *Let $\Delta_n(x)$ denote the set of all functions defined on an open set $\mathcal{O} \subset R^n$ and quasidifferentiable at a point $x \in \mathcal{O}$. Then, the following hold.*

(1) *If $f_1, f_2 \in \Delta_n(x)$, c_1, c_2 are real numbers, then $c_1 f_1 + c_2 f_2 \in \Delta_n(x)$, and*

$$D(c_1 f_1 + c_2 f_2)(x) = c_1 Df_1(x) + c_2 Df_2(x). \quad (17)$$

Note that, in particular, $D(-f(x)) = -Df(x)$.

(2) *Let $f_1, f_2 \in \Delta_n(x)$. Then, $f_1 \cdot f_2 \in \Delta_n(x)$ and*

$$D(f_1 \cdot f_2)(x) = f_1(x) Df_2(x) + f_2(x) Df_1(x). \quad (18)$$

(3) *If $f \in \Delta_n(x)$, $f(x) \neq 0$, then $1/f$ is quasidifferentiable at x and*

$$Df^{-1}(x) = -f^{-2}(x) Df(x). \quad (19)$$

(4) *Let $f_1, f_2, \ldots, f_n \in \Delta_n(x)$ and*

$$\begin{aligned}
g(y) &= \max(f_1(y), \ldots, f_n(y)), \quad \forall y \in \mathcal{O}, \\
h(y) &= \min(f_1(y), \ldots, f_n(y)), \quad \forall y \in \mathcal{O}.
\end{aligned} \quad (20)$$

Then, $g \in \Delta_n(x)$, $h \in \Delta_n(x)$, and

$$Dg(x) = [\underline{\partial} g(x), \overline{\partial} g(x)], \qquad Dh(x) = [\underline{\partial} h(x), \overline{\partial} h(x)], \quad (21)$$

where

$$\begin{aligned}
\underline{\partial} g(x) &= \text{co} \bigcup_{k \in R(x)} \left(\underline{\partial} f_k(x) - \sum_{i \in R(x), i \neq k} \overline{\partial} f_i(x)\right), \\
\overline{\partial} g(x) &= \sum_{k \in R(x)} \overline{\partial} f_k(x), \qquad \underline{\partial} h(x) = \sum_{k \in S(x)} \underline{\partial} f_k(x), \\
\overline{\partial} h(x) &= \text{co} \bigcup_{k \in S(x)} \left(\overline{\partial} f_k(x) - \sum_{i \in S(x), i \neq k} \underline{\partial} f_i(x)\right).
\end{aligned} \quad (22)$$

Here, $R(x) = \{i \mid f_i(x) = g(x)\}$, $S(x) = \{i \mid f_i(x) = h(x)\}$.

3. Adjoint Kernelled Quasidifferential

The kernelled quasidifferential is known to have good algebraic properties (see [25]), but it is still not very convenient for calculating the kernelled quasidifferentials of $-f$ and $\min\{f_i \mid i \in \text{a finite index set } I\}$, where f and f_i are kernelled quasidifferentiable functions. So it is natural and necessary to explore the pair of sets $[\underline{S}, S]$, where S is defined as in (3) and

$$\underline{S} = \bigcap_{Df(x) \in \mathscr{D} f(x)} (\underline{\partial} f(x) - \underline{\partial} f(x)). \quad (23)$$

Obviously, $\underline{S}$ is nonempty and symmetric. Since having the similar structure to the quasi-kernel of $\mathscr{D} f(x)$, $[\underline{S}, S]$ is called an adjoint quasi-kernel of $\mathscr{D} f(x)$, where $\underline{S}$ and S are called adjoint sub-kernel and adjoint super-kernel, respectively. Of course $\underline{S}$ and S are compact convex. This motivates the introduction of the following notions.

Definition 6. Let $f \in \Delta_n(x)$. The adjoint quasi-kernel is said to be an adjoint kernelled quasidifferential of f at x if and only if

$$[\underline{S}, S] \in \mathscr{D} f(x). \tag{24}$$

If f has an adjoint kernelled quasidifferential at $x \in R^n$, then f is said to be an adjoint kernelled quasidifferentiable function at x. The adjoint kernel $[\underline{S}, S]$ is a quasidifferential, denoted by $D_{k^*} f(x) = [\underline{\partial}_{k^*} f(x), \overline{\partial}_{k^*} f(x)]$.

From the definition of quasidifferential and Proposition 5, the following proposition can be obtained immediately, which is especially useful in the study of the operation rules of adjoint kernelled quasidifferential.

Proposition 7. *(1) If $f_1, f_2 \in \Delta_n(x)$, $c_1, c_2 \in R$, then*

$$\mathscr{D}(c_1 f_1 + c_2 f_2)(x) = c_1 \mathscr{D} f_1(x) + c_2 \mathscr{D} f_2(x). \tag{25}$$

Note that, in particular, $\mathscr{D}(-f(x)) = -\mathscr{D} f(x)$.
(2) Let $f_1, f_2 \in \Delta_n(x)$. Then,

$$\mathscr{D}(f_1 \cdot f_2)(x) = f_1(x) \mathscr{D} f_2(x) + f_2(x) \mathscr{D} f_1(x). \tag{26}$$

(3) If $f \in \Delta_n(x)$, $f(x) \neq 0$, then

$$\mathscr{D} f^{-1}(x) = -f^{-2}(x) \mathscr{D} f(x). \tag{27}$$

If the adjoint kernelled quasidifferential exists, some operation rules of adjoint kernelled quasidifferential are presented as follows.

Theorem 8. *Let $\Delta_{n,k^*}(x)$ denote the set of all functions in $\Delta_n(x)$ and having adjoint kernelled quasidifferential at x. Then, the following hold.*

(1) *If $f_1, f_2 \in \Delta_{n,k^*}(x)$, then $f_1 + f_2 \in \Delta_{n,k^*}(x)$ and*

$$D_{k^*}(f_1 + f_2)(x) = D_{k^*} f_1(x) + D_{k^*} f_2(x). \tag{28}$$

(2) *If $f, -f \in \Delta_{n,k^*}(x)$, $c \in R$, then $cf \in \Delta_{n,k^*}(x)$ and*

$$D_{k^*} cf(x) = |c| D_{k^*} (\operatorname{sign} c) f(x). \tag{29}$$

(3) *If $f_1, f_2, -f_1, -f_2 \in \Delta_{n,k^*}(x)$, then $f_1 \cdot f_2 \in \Delta_{n,k^*}(x)$ and*

$$\begin{aligned} D_{k^*}(f_1 \cdot f_2)(x) = {} & |f_1(x)| D_{k^*} (\operatorname{sign} f_1(x)) f_2(x) \\ & + |f_2(x)| D_{k^*} (\operatorname{sign} f_2(x)) f_1(x). \end{aligned} \tag{30}$$

(4) *If $f, -f \in \Delta_{n,k^*}(x)$, $f(x) \neq 0$, then $1/f \in \Delta_{n,k^*}(x)$ and*

$$D_{k^*} f^{-1}(x) = f^{-2}(x) D_{k^*}(-f(x)). \tag{31}$$

Proof. We will prove only Properties (1) and (2). Properties (3) and (4) can be proved in an analogous manner.

(1) Since $f_1, f_2 \in \Delta_{n,k^*}(x)$, then

$$\begin{aligned} \bigcap_{Df_1(x) \in \mathscr{D} f_1(x)} (\underline{\partial} f_1(x) - \underline{\partial} f_1(x)) &= \underline{\partial}_{k^*} f_1(x), \\ \bigcap_{Df_2(x) \in \mathscr{D} f_2(x)} (\underline{\partial} f_2(x) - \underline{\partial} f_2(x)) &= \underline{\partial}_{k^*} f_2(x). \end{aligned} \tag{32}$$

From Propositions 5 and 7 and (32), it follows that

$$\begin{aligned} & \bigcap_{D(f_1+f_2)(x) \in \mathscr{D}(f_1+f_2)(x)} (\underline{\partial}(f_1 + f_2)(x) - \underline{\partial}(f_1 + f_2)(x)) \\ & = \bigcap_{Df_1(x) + Df_2(x) \in \mathscr{D} f_1(x) + \mathscr{D} f_2(x)} (\underline{\partial} f_1(x) - \underline{\partial} f_1(x) \\ & \qquad + \underline{\partial} f_2(x) - \underline{\partial} f_2(x)) \\ & = \bigcap_{Df_1(x) \in \mathscr{D} f_1(x)} (\underline{\partial} f_1(x) - \underline{\partial} f_1(x)) \\ & \quad + \bigcap_{Df_2(x) \in \mathscr{D} f_2(x)} (\underline{\partial} f_2(x) - \underline{\partial} f_2(x)) \\ & = \underline{\partial}_{k^*} f_1(x) + \underline{\partial}_{k^*} f_2(x). \end{aligned} \tag{33}$$

By the similar way, we can prove that

$$\begin{aligned} & \bigcap_{D(f_1+f_2)(x) \in \mathscr{D}(f_1+f_2)(x)} (\underline{\partial}(f_1 + f_2)(x) + \overline{\partial}(f_1 + f_2)(x)) \\ & \qquad = \overline{\partial}_{k^*} f_1(x) + \overline{\partial}_{k^*} f_2(x). \end{aligned} \tag{34}$$

Since $[\underline{\partial}_{k^*} f_1(x) + \underline{\partial}_{k^*} f_2(x), \overline{\partial}_{k^*} f_1(x) + \overline{\partial}_{k^*} f_2(x)] \in \mathscr{D} f_1(x) + \mathscr{D} f_2(x) = \mathscr{D}(f_1 + f_2)(x)$, hence, together with (33) and (34), one has that $f_1 + f_2 \in \Delta_{n,k^*}(x)$.

(2) Since $f, -f \in \Delta_{n,k^*}(x)$, then, together with Propositions 5 and 7, one has that

$$\begin{aligned} & \bigcap_{Dcf(x) \in \mathscr{D} cf(x)} (\underline{\partial} cf(x) - \underline{\partial} cf(x)) \\ & = \bigcap_{|c| D(\operatorname{sign} c) f(x) \in |c| \mathscr{D}(\operatorname{sign} c) f(x)} |c| (\underline{\partial}(\operatorname{sign} c) f(x) \\ & \qquad - \underline{\partial}(\operatorname{sign} c) f(x)) \\ & = |c| \bigcap_{D(\operatorname{sign} c) f(x) \in \mathscr{D}(\operatorname{sign} c) f(x)} (\underline{\partial} f(x) - \underline{\partial} f(x)) \\ & = |c| \underline{\partial}_{k^*} (\operatorname{sign} c) f(x). \end{aligned} \tag{35}$$

Similarly, we can prove that

$$\bigcap_{Dcf(x)\in\mathscr{D}cf(x)} \left(\underline{\partial}cf(x)+\overline{\partial}cf(x)\right)=|c|\,\overline{\partial}_{k^*}(\operatorname{sign}c)\,f(x). \tag{36}$$

Combining (35) with (36) leads to

$$\left[\bigcap_{Dcf(x)\in\mathscr{D}cf(x)} \left(\underline{\partial}cf(x)-\underline{\partial}cf(x)\right),\ \bigcap_{Dcf(x)\in\mathscr{D}cf(x)} \left(\underline{\partial}cf(x)+\overline{\partial}cf(x)\right)\right] \in |c|\,\mathscr{D}(\operatorname{sign}c)\,f(x)=\mathscr{D}cf(x). \tag{37}$$

Hence, $cf\in\Delta_{n,k^*}(x)$. □

By $\Delta_{n,k}(x)$ we denote the set of all functions in $\Delta_n(x)$ and having kernelled quasidifferential at x. Obviously, one has that $\Delta_{n,k}(x)\subset\Delta_n(x)$. The adjoint kernelled quasidifferential is convenient for calculating $D_{k^*}\min\{f_i \mid i\in I\}$ and can calculate the adjoint kernelled quasidifferential of $-f$ with kernelled quasidifferential, where $f, f_i\in\Delta_{n,k}(x)$, $i\in$ a finite index set I.

Theorem 9. *If $f\in\Delta_{n,k}(x)$, then $-f\in\Delta_{n,k^*}(x)$ and*

$$D_{k^*}(-f)=-D_kf(x). \tag{38}$$

If $f\in\Delta_{n,k^*}(x)$, then $-f\in\Delta_{n,k}(x)$ and

$$D_k(-f)=-D_{k^*}f(x). \tag{39}$$

Proof. Since $f\in\Delta_{n,k}(x)$, then $D_kf(x)=[\underline{\partial}_kf(x),\overline{\partial}_kf(x)]\in\mathscr{D}f(x)$, where

$$\underline{\partial}_kf(x)=\bigcap_{Df(x)\in\mathscr{D}f(x)}\left(\underline{\partial}f(x)+\overline{\partial}f(x)\right), \tag{40}$$

$$\overline{\partial}_kf(x)=\bigcap_{Df(x)\in\mathscr{D}f(x)}\left(\overline{\partial}f(x)-\overline{\partial}f(x)\right). \tag{41}$$

By Propositions 5 and 7 and (41), we obtain

$$\begin{aligned}&\bigcap_{D(-f)(x)\in\mathscr{D}(-f)(x)}\underline{\partial}(-f)(x)-\underline{\partial}(-f)(x)\\&=\bigcap_{Df(x)\in\mathscr{D}f(x)}-\left(\overline{\partial}f(x)-\overline{\partial}f(x)\right)=-\overline{\partial}_kf(x).\end{aligned} \tag{42}$$

From Propositions 5 and 7 and (40), it follows that

$$\begin{aligned}&\bigcap_{D(-f)(x)\in\mathscr{D}(-f)(x)}\underline{\partial}(-f)(x)+\overline{\partial}(-f)(x)\\&=\bigcap_{Df(x)\in\mathscr{D}f(x)}-\left(\underline{\partial}f(x)+\overline{\partial}f(x)\right)=-\underline{\partial}_kf(x).\end{aligned} \tag{43}$$

Obviously, $[-\overline{\partial}_kf(x),-\underline{\partial}_kf(x)]=-D_kf(x)\in-\mathscr{D}f(x)=\mathscr{D}(-f)(x)$. This fact, together with (42) and (43), implies that

$$\left[\bigcap_{D(-f)(x)\in\mathscr{D}(-f)(x)}\underline{\partial}(-f)(x)-\underline{\partial}(-f)(x),\ \bigcap_{D(-f)(x)\in\mathscr{D}(-f)(x)}\underline{\partial}(-f)(x)+\overline{\partial}\ (-f)(x)\right]\in\mathscr{D}(-f)(x). \tag{44}$$

Then, $-f\in\Delta_{n,k^*}(x)$ and $D_{k^*}(-f)(x)=-D_kf(x)$. Similarly, it can be proved that if $f\in\Delta_{n,k^*}(x)$, then $-f\in\Delta_{n,k}(x)$ and $D_k(-f)=-D_{k^*}f(x)$. The proof is completed. □

Theorem 10. *Let $f_1,f_2,\ldots,f_n\in\Delta_{n,k^*}(x)$ and*

$$f(y)=\min\left(f_1(y),\ldots,f_n(y)\right),\quad\forall y\in\mathscr{O}. \tag{45}$$

Then, $f\in\Delta_{n,k^*}(x)$ and $D_{k^*}f(x)=[\underline{\partial}_{k^*}f(x),\overline{\partial}_{k^*}f(x)]$, where

$$\begin{aligned}\underline{\partial}_{k^*}f(x)&=\sum_{i\in S(x)}\underline{\partial}_{k^*}f_i(x),\\ \overline{\partial}_{k^*}f(x)&=\operatorname{co}\bigcup_{i\in S(x)}\left(\overline{\partial}_{k^*}f_i(x)-\sum_{j\in S(x),j\neq i}\underline{\partial}_{k^*}f_j(x)\right).\end{aligned} \tag{46}$$

Here, $S(x)=\{i\mid f_i(x)=f(x)\}$.

Proof. Since $f_1,f_2,\ldots,f_n\in\Delta_{n,k^*}(x)$ and $f(y)=\min(f_1(y),\ldots,f_n(y))$, $\forall y\in\mathscr{O}$, then, according to Propositions 5 and 7, we have

$$\begin{aligned}&\bigcap_{Df(x)\in\mathscr{D}f(x)}\left(\underline{\partial}f(x)-\underline{\partial}f(x)\right)\\&=\bigcap_{Df(x)\in\mathscr{D}f(x)}\left(\sum_{i\in S(x)}\underline{\partial}f_i(x)-\sum_{i\in S(x)}\underline{\partial}f_i(x)\right)\\&=\sum_{i\in S(x)}\bigcap_{Df_i(x)\in\mathscr{D}f_i(x)}\left(\underline{\partial}f_i(x)-\underline{\partial}f_i(x)\right)\\&=\sum_{i\in S(x)}\underline{\partial}_{k^*}f_i(x).\end{aligned} \tag{47}$$

Since, for $C_i\in Y_n$, $i\in I$, where I denotes a finite index set, one has that

$$\operatorname{co}\bigcup_{i\in I}C_i=\bigcup_{\lambda_i\geq0,\sum_{i\in I}\lambda_i=1}\ \sum_{i\in I}\lambda_iC_i, \tag{48}$$

where $\lambda_i \in R$, $i \in I$. Hence, together with Proposition 5, it follows that

$$\begin{aligned}
&\underline{\partial} f(x) + \overline{\partial} f(x) \\
&= \sum_{i\in S(x)} \underline{\partial} f_i(x) + \operatorname{co} \bigcup_{i\in S(x)} \left(\overline{\partial} f_i(x) - \sum_{j\in S(x), j\neq i} \underline{\partial} f_j(x) \right) \\
&= \sum_{i\in S(x)} \underline{\partial} f_i(x) \\
&\quad + \bigcup_{\lambda_i\ge 0, \sum_{i\in S(x)} \lambda_i = 1} \left(\sum_{i\in S(x)} \lambda_i \left(\overline{\partial} f_i(x) - \sum_{\substack{j\in S(x)\\ j\neq i}} \underline{\partial} f_j(x) \right)\right) \\
&= \bigcup_{\lambda_i\ge 0, \sum_{i\in S(x)} \lambda_i = 1} \sum_{i\in S(x)} \lambda_i \left(\underline{\partial} f_i(x) + \overline{\partial} f_i(x) - \sum_{\substack{j\in S(x)\\ j\neq i}} \left(\underline{\partial} f_j(x) - \underline{\partial} f_j(x) \right) \right).
\end{aligned} \tag{49}$$

By Propositions 5 and 7 and (49), we obtain

$$\begin{aligned}
&\bigcap_{Df(x)\in\mathscr{D}f(x)} \left(\underline{\partial} f(x) + \overline{\partial} f(x) \right) \\
&= \bigcap_{Df(x)\in\mathscr{D}f(x)} \bigcup_{\lambda_i\ge 0, \sum_{i\in S(x)} \lambda_i = 1} \sum_{i\in S(x)} \lambda_i \left(\underline{\partial} f_i(x) + \overline{\partial} f_i(x) - \sum_{\substack{j\in S(x)\\ j\neq i}} \left(\underline{\partial} f_j(x) - \underline{\partial} f_j(x) \right) \right) \\
&= \bigcup_{\lambda_i\ge 0, \sum_{i\in S(x)} \lambda_i = 1} \sum_{i\in S(x)} \lambda_i \left(\bigcap_{Df_i(x)\in\mathscr{D}f_i(x)} \left(\underline{\partial} f_i(x) + \overline{\partial} f_i(x) \right) - \sum_{\substack{j\in S(x)\\ j\neq i}} \bigcap_{Df_j(x)\in\mathscr{D}f_j(x)} \left(\underline{\partial} f_j(x) - \underline{\partial} f_j(x) \right) \right) \\
&= \operatorname{co} \bigcup_{i\in S(x)} \left(\overline{\partial}_{k^*} f_i(x) - \sum_{j\in S(x), j\neq i} \underline{\partial}_{k^*} f_j(x) \right).
\end{aligned} \tag{50}$$

Based on Propositions 5 and 7 and (47) and (50), one has that

$$\left[\bigcap_{Df(x)\in\mathscr{D}f(x)} \left(\underline{\partial} f(x) - \underline{\partial} f(x) \right), \bigcap_{Df(x)\in\mathscr{D}f(x)} \left(\underline{\partial} f(x) + \overline{\partial} f(x) \right) \right] \in \mathscr{D} f(x); \tag{51}$$

hence $f \in \Delta_{n,k^*}(x)$. The demonstration is completed. □

4. Existence of the Adjoint Kernelled Quasidifferential

In this section, the existence of the adjoint kernelled quasidifferential of a quasidifferentiable function is established. In one- and two-dimensional cases, we prove that the adjoint kernelled quasidifferential exists and give its expression by using of a minimal quasidifferential. We also develop the existence of the adjoint kernelled quasidifferential for a quasidifferentiable function on R^n ($n \ge 3$) under some conditions.

Theorem 11. *Suppose that $f \in \Delta_n(x)$, $n = 1, 2$, and $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)]$ is a minimal quasidifferential of f at x. Then, the relations below hold*

$$\underline{S} = \underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x), \qquad S = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x). \tag{52}$$

Furthermore, $f \in \Delta_{n,k^*}(x)$; that is, $[\underline{S}, S] \in \mathscr{D} f(x)$.

Proof. Let $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$. From the existence of the minimal quasidifferentials, see [18], it follows that there exists a minimal quasidifferential of f at x, denoted by $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$, such that $\underline{\partial}^m f(x) \subset \underline{\partial} f(x)$, $\overline{\partial}^m f(x) \subset \overline{\partial} f(x)$. Consequently,

$$\underline{\partial}^m f(x) - \underline{\partial}^m f(x) \subset \underline{\partial} f(x) - \underline{\partial} f(x), \tag{53a}$$

$$\underline{\partial}^m f(x) + \overline{\partial}^m f(x) \subset \underline{\partial} f(x) + \overline{\partial} f(x). \tag{53b}$$

Note that both $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$ and $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)]$ are the minimal quasidifferentials of f at x. According to the translation property of the equivalent minimal quasidifferentials in the one- and two-dimensional case, see [15, 18], there exists $c \in R^n$, $n = 1, 2$, such that the minimal quasidifferential $[\underline{\partial}^m f(x), \overline{\partial}^m f(x)]$ can be expressed as

$$\left[\underline{\partial}^m f(x), \overline{\partial}^m f(x)\right] = \left[\underline{\partial}_0^m f(x) + \{c\}, \overline{\partial}_0^m f(x) - \{c\}\right]. \tag{54}$$

This leads to

$$\underline{\partial}^m f(x) - \underline{\partial}^m f(x) = \underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x), \tag{55a}$$

$$\underline{\partial}^m f(x) + \overline{\partial}^m f(x) = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x). \tag{55b}$$

It follows from (53a), (53b), (55a), and (55b) that

$$\underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x) \subset \underline{\partial} f(x) - \underline{\partial} f(x), \tag{56a}$$

$$\underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x) \subset \underline{\partial} f(x) + \overline{\partial} f(x). \tag{56b}$$

Taking the intersection on the right hands of (56a) and of (56b) for all quasidifferentials of f at x, we have that

$$\underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x) \subset \underline{S}, \tag{57a}$$

$$\underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x) \subset S. \tag{57b}$$

On the other hand, $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)] \in \mathscr{D} f(x)$ implies that

$$\underline{S} \subset \underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x), \tag{58a}$$

$$S \subset \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x). \tag{58b}$$

The relations (57a), (57b), (58a), and (58b) lead to that

$$\underline{S} = \underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x), \tag{59a}$$

$$S = \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x). \tag{59b}$$

Note that $[\underline{\partial}_0^m f(x), \overline{\partial}_0^m f(x)] \in \mathscr{D} f(x)$ and $\underline{\partial}_0^m f(x) \in Y_n$, $n = 1, 2$. Hence,

$$\left[\underline{\partial}_0^m f(x) - \underline{\partial}_0^m f(x), \underline{\partial}_0^m f(x) + \overline{\partial}_0^m f(x)\right] \in \mathscr{D} f(x). \tag{60}$$

Equations (59a), (59b), and (60) show that $[\underline{S}, S] \in \mathscr{D} f(x)$. The proof is completed. □

The conclusion of Theorem 11 strongly depends upon the translation of minimal quasidifferentials. Unfortunately, the minimal quasidifferential is not uniquely determined up to a translation in R^n if $n \geq 3$ [15]. But. by the tool of Demyanov difference of compact convex sets, we get the following interesting result about minimal quasidifferential.

Proposition 12. *Suppose that $f \in \Delta_n(x)$ and there exists a quasidifferential $[\underline{\partial}_0 f(x), \overline{\partial}_0 f(x)] \in \mathscr{D} f(x)$ such that*

$$\underline{\partial}_0 f(x) \dot{-} \left(-\overline{\partial}_0 f(x)\right) = \underline{\partial}_0 f(x) - \left(-\overline{\partial}_0 f(x)\right). \tag{61}$$

Then $[\underline{\partial}_0 f(x), \overline{\partial}_0 f(x)]$ is a minimal quasidifferential of f at x.

Proof. Let $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$ and

$$\underline{\partial} f(x) \subset \underline{\partial}_0 f(x), \qquad \overline{\partial} f(x) \subset \overline{\partial}_0 f(x). \tag{62}$$

Obviously, one has

$$\underline{\partial} f(x) + \overline{\partial} f(x) \subset \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x). \tag{63}$$

By Proposition 4 and (61), we obtain

$$\begin{aligned} \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) &= \underline{\partial}_0 f(x) \dot{-} \left(-\overline{\partial}_0 f(x)\right) \\ &= \underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right). \end{aligned} \tag{64}$$

From (13) and (64), it follows that

$$\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) \subset \underline{\partial} f(x) + \overline{\partial} f(x). \tag{65}$$

Combining (63) with (65) leads to

$$\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) = \underline{\partial} f(x) + \overline{\partial} f(x). \tag{66}$$

According to (62) and (66), we conclude that

$$\underline{\partial} f(x) = \underline{\partial}_0 f(x), \qquad \overline{\partial} f(x) = \overline{\partial}_0 f(x). \tag{67}$$

Then, by the definition of the minimal quasidifferential, $[\underline{\partial}_0 f(x), \overline{\partial}_0 f(x)]$ is a minimal quasidifferential of f at x. □

Inspired by Proposition 12, we present the following theorem, which gives a sufficient condition for the existence of the adjoint kernelled quasidifferential in R^n $(n \geq 3)$.

Theorem 13. *Suppose that $f \in \Delta_n(x)$ and there exists a quasidifferential $[\underline{\partial}_0 f(x), \overline{\partial}_0 f(x)] \in \mathscr{D} f(x)$ such that*

$$\underline{\partial}_0 f(x) \dot{-} \left(-\overline{\partial}_0 f(x)\right) = \underline{\partial}_0 f(x) - \left(-\overline{\partial}_0 f(x)\right). \tag{68}$$

Then, one has

$$\underline{S} = \underline{\partial}_0 f(x) - \underline{\partial}_0 f(x), \tag{69a}$$

$$S = \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x). \tag{69b}$$

Furthermore, $[\underline{S}, S] \in \mathscr{D} f(x)$; that is, $f \in \Delta_{n,k^*}(x)$.

Proof. Let $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$. From Proposition 4 and (68), it follows that

$$\begin{aligned} \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) &= \underline{\partial}_0 f(x) \dot{-} \left(-\overline{\partial}_0 f(x)\right) \\ &= \underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right) \subset \underline{\partial} f(x) + \overline{\partial} f(x). \end{aligned} \tag{70}$$

By the definition of the quasidifferential, it is easy to check that $[U, V] \in \mathscr{D} f(x)$ implies $[U - U, U + V] \in \mathscr{D} f(x)$. Therefore, we have $[\underline{\partial} f(x) - \underline{\partial} f(x), \underline{\partial} f(x) + \overline{\partial} f(x)] \in \mathscr{D} f(x)$ and $[\underline{\partial}_0 f(x) - \underline{\partial}_0 f(x), \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)] \in \mathscr{D} f(x)$. These give

$$\begin{aligned} &\delta^*\left(y \mid \underline{\partial} f(x) - \underline{\partial} f(x)\right) - \delta^*\left(y \mid -\left(\underline{\partial} f(x) + \overline{\partial} f(x)\right)\right) \\ &\quad = \delta^*\left(y \mid \underline{\partial}_0 f(x) - \underline{\partial}_0 f(x)\right) \\ &\qquad - \delta^*\left(y \mid -\left(\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)\right)\right), \quad \forall y \in R^n. \end{aligned} \tag{71}$$

By (70) and Proposition 1, we obtain

$$\delta^*\left(y \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)\right) \leq \delta^*\left(y \mid \underline{\partial} f(x) + \overline{\partial} f(x)\right), \quad \forall y \in R^n. \tag{72}$$

Evidently, (72) is equivalent to the following:

$$-\delta^*\left(-y \mid \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)\right) \geq -\delta^*\left(-y \mid \underline{\partial} f(x) + \overline{\partial} f(x)\right), \quad \forall y \in R^n. \tag{73}$$

Combining (71) with (73) leads to

$$\delta^*\left(y \mid \underline{\partial}_0 f(x) - \underline{\partial}_0 f(x)\right) \leq \delta^*\left(y \mid \underline{\partial} f(x) - \underline{\partial} f(x)\right), \quad \forall y \in R^n. \tag{74}$$

Based on (74) and Proposition 1, one has that

$$\underline{\partial}_0 f(x) - \underline{\partial}_0 f(x) \subset \underline{\partial} f(x) - \underline{\partial} f(x). \tag{75}$$

Notice that both (70) and (75) hold for any $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$. Taking the intersection on the right-hand sides of (70) and of (75), respectively, for all quasidifferentials of f at x, it is obtained that

$$\underline{\partial}_0 f(x) - \underline{\partial}_0 f(x) \subset \underline{S}, \tag{76a}$$

$$\underline{\partial}_0 f(x) + \overline{\partial}_0 f(x) \subset S. \tag{76b}$$

On the other hand, $[\underline{\partial}_0 f(x), \overline{\partial}_0 f(x)] \in \mathscr{D} f(x)$ implies

$$\underline{S} \subset \underline{\partial}_0 f(x) - \underline{\partial}_0 f(x), \tag{77a}$$

$$S \subset \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x). \tag{77b}$$

Combining (76a) with (77b) yields (69a). Likewise, (76b) and (77b) yield (69b). Notice that the relation $[\underline{\partial}_0 f(x) - \underline{\partial}_0 f(x), \underline{\partial}_0 f(x) + \overline{\partial}_0 f(x)] \in \mathscr{D} f(x)$ has been claimed. We thus complete the proof of the theorem. □

A decomposition structure of $f'(x;\cdot)$ is defined by

$$f'(x;\cdot) = \underline{f}'(x;\cdot) - \overline{f}'(x;\cdot), \tag{78}$$

where $\underline{f}'(x;\cdot)$ and $\overline{f}'(x;\cdot)$ are defined by

$$\underline{f}'(x;\cdot) = \inf_{Df(x) \in \mathscr{D} f(x)} \delta^*\left(\cdot \mid \underline{\partial} f(x) - \underline{\partial} f(x)\right),$$
$$\overline{f}'(x;\cdot) = \inf_{Df(x) \in \mathscr{D} f(x)} \delta^*\left(\cdot \mid -\left(\underline{\partial} f(x) + \overline{\partial} f(x)\right)\right), \tag{79}$$

respectively. Generally, $\underline{f}'$ and $\overline{f}'$ are positively homogeneous, but not sublinear. It is easy to be seen that

$$\delta^*(\cdot \mid \underline{S}) \leq \underline{f}'(x;\cdot), \qquad \delta^*(\cdot \mid -S) \leq \overline{f}'(x;\cdot). \tag{80}$$

It is easy to be seen that, for any $u \in \underline{S}$, there exists at least one sequence $\{u_i \mid u_i \in \underline{\partial}_i f(x) - \underline{\partial}_i f(x)\}$ convergent to u, where $[\underline{\partial}_i f(x), \overline{\partial}_i f(x)] \in \mathscr{D} f(x)$. According to Proposition 2, if $u \in \underline{S}$ and $d \in R^n$ such that there exist sequences $\{u_i \mid u_i \in \underline{\partial}_i f(x) - \underline{\partial}_i f(x)\}_{i=1}^{\infty} \to u$ and $\{d_i \mid d_i \in N_{\underline{\partial}_i f(x) - \underline{\partial}_i f(x)}(u_i)\} \to d$, then $d \in N_{\underline{S}}(u)$ and $\delta^*(d \mid \underline{S}) = \underline{f}'(x; d)$.

The above lines enable us to give the following theorem which provides a sufficient condition for $[\underline{S}, S]$ to be an adjoint kernelled quasidifferential.

Let $\mathscr{F}(\underline{S}, -S)$ be a shape of $(\underline{S}, -S)$ that is defined by a similar way according to [18], such that

$$\operatorname{cl}\operatorname{co} \bigcup_{d \in \mathscr{F}(\underline{S}, -S)} \underline{S}(d) = \underline{S}, \qquad \operatorname{cl}\operatorname{co} \bigcup_{d \in \mathscr{F}(\underline{S}, -S)} -S(d) = -S. \tag{81}$$

Theorem 14. *Let $f \in \Delta_n(x)$ and suppose that $\underline{f}'(x;\cdot)$ and $\overline{f}'(x;\cdot)$ are continuous with respect to direction, and, furthermore, there exists a shape $\mathscr{F}(\underline{S}, -S)$ of $(\underline{S}, -S)$ such that, for any $u \in \underline{S}$ and $v \in -S$, one has that*

$$N_{\underline{S}}(u) = \overline{\operatorname{cone}}\left\{N_{\underline{S}}(u) \cap \mathscr{F}(\underline{S}, -S)\right\},$$
$$N_{-S}(v) = \overline{\operatorname{cone}}\left\{N_{-S}(v) \cap \mathscr{F}(\underline{S}, -S)\right\}, \tag{82}$$

where $\overline{\operatorname{cone}}$ denotes the closed convex conical hull. If, for any $d \in \mathscr{F}(\underline{S}, -S)$, $u \in \underline{S}(d)$, and $v \in -S(d)$, there exist sequences

$$\{u_i \mid u_i \in \underline{\partial}_i f(x) - \underline{\partial}_i f(x)\}_{i=1}^{\infty} \longrightarrow u, \tag{83}$$

$$\left\{v_i \mid v_i \in -\left(\underline{\partial}_i f(x) + \overline{\partial}_i f(x)\right)\right\}_{i=1}^{\infty} \longrightarrow v, \tag{84}$$

$$\left\{d_i \mid d_i \in N_{\underline{\partial}_i f(x) - \underline{\partial}_i f(x)}(u_i) \cap N_{-(\underline{\partial}_i f(x) + \overline{\partial}_i f(x))}(v_i)\right\}, \tag{85}$$

such that d is one of clusters of $\{d_i\}_{i=1}^{\infty}$; then $[\underline{S}, S] \in \mathscr{D} f(x)$, that is, $f \in \Delta_{n,k^*}(x)$.

Proof. Let $d \in R^n$ be an arbitrary nonzero vector. There exist $u \in \underline{S}$ and $v \in -S$ such that $d \in N_{\underline{S}}(u) \cap N_{-S}(v)$. According to (82), there exists a sequence

$$d_i \in \overline{\operatorname{cone}}\left\{N_{\underline{S}}(u) \cap \mathscr{F}(\underline{S}, -S)\right\} \cap \overline{\operatorname{cone}}\left\{N_{-S}(v) \cap \mathscr{F}(\underline{S}, -S)\right\}, \tag{86}$$

$i = 1, 2, \ldots$, convergent to d. For each i, there are two index sets $\underline{J}_i$ and $\overline{J}_i$, with finite indices such that

$$\underline{d}_{ij} \in N_{\underline{S}}(u_i) \cap \mathscr{F}(\underline{S}, -S), \quad j \in \underline{J}_i,$$
$$\overline{d}_{ij} \in N_{-S}(v_i) \cap \mathscr{F}(\underline{S}, -S), \quad j \in \overline{J}_i,$$
$$d_i \in \operatorname{co}\left\{\underline{d}_{ij} \mid j \in \underline{J}_i\right\} \cap \operatorname{co}\left\{\overline{d}_{ij} \mid j \in \overline{J}_i\right\}. \tag{87}$$

It follows from (83)–(85) and (87) that, for each ij, there exist $\{\underline{d}_{ij_k}\}_{k=1}^{\infty}$, $\{\overline{d}_{ij_k}\}_{k=1}^{\infty}$, $\{u_{ij_k}\}_{k=1}^{\infty}$, and $\{v_{ij_k}\}_{k=1}^{\infty}$ such that

$$\left\{u_{ij_k} \in \underline{\partial}_{ij_k} f(x) - \underline{\partial}_{ij_k} f(x)\right\}_{k=1}^{\infty} \longrightarrow u_i,$$
$$\left\{v_{ij_k} \in -\left(\underline{\partial}_{ij_k} f(x) + \overline{\partial}_{ij_k} f(x)\right)\right\}_{k=1}^{\infty} \longrightarrow v_i,$$
$$\left\{\underline{d}_{ij_k} \in N_{\underline{\partial}_{ij_k} f(x) - \underline{\partial}_{ij_k} f(x)}\left(u_{ij_k}\right)\right\}_{k=1}^{\infty} \longrightarrow \underline{d}_{ij}, \quad j \in \underline{J}_i, \quad i = 1, 2, \ldots,$$
$$\left\{\overline{d}_{ij_k} \in N_{-(\underline{\partial}_{ij_k} f(x) + \overline{\partial}_{ij_k} f(x))}\left(v_{ij_k}\right)\right\}_{k=1}^{\infty} \longrightarrow \overline{d}_{ij}, \quad j \in \overline{J}_i, \quad i = 1, 2, \ldots. \tag{88}$$

Since each d_i is a convex combination of $\underline{d}_{ij}$, $j \in \underline{J}_i$, or of $\overline{d}_{ij}$, $j \in \overline{J}_i$, one has that there are $\underline{\lambda}_{ij} \geq 0$ and $\overline{\lambda}_{ij} \geq 0$ such that

$$\sum_{j\in \underline{J}_i} \underline{\lambda}_{ij} = 1, \qquad \sum_{j\in \overline{J}_i} \overline{\lambda}_{ij} = 1 \tag{89}$$

satisfying

$$\begin{gathered} d_i = \sum_{j\in \underline{J}_i} \underline{\lambda}_{ij}\underline{d}_{ij} = \sum_{j\in \overline{J}_i} \overline{\lambda}_{ij}\overline{d}_{ij}, \\ \delta^* (d_i \mid \underline{S}) = \sum_{j\in \underline{J}_i} \underline{\lambda}_{ij} \langle \underline{d}_{ij}, u_i\rangle = \sum_{j\in \underline{J}_i} \underline{\lambda}_{ij} \lim_{k\to\infty} \langle \underline{d}_{ij_k}, u_{ij_k}\rangle \end{gathered} \tag{90}$$

from (83) and (84), where $\underline{d}_{ij_k} \in N_{\underline{\partial}_{ij_k} f(x) - \underline{\partial}_{ij_k} f(x)}(u_{ij_k})$. Since $\{u_{ij_k} \in \underline{\partial}_{ij_k} f(x) - \underline{\partial}_{ij_k} f(x)\}_{k=1}^{\infty} \to u_i$, $\{\underline{d}_{ij_k} \in N_{\underline{\partial}_{ij_k} f(x) - \underline{\partial}_{ij_k} f(x)}(u_{ij_k})\}_{k=1}^{\infty} \to \underline{d}_{ij}$, it follows, from the sufficient condition for $\delta^*(d \mid \underline{S}) = \underline{f}'(x; d)$ given before the theorem, that

$$\delta^* \left(\underline{d}_{ij} \mid \underline{S}\right) = \underline{f}' \left(x; \underline{d}_{ij}\right) = \lim_{k\to\infty} \left\langle \underline{d}_{ij_k}, u_{ij_k}\right\rangle = \left\langle \underline{d}_{ij}, u_i\right\rangle. \tag{91}$$

Thus, it follows from (91) that

$$\delta^* \left(d_i \mid \underline{S}\right) = \left\langle \sum_{j\in \underline{J}_i} \underline{\lambda}_{ij}\underline{d}_{ij}, u_i \right\rangle = \underline{f}' \left(x; \underline{d}_i\right). \tag{92}$$

Without loss of generality, assume $\{d_i\}_{i=1}^{\infty} \to d$. Taking the limit to (92), one has that

$$\delta^* \left(d \mid \underline{S}\right) = \langle d, u\rangle = \lim_{i\to\infty} \underline{f}' \left(x; \underline{d}_i\right). \tag{93}$$

According to the continuity of $\underline{f}'(x; \cdot)$, (93) becomes

$$\delta^* \left(d \mid \underline{S}\right) = \underline{f}' \left(x; d\right). \tag{94}$$

Similarly, it can be proved that

$$\delta^* \left(d \mid -S\right) = \overline{f}' \left(x; d\right). \tag{95}$$

According to (94) and (95), we conclude

$$f' \left(x; d\right) = \underline{f}' \left(x; d\right) - \overline{f}' \left(x; d\right) = \delta^* \left(d \mid \underline{S}\right) - \delta^* \left(d \mid -S\right). \tag{96}$$

Then, by the definition of the quasidifferential, one has $[\underline{S}, S] \in \mathscr{D} f(x)$, that is, $f \in \Delta_{n,k^*}(x)$. The demonstration is completed. □

5. Formula of Representative for Quasidifferentials

Theorem 13 only gives the existence of the adjoint kernelled quasidifferential but does not show us how to calculate it. For the practical purpose, we expect to find a way to calculate a representative of the equivalent class of quasidifferentials for a given quasidifferential. The present section is devoted to this topic.

