# New Developments in Time-Delay Systems and Its Applications in Engineering 

Guest Editors: Guangdeng Zong, Wei Xing Zheng, Ligang Wu, and Yang Yi


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## Mathematical Problems in Engineering

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## Editorial

# New Developments in Time-Delay Systems and Its Applications in Engineering 

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A system is said to have a delay when the rate of variation in the system state depends on past states. Such a system is called a time-delay system. Over the past several decades, there has been extensive concern in time-delay systems in the field of control theory and engineering. On one hand, time delays are inherent in various engineering systems such as long transmission lines in pneumatic systems, networked control systems, nuclear reactors, rolling mills, hydraulic systems, and manufacturing. On the other hand, it has been recognized that time delay is often a source of generation of oscillation, instability, or poor control performance of underlying control systems. Therefore, for time-delay systems, there are many challenging issues, for example, stability analysis, stabilization, $H_{\infty}$ control, robust performance analysis, model identification problem, and antidisturbance control, that need to be solved.

This special issue contains twenty-one papers, of which seven are related to stability analysis and stabilization of different time-delay systems and four concern networked control systems with time-delay. Six papers discuss antidisturbance problem and H -infinite robust control for various time-delay systems. There is also a single paper focusing on collaboration control for multiagent systems with sampling delay. Another paper deals with online identification of multivariable discrete time-delay systems. Finally, two papers cover the switched models with state delays.
"Stability and $L_{1}$ gain analysis for positive 2D systems with state delays in the Roesser model" by Z. Duan et al. concerns
the state-delay term described by the Roesser model and shows the sufficient conditions of delay-dependent stability for positive 2D systems. "New exponential stability conditions of switched BAM neural networks with delays" by Y. Yang constructs a new switching dependent Lyapunov-Krasovskii function and solves the exponential stability problem for a class of discrete-time switched BAM neural networks with time delay, while "Stability analysis for delayed neural networks: reciprocally convex approach" by H. Yu et al. investigates the global stability problem for a class of continuous neural networks with time-varying delay by using reciprocally convex combination approaches. "Stability analysis of a harvested prey-predator model with stage structure and maturation delay" by C. Liu et al. proposes a harvested prey-predator model to investigate the effects of density dependent maturation delay and discusses the global stability of positive equilibrium by utilizing an iterative technique. " $P$ moment stability of stochastic differential delay systems with impulsive jump and Markovian switching" by L. Gao gives a novel P-moment stability criteria for stochastic differential delay systems. "A simplified descriptor system approach to delay-dependent stability and robust performance analysis for discrete-time systems with time delays" by F. Xu and D. Li reduces the conservatism of the existing results of time-delay discrete-time systems by removing the redundant matrix variables. "Directly solving special second order delay differential equations using Runge-Kutta-Nyström method" by M. Mechee et al. studies a novel stability analysis method for
the second order delay differential equations based on the designed Runge-Kutta-Nyström algorithm.
"Robust fault tolerant control for a class of time-delay systems with multiple disturbances" by S. Cao and J. Qiao addresses a robust fault tolerant control approach for a class of nonlinear systems with time-delay, actuator faults, and multiple disturbances using a composite strategy consisting of disturbance observer-based control and fault accommodation. "Enhanced disturbance-observer-based control for a class of time-delay system with uncertain sinusoidal disturbances" by X. Wen provides a thorough study on disturbance-observerbased control (DOBC) for a class of time-delay systems under uncertain sinusoidal disturbances. "Composite disturbance observer-based control and $H_{\infty}$ output tracking control for discrete-time switched systems with time-varying delay" by H. Sun and L. Hou concerns the problem of $H_{\infty}$ output tracking control for discrete-time switched systems with time-varying delay and external disturbances. Furthermore, "Robust $H_{\infty}$ control of uncertain T-S fuzzy time-delay system: a delay decomposition approach" by C. Gong and C. Han concerns with the problem of robust $H_{\infty}$ control of a class of uncertain time-delay fuzzy systems by utilizing the instrumental idea of delay decomposition. "A novel approach to $H_{\infty}$ control design for linear neutral time-delay systems" by H. Xia et al. explores some delay-dependent sufficient conditions of linear time-varying neutral systems by designing the state feedback controller with $H_{\infty}$ performance level. " $H_{\infty}$ control for flexible spacecraft with time-varying input delay" by R. Zhang et al. provides an effective $H_{\infty}$ control algorithm for flexible spacecraft with time-varying control input delay and obtains a more flexible result.
"Performance of networked control systems" by Y. Zhang et al. designs an optimal controller for the new NCSs with data packets dropout and minimizes the infinite performance index at each sampling time. "Control for networked control systems with time delays and packet dropouts" by Y. Wang et al. explores the mean square exponential $H_{\infty}$ performance for NCSs with random delay and packet dropout. "Improving the performance metric of wireless sensor networks with clustering Markov chain model and multilevel fusion" by S. Havedanloo and H. R. Karimi proposes a performance metric evaluation of a distributed detection wireless sensor network with respect to IEEE 802.15 .4 standard and improves the performance of network in terms of reliability, packet failure, average delay, and power consumption. "Robust filtering for networked stochastic systems subject to sensor nonlinearity" by G . Wu et al. considers the effects of the sensor saturation, output quantization, and network-induced delay in the network environment and models the random delays as a linear function of the stochastic variable described by a Bernoulli random binary distribution.
"Collaboration control of fractional-order multiagent systems with sampling delay" by H.-Y. et al. Yang investigates the novel collaboration control problems of continuous-time networked fractional-order multiagent systems via sampled control and sampling delay and many sufficient conditions for reaching consensus with sampled data, and sampling delay can be obtained. "Online identification of multivariable discrete time delay systems using a cursive least square algorithm"
by S. Bedoui identifies the time delays and the parameters of linear discrete-time delay multivariable systems by using the least square approach. Moreover, "Robust reliable control of uncertain discrete impulsive switched systems with state delays" by X. Li et al. presents the problem of robust reliable control for a class of uncertain discrete impulsive switched systems with state delays and actuator failures. "Sliding mode control based on observer for a class of state-delayed switched systems with uncertain perturbation" by Z . He et al. proposes a state observer-based sliding mode control design methodology for a class of continuous-time state-delayed switched systems with unmeasurable states and nonlinear uncertainties.

It is noted that both the stability analyze method and the $H_{\infty}$ control for complex time-delay systems have always been hot issues in the field of control theory for the last 20 years. Recently, networked control systems (NCSs) have been extensively investigated due to its broad applications in engineering, in which the phenomenon of networkedinduced communication time-delay has received more and more research interests. It is well known that external disturbances originating from various sources exist in almost all controlled systems accompanied by increasingly largescale and complicated industrial processes. Thus, the research of antidisturbance control and disturbance attenuation performance for complex time-delay systems is a challenging problem. On the other hand, the performance analyze for switched systems with time-delay has also have received considerable attention because of their applicability and significance in various areas. In summary, almost all papers in this special issue concern those recent focus and some new developments emerged in the time-delay systems. Moreover, many practical applications can also be found in this special issue, such as truck trailer systems, flexible spacecraft, A4D aircraft, and NCSs.

Of course, the selected topics and papers are not a comprehensive representation of the area of this special issue. Nonetheless, they represent the rich and many faceted knowledge that we have the pleasure of sharing with the readers.

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Guangdeng Zong
Wei Xing Zheng
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## Research Article

# Online Identification of Multivariable Discrete Time Delay Systems Using a Recursive Least Square Algorithm 

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#### Abstract

This paper addresses the problem of simultaneous identification of linear discrete time delay multivariable systems. This problem involves both the estimation of the time delays and the dynamic parameters matrices. In fact, we suggest a new formulation of this problem allowing defining the time delay and the dynamic parameters in the same estimated vector and building the corresponding observation vector. Then, we use this formulation to propose a new method to identify the time delays and the parameters of these systems using the least square approach. Convergence conditions and statistics properties of the proposed method are also developed. Simulation results are presented to illustrate the performance of the proposed method. An application of the developed approach to compact disc player arm is also suggested in order to validate simulation results.


## 1. Introduction

Time delay system identification has received great attention in the last years since time delay is a physical phenomenon which arises in most control loops industrial systems [1,2]. Several reasons cause the presence of time delay in control loops. In fact, it may be an inherent feature of the system such as processes of transport (mass, energy, and information), higher order processes, and accumulation of time lags in several systems that are connected in series. It may also be introduced by the devices of control loops, such as response times of sensors and actuators, computation time of control laws, and information transmission time in networks. This delay can be neglected if its value is too small for the system time constants. Otherwise, it cannot be neglected, and the dynamic representation of the system must be described by a time delay model. This model is, generally, constructed using the identification approach which allows building a mathematical model from input-output data.

The identification of time delay systems is known to be a challenging identification problem because it involves both the estimation of dynamic parameters and time delay. Numerous methods have been proposed in the literature for the identification of time delay systems [3-15].

Among these methods, the graphical approach has been the most popular since it represents the first method proposed in the literature for the identification of continuous time delay systems [16]. Moreover, it is frequently used to compute the parameters of PID controllers for industrial processes. It consists in determining the time delay and the dynamic parameters of the system from its step response. The main advantage of this approach lies in the simplicity of its implementation. However, it produces inaccurate results because it is very sensitive to measurement noises. Another popular approach is proposed in [9]. It is based on the approximation of the time delay by a rational transfer function using classical approximations such as Pade or Laguerre. This method can insure very satisfactory results in the case of linear systems with constant time delays and lower order. However, its performance degrades rapidly in the case of higher order systems or important time delays. Moreover, it raises the computational complexity because it increases the number of parameters to be estimated. The parametrisation approach can be considered as one of the interesting methods because it is based on theoretical concepts of discrete time systems [17]. It consists, firstly, in inserting a known time delay in the numerator of the discrete time model, secondly, in estimating the dynamic
parameters of the system using a recursive algorithm, and, finally, in deducing the time delay from zero coefficients of the numerator. In practice, it is difficult or rather impossible to have zero coefficients from experimental data. Indeed, we must set a threshold, which is a delicate task, mainly in the case of a noisy output. The method developed in [3] consists, It consists, firstly, in using the recursive least square approach to identify the parameters assuming that the time delay is known, and secondly, in estimating the time delay, taking into account the results of the first step. The time delay may be identified either by maximizing the correlation function or by minimizing the quadratic error. This method assumes that the domain range of the time delay is a priori known. We also mention the method presented in [18]. It allows the identification of the time delay and the system parameters using the Levenberg-Marquaydt optimization approach to minimize the prediction error. An online identification algorithm for continuous-time singleinput single-output (SISO) linear time delay systems with uncertain time invariant parameters is developed in [10]. It consists in constructing a sliding mode-based observer of an underlying system with uncertain parameters. This observer is then used to design an adaptive identifier of system parameters. A linear filtering method is introduced in [19] for simultaneous parameter and time delay estimation of transfer function models. This method estimates the time delay along other model using an iterative way through simple linear regression. Another method that identifies the time delay and the system parameters is presented in [20] which is based on the correlation technique. The method developed in [21] allows the identification of time delay and the parameters of a system operating in the presence of colored noise. This method is based on correlation analysis. The method developed in [12, 22] allows the identification of time delay and the parameters. It minimizes the error between the process output and the process predictive model output, and then the variable time delay parameter is identified. In our previous work, we have proposed two methods for the simultaneous identification of the time delay and dynamic parameters of monovariable time delay discrete systems. The first method is based on the least square approach [23, $24]$. The second method consists in minimizing a quadratic criterion using the gradient approach [25].

Most of these approaches deal with the problem of the identification of single-input single-output (SISO) time delay systems. However, the problem of multi-input multioutput (MIMO) time delay systems is one of the most difficult problems that represents an area of research where few efforts have been devoted in the past. The use of time delay approximation is extended to the MIMO case [26]. In fact, the authors deal with the problem of the identification of time delay processes using an overparameterization method. In [27], a method is developed for time delay estimation in the frequency domain of MIMO systems based on the combination of continuous wavelet transform (CWT) and cross-correlation. During the estimation, cross-correlation computations are carried out between the CWT coefficients of the input and the output data. The authors of [28] have proposed a simple method based on the combination of two
well-known approaches: time delay estimation from impulse response and subspace identification.

In this paper, we propose an alternative approach for the problem of simultaneous identification of linear discrete time delay multivariable systems. Indeed, we develop a new formulation of the problem allowing to define the time delay and the dynamic parameters in the same estimated vector and to build the corresponding observation vector. Then, we use this formulation to propose a new method to identify the time delays and the parameters of these systems using the least square approach. Convergence conditions and statistics properties of the proposed method are also developed. Simulation and experimental examples are presented to illustrate the effectiveness of the proposed methods and to compare their performance in terms of convergence and speed. Our approach presents several interesting properties which can be summarized as follows.
(i) The simultaneous identification of the time delays and parameters matrices is achieved by a new formulation of the parameters matrices.
(ii) No a priori knowledge of the time delay is required. In fact, most of the publications assume the knowledge of the time delay variation range or the initial condition.
(iii) The consistency of recursive least square methods has received much attention in the identification literature. In this paper, the proof of the consistency of the estimates is established.
(iv) It can be used to deal with control adaptive purposes.

This paper is organized as follows. Section 2 presents the model and its assumptions. In Section 3, we propose an extended least square algorithm for simultaneous online identification of unknown time delays and parameters of multivariable discrete time delay systems. Moreover, we develop the convergence properties of the estimates in order to show that the obtained estimates are unbiased. Simulation results and experimental test are provided in the last section.

## 2. Problem Statement

In this paper, we address the problem of identification of square linear multivariable delay system with $p$ inputs and $p$ ARX model:

$$
\begin{equation*}
A\left(q^{-1}\right) Y(k)=B\left(q^{-1}\right) \underline{U}(k)+V(k) \tag{1}
\end{equation*}
$$

where $Y(k)=\left[y_{1}(k), \ldots, y_{p}(k)\right]^{T}$ and $\underline{U}(k)=\left[U_{1}(k-\right.$ $\left.\left.d_{1}\right), \ldots, U_{p}\left(k-d_{p}\right)\right]^{T}$ are the outputs and the delayed inputs of the system at time $k$ and $V(k)=\left[V_{1}(k), \ldots, V_{p}(k)\right]^{T}$ is a vector of independent random variables sequences. Let $\{D=$ $\left.\operatorname{diag}\left[d_{1}, \ldots, d_{p}\right] / d_{i} \in \mathbb{N}, i=1, \ldots, p\right\}$ be the time delay diagonal matrix, also called the interactive matrix, and $A\left(q^{-1}\right)$ and $B\left(q^{-1}\right)$ two polynomial matrices in the unit backward
shift operator $q^{-1}$ (i.e., $\left.q^{-1} y_{i}(k)=y_{i}(k-1), i=1, \ldots, p\right)$, defined by

$$
\begin{gather*}
A\left(q^{-1}\right)=I_{p}+A_{1} q^{-1}+\cdots+A_{n_{a}} q^{-n_{a}}, \\
\operatorname{dim} A_{r}=(p, p), \quad r \in\left[1, n_{a}\right] \\
B\left(q^{-1}\right)=B_{1} q^{-1}+\cdots+B_{n_{b}} q^{-n_{b}}  \tag{2}\\
\operatorname{dim} B_{r}=(p, p), \quad r \in\left[1, n_{b}\right]
\end{gather*}
$$

The delayed inputs $\underset{\sim}{U}(k)$ can be expressed as

$$
\begin{equation*}
\underline{U}(k)=\Omega U(k) \tag{3}
\end{equation*}
$$

where $\Omega$ is a diagonal matrix defined as

$$
\Omega=\left(\begin{array}{cccc}
q^{-d_{1}} & 0 & \cdots & 0  \tag{4}\\
0 & q^{-d_{2}} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & q^{-d_{p}}
\end{array}\right)
$$

The following assumptions are made.
(A1) The two polynomial matrices $A\left(q^{-1}\right)$ and $B\left(q^{-1}\right)$ have no common left factor.
(A2) The orders $n_{a}$ and $n_{b}$ of the model are known.
(A3) The input sequences $\left\{\underset{U}{U}=\left[U_{1}\left(k-d_{1}\right), \ldots, U_{p}(k-\right.\right.$ $\left.\left.\left.d_{p}\right)\right]^{T}\right\}$ are independent of $V(k)$, mutually independent and identically distributed with $E[U(k)]=0$ and $E\left[U(k) U(k)^{T}\right]=I$, and are persistently exciting (PE).
(A4) The disturbance $V(k)=\left[V_{1}(k), \ldots, V_{p}(k)\right]^{T}$ is sequences of independent, identically distributed random variables with zero mean and finite variance $\Sigma=$ $\left\{\sigma_{1}^{2}, \ldots, \sigma_{p}^{2}\right\}$.
(A5) The inputs, the outputs, and the noises are causal; that is, $U(k)=[0], Y(k)=[0]$, and $V(k)=[0]$ for $k \leqslant 0$.

Problem Statement. The goal is to develop a recursive algorithm to estimate, simultaneously, the time delay matrix $D$ and the matrices $\left\{A_{i}(k), B_{i}(k)\right\}$ using the input/output measurement data $\{U(k), Y(k)\}$.

In the following, we present three necessary definitions.
Definition 1. Operator round (d) is defined by

$$
\operatorname{round}\left(d_{i}\right)= \begin{cases}\operatorname{int}\left(d_{i}\right)+1 & \text { if } d_{i}-\operatorname{int}\left(d_{i}\right) \geqslant 0.5  \tag{5}\\ \operatorname{int}\left(d_{i}\right) & \text { if } d_{i}-\operatorname{int}\left(d_{i}\right)<0.5\end{cases}
$$

where int (d) denotes the integer part of $d_{i}, i=1, \ldots, p$.
Definition 2. Operator $\widetilde{d}(\cdot)$ is defined by

$$
\begin{equation*}
\widetilde{d}_{i}(\cdot)=\operatorname{round}\left(\widehat{d}_{i}(\cdot)\right), \quad i=1, \ldots, p \tag{6}
\end{equation*}
$$

Definition 3. Operator $\widetilde{D}(\cdot)$ is defined by

$$
\widetilde{D}(\cdot)=\left(\begin{array}{cccc}
\tilde{d}_{1} & 0 & \cdots & 0  \tag{7}\\
0 & \tilde{d}_{2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \tilde{d}_{p}
\end{array}\right)
$$

## 3. The Proposed Approach

In this section, an extended least square algorithm for simultaneous online identification of time delays and parameter matrices is developed.

Equation (1) can be rewritten as

$$
\begin{equation*}
Y(k)=\Theta^{T} \varphi(k, D)+V(k) \tag{8}
\end{equation*}
$$

where $\Theta$ is the parameter matrix and $\varphi(k, D)$ is the observation vector defined as

$$
\begin{gather*}
\Theta^{T}=\left[A_{1}, A_{2}, \ldots, A_{n_{a}}, B_{1}, B_{2}, \ldots, B_{n_{b}}\right], \\
\varphi(k, D)=\left[-Y^{T}(k-1),-Y^{T}(k-2), \ldots,-Y^{T}\left(k-n_{a}\right),\right. \\
\left.\underline{U}^{T}(k-1), \ldots, \underline{U}^{T}\left(k-n_{b}\right)\right]^{T} . \tag{9}
\end{gather*}
$$

On the other hand, the estimated output is described by the following relation:

$$
\begin{equation*}
\widehat{Y}(k)=\widehat{\Theta}^{T} \varphi(k, \widetilde{D}) \tag{10}
\end{equation*}
$$

where $\widehat{\Theta}$ and $\widetilde{D}=\operatorname{diag}\left[\widetilde{d}_{1}, \ldots, \widetilde{d}_{p}\right]$ represent, respectively, the estimated parameter matrix and the estimated time delay matrix.

Let us consider the prediction error $\Upsilon(k)$ given by

$$
\begin{equation*}
\Upsilon(k)=Y(k)-\widehat{\Theta}^{T} \varphi(k, \widetilde{D}) \tag{11}
\end{equation*}
$$

Since parameter matrix $\widehat{\Theta}$ does not contain the unknown time delays $\widetilde{D}$, then consequently it is not directly applicable to achieve our objective which is the simultaneous identification of the time delays and the parameter matrices of the multivariable discrete time delay systems (1).

To overcome this problem, we suggest considering the time delay matrix in parameter matrices $\Theta$ to be estimated. Indeed, the new matrix, called generalized matrix, is given by

$$
\begin{equation*}
\Theta_{G}^{T}=[\Theta, D] \tag{12}
\end{equation*}
$$

Moreover, we propose the use of the negative gradient of the error to obtain an appropriate observation vector which is given by

$$
\begin{equation*}
\Phi\left(k, \widehat{\Theta}_{G}\right)=-\frac{\partial \Upsilon}{\partial \widehat{\Theta}_{G}} \tag{13}
\end{equation*}
$$

Then,

$$
\begin{gather*}
\Phi\left(k, \widehat{\Theta}_{G}\right)=\left[\varphi^{T}(k, \widetilde{D}),-\frac{\partial Y}{\partial \widetilde{D}}\right]^{T} \\
\Phi\left(k, \widehat{\Theta}_{G}\right)=\left[\varphi^{T}(k, \widetilde{D}), \frac{\partial \widehat{Y}}{\partial \widetilde{D}}\right]^{T}  \tag{14}\\
\Phi\left(k, \widehat{\Theta}_{G}\right)=\left[\varphi^{T}(k, \widetilde{D}), \frac{\partial}{\partial \widehat{D}} \sum_{i=1}^{n_{b}} U^{T}(k-i) \widehat{B}_{i} \widehat{\Omega}\right]^{T} .
\end{gather*}
$$

The use of the approximation of $\operatorname{Ln}(q) \approx 1-q^{-1}$ (see the appendix) leads to

$$
\begin{equation*}
\Phi\left(k, \widehat{\Theta}_{G}\right)=\left[\varphi^{T}(k, \widetilde{D}),-\sum_{i=1}^{n_{b}} \Delta U^{T}(k-i) \widehat{B}_{i} \widehat{\Omega}\right]^{T} \tag{15}
\end{equation*}
$$

where $\Delta U(k)=U(k)-U(k-1)$.
Replacing $\varphi^{T}(k, \widetilde{D})$ by its expression, we obtain the generalized observation vector:

$$
\Phi\left(k, \widehat{\Theta}_{G}\right)=\left[\begin{array}{c}
-Y^{T}(k-1)  \tag{16}\\
\vdots \\
-Y^{T}\left(k-n_{a}\right) \\
\widehat{\Omega} U^{T}(k-1) \\
\vdots \\
\widehat{\Omega} U^{T}\left(k-n_{b}\right) \\
-\sum_{i=1}^{n_{b}} \Delta U^{T}(k-i) \widehat{B_{i}} \widehat{\Omega}
\end{array}\right]
$$

An estimation $\widehat{\Theta}_{G}$ of $\Theta_{G}$ is denoted by the minimization of the following criterion:

$$
\begin{equation*}
J\left(k, \Theta_{G}\right)=\frac{1}{2} \sum_{i=0}^{k} \Upsilon(i)^{2} \tag{17}
\end{equation*}
$$

Then, the partial derivative of the criterion with respect to the generalized matrix is

$$
\begin{equation*}
\frac{\partial J}{\partial \widehat{\Theta}_{G}}=\sum_{i=0}^{k} \frac{\partial \Upsilon(i)}{\partial \widehat{\Theta}_{G}} \Upsilon(i)=-\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right) \Upsilon(i) \tag{18}
\end{equation*}
$$

So

$$
\begin{equation*}
\frac{\partial J}{\partial \widehat{\Theta}_{G}}=-\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right)\left[Y(i)-\widehat{\Theta}^{T} \varphi(i, \widetilde{D})\right] \tag{19}
\end{equation*}
$$

Let us consider

$$
\begin{equation*}
\psi=-\sum_{j=1}^{n_{b}} \widetilde{D} \Delta U^{T}(i-j) \widehat{B}_{j} \widehat{\Omega} \tag{20}
\end{equation*}
$$

Adding and subtracting from (19) the term $\psi$, given by (20), we have

$$
\begin{align*}
\frac{\partial J}{\partial \widehat{\Theta}_{G}} & =-\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right)\left[Y(i)-\Phi^{T}\left(i, \widehat{\Theta}_{G}\right) \widehat{\Theta}_{G}+\psi\right]  \tag{21}\\
& =-\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right)\left[Y(i)+\psi-\Phi^{T}\left(i, \widehat{\Theta}_{G}\right) \widehat{\Theta}_{G}\right]
\end{align*}
$$

Canceling the partial derivative of the criterion, we obtain

$$
\begin{align*}
\widehat{\Theta}_{G}(k)= & {\left[\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right) \Phi^{T}\left(i, \widehat{\Theta}_{G}\right)\right]^{-1} \sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right) }  \tag{22}\\
& \times[Y(i)+\psi] .
\end{align*}
$$

Let,

$$
\begin{equation*}
R(k)=\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right) \Phi^{T}\left(i, \widehat{\Theta}_{G}\right) . \tag{23}
\end{equation*}
$$

Based on assumption A3 which ensures that matrix $R(k)$ is invertible [29], (22) can be rewritten as

$$
\begin{equation*}
\widehat{\Theta}_{G}(k)=R(k)^{-1} \sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right)[Y(i)+\psi] . \tag{24}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\widehat{\Theta}_{G}(k)=R(k)^{-1}( & \sum_{i=0}^{k-1} \Phi\left(i, \widehat{\Theta}_{G}\right)[Y(i)+\psi] \\
& \left.+\Phi\left(k, \widehat{\Theta}_{G}\right)[Y(k)+\psi]\right) \tag{25}
\end{align*}
$$

Using (22), we have

$$
\begin{align*}
\widehat{\Theta}_{G}(k)=R(k)^{-1} & \left(R(k-1) \widehat{\Theta}_{G}(k-1)\right. \\
& \left.+\Phi\left(k, \widehat{\Theta}_{G}\right)[Y(k)+\psi]\right) \\
=R(k)^{-1} & \left(R(k) \widehat{\Theta}_{G}(k-1)\right.  \tag{26}\\
& -\Phi\left(k, \widehat{\Theta}_{G}\right) \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) \widehat{\Theta}_{G}(k-1) \\
& \left.+\Phi\left(k, \widehat{\Theta}_{G}\right)[Y(k)+\psi]\right) .
\end{align*}
$$

So

$$
\begin{align*}
\widehat{\Theta}_{G}(k)= & \widehat{\Theta}_{G}(k-1)+R(k)^{-1} \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& \times\left(-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) \widehat{\Theta}_{G}(k-1)+[Y(k)+\psi]\right) . \tag{27}
\end{align*}
$$

It follows from (27) that

$$
\begin{align*}
\widehat{\Theta}_{G}(k)= & \widehat{\Theta}_{G}(k-1)+R(k)^{-1} \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& \times\left(Y(k)-\widehat{\Theta}^{T} \varphi(k, \widetilde{D})\right) . \tag{28}
\end{align*}
$$

Thus,

$$
\begin{align*}
\widehat{\Theta}_{G}(k) & =\widehat{\Theta}_{G}(k-1)+P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \Upsilon(k)  \tag{29}\\
P(k) & =\left[\sum_{i=0}^{k} \Phi\left(i, \widehat{\Theta}_{G}\right) \Phi^{T}\left(i, \widehat{\Theta}_{G}\right)\right]^{-1} \\
& =\left[R(k-1)+\Phi\left(k, \widehat{\Theta}_{G}\right) \Phi^{T}\left(k, \widehat{\Theta}_{G}\right)\right]^{-1} \tag{30}
\end{align*}
$$

Using the matrix inversion lemma given by [29],

$$
\begin{equation*}
\left[B+C D^{T}\right]^{-1}=B^{-1}-B^{-1} C D^{T} B^{-1}\left[1+D^{T} B^{-1} C\right]^{-1} \tag{31}
\end{equation*}
$$

Let $B=R(k-1), C=\Phi\left(k, \Theta_{G}\right)$, and $D=\Phi\left(k, \Theta_{G}\right)^{T}$, and then we have
$P(k)=P(k-1)-\frac{P(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k-1)}{1+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)}$.

The previous approach can be summarized by Algorithm 1.

## 4. Convergence Properties

4.1. Consistency. For the system in (1), assume that (A3) and (A4) hold. Then for any $c>1$, the parameter estimation error, $\widehat{\Theta}_{G} \rightarrow \Theta_{G}$, associated with the LS algorithm in (29) and (30) satisfies

$$
\begin{equation*}
\left\|\widehat{\Theta}_{G}-\Theta_{G}\right\|^{2}=O\left(\frac{[\operatorname{Ln} r(k)]^{c}}{\lambda_{\min }\left[P^{-1}(k)\right]}\right) \tag{33}
\end{equation*}
$$

where $r(k)$ is the trace of the covariance matrix $P^{-1}(k)$ and $\lambda_{\text {min }}$ represents the minimum eigenvalues of $P^{-1}(k)$.
Proof. Define the parameter estimation error vector:

$$
\begin{equation*}
\bar{\Theta}_{G}:=\widehat{\Theta}_{G}-\Theta_{G} . \tag{34}
\end{equation*}
$$

Using (27) and (11) we have

$$
\begin{align*}
\bar{\Theta}_{G}(k)= & \bar{\Theta}_{G}(k-1)+P \Phi\left(k, \Theta_{G}\right) \\
\times & {\left[\widehat{\Theta}^{T}(k-1) \varphi(k, \widehat{D})\right.}  \tag{35}\\
& \left.+V(k)-\Theta^{T}(k-1) \varphi(k, D)\right] \\
= & \bar{\Theta}_{G}(k-1)+P \Phi\left(k, \Theta_{G}\right)[-\bar{\rho}(k)+V(k)]
\end{align*}
$$

where

$$
\begin{equation*}
\bar{\rho}(k)=\widehat{\Theta}^{T}(k-1) \varphi(k, \widehat{D})-\Theta^{T}(k-1) \varphi(k, D) \tag{36}
\end{equation*}
$$

Equation (35) can be rewritten as (see the appendix)

$$
\begin{equation*}
\bar{\Theta}_{G}(k)=\bar{\Theta}_{G}(k-1)+P \Phi\left(k, \Theta_{G}\right)[-\bar{Y}(k)-\xi(k)+V(k)] \tag{37}
\end{equation*}
$$

where

$$
\begin{align*}
\xi(k)= & \sum_{i=1}^{n b} B_{i}(\widehat{\Omega}-\Omega) U^{T}(k-i) \\
& +\sum_{i=1}^{n b}\left(\widehat{D} \widehat{B}_{i}-D B_{i}\right) \widehat{\Omega} \Delta U^{T}(k-i),  \tag{38}\\
\bar{Y}(k)= & \bar{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) .
\end{align*}
$$

Let us define now a nonnegative definite function:

$$
\begin{equation*}
S(k):=\bar{\Theta}_{G}^{T}(k) P^{-1}(k) \bar{\Theta}_{G}(k) . \tag{39}
\end{equation*}
$$

Replacing $\bar{\Theta}_{G}^{T}(k)$ by its expression, we obtain

$$
\begin{align*}
S(k)= & {\left[\bar{\Theta}_{G}(k-1)+P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right.} \\
& \times(-\bar{Y}(k)-\xi(k)+V(k))]^{T} P^{-1}(k) \\
& \times\left[\bar{\Theta}_{G}(k-1)+P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right.  \tag{40}\\
& \times(-\bar{Y}(k)-\xi(k)+V(k))] .
\end{align*}
$$

So

$$
\begin{align*}
S(k)= & \bar{\Theta}_{G}^{T}(k-1) P^{-1}(k) \bar{\Theta}_{G}(k-1) \\
& +2 \bar{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& \times[-\bar{Y}(k)-\xi(k)+V(k)] \\
& +\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)  \tag{41}\\
& \times[-\bar{Y}(k)-\xi(k)+V(k)] \\
& \times[-\bar{Y}(k)-\xi(k)+V(k)]^{T} .
\end{align*}
$$

Substituting $P^{-1}(k)$ by (30), we get

$$
\begin{align*}
S(k)= & \bar{\Theta}_{G}^{T}(k-1)\left[P^{-1}(k-1)\right. \\
& \left.+\Phi\left(k, \widehat{\Theta}_{G}\right) \Phi^{T}\left(k, \widehat{\Theta}_{G}\right)\right] \\
\times & \bar{\Theta}_{G}(k-1)+2 \bar{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
\times & {[-\bar{Y}(k)-\xi(k)+V(k)] }  \tag{42}\\
+ & \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
\times & {[-\bar{Y}(k)-\xi(k)+V(k)] } \\
\times & {[-\bar{Y}(k)-\xi(k)+V(k)]^{T} . }
\end{align*}
$$

Then,

$$
\begin{align*}
S(k)= & S(k-1)+\bar{Y}(k) \bar{Y}(k)^{T} \\
& -2 \bar{Y}(k) \bar{Y}(k)^{T}-2 \bar{Y}(k) \xi^{T}(k) \\
& +2 \bar{Y}(k) V^{T}(k)+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi \\
& \times\left(k, \widehat{\Theta}_{G}\right) \bar{Y}(k) \bar{Y}(k)^{T} \\
& -2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \bar{Y}(k)  \tag{43}\\
& \times(-\xi(k)+V(k))^{T}+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) \\
& \times P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\left(\xi(k) \xi(k)^{T}+V(k) V(k)^{T}\right. \\
& \left.\quad-2 \xi(k) V^{T}(k)\right) .
\end{align*}
$$

Step 1. Data acquisition $\{U(k), Y(k)\}$ and initialization: set $\widehat{\Theta}_{G}=\Theta_{G_{0}}$ and $P=\beta I$ where $\beta$ is a scalar and $I$ is the identity matrix of size $\left(p, p\left(n_{a}+n_{b}+n_{c}+1\right)\right)$ and $k=0$,
Step 2. Increment $k$ and construct the observation vector $\varphi(k, \widetilde{D})$, the generalized observation vector $\Phi^{T}\left(k, \widehat{\Theta}_{G}\right)$ using (9) and (16),
Step 3. Estimate $\widehat{\Theta}_{G}$ using the developed identification method:
$\Upsilon(k)=Y(k)-\widehat{\Theta}^{T}(k-1) \varphi(k, \widetilde{D})$
$P(k)=P(k-1)-\frac{P(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k-1)}{1+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)}$
$\widehat{\Theta}_{G}(k)=\widehat{\Theta}_{G}(k-1)+P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \Upsilon(k)$
Step 4. Return to Step 2 until $k=N$ where $N$ is the number of input/output data.

It follows from (43) that

$$
\begin{aligned}
S(k)= & S(k-1) \\
& -\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times \bar{Y}(k) \bar{Y}(k)^{T}+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) \\
& \times P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) V(k) V(k)^{T} \\
& +2\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times \bar{Y}(k) V^{T}(k)-2 \\
& \times\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times \bar{Y}(k) \xi^{T}(k)+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi \\
& \times\left(k, \widehat{\Theta}_{G}\right) \xi(k) \xi(k)^{T} \\
& -2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi \\
& \times\left(k, \widehat{\Theta}_{G}\right) \xi(k) V^{T}(k) .
\end{aligned}
$$

We have

$$
\begin{aligned}
- & 2\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times \bar{Y}(k) \xi^{T}(k) \\
= & {\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] } \\
& \times \bar{Y}(k) \bar{Y}(k)^{T} \\
& +\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times \xi(k) \xi(k)^{T}
\end{aligned}
$$

$$
\begin{align*}
& -\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right] \\
& \times(\bar{Y}(k)+\xi(k))(\bar{Y}(k)+\xi(k))^{T} \tag{45}
\end{align*}
$$

and the use of the relation

$$
\begin{align*}
1- & \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& =\left[1+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)\right]^{-1} \geqslant 0 \tag{46}
\end{align*}
$$

leads to

$$
\begin{align*}
S(k) \leqslant & S(k-1)+\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi \\
& \times\left(k, \widehat{\Theta}_{G}\right) V(k) V(k)^{T}+2 \\
& \times\left[1-\Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)\right]  \tag{47}\\
& \times \bar{Y}(k) V^{T}(k)+\xi(k) \xi(k)^{T} \\
& -2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \xi(k) V^{T}(k) .
\end{align*}
$$

Since $\bar{Y}(k), \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right), \xi(k)$, and $S(k-1)$ are uncorrelated with $V(k)$, taking the conditional expectation on both sides of (47) and using (A3) and (A4), we obtain

$$
\begin{equation*}
E[S(k)] \leqslant S(k-1)+2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right) \Sigma+\beta . \tag{48}
\end{equation*}
$$

Define

$$
\begin{equation*}
W(k):=\frac{S(k)}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}} . \tag{49}
\end{equation*}
$$

Since $\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}$ is nondecreasing, we have

$$
\begin{align*}
E[W(k)] \leqslant & \frac{S(k-1)}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}} \\
& +\frac{2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}} \Sigma \\
& +\frac{\beta}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}}  \tag{50}\\
\leqslant & W(k-1) \\
& +\frac{2 \Phi^{T}\left(k, \widehat{\Theta}_{G}\right) P(k) \Phi\left(k, \widehat{\Theta}_{G}\right)}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}} \Sigma \\
& +\frac{\beta}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}}
\end{align*}
$$

Using this property $\sum_{i=1}^{\infty}\left(\Phi^{T}\left(i, \widehat{\Theta}_{G}\right) P(i) \Phi\left(i, \widehat{\Theta}_{G}\right) /[\operatorname{Ln}]\right.$ $\left.\left.P^{-1}(i) \mid\right]^{c}\right)<\infty$ (the proof in the same way as the proof of Lemma 1 in [30]), we can see that the sum of the right-hand second term of equation (50) for $k$ from 1 to $\infty$ is finite. Applying the martingale convergence theorem [30] to the previous inequality, we conclude that $W(k)$ converges a.s. to a finite random variable, say $W_{0}$; that is,

$$
\begin{equation*}
W(k)=\frac{S(k)}{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}} \longrightarrow W_{0}<\infty \tag{51}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
S(k)=O\left(\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}\right) \tag{52}
\end{equation*}
$$

From the definition of $V(k)$, we obtain

$$
\begin{equation*}
\left\|\bar{\Theta}_{G}\right\| \leqslant \frac{\bar{\Theta}_{G}(k)^{T} P^{-1}(k) \bar{\Theta}_{G}(k)}{\lambda_{\min }\left[P^{-1}(k)\right]}=\frac{S(k)}{\lambda_{\min }\left[P^{-1}(k)\right]} \tag{53}
\end{equation*}
$$

Now, let us define the matrix trace

$$
\begin{equation*}
r(k)=\operatorname{tr}\left[P^{-1}(k)\right] \tag{54}
\end{equation*}
$$

It follows that:

$$
\begin{gather*}
\left|P^{-1}(k)\right| \leqslant r^{n}(k), \\
r(k) \leqslant n \lambda_{\max }\left|P^{-1}(k)\right|, \tag{55}
\end{gather*}
$$

$\operatorname{Ln}\left|P^{-1}(k)\right|=O(\operatorname{Ln} r(k))=O\left(\operatorname{Ln} \lambda_{\max }\left|P^{-1}(k)\right|\right)$.
We obtain, finally,

$$
\begin{aligned}
\left\|\widehat{\Theta}_{G}-\Theta_{G}\right\|^{2} & =O\left(\frac{\left[\operatorname{Ln}\left|P^{-1}(k)\right|\right]^{c}}{\lambda_{\min }\left|P^{-1}(k)\right|}\right) \\
& =O\left(\frac{[\operatorname{Ln} r(k)]^{c}}{\lambda_{\min }\left|P^{-1}(k)\right|}\right) \\
& =O\left(\frac{\left[\lambda_{\max }\left|P^{-1}(k)\right|\right]^{c}}{\lambda_{\min }\left|P^{-1}(k)\right|}\right)
\end{aligned}
$$

4.2. Lemma. For the estimate (22) with the assumption (A4), the following proprieties hold.
(P1) $\widehat{\Theta}_{G}$ is an unbiased estimate of $\Theta_{G}$.
(P2) The covariance matrix of $\widehat{\Theta}_{G}$ is given by

$$
\begin{align*}
E & {\left[\left(\widehat{\Theta}_{G}-\Theta_{G}\right)\left(\widehat{\Theta}_{G}-\Theta_{G}\right)^{T}\right] } \\
& =\left(\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi\left(i, \Theta_{G}\right)^{T}\right)^{-1} \Sigma \tag{57}
\end{align*}
$$

where

$$
\Sigma=\left(\begin{array}{cccc}
\sigma_{1}^{2} & 0 & \cdots & 0  \tag{58}\\
0 & \sigma_{2}^{2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \sigma_{p}^{2}
\end{array}\right)
$$

Proof. If we replace (8) in (22), we have

$$
\begin{align*}
\widehat{\Theta}_{G}= & {\left[\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right]^{-1} }  \tag{59}\\
& \times \sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right)\left[\Theta^{T} \varphi(i, D)+V(i)+\psi\right]
\end{align*}
$$

Then,

$$
\begin{align*}
\widehat{\Theta}_{G}= & {\left[\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right]^{-1} } \\
& \times \sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right)\left[\Theta^{T} \varphi(i, D)+\psi\right]  \tag{60}\\
& +\left[\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right]^{-1} \\
& \times \sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) V(i)
\end{align*}
$$

So

$$
\begin{align*}
E\left[\widehat{\Theta}_{G}\right]=\Theta_{G}+E([ & \left.\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right]^{-1}  \tag{61}\\
& \left.\times \sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) V(i)\right) .
\end{align*}
$$

Since $V(i)$ is uncorrelated with the elements of $\Phi\left(i, \Theta_{G}\right)(13)$, then

$$
\begin{equation*}
E\left[\widehat{\Theta}_{G}\right]=\Theta_{G} \tag{62}
\end{equation*}
$$

which proves (P1).