Lemma 15. *Let $[A, B]$, $[U, V] \in \mathscr{D} f(x)$. Then, U has the following form:*

$$U = (A - V) \dot{-} (-B). \tag{97}$$

Proof. Evidently,

$$\begin{gathered} \delta^* \left(y \mid A\right) - \delta^* \left(y \mid -B\right) = \delta^* \left(y \mid U\right) - \delta^* \left(y \mid -V\right), \\ \forall y \in R^n. \end{gathered} \tag{98}$$

This leads to

$$\delta^* \left(y \mid A - V\right) - \delta^* \left(y \mid -B\right) = \delta^* \left(y \mid U\right), \quad \forall y \in R^n. \tag{99}$$

Taking the Clarke subdifferential at $y = 0$, (99) becomes

$$\begin{aligned} &\partial_{\mathrm{Cl}} \left(\delta^* \left(y \mid A - V\right) - \delta^* \left(y \mid -B\right)\right)\big|_{y=0} \\ &\quad = \partial_{\mathrm{Cl}} \left(\delta^* \left(y \mid U\right)\right)\big|_{y=0}. \end{aligned} \tag{100}$$

Based on the definition of the Demyanov difference, (100) yields $(A - V) \dot{-} (-B) = U$; that is, (97) holds. □

Theorem 16. *Let $f \in \Delta_n(x)$ and $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$. If there exists $W \in Y_n$ such that $[W, \underline{\partial} f(x) \dot{-} (-\overline{\partial} f(x))] \in \mathscr{D} f(x)$, then*

$$W = \left\{\underline{\partial} f(x) - \left(\underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right)\right\} \dot{-} \left(-\overline{\partial} f(x)\right). \tag{101}$$

Proof. Setting $[A, B] = [\underline{\partial} f(x), \overline{\partial} f(x)]$ and $[U, V] = [W, \underline{\partial} f(x) \dot{-} (-\overline{\partial} f(x))]$ in Lemma 15, we have

$$\begin{aligned} W = (A - V) \dot{-} (-B) = &\left\{\underline{\partial} f(x) - \left(\underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right)\right\} \\ &\dot{-} \left(-\overline{\partial} f(x)\right). \end{aligned} \tag{102}$$

This completes the proof of the theorem. □

Theorem 17. *Let $f \in \Delta_n(x)$. If there exists $[A, B] \in \mathscr{D} f(x)$ satisfying $A \dot{-} (-B) = A - (-B)$, then, for any $[\underline{\partial} f(x), \overline{\partial} f(x)] \in \mathscr{D} f(x)$, the pair of sets*

$$\begin{gathered} \left[\left\{\underline{\partial} f(x) - \left(\underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right)\right\}\right. \\ \left.\dot{-} \left(-\overline{\partial} f(x)\right), \underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right] \end{gathered} \tag{103}$$

is the adjoint kernelled quasidifferential of f.

Proof. By Theorem 13 and Proposition 4,

$$\left[A - A, A \dot{-} (-B)\right] = \left[A - A, \underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right] \tag{104}$$

is the kernelled quasidifferential. According to Theorem 16, $[A - A, \underline{\partial} f(x) \dot{-} (-\overline{\partial} f(x))] \in \mathscr{D} f(x)$ leads to

$$A - A = \left\{\underline{\partial} f(x) - \left(\underline{\partial} f(x) \dot{-} \left(-\overline{\partial} f(x)\right)\right)\right\} \dot{-} \left(-\overline{\partial} f(x)\right). \tag{105}$$

This means that (103) is the kernelled quasidifferential. The proof is concluded. □

Noticing that the Demyanov difference and the Minkowski difference of polyhedra are polyhedra, we have the following corollary.

Corollary 18. *Suppose that there exist* $[A, B] \in \mathscr{D} f(x)$ *satisfying* $A \dot{-} (-B) = A - (-B)$ *and a pair of polyhedra* $[U, V] \in \mathscr{D} f(x)$. *Then, the kernelled quasidifferential is a pair of polyhedra.*

Based on above two theorems, given a quasidifferential, the adjoint kernelled quasidifferential can be formulated under some conditions, for instance, the condition in Theorem 13. In particular, if a polyhedral quasidifferential is given, the adjoint kernelled quasidifferential can be calculated because the Demyanov difference of polyhedra can be calculated (for instance, see [9]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper is supported by the National Natural Science Foundation of China under Grant no. 11171049.

References

[1] R. Baier, E. Farkhi, and V. Roshchina, "The directed and Rubinov subdifferentials of quasidifferentiable functions—part I: definition and examples," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 3, pp. 1074–1088, 2012.

[2] R. Baier, E. Farkhi, and V. Roshchina, "The directed and Rubinov subdifferentials of quasidifferentiable functions—part II: calculus," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 3, pp. 1058–1073, 2012.

[3] V. F. Demyanov, "On a relation between the Clarke subdifferential and quasidifferential," *Vestnik Leningrad University*, vol. 13, pp. 183–189, 1981.

[4] V. F. Demyanov and Z. Q. Xia, *Minimal Quasidifferentials*, International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria, 1984.

[5] V. F. Demyanov and A. M. Rubinov, *Quasidifferential Calculus*, Optimization Software, New York, NY, USA, 1986.

[6] V. F. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, vol. 7 of *Approximation & Optimization*, Peter Lang, Frankfurt, Germany, 1995.

[7] V. F. Dem'yanov, G. E. Stavroulakis, L. N. Polyakova, and P. D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, vol. 10 of *Nonconvex Optimization and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1996.

[8] V. F. Demyanov and A. M. Rubinov, *Quasidifferentiability and Related Topics*, Kluwer Academic, Dodrecht, The Netherlands, 2000.

[9] Y. Gao, "Demyanov difference of two sets and optimality conditions of Lagrange multiplier type for constrained quasidifferentiable optimization," *Journal of Optimization Theory and Applications*, vol. 104, no. 2, pp. 377–394, 2000.

[10] Y. Gao, "Representation of the Clarke generalized Jacobian via the quasidifferential," *Journal of Optimization Theory and Applications*, vol. 123, no. 3, pp. 519–532, 2004.

[11] Y. Gao, "Representative of quasidifferentials and its formula for a quasidifferentiable function," *Set-Valued Analysis*, vol. 13, no. 4, pp. 323–336, 2005.

[12] Y. Gao, "Calculating the proximal subdifferential via the quasidifferential," *Applied Mathematics Letters*, vol. 21, no. 11, pp. 1172–1176, 2008.

[13] B. Luderer and D. Wagner, "Algorithms of quasidifferentiable optimization for the separation of point sets," in *Optimization and Optimal Control: Theory and Applications*, vol. 39 of *Springer Optimization and Its Applications*, pp. 157–167, Springer, New York, NY, USA, 2010.

[14] M. R. Deng and Y. Gao, "A property on quasidifferentials," *Chinese Journal of Operations Research*, vol. 10, no. 1, pp. 65–67, 1991 (Chinese).

[15] J. Grzybowski, "Minimal pairs of convex compact sets," *Archiv der Mathematik*, vol. 63, no. 2, pp. 173–181, 1994.

[16] Y. Gao, "The star-kernel for a quasidifferentiable function in one dimensional space," *Journal of Mathematical Research and Exposition*, vol. 8, no. 1, p. 152, 1988.

[17] Y. Gao, Z. Q. Xia, and L. W. Zhang, "Kernelled quasidifferential for a quasidifferentiable function in two-dimensional space," *Journal of Convex Analysis*, vol. 8, no. 2, pp. 401–408, 2001.

[18] D. Pallaschke, S. Scholtes, and R. Urbański, "On minimal pairs of convex compact sets," *Bulletin of the Polish Academy of Sciences. Mathematics*, vol. 39, no. 1-2, pp. 105–109, 1991.

[19] D. Pallaschke and R. Urbański, "A continuum of minimal pairs of compact convex sets which are not connected by translations," *Journal of Convex Analysis*, vol. 3, no. 1, pp. 83–95, 1996.

[20] D. Pallaschke and R. Urbański, "Some criteria for the minimality of pairs of compact convex sets," *Zeitschrift für Operations Research*, vol. 37, no. 2, pp. 129–150, 1993.

[21] D. Pallaschke and R. Urbański, "Reduction of quasidifferentials and minimal representations," *Mathematical Programming*, vol. 66, no. 2, pp. 161–180, 1994.

[22] S. Scholtes, "Minimal pairs of convex bodies in two dimensions," *Mathematika*, vol. 39, no. 2, pp. 267–273, 1992.

[23] C. L. Song and Z. Q. Xia, "A note on the kernelled quasidifferential in the n-dimensional space," *Fuzzy Engineering and Operations Research*, vol. 147, pp. 541–547, 2012.

[24] Z. Q. Xia, *The ∗-Kernel for Quasidiffertiable Functions*, WP-87-89, IIASA, Laxenburg, Austria, 1987.

[25] Z. Q. Xia, "On quasidifferential kernels," *Demonstratio Mathematica*, vol. 26, no. 1, pp. 159–182, 1993.

[26] L. W. Zhang, Z. Q. Xia, Y. Gao, and M. Z. Wang, "Star-kernels and star-differentials in quasidifferential analysis," *Journal of Convex Analysis*, vol. 9, no. 1, pp. 139–158, 2002.

[27] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, USA, 1970.

[28] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley-Interscience, New York, NY, USA, 1983.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 945985, 18 pages
http://dx.doi.org/10.1155/2014/945985

Research Article

Hybrid Viscosity Approaches to General Systems of Variational Inequalities with Hierarchical Fixed Point Problem Constraints in Banach Spaces

Lu-Chuan Ceng,[1] Saleh A. Al-Mezel,[2] and Abdul Latif[2]

[1] *Department of Mathematics, Shanghai Normal University and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Saleh A. Al-Mezel; mathsaleh@yahoo.com

Received 4 August 2013; Revised 4 December 2013; Accepted 20 December 2013; Published 17 February 2014

Academic Editor: Hichem Ben-El-Mechaiekh

The purpose of this paper is to introduce and analyze hybrid viscosity methods for a general system of variational inequalities (GSVI) with hierarchical fixed point problem constraint in the setting of real uniformly convex and 2-uniformly smooth Banach spaces. Here, the hybrid viscosity methods are based on Korpelevich's extragradient method, viscosity approximation method, and hybrid steepest-descent method. We propose and consider hybrid implicit and explicit viscosity iterative algorithms for solving the GSVI with hierarchical fixed point problem constraint not only for a nonexpansive mapping but also for a countable family of nonexpansive mappings in X, respectively. We derive some strong convergence theorems under appropriate conditions. Our results extend, improve, supplement, and develop the recent results announced by many authors.

1. Introduction

Let X be a real Banach space whose dual space is denoted by X^*. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. A Banach space X is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \Longrightarrow \frac{\|x + y\|}{2} \leq 1 - \delta. \quad (1)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. In particular, if $L = 1$, then T is said to be nonexpansive. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence. A mapping $A : C \to X$ is said to be

(i) accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0, \quad (3)$$

where J is the normalized duality mapping of X,

(ii) α-inverse-strongly accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha\|x - y\|^2, \quad (4)$$

for some $\alpha \in (0, 1)$,

(iii) pseudocontractive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \leq \|x - y\|^2, \quad (5)$$

(iv) β-strongly pseudocontractive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \leq \beta\|x-y\|^2 \tag{6}$$

for some $\beta \in (0,1)$,

(v) λ-strictly pseudocontractive if, for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Ax-Ay)\|^2 \tag{7}$$

for some $\lambda \in (0,1)$.

It is worth emphasizing that the definition of the inverse-strongly accretive mapping is based on that of the inverse-strongly monotone mapping, which was studied by so many authors; see, for example, [1–7].

A Banach space X is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{8}$$

exists for all $x, y \in X$; in this case, X is also said to have a Gateaux differentiable norm. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$; in this case, X is also said to have a uniformly Frechet differentiable norm. The norm of X is said to be the Frechet differential if, for each $x \in U$, this limit is attained uniformly for $y \in U$. In the meantime, we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau\right\}. \tag{9}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$, for all $\tau > 0$. As pointed out in [8], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas, if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X.

In a real smooth Banach space X, we say that an operator A is strongly positive (see [9]), if there exists a constant $\overline{\gamma} > 0$ with the property

$$\begin{gathered}\langle Ax, J(x)\rangle \geq \overline{\gamma}\|x\|^2,\\ \|aI - bA\| = \sup_{\|x\|\leq 1}|\langle (aI-bA)x, J(x)\rangle|,\\ a \in [0,1],\ b \in [-1,1],\end{gathered} \tag{10}$$

where I is the identity mapping.

Proposition CB (see [9, Lemma 2.5]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T : C \to C$ be a continuous pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0,1)$ and Lipschitzian constant $L > 0$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $C \pm C \subset C$ and $0 < \beta < \overline{\gamma}$. Let $\{x_t\}$ be defined by*

$$x_t = tf(x_t) + (I - tA)Tx_t. \tag{11}$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point p of T such that p is the unique solution in $\mathrm{Fix}(T)$ *to the VIP:*

$$\langle (A-f)p, J(p-u)\rangle \leq 0, \quad \forall u \in \mathrm{Fix}(T). \tag{12}$$

On the other hand, Cai and Bu [10] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered}\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C,\\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, \quad \forall x \in C,\end{gathered} \tag{13}$$

where C is a nonempty, closed, and convex subset of X, $B_1, B_2 : C \to X$ are two nonlinear mappings, and μ_1 and μ_2 are two positive constants. Here the set of solutions of GSVI (13) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. Very recently, Cai and Bu [10] constructed an iterative algorithm for solving GSVI (13) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the proposed algorithm by virtue of the following inequality in a 2-uniformly smooth Banach space X.

Lemma 1 (see [11]). *Let X be a 2-uniformly smooth Banach space. Then, there exists a best smooth constant $\kappa > 0$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|\kappa y\|, \quad \forall x, y \in X, \tag{14}$$

where J is the normalized duality mapping from X into X^*.

The authors [10] have used the following inequality in a real smooth and uniform convex Banach space X.

Proposition 2 (see [12]). *Let X be a real smooth and uniform convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x-y\|) \leq \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall x, y \in B_r, \tag{15}$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 3 can be found in [13]. Lemma 4 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 3. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n) a_n + b_n c_n, \quad \forall n \geq 0, \tag{16}$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} b_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4. *In a smooth Banach space X, there holds the inequality*

$$\begin{aligned} \|x\|^2 + 2\langle y, J(x)\rangle &\leq \|x+y\|^2 \\ &\leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall x, y \in X, \end{aligned} \tag{17}$$

where J is the normalized duality mapping of X.

Let μ be a mean if μ is a continuous linear functional on l^{∞} satisfying $\|\mu\| = 1 = \mu(1)$. Then, we know that μ is a mean on $\mathbf{N}$ if and only if

$$\inf \{a_n : n \in \mathbf{N}\} \leq \mu(a) \leq \sup \{a_n : n \in \mathbf{N}\} \tag{18}$$

for every $a = (a_1, a_2, \ldots) \in l^{\infty}$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbf{N}$ is called a Banach limit if and only if

$$\mu_n(a_n) = \mu_n(a_{n+1}) \tag{19}$$

for every $a = (a_1, a_2, \ldots) \in l^{\infty}$. We know that, if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n \tag{20}$$

for every $a = (a_1, a_2, \ldots) \in l^{\infty}$. So, if $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in l^{\infty}$, and $a_n \to c$ (resp., $a_n - b_n \to 0$), as $n \to \infty$, we have

$$\mu_n(a_n) = \mu(a) = c \quad (\text{resp., } \mu_n(a_n) = \mu_n(b_n)). \tag{21}$$

Further, it is well known that there holds the following result.

Lemma 5 (see [14]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $\{x_n\}$ be a bounded sequence of X; let μ be a mean on $\mathbf{N}$ and let $z \in C$. Then,*

$$\mu_n \|x_n - z\|^2 = \min_{y\in C} \mu_n \|x_n - y\|^2 \tag{22}$$

if and only if

$$\mu_n \langle y - z, J(x_n - z)\rangle \leq 0, \quad \forall y \in C, \tag{23}$$

where J is the normalized duality mapping of X.

Lemma 6 (see [9, Lemma 2.6]). *Let C be a nonempty closed convex subset of a real Banach space X which has uniformly Gateaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0,1)$ and Lipschitzian constant $L > 0$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $C \pm C \subset C$ and that $\{x_t\}$ converges strongly to $p \in$* $\mathrm{Fix}(T)$ *as $t \to 0$, where x_t is defined by $x_t = tf(x_t) + (I - tA)Tx_t$. Suppose that $\{x_n\} \subset C$ is bounded and that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then,* $\limsup_{n\to\infty} \langle (f - A)p, J(x_n - p)\rangle \leq 0$.

Lemma 7. *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let $F : C \to X$ be an α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \geq 1$. Then, $I - F$ is nonexpansive and F is Lipschitz continuous with constant $(1 + \sqrt{(1-\alpha)/\lambda})$. Further, for any fixed $\tau \in (0,1)$, $I - \tau F$ is contractive with coefficient $1 - \tau(1 - \sqrt{(1-\alpha)/\lambda})$.*

Proof. From the λ-strictly pseudocontractivity and α-strongly accretivity of F, we have, for all $x, y \in C$,

$$\begin{aligned} &\lambda \|(I-F)x - (I-F)y\|^2 \\ &\quad \leq \|x-y\|^2 - \langle F(x) - F(y), J(x-y)\rangle \\ &\quad \leq (1-\alpha)\|x-y\|^2, \end{aligned} \tag{24}$$

which implies that

$$\|(I-F)x - (I-F)y\| \leq \sqrt{\frac{1-\alpha}{\lambda}} \|x-y\|. \tag{25}$$

Because $\alpha + \lambda \geq 1 \Leftrightarrow \sqrt{(1-\alpha)/\lambda} \leq 1$, we know that $I - F$ is nonexpansive. Also note that

$$\begin{aligned} \|F(x) - F(y)\| &\leq \|(I-F)x - (I-F)y\| + \|x-y\| \\ &\leq \left(1 + \sqrt{\frac{1-\alpha}{\lambda}}\right)\|x-y\|. \end{aligned} \tag{26}$$

Now, take a fixed $\tau \in (0,1)$ arbitrarily. Observe that, for all $x, y \in C$,

$$\begin{aligned} &\|(I-\tau F)x - (I-\tau F)y\| \\ &\quad = \|(1-\tau)(x-y) + \tau[(I-F)x - (I-F)y]\| \\ &\quad \leq (1-\tau)\|x-y\| + \tau\|(I-F)x - (I-F)y\| \\ &\quad \leq (1-\tau)\|x-y\| + \tau\left(\sqrt{\frac{1-\alpha}{\lambda}}\right)\|x-y\| \\ &\quad = \left(1 - \tau\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|x-y\|. \end{aligned} \tag{27}$$

Because $\alpha + \lambda > 1 \Leftrightarrow \sqrt{(1-\alpha)/\lambda} < 1$, we know that $I - \tau F$ is contractive with coefficient $1 - \tau(1 - \sqrt{(1-\alpha)/\lambda})$. ☐

Let D be a subset of C and let Π be a mapping of C into D. Then, Π is said to be sunny if

$$\Pi\left[\Pi(x)+t(x-\Pi(x))\right]=\Pi(x), \tag{28}$$

whenever $\Pi(x)+t(x-\Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2=\Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z)=z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 8 (see [15]). *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then, the following are equivalent:*

(i) *Π is sunny and nonexpansive;*

(ii) $\|\Pi(x)-\Pi(y)\|^2 \leq \langle x-y, J(\Pi(x)-\Pi(y))\rangle$, *for all* $x, y \in C$;

(iii) $\langle x-\Pi(x), J(y-\Pi(x))\rangle \leq 0$, *for all* $x \in C$, $y \in D$.

It is well known that, if $X=H$ is a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C=P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T: C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 9. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let $B_1, B_2 : C \to X$ be nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI* (13) *if and only if $x^*=\Pi_C(y^*-\mu_1 B_1 y^*)$, where $y^*=\Pi_C(x^*-\mu_2 B_2 x^*)$.*

Proof. We can rewrite GSVI (13) as

$$\begin{aligned} &\left\langle x^*-\left(y^*-\mu_1 B_1 y^*\right), J\left(x-x^*\right)\right\rangle \geq 0, \quad \forall x \in C,\\ &\left\langle y^*-\left(x^*-\mu_2 B_2 x^*\right), J\left(x-y^*\right)\right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{29}$$

which is obviously equivalent to

$$\begin{aligned} x^* &= \Pi_C\left(y^*-\mu_1 B_1 y^*\right),\\ y^* &= \Pi_C\left(x^*-\mu_2 B_2 x^*\right), \end{aligned} \tag{30}$$

because of Lemma 8. This completes the proof. □

In terms of Lemma 9, define the mapping $G: C \to C$ as follows:

$$G(x) := \Pi_C\left(I-\mu_1 B_1\right) \Pi_C\left(I-\mu_2 B_2\right) x, \quad \forall x \in C. \tag{31}$$

Then, we observe that

$$x^*=\Pi_C\left[\Pi_C\left(x^*-\mu_2 B_2 x^*\right)-\mu_1 B_1 \Pi_C\left(x^*-\mu_2 B_2 x^*\right)\right], \tag{32}$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

Lemma 10 (see [16]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n=1$. Then, a mapping T on C defined by $Tx=\sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is well-defined, nonexpansive and $\mathrm{Fix}(T)=\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 11 (see [17]). *Let C be a nonempty closed convex subset of a Banach space X. Let $S_1, S_2, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x-S_n x\| : x \in C\}<\infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy=\lim_{n\to\infty} S_n y$, for all $y \in C$. Then, $\lim_{n\to\infty}\sup\{\|Sx-S_n x\| : x \in C\}=0$.*

3. GSVI with Hierarchical Fixed Point Problem Constraint for a Nonexpansive Mapping

In this section, we introduce our hybrid implicit viscosity scheme for solving the GSVI (13) with hierarchical fixed point problem constraint for a nonexpansive mapping and show the strong convergence theorem. First, we list several useful and helpful lemmas.

Lemma 12 (see [10, Lemma 2.8]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive. Then, one has*

$$\begin{aligned} &\left\|\left(I-\mu_i B_i\right) x-\left(I-\mu_i B_i\right) y\right\|^2\\ &\quad \leq\|x-y\|^2+2 \mu_i\left(\mu_i \kappa^2-\alpha_i\right)\left\|B_i x-B_i y\right\|^2, \quad \forall x, y \in C, \end{aligned} \tag{33}$$

for $i=1,2$, where $\mu_i>0$. In particular, if $0<\mu_i \leq \alpha_i/\kappa^2$ (where κ is the best constant of X as in Lemma 1), then $I-\mu_i B_i$ is nonexpansive for $i=1,2$.

Lemma 13 (see [10, Lemma 2.9]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $G: C \to C$ be the mapping defined by*

$$\begin{aligned} Gx=\Pi_C\left[\Pi_C\left(x-\mu_2 B_2 x\right)-\mu_1 B_1 \Pi_C\left(x-\mu_2 B_2 x\right)\right],&\\ \forall x \in C.& \end{aligned} \tag{34}$$

If $0<\mu_i \leq \alpha_i/\kappa^2$, for $i=1,2$, then $G: C \to C$ is nonexpansive.

Lemma 14 (see [18]). *Let X be a Banach space, C a nonempty closed and convex subset of X, and $T: C \to C$ a continuous* *and strong pseudocontraction. Then, T has a unique fixed point in C.*

Lemma 15 (see [19]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then, $\|I - \rho A\|^2 \le 1 - \rho\overline{\gamma}$.*

We now state and prove our first result.

Theorem 16. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X such that $C \pm C \subset C$. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $T : C \to C$ be a nonexpansive mapping such that $\Lambda = \mathrm{Fix}(T) \cap \Omega \neq \emptyset$ where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0, 1)$ and Lipschitzian constant $L > 0$, let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $0 < \overline{\gamma} - \beta \le 1$. Let $\{x_t\}$ be defined by*

$$x_t = tf(x_t) + (I - tA)[G(Tx_t) - \theta_t FG(Tx_t)], \tag{35}$$

where $\{\theta_t : t \in (0, 1)\} \subset [0, 1)$ with $\lim_{t\to 0}\theta_t/t = 0$. Then, as $t \to 0$, $\{x_t\}$ converges strongly to a point $p \in \Lambda$, which is the unique solution in Λ to the VIP,

$$\langle (A - f)p, J(p - u)\rangle \le 0, \quad \forall u \in \Lambda. \tag{36}$$

Proof. First, let us show that the net $\{x_t\}$ is defined well. As a matter of fact, define the mapping $S_t : C \to C$ as follows:

$$S_t x = tf(x) + (I - tA)[G(Tx) - \theta_t FG(Tx)], \quad \forall x \in C. \tag{37}$$

We may assume, without loss of generality, that $t \le \|A\|^{-1}$. Utilizing Lemmas 7, 13, and 15, we have

$$\begin{aligned}
&\langle S_t x - S_t y, J(x - y)\rangle \\
&\quad = t\langle f(x) - f(y), J(x - y)\rangle \\
&\qquad + \langle (I - tA)[(I - \theta_t F)G(Tx) - (I - \theta_t F)G(Ty)], \\
&\qquad\quad J(x - y)\rangle \\
&\quad \le t\beta\|x - y\|^2 + (1 - t\overline{\gamma}) \\
&\qquad \times \|(I - \theta_t F)G(Tx) - (I - \theta_t F)G(Ty)\|\,\|x - y\| \\
&\quad \le t\beta\|x - y\|^2 + (1 - t\overline{\gamma})\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \alpha}{\lambda}}\right)\right) \\
&\qquad \times \|G(Tx) - G(Ty)\|\,\|x - y\| \\
&\quad \le t\beta\|x - y\|^2 + (1 - t\overline{\gamma})\|Tx - Ty\|\,\|x - y\| \\
&\quad \le t\beta\|x - y\|^2 + (1 - t\overline{\gamma})\|x - y\|^2 \\
&\quad = (1 - t(\overline{\gamma} - \beta))\|x - y\|^2.
\end{aligned} \tag{38}$$

Hence, it is known that $S_t : C \to C$ is a continuous and strongly pseudocontractive mapping with pseudocontractive coefficient $1 - t(\overline{\gamma} - \beta) \in (0, 1)$ Thus, by Lemma 14, we deduce that there exists a unique fixed point in C, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = tf(x_t) + (I - tA)[G(Tx_t) - \theta_t FG(Tx_t)]. \tag{39}$$

Let us show the uniqueness of the solution of VIP (36). Suppose that both $p_1 \in \Lambda$ and $p_2 \in \Lambda$ are solutions to VIP (36). Then, we have

$$\begin{aligned}
&\langle (A - f)p_1, J(p_1 - p_2)\rangle \le 0, \\
&\langle (A - f)p_2, J(p_2 - p_1)\rangle \le 0.
\end{aligned} \tag{40}$$

Adding up the above two inequalities, we obtain

$$\langle (A - f)p_1 - (A - f)p_2, J(p_1 - p_2)\rangle \le 0. \tag{41}$$

Note that

$$\begin{aligned}
&\langle (A - f)p_1 - (A - f)p_2, J(p_1 - p_2)\rangle \\
&\quad = \langle A(p_1 - p_2), J(p_1 - p_2)\rangle \\
&\qquad - \langle f(p_1) - f(p_2), J(p_1 - p_2)\rangle \\
&\quad \ge \overline{\gamma}\|p_1 - p_2\|^2 - \beta\|p_1 - p_2\|^2 \\
&\quad = (\overline{\gamma} - \beta)\|p_1 - p_2\|^2 \ge 0.
\end{aligned} \tag{42}$$

Consequently, we have $p_1 = p_2$, and the uniqueness is proved.

Next, let us show that, for some $a \in (0, 1)$, $\{x_t : t \in (0, a]\}$ is bounded. Indeed, since $\{\theta_t : t \in (0, 1)\} \subset [0, 1)$ with $\lim_{t\to 0}(\theta_t/t) = 0$, there exists some $a \in (0, 1)$ such that $0 \le \theta_t/t < 1$ for all $t \in (0, a]$. Take a fixed $p \in \mathrm{Fix}(\Lambda)$ arbitrarily. Utilizing Lemma 7, we have

$$\begin{aligned}
&\|x_t - p\|^2 \\
&\quad = \langle t(f(x_t) - f(p)) + (I - tA)[G(Tx_t) - \theta_t FG(Tx_t) - p] \\
&\qquad - t(Ap - f(p)), J(x_t - p)\rangle \\
&\quad = t\langle f(x_t) - f(p), J(x_t - p)\rangle \\
&\qquad + \langle (I - tA)[G(Tx_t) - \theta_t FG(Tx_t) - p], J(x_t - p)\rangle \\
&\qquad - t\langle (A - f)p, J(x_t - p)\rangle
\end{aligned}$$

$$\begin{aligned}
&\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\|G(Tx_t)-\theta_t FG(Tx_t)-p\|\\
&\quad\times\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\\
&\quad\times[\|(I-\theta_t F)G(Tx_t)-(I-\theta_t F)G(Tp)\|\\
&\qquad+\|(I-\theta_t F)G(Tp)-p\|]\|x_t-p\|\\
&\quad+t\|(A-f)p\|\|x_t-p\|\\
&\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\left(1-\theta_t\left(1-\sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\\
&\quad\times\|G(Tx_t)-G(Tp)\|\|x_t-p\|\\
&\quad+(1-t\overline{\gamma})\theta_t\|Fp\|\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\left(1-\theta_t\left(1-\sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\\
&\quad\times\|Tx_t-Tp\|\|x_t-p\|\\
&\quad+\theta_t\|Fp\|\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\|x_t-p\|^2\\
&\quad+\theta_t\|Fp\|\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&=(1-t(\overline{\gamma}-\beta))\|x_t-p\|^2+\theta_t\|Fp\|\|x_t-p\|\\
&\quad+t\|(A-f)p\|\|x_t-p\|,
\end{aligned}\tag{43}$$

and, hence, for all $t\in(0,a]$,

$$\begin{aligned}
\|x_t-p\|&\le\frac{1}{\overline{\gamma}-\beta}\left(\|(A-f)p\|+\frac{\theta_t}{t}\|Fp\|\right)\\
&\le\frac{1}{\overline{\gamma}-\beta}(\|(A-f)p\|+\|Fp\|).
\end{aligned}\tag{44}$$

Thus, this implies that $\{x_t:t\in(0,a]\}$ is bounded and so are $\{f(x_t):t\in(0,a]\}$, $\{Tx_t:t\in(0,a]\}$, and $\{G(Tx_t):t\in(0,a]\}$.

Let us show that $\|Tx_t-G(Tx_t)\|\to 0$ as $t\to 0$.