Consider the first-order Taylor series expansion around the real matrix of $\Theta_{G}$ :

$$
\begin{equation*}
\frac{\partial J\left(k, \widehat{\Theta}_{G}\right)}{\partial \widehat{\Theta}_{G}}=\frac{\partial J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}}+\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}}\left(\widehat{\Theta}_{G}-\Theta_{G}\right) . \tag{63}
\end{equation*}
$$

Since $\partial J\left(k, \widehat{\Theta}_{G}\right) / \partial \widehat{\Theta}_{G}=0$, it derives from (63) that

$$
\begin{align*}
\left(\widehat{\Theta}_{G}\right. & \left.-\Theta_{G}\right)\left(\widehat{\Theta}_{G}-\Theta_{G}\right)^{T} \\
& =\left[\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}}\right]^{-1} \frac{\partial J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}}  \tag{64}\\
& \quad \times\left[\frac{\partial J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}}\right]^{T}\left[\left(\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}}\right)^{-1}\right]^{T}
\end{align*}
$$

The second partial derivative of the criterion with respect to the generalized matrix is

$$
\begin{equation*}
\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}}=\sum_{i=0}^{k}\left(\Upsilon(i) \frac{\partial^{2} \Upsilon(i)}{\partial^{2} \Theta_{G}}-\frac{\partial \Upsilon(i)}{\partial \Theta_{G}} \Phi\left(i, \Theta_{G}\right)\right) \tag{65}
\end{equation*}
$$

So

$$
\begin{equation*}
\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}}=\sum_{i=0}^{k}\left(\Upsilon(i) \frac{\partial^{2} \Upsilon(i)}{\partial^{2} \Theta_{G}}+\Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right) \tag{66}
\end{equation*}
$$

The use of the small residual algorithms [31] leads to neglect the following term, then:

$$
\begin{equation*}
\sum_{i=0}^{k} \Upsilon(i) \frac{\partial^{2} \Upsilon(i)}{\partial^{2} \Theta_{G}} \longrightarrow 0 \tag{67}
\end{equation*}
$$

Hence, an approach of $\partial^{2} J\left(k, \Theta_{G}\right) / \partial \Theta_{G}^{2}$ is obtained:

$$
\begin{equation*}
\frac{\partial^{2} J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}^{2}} \simeq \sum_{i=0}^{k}\left(\Phi\left(i, \Theta_{G}\right) \Phi^{T}\left(i, \Theta_{G}\right)\right) \tag{68}
\end{equation*}
$$

Applying the mean value of $\left(\partial J\left(k, \Theta_{G}\right) / \partial \Theta_{G}\right)\left(\partial J\left(k, \Theta_{G}\right) /\right.$ $\partial \Theta_{G}{ }^{T}$, we get:

$$
\begin{align*}
E & {\left[\frac{\partial J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}} \frac{\partial J\left(k, \Theta_{G}\right)^{T}}{\partial \Theta_{G}}\right] }  \tag{69}\\
& =\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi\left(i, \Theta_{G}\right)^{T} E\left(\Upsilon(i) \Upsilon(i)^{T}\right)
\end{align*}
$$

So

$$
\begin{equation*}
E\left[\frac{\partial J\left(k, \Theta_{G}\right)}{\partial \Theta_{G}} \frac{\partial J\left(k, \Theta_{G}\right)^{T}}{\partial \Theta_{G}}\right]=\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi\left(i, \Theta_{G}\right)^{T} \Sigma \tag{70}
\end{equation*}
$$

Then, we have

$$
\begin{align*}
& E\left[\left(\widehat{\Theta}_{G}-\Theta_{G}\right)\left(\widehat{\Theta}_{G}-\Theta_{G}\right)^{T}\right] \\
& \quad=E\left[\left(\frac{\partial^{2} J}{\partial \Theta_{G}^{2}}\right)^{-1} \sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi\left(i, \Theta_{G}\right)^{T} \Sigma\left[\left(\frac{\partial^{2} J}{\partial \Theta_{G}^{2}}\right)^{-1}\right]^{T}\right] . \tag{71}
\end{align*}
$$

Finally, we obtain

$$
\begin{align*}
& E\left[\left(\widehat{\Theta}_{G}-\Theta_{G}\right)\left(\widehat{\Theta}_{G}-\Theta_{G}\right)^{T}\right] \\
& \quad=\left(\sum_{i=0}^{k} \Phi\left(i, \Theta_{G}\right) \Phi\left(i, \Theta_{G}\right)^{T}\right)^{-1} \Sigma \tag{72}
\end{align*}
$$

which proves (P2).

## 5. Results

We now present a simulation example and an experimental validation to illustrate the performance of the proposed approach for the simultaneous identification of time delays and parameter matrices of square multivariable systems.
5.1. Simulation Example. The objective of the simulation is to compare the efficiency of the proposed method (DRLS) with that of the classic recursive least square approach (RLS) [29] which assumes that the delays are a priori known. In fact, we consider the following cases.

Case 1. The output is noise-free and the RLS method uses the true time delays.

Case 2. The output is noise free and the RLS method uses the misestimated time delays.

Case 3. The output is contaminated by additive noise and the RLS method uses the true time delays.

We consider a square linear multivariable discrete time delay system with two inputs and two outputs described by the following equation [32]:

$$
\begin{equation*}
A\left(q^{-1}\right) Y(k)=B\left(q^{-1}\right) \underset{\sim}{U}+V(k) \tag{73}
\end{equation*}
$$

where the delayed inputs and the outputs are defined, respectively, by $\underset{\sim}{U}(k)=\binom{u_{1}\left(k-d_{1}\right)}{u_{2}\left(k-d_{2}\right)}$ and $Y(k)=\binom{y_{1}(k)}{y_{2}(k)}$.

The two polynomials matrices $A\left(q^{-1}\right)$ and $B\left(q^{-1}\right)$ are given by

$$
\begin{gather*}
A\left(q^{-1}\right)=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
0.9048 & 0 \\
0 & 0.9048
\end{array}\right) q^{-1} \\
B\left(q^{-1}\right)=\left(\begin{array}{cc}
0.09516 & 0.03807 \\
-0.0297 & 0.0475
\end{array}\right) q^{-1} . \tag{74}
\end{gather*}
$$

The time delay matrix $D$ is given by

$$
D=\left(\begin{array}{ll}
3 & 0  \tag{75}\\
0 & 1
\end{array}\right)
$$



Figure 1: Evolution of the true (-) and the estimated (--) delays: Case 1.


Figure 2: Evolution of the true and the estimated parameters: $A_{i}$ : Case 1.
5.1.1. Case 1. The proposed approach (DLSR) and the RLS algorithm are applied to estimate time delays and parameter matrices. The estimation starts with zero initial conditions. The obtained results are illustrated in Table 1 and Figures 1, 2, and 3.

Figures $1-3$ show the evolution of the estimated and the true parameter matrices.

A validation of the obtained model is presented in Figures 4 and 5 which show that the estimated outputs track fast and accurately the true outputs.


Figure 3: Evolution of the true and the estimated parameters: $B_{i}$ : Case 1.

TAble 1: Simulation results: Case 1.

|  | True |  | Developed RLS |  | RLS |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.9048 | 0 | 0.9 | -0.0022 | 0.9048 | -0.0000 |
|  | 0 | 0.9048 | -0.0043 | 0.9021 | -0.0000 | 0.9048 |
| $B_{1}$ | 0.09516 | 0.03807 | 0.0947 | 0.0378 | 0.0952 | 0.0381 |
|  | -0.02974 | 0.04758 | -0.0297 | 0.0475 | -0.0297 | 0.0476 |
| $D$ | 3 | 0 | 3 | 0 | Known |  |
|  | 0 | 1 | 0 | 1 |  |  |

5.1.2. Case 2. We apply the proposed approach (DLSR) and the RLS algorithm to estimate time delays and parameter matrices. The RLS algorithm uses misestimated time delays $\left(\begin{array}{ll}2 & 0 \\ 0 & 1\end{array}\right)$. The obtained results are illustrated in Table 2 and Figures 6 and 7.

Figures 6 and 7 show the evolution of the estimated and the true parameter matrices.

Table 2: Simulation results: Case 2.

|  | True | Developed RLS | RLS |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.9048 | 0 | 0.9 |  | -0.0022 | 0.5829 |$-0.2376$

5.1.3. Case 3. The system's output is corrupted by additive zero mean white noises $V(k)=\left[v_{1}(k), v_{2}(k)\right]$ with variances $\sigma_{1}=0.0737, \sigma_{2}=0.086$.

The result of the simulation is given in Table 3.
Figures 8,9 , and 10 show the evolution of the estimated and the true parameter matrices.


* Estimated output
-_ True output
Figure 4: The evolution of the true and the estimated outputs $y_{1}(k)$ : Case 1.

* Estimated output
- True output

Figure 5: The evolution of the true and the estimated outputs $y_{2}(k)$ : Case 1.

Table 3: Simulation results: Case 3.

|  | True | Developed RLS |  | RLS |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $A_{1}$ | 0.9048 | 0 | 0.8949 | -0.0084 | 0.9032 | -0.0016 |
|  | 0 | 0.9048 | -0.0084 | 0.9 | 0.0003 | 0.951 |
| $B_{1}$ | 0.09516 | 0.03807 | 0.0941 | 0.073 | 0.0952 | 0.0381 |
|  | -0.02974 | 0.04758 | -0.0303 | 0.0476 | -0.0297 | 0.0476 |
| $D$ | 3 | 0 | 3 | 0 |  | Known |
|  | 0 | 1 | 0 | 1 |  |  |

A validation of the obtained model is presented in Figures 11 and 12 which show that the estimated outputs track fast and accurately the true outputs.
5.1.4. Observations. Based on Tables 1-3 and Figures 1-12, we observe that
(i) the RLS method gives the better performance when the true time delays are used. However, it poorly performs for misestimated delays;
(ii) the proposed approach converges to the true delays with acceptable speed for the considered cases.
5.2. Experiment Example. The experimental data from a mechanical construction of a CD player arm is considered. The system has two inputs that are forces of the mechanical actuators $(U)$ and two outputs that are related to the tracking accuracy of the arm $(Y)$.

The data set contains 2048 sample points out of which 1000 were used for the identification procedure and the rest to validate the identified models. The input/output identification signals [33] are given in Figures 13 and 14.

The system is described by the following equation:

$$
\begin{equation*}
A\left(q^{-1}\right) Y(k)=B\left(q^{-1}\right) \underset{\sim}{U}(k)+V(k) \tag{76}
\end{equation*}
$$

where $Y(k)$ and $\underset{\sim}{U}(k)$ are the output and the delayed input of the system at time $k, p=2$ and the orders of $A\left(q^{-1}\right), B\left(q^{-1}\right)$ are $n_{a}=n_{b}=2$.

The estimation starts with zero initial values for the parameter and the time delay matrices. Applying the proposed algorithm, we obtain

$$
\begin{align*}
\widehat{A}\left(q^{-1}\right)= & \left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{ll}
-0.9464 & -0.6600 \\
-0.1263 & -0.5278
\end{array}\right) q^{-1} \\
& +\left(\begin{array}{cc}
0.0809 & 0.5506 \\
-0.1066 & 0.2243
\end{array}\right) q^{-2}, \\
\widehat{B}\left(q^{-1}\right)= & \left(\begin{array}{cc}
-1.4839 & 1.0274 \\
0.2900 & -1.1819
\end{array}\right) q^{-1}  \tag{77}\\
& +\left(\begin{array}{cc}
1.2452 & -0.7638 \\
-0.1866 & 0.8853
\end{array}\right) q^{-2} .
\end{align*}
$$

The estimated time delay matrix is

$$
\widehat{D}=\left(\begin{array}{ll}
0 & 0  \tag{78}\\
0 & 2
\end{array}\right) .
$$

Another data set is used for the validation test which is illustrated by Figures 15 and 16.

We can see clearly that the estimated output tracks fast and accurately the true output.

## 6. Conclusions

In this paper, we have addressed the problem of identification of linear discrete time delay multivariable systems. In fact, we have proposed a novel approach for the simultaneous identification of the unknown time delays and the parameter matrices of these systems. The proposed approach consists in constructing a linear-parameter formulation that is used to


Figure 6: Evolution of the true and the estimated parameters: $A_{i}$ : Case 2.
estimate the time delays and the polynomial matrices using recursive least square algorithm. The obtained estimates were shown to be unbiased, and an expression for their covariance matrix was given. Numerical simulation and experimental test are presented to demonstrate the performance of the proposed approach.

## Appendix

## A. Approximation of $\operatorname{Ln}(q)$

Let us consider the shift operator and the backward difference given, respectively, by

$$
\begin{gather*}
q u(k)=u(k+1),  \tag{A.1}\\
\Delta u(k)=u(k)-u(k-1) . \tag{A.2}
\end{gather*}
$$

So

$$
\begin{equation*}
\Delta u(k)=\left(1-q^{-1}\right) u(k) . \tag{A.3}
\end{equation*}
$$

We can infer the identity between the shift operator and the backward difference [34], and then

$$
\begin{equation*}
\Delta=1-q^{-1} \tag{A.4}
\end{equation*}
$$

It is equivalent to

$$
\begin{equation*}
q^{-1}=1-\Delta . \tag{A.5}
\end{equation*}
$$

Applying the logarithm function of both sides of (A.1), we get

$$
\begin{equation*}
\operatorname{Ln}(q)=-\operatorname{Ln}(1-\Delta) \tag{A.6}
\end{equation*}
$$

Using the series expansion of $\operatorname{Ln}(1-x)$, we have

$$
\begin{equation*}
\operatorname{Ln}(q)=\Delta+\frac{1}{2} \Delta^{2}+\frac{1}{3} \Delta^{3}+\cdots \tag{A.7}
\end{equation*}
$$



Figure 7: Evolution of the true and the estimated parameters: $B_{i}$ : Case 2.

Finally, we use a first-order approximation of the shift It is equivalent to operator given by

$$
\begin{equation*}
\operatorname{Ln}(q)=\Delta=1-q^{-1} \tag{A.8}
\end{equation*}
$$

## B. Expression of $\partial \widehat{\Omega} / \partial \widehat{D}$

The partial derivative of $\widehat{\Omega}$ with respect to the estimated time delay matrix is given by

$$
\frac{\partial \widehat{\Omega}}{\partial \widehat{D}}=\frac{\partial}{\partial \widehat{D}}\left(\begin{array}{cccc}
q^{-\widehat{d}_{1}} & 0 & \cdots & 0  \tag{B.1}\\
0 & q^{-\widehat{d}_{2}} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & q^{-\widehat{d}_{p}}
\end{array}\right) .
$$

We then obtain

$$
\begin{equation*}
\frac{\partial \widehat{\Omega}}{\partial \widehat{D}}=-\operatorname{Ln}(q) \frac{\partial e^{-\widehat{D} \operatorname{Ln}(q)}}{\partial \widehat{D}}=-\operatorname{Ln}(q) \widehat{\Omega} \tag{B.4}
\end{equation*}
$$



Figure 8: Evolution of the true (-) and the estimated (--) delays: Case 3.


Figure 9: Evolution of the true and the estimated parameters: $A_{i}$ : Case 3.

Using the approximation $\operatorname{Ln}(q) \simeq\left(1-q^{-1}\right)$, we get the partial derivative of $\widehat{\Omega}$ with respect to the estimated time delay matrix:

$$
\begin{equation*}
\frac{\partial \widehat{\Omega}}{\partial \widehat{D}} \simeq-\left(1-q^{-1}\right) \widehat{\Omega} . \tag{B.5}
\end{equation*}
$$

## C. Expression of $\bar{\rho}(k)$

We have

$$
\begin{equation*}
\bar{\rho}(k)=\widehat{\Theta}^{T}(k-1) \varphi(k, \widehat{D})-\Theta^{T} \varphi(k, D) . \tag{C.1}
\end{equation*}
$$

In the same way as before (see the equations (22) and (21)), the estimated generalized vector parameters $\widehat{\Theta}_{G}$ and the


Figure 10: Evolution of the true and the estimated parameters: $B_{i}$ : Case 3.


* Estimated output
- True output

Figure 11: The evolution of the true and the estimated outputs $y_{1}(k)$ : Case 3.


* Estimated output
- True output

Figure 12: The evolution of the true and the estimated outputs $y_{2}(k)$ : Case 3.


Figure 13: Evolution of input identification signals of CD arm.


Figure 14: Evolution of output identification signals of $C D$ arm.

$\begin{array}{ll}--\hat{y}_{1}(k) \\ - & y_{1}(k)\end{array}$
Figure 15: The evolution of the true (-) and the estimated (- -) outputs.
generalized vector parameters $\Theta_{G}$ can appear after adding and subtracting the appropriate term of the equation (C.1):

$$
\begin{align*}
\bar{\rho}(k)= & \widehat{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& +\sum_{i=1}^{n b} \widetilde{D} \widehat{B}_{i} \widehat{\Omega} \Delta U^{T}(k-i) \\
& -\Theta_{G}^{T}(k-1) \Phi\left(k, \Theta_{G}\right)  \tag{C.2}\\
& -\sum_{i=1}^{n b} D B_{i} \Omega \Delta U^{T}(k-i) .
\end{align*}
$$




Figure 16: The evolution of the true (-) and the estimated (- -) outputs.

Adding and subtracting the term $\Theta_{G}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)$ from (C.2), we obtain

$$
\begin{align*}
\bar{\rho}(k)= & \widehat{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& -\Theta_{G}^{T}(k-1) \Phi\left(k, \Theta_{G}\right)+\alpha \\
& +\Theta_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)  \tag{C.3}\\
& -\Theta_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right)
\end{align*}
$$

where

$$
\begin{align*}
\alpha=\sum_{i=1}^{n b} \widehat{D} \widehat{B}_{i} \widehat{\Omega} \Delta U^{T} & (k-i)-\sum_{i=1}^{n b} D B_{i} \Omega \Delta U^{T}(k-i) \\
\bar{\rho}(k)= & \bar{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) \\
& -\sum_{i=1}^{n b} B_{i} \Omega U(k-i) \\
& +\sum_{i=1}^{n b} B_{i} \widehat{\Omega} U^{T}(k-i)  \tag{C.4}\\
& +\sum_{i=1}^{n b} \widehat{D} \widehat{B_{i}} \widehat{\Omega} \Delta U^{T}(k-i) \\
& -\sum_{i=1}^{n b} D B_{i} \widehat{\Omega} \Delta U^{T}(k-i)
\end{align*}
$$

So

$$
\begin{equation*}
\bar{\rho}(k)=\bar{Y}(k)+\xi(k), \tag{C.5}
\end{equation*}
$$

where

$$
\begin{align*}
\xi(k)= & \sum_{i=1}^{n b} B_{i}(\widehat{\Omega}-\Omega) U^{T}(k-i) \\
& +\sum_{i=1}^{n b}\left(\widehat{D} \widehat{B}_{i}-D B_{i}\right) \widehat{\Omega} \Delta U^{T}(k-i),  \tag{C.6}\\
& \bar{Y}(k)=\bar{\Theta}_{G}^{T}(k-1) \Phi\left(k, \widehat{\Theta}_{G}\right) .
\end{align*}
$$

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# Robust Fault Tolerant Control for a Class of Time-Delay Systems with Multiple Disturbances 

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#### Abstract

A robust fault tolerant control (FTC) approach is addressed for a class of nonlinear systems with time delay, actuator faults, and multiple disturbances. The first part of the multiple disturbances is supposed to be an uncertain modeled disturbance and the second one represents a norm-bounded variable. First, a composite observer is designed to estimate the uncertain modeled disturbance and actuator fault simultaneously. Then, an FTC strategy consisting of disturbance observer based control (DOBC), fault accommodation, and a mixed $H_{2} / H_{\infty}$ controller is constructed to reconfigure the considered systems with disturbance rejection and attenuation performance. Finally, simulations for a flight control system are given to show the efficiency of the proposed approach.


## 1. Introduction

To reduce the influence of model uncertainties and system disturbances, there are several control approaches focusing on nonlinear systems with unknown disturbances (see the survey paper [1] and references therein). The methodologies can be mainly classified into disturbance attenuation methods (such as $H_{\infty}$ and $H_{2}$ control) and disturbance rejection approaches (such as output regulation theory, disturbance observer based control). The disturbance attenuation approaches have conservativeness for bounded stochastic disturbance. The disturbance rejection methods are established based on model-matching conditions. It has been shown that multiple disturbances exist in most practical systems. The idea of disturbance observer based control (DOBC) is to construct an observer to estimate and compensate some external disturbances [2]. A composite control scheme combining DOBC and PD (proportional derivative) control for flexible spacecraft attitude control was proposed in the presence of model uncertainty, elastic vibration, and external disturbances [3]. For nonlinear systems with multiple disturbances, it has been seen that the $H_{\infty}$ and
variable structure control have been integrated with DOBC in $[4,5]$. In [6], a composite DOBC and adaptive control approach were proposed for a class of nonlinear systems with multiple disturbances. By constructing a disturbance compensation gain vector in the composite control law, a nonlinear robust DOBC was proposed to attenuate the mismatched disturbances and the influence of parameter variations from output channels [7].

In order to increase the reliability and safety of practical engineering, the issues of fault diagnosis and fault tolerant control (FTC) have become an attractive topic and have been paid much attention in recent years (see [8-13] and references therein). It is difficult to accommodate faults if the disturbances and faults exist simultaneously in the controlled systems. In [14], an optimal fault tolerant control approach was proposed for the nonlinear systems, where generalized $H_{\infty}$ optimization was applied to estimate the fault and attenuate the disturbances. In [15], a robust observer was proposed to simultaneously estimate system states, faults, and their finite time derivatives and attenuate disturbances; then an FTC approach was designed based on their estimations. For systems with modeled disturbance, a fault diagnosis
approach based on disturbance observer was firstly proposed in [16] with disturbance rejection performance. For the nonlinear system with multiple disturbances, [17] addressed a fault tolerant control approach with disturbance rejection and attenuation performances. It is well known that time delay frequently occurs in many practical systems, such as manufacturing systems, telecommunication, and economic systems. Therefore, the problem of fault accommodation for time-delay systems has been a hot topic in the control field. Many important results have been reported in the literature (see [18-21] and references therein). In [18], an adaptive fuzzy fault accommodation control approach was proposed for nonlinear time-delay system. In [19], a fault accommodation was addressed for time-varying delay system using adaptive fault diagnosis observer. [20] dealt with fault tolerant guaranteed cost controller design problem for linear time-delay system against actuator faults.

In this paper, FTC problem is discussed for a class of time-delay systems with actuator fault and multiple disturbances. The first part of multiple disturbances is modeled disturbance formulated by an exogenous system and the second one is norm bounded uncertain variable. A composite observer is designed to estimate the modeled disturbance and time-varying fault. Then, an FTC scheme is addressed with disturbance rejection and attenuation performance by combining fault accommodation and DOBC with a robust $H_{2} / H_{\infty}$ controller.

## 2. Model Description

In this paper, we consider the following nonlinear system with time-varying faults, time-delay, and multiple disturbances simultaneously:

$$
\begin{gather*}
\dot{x}(t)=A x(t)+A_{d} x(t-\tau)+G g(x(t)) \\
+J[u(t)+F(t)]+J_{1} d_{1}(t)+J_{2} d_{2}(t)  \tag{1}\\
y(t)=C x(t),
\end{gather*}
$$

where $x(t) \in R^{n}$ is system state, $u(t)$ is control input, and $y(t) \in R^{m}$ represents output variable. $F(t)$ is timevarying actuator fault to be diagnosed. $A, A_{d}, C, G, J, J_{1}$, and $J_{2}$ represent coefficient matrices of the system with suitable dimensions. $\tau$ is a known constant delay. The modeled external disturbance $d_{1}(t)$ is supposed to be generated by a linear exogenous system described by

$$
\begin{gather*}
\dot{\omega}(t)=W \omega(t)+J_{3} \delta(t) \\
d_{1}(t)=V \omega(t) \tag{2}
\end{gather*}
$$

where $\omega(t)$ is state variable, $W \in R^{p \times p}$, and $V$ and $J_{3}$ are known parameter matrices of the exogenous system. $\delta(t)$ is additional disturbance which results from perturbations and uncertainties in the exogenous system. The disturbances $d_{2}(t)$ and $\delta(t)$ are supposed to have the bounded $\mathrm{H}_{2}$ norm.

For a known matrix $U_{1}, g(x(t))$ is a known nonlinear vector function that is supposed to satisfy $g(0)=0$ and the following norm condition:

$$
\begin{equation*}
\left\|g\left(x_{1}(t)\right)-g\left(x_{2}(t)\right)\right\| \leq\left\|U_{1}\left(x_{1}(t)-x_{2}(t)\right)\right\| \tag{3}
\end{equation*}
$$

for any $x_{1}(t)$ and $x_{2}(t)$.
The following assumptions are required so that the considered problem can be well-posed in this paper.

Assumption 1. $(A, J)$ is controllable; $\left(W, J_{1} V\right)$ is observable.
Assumption 2. $\operatorname{rank}\left(J, J_{1}\right)=\operatorname{rank}(J)$.
Remark 1. In practical engineering, the exogenous model (2) can represent many kinds of disturbances including harmonic disturbance signal caused by vibration, unknown constant load in the motor, inertial sensor drift represented by first-order Gaussian Markov process, and so on. Compared with the previous works [17, 19, 20], both the time-delay and multiple disturbances are simultaneously considered in this paper. Furthermore, the modeled disturbance $d_{1}(t)$ and control input are assumed in different channels, while in [17] are in the same channel.

## 3. Robust Fault Tolerant Controller Design

3.1. Disturbance Observer. In order to reject the modeled external disturbance, disturbance observer should be designed in this subsection. In this paper, we only consider the case of available states. The disturbance observer is formulated as

$$
\begin{gather*}
\widehat{\omega}(t)=\xi(t)-L x(t), \\
\widehat{d}_{1}(t)=V \widehat{\omega}(t) \tag{4}
\end{gather*}
$$

where $\xi(t)$ is auxiliary variable generated by

$$
\begin{align*}
\dot{\xi}(t)=( & \left.W+L J_{1} V\right)[\xi(t)-L x(t)]+L \\
\times & {\left[A x(t)+A_{d} x(t-\tau)+G g(x(t))\right.}  \tag{5}\\
& \left.+J u(t)+J u_{f c}(t)\right]
\end{align*}
$$

$\widehat{\omega}(t)$ is estimation of $\omega(t), \widehat{d}_{1}(t)$ is estimation of modeled disturbance $d_{1}(t)$, matrix $L$ is the disturbance observer gain to be determined later, and $u_{f c}(t)=\widehat{F}(t)$ is compensation term to be designed in fault diagnosis observer, where $\widehat{F}(t)$ is denoted as an estimation of fault $F(t)$.

By defining $e_{\omega}(t)=\omega(t)-\widehat{\omega}(t)$ and $e_{F}(t)=F(t)-\widehat{F}(t)$, estimation error system can be obtained from (1), (2), (4), and (5) to show the following:

$$
\begin{equation*}
\dot{e}_{\omega}(t)=\left(W+L J_{1} V\right) e_{\omega}(t)+L J e_{F}(t)+L J_{2} d_{2}(t)+J_{3} \delta(t) . \tag{6}
\end{equation*}
$$

In the following subsection, we will construct a fault diagnosis observer with disturbance estimation so that the modeled disturbance can be rejected and fault can be diagnosed.
3.2. Fault Diagnosis Observer. The following fault diagnosis observer is constructed to diagnose the time-varying actuator fault:

$$
\begin{gather*}
\widehat{F}(t)=\eta(t)-K x(t) \\
\dot{\eta}(t)=K J[\eta(t)-K x(t)]  \tag{7}\\
+K\left[A x(t)+A_{d} x(t-\tau)\right. \\
\left.+G g(x(t))+J u(t)+J_{1} u_{d c}(t)\right]
\end{gather*}
$$

where $\widehat{F}(t)$ is estimation of $F(t)$. The disturbance observer based control term $u_{d c}(t)=\widehat{d}_{1}(t)$ is applied to reject modeled disturbance $d_{1}(t)$ by its estimation from disturbance observer. $K$ is the fault diagnosis observer gain to be determined later.

The fault estimation error system yields

$$
\begin{align*}
\dot{e}_{F}(t) & =\dot{F}(t)-\dot{\widehat{F}}(t)  \tag{8}\\
& =\dot{F}(t)+K J e_{F}(t)+K J_{1} V e_{\omega}(t)+K J_{2} d_{2}(t)
\end{align*}
$$

In the next subsection, a composite fault tolerant controller should be determined for reconfiguring the systems with disturbance rejection and attenuation performance.
3.3. Composite Fault Tolerant Controller. In this section, the object is to construct a control approach to guarantee that the system (1) is stable in the presence (or absence) of faults and multiple disturbances simultaneously for the considered time-delay system. The structure of composite fault tolerant controller is formulated as

$$
\begin{equation*}
u(t)=u_{s c}(t)-u_{f c}(t)-J^{*} J_{1} u_{d c}(t), \tag{9}
\end{equation*}
$$

where $u_{f c}(t)=\widehat{F}(t), u_{d c}(t)=\widehat{d}_{1}(t)$, and $u_{s c}(t)=S x(t), S$ is the state feedback controller gain to be determined later. Substituting (9) into (1), it can be seen that

$$
\begin{align*}
\dot{x}(t)= & (A+J S) x(t)+A_{d} x(t-\tau)+G g(x(t)) \\
& +J e_{F}(t)+J_{1} d_{1}(t)-J J^{*} J_{1} \hat{d}_{1}(t)+J_{2} d_{2}(t) . \tag{10}
\end{align*}
$$

From Assumption 2, it can be seen that the vector space spanned by the columns of $J_{1}$ is a subset of the space spanned by the column vectors of $J$ [19]; that is, span $\left(J_{1}\right) \subset \operatorname{span}(J)$, which is equivalent to the existence of $J$, such that

$$
\begin{equation*}
J_{1}-J J^{*} J_{1}=0 \tag{11}
\end{equation*}
$$

Then, it can be concluded that

$$
\begin{align*}
\dot{x}(t)= & (A+J S) x(t)+A_{d} x(t-\tau)+G g(x(t)) \\
& +J e_{F}(t)+J_{1} V e_{\omega}(t)+J_{2} d_{2}(t) \tag{12}
\end{align*}
$$

Combing estimation error equations (6) and (8) with (12) yields

$$
\begin{gather*}
\dot{\bar{x}}(t)=\bar{A} \bar{x}(t)+\bar{A}_{d} \bar{x}(t-\tau)+\bar{G} g(\bar{x}(t))+\bar{J} \bar{d}(t), \\
z_{2}(t)=\bar{C}_{2} \bar{x}(t)+\bar{C}_{d 2} \bar{x}(t-\tau),  \tag{13}\\
z_{\infty}(t)=\bar{C}_{\infty} \bar{x}(t)+\bar{C}_{d \infty} \bar{x}(t-\tau)+D \bar{d}(t),
\end{gather*}
$$

where

$$
\begin{gather*}
\bar{x}(t)=\left[\begin{array}{c}
x(t) \\
e_{\omega}(t) \\
e_{F}(t)
\end{array}\right], \quad \bar{A}=\left[\begin{array}{ccc}
A+J S & J_{1} V & J \\
0 & W+L J_{1} V & L J \\
0 & K J_{1} V & K J
\end{array}\right], \\
\bar{G}=\left[\begin{array}{c}
G \\
0 \\
0
\end{array}\right], \\
\bar{A}_{d}=\left[\begin{array}{ccc}
A_{d} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \quad \bar{J}=\left[\begin{array}{ccc}
J_{2} & 0 & 0 \\
L J_{2} & J_{3} & 0 \\
K J_{2} & 0 & I
\end{array}\right] \\
\bar{d}(t)=\left[\begin{array}{c}
d_{2}(t) \\
\delta(t) \\
\dot{F}(t)
\end{array}\right] \\
g(\bar{x}(t))=g(x(t)) \tag{14}
\end{gather*}
$$

$z_{2}(t)$ is $H_{2}$ reference output,

$$
\begin{equation*}
z_{2}(t)=C_{21} x(t)+C_{22}\left[e_{\omega}^{T}(t) e_{F}^{T}(t)\right]^{T}+C_{d 2} x(t-\tau) \tag{15}
\end{equation*}
$$

$z_{\infty}(t)$ is $H_{\infty}$ reference output,

$$
\begin{align*}
z_{\infty}(t)= & C_{\infty 1} x(t)+C_{\infty 2}\left[e_{\omega}^{T}(t) e_{F}^{T}(t)\right]^{T}  \tag{16}\\
& +C_{d \infty} x(t-\tau)+D \bar{d}(t),
\end{align*}
$$

where $C_{21}, C_{22}, C_{d 2}, C_{\infty 1}, C_{\infty 2}, C_{d \infty}$, and $D$ are the selected weighting matrices.

Definition 2. For constants $\gamma_{1}>0, \gamma_{2}>0$, and $\gamma_{3}>0$, the $H_{\infty}$ performance is denoted as follows:

$$
\begin{align*}
J_{\infty}= & \left\|z_{\infty}(t)\right\|^{2}-\gamma_{1}^{2}\left\|d_{2}(t)\right\|^{2}-\gamma_{2}^{2}\|\delta(t)\|^{2} \\
& -\gamma_{3}^{2}\|\dot{F}(t)\|^{2}-\delta\left(P_{1}, Q\right), \tag{17}
\end{align*}
$$

where

$$
\begin{equation*}
\delta\left(P_{1}, Q\right)=\phi^{T}(0) P_{1} \phi(0)+\int_{-d}^{0} \phi^{T}(\tau) Q \phi(\tau) d \tau \tag{18}
\end{equation*}
$$

Definition 3. The $H_{2}$ performance measure for (13) is defined as $J_{2}=\left\|z_{2}(t)\right\|^{2}$.

Remark 4. Compared with [17], $H_{2} / H_{\infty}$ mixed multiobjective optimization technique is used for the composite system (13). In the proposed approach, the modeled disturbance and fault are rejected by their estimations, while $H_{\infty}$ performance is adopted to attenuate norm bounded uncertain disturbances and $\mathrm{H}_{2}$ performance index is applied to optimize estimation error.

At this stage, the objective is to find $K, L$, and $S$ such that system (13) is stable. The following result provides a design method based on convex optimization technology [22].

Theorem 5. If for the parameter $\lambda>0, r_{i}(i=1, \ldots, 4)$, matrices $C_{21}, C_{22}, C_{d 2}, C_{\infty 1}, C_{\infty 2}, C_{d \infty}$, and $D$, there exist matrices $P_{0}>0, P_{2}>0, Q_{0}>0, R_{0}$, and $R_{2}$ and constants $\gamma_{1}>0, \gamma_{2}>0$, and $\gamma_{3}>0$, if the following LMI-based optimization problem holds:

$$
\begin{equation*}
\min \left\{r_{1} \gamma_{1}+r_{2} \gamma_{2}+r_{3} \gamma_{3}+r_{4} \phi^{T}(0) P_{0}^{-1} \phi(0)\right\} \tag{19}
\end{equation*}
$$

subject to

$$
\left[\begin{array}{ccccccccc}
\Xi_{1} & A_{d} Q_{0} & G & \bar{V} & \Xi_{15} & P_{0} U_{1}^{T} & P_{0} C_{\infty 1}^{T} & P_{0} C_{21}^{T} & P_{0} \\
* & -Q_{0} & 0 & 0 & 0 & 0 & C_{d \infty}^{T} & C_{d 2}^{T} & 0  \tag{20}\\
* & * & -\frac{1}{\lambda} I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{4} & \Xi_{45} & 0 & C_{\infty 2}^{T} & C_{22}^{T} & 0 \\
* & * & * & * & \Xi_{5} & 0 & D^{T} & 0 & 0 \\
* & * & * & * & * & -\lambda I & 0 & 0 & 0 \\
* & * & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & * & -Q_{0}
\end{array}\right]
$$

where

$$
\begin{gathered}
\Xi_{1}=\operatorname{sym}\left(A P_{0}+J R_{0}\right), \quad \Xi_{15}=\left[\begin{array}{lll}
J_{2} & 0 & 0
\end{array}\right] \\
\Xi_{4}=\operatorname{sym}\left(P_{2} \bar{W}+R_{2} \bar{V}\right)
\end{gathered}
$$

$$
\begin{gather*}
\Xi_{45}=\left[\begin{array}{lll}
R_{2} J_{2} & P_{2} \bar{J}_{3} & P_{2} \bar{J}_{1}
\end{array}\right] \\
\Xi_{5}=\left[\begin{array}{ccc}
-\gamma_{1}^{2} I & 0 & 0 \\
0 & -\gamma_{2}^{2} I & 0 \\
0 & 0 & -\gamma_{3}^{2} I
\end{array}\right] \\
\bar{W}=\left[\begin{array}{cc}
W & 0 \\
0 & 0
\end{array}\right], \quad \bar{V}=\left[\begin{array}{ll}
J_{1} V & J
\end{array}\right] \\
\bar{J}_{1}=\left[\begin{array}{l}
0 \\
I
\end{array}\right], \quad \bar{J}_{3}=\left[\begin{array}{c}
J_{3} \\
0
\end{array}\right], \quad \bar{C}=\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right] \tag{21}
\end{gather*}
$$

then with gains $S=R_{0} P_{0}^{-1}$ and $\bar{L}=\left[\begin{array}{c}L \\ K\end{array}\right]=P_{2}^{-1} R_{2}$, error system (13) is stable and satisfies $J_{\infty}<0$ and $J_{2} \leq \delta\left(P_{1}, Q\right)$. The symmetric terms in a symmetric matrix are denoted by $*$. The symbol $\operatorname{sym}()$ represents $\operatorname{sym}(\Theta):=\Theta+\Theta^{T}$.