Indeed, for simplicity, we put $q=\Pi_C(I-\mu_2B_2)p$, $\widehat{x}_t=Tx_t$, $u_t=\Pi_C(I-\mu_2B_2)\widehat{x}_t$, and $v_t=\Pi_C(I-\mu_1B_1)u_t$. Then, it is clear that $p=\Pi_C(I-\mu_1B_1)q$ and $v_t=G(\widehat{x}_t)=G(Tx_t)$. Hence, from (43), it follows that

$$\begin{aligned}
&\|x_t-p\|^2\\
&\quad\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\left(1-\theta_t\left(1-\sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\\
&\qquad\times\|G(Tx_t)-G(Tp)\|\|x_t-p\|\\
&\qquad+(1-t\overline{\gamma})\theta_t\|Fp\|\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&\quad\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\|v_t-p\|\|x_t-p\|\\
&\qquad+\theta_t\|Fp\|\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|.
\end{aligned}\tag{45}$$

From Lemma 12, we have

$$\begin{aligned}
\|u_t-q\|^2&=\|\Pi_C(\widehat{x}_t-\mu_2B_2\widehat{x}_t)-\Pi_C(p-\mu_2B_2p)\|^2\\
&\le\|\widehat{x}_t-p-\mu_2(B_2\widehat{x}_t-B_2p)\|^2\\
&\le\|\widehat{x}_t-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2\widehat{x}_t-B_2p\|^2,\\
\|v_t-p\|^2&=\|\Pi_C(u_t-\mu_1B_1u_t)-\Pi_C(q-\mu_1B_1q)\|^2\\
&\le\|u_t-q-\mu_1(B_1u_t-B_1q)\|^2\\
&\le\|u_t-q\|^2-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_t-B_1q\|^2.
\end{aligned}\tag{46}$$

From the last two inequalities, we obtain

$$\begin{aligned}
\|v_t-p\|^2&\le\|\widehat{x}_t-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\\
&\quad\times\|B_2\widehat{x}_t-B_2p\|^2-2\mu_1(\alpha_1-\kappa^2\mu_1)\\
&\quad\times\|B_1u_t-B_1q\|^2\\
&\le\|x_t-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2\widehat{x}_t-B_2p\|^2\\
&\quad-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_t-B_1q\|^2,
\end{aligned}\tag{47}$$

which together with (45) implies that

$$\begin{aligned}
&\|x_t-p\|^2\\
&\quad\le t\beta\|x_t-p\|^2+(1-t\overline{\gamma})\|v_t-p\|\|x_t-p\|+\theta_t\|Fp\|\\
&\qquad\times\|x_t-p\|+t\|(A-f)p\|\|x_t-p\|\\
&\quad\le t\beta\|x_t-p\|^2+t\|(A-f)p\|\|x_t-p\|+(1-t\overline{\gamma})\\
&\qquad\times\frac{1}{2}\left(\|x_t-p\|^2+\|v_t-p\|^2\right)+\theta_t\|Fp\|\|x_t-p\|\\
&\quad\le t\beta\|x_t-p\|^2+t\|(A-f)p\|\|x_t-p\|\\
&\qquad+(1-t\overline{\gamma})\frac{1}{2}\{\|x_t-p\|^2+\|x_t-p\|^2\\
&\qquad\qquad-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2\widehat{x}_t-B_2p\|^2\\
&\qquad\qquad-2\mu_1(\alpha_1-\kappa^2\mu_1)\\
&\qquad\qquad\times\|B_1u_t-B_1q\|^2\}+\theta_t\|Fp\|\|x_t-p\|\\
&\quad=(1-t(\overline{\gamma}-\beta))\|x_t-p\|^2+t\|(A-f)p\|\|x_t-p\|\\
&\qquad-(1-t\overline{\gamma})[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2\widehat{x}_t-B_2p\|^2\\
&\qquad\qquad+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_t-B_1q\|^2]
\end{aligned}$$

$$
\begin{aligned}
&+\theta_t \|Fp\| \|x_t - p\| \\
&\leq \|x_t - p\|^2 + t \|(A - f) p\| \|x_t - p\| - (1 - t\overline{\gamma}) \\
&\quad \times \left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_t - B_2 p\|^2 \right. \\
&\quad \left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_t - B_1 q\|^2\right] + \theta_t \|Fp\| \|x_t - p\|.
\end{aligned} \tag{48}
$$

So, it immediately follows that

$$
\begin{aligned}
&(1 - t\overline{\gamma}) \left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_t - B_2 p\|^2 + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \right. \\
&\quad \left. \times \|B_1 u_t - B_1 q\|^2\right] \\
&\leq t \|(A - f) p\| \|x_t - p\| + \theta_t \|Fp\| \|x_t - p\|.
\end{aligned} \tag{49}
$$

Since $0 < \mu_i < \alpha_i/\kappa^2$, for $i = 1, 2$, we have

$$
\lim_{t \to 0} \|B_2 \widehat{x}_t - B_2 p\| = 0, \qquad \lim_{t \to 0} \|B_1 u_t - B_1 q\| = 0. \tag{50}
$$

Utilizing Proposition 2 and Lemma 8, we have that there exists g_1 such that

$$
\begin{aligned}
\|u_t - q\|^2 &= \|\Pi_C (\widehat{x}_t - \mu_2 B_2 \widehat{x}_t) - \Pi_C (p - \mu_2 B_2 p)\|^2 \\
&\leq \langle \widehat{x}_t - \mu_2 B_2 \widehat{x}_t - (p - \mu_2 B_2 p), J (u_t - q)\rangle \\
&= \langle \widehat{x}_t - p, J (u_t - q)\rangle + \mu_2 \langle B_2 p - B_2 \widehat{x}_t, J (u_t - q)\rangle \\
&\leq \frac{1}{2} \left[\|\widehat{x}_t - p\|^2 + \|u_t - q\|^2 \right. \\
&\quad \left. - g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \right] \\
&\quad + \mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\|,
\end{aligned} \tag{51}
$$

which implies that

$$
\begin{aligned}
\|u_t - q\|^2 \leq{}& \|\widehat{x}_t - p\|^2 - g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \\
&+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\|.
\end{aligned} \tag{52}
$$

In the same way, we derive that there exists g_2:

$$
\begin{aligned}
\|v_t - p\|^2 &= \|\Pi_C (u_t - \mu_1 B_1 u_t) - \Pi_C (q - \mu_1 B_1 q)\|^2 \\
&\leq \langle u_t - \mu_1 B_1 u_t - (q - \mu_1 B_1 q), J (v_t - p)\rangle \\
&= \langle u_t - q, J (v_t - p)\rangle + \mu_1 \langle B_1 q - B_1 u_t, J (v_t - p)\rangle \\
&\leq \frac{1}{2} \left[\|u_t - q\|^2 + \|v_t - p\|^2 \right. \\
&\quad \left. - g_2 (\|u_t - v_t + (p - q)\|) \right] \\
&\quad + \mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\|,
\end{aligned} \tag{53}
$$

which implies that

$$
\begin{aligned}
\|v_t - p\|^2 \leq{}& \|u_t - q\|^2 - g_2 (\|u_t - v_t + (p - q)\|) \\
&+ 2\mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\|.
\end{aligned} \tag{54}
$$

Substituting (52) for (54), we get

$$
\begin{aligned}
\|v_t - p\|^2 \leq{}& \|\widehat{x}_t - p\|^2 - g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \\
&- g_2 (\|u_t - v_t + (p - q)\|) \\
&+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \\
&+ 2\mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\| \\
\leq{}& \|x_t - p\|^2 - g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \\
&- g_2 (\|u_t - v_t + (p - q)\|) \\
&+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \\
&+ 2\mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\|,
\end{aligned} \tag{55}
$$

which together with (45) implies that

$$
\begin{aligned}
&\|x_t - p\|^2 \\
&\leq t\beta \|x_t - p\|^2 + (1 - t\overline{\gamma}) \|v_t - p\| \|x_t - p\| + \theta_t \|Fp\| \\
&\quad \times \|x_t - p\| + t \|(A - f) p\| \|x_t - p\| \\
&\leq t\beta \|x_t - p\|^2 + t \|(A - f) p\| \|x_t - p\| + (1 - t\overline{\gamma}) \\
&\quad \times \frac{1}{2} \left(\|x_t - p\|^2 + \|v_t - p\|^2\right) + \theta_t \|Fp\| \|x_t - p\| \\
&\leq t\beta \|x_t - p\|^2 + t \|(A - f) p\| \|x_t - p\| + (1 - t\overline{\gamma}) \\
&\quad \times \frac{1}{2} \left\{\|x_t - p\|^2 + \|x_t - p\|^2 \right. \\
&\qquad - g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \\
&\qquad - g_2 (\|u_t - v_t + (p - q)\|) \\
&\qquad + 2\mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \\
&\qquad \left. + 2\mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\|\right\} \\
&\quad + \theta_t \|Fp\| \|x_t - p\| \\
&= (1 - t (\overline{\gamma} - \beta)) \|x_t - p\|^2 + t \|(A - f) p\| \|x_t - p\| \\
&\quad - (1 - t\overline{\gamma}) \\
&\quad \times \frac{1}{2} \left[g_1 (\|\widehat{x}_t - u_t - (p - q)\|) \right. \\
&\qquad \left. + g_2 (\|u_t - v_t + (p - q)\|)\right] + (1 - t\overline{\gamma}) \\
&\quad \times \left[\mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \right. \\
&\qquad \left. + \mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\|\right]
\end{aligned}
$$

$$\begin{aligned}
&+\theta_t \|Fp\| \|x_t - p\| \\
&\le \|x_t - p\|^2 + t\|(A-f)p\| \|x_t - p\| - (1-t\overline{\gamma}) \\
&\quad \times \frac{1}{2}\left[g_1\left(\|\widehat{x}_t - u_t - (p-q)\|\right)\right. \\
&\qquad \left.+ g_2\left(\|u_t - v_t + (p-q)\|\right)\right] \\
&\quad + \mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \\
&\quad + \mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\| + \theta_t \|Fp\| \|x_t - p\|.
\end{aligned} \tag{56}$$

So, it immediately follows that

$$\begin{aligned}
&(1-t\overline{\gamma})\frac{1}{2}\left[g_1\left(\|\widehat{x}_t - u_t - (p-q)\|\right)\right. \\
&\qquad \left.+ g_2\left(\|u_t - v_t + (p-q)\|\right)\right] \\
&\le t\|(A-f)p\| \|x_t - p\| + \mu_2 \|B_2 p - B_2 \widehat{x}_t\| \|u_t - q\| \\
&\quad + \mu_1 \|B_1 q - B_1 u_t\| \|v_t - p\| + \theta_t \|Fp\| \|x_t - p\|.
\end{aligned} \tag{57}$$

Hence, from (50), we conclude that

$$\begin{aligned}
&\lim_{t\to 0} g_1\left(\|\widehat{x}_t - u_t - (p-q)\|\right) = 0, \\
&\lim_{t\to 0} g_2\left(\|u_t - v_t + (p-q)\|\right) = 0.
\end{aligned} \tag{58}$$

Utilizing the properties of g_1 and g_2, we get

$$\begin{aligned}
&\lim_{t\to 0} \|\widehat{x}_t - u_t - (p-q)\| = 0, \\
&\lim_{t\to 0} \|u_t - v_t + (p-q)\| = 0,
\end{aligned} \tag{59}$$

which leads to

$$\begin{aligned}
\|\widehat{x}_t - v_t\| &\le \|\widehat{x}_t - u_t - (p-q)\| \\
&\quad + \|u_t - v_t + (p-q)\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0.
\end{aligned} \tag{60}$$

That is,

$$\lim_{t\to 0} \|Tx_t - G(Tx_t)\| = \lim_{t\to 0} \|\widehat{x}_t - v_t\| = 0. \tag{61}$$

Note that $\{x_t : t \in (0,a]\}$ is bounded and so are $\{f(x_t) : t \in (0,a]\}$, $\{Tx_t : t \in (0,a]\}$, and $\{G(Tx_t) : t \in (0,a]\}$. Hence, we have

$$\begin{aligned}
&\|x_t - G(Tx_t)\| \\
&\quad = t\left\|f(x_t) - AG(Tx_t) - \frac{\theta_t}{t}(I - tA)FG(Tx_t)\right\| \longrightarrow 0,
\end{aligned} \tag{62}$$

as $t \to 0$. Also, observe that

$$\|x_t - Tx_t\| \le \|x_t - G(Tx_t)\| + \|G(Tx_t) - Tx_t\|. \tag{63}$$

This together with (61) and (62) implies that

$$\lim_{t\to 0} \|x_t - Tx_t\| = 0. \tag{64}$$

Utilizing the nonexpansivity of G, we obtain

$$\begin{aligned}
\|x_t - Gx_t\| &\le \|x_t - G(Tx_t)\| + \|G(Tx_t) - Gx_t\| \\
&\le \|x_t - G(Tx_t)\| + \|Tx_t - x_t\|,
\end{aligned} \tag{65}$$

which together with (62) and (64) implies that

$$\lim_{t\to 0} \|x_t - Gx_t\| = 0. \tag{66}$$

Now, let $\{t_k\}$ be a sequence in $(0,a]$ that converges to 0 as $k \to \infty$, and define a function g on C by

$$g(x) = \mu_k \frac{1}{2}\left\|x_{t_k} - x\right\|^2, \quad \forall x \in C, \tag{67}$$

where μ is a Banach limit. Define the set

$$K := \{w \in C : g(w) = \min\{g(y) : y \in C\}\} \tag{68}$$

and the mapping

$$Wx = (1-\theta)Tx + \theta Gx, \quad \forall x \in C, \tag{69}$$

where θ is a constant in $(0,1)$. Then, by Lemma 10, we know that $\mathrm{Fix}(W) = \mathrm{Fix}(T) \cap \mathrm{Fix}(G) = \Lambda$. We observe that

$$\begin{aligned}
\|x_t - Wx_t\| &= \|(1-\theta)(x_t - Tx_t) + \theta(x_t - Gx_t)\| \\
&\le (1-\theta)\|x_t - Tx_t\| + \theta\|x_t - Gx_t\|.
\end{aligned} \tag{70}$$

So, from (64) and (66), we obtain

$$\lim_{n\to\infty} \|x_t - Wx_t\| = 0. \tag{71}$$

Since X is a uniformly smooth Banach space, K is a nonempty bounded closed convex subset of C; for more details, see [14]. We claim that K is also invariant under the nonexpansive mapping W. Indeed, noticing (71), we have, for $w \in K$,

$$\begin{aligned}
g(Ww) &= \mu_k \frac{1}{2}\left\|x_{t_k} - Ww\right\|^2 = \mu_k \frac{1}{2}\left\|Wx_{t_k} - Ww\right\|^2 \\
&\le \mu_k \frac{1}{2}\left\|x_{t_k} - w\right\|^2 = g(w).
\end{aligned} \tag{72}$$

Since every nonempty closed bounded convex subset of a uniformly smooth Banach space X has the fixed point property for nonexpansive mappings and W is a nonexpansive mapping of K, W has a fixed point in K, say p. Utilizing Lemma 5, we get

$$\mu_k \left\langle x - p, J\left(x_{t_k} - p\right)\right\rangle \le 0, \quad \forall x \in C. \tag{73}$$

Putting $x = (f - A)p + p \in C$, we have

$$\mu_k \left\langle (f - A)p, J\left(x_{t_k} - p\right)\right\rangle \le 0, \quad \forall x \in C. \tag{74}$$

Since $x_{t_k} - p = t_k(f(x_{t_k}) - f(p)) + (I - t_k A)[G(Tx_{t_k}) - \theta_{t_k} FG(Tx_{t_k}) - p] - t_k(A - f)p$, we get

$$\begin{aligned}
&\left\|x_{t_k} - p\right\|^2 \\
&\quad = t_k \left\langle f\left(x_{t_k}\right) - f(p), J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad + \left\langle (I - t_k A)\left(G\left(Tx_{t_k}\right) - p\right), J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad - \theta_{t_k} \left\langle (I - t_k A) FG\left(Tx_{t_k}\right), J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad - t_k \left\langle (A - f) p, J\left(x_{t_k} - p\right)\right\rangle \\
&\quad \le t_k \beta\left\|x_{t_k} - p\right\|^2 + t_k \left\langle (f - A) p, J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad + (1 - t_k\overline{\gamma}) \left\|G\left(Tx_{t_k}\right) - p\right\| \left\|x_{t_k} - p\right\| \\
&\qquad + (1 - t_k\overline{\gamma}) \theta_{t_k} \left\|FG\left(Tx_{t_k}\right)\right\| \left\|x_{t_k} - p\right\| \\
&\quad \le t_k \beta\left\|x_{t_k} - p\right\|^2 + t_k \left\langle (f - A) p, J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad + (1 - t_k\overline{\gamma}) \left\|x_{t_k} - p\right\|^2 + \theta_{t_k} \left\|FG\left(Tx_{t_k}\right)\right\| \left\|x_{t_k} - p\right\|.
\end{aligned} \tag{75}$$

It follows that

$$\begin{aligned}
\left\|x_{t_k} - p\right\|^2 \le \frac{1}{\overline{\gamma} - \beta} \Bigg[&\left\langle (f - A) p, J\left(x_{t_k} - p\right)\right\rangle \\
&+ \frac{\theta_{t_k}}{t_k} \left\|FG\left(Tx_{t_k}\right)\right\| \left\|x_{t_k} - p\right\| \Bigg].
\end{aligned} \tag{76}$$

Since $\lim_{k\to\infty}(\theta_{t_k}/t_k) = 0$, from (74) and the boundedness of sequences $\{FG(Tx_{t_k})\}$, $\{x_{t_k}\}$, it follows that

$$\begin{aligned}
\mu_k\left\|x_{t_k} - p\right\|^2 &\le \frac{1}{\overline{\gamma} - \beta}\mu_k \Bigg[\left\langle (f - A) p, J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad + \frac{\theta_{t_k}}{t_k} \left\|FG\left(Tx_{t_k}\right)\right\| \left\|x_{t_k} - p\right\| \Bigg] \\
&= \frac{1}{\overline{\gamma} - \beta} \Bigg[\mu_k \left\langle (f - A) p, J\left(x_{t_k} - p\right)\right\rangle \\
&\qquad + \mu_k \left(\frac{\theta_{t_k}}{t_k} \left\|FG\left(Tx_{t_k}\right)\right\| \left\|x_{t_k} - p\right\| \right) \Bigg] \le 0.
\end{aligned} \tag{77}$$

Therefore, for the sequence $\{x_{t_k}\}$ in $\{x_t : t \in (0, a]\}$, there exists a subsequence which is still denoted by $\{x_{t_k}\}$ that converges strongly to some fixed point p of W.

Now, we claim that such a p is the unique solution in Λ to the VIP (36).

Indeed, from (35), it follows that for all $u \in \Lambda = \text{Fix}(T) \cap \Omega$

$$\begin{aligned}
&\left\|x_t - u\right\|^2 \\
&= t \left\langle f\left(x_t\right) - f(u), J\left(x_t - u\right)\right\rangle \\
&\quad + \left\langle (I - tA)\left[G\left(Tx_t\right) - \theta_t FG\left(Tx_t\right) - u\right], J\left(x_t - u\right)\right\rangle \\
&\quad - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&= \big\langle (I - tA)\left[(I - \theta_t F) G\left(Tx_t\right) - (I - \theta_t F) u\right. \\
&\qquad\qquad \left. + (I - \theta_t F) u - u\right], J\left(x_t - u\right)\big\rangle \\
&\quad + t \left\langle f\left(x_t\right) - f(u), J\left(x_t - u\right)\right\rangle - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&\le (1 - t\overline{\gamma}) \left[\left\|(I - \theta_t F) G\left(Tx_t\right) - (I - \theta_t F) u\right\|\right. \\
&\qquad\qquad \left. + \left\|(I - \theta_t F) u - u\right\|\right] \\
&\quad \times \left\|x_t - u\right\| + t\beta\left\|x_t - u\right\|^2 - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&\le (1 - t\overline{\gamma}) \left[\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \alpha}{\lambda}}\right)\right) \left\|x_t - u\right\| + \theta_t \left\|Fu\right\|\right] \\
&\quad \times \left\|x_t - u\right\| + t\beta\left\|x_t - u\right\|^2 - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&\le (1 - t\overline{\gamma}) \left[\left\|x_t - u\right\| + \theta_t \left\|Fu\right\|\right] \left\|x_t - u\right\| \\
&\quad + t\beta\left\|x_t - u\right\|^2 - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&\le (1 - t(\overline{\gamma} - \beta)) \left\|x_t - u\right\|^2 + \theta_t \left\|Fu\right\| \\
&\quad \times \left\|x_t - u\right\| - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \\
&\le \left\|x_t - u\right\|^2 + \theta_t \left\|Fu\right\| \left\|x_t - u\right\| - t \left\langle (A - f) u, J\left(x_t - u\right)\right\rangle,
\end{aligned} \tag{78}$$

which hence implies that

$$\left\langle (A - f) u, J\left(x_t - u\right)\right\rangle \le \frac{\theta_t}{t} \left\|Fu\right\| \left\|x_t - u\right\|, \quad \forall u \in \Lambda. \tag{79}$$

Since $x_{t_k} \to p$ as $t_k \to 0$ and $\lim_{t\to 0}(\theta_t/t) = 0$, we obtain from the last inequality that

$$\left\langle (A - f) u, J(p - u)\right\rangle \le 0, \quad \forall u \in \Lambda. \tag{80}$$

Utilizing the well-known Minty-type Lemma, we get

$$\left\langle (A - f) p, J(p - u)\right\rangle \le 0, \quad \forall u \in \Lambda. \tag{81}$$

So, p is a solution in Λ to the VIP (36).

In order to prove that the net $\{x_t : t \in (0, a]\}$ converges strongly to p as $t \to 0$, suppose that there exists another subsequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to q$ as $s_k \to 0$; then we also have $q \in \text{Fix}(W) = \text{Fix}(T) \cap \Omega =: \Lambda$ due to (71). Repeating the same argument as above, we know that q is another solution in Λ to the VIP (36). In terms of the uniqueness of solutions in Λ to the VIP (36), we immediately get $p = q$. This completes the proof. □

Remark 17. It is worth emphasizing that, in the assertion of Theorem 16, "as $t \to 0, \{x_t\}$ converges strongly to a point $p \in \Lambda$," this p depends on no one of the mappings f, A, and F. Indeed, although $\{x_t\}$ is defined by

$$x_t = tf(x_t)+(I-tA)[G(Tx_t)-\theta_t FG(Tx_t)], \quad \forall t \in (0,1), \tag{82}$$

in the proof of Theorem 16, it can be readily seen that p is first found out as a fixed point of the nonexpansive self-mapping W of K. This shows that p depends on no one of the mappings f, A, and F.

Remark 18. Theorem 16 improves, extends, supplements, and develops Cai and Bu [9, Lemma 2.5] in the following aspects.

(i) The GSVI (13) with hierarchical fixed point problem constraint for a nonexpansive mapping is more general and more subtle than the problem in Cai and Bu [9, Lemma 2.5] because our problem is to find a point $p \in \Lambda = \mathrm{Fix}(T) \cap \Omega$, which is the unique solution in Λ to the VIP:

$$\langle (A-f)p, J(p-u)\rangle \leq 0, \quad \forall u \in \Lambda. \tag{83}$$

(ii) The iterative scheme in [9, Lemma 2.5] is extended to develop the iterative scheme in Theorem 16 by virtue of hybrid steepest-descent method. The iterative scheme in Theorem 16 is more advantageous and more flexible than the iterative scheme of [9, Lemma 2.5] because our iterative scheme involves solving two problems: the GSVI (13) and the fixed point problem of a nonexpansive mapping T.

(iii) The iterative scheme in Theorem 16 is very different from the iterative scheme in [9, Lemma 2.5] because our iterative scheme involves hybrid steepest-descent method (namely, we add a strongly accretive and strictly pseudocontractive mapping F in our iterative scheme) and because the mapping T in [9, Lemma 2.5] is replaced by the composite mapping $G \circ T$ in the iterative scheme of Theorem 16.

(iv) The argument techniques of Theorem 16 are very different from Cai and Bu's ones of [9, Lemma 2.5]. Because the composite mapping $G \circ T$ appears in the iterative scheme of Theorem 16, the proof of Theorem 16 depends on the argument techniques in [18], the inequality in 2-uniformly smooth Banach spaces (see Lemma 1), the inequality in smooth and uniform convex Banach spaces (see Proposition 2), and the properties of the strongly positive linear bounded operator (see Lemmas 15), the Banach limit (see Lemma 5), and the strongly accretive and strictly pseudocontractive mapping (see Lemma 7).

4. GSVI with Hierarchical Fixed Point Problem Constraint for a Countable Family of Nonexpansive mappings

In this section, we propose our hybrid explicit viscosity scheme for solving the GSVI (13) with hierarchical fixed point problem constraint for a countable family of nonexpansive mappings and show the strong convergence theorem.

Theorem 19. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X such that $C \pm C \subset C$. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $\Delta = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Let $f : C \to C$ be a fixed contractive map with coefficient $\beta \in (0,1)$, let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha+\lambda > 1$, and let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $0 < \overline{\gamma} - \beta \leq 1$. Given sequences $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ in $[0,1]$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1]$, suppose that there hold the following conditions:*

(i) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;

(ii) $\lim_{n\to\infty}(\lambda_n\mu_n)/\beta_n = 0$;

(iii) $\{\alpha_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$;

(iv) $\sum_{n=0}^{\infty}(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|+|\lambda_{n+1}-\lambda_n|+|\mu_{n+1}-\mu_n|) < \infty$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty}S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then, for any given point $x_0 \in C$, the sequence $\{x_n\}$ generated by

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) G(S_n x_n), \\ x_{n+1} &= \beta_n f(x_n) + (I-\beta_n A) \\ &\quad \times [G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n)], \\ &\quad \forall n \geq 0, \end{aligned} \tag{84}$$

converges strongly to $p \in \Delta$, which is the unique solution in Δ to the VIP:

$$\langle (A-f)p, J(p-u)\rangle \leq 0, \quad \forall u \in \Delta. \tag{85}$$

Proof. First, let us show that $\{x_n\}$ is bounded. Indeed, taking a fixed $u \in \Delta$ arbitrarily, we have

$$\begin{aligned} \|y_n - u\| &= \|\alpha_n x_n + (1-\alpha_n) G(S_n x_n) - u\| \\ &\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|G(S_n x_n) - u\| \\ &\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|S_n x_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|x_n - u\| = \|x_n - u\|. \end{aligned} \tag{86}$$

So, $\|y_n - u\| \leq \|x_n - u\|$ for all $n \geq 0$. Taking into account $\lim_{n\to\infty}(\lambda_n\mu_n)/\beta_n = 0$, we may assume, without loss of generality, that $\lambda_n\mu_n \leq \beta_n \leq \|A\|^{-1}$ for all $n \geq 0$. Thus, by Lemma 7 (ii), we have

$$\begin{aligned}
&\|x_{n+1} - u\| \\
&= \|\beta_n (f(x_n) - f(u)) + (I - \beta_n A) \\
&\quad \times [G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - u] - \beta_n (A - f) u\| \\
&\leq \beta_n \|f(x_n) - f(u)\| \\
&\quad + \|(I - \beta_n A)[G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - u]\| \\
&\quad + \beta_n \|(A - f) u\| \\
&\leq \beta_n\beta \|x_n - u\| + (1 - \beta_n\overline{\gamma}) \\
&\quad \times [\|(I - \lambda_n\mu_n F) G(S_n y_n) - (I - \lambda_n\mu_n F) G(S_n u)\| \\
&\quad\quad + \|(I - \lambda_n\mu_n F) G(S_n u) - u\|] + \beta_n \|(A - f) u\| \\
&\leq \beta_n\beta \|x_n - u\| + (1 - \beta_n\overline{\gamma}) \\
&\quad \times \left(1 - \lambda_n\mu_n \left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \|G(S_n y_n) - G(S_n u)\| \\
&\quad + (1 - \beta_n\overline{\gamma}) \lambda_n\mu_n \|Fu\| + \beta_n \|(A - f) u\| \\
&\leq \beta_n\beta \|x_n - u\| + (1 - \beta_n\overline{\gamma}) \left(1 - \lambda_n\mu_n \left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \\
&\quad \times \|S_n y_n - S_n u\| + \lambda_n\mu_n \|Fu\| + \beta_n \|(A - f) u\| \\
&\leq \beta_n\beta \|x_n - u\| + (1 - \beta_n\overline{\gamma}) \|y_n - u\| \\
&\quad + \lambda_n\mu_n \|Fu\| + \beta_n \|(A - f) u\| \\
&\leq \beta_n\beta \|x_n - u\| + (1 - \beta_n\overline{\gamma}) \|x_n - u\| \\
&\quad + \beta_n \|Fu\| + \beta_n \|(A - f) u\| \\
&= (1 - \beta_n (\overline{\gamma} - \beta)) \|x_n - u\| \\
&\quad + \beta_n (\overline{\gamma} - \beta) \frac{\|(A - f) u\| + \|Fu\|}{\overline{\gamma} - \beta} \\
&\leq \max\left\{\|x_n - u\|, \frac{\|(A - f) u\| + \|Fu\|}{\overline{\gamma} - \beta}\right\},
\end{aligned} \tag{87}$$

By induction,

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{\|(A - f) u\| + \|Fu\|}{\overline{\gamma} - \beta}\right\}, \quad \forall n \geq 0. \tag{88}$$

Thus, $\{x_n\}$ is bounded and so is $\{y_n\}$. Because G and S_n are nonexpansive for all $n \geq 0$, f is contractive, and F is Lipschitzian, $\{S_n x_n\}$, $\{S_n y_n\}$, $\{G(S_n x_n)\}$, $\{G(S_n y_n)\}$, $\{f(x_n)\}$, and $\{FG(S_n y_n)\}$ are bounded. From conditions (i) and (ii) we have

$$\begin{aligned}
&\|x_{n+1} - G(S_n y_n)\| \\
&\quad = \beta_n \|(f(x_n) - AG(S_n y_n)) + (I - \beta_n A) \\
&\quad\quad \times (G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - G(S_n y_n))\| \\
&\quad \leq \beta_n \|f(x_n) - AG(S_n y_n)\| \\
&\quad\quad + (1 - \beta_n\overline{\gamma}) \lambda_n\mu_n \|FG(S_n y_n)\| \\
&\quad \leq \beta_n \|f(x_n) - AG(S_n y_n)\| \\
&\quad\quad + \lambda_n\mu_n \|FG(S_n y_n)\| \longrightarrow 0\,2 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{89}$$

Now, we claim that

$$\|x_{n+1} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{90}$$

In order to prove (90), we estimate $\|x_{n+1} - x_n\|$ first. From (84), we have

$$\begin{gathered}
y_n = \alpha_n x_n + (1 - \alpha_n) G(S_n x_n), \\
y_{n-1} = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1}) G(S_{n-1} x_{n-1}).
\end{gathered} \tag{91}$$

Simple calculations show that

$$\begin{aligned}
y_n - y_{n-1} = {}& (1 - \alpha_n) (G(S_n x_n) - G(S_{n-1} x_{n-1})) \\
& + \alpha_n (x_n - x_{n-1}) + (x_{n-1} - G(S_{n-1} x_{n-1})) \\
& \times (\alpha_n - \alpha_{n-1}).
\end{aligned} \tag{92}$$

It follows that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \leq (1 - \alpha_n) \|G(S_n x_n) - G(S_{n-1} x_{n-1})\| + \alpha_n \|x_n - x_{n-1}\| \\
&\quad\quad + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| \\
&\quad \leq (1 - \alpha_n) \|S_n x_n - S_{n-1} x_{n-1}\| + \alpha_n \|x_n - x_{n-1}\| \\
&\quad\quad + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| \\
&\quad \leq (1 - \alpha_n) (\|S_n x_n - S_n x_{n-1}\| + \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) \\
&\quad\quad + \alpha_n \|x_n - x_{n-1}\| + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| \\
&\quad \leq (1 - \alpha_n) (\|x_n - x_{n-1}\| + \|S_n x_{n-1} - S_{n-1} x_{n-1}\|) \\
&\quad\quad + \alpha_n \|x_n - x_{n-1}\| + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| \\
&\quad \leq \|x_n - x_{n-1}\| + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| \\
&\quad\quad + \|S_n x_{n-1} - S_{n-1} x_{n-1}\|.
\end{aligned} \tag{93}$$

In the meantime, it follows from (84) that

$$\begin{aligned} x_{n+1} &= \beta_n f(x_n) + (I - \beta_n A)\left[G(S_n y_n) - \lambda_n \mu_n FG(S_n y_n)\right], \\ x_n &= \beta_{n-1} f(x_{n-1}) + (I - \beta_{n-1} A) \\ &\quad \times \left[G(S_{n-1} y_{n-1}) - \lambda_{n-1}\mu_{n-1} FG(S_{n-1} y_{n-1})\right]. \end{aligned} \tag{94}$$

Simple calculations show that

$$\begin{aligned} & x_{n+1} - x_n \\ &\quad = (\beta_n - \beta_{n-1}) f(x_{n-1}) + \beta_n (f(x_n) - f(x_{n-1})) \\ &\qquad + (\beta_{n-1} - \beta_n) A (I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1}) \\ &\qquad + (I - \beta_n A)\left[(I - \lambda_n \mu_n F) G(S_n y_n)\right. \\ &\qquad\qquad \left. - (I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\right] \\ &\quad = (\beta_n - \beta_{n-1}) f(x_{n-1}) + \beta_n (f(x_n) - f(x_{n-1})) \\ &\qquad + (\beta_{n-1} - \beta_n) A (I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1}) \\ &\qquad + (I - \beta_n A)\left[(I - \lambda_n \mu_n F) G(S_n y_n)\right. \\ &\qquad\qquad - (I - \lambda_n \mu_n F) G(S_{n-1} y_{n-1}) \\ &\qquad\qquad \left. + (\lambda_{n-1}\mu_{n-1} - \lambda_n \mu_n) FG(S_{n-1} y_{n-1})\right]. \end{aligned} \tag{95}$$

It follows from Lemma 7 (ii) and (93) that

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \|f(x_n) - f(x_{n-1})\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\|(I - \lambda_n \mu_n F) G(S_n y_n)\right. \\ &\qquad\qquad - (I - \lambda_n \mu_n F) G(S_{n-1} y_{n-1})\| \\ &\qquad\qquad \left. + |\lambda_{n-1}\mu_{n-1} - \lambda_n \mu_n| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \beta \|x_n - x_{n-1}\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\right. \\ &\qquad\qquad \times \|G(S_n y_n) - G(S_{n-1} y_{n-1})\| \\ &\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \beta \|x_n - x_{n-1}\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\|S_n y_n - S_{n-1} y_{n-1}\|\right. \\ &\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \beta \|x_n - x_{n-1}\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\|S_n y_n - S_n y_{n-1}\| + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|\right. \\ &\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \beta \|x_n - x_{n-1}\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\|y_n - y_{n-1}\| + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|\right. \\ &\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \beta \|x_n - x_{n-1}\| \\ &\qquad + |\beta_{n-1} - \beta_n| \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\| \\ &\qquad + (1 - \beta_n \overline{\gamma})\left[\|x_n - x_{n-1}\| + \|x_{n-1} - G(S_{n-1} x_{n-1})\|\right. \\ &\qquad\qquad \times |\alpha_n - \alpha_{n-1}| + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\ &\qquad\qquad + \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\ &\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \|FG(S_{n-1} y_{n-1})\|\right] \\ &\quad \le (1 - \beta_n(\overline{\gamma} - \beta)) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \\ &\qquad \times \left(\|f(x_{n-1})\| + \|A(I - \lambda_{n-1}\mu_{n-1} F) G(S_{n-1} y_{n-1})\|\right) \\ &\qquad + \|x_{n-1} - G(S_{n-1} x_{n-1})\| |\alpha_n - \alpha_{n-1}| + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| \\ &\qquad \times \|FG(S_{n-1} y_{n-1})\| + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\ &\qquad + \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\ &\quad \le (1 - \beta_n(\overline{\gamma} - \beta)) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_0 \\ &\qquad + M_0 |\alpha_n - \alpha_{n-1}| \\ &\qquad + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}| M_0 + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\ &\qquad + \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\ &\quad = (1 - \beta_n(\overline{\gamma} - \beta)) \|x_n - x_{n-1}\| \\ &\qquad + M_0 \left(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n \mu_n - \lambda_{n-1}\mu_{n-1}|\right) \\ &\qquad + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|, \\ &\quad \le (1 - \beta_n(\overline{\gamma} - \beta)) \|x_n - x_{n-1}\| \\ &\qquad + M_0 \left(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}|\right. \\ &\qquad\qquad \left. + |\mu_n - \mu_{n-1}|\right) + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\ &\qquad + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|, \end{aligned} \tag{96}$$

where $\sup_{n\geq 0}\{\|f(x_n)\|+\|A(I-\lambda_n\mu_n F)G(S_n y_n)\|+\|FG(S_n y_n)\|+\|x_n - G(S_n x_n)\|\} \leq M_0$ for some $M_0 > 0$. Since it follows from conditions (i) and (iv) that $\sum_{n=0}^{\infty}\beta_n(\overline{\gamma}-\beta) = \infty$ and

$$\sum_{n=0}^{\infty} M_0 \left(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|\right) < \infty, \tag{97}$$

applying Lemma 3 to (96), we obtain from the assumption on $\{S_n\}$ that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{98}$$

By condition (iii) and (84), we have

$$\begin{aligned} \|y_n - x_n\| &= (1-\alpha_n)\|G(S_n x_n) - x_n\| \\ &\leq (1-a)\left(\|G(S_n x_n) - G(S_n y_n)\| + \|G(S_n y_n) - x_{n+1}\| + \|x_{n+1} - x_n\|\right) \\ &\leq (1-a)\left(\|x_n - y_n\| + \|G(S_n y_n) - x_{n+1}\| + \|x_{n+1} - x_n\|\right), \end{aligned} \tag{99}$$

which implies that

$$\|y_n - x_n\| \leq \frac{1-a}{a}\left(\|G(S_n y_n) - x_{n+1}\| + \|x_{n+1} - x_n\|\right). \tag{100}$$

This together with (89)-(90) implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{101}$$

So, we obtain

$$\begin{aligned} \|x_n - G(S_n x_n)\| &\leq \|x_n - y_n\| + \|y_n - G(S_n x_n)\| \\ &\leq \|x_n - y_n\| + \alpha_n \|x_n - G(S_n x_n)\| \\ &\leq \|x_n - y_n\| + b\|x_n - G(S_n x_n)\|, \end{aligned} \tag{102}$$

which implies that

$$\|x_n - G(S_n x_n)\| \leq \frac{1}{1-b}\|x_n - y_n\|, \tag{103}$$

and hence

$$\lim_{n\to\infty} \|x_n - G(S_n x_n)\| = 0. \tag{104}$$

Let $u \in \Delta$. Now, we show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$.

Indeed, for simplicity, put $v = \Pi_C(u - \mu_2 B_2 u)$, $\widehat{x}_n = S_n x_n$, $u_n = \Pi_C(\widehat{x}_n - \mu_2 B_2 \widehat{x}_n)$, and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then, $u = \Pi_C(v - \mu_1 B_1 v)$ and $v_n = G\widehat{x}_n = G(S_n x_n)$ for all $n \geq 0$. It is clear from (84) that

$$\begin{aligned} \|y_n - u\|^2 &\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|G(S_n x_n) - u\|^2 \\ &= \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|v_n - u\|^2. \end{aligned} \tag{105}$$

Utilizing Lemma 12, we have

$$\begin{aligned} \|u_n - v\|^2 &= \|\Pi_C(\widehat{x}_n - \mu_2 B_2 \widehat{x}_n) - \Pi_C(u - \mu_2 B_2 u)\|^2 \\ &\leq \|\widehat{x}_n - u - \mu_2(B_2 \widehat{x}_n - B_2 u)\|^2 \\ &\leq \|\widehat{x}_n - u\|^2 - 2\mu_2(\alpha_2 - \kappa^2 \mu_2)\|B_2 \widehat{x}_n - B_2 u\|^2, \end{aligned} \tag{106}$$

$$\begin{aligned} \|v_n - u\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(v - \mu_1 B_1 v)\|^2 \\ &\leq \|u_n - v - \mu_1(B_1 u_n - B_1 v)\|^2 \\ &\leq \|u_n - v\|^2 - 2\mu_1(\alpha_1 - \kappa^2 \mu_1)\|B_1 u_n - B_1 v\|^2. \end{aligned} \tag{107}$$

Substituting (106) for (107), we obtain

$$\begin{aligned} \|v_n - u\|^2 &\leq \|\widehat{x}_n - u\|^2 - 2\mu_2(\alpha_2 - \kappa^2\mu_2) \times \|B_2 \widehat{x}_n - B_2 u\|^2 - 2\mu_1(\alpha_1 - \kappa^2\mu_1) \times \|B_1 u_n - B_1 v\|^2 \\ &\leq \|x_n - u\|^2 - 2\mu_2(\alpha_2 - \kappa^2\mu_2) \times \|B_2 \widehat{x}_n - B_2 u\|^2 - 2\mu_1(\alpha_1 - \kappa^2\mu_1) \times \|B_1 u_n - B_1 v\|^2, \end{aligned} \tag{108}$$

which together with (105) implies that

$$\begin{aligned} \|y_n - u\|^2 &\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|v_n - u\|^2 \\ &\leq \alpha_n \|x_n - u\|^2 + (1-\alpha_n) \times \left[\|x_n - u\|^2 - 2\mu_2(\alpha_2 - \kappa^2\mu_2) \times \|B_2 \widehat{x}_n - B_2 u\|^2 - 2\mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1 u_n - B_1 v\|^2\right] \\ &= \|x_n - u\|^2 - 2(1-\alpha_n) \times \left[\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2 \widehat{x}_n - B_2 u\|^2 + \mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1 u_n - B_1 v\|^2\right]. \end{aligned} \tag{109}$$

It immediately follows that

$$\begin{aligned} &2(1-\alpha_n)\left[\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2 \widehat{x}_n - B_2 u\|^2 + \mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1 u_n - B_1 v\|^2\right] \\ &\leq \|x_n - u\|^2 - \|y_n - u\|^2 \\ &\leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|. \end{aligned} \tag{110}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$, we deduce from (101) and condition (iii) that

$$\lim_{n\to\infty} \|B_2\widehat{x}_n - B_2u\| = 0, \qquad \lim_{n\to\infty} \|B_1u_n - B_1v\| = 0. \tag{111}$$

Utilizing Proposition 2 and Lemma 8, we have that there exists g_1 such that

$$\begin{aligned}
\|u_n - v\|^2 &= \|\Pi_C(\widehat{x}_n - \mu_2 B_2\widehat{x}_n) - \Pi_C(u - \mu_2 B_2 u)\|^2 \\
&\leq \langle \widehat{x}_n - \mu_2 B_2\widehat{x}_n - (u - \mu_2 B_2 u), J(u_n - v)\rangle \\
&= \langle \widehat{x}_n - u, J(u_n - v)\rangle \\
&\quad + \mu_2 \langle B_2u - B_2\widehat{x}_n, J(u_n - v)\rangle \\
&\leq \frac{1}{2}\left[\|\widehat{x}_n - u\|^2 + \|u_n - v\|^2 \right.\\
&\qquad \left. - g_1(\|\widehat{x}_n - u_n - (u - v)\|)\right] \\
&\quad + \mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\|,
\end{aligned} \tag{112}$$

which implies that

$$\begin{aligned}
\|u_n - v\|^2 \leq{}& \|\widehat{x}_n - u\|^2 - g_1(\|\widehat{x}_n - u_n - (u - v)\|) \\
&+ 2\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\|.
\end{aligned} \tag{113}$$

In the same way, we derive that there exists g_2 such that

$$\begin{aligned}
\|v_n - u\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(v - \mu_1 B_1 v)\|^2 \\
&\leq \langle u_n - \mu_1 B_1 u_n - (v - \mu_1 B_1 v), J(v_n - u)\rangle \\
&= \langle u_n - v, J(v_n - u)\rangle + \mu_1 \langle B_1 v - B_1 u_n, J(v_n - u)\rangle \\
&\leq \frac{1}{2}\left[\|u_n - v\|^2 + \|v_n - u\|^2 \right.\\
&\qquad \left. - g_2(\|u_n - v_n + (u - v)\|)\right] \\
&\quad + \mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|,
\end{aligned} \tag{114}$$

which implies that

$$\begin{aligned}
\|v_n - u\|^2 \leq{}& \|u_n - v\|^2 - g_2(\|u_n - v_n + (u - v)\|) \\
&+ 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|.
\end{aligned} \tag{115}$$

Substituting (113) for (115), we get

$$\begin{aligned}
&\|v_n - u\|^2 \\
&\quad \leq \|\widehat{x}_n - u\|^2 - g_1(\|\widehat{x}_n - u_n - (u - v)\|) \\
&\qquad - g_2(\|u_n - v_n + (u - v)\|) \\
&\qquad + 2\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad + 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\| \\
&\quad \leq \|x_n - u\|^2 - g_1(\|\widehat{x}_n - u_n - (u - v)\|) \\
&\qquad - g_2(\|u_n - v_n + (u - v)\|) \\
&\qquad + 2\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad + 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|,
\end{aligned} \tag{116}$$

which together with (105) implies that

$$\begin{aligned}
\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|v_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n) \\
&\quad \times \left[\|x_n - u\|^2 - g_1(\|\widehat{x}_n - u_n - (u - v)\|)\right. \\
&\qquad - g_2(\|u_n - v_n + (u - v)\|) \\
&\qquad + 2\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad \left. + 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|\right] \\
&= \|x_n - u\|^2 - (1 - \alpha_n) \\
&\quad \times \left[g_1(\|\widehat{x}_n - u_n - (u - v)\|)\right. \\
&\qquad \left. + g_2(\|u_n - v_n + (u - v)\|)\right] + 2(1 - \alpha_n) \\
&\quad \times (\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad + \mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|).
\end{aligned} \tag{117}$$

It immediately follows that

$$\begin{aligned}
&(1 - \alpha_n)\left[g_1(\|\widehat{x}_n - u_n - (u - v)\|)\right. \\
&\qquad \left. + g_2(\|u_n - v_n + (u - v)\|)\right] \\
&\quad \leq \|x_n - u\|^2 - \|y_n - u\|^2 + 2(1 - \alpha_n) \\
&\qquad \times (\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad\quad + \mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|) \\
&\quad \leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\| \\
&\qquad + 2\mu_2 \|B_2u - B_2\widehat{x}_n\| \|u_n - v\| \\
&\qquad + 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|.
\end{aligned} \tag{118}$$

Since $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ are bounded, we deduce from (101), (111), and condition (iii) that

$$\lim_{n\to\infty} g_1\left(\|\hat{x}_n - u_n - (u-v)\|\right) = 0,$$
$$\lim_{n\to\infty} g_2\left(\|u_n - v_n + (u-v)\|\right) = 0. \tag{119}$$

Utilizing the properties of g_1 and g_2, we get

$$\lim_{n\to\infty} \|\hat{x}_n - u_n - (u-v)\| = 0,$$
$$\lim_{n\to\infty} \|u_n - v_n + (u-v)\| = 0, \tag{120}$$

which hence yields

$$\begin{aligned}\|\hat{x}_n - v_n\| &\le \|\hat{x}_n - u_n - (u-v)\| \\ &\quad + \|u_n - v_n + (u-v)\| \longrightarrow 0 \\ &\quad \text{as } n \to \infty.\end{aligned} \tag{121}$$

That is,

$$\lim_{n\to\infty} \|S_n x_n - G(S_n x_n)\| = \lim_{n\to\infty} \|\hat{x}_n - v_n\| = 0. \tag{122}$$

Note that

$$\|x_n - S_n x_n\| \le \|x_n - G(S_n x_n)\| + \|G(S_n x_n) - S_n x_n\|. \tag{123}$$

So, from (104) and (122), we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0, \tag{124}$$

which together with (104) and the assumption on $\{S_n\}$ implies that

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \longrightarrow 0 \\ &\quad \text{as } n \longrightarrow \infty, \\ \|x_n - Gx_n\| &\le \|x_n - G(S_n x_n)\| + \|G(S_n x_n) - Gx_n\| \\ &\le \|x_n - G(S_n x_n)\| + \|S_n x_n - x_n\| \longrightarrow 0 \\ &\quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{125}$$

That is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{126}$$

Define a mapping

$$Wx = (1-\theta) Sx + \theta Gx, \quad \forall x \in C, \tag{127}$$

where θ is a constant in $(0,1)$. Then, by Lemma 10, we know that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = \Delta$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \|(1-\theta)(x_n - Sx_n) + \theta(x_n - Gx_n)\| \\ &\le (1-\theta)\|x_n - Sx_n\| + \|\theta x_n - Gx_n\|.\end{aligned} \tag{128}$$

So, from (126), we get

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0, \tag{129}$$

where p is defined below. Now, we claim that

$$\limsup_{n\to\infty} \langle (f-A)p, J(x_n - p)\rangle \le 0. \tag{130}$$

Indeed, let $\{x_t\}$ be defined by

$$x_t = tf(x_t) + (I - tA) Wx_t. \tag{131}$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to $p \in \mathrm{Fix}(W) = \Delta$, which by Proposition CB is the unique solution in Δ to the VIP:

$$\langle (A-f)p, J(p-u)\rangle \le 0, \quad \forall u \in \Delta. \tag{132}$$

In terms of Lemma 6, we conclude from (129) that (130) holds. It is clear that

$$\limsup_{n\to\infty} \langle (f-A)p, J(x_{n+1} - p)\rangle \le 0. \tag{133}$$

Finally, let us show that $x_n \to p$ as $n \to \infty$. We observe that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|G(S_n x_n) - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2,\end{aligned} \tag{134}$$

and hence

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \|\beta_n(f(x_n) - f(p)) + (I - \beta_n A) \\ &\qquad \times [G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - p] + \beta_n(f-A)p\|^2 \\ &\quad \le \|\beta_n(f(x_n) - f(p)) + (I - \beta_n A) \\ &\qquad \times [G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - p]\|^2 \\ &\qquad + 2\beta_n \langle (f-A)p, J(x_{n+1} - p)\rangle \\ &\quad \le [\beta_n \|f(x_n) - f(p)\| + (1 - \beta_n\bar{\gamma}) \\ &\qquad\quad \times \|G(S_n y_n) - \lambda_n\mu_n FG(S_n y_n) - p\|]^2 \\ &\qquad + 2\beta_n \langle (f-A)p, J(x_{n+1} - p)\rangle \\ &\quad = [\beta_n \|f(x_n) - f(p)\| + (1 - \beta_n\bar{\gamma}) \\ &\qquad\quad \times \|(I - \lambda_n\mu_n F) G(S_n y_n) \\ &\qquad\qquad - (I - \lambda_n\mu_n F)p - \lambda_n\mu_n Fp\|]^2 \\ &\qquad + 2\beta_n \langle (f-A)p, J(x_{n+1} - p)\rangle \\ &\quad \le [\beta_n \|\beta x_n - p\| + (1 - \beta_n\bar{\gamma}) \\ &\qquad\quad \times (\|(I - \lambda_n\mu_n F) G(S_n y_n) - (I - \lambda_n\mu_n F)p\| \\ &\qquad\qquad + \lambda_n\mu_n \|Fp\|)]^2 + 2\beta_n \langle (f-A)p, J(x_{n+1} - p)\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \Bigg[\beta_n\beta\left\|x_n-p\right\|+\left(1-\beta_n\overline{\gamma}\right) \\
&\qquad\times\Bigg(\Bigg(1-\lambda_n\mu_n\Bigg(1-\sqrt{\frac{1-\alpha}{\lambda}}\Bigg)\Bigg) \\
&\qquad\qquad\times\left\|G\left(S_ny_n\right)-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\Bigg)\Bigg]^2 \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&\leq \left[\beta_n\beta\left\|x_n-p\right\|\right. \\
&\qquad\left.+\left(1-\beta_n\overline{\gamma}\right)\left(\left\|y_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right)\right]^2 \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&\leq \left[\beta_n\beta\left\|x_n-p\right\|+\left(1-\beta_n\overline{\gamma}\right)\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right]^2 \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&= \left[\left(1-\beta_n\left(\overline{\gamma}-\beta\right)\right)\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right]^2 \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&= \left(1-\beta_n\left(\overline{\gamma}-\beta\right)\right)^2\left\|x_n-p\right\|^2+\lambda_n\mu_n\left\|Fp\right\| \\
&\qquad\times\left[2\left(1-\beta_n\left(\overline{\gamma}-\beta\right)\right)\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right] \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&\leq \left(1-\beta_n\left(\overline{\gamma}-\beta\right)\right)\left\|x_n-p\right\|^2 \\
&\quad+\lambda_n\mu_n\left\|Fp\right\|\left(2\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right) \\
&\quad+2\beta_n\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle \\
&= \left(1-\beta_n\left(\overline{\gamma}-\beta\right)\right)\left\|x_n-p\right\|^2+\beta_n\left(\overline{\gamma}-\beta\right)\cdot\frac{1}{\overline{\gamma}-\beta} \\
&\quad\times\left[\frac{\lambda_n\mu_n}{\beta_n}\left\|Fp\right\|\left(2\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right)\right. \\
&\qquad\left.+2\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle\right].
\end{aligned}
\tag{135}
$$

Taking into account (133) and conditions (i) and (ii), we obtain that $\sum_{n=0}^{\infty}(\overline{\gamma}-\beta)\beta_n=\infty$ and

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{1}{\overline{\gamma}-\beta}&\left[\frac{\lambda_n\mu_n}{\beta_n}\left\|Fp\right\|\left(2\left\|x_n-p\right\|+\lambda_n\mu_n\left\|Fp\right\|\right)\right. \\
&\left.+2\left\langle\left(f-A\right)p,J\left(x_{n+1}-p\right)\right\rangle\right]\leq 0.
\end{aligned}
\tag{136}
$$

Therefore, applying Lemma 3 to (135), we infer that

$$
\lim_{n\to\infty}\left\|x_n-p\right\|=0. \tag{137}
$$

This completes the proof. □

Remark 20. It is worth pointing out that the sequences $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\beta_n\}$ can be taken, which satisfy the conditions in Theorem 19. As a matter of fact, put $\lambda_n=(1+n)^{-5/6}$, $\mu_n=1$, and $\beta_n=(1+n)^{-2/3}$ for all $n\geq 0$. Then, there hold the following statements:

(i) $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=0}^{\infty}\beta_n=\infty$;

(ii) $\lim_{n\to\infty}(\lambda_n\mu_n)/\beta_n=0$;

(iii) $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, and $\sum_{n=0}^{\infty}|\mu_{n+1}-\mu_n|<\infty$.

By the careful analysis of the proof of Theorem 19, we can obtain the following result. Because its proof is much simpler than that of Theorem 19, we omit its proof.

Theorem 21. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X such that $C\pm C\subset C$. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i: C\to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $\Delta=\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\Omega\neq\emptyset$, where Ω is the fixed point set of the mapping $G=\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$ with $0<\mu_i<\alpha_i/\kappa^2$ for $i=1,2$. Let $f: C\to C$ be a fixed contractive map with coefficient $\beta\in(0,1)$, let $F: C\to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha+\lambda>1$, and let $A: C\to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $0<\overline{\gamma}-\beta\leq 1$. Given sequences $\{\lambda_n\}_{n=0}^{\infty}$ in $[0,1]$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1]$, suppose that there hold the following conditions:*

(i) *$\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=0}^{\infty}\beta_n=\infty$;*

(ii) *$\lim_{n\to\infty}\lambda_n/\beta_n=0$ and $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$;*

(iii) *$\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$;*

(iv) *$\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$.*

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x-S_nx\|<\infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx=\lim_{n\to\infty}S_nx$ for all $x\in C$ and suppose that $\mathrm{Fix}(S)=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then, for any given point $x_0\in C$, the sequence $\{x_n\}$ generated by

$$
\begin{aligned}
y_n&=\alpha_nx_n+\left(1-\alpha_n\right)G\left(S_nx_n\right), \\
x_{n+1}&=\beta_nf\left(x_n\right)+\left(I-\beta_nA\right)\left[y_n-\lambda_nF\left(y_n\right)\right],\quad\forall n\geq 0,
\end{aligned}
\tag{138}
$$

converges strongly to $p\in\Delta$, which is the unique solution in Δ to the VIP (85).

Remark 22. Theorems 19 and 21 improve, extend, supplement and develop Cai and Bu [10, Theorem 3.1] and Cai and Bu [9, Theorems 3.1] in the following aspects.

(i) The GSVI (13) with hierarchical fixed point problem constraint for a countable family of nonexpansive mappings is more general and more subtle than every problem in Cai and Bu [10, Theorems 3.1] and Cai and Bu [9, Theorem 3.1] because our problem is to find a point $p \in \Delta = \bigcap_n \mathrm{Fix}(S_n) \cap \Omega$, which is the unique solution in Δ to the VIP:

$$\langle (A - f) p, J(p - u) \rangle \leq 0, \quad \forall u \in \Delta. \tag{139}$$

(ii) The iterative scheme in [10, Theorem 3.1] is extended to develop the iterative schemes in Theorems 19 and 21 by virtue of hybrid steepest-descent method. The iterative schemes in Theorems 19 and 21 are more advantageous and more flexible than the iterative scheme of [9, Theorem 3.1] because the iterative scheme of [9, Theorem 3.1] is implicit and our iterative schemes involve solving two problems: the GSVI (13) and the fixed point problem of a countable family of nonexpansive mappings $\{S_n\}$.

(iii) The iterative schemes in Theorems 19 and 21 are very different from everyone in both [10, Theorem 3.1] and [9, Theorem 3.1] because our iterative schemes involve hybrid steepest-descent method (namely, we add a strongly accretive and strictly pseudocontractive mapping F in our iterative schemes) and because the mappings G and S_n in [10, Theorem 3.1] and the mapping S_n in [9, Theorem 3.1] are replaced by the same composite mapping $G \circ S_n$ in the iterative schemes of Theorems 19 and 21.

(iv) Cai and Bu's proof in [10, Theorem 3.1] depends on the argument techniques in [20], the inequality in 2-uniformly smooth Banach spaces (see Lemma 1), and the inequality in smooth and uniform convex Banach spaces (see Proposition 2). Because the composite mapping $G \circ S_n$ appears in the iterative schemes in Theorems 19 and 21, the proof of Theorems 19 and 21 depends on the argument techniques in [20], the inequality in 2-uniformly smooth Banach spaces (see Lemma 1), the inequality in smooth and uniform convex Banach spaces (see Proposition 2), and the properties of the strongly positive linear bounded operator (see Lemmas 15), the Banach limit (see Lemma 5), and the strongly accretive and strictly pseudocontractive mapping (see Lemma 7).

Remark 23. Theorems 19 and 21 extend and improve Theorem 16 of Yao et al. [21] to a great extent in the following aspects:

(i) the u is replaced by a fixed contractive mapping;

(ii) one continuous pseudocontractive mapping (including nonexpansive mapping) is replaced by a countable family of nonexpansive mappings;

(iii) we add a strongly positive linear bounded operator A and a strongly accretive and strictly pseudocontractive mapping F in our iterative algorithms.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for the technical and financial support. This research was partially supported to first author by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), and Ph.D. Program Foundation of Ministry of Education of China (20123127110002). Finally, the authors thank the referees for their valuable comments and suggestions for improvement of the paper.

References

[1] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings," *Nonlinear Analysis A*, vol. 61, no. 3, pp. 341–350, 2005.

[2] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "An extragradient method for solving split feasibility and fixed point problems," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 633–642, 2012.

[3] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem," *Nonlinear Analysis A*, vol. 75, no. 4, pp. 2116–2125, 2012.

[4] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Relaxed extragradient iterative methods for variational inequalities," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1112–1123, 2011.

[5] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 29, no. 9-10, pp. 987–1033, 2008.

[6] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[7] H. Iiduka, W. Takahashi, and M. Toyoda, "Approximation of solutions of variational inequalities for monotone mappings," *Panamerican Mathematical Journal*, vol. 14, no. 2, pp. 49–61, 2004.

[8] Y. Takahashi, K. Hashimoto, and M. Kato, "On sharp uniform convexity, smoothness, and strong type, cotype inequalities," *Journal of Nonlinear and Convex Analysis*, vol. 3, no. 2, pp. 267–281, 2002.

[9] G. Cai and S. Bu, "Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 1998–2004, 2011.

[10] G. Cai and S. Bu, "Convergence analysis for variational inequality problems and fixed point problems in 2-uniformly smooth and uniformly convex Banach spaces," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 538–546, 2012.

[11] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis A*, vol. 16, no. 12, pp. 1127–1138, 1991.

[12] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[13] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[14] W. Takahashi, *Nonlinear Functional Analysis—Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000, Japanese.

[15] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[16] R. E. Bruck, Jr., "Properties of fixed-point sets of nonexpansive mappings in Banach spaces," *Transactions of the American Mathematical Society*, vol. 179, pp. 251–262, 1973.

[17] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis A*, vol. 67, no. 8, pp. 2350–2360, 2007.

[18] L.-C. Ceng and J.-C. Yao, "An extragradient-like approximation method for variational inequality problems and fixed point problems," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 205–215, 2007.

[19] Y. Censor, A. Gibali, and S. Reich, "Two extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space," Technical Report, 2010.

[20] L.-C. Ceng, C.-y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[21] Y. Yao, Y.-C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis A*, vol. 67, no. 12, pp. 3311–3317, 2007.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 283158, 12 pages
http://dx.doi.org/10.1155/2014/283158

Research Article

$\mathcal{J}\mathcal{H}$-Operator Pairs with Application to Functional Equations Arising in Dynamic Programming

A. Razani and B. Moeini

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

Correspondence should be addressed to A. Razani; razani@ipm.ir

Received 16 July 2013; Revised 21 December 2013; Accepted 21 December 2013; Published 13 February 2014

Academic Editor: Hichem Ben-El-Mechaiekh

Some common fixed point theorems for $\mathcal{J}\mathcal{H}$-operator pairs are proved. As an application, the existence and uniqueness of the common solution for systems of functional equations arising in dynamic programming are discussed. Also, an example to validate all the conditions of the main result is presented.

1. Introduction and Preliminaries

Jungck [1] introduced compatible mappings as a generalization of weakly commuting mappings. Jungck and Pathak [2] defined the concept of the biased mappings in order to generalize the concept of compatible mappings. Also, several authors [3–6] studied various classes of compatible mappings and proved common fixed point theorems for these classes. Recently, Hussain et al. [7] introduced $\mathcal{J}\mathcal{H}$-operator pairs as a new class of noncommuting self-mappings that contains the occasionally weakly compatible, and Sintunavarat and Kumam [8] introduced generalized $\mathcal{J}\mathcal{H}$-operator pairs that contain $\mathcal{J}\mathcal{H}$-operator pairs. On the other hand, fixed point theory has various applications in other fields, for instance, obtaining a solution of several classes of functional equations (or a system of functional equations) arising in dynamic programming (see [9–12]). Bellman and Lee [13], Zhang [14], and Chang and Ma [15] point out that the basic form of the functional equations of dynamic programming and the system of functional equations of dynamic programming are as follows:

$$f(x) = \sup_{y\in D} H(x, y, f(T(x, y))), \quad \forall x \in S,$$

$$f(x) = \sup_{y\in D} \{u(x, y) + G(x, y, g(T(x, y)))\}, \quad \forall x \in S,$$

$$g(x) = \sup_{y\in D} \{u(x, y) + F(x, y, f(T(x, y)))\}, \quad \forall x \in S. \tag{1}$$

In this presented work, $\mathcal{J}\mathcal{H}$-operator pairs are compared with the various type of compatible mappings and it is shown that the $\mathcal{J}\mathcal{H}$-operator pairs reduce to symmetric Banach operator pairs under relaxed conditions. We omit the completeness condition of the space. Then some common fixed point theorems are proved for $\mathcal{J}\mathcal{H}$-operator pairs. Eventually, the results are used to show the existence and uniqueness of common solution for systems of functional equations without completeness of the space.

The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of S and T if $Sx = Tx(x = Sx = Tx)$. Let $C(S,T)$, $PC(S,T)$ denote the sets of all coincidence points and points of coincidence, respectively, of the pair (S,T). The pair (S,T) is called a Banach operator pair if the set $F(T)$ is S-invariant, namely, $S(F(T)) \subseteq F(T)$. If (S,T) is a Banach operator pair, then (T,S) need not be a Banach operator pair. Let (X,d) be a metric space and f,S self-mappings on X; the pair (f,S) is called as follows:

(0) symmetric Banach operator if both (f,S) and (S,f) are Banach operator pairs [16];

(1) compatible if $d(fSx_n, Sfx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that fx_n and $Sx_n \to t \in X$ [1];

(2) $\mathscr{P}$-operator pair if $d(x, Sx) \leq \operatorname{diam}(C(f,S))$, for some $x \in C(f,S)$ [17];

(3) $\mathscr{J}\mathscr{H}$-operator pair if there exists a point $w = fx = Sx$ in $PC(f,S)$ such that

$$d(w,x) \leq \operatorname{diam}(PC(f,S)); \tag{2}$$

see [7];

(4) compatible of type (A) if

$$d(fSx_n, SSx_n) \longrightarrow 0, \quad d(Sfx_n, ffx_n) \longrightarrow 0, \tag{3}$$

whenever $\{x_n\}$ is a sequence in X such that fx_n and $Sx_n \to t \in X$ [6];

(5) weakly S-biased of type (A) if $fp = Sp$ implies that

$$d(SSp, fp) \leq d(fSp, Sp); \tag{4}$$

see [18];

(6) compatible of type (B) if

$$\begin{aligned}
&\lim_{n\to\infty} d(Sfx_n, ffx_n) \\
&\quad \leq \frac{1}{2}\left[\lim_{n\to\infty} d(Sfx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right], \\
&\lim_{n\to\infty} d(fSx_n, SSx_n) \\
&\quad \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fSx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\right],
\end{aligned} \tag{5}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [3];

(7) compatible of type (P) if

$$\lim_{n\to\infty} d(ffx_n, SSx_n) = 0, \tag{6}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [4];

(8) compatible of type (C) if

$$\begin{aligned}
&\lim_{n\to\infty} d(Sfx_n, ffx_n) \\
&\quad \leq \frac{1}{3}\left[\lim_{n\to\infty} d(Sfx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right. \\
&\qquad \left. + \lim_{n\to\infty} d(St, ffx_n)\right], \\
&\lim_{n\to\infty} d(fSx_n, SSx_n) \\
&\quad \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fSx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\right. \\
&\qquad \left. + \lim_{n\to\infty} d(ft, SSx_n)\right],
\end{aligned} \tag{7}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [5].

2. $\mathscr{J}\mathscr{H}$-Operator Pair

Proposition 1. *Let f and S be self-mappings of metric space (X,d), and $C(f,S) \neq \emptyset$. If f and S are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair.*

Proof. If f and S are one of the assumptions listed, then f and S are weakly compatible and, hence, they are occasionally weakly compatible; then (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair. □

Notation 1. The following example shows that the converse of Proposition 1 is not true, in general.

Example 2. Suppose that $(X = [0,1], d)$ is a metric space with $d(x,y) = |x - y|$ and f, S are defined by

$$\begin{aligned}
fx &= \begin{cases} x^2, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases} \\
Sx &= \begin{cases} \dfrac{x}{2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}
\end{aligned} \tag{8}$$

Then, $C(f,S) = \{0, 1/2\}$, $PC(f,S) = \{1, 1/4\}$. On the other hand, for $w = 1/4 \in PC(f,S)$ we have $f(1/2) = S(1/2) = 1/4$ and

$$d\left(\frac{1}{2}, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| \leq \operatorname{diam}(PC(f,S)) = \left|1 - \frac{1}{4}\right|. \tag{9}$$

Thus, (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair.

Now, suppose that $\{x_n\}$ is a sequence in $[0,1]$ defined by $x_n = 1/2$. Then, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t = 1/4$, $fx_n = Sx_n = 1/4$, and $fSx_n = 1/16$, $Sfx_n = 1/8$. Since

$$\lim_{n\to\infty} |fSx_n - Sfx_n| = \frac{1}{16} \neq 0, \tag{10}$$

so (f,S) is not compatible.

$SSx_n = 1/8$, $ffx_n = 1/16$. Since

$$\lim_{n\to\infty} |fSx_n - SSx_n| = \frac{1}{16} \neq 0, \tag{11}$$

thus (f,S) is not compatible of type (A).

Since

$$\begin{aligned}
&\lim_{n\to\infty} |fSx_n - SSx_n| \\
&\quad = \frac{1}{16} > \frac{1}{2}\left[\lim_{n\to\infty} |fSx_n - ft| + \lim_{n\to\infty} |ft - ffx_n|\right] = 0,
\end{aligned} \tag{12}$$

then (f,S) is not compatible of type (B).

Since

$$\lim_{n\to\infty} |SSx_n - ffx_n| = \frac{1}{16} \neq 0, \tag{13}$$

thus (f,S) is not compatible of type (P).

Since

$$\lim_{n\to\infty} |fSx_n - SSx_n| = \frac{1}{16} > \frac{1}{3}\left[\lim_{n\to\infty}|fSx_n - ft| + \lim_{n\to\infty}|ft - ffx_n| + \lim_{n\to\infty}|ft - SSx_n|\right] = \frac{1}{48}, \quad (14)$$

therefore, (f, S) is not compatible of type (C).

Proposition 3. *Let f and S be self-mappings of metric space (X, d). If (f, S) is a $\mathcal{JH}$-operator pair such that $PC(f, S)$ is singleton, then (f, S) is symmetric Banach operator pair.*

Proof. By hypothesis, there is a point $fx = Sx = w \in PC(f, S)$ such that $d(x, w) \le \operatorname{diam}(PC(f, S)) = 0$. Thus, $x = w = fx = Sx$ and x is a unique point of $C(f, S)$. Also, by Proposition 2.4 [19] (f, S) is weakly compatible and hence, by Lemma 2.1 [19], $w = x$ is a unique common fixed point of f and S. Now, since the sets $PC(f, S)$ and $C(f, S)$ are singleton, then $F(f) = F(S) = \{x\}$, $f(F(S)) \subseteq F(S)$ and $S(F(f)) \subseteq F(f)$; that is, (f, S) is symmetric Banach operator pair. □

Example 4. Suppose that $(X = [0, 5], d)$ is a metric space with $d(x, y) = |x - y|$ and f, S are defined by

$$fx = \begin{cases} 0, & \text{if } x = 0, \\ x + 4, & \text{if } x \in (0, 1], \\ x - 1, & \text{if } x \in (1, 5], \end{cases} \qquad Sx = \begin{cases} 2, & \text{if } x \in (0, 1], \\ 0, & \text{if } x \in \{0\} \cup (1, 5]. \end{cases} \quad (15)$$

Then $C(f, S) = PC(f, S) = \{0\}$. Clearly (f, S) is $\mathcal{JH}$-operator pair and symmetric Banach operator pair.

Proposition 5. *Let f and S be self-mappings of metric space (X, d). If (f, S) is a $\mathcal{JH}$-operator pair and for all $x, y \in X$ we have*

$$d(fx, fy) \le \phi\left(\max\left\{d(Sx, Sy), d(Sx, fx), d(fy, Sy), \frac{1}{2}(d(Sx, fy) + d(Sy, fx))\right\}\right), \quad (16)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying the condition $\phi(t) < t$ for $t > 0$, then (f, S) is symmetric Banach operator pair.

Proof. Since (f, S) is a $\mathcal{JH}$-operator pair, there is a point $fx = Sx = w$ in $PC(f, S)$ such that $d(x, w) \le \operatorname{diam}(PC(f, S))$. Now, if there is another point $fy = Sy = z$ in $PC(f, S)$ and $z \ne w$, then, by (16),

$$d(w, z) = d(fx, fy) \le \phi\left(\max\left\{d(w, z), 0, 0, \frac{1}{2}(d(w, z) + d(w, z))\right\}\right); \quad (17)$$

therefore, $d(w, z) \le \phi(d(w, z)) < d(w, z)$ which is a contradiction. Then $w = z$, that is, $PC(f, S)$ is singleton and, hence, by Proposition 3 (f, S) is symmetric Banach operator pair. □

Proposition 6. *Let f and S be self-mappings of metric space (X, d). If (f, S) is a $\mathcal{P}$-operator pair such that $C(f, S)$ is singleton, then (f, S) is symmetric Banach operator pair.*

Corollary 7. *Let (f, S) be an occasionally weakly compatible pair of self-mappings on X that $C(f, S)$ is singleton; then (f, S) is symmetric Banach operator pair.*

Proof. Clearly, occasionally weakly compatible mappings are $\mathcal{P}$-operators; then by Proposition 6 the result is obtained. □

3. Common Fixed Point

Definition 8 (see [20]). A function $\psi : [0, \infty] \to [0, \infty]$ is called an altering distance function if

(i) ψ is monotone increasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 9. *Suppose that S and T are self-mappings of metric space (X, d). The pair (S, T) is a $\mathcal{JH}$-operator pair and, for all $x, y \in X$,*

$$\psi(d(Sx, Ty)) \le \psi\left(\max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{1}{2}[d(Tx, Ty) + d(Sy, Tx)]\right\}\right) - \phi(\max\{d(Sx, Sy), d(Tx, Sx), d(Sy, Ty)\}), \quad (18)$$

where ψ is an altering distance function and $\phi : [0, \infty] \to [0, \infty]$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then S and T have a unique common fixed point. Moreover, any fixed point of S is a fixed point of T and conversely.

Proof. By hypothesis, there exists a point $w \in X$ such that $w = Sx = Tx$ and

$$d(w, x) \le \operatorname{diam}(PC(S, T)). \quad (19)$$

Suppose that there exists another point $z \in X$ and $z \ne w$, for which $z = Sy = Ty$. Then, from (18), we get

$$\psi(d(w, z)) \le \psi\left(\max\left\{d(w, z), 0, 0, \frac{1}{2}[d(w, z) + d(z, w)]\right\}\right) - \phi(\max\{d(w, z), 0, 0\}); \quad (20)$$

accordingly, $\psi(d(w, z)) \le \psi(d(w, z)) - \phi(d(w, z))$, which is a contradiction with definition of ϕ. Therefore, $PC(S, T)$ is singleton so $\operatorname{diam}(PC(S,T)) = 0$. By using (19), $d(w,x) \leq \operatorname{diam}(PC(S,T)) = 0$; thus, $w = x$; that is, x is a unique common fixed point of S and T.

Now, suppose that u is a fixed point of S but $u \neq Tu$, from (18),

$$\begin{aligned}
&\psi(d(u,Tu)) \\
&\quad = \psi(d(Su,Tu)) \\
&\quad \leq \psi\left(\max\left\{0, d(Tu,u), d(u,Tu), \frac{1}{2}[0 + d(u,Tu)]\right\}\right) \\
&\qquad - \phi(\max\{0, d(u,Tu), d(u,Tu)\});
\end{aligned} \tag{21}$$

thus, $\psi(d(u,Tu)) \leq \psi(d(u,Tu)) - \phi(d(u,Tu))$, which is a contradiction with definition of ϕ. Hence, $u = Tu$. By using a similar argument, the conclusion will be obtained. □

Example 10. Suppose that $X = \{0,2,4,6,\ldots\}$ and $d : X \times X \to \mathbb{R}$ is given by

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{22}$$

Then (X,d) is a metric space.

Let $\psi : [0,\infty) \to [0,\infty)$ be defined as

$$\psi(t) = 2t^2, \quad \text{for } t \in [0,\infty). \tag{23}$$

Suppose that $\phi : [0,\infty) \to [0,\infty)$ is defined as

$$\phi(s) = \begin{cases} s, & \text{if } s \leq 1, \\ 1, & \text{if } s > 1. \end{cases} \tag{24}$$

Then $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Let $S, T : X \to X$ be defined as

$$\begin{aligned}
Sx &= \begin{cases} 2x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \\
Tx &= \begin{cases} 2x-2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}
\end{aligned} \tag{25}$$

Now, we have the following cases for $x, y \in X$.

Case 1. $x \neq y$.

(i) If $y \neq 0$ and $x > y$, then

$$\begin{aligned}
&\psi(d(Sx,Ty)) \\
&\quad = \psi(d(2x,2y-2)) = \psi(2x+2y-2) \\
&\quad = 8(x+y-1)^2, \\
&\psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty),\right.\right. \\
&\qquad \left.\left.\frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\right\}\right) \\
&\quad = \psi(\max\{2x+2y, 4x-2, 4y-2, \\
&\qquad 2x+2y-3\}) \\
&\quad = \psi(4x-2) = 2(4x-2)^2, \\
&\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\
&\quad = \phi(\max\{2x+2y, 4x-2, 4y-2\}) \\
&\quad = \phi(4x-2) = 1.
\end{aligned} \tag{26}$$

Since, $8(x+y-1)^2 \leq 2(4x-2)^2 - 1$, then relation (18) is established.

(ii) If $x \neq 0$ and $y > x$, then

$$\begin{aligned}
&\psi(d(Sx,Ty)) \\
&\quad = \psi(d(2x,2y-2)) = \psi(2x+2y-2) = 8(x+y-1)^2, \\
&\psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty),\right.\right. \\
&\qquad \left.\left.\frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\right\}\right) \\
&\quad = \psi(\max\{2x+2y, 4x-2, 4y-2, 2x+2y-3\}) \\
&\quad = \psi(4y-2) = 2(4y-2)^2, \\
&\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\
&\quad = \phi(\max\{2x+2y, 4x-2, 4y-2\}) \\
&\quad = \phi(4y-2) = 1.
\end{aligned} \tag{27}$$

Since, $8(x+y-1)^2 \leq 2(4y-2)^2 - 1$, then relation (18) is established.

(iii) $y = 0$; then

$$\begin{aligned}
&\psi(d(Sx,Ty)) = \psi(d(2x,0)) = \psi(2x) = 8x^2, \\
&\psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty),\right.\right. \\
&\qquad \left.\left.\frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\right\}\right) \\
&\quad = \psi(\max\{2x, 4x-2, 0, 2x-2\}) \\
&\quad = \psi(4x-2) = 2(4x-2)^2, \\
&\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\
&\quad = \phi(\max\{2x, 4x-2, 0\}) = \phi(4x-2) = 1.
\end{aligned} \tag{28}$$

Since, $8x^2 \leq 2(4x-2)^2 - 1$, then relation (18) is established.

(iv) $x = 0$; then

$$\begin{aligned}
&\psi(d(Sx,Ty)) \\
&\quad = \psi(d(0,2y-2)) = \psi(2y-2) = 2(2y-2)^2,
\end{aligned}$$

$$\begin{aligned}
&\psi\Big(\max\Big\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),\\
&\qquad\frac{1}{2}\left[d(Tx,Ty)+d(Sy,Tx)\right]\Big\}\Big)\\
&\quad=\psi\left(\max\{2y,0,4y-2,2y-1\}\right)\\
&\quad=\psi\left(4y-2\right)=2(4y-2)^2,\\
&\phi\left(\max\{d(Sx,Sy),d(Tx,Sx),d(Sy,Ty)\}\right)\\
&\quad=\phi\left(\max\{2y,0,4y-2\}\right)=\phi\left(4y-2\right)=1.
\end{aligned} \tag{29}$$

Since, $2(2y-2)^2 \le 2(4y-2)^2-1$, then relation (18) is established.

Case 2. $x = y$.

In this case, it is easy to see that the relation (18) is hold.

Therefore, for all $x, y \in X$,

$$\begin{aligned}
&\psi\left(d(Sx,Ty)\right)\\
&\quad\le\psi\Big(\max\Big\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),\\
&\qquad\frac{1}{2}\left[d(Tx,Ty)+d(Sy,Tx)\right]\Big\}\Big)\\
&\qquad-\phi\left(\max\{d(Sx,Sy),d(Tx,Sx),d(Sy,Ty)\}\right).
\end{aligned} \tag{30}$$

Accordingly, the conditions of Theorem 9 are satisfied and 0 is the unique common fixed point of S and T.

Suppose that Φ is the collection of mappings $\phi : [0,\infty) \to [0,\infty)$ which are upper semicontinuous, nondecreasing in each coordinate variable and $\phi(t) < t$ for all $t > 0$ [21].

Lemma 11 (see [21]). *If $\phi_i \in \Phi$ and $i \in I$ where I is a finite index set, then there exists some $\phi \in \Phi$ such that $\max\{\phi_i(t) : i \in I\} \le \phi(t)$ for all $t > 0$.*

Let $f, g, S,$ and T be self-mappings of a metric space (X, d) such that

$$\begin{aligned}
&d(fx,gy)\\
&\quad\le a\phi_0\left(d(Sx,Ty)\right)+(1-a)\\
&\qquad\times\max\Big\{\phi_1\left(d(Sx,Ty)\right),\\
&\qquad\quad\phi_2\left(\frac{1}{2}\left[d(Sx,fx)+d(Ty,gy)\right]\right),\\
&\qquad\quad\phi_3\left(\frac{1}{2}\left[d(Sx,gy)+d(Ty,fx)\right]\right),\\
&\qquad\quad\phi_4\left(\frac{1}{2}\left[d(Sx,fx)+d(Ty,fx)\right]\right),\\
&\qquad\quad\phi_5\left(\frac{1}{2}\left[d(Sx,gy)+d(Ty,gy)\right]\right)\Big\},
\end{aligned} \tag{31}$$

for all $x, y \in X$, where $\phi_i \in \Phi$, $i = 0, 1, 2, 3, 4, 5$, $0 \le a \le 1$.