Proof. Consider the following Lyapunov function:

$$
\begin{align*}
\Pi(t)= & x^{T}(t) P_{1} x(t)+\int_{t-\tau}^{t} x^{T}(\tau) Q x(\tau) d \tau+\bar{e}^{T}(t) P_{2} \bar{e}(t) \\
& +\lambda \int_{0}^{t}\left[\|U x(\tau)\|^{2}-\|g(\tau)\|^{2}\right] d \tau \tag{22}
\end{align*}
$$

It is verified that $\Pi(t) \geq 0$ holds for all arguments. Along with the trajectories of (13), it can be shown that

$$
\begin{align*}
\dot{\Pi}(t)= & 2 x^{T}(t) P_{1} \dot{x}(t)+x^{T}(t) Q x(t)-x(t-\tau)^{T} Q x(t-\tau) \\
& +2 \bar{e}^{T}(t) P_{2} \dot{\bar{e}}(t)+\lambda\left[\|U x(t)\|^{2}-\|g(x(t))\|^{2}\right] \\
= & 2 x^{T}(t) P_{1}\left[(A+J S) x(t)+A_{d} x(t-\tau)+G g(x(t))\right. \\
& \left.+J_{1} V e_{\omega}(t)+J e_{F}(t)+J_{2} d_{2}(t)\right] \\
& +x^{T}(t) Q x(t)-x(t-\tau)^{T} Q x(t-\tau) \\
& +\lambda x^{T}(t) U^{T} U x(t)-\lambda\|g(x(t))\|^{2}+2 \bar{e}^{T}(t) \\
& \times P_{2}\left[(\bar{W}+\bar{L} \bar{V}) \bar{e}(t)+\bar{L} J_{2} d_{2}(t)+\bar{J}_{3} \delta(t)+\bar{J}_{1} \dot{F}(t)\right] . \tag{23}
\end{align*}
$$

In the absence of $\bar{d}(t)$ (i.e., $\bar{d}(t)=0$ ), it can be seen that

$$
\begin{equation*}
\dot{V}(t)=s^{T}(t)\left[\Phi-\zeta \zeta^{T}\right] s(t), \tag{24}
\end{equation*}
$$

where

$$
s^{T}(t)=\left[\begin{array}{llll}
x^{T}(t) & x^{T}(t-\tau) & g^{T}(x(t)) & \bar{e}^{T}(t)
\end{array}\right]
$$

$$
\begin{gathered}
\zeta^{T}=\left[\begin{array}{llll}
C_{21} & C_{d 2} & 0 & C_{22}
\end{array}\right], \\
\Phi=\left[\begin{array}{cccc}
\Phi_{11} & P_{1} A_{d}+C_{21}^{T} C_{d 2} & P_{1} G & P_{1} \bar{V}+C_{21}^{T} C_{22} \\
* & -Q+C_{d 2}^{T} C_{d 2} & 0 & 0 \\
* & * & -\lambda I & 0 \\
* & * & * & \Phi_{44}
\end{array}\right],
\end{gathered}
$$

$$
\begin{gather*}
\Phi_{11}=\operatorname{sym}\left(P_{1}(A+J S)\right)+\lambda U^{T} U+Q+C_{21}^{T} C_{21}  \tag{25}\\
\Phi_{44}=\operatorname{sym}\left(P_{2}(W+\bar{L} \bar{V})\right)+C_{22}^{T} C_{22}
\end{gather*}
$$

$$
\left[\begin{array}{ccccccc}
\Xi_{1}+P_{0} Q_{0}^{-1} P_{0}+\frac{1}{\lambda} P_{0} U_{0}^{T} U_{0} P_{0} & Q_{0} A_{d} & G & \bar{V} & \Xi_{15} & P_{0} C_{\infty 1}^{T} & P_{0} C_{21}^{T} \\
* & -Q_{0} & 0 & 0 & 0 & C_{d \infty}^{T} & C_{d 2}^{T} \\
* & * & -\frac{1}{\lambda} I & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{4} & \Xi_{45} & C_{\infty}^{T} & C_{22}^{T} \\
* & * & * & * & \Xi_{5} & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & -I
\end{array}\right]<0 .
$$

From the first, second, third, fourth, and seventh columns and rows of the left matrix in inequality (26) it can be verified that

$$
\begin{aligned}
& \Phi_{0}= \\
& {\left[\begin{array}{ccccc}
\Xi_{1}+P_{0} Q_{0}^{-1} P_{0}+\frac{1}{\lambda} P_{0} U_{0}^{T} U_{0} P_{0} & Q_{0} A_{d} & G & \bar{V} & P_{0} C_{21}^{T} \\
* & -Q_{0} & 0 & 0 & C_{d 2}^{T} \\
* & * & -\frac{1}{\lambda} I & 0 & 0 \\
* & * & * & \Xi_{4} & C_{22}^{T} \\
* & * & * & * & -I
\end{array}\right]}
\end{aligned}
$$

$$
\begin{equation*}
<0 \tag{27}
\end{equation*}
$$

Defining $P_{0}=P_{1}^{-1}, Q^{-1}=Q_{0}$, premultiplied and postmultiplied simultaneously by $\operatorname{diag}\left\{P_{0}^{-1}, Q_{0}^{-1}, I, I, I\right\}$, it can be seen by using Schur complement formula that $\Phi_{0}<0$ leads to $\Phi<0$. When $\Phi<0$ holds, we have

$$
\begin{equation*}
\dot{V}(t)<-s^{T}(t) \zeta \zeta^{T} s(t)=-z_{2}^{T} z_{2} \leq 0 . \tag{28}
\end{equation*}
$$

It follows that (13) is asymptotically stable in the absence of the exogenous input $\bar{d}(t)$. Next, we consider the performances to be optimized for (13). Consider two auxiliary functions as follows:

$$
\begin{gather*}
J_{0}=z_{2}^{T}(t) z_{2}(t)+\dot{V}(t) \\
J_{1}=z_{\infty}^{T}(t) z_{\infty}(t)-\gamma_{1}^{2} d_{2}(t)^{T} d_{2}(t)-\gamma_{2}^{2} \delta(t)^{T} \delta(t)  \tag{29}\\
-\gamma_{3}^{2} \dot{F}(t)^{T} \dot{F}(t)+\dot{V}(t)
\end{gather*}
$$

Following the definition of the $H_{2}$ performance, we only consider the case in the absence of $\bar{d}(t)$. It can be verified that $J_{0} \leq s^{T}(t) \Phi s(t)$.

From (20), it can be seen that

In the presence of $\bar{d}(t)$, it can be seen that $J_{1}=q^{T}(t) \Psi q(t)$, where

$$
q^{T}(t)=\left[\begin{array}{llll}
x^{T}(t) & x^{T}(t-\tau) & g^{T}(x(t)) & \bar{e}^{T}(t)
\end{array} \bar{d}^{T}(t)\right]
$$

$$
\begin{align*}
\Psi= & {\left[\begin{array}{ccccc}
\Psi_{11} & P_{1} A_{d} & P_{1} G & P_{1} \bar{V} & \Psi_{15} \\
* & -Q & 0 & 0 & 0 \\
* & * & -\frac{1}{\lambda} I & 0 & 0 \\
* & * & * & \Psi_{44} & \Psi_{45} \\
* & * & * & * & \Xi_{5}
\end{array}\right] } \\
& +\left[\begin{array}{c}
C_{\infty 1}^{T} \\
C_{d \infty}^{T} \\
0 \\
\\
C_{\infty 2}^{T} \\
0
\end{array}\right]\left[\begin{array}{c}
C_{\infty 11}^{T} \\
C_{d \infty}^{T} \\
0 \\
C_{\infty 2}^{T} \\
0
\end{array}\right], \tag{30}
\end{align*}
$$

where

$$
\begin{gather*}
\Psi_{11}=\operatorname{sym}\left(P_{1} A+P_{1} J S\right)+\frac{1}{\lambda} U^{T} U+Q \\
\Psi_{15}=\left[\begin{array}{lll}
P_{1} J_{2} & 0 & 0
\end{array}\right]  \tag{31}\\
\Psi_{44}=\operatorname{sym}\left(P_{2} \bar{W}+P_{2} \bar{L} \bar{V}\right) \\
\Psi_{45}=\left[\begin{array}{lll}
P_{2} \bar{L} J_{2} & P_{2} \bar{J}_{3} & P_{2} \bar{J}_{1}
\end{array}\right]
\end{gather*}
$$

Denote

$$
\Psi_{0}=\Psi+\left[\begin{array}{lllll}
C_{21} & C_{d 2} & 0 & C_{22} & 0
\end{array}\right]^{T}\left[\begin{array}{lllll}
C_{21} & C_{d 2} & 0 & C_{22} & 0 \tag{32}
\end{array}\right]
$$

It can be seen by using Schur complement formula that (20) leads to $\Psi_{0}<0$, and then $\Psi<0$ holds. It can be verified that both $J_{0}<0$ and $J_{1}<0$ hold. This completes the proof.

## 4. Simulation Examples

In this section, we consider the longitudinal dynamics of A4D aircraft at a flight condition of 15000 ft altitude and 0.9 Mach given in [2]. The longitudinal dynamics can be denoted as

$$
\begin{align*}
\dot{x}(t)= & A x(t)+G g(x(t))+J(u(t)+F(t))  \tag{33}\\
& +J_{1} d_{1}(t)+J_{2} d_{2}(t),
\end{align*}
$$

where $x(t)$ are measurable by using sensors technique. The coefficient matrices of aircraft model are given by

$$
\begin{gather*}
A=\left[\begin{array}{cccc}
-0.0605 & 32.37 & 0 & 32.2 \\
-0.00014 & -1.475 & 1 & 0 \\
-0.0111 & -34.72 & -2.793 & 0 \\
0 & 0 & 1 & 0
\end{array}\right], \\
J=\left[\begin{array}{c}
0 \\
-0.1064 \\
-33.8 \\
0
\end{array}\right], \quad J_{1}=\left[\begin{array}{c}
0 \\
-0.0532 \\
-16.9000 \\
0
\end{array}\right], \quad J_{2}=\left[\begin{array}{c}
0.1 \\
0 \\
-3 \\
0.1
\end{array}\right] . \tag{34}
\end{gather*}
$$

It is supposed that

$$
G=\left[\begin{array}{llll}
0 & 0 & 50 & 0 \tag{35}
\end{array}\right]^{T}, \quad g(x(t))=\sin (2 \pi 5 t) x_{2}(t)
$$

then, the matrix $U$ can be selected as $U=\operatorname{diag}\{0,1,0,0\}$ and the norm condition (3) can be satisfied. $A_{d}=0.5 \times$ $A, \tau=2$. Periodic disturbance $d_{1}(t)$ caused by rotating aerial propeller is assumed to be an unknown harmonic disturbance described by (2) with

$$
W=\left[\begin{array}{cc}
0 & 5  \tag{36}\\
-5 & 0
\end{array}\right], \quad E=\left[\begin{array}{ll}
25 & 0
\end{array}\right], \quad H_{3}=\left[\begin{array}{c}
0.1 \\
0.1
\end{array}\right]
$$

$\delta(t)$ is the additional disturbance signal resulting from the perturbations and uncertainties in the exogenous system (2) and satisfies 2 -norm boundedness. In simulation, we select $\delta(t)$ as the random signal with upper 2 -norm bound 1 . Wind gust and system noises $d_{2}(t)$ can also be considered as the random signal with upper 2-norm bounded.

The initial values of the states are supposed to be $x^{T}(0)=$ $\left[\begin{array}{llll}2 & -2 & 3 & 2\end{array}\right]$. For the reference output, it is denoted that

$$
\begin{align*}
& C_{\infty 1}=\left[\begin{array}{lll}
1 & 1 & 1
\end{array}\right], \quad C_{\infty 2}=\left[\begin{array}{lll}
1 & 1 & 1
\end{array}\right], \\
& C_{d \infty}=\left[\begin{array}{llll}
0.1 & 0.1 & 0.1 & 0.1
\end{array}\right], \\
& C_{21}=\left[\begin{array}{llll}
0.1 & 0.1 & 0.1 & 0.1
\end{array}\right], \quad C_{22}=\left[\begin{array}{lll}
0.1 & 0.1 & 0.1
\end{array}\right] \text {, }  \tag{37}\\
& C_{d 2}=\left[\begin{array}{llll}
0.01 & 0.01 & 0.01 & 0.01
\end{array}\right] .
\end{align*}
$$

For $\lambda=1, \gamma_{1}=1, \gamma_{2}=1$, and $\gamma_{3}=1$, it can be solved via LMI related to (20) that the gain of fault diagnosis observer (7) is

$$
K=\left[\begin{array}{llll}
20.9299 & 10.0231 & 1.3553 & 20.9682 \tag{38}
\end{array}\right],
$$

the gain of disturbance observer (4) is

$$
L=\left[\begin{array}{cccc}
-0.2584 & -0.5402 & -0.0142 & -0.2610  \tag{39}\\
2.3246 & 0.7523 & 0.1526 & 2.3269
\end{array}\right]
$$



Figure 1: Disturbances estimation error in disturbance observer.


Figure 2: Bias fault, its estimation, and error.
and the gain of state feedback controller is

$$
S=\left[\begin{array}{lllll}
89.6396 & 471.0670 & 74.2419 & 465.7941 \tag{40}
\end{array}\right]
$$

When the disturbance observer is constructed based on (4) and (5), the estimation error of exogenous disturbances is shown in Figure 1. The actuator bias fault is supposed to occur at 15th second as $F=4$. The estimation and its error of fault with system disturbances are demonstrated in Figure 2, where the solid line represents the real fault signal, the dash-dotted line is its estimation, and the dash line denotes its estimation error. In Figure 3, the state response signals of the control system are illustrated. It can be seen that the proposed fault tolerant controller has a good control ability for configuring fault and rejecting and attenuating disturbances simultaneously. In Figure 4, the ramp fault is assumed to occur at 15 th second with slope 0.1 . From


Figure 3: The responses of state variable.


Figure 4: Ramp fault, its estimation, and error.

Figure 4, it is shown that the time-varying fault can also be well estimated.

## 5. Conclusion

In this paper, a robust antidisturbance fault tolerant control problem is investigated for nonlinear time delay systems with faults and multiple disturbances. There are the following features of the proposed algorithm compared with the previous results. First, the multiple disturbances and state timedelay are considered simultaneously in this paper. Second, an FTC scheme is addressed with disturbance rejection and attenuation performance by combining fault accommodation and DOBC with a mixed $H_{2} / H_{\infty}$ controller, with which the fault can be accommodated and the disturbances can be rejected and attenuated simultaneously. Finally, simulation
for a flight control system is given to show the efficiency of the proposed approach.

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## Research Article

# New Exponential Stability Conditions of Switched BAM Neural Networks with Delays 

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#### Abstract

The exponential stability problem is considered in this paper for discrete-time switched BAM neural networks with time delay. The average dwell time method is introduced to deal with the exponential stability analysis of the systems for the first time. By constructing a new switching-dependent Lyapunov-Krasovskii functional, some new delay-dependent criteria are developed, which guarantee the exponential stability. A numerical example is provided to demonstrate the potential and effectiveness of the proposed algorithms.


## 1. Introduction

It is well known that bidirectional associative memory (BAM) neural networks have been proposed by Kosko [1, 2], which include two layers: the $X$-layer and the $Y$-layer. The neurons in one layer are fully interconnected to the neurons in another layer. Recently, the dynamics analysis for BAM neural networks has received much attention due to their extensive applications in pattern recognition, solving optimization, automatic control engineering, and so forth. It is known that time delay, which will inevitably occur in the communication owing to the unavoidable finite switching speed of amplifiers, is the main cause of instability and poor performance of neural networks. Hence, it is of great importance to study the stability of BAM neural networks with time delay. Many asymptotic or exponential stability conditions for BAM neural networks with time delay were developed, see, for example [3-10] and the references therein.

On the other hand, switched systems are an important class of hybrid dynamical systems which consist of a family of continuous-time or discrete-time subsystems and a rule that orchestrates the switching among them. Switched systems provide a natural and convenient unified framework for mathematical modeling of many physical phenomena and practical applications such as autonomous transmission
systems, computer disc driver, room temperature control, power electronics, and chaos generators, to name a few. Lots of valuable results concerning the stability analysis and stabilization for linear or nonlinear hybrid and switched systems were established, see, for example [11-14] and the references cited therein.

Recently, the switched neural networks, whose individual subsystems are a set of neural networks, have found their applications in the field of high-speed signal processing and artificial intelligence. Many researchers have been devoted to studying the stability issues for switched neural networks; see, for example, [15-17]. In [15], by using switched Lyapunov function method and a generalized Halanay inequality technique, the authors illustrated the asymptotic and exponential stability conditions for hybrid impulsive and switching Hopfield neural networks. While the switched Hopfield neural networks with time-varying delay were considered in [16], a robust stability condition was proposed based on the Lyapunov-Krasovskii functional approach. By combining Cohen-Grossberg neural networks with an arbitrary switching rule, the model of the switched Cohen-Grossberg neural networks with mixed time-varying delays was established in [17], and the robust stability criteria were established for these systems. However, all these results are related to the continuous-time switched neural networks. To the best of
the authors' knowledge, stability issues of the discrete-time switched neural networks have not been fully investigated to date. Particularly for the exponential stability analysis of the discrete-time switched BAM neural networks under some constrained switching, few results have been available in the literature so far, which motivates us to carry out the present study.

In this paper, the exponential stability analysis of discretetime switched BAM neural networks with time delay is considered. To begin with, the mathematical model of the discrete-time switched BAM neural networks with time delay is established. Then by constructing a new switchingdependent Lyapunov-Krasovskii functional, some sufficient criteria are developed to guarantee the discrete-time switched BAM neural networks to be exponentially stable based on the average dwell time approach and finite sum inequality technology. Finally, A numerical example is provided to demonstrate the potential and effectiveness of the proposed algorithms.

Notations. In this paper, we use $A>0(A<0)$ to denote a positive- (negative-) definite matrix $A ; A^{T}$ represents the transpose of matrix $A ; \lambda_{M}(\cdot)$ (resp., $\left.\lambda_{m}(\cdot)\right)$ means the maximum (resp., minimum) eigenvalue of $(\cdot)$. Let $\mathbb{R}$ denote the set of real numbers; $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space; $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices; $\mathbb{R}^{+}$denotes the set of $\{0,1,2, \ldots\} . \mathcal{N}=\{1,2, \ldots, N\}$ means a set of positive integers; $\mathbb{N}=\{1,2, \ldots, n\}$. The notation $\operatorname{diag}(\cdot)$ denotes a diagonal matrix. For given $\tau>0$ and $\theta \in[-\tau, 0],\|x(t)\|$ denotes vector norm defined by $\|x(t)\|=\sup _{-\tau \leq \theta \leq 0} \| x(t+$ $\theta) \|$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

## 2. Problem Formulation and Preliminaries

In this section, firstly, we will establish the model of discretetime switched BAM neural networks. Consider the following discrete-time BAM neural networks with time delay $\left(\Sigma_{1}\right)$ :

$$
\begin{align*}
& \tilde{x}_{p}(k+1) \\
& \quad=a_{p} \tilde{x}_{p}(k)+\sum_{q=1}^{n} w_{q p} \tilde{f}_{q}\left(\tilde{y}_{q}(k-d)\right)+I_{p}, \quad p \in \mathbb{N}, \\
& \tilde{y}_{q}(k+1)  \tag{1}\\
& \quad=b_{q} \widetilde{y}_{q}(k)+\sum_{p=1}^{n} v_{p q} \widetilde{g}_{p}\left(\tilde{x}_{p}(k-\tau)\right)+J_{q}, \quad q \in \mathbb{N},
\end{align*}
$$

where $\tilde{x}_{p}(k), \tilde{y}_{q}(k)$ are states of the $p$ th neuron from the neural field $F_{X}$ and the $q$ th neuron from the neural field $F_{Y}$ at time $k$, respectively. $a_{p}, b_{q} \in(0,1)$ describe the stability of internal neuron processes on the $X$-layer and the $Y$-layer, respectively. $w_{q p}, v_{p q}$ are constants and denote the synaptic connection weights. $\tilde{f}_{q}(\cdot)$ and $\widetilde{g}_{p}(\cdot)$ denote the activation functions of the $q$ th neuron from the neural field $F_{Y}$ and the $p$ th neuron from the neural field $F_{X}$, respectively. $I_{p}$ and $J_{q}$ are the external constant inputs from outside of the network acting on the $p$ th
neuron from the neural field $F_{X}$ and the $q$ th neuron from the neural field $F_{Y}$, respectively. $d$ and $\tau$ are constant delays.

The system $\left(\Sigma_{1}\right)$ can be rewritten as the vector form $\left(\Sigma_{2}\right)$ :

$$
\begin{gather*}
\widetilde{x}(k+1)=A \widetilde{x}(k)+W^{T} \tilde{f}(\tilde{y}(k-d))+I,  \tag{2}\\
\widetilde{y}(k+1)=B \tilde{y}(k)+V^{T} \widetilde{g}(\widetilde{x}(k-\tau))+J,
\end{gather*}
$$

where

$$
\begin{gather*}
\tilde{x}(k)=\left[\tilde{x}_{1}(k), \tilde{x}_{2}(k), \ldots, \tilde{x}_{n}(k)\right]^{T}, \\
\tilde{y}(k)=\left[\tilde{y}_{1}(k), \tilde{y}_{2}(k), \ldots, \tilde{y}_{n}(k)\right]^{T}, \\
A=\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right), \\
B=\operatorname{diag}\left(b_{1}, b_{2}, \ldots, b_{n}\right), \\
W=\left(w_{q p}\right)_{n \times n}, \quad V=\left(v_{p q}\right)_{n \times n} \tag{3}
\end{gather*}
$$

$\tilde{f}(\widetilde{y}(k))$

$$
=\left[\tilde{f}_{1}\left(\tilde{y}_{1}(k)\right), \tilde{f}_{2}\left(\tilde{y}_{2}(k)\right), \ldots, \tilde{f}_{n}\left(\tilde{y}_{n}(k)\right)\right]^{T}
$$

$$
\tilde{g}(\tilde{x}(k))
$$

$$
=\left[\widetilde{g}_{1}\left(\widetilde{x}_{1}(k)\right), \tilde{g}_{2}\left(\widetilde{x}_{2}(k)\right), \ldots, \tilde{g}_{n}\left(\widetilde{x}_{n}(k)\right)\right]^{T},
$$

$$
I=\left[I_{1}, I_{2}, \ldots, I_{n}\right], \quad J=\left[J_{1}, J_{2}, \ldots, J_{n}\right]
$$

Throughout this paper, we always assume the following.
$\left(\mathrm{G}_{1}\right)$ The neurons activation functions $\tilde{f}_{q}(\cdot)$ and $\tilde{g}_{p}(\cdot)(p, q \in \mathbb{N})$ are bounded on $\mathbb{R}$.
$\left(\mathrm{G}_{2}\right)$ There exist constants $\ell_{q}^{(1)}>0$ and $\ell_{p}^{(2)}>0$ such that

$$
\begin{array}{r}
\left|\tilde{f}_{q}\left(\xi_{1}\right)-\tilde{f}_{q}\left(\xi_{2}\right)\right| \leq \ell_{q}^{(1)}\left|\xi_{1}-\xi_{2}\right|, \\
\left|\widetilde{g}_{p}\left(\xi_{1}\right)-\tilde{g}_{p}\left(\xi_{2}\right)\right| \leq \ell_{p}^{(2)}\left|\xi_{1}-\xi_{2}\right|,  \tag{4}\\
\forall \xi_{1}, \xi_{2} \in \mathbb{R}, p, q \in \mathbb{N} .
\end{array}
$$

Then, under the assumptions $\left(\mathrm{G}_{1}\right)$ and $\left(\mathrm{G}_{2}\right)$, system $\left(\Sigma_{2}\right)$ has at least one equilibrium.

Now, we shift equilibrium point $\widetilde{x}^{*}=\left[\begin{array}{lll}\tilde{x}_{1}^{*} & \tilde{x}_{2}^{*} & \ldots\end{array}\right.$ $\left.\tilde{x}_{n}^{*}\right], \quad \tilde{y}^{*}=\left[\begin{array}{llll}\tilde{y}_{1}^{*} & \tilde{y}_{2}^{*} & \cdots & \tilde{y}_{n}^{*}\end{array}\right]$ of system $\left(\Sigma_{2}\right)$ to the origin. Let $x(k)=\tilde{x}(k)-\tilde{x}^{*}, y(k)=\widetilde{y}(k)-\widetilde{y}^{*}$; then the system $\left(\Sigma_{2}\right)$ can be transformed to the following system $\left(\Sigma_{3}\right)$ :

$$
\begin{gather*}
x(k+1)=A x(k)+W^{T} f(y(k-d))  \tag{5}\\
y(k+1)=B y(k)+V^{T} g(x(k-\tau))
\end{gather*}
$$

where

$$
\begin{gather*}
x(k)=\left[x_{1}(k), x_{2}(k), \ldots, x_{n}(k)\right]^{T}, \\
y(k)=\left[y_{1}(k), y_{2}(k), \ldots, y_{n}(k)\right]^{T}, \\
f(y(k))=\left[f_{1}\left(y_{1}(k)\right), f_{2}\left(y_{2}(k)\right), \ldots, f_{n}\left(y_{n}(k)\right)\right]^{T}, \\
g(x(k))=\left[g_{1}\left(x_{1}(k)\right), g_{2}\left(x_{2}(k)\right), \ldots, g_{n}\left(x_{n}(k)\right)\right]^{T},  \tag{6}\\
f_{q}\left(y_{q}(k)\right)=\tilde{f}_{q}\left(\tilde{y}_{q}(k)\right)-\tilde{f}_{q}\left(\tilde{y}_{q}^{*}\right), \quad q \in \mathbb{N}, \\
g_{p}\left(x_{p}(k)\right)=\widetilde{g}_{p}\left(\widetilde{x}_{p}(k)\right)-\widetilde{g}_{p}\left(\tilde{g}_{p}^{*}\right), \quad p \in \mathbb{N} .
\end{gather*}
$$

Obviously, the activation functions $f_{q}(\cdot)$ and $g_{p}(\cdot)$ satisfy the following conditions.
$\left(\mathrm{G}_{3}\right)$ There exist constants $\ell_{q}^{(1)}>0$ and $\ell_{p}^{(2)}>0$ such that

$$
\begin{align*}
&\left|f_{q}(\xi)\right| \leq \ell_{q}^{(1)}|\xi|,\left|g_{p}(\xi)\right| \leq \ell_{p}^{(2)}|\xi|  \tag{7}\\
& \forall \xi \in \mathbb{R}, p, q \in \mathbb{N}
\end{align*}
$$

With the rapid development of intelligent control, hybrid systems have been investigated due to their extensive applications. In recent years, considerable efforts have been focused on analysis and design of switched systems. The discretetime switched system can be characterized by the following difference equation $\left(\Sigma_{4}\right)$ :

$$
\begin{equation*}
x(k+1)=\Gamma_{\sigma(k)} x(k), \tag{8}
\end{equation*}
$$

where $\sigma(k)$ is a switching signal which takes its values in the finite set $\mathcal{N}=\{1,2, \ldots, N\} . \Gamma_{\sigma(k)}=\Gamma_{i}$, when $\sigma(k)=i$, are the functions of the switching signals.

Combining the theories of switched systems and discretetime BAM neural networks, the discrete-time switched BAM neural networks can be formulated as the following system $(\Sigma)$ :

$$
\begin{gather*}
x(k+1)=A_{\sigma(k)} x(k)+W_{\sigma(k)}^{T} f(y(k-d)), \\
y(k+1)=B_{\sigma(k)} y(k)+V_{\sigma(k)}^{T} g(x(k-\tau)), \tag{9}
\end{gather*}
$$

where $\sigma(k)$ is a switching signal which takes its values in the finite set $\mathcal{N}=\{1,2, \ldots, N\}$.

For the discrete-time switched BAM neural networks ( $\Sigma$ ), we have the following assumptions.
$\left(\mathrm{H}_{1}\right)$ The initial value is $x(s)=\phi(s), y(s)=\psi(s), s \in$ $[-h, 0]$, where $h=\max \{d, \tau\}$.
$\left(\mathrm{H}_{2}\right)$ There exist matrices $L_{1}>0$ and $L_{2}>0$ such that

$$
\begin{equation*}
|f(\xi)| \leq L_{1}|\xi|, \quad|g(\xi)| \leq L_{2}|\xi|, \quad \forall \xi \in \mathbb{R} \tag{10}
\end{equation*}
$$

where $L_{1}=\operatorname{diag}\left(\ell_{1}^{(1)}, \ell_{2}^{(1)}, \ldots, \ell_{n}^{(1)}\right)$ and $L_{2}=$ $\operatorname{diag}\left(\ell_{1}^{(2)}, \ell_{2}^{(2)}, \ldots, \ell_{n}^{(2)}\right)$.
$\left(\mathrm{H}_{3}\right)$ Switching sequence is defined as $\zeta=\left\{\left[\begin{array}{ll}x_{k_{0}} & y_{k_{0}}\end{array}\right]^{T}\right.$; $\left.\left(i_{0}, k_{0}\right),\left(i_{1}, k_{1}\right), \ldots,\left(i_{m}, k_{m}\right), \ldots, \mid i_{m} \in \mathcal{N}, m \in \mathbb{R}^{+}\right\}$. When $k \in\left[k_{m}, k_{m+1}\right)$, the $k_{m}$ th subsystem is activated and the states of system $(\Sigma)$ do not jump when switch occurs.

Remark 1. By combining the switched systems theory and the discrete-time BAM neural networks model, the mathematical model of discrete-time switched BAM neural networks is introduced as above. A set of discrete-time BAM neural networks with time delay are used as the subsystems, and an arbitrary switching rule is assumed to coordinate the switching between these neural networks.

To present the main results of this paper more precisely, the following definitions and lemmas are introduced, which will be essential for the later development.

Definition 2 (see [12]). For any $k \geq k_{0}$ and any switched signal $\sigma(\varsigma), k_{0} \leq \varsigma<k$, let $N_{\sigma}$ denote the switching numbers of $\sigma(\varsigma)$ during the interval $\left[k_{0}, k\right]$. If there exist $N_{0} \geq 0$ and $T_{a}>0$ such that $N_{\sigma}\left(k_{0}, k\right) \leq N_{0}+\left(k-k_{0}\right) / T_{a}$, then $T_{a}$ and $N_{0}$ are called average dwell time and the chatter bound, respectively.

Definition 3. The discrete-time switched BAM neural network $(\Sigma)$ is said to be exponentially stable if its solution satisfies

$$
\begin{equation*}
\|x(k)\|^{2}+\|y(k)\|^{2} \leq K\left(\|\phi\|_{L}^{2}+\|\psi\|_{L}^{2}\right) \lambda^{-\left(k-k_{0}\right)}, \quad \forall k \geq k_{0} \tag{11}
\end{equation*}
$$

for any initial condition $\left(k_{0}, \phi\right) \in \mathbb{R}^{+} \times C^{n}$ and $\left(k_{0}, \psi\right) \in$ $\mathbb{R}^{+} \times C^{n} \cdot\|\phi\|_{L}=\sup _{k_{0}-h \leq \ell \leq k_{0}}\|\phi(\ell)\|$, and $\|\psi\|_{L}=$ $\sup _{k_{0}-h \leq \ell \leq k_{0}}\|\psi(\ell)\|, h=\max \{d, \tau\} . \quad K>0$ is the decay coefficient, and $\lambda>1$ is the decay rate.

Remark 4. Without loss of generality, in this paper, we assume $N_{0}=0$ for simplicity as commonly used in the literature.

Remark 5. Based on the definition of exponential stability for BAM neural networks in [5] and the definition of exponential stability for switched systems in [13], we give the above definition of exponential stability for discrete-time switched BAM neural networks.

Lemma 6 (the Schur complement [18]). For any symmetric matrix $S=\left[\begin{array}{ll}s_{11} & S_{12} \\ s_{12}^{T} & S_{22}\end{array}\right]<0$, the following conditions are equivalent:
(i) $S_{11}<0$, and $S_{22}-S_{12}^{T} S_{11}^{-1} S_{12}<0$,
(ii) $S_{22}<0$, and $S_{11}-S_{12} S_{22}^{-1} S_{12}^{T}<0$.

Lemma 7 (finite sum inequality [13]). For any constant matrix $Y=\left[\begin{array}{ll}M_{1} & M_{2}\end{array}\right] \in \mathbb{R}^{n \times 2 n}, R>0, h \geq 0$, the following inequality holds:

$$
\begin{align*}
-\sum_{j=k-h}^{k-1} \ell^{T}(j) R \ell(j) \leq & \xi^{T}(k)\left[\begin{array}{cc}
M_{1}^{T}+M_{1} & -M_{1}^{T}+M_{2} \\
* & -M_{2}^{T}-M_{2}
\end{array}\right] \\
& \times \xi(k)+h \xi^{T}(k) Y^{T} R^{-1} Y \xi(k), \tag{12}
\end{align*}
$$

where $\ell(k)=x(k+1)-x(k)$ and $\xi(k)=\left[x^{T}(k) x^{T}(k-h)\right]^{T}$.

## 3. Main Result

In this section, the exponential stability condition for the discrete-time switched BAM neural networks ( $\Sigma$ ) will be presented using the average dwell time method.

When $\sigma(k)=i$, we have the following subsystem $\left(\Sigma^{i}\right)$ :

$$
\begin{gather*}
x(k+1)=A_{i} x(k)+W_{i}^{T} f(y(k-d)) \\
y(k+1)=B_{i} y(k)+V_{i}^{T} g(x(k-\tau)) \tag{13}
\end{gather*}
$$

Choose the Lyapunov-Krasovskii functional candidate for the subsystem ( $\Sigma^{i}$ ) as

$$
\begin{equation*}
V_{i}(k)=V_{1 i}(k)+V_{2 i}(k)+V_{3 i}(k), \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1 i}(k)= & x^{T}(k) P_{1 i} x(k)+y^{T}(k) P_{2 i} y(k), \\
V_{2 i}(k)= & \sum_{\theta=k-\tau}^{k-1} r^{\theta-k+1} x^{T}(\theta) Q_{1 i} x(\theta) \\
& +\sum_{\theta=k-d}^{k-1} r^{\theta-k+1} y^{T}(\theta) Q_{2 i} y(\theta), \\
V_{3 i}(k)= & \sum_{s=-\tau}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{1}^{T}(\theta) R_{1 i} \ell_{1}(\theta)  \tag{15}\\
& +\sum_{s=-d}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{2}^{T}(\theta) R_{2 i} \ell_{2}(\theta), \\
& \ell_{1}(k)=x(k+1)-x(k), \\
& \ell_{2}(k)=y(k+1)-y(k) .
\end{align*}
$$

Now we give the following theorem, which plays an important role in the derivation of the exponential stability condition for the discrete-time switched BAM neural networks ( $\Sigma$ ).

Theorem 8. Under the assumptions $\left(H_{1}\right)-\left(H_{3}\right)$, for given scalar $r>1$, the decay estimation

$$
\begin{equation*}
V_{i}(k) \leq r^{-\left(k-k_{0}\right)} V_{i}\left(k_{0}\right) \tag{16}
\end{equation*}
$$

is satisfied along any trajectory of system $\left(\Sigma^{i}\right)$ if there exist matrices $P_{1 i}>0, P_{2 i}>0, Q_{1 i}>0, Q_{2 i}>0, R_{1 i}>0, R_{2 i}>$ $0, N_{1 i}, N_{2 i}, M_{1 i}, M_{2 i}, T_{1 i}>0$, and $T_{2 i}>0, i \in \mathcal{N}$, such that the following linear matrix inequality holds:

$$
\left[\begin{array}{ccc}
\Omega_{i} & \Gamma_{1 i}^{T} & \Gamma_{2 i}^{T}  \tag{17}\\
* & -\tau r^{\tau} R_{1 i} & 0 \\
* & * & -d r^{d} R_{2 i}
\end{array}\right]<0
$$

where

$$
\begin{aligned}
& \Omega_{i}=\left[\begin{array}{cccccc}
\Omega_{11} & \Omega_{12} & 0 & 0 & 0 & \Omega_{16} \\
* & \Omega_{22} & 0 & 0 & 0 & 0 \\
* & * & \Omega_{33} & \Omega_{34} & 0 & 0 \\
* & * & * & \Omega_{44} & \Omega_{45} & 0 \\
* & * & * & * & \Omega_{55} & 0 \\
* & * & * & * & * & \Omega_{66}
\end{array}\right], \\
& \Gamma_{1 i}=\left[\begin{array}{llllll}
\tau N_{1 i} & \tau N_{2 i} & 0 & 0 & 0 & 0
\end{array}\right], \\
& \Gamma_{2 i}=\left[\begin{array}{llllll}
0 & 0 & 0 & d M_{1 i} & d M_{2 i} & 0
\end{array}\right], \\
& \begin{aligned}
\Omega_{11}= & A_{i} P_{1 i} A_{i}-r^{-1} P_{1 i}+Q_{1 i} \\
& +\left(A_{i}-I\right)\left(\tau R_{1 i}\right)\left(A_{i}-I\right)+r^{-\tau}\left(N_{1 i}^{T}+N_{1 i}\right),
\end{aligned} \\
& \Omega_{12}=r^{-\tau}\left(-N_{1 i}^{T}+N_{2 i}\right) \text {, } \\
& \Omega_{22}=-r^{-\tau}\left(Q_{1 i}+N_{2 i}^{T}+N_{2 i}\right)+T_{1 i}, \\
& \Omega_{33}=V_{i}\left(P_{2 i}+d R_{2 i}\right) V_{i}^{T}-L_{1 i}^{-1} T_{1 i} L_{1 i}^{-1}, \\
& \Omega_{34}=V_{i} P_{2 i} B_{i}+V_{i}\left(d R_{2 i}\right)\left(B_{i}-I\right), \\
& \Omega_{44}=B_{i} P_{2 i} B_{i}-r^{-1} P_{2 i}+Q_{2 i}+\left(B_{i}-I\right)\left(d R_{2 i}\right)\left(B_{i}-I\right) \\
& +r^{-d}\left(M_{1 i}^{T}+M_{1 i}\right), \\
& \Omega_{45}=r^{-d}\left(-M_{1 i}^{T}+M_{2 i}\right), \\
& \Omega_{55}=-r^{-d}\left(Q_{2 i}+M_{2 i}^{T}+M_{2 i}\right)+T_{2 i}, \\
& \Omega_{16}=A_{i} P_{1 i} W_{i}^{T}+\left(A_{i}-I\right)\left(\tau R_{1 i}\right) W_{i}^{T}, \\
& \Omega_{66}=W_{i}\left(P_{1 i}+\tau R_{1 i}\right) W_{i}^{T}-L_{2 i}^{-1} T_{2 i} L_{2 i}^{-1} .
\end{aligned}
$$

Proof. Calculating the differential of $V_{i}(k)$ along the trajectory of system $\left(\Sigma^{i}\right)$, we obtain

$$
\begin{align*}
\Delta V_{1 i}(k)= & x^{T}(k+1) P_{1 i} x(k+1) \\
& +y^{T}(k+1) P_{2 i} y(k+1) \\
& -x^{T}(k) P_{1 i} x(k)-y^{T}(k) P_{2 i} y(k) \\
= & x^{T}(k+1) P_{1 i} x(k+1) \\
& -r^{-1} x^{T}(k) P_{1 i} x(k)+y^{T}(k+1) P_{2 i} y(k+1) \\
& -r^{-1} y^{T}(k) P_{2 i} y(k)+\left(r^{-1}-1\right) V_{1 i}(k), \tag{19}
\end{align*}
$$

$$
\begin{aligned}
\Delta V_{2 i}(k)= & \sum_{\theta=k+1-\tau}^{k} r^{\theta-k} x^{T}(\theta) Q_{1 i} x(\theta) \\
& +\sum_{\theta=k+1-d}^{k} r^{\theta-k} y^{T}(\theta) Q_{2 i} y(\theta) \\
& -\sum_{\theta=k-\tau}^{k-1} r^{\theta-k+1} x^{T}(\theta) Q_{1 i} x(\theta) \\
& -\sum_{\theta=k-d}^{k-1} r^{\theta-k+1} y^{T}(\theta) Q_{2 i} y(\theta) \\
= & x^{T}(k) Q_{1 i} x(k)-r^{-\tau} x^{T}(k-\tau) Q_{1 i} x(k-\tau) \\
& +y^{T}(k) Q_{2 i} y(k) \\
& -r^{-d} y^{T}(k-d) Q_{2 i} y(k-d)+\left(r^{-1}-1\right) V_{2 i}(k)
\end{aligned}
$$

$$
\Delta V_{3 i}(k)=\sum_{s=-\tau}^{-1} \sum_{\theta=k+1+s}^{k} r^{\theta-k} \ell_{1}^{T}(\theta) R_{1 i} \ell_{1}(\theta)
$$

$$
+\sum_{s=-d}^{-1} \sum_{\theta=k+1+s}^{k} r^{\theta-k} \ell_{2}^{T}(\theta) R_{2 i} \ell_{2}(\theta)
$$

$$
-\sum_{s=-\tau}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{1}^{T}(\theta) R_{1 i} \ell_{1}(\theta)
$$

$$
-\sum_{s=-d}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{2}^{T}(\theta) R_{2 i} \ell_{2}(\theta)
$$

$$
=\sum_{s=-\tau}^{-1}\left(\ell_{1}^{T}(k) R_{1 i} \ell_{1}(k)\right.
$$

$$
+\sum_{\theta=k+s}^{k-1} r^{\theta-k} \ell_{1}^{T}(\theta) R_{1 i} \ell_{1}(\theta)
$$

$$
\left.-r^{s} \ell_{1}^{T}(k+s) R_{1 i} \ell_{1}(k+s)\right)
$$

$$
+\sum_{s=-d}^{-1}\left(\ell_{2}^{T}(k) R_{2 i} \ell_{2}(k)\right.
$$

$$
+\sum_{\theta=k+s}^{k-1} r^{\theta-k} \ell_{2}^{T}(\theta) R_{2 i} \ell_{2}(\theta)
$$

$$
\left.-r^{s} \ell_{2}^{T}(k+s) R_{2 i} \ell_{2}(k+s)\right)-V_{3 i}(k)
$$

$$
=\tau \ell_{1}^{T}(k) R_{1 i} \ell_{1}(k)
$$

$$
-\sum_{s=k-\tau}^{k-1} r^{s-k} \ell_{1}^{T}(s) R_{1 i} \ell_{1}(s)+d \ell_{2}^{T}(k) R_{2 i} \ell_{2}(k)
$$

$$
\begin{align*}
& -\sum_{s=k-d}^{k-1} r^{s-k} \ell_{2}^{T}(s) R_{2 i} \ell_{2}(s)+\left(r^{-1}-1\right) V_{3 i}(k) \\
\leq & \tau \ell_{1}^{T}(k) R_{1 i} \ell_{1}(k)+d \ell_{2}^{T}(k) R_{2 i} \ell_{2}(k) \\
& -\sum_{s=k-\tau}^{k-1} r^{-\tau} \ell_{1}^{T}(s) R_{1 i} \ell_{1}(s) \\
& -\sum_{s=k-d}^{k-1} r^{-d} \ell_{2}^{T}(s) R_{2 i} \ell_{2}(s)+\left(r^{-1}-1\right) V_{3 i}(k) . \tag{20}
\end{align*}
$$

Note that

$$
\begin{align*}
& -r^{-\tau} \sum_{s=k-\tau}^{k-1} \ell_{1}^{T}(s) R_{1 i} \ell_{1}(s) \\
& \leq r^{-\tau} \eta_{1}^{T}(k)\left[\begin{array}{cc}
N_{1 i}^{T}+N_{1 i} & -N_{1 i}^{T}+N_{2 i} \\
* & -N_{2 i}^{T}-N_{2 i}
\end{array}\right] \eta_{1}(k) \\
& \quad+r^{-\tau} \eta_{1}^{T}(k)\left[\begin{array}{c}
N_{1 i}^{T} \\
N_{2 i}^{T}
\end{array}\right] \tau R_{1 i}^{-1}\left[\begin{array}{ll}
N_{1 i} & N_{2 i}
\end{array}\right] \eta_{1}(k) \\
& -r^{-d} \sum_{s=k-d}^{k-1} \ell_{2}^{T}(s) R_{2 i} \ell_{2}(s)  \tag{21}\\
& \leq \\
& \left.\quad r^{-d} \eta_{2}^{T}(k)\left[\begin{array}{c}
M_{1 i}^{T}+M_{1 i} \\
* \\
\\
\quad+r^{-d} \eta_{2 i}^{T}(k)\left[\begin{array}{c}
M_{2 i}^{T} \\
M_{1 i}^{T} \\
M_{2 i}^{T}
\end{array}\right] d R_{2 i}^{-1}\left[M_{1 i}\right.
\end{array}\right] M_{2 i}\right] \eta_{2}(k)
\end{align*}
$$

where

$$
\begin{align*}
& \eta_{1}(k)=\left[\begin{array}{c}
x(k) \\
x(k-\tau)
\end{array}\right], \quad \eta_{2}(k)=\left[\begin{array}{c}
y(k) \\
y(k-d)
\end{array}\right]  \tag{22}\\
& x^{T}(k-\tau) T_{1 i} x(k-\tau)  \tag{23}\\
& \quad-g^{T}(x(k-\tau)) L_{1 i}^{-1} T_{1 i} L_{1 i}^{-1} g(x(k-\tau)) \geq 0 \\
& y^{T}(k-d) T_{2 i} x(k-d) \\
& \quad-f^{T}(y(k-d)) L_{2 i}^{-1} T_{2 i} L_{2 i}^{-1} f(y(k-d)) \geq 0 \tag{24}
\end{align*}
$$

From (20) to (24), the following inequality is satisfied:
$\Delta V_{i}(k)$

$$
\begin{aligned}
\leq & {\left[\begin{array}{c}
x(k) \\
f(y(k-d))
\end{array}\right]^{T}\left[\begin{array}{c}
A_{i} \\
W_{i}
\end{array}\right] P_{1 i}\left[\begin{array}{ll}
A_{i} & W_{i}^{T}
\end{array}\right]\left[\begin{array}{c}
x(k) \\
f(y(k-d))
\end{array}\right] } \\
& -x^{T}(k) r^{-1} P_{1 i} x(k)+x^{T}(k) Q_{1 i} x(k) \\
& -x^{T}(k-\tau) r^{-\tau} Q_{1 i} x(k-\tau) \\
& +\left[\begin{array}{c}
y(k) \\
g(x(k-\tau))
\end{array}\right]^{T}\left[\begin{array}{c}
B_{i} \\
V_{i}
\end{array}\right] P_{2 i}\left[\begin{array}{ll}
B_{i} & V_{i}^{T}
\end{array}\right]\left[\begin{array}{c}
y(k) \\
g(x(k-\tau))
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
& -y^{T}(k) r^{-1} P_{2 i} y(k)+y^{T}(k) Q_{2 i} y(k) \\
& -y^{T}(k-d) r^{-d} Q_{2 i} y(k-d) \\
& +\left[\begin{array}{c}
x(k) \\
f(y(k-d))
\end{array}\right]^{T}\left[\begin{array}{c}
A_{i}-I \\
W_{i}
\end{array}\right] \\
& \times \tau R_{1 i}\left[\begin{array}{ll}
A_{i}-I & W_{i}^{T}
\end{array}\right]\left[\begin{array}{c}
x(k) \\
f(y(k-d))
\end{array}\right] \\
& +\left[\begin{array}{c}
y(k) \\
g(x(k-\tau))
\end{array}\right]^{T}\left[\begin{array}{c}
B_{i}-I \\
V_{i}
\end{array}\right] \\
& \times d R_{2 i}\left[\begin{array}{ll}
B_{i}-I & V_{i}^{T}
\end{array}\right]\left[\begin{array}{c}
y(k) \\
g(x(k-\tau))
\end{array}\right] \\
& +r^{-\tau} \eta_{1}^{T}(k)\left[\begin{array}{cc}
N_{1 i}^{T}+N_{1 i} & -N_{1 i}^{T}+N_{2 i} \\
* & -N_{2 i}^{T}-N_{2 i}
\end{array}\right] \eta_{1}(k) \\
& +r^{-\tau} \eta_{1}^{T}(k)\left[\begin{array}{l}
N_{1 i}^{T} \\
N_{2 i}^{T}
\end{array}\right] \tau R_{1 i}^{-1}\left[\begin{array}{ll}
N_{1 i} & N_{2 i}
\end{array}\right] \eta_{1}(k) \\
& +r^{-d} \eta_{2}^{T}(k)\left[\begin{array}{cc}
M_{1 i}^{T}+M_{1 i} & -M_{1 i}^{T}+M_{2 i} \\
* & -M_{2 i}^{T}-M_{2 i}
\end{array}\right] \eta_{2}(k)
\end{aligned}
$$

$$
\begin{align*}
& +r^{-d} \eta_{2}^{T}(k)\left[\begin{array}{l}
M_{1 i}^{T} \\
M_{2 i}^{T}
\end{array}\right] d R_{2 i}^{-1}\left[\begin{array}{ll}
M_{1 i} & M_{2 i}
\end{array}\right] \eta_{2}(k) \\
& +x^{T}(k-\tau) T_{1 i} x(k-\tau) \\
& -g^{T}(x(k-\tau)) L_{1 i}^{-1} T_{1 i} L_{1 i}^{-1} g(x(k-\tau)) \\
& +y^{T}(k-d) T_{2 i} x(k-d) \\
& -f^{T}(y(k-d)) L_{2 i}^{-1} T_{2 i} L_{2 i}^{-1} f(y(k-d)) \\
& +\left(r^{-1}-1\right) V_{i}(k) \\
& =\zeta^{T}(k) \Omega_{i} \zeta(k)+r^{-\tau} \eta_{1}^{T}(k)\left[\begin{array}{c}
N_{1 i}^{T} \\
N_{2 i}^{T}
\end{array}\right] \\
& \times \tau R_{1 i}^{-1}\left[\begin{array}{ll}
N_{1 i} & N_{2 i}
\end{array}\right] \eta_{1}(k) \\
& +r^{-d} \eta_{2}^{T}(k)\left[\begin{array}{l}
M_{1 i}^{T} \\
M_{2 i}^{T}
\end{array}\right] d R_{2 i}^{-1}\left[\begin{array}{ll}
M_{1 i} & M_{2 i}
\end{array}\right] \eta_{2}(k) \\
& +\left(r^{-1}-1\right) V_{i}(k), \tag{25}
\end{align*}
$$

where

$$
\zeta(k)=\left[\begin{array}{llllll}
x^{T}(k) & x^{T}(k-\tau) & g^{T}(x(k-\tau)) & y^{T}(k) & y^{T}(k-d) & f^{T}(y(k-d)) \tag{26}
\end{array}\right]^{T}
$$

Therefore, from (17), we have

$$
\begin{equation*}
\Delta V_{i}(k) \leq\left(r^{-1}-1\right) V_{i}(k) \tag{27}
\end{equation*}
$$

which implies (16) is true. This completes the proof of Theorem 8.

In what follows, we are in a position to derive the delaydependent exponential stability condition for the discretetime switched BAM neural networks ( $\Sigma$ ), and the results are given in the following theorem.

Theorem 9. Under the assumptions $\left(H_{1}\right)-\left(H_{3}\right)$, for given scalars $r>1, \mu \geq 1$, the system $(\Sigma)$ is exponentially stable and ensures a decay rate $\lambda$, where $\lambda=r^{\left(-\ln \mu /\left(T_{a} \ln r\right)\right)+1}$, if there exist matrices $P_{1 i}>0, P_{2 i}>0, Q_{1 i}>0, Q_{2 i}>0, R_{1 i}>0, R_{2 i}>$ $0, N_{1 i}, N_{2 i}, M_{1 i}, M_{2 i}$, and $T_{1 i}>0, T_{2 i}>0, i \in \mathcal{N}$, such that (17) and the following inequalities hold:

$$
\begin{array}{ccc}
P_{1 \alpha} \leq \mu P_{1 \beta}, & P_{2 \alpha} \leq \mu P_{2 \beta}, & Q_{1 \alpha} \leq \mu Q_{1 \beta} \\
Q_{2 \alpha} \leq \mu Q_{2 \beta}, & R_{1 \alpha} \leq \mu R_{1 \beta}, & R_{2 \alpha} \leq \mu R_{2 \beta} \\
T_{a} \geq T_{a}^{*}=\operatorname{ceil}\left[\frac{\ln \mu}{\ln r}\right], & \alpha, \beta \in \mathscr{N} . \tag{30}
\end{array}
$$

Proof. Choose the Lyapunov-Krasovskii functional candidate of system ( $\Sigma$ ) as

$$
\begin{equation*}
V_{\sigma(k)}(k)=V_{1 \sigma(k)}(k)+V_{2 \sigma(k)}(k)+V_{3 \sigma(k)}(k), \tag{31}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1 \sigma(k)}(k)= & x^{T}(k) P_{1 \sigma(k)} x(k)+y^{T}(k) P_{2 \sigma(k)} y(k), \\
V_{2 \sigma(k)}(k)= & \sum_{\theta=k-\tau}^{k-1} r^{\theta-k+1} x^{T}(\theta) Q_{1 \sigma(k)} x(\theta) \\
& +\sum_{\theta=k-d}^{k-1} r^{\theta-k+1} y^{T}(\theta) Q_{2 \sigma(k)} y(\theta),  \tag{32}\\
V_{3 \sigma(k)}(k)= & \sum_{s=-\tau}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{1}^{T}(\theta) R_{1 \sigma(k)} \ell_{1}(\theta) \\
& +\sum_{s=-d}^{-1} \sum_{\theta=k+s}^{k-1} r^{\theta-k+1} \ell_{2}^{T}(\theta) R_{2 \sigma(k)} \ell_{2}(\theta) .
\end{align*}
$$

From (16), (28), and (29), when $k \in\left[k_{m}, k_{m+1}\right)$, there holds

$$
\begin{align*}
V_{\sigma(k)}(k) & \leq r^{-\left(k-k_{m}\right)} V_{\sigma\left(k_{m}\right)}\left(k_{m}\right) \\
& \leq \mu r^{-\left(k-k_{m}\right)} V_{\sigma\left(k_{m}-1\right)}\left(k_{m}\right) \\
& \leq \mu r^{-\left(k-k_{m}\right)} r^{-\left(k_{m}-k_{m-1}\right)} V_{\sigma\left(k_{m}-1\right)}\left(k_{m-1}\right)  \tag{33}\\
& =\mu r^{-\left(k-k_{m-1}\right)} V_{\sigma\left(k_{m}-1\right)}\left(k_{m-1}\right) \\
& \leq \cdots \leq \mu^{N_{\sigma}} r^{-\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) .
\end{align*}
$$

Observe that

$$
\begin{equation*}
N_{\sigma} \leq \frac{k-k_{0}}{T_{a}}, \quad \mu=r^{\ln \mu / \ln r} \tag{34}
\end{equation*}
$$

This together with (30) and (33) yields

$$
\begin{equation*}
\mu^{N_{\sigma}} r^{-\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \leq \lambda^{-\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) . \tag{35}
\end{equation*}
$$

This further implies

$$
\begin{equation*}
V_{\sigma(k)}(k) \leq \lambda^{-\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \tag{36}
\end{equation*}
$$

Let

$$
\begin{align*}
& \beta_{1}=\min \left\{\min _{i \in \mathcal{N}}\left\{\lambda_{m}\left(P_{1 i}\right)\right\}, \min _{i \in \mathcal{N}}\left\{\lambda_{m}\left(P_{2 i}\right)\right\}\right\},  \tag{37}\\
& \beta_{2}=\max \left\{\beta_{21}, \beta_{22}\right\},
\end{align*}
$$

where

$$
\begin{align*}
\beta_{21}= & \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(P_{1 i}\right)\right\}+\frac{r\left(1-r^{-\tau}\right)}{r-1} \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(Q_{1 i}\right)\right\} \\
& +2 \frac{r \tau(r-1)-r\left(1-r^{-\tau}\right)}{(r-1)^{2}} \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(R_{1 i}\right)\right\},  \tag{38}\\
\beta_{22}= & \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(P_{2 i}\right)\right\}+\frac{r\left(1-r^{-d}\right)}{r-1} \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(Q_{2 i}\right)\right\} \\
& +2 \frac{r d(r-1)-r\left(1-r^{-d}\right)}{(r-1)^{2}} \max _{i \in \mathcal{N}}\left\{\lambda_{M}\left(R_{2 i}\right)\right\} . \tag{39}
\end{align*}
$$

It can be verified from (31) that

$$
\begin{gather*}
V_{\sigma(k)}(k) \geq \beta_{1}\left(\|x(k)\|^{2}+\|y(k)\|^{2}\right)  \tag{40}\\
V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \leq \beta_{21}\|\phi(k)\|_{L}^{2}+\beta_{22}\|\psi(k)\|_{L}^{2} \tag{41}
\end{gather*}
$$

which gives rise to

$$
\begin{align*}
\beta_{1}\left(\|x(k)\|^{2}+\|y(k)\|^{2}\right) & \leq V_{\sigma(k)}(k) \leq \lambda^{-\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& \leq \lambda^{-\left(k-k_{0}\right)} \beta_{2}\left(\|\phi\|_{L}^{2}+\|\psi\|_{L}^{2}\right) \tag{42}
\end{align*}
$$

Therefore, we have

$$
\begin{equation*}
\|x(k)\|^{2}+\|y(k)\|^{2} \leq \frac{\beta_{2}}{\beta_{1}}\left(\|\phi\|_{L}^{2}+\|\psi\|_{L}^{2}\right) \lambda^{-\left(k-k_{0}\right)} \tag{43}
\end{equation*}
$$

which implies that the discrete-time switched BAM neural networks ( $\Sigma$ ) are exponentially stable. This completes the proof of Theorem 9 .

Remark 10. In (30), the function $\operatorname{ceil}(t)$ is used, which represents rounding real number $t$ to the nearest integer greater than or equal to $t$. The reason that we introduce the function ceil is that the dwell time length of the currently active subsystem is the number of sampling periods between the two consecutive switching times.

Remark 11. In [3-6], the asymptotic or exponential stability problem is considered for continuous-time BAM neural networks with time delay

$$
\begin{gather*}
\dot{x}(t)=-A x(t)+W f(y(t-d))  \tag{44}\\
\dot{y}(t)=-B x(t)+\operatorname{Vg}(x(t-\tau)) \tag{45}
\end{gather*}
$$

However, the dynamics of discrete-time neural networks may be quite different from those of continuous-time ones, and the stability criteria established for continuous-time BAM neural networks model are not necessarily applicable to discrete-time systems. Considering the importance in both theory and practice, it is necessary to study the dynamics of the discrete-time BAM neural networks.

Remark 12. There are few references concerning exponential stability analysis for discrete-time switched BAM neural networks. In this paper, some delay-dependent sufficient conditions checking the exponential stability of discrete-time switched BAM neural networks using average dwell time approach are presented. These conditions are proposed in the form of linear matrix inequalities, which can be easily solved by using the recently developed interior algorithms.

## 4. An Illustrative Example

Consider the discrete-time switched BAM neural networks $(\Sigma)$ combining two subsystems with the following parameters:

$$
\begin{gather*}
A_{1}=\left[\begin{array}{cc}
0.5 & 0 \\
0 & 0.5
\end{array}\right]  \tag{46}\\
W_{1}=\left[\begin{array}{cc}
-0.0483 & -0.01 \\
-0.03 & -0.04
\end{array}\right], \quad B_{1}=\left[\begin{array}{cc}
0.3 & 0 \\
0 & 0.2
\end{array}\right] \\
V_{1}=\left[\begin{array}{cc}
-0.05 & -0.01 \\
-0.05 & -0.06
\end{array}\right] \\
A_{2}=\left[\begin{array}{cc}
0.7 & 0 \\
0 & 0.4
\end{array}\right], \quad B_{2}=\left[\begin{array}{cc}
0.6 & 0 \\
0 & 0.7
\end{array}\right]  \tag{47}\\
W_{2}=\left[\begin{array}{cc}
-0.032 & -0.06 \\
-0.01 & -0.05
\end{array}\right], \quad V_{2}=\left[\begin{array}{cc}
-0.05 & -0.01 \\
-0.05 & -0.06
\end{array}\right] \tag{48}
\end{gather*}
$$

The activation functions are taken as

$$
\begin{align*}
& f(y)=\frac{1}{2}(|y+1|-|y-1|)  \tag{49}\\
& g(x)=\frac{1}{2}(|x+1|-|x-1|)
\end{align*}
$$

Let $d=1$ and $\tau=1$. Solving LMI (17), (28), and (29), it is found that the LMIs are feasible for all $r \leq 1.47$.

Table 1: The maximum delay bound $\tau, d$ and decay rate $\lambda$.

| $r$ | 1.45 | 1.4 | 1.35 | 1.3 | 1.25 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\lambda$ | 1.2083 | 1.1667 | 1.1250 | 1.0833 | 1.0417 |
| $d=2, \tau_{\max }$ | 10 | 14 | 18 | 22 | 27 |
| $d=3, \tau_{\max }$ | 9 | 13 | 17 | 21 | 19 |
| $d=5, \tau_{\max }$ | 7 | 11 | 15 | 16 | 25 |
| $d=8, \tau_{\max }$ | 4 | 8 | 12 | 13 | 22 |
| $d=11, \tau_{\max }$ | 1 | 5 | 9 | 10 | 19 |
| $d=14, \tau_{\max }$ | - | 2 | 3 | 4 | 16 |
| $d=17, \tau_{\max }$ | - | - | - | 4 | 13 |
| $d=20, \tau_{\max }$ | - | - | 10 | 12 | 10 |
| $\max \{d=\tau\}$ | 6 | 8 |  | 15 |  |



Figure 1: State response of the given system.

The calculated values of the delay upper bound $\tau$ and decay rate $\lambda$ for different values of $d$ and $r$ are given in Table 1 when $T_{a}=1$. From Table 1, we can see that the delay is related to the decay rate. For a given $d$, a smaller decay rate $\lambda$ allows a larger delay $\tau_{\text {max }}$. Moreover, for every $r$, the delay $\tau_{\max }$ decreases when the delay $d$ increases.

Letting $r=1.4, d=3$, and $\mu=1.2$, we obtain that $T_{a}^{*}=\operatorname{ceil}[0.5419]$. Based on (30), $T_{a}=1$ is satisfied. Then we can calculate that the decay rate $\lambda=r^{\left(-\ln \mu / T_{a} \ln r\right)+1}=1.1667$. Therefore, the discrete-time switched BAM neural networks with time delay are exponentially stable with the decay rate $\lambda=1.1667$ if the delay $\tau$ is not larger than 13 based on Table 1.

For $r=1.4, d=3$, and $\tau=2$, based on Definition 3, the discrete-time switched BAM neural networks ensure the following exponential decay estimation:

$$
\begin{align*}
& \|x(k)\|^{2}+\|y(k)\|^{2} \\
& \quad \leq 2.2866 \times 1.1667^{-\left(k-k_{0}\right)}\left(\|\phi\|_{L}^{2}+\|\psi\|_{L}^{2}\right), \quad \forall k \geq k_{0} . \tag{50}
\end{align*}
$$

Let $k_{0}=0$. Suppose the switching sequence is: $121212 \ldots$. It can be seen from switched sequence that $T_{a}=1$. Choose initial value as $\phi(s)=\left[\begin{array}{ll}0.5 & -0.6\end{array}\right]^{T}$ and $\psi(s)=\left[\begin{array}{ll}0.8 & -0.7\end{array}\right]^{T}$; then we obtain Figure 1, which depicts the trajectories of the system state.

## 5. Conclusions

In this paper, the exponential stability problem for the discrete-time switched BAM neural networks with time delay has been proposed. At first, the mathematical model of the discrete-time switched BAM neural networks with time delay has been established. And then by constructing a new switching-dependent Lyapunov-Krasovskii functional, some sufficient criteria have been developed to guarantee the discrete-time switched BAM neural networks to be exponentially stable based on the average dwell time approach and finite sum inequality technology. Finally, a numerical example has been provided to demonstrate the potential and effectiveness of the proposed algorithms.

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## Research Article

# Directly Solving Special Second Order Delay Differential Equations Using Runge-Kutta-Nyström Method 

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#### Abstract

Runge-Kutta-Nyström (RKN) method is adapted for solving the special second order delay differential equations (DDEs). The stability polynomial is obtained when this method is used for solving linear second order delay differential equation. A standard set of test problems is solved using the method together with a cubic interpolation for evaluating the delay terms. The same set of problems is reduced to a system of first order delay differential equations and then solved using the existing Runge-Kutta (RK) method. Numerical results show that the RKN method is more efficient in terms of accuracy and computational time when compared to RK method. The methods are applied to a well-known problem involving delay differential equations, that is, the Mathieu problem. The numerical comparison shows that both methods are in a good agreement.


## 1. Introduction

A special second order differential equations (ODEs) of the form

$$
\begin{equation*}
y^{\prime \prime}(t)=f(t, y(t)), \quad y\left(t_{0}\right)=\alpha, \quad y^{\prime}\left(t_{0}\right)=\beta, \quad t \geq t_{0} \tag{1}
\end{equation*}
$$

which is not explicitly dependent on the first derivative of the solution are frequently found in many physical problems such as electromagnetic waves, thin film flow, and gravity driven flow. Most researchers, scientists, and engineers used to solve (1) by converting the second order differential equations to a system of first order equations twice the dimension. However there are also studies on numerical methods which directly solve (1) using one-step methods or multistep methods. Such work can be seen in [1-6].

Most of the methods for solving special second order ODEs can be adapted for solving special second order delay differential equations (DDEs). In recent years there has been a growing interest in numerical solutions of DDEs. This is due to the appearance of such equations in various areas such as
neural network theory, epidemiology, and time lag control processes. DDEs also provide us with realistic model of many phenomena arising in real world problems. For example, DDEs can be used in modelling of population dynamics and spread of infectious diseases and two body problems of electrodynamics [7-12]. Most of the work concerning DDEs in the literature involved first order delay differential equations. Hence, in this research, we are going to focus on numerical methods for solving special second order DDEs. Special second order delay differential equations with multiple delays can be written in the following form:

$$
\begin{array}{r}
y^{\prime \prime}(t)=f\left(t, y(t), y\left(t-\tau_{1}\right), y\left(t-\tau_{2}\right), \ldots, y\left(t-\tau_{n}\right)\right) \\
t>t_{0} \tag{2}
\end{array}
$$

with initial conditions

$$
\begin{equation*}
y(t)=\varphi(t), \quad y^{\prime}(t)=\varphi^{\prime}(t), \quad t \leq t_{0} \tag{3}
\end{equation*}
$$

Runge-Kutta-Nyström method of order four will be adapted for solving second order DDEs (2). Stability polynomial of the method is also presented when applied to linear
second order DDE. Numerical results on a set of test problems are given and compared with the numerical results when the problems are reduced to a system of first order DDEs and solved using Runge-Kutta methods. We are also going to solve a well-known problem in engineering which involves second order delay differential equations, that is, the Mathieu problem.

## 2. Numerical Methods for Second Order ODEs

Runge-Kutta methods are designed for special second order differential equations:

$$
\begin{equation*}
y^{\prime \prime}=f(t, y(t)), \quad y\left(t_{0}\right)=\alpha, \quad y^{\prime}\left(t_{0}\right)=\beta, \quad t>t_{0} \tag{4}
\end{equation*}
$$

and are usually termed as Runge-Kutta-Nyström formula (RKN) since their introduction in 1925 by Nyström. An sstage RKN method for the numerical integration of the IVP in (4) is given by

$$
\begin{gather*}
y_{n+1}=y_{n}+h y_{n}^{\prime}+h^{2} \sum_{i=1}^{s} b_{i} k_{i}  \tag{5}\\
y_{n+1}^{\prime}=y_{n}^{\prime}+h \sum_{i=1}^{s} b_{i}^{\prime} k_{i}
\end{gather*}
$$

where

$$
\begin{equation*}
k_{i}=f\left(t_{n}+c_{i} h, y_{n}+c_{i} h y_{n}^{\prime}+h^{2} \sum_{j=1}^{s} a_{i j} k_{j}\right) \tag{6}
\end{equation*}
$$

for $i=1,2, \ldots, s$. The RKN parameters $c_{j}, a_{i j}, b_{j}$, and $b_{j}^{\prime}$ are assumed to be real for $i, j=1,2,3, \ldots, s$ and $s$ is the number of stages of the method. Let us introduce the $s$ dimensional vectors $c, b$, and $b^{\prime}$; moreover, the matrix $A$ is $s \times s$, where $c=\left[c_{1}, c_{2}, c_{3}, \ldots, c_{s}\right], b=\left[b_{1}, b_{2}, b_{3}, \ldots, b_{s}\right]$, $b^{\prime}=\left[b_{1}^{\prime}, b_{2}^{\prime}, b_{3}^{\prime}, \ldots, b_{s}^{\prime}\right]$, and $A=\left[a_{i j}\right]$, respectively. RKN method can be expressed in Butcher notation using the table of coefficients as follows:

$$
\begin{array}{c|c}
c & A  \tag{7}\\
\hline & \\
& b^{T} \\
& b^{\prime T}
\end{array}
$$

RKN methods can be divided into two classes explicit methods when $a_{i j}=0$ for $i \leq j$ and implicit methods otherwise.

## 3. Runge-Kutta-Nyström for DDEs

Consider second order DDE

$$
\begin{align*}
& y^{\prime \prime}=f\left(t, y(t), y\left(t-\tau_{1}\right), y\left(t-\tau_{2}\right),\right.  \tag{8}\\
& \\
& \left.\quad y\left(t-\tau_{3}\right), \ldots, y\left(t-\tau_{m}\right)\right), \quad t \geq t_{0}
\end{align*}
$$

with initial conditions

$$
\begin{equation*}
y(t)=\varphi(t), \quad y^{\prime}(t)=\varphi^{\prime}(t), \quad t \leq t_{0} \tag{9}
\end{equation*}
$$

$y_{n}$ is the numerical solution of (8) for all $t_{n}>t_{0}, n=1,2$, $3, \ldots$..

The RKN for second order ODE has been adapted to solve second order DDE and the formula can be written as follows:

$$
\begin{align*}
& y_{n+1}=y_{n}+h y_{n}^{\prime} \\
& +h^{2} \sum_{i=1}^{s} b_{i} k_{i}\left(t_{n}+c_{i} h, Y_{i}, y\left(t_{n}+c_{i} h-\tau_{1}\right),\right. \\
& \left.y\left(t_{n}+c_{i} h-\tau_{2}\right), \ldots, y\left(t_{n}+c_{i} h-\tau_{m}\right)\right), \\
& y_{n+1}^{\prime}=y_{n}^{\prime} \\
& +h \sum_{i=1}^{s} b_{i}^{\prime} k_{i}\left(t_{n}+c_{i} h, Y_{i}, y\left(t_{n}+c_{i} h-\tau_{1}\right),\right. \\
& \left.y\left(t_{n}+c_{i} h-\tau_{2}\right), \ldots, y\left(t_{n}+c_{i} h-\tau_{m}\right)\right), \tag{10}
\end{align*}
$$

where

$$
k_{i}=f\left(t_{n}+c_{i} h, y_{n}+c_{i} h y_{n}^{\prime}+h^{2} \sum_{j=1}^{s} a_{i j} k_{j}, y\left(x_{n}+c_{i} h-\tau_{1}\right),\right.
$$

$$
\begin{equation*}
\left.y\left(t_{n}+c_{i} h-\tau_{2}\right), \ldots, y\left(t_{n}+c_{i} h-\tau_{m}\right)\right) \tag{11}
\end{equation*}
$$

for $i=1,2, \ldots, s$. It can be written as follows:

$$
\begin{gather*}
y_{n+1}=y_{n}+h y_{n}^{\prime}+h^{2} \sum_{i=1}^{s} b_{i} f\left(t_{n}+c_{i} h, Y_{i}, z_{i}\right),  \tag{12}\\
y_{n+1}^{\prime}=y_{n}^{\prime}+h \sum_{i=1}^{s} b_{i}^{\prime} f\left(t_{n}+c_{i} h, Y_{i}, z_{i}\right),
\end{gather*}
$$

where

$$
\begin{align*}
& Y_{i}=y_{n}+c_{i} h y_{i}^{\prime}+h^{2} \sum_{j=1}^{s} a_{i j} k_{j}\left(t_{n}+c_{i} h, Y_{j}, z_{i}\right) \\
& z_{i}=\left(y\left(t_{n}+c_{i} h-\tau_{1}\right), y\left(t_{n}+c_{i} h-\tau_{2}\right), \ldots,\right.  \tag{13}\\
& \left.y\left(t_{n}+c_{i} h-\tau_{m}\right)\right) .
\end{align*}
$$

3.1. Time Delay Interpolation. Let the interval of the definition of delay differential equation (8) be $I=[a, b]$ and $t_{i}=a+i h$ for $i=0,1, \ldots, n$ and $h=(b-a) / n$, where $n$ is the number of points in interval $I$. The numerical method approximates the solutions at point $t_{i}$ for $i=1,2, \ldots, n$, to approximate the solution $y_{i+1}$ at point $t_{i+1}$ for $i=0,1, \ldots, n-$ 1. Hence,

$$
\begin{equation*}
y_{i+1}=y_{i+1}\left(t_{i}, y\left(t_{i}-\tau_{1}\right), y\left(t_{i}-\tau_{2}\right), \ldots, y\left(t_{i}-\tau_{m}\right)\right) . \tag{14}
\end{equation*}
$$

The method of interpolation has the following cases.

Case 1. If time delay $\tau_{1}, \tau_{2}, \ldots, \tau_{m}$ are constants and suppose $\tau_{i_{1}}<\tau_{i_{2}}<\cdots<\tau_{i_{m}}$ for $i_{1}, i_{2}, \ldots, i_{m}=1,2, \ldots, m$, therefore we can interpolate $y\left(t_{i}-\tau_{1}\right), y\left(t_{i}-\tau_{2}\right), \ldots, y\left(t_{i}-\tau_{m}\right)$ as follows:

$$
\begin{gather*}
y\left(t_{i}-\tau_{1}\right)=f_{\tau_{1}}\left(t_{i-d}, t_{i-d+1}, \ldots, t_{i}, y\left(t_{i-d}\right), y\left(t_{i-d+1}\right),\right. \\
\left.\ldots, y\left(t_{i}\right)\right), \\
y\left(t_{i}-\tau_{2}\right)=f_{\tau_{2}}\left(t_{i-d+1}, t_{i-d+2}, \ldots, t_{i}, y\left(t_{i-d+1}\right), y\left(t_{i-d+2}\right),\right. \\
\left.\ldots, y\left(t_{i}\right), y\left(t_{i}-\tau_{1}\right)\right), \\
y\left(t_{i}-\tau_{3}\right)=f_{\tau_{3}}\left(t_{i-d+2}, t_{i-d+3}, \ldots, t_{i}, y\left(t_{i-d+2}\right), y\left(t_{i-d+3}\right),\right. \\
\left.\ldots, y\left(t_{i}\right), y\left(t_{i}-\tau_{1}\right), y\left(t_{i}-\tau_{2}\right)\right), \\
\vdots \\
\vdots \\
y\left(t_{i}-\tau_{d}\right)=f_{\tau_{d}}\left(t_{i-1}, t_{i}, y\left(t_{i-1}\right), y\left(t_{i}\right), y\left(t_{i}-\tau_{1}\right),\right. \\
\left.y\left(t_{i}-\tau_{2}\right), \ldots, y\left(t_{i}-\tau_{d-1}\right)\right), \\
y\left(t_{i}-\tau_{j}\right)=f_{\tau_{j}}\left(t_{i}, y\left(t_{i}\right), y\left(t_{i}-\tau_{1}\right), y\left(t_{i}-\tau_{2}\right),\right. \\
\left.\ldots, y\left(t_{i}-\tau_{d}\right)\right),  \tag{15}\\
m \geq j \geq d .
\end{gather*}
$$

The functions $f_{\tau_{j}}$ for $j=1,2, \ldots, m$ depend on the interpolation which is used in the numerical method which has degree $d$.

Case 2. If $\tau_{1}, \tau_{2}, \ldots, \tau_{m}$ are the variables of time delays, that is, $\tau_{j}=\tau_{j}(t)$ for $j=1,2, \ldots, m$, however, we can consider

$$
\begin{gather*}
y\left(t_{i}-\tau_{j}\right)=f_{\tau_{j}}\left(t_{i-d}, t_{i-d+1}, \ldots, t_{i}, y\left(t_{i-d}\right), y\left(t_{i-d+1}\right),\right. \\
\left.\ldots, y\left(t_{i}\right)\right) \tag{16}
\end{gather*}
$$

for $j=1,2, \ldots, m$, knowing that $y_{k}=y\left(t_{k}\right)$ for $k=i-d$, $i-d+1, \ldots, i$. The function $f_{\tau_{j}}$ depends on the type and degree of the interpolation. In this paper, we consider the cubic interpolation to approximate DDEs with three time delays.

## 4. Stability of the Method

Stability aspect of numerical methods for delay differential equations has been introduced in [13-17]. To study the stability of numerical method (10), consider the linear test equation

$$
\begin{equation*}
y^{\prime \prime}=\lambda^{2} y(t)+\mu^{2} y(t-\tau) \tag{17}
\end{equation*}
$$

When the method is applied to the linear test equation (17), we have

$$
\begin{aligned}
y_{n+1} & =y_{n}+h y_{n}^{\prime}+h^{2} \sum_{i=1}^{s} b_{i} f\left(t_{n}+c_{i} h, Y_{i}, y\left(t_{n}+c_{i} h-\tau\right)\right) \\
& =y_{n}+h y_{n}^{\prime}+h^{2} \sum_{i=1}^{s} b_{i}\left(\lambda^{2} Y_{i}+\mu^{2} y\left(t_{n}+c_{i} h-\tau\right)\right)
\end{aligned}
$$

$$
\begin{align*}
y_{n+1}^{\prime} & =y_{n}^{\prime}+h \sum_{i=1}^{s} b_{i}^{\prime} k_{i}\left(t_{n}+c_{i} h, Y_{i}, y\left(t_{n}+c_{i} h-\tau\right)\right) \\
& =y_{n}^{\prime}+h \sum_{i=1}^{s} b_{i}^{\prime} k_{i}\left(\lambda^{2} Y_{i}+\mu^{2} y\left(t_{n}+c_{i} h-\tau\right)\right) \tag{18}
\end{align*}
$$

where

$$
\begin{gather*}
Y_{i}=y_{n}+c_{i} h y_{n}^{\prime}+h^{2} \sum_{j=1}^{s} a_{i j} f\left(t_{n}+c_{i} h, Y_{j}, y\left(t_{n}+c_{i} h-\tau\right)\right) \\
=y_{n}+c_{i} h y_{n}^{\prime}+h^{2} \sum_{j=1}^{s} a_{i j}\left(\lambda^{2} Y_{j}+\mu^{2} y\left(t_{n}+c_{i} h-\tau\right)\right) \\
Z_{n+1}=T Z_{n}+\lambda^{2} h^{2} B Y+\mu^{2} h^{2} B Z_{n}(\tau) \tag{19}
\end{gather*}
$$

such that

$$
\begin{gather*}
Z_{n}=\binom{y_{n}}{h y_{n}^{\prime}}, \quad T=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right), \quad B=\left(\begin{array}{cccc}
b_{1} & b_{2} & \cdots & b_{s} \\
b_{1}^{\prime} & b_{2}^{\prime} & \cdots & b_{s}^{\prime}
\end{array}\right), \\
Z_{n}(\tau)=\left(\begin{array}{c}
y\left(t_{n}+c_{1} h-\tau\right) \\
y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
y\left(t_{n}+c_{s} h-\tau\right)
\end{array}\right) . \tag{20}
\end{gather*}
$$

So

$$
\begin{aligned}
\left(\begin{array}{c}
Y_{1} \\
Y_{2} \\
\vdots \\
Y_{s}
\end{array}\right)= & \left(\begin{array}{c}
y_{n} \\
y_{n} \\
\vdots \\
y_{n}
\end{array}\right)+\left(\begin{array}{c}
c_{1} \\
c_{2} \\
\vdots \\
c_{s}
\end{array}\right) h y_{n}^{\prime} \\
& +h^{2}\left(\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 s} \\
a_{21} & a_{22} & \cdots & a_{2 s} \\
\vdots & & \ddots & \vdots \\
a_{s 1} & a_{s 2} & \cdots & a_{s s}
\end{array}\right) \\
& \times\left(\begin{array}{c}
\lambda^{2} Y_{1}+\mu^{2} y\left(t_{n}+c_{1} h-\tau\right) \\
\lambda^{2} Y_{2}+\mu^{2} y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
\lambda^{2} Y_{s}+\mu^{2} y\left(t_{n}+c_{s} h-\tau\right)
\end{array}\right) \\
& \left(\begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array}\right) y_{n}+\left(\begin{array}{c}
c_{1} \\
c_{2} \\
\vdots \\
c_{s}
\end{array}\right) h y_{n}^{\prime} \\
& +h^{2} \lambda^{2} A\left(\begin{array}{c}
Y_{1} \\
Y_{2} \\
\vdots \\
Y_{s}
\end{array}\right)
\end{aligned}
$$

$$
+\mu^{2} h^{2} A\left(\begin{array}{c}
\mu^{2} y\left(t_{n}+c_{1} h-\tau\right)  \tag{21}\\
\mu^{2} y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
\mu^{2} y\left(x_{n}+c_{s} h-\tau\right)
\end{array}\right)
$$

This implies that

$$
\begin{align*}
\left(I-H_{\lambda} A\right)\left(\begin{array}{c}
Y_{1} \\
Y_{2} \\
\vdots \\
Y_{s}
\end{array}\right)= & \left(\begin{array}{cc}
1 & c_{1} \\
1 & c_{2} \\
\vdots & \vdots \\
1 & c_{s}
\end{array}\right) Z_{n} \\
& +H_{\mu} A\left(\begin{array}{c}
\mu^{2} y\left(t_{n}+c_{1} h-\tau\right) \\
\mu^{2} y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
\mu^{2} y\left(x_{n}+c_{s} h-\tau\right)
\end{array}\right) \tag{22}
\end{align*}
$$

where $H_{\lambda}=(\lambda h)^{2}, H_{\mu}=(\mu h)^{2}$.
So

$$
\left.\begin{array}{rl}
\left(\begin{array}{c}
Y_{1} \\
Y_{2} \\
\vdots \\
Y_{s}
\end{array}\right)= & \left(I-H_{\lambda} A\right)^{-1} \\
& \times\left(\begin{array}{c}
\mu^{2} y\left(t_{n}+c_{1} h-\tau\right) \\
\mu^{2} y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
\mu^{2} y\left(t_{n}+c_{s} h-\tau\right)
\end{array}\right) \tag{23}
\end{array}\right),
$$

where

$$
\begin{gather*}
A=\left(\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1 s} \\
a_{21} & a_{22} & \cdots & a_{2 s} \\
\vdots & & \ddots & \vdots \\
a_{s 1} & a_{s 2} & \cdots & a_{s s}
\end{array}\right), \\
C=\left(\begin{array}{cc}
1 & c_{1} \\
1 & c_{2} \\
\vdots & \vdots \\
1 & c_{s}
\end{array}\right) . \tag{24}
\end{gather*}
$$

Hence, the stability polynomial of the method is

$$
\begin{aligned}
Z_{n+1}= & T Z_{n}+H_{\lambda} B\left(I-H_{\lambda} A\right)^{-1} \\
& \times\left(C Z_{n}+H_{\mu} A Z_{n}(\tau)\right)+H_{\mu} B Z_{n}(\tau) \\
& Z_{n+1}=T_{1} Z_{n}+T_{2} Z_{n}(\tau)
\end{aligned}
$$

$$
\begin{gather*}
Z_{n}(\tau)=\left(\begin{array}{c}
y\left(t_{n}+c_{1} h-\tau\right) \\
y\left(t_{n}+c_{2} h-\tau\right) \\
\vdots \\
y\left(x_{n}+c_{s} h-\tau\right)
\end{array}\right), \\
T_{1}=T+H_{\lambda} B\left(I-H_{\lambda} A\right)^{-1} C, \\
T_{2}=H_{\mu} B\left[H_{\lambda}\left(I-H_{\lambda} A\right)^{-1} A+I\right] . \tag{25}
\end{gather*}
$$

## 5. Numerical Results

In this section, some of the problems involving second order DDEs are solved using RKN methods. Then the same set of problems is reduced to a first order DDEs system and solved using RK methods of the same order. Then numerical results are given.