Theorem 12. *Let $f, g, S,$ and T be self-mappings of a metric space (X, d) satisfying (31). If (f, S) and (g, T) are each $\mathcal{JH}$-operator pairs, then $f, g, S,$ and T have a unique common fixed point.*

Proof. By hypothesis there exist points $x, y \in X$ such that $fx = Sx = w$ and $gy = Ty = z$. If $fx \ne gy$, then, from (31), we get

$$\begin{aligned}
&d(fx,gy)\\
&\quad\le a\phi_0\left(d(fx,gy)\right)+(1-a)\\
&\qquad\times\max\Big\{\phi_1\left(d(fx,gy)\right),\ \phi_2(0),\\
&\qquad\quad\phi_3\left(\frac{1}{2}\left[d(fx,gy)+d(gy,fx)\right]\right),\\
&\qquad\quad\phi_4\left(\frac{1}{2}\left[d(gy,fx)\right]\right),\phi_5\left(\frac{1}{2}\left[d(fx,gy)\right]\right)\Big\},
\end{aligned} \tag{32}$$

which implies that $d(fx, gy) \le a\phi(d(fx, gy)) + (1 - a)\phi(d(fx, gy)) = \phi(d(fx, gy)) < d(fx, gy)$, a contradiction. Thus, $w = fx = gy = z$. Suppose that there exists another point u such that $fu = Su$. Then condition (31) implies that $fu = Su = gy = Ty = fx = Sx$. Hence, $w = fx = fu$. That is, $PC(f, S)$ is singleton. Since $d(x, w) \le \operatorname{diam}(PC(f, S)) = 0$, so $d(x, w) = 0$ and $x = w$ is a unique common fixed point of f and S. Similarly, $y = z$ is a unique common fixed point of g and T. Therefore, $w = z$ is a unique common fixed point of $f, g, S,$ and T. □

Corollary 13. *Let $f, g, S,$ and T be self-mappings of a metric space (X, d) satisfying $d(fx, gy) \le kd(Sx, Ty)$, for all $x, y \in X$ where $0 < k < 1$. If (f, S) and (g, T) are each $\mathcal{JH}$-operator pairs, then $f, g, S,$ and T have a unique common fixed point.*

Proof. It is sufficient to set $a = 1$ and take $\phi_0(t) = kt \in \Phi$ in Theorem 12. □

Corollary 14. *Let f, S be self-mappings of a metric space (X, d) satisfying the following condition:*

$$\begin{aligned}
&d(fx,fy)\\
&\quad\le a\phi_0\left(d(Sx,Sy)\right)+(1-a)\\
&\qquad\times\max\Big\{\phi_1\left(d(Sx,Sy)\right),\\
&\qquad\quad\phi_2\left(\frac{1}{2}\left[d(Sx,fx)+d(Sy,fy)\right]\right),
\end{aligned}$$

$$\phi_3\left(\frac{1}{2}\left[d(Sx,fy)+d(Sy,fx)\right]\right),$$
$$\phi_4\left(\frac{1}{2}\left[d(Sx,fx)+d(Sy,fx)\right]\right),$$
$$\phi_5\left(\frac{1}{2}\left[d(Sx,fy)+d(Sy,fy)\right]\right)\Big\}. \quad (33)$$

If (f,S) is $\mathcal{JH}$-operator pair, then f and S have a unique common fixed point.

Proof. Considering that $g := f$ and $T := S$ in Theorem 12, the result is obtained. □

Theorem 15. *Let f,S be self-mappings of a metric space (X,d) satisfying (33). Suppose that (f,S) is nontrivial Banach operator pair on X, then f and S have a unique common fixed point.*

Proof. By hypothesis $F(S)\neq\emptyset$ and $f(F(S))\subseteq F(S)$. From (33), for any $x,y\in F(S)$

$$\begin{aligned} d(fx,fy) &\le a\phi_0(d(x,y))+(1-a)\\ &\quad\times\max\Big\{\phi_1(d(x,y)),\\ &\quad\phi_2\left(\frac{1}{2}\left[d(x,fx)+d(y,fy)\right]\right),\\ &\quad\phi_3\left(\frac{1}{2}\left[d(x,fy)+d(y,fx)\right]\right),\\ &\quad\phi_4\left(\frac{1}{2}\left[d(x,fx)+d(y,fx)\right]\right),\\ &\quad\phi_5\left(\frac{1}{2}\left[d(x,fy)+d(y,fy)\right]\right)\Big\}. \end{aligned} \quad (34)$$

By Corollary 14 (with S as identity map on X), f has a unique fixed point on $F(S)$ and hence f and S have a unique common fixed point. □

Corollary 16. *Let f,S be self-mappings of a metric space (X,d) satisfying $d(fx,fy)\le kd(Sx,Sy)$, for all $x,y\in X$ where $0<k<1$. If (f,S) is a nontrivial Banach operator pair, then f and S have a unique common fixed point.*

Proof. It is sufficient to set $a=1$ and take $\phi_0(t)=kt\in\Phi$ in Theorem 15. □

Example 17. Let $X=[0,1]$ be a metric space with the usual metric $d(x,y)=|x-y|$ for all $x,y\in X$. Define $fx=gx=-1+\sqrt{3}$, $Sx=1-(1/2)x^2$, and $Tx=x$ for all $x\in[0,1]$. Obviously, $|fx-gy|=0$ and $|fx-gy|\le k|Sx-Ty|$ for all $x,y\in X$ and $0<k<1$. Also, $C(f,S)=\{-1+\sqrt{3}\}$, $PC(f,S)=\{-1+\sqrt{3}\}$, $C(g,T)=\{-1+\sqrt{3}\}$, and $PC(g,T)=\{-1+\sqrt{3}\}$. So, clearly (f,S) and (g,T) are each $\mathcal{JH}$-operator pairs. Thus, all the conditions of Corollary 13 are satisfied and $-1+\sqrt{3}$ is the unique common fixed point of f,g,S, and T.

4. Applications

In this section, we utilize the common fixed point theorems and their results to deduce the existence and uniqueness of the common solution for the system of functional equations in dynamic programming.

Remark 18. Many authors (e.g., see [9, 11–15, 22], or [3–5, 8–12, 17, 22] in [22]) used the fixed point theory to solve functional equations arising in dynamic programming on complete metric spaces such as Banach spaces. But, in the final section, we omit the completeness of the space and we state the result in the normed vector spaces and metric spaces setting.

Let X,Y be normed vector spaces, $S\subseteq X$ the state space, and $D\subseteq Y$ the decision space. Denote by $B(S)$ the set of all bounded real-valued functions on S and $d(f,g)=\sup\{|f(x)-g(x)|:x\in S\}$. It is clear that $(B(S),d)$ is a metric space:

$$\begin{aligned} f_i(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_i(x,y,f_i(T(x,y)))\},\\ &\qquad \forall x\in S,\quad i=1,2,\\ g_i(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+F_i(x,y,,g_i(T(x,y)))\},\\ &\qquad \forall x\in S,\quad i=1,2, \end{aligned} \quad (35)$$

where opt stands for sup or inf, $u:S\times D\to\mathbb{R}$, $T:S\times D\to S$, and $H_i,F_i:S\times D\times\mathbb{R}\to\mathbb{R}$ for $i=1,2$. Suppose that the mappings A_i and T_i $(i=1,2)$ are defined:

$$\begin{aligned} A_ih(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+H_i(x,y,h(T(x,y)))\},\\ &\qquad \forall x\in S,\quad h\in B(S),\quad i=1,2,\\ T_ik(x) &= \operatorname*{opt}_{y\in D}\{u(x,y)+F_i(x,y,k(T(x,y)))\},\\ &\qquad \forall x\in S,\quad k\in B(S),\quad i=1,2. \end{aligned} \quad (36)$$

Theorem 19. *Suppose that the following conditions are satisfied:*

(i) for given $h\in B(S)$, there exist $r(h)>0$ such that

$$\begin{aligned} &|u(x,y)|+\max\{|H_i(x,y,h(T(x,y)))|,\\ &\qquad |F_i(x,y,h(T(x,y)))|\ i=1,2\}\\ &\le r(h),\quad \forall(x,y)\in S\times D; \end{aligned} \quad (37)$$

(ii)

$$\begin{aligned} &|H_1(x,y,h(t))-H_2(x,y,k(t))|\\ &\le a\phi_0(d(T_1h(t),T_2k(t)))+(1-a)\\ &\quad\times\max\Big\{\phi_1(d(T_1h(t),T_2k(t))), \end{aligned}$$

$$\phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right)\right.\right.$$
$$\left.\left.+\, d\left(T_2k(t), A_2k(t)\right)\right]\right),$$
$$\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_2k(t)\right)\right.\right.$$
$$\left.\left.+d\left(T_2k(t), A_1h(t)\right)\right]\right),$$
$$\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right)\right.\right.$$
$$\left.\left.+d\left(T_2k(t), A_1h(t)\right)\right]\right),$$
$$\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_2k(t)\right)\right.\right.$$
$$\left.\left.\left.+d\left(T_2k(t), A_2k(t)\right)\right]\right)\right\}, \tag{38}$$

for all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$, *where* $\phi_i \in \Phi$, $i = 0, 1, 2, 3, 4, 5$, $0 \le a \le 1$;

(iii) for $i = 1, 2$, $\emptyset \ne \Gamma_i = \{\tau_{p_i} : A_i\tau_{p_i} = T_i\tau_{p_i} = \Theta_{p_i}\} \subseteq B(S)$;

(iv) there exist $\tau_{p_i} \in \Gamma_i (i = 1, 2)$, *such that*

$$\left|H_i\left(x, y, \tau_{q_i}(t)\right) - F_i\left(x, y, \tau_{r_i}(t)\right)\right| \ge \left|\Theta_{p_i} - \tau_{p_i}\right|, \tag{39}$$

for some $\tau_{q_i}, \tau_{r_i} \in \Gamma_i$ *and for all* $(x, y) \in S \times D$, $t \in S$.

Then the system of functional equations (35) *possesses a unique common solution in* $B(S)$.

Proof. Assume that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. By condition (i) and (36), A_i and T_i are self-mappings of $B(S)$. Using (i) and (36), one can deduce that there exist $y, z \in D$ such that

$$A_1h(x) > u(x, y) + H_1(x, y, h(T(x, y))) - \epsilon, \tag{40}$$
$$A_2k(x) > u(x, y) + H_2(x, z, k(T(x, z))) - \epsilon. \tag{41}$$

Note that

$$A_1h(x) \le u(x, z) + H_1(x, z, h(T(x, z))), \tag{42}$$
$$A_2k(x) \le u(x, y) + H_2(x, y, h(T(x, y))). \tag{43}$$

By virtue of (41) and (42),

$$\begin{aligned} &A_1h(x) - A_2k(x) \\ &\quad < H_1(x, z, h(T(x, z))) - H_2(x, z, k(T(x, z))) + \epsilon \\ &\quad \le \left|H_1(x, z, h(T(x, z))) - H_2(x, z, k(T(x, z)))\right| + \epsilon. \end{aligned} \tag{44}$$

From (40) and (43), we conclude that

$$\begin{aligned} &A_1h(x) - A_2k(x) \\ &\quad > H_1(x, y, h(T(x, y))) - H_2(x, y, k(T(x, y))) - \epsilon \\ &\quad \ge \left|H_1(x, y, h(T(x, y))) - H_2(x, y, k(T(x, y)))\right| - \epsilon. \end{aligned} \tag{45}$$

It follows from (44) and (45) that

$$\begin{aligned} &\left|A_1h(x) - A_2k(x)\right| \\ &\quad \le \max\left\{\left|H_1(x, y, h(T(x, y))) - H_2(x, y, k(T(x, y)))\right|,\right. \\ &\qquad \left.\left|H_1(x, z, h(T(x, z))) - H_2(x, z, k(T(x, z)))\right|\right\} + \epsilon. \end{aligned} \tag{46}$$

Equation (46) and (ii) lead to

$$\begin{aligned} &\left|A_1h(x) - A_2k(x)\right| \\ &\quad \le \max\left\{\left|H_1(x, y, h(T(x, y))) - H_2(x, y, k(T(x, y)))\right|,\right. \\ &\qquad \left.\left|H_1(x, z, h(T(x, z))) - H_2(x, z, k(T(x, z)))\right|\right\} + \epsilon \\ &\quad \le a\phi_0\left(d\left(T_1h(t), T_2k(t)\right)\right) + (1 - a) \\ &\qquad \times \max\left\{\phi_1\left(d\left(T_1h(t), T_2k(t)\right)\right),\right. \\ &\qquad \phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_2k(t), A_2k(t)\right)\right]\right), \\ &\qquad \phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_2k(t)\right) + d\left(T_2k(t), A_1h(t)\right)\right]\right), \\ &\qquad \phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_2k(t), A_1h(t)\right)\right]\right), \\ &\qquad \left.\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_2k(t)\right) + d\left(T_2k(t), A_2k(t)\right)\right]\right)\right\} + \epsilon \end{aligned} \tag{47}$$

which yields that

$$\begin{aligned} &d\left(A_1h, A_2k\right) \\ &\quad = \sup_{x\in S}\left|A_1h(x) - A_2k(x)\right| \\ &\quad \le a\phi_0\left(d\left(T_1h(t), T_2k(t)\right)\right) + (1 - a) \\ &\qquad \times \max\left\{\phi_1\left(d\left(T_1h(t), T_2k(t)\right)\right),\right. \\ &\qquad \phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_2k(t), A_2k(t)\right)\right]\right), \end{aligned}$$

$$\begin{aligned}&\phi_3\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))\\&\qquad+d(T_2k(t),A_1h(t))]\Big),\\&\phi_4\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))\\&\qquad+d(T_2k(t),A_1h(t))]\Big),\\&\phi_5\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))\\&\qquad+d(T_2k(t),A_2k(t))]\Big)\Big\}+\epsilon.\end{aligned}\tag{48}$$

Let $\epsilon \to 0$ in (48); then

$$\begin{aligned}d(A_1h,A_2k)&=\sup_{x\in S}|A_1h(x)-A_2k(x)|\\&\leq a\phi_0(d(T_1h(t),T_2k(t)))+(1-a)\\&\quad\times\max\Big\{\phi_1(d(T_1h(t),T_2k(t))),\\&\qquad\phi_2\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))\\&\qquad\quad+d(T_2k(t),A_2k(t))]\Big),\\&\qquad\phi_3\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))\\&\qquad\quad+d(T_2k(t),A_1h(t))]\Big),\\&\qquad\phi_4\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))\\&\qquad\quad+d(T_2k(t),A_1h(t))]\Big),\\&\qquad\phi_5\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))\\&\qquad\quad+d(T_2k(t),A_2k(t))]\Big)\Big\}.\end{aligned}\tag{49}$$

Now, we shall show that (A_1,T_1) and (A_2,T_2) are $\mathcal{JH}$-operator pairs. By (iii) there exists $\tau_{p_1}\in\Gamma_1$; thus, $A_1\tau_{p_1}=T_1\tau_{p_1}=\Theta_{p_1}$ and by (iv) for all $(x,y)\in S\times D$, $t\in S$, we have

$$\left|\Theta_{p_1}-\tau_{p_1}\right|\leq\left|H_1(x,y,\tau_{q_1}(t))-F_1(x,y,\tau_{r_1}(t))\right|,\tag{50}$$

for some $\tau_{q_1},\tau_{r_1}\in\Gamma_1$. Therefore, for all $x\in S$

$$\begin{aligned}&\left|\Theta_{p_1}(x)-\tau_{p_1}(x)\right|\\&\leq\left|\inf_{y\in D}\left(H_1(x,y,\tau_{q_1}(T(x,y)))-F_1(x,y,\tau_{r_1}(T(x,y)))\right)\right|\\&=\Big|\inf_{y\in D}\big(u(x,y)+H_1(x,y,\tau_{q_1}(T(x,y)))-u(x,y)\\&\qquad-F_1(x,y,\tau_{r_1}(T(x,y)))\big)\Big|\\&=\left|A_1\tau_{q_1}-T_1\tau_{r_1}\right|\\&=\left|\Theta_{q_1}-\Theta_{r_1}\right|\\&\leq\sup_{x\in S}\left|\Theta_{q_1}(x)-\Theta_{r_1}(x)\right|\\&=d(\Theta_{q_1},\Theta_{r_1})\\&\leq\operatorname{diam}(PC(A_1,T_1)).\end{aligned}\tag{51}$$

So

$$\sup_{x\in S}\left|\Theta_{p_1}(x)-\tau_{p_1}(x)\right|\leq\operatorname{diam}(PC(A_1,T_1)),\tag{52}$$

and, hence, $d(\Theta_{p_1},\tau_{p_1})\leq\operatorname{diam}(PC(A_1,T_1))$. That is, (A_1,T_1) is $\mathcal{JH}$-operator pair. Similarly, (A_2,T_2) is also $\mathcal{JH}$-operator pair. Clearly, all the above process also holds for $\text{opt}_{y\in D}=\sup_{y\in D}$. Then all of the conditions of Theorem 15 are satisfied and $h\in B(S)$ is a unique common fixed point of A_1,T_1,A_2, and T_2; that is, $h(x)$ is a unique common solution of functional equations (35). □

Corollary 20. *Suppose that the conditions* (*i*), (*iii*), *and* (*iv*) *of Theorem 19 are satisfied. Moreover, if the following condition also holds:*

$$\left|H_1(x,y,h(t))-H_2(x,y,k(t))\right|\leq\alpha d(T_1h(t),T_2k(t)),\tag{53}$$

for all $(x,y)\in S\times D$, $h,k\in B(S)$, $t\in S$, *where* $0<\alpha<1$, *then the system of functional equations* (35) *possesses a unique common solution in* $B(S)$.

Proof. It is sufficient to set $a=1$ and take $\phi_0(t)=\alpha t\in\Phi$ in Theorem 19. □

Example 21. Let $X=Y=\mathbb{R}$ be normed vector spaces endowed with the usual norm $\|\cdot\|$ defined by $\|x\|=|x|$ for all $x\in\mathbb{R}$. Let $S=[0,1]\subseteq X$ be the state space and $D=[1,\infty)\subseteq Y$ the decision space. Define $T:S\times D\to S$ and $u:S\times D\to\mathbb{R}$ by

$$T(x,y)=\frac{x+1}{y^2+2},\quad u(x,y)=0,\quad\forall x\in S,\ y\in D.\tag{54}$$

Define $f_i,g_i:S\to\mathbb{R}$ $(i=1,2)$ by

$$\begin{aligned}f_i(x)&=\frac{1}{16}\left[x-x^2\right],\\g_1(x)&=\frac{1}{2}\sqrt{x},\qquad g_2(x)=\frac{1}{2}x^3.\end{aligned}\tag{55}$$

Now, for all $h, k \in B(S)$; $x \in S$, we define mappings A_i and T_i $(i = 1, 2)$ by

$$\begin{aligned} A_1 h(x) &= \sup_{y \in D} \{u(x,y) + H_1(x, y, h(T(x,y)))\}, \\ A_2 k(x) &= \sup_{y \in D} \{u(x,y) + H_2(x, y, k(T(x,y)))\}, \\ T_1 h(x) &= \sup_{y \in D} \{u(x,y) + F_1(x, y, h(T(x,y)))\}, \\ T_2 k(x) &= \sup_{y \in D} \{u(x,y) + F_2(x, y, k(T(x,y)))\}, \end{aligned} \tag{56}$$

for which $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are defined as follows:

$$\begin{aligned} H_i(x, y, t) &= \frac{1}{16}\left[\left(x - x^2\right)\cos\left(t \cdot \left(1 - \frac{1}{y+2}\right)\right)\right], \\ F_1(x, y, t) &= \frac{1}{2}\left[\sqrt{x}\sin\left(t \cdot \left(1 - \frac{1}{y+2}\right)\right)\right], \\ F_2(x, y, t) &= \frac{1}{2}\left[x^3 \sin\left(t \cdot \left(1 - \frac{1}{y+2}\right)\right)\right]. \end{aligned} \tag{57}$$

So,

$$\begin{aligned} A_1 h(x) &= \sup_{y \in D} H_1\left(x, y, h\left(\frac{x+1}{y^2+1}\right)\right) \\ &= \sup_{y \in D} \frac{1}{16}\left[\left(x - x^2\right)\cos\left(h\left(\frac{x+1}{y^2+2}\right) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right] \\ &= \frac{1}{16}\left[x - x^2\right] = f_1(x), \\ A_2 k(x) &= \sup_{y \in D} H_2\left(x, y, k\left(\frac{x+1}{y^2+2}\right)\right) \\ &= \sup_{y \in D} \frac{1}{16}\left[\left(x - x^2\right)\cos\left(k\left(\frac{x+1}{y^2+2}\right) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right] \\ &= \frac{1}{16}\left[x - x^2\right] = f_2(x), \\ T_1 h(x) &= \sup_{y \in D} F_1\left(x, y, h\left(\frac{x+1}{y^2+1}\right)\right) \\ &= \sup_{y \in D} \frac{1}{2}\left[\sqrt{x}\sin\left(h\left(\frac{x+1}{y^2+2}\right) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right] \\ &= \frac{1}{2}\sqrt{x} = g_1(x), \\ T_2 k(x) &= \sup_{y \in D} F_2\left(x, y, k\left(\frac{x+1}{y^2+2}\right)\right) \\ &= \sup_{y \in D} \frac{1}{2}\left[x^3 \sin\left(k\left(\frac{x+1}{y^2+2}\right) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right] \\ &= \frac{1}{2}x^3 = g_2(x), \end{aligned} \tag{58}$$

for all $x \in S$, $h, k \in B(S)$. Also, $\|H_i\| \le 1/16$ and $\|F_i\| \le 1/2$, $(i = 1, 2)$; then easily we have the following:

(i) for given $h \in B(S)$, there exist $r(h) > 0$ such that

$$\begin{aligned} &|u(x,y)| + \max\{|H_i(x, y, h(T(x,y)))|, \\ &\qquad |F_i(x, y, h(T(x,y)))| \ i = 1, 2\} \\ &\quad \le r(h), \quad \forall (x,y) \in S \times D, \end{aligned} \tag{59}$$

if choose $a = 1$, and $\phi_0(n) = (1/4)n$, for $n \in [0, \infty)$ we have

(ii)

$$\begin{aligned} &|H_1(x, y, h(t)) - H_2(x, y, k(t))| \\ &\quad = \left|\frac{1}{16}\left[\left(x - x^2\right)\cos\left(h(t) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right] - \frac{1}{16}\left[\left(x - x^2\right)\cos\left(k(t) . \left(1 - \frac{1}{y+2}\right)\right)\right]\right| \\ &\quad = \frac{1}{16}\left|x - x^2\right|\left|\cos\left(h(t) \cdot \left(1 - \frac{1}{y+2}\right)\right) - \cos\left(k(t) \cdot \left(1 - \frac{1}{y+2}\right)\right)\right| \\ &\quad = \frac{1}{16}\left|x - x^2\right| \cdot 2\left|\sin\left(\left(h(t) \cdot \left(1 - \frac{1}{y+2}\right) - k(t) \cdot \left(1 - \frac{1}{y+2}\right)\right) \times (2)^{-1}\right)\right| \\ &\qquad \times \left|\sin\left(\left(h(t) \cdot \left(1 - \frac{1}{y+2}\right) + k(t) \cdot \left(1 - \frac{1}{y+2}\right)\right) \times (2)^{-1}\right)\right| \\ &\quad \le \frac{1}{8}\left|x - x^2\right| \le \frac{1}{8}\left[\sqrt{x} - x^3\right] = \frac{1}{4}\left[\frac{\sqrt{x}}{2} - \frac{x^3}{2}\right] \\ &\quad = a\phi_0\left(\left|\frac{\sqrt{x}}{2} - \frac{x^3}{2}\right|\right) \\ &\quad = a\phi_0\left(d\left(T_1 h(x), T_2 k(x)\right)\right). \end{aligned} \tag{60}$$

Therefore, $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \le a\phi_0(d(T_1 h(t), T_2 k(t)))$, for all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$.

(iii) $\Gamma_1 = \{\tau_{p_1} : A_1\tau_{p_1} = T_1\tau_{p_1} = \Theta_{p_1}\} = B(\{0\}) \neq \emptyset$ and $\Gamma_2 = \{\tau_{p_2} : A_2\tau_{p_2} = T_2\tau_{p_2} = \Theta_{p_2}\} = B(\{0\}) \neq \emptyset$.

(iv) Clearly, there exist $\tau_{p_i} \in \Gamma_i (i = 1, 2)$, such that

$$\left|H_i\left(x, y, \tau_{q_i}(t)\right) - F_i\left(x, y, \tau_{r_i}(t)\right)\right| \geq \left|\Theta_{p_i} - \tau_{p_i}\right|, \tag{61}$$

for some $\tau_{q_i}, \tau_{r_i} \in \Gamma_i$ and for all $(x, y) \in S \times D$.

Thus, all the assumptions of Theorem 19 are satisfied. So, the system of (35) has a unique common solution in $B(S)$.

Let

$$\begin{aligned} f(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + H(x, y, f(T(x, y)))\}, \quad \forall x \in S, \\ g(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + F(x, y, g(T(x, y)))\}, \quad \forall x \in S, \end{aligned} \tag{62}$$

where $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$ and $H, F : S\times D\times\mathbb{R} \to \mathbb{R}$. Suppose that the mappings A_1 and T_1 are defined:

$$\begin{aligned} A_1h(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + H(x, y, h(T(x, y)))\}, \\ &\qquad \forall x \in S, \quad h \in B(S), \\ T_1k(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + F(x, y, k(T(x, y)))\}, \\ &\qquad \forall x \in S, \quad k \in B(S), \end{aligned} \tag{63}$$

Theorem 22. *Suppose that the following conditions are satisfied:*

(i) for given $h \in B(S)$, there exist $r(h) > 0$ such that

$$\begin{aligned} &|u(x, y)| + \max\{|H(x, y, h(T(x, y)))|, \\ &\qquad |F(x, y, h(T(x, y)))|\} \\ &\quad \leq r(h), \quad \forall (x, y) \in S \times D; \end{aligned} \tag{64}$$

(ii)

$$\begin{aligned} &|H(x, y, h(t)) - H(x, y, k(t))| \\ &\quad \leq a\phi_0(d(T_1h(t), T_1k(t))) + (1 - a) \\ &\quad\times \max\Big\{\phi_1(d(T_1h(t), T_1k(t))), \\ &\qquad \phi_2\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) \\ &\qquad\qquad + d(T_1k(t), A_1k(t))]\Big), \\ &\qquad \phi_3\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) \\ &\qquad\qquad + d(T_1k(t), A_1h(t))]\Big), \\ &\qquad \phi_4\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) \\ &\qquad\qquad + d(T_1k(t), A_1h(t))]\Big), \\ &\qquad \phi_5\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) \\ &\qquad\qquad + d(T_1k(t), A_1k(t))]\Big)\Big\}, \end{aligned} \tag{65}$$

for all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$, where $\phi_i \in \Phi$, $i = 0, 1, 2, 3, 4, 5$, $0 \leq a \leq 1$;

(iii) $\emptyset \neq \Gamma = \{\tau_p : A_1\tau_p = T_i\tau_p = \Theta_p\} \subseteq B(S)$;

(iv) there exist $\tau_p \in \Gamma$, such that

$$\left|H\left(x, y, \tau_q(t)\right) - F\left(x, y, \tau_r(t)\right)\right| \geq \left|\Theta_p - \tau_p\right|, \tag{66}$$

for some $\tau_q, \tau_r \in \Gamma$ and for all $(x, y) \in S \times D$, $t \in S$.

Then the functional equations (62) have a unique common solution in $B(S)$.

Proof. Assume that $\operatorname{opt}_{y\in D} = \sup_{y\in D}$. By conditions (i) and (63), A_1 and T_1 are self-mappings of $B(S)$. Let h, k be any two points of $B(S)$, $x \in S$, and $\epsilon > 0$ any positive number; using (i) and (63), we deduce that there exist $y, z \in D$ such that

$$A_1h(x) < u(x, y) + H(x, y, h(T(x, y))) + \epsilon, \tag{67}$$

$$A_1k(x) < u(x, y) + H(x, z, k(T(x, z))) + \epsilon, \tag{68}$$

$$A_1h(x) \geq u(x, y) + H(x, z, h(T(x, z))), \tag{69}$$

$$A_1k(x) \geq u(x, y) + H(x, y, k(T(x, y))). \tag{70}$$

Subtracting (70) from (67) and using (ii),

$$\begin{aligned} &A_1h(x) - A_1k(x) \\ &\quad < H(x, y, h(T(x, y))) - H(x, y, k(T(x, y))) + \epsilon \\ &\quad \leq |H(x, y, h(T(x, y))) - H(x, y, k(T(x, y)))| + \epsilon \\ &\quad \leq a\phi_0(d(T_1h(t), T_1k(t))) + (1 - a) \\ &\qquad \times \max\Big\{\phi_1(d(T_1h(t), T_1k(t))), \\ &\qquad\quad \phi_2\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) \\ &\qquad\qquad + d(T_1k(t), A_1k(t))]\Big), \\ &\qquad\quad \phi_3\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) \\ &\qquad\qquad + d(T_1k(t), A_1h(t))]\Big), \\ &\qquad\quad \phi_4\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) \\ &\qquad\qquad + d(T_1k(t), A_1h(t))]\Big), \end{aligned}$$

$$\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\Big\} + \epsilon. \tag{71}$$

From (68) and (69), we get

$$\begin{aligned} A_1h(x) - A_1k(x) &> H(x,z,h(T(x,z))) - H(x,z,k(T(x,z))) - \epsilon \\ &\geq |H(x,z,h(T(x,z))) - H(x,z,k(T(x,z)))| - \epsilon \\ &\geq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\ &\quad \times \max\Big\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\ &\quad \phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\ &\quad \phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\ &\quad \phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right)\right]\right) + d\left(T_1k(t), A_1h(t)\right)\Big]\Big), \\ &\quad \phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\Big\} - \epsilon. \end{aligned} \tag{72}$$

Hence, from (71) and (72)

$$\begin{aligned} |A_1h(x) - A_1k(x)| &\leq |H(x,y,h(T(x,y))) - H(x,y,k(T(x,y)))| + \epsilon \\ &\leq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\ &\quad \times \max\Big\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\ &\quad \phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\ &\quad \phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\ &\quad \phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\ &\quad \phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\Big\} + \epsilon. \end{aligned} \tag{73}$$

So

$$\begin{aligned} d\left(A_1h, A_1k\right) &= \sup_{x\in S}|A_1h(x) - A_1k(x)| \\ &\leq |H(x,y,h(T(x,y))) - H(x,y,k(T(x,y)))| + \epsilon \\ &\leq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\ &\quad \times \max\Big\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\ &\quad \phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\ &\quad \phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\ &\quad \phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\ &\quad \phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\Big\} + \epsilon. \end{aligned} \tag{74}$$

Also from (iii) and (iv) and similar to Theorem 19, it is easy to prove that the pair (A_1, T_1) is $\mathcal{JH}$-operator pair. Therefore, by Corollary 14, A_1 and T_1 have a unique common fixed point in $B(S)$ and hence the functional equations (62) have a unique common solution in $B(S)$. □

Corollary 23. *Suppose that the conditions (i), (iii), and (iv) of Theorem 22 are satisfied. Moreover, if the following condition also holds:*

$$|H(x,y,h(t)) - H(x,y,k(t))| \leq \alpha d\left(T_1h(t), T_1k(t)\right), \tag{75}$$

for all $(x,y) \in S \times D$, $h,k \in B(S)$, $t \in S$, where $0 < \alpha < 1$, then the functional equations (62) have a unique common solution in $B(S)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.

[2] G. Jungck and H. K. Pathak, "Fixed points via 'biased maps,'" *Proceedings of the American Mathematical Society*, vol. 123, no. 7, pp. 2049–2060, 1995.

[3] H. K. Pathak and M. S. Khan, "Compatible mappings of type (B) and common fixed point theorems of Greguš type," *Czechoslovak Mathematical Journal*, vol. 45, no. 120, pp. 685–698, 1995.

[4] H. K. Pathak, Y. J. Cho, S. M. Kang, and B. S. Lee, "Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming," *Le Matematiche*, vol. 50, no. 1, pp. 15–33, 1995.

[5] H. K. Pathak, Y. J. Cho, S. M. Kang, and B. Madharia, "Compatible mappings of type (C) and common fixed point theorems of Greguš type," *Demonstratio Mathematica*, vol. 31, no. 3, pp. 499–518, 1998.

[6] G. Jungck, P. P. Murthy, and Y. J. Cho, "Compatible mappings of type (A) and common fixed points," *Mathematica Japonica*, vol. 38, no. 2, pp. 381–390, 1993.

[7] N. Hussain, M. A. Khamsi, and A. Latif, "Common fixed points for $\mathcal{JH}$-operators and occasionally weakly biased pairs under relaxed conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 6, pp. 2133–2140, 2011.

[8] W. Sintunavarat and P. Kumam, "Common fixed point theorems for generalized $\mathcal{JH}$-operator classes and invariant approximations," *Journal of Inequalities and Applications*, vol. 2011, article 67, 2011.

[9] N.-J. Huang, B. S. Lee, and M. K. Kang, "Fixed point theorems for compatible mappings with applications to the solutions of functional equations arising in dynamic programmings," *International Journal of Mathematics and Mathematical Sciences*, vol. 20, no. 4, pp. 673–680, 1997.

[10] A. Bhatt, H. Chandra, and D. R. Sahu, "Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 1, pp. 176–182, 2010.

[11] Z. Liu, J. S. Ume, and S. M. Kang, "Some existence theorems for functional equations arising in dynamic programming," *Journal of the Korean Mathematical Society*, vol. 43, no. 1, pp. 11–28, 2006.

[12] Z. Liu, J. S. Ume, and S. M. Kang, "Some existence theorems for functional equations and system of functional equations arising in dynamic programming," *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1517–1536, 2010.

[13] R. Bellman and E. S. Lee, "Functional equations in dynamic programming," *Aequationes Mathematicae*, vol. 17, no. 1, pp. 1–18, 1978.

[14] S. S. Zhang, "Some existence theorems of common and coincidence solutions for a class of systems of functional equations arising in dynamic programming," *Applied Mathematics and Mechanics*, vol. 12, no. 1, pp. 31–37, 1991.

[15] S.-S. Chang and Y. H. Ma, "Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming," *Journal of Mathematical Analysis and Applications*, vol. 160, no. 2, pp. 468–479, 1991.

[16] N. Hussain, M. A. Khamsi, and A. Latif, "Banach operator pairs and common fixed points in hyperconvex metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 17, pp. 5956–5961, 2011.

[17] H. K. Pathak and N. Hussain, "Common fixed points for $\mathcal{P}$-operator pair with applications," *Applied Mathematics and Computation*, vol. 217, no. 7, pp. 3137–3143, 2010.

[18] H. K. Pathak, Y. J. Cho, and S. M. Kang, "Common fixed points of biased maps of type (A) and applications," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 4, pp. 681–693, 1998.

[19] M. A. Alghamdi, S. Radenović, and N. Shahzad, "On some generalizations of commuting mappings," *Abstract and Applied Analysis*, vol. 2012, Article ID 952052, 6 pages, 2012.

[20] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.

[21] H. K. Pathak, M. S. Khan, and R. Tiwari, "A common fixed point theorem and its application to nonlinear integral equations," *Computers & Mathematics with Applications*, vol. 53, no. 6, pp. 961–971, 2007.

[22] Z. Liu, M. Liu, C. Y. Jung, and S. M. Kang, "Common fixed point theorems with applications in dynamic programming," *Applied Mathematical Sciences*, vol. 2, no. 41-44, pp. 2173–2186, 2008.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 342687, 8 pages
http://dx.doi.org/10.1155/2014/342687

Research Article

On Sufficient Conditions for the Existence of Past-Present-Future Dependent Fixed Point in the Razumikhin Class and Application

Marwan Amin Kutbi[1] and Wutiphol Sintunavarat[2]

[1] *Department of Mathematics, King AbdulAziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand*

Correspondence should be addressed to Wutiphol Sintunavarat; wutiphol@mathstat.sci.tu.ac.th

Received 17 September 2013; Revised 19 December 2013; Accepted 20 December 2013; Published 5 February 2014

Academic Editor: Hichem Ben-El-Mechaiekh

We introduce the new type of nonself mapping and study sufficient conditions for the existence of past-present-future (for short PPF) dependent fixed point for such mapping in the Razumikhin class. Also, we apply our result to prove the PPF dependent coincidence point theorems. Finally, we use PPF dependence techniques to obtain solution for a nonlinear integral problem with delay.

1. Introduction

It is well known that many problems in many branches of mathematics, such as optimization problems, equilibrium problems, and variational problems, can be transformed to fixed point problem of the form $Tx = x$ for self-mapping T defined on framework of metric space (X, d) or vector space $(X, \|\cdot\|)$. Therefore, the applications of fixed point theory are very important in diverse disciplines of mathematics. The famous Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. From inspiration of this work, several researchers heavily studied this field. For example, see works of Kannan [1], Chatterjea [2], Berinde [3], Ćirić [4], Geraghty [5], Meir and Keeler [6], Suzuki [7], Mizogushi and Takahashi [8], Dass and Gupta [9], Jaggi [10], Lou [11], and so forth.

On the other hand, Bernfeld et al. [12] introduced the concept of Past-Present-Future (for short PPF) dependent fixed point or the fixed point with PPF dependence which is one type of fixed points for mappings that have different domains and ranges. They also established the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. These results are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data, and future consideration. The generalizations of this result have been investigated heavily by many mathematicians (see [13–18] and references therein).

In this paper, we will introduce the new type of nonself mapping called Ciric-rational type contraction mapping. Also, we will study the sufficient conditions for the existence of PPF dependent fixed point theorems for such mapping in Razumikhin class. Furthermore, we apply the main result to the existence of PPF dependence coincidence point theorems. In the last section, an application to an integral problem with delay is also given.

2. Preliminaries

In this section, we recall some concepts and definitions that will be required in the sequel. Throughout this paper, let E denote a Banach space with the norm $\|\cdot\|_E$, I denote a closed interval $[a, b]$ in $\mathbb{R}$, and $E_0 = C(I, E)$ denote the set of all continuous E-valued functions on I equips with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I}\|\phi(t)\|_E. \tag{1}$$

A point $\phi \in E_0$ is said to be a PPF *dependent fixed point* or *a fixed point with PPF dependence* of a nonself mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathscr{R}_c := \left\{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\right\}. \tag{2}$$

It is easy to see that constant functions are member of $\mathscr{R}_c$.

The class $\mathscr{R}_c$ is algebraically closed with respect to difference if $\phi - \xi \in \mathscr{R}_c$ whenever $\phi, \xi \in \mathscr{R}_c$. Similarly, $\mathscr{R}_c$ is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 1 (see Bernfeld et al. [12]). The mapping $T : E_0 \to E$ is said to be Banach type contraction if there exists a real number $\alpha \in [0,1)$ such that

$$\|T\phi - T\xi\|_E \le \alpha\|\phi - \xi\|_{E_0} \tag{3}$$

for all $\phi, \xi \in E_0$.

The following PPF dependent fixed point theorem is proved by Bernfeld et al. [12].