The following notations are used in Figures 1, 2, 3, 4, 5, and 6:
(i) $h$ : stepsize used.
(ii) RKN4-N: the existing Rung-Kutta-Nyrström method of order four derived by Senu et al. [18].
(iii) RKN4-D: the existing Rung-Kutta-Nyrström method of order four as in [19].
(iv) RK4: existing Runge-Kutta method order four as given in [20].
(v) DOPRI: existing Runge-Kutta method order fifth as in Dormand [19].
(vi) Total time: the total time in second to solve the problems.
(vii) MAX ERROR: $\operatorname{Max}_{n}\left|y\left(x_{n}\right)-y_{n}\right|$ Absolute value of the true solution minus the computed solution.

Problem 1 (nonlinear). Consider the following:

$$
\begin{gather*}
y^{\prime \prime}(t)=e^{-2 \tau} \frac{y^{2}(t-\tau)}{y(t)}, \quad t \geq t_{0} \\
y(t)=e^{-t}, \quad y^{\prime}(t)=-e^{-t}, \quad t \leq t_{0}  \tag{26}\\
\tau=\frac{h}{10} .
\end{gather*}
$$

Exact solution: $y(t)=e^{-t}$.
Problem 2 (nonlinear). Consider the following:

$$
\begin{gather*}
y^{\prime \prime}(t)=y^{2}(t-\tau)-\frac{1}{4 \sqrt{(1+t)^{3}}}-(1+t)+\tau, \quad t \geq t_{0} \\
y(t)=\sqrt{1+t}, \quad y^{\prime}(t)=\frac{1}{2 \sqrt{1+t}}, \quad t \leq t_{0} \\
\tau=\frac{h}{10} . \tag{27}
\end{gather*}
$$

Exact solution: $y(t)=\sqrt{1+t}$.


Figure 1: The efficiency curves for all methods for Problem 1 with $t_{\text {end }}=1$ and $h=1 / 4^{i}, i=1, \ldots, 5$.


Figure 2: The efficiency curves for all methods for Problem 2 with $t_{\text {end }}=1$ and $h=1 / 4^{i}, i=1, \ldots, 5$.

Problem 3 (linear example). Consider the following:

$$
\begin{gather*}
y^{\prime \prime}(t)=y(t-\tau)-\frac{1}{(1+t)^{2}}-\ln (1+t-\tau), \quad t \geq t_{0} \\
y(t)=\ln (1+t), \quad y^{\prime}(t)=\frac{1}{1+t}, \quad t \leq t_{0} \\
\tau=\frac{h}{10} \tag{28}
\end{gather*}
$$

Exact solution: $y(t)=\ln (1+t)$.


Figure 3: The efficiency curves for all methods for Problem 3 with $t_{\text {end }}=1$ and $h=1 / 4^{i}, i=1, \ldots, 5$.


Figure 4: The efficiency curves for all methods for Problem 4 with $t_{\text {end }}=1$ and $h=1 / 4^{i}, i=1, \ldots, 5$.

Problem 4 (nonlinear). Consider the following:

$$
\begin{gather*}
y^{\prime \prime}(t)=\frac{1}{3}\left(e^{-\tau_{1}} y\left(t-\tau_{1}\right)+e^{-\tau_{2}} y\left(t-\tau_{2}\right)\right. \\
\left.+e^{-\tau_{3}} y\left(t-\tau_{3}\right)\right), \quad t \geq t_{0}, \\
y(t)=e^{-t}, \quad y^{\prime}(t)=-e^{-t}, \quad t \leq t_{0}, \\
\tau_{1}=\frac{h}{10}, \quad \tau_{2}=\frac{h}{20}, \quad \tau_{3}=\frac{h}{30} . \tag{29}
\end{gather*}
$$

Exact solution: $y(t)=e^{-t}$.
5.1. An Application to Mathieu Equation. In this section we will apply the RKN and RK methods to solve a well-known


Figure 5: The numerical solution for all methods for Linear Problem with $t_{\text {end }}=10$ and $h=1 / 10$.


Figure 6: The numerical solution for all methods for Nonlinear Problem with $t_{\text {end }}=10$ and $h=1 / 10$.
equation in engineering, the Matheiu's equation, which is defined as follows:

$$
\begin{equation*}
y^{\prime \prime}(t)+(\delta+a \cos t) y(t)+c y^{3}(t)=b y(t-T) \tag{30}
\end{equation*}
$$

which is a nonlinear delay differential equation.
Where $\delta, a, b, c$, and $T$ are parameters.
$\delta$ is the frequency squared of the simple harmonic oscillator, and $a$ is the amplitude of the parametric resonance, and $b$ is the amplitude of delay while $c$ is the amplitude of the cubic nonlinearity and $T$ is the time delay.

Equation (30) is a model for high speed milling, a kind of parametrically interrupted cutting as opposed to the selfinterrupted cutting arising in an unstable turning process. More information on the problem can be found in [21]. Various special cases of (30) have been studied, depending on which parameters are zero.

When $\delta=a=b=1$ and $c=0$ we obtained the following linear Mathieu equation:

$$
\begin{gather*}
y^{\prime \prime}(t)=(1+\cos t) y(t)=y(t-T), \quad t \in[0,10] \\
y(t)=\sin (t)  \tag{31}\\
y^{\prime}(t)=\cos (t), \quad t \leq 0
\end{gather*}
$$

where $T=\tau=h / 10$ is the delay term, the exact solution does not exist.

When $\delta=a=b=c=1$, we obtained the following nonlinear Mathieu equation:

$$
\begin{gather*}
y^{\prime \prime}(t)=(1+\cos t) y(t)+y^{3}(t) \\
=y(t-T), \quad t \in[0,10], \\
y(t)=\sin (t),  \tag{32}\\
y^{\prime}(t)=\cos (t), \quad t \leq 0,
\end{gather*}
$$

where $T=\tau=h / 10$ is the delay term, the exact solution does not exist.

Both the linear and nonlinear Mathieu equations are solved using RKN and RK methods and the results are plotted in Figures 5 and 6.

## 6. Discussion and Conclusion

In this paper we adapted the Runge-Kutta-Nyström method for solving special second order delay differential equations in which cubic interpolation is used to evaluate the delay term. We also presented the stability of the method when applied to linear second order DDEs. We solved a set of DDEs using RKN4-N by Senu et al. [6] and RKN4-D by Dormand [19]. For comparison purposes the same set of problems a reduced to a system of first order DDEs and solved using classical fourth order Runge-Kutta method and fifth order RungeKutta method by Dormand [19]. The log of maximum error is plotted against time. From the numerical results, we observed that both RKN methods are just as efficient. However they are more efficient compared to the fourth order classical RungeKutta method followed by the fifth order Runge-Kutta method by Dormand [19]. This is because RKN method directly solved the equations whereas in RK method the equations are reduced to a system of first order DDEs. The fifth order RK method has more stages compared to the fourth order RK method which required more function evaluations at each step. Both RKN and RK methods are also used to solve the linear and nonlinear Mathieu equations. Since they do not have the exact solution, we just plot the numerical results in which both methods show a good agreement.

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## Research Article

# $p$-Moment Stability of Stochastic Differential Delay Systems with Impulsive Jump and Markovian Switching 

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#### Abstract

This paper investigates $p$-moment stability of the stochastic differential delay systems with impulsive jump and Markovian switching. Some stability criteria are obtained based on Lyapunov functional method and stochastic theory. It is shown that, even if all the subsystems governing the continuous dynamics without impulse are not stable, as impulsive and switching signal satisfies a dwell-time upper bound condition, impulses can stabilize the systems in the $p$-moment stability sense. The opposite situation is also developed for which all the subsystems governing the continuous dynamics are $p$-moment stable. The results can be easily applied to stochastic systems with arbitrarily large delays. The efficiency of the proposed results is illustrated by two numerical examples.


## 1. Introduction

Stochastic differential systems which include stochastic delay differential systems have attracted much attention, owing to stochastic modeling having played an important role in many ways such as science and engineering and forecast of the growth of population [1,2]. Stability analysis of different stochastic systems has been a subject of intense activities in the literature [3-9]. Switched systems are an important class of hybrid dynamical systems which are composed of a family of continuous time or discrete-time dynamical systems and a rule that orchestrates the switching among them [10]. A particular class of switched systems is named as markovian switched systems, whose system mode is governed by a Markov process. During the past few decades, many issues on Markovian switched systems such as stability and stabilization [11-14], $H_{\infty}$ control and filtering [15], and adaptive control problem $[16,17]$ have been well investigated.

Beside stochastic effects and markovian switching, impulsive effect likewise exists in many evolution processed in which systems states change abruptly at certain moments of time, involving such fields as biology, engineering, and information science [18]. Therefore, the stability investigation of stochastic differential systems with impulsive jump is interesting to many investigators. The $p$-moment stability
of stochastic differential systems is studied in [19] for the systems with impulsive jump, nonswitched, and no time delay. Thus, the results in [19] cannot be easily applied to the class of impulsive systems with markovian switching and time-varying delay. Liu and Peng [20] discussed the $p$ moment stability of stochastic differential delay systems with impulsive jump and markovian switching. It is shown that when the delayed continuous dynamics are $p$-moment stable, the stability properties is not destroyed by impulse at discrete instants irrespective of the length of delay. It is also noticed that the conclusions received in [20] are restricted to the case that are all the subsystems that govern the continuous dynamics are stable in $p$-moment sense. There is no attention has been paid to the class of hybrid systems in which all the subsystems that govern the continuous dynamics is not stable. Thus, how to establish a sufficient condition of $p$-moment stability for the class of stochastic delays hybrid systems is the key problem to be solved in future research.

Based on the above analysis, in this work, the $p$-moment stability of stochastic differential delay systems with impulsive jump and markovian switching is investigated. Motivated by the work of [20], we first relax the global Lipschitz condition of impulsive control law $\Delta x\left(t_{k}\right)$. Then, we will propose some conditions of the $p$-moment stability for two classes of stochastic hybrid delay systems, that is, all the
subsystems that govern the continuous dynamics are stable and not. Compared with [20], the criteria for the stable case in this paper are more general. Further, the results are applied to linear stochastic delay systems with arbitrarily large delays.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, we let $\left(\Omega, F,\left\{F_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\bar{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions, that is, it is right continuous and $F_{0}$ contains all $P$-null sets. $\omega(t)$ is an $m$ dimensional Brownian motion defined on ( $\left.\Omega, F,\left\{F_{t}\right\}_{t \geq 0}, P\right)$. Let $Z^{+}$define the set of nonnegative real numbers and $R^{n}$ the $n$-dimensional real Euclidean space. $|\cdot|$ denotes the Euclidean norm for vectors or the spectral norm for matrices. For $r>0$, let $\mathrm{PC}\left([-r, 0], R^{n}\right)$ denote the class of functions from $[-r, 0]$ to $R^{n}$ satisfying the following: (i) it has at most a finite number of jump discontinuities on $(-r, 0$ ]; (ii) it is continuous from the right at all points in $[-r, 0)$. For simplicity, PC is used for $\operatorname{PC}\left([-r, 0], R^{n}\right)$ for the rest of this paper. For function $\phi$ : $[-r, 0] \rightarrow R^{n}$, a norm is defined as $\|\phi\|_{r}=\sup _{-r \leq \theta \leq 0}|\phi(\theta)|$. Given $x \in \operatorname{PC}\left([-r, \infty], R^{n}\right)$ and for each $t \in R^{+}$, define $x_{t}, x_{t^{-}} \in \mathrm{PC}$ by $x_{t}(s)=x(t+s)$ for $-r \leq s \leq 0$ and $x_{t^{-}}(s)=$ $x(t+s)$ for $-r \leq s<0$, respectively. Denote by $\mathrm{PC}_{F_{t}}^{P}$ the family of all $F_{t}$-measurable PC-valued random variables $\phi=\{\phi(\theta)$ : $-r \leq \theta \leq 0\}$, satisfying $\sup _{-r \leq \theta \leq 0} E\|\phi(\theta)\|^{p}<\infty$, where $E$ stands for the mathematical expectation. Let $\mathrm{PC}_{F_{t}}^{b}(\delta)=\{\phi$ : $\phi \in \mathrm{PC}_{F_{t}}^{p}\left([-r, 0], R^{n}\right)$ and $\left.\sup _{-r \leq \theta \leq 0} E\|\phi(\theta)\|^{p}<\delta\right\}$.

The Markov process $\{r(t), t \geq 0\}$ represents the switching between the different modes taking values in a finite state space $S=\{1,2, \ldots, N\}$ with generator $\pi=\left(\pi_{i j}\right)_{N \times N}$ given by

$$
\operatorname{Pr}\{r(t+\Delta)=j \mid r(t)=i\}= \begin{cases}\pi_{i j} \Delta+o(\Delta), & \text { if } i \neq j  \tag{1}\\ 1+\pi_{i i} \Delta+o(\Delta), & \text { if } i=j\end{cases}
$$

where $\pi_{i j}$ is the transition rate from mode $i$ to $j$ and satisfies the following relations:

$$
\begin{equation*}
\pi_{i j} \geq 0, \quad \pi_{i i}=-\sum_{j \neq i} \pi_{i j} \tag{2}
\end{equation*}
$$

and $o(\Delta)$ is such that $\lim _{\Delta \rightarrow 0} o(\Delta) / \Delta=0$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $\omega(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $R^{+}=[0,+\infty)$.

Consider the following stochastic nonlinear delay system with impulsive jump and markovian switching

$$
\begin{aligned}
& d x(t)= f\left(x(t), x_{t}, t, r(t)\right) d t \\
&+g\left(x(t), x_{t}, t, r(t)\right) d \omega(t), \quad t \neq t_{k}, \\
& \Delta x(t)=I_{k}\left(x\left(t^{-}\right), t\right), \quad t=t_{k}, \\
& x\left(t_{0}+\theta\right)=\phi(\theta), \quad \theta \in[-r, 0],
\end{aligned}
$$

where $\phi(\theta) \in \mathrm{PC}$ is the initial data, $x(t) \in R^{n}$ is the systems state, $x\left(t^{+}\right)$and $x\left(t^{-}\right)$denote the limit from the right and the left at point $t$, respectively, and $\left\{t_{k}: k \in Z^{+}\right\} \subset R^{+}$a strictly increasing sequence. We assume that for each $i \in S$, given functional $f: R^{n} \times \mathrm{PC} \times S \rightarrow R^{n}, g: R^{n} \times \mathrm{PC} \times S \rightarrow R^{n \times m}$ satisfying $f(0,0, t, i) \equiv g(0,0, t, i) \equiv 0 . I_{k}: R^{n} \times R \rightarrow R^{n}$ with $I_{k}(0, t) \equiv 0$ is the change of state variable at instant $t_{k}$. In this paper, we always assume that there exists a unique stochastic process satisfying systems (3), and all solutions of systems (3) are continuous on the right-hand side and on the left-hand side. Moreover, by $f(0,0, t, i) \equiv g(0,0, t, i) \equiv 0$ and $I_{k}(0, t) \equiv$ 0 , it is easily obtained that system (3) admits a trivial solution.

Definition 1 (see [20]). The trivial solution of system (3) is
(1) $p$-moment stable if for any initial data $x_{t_{0}}=\phi$ and any $\varepsilon>0$, there exists a $\delta>0$ such that

$$
\begin{equation*}
E\|x(t)\|^{p}<\varepsilon, \quad t \geq t_{0} \tag{4}
\end{equation*}
$$

whenever $\|\phi\|_{r}^{p}<\delta$;
(2) uniformly $p$-moment stable if the $\delta$ in (1) is independent of $t_{0}$.

Definition 2. Let $C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$denote the family of all nonnegative functions $V(x, t, i)$ that are continuously twice differentiable in $x$ and once differentiable with respect to $t$. For each $V(x, t, i) \in C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$, define an operator $L V$ from $R^{n} \times\left[t_{0}-r, \infty\right] \times S$ to $R^{+}$as follows:

$$
\begin{align*}
\operatorname{LV}(x, t, i)= & V_{t}(x, t, i)+V_{x}(x, t, i) f(x, y, t, i) \\
& +\frac{1}{2} \operatorname{trace}\left[g^{T}(x, y, t, i) V_{x x}(x, t, i) g(x, y, t, i)\right] \\
& +\sum_{j=1}^{N} q_{i j} V(x, t, j), \tag{5}
\end{align*}
$$

where

$$
\begin{gather*}
V_{t}(x, t, i)=\frac{\partial V(x, t, i)}{\partial t} \\
V_{x}(x, t, i)=\left(\frac{\partial V(x, t, i)}{\partial x_{1}}, \ldots, \frac{\partial V(x, t, i)}{\partial x_{n}}\right),  \tag{6}\\
V_{x x}(x, t, i)=\left(\frac{\partial^{2} V(x, t, i)}{\partial x_{i} \partial x_{j}}\right)_{n \times n} .
\end{gather*}
$$

Before giving the results, we need a lemma.
Lemma 3 (see [20]). If $V \in C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$, then for any stopping times $0 \leq t_{1} \leq t_{2}<+\infty$,

$$
\begin{align*}
E\left(V\left(x\left(t_{2}\right), t_{2}, r\left(t_{2}\right)\right)\right)= & E\left(V\left(x\left(t_{1}\right), t_{1}, r\left(t_{1}\right)\right)\right) \\
& +E\left(\int_{t_{1}}^{t_{2}} L V(x(s), s, r(s)) d s\right), \tag{7}
\end{align*}
$$

as long as the integration involved exist and finite.

## 3. Main Results

In this section, Lyapunov-based sufficient conditions for $p$ moment stability of system (3) are developed. The first result is concerned with $p$-moment stability of system (3), in the case when all the subsystems governing the continuous dynamics of (3) are stable. Intuitively, the conditions in the following theorem consist of three aspects: (i) the LyapunovKrasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some negative estimates of the upper right-hand derivatives of the functionals with respect to each stable mode of system (3); (iii) the jumps induced by the impulses and the estimates on the decay rate of continuous dynamics satisfy certain conditions.

Theorem 4. Assume that there exist function $V(x(t), t, i) \in$ $C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$and some positive constants $p, a$, $b, e_{i}$, the nonnegative and continuous functions $c_{i}(t)$ and $d_{i}(t)$, such that
(i) $a|x|^{p} \leq V(x(t), t, i) \leq b\left|x_{t}\right|_{r}^{p}$;
(ii) $E L V(x(t), t, i) \leq-c_{i}(t) E V(x(t), t, i)+d_{i}(t) E V\left(x_{t}, t, i\right)$, $t \in\left[t_{k-1}, t_{k}\right), k=1,2, \ldots ;$
(iii) $E V\left(\phi(0)+I_{k}\left(\phi, t_{k}, \widetilde{i}\right) \leq e_{i} V\left(\phi(0), t_{k}^{-}, i\right)\right.$;
(iv) $0<c^{*}<1$, where $c^{*}=\sup \left\{c_{i}^{*} \mid c_{i}^{*}=b e_{i} / a, i=1\right.$, $2, \ldots\} ;$
(v) $c_{i}(t)>d_{i}(t) / c^{*}, t \in\left[t_{k}, t_{k+1}\right)$;
then the trivial solution of system (3) is uniformly p-moment stable.

Proof. For any $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that $\delta<\left(a c^{*} / b\right) \varepsilon$ independent of $t_{0}$. For any $t_{0} \geq 0$ and $x_{t_{0}}=\phi \in$ $\mathrm{PC}_{F_{t}}^{b}(\delta)$, let $x(t)=x\left(t, t_{0}, \phi\right)$ be the solution of (3).

In the following, we will show that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{k-1}, t_{k}\right), k \in Z^{+} . \tag{8}
\end{equation*}
$$

It will be firstly proved that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}, t_{1}\right) \tag{9}
\end{equation*}
$$

For any initial data $x_{t_{0}} \in \operatorname{PC}_{F_{t}}^{b}(\delta)$, it follows from assumption (i) that

$$
\begin{align*}
E V(x(t), t, r(t)) & \leq b E\left\|x_{t_{0}}\right\|_{r}^{p} \\
& \leq b \delta<\frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}-r, t_{0}\right] . \tag{10}
\end{align*}
$$

Suppose that (9) is not true, then there exists some $t \in\left(t_{0}, t_{1}\right)$ such that

$$
\begin{equation*}
E V(x(t), t, r(t))>\frac{b}{c^{*}} \delta>b \delta \geq E V\left(x\left(t_{0}\right), t_{0}, r\left(t_{0}\right)\right) \tag{11}
\end{equation*}
$$

Set $t^{*}=\inf \left\{t \in\left[t_{0}, t_{1}\right): E V(x(t), t, r(t))>\left(b / c^{*}\right) \delta\right\}$. Observe that $E V(x(t), t, r(t))$ is continuous on $t \in\left[t_{0}, t_{1}\right)$, then $t^{*} \in$ ( $t_{0}, t_{1}$ ) and

$$
\begin{gather*}
E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right)=\frac{b}{c^{*}} \delta,  \tag{12}\\
E V(x(t), t, r(t))<\frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}-r, t^{*}\right) . \tag{13}
\end{gather*}
$$

In view of (10), define $t^{* *}=\sup \left\{t \in\left[t_{0}-r, t^{*}\right]:\right.$ $E V(x(t), t, r(t)) \leq b \delta\}$. Then $t^{* *} \in\left(t_{0}, t^{*}\right)$ and

$$
\begin{gather*}
E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right)=b \delta,  \tag{14}\\
E V(x(t), t, r(t))>b \delta, \quad t \in\left(t^{* *}, t^{*}\right] . \tag{15}
\end{gather*}
$$

Consequently, in view of (10)-(15), for all $t \in\left[t^{*}, t^{* *}\right]$ and $\theta \in[-r, 0]$, one has

$$
\begin{align*}
E V & (x(t+\theta), t+\theta, r(t+\theta)) \\
& \leq \frac{b}{c^{*}} \delta \leq \frac{1}{c^{*}} E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right)  \tag{16}\\
& \leq \frac{1}{c^{*}} E V(x(t), t, r(t))
\end{align*}
$$

Now, combining (16) and condition (ii), we have

$$
\begin{align*}
E L V & (x(t), t, r(t)) \\
& \leq-c_{i}(t) E V(x(t), t, r(t))+d_{i}(t) E V\left(x_{t}, t, r(t)\right)  \tag{17}\\
& \leq\left(-c_{i}(t)+\frac{d_{i}(t)}{c^{*}}\right) E V(x(t), t, r(t)) .
\end{align*}
$$

Applying Lemma 3, integrating (17) on $\left[t^{* *}, t^{*}\right]$, one obtains that

$$
\begin{align*}
& E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right) \\
& \leq \leq E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right)  \tag{18}\\
& \quad+\int_{t^{* *}}^{t^{*}}\left(-c_{i}(t)+\frac{d_{i}(t)}{c^{*}}\right) E V(x(s), s, r(s)) d s .
\end{align*}
$$

By condition (v) and (14) and the Gronwall inequality, one has

$$
\begin{align*}
& E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right) \\
& \quad \leq E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right) e^{\left(-c_{i}(t)+\left(d_{i}(t) / c^{*}\right)\right)\left(t^{*}-t^{* *}\right)}  \tag{19}\\
& \quad \leq E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right)=b \delta<\frac{b}{c^{*}} \delta .
\end{align*}
$$

Since (19) contradicts with (12), inequality (9) holds and (8) is true for $k=1$.

Now, assume that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{k-1}, t_{k}\right) \tag{20}
\end{equation*}
$$

for all $k \leq m$, where $k, m \in Z^{+}$. We proceed to show that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{m}, t_{m+1}\right) . \tag{21}
\end{equation*}
$$

Suppose that (21) is not true, set $\bar{t}=\inf \left\{t \in\left[t_{m}, t_{m+1}\right)\right.$ : $\left.E V(x(t), t, r(t))>\left(b / c^{*}\right) \delta\right\}$. From condition (i), (iv), and (21), we know that

$$
\begin{align*}
E V & \left(x\left(t_{m}\right), t_{m}, r\left(t_{m}\right)\right) \\
& \leq e_{i_{m}} E V\left(x\left(t_{m}^{-}\right), t_{m}^{-}, r\left(t_{m}^{-}\right)\right) \\
& \leq b e_{i_{m}}\left\|x_{t_{m}^{-}}\right\|_{r}^{p} \\
& \leq \frac{b e_{i_{m}}}{a} \sup _{-r \leq \theta \leq 0} E V\left(x\left(t_{m}^{-}+\theta\right), t_{m}^{-}+\theta, r\left(t_{m}^{-}+\theta\right)\right)  \tag{22}\\
& \leq \frac{b e_{i_{m}}}{a} \frac{b}{c^{*}} \delta \leq b \delta<\frac{b}{c^{*}} \delta .
\end{align*}
$$

Owing to $E V(x(t), t, r(t))$ is continuous on $t \in\left[t_{m}, t_{m+1}\right)$, then $\bar{t} \in\left(t_{m}, t_{m+1}\right)$ and

$$
\begin{gather*}
E V(x(\bar{t}), \bar{t}, r(\bar{t}))=\frac{b}{c^{*}} \delta,  \tag{23}\\
E V(x(t), t, r(t))<\frac{b}{c^{*}} \delta, \quad t \in\left[t_{m}, \bar{t}\right) . \tag{24}
\end{gather*}
$$

Define $\underline{t}=\sup \left\{t \in\left[t_{0}-r, \bar{t}\right]: E V(x(t), t, r(t)) \leq b \delta\right\}$, then $\underline{t} \in\left(t_{m}, \overline{\bar{t}}\right)$ and

$$
\begin{gather*}
E V(x(\underline{t}), \underline{t}, r(\underline{t}))=b \delta,  \tag{25}\\
E V(x(t), t, r(t))>b \delta, \quad t \in(\underline{t}, \bar{t}] . \tag{26}
\end{gather*}
$$

Fix any $t \in[\underline{t}, \bar{t}]$, when $t+\theta \geq t_{m}$ for all $\theta \in[-r, 0]$, then from (22)-(26), one has

$$
\begin{align*}
E V & (x(t+\theta), t+\theta, r(t+\theta)) \\
& \leq \frac{b}{c^{*}} \delta \leq \frac{1}{c^{*}} E V(x(\bar{t}), \bar{t}, r(\bar{t}))  \tag{27}\\
& \leq \frac{1}{c^{*}} E V(x(t), t, r(t))
\end{align*}
$$

When $t+\theta<t_{m}$ for some $\theta \in[-r, 0]$, without loss of generality, we assume that $t+\theta \in\left[t_{l-1}, t_{l}\right)$ for some $l \in Z^{+}$, $l<m$, then from (20) and (25), we obtain that

$$
\begin{align*}
E V( & x(t+\theta), t+\theta, r(t+\theta)) \\
& \leq \frac{b}{c^{*}} \delta \leq \frac{1}{c^{*}} E V(x(\bar{t}), \bar{t}, r(\bar{t}))  \tag{28}\\
& \leq \frac{1}{c^{*}} E V(x(t), t, r(t)) .
\end{align*}
$$

Therefore, from condition (ii), (27), and (28), one gets

$$
\begin{align*}
E L V & (x(t+\theta), t+\theta, r(t+\theta)) \\
& \leq\left(-c_{i_{m}}(t)+\frac{d_{i_{m}}(t)}{c^{*}}\right) E V(x(t), t, r(t)) . \tag{29}
\end{align*}
$$

Similar to the argument on $\left[t^{* *}, t^{*}\right]$, an application of Ito's formula on $[\underline{t}, \bar{t}]$ will lead to $E V(x(\bar{t}), \bar{t}, r(\bar{t}))<\left(b / c^{*}\right) \delta$, which would contradict with (23). So the inequality (21) is true. Therefore, by mathematical induction, one can obtain that (8) holds for all $k \in Z^{+}$. Then from condition (i) and the definition of $\delta$, we have

$$
\begin{equation*}
E\|x(t)\|^{p} \leq \frac{b}{a c^{*}} \delta<\varepsilon, \quad t \geq t_{0} \tag{30}
\end{equation*}
$$

According to Definition 1, it is concluded that the trivial solution of system (3) is uniformly $p$-moment stable. The proof is complete.

Corollary 5. Assume that there exist function $V(x(t), t, i) \in$ $C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$and some positive constants $p, a$, $b, e_{i}, c_{i}$, and $d_{i}$ such that
(i) $a|x|^{p} \leq V(x(t), t, i) \leq b\left|x_{t}\right|_{r}^{p}$;
(ii) $\operatorname{ELV}(x(t), t, i) \leq-c_{i} E V(x(t), t, i)+d_{i} E V\left(x_{t}, t, i\right), t \in$ $\left[t_{k-1}, t_{k}\right), k=1,2, \ldots ;$
(iii) $E V\left(\phi(0)+I_{k}\left(\phi, t_{k}, \widetilde{i}\right) \leq e_{i} V\left(\phi(0), t_{k}^{-}, i\right)\right.$;
(iv) $0<c^{*}<1$, where $c^{*}=\sup \left\{c_{i}^{*} \mid c_{i}^{*}=\left(b e_{i} / a\right), i=\right.$ $1,2, \ldots\} ;$
(v) $c_{i}>d_{i} / c^{*}, t \in\left[t_{k}, t_{k+1}\right) ;$
then the trivial solution of system (3) is uniformly p-moment stable.

Proof. Replacing $c_{i}(t)$ and $d_{i}(t)$ in Theorem 4 with $c_{i}$ and $d_{i}$, respectively, we find that it is a direct conclusion of Theorem 4.

Remark 6. From condition (ii) in Theorem 4, it is seen that each of the continuous dynamics is stable. On the other hand, from condition (iv), we see that each of the discrete dynamics is also stabilizing $\left(0<c^{*}<1\right.$ implies $\left.e_{i}<1\right)$. It implies that the stability properties of a time-delay impulsive markovian switched system with stable continuous dynamics can be preserved under stabilizing impulsive perturbations irrespective of the times of impulses and switching. Hence, the bound of dwell-time is not necessary.

Remark 7. If condition (iii) is omitted in Corollary 5, the result is consistent with that of Theorem 1 in [20]. Thus, the results in this work are an extension of that in [20].

We proceed to consider in next the $p$-moment stability of systems (3). It is supposed that all the subsystems governing the continuous dynamics of (3) can be unstable while the impulses are stabilizing. Intuitively, the conditions in the following theorem consist of four aspects: (i) the Lyapunov functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some positive estimates of the upper right-hand derivatives of the functionals with respect to each unstable mode of system (3); (iii) the dwell time of each mode of system (3) satisfies some supper bounds; (iv) the jumps induced by the stabilizing impulses satisfy certain diminishing conditions.

Theorem 8. Assume that there exist function $V(x(t), t, i) \in$ $C^{2,1}\left(R^{n} \times\left[t_{0}-r, \infty\right] \times S ; R^{+}\right)$and some positive scalars $p, a, b$, $e_{i}, \rho<1, \alpha$, and $\beta$ such that
(i) $a|x|^{p} \leq V(x(t), t, i) \leq b\left|x_{t}\right|_{r}^{p}$;
(ii) $E V\left(\phi(0)+I_{k}\left(\phi, t_{k}, \widetilde{i}\right) \leq \rho e_{i} V\left(\phi(0), t_{k}^{-}, i\right)\right.$;
(iii) $0<c^{*}<1$, where $c^{*}=\sup \left\{c_{i}^{*} \mid c_{i}^{*}=(b / a) \rho e_{i}, i=\right.$ $1,2, \ldots\} ;$
(iv) there exist nonnegative and piecewise continuous functions $c_{i}(t):\left[t_{0}, \infty\right) \rightarrow R^{n}$ satisfying $\int_{t}^{t+\alpha} c_{i}(s) d s \leq \alpha \beta$ for all $t \geq t_{0}$, such that

$$
\begin{gather*}
E L V(x(t), t, i) \leq c_{i}(t) E V(x(t), t, i), \\
t \in\left[t_{k-1}, t_{k}\right), \quad k=1,2, \ldots \tag{31}
\end{gather*}
$$

whenever $t \geq t_{0}$ and $\phi \in \mathrm{PC}_{F_{t}}^{b}$ are such that $c^{*} E L V$ $(\phi(s), t+s, \widetilde{i}) \leq E V(\phi(0), t, i) ;$
(v) $\sup _{k \in Z^{+}}\left\{t_{k}-t_{k-1}\right\}=\alpha<-(1 / \beta) \ln \left((b / a) \rho e_{i}\right), k \in Z^{+} ;$ then the trivial solution of system (3) is uniformly $p$ moment stable.

Proof. For any $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that $\delta<\left(a c^{*} / b\right) \varepsilon$ independent of $t_{0}$. For any $t_{0} \geq 0$ and $x_{t_{0}}=\phi \in$ $\mathrm{PC}_{F_{t}}^{b}(\delta)$, let $x(t)=x\left(t, t_{0}, \phi\right)$ be the solution of (3).

By generalized Ito's formula (Lemma 3), when $t \in\left[t_{k}\right.$, $t_{k+1}$ ), we get

$$
\begin{align*}
& E V(x(t), t, r(t)) \\
&= E V\left(x\left(t_{k}\right), t_{k}, r\left(t_{k}\right)\right) \\
&+E\left(\int_{t_{k}}^{t} L V(x(s), s, r(s)) d s\right) \\
&+E\left(\int_{t_{k}}^{t} \frac{\partial V(x(s), s, r(s))}{\partial x(s)} g(x(s), s, r(s)) d \omega(s)\right) \\
&= E V\left(x\left(t_{k}\right), t_{k}, r\left(t_{k}\right)\right) \\
&+E\left(\int_{t_{k}}^{t} L V(x(s), s, r(s)) d s\right) . \tag{32}
\end{align*}
$$

Given small enough $\Delta t>0$ such that $t+\Delta t \in\left[t_{k}, t_{k+1}\right)$, one has

$$
\begin{aligned}
& E V(x(t+\Delta t), t+\Delta t, r(t+\Delta t)) \\
& =E V\left(x\left(t_{k}\right), t_{k}, r\left(t_{k}\right)\right) \\
& +E\left(\int_{t_{k}}^{t+\Delta t} L V(x(s), s, r(s)) d s\right) \\
& \quad t \in\left[t_{k}, t_{k+1}\right) .
\end{aligned}
$$

In view of condition (iv), (32), and (33), it follows that

$$
\begin{align*}
& E V(x( t+\Delta t), t+\Delta t, r(t+\Delta t)) \\
&-E V(x(t), t, r(t)) \\
&= \int_{t}^{t+\Delta t} E L V(x(s), s, r(s)) d s  \tag{34}\\
& \leq \int_{t}^{t+\Delta t} c_{i}(t) E V(x(s), s, r(s)) d s \\
& t \in\left[t_{k}, t_{k+1}\right)
\end{align*}
$$

From (34), one gets

$$
\begin{array}{r}
D^{+} E V(x(t), t, r(t)) \leq c_{i}(t) E V(x(t), t, r(t)), \\
t \in\left[t_{k}, t_{k+1}\right), \tag{35}
\end{array}
$$

where $D^{+}$is the Dini-derivative defined as

$$
\begin{equation*}
D^{+} V(t)=\lim _{h \rightarrow 0^{+}} \sup \frac{V(t+h)-V(t)}{h} \tag{36}
\end{equation*}
$$

in which $V(t)$ is continuous function.
In the following, we will show that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{k-1}, t_{k}\right), k \in Z^{+} \tag{37}
\end{equation*}
$$

For this purpose, we first prove that

$$
\begin{equation*}
E V(x(t), t, i) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}, t_{1}\right) \tag{38}
\end{equation*}
$$

From condition (i) and $x_{t_{0}} \in \operatorname{PC}_{F_{t}}^{b}(\delta)$, it follows that

$$
\begin{align*}
E V(x(t), t, r(t)) & \leq b E\left\|x_{t_{0}}\right\|_{r}^{p} \\
& \leq b \delta<\frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}-r, t_{0}\right] \tag{39}
\end{align*}
$$

If (38) is not true, there must exist some $t \in\left[t_{0}, t_{1}\right)$ such that

$$
\begin{equation*}
E V(x(t), t, r(t))>\frac{b}{c^{*}} \delta>b \delta \geq E V\left(x\left(t_{0}\right), t_{0}, r\left(t_{0}\right)\right) \tag{40}
\end{equation*}
$$

Let $t^{*}=\inf \left\{t \in\left[t_{0}, t_{1}\right): E V(x(t), t, r(t))>\left(b / c^{*}\right) \delta\right\}$. From (39), one has $t^{*} \in\left(t_{0}, t_{1}\right)$ and

$$
\begin{gather*}
E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right)=\frac{b}{c^{*}} \delta, \\
E V(x(t), t, r(t))<\frac{b}{c^{*}} \delta, \quad t \in\left[t_{0}-r, t^{*}\right),  \tag{41}\\
D^{+} E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right) \geq 0
\end{gather*}
$$

Owing to $\left(b / c^{*}\right) \delta>b \delta$ and $E V(x(t), t, r(t)) \leq b \delta$ for $t \in$ $\left[t_{0}-r, t_{0}\right]$, then there exists a $t^{* *} \in\left[t_{0}, t^{*}\right)$ such that

$$
\begin{gather*}
E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right)=b \delta, \\
E V(x(t), t, r(t))>b \delta, \quad t \in\left(t^{* *}, t^{*}\right],  \tag{42}\\
D^{+} E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right) \geq 0 .
\end{gather*}
$$

Consequently, by (41) and (42), for $t \in\left[t^{*}, t^{* *}\right]$, we have

$$
\begin{equation*}
E V(x(t+s), t+s, r(t+s)) \leq \frac{b \delta}{c^{*}} \leq \frac{1}{c^{*}} E V(x(t), t, r(t)) . \tag{43}
\end{equation*}
$$

So, for $t \in\left[t^{*}, t^{* *}\right]$, it has

$$
\begin{equation*}
D^{+} E V(x(t), t, r(t)) \leq c_{i}(t) E V(x(t), t, r(t)) . \tag{44}
\end{equation*}
$$

Integrating (44) from $t^{* *}$ to $t^{*}$

$$
\begin{equation*}
\int_{t^{* *}}^{t^{*}} \frac{D^{+} E V(x(s), s, r(s))}{E V(x(s), s, r(s))} d s \leq \int_{t_{0}}^{t_{1}} c_{i}(t) d s \leq \alpha \beta \tag{45}
\end{equation*}
$$

Meanwhile, by condition (v), one has

$$
\begin{align*}
\int_{t^{* *}}^{t^{*}} & \frac{D^{+} E V(x(s), s, r(s))}{E V(x(s), s, r(s))} d s \\
& =\int_{E V\left(x\left(t^{* *}\right), t^{* *}, r\left(t^{* *}\right)\right)}^{E V\left(x\left(t^{*}\right), t^{*}, r\left(t^{*}\right)\right)} \frac{1}{u} d u  \tag{46}\\
& =\int_{b \delta}^{\left(b / c^{*}\right) \delta} \frac{1}{u} d u=-\ln c^{*}>c_{i}\left(t_{1}-t_{0}\right),
\end{align*}
$$

which is contradictory with (45), so (38) is true.
By condition (i) and (38), we have

$$
\left.\begin{array}{rl}
E V & (x
\end{array} \quad\left(t_{1}\right), t_{1}, r\left(t_{1}\right)\right) .
$$

Now, we assume that (37) holds for $k=1,2, \ldots, m$, that is,

$$
\begin{equation*}
E V(x(t), t, r(t)) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{k-1}, t_{k}\right] . \tag{48}
\end{equation*}
$$

We will prove that, for $k=m+1$,

$$
\begin{equation*}
E V(x(t), t, r(t)) \leq \frac{b}{c^{*}} \delta, \quad t \in\left[t_{m}, t_{m+1}\right) . \tag{49}
\end{equation*}
$$

Suppose that (49) does not hold, by the same procedure as in [ $t_{0}, t_{1}$ ), we can get a contradiction and (49) follows. Finally, we get

$$
\begin{equation*}
E\|x(t)\|^{p} \leq \frac{b}{a c^{*}} \delta<\varepsilon, \quad t \geq t_{0} \tag{50}
\end{equation*}
$$

From Definition 1, it is concluded that the trivial solution of system (3) is uniformly $p$-moment stable. The proof is complete.

Corollary 9. Assume that there exist positive scalars $p, a, b$, $\rho<1, e_{i}, c_{i}(i \in S), \alpha$, and $\beta$ such that conditions (iv) in Theorem 8 are replaced by the following:
(iv)* there exists positive number $c_{i}$ such that

$$
\begin{gather*}
E L V(x(t), t, i) \leq c_{i} E V(x(t), t, i), \\
t \in\left[t_{k-1}, t_{k}\right), \quad k=1,2, \ldots, \tag{51}
\end{gather*}
$$

whenever $t \geq t_{0}$ and $\varphi \in \mathrm{PC}_{F_{t}}^{b}\left([-r, 0] ; R^{n}\right)$ are such that $c^{*} E L V(\varphi(s), t+s, \widetilde{i}) \leq E V(\varphi(0), t, i) ;$ and all other conditions remain the same, then the trivial solution of system (3) is uniformly $p$-moment stable.

Proof. Similar to the procedure of Theorem 8, correspondingly, the item $c_{i}(t)$ in the proof is replaced by $c_{i}$.

Remark 10. It is clear that Theorem 8 implies that each of the continuous dynamics can be unstable, since the item $c_{i}(t)$ in condition (iv), which characterizes the changing rate of Lyapunov function $V(x(t), t, r(t))$ at $t$, is assumed to be nonnegative. Theorem 8 shows that an unstable stochastic delay system can be successfully stabilized by impulse.

Remark 11. Constant $e_{i}$ in condition (ii) characterizes certain perturbations in the overall impulsive stabilization process, that is, it is not strictly required by Theorem 8 that each impulse contributes to stabilize the system, as long as the overall contribution of the impulses is stabilizing. When $e_{i} \equiv$ 1 , it is easily obtained that each impulse must be stabilizing ( $\rho<1$ ), which is more restrictive.

Now consider the linear stochastic system with impulses and markovian switching

$$
\begin{gather*}
d x(t)=[A(r(t)) x(t)+B(r(t)) x(t-r)] d t \\
+[C(r(t)) x(t)+D(r(t)) x(t-r)] d \omega(t), \\
t \neq t_{k}, \\
\Delta x(t)=E_{k}\left(x\left(t^{-}\right), r(t)\right), \quad t=t_{k}, \\
x\left(t_{0}+\theta\right)=\phi(\theta), \quad \theta \in[-r, 0], \tag{52}
\end{gather*}
$$

where $A_{i}, B_{i}, C_{i}, D_{i}$, and $E_{k}$ are all $n \times n$ matrices and $\omega$ is a one-dimensional standard wiener process. $r(t)$ is a Markov chain taking values in $S=\{1,2, \ldots, N\}$ in which the transition rate is $\pi_{i j}$.

Theorem 12. When $r(t)=i \in S$, define $A(r(t))=A_{i}$, $B(r(t))=B_{i}, C(r(t))=C_{i}, D(r(t))=D_{i}$.
(i) If there exist constants $p_{i}, c_{i}, d_{i}, \varepsilon_{1}$, and $\varepsilon_{2}$ such that

$$
p_{i} \lambda_{\max }\left(A_{i}^{T}+A_{i}+\varepsilon_{1} I\right)+p_{i}\left(1+\varepsilon_{2}\right)\left\|D_{i}\right\|^{2}+\sum_{j=1}^{N} \pi_{i j} p_{j}
$$

$$
\begin{gather*}
\leq-c_{i}, p_{i}\left\{\varepsilon_{1}^{-1}\left\|B_{i}\right\|^{2}+\left(1+\varepsilon_{2}^{-1}\right)\left\|G_{i}\right\|^{2}\right\} \leq d_{i} ; \\
p_{i} c_{i}\left\|I+E_{k}\right\|^{2}>d_{i}, \tag{53}
\end{gather*}
$$

then system (52) is uniformly $p$-moment stable.
(ii) If there exist constants $c_{i}$, $\varepsilon_{1}$, and $\varepsilon_{2}$ such that

$$
\begin{gather*}
p_{i} \lambda_{\max }\left(A_{i}^{T}+A_{i}+\varepsilon_{1} I\right)+p_{i}\left(1+\varepsilon_{2}\right)\left\|D_{i}\right\|^{2} \\
+\sum_{j=1}^{N} \pi_{i j} p_{j}+2 p_{i} \ln \left\|I+E_{k}\right\|  \tag{54}\\
\times\left[\varepsilon_{1}^{-1}\left\|B_{i}\right\|^{2}+\left(1+\varepsilon_{2}^{-1}\right)\left\|G_{i}\right\|^{2}\right] \leq c_{i} ; \\
\ln \left(p_{i}\left\|I+E_{k}\right\|\right) \leq-\frac{1}{2} c_{i}\left(t_{k+1}-t_{k}\right),
\end{gather*}
$$

then system (52) is uniformly p-moment stable.
Proof. It is a direct consequence of Corollaries 5 and 9 with $V(x(t), r(t))=p_{i}\|x(t)\|^{2}$.

## 4. Numerical Examples

The applicability of the results derived in the preceding section is illustrated by the following two examples.

Example 1. Consider the following linear impulsive switching delay system:

$$
\begin{align*}
& d x(t)=[A(r(t)) x(t)+B(r(t)) x(t-1)] d t \\
& +[C(r(t)) x(t)+D(r(t)) x(t-1)] d \omega(t), \\
& t \neq t_{k},  \tag{55}\\
& \Delta x(t)=E_{k}\left(x\left(t^{-}\right), r(t)\right), \quad t=t_{k},
\end{align*}
$$

where

$$
\begin{array}{cc}
A_{1}=\left[\begin{array}{ccc}
0.1 & 0.2 & -0.1 \\
0.2 & 0.15 & 0.3 \\
0 & 0.24 & 0.1
\end{array}\right], & B_{1}=\left[\begin{array}{ccc}
-0.24 & 0.04 & 0 \\
0.24 & -0.4 & 0.1 \\
0 & 0.24 & -0.2
\end{array}\right], \\
D_{1}=\left[\begin{array}{ccc}
0.5 & 0.4 & 0.2 \\
-0.24 & 0.32 & 0.3 \\
0.6 & 0.26 & -0.5
\end{array}\right], & G_{1}=\left[\begin{array}{ccc}
0.14 & 0 & -0.21 \\
0.2 & 0.1 & 0.16 \\
0.1 & 0.15 & 0.17
\end{array}\right], \\
A_{2}=\left[\begin{array}{ccc}
0.2 & 0.3 & -0.1 \\
0.3 & 0.1 & 0.3 \\
0 & 0.2 & 0.1
\end{array}\right], & B_{2}=\left[\begin{array}{ccc}
-0.2 & 0.04 & 0 \\
0.2 & -0.4 & 0.1 \\
0 & 0.2 & -0.2
\end{array}\right], \\
D_{2}=\left[\begin{array}{ccc}
0.5 & 0.4 & 0.2 \\
-0.2 & 0.3 & 0.3 \\
0.6 & 0.2 & -0.5
\end{array}\right], & G_{2}=\left[\begin{array}{ccc}
0.24 & 0 & -0.21 \\
0.2 & 0.1 & 0.16 \\
0.1 & 0.1 & 0.17
\end{array}\right], \\
E_{k}=\left[\begin{array}{ccc}
-0.5 & 0 & 0 \\
0 & -0.4 & 0 \\
0 & 0 & -0.3
\end{array}\right], \quad \pi=\left[\begin{array}{cc}
-\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{2}{3}
\end{array}\right], \tag{56}
\end{array}
$$



Figure 1: State response of system (55) under impulsive perturbations.

Set $\varepsilon_{1}=\varepsilon_{2}=0.5, p_{i}=1$. It is easy to see that $\lambda_{\max }\left(A_{1}^{T}+\right.$ $\left.A_{1}+0.5 I\right)=1.3819, \lambda_{\max }\left(A_{2}^{T}+A_{2}+0.5 I\right)=1.5020,\left\|B_{1}\right\|^{2}=$ $0.3141,\left\|B_{2}\right\|^{2}=0.2764,\left\|D_{1}\right\|^{2}=0.9175,\left\|D_{2}\right\|^{2}=0.8737$, $\left\|G_{1}\right\|^{2}=0.1360,\left\|G_{2}\right\|^{2}=0.1219$, and $\left\|I+E_{k}\right\|^{2}=4.9$. Taking $c_{1}=3.1, c_{2}=3.2, t_{k}-t_{k-1}=0.22$, we can verify that all the conditions (ii) of Theorem 12 are satisfied:

$$
\begin{align*}
& \lambda_{\max }\left(A_{1}^{T}+A_{1}+\varepsilon_{1} I\right)+\left(1+\varepsilon_{2}\right)\left\|D_{1}\right\|^{2} \\
&+\sum_{j=1}^{N} \pi_{1 j}+2 \ln \left\|I+E_{k}\right\| \\
& \times {\left[\varepsilon_{1}^{-1}\left\|B_{1}\right\|^{2}+\left(1+\varepsilon_{2}^{-1}\right)\left\|G_{1}\right\|^{2}\right] } \\
&= 3.0190 \leq c_{1}=3.1 ; \\
& \lambda_{\max }\left(A_{2}^{T}+A_{2}+\varepsilon_{1} I\right)+\left(1+\varepsilon_{2}\right)\left\|D_{1}\right\|^{2}  \tag{57}\\
&+\sum_{j=1}^{N} \pi_{2 j}+2 \ln \left\|I+E_{k}\right\| \\
& \times\left[\varepsilon_{1}^{-1}\left\|B_{2}\right\|^{2}+\left(1+\varepsilon_{2}^{-1}\right)\left\|G_{2}\right\|^{2}\right] \\
&= 3.1574 \leq c_{2}=3.2 ; \\
& \ln \left\|I+E_{k}\right\|^{2}=-0.7133 \leq-c_{i}\left(t_{k+1}-t_{k}\right) \\
& \quad=-3.2 \times 0.22=-0.7040 .
\end{align*}
$$

So, system (55) is $p$-moment stable. Numerical simulations for this example are shown in Figures 1 and 2. It is clearly demonstrated that impulses can successfully stabilize an otherwise unstable stochastic delay system.


Figure 2: State response of system (55) without impulsive perturbations.

Example 2. Consider the following linear impulsive switching delay system:

$$
\begin{aligned}
& d x(t)= {[A(r(t)) x(t)+B(r(t)) x(t-1)] d t } \\
&+[C(r(t)) x(t)+D(r(t)) x(t-1)] d \omega(t), \\
& t \neq t_{k}, \\
& \Delta x(t)=E_{k}\left(x\left(t^{-}\right), r(t)\right), \quad t=t_{k},
\end{aligned}
$$

where

$$
\begin{array}{cl}
A_{1}=\left[\begin{array}{cc}
-4 & 0 \\
0 & -5
\end{array}\right], \quad B_{1}=\left[\begin{array}{ll}
0.2 & 0.3 \\
0.5 & 0.1
\end{array}\right] \\
D_{1}=\left[\begin{array}{cc}
0.3 & 0.4 \\
0.5 & 0.2
\end{array}\right], \quad G_{1}=\left[\begin{array}{ll}
0.8 & 0.1 \\
0.1 & 0.3
\end{array}\right], \\
E_{k}=\left[\begin{array}{cc}
-0.6 & 0.5 \\
0.3 & -0.5
\end{array}\right], \\
A_{2}=\left[\begin{array}{cc}
-3 & 0 \\
0 & -4
\end{array}\right], \quad B_{2}=\left[\begin{array}{ll}
0.3 & 0.5 \\
0.4 & 0.3
\end{array}\right], \\
D_{2}=\left[\begin{array}{cc}
0.5 & 0.1 \\
0.4 & 0.1
\end{array}\right], \quad G_{2}=\left[\begin{array}{ll}
0.2 & 0.2 \\
0.1 & 0.4
\end{array}\right], \\
P=\left[\begin{array}{ll}
0.5 & 0.5 \\
0.5 & 0.5
\end{array}\right] . \tag{60}
\end{array}
$$

Choosing $\varepsilon_{1}=\varepsilon_{2}=2, p_{i}=1$. It is easy to see that $\lambda_{\max }\left(A_{1}^{T}+\right.$ $\left.A_{1}+2 I\right)=-6, \lambda_{\max }\left(A_{2}^{T}+A_{2}+2 I\right)=-4,\left\|B_{1}\right\|^{2}=0.3403$, $\left\|B_{2}\right\|^{2}=0.5687,\left\|D_{1}\right\|^{2}=0.5009,\left\|D_{2}\right\|^{2}=0.4298,\left\|G_{1}\right\|^{2}=$ $0.6712,\left\|G_{2}\right\|^{2}=0.2347,\left\|I+E_{k}\right\|^{2}=0.7467$. Taking $c_{1}=3.3$,
$c_{2}=1.6, d_{1}=1.2, d_{2}=0.65$, we can verify that all the conditions (i) of Theorem 12 are satisfied:

$$
\begin{gather*}
\lambda_{\max }\left(A_{i}^{T}+A_{i}+\varepsilon_{1} I\right)+\left(1+\varepsilon_{2}\right)\left\|D_{i}\right\|^{2} \\
+\sum_{j=1}^{N} \pi_{i j}=\left\{\begin{array}{l}
-3.4973<-3.3=c_{1} ; \\
-1.7166<-1.6=c_{2} ;
\end{array}\right.  \tag{61}\\
\varepsilon_{1}^{-1}\left\|B_{i}\right\|^{2}+\left(1+\varepsilon_{2}^{-1}\right)\left\|G_{i}\right\|^{2}=\left\{\begin{array}{l}
1.1770<1.2=d_{1} ; \\
0.6364<0.65=d_{2} ;
\end{array}\right.
\end{gather*}
$$

$$
\left\|I+E_{k}\right\|^{2} c_{i}=0.7467 \times\left\{\begin{array}{l}
3.3=2.4641>1.2=d_{1}  \tag{62}\\
1.6=1.1947>0.65=d_{2}
\end{array}\right.
$$

Theorem 12 guarantees that the trivial solution of system (58) is $p$-moment stable. Numerical simulations are shown in Figure 3. It is clearly demonstrated that the $p$-moment stability properties can be preserved irrespective of the length of the delay.

## 5. Conclusions

In this paper, a method of multiple Lyapunov functionals has been applied to deal with the effects of impulses on $p$ moment stability of stochastic differential delay systems with impulsive jump and markovian switching. Some stability criteria are obtained based on Lyapunov functional method and stochastic theory. It is shown that, even if all the subsystems governing the continuous dynamics without impulse are not stable, as impulsive and switching signal satisfies a dwell-time upper bound condition, impulses can stabilize the systems in the $p$-moment stability sense. The opposite situation is also developed for which all the subsystems governing the


Figure 3: State response of system (58) under impulsive perturbations
continuous dynamics without impulse are $p$-moment stable. Applying the derived results to a class of stochastic systems with arbitrarily large delays, the deduced new stability criteria can relax some restrictions on impulses imposed by the existing results. Two illustrative examples have been provided to demonstrate the main theoretical analysis.

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## Research Article

# Performance of Networked Control Systems 

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Data packet dropout is a special kind of time delay problem. In this paper, predictive controllers for networked control systems (NCSs) with dual-network are designed by model predictive control method. The contributions are as follows. (1) The predictive control problem of the dual-network is considered. (2) The predictive performance of the dual-network is evaluated. (3) Compared to the popular networked control systems, the optimal controller of the new NCSs with data packets dropout is designed, which can minimize infinite performance index at each sampling time and guarantee the closed-loop system stability. Finally, the simulation results show the feasibility and effectiveness of the controllers designed.

## 1. Introduction

With the rapid development of computer networks technology, the control system based on NCSs has become one of the hot research tasks in the current international control field. As stability analysis of NCSs subjected to packet dropping has received much attention in [1-3], various approaches for the delay issue in NCSs have been presented in [4-8], and so on. Compared to the traditional control systems, the main advantages of NCSs are lower cost, simpler installation, and higher reliability [9-11]. Because of these attractive advantages, typical application of these systems ranges from a wide field, such as automotive [12], mobile [13], and advanced aircraft [14]. However, the introduction of communication networks in the control loops makes the analysis and design of NCSs complex. For example, network-induced delays and data packet dropout problem may be inevitable during transmitting communication.

Data packet dropout is a special kind of time delay problems. Data packet dropout which is a kind of uncertainty that may happen due to node failures or network congestion is a common problem in networked systems. This loss will deteriorate the performance and may even cause the system to be unstable. Recently, the effect of data packet dropout on the stability and performance of NCSs has received great
attention. In [15], the stability of a linear networked control system in the presence of dropped packets was studied. A stability analysis of model-based NCSs can be found in [2, 16-19], where an additional model was used for estimating the plant state between transmission times and generating a control signal. In [20], though turning the model of NCSs into an asynchronous dynamic system, hybrid system technology has been used to handle the system with time delay and data dropouts. A stability condition was obtained for the Try-Once-Discard networked protocol in [3]. In [21], Hadjicostis and Touri analyzed the performance when lost data were replaced by zeros. In [22, 23], Ling and Lemmon posed the problem of optimal compensator design for the case when data loss was independent and identically distributed. Reference [24] addresses the random time delays and packet losses issues of NCSs within the framework of jump linear systems with mode-dependent time delays. Jump linear systems with Markov chains [25, 26] also were used to analyze the effect of dropouts on system stability and performance. In [27], a delay-dependent stability condition was presented for discrete-time jump time delay system where the time delay was dependent on the system mode. In [28], the problem of stability analysis and controller design has been proposed based on a new model with packet dropouts.

Model predictive control (MPC) can now be found in a wide variety of application areas including chemicals, food
processing, and so on. MPC is also an effective method to incorporate the input and output constraints into online optimization, which increases the possibility of its application in the synthesis and analysis of NCSs [29]. In [30], the MPC strategy for multivariable plants was presented. Wu et al. [31] introduced MPC into NCSs with time delay and designed an optimal control rule. In [4], the networked predictive control with modified MPC was proposed.

In this paper, the contributions are as follows. (1) The predictive control problem of the dual-network is considered. (2) The predictive performance of the dual-network is evaluated. (3) Compared to the popular networked control systems, the optimal controller of the new NCSs with data packets dropout is designed, which can minimize infinite performance index at each sampling time and guarantee the closed-loop system stability.

The remainder of this paper is organized as follows. The problem descriptions and model of NCSs with packet dropouts are given in Section 2. The optimization method of the new closed-loop systems is proposed in Section 3. The stability analysis is given in Section 4. A numerical example and an industrial example show the effectiveness of the proposed method in Section 5, and some conclusions are given in Section 6.

## 2. Problem Descriptions and Modeling of NCSs with Packet Dropouts

Consider that NCSs model which is shown in Figure 1, sensors, controllers, and actuators are connected by networks, and we suppose that the communication link between primary sensor and controller $\left(S_{1} / C_{1}\right)$ is ideal. Based on such a structure, the problem of packet dropouts in network transmission mainly exists in the actuator and secondary sensor and controller $\left(S_{2} / C_{2}\right.$ and $\left.C_{2} / A\right)$.

In Figure 1, $S_{i}(i=1,2), C_{i}(i=1,2)$, and $P_{i}(i=$ $1,2)$ denote the sensor, controller, and plant of primary and secondary, respectively, and $A$ is actuator. $u_{1}(k)$ and $\bar{u}_{2}(k)$ are outputs of primary and secondary controllers at sampling time $k . u_{2}(k)$ is input of actuator at sampling time $k, d_{k}^{C_{2} A}$ is the quantity of packet dropouts between sampling current time $k$ and the latest communicate time successfully $\left(k-d_{k}^{\mathrm{C}_{2} A}\right)$ on $C_{2} / A$ side, and $d_{k}^{S_{2} C_{2}}$ is the quantity of packet dropouts at current time $k$ and the latest communicate successfully $\left(k-d_{k}^{S_{2} C_{2}}\right)$ on $S_{2} / C_{2}$ side. The quantity of packet dropouts on $S_{2} / C_{2}$ and $C_{2} / A$ sides is assumed to satisfy

$$
\begin{equation*}
d_{m} \leq d_{k}^{S_{2} C_{2}} \leq d_{M}, \quad \tau_{m} \leq d_{k}^{C_{2} A} \leq \tau_{M} \tag{1}
\end{equation*}
$$

where $d_{m}, d_{M}, \tau_{m}$, and $\tau_{M}$ are constant positive scalars representing the minimum and maximum quantities of packet dropouts on $S_{2} / C_{2}$ and $C_{2} / A$ sides, respectively, where $d_{k}^{S_{2} C_{2}}$ and $d_{k}^{C_{2} A}$ are two independent Markov chains. Without loss of generality, define

$$
\begin{equation*}
0 \leq d_{k}^{S_{2} C_{2}} \leq d_{M}, \quad 0 \leq d_{k}^{C_{2} A} \leq \tau_{M} \tag{2}
\end{equation*}
$$

Suppose that under the condition $d_{k}^{S_{2} C_{2}}=i$, the probability that the state of packet dropouts at time $k+1$ is $j\left(d_{k+1}^{S_{2} C_{2}}=j\right)$ is

$$
\begin{equation*}
\operatorname{Pr}\left\{d_{k+1}^{S_{2} C_{2}}=j \mid d_{k}^{S_{2} C_{2}}=i\right\}=\pi_{i j} \quad \forall i, j \in \chi_{1} \tag{3}
\end{equation*}
$$

where $\chi_{1}=\left\{0,1,2, \ldots, d_{M}\right\}$. That is also to say that the transition probability of $d_{k}^{S_{2} C_{2}}$ jumping from mode $i$ to $j$ is (3), and the transition probability of $d_{k}^{C_{2} A}$ jumping from mode $r$ to $s$ is

$$
\begin{equation*}
\operatorname{Pr}\left\{d_{k+1}^{C_{2} A}=s \mid d_{k}^{C_{2} A}=r\right\}=\lambda_{r s} \quad \forall r, s \in \chi_{2} \tag{4}
\end{equation*}
$$

where $\chi_{2}=\left\{0,1,2, \ldots, \tau_{M}\right\}, \pi_{i j} \geq 0, \sum_{j=0}^{d_{M}} \pi_{i j}=1$, $\sum_{j=0, j \neq i}^{d_{M}} \pi_{i j}=1-\pi_{i i}$, and $\lambda_{r s} \geq 0, \sum_{s=0}^{\tau_{M}} \lambda_{r s}=1$.

For analysis convenience and without loss of generality, we suppose that Markov chains jump no more than one step. Thus, Markov chain transferring probability matrix is satisfied [28]. Consider

$$
\begin{array}{ll}
\pi_{i j}=0, & \text { if } j \neq i+1, \quad j \neq 0 \\
\lambda_{r s}=0, & \text { if } s \neq r+1, s \neq 0 \tag{5}
\end{array}
$$

And the two equations are used to produce the definition of data packet dropout. $i$ denotes the quantity of the packet dropouts at time $k-1$. $j$ denotes the quantity of packet dropouts at time $k$. The system's conditional probability of data packets dropout is equal to zero when the difference of the quantity of packet dropouts between time $k$ and $k-1$ is not equal to 1 or there is no dropout at time $k$.

Some assumptions of NCSs are introduced as follows.
(1) The sensors, controllers, and actuators are clockdriven.
(2) The buffers are big enough to hold all the data arrived, and rule of the buffer is last-in-first-out.
(3) Transmission link of primary control loop is ideal and data packet dropouts only happen in secondary control loop.
(4) The transition of NCSs is single-packet transmission and no timing sequence disordered.

The linear time-invariant discrete-time models of primary and secondary plants in Figure 1 are described as follows:

$$
\begin{align*}
& P_{1}:\left\{\begin{array}{l}
x_{1}(k+1)=\Phi_{1} x_{1}(k)+\Gamma_{1} y_{2}(k), \\
y_{1}(k)=C_{1} x_{1}(k),
\end{array}\right.  \tag{6}\\
& P_{2}:\left\{\begin{array}{l}
x_{2}(k+1)=\Phi_{2} x_{2}(k)+\Gamma_{2} u_{2}(k), \\
y_{2}(k)=C_{2} x_{2}(k),
\end{array}\right.
\end{align*}
$$

where $x_{1}(k)$ and $x_{2}(k)$ are the states of primary and secondary plants, respectively, at sampling time $k, u_{2}(k)$ is the input of secondary plant $P_{2}$ at sampling time $k$, and $\Phi_{1}, \Gamma_{1}, C_{1}$, $\Phi_{2}, \Gamma_{2}, C_{2}$ are known constant matrixes with appropriate dimensions.


Figure 1: Networked control system model.

Due to the existence of data packet dropouts, transmission link in network cannot normally communicate. The controller and actuator can use the buffer rule, named first-in-last-out, to pick up signal [28]. Take the controller, for example; when a secondary sensor data $x_{2}(k)$ is false to transmit, the secondary controller gets the latest data $\bar{x}_{2}(k-1)$ from buffer and uses it as $\bar{x}_{2}(k)$ to calculate new control input. Otherwise, the new sensor data $x_{2}(k)$ will be saved to buffer and used by the secondary controller as $\bar{x}_{2}(k)$. Thus,

$$
\bar{x}_{2}(k)= \begin{cases}x_{2}(k), & d_{k}^{S_{2} C_{2}}=0  \tag{7}\\ \bar{x}_{2}(k-1), & d_{k}^{S_{2} C_{2}}>0\end{cases}
$$

Similarly

$$
u_{2}(k)= \begin{cases}\bar{u}_{2}(k), & d_{k}^{C_{2} A}=0  \tag{8}\\ u_{2}(k-1), & d_{k}^{C_{2} A}>0\end{cases}
$$

It can easily derive

$$
\begin{equation*}
\bar{x}_{2}(k)=x_{2}\left(k-d_{k}^{S_{2} C_{2}}\right) \tag{9}
\end{equation*}
$$

From the model of NCSs, as shown in Figure $1, u_{1}(k)$ is the output of primary controller at sampling time $k$ as

$$
\begin{equation*}
u_{1}(k)=F_{1} x_{1}(k) \tag{10}
\end{equation*}
$$

where $F_{1}$ is to be designed by MPC method.
$\bar{u}_{2}(k)$ is the output of secondary controller as

$$
\begin{equation*}
\bar{u}_{2}(k)=u_{1}(k)+F_{2}\left(d_{k}^{S_{2} C_{2}}\right) \bar{x}_{2}(k) \tag{11}
\end{equation*}
$$

where $F_{2}\left(d_{k}^{S_{2} C_{2}}\right)$ is to be designed by MPC method.
Substituting (9) and (10) into (11), we have

$$
\begin{equation*}
\bar{u}_{2}(k)=F_{1} x_{1}(k)+F_{2}\left(d_{k}^{S_{2} C_{2}}\right) x_{2}\left(k-d_{k}^{S_{2} C_{2}}\right) . \tag{12}
\end{equation*}
$$

Therefore, (8) can be rewritten as

$$
u_{2}(k)= \begin{cases}F_{1} x_{1}(k)+F_{2}\left(d_{k}^{S_{2} C_{2}}\right) x_{2}\left(k-d_{k}^{S_{2} C_{2}}\right), & d_{k}^{C_{2} A}=0  \tag{13}\\ u_{2}(k-1), & d_{k}^{C_{2} A}>0\end{cases}
$$

In order to simplify the expression of the closed-loop control systems, a function $\alpha=\left\{\begin{array}{l}0, d_{k}^{C_{2} A}=0 \\ 1, d_{k}^{C_{2} A}>0\end{array}\right.$ is introduced, which is dependent on whether packet dropped or not, instead of the quantity of packet dropouts. Combining (6) and (13), it can obtain that

$$
\begin{align*}
& u_{2}(k)= \alpha u_{2}(k-1)+[1-\alpha] F_{1} x_{1}(k) \\
&+[1-\alpha] F_{2}\left(d_{k}^{S_{2} C_{2}}\right) x_{2}\left(k-d_{k}^{S_{2} C_{2}}\right),  \tag{14}\\
& x_{1}(k+1)=\Phi_{1} x_{1}(k)+\Gamma_{1} C_{2} x_{2}(k),  \tag{15}\\
& x_{2}(k+1)= \Phi_{2} x_{2}(k)+\Gamma_{2} \alpha u_{2}(k-1) \\
&+\Gamma_{2}[1-\alpha] F_{1} x_{1}(k)+\Gamma_{2}[1-\alpha]  \tag{16}\\
& \times F_{2}\left(d_{k}^{S_{2} C_{2}}\right) x_{2}\left(k-d_{k}^{S_{2} C_{2}}\right) .
\end{align*}
$$

Combining (10), (14), (15), and (16) and augmenting the state vectors, the new resulting closed-loop control systems and the augmenting vector (18) formed by predictive controller (10) and (14) designed by MPC method are as follows:

$$
\begin{align*}
x(k+1) & =A\left(d_{k}^{C_{2} A}\right) x(k)+B\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right) x\left(k-d_{k}^{S_{2} C_{2}}\right),  \tag{17}\\
u(k) & =\binom{u_{1}(k)}{u_{2}(k)} \\
& =C\left(d_{k}^{C_{2} A}\right) x(k)+D\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right) x\left(k-d_{k}^{S_{2} C_{2}}\right), \tag{18}
\end{align*}
$$

where $x(k)=\left(x_{1}^{\mathrm{T}}(k) x_{2}^{\mathrm{T}}(k) u_{2}^{\mathrm{T}}(k-1)\right)^{\mathrm{T}}$,

$$
\left.\begin{array}{c}
x\left(k-d_{k}^{S_{2} C_{2}}\right) \\
=\left(x_{1}^{\mathrm{T}}\left(k-d_{k}^{S_{2} C_{2}}\right)\right. \\
x_{2}^{\mathrm{T}}\left(k-d_{k}^{S_{2} C_{2}}\right)
\end{array} u_{2}^{\mathrm{T}}\left(k-d_{k}^{S_{2} C_{2}}-1\right)\right)^{\mathrm{T}}, ~\left(\begin{array}{ccc}
\Phi_{1} & \Gamma_{1} C_{2} & 0 \\
A\left(d_{k}^{\mathrm{C}_{2} A}\right)=\left(\begin{array}{ccc}
\Gamma_{2}[1-\alpha] F_{1} & \Phi_{2} & \Gamma_{2} \alpha \\
{[1-\alpha] F_{1}} & 0 & \alpha
\end{array}\right), \\
B\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \Gamma_{2}[1-\alpha] F_{2}\left(d_{k}^{S_{2} C_{2}}\right) & 0 \\
0 & {[1-\alpha] F_{2}\left(d_{k}^{S_{2} C_{2}}\right)} & 0
\end{array}\right) \\
C\left(d_{k}^{C_{2} A}\right) & =\left(\begin{array}{ccc}
F_{1} & 0 & 0 \\
{[1-\alpha] F_{1}} & 0 & \alpha
\end{array}\right),  \tag{19}\\
D\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right) & =\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & {[1-\alpha] F_{2}\left(d_{k}^{S_{2} C_{2}}\right)} & 0
\end{array}\right)
\end{array}\right.
$$

Remark 1. The new closed-loop control systems (17) and the augmenting vector (18) formed by predictive controller (10) and (14) are linear jumping systems, where their communications are described by Markov chain which is the description of the quantity of packet dropouts at sampling time $k$ on $C_{2} / A$ and $S_{2} / C_{2}$ sides. The value of $\alpha$ depends on whether the designed control signal is successfully transmitted or not. Therefore we can use the results of linear jumping system with delay to analyze this class of NCSs with packets dropped.

## 3. Optimum Analysis Based on MPC

Assume that predictive horizon $p=\infty$, control horizon $q=\infty$, and the $m$ steps control sequences $u(k+m \mid k)$, $m=0,1,2, \ldots, \infty$, are computed by minimizing the following performance function:

$$
\begin{equation*}
J_{\infty}=\sum_{m=0}^{\infty}\left[\|x(k+m \mid k)\|_{\mathrm{Q}}^{2}+\|u(k+m \mid k)\|_{R}^{2}\right] . \tag{20}
\end{equation*}
$$

The norm terms in the performance function are defined as

$$
\begin{equation*}
\|x\|_{\mathrm{Q}}^{2}=x^{\mathrm{T}} \mathrm{Q} x \tag{21}
\end{equation*}
$$

The exact measurement state of NCSs at each sampling time $k$ is

$$
\begin{equation*}
x(k \mid k)=x(k) . \tag{22}
\end{equation*}
$$

By using the control input given in (18), the first control move is

$$
\begin{align*}
u(k)= & u(k \mid k) \\
= & C\left(d_{k \mid k}^{C_{2} A}\right) x(k \mid k)  \tag{23}\\
& +D\left(d_{k \mid k}^{S_{2} C_{2}}, d_{k \mid k}^{C_{2} A}\right) x\left(k-d_{k \mid k}^{S_{2} C_{2}} \mid k\right) .
\end{align*}
$$

According to (17) and (22), the predicted state at time $k+$ $m$ is obtained as (24) which is predicted based on the exact measurement state $x(k \mid k)$. Consider

$$
\begin{align*}
& x(k+m+1 \mid k) \\
&= A\left(d_{k+m \mid k}^{C_{2} A}\right) x(k+m \mid k)+B\left(d_{k+m \mid k}^{S_{2} C_{2}}, d_{k+m \mid k}^{C_{2} A}\right)  \tag{24}\\
& \times x\left(k+m-d_{k+m \mid k}^{S_{2} C_{2}} \mid k\right),
\end{align*}
$$

where $d_{k+m \mid k}^{S_{2} C_{2}}$ is model-dependent time invariable delay on $S_{2} / C_{2}$ side, which is $m$-step ahead prediction based on the measurement time $k . d_{k+m \mid k}^{C_{2} A}$ is the predicted quantity of packet dropouts at time $k+m$.

And the key of predictive controller rule is to compute the matrixes $F_{1}$ and $F_{2}\left(d_{k}^{S_{2} C_{2}}\right)$ in (14). The predicted controller at time $k+m$ based on the first control move $u(k \mid k)$ is

$$
\begin{align*}
u(k+m \mid k)= & C\left(d_{k+m \mid k}^{C_{2} A}\right) x(k+m \mid k) \\
& +D\left(d_{k+m \mid k}^{S_{2} C_{2}}, d_{k+m \mid k}^{C_{2} A}\right) x\left(k+m-d_{k+m \mid k}^{S_{2} C_{2}} \mid k\right) \tag{25}
\end{align*}
$$

where $C\left(d_{k}^{C_{2} A}\right)$ and $D\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right)$ are matrixes which contain the control gain matrixes $F_{1}$ and $F_{2}\left(d_{k}^{S_{2} C_{2}}\right)$, respectively.

Next we will introduce a method to solve the optimal control sequence which can minimize the following performance index at each sampling time:

$$
\begin{align*}
& \min _{u(k+m \mid k), m=0,1,2, \ldots, \infty} J_{\infty} \\
& =\sum_{m=0}^{\infty}[ \tag{26}
\end{align*} x^{\mathrm{T}}(k+m \mid k) Q x(k+m \mid k) .
$$

$Q$ and $R$ are symmetric positive definite weight matrixes.
In order to simplify, let $d_{k+m \mid k}^{S_{2} C_{2}}=v, d_{k+m \mid k}^{C_{2} A}=\varsigma, d_{k+m+1 \mid k}^{S_{2} C_{2}}=$ $v_{1}$.

First, consider a Lyapunov function candidate $V(X(k+$ $m \mid k), v)$ with $X(k+m \mid k)=\left[x^{\mathrm{T}}(k+m \mid k), x^{\mathrm{T}}(k+m-1 \mid\right.$ $\left.k), \ldots, x^{\mathrm{T}}(k+m-v \mid k)\right]^{\mathrm{T}}$ as follows:

$$
\begin{equation*}
V(X(k+m \mid k), v)=V_{1}+V_{2}+V_{3}, \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
& V_{1}(X(k+m \mid k), v)=x^{\mathrm{T}}(k+m \mid k) P(v) x(k+m \mid k), \\
& \quad V_{2}(X(k+m \mid k), v)=\sum_{l=k+m-v}^{k+m-1} x^{\mathrm{T}}(l \mid k) S x(l \mid k), \\
& V_{3}(X(k+m \mid k), v) \\
& \quad=\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+m+\theta}^{k+m-1} x^{\mathrm{T}}(l \mid k) S x(l \mid k) . \tag{28}
\end{align*}
$$

Matrixes $P(v)$ and $S$ are positive definite with appropriate dimensions.

Second, suppose that at sampling time $k, V(X(k+m$ | $k), v$ ) satisfies (29) for any $x(k+m \mid k)$ and $u(k+m \mid k)$. Consider

$$
\begin{align*}
& V\left(X(k+m+1 \mid k), v_{1}\right)-V(X(k+m \mid k), v) \\
& \leq-\left[x^{\mathrm{T}}(k+m \mid k) Q x(k+m \mid k)\right.  \tag{29}\\
& \left.\quad+u^{\mathrm{T}}(k+m \mid k) R u(k+m \mid k)\right]
\end{align*}
$$

For the infinite performance index $J_{\infty}(k)$ to be finite, we must set $x(\infty \mid k)=0$; hence, $V\left(x(\infty \mid k), d_{\infty \mid k}^{S_{2} C_{2}}\right)=0$. Summing (29) from $m=0$ to $m=\infty$, it can be obtained that

$$
\begin{equation*}
J_{\infty}(k) \leq V\left(X(k \mid k), d_{k \mid k}^{S_{2} C_{2}}\right) \tag{30}
\end{equation*}
$$

Therefore, the infinite optimization problem at time $k$ has transformed into optimizing upper function $V(X(k \mid$ $k), d_{k \mid k}^{S_{2} C_{2}}$ ) at each sampling time $k$. As a standard in MPC, only the first control move $u(k \mid k)$ is implemented, and at the next sampling time $k+1$, the state $x(k+1)$ is measured and the optimization is repeated to compute matrixes $F_{1}$ and $F_{2}\left(d_{k}^{S_{2} C_{2}}\right)$.

Third, a sufficient condition for existing of a set of control sequence is

$$
\begin{equation*}
J_{\infty}<\Upsilon \tag{31}
\end{equation*}
$$

where $\Upsilon$ is a suitable nonnegative coefficient to be minimized.
Before proceeding, the following lemma needs to be introduced.

Lemma 2 (Schur Complement Lemma). Given constant matrixes $Z_{1}, Z_{2}, Z_{3}$, where $Z_{1}=Z_{1}^{\mathrm{T}}, Z_{2}=Z_{2}^{\mathrm{T}}$, then $Z_{1}+$ $Z_{3}^{\mathrm{T}} Z_{2}^{-1} Z_{3}<0$ holds if $\left(\begin{array}{cc}Z_{1} & Z_{3}^{\mathrm{T}} \\ Z_{3} & -Z_{2}\end{array}\right)<0$ or $\left(\begin{array}{cc}-Z_{2} & Z_{3} \\ Z_{3}^{\mathrm{T}} & Z_{1}\end{array}\right)<0$.
Assumption 3. The values of $d_{k}^{S_{2} C_{2}}$ and $\alpha$ are known to the controller.