Theorem 2 (see Bernfeld et al. [12]). *Let $T : E_0 \to E$ be a Banach type contraction. If $\mathscr{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathscr{R}_c$.*

3. PPF Dependent Fixed Point Theorems

In this section, we introduce the concept of the Ciric-rational type contraction mappings. Also, we study sufficient condition for the existence of PPF dependent fixed point for such mapping.

Definition 3. The mapping $T : E_0 \to E$ is called Ciric-rational type contraction if there exist real numbers $\alpha, \beta, \gamma, \delta, \kappa \in [0,1)$ with $\alpha + \beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$\begin{aligned}
&\|T\phi - T\xi\|_E \\
&\quad\le \alpha \max\Bigg\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E, \\
&\qquad\qquad \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2}\Bigg\} \\
&\qquad + \frac{\beta\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
&\qquad + \frac{\gamma\|\phi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi(c) - T\phi\|_E\|\phi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
&\qquad + \frac{\kappa\|\xi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}}
\end{aligned} \tag{4}$$

for all $\phi, \xi \in E_0$.

Remark 4. (i) All Banach type, Kannan type, and Chatterjea type mappings are Ciric-rational type contraction mapping.

(ii) If $\beta = \gamma = \delta = \kappa = 0$, then Ciric-rational type contraction mapping reduces to Ciric-type contraction.

(iii) If $\alpha = 0$, then T is a generalization and improvement of rational type contraction mapping.

Here, we prove PPF dependent fixed point theorems for Ciric-rational type contraction mappings.

Theorem 5. *Let $T : E_0 \to E$ be a Ciric-rational type contraction mapping. If $\mathscr{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathscr{R}_c$.*

Moreover, for a fixed $\phi_0 \in \mathscr{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathscr{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{5}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathscr{R}_c$.

Proof. Let ϕ_0 be an arbitrary function in $\mathscr{R}_c \subseteq E_0$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathscr{R}_c$ such that

$$x_1 = \phi_1(c). \tag{6}$$

Since $\phi_1 \in \mathscr{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathscr{R}_c$ such that

$$x_2 = \phi_2(c). \tag{7}$$

By continuing this process, we can construct the sequence $\{\phi_n\}$ in $\mathscr{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c) \tag{8}$$

for all $n \in \mathbb{N}$. Since $\mathscr{R}_c$ is algebraically closed with respect to difference, we have

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E \tag{9}$$

for all $n \in \mathbb{N}$.

Next, we will show that $\{\phi_n\}$ is a Cauchy sequence in $\mathscr{R}_c$.

For each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|\phi_n - \phi_{n+1}\|_{E_0} \\
&\quad = \|\phi_n(c) - \phi_{n+1}(c)\|_E \\
&\quad = \|T\phi_{n-1} - T\phi_n\|_E \\
&\quad \le \alpha \max\Bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \\
&\qquad\qquad \|\phi_n(c) - T\phi_n\|_E, \\
&\qquad\qquad \frac{\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}{2}\Bigg\} \\
&\qquad + \frac{\beta\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\qquad + \frac{\gamma\|\phi_{n-1}(c) - T\phi_n\|_E\|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_{n-1}(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\qquad + \frac{\kappa\|\phi_n(c) - T\phi_n\|_E\|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\quad = \alpha \max\Bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - \phi_n(c)\|_E, \\
&\qquad\qquad \|\phi_n(c) - \phi_{n+1}(c)\|_E, \frac{\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{2}\Bigg\} \\
&\qquad + \frac{\beta\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
&\quad = \alpha \max\Bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - \phi_n(c)\|_E, \\
&\qquad\qquad \|\phi_n(c) - \phi_{n+1}(c)\|_E, \frac{\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{2}\Bigg\} \\
&\qquad + \beta\|\phi_{n-1}(c) - \phi_n(c)\|_E\left(\frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_n(c)\|_E}\right) \\
&\qquad + \delta\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E\left(\frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_n(c)\|_E}\right) \\
&\quad \le \alpha \max\Bigg\{\|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \\
&\qquad\qquad \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E}{2}\Bigg\} \\
&\qquad + \beta\|\phi_{n-1}(c) - \phi_n(c)\|_E + \delta\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E \\
&\quad \le \alpha \max\left\{\|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E\right\} \\
&\qquad + \beta\|\phi_{n-1}(c) - \phi_n(c)\|_E + \delta\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E \\
&\quad = \alpha \max\left\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\right\} \\
&\qquad + \beta\|\phi_{n-1} - \phi_n\|_{E_0} + \delta\|\phi_{n-1} - \phi_{n+1}\|_{E_0} \\
&\quad \le \alpha \max\left\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\right\} \\
&\qquad + \beta\|\phi_{n-1} - \phi_n\|_{E_0} + \delta\|\phi_{n-1} - \phi_n\|_{E_0} + \delta\|\phi_n - \phi_{n+1}\|_{E_0}.
\end{aligned} \tag{10}
$$

For fixed $n \in \mathbb{N}$, if $\max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\} = \|\phi_{n-1} - \phi_n\|_{E_0}$, then we get

$$
\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &\le \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \beta\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \delta\|\phi_{n-1} - \phi_n\|_{E_0} + \delta\|\phi_n - \phi_{n+1}\|_{E_0}.
\end{aligned} \tag{11}
$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le \left(\frac{\alpha + \beta + \delta}{1 - \delta}\right)\|\phi_{n-1} - \phi_n\|_{E_0}. \tag{12}$$

On the other hand, if $\max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\} = \|\phi_n - \phi_{n+1}\|_{E_0}$, then we get

$$
\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &\le \alpha\|\phi_n - \phi_{n+1}\|_{E_0} + \beta\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \delta\|\phi_{n-1} - \phi_n\|_{E_0} + \delta\|\phi_n - \phi_{n+1}\|_{E_0}.
\end{aligned} \tag{13}
$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le \left(\frac{\beta + \delta}{1 - \alpha - \delta}\right)\|\phi_{n-1} - \phi_n\|_{E_0}. \tag{14}$$

Now, we let

$$k := \max\left\{\frac{\alpha + \beta + \delta}{1 - \delta}, \frac{\beta + \delta}{1 - \alpha - \delta}\right\}. \tag{15}$$

From (12) and (14), we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le k\|\phi_{n-1} - \phi_n\|_{E_0} \tag{16}$$

for all $n \in \mathbb{N}$. Repeated application of the above relation yields

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le k^n\|\phi_0 - \phi_1\|_{E_0} \tag{17}$$

for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$
\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\le \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} \\
&\quad + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\le \left(k^n + k^{n+1} + \cdots + k^{m-1}\right)\|\phi_0 - \phi_1\|_{E_0} \\
&\le \frac{k^n}{1 - k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned} \tag{18}
$$

Since $0 \le \alpha+\beta+\gamma+2\delta+\kappa < 1$, we have $0 \le k < 1$. This shows that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathscr{R}_c \subseteq E_0$. By the completeness of E_0, we get $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$. Therefore, $\lim_{n\to\infty}\phi_n = \phi^*$; that is,

$$\lim_{n\to\infty}\|\phi_n - \phi^*\|_{E_0} = 0_{E_0}. \tag{19}$$

Further, since $\mathscr{R}_c$ is topologically closed, we have $\phi^* \in \mathscr{R}_c$ and thus

$$\lim_{n\to\infty}\|\phi_n(c) - \phi^*(c)\|_E = 0_E. \tag{20}$$

Now we prove that ϕ^* is a PPF dependent fixed point of T. From the assumption of Ciric-rational type contraction of T, we get

$$\begin{aligned}
&\|T\phi^* - \phi^*(c)\|_E \\
&\quad\le \|T\phi^* - \phi_{n+1}(c)\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_E \\
&\quad= \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1} - \phi^*\|_{E_0} \\
&\quad\le \alpha\max\Big\{\|\phi^* - \phi_n\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \\
&\qquad\qquad \|\phi_n(c) - T\phi_n\|_E, \\
&\qquad\qquad \frac{\|\phi^*(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi^*\|_E}{2}\Big\} \\
&\qquad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\phi_n(c) - T\phi_n\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\gamma\|\phi^*(c) - T\phi_n\|_E\|\phi_n(c) - T\phi^*\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi^*(c) - T\phi^*\|_E\|\phi^*(c) - T\phi_n\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\kappa\|\phi_n(c) - T\phi_n\|_E\|\phi_n(c) - T\phi^*\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \|\phi_{n+1} - \phi^*\|_{E_0} \\
&\quad= \alpha\max\Big\{\|\phi^* - \phi_n\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \\
&\qquad\qquad \|\phi_n(c) - \phi_{n+1}(c)\|_E, \\
&\qquad\qquad \frac{\|\phi^*(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - T\phi^*\|_E}{2}\Big\} \\
&\qquad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\phi_n(c) - \phi_{n+1}(c)\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\gamma\|\phi^*(c) - \phi_{n+1}(c)\|_E\|\phi_n(c) - T\phi^*\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi^*(c) - T\phi^*\|_E\|\phi^*(c) - \phi_{n+1}(c)\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \frac{\kappa\|\phi_n(c) - \phi_{n+1}(c)\|_E\|\phi_n(c) - T\phi^*\|_E}{1+\|\phi^* - \phi_n\|_{E_0}} \\
&\qquad + \|\phi_{n+1} - \phi^*\|_{E_0}
\end{aligned} \tag{21}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, by (19) and (20), we have

$$\|T\phi^* - \phi^*(c)\|_E \le \alpha\|T\phi^* - \phi^*(c)\|_E. \tag{22}$$

This implies that

$$\|T\phi^* - \phi^*(c)\|_E = 0 \tag{23}$$

and then

$$T\phi^* = \phi^*(c). \tag{24}$$

Therefore, ϕ^* is a PPF dependent fixed point of T in $\mathscr{R}_c$.

Finally, we prove the uniqueness of PPF dependent fixed point of T in $\mathscr{R}_c$. Let ϕ^* and ξ^* be two PPF dependent fixed points of T in $\mathscr{R}_c$. Therefore,

$$\begin{aligned}
&\|\phi^* - \xi^*\|_{E_0} \\
&\quad= \|\phi^*(c) - \xi^*(c)\|_E \\
&\quad= \|T\phi^* - T\xi^*\|_E \\
&\quad\le \alpha\max\Big\{\|\phi^* - \xi^*\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \\
&\qquad\qquad \|\xi^*(c) - T\xi^*\|_E, \\
&\qquad\qquad \frac{\|\phi^*(c) - T\xi^*\|_E + \|\xi^*(c) - T\phi^*\|_E}{2}\Big\} \\
&\qquad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\xi^*(c) - T\xi^*\|_E}{1+\|\phi^* - \xi^*\|_{E_0}} \\
&\qquad + \frac{\gamma\|\phi^*(c) - T\xi^*\|_E\|\xi^*(c) - T\phi^*\|_E}{1+\|\phi^* - \xi^*\|_{E_0}} \\
&\qquad + \frac{\delta\|\phi^*(c) - T\phi^*\|_E\|\phi^*(c) - T\xi^*\|_E}{1+\|\phi^* - \xi^*\|_{E_0}} \\
&\qquad + \frac{\kappa\|\xi^*(c) - T\xi^*\|_E\|\xi^*(c) - T\phi^*\|_E}{1+\|\phi^* - \xi^*\|_{E_0}} \\
&\quad= \alpha\max\Big\{\|\phi^* - \xi^*\|_{E_0}, \\
&\qquad\qquad \frac{\|\phi^*(c) - \xi^*(c)\|_E + \|\xi^*(c) - \phi^*(c)\|_E}{2}\Big\}
\end{aligned}$$

$$+\frac{\gamma\|\phi^*(c)-T\xi^*\|_E\|\xi^*(c)-T\phi^*\|_E}{1+\|\phi^*-\xi^*\|_{E_0}}$$

$$=\alpha\max\left\{\|\phi^*-\xi^*\|_{E_0},\frac{\|\phi^*-\xi^*\|_{E_0}+\|\xi^*-\phi^*\|_{E_0}}{2}\right\}$$

$$+\frac{\gamma\|\phi^*(c)-\xi^*(c)\|_E\|\xi^*(c)-\phi^*(c)\|_E}{1+\|\phi^*-\xi^*\|_{E_0}}$$

$$=\alpha\|\phi^*-\xi^*\|_{E_0}+\frac{\gamma\|\phi^*-\xi^*\|_{E_0}\|\xi^*-\phi^*\|_{E_0}}{1+\|\phi^*-\xi^*\|_{E_0}}$$

$$=\alpha\|\phi^*-\xi^*\|_{E_0}+\gamma\|\phi^*-\xi^*\|_{E_0}\left(\frac{\|\phi^*-\xi^*\|_{E_0}}{1+\|\phi^*-\xi^*\|_{E_0}}\right)$$

$$\leq\alpha\|\phi^*-\xi^*\|_{E_0}+\gamma\|\phi^*-\xi^*\|_{E_0}. \tag{25}$$

Since $0 \leq \alpha+\gamma < 1$, we have $\|\phi^*-\xi^*\|_{E_0} = 0$ and hence $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in $\mathscr{R}_c$. This completes the proof. □

Remark 6. If the Razumikhin class $\mathscr{R}_c$ is not topologically closed, then the limit of the sequence $\{\phi_n\}$ in Theorem 5 may be outside of $\mathscr{R}_c$. Therefore, a PPF dependent fixed point of T may not be unique.

By applying Theorem 5, we obtain the following result.

Corollary 7. *Let $T: E_0 \to E$ be a nonself mapping and there exists a real number $\alpha \in [0,1)$ such that*

$$\|T\phi-T\xi\|_E \leq \alpha\|\phi-\xi\|_{E_0} \tag{26}$$

for all $\phi,\xi \in E_0$.

If there exists $c \in I$ such that $\mathscr{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathscr{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathscr{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathscr{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{27}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathscr{R}_c$.

If we set $\beta=\gamma=\delta=\kappa=0$ in Theorem 5, we get the PPF dependent fixed point result for Ciric-type contraction mapping.

Corollary 8. *Let $T: E_0 \to E$ be a nonself mapping and there exist real number $0 \leq \alpha < 1$ and $c \in I$ such that*

$$\begin{aligned}\|T\phi-T\xi\|_E &\leq \alpha\max\Bigg\{\|\phi-\xi\|_{E_0},\|\phi(c)-T\phi\|_E\ \|\xi(c)-T\xi\|_E,\\ &\qquad\frac{\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E}{2}\Bigg\}\end{aligned} \tag{28}$$

for all $\phi,\xi \in E_0$.

If $\mathscr{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathscr{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathscr{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathscr{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{29}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathscr{R}_c$.

If we set $\alpha = 0$ in Theorem 5, we get the PPF dependent fixed point result for generalized ratio type contraction mapping.

Corollary 9. *Let $T: E_0 \to E$ be a nonself mapping and there exist real numbers $\beta,\gamma,\delta,\kappa \in [0,1)$ with $\beta+\gamma+2\delta+\kappa < 1$ and $c \in I$ such that*

$$\begin{aligned}\|T\phi-T\xi\|_E \leq{}& \frac{\beta\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}}\\ &+\frac{\gamma\|\phi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi-\xi\|_{E_0}}\\ &+\frac{\delta\|\phi(c)-T\phi\|_E\|\phi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}}\\ &+\frac{\kappa\|\xi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi-\xi\|_{E_0}}\end{aligned} \tag{30}$$

for all $\phi,\xi \in E_0$.

If $\mathscr{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathscr{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathscr{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathscr{R}_c$ is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{31}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathscr{R}_c$.

4. PPF Dependent Coincidence Point Theorems

Definition 10. Let $T: E_0 \to E$ and $S: E_0 \to E_0$ be two nonself mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of T and S if $T\phi = (S\phi)(c)$ for some $c \in I$.

Next, we introduce the condition of the Ciric-rational type contraction for a pair of two nonself mappings.

Definition 11. Let $T: E_0 \to E$ and $S: E_0 \to E_0$ be two nonself mappings. The ordered pair (T,S) is said to satisfy the condition of Ciric-rational type contraction if there exist real numbers $\alpha, \beta, \gamma, \delta, \kappa \in [0,1)$ with $\alpha + \beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$
\begin{aligned}
&\|T\phi - T\xi\|_E \\
&\quad \le \alpha \max \left\{ \|S\phi - S\xi\|_{E_0}, \right. \\
&\qquad \|(S\phi)(c) - T\phi\|_E \ \|(S\xi)(c) - T\xi\|_E, \\
&\qquad \left. \frac{\|(S\phi)(c) - T\xi\|_E + \|(S\xi)(c) - T\phi\|_E}{2} \right\} \\
&\quad + \frac{\beta \|(S\phi)(c) - T\phi\|_E \|(S\xi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\gamma \|(S\phi)(c) - T\xi\|_E \|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\delta \|(S\phi)(c) - T\phi\|_E \|(S\phi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\kappa \|(S\xi)(c) - T\xi\|_E \|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}}
\end{aligned}
\tag{32}
$$

for all $\phi, \xi \in E_0$.

Remark 12. It is easy to see that

(T, S) satisfies the condition of Ciric-rational type contraction and S is identity mapping

$\Downarrow$

T is a Ciric-rational type contraction mapping.

Now, we apply our result to the previous section to the PPF dependent coincidence point theorem.

Theorem 13. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two nonself mappings. Suppose that the following conditions hold:*

$(\star_1)$ *(T, S) satisfies the condition of Ciric-rational type contraction;*

$(\star_2)$ *$S(\mathscr{R}_c) \subseteq \mathscr{R}_c$.*

If $S(\mathscr{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point.

Proof. For self-mapping $S : E_0 \to E_0$, it is well know that there exists $F_0 \subset E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since

$$T(F_0) \subseteq T(E_0) \subseteq E, \tag{33}$$

we can define a nonself mapping $\mathscr{H} : S(F_0) \to E$ by

$$\mathscr{H}(S\phi) = T\phi \tag{34}$$

for all $\phi \in F_0$. Since S_{F_0} is one-to-one mapping, we have $\mathscr{H}$ is well-defined.

By the condition of Ciric-rational type contraction of (T, S) and the construction of $\mathscr{H}$, we get

$$
\begin{aligned}
&\|\mathscr{H}(S\phi) - \mathscr{H}(S\xi)\|_E \\
&\quad \le \alpha \max \left\{ \|S\phi - S\xi\|_{E_0}, \right. \\
&\qquad \|(S\phi)(c) - \mathscr{H}(S\phi)\|_E \ \|(S\xi)(c) - \mathscr{H}(S\xi)\|_E, \\
&\qquad \left. \frac{\|(S\phi)(c) - \mathscr{H}(S\xi)\|_E + \|(S\xi)(c) - \mathscr{H}(S\phi)\|_E}{2} \right\} \\
&\quad + \frac{\beta \|(S\phi)(c) - \mathscr{H}(S\phi)\|_E \|(S\xi)(c) - \mathscr{H}(S\xi)\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\gamma \|(S\phi)(c) - \mathscr{H}(S\xi)\|_E \|(S\xi)(c) - \mathscr{H}(S\phi)\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\delta \|(S\phi)(c) - \mathscr{H}(S\phi)\|_E \|(S\phi)(c) - \mathscr{H}(S\xi)\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\
&\quad + \frac{\kappa \|(S\xi)(c) - \mathscr{H}(S\xi)\|_E \|(S\xi)(c) - \mathscr{H}(S\phi)\|_E}{1 + \|S\phi - S\xi\|_{E_0}}
\end{aligned}
\tag{35}
$$

for all $S\phi, S\xi \in S(E_0)$. This implies that $\mathscr{H}$ is a Ciric-rational type contraction mapping.

Using Theorem 5 with a mapping $\mathscr{H}$, we can find a unique PPF dependent fixed point of $\mathscr{H}$. Let a unique PPF dependent fixed point of $\mathscr{H}$ be $\zeta \in S(F_0)$; that is, $\mathscr{H}\zeta = \zeta(c)$. Since $\zeta \in S(F_0)$, we can find $\varpi \in F_0$ such that $\zeta = S\varpi$. Now, we have

$$T\varpi = \mathscr{H}(S\varpi) = \mathscr{H}\zeta = \zeta(c) = (S\varpi)(c). \tag{36}$$

Therefore, ϖ is a PPF dependent coincidence point of T and S. This completes the proof. □

By applying Theorem 13, we obtain the following corollaries.

Corollary 14. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two nonself mappings. Suppose that the following conditions hold:*

$(\star_1)$ *there exists a real number $\alpha \in [0,1)$ such that*

$$\|T\phi - T\xi\|_E \le \alpha \|S\phi - S\xi\|_{E_0} \tag{37}$$

for all $\phi, \xi \in E_0$;

$(\star_2)$ *there exists $c \in I$ such that $S(\mathscr{R}_c) \subseteq \mathscr{R}_c$.*

If $S(\mathscr{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in $\mathscr{R}_c$.

Corollary 15. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two nonself mappings. Suppose that the following conditions hold:*

($\star_1$) *there exist real numbers* $0 \le \alpha < 1$ *and* $c \in I$ *such that*

$$\begin{aligned}\|T\phi - T\xi\|_E &\\ \le \alpha \max \Big\{ &\|S\phi - S\xi\|_{E_0}, \\ &\|(S\phi)(c) - T\phi\|_E \ \|(S\xi)(c) - T\xi\|_E, \\ &\frac{\|(S\phi)(c) - T\xi\|_E + \|(S\xi)(c) - T\phi\|_E}{2} \Big\}\end{aligned} \tag{38}$$

for all $\phi, \xi \in E_0$;

($\star_2$) $S(\mathscr{R}_c) \subseteq \mathscr{R}_c$.

If $S(\mathscr{R}_c)$ *is topologically closed and algebraically closed with respect to difference, then* T *and* S *have a PPF dependent coincidence point in* $\mathscr{R}_c$.

Corollary 16. *Let* $T : E_0 \to E$ *and* $S : E_0 \to E_0$ *be two nonself mappings. Suppose that the following conditions hold:*

($\star_1$) *there exist real numbers* $\beta, \gamma, \delta, \kappa \in [0, 1)$ *with* $\beta + \gamma + 2\delta + \kappa < 1$ *and* $c \in I$ *such that*

$$\begin{aligned}\|T\phi - T\xi\|_E \le + &\frac{\beta\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ + &\frac{\gamma\|(S\phi)(c) - T\xi\|_E\|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ + &\frac{\delta\|(S\phi)(c) - T\phi\|_E\|(S\phi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ + &\frac{\kappa\|(S\xi)(c) - T\xi\|_E\|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}}\end{aligned} \tag{39}$$

for all $\phi, \xi \in E_0$;

($\star_2$) $S(\mathscr{R}_c) \subseteq \mathscr{R}_c$.

If $S(\mathscr{R}_c)$ *is topologically closed and algebraically closed with respect to difference, then* T *and* S *have a PPF dependent coincidence point in* $\mathscr{R}_c$.

5. Application to a Nonlinear Integral Equation

In this section, we apply our result to study the existence and uniqueness of solution of a nonlinear integral equation.

Given a closed interval $J := [j, 0]$ such that $j \in \mathbb{R}^-$, let Ω_0 denote the space of continuous real-valued functions defined on J. We equip the space Ω_0 with supremum normed $\|\cdot\|_{\Omega_0}$ defined by

$$\|\phi\|_{\Omega_0} = \sup_{t \in J} |\phi(t)|. \tag{40}$$

It well known that Ω_0 is a Banach space with this normed.

For fixed $T \in \mathbb{R}^+$, for each $t \in I := [0, T]$, define a function $t \mapsto \phi_t$ by

$$\phi_t(a) = \phi(t + a), \quad \text{for } a \in J, \tag{41}$$

where the argument a represents the delay in the argument of solutions.

Given $\varsigma \in C(I, \mathbb{R})$, we will consider the following nonlinear integral problem:

$$\phi(t) = \varsigma(0) + \int_0^T G(T, s) f(s, \phi_s)\, ds \tag{42}$$

for all $t \in I$, where $\phi \in C(I, \mathbb{R})$, $f \in C(I \times C(J, \mathbb{R}), \mathbb{R})$, and $G \in C(I \times I, \mathbb{R}_+)$.

Theorem 17. *Problem* (42) *has only one solution defined on* $J \cup I$ *if the following conditions hold:*

($\heartsuit_1$) $\sup_{t \in I}(\int_0^t G(t, s) ds) \le 1$,

($\heartsuit_2$) *there exist nonnegative real number* $\alpha < 1$ *such that, for all* $t \in I$ *and* $\phi, \xi \in C(I, \mathbb{R})$, *one has*

$$|f(t, \phi) - f(t, \xi)| \le \alpha |\phi(0) - \xi(0)|. \tag{43}$$

Proof. Define the following set:

$$\widehat{E} := \left\{\widehat{\phi} = (\phi_t)_{t \in I} : \phi_t \in \Omega_0, \phi \in C(I, \mathbb{R})\right\}. \tag{44}$$

Also, define the normed $\|\cdot\|_{\widehat{E}}$ in $\widehat{E}$ by

$$\|\widehat{\phi}\|_{\widehat{E}} := \sup_{t \in I} \|\phi_t\|_{\Omega_0}. \tag{45}$$

We obtain that $\widehat{\phi} \in C(J, \mathbb{R})$. Next, we show that $\widehat{E}$ is complete. Consider a Cauchy sequence $\{\widehat{\phi}_n\}$ in $\widehat{E}$. It is easy to see that $\{\phi_{n_t}\}_{t \in I}$ is a Cauchy sequence in $C(J, \mathbb{R})$ for all $t \in I$. This implies that $\{\phi_{n_t}(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in J$. So $\phi_{n_t}(s)$ converges to $\phi_t(s)$ for each $s \in J$. Since $\{\phi_{n_t}\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $\phi_t(s)$ is also continuous in $s \in J$. Thus going backwards we get that $\widehat{\phi}_n$ converges to $\widehat{\phi}$ in $\widehat{E}$. Therefore, $\widehat{E}$ is complete.

Next, we define the function $T : \widehat{E} \to \mathbb{R}$ by

$$T\widehat{\phi} \equiv T(\phi_t)_{t \in I} := \varsigma(0) + \int_0^T G(T, s) f(s, \phi_s)\, ds. \tag{46}$$

For $\widehat{\phi}, \widehat{\xi} \in \widehat{E}$, we have

$$\begin{aligned}
\left|T\widehat{\phi} - T\widehat{\xi}\right| &= \left|\int_0^T G(T,s) f(s,\phi_s)\, ds - \int_0^T G(T,s) f(s,\xi_s)\, ds\right| \\
&= \left|\int_0^T \left(G(T,s) f(s,\phi_s) - G(T,s) f(s,\xi_s)\right) ds\right| \\
&= \int_0^T \left(G(T,s) f(s,\phi_s) - G(T,s) f(s,\xi_s)\right) ds \\
&\le \int_0^T \left|G(T,s) f(s,\phi_s) - G(T,s) f(s,\xi_s)\right| ds \\
&= \int_0^T G(T,s) \left|f(s,\phi_s) - f(s,\xi_s)\right| ds \\
&\le \int_0^T G(T,s)\, \alpha \left|\phi_s(0) - \xi_s(0)\right| ds \\
&= \int_0^T G(T,s)\, \alpha \left|\phi(s) - \xi(s)\right| ds \\
&\le \int_0^T G(T,s)\, \alpha \left\|\widehat{\phi} - \widehat{\xi}\right\|_{\widehat{E}} ds \\
&= \alpha \left\|\widehat{\phi} - \widehat{\xi}\right\|_{\widehat{E}} \left(\int_0^T G(T,s)\, ds\right) \\
&\le \alpha \left\|\widehat{\phi} - \widehat{\xi}\right\|_{\widehat{E}} \left[\sup_{t\in I} \left(\int_0^t G(t,s)\, ds\right)\right] \\
&\le \alpha \left\|\widehat{\phi} - \widehat{\xi}\right\|_{\widehat{E}}.
\end{aligned} \tag{47}$$

This implies that T is a Ciric-rational type contraction.

Moreover, the Razumikhin $\mathscr{R}_0$ is $C(I,\mathbb{R})$ which is topologically closed and algebraically closed with respect to difference. Now all hypotheses of Theorem 5 are automatically satisfied with $c = 0$. Therefore, there exists PPF dependence fixed point $\widehat{\phi}^*$ of T; that is, $T\widehat{\phi}^* = \widehat{\phi}^*(0)$. This implies that

$$\varsigma(0) + \int_0^T G(T,s) f\left(s, \widehat{\phi}_s^*\right) ds = \left(\widehat{\phi}_t^*(0)\right)_{t\in I} = \left(\widehat{\phi}^*(t)\right)_{t\in I}. \tag{48}$$

Hence, the integral equation (42) has a solution. This completes the proof. □

Conflict of Interests

The authors declare that they have no competing interests.

Acknowledgment

The first author gratefully acknowledges the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research.

References

[1] R. Kannan, "Some results on fixed points—II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[2] S. K. Chatterjea, "Fixed-point theorems," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 25, pp. 727–730, 1972.

[3] V. Berinde, "Approximating fixed points of weak contractions using the Picard iteration," *Nonlinear Analysis Forum*, vol. 9, no. 1, pp. 43–53, 2004.

[4] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.

[5] M. A. Geraghty, "On contractive mappings," *Proceedings of the American Mathematical Society*, vol. 40, pp. 604–608, 1973.

[6] A. Meir and E. Keeler, "A theorem on contraction mappings," *Journal of Mathematical Analysis and Applications*, vol. 28, pp. 326–329, 1969.

[7] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[8] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989.

[9] B. K. Dass and S. Gupta, "An extension of Banach contraction principle through rational expression," *Indian Journal of Pure and Applied Mathematics*, vol. 6, no. 12, pp. 1455–1458, 1975.

[10] D. S. Jaggi, "Some unique fixed point theorems," *Indian Journal of Pure and Applied Mathematics*, vol. 8, no. 2, pp. 223–230, 1977.

[11] B. Lou, "Fixed points for operators in a space of continuous functions and applications," *Proceedings of the American Mathematical Society*, vol. 127, no. 8, pp. 2259–2264, 1999.

[12] S. R. Bernfeld, V. Lakshmikantham, and Y. M. Reddy, "Fixed point theorems of operators with PPF dependence in Banach spaces," *Applicable Analysis*, vol. 6, no. 4, pp. 271–280, 1977.

[13] R. P. Agarwal, W. Sintunavarat, and P. Kumam, "PPF dependent fixed point theorems for an α_c-admissible non-self mapping in the Razumikhin class," *Fixed Point Theory and Applications*, vol. 2013, article 280, 2013.

[14] B. C. Dhage, "On some common fixed point theorems with PPF dependence in Banach spaces," *Journal of Nonlinear Science and Its Applications*, vol. 5, no. 3, pp. 220–232, 2012.

[15] B. C. Dhage, "Fixed point theorems with PPF dependence and functional differential equations," *Fixed Point Theory*, vol. 13, no. 2, pp. 439–452, 2012.

[16] Z. Drici, F. A. McRae, and J. Vasundhara Devi, "Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 67, no. 2, pp. 641–647, 2007.

[17] Z. Drici, F. A. McRae, and J. Vasundhara Devi, "Fixed point theorems for mixed monotone operators with PPF dependence," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 2, pp. 632–636, 2008.

[18] W. Sintunavarat and P. Kumam, "PPF dependent fixed point theorems for rational type contraction mappings in Banach spaces," *Journal of Nonlinear Analysis and Optimization*, vol. 4, no. 2, pp. 157–162, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 825130, 8 pages
http://dx.doi.org/10.1155/2013/825130

Research Article

Strong Convergence Results for Equilibrium Problems and Fixed Point Problems for Multivalued Mappings

J. Vahidi,[1] A. Latif,[2] and M. Eslamian[3]

[1] *Department of Mathematics, Iran University of Science and Technology, Behshahr, Iran*
[2] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[3] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

Correspondence should be addressed to A. Latif; latifmath@yahoo.com

Received 21 July 2013; Accepted 10 October 2013

Academic Editor: Mohamed Amine Khamsi

Using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition (E) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset H. A subset $C \subset H$ is called proximal if, for each $x \in H$, there exists an element $y \in C$ such that

$$\|x-y\| = \operatorname{dist}(x, C) = \inf\{\|x-z\| : z \in C\}. \quad (1)$$

A single-valued mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x, y \in C. \quad (2)$$

Let P_C be a nearest point projection of H into C; that is, for $x \in H$, $P_C x$ is a unique nearest point in C with the property

$$\|x - P_C x\| := \inf\{\|x-y\| : y \in C\}. \quad (3)$$

We denote by $CB(C)$, $K(C)$, and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C respectively. The Hausdorff metric H on $CB(H)$ is defined by

$$H(A,B) := \max\left\{\sup_{x\in A}\operatorname{dist}(x,B), \sup_{y\in B}\operatorname{dist}(y,A)\right\}, \quad (4)$$

for all $A, B \in CB(H)$.

Let $T : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of T, if $x \in Tx$ and the set of fixed points of T is denoted by $F(T)$.

A multivalued mapping $T : H \to CB(H)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \le \|x-y\|, \quad x, y \in H; \quad (5)$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \le \|x-p\|$ for all $x \in H$ and all $p \in F(T)$.

Recently, García-Falset et al. [1] introduced a new condition on single-valued mappings, called condition (E), which is weaker than nonexpansiveness.

Definition 1. A mapping $T : H \to H$ is said to satisfy condition (E_μ) provided that

$$\|x - Ty\| \le \mu\ \|x - Tx\| + \|x-y\|, \quad x, y \in H. \quad (6)$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Recently, Abkar and Eslamian [2, 3] generalized this condition for multivalued mappings as follows.

Definition 2. A multivalued mapping $T : H \to CB(H)$ is said to satisfy condition (E) provided that

$$H(Tx, Ty) \le \mu \ \operatorname{dist}(x, Tx) + \|x - y\|, \quad x, y \in H, \tag{7}$$

for some $\mu \ge 1$.

It is obvious that every nonexpansive multivalued mapping $T : H \to CB(H)$ satisfies the condition (E), and every mapping $T : H \to CB(H)$ which satisfies the condition (E) with nonempty fixed point set $F(T)$ is quasi-nonexpansive.

Example 3. Let us define a mapping T on $[0, 3]$ by

$$T(x) = \begin{cases} \left[0, \dfrac{x}{3}\right], & x \ne 3 \\ [1, 2] & x = 3. \end{cases} \tag{8}$$

It is easy to see that T satisfies the condition (E) but is not nonexpansive. Indeed, for $x, y \in [0, 3)$, $H(Tx, Ty) = |(x - y)/3| \le |x - y|$. Let $x = 0$ and $y = 3$. Then $H(Tx, Ty) = 2 \le 3 = |x - y|$. If $x \in (0, 3)$ and $y = 3$, then, we have $\operatorname{dist}(x, Tx) = 2x/3$ and $\operatorname{dist}(y, Ty) = 1$; hence

$$H(Tx, Ty) = 2 - \frac{x}{3} \le 3 - x + \frac{4x}{3} = |x - y| + 2 \operatorname{dist}(x, Tx). \tag{9}$$

Thus, T satisfies the condition (E). However, T is not nonexpansive; indeed for $x = 3$ and $y = 7/3$, $H(Tx, Ty) = 11/9 > 2/3 = |x - y|$.

Let $\Psi : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem associated with the bifunction Ψ and the set C is:

$$\text{find } x \in C \quad \text{such that } \Psi(x, y) \ge 0, \ \forall y \in C. \tag{10}$$

Such a point $x \in C$ is called the solution of the equilibrium problem. The set of solutions is denoted by $EP(\Psi)$.

A broad class of problems in optimization theory, such as variational inequality, convex minimization, and fixed point problems, can be formulated as an equilibrium problem; see [4, 5]. In the literature, many techniques and algorithms have been proposed to analyze the existence and approximation of a solution to equilibrium problem; see [6]. Many researchers have studied various iteration processes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of a class of nonlinear mappings. For example, see [7–22].

Fixed points and fixed point iteration process for nonexpansive mappings have been studied extensively by many authors to solve nonlinear operator equations, as well as variational inequalities; see, for example, [23–28]. In the recent years, fixed point theory for multivalued mappings has been studied by many authors; see [29–40] and the references therein.

In this paper, using viscosity approximation method, we study strong convergence to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of multivalued mappings satisfying the condition (E) in the setting of Hilbert space. Our results improve and extend some recent results in the literature.

2. Preliminaries

For solving the equilibrium problem, we assume that the bifunction Ψ satisfies the following conditions:

(A1) $\Psi(x, x) = 0$ for any $x \in C$;

(A2) Ψ is monotone; that is, $\Psi(x, y) + \Psi(y, x) \le 0$ for any $x, y \in C$;

(A3) Ψ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Psi(tz + (1 - t)x, y) \le \Psi(x, y); \tag{11}$$

(A4) $\Psi(x, .)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 4 (see [4]). *Let C be a nonempty closed convex subset of H and let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \forall y \in C. \tag{12}$$

Lemma 5 (see [6]). *Assume that $\Psi : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r x = \left\{z \in C : \Psi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\right\}. \tag{13}$$

Then, the following hold:

(i) *S_r is single valued;*

(ii) *S_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|S_r x - S_r y\|^2 \le \langle S_r x - S_r y, x - y\rangle; \tag{14}$$

(iii) *$F(S_r) = EP(\Psi)$;*

(iv) *$EP(\Psi)$ is closed and convex.*

Lemma 6 (see [41]). *Let H be a real Hilbert space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$ one has*

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|z - y\|^2. \end{aligned} \tag{15}$$

Lemma 7. *For every x and y in a Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{16}$$

Lemma 8 (see [42]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\gamma_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\{\beta_n\}$ a sequence of real numbers with* $\limsup_{n \to \infty} \beta_n \le 0$. *Suppose that the following inequality holds:*

$$a_{n+1} \le (1 - \alpha_n) a_n + \alpha_n \beta_n + \gamma_n, \quad n \ge 0. \tag{17}$$

Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 9 (see [43]). *Let $\{u_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} < u_{n_i+1}$ for all $i \geq 0$. For every sufficiently large number $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : u_k < u_{k+1}\}. \quad (18)$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\max\{u_{\tau(n)}, u_n\} \leq u_{\tau(n)+1}. \quad (19)$$

Lemma 10 (see [20]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a quasi-nonexpansive multivalued mapping. If $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.*

Lemma 11 (see [20]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is quasi-nonexpansive and $F(T) \neq \emptyset$, where $P_T(x) = \{y \in Tx : \|x - y\| = \operatorname{dist}(x, Tx)\}$. Then, $F(T)$ is closed and convex.*

Lemma 12 (see [16, 20]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a multivalued mapping satisfying the condition (E). If x_n converges weakly to v and $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$, then $v \in Tv$.*

3. A Strong Convergence Theorem

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $T_i : C \to CB(C)$ $(i = 1, 2, \ldots, m)$ be a finite family of multivalued mappings, each satisfying condition (E). Assume further that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \bigcap EP(\Psi) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \ldots, m)$ for each $p \in \mathcal{F}$. Let f be a k-contraction of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{gathered} x_0 \in C, \\ u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \\ y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1}, \\ y_{n,2} = a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2}, \\ y_{n,3} = a_{n,3}u_n + b_{n,3}z_{n,2} + c_{n,3}z_{n,3} \\ \vdots \\ y_{n,m} = a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m}, \\ x_{n+1} = \vartheta_n f(x_n) + (1 - \vartheta_n) y_{n,m}, \quad \forall n \geq 0, \end{gathered} \quad (20)$$

where $z_{n,1} \in T_1(u_n)$, $z_{n,k} \in T_k(y_{n,k-1})$ for $k = 2, \ldots, m$, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{c_{n,i}\}$, $\{\vartheta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a, b] \subset (0, 1), a_{n,i} + b_{n,i} + c_{n,i} = 1, (i = 1, 2, \ldots, m)$,

(ii) $\{\vartheta_n\} \subset (0, 1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(iii) $\{r_n\} \subset (0, \infty)$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}} f(q)$.