Theorem 4. Suppose the states $x(k \mid k), x(k-1 \mid k), \ldots, x(k-$ $\left.d_{M} \mid k\right)$ of system (17) are measured; then, the control rule in (18) makeing (29) and $J_{\infty}<\Upsilon$ hold, if existing matrixes $P(v)>$ $0, S>0, C(\varsigma), X(v)>0, D(v, \varsigma), F$, and scalar $\Upsilon>0$ make the following optimization feasible:

$$
\begin{equation*}
\min \Upsilon \tag{32}
\end{equation*}
$$

subject to

$$
\begin{gather*}
x_{k+m \mid k}^{\mathrm{T}} P(v) x_{k+m \mid k}+\sum_{l=k+m-d_{k}^{S C}}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} \\
+\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+m+\theta}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} \leq \Upsilon,  \tag{33}\\
\left(\begin{array}{cccc}
-P(v)+(1+\mu) S+Q & 0 & A^{\mathrm{T}}(\varsigma) & C^{\mathrm{T}}(\varsigma) \\
0 & -S & B^{\mathrm{T}}(v, \varsigma) & D^{\mathrm{T}}(v, \varsigma) \\
A(\varsigma) & B(v, \varsigma) & -\widehat{P}^{-1}(v) & 0 \\
C(\varsigma) & D(v, \varsigma) & 0 & -R^{-1}
\end{array}\right)<0, \tag{34}
\end{gather*}
$$

or

$$
\begin{gather*}
\left(\begin{array}{cccc}
-P(v)+(1+\mu) S+Q & * & * & * \\
0 & -S & * & * \\
\widehat{A}(\varsigma) & \widehat{B}(\varsigma) & -\Lambda & * \\
\widetilde{I}(\varsigma)+\widetilde{J}(\varsigma) F \widetilde{H}_{1} & \widetilde{K}(\varsigma) F \widetilde{H} & 0 & -R^{-1}
\end{array}\right)<0 \\
\left(\begin{array}{cc}
P & I \\
I & X
\end{array}\right) \geq 0,  \tag{35}\\
P X=I
\end{gather*}
$$

where $\pi_{m}=\min \left\{\pi_{i i}, i \in \chi_{1}\right\}, \widehat{P}(v)=\sum_{j=d_{m}, j \neq i}^{d_{M}} \pi_{i j} P(j), i$ denotes the quantity of packet dropouts, $i \in \chi_{1}, \boldsymbol{\chi}_{1}=\{0,1$, $\left.2, \ldots, d_{M}\right\}, \mu=d_{M}\left(1-\pi_{m}\right), \widehat{A}(\varsigma)=\left[\sqrt{\pi_{i 0}} A(\varsigma), \sqrt{\pi_{i 1}} A(\varsigma), \ldots\right.$, $\left.\sqrt{\pi_{i d_{M}}} A(\varsigma)\right]^{\mathrm{T}}, \widehat{B}(\varsigma)=\left[\sqrt{\pi_{i 0}} B(\varsigma), \sqrt{\pi_{i 1}} B(\varsigma), \ldots, \sqrt{\pi_{i d_{M}}} B(\varsigma)\right]^{\mathrm{T}}$, $\Lambda=\operatorname{diag}\left(X_{0}, X_{1}, \ldots, X_{i}\right), \quad i \quad \in \quad\left(0,1,2, \ldots, d_{M}\right)$, $X(v)=P^{-1}(v), \Theta=\operatorname{diag}\left[-\left[1+\left(1-\pi_{m}\right)\left(d_{M}-1\right) S\right]^{-1}\right.$, $\left.-\left[1+\left(1-\pi_{m}\right)\left(d_{M}-2\right) S\right]^{-1}, \ldots,-\left[\left(1-\pi_{m}\right) S\right]^{-1}\right], x_{p}=$ $\left[x(k-1 \mid k), x(k-2 \mid k), \ldots, x\left(k-d_{M}+1 \mid k\right)\right]^{\mathrm{T}}$, $F=\left[F_{1}, F_{2}\right]$.

In order to express conveniently, let $x_{k+m \mid k}=x(k+m \mid$ $k), x_{k+m-v \mid k}=x(k+m-v \mid k), u_{k+m \mid k}=u(k+m \mid k)$, $x_{l \mid k}=x(l \mid k)$. Thus, (24) becomes $x_{k+m+1 \mid k}=A(\varsigma) x_{k+m \mid k}+$ $B(v, \varsigma) x_{k+m-v \mid k}$.

The proof of the previous theorem is divided into the following two steps: Step 1, we design a Lyapunov function $V(X(k+m \mid k), v)$ for the systems; Step 2, an optimal controller is designed such that the closed-loop systems are stable and the optimal characteristics are satisfied.

## Proof

Step 1. Consider a Lyapunov function candidate $V(X(k+m)$ $k), v)=V_{1}+V_{2}+V_{3}$ with $\mathbf{x} \neq 0$, where

$$
\begin{align*}
& V_{1}(X(k+m \mid k), v)=x_{k+m \mid k}^{\mathrm{T}} P(v) x_{k+m \mid k},  \tag{36}\\
& V_{2}(X(k+m \mid k), v)=\sum_{l=k+m-v}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k}, \\
& V_{3}(X(k+m \mid k), v)=\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+m+\theta}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k}, \\
& \Delta V_{1}=\mathrm{E}\left[V_{1}\left(X_{k+m+1 \mid k}, v\right)\right]-V_{1}\left(X_{k+m \mid k}, v\right) \\
& =\mathrm{E}\left[x_{k+m+1 \mid k}^{\mathrm{T}} P(v) x_{k+m+1 \mid k}\right]-x_{k+m \mid k}^{\mathrm{T}} P(v) x_{k+m \mid k} \\
& =x_{k+m+1 \mid k}^{\mathrm{T}} \widehat{P}(v) x_{k+m+1 \mid k}-x_{k+m \mid k}^{\mathrm{T}} P(v) x_{k+m \mid k} \\
& =\eta_{k+m \mid k}^{\mathrm{T}}\left(\begin{array}{cc}
A^{\mathrm{T}}(\varsigma) \widehat{P}(v) & A^{\mathrm{T}}(\varsigma) \widehat{P}(v) B(v, \varsigma) \\
x A(\varsigma)-P(v) & B^{\mathrm{T}}(v, \varsigma) \widehat{P}(v) B(v, \varsigma)
\end{array}\right) \eta_{k+m \mid k}, \tag{37}
\end{align*}
$$

where $\eta_{k+m \mid k}=\left(\begin{array}{ll}x_{k+m \mid k} & x_{k+m-v}\end{array}\right)^{\mathrm{T}}$, and E is the mathematical expectation. Consider

$$
\begin{align*}
\Delta V_{2}= & \mathrm{E}\left[V_{2}\left(X_{k+m+1 \mid k}, v_{1}\right)\right]-V_{2}\left(X_{k+m \mid k}, v\right) \\
= & \mathrm{E}\left[\sum_{l=k+m+1-d_{k}^{S C}}^{k+m} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k}\right]-\sum_{l=k+m-d_{k}^{S C}}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} \\
= & x_{k+m \mid k}^{\mathrm{T}} S x_{k+m \mid k}-x_{k+m-d_{k}^{\mathrm{SC}} \mid k} S x_{k+m-d_{k}^{S C} \mid k}^{\mathrm{T}}  \tag{38}\\
& +\sum_{j=0, i \neq j}^{d_{M}} \pi_{i j}\left(\sum_{l=k+m+1-d_{k+1}^{\mathrm{SC}}}^{k+m-1}-\sum_{l=k+m-d_{k}^{S C}}^{k+m-1}\right) x_{l \mid k}^{\mathrm{T}} S x_{l \mid k},
\end{align*}
$$

and $\sum_{j=0, j \neq i}^{d_{M}} \pi_{i j}=1-\pi_{i i} \leq 1-\pi_{m}, 0 \leq v \leq d_{M}$; thus,

$$
\begin{align*}
\Delta V_{2}= & \mathrm{E}\left[V_{2}\left(X_{k+m+1 \mid k}, v_{1}\right)\right]-V_{2}\left(X_{k+m \mid k}, v\right) \\
\leq & x_{k+m \mid k}^{\mathrm{T}} S x_{k+m \mid k}-x_{k+m-v \mid k}^{\mathrm{T}} S x_{k+m-v \mid k}  \tag{39}\\
& +\left(1-\pi_{m}\right) \sum_{l=k+m-d_{M}+1}^{k+m} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} .
\end{align*}
$$

Similarly,

$$
\begin{align*}
\Delta V_{3} & =\mathrm{E}\left[V_{3}\left(X_{k+m+1 \mid k}, v_{1}\right)\right]-V_{3}\left(X_{k+m \mid k}, v\right) \\
& \leq d_{M}\left(1-\pi_{m}\right) x_{l \mid k}^{\mathrm{T}} S x_{l \mid k}-\left(1-\pi_{m}\right) \sum_{l=k+m-d_{M}+1}^{k+m} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} . \tag{40}
\end{align*}
$$

Therefore, $\Delta V=\Delta V_{1}+\Delta V_{2}+\Delta V_{3}$ and

$$
\leq \eta_{k+m \mid k}^{\mathrm{T}}\left(\begin{array}{cc}
A^{\mathrm{T}}(\varsigma) \widehat{P}(v) A(\varsigma) & A^{\mathrm{T}}(\varsigma) \widehat{P}(v) B(v, \varsigma)  \tag{A1}\\
-P(v)+(1+\mu) S & B^{\mathrm{T}}(v, \varsigma) \widehat{P}(v) B(v, \varsigma)-S
\end{array}\right) \eta_{k+m \mid k}
$$

where $*$ denotes the block determined by symmetry.
Substituting (25) and (A1) into (29), we have

$$
\begin{gather*}
V\left(X_{k+m+1 \mid k}, v_{1}\right)-V\left(X_{k+m \mid k}, v\right)+x_{k+m \mid k}^{\mathrm{T}} Q x_{k+m \mid k} \\
+u_{k+m \mid k}^{\mathrm{T}} R u_{k+m \mid k} \leq \eta_{k+m \mid k}^{\mathrm{T}} \Xi \eta_{k+m \mid k}, \\
\Xi=\left(\begin{array}{cr}
\Phi & A^{\mathrm{T}}(\varsigma) \widehat{P}(v) B(v, \varsigma)+C^{\mathrm{T}}(\varsigma) R D(v, \varsigma) \\
* & B^{\mathrm{T}}(v, \varsigma) \widehat{P}(v) B(v, \varsigma)-S+D^{\mathrm{T}}(v, \varsigma) R D(v, \varsigma)
\end{array}\right), \tag{41}
\end{gather*}
$$

where $\Phi=A^{\mathrm{T}}(\varsigma) \widehat{P}(v) A(\varsigma)-P(v)+(1+\mu) S+Q+C^{\mathrm{T}}(\varsigma) R C(\varsigma)$.
According to Lemma 2, we obtain

$$
\begin{align*}
& \Xi= \\
& \left(\begin{array}{cccc}
-P(v)+(1+\mu) S+Q & 0 & A^{\mathrm{T}}(\varsigma) & C^{\mathrm{T}}(\varsigma) \\
0 & -S & B^{\mathrm{T}}(v, \varsigma) & D^{\mathrm{T}}(v, \varsigma) \\
A(\varsigma) & B(v, \varsigma) & -\widehat{P}^{-1}(v) & 0 \\
C(\varsigma) & D(v, \varsigma) & 0 & -R^{-1}
\end{array}\right) . \tag{42}
\end{align*}
$$

According to (34), then

$$
\begin{equation*}
\eta_{k+m \mid k}^{\mathrm{T}} \mathrm{\Xi} \eta_{k+m \mid k}<0 . \tag{43}
\end{equation*}
$$

Hence, (29) holds.
And

$$
\begin{align*}
& x_{k+m \mid k}^{\mathrm{T}} P(v) x_{k+m \mid k}+\sum_{l=k+m-v}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k}  \tag{44}\\
& \quad+\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+m+\theta}^{k+m-1} x_{l \mid k}^{\mathrm{T}} S x_{l \mid k} \leq \Upsilon .
\end{align*}
$$

That is,

$$
\begin{equation*}
V\left(X_{k+m \mid k}, v\right) \leq \Upsilon \tag{45}
\end{equation*}
$$

From (30), we have

$$
\begin{equation*}
J_{\infty}(k)<\Upsilon . \tag{46}
\end{equation*}
$$

Step 2. A congruence transformation to (17) and (18) leads to

$$
\begin{array}{ll}
A(\varsigma)=\widetilde{D}(\varsigma)+\widetilde{E}(\varsigma) F \widetilde{H}_{1}, & B(v, \varsigma)=\widetilde{E}(\varsigma) F \widetilde{H},  \tag{A2}\\
C(\varsigma)=\widetilde{I}(\varsigma)+\widetilde{J}(\varsigma) F \widetilde{H}_{1}, & D(v, \varsigma)=\widetilde{K}(\varsigma) F \widetilde{H},
\end{array}
$$

where

$$
\widetilde{D}(\varsigma)=\left(\begin{array}{ccc}
\Phi_{1} & \Gamma_{1} C_{2} & 0 \\
0 & \Phi_{2} & \Gamma_{2} \alpha \\
0 & 0 & \alpha
\end{array}\right)
$$

$$
\widetilde{E}(\varsigma)=\left(\begin{array}{c}
0 \\
\Gamma_{2}[1-\alpha] \\
{[1-\alpha]}
\end{array}\right), \quad F=\left[F_{1}, F_{2}\right]
$$

$$
\widetilde{H}=\left(\begin{array}{lll}
0 & 0 & 0  \tag{47}\\
0 & \mathrm{I} & 0
\end{array}\right), \quad \widetilde{I}(\varsigma)=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & \alpha
\end{array}\right),
$$

$$
\widetilde{J}(\varsigma)=\binom{\mathrm{I}}{[1-\alpha]}, \quad \widetilde{H}_{1}=\left(\begin{array}{lll}
\mathrm{I} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

$$
\widetilde{K}(\varsigma)=\binom{0}{[1-\alpha]} .
$$

Substituting (A2) into (34), (35) can be obtained.
The proof of Theorem 4 is completed.
Remark 5. According to the proposed method, the optimal input is unique. $\Upsilon$ is a scalar and $\Upsilon>0$. Unique resolution is given finally by using Toolbox of Matlab. Its convergence can be found in Figure 3.

## 4. Stability Analysis of NCSs with Packet Dropouts

Theorem 6. Suppose that the optimization problem in Theorem 4 is feasible at time $k$; then, the new resulting closedloop systems in (17) are asymptotically stable by the optimal control law in (18) which is obtained from Theorem 4.


Figure 2: The number of data packet dropouts of $S$ to $C$ and $C_{2}$ to $A$.


Figure 3: The upper of performance index.

Proof. First, the Lyapunov function, $V(X(k+m \mid k), v)=$ $V_{1}+V_{2}+V_{3}$ is discussed and $V\left(X(k+m \mid k), d_{k+1 \mid k}^{S_{2} C_{2}}\right)-V(X(k \mid$ $\left.k), d_{k \mid k}^{S_{2} C_{2}}\right)<0$ holds only when $x \neq 0$.

Next, in the following we will consider the situation at time $k$ (i.e., $m=0$ ):

$$
\begin{align*}
V\left(X(k \mid k), d_{k \mid k}^{S C}\right)= & x_{k \mid k}^{\mathrm{T}} P_{k} x_{k \mid k}+\sum_{l=k-d_{k \mid k}^{S C}}^{k-1} x_{l \mid k}^{\mathrm{T}} S_{k} x_{l \mid k} \\
& +\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+\theta}^{k-1} x_{l \mid k}^{\mathrm{T}} S_{k} x_{l \mid k} \tag{48}
\end{align*}
$$

where $P_{k}>0, S_{k}>0$ are obtained from the optimal solution at time $k$. Suppose that the optimization problem in Theorem 4 is feasible at time $k$, and according to the introduction in [8], these optimization problems are also
feasible for all $k+1$. Let us note values of $P_{k}>0, S_{k}>0$, $F_{1 k}, F_{2 k}, \Upsilon_{k}$, and $P_{k+1}>0, S_{k+1}>0, F_{1 k+1}, F_{2 k+1}$ obtained from the optimal solution at time $k$ and $k+1$, respectively. Therefore, we have

$$
\begin{align*}
& x_{k+1 \mid k+1}^{\mathrm{T}} P_{k+1} x_{k+1 \mid k+1}+\sum_{l=k+1-d_{k+1 \mid k+1}^{S C}}^{k} x_{l \mid k+1}^{\mathrm{T}} S_{k+1} x_{l \mid k+1} \\
& \quad+\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+1+\theta}^{k} x_{l \mid k+1}^{\mathrm{T}} S_{k+1} x_{l \mid k+1}  \tag{49}\\
& \leq x_{k+1 \mid k+1}^{\mathrm{T}} P_{k} x_{k+1 \mid k+1}+\sum_{l=k+1-d_{k \mid k}^{S C}}^{k} x_{l \mid k+1}^{\mathrm{T}} S_{k} x_{l \mid k+1} \\
& \quad+\left(1-\pi_{m}\right) \sum_{\theta=-d_{M}+1}^{0} \sum_{l=k+1+\theta}^{k} x_{l \mid k+1}^{\mathrm{T}} S_{k} x_{l \mid k+1}
\end{align*}
$$

That is because $P_{k+1}>0, S_{k+1}>0$ are optimal, whereas $P_{k}>$ $0, S_{k}>0$ are only feasible at time $k$.

From (29), it can be obtained that

$$
\begin{align*}
V & \left(X(k+1 \mid k), d_{k+1 \mid k}^{S_{2} C_{2}}\right)-V\left(X(k \mid k), d_{k \mid k}^{S_{2} C_{2}}\right) \\
& \leq-\left[x^{\mathrm{T}}(k \mid k) Q x(k \mid k)+u^{\mathrm{T}}(k \mid k) R u(k \mid k)\right] \\
& \leq-x^{\mathrm{T}}(k \mid k) Q x(k \mid k) \leq-\lambda_{\min }(Q)\left\{\|x(k \mid k)\|^{2}\right\} . \tag{50}
\end{align*}
$$

Because $Q$ is symmetric positive definite, we have $\lambda_{\text {min }}(Q)>0$ (the minimal eigenvalue of matrix $Q$ ), and then

$$
\begin{equation*}
V\left(X(k+1 \mid k), d_{k+1 \mid k}^{S_{2} C_{2}}\right)-V\left(X(k \mid k), d_{k \mid k}^{S_{2} C_{2}}\right)<0 \tag{51}
\end{equation*}
$$

When $m=0$, (24) is becoming

$$
\begin{align*}
x(k+1 \mid k)= & A\left(d_{k \mid k}^{C_{2} A}\right) x(k \mid k) \\
& +B\left(d_{k \mid k}^{S_{2} C_{2}}, d_{k \mid k}^{C_{2} A}\right) x\left(k-d_{k \mid k}^{S_{2} C_{2}} \mid k\right) . \tag{52}
\end{align*}
$$

Because of the measured state,

$$
\begin{align*}
x & (k+1 \mid k+1) \\
& =x(k+1) \\
& =A\left(d_{k}^{C_{2} A}\right) x(k)+B\left(d_{k}^{S_{2} C_{2}}, d_{k}^{C_{2} A}\right) x\left(k-d_{k}^{S_{2} C_{2}}\right) . \tag{53}
\end{align*}
$$

So we have

$$
\begin{equation*}
x(k+1 \mid k+1)=x(k+1 \mid k) . \tag{54}
\end{equation*}
$$

Comparing with (A1), it can be obtained that

$$
\begin{equation*}
V\left(X(k+1 \mid k+1), d_{k+1 \mid k+1}^{S_{2} C_{2}}\right)-V\left(X(k \mid k), d_{k \mid k}^{S_{2} C_{2}}\right)<0 \tag{55}
\end{equation*}
$$



Figure 4: (a) State response of state $x_{2}$ with designed optimal controller, (b) state response of state $x_{1}$ with designed optimal controller, (c) state trajectories of state $x_{2}$, (d) state trajectories of state $x_{1}$, (e) state response of state $x_{2}$, and (f) state trajectories of state $x_{1}$.


Figure 5: The number of data packet dropouts of $S$ to $C$ side.

Therefore, the Lyapunov function $V\left(X(k \mid k), d_{k \mid k}^{S_{2} C_{2}}\right)$ is decreasing for the new closed loop, and $\lim _{k \rightarrow \infty} x(k \mid k)=0$ is concluded. The stochastic stability is obtained.

## 5. Simulation Examples

In order to show the effectiveness of the proposed method, we will give some cases and simulations.
5.1. A Simple Example. Considering the coefficient matrixes of $P_{1}$ and $P_{2}$

$$
\begin{gather*}
\Phi_{1}=\left(\begin{array}{cc}
0.9512 & 1.051 \\
-0.6065 & -1.1618
\end{array}\right), \quad \Gamma_{1}=\binom{8.5530}{29.4735},  \tag{56}\\
C_{1}=\left(\begin{array}{ll}
-0.0045 & 0.1004
\end{array}\right), \\
\Phi_{2}=\left(\begin{array}{cc}
-0.3667 & -0.0111 \\
1 & 0
\end{array}\right), \quad \Gamma_{2}=\binom{1}{0},  \tag{57}\\
C_{2}=\left(\begin{array}{ll}
0 & 0.0111
\end{array}\right) . \tag{58}
\end{gather*}
$$

Markov models of the whole closed control systems are consisted. Suppose that $d_{k}^{S_{2} C_{2}}=i \in(0,1)$ and $d_{k}^{C_{2} A}=r \in$ $(0,1)$, as shown in Figures 2 and 5, which means that the number of data packet dropouts is $i$ at sensor to controller of secondary side and, similarly, is $r$ at secondary controller to actuator side at time $k$. In the method, $\Upsilon$ is a scalar and $\Upsilon>$ 0 . Its convergence is shown in Figure 3. Unique resolution is given finally by using Toolbox of Matlab. The simulation results under the optimal state feedback controller are shown in Figure 4.

From the simulation (1) Figure 3 is the upper of performance index. (2) In Figures 4(a) and 4(b), system state trajectories will eventually be more stable. It is shown that

NCSs with data packet dropouts are stable with the optimal controller which is designed by Theorem 4. (3) Through comparing to LQR controller, which is showed in Figures $4(\mathrm{c}), 4(\mathrm{~d}), 4(\mathrm{e})$, and $4(\mathrm{f})$, the system overshoots are shortened, and state trajectory paths are more superior. Thus, the optimal controller designed in our paper is effective.
5.2. Industrial Systems. In order to verify the method proposed earlier, a networked control system for main furnace temperature in industrial systems is taken for an example. It is assumed that the transfer functions of the inertial and leading sections are $G_{p 1}(s)=(1 /[(30 s+1)(3 s+1)])$ and $G_{p 2}(s)=$ $\left(1 /\left[(s+1)^{2}(10 s+1)\right]\right)$, respectively. After discretization, the following state space models are available:

$$
\begin{gather*}
P_{1}:\left\{\begin{array}{r}
x_{1}(k+1)=\left(\begin{array}{rr}
0.6887 & -0.0093 \\
0.8356 & 0.9951
\end{array}\right) x_{1}(k) \\
+\binom{0.8356}{0.4437} y_{2}(k), \\
y_{1}(k)=\left(\begin{array}{ll}
0 & 0.0111)
\end{array} x_{1}(k),\right.
\end{array}\right. \\
P_{2}:\left\{\begin{array}{r}
-0.0342
\end{array}-0.4364\right.  \tag{59}\\
x_{2}(k+1)=\left(\begin{array}{rrr}
0.3425 & 0.6849 & -0.0342 \\
0.2542 & 0.8762 & 0.9899
\end{array}\right) x_{2}(k) \\
+\left(\begin{array}{c}
0.3425 \\
0.2542 \\
0.1008
\end{array}\right) u_{2}(k), \\
y_{2}(k)=\left(\begin{array}{lll}
0 & 0 & 0.1
\end{array}\right) x_{2}(k) .
\end{gather*}
$$

Figures 6(a) and 6(b) can be obtained, from which the systems in (17) are stabilized by our designed controller. Through comparing to LQR controller, as shown in Figures $6(c), 6(d), 6(e)$, and 6(f), and the system overshoots shortened, state trajectories are more superior and performance of the control systems is further improved. Thus, the proposed optimal controller designed is effective and feasible.

## 6. Conclusions

In this paper, the modeling, optimal, and control problems for a class of NCSs under data packet dropout effect have been studied. The data packet dropouts are described by Markov chains. The Markov chains describe that the quantity of packet dropouts between current time $k$ and its latest communicates successfully instead of whether the data packets dropped or not. Though augmenting the state vectors, the resulting closed-loop control system is transformed into a jump system with time delays. Model predictive control method is applied to study optimization and stability problems of the resulting closed-loop NCSs. Sufficient conditions are proposed for the optimization and stability of the resulting closed-loop NCSs. Some simulations are given in the last section and it can be seen that the designed controllers are feasible and effective. In this work, the full state feedback controllers are designed. However, it is difficult to obtain


Figure 6: (a) State response of state $x_{2}$ with designed optimal controller, (b) state response of state $x_{1}$ with designed optimal controller, (c) state trajectories of state $x_{2}$, (d) state trajectories of state $x_{1}$, (e) state response of state $x_{2}$, and (f) state trajectories of state $x_{1}$.
the full state. Output feedback controllers should be considered. Dynamic output feedback can be investigated in the future work.

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## Research Article

# Stability Analysis of a Harvested Prey-Predator Model with Stage Structure and Maturation Delay 

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#### Abstract

A harvested prey-predator model with density-dependent maturation delay and stage structure for prey is proposed, where selective harvest effort on predator population is considered. Conditions which influence positiveness and boundedness of solutions of model system are analytically investigated. Criteria for existence of all equilibria and uniqueness of positive equilibrium are also studied. In order to discuss effects of maturation delay and harvesting on model dynamics, local stability analysis around all equilibria of the proposed model system is discussed due to variation of maturation delay and harvest effort level. Furthermore, global stability of positive equilibrium is investigated by utilizing an iterative technique. Finally, numerical simulations are carried out to show consistency with theoretical analysis.


## 1. Introduction

In the natural world, many species have a life history that takes them through two stages, juvenile stage and adult stage. Individuals in each stage are identical in biological characteristics, and some vital rates (rates of survival, development, and reproduction) of individuals in a population almost always depend on stage structure. Furthermore, many complex biological phenomena arising in prey-predator ecosystem always depend on the past history of system, and it has been recognized that time delay may have complicated impact on dynamics of prey-predator ecosystem [1]. In the past several decades, there has been an increasing interest in prey-predator model system with stage structure and time delay (see [2-26] and the references therein).

In the model proposed by Aiello and Freedman [2], stage structure of single population growth with stage structure and time delay representing for maturation of population is considered. Their model predicts a positive steady state as the global attractor, thereby suggesting that stage structure does not generate sustained oscillations frequently observed
in single population in the real world. Subsequent work made by other authors [ $3,6,7,12-14$ ] suggests that time delay to adulthood should be state dependent. Generally, boundedness and persistence of solutions of model system may be affected by introduction of time delay into preypredator system with stage structure $[14,15,20-22,24-$ 26]. Time delay can also cause loss of stability and other complicated dynamical behavior [27]. Especially, there is a well-developed theory of stage-structured models which incorporate time delay into maturity of population [4].

It is well known that harvesting has a strong impact on dynamic evolution of a population; there has been considerable interest in the modeling of harvesting of biological resources [1]. In these models, the harvesting effort is considered to be a dynamic variable; several kinds of harvesting policies are utilized to study the dynamical behavior of the model system. In recent years, there has been growing interest in the study of stage-structured prey-predator system with harvesting. Several prey-predator models with stage structure and harvest effort on predator have been investigated in [2833] and the references therein.

Recently, Huo et al. [24] investigated dynamical behavior and stability of the following stage-structured system with time delay:

$$
\begin{gather*}
\dot{x}_{1}(t)=r_{1} x_{2}(t)-d x_{1}(t)-r_{1} e^{-d \tau} x_{2}(t-\tau) \\
\dot{x}_{2}(t)=r_{1} e^{-d \tau} x_{2}(t-\tau)-b x_{2}^{2}(t)-\frac{a_{1} y(t) x_{2}(t)}{x_{2}(t)+k_{1}},  \tag{1}\\
\dot{y}(t)=y(t)\left(r_{2}-\frac{a_{2} y(t)}{x_{2}(t)+k_{2}}\right)
\end{gather*}
$$

where $x_{1}(t), x_{2}(t)$, and $y(t)$ represent the density of immature prey population, mature prey population and predator population, at time $t$, respectively; $r_{1}$ is the intrinsic growth rate of mature prey population, and $d$ is the death rate of immature prey population. Constant $\tau \geq 0$ denotes maturation delay of immature prey population to mature prey population, and the term $r_{1} e^{-d \tau} x_{2}(t-\tau)$ represents the immature prey population who were born at time $t-\tau$ and survived at time $t . b$ denotes the intracompetition rate for mature prey population due to overcrowding phenomenon with mature prey population. $a_{1}$ is the maximum value of the per capita reduction rate of mature prey population due to predator population, and $a_{2}$ is the maximum value of the per capita reduction rate of predator population due to mature prey population. $k_{1}$ measures the extent to which the environment provides protection to mature prey population, and $k_{2}$ measures the extent to which the environment provides protection to predator population. $r_{2}$ represents the maximal per capita growth rate of predator population. All the parameters mentioned previously are all positive constants. Furthermore, global stability of positive equilibrium of model system (1) is investigated in [26].

It is well known that the length of time for prey population to maturity is density dependent; that is, maturation time depends on the total population amount of prey population within prey predator ecosystem, and prey population takes less time to reach maturity with depletion of predator population [23,34-36]. Density-dependent maturity of population in prey predator ecosystem is discussed in their work, which reveals that density-dependent effects of the predators' counterparts to prey defenses and the density dependence effect of each type of predator offense are analogous to the corresponding type of prey defense. Dynamical behavior and stability switch is investigated in [23, 34-36]. However, harvest effort on population within prey-predator ecosystem is not considered in [23,34-36].

By assuming maturity delay of prey population is density dependent and predator population is harvested; work done in [24] is extended in this paper, and a harvested prey predator model with density-dependent maturation delay and stage structure for prey population is proposed in the second section of this paper. In the third section of this paper, positiveness and boundedness of solution of the proposed model are studied, and the conditions for existence of equilibria and uniqueness of positive equilibrium are also investigated. Local stability analysis around all equilibria is discussed due to variation of maturation delay as well as harvest effort level. Furthermore, global stability of the positive equilibrium of the proposed model system is studied
by utilizing an iterative technique. In the fourth section of this paper, numerical simulations are carried out to show consistency with theoretical analysis. Finally, this paper ends with a conclusion.

## 2. Model Formulation

Based on the previous analysis, the model proposed by Huo et al. in [24] is extended by incorporating harvest effort on predator population and assuming that maturation delay of prey population is density dependent, and the model can be governed by the following differential equations:

$$
\begin{gather*}
\dot{x}_{1}(t)=r_{1} x_{2}(t)-d x_{1}(t)-r_{1} e^{-d \tau(z(t))} x_{2}(t-\tau(z(t))), \\
\dot{x}_{2}(t)=r_{1} e^{-d \tau(z(t))} x_{2}(t-\tau(z(t)))-b x_{2}^{2}(t)-\frac{a_{1} y(t) x_{2}(t)}{x_{2}(t)+k_{1}}, \\
\dot{y}(t)=y(t)\left(r_{2}-\frac{a_{2} y(t)}{x_{2}(t)+k_{2}}\right)-q E y(t) . \tag{2}
\end{gather*}
$$

The initial conditions for model system (2) take the following form:

$$
\begin{gather*}
x_{1}(0)>0, \quad y(0)>0 \\
x_{2}(\theta)=\phi(\theta)>0, \quad \theta \in[-\widehat{\tau}, 0) \tag{3}
\end{gather*}
$$

where $z(t)=x_{1}(t)+x_{2}(t)+y(t)$, a scalar $E \geq 0$ denotes the harvesting effort to predator population, constant $q$ is the catchability coefficient of predator, and the harvesting term $q E y(t)$ follows the catch per unit effort hypothesis [1]. Furthermore, $r_{1}, r_{2}, d, b, a_{1}, a_{2}, k_{1}$, and $k_{2}$ in model system (2) share the same interpretations mentioned in model system (1).

In the following section of this paper, model system (2) is derived under the following hypotheses.
(H1) Prey population is divided into two-stage groups, that is, immature and mature. The term $r_{1} e^{-d \tau(z(t))} x_{2}(t-$ $\tau(z(t)))$ represents the immature prey population born at time $t-\tau(z(t))$ and survive at time $t$ with death rate $d$, which represents transformation term from immature prey to mature prey.
(H2) Density-dependent time delay $\tau(z(t))$ is taken to be an increasing differentiable bounded function of the total population (immature prey, mature prey, and predator population), which satisfies

$$
\begin{gather*}
\frac{d[\tau(z(t))]}{d t} \geq 0, \quad 0 \leq \tau_{0} \leq \tau(z(t)) \leq \tau_{1}  \tag{4}\\
\lim _{z(t) \rightarrow 0+} \tau(z(t))=\tau_{0}, \quad \lim _{z(t) \rightarrow+\infty} \tau(z(t))=\tau_{1}
\end{gather*}
$$

(H3) For the continuity of initial conditions, it is required that

$$
\begin{equation*}
x_{1}(0)=\int_{-\hat{\tau}}^{0} r_{1} e^{d s} \phi(s) d s \tag{5}
\end{equation*}
$$

where $\phi(s)$ is assumed to be continuous function (for mathematical reason) and nonnegative (for biological reason).
(H4) In order to exclude the possibility of immature prey becoming mature prey except by birth, $t-\tau(z(t))$ is assumed to be a strictly increasing function of $t$. Otherwise, there are two different times at which the same individual immature prey turns to be mature prey twice at the same instant of time, which is absurd to practical biological interpretations. (For detailed methodology, see [3].)

## 3. Qualitative Analysis of Model System

In this section, positiveness and boundedness of solution of model system (2) are analytically investigated. Criteria for existence of equilibria and uniqueness of positive equilibrium are also studied. By using differential dynamical system theory and stability theory, local stability analysis around all equilibria of model system is discussed. Furthermore, global stability of the positive equilibrium of the proposed model system is studied by utilizing an iterative technique.

### 3.1. Positiveness and Boundness of Solutions

Theorem 1. Under hypotheses (H1)-(H4), solutions of model system (2) with given initial conditions are positive for all $t>0$.

Proof. Assume that there exists $t_{0}=\inf \left\{t>0 \mid x_{2}(t)=\right.$ $0\}$. Based on the continuity $t_{0}>0$, it can be computed by evaluating the model system (2) at time $t_{0}$ :

$$
\dot{x}_{2}\left(t_{0}\right)= \begin{cases}r_{1} e^{-d \tau\left(z\left(t_{0}\right)\right)} \phi\left(t_{0}-\tau\left(z\left(t_{0}\right)\right)\right), & 0 \leq t_{0} \leq \widehat{\tau}  \tag{6}\\ r_{1} e^{-d \tau\left(z\left(t_{0}\right)\right)} x_{2}\left(t_{0}-\tau\left(z\left(t_{0}\right)\right)\right), & t_{0}>\widehat{\tau}\end{cases}
$$

According to (6) and the initial conditions of model system (2), it is easy to show that $\dot{x}_{2}\left(t_{0}\right)>0$. On the other hand, it follows from the definition of $t_{0}$ that $\dot{x}_{2}\left(t_{0}\right)=0$, which is a contradiction. Consequently, $x_{2}(t)>0$ for all $t>0$.

Based on the positiveness of $x_{2}(t)$ and the third equation of model system (2), it is easy to show that $y(t)>0$ for $y(0)>$ $0, t>0$.

Consider the equation

$$
\begin{gather*}
\dot{u}(t)=-d u(t)-r_{1} e^{-d \tau(z)} x_{2}(t-\tau(z)),  \tag{7}\\
u(0)=x_{1}(0)>0 .
\end{gather*}
$$

It is obvious to show that $\dot{u}(t)<0$; that is, $u(t)$ is strictly decreasing. By virtue of positiveness of $x_{2}(t), y(t)$, it derives that

$$
\begin{equation*}
x_{1}(t)>u(t), \quad 0<t \leq \widehat{\tau} . \tag{8}
\end{equation*}
$$

By solving (7), it gives that

$$
\begin{equation*}
u(t)=e^{-d t} u(0)-r_{1} e^{-d t} \int_{0}^{t} e^{d q} e^{-d \tau(z(q))} x_{2}(q-\tau(z(q))) d q \tag{9}
\end{equation*}
$$

According to (5), it derives that

$$
\begin{align*}
u(t)= & e^{-d t} \int_{-\hat{\tau}}^{0} r_{1} e^{d s} \phi(s) d s  \tag{10}\\
& -r_{1} e^{-d t} \int_{0}^{t} e^{d q} e^{-d \tau(z(q))} x_{2}(q-\tau(z(q))) d q .
\end{align*}
$$

By substituting $p=q-\tau(z(q))$ in the above equation, it can be obtained that

$$
\begin{align*}
u(t)= & r_{1} e^{-d t} \int_{-\hat{\tau}}^{0} e^{d s} \phi(s) d s \\
& -r_{1} e^{-d t} \int_{-\hat{\tau}}^{t-\tau(z(t))} \frac{e^{d p} x_{2}(p)}{1-\tau^{\prime}(z) \dot{z}(p)} d p \tag{11}
\end{align*}
$$

which implies that

$$
\begin{align*}
u(\hat{\tau})= & r_{1} e^{-d \hat{\tau}} \int_{-\hat{\tau}}^{0} e^{d s} \phi(s) d s \\
& -r_{1} e^{-d \hat{\tau}} \int_{-\hat{\tau}}^{\hat{\tau}-\tau(z(\hat{\tau}))} \frac{e^{d p} x_{2}(p)}{1-\tau^{\prime}(z) \dot{z}(p)} d p \tag{12}
\end{align*}
$$

According to $x_{2}(t)>0, y(t)>0$ for all $t>0$ and $t-$ $\tau(z(t))$ is an increasing function based on (H4). $1-\tau^{\prime}(z) \dot{z}(t)>$ 0 holds for $-\widehat{\tau} \leq t \leq \widehat{\tau}-\tau(z(\widehat{\tau}))$, and the following inequality can be obtained:

$$
\begin{equation*}
u(\hat{\tau}) \geq r_{1} e^{-d \hat{\tau}} \int_{-\hat{\tau}}^{0} e^{d p} \phi(p)\left(1-\frac{1}{1-\tau^{\prime}(z) \dot{z}(p)}\right) d p>0 \tag{13}
\end{equation*}
$$

since $u(\hat{\tau})>0$ and $u(t)$ is strictly decreasing, $x_{1}(t)>0,-\widehat{\tau} \leq$ $t \leq 0$. By repeating this argument to include all positive time, it can be shown that $x_{1}(t)>0$ for all $t>0$. Hence, solutions of model system (2) with given initial conditions are positive for all $t>0$.

Theorem 2. If the hypotheses (H1)-(H4) hold and $r_{2}>q E$, all solutions of model system (2) are bounded within a region $\Omega$ :

$$
\begin{align*}
\Omega=\{ & \left(x_{1}(t), x_{2}(t), y(t)\right) \mid \\
& \left.0<x_{1}(t)+x_{2}(t) \leq R_{1}, 0<y(t) \leq R_{2}\right\}, \tag{14}
\end{align*}
$$

where $R_{1}=\left(r_{1}+d\right)^{2} / 4 b d, R_{2}=\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right) / a_{2}$.
Proof. Let $v(t)=x_{1}(t)+x_{2}(t)$, and it is easy to show that $v(t)>0$ based on positiveness of solutions of model system (2). By calculating the derivative of $v(t)$ along the solutions, it gives that

$$
\begin{align*}
\dot{v}(t) & =r_{1} x_{2}(t)-d x_{1}(t)-b x_{2}^{2}(t)-\frac{a_{1} x_{2}(t) y(t)}{x_{2}(t)+k_{1}}  \tag{15}\\
& <\left(r_{1}+d\right) x_{2}(t)-b x_{2}^{2}-d v(t)
\end{align*}
$$

By using the standard comparison principle in (15), it derives that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup (v(t)) \leq \frac{\left(r_{1}+d\right)^{2}}{4 b d}=R_{1} \tag{16}
\end{equation*}
$$

It follows from the third equation of model system (2) that

$$
\begin{align*}
\dot{y}(t) & =\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{x_{2}(t)+k_{2}}  \tag{17}\\
& \leq\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{R_{1}+k_{2}}
\end{align*}
$$

By utilizing the standard comparison principle in inequality (17) and $r_{2}-q E>0$, it gives that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup y(t) \leq \frac{\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right)}{a_{2}}=R_{2} \tag{18}
\end{equation*}
$$

Consequently, all solutions of model system (2) are bounded within a region $\Omega$ :

$$
\begin{align*}
\Omega=\{ & \left(x_{1}(t), x_{2}(t), y(t)\right) \mid \\
& \left.0<x_{1}(t)+x_{2}(t) \leq R_{1}, 0<y(t) \leq R_{2}\right\}, \tag{19}
\end{align*}
$$

where $R_{1}=\left(r_{1}+d\right)^{2} / 4 b d, R_{2}=\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right) / a_{2}$.

### 3.2. Existence of Equilibria and Uniqueness of Positive Equilib-

 rium. The existence of biologically reasonable equilibria of model system (2) is investigated in this subsection. Since the biological interpretation of the positive equilibrium implies that immature prey, mature prey, and predator population all exist, uniqueness of positive equilibrium is also studied.By simple computation, there are two equilibria denoted by $P_{0}(0,0,0)$ and $P_{1}\left(0,0, k_{2}\left(r_{2}-q E\right) / a_{2}\right)$. The biological interpretations of $P_{0}, P_{1}$ are as follows. For $P_{0}(0,0,0)$, it implies that all population in harvested prey predator ecosystem does not exist. For $P_{1}\left(0,0, k_{2}\left(r_{2}-q E\right) / a_{2}\right)$, it implies that there is not any predation source for predator population. It follows from the previous biological interpretations that population in such ecosystem cannot be maintained at an ideal level for sustainable development, which are not relevant to major investigation in this paper.