Proof. Let $Q = P_{\mathcal{F}}$. It is easy to see that Qf is a contraction. By Banach contraction principle, there exists a $q \in \mathcal{F}$ such that $q = P_{\mathcal{F}} f(q)$. From Lemma 5 for all $n \geq 0$, we have

$$\|u_n - q\| = \|S_{r_n}x_n - S_{r_n}q\| \leq \|x_n - q\|. \quad (21)$$

We show that $\{x_n\}$ is bounded. Since, for each $i = 1, 2, \ldots, m$, T_i satisfies the condition (E) and we have

$$\begin{aligned} \|y_{n,1} - q\| &= \|a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1} - q\| \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|z_{n,1} - q\| \\ &= a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\operatorname{dist}(z_{n,1}, T_1q) \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}H(T_1u_n, T_1q) \\ &\leq a_{n,1}\|u_n - q\| + b_{n,1}\|x_n - q\| + c_{n,1}\|u_n - q\| \\ &\leq \|x_n - q\|, \end{aligned} \quad (22)$$

$$\begin{aligned} \|y_{n,2} - q\| &= \|a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2} - q\| \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}\|z_{n,1} - q\| + c_{n,2}\|z_{n,2} - q\| \\ &= a_{n,2}\|u_n - q\| + b_{n,2}\operatorname{dist}(z_{n,1}, T_1q) + c_{n,2}\operatorname{dist}(z_{n,2}, T_2q) \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}H(T_1u_n, T_1q) + c_{n,2}H(T_2y_{n,1}, T_2q) \\ &\leq a_{n,2}\|u_n - q\| + b_{n,2}\|u_n - q\| + c_{n,2}\|y_{n,1} - q\| \\ &\leq \| x_n - q \|. \end{aligned} \quad (23)$$

By continuing this process, we obtain

$$\|y_{n,m} - q\| \leq \|x_n - q\|. \quad (24)$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-q\| \\
&\quad = \|\vartheta_n f x_n + (1-\vartheta_n) y_n - q\| \\
&\quad \le \vartheta_n \|f x_n - q\| + (1-\vartheta_n)\|y_n - q\| \\
&\quad \le \vartheta_n (\|f x_n - f q\| + \|f q - q\|) + (1-\vartheta_n)\|x_n - q\| \\
&\quad \le \vartheta_n k \|x_n - q\| + \vartheta_n \|f q - q\| + (1-\vartheta_n)\|x_n - q\| \\
&\quad = (1-\vartheta_n(1-k))\|x_n - q\| + \vartheta_n \|f q - q\| \\
&\quad \le \max\left\{\|x_n - q\|, \frac{\|f q - q\|}{1-k}\right\}.
\end{aligned} \tag{25}$$

By induction, we get

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{\|f q - q\|}{1-k}\right\}, \tag{26}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and we also obtain that $\{u_n\}, \{y_n\}, \{f x_n\}$, and $\{z_{n,i}\}$ are bounded. Next, we show that $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$ for each $i \in \mathbb{N}$. By Lemma 6, we have

$$\begin{aligned}
&\|y_{n,1}-q\|^2 \\
&\quad = \|a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1} - q\|^2 \\
&\quad \le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\qquad + c_{n,1}\|z_{n,1} - q\|^2 \\
&\qquad - a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\quad = a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\qquad + c_{n,1}\operatorname{dist}(z_{n,1}, T_1 q)^2 \\
&\qquad - a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\quad \le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\qquad + c_{n,1}H(T_1 u_n, T_1 q)^2 \\
&\qquad - a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\quad \le a_{n,1}\|u_n - q\|^2 + b_{n,1}\|x_n - q\|^2 \\
&\qquad + c_{n,1}\|u_n - q\|^2 \\
&\qquad - a_{n,1}b_{n,1}\|x_n - u_n\|^2 - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2 \\
&\quad \le \|x_n - q\|^2 - a_{n,1}b_{n,1}\|x_n - u_n\|^2 \\
&\qquad - a_{n,1}c_{n,1}\|u_n - z_{n,1}\|^2.
\end{aligned} \tag{27}$$

Applying Lemma 6 once more, we have

$$\begin{aligned}
&\|y_{n,2}-q\|^2 \\
&\quad = \|a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2} - q\|^2 \\
&\quad \le a_{n,2}\|u_n - q\|^2 + b_{n,2}\|z_{n,1} - q\|^2 + c_{n,2}\|z_{n,2} - q\|^2 \\
&\qquad - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\quad = a_{n,2}\|u_n - q\|^2 + b_{n,2}\operatorname{dist}(z_{n,1}, T_1 q)^2 \\
&\quad + c_{n,2}\operatorname{dist}(z_{n,2}, T_2 q)^2 - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\quad \le a_{n,2}\|u_n - q\|^2 + b_{n,2}H(T_1 u_n, T_1 q)^2 \\
&\qquad + c_{n,2}H(T_1 y_{n,1}, T_2 q)^2 - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\quad \le a_{n,2}\|u_n - q\|^2 + b_{n,2}\|u_n - q\|^2 + c_{n,2}\|y_{n,1} - q\|^2 \\
&\qquad - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\quad \le \|x_n - q\|^2 - a_{n,2}c_{n,2}\|u_n - z_{n,2}\|^2 \\
&\qquad - a_{n,1}c_{n,1}c_{n,2}\|u_n - z_{n,1}\|^2 - a_{n,1}b_{n,1}c_{n,2}\|x_n - u_n\|^2.
\end{aligned} \tag{28}$$

By continuing this process we have

$$\begin{aligned}
&\|y_{n,m}-q\|^2 \\
&\quad = \|a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m} - q\|^2 \\
&\quad \le a_{n,m}\|u_n - q\|^2 + b_{n,m}\|z_{n,m-1} - q\|^2 + c_{n,m}\|z_{n,m} - q\|^2 \\
&\qquad - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\quad = a_{n,m}\|u_n - q\|^2 + b_{n,m}\operatorname{dist}(z_{n,m-1}, T_{m-1} q)^2 \\
&\qquad + c_{n,m}\operatorname{dist}(z_{n,m}, T_m q)^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\quad \le a_{n,m}\|u_n - q\|^2 + b_{n,m}H(T_{m-1} y_{n,m-2}, T_{m-1} q)^2 \\
&\qquad + c_{n,m}H(T_m y_{n,m-1}, T_m q)^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\quad \le a_{n,m}\|u_n - q\|^2 + b_{n,m}\|y_{n,m-2} - q\|^2 \\
&\qquad + c_{n,m}\|y_{n,m-1} - q\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\quad \le \|u_n - q\|^2 - a_{n,m}c_{n,m}\|u_n - z_{n,m}\|^2 \\
&\qquad - a_{n,m-1}c_{n,m-1}c_{n,m}\|u_n - z_{n,m-1}\|^2 \\
&\qquad - \cdots - a_{n,1}c_{n,1}c_{n,2}\cdots c_{n,m}\|u_n - z_{n,1}\|^2 \\
&\qquad - a_{n,1}b_{n,1}c_{n,2}\cdots c_{n,m}\|u_n - x_n\|^2,
\end{aligned} \tag{29}$$

which implies that

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\vartheta_n f x_n + (1-\vartheta_n) y_{n,m} - q\|^2 \\ &\le \vartheta_n \|f x_n - q\|^2 + (1-\vartheta_n)\|y_{n,m}-q\|^2 \\ &\le \vartheta_n \|f x_n - q\|^2 + (1-\vartheta_n)\|u_n-q\|^2 \\ &\quad - (1-\vartheta_n) a_{n,m} c_{n,m} \|u_n - z_{n,m}\|^2 \\ &\quad - (1-\vartheta_n) a_{n,m-1} c_{n,m-1} c_{n,m} \|u_n - z_{n,m-1}\|^2 \\ &\quad - \cdots - (1-\vartheta_n) a_{n,1} c_{n,1} c_{n,2} \dots c_{n,m} \|u_n - z_{n,1}\|^2 \\ &\quad - (1-\vartheta_n) a_{n,1} b_{n,1} c_{n,2} \dots c_{n,m} \|u_n - x_n\|^2. \end{aligned} \tag{30}$$

Therefore, we have that

$$\begin{aligned}&(1-\vartheta_n) a_{n,1} b_{n,1} c_{n,2} \dots c_{n,m} \|u_n - x_n\|^2 \\ &\quad \le \|x_n - q\|^2 - \|x_{n+1}-q\|^2 + \vartheta_n \|\gamma f x_n - q\|. \end{aligned} \tag{31}$$

In order to prove that $x_n \to q$ as $n \to \infty$, we consider the following two cases.

Case 1. Suppose that there exists n_0 such that $\{\|x_n - q\|\}$ is nonincreasing, for all $n \ge n_0$. Boundedness of $\{\|x_n - q\|\}$ implies that $\|x_n - q\|$ is convergent. From (31) and conditions (i), (ii) we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{32}$$

By a similar argument, for $k = 1, 2, \dots, m$, we obtain that

$$\lim_{n\to\infty} \|u_n - z_{n,k}\| = 0. \tag{33}$$

Hence,

$$\begin{aligned}&\lim_{n\to\infty} \operatorname{dist}(u_n, T_1 u_n) \le \lim_{n\to\infty} \|u_n - z_{n,1}\| = 0, \\ &\lim_{n\to\infty} \operatorname{dist}(u_n, T_k y_{n,k-1}) \le \lim_{n\to\infty} \|u_n - z_{n,k}\| = 0, \\ &\qquad (k = 2, \dots, m). \end{aligned} \tag{34}$$

Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} \|u_n - y_{n,1}\| &\le \lim_{n\to\infty} b_{n,1} \|u_n - x_n\| \\ &\quad + \lim_{n\to\infty} c_{n,1} \|u_n - z_{n,1}\| = 0. \end{aligned} \tag{35}$$

For $k = 2, \dots, m$, we have

$$\begin{aligned}\lim_{n\to\infty} \|u_n - y_{n,k}\| &\le \lim_{n\to\infty} b_{n,k} \|u_n - z_{n,k-1}\| \\ &\quad + \lim_{n\to\infty} c_{n,k} \|u_n - z_{n,k}\| = 0. \end{aligned} \tag{36}$$

Using the previous inequality for $k = 2, \dots, m$, we have

$$\begin{aligned}\operatorname{dist}(u_n, T_k u_n) &\le \operatorname{dist}(u_n, T_k y_{n,k-1}) + H(T_k y_{n,k-1}, T_k u_n) \\ &\le \operatorname{dist}(u_n, T_k y_{n,k-1}) + \mu \operatorname{dist}(y_{n,k-1}, T_k y_{n,k-1}) \\ &\quad + \|y_{n,k-1} - u_n\| \\ &\le (\mu+1)\operatorname{dist}(u_n, T_k y_{n,k-1}) + (\mu+1)\|y_{n,k-1} - u_n\| \\ &\le (\mu+1)\|u_n - z_{n,k}\| + (\mu+1)\|y_{n,k-1} - u_n\| \longrightarrow 0, \\ &\qquad n \longrightarrow \infty. \end{aligned} \tag{37}$$

Next, we show that

$$\limsup_{n\to\infty} \langle q - fq, q - x_n \rangle \le 0, \tag{38}$$

where $q = P_{\mathcal{F}} f(q)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle q - fq, q - x_{n_i} \rangle = \limsup_{n\to\infty} \langle q - fq, q - x_n \rangle. \tag{39}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to v. Without loss of generality, we can assume that x_{n_i} converges weakly to v. Since $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have u_{n_i} converges weakly to v. We show that $v \in \mathcal{F}$. Let us show $v \in EP(\Psi)$. Since $u_n = S_{r_n} x_n$, we have

$$\Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0 \quad \forall y \in C. \tag{40}$$

From (A2), we have

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge \Psi(y, u_n). \tag{41}$$

Replacing n with n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge \Psi(y, u_{n_i}). \tag{42}$$

From (A4), we have

$$0 \ge \Psi(y, v), \quad \forall\ y \in C. \tag{43}$$

For $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)v$. Since $y, v \in C$, and C is convex, we have $y_t \in C$ and hence $\Psi(y_t, v) \le 0$. So, from (A1) and (A4) we have

$$0 = \Psi(y_t, y_t) \le t\Psi(y_t, y) + (1-t)\Psi(y_t, v) \le t\Psi(y_t, y), \tag{44}$$

which gives $0 \le \Psi(y_t, y)$. Letting $t \to 0$, we have, for each $y \in C$, $0 \le \Psi(v, y)$ Also, since $u_{n_i} \rightharpoonup v$ and $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$, by Lemma 12 we have $v \in \bigcap_{i=1}^m F(T_i)$. Hence, $v \in \mathcal{F}$. Since $q = P_{\mathcal{F}} f(q)$ and $v \in \mathcal{F}$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \langle q - fq, q - x_n \rangle &= \lim_{i\to\infty} \langle q - fq, q - x_{n_i} \rangle \\ &= \langle q - fq, q - v \rangle \le 0. \end{aligned} \tag{45}$$

By using Lemma 7 and inequality (31) we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\leq \|(1-\vartheta_n)(y_{n,m}-q)\|^2 + 2\vartheta_n\langle fx_n - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|y_{n,m}-q\|^2 + 2\vartheta_n\langle fx_n - fq, x_{n+1}-q\rangle \\
&\quad + 2\vartheta_n\langle fq - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|x_n-q\|^2 + 2\vartheta_n k\|x_n-q\|\|x_{n+1}-q\| \\
&\quad + 2\vartheta_n\langle fq - q, x_{n+1}-q\rangle \\
&\leq (1-\vartheta_n)^2\|x_n-q\|^2 + \vartheta_n k\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right) \\
&\quad + 2\vartheta_n\langle fq - q, x_{n+1}-q\rangle \\
&\leq \left((1-\vartheta_n)^2+\vartheta_n k\right)\|x_n-q\|^2 + \vartheta_n k\|x_{n+1}-q\|^2 \\
&\quad + 2\vartheta_n\langle fq - q, x_{n+1}-q\rangle .
\end{aligned} \tag{46}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \left(1-\frac{2(1-k)\vartheta_n}{1-\vartheta_n k}\right)\|x_n-q\|^2 \\
&\quad + \frac{\vartheta_n^2}{1-\vartheta_n k}\|x_n-q\|^2 \\
&\quad + \frac{2\vartheta_n}{1-\vartheta_n k}\langle fq-q, x_{n+1}-q\rangle .
\end{aligned} \tag{47}$$

From Lemma 8, we conclude that the sequence $\{x_n\}$ converges strongly to q.

Case 2. Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\left\|x_{n_j}-q\right\| < \left\|x_{n_{j+1}}-q\right\|, \tag{48}$$

for all $j \in \mathbb{N}$. In this case, from Lemma 9, there exists a nondecreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ for all $n \geq n_0$ (for some n_0 large enough) such that $\tau(n) \to \infty$ as $n \to \infty$ and the following inequalities hold for all $n \geq n_0$:

$$\|x_{\tau(n)}-q\| \leq \|x_{\tau(n)+1}-q\|, \qquad \|x_n-q\| \leq \|x_{\tau(n)+1}-q\| . \tag{49}$$

From (31) we obtain $\lim_{n\to\infty}\|u_{\tau(n)} - T_i u_{\tau(n)}\| = 0$, and $\lim_{n\to\infty}\|u_{\tau(n)} - x_{\tau(n)}\| = 0$. Following an argument similar to that in Case 1, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-q\| = 0, \quad \lim_{n\to\infty}\|x_{\tau(n)+1}-q\| = 0. \tag{50}$$

Thus, by Lemma 9 we have

$$0 \leq \|x_n-q\| \leq \max\left\{\|x_{\tau(n)}-q\|, \|x_n-q\|\right\} \leq \|x_{\tau(n)+1}-q\| . \tag{51}$$

Therefore, $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}f(q)$. This completes the proof. □

Now, we remove the condition that $T(p) = \{p\}$ for all $p \in \mathcal{F}$, and state the following theorem.

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let, for each $1 \leq i \leq m$, $T_i : C \to P(C)$ be multivalued mappings such that P_{T_i} satisfies the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i)\bigcap EP(\Psi) \neq \emptyset$. Let f be a k-contraction of C into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{gathered}
x_0 \in C, \\
u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \\
\forall y \in C \\
y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}z_{n,1}, \\
y_{n,2} = a_{n,2}u_n + b_{n,2}z_{n,1} + c_{n,2}z_{n,2}, \\
y_{n,3} = a_{n,3}u_n + b_{n,3}z_{n,2} + c_{n,3}z_{n,3} \\
\vdots \\
y_{n,m} = a_{n,m}u_n + b_{n,m}z_{n,m-1} + c_{n,m}z_{n,m}, \\
x_{n+1} = \vartheta_n f x_n + (1-\vartheta_n) y_{n,m}, \quad \forall n \geq 0,
\end{gathered} \tag{52}$$

where $z_{n,1} \in P_{T_1}(u_n)$, $z_{n,k} \in P_{T_k}(y_{n,k-1})$ for $k = 2,\ldots,m$, and $\{a_{n,i}\}$, $\{b_{n,i}\}$, $\{c_{n,i}\}$, $\{\vartheta_n\}$ and, $\{r_n\}$ satisfy the following conditions:

(i) $\{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a,b] \subset (0,1), a_{n,i}+b_{n,i}+c_{n,i} = 1, (i = 1,2,\ldots,m)$,

(ii) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty}\vartheta_n = \infty$,

(iii) $\{r_n\} \subset (0,\infty)$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}f(q)$.

Proof. Let $p \in \mathcal{F}$; then $P_{T_i}(p) = \{p\}, (i = 1,2,\ldots,m)$. Now by substituting P_{T_i} instead of T_i, and using a similar argument as in the proof of Theorem 13, the desired result follows. □

As a corollary for single-valued mappings, we obtain the following result.

Corollary 15. *Let C be a nonempty closed convex subset of a real Hilbert space H and Ψ a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let, for each $1 \leq i \leq m$, $T_i : C \to C$ be a finite family of mappings satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i)\bigcap EP(\Psi) \neq \emptyset$. Let f be a k-contraction* *of C into itself. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated the following algorithm:*

$$\begin{aligned}
&x_0 \in C,\\
&u_n \in C \text{ such that } \Psi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y \in C\\
&y_{n,1} = a_{n,1}u_n + b_{n,1}x_n + c_{n,1}T_1u_n,\\
&y_{n,2} = a_{n,2}u_n + b_{n,2}T_1u_n + c_{n,2}T_2y_{n,1}\\
&\vdots\\
&y_{n,m} = a_{n,m}u_n + b_{n,m}T_{m-1}y_{n,m-2} + T_my_{n,m-1},\\
&x_{n+1} = \vartheta_n f x_n + (1 - \vartheta_n)\,y_{n,m}, \quad \forall n \geq 0,
\end{aligned} \tag{53}$$

where $\{a_{n,i}\}$, $\{b_{n,i}\}$,$\{c_{n,i}\}$, $\{\vartheta_n\}$, *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\{a_{n,i}\}, \{b_{n,i}\}, \{c_{n,i}\} \subset [a,b] \subset (0,1), a_{n,i} + b_{n,i} + c_{n,i} = 1$, $(i = 1, 2, \ldots, m)$,

(ii) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty}\vartheta_n = 0$, $\sum_{n=1}^{\infty}\vartheta_n = \infty$

(iii) $\{r_n\} \subset (0,\infty)$, *and* $\lim\inf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \mathcal{F}$, *where* $q = P_{\mathcal{F}}f(q)$.

Remark 16. Our results generalize the corresponding results of S. Takahashi and W. Takahashi [9] from a single valued nonexpansive mapping to a finite family of multivalued mappings satisfying the condition (E). Our results also improve the recent results of Eslamian [16].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors therefore, acknowledge with thanks DSR for technical and financial support. The authors are also thankful to the referees for their valuable suggestion/comments.

References

[1] J. García-Falset, E. Llorens-Fuster, and T. Suzuki, "Fixed point theory for a class of generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 1, pp. 185–195, 2011.

[2] A. Abkar and M. Eslamian, "Common fixed point results in CAT(0) spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 5, pp. 1835–1840, 2011.

[3] M. Eslamian and A. Abkar, "Fixed point theorems for suzuki generalized nonexpansive multivalued mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 457935, 10 pages, 2010.

[4] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1-4, pp. 123–145, 1994.

[5] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, Ser. A, pp. 29–41, 1997.

[6] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[7] A. Latif, L.-C. Ceng, and Q. H. Ansari, "Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of an equilibrium problem and fixed point problems," *Fixed Point Theory and Applications*, vol. 2012, article 186, 26 pages, 2012.

[8] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[9] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[10] L.-C. Ceng, S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, "An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 967–974, 2009.

[11] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems," *Journal of Global Optimization*, vol. 43, no. 4, pp. 487–502, 2009.

[12] L.-C. Zeng, Q. H. Ansari, and S. Al-Homidan, "Hybrid proximal-type algorithms for generalized equilibrium problems, maximal monotone operators, and relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2011, Article ID 973028, 23 pages, 2011.

[13] S. Plubtieng and R. Punpaeng, "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455–469, 2007.

[14] L.-C. Ceng, N. Hadjisavvas, and N.-C. Wong, "Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems," *Journal of Global Optimization*, vol. 46, no. 4, pp. 635–646, 2010.

[15] F. Cianciaruso, G. Marino, and L. Muglia, "Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 146, no. 2, pp. 491–509, 2010.

[16] M. Eslamian, "Convergence theorems for nonspreading mappings and nonexpansive multivalued mappings and equilibrium problems," *Optimization Letters*, vol. 7, no. 3, pp. 547–557, 2013.

[17] M. Eslamian, "Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A*, vol. 107, no. 2, pp. 299–307, 2013.

[18] A. Moudafi, "Weak convergence theorems for nonexpansive mappings and equilibrium problems," *Journal of Nonlinear and Convex Analysis*, vol. 9, no. 1, pp. 37–43, 2008.

[19] U. Kamraksa and R. Wangkeeree, "Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces," *Journal of Global Optimization*, vol. 51, no. 4, pp. 689–714, 2011.

[20] A. Abkar and M. Eslamian, "Strong convergence theorems for equilibrium problems and fixed point problem of multivalued nonexpansive mappings via hybrid projection method," *Journal of Inequalities and Applications*, vol. 2012, article 164, 13 pages, 2012.

[21] S. Wang, G. Marino, and Y.-C. Liou, "Strong convergence theorems for variational inequality, equilibrium and fixed point problems with applications," *Journal of Global Optimization*, vol. 54, no. 1, pp. 155–171, 2012.

[22] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 99–112, 2010.

[23] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[24] C. I. Podilchuk and R. J. Mammone, "Image recovery by convex projections using a least-squares constraint," *Journal of the Optical Society of America A*, vol. 7, pp. 517–521, 1990.

[25] D. C. Youla, "On deterministic convergence of iterations of relaxed projection operators," *Journal of Visual Communication and Image Representation*, vol. 1, no. 1, pp. 12–20, 1990.

[26] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[27] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[28] R. P. Agarwal, D. O'Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 61–79, 2007.

[29] E. Lami Dozo, "Multivalued nonexpansive mappings and Opial's condition," *Proceedings of the American Mathematical Society*, vol. 38, pp. 286–292, 1973.

[30] T. C. Lim, "A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space," *Bulletin of the American Mathematical Society*, vol. 80, pp. 1123–1126, 1974.

[31] S. Dhompongsa, A. Kaewkhao, and B. Panyanak, "Browder's convergence theorem for multivalued mappings without endpoint condition," *Topology and Its Applications*, vol. 159, no. 10-11, pp. 2757–2763, 2012.

[32] M. A. Khamsi and W. A. Kirk, "On uniformly Lipschitzian multivalued mappings in Banach and metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 2080–2085, 2010.

[33] J. Garcia-Falset, E. Llorens-Fuster, and E. Moreno-Galvez, "Fixed point theory for multivalued generalized nonexpansive mappings," *Applicable Analysis and Discrete Mathematics*, vol. 6, no. 2, pp. 265–286, 2012.

[34] A. Latif, T. Husain, and I. Beg, "Fixed point of nonexpansive type and K-multivalued maps," *International Journal of Mathematics and Mathematical Sciences*, vol. 17, no. 3, pp. 429–435, 1994.

[35] S. Dhompongsa, W. A. Kirk, and B. Panyanak, "Nonexpansive set-valued mappings in metric and Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 35–45, 2007.

[36] A. Abkar and M. Eslamian, "Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 457935, 10 pages, 2010.

[37] Y. Song and H. Wang, "Convergence of iterative algorithms for multivalued mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 4, pp. 1547–1556, 2009.

[38] A. Latif and I. Beg, "Geometric fixed points for single and multivalued mappings," *Demonstratio Mathematica*, vol. 30, no. 4, pp. 791–800, 1997.

[39] M. Eslamian and A. Abkar, "One-step iterative process for a finite family of multivalued mappings," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 105–111, 2011.

[40] S. Dhompongsa, A. Kaewkhao, and B. Panyanak, "On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 2, pp. 459–468, 2012.

[41] M. O. Osilike and D. I. Igbokwe, "Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations," *Computers & Mathematics with Applications*, vol. 40, no. 4-5, pp. 559–567, 2000.

[42] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[43] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 612576, 4 pages
http://dx.doi.org/10.1155/2013/612576

Research Article

A Fixed Point Approach to the Stability of an Integral Equation Related to the Wave Equation

Soon-Mo Jung

Mathematics Section, College of Science and Technology, Hongik University, Sejong 339-701, Republic of Korea

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

Received 24 July 2013; Accepted 16 October 2013

Academic Editor: Hichem Ben-El-Mechaiekh

We will apply the fixed point method for proving the generalized Hyers-Ulam stability of the integral equation $(1/2c)\int_{x-ct}^{x+ct} u(\tau, t_0)d\tau = u(x,t)$ which is strongly related to the wave equation.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are the Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1}$$

and proved the Hyers theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. (Aoki [4] has provided a proof of a special case of Rassias' theorem just for the stability of the additive function. Aoki did not prove the stability of the linear function, which was implied by Rassias' theorem.) Since then, the stability of several functional equations has been extensively investigated [5–12].

The terminologies generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability, and Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and various integral equations.

Let c and t_0 be fixed real numbers with $c > 0$. For any differentiable function $h : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, the function defined as

$$u(x,t) := \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau, t_0)\,d\tau \tag{2}$$

is a solution of the wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \tag{3}$$

as we see

$$\begin{aligned} u_t(x,t) &= \frac{1}{2c}\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} h(\tau,t_0)\,d\tau \\ &= \frac{1}{2}h(x+ct,t_0) + \frac{1}{2}h(x-ct,t_0), \\ u_{tt}(x,t) &= \frac{c}{2}h_x(x+ct,t_0) - \frac{c}{2}h_x(x-ct,t_0), \end{aligned}$$

$$u_x(x,t) = \frac{1}{2c}h(x+ct,t_0) - \frac{1}{2c}h(x-ct,t_0),$$
$$u_{xx}(x,t) = \frac{1}{2c}h_x(x+ct,t_0) - \frac{1}{2c}h_x(x-ct,t_0), \tag{4}$$

from which we know that $u(x,t)$ satisfies the wave equation (3).

Conversely, we know that every solution $u : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ of the wave equation (3) can be expressed by

$$u(x,t) = f(x+ct) + g(x-ct), \tag{5}$$

where $f, g : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ are arbitrary twice differentiable functions. If these $f(x,t)$ and $g(x,t)$ satisfy

$$\frac{1}{2c}\int_{x-ct}^{x+ct} f(\tau)\,d\tau = f(x+ct),$$
$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau = g(x-ct) \tag{6}$$

for all $x, t \in \mathbb{R}$, then $u(x,t)$ expressed by (5) satisfies the integral equation (7). These facts imply that the integral equation (7) is strongly connected with the wave equation (3).

Cădariu and Radu [13] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such a clever idea, they could present another proof for the Hyers-Ulam stability of that equation [14–19].

In this paper, we introduce the integral equation:

$$\frac{1}{2c}\int_{x-ct}^{x+ct} u(\tau,t_0)\,d\tau = u(x,t), \tag{7}$$

which may be considered as a special form of (2), and prove the generalized Hyers-Ulam stability of the integral equation (7) by using ideas from [13, 15, 19, 20]. More precisely, assume that $\varphi(x,t)$ is a given function and $u(x,t)$ is an arbitrary and continuous function which satisfies the integral inequality:

$$\left|\frac{1}{2c}\int_{x-ct}^{x+ct} u(\tau,t_0)\,d\tau - u(x,t)\right| \le \varphi(x,t). \tag{8}$$

If there exist a function $u_0(x,t)$ and a constant $C > 0$ such that

$$\frac{1}{2c}\int_{x-ct}^{x+ct} u_0(\tau,t_0)\,d\tau = u_0(x,t),$$
$$\left|u(x,t) - u_0(x,t)\right| \le C\varphi(x,t), \tag{9}$$

then we say that the integral equation (7) has the generalized Hyers-Ulam stability.

2. Preliminaries

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X\times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x,y) = 0$ if and only if $x = y$;

(M_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(M_3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [21]. This theorem will play an important role in proving our main theorems.

Theorem 1. *Let (X,d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*

(b) *x^* is the unique fixed point of Λ in*

$$X^* = \left\{y \in X \mid d\left(\Lambda^k x, y\right) < \infty\right\}; \tag{10}$$

(c) *if $y \in X^*$, then*

$$d(y,x^*) \le \frac{1}{1-L}d(\Lambda y, y). \tag{11}$$

3. The Generalized Hyers-Ulam Stability

In the following theorem, for given real numbers a, b, c, and t_0 satisfying $c > 0$, $t_0 > 0$, and $a + ct_0 < b - ct_0$, let $I := [a,b]$, $T := (0,t_0]$, and $I_0 := [a+ct_0, b-ct_0]$ be finite intervals. Assume that L and M are positive constants with $0 < L < 1$. Moreover, let $\varphi : I\times T \to (0,1]$ be a continuous function satisfying

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(\tau,t_0)\,d\tau \le L\varphi(x,t) \tag{12}$$

for all $x \in I_0$ and $t \in T$.

We denote by X the set of all functions $f : I\times T \to \mathbb{C}$ with the following properties:

(a) $f(x,t)$ is continuous for all $x \in I_0$ and $t \in T$;

(b) $f(x,t) = 0$ for all $x \in I \setminus I_0$ and $t \in T$;

(c) $|f(x,t)| \le M\varphi(x,t)$ for all $x \in I_0$ and $t \in T$.

Moreover, we introduce a generalized metric on X as follows:

$$d(f,g) := \inf\{C \in [0,\infty] \mid |f(x,t) - g(x,t)| \le C\varphi(x,t)\ \forall x \in I_0, t \in T\}. \tag{13}$$

Theorem 2. *If a function $u \in X$ satisfies the integral inequality:*

$$\left|\frac{1}{2c}\int_{x-ct}^{x+ct} u(\tau,t_0)\,d\tau - u(x,t)\right| \le \varphi(x,t) \tag{14}$$

for all $x \in I_0$ and $t \in T$, then there exists a unique function $u_0 \in X$ which satisfies

$$\frac{1}{2c}\int_{x-ct}^{x+ct} u_0(\tau, t_0)\, d\tau = u_0(x,t), \tag{15}$$

$$|u(x,t) - u_0(x,t)| \le \frac{1}{1-L}\varphi(x,t) \tag{16}$$

for all $x \in I_0$ and $t \in T$.

Proof. First, we show that (X, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (X, d). Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) \le \varepsilon$ for all $m, n \ge N_\varepsilon$. In view of (13), we have

$$\begin{gathered} \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall m, n \ge N_\varepsilon,\ \forall x \in I_0,\ \forall t \in T: \\ |h_m(x,t) - h_n(x,t)| \le \varepsilon\varphi(x,t). \end{gathered} \tag{17}$$

If x and t are fixed, (17) implies that $\{h_n(x,t)\}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\{h_n(x,t)\}$ converges for any $x \in I_0$ and $t \in T$. Thus, considering (b), we can define a function $h : I \times T \to \mathbb{C}$ by

$$h(x,t) := \lim_{n\to\infty} h_n(x,t), \quad (x \in I,\ t \in T). \tag{18}$$

Since φ is bounded on $I_0 \times T$, (17) implies that $\{h_n|_{I_0\times T}\}$ converges uniformly to $h|_{I_0\times T}$ in the usual topology of $\mathbb{C}$. Hence, h is continuous and $|h|$ is bounded on $I_0 \times T$ with an upper bound $M\varphi(x,t)$; that is, $h \in X$. (It has not been proved yet that $\{h_n\}$ converges to h in (X, d).)

If we let m increase to infinity, it follows from (17) that

$$\begin{gathered} \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall n \ge N_\varepsilon,\ \forall x \in I_0,\ \forall t \in T: \\ |h(x,t) - h_n(x,t)| \le \varepsilon\varphi(x,t). \end{gathered} \tag{19}$$

By considering (13), we get

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall n \ge N_\varepsilon : d(h, h_n) \le \varepsilon. \tag{20}$$

This implies that the Cauchy sequence $\{h_n\}$ converges to h in (X, d). Hence, (X, d) is complete.

We now define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x,t) := \begin{cases} \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} h(\tau, t_0)\, d\tau & (x \in I_0,\ t \in T), \\ 0 & (\text{otherwise}) \end{cases} \tag{21}$$

for all $h \in X$. Then, according to the fundamental theorem of calculus, Λh is continuous on $I_0 \times T$. Furthermore, it follows from (12), (c), and (21) that

$$\begin{aligned} |(\Lambda h)(x,t)| &\le \frac{1}{2c}\int_{x-ct}^{x+ct} |h(\tau, t_0)|\, d\tau \\ &\le \frac{1}{2c}\int_{x-ct}^{x+ct} M\varphi(\tau, t_0)\, d\tau \\ &\le ML\varphi(x,t) < M\varphi(x,t) \end{aligned} \tag{22}$$

for any $x \in I_0$ and $t \in T$. Hence, we conclude that $\Lambda h \in X$.

We assert that Λ is strictly contractive on X. Given any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \le C_{fg}$. That is,

$$|f(x,t) - g(x,t)| \le C_{fg}\varphi(x,t) \tag{23}$$

for all $x \in I_0$ and $t \in T$. Then, it follows from (12), (21), and (23) that

$$\begin{aligned} &|(\Lambda f)(x,t) - (\Lambda g)(x,t)| \\ &\quad = \frac{1}{2c}\left|\int_{x-ct}^{x+ct} (f(\tau, t_0) - g(\tau, t_0))\, d\tau\right| \\ &\quad \le \frac{1}{2c}\int_{x-ct}^{x+ct} |f(\tau, t_0) - g(\tau, t_0)|\, d\tau \\ &\quad \le \frac{C_{fg}}{2c}\int_{x-ct}^{x+ct} \varphi(\tau, t_0)\, d\tau \\ &\quad \le LC_{fg}\varphi(x,t) \end{aligned} \tag{24}$$

for all $x \in I_0$ and $t \in T$. That is, $d(\Lambda f, \Lambda g) \le LC_{fg}$. Hence, we may conclude that $d(\Lambda f, \Lambda g) \le Ld(f, g)$ for any $f, g \in X$ and we note that $0 < L < 1$.

We prove that the distance between the first two successive approximations of Λ is finite. Let $h_0 \in X$ be given. By (b), (c), and (13) and from the fact that $\Lambda h_0 \in X$, we have

$$\begin{aligned} |(\Lambda h_0)(x,t) - h_0(x,t)| &\le |(\Lambda h_0)(x,t)| + |h_0(x,t)| \\ &\le 2M\varphi(x,t) \end{aligned} \tag{25}$$

for any $x \in I_0$ and $t \in T$. Thus, (13) implies that

$$d(\Lambda h_0, h_0) \le 2M < \infty. \tag{26}$$

Therefore, it follows from Theorem 1(a) that there exists a $u_0 \in X$ such that $\Lambda^n h_0 \to u_0$ in (X, d) and $\Lambda u_0 = u_0$.

In view of (c) and (13), it is obvious that $\{f \in X \mid d(h_0, f) < \infty\} = X$, where h_0 was chosen with the property (26). Now, Theorem 1(b) implies that u_0 is the unique element of X which satisfies $(\Lambda u_0)(x,t) = u_0(x,t)$ for any $x \in I_0$ and $t \in T$.

Finally, Theorem 1(c), together with (13) and (14), implies that

$$d(u, u_0) \le \frac{1}{1-L}d(\Lambda u, u) \le \frac{1}{1-L}, \tag{27}$$

since (14) means that $d(\Lambda u, u) \le 1$. In view of (13), we can conclude that (16) holds for all $x \in I_0$ and $t \in T$. □

Remark 3. Even though condition (12) seems to be strict, the condition can be satisfied provided that a and b are chosen so that $|b - a|$ is small enough and c is a large number.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to express his cordial thanks to the referees for useful remarks. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2013R1A1A2005557).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[3] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297–300, 1978.

[4] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, no. 1-2, pp. 64–66, 1950.

[5] L. Cădariu, "The generalized Hyers-Ulam stability for a class of the Volterra nonlinear integral equations," *Buletinul Stiintific al Universitatii "Politehnica" din Timisoara*, vol. 56, no. 70, pp. 30–38, 2011.

[6] L. Cădariu, L. Găvruţa, and P. Găvruţa, "Weighted space method for the stability of some nonlinear equations," *Applicable Analysis and Discrete Mathematics*, vol. 6, no. 1, pp. 126–139, 2012.

[7] G. L. Forti, "Hyers-Ulam stability of functional equations in several variables," *Aequationes Mathematicae*, vol. 50, no. 1-2, pp. 143–190, 1995.

[8] P. Găvrută, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[9] D. H. Hyers, G. Isac, and M. Th. Rassias, *Stability of Functional Equations of Several Variables*, Birkhäuser, Basel, Switzerland, 1998.

[10] D. H. Hyers and T. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[11] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2011.

[12] T. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[13] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," *Grazer Mathematische Berichte*, vol. 346, pp. 43–52, 2004.

[14] M. Akkouchi, "Hyers-Ulam-Rassias stability of nonlinear Volterra integral equations via a fixed point approach," *Acta Universitatis Apulensis*, vol. 26, pp. 257–266, 2011.

[15] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, Article ID 4, 2003.

[16] L. P. Castro and A. Ramos, "Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations," *Banach Journal of Mathematical Analysis*, vol. 3, no. 1, pp. 36–43, 2009.

[17] K. Ciepliński, "Applications of fixed point theorems to the Hyers-Ulam stability of functional equations-a survey," *Annals of Functional Analysis*, vol. 3, no. 1, pp. 151–164, 2012.

[18] S.-M. Jung, "A fixed point approach to the stability of isometries," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 879–890, 2007.

[19] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.

[20] S.-M. Jung, "A fixed point approach to the stability of a volterra integral equation," *Fixed Point Theory and Applications*, vol. 2007, Article ID 57064, 9 pages, 2007.

[21] B. Margolis and J. Diaz, "A fixed point theorem of the alternative for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 672069, 8 pages
http://dx.doi.org/10.1155/2013/672069

Research Article

One-Local Retract and Common Fixed Point in Modular Metric Spaces

Afrah A. N. Abdou

Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Afrah A. N. Abdou; fixedmathstar@hotmail.com

Received 8 July 2013; Accepted 19 August 2013

Academic Editor: Mohamed A. Khamsi

The notion of a modular metric on an arbitrary set and the corresponding modular spaces, generalizing classical modulars over linear spaces like Orlicz spaces, were recently introduced. In this paper we introduced and study the concept of one-local retract in modular metric space. In particular, we investigate the existence of common fixed points of modular nonexpansive mappings defined on nonempty ω-closed ω-bounded subset of modular metric space.

1. Introduction

The purpose of this paper is to give an outline of a common fixed-point theory for nonexpansive mappings (i.e., mappings with the modular Lipschitz constant 1) on some subsets of modular metric spaces which are natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii, and many other spaces. Modular metric spaces were introduced in [1, 2]. The main idea behind this new concept is the physical interpretation of the modular. Informally speaking, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities" to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But the way we approached the concept of modular metric spaces is different. Indeed we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [3] on vector spaces and Musielak-Orlicz spaces introduced by Musielak [4] and Orlicz [5].

In recent years, there was an uptake interest in the study of electrorheological fluids, sometimes referred to as "smart fluids" (for instance, lithium polymethacrylate). For these fluids, modeling with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant is not adequate, but rather the exponent p should be able to vary [6, 7]. One of the most interesting problems in this setting is the famous Dirichlet energy problem [8, 9]. The classical technique used so far in studying this problem is to convert the energy function, naturally defined by a modular, to a convoluted and complicated problem which involves a norm (the Luxemburg norm). The modular metric approach is more natural and has not been used extensively.

In many cases, particularly in applications to integral operators, approximation, and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. In recent years, there was a strong interest to study the fixed point property in modular function spaces after the first paper [10] was published in 1990. More recently, the authors presented a fixed point result for pointwise nonexpansive and asymptotic pointwise nonexpansive acting in modular functions spaces [11]. The theory of nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s (see, e.g., Belluce and Kirk [12], Browder [13], Bruck [14], and Lim [15]), and generalized to other metric spaces (see e.g., [16–18]), and modular function spaces (see e.g., [10]). The corresponding fixed-point results were then extended to larger classes of mappings like pointwise contractions, asymptotic pointwise contractions [18–22], and asymptotic pointwise nonexpansive mappings

[11]. In [23], Penot presented an abstract version of Kirk's fixed point theorem [24] for nonexpansive mappings. Many results of fixed point in metric spaces were developed after Penot's formulation. Using Penot's work, the author in [25] proved some results in metric spaces with uniform normal structure similar to the ones known in Banach spaces. In [26], Khamsi introduced the concept of one-local retract in metric spaces and proved that any commutative family of nonexpansive mappings defined on a metric space with a compact and normal convexity structure has a common fixed point. Recently in [27], the authors introduced the concept of one-local retract in modular function spaces and proved the existence of common fixed points for commutative mappings.

In this paper, we study the concept of one-local retract in more general setting in modular metric space; therefore, we prove the existence of common fixed points for a family of modular nonexpansive mappings defined on nonempty ω-closed ω-bounded subsets in modular metric space.

For more on metric fixed point theory, the reader may consult the book [28] and for modular function spaces the book [29].

2. Basic Definitions and Properties

Let X be a nonempty set. Throughout this paper for a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, we will write

$$\omega_\lambda (x, y) = \omega (\lambda, x, y), \tag{1}$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1 (see [1, 2]). A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$, and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (i), we have only the condition (i$'$)

$$\omega_\lambda (x, x) = 0, \quad \forall \lambda > 0,\ x \in X, \tag{2}$$

then ω is said to be a pseudomodular (metric) on X. A modular metric ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$\begin{aligned} x = y \qquad &\text{iff } \omega_\lambda (x, y) = 0, \\ &\text{for some } \lambda > 0. \end{aligned} \tag{3}$$

Finally, ω is said to be convex if, for $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu} (x, y) \le \frac{\lambda}{\lambda + \mu}\omega_\lambda (x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu (z, y). \tag{4}$$

Note that, for a metric pseudomodular ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda (x, y) \le \omega_{\lambda-\mu} (x, x) + \omega_\mu (x, y) = \omega_\mu (x, y). \tag{5}$$

Definition 2 (see [1, 2]). Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets:

$$\begin{aligned} X_\omega = X_\omega (x_0) &= \{x \in X : \omega_\lambda (x, x_0) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty\}, \\ X^*_\omega = X^*_\omega(x_0) &= \{x \in X : \exists\lambda = \lambda (x) > 0 \\ &\qquad \text{such that } \omega_\lambda (x, x_0) < \infty\} \end{aligned} \tag{6}$$

are said to be modular spaces (around x_0).

It is clear that $X_\omega \subset X^*_\omega$ but this inclusion may be proper in general. It follows from [1, 2] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) metric, generated by ω and given by

$$d_\omega (x, y) = \inf \{\lambda > 0 : \omega_\lambda (x, y) \le \lambda\}, \tag{7}$$

for any $x, y \in X_\omega$. If ω is a convex modular on X, according to [1, 2] the two modular spaces coincide, that is $X^*_\omega = X_\omega$, and this common set can be endowed with the metric d^*_ω given by

$$d^*_\omega (x, y) = \inf \{\lambda > 0 : \omega_\lambda (x, y) \le 1\}, \tag{8}$$

for any $x, y \in X_\omega$. These distances will be called Luxemburg distances (see example below for the justification).

Definition 3. Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to \infty$. x will be called the ω-limit of (x_n).

(2) The sequence $(x_n)_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to \infty$.

(3) A subset C of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of C always belongs to C.

(4) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence in C is a ω-convergent sequence and its ω-limit is in C.

(5) Let $x \in X_\omega$ and $C \subset X_\omega$. The ω-distance between x and C is defined as

$$d_\omega (x, C) = \inf \{\omega_1 (x, y) ; y \in C\}. \tag{9}$$

(6) A subset C of X_ω is said to be ω-bounded if we have

$$\delta_\omega (C) = \sup \{\omega_1 (x, y) ; x, y \in C\} < \infty. \tag{10}$$

In general if $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that ω satisfies Δ_2 condition if this is the case; that is $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. In [1, 2], one will find a discussion about the connection between ω-convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega (x_n, x) = 0 \quad \text{iff } \lim_{n\to\infty} \omega_\lambda (x_n, x) = 0, \quad \forall\lambda > 0, \tag{11}$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. And in particular we have that ω-convergence and d_ω-convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d_ω^* and d_ω are equivalent which implies that

$$\lim_{n\to\infty} d_\omega^*(x_n, x) = 0 \quad \text{iff} \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0, \quad \forall \lambda > 0, \tag{12}$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$ [1, 2]. Another question that arises in this setting is the uniqueness of the ω-limit. Assume ω is regular, and let $\{x_n\} \in X_\omega$ be a sequence such that $\{x_n\}$ ω-converges to $a \in X_\omega$ and $b \in X_\omega$. Then we have

$$\omega_2(a, b) \le \omega_1(a, x_n) + \omega_1(x_n, b), \tag{13}$$

for any $n \ge 1$. Our assumptions will imply $\omega_2(a, b) = 0$. Since ω is regular, we get $a = b$; that is, the ω-limit of a sequence is unique.

Let (X, ω) be a modular metric space. Throughout the rest of this work, we will assume that ω satisfies the Fatou property; that is, if $\{x_n\}$ ω-converges to x and $\{y_n\}$ ω-converges to y, then we must have

$$\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y_n). \tag{14}$$

For any $x \in X_\omega$ and $r \ge 0$, we define the modular ball

$$B_\omega(x, r) = \{y \in X_\omega; \omega_1(x, y) \le r\}. \tag{15}$$

Note that if ω satisfies the Fatou property, then modular balls (ω-balls) are ω-closed. An admissible subset of X_ω is defined as an intersection of modular balls. We say A is an admissible subset of C if

$$A = \bigcap_{i\in I} B_\omega(b_i, r_i) \cap C, \tag{16}$$

where $b_i \in C$, $r_i \ge 0$, and I is an arbitrary index set. Denote by $\mathcal{A}_\omega(X_\omega)$ the family of admissible subsets of X_ω. Note that $\mathcal{A}_\omega(X_\omega)$ is stable by intersection. At this point we will need to define the concept of Chebyshev center and radius in modular metric spaces. Let $A \subset X$ be a nonempty ω-bounded subset. For any $x \in A$, define

$$r_x(A) = \sup\{\omega_1(x, y); \ y \in A\}. \tag{17}$$

The Chebyshev radius of A is defined by

$$R_\omega(A) = \inf\{r_x(A); \ x \in A\}. \tag{18}$$

Obviously we have $R_\omega(A) \le r_x(A) \le \delta_\omega(A)$, for any $x \in A$. The Chebyshev center of A is defined as

$$\mathcal{C}_\omega(A) = \{x \in A; \ r_x(A) = R_\omega(A)\}. \tag{19}$$

Definition 4. Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω.

(i) We will say that $\mathcal{A}_\omega(C)$ is compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathcal{A}_\omega(C)$ has a nonempty intersection provided $\cap_{\alpha\in F} A_\alpha \ne \emptyset$, for any finite subset $F \subset \Gamma$.

(ii) We will say that $\mathcal{A}_\omega(C)$ is normal if for any $A \in \mathcal{A}_\omega(C)$, not reduced to one point, ω-bounded, we have $R_\omega(A) < \delta_\omega(A)$.

Remark 5. Note that if $\mathcal{A}_\omega(X_\omega)$ is compact, then X_ω is ω-complete.

Definition 6. Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω. A mapping $T : C \to C$ is said to be ω-nonexpansive if

$$\omega_1(T(x), T(y)) \le \omega_1(x, y) \quad \text{for any } x, y \in C. \tag{20}$$

For such mapping we will denote by $\operatorname{Fix}(T)$ the set of its fixed points; that is, $\operatorname{Fix}(T) = \{x \in C; T(x) = x\}$.

In [1, 2] the author defined Lipschitzian mappings in modular metric spaces and proved some fixed point theorems. Our definition is more general. Indeed, in the case of modular function spaces, it is proved in [10] that

$$\omega_\lambda(T(x), T(y)) \le \omega_\lambda(x, y), \quad \text{for any } \lambda > 0 \tag{21}$$

if and only if $d_\omega(T(x), T(y)) \le d_\omega(x, y)$, for any $x, y \in C$. Next we give an example, which first appeared in [10], of a mapping which is ω-nonexpansive in our sense but fails to be nonexpansive with respect to d_ω.

Example 7. Let $X = (0, \infty)$. Define the Musielak-Orlicz function modular on the space of all Lebesgue measurable functions by

$$\rho(f) = \frac{1}{e^2}\int_0^\infty |f(x)|^{x+1} dm(x). \tag{22}$$

Let B be the set of all measurable functions $f : (0, \infty) \to \mathbb{R}$ such that $0 \le f(x) \le 1/2$. Consider the map

$$T(f)(x) = \begin{cases} f(x-1), & \text{for } x \ge 1 \\ 0, & \text{for } x \in [0, 1]. \end{cases} \tag{23}$$

Clearly, $T(B) \subset B$. In [10], it was proved that, for every $\lambda \le 1$ and for all $f, g \in B$, we have

$$\rho(\lambda(T(f) - T(g))) \le \lambda\rho(\lambda(f - g)). \tag{24}$$

This inequality clearly implies that T is ω-nonexpansive. On the other hand, if we take $f = 1_{[0,1]}$, then

$$\|T(f)\|_\rho > e \ge \|f\|_\rho, \tag{25}$$

which clearly implies that T is not d_ω-nonexpansive.

Next we present the analog of Kirk's fixed point theorem [24] in modular metric spaces.

Theorem 8 (see [30]). *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that the family $\mathcal{A}_\omega(C)$ is normal and compact. Let $T : C \to C$ be ω-nonexpansive. Then T has a fixed point.*

3. One-Local Retract Subsets in Modular Metric Spaces

Let C be a nonempty subset of X_ω. A nonempty subset D of C is said to be a one-local retract of C if, for every family $\{B_i; i \in I\}$ of ω-balls centered in D such that $C \cap (\cap_{i\in I} B_i) \neq \emptyset$, it is the case that $D \cap (\cap_{i\in I} B_i) \neq \emptyset$. It is immediate that each ω-nonexpansive retract of X_ω is a one-local retract (but not conversely). Recall that $D \subset C$ is a ω-nonexpansive retract of C if there exists a ω-nonexpansive map $R : C \to D$ such that $R(x) = x$, for every $x \in D$.

The result in [26] may be stated in modular metric spaces as follows.

Theorem 9. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Then, for any ω-nonexpansive mapping $T : C \to C$, the fixed point set $\operatorname{Fix}(T)$ is nonempty one-local retract of C.*

Proof. Theorem 8 shows that $\operatorname{Fix}(T)$ is nonempty. Let us complete the proof by showing that it is a one-local retract of C. Let $\{B_\omega(x_i, r_i)\}_{i\in I}$ be any family of ω-closed balls such that $x_i \in \operatorname{Fix}(T)$, for any $i \in I$, and

$$C_0 = C \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{26}$$

Let us prove that $\operatorname{Fix}(T) \cap (\cap_{i\in I} B_\omega(x_i, r_i)) \neq \emptyset$. Since $\{x_i\}_{i\in I} \subset \operatorname{Fix}(T)$ and T is ω-nonexpansive, then $T(C_0) \subset C_0$. Clearly, $C_0 \in \mathscr{A}_\omega(C)$ and is nonempty. Then we have $\mathscr{A}_\omega(C_0) \subset \mathscr{A}_\omega(C)$. Therefore, $\mathscr{A}_\omega(C_0)$ is compact and normal. Theorem 8 will imply that T has a fixed point in C_0 which will imply

$$\operatorname{Fix}(T) \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{27}$$

□

Now, we discuss some properties of one-local retract subsets.

Theorem 10. *Let (X, ω) be a modular metric space. Let C be a nonempty ω-closed ω-bounded subset of X_ω. Let D be a nonempty subset of C. The following are equivalent.*

(i) *D is a one-local retract of C.*

(ii) *D is a ω-nonexpansive retract of $D \cup \{x\} \to D$, for every $x \in C$.*

Proof. Let us prove (i) ⇒ (ii). Let $x \in C$. We may assume that x does not belong to D. In order to construct a ω-nonexpansive retract $R : D \cup \{x\} \to D$, we only need to find $R(x) \in D$ such that

$$\omega_1(R(x), y) \leq \omega_1(x, y), \quad \text{for every } y \in D. \tag{28}$$

Since $x \in \cap_{y\in D} B_\omega(y, \omega_1(x, y))$ and $x \in C$, then

$$C \cap \left(\bigcap_{y\in D} B_\omega(y, \omega_1(x, y)) \right) \neq \emptyset. \tag{29}$$

Since D is one-local retract of C, we get

$$D_0 = D \cap \left(\bigcap_{y\in D} B_\omega(y, \omega_1(x, y)) \right) \neq \emptyset. \tag{30}$$

Any point in D_0 will work as $R(x)$. □

Next, we prove that (ii) ⇒ (i). In order to prove that D is a one-local retract of C, let $\{B_\omega(x_i, r_i)\}_{i\in I}$ be any family of ω-closed balls such that $x_i \in D$, for any $i \in I$, and

$$C_0 = C \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{31}$$

Let us prove that $D \cap (\cap_{i\in I} B_\omega(x_i, r_i)) \neq \emptyset$. Let $x \in C_0$. If $x \in D$, we have nothing to prove. Assume otherwise that x does not belong to D. Property (ii) implies the existence of a ω-nonexpansive retract $R : D \cup \{x\} \to C$. It is easy to check that $R(x) \in D \cap (\cap_{i\in I} B_\omega(x_i, r_i)) = \emptyset$, which completes the proof of our theorem.

For the rest of this work, we will need the following technical result.

Lemma 11. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Let D be a nonempty one-local retract of C. Set $\operatorname{co}_C(D) = C \cap (\cap\{A; A \in \mathscr{A}_\omega(C)$ and $D \subset A\})$. Then*

(i) $r_x(D) = r_x(\operatorname{co}_C(D))$, *for any $x \in C$;*

(ii) $R_\omega(\operatorname{co}_C(D)) = R_\omega(D)$;

(iii) $\delta_\omega(\operatorname{co}_C(D)) = \delta_\omega(D)$.

Proof. Let us first prove (i). Fix $x \in C$. Since $D \subset \operatorname{co}_C(D)$, we get $r_x(D) \leq r_x(\operatorname{co}_C(D))$. On the other hand we have $D \subset B_\omega(x, r_x(D)) \in \mathscr{A}_\omega(C)$. The definition of $\operatorname{co}_C(D)$ implies $\operatorname{co}_C(D) \subset B_\omega(x, r_x(D))$. Hence $r_x(\operatorname{co}_C(D)) \leq r_x(D)$, which implies

$$r_x(\operatorname{co}_C(D)) = r_x(D). \tag{32}$$

Next, we prove (ii). Let $x \in D$. We have $x \in \operatorname{co}_C(D)$. Using (i), we get

$$r_x(\operatorname{co}_C(D)) = r_x(D) \geq R_\omega(\operatorname{co}_C(D)). \tag{33}$$

Hence, $R_\omega(D) \geq R_\omega(\operatorname{co}_C(D))$. Next, let $x \in \operatorname{co}_C(D)$. We have $D \subset \operatorname{co}_C(D) \subset B_\omega(x, r_x(\operatorname{co}_C(D)))$. Hence, $x \in \cap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D)))$. Hence

$$C \cap \left(\bigcap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D))) \right) = \emptyset. \tag{34}$$

Since D is a one-local retract of C, we get

$$D_0 = D \cap \left(\bigcap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D))) \right) = \emptyset. \tag{35}$$

Let $y \in D_0$. Then it is easy to see that $r_y(D) \le r_x(\mathrm{co}_C(D))$. Hence $R_\omega(D) \le r_x(\mathrm{co}_C(D))$. Since x was arbitrary taken in $\mathrm{co}_C(D)$, we get

$$R_\omega(D) \le R_\omega(\mathrm{co}_C(D)), \tag{36}$$

which implies

$$R_\omega(D) = R_\omega(\mathrm{co}_C(D)). \tag{37}$$

Finally, let us prove (iii). Since $D \subset \mathrm{co}_C(D)$, we get

$$\delta_\omega(D) \le \delta_\omega(\mathrm{co}_C(D)). \tag{38}$$

Now, for any $x \in D$, we have

$$D \subset B_\omega(x, \delta_\omega(D)). \tag{39}$$

Hence

$$\mathrm{co}_C(D) \subset B_\omega(x, \delta_\omega(D)). \tag{40}$$

This implies

$$x \in \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{41}$$

Since x was taken arbitrary in D, we get

$$D \subset \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{42}$$

The definition of $\mathrm{co}_C(D)$ implies

$$\mathrm{co}_C(D) \in \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{43}$$

So for any $x, y \in \mathrm{co}_C(D)$, we have

$$\omega_1(x, y) \le \delta_\omega(D). \tag{44}$$

Hence

$$\delta_\omega(\mathrm{co}_C(D)) \le \delta_\omega(D), \tag{45}$$

which implies

$$\delta_\omega(\mathrm{co}_C(D)) = \delta_\omega(D). \tag{46}$$

□

As an application of this lemma we have the following result.

Theorem 12. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. If D is a nonempty one-local retract of C, then $\mathscr{A}_\omega(D)$ is compact and normal.*

Proof. Using the definition of one-local retract, it is easy to see that $\mathscr{A}_\omega(D)$ is compact. Let us show that $\mathscr{A}_\omega(D)$ is normal. Let $A_0 \in \mathscr{A}_\omega(D)$ be nonempty and reduced to one point. Set

$$\mathrm{co}_C(A_0) = C \cap (\cap\{A; A \in \mathscr{A}_\omega(C) \text{ and } A_0 \subset A\}). \tag{47}$$

Then from Lemma 11, we get

$$\begin{aligned} R_\omega(\mathrm{co}_C(A_0)) &= R_\omega(A_0), \\ \delta_\omega(\mathrm{co}_C(A_0)) &= \delta_\omega(A_0). \end{aligned} \tag{48}$$

Since $\mathrm{co}_C(A_0) \in \mathscr{A}_\omega(C)$, then we must have

$$R_\omega(\mathrm{co}_C(A_0)) < \delta_\omega(\mathrm{co}_C(A_0)), \tag{49}$$

because $\mathscr{A}_\omega(C)$ is normal. Therefore, we have

$$R_\omega(A_0) < \delta_\omega(A_0), \tag{50}$$

which completes the proof of our claim. □

The following result has found many application in metric spaces. Most of the ideas in its proof go back to Baillon's work [31].

Theorem 13. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $(C_\beta)_{\beta\in\Gamma}$ be a decreasing family of one-local retracts of C, where $(\Gamma, \prec)$ is totally ordered. Then $\cap_{\beta\in\Gamma} C_\beta$ is not empty and is one-local retract of C.*

Proof. Consider the family

$$\mathscr{F} = \left\{ \prod_{\beta\in\Gamma} A_\beta : A_\beta \in \mathscr{A}_\omega(C_\beta), (A_\beta) \text{ is decreasing} \right\}. \tag{51}$$

$\mathscr{F}$ is not empty since $\prod_{\beta\in\Gamma} C_\beta \in \mathscr{F}$. $\mathscr{F}$ will be ordered by inclusion; that is, $\prod_{\beta\in\Gamma} A_\beta \subset \prod_{\beta\in\Gamma} B_\beta$ if and only if $A_\beta \subset B_\beta$ for any $\beta \in \Gamma$. From Theorem 12, we know that $\mathscr{A}_\omega(C_\beta)$ is compact, for every $\beta \in \Gamma$. Therefore, $\mathscr{F}$ satisfies the hypothesis of Zorn's Lemma. Hence for every $D \in \mathscr{F}$, there exists a minimal element $A \in \mathscr{F}$ such that $A \subset D$. We claim that if $A = \prod_{\beta\in\Gamma} A_\beta$ is minimal, then there exists $\beta_0 \in \Gamma$ such that $\delta_\omega(A_\beta) = 0$, for every $\beta > \beta_0$. Assume not, that is, $\delta_\omega(A_\beta) > 0$, for every $\beta \in \Gamma$. Fix $\beta \in \Gamma$. For every $K \subset C$, set

$$\mathrm{co}_\beta(K) = \bigcap_{x \in C_\beta} B_\omega(x, r_x(K)). \tag{52}$$

Consider, $A' = \prod_{\alpha\in\Gamma} A'_\alpha$ where

$$\begin{aligned} A'_\alpha &= \mathrm{co}_\beta(A_\beta) \cap A_\alpha \quad \text{if } \alpha \le \beta, \\ A'_\alpha &= A_\alpha \quad \text{if } \alpha \ge \beta. \end{aligned} \tag{53}$$

The family $(A'_{\alpha\ge\beta})$ is decreasing since $A \in \mathscr{F}$. Let $\alpha \le \gamma \le \beta$. Then $A'_\gamma \subset A'_\alpha$, since $A_\gamma \subset A_\alpha$ and $A_\beta = \mathrm{co}_\beta(A_\beta) \cap A_\beta$. Hence the family (A'_α) is decreasing. On the other hand if $\alpha \prec \beta$, then $\mathrm{co}_\beta(A_\beta) \cap A_\alpha \in \mathscr{A}_\omega(C_\alpha)$ since $C_\beta \subset C_\alpha$. Hence $A'_\alpha \in \mathscr{A}_\omega(C_\alpha)$. Therefore, we have $A' \in \mathscr{F}$. Since A is minimal, then $A = A'$. Hence

$$A_\alpha = \mathrm{co}_\beta(A_\beta) \cap A_\alpha, \quad \text{for every } \alpha < \beta. \tag{54}$$

Let $x \in C_\beta$ and $\alpha < \beta$. Since $A_\beta \subset A_\alpha$, then

$$r_x\left(A_\beta\right) \le r_x\left(A_\alpha\right). \quad (55)$$

Because $\mathrm{co}_\beta(A_\beta) = \cap_{y\in C_\beta} B_\omega(y, r_y(A_\beta))$, then we have

$$\mathrm{co}_\beta\left(A_\beta\right) \subset B_\omega\left(y, r_y\left(A_\beta\right)\right), \quad (56)$$

which implies

$$r_y\left(A_\beta\right) \le r_y\left(A_\alpha\right). \quad (57)$$

Since $A_\alpha \subset \mathrm{co}_\beta(A_\beta)$, then

$$r_y\left(A_\beta\right) \le r_y\left(A_\alpha\right) \le r_y\left(\mathrm{co}_\beta\left(A_\beta\right)\right) \le r_y\left(A_\beta\right). \quad (58)$$

Therefore, we have

$$r_y\left(A_\alpha\right) \le r_y\left(A_\beta\right), \quad \text{for every } y \in C_\beta. \quad (59)$$

Using the definition of Chebyshev radius R_ω, we get

$$R_\omega\left(A_\alpha\right) \le R_\omega\left(A_\beta\right). \quad (60)$$

Let $x \in A_\alpha$ and set $s = r_x(A_\alpha)$. Then $x \in \mathrm{co}_\beta(A_\beta)$ since $A_\alpha \subset \mathrm{co}_\beta(A_\beta)$. Hence,

$$x \in \left(\bigcap_{y\in A_\beta} B_\omega\left(y, s\right)\right) \cap \mathrm{co}_\beta\left(A_\beta\right). \quad (61)$$

Since C_β is one-local retract of C, then

$$S_\beta = C_\beta \cap \left(\bigcap_{y\in A_\beta} B_\omega\left(y, s\right)\right) \cap \mathrm{co}_\beta\left(A_\beta\right) \ne \emptyset. \quad (62)$$

Since $A_\beta = C_\beta \cap \mathrm{co}_\beta(A_\beta)$, then we have

$$S_\beta = A_\beta \cap \left(\bigcap_{y\in A_\beta} B_\omega\left(y, s\right)\right). \quad (63)$$

Let $h \in S_\beta$, then $h \in \cap_{y\in A_\beta} B_\omega(y, s)$. Hence, $r_h(A_\beta) \le s$, which implies

$$R_\omega\left(A_\beta\right) \le s = r_x\left(A_\alpha\right), \quad \text{for every } x \in A_\alpha. \quad (64)$$

Hence, $R_\omega(A_\beta) \le R_\omega(A_\alpha)$. Therefore, we have

$$R_\omega\left(A_\beta\right) = R_\omega\left(A_\alpha\right), \quad \text{for every } \alpha, \beta \in \Gamma. \quad (65)$$

Since $\delta_\omega(A_\beta) > 0$, for every $\beta \in \Gamma$, set A''_β to the Chebyshev center of A_β, that is, $A''_\beta = C_\omega(A_\beta)$, for every $\beta \in \Gamma$. Since $R_\omega(A_\beta) = R_\omega(A_\alpha)$, for every $\alpha, \beta \in \Gamma$, then the family (A''_β) is decreasing. Indeed, let $\alpha \prec \beta$ and $x \in A''_\beta$. Then we have $r_x(A_\beta) = R_\omega(A_\beta)$. Since we proved that

$$r_y\left(A_\beta\right) = r_y\left(A_\alpha\right), \quad \text{for every } a \in C_\beta, \quad (66)$$

then

$$r_x\left(A_\alpha\right) = r_x\left(A_\beta\right) = R_\omega\left(A_\beta\right) = R_\omega\left(A_\alpha\right), \quad (67)$$

which implies that $x \in A''_\alpha$. Therefore, we have $A'' = \prod_{\beta\in\Gamma} A''_\beta \in \mathcal{F}$. Since $A'' \subset A$ and A is minimal, we get $A = A''$. Therefore, we have $C_\omega(A_\beta) = A_\beta$ for every $\beta \in \Gamma$. This contradicts the fact that $\mathcal{A}_\omega(C_\beta)$ is normal for every $\beta \in \Gamma$. Hence there exists $\beta_0 \in \Gamma$ such that

$$\delta_\omega\left(A_\beta\right) = 0, \quad \text{for every } \beta \succ \beta_0. \quad (68)$$

The proof of our claim is therefore complete. Then we have $A_\beta = \{x\}$, for every $\beta \succ \beta_0$. This clearly implies that $x \in \cap_{\beta\in\Gamma} C_\beta \ne \emptyset$. In order to complete the proof, we need to show that $S = \cap_{\beta\in\Gamma} C_\beta$ is one-local retract of C. Let $(B_i)_{i\in I}$ be a family of ω-balls centered in S such that $\cap_{i\in I}(B_i) \ne \emptyset$. Set

$$D_\beta = \left(\bigcap_{i\in I} B_i\right) \cap C_\beta, \quad \text{for any } \beta \in \Gamma. \quad (69)$$

Since C_β is a one-local retract of C and the family (B_i) is centered in C_β, then D_β is not empty and $D_\beta \in \mathcal{A}_\omega(C_\beta)$. Therefore,

$$D = \prod_{\beta\in\Gamma} D_\beta \in \mathcal{F}. \quad (70)$$

Let $A = \prod_{\beta\in\Gamma} A_\beta \subset D$ be a minimal element of $\mathcal{F}$. The above proof shows that

$$\bigcap_{\beta\in\Gamma} A_\beta \subset \bigcap_{\beta\in\Gamma} D_\beta \ne \emptyset. \quad (71)$$

The proof of our theorem is complete. □

The next theorem will be useful to prove the main result of the next section.

Theorem 14. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathcal{A}_\omega(C)$ is normal and compact. Let $(C_\beta)_{\beta\in\Gamma}$ be a family of one-local retracts of C such that for any finite subset I of Γ. Then $\cap_{\beta\in\Gamma} C_\beta$ is not empty and is one-local retract of C.*

Proof. Consider the family $\mathcal{F}$ of subsets $I \subset \Gamma$ such that, for any finite subset $J \subset \Gamma$ (empty or not), we have $\cap_{\alpha\in I\cup J} C_\alpha$ that is nonempty one-local retract of C. Note that $\mathcal{F}$ is not empty since any finite subset of Γ is in $\mathcal{F}$. Using Theorem 13, we can show that $\mathcal{F}$ satisfies the hypothesis of Zorn's lemma. Hence $\mathcal{F}$ has a maximal element $I \subset \Gamma$. Assume $I \ne \Gamma$. Let $\alpha \in \Gamma \setminus I$. Obviously we have $I \cup \{\alpha\} \in \mathcal{F}$. This is a clear contradiction with the maximality of I. Therefore we have $I = \Gamma \in \mathcal{F}$; that is, $\cap_{\beta\in\Gamma} C_\beta$ is not empty and is a one-local retract of C. □

4. Common Fixed Point Result

In this section we discuss the existence of a common fixed point of a family of commutative ω-nonexpansive mappings in modular metric space which either generalize or improve the corresponding recent common fixed point results of [26, 27].

First, we will need to discuss the case of finite families.

Theorem 15. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $\mathscr{F} = \{T_1, T_2, \ldots, T_n\}$ be a family of commutative ω-nonexpansive mappings defined on C. Then the family $\mathscr{F}$ has a common fixed point. Moreover, the common fixed point set $\operatorname{Fix}(\mathscr{F})$ is a one-local retract of C.*

Proof. First, let us prove that $\operatorname{Fix}(\mathscr{F})$ is not empty. Using Theorem 9, $\operatorname{Fix}(T_1)$ is nonempty one-local retract of C, and then Theorem 12 implies that $\mathscr{A}_\omega(\operatorname{Fix}(T_1))$ is compact and normal. On the other hand since T_1 and T_2 are commutative, we have

$$T_2(\operatorname{Fix}(T_1)) \subset \operatorname{Fix}(T_1). \tag{72}$$

Hence T_2 has a fixed point in $\operatorname{Fix}(T_1)$. If we restrict ourselves to $\operatorname{Fix}(T_1, T_2)$, the common fixed point set of T_1 and T_2, then one can prove in an identical argument that T_3 has a fixed point in $\operatorname{Fix}(T_1, T_2)$. Step by step, we can prove that the common fixed point set $\operatorname{Fix}(\mathscr{F})$ of $T_1, T_2, \ldots, T_n$ is not empty. The same argument used to prove that the fixed point set of ω-nonexpansive map is a one-local retract can be reduced here to prove that $\operatorname{Fix}(\mathscr{F})$ is one-local retract. □

The following result extends [26, Theorem 8] to the setting of modular metric space.

Theorem 16. *Let (X, ω) be a modular metric space and let C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $\mathscr{F} = (T_i)_{i \in I}$ be a family of commutative ω-nonexpansive mappings defined on C. Then the family $\mathscr{F}$ has a common fixed point. Moreover, the common fixed point set $\operatorname{Fix}(\mathscr{F})$ is a one-local retract of C.*

Proof. Let $\Gamma = \{\beta : \beta$ be a nonempty finite subset of $I\}$. Theorem 15 implies that, for every $\beta \in \Gamma$, the set $F_\beta = \cap_{i \in \beta} \operatorname{Fix}(T_i)$ of common fixed point set of the mappings T_i, $i \in \beta$, is nonempty one-local retract of C. Clearly the family $(F_\beta)_{\beta \in \Gamma}$ is decreasing and satisfies the assumptions of Theorem 14. Therefore, we deduced that $\cap_{\beta \in \Gamma} F_\beta$ is nonempty and is a one-local retract of C. □

Acknowledgment

The author would like to thank Professor Mohamed A. Khamsi with whom the author had many fruitful discussions regarding this work.

References

[1] V. V. Chistyakov, "Modular metric spaces. I. Basic concepts," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 72, no. 1, pp. 1–14, 2010.

[2] V. V. Chistyakov, "Modular metric spaces. II. Application to superposition operators," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 72, no. 1, pp. 15–30, 2010.

[3] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, Japan, 1950.

[4] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.

[5] W. Orlicz, *Collected Papers, Part I, II*, PWN Polish Scientific Publishers, Warsaw, Poland, 1988.

[6] L. Diening, *Theoretical and numerical results for electrorheological fluids [Ph.D. thesis]*, University of Freiburg, Freiburg, Germany, 2002.

[7] M. Ruzicka, *Electrorheological Fluids Modeling and Mathematical Theory*, Springer, Berlin, Germany, 2002.

[8] P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen, "The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values," *Potential Analysis*, vol. 25, no. 3, pp. 205–222, 2006.

[9] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, Oxford, UK, 1993.

[10] M. A. Khamsi, W. M. Kozłowski, and S. Reich, "Fixed point theory in modular function spaces," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 14, no. 11, pp. 935–953, 1990.

[11] M. A. Khamsi and W. M. Kozlowski, "On asymptotic pointwise nonexpansive mappings in modular function spaces," *Journal of Mathematical Analysis and Applications*, vol. 380, no. 2, pp. 697–708, 2011.

[12] L. P. Belluce and W. A. Kirk, "Nonexpansive mappings and fixed-points in Banach spaces," *Illinois Journal of Mathematics*, vol. 11, pp. 474–479, 1967.

[13] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, pp. 1041–1044, 1965.

[14] R. E. Bruck,, "A common fixed point theorem for a commuting family of nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 53, pp. 59–71, 1974.

[15] T. C. Lim, "A fixed point theorem for families on nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 53, pp. 487–493, 1974.

[16] K. Goebel and S. Reich, *Reich Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1984.

[17] K. Goebel, T. Sekowski, and A. Stachura, "Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 1011–1021, 1980.

[18] W. A. Kirk, "Fixed point theorems in *CAT(0)* spaces and $\mathbb{R}$-*trees*," *Fixed Point Theory and Applications*, no. 4, pp. 309–316, 2004.

[19] N. Hussain and M. A. Khamsi, "On asymptotic pointwise contractions in metric spaces," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 71, no. 10, pp. 4423–4429, 2009.

[20] W. A. Kirk, "Fixed points of asymptotic contractions," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 645–650, 2003.

[21] W. A. Kirk, "Asymptotic pointwise contractions, plenary lecture," in *Proceedings of the 8th International Conference on Fixed Point Theory and Its Applications*, pp. 16–22, ChiangMai University, Chiang Mai, Thailand, July 2007.

[22] W. A. Kirk and H.-K. Xu, "Asymptotic pointwise contractions," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 12, pp. 4706–4712, 2008.

[23] J.-P. Penot, "Fixed point theorems without convexity," *Mémoires de la Société Mathématique de France*, vol. 60, pp. 129–152, 1979, Analyse non convexe.

[24] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, pp. 1004–1006, 1965.

[25] M. A. Khamsi, "On metric spaces with uniform normal structure," *Proceedings of the American Mathematical Society*, vol. 106, no. 3, pp. 723–726, 1989.

[26] M. A. Khamsi, "One-local retract and common fixed point for commuting mappings in metric spaces," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 27, no. 11, pp. 1307–1313, 1996.

[27] S. A. Al-Mezel, A. Al-Roqi, and M. A. Khamsi, "One-local retract and common fixed point in modular function spaces," *Fixed Point Theory and Applications*, vol. 2012, article 109, 13 pages, 2012.

[28] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley, New York, NY, USA, 2001.

[29] W. M. Kozlowski, *Modular Function Spaces*, vol. 122 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1988.

[30] A. A. N. Abdou and M. A. Khamsi, "On the fixed points of nonexpansive maps in modular metric spaces," Preprint.

[31] J. B. Baillon, "Nonexpansive mappings and hyperconvex spaces," *Contemporary Mathematics*, vol. 72, pp. 11–19, 1988.