Furthermore, there is one or more positive equilibria denoted by $P^{*}\left(x_{1}^{*}, x_{2}^{*}, y^{*}\right)$. In order to discuss the existence of $P^{*}$, it is equivalent to show that the following equations always have at least one positive solution:

$$
\begin{gather*}
r_{1} x_{2}-d x_{1}-r_{1} e^{-d \tau\left(x_{1}+x_{2}+y\right)} x_{2}=0 \\
r_{1} e^{-d \tau\left(x_{1}+x_{2}+y\right)}-b x_{2}-\frac{a_{1} y}{x_{2}+k_{1}}=0  \tag{20}\\
r_{2}-q E-\frac{a_{2} y}{x_{2}+k_{2}}=0
\end{gather*}
$$

It follows from (20) that

$$
\begin{gather*}
y=\frac{\left(r_{2}-q E\right)\left(x_{2}+k_{2}\right)}{a_{2}}=f\left(x_{2}\right)  \tag{21}\\
r_{1} x_{2}-d x_{1}=r_{1} x_{2} e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)}  \tag{22}\\
r_{1} e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)}=b x_{2}+a_{1} h\left(x_{2}\right) \tag{23}
\end{gather*}
$$

where $g\left(x_{2}\right)=x_{2}+y=x_{2}+f\left(x_{2}\right)$ and $h\left(x_{2}\right)=y /\left(x_{2}+k_{1}\right)=$ $f\left(x_{2}\right) /\left(x_{2}+k_{1}\right)$.

Theorem 3 (existence of positive equilibrium). Supposing that hypotheses (H1)-(H4) hold, if $k_{1} \geq k_{2}, r_{2}>q E$, and $a_{2} r_{1} e^{-d \tau_{1}}>a_{1} r_{2}$, then there exists at least one positive equilibrium $P^{*}$.

Proof. Let $\Gamma_{1}$ and $\Gamma_{2}$ be the solution curves of (22) and (23) for $x_{1} \geq 0, x_{2} \geq 0$, respectively. The analytical properties of curve $\Gamma_{1}$ and $\Gamma_{2}$ are as follows.

For $\Gamma_{1}$ : by simple computing, it can be found that $(0,0) \in$ $\Gamma_{1}$.

According to (H2) and positiveness of all solutions of model system (2), it is easy to show that $\lim _{x_{1} \rightarrow+\infty} \tau\left(x_{1}(t)+\right.$ $\left.g\left(x_{2}\right)\right)=\tau_{1}$, and

$$
\begin{align*}
\lim _{x_{1} \rightarrow+\infty} x_{2}\left(x_{1}\right) & =\lim _{x_{1} \rightarrow+\infty} \frac{d x_{1}}{r_{1}\left(1-e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right.}\right)} \\
& =\lim _{x_{1} \rightarrow+\infty} \frac{d x_{1}}{r_{1}\left(1-e^{-d \tau_{1}}\right)}=+\infty . \tag{24}
\end{align*}
$$

For $\Gamma_{2}$ : by differentiating $x_{2}$ against $x_{1}$ along $\Gamma_{2}$, it can be obtained that

$$
\begin{align*}
\frac{d x_{2}}{d x_{1}}= & -d r_{1} e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)} \\
& \times\left(d r_{1} e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)} \tau^{\prime}\left(x_{1}+g\left(x_{2}\right)\right) \frac{a_{2}+r_{2}-q E}{a_{2}}\right. \\
& \left.+b+\frac{a_{1}\left(r_{2}-q E\right)\left(k_{1}-k_{2}\right)}{a_{2}\left(x_{2}+k_{1}\right)^{2}}\right)^{-1} . \tag{25}
\end{align*}
$$

It can be shown that $\left(d x_{2} / d x_{1}\right)<0$, provided that $k_{1} \geq$ $k_{2}, r_{2}>q E$, and then $\Gamma_{2}$ is strictly decreasing.

Furthermore, according to $k_{1} \geq k_{2}, r_{2}>q E$, and $a_{2} r_{1} e^{-d \tau_{1}}>a_{1}\left(r_{2}-q E\right)$,

$$
\begin{equation*}
\lim _{x_{1} \rightarrow \infty} x_{2}\left(x_{1}\right)=\frac{1}{b}\left(r_{1} e^{-d \tau_{1}}-\frac{a_{1}\left(r_{2}-q E\right)\left(x_{2}+k_{2}\right)}{a_{2}\left(x_{2}+k_{1}\right)}\right)>0 . \tag{26}
\end{equation*}
$$

Based on the above analysis, $\Gamma_{1}$ and $\Gamma_{2}$ intersect at some positive values, which proves the existence of positive equilibrium $P^{*}$.

Theorem 4 (uniqueness of positive equilibrium). Supposing that hypotheses (H1)-(H4) hold, if the following inequality holds

$$
\begin{align*}
& 1- d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right) \\
& \times\left(a_{1} h^{\prime}\left(x_{2}^{*}\right)+d g^{\prime}\left(x_{2}^{*}\right) \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\right. \\
&\left.\times\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right)\right)+a_{1} d h^{\prime}\left(x_{2}^{*}\right)  \tag{27}\\
&+d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right) g^{\prime}\left(x_{2}^{*}\right)\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right)\right. \\
& \times\left(d+b+d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\right)>0,
\end{align*}
$$

then there exists a unique positive equilibrium.

Proof. Based on (22) and (23), $x_{2}$ can be defined as the function of $x_{1}$ :

$$
\begin{align*}
& \Gamma_{1}: x_{2}=g_{1}\left(x_{1}\right),  \tag{28}\\
& \Gamma_{2}: x_{2}=g_{2}\left(x_{1}\right) .
\end{align*}
$$

The positive equilibrium $P^{*}$ will be unique, provided that $g_{1}^{\prime}\left(x_{1}\right)>g_{2}^{\prime}\left(x_{1}\right)$ for every such $P^{*}$ otherwise reverse inequality holds.

By differentiating (22) with respect to $x_{1}$, it can be obtained that

$$
\begin{equation*}
g_{1}^{\prime}\left(x_{1}\right)=\frac{d+r_{1} e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)}\left(1-d \tau^{\prime}\left(x_{1}+g\left(x_{2}\right)\right)\right)}{r_{1}\left(1+e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)} d \tau^{\prime}\left(x_{1}+g\left(x_{2}\right)\right) g^{\prime}\left(x_{2}\right)\right)} . \tag{29}
\end{equation*}
$$

By differentiating (23) with respect to $x_{1}$, it can be obtained that

$$
\begin{align*}
& g_{2}^{\prime}\left(x_{1}\right) \\
& \quad=-\frac{b+d r_{1} \tau^{\prime}\left(x_{1}+g\left(x_{2}\right)\right) e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)}}{a_{1} h^{\prime}\left(x_{2}\right)+d r_{1} g^{\prime}\left(x_{2}\right) \tau^{\prime}\left(x_{1}+g\left(x_{2}\right)\right) e^{-d \tau\left(x_{1}+g\left(x_{2}\right)\right)}} . \tag{30}
\end{align*}
$$

On the other hand, some expressions about positive equilibrium $P^{*}\left(x_{1}^{*}, x_{2}^{*}, y^{*}\right)$ can be obtained based on (22) and (23),

$$
\begin{gather*}
r_{1} x_{2}^{*}-d x_{1}^{*}=r_{1} x_{2}^{*} e^{-d \tau\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)} \\
r_{1} e^{-d \tau\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)}=b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right) \tag{31}
\end{gather*}
$$

According to (31), $g_{1}^{\prime}\left(x_{1}^{*}\right)>g_{1}^{\prime}\left(x_{2}^{*}\right)$ is equivalent to the following inequality:

$$
\begin{align*}
& 1-d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right) \\
& \times\left(a_{1} h^{\prime}\left(x_{2}^{*}\right)+d g^{\prime}\left(x_{2}^{*}\right) \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\right. \\
& \left.\quad \times\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right)\right)+a_{1} d h^{\prime}\left(x_{2}^{*}\right) \\
& \quad+d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right) g^{\prime}\left(x_{2}^{*}\right)\right. \\
& \times\left(b x_{2}^{*}+a_{1} h\left(x_{2}^{*}\right)\right)\left(d+b+d \tau^{\prime}\left(x_{1}^{*}+g\left(x_{2}^{*}\right)\right)\right)>0 . \tag{32}
\end{align*}
$$

This completes the proof.
3.3. Local Stability Analysis around Equilibria. Local stability of model system (2) around all equilibria of model system (2) is investigated. Furthermore, stability switch due to variation of maturity delay and harvest effort level is also studied in this subsection.

The characteristic equation of model system (2) about some equilibrium $\widetilde{P}=\left(\widetilde{x}_{1}, \widetilde{x}_{2}, \widetilde{y}\right)$ takes the following form:

$$
\left|\begin{array}{ccc}
\lambda+(d-\widetilde{A}) e^{-d \tau(\tilde{z})} & r_{1}\left(1+e^{-(\lambda+d) \tau(\tilde{z})}\right)-\widetilde{A} e^{-d \tau(\tilde{z})} & -\widetilde{A} e^{-d \tau(\tilde{z})}  \tag{33}\\
\widetilde{A} e^{-d \tau(\tilde{z})} & \lambda+2 b \widetilde{x}+\frac{a_{1} k_{1} \tilde{y}}{\left(\widetilde{x}_{2}+k_{1}\right)^{2}}+\widetilde{A} e^{-d \tau(\tilde{z})}+r_{1} e^{-(\lambda+d) \tau(\tilde{z})} & \widetilde{A} e^{-d \tau(\tilde{z})}+\frac{a_{1} \widetilde{x}_{2}}{k_{1}+\widetilde{x}_{2}} \\
0 & -\frac{a_{2} \widetilde{y}^{2}}{\left(k_{2}+\widetilde{x}_{2}\right)^{2}} & \lambda-r_{2}+q E+\frac{2 a_{2} \tilde{y}}{k_{2}+\widetilde{x}_{2}}
\end{array}\right|=0
$$

where $\widetilde{A}=d r_{1} \widetilde{x}_{2} \tau^{\prime}(\widetilde{z})$.
Theorem 5. Local stability analysis of model system (2) around $P_{0}$ and $P_{1}$ is as follows:
(a) if $r_{2}<q E$, then model system is locally stable around $P_{0}$, and $P_{1}$ is a saddle point which is unstable in the $y$ direction and stable in the $x_{1}-x_{2}$ plane;
(b) if $r_{2}>q E$, then model system is locally stable around $P_{1}$, and $P_{0}$ is a saddle point which is unstable in the $y$ direction and stable in the $x_{1}-x_{2}$ plane.

Proof. For $P_{0}(0,0,0),(33)$ reduces to

$$
\begin{equation*}
\left(\lambda+d e^{-d \tau(0)}\right)\left(\lambda+r_{1} e^{-(\lambda+d) \tau(0)}\right)\left(\lambda-\left(r_{2}-q E\right)\right)=0 \tag{34}
\end{equation*}
$$

By solving (34), it can be found that there are two negative eigenvalues and only one positive eigenvalue, provided $r_{2}>$
$q E$, which implies that $P_{0}$ is a saddle point which is unstable in the $y$-direction and stable in the $x_{1}-x_{2}$ plane. On the other hand, there are three negative eigenvalues, provided that $r_{2}<$ $q E$, which implies that $P_{0}$ is a stable point.

For $P_{1}\left(0,0,\left(k_{2}\left(r_{2}-q E\right) / a_{2}\right)\right),(33)$ reduces to

$$
\begin{equation*}
\left(\lambda+d e^{-d \tau(\tilde{y})}\right)\left(\lambda+r_{1} e^{-(\lambda+d) \tau(\tilde{y})}+\frac{a_{1} \tilde{y}}{k_{1}}\right)\left(\lambda+\left(r_{2}-q E\right)\right)=0 \tag{35}
\end{equation*}
$$

where $\tilde{y}=k_{2}\left(r_{2}-q E\right) / a_{2}$. It follows from (35) that there are two negative eigenvalues and only one positive eigenvalue, provided $r_{2}<q E$, which implies that $P_{1}$ is a saddle point which is unstable in the $y$-direction and stable in the $x_{1}-x_{2}$ plane. On the other hand, there are three negative
eigenvalues, provided that $r_{2}>q E$, which implies that $P_{1}$ is a stable point.

In order to discuss the local stability of model system (2) around the positive equilibrium $P^{*}\left(x_{1}^{*}, x_{2}^{*}, y^{*}\right)$, (33) reduces to

$$
\left|\begin{array}{ccc}
\lambda+d B^{*}-A^{*} & r_{1}\left(1+B^{*} e^{-\lambda \tau\left(z^{*}\right)}\right)-A^{*} & -A^{*}  \tag{36}\\
A^{*} & \lambda+A^{*}+B^{*} e^{-\lambda \tau\left(z^{*}\right)}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}} & A^{*}+\frac{a_{1} x_{2}^{*}}{x_{2}^{*}+k_{1}} \\
0 & -\frac{a_{2} y^{* 2}}{\left(x_{2}^{*}+k_{2}\right)^{2}} & \lambda-r_{2}+q E+\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}
\end{array}\right|=0,
$$

where $A^{*}=d\left(r_{1} x_{2}^{*}-d x_{1}^{*}\right) \tau^{\prime}\left(z^{*}\right), B^{*}=\left(r_{1} x_{2}^{*}-d x_{1}^{*}\right) / r_{1} x_{2}^{*}$ and $z^{*}=x_{1}^{*}+x_{2}^{*}+y^{*}$.

It can be computed that

$$
\begin{equation*}
M(\lambda)+N(\lambda) e^{-\lambda \tau\left(z^{*}\right)}=0 \tag{37}
\end{equation*}
$$

where $M(\lambda)=\lambda^{3}+m_{1} \lambda^{2}+m_{2} \lambda+m_{3}$ and $N(\lambda)=n_{1} \lambda^{2}+$ $n_{2} \lambda+n_{3}$,

$$
\begin{align*}
m_{1}= & d B^{*}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}-r_{2}+q E+\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}, \\
m_{2}= & \left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right)\left(A^{*}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}\right) \\
& -A^{*}\left(r_{1}-A^{*}\right), \\
m_{3}= & \frac{a_{2} y^{* 2} A^{* 2}}{\left(x_{2}^{*}+k_{2}\right)^{2}}+d B^{*}\left(A^{*}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}\right) \\
& \times\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right) \\
& -A^{*}\left(r_{1}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}\right) \\
& \times\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right), \\
n_{1}= & B^{*}, \\
n_{2}= & B^{*}\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E-r_{1} A^{*}\right), \\
n_{3}= & B^{*}\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right)\left(d B^{*}-A^{*}-r_{1} A^{*}\right) . \tag{38}
\end{align*}
$$

In the following part, dynamical behavior of model system (2) around the positive equilibrium $P^{*}$ is investigated. Furthermore, local stability analysis is discussed due to the variation of maturation delay and harvest effort level. By taking $\tau^{\prime}\left(z^{*}\right)$ as a bifurcation parameter, conditions for local stability switch are discussed with the increase of $\tau^{\prime}\left(z^{*}\right)$ from zero.

Case $1\left(\tau^{\prime}\left(z^{*}\right)=0\right)$. In the case of $\tau^{\prime}\left(z^{*}\right)=0$, it derives that $\tau\left(z^{*}\right)$ remains as a constant (zero or a positive constant) for all time $t>0$ based on (H2). In the following part, $\tau\left(z^{*}\right)$ is denoted as $\tau^{*}$ for simplifying. Furthermore, it can be computed that $A^{*}=0$, and $m_{i}, n_{i}(i=1,2,3)$ in (37) can be rewritten as follows:

$$
\begin{align*}
& \widehat{m}_{1}=d B^{*}+2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}-r_{2}+q E+\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}, \\
& \widehat{m}_{2}=\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right)\left(2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}\right), \\
& \widehat{m}_{3}=d B^{*}\left(2 b x_{2}^{*}+\frac{a_{1} k_{1} y^{*}}{\left(x_{2}^{*}+k_{1}\right)^{2}}\right)\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right), \\
& \widehat{n}_{1}=B^{*}, \\
& \widehat{n}_{2}=B^{*}\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right), \\
& \widehat{n}_{3}=d B^{* 2}\left(\frac{2 a_{2} y^{*}}{x_{2}^{*}+k_{2}}-r_{2}+q E\right) . \tag{39}
\end{align*}
$$

Theorem 6. Supposing that hypotheses (H1)-(H4) hold, if $r_{2}-$ $q E>0$, then model system (2) is stable around the positive equilibrium $P^{*}$ in the case of $\tau^{*}=0$.

Proof. When $\tau^{*}=0$, (37) can be rewritten as follows:

$$
\begin{equation*}
\lambda^{3}+\left(\widehat{m}_{1}+\widehat{n}_{1}\right) \lambda^{2}+\left(\widehat{m}_{2}+\widehat{n}_{2}\right) \lambda+\widehat{m}_{3}+\widehat{n}_{3}=0 . \tag{40}
\end{equation*}
$$

Based on the above analysis, it can be concluded that the roots of (40) have negative real parts by using the RouthHurwitz criteria [1]. Consequently, $P^{*}$ is locally stable in the case of $\tau^{*}=0$.

When $\tau^{*}>0$, let $\lambda=i \omega$ be a root of (37), where $\omega$ is positive. Substitute $\lambda=i \omega$ into (37) and separate the real and imaginary parts, and then two transcendental equations can be obtained as follows:

$$
\begin{align*}
& \omega^{3}-\widehat{m}_{2} \omega=\left(\widehat{n}_{1} \omega^{2}-\widehat{n}_{3}\right) \sin \left(\omega \tau^{*}\right)+\widehat{n}_{2} \omega \cos \left(\omega \tau^{*}\right), \\
& \widehat{m}_{1} \omega^{2}-\widehat{m}_{3}=\widehat{n}_{2} \omega \sin \left(\omega \tau^{*}\right)-\left(\widehat{n}_{1} \omega^{2}-\widehat{n}_{3}\right) \cos \left(\omega \tau^{*}\right) . \tag{41}
\end{align*}
$$

By squaring and adding (41), it can be obtained that

$$
\begin{equation*}
\omega^{6}+B_{1} \omega^{4}+B_{2} \omega^{2}+B_{3}=0 \tag{42}
\end{equation*}
$$

where $B_{1}=\widehat{m}_{1}^{2}-2 \widehat{m}_{2}-\widehat{n}_{1}^{2}, B_{2}=\widehat{m}_{2}^{2}-2 \widehat{m}_{1} \widehat{m}_{3}+2 \widehat{n}_{1} \widehat{n}_{3}-\widehat{n}_{2}^{2}$, $B_{3}=\widehat{m}_{3}^{2}-\widehat{n}_{3}^{2}$, and $\widehat{m}_{i}, \widehat{n}_{i}(i=1,2,3)$ have been defined in (40).

According to the values of $B_{i}(i=1,2,3)$ and the RouthHurwitz criteria [1], a simple assumption of the existence of a positive root for (42) is $B_{3}<0$.

If $B_{3}<0$ holds, then (42) has a positive root $\omega_{0}$, and (37) has a pair of purely imaginary roots of the form $\pm i \omega_{0}$. Consequently, it can be obtained by eliminating $\sin \left(\omega \tau^{*}\right)$ from (41):

$$
\begin{align*}
\cos & \left(\omega \tau^{*}\right) \\
& =\frac{\left(\widehat{n}_{2}-\widehat{m}_{1} \widehat{n}_{1}\right) \omega^{4}+\left(\widehat{m}_{1} \widehat{n}_{3}+\widehat{m}_{3} \widehat{n}_{1}-\widehat{m}_{2} \widehat{n}_{2}\right) \omega^{2}-\widehat{m}_{3} \widehat{n}_{3}}{\left(\widehat{n}_{2} \omega^{2}\right)^{2}+\left(\widehat{n}_{3}-\widehat{n}_{1} \omega^{2}\right)^{2}}, \tag{43}
\end{align*}
$$

The $\widetilde{\tau}_{k}$ corresponding to $\omega_{0}$ is as follows:

$$
\begin{align*}
\widetilde{\tau}_{k}=\frac{1}{\omega_{0}} \arccos [ & \left(\left(\widehat{n}_{2}-\widehat{m}_{1} \widehat{n}_{1}\right) \omega^{4}\right. \\
& \left.+\left(\widehat{m}_{1} \widehat{n}_{3}+\widehat{m}_{3} \widehat{n}_{1}-\widehat{m}_{2} \widehat{n}_{2}\right) \omega^{2}-\widehat{m}_{3} \widehat{n}_{3}\right) \\
& \left.\times\left(\left(\widehat{n}_{2} \omega^{2}\right)^{2}+\left(\widehat{n}_{3}-\widehat{n}_{1} \omega^{2}\right)^{2}\right)^{-1}\right]+\frac{2 k \pi}{\omega_{0}} \tag{44}
\end{align*}
$$

$k=0,1,2, \ldots$. By virtue of Butler's lemma [37], it can be concluded that the positive equilibrium $P^{*}$ remains locally stable for $\tau^{*}<\tilde{\tau}_{0}$, as $k=0$.

Case $2\left(\tau^{\prime}\left(z^{*}\right)>0\right)$. In the case of $\tau^{\prime}\left(z^{*}\right)>0$, local stability of model system (2) around the positive equilibrium $P^{*}$ can change only if there exists at least one root of (37) such that $\operatorname{Re} \lambda=0$.

Let $\lambda=i \nu$ be one such root, where $\nu$ is positive. Substitute $\lambda=i \nu$ into (37) and separate the real and imaginary parts, and then two transcendental equations can be obtained as follows:

$$
\begin{align*}
v^{3}-m_{2} \nu & =\left(n_{1} v^{2}-n_{3}\right) \sin \left(v \tau\left(z^{*}\right)\right) \\
& +n_{2} v \cos \left(v \tau\left(z^{*}\right)\right) \\
m_{1} v^{2}-m_{3}= & n_{2} v \sin \left(\nu \tau\left(z^{*}\right)\right)  \tag{45}\\
& -\left(n_{1} v^{2}-n_{3}\right) \cos \left(v \tau\left(z^{*}\right)\right) .
\end{align*}
$$

By squaring and adding (45), it can be obtained that

$$
\begin{equation*}
\nu^{6}+C_{1} \nu^{4}+C_{2} v^{2}+C_{3}=0 \tag{46}
\end{equation*}
$$

where $C_{1}=m_{1}^{2}-2 m_{2}-n_{1}^{2}, C_{2}=m_{2}^{2}-2 m_{1} m_{3}+2 n_{1} n_{3}-n_{2}^{2}$, $C_{3}=m_{3}^{2}-n_{3}^{2}$, and $m_{i}, n_{i}(i=1,2,3)$ have been defined in (37).

According to the values of $C_{i}(i=1,2,3)$ and the RouthHurwitz criteria [1], a simple assumption of the existence of a positive root for (42) is $C_{3}<0$, which derives that

$$
\begin{align*}
& \tau^{\prime}\left(z^{*}\right) \\
& \quad \begin{aligned}
& \\
&\left.\quad+a_{1} k_{1} r_{1}\left(r_{2}-q E\right) x_{2}^{*}\left(x_{2}^{*}+k_{1}\right)^{2}\left(r_{1} x_{2}^{*}\left(2 b x_{2}^{*}-1\right)+d x_{1}^{*}\right)\right) \\
& \quad \times\left(r _ { 1 } x _ { 2 } ^ { * } \left[a_{1} k_{1} r_{1}\left(r_{2}-q E\right) x_{2}^{*}\left(x_{2}^{*}+k_{2}\right)\right.\right. \\
&+a_{2}\left(x_{2}^{*}+k_{1}\right)^{2} \\
&\left.\left.\times\left(d x_{1}^{*}(d+2)+r_{1} x_{2}^{*}\left(2 b x_{2}^{*}-d-1\right)\right)\right]\right)^{-1}
\end{aligned}
\end{align*}
$$

If the above inequality holds, then model system (2) is unstable around the positive equilibrium $P^{*}$ in the case of $\tau^{\prime}\left(z^{*}\right)>0$.
3.4. Global Stability Analysis of Positive Equilibrium. In this section, global stability of the positive equilibrium $P^{*}$ is discussed by using an iterative technique in the case of $\tau^{\prime}\left(z^{*}\right)=0$.

Lemma 7 (see [29]). Consider the following equation:

$$
\begin{equation*}
\dot{x}=a x(t-\tau)-b x(t)-c x^{2}(t) \tag{48}
\end{equation*}
$$

where $a, b, c$, and $\tau$ are positive constants, and $x(t)>0$ for $t \in[-\tau, 0]$; it follows that
(i) If $a>b$, then $\lim _{t \rightarrow+\infty} x(t)=(a-b) / c$;
(ii) If $a<b$, then $\lim _{t \rightarrow+\infty} x(t)=0$.

Theorem 8. Supposing that hypotheses (H1)-(H4) and $r_{2}-$ $q E>0$ hold, if the following inequalities hold

$$
\begin{gather*}
b k_{2}+r_{1} e^{-d \tau^{*}}<a_{2} b y^{*}  \tag{49}\\
a_{2} k_{1} r_{1} e^{-d \tau^{*}}>a_{1}\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right)
\end{gather*}
$$

then the positive equilibrium $P^{*}$ is globally asymptotically stable in the case of $\tau^{\prime}\left(z^{*}\right)=0$.

Proof. In the case of $\tau^{\prime}\left(z^{*}\right)=0$, it derives that $\tau\left(z^{*}\right)$ remains as a constant (zero or a positive constant) for all time $t>0$ based on (H2). In the following part, $\tau\left(z^{*}\right)$ is denoted as $\tau^{*}$ for simplifying. Let

$$
\begin{array}{ll}
U_{1}=\lim _{t \rightarrow+\infty} \sup x_{2}(t), & V_{1}=\lim _{t \rightarrow+\infty} \inf x_{2}(t),  \tag{50}\\
U_{2}=\lim _{t \rightarrow+\infty} \sup y(t), & V_{2}=\lim _{t \rightarrow+\infty} \inf y(t) .
\end{array}
$$

In the following, we will claim that $U_{1}=V_{1}=x_{2}^{*}, U_{2}=$ $V_{2}=y^{*}$.

It follows from Theorem 2 that $x_{2}(t) \leq R_{1}\left(R_{1}\right.$ has been defined in Theorem 2). From model system (2),

$$
\begin{equation*}
\dot{y}(t) \leq\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{R_{1}+k_{2}} . \tag{51}
\end{equation*}
$$

By standard comparison argument, it derives that

$$
\begin{equation*}
U_{2} \leq \frac{\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right)}{a_{2}}:=J_{1}^{y} \tag{52}
\end{equation*}
$$

and then for sufficiently small $\epsilon>0$, there exists a $T_{11}>0$ such that if $t>T_{11}, y(t) \leq J_{1}^{y}+\epsilon$. Based on Theorem $1, x_{2}(t)+$ $k_{1}>k_{1}$, it can be obtained that for $t>T_{11}+\tau^{*}$,

$$
\begin{align*}
\dot{x}_{2}(t) & \geq r_{1} e^{-d \tau^{*}} x_{2}\left(t-\tau^{*}\right)-b x_{2}^{2}(t)-\frac{a_{1}\left(J_{1}^{y}+\epsilon\right) x_{2}(t)}{x_{2}(t)+k_{1}} \\
& >r_{1} e^{-d \tau^{*}} x_{2}\left(t-\tau^{*}\right)-b x_{2}^{2}(t)-\frac{a_{1}\left(J_{1}^{y}+\epsilon\right) x_{2}(t)}{k_{1}} . \tag{53}
\end{align*}
$$

Consider the following auxiliary equation:

$$
\begin{equation*}
\dot{v}(t)=r_{1} e^{-d \tau^{*}} v\left(t-\tau^{*}\right)-b v^{2}(t)-\frac{a_{1}\left(J_{1}^{y}+\epsilon\right) v(t)}{k_{1}} . \tag{54}
\end{equation*}
$$

Under the condition $a_{2} k_{1} r_{1} e^{-d \tau^{*}}>a_{1}\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right)$, it follows from Lemma 7 that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} v(t)=\frac{k_{1} r_{1} e^{-d \tau^{*}}-a_{1}\left(J_{1}^{y}+\epsilon\right)}{b k_{1}}:=I_{1}^{x} . \tag{55}
\end{equation*}
$$

Hence, $V_{1} \geq I_{1}^{x}$. For sufficiently small $\epsilon>0$, there exits $T_{12} \geq T_{11}+\tau^{*}$ such that if $t>T_{22}$, then $x_{2}(t) \geq I_{1}^{x}-\epsilon$.

We derive from the model system (2) that for $t>T_{12}$,

$$
\begin{equation*}
\dot{y}(t) \geq\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{k_{2}+I_{1}^{x}-\epsilon} \tag{56}
\end{equation*}
$$

A standard comparison argument shows that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} y(t)=\frac{\left(r_{2}-q E\right)\left(k_{2}+I_{1}^{x}-\epsilon\right)}{a_{2}}:=I_{1}^{y} \tag{57}
\end{equation*}
$$

Hence, for sufficiently small $\epsilon>0$, there is a $T_{21} \geq T_{12}$ satisfying if $t>T_{21}$, then $y(t) \geq I_{1}^{y}-\epsilon$. Consequently, for $t>T_{21}+\tau^{*}$,

$$
\begin{equation*}
\dot{x}_{2}(t) \leq r_{1} e^{-d \tau^{*}} x_{2}\left(t-\tau^{*}\right)-b x_{2}^{2}(t)-\frac{a_{1}\left(I_{1}^{y}-\epsilon\right) x_{2}(t)}{R_{1}+k_{1}} . \tag{58}
\end{equation*}
$$

Consider the following auxiliary equation:
$\dot{v}(t)=r_{1} e^{-d \tau^{*}} v\left(t-\tau^{*}\right)-b v^{2}(t)-\frac{a_{1}\left(I_{1}^{y}-\epsilon\right) v(t)}{R_{1}+k_{1}}$.
It follows from Lemma 7 that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} v(t)=\frac{\left(r_{1} e^{-d \tau^{*}}-b\right)\left(R_{1}+k_{1}\right)}{a_{1}\left(I_{1}^{y}-\epsilon\right)}:=J_{1}^{x} \tag{60}
\end{equation*}
$$

Hence, $U_{1} \leq J_{1}^{x}$. For sufficiently small $\epsilon>0$, there exists a $T_{22} \geq T_{21}+\tau^{*}$ satisfying that if $t>T_{22}$, then $x_{2}(t) \leq J_{1}^{x}+\epsilon$. For $t>T_{22}$, it gives that

$$
\begin{equation*}
\dot{y}(t) \leq\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{k_{2}+J_{1}^{x}+\epsilon} . \tag{61}
\end{equation*}
$$

By standard comparison argument, it derives that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} y(t)=\frac{\left(r_{2}-q E\right)\left(J_{1}^{x}+k_{2}+\epsilon\right)}{a_{2}}:=J_{2}^{y} \tag{62}
\end{equation*}
$$

Hence, for sufficiently small $\epsilon>0$, there exists $T_{31} \geq T_{22}$ satisfying that if $t>T_{31}, y(t) \leq J_{2}^{y}+\epsilon$, the for $t>T_{31}+\tau^{*}$

$$
\begin{equation*}
\dot{x}_{2}(t) \geq r_{1} e^{-d \tau^{*}} x_{2}\left(t-\tau^{*}\right)-b x_{2}^{2}(t)-\frac{a_{1}\left(J_{2}^{y}+\epsilon\right) x_{2}(t)}{k_{1}} . \tag{63}
\end{equation*}
$$

Consider the following auxiliary equation:
$\dot{v}(t)=r_{1} e^{-d \tau^{*}} v\left(t-\tau^{*}\right)-b v^{2}(t)-\frac{a_{1}\left(J_{2}^{y}+\epsilon\right) v(t)}{k_{1}}$.
By using Lemma 7, it can be obtained that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} v(t)=\frac{k_{1} r_{1} e^{-d \tau^{*}}-a_{1}\left(J_{2}^{y}+\epsilon\right)}{b k_{1}}:=I_{2}^{x} \tag{65}
\end{equation*}
$$

Since it is true for any sufficiently small $\epsilon>0, V_{1} \geq I_{2}^{x}$. Therefore, there exists $T_{32} \geq T_{31}+\tau^{*}$ such that if $t>T_{32}$, then $x_{2}(t) \geq I_{2}^{x}-\epsilon$.

It follows from model system (2) that for $t>T_{32}$,

$$
\begin{equation*}
\dot{y}(t) \geq\left(r_{2}-q E\right) y(t)-\frac{a_{2} y^{2}(t)}{I_{2}^{x}-\epsilon+k_{2}} . \tag{66}
\end{equation*}
$$

By using standard comparison argument, it derives that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} y(t)=\frac{\left(r_{2}-q E\right)\left(I_{2}^{x}-\epsilon+k_{2}\right)}{a_{2}}:=I_{2}^{y} \tag{67}
\end{equation*}
$$

Since this is true for any sufficiently small $\epsilon>0, V_{2} \geq I_{2}^{y}$. Consequently, there exists $T_{41} \geq T_{32}$ satisfying if $t>T_{41}$, then $y(t) \geq I_{2}^{y}-\epsilon$.

It follows from model system (2) that for $t>T_{41}+\tau^{*}$,
$\dot{x}_{2}(t) \leq r_{1} e^{-d \tau^{*}} x_{2}\left(t-\tau^{*}\right)-b x_{2}^{2}(t)-\frac{a_{1}\left(I_{2}^{y}-\epsilon\right) x_{2}(t)}{R_{1}+k_{1}}$.

Consider the following auxiliary equation,

$$
\begin{equation*}
\dot{v}(t) \leq r_{1} e^{-d \tau^{*}} v\left(t-\tau^{*}\right)-b v^{2}(t)-\frac{a_{1}\left(I_{2}^{y}-\epsilon\right) v(t)}{R_{1}+k_{1}} \tag{69}
\end{equation*}
$$

By using Lemma 7, it derives that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} x_{2}(t)=\frac{r_{1} e^{-d \tau^{*}}\left(R_{1}+k_{1}\right)-a_{1}\left(I_{2}^{y}-\epsilon\right)}{b\left(R_{1}+k_{1}\right)}:=J_{2}^{x} \tag{70}
\end{equation*}
$$

Continuing the above process, four sequences $\left\{I_{n}^{x}\right\},\left\{I_{n}^{y}\right\}$, $\left\{J_{n}^{x}\right\},\left\{J_{n}^{y}\right\}, n=1,2, \ldots$, are obtained which take the following form

$$
\begin{align*}
& J_{n}^{x}=\frac{r_{1} e^{-d \tau^{*}}\left(R_{1}+k_{1}\right)+a_{1} \epsilon-a_{1} I_{n}^{y}}{b\left(R_{1}+k_{1}\right)} \\
& J_{n}^{y}=\frac{\left(\epsilon+k_{2}\right)\left(r_{2}-q E\right)+\left(r_{2}-q E\right) J_{n-1}^{x}}{a_{2}} \\
& I_{n}^{x}=\frac{k_{1} r_{1} e^{-d \tau^{*}}-a_{1} \epsilon-a_{1} J_{n}^{y}}{b k_{1}}  \tag{71}\\
& I_{n}^{y}=\frac{\left(r_{2}-q E\right)\left(k_{2}-\epsilon\right)+\left(r_{2}-q E\right) I_{n}^{x}}{a_{2}}
\end{align*}
$$

It is easy to show that

$$
\begin{equation*}
I_{n}^{x} \leq V_{1} \leq U_{1} \leq J_{n}^{x}, \quad I_{n}^{y} \leq V_{2} \leq U_{2} \leq J_{n}^{y} \tag{72}
\end{equation*}
$$

By virtue of (71), it derives that

$$
\begin{align*}
& J_{n}^{y} \\
& \qquad \begin{array}{l}
=\frac{\left(r_{2}-q E\right)\left[b\left(\epsilon+k_{2}\right)+r_{1} e^{-d \tau^{*}}\right]}{a_{2} b}+\frac{a_{1} \epsilon\left(r_{2}-q E\right)}{a_{2} b\left(R_{1}+k_{1}\right)} \\
\\
-\frac{a_{1}\left(r_{2}-q E\right)^{2}\left[k_{1}\left(r_{1} e^{-d \tau^{*}}+b k_{2}\right)+\epsilon\left(b k_{1}-a_{1}\right)-a_{1} J_{n-1}^{y}\right]}{a_{2}^{2} b^{2} k_{1}\left(R_{1}+k_{1}\right)} .
\end{array} .
\end{align*}
$$

Furthermore,

$$
\begin{align*}
J_{n}^{y}- & J_{n-1}^{y} \\
& =\frac{\left(r_{2}-q E\right)\left[b\left(\epsilon+k_{2}\right)+r_{1} e^{-d \tau^{*}}\right]}{a_{2} b}+\frac{a_{1} \epsilon\left(r_{2}-q E\right)}{a_{2} b\left(R_{1}+k_{1}\right)} \\
& -\frac{a_{1}\left(r_{2}-q E\right)^{2}\left[k_{1}\left(r_{1} e^{-d \tau^{*}}+b k_{2}\right)+\epsilon\left(b k_{1}-a_{1}\right)-a_{1} J_{n-1}^{y}\right]}{a_{2}^{2} b^{2} k_{1}\left(R_{1}+k_{1}\right)} \\
& -J_{n-1}^{y} . \tag{74}
\end{align*}
$$

If the following inequalities hold

$$
\begin{gather*}
b k_{2}+r_{1} e^{-d \tau^{*}}<a_{2} b y^{*} \\
a_{2} k_{1} r_{1} e^{-d \tau^{*}}>a_{1}\left(r_{2}-q E\right)\left(R_{1}+k_{2}\right) \tag{75}
\end{gather*}
$$

then $J_{n}^{y}-J_{n-1}^{y} \leq 0$, which implies that $\left\{J_{n}^{y} \mid J_{n}^{y} \geq y^{*}, n=\right.$ $1,2, \ldots\}$ is monotonically decreasing. Hence, it can be shown that limitation of sequence $\left\{J_{n}^{y}\right\}$ exists. Taking $n \rightarrow+\infty$, it follows from (73) that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} J_{n}^{y}=y^{*} \tag{76}
\end{equation*}
$$

By using (71) and (76), it can be shown that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} J_{n}^{x}=x_{2}^{*}, \quad \lim _{n \rightarrow+\infty} I_{n}^{y}=y^{*}, \quad \lim _{n \rightarrow+\infty} I_{n}^{x}=x_{2}^{*} \tag{77}
\end{equation*}
$$

According to the definition of $U_{1}, U_{2}, V_{1}$, and $V_{2}$, it derives that

$$
\begin{equation*}
U_{1}=V_{1}=x_{2}^{*}, \quad U_{2}=V_{2}=y^{*} \tag{78}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} x_{2}(t)=x_{2}^{*}, \quad \lim _{t \rightarrow+\infty} y(t)=y^{*} \tag{79}
\end{equation*}
$$

Based on (5), it derives that

$$
\begin{equation*}
x_{1}(t)=\int_{t-\hat{\tau}}^{t} r_{1} e^{-d(t-s)} \phi(s) d s \tag{80}
\end{equation*}
$$

By using L'Hospital's rule, it derives that

$$
\begin{align*}
\lim _{t \rightarrow+\infty} x_{1}(t) & =\lim _{t \rightarrow+\infty} \frac{r_{1}\left[e^{d t} \phi(t)-e^{d(t-\hat{\tau})} \phi(t-\hat{\tau})\right]}{d e^{d t}} \\
& =\lim _{t \rightarrow+\infty} \frac{r_{1}}{d}\left[\phi(t)-e^{-d \hat{\tau}} \phi(t-\hat{\tau})\right]  \tag{81}\\
& =\lim _{t \rightarrow+\infty} \frac{r_{1}}{d}\left(1-e^{-d \hat{\tau}}\right) x_{2}^{*} .
\end{align*}
$$

According to (20), it is easy to show that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} x_{1}(t)=\lim _{t \rightarrow+\infty} \frac{r_{1}}{d}\left(1-e^{-d \hat{\tau}}\right) x_{2}^{*}=x_{1}^{*} . \tag{82}
\end{equation*}
$$

This completes the proof.

## 4. Numerical Simulation

With the help of MATLAB, numerical simulations are provided to understand the theoretical results which have been established in the previous sections of this paper. In order to facilitate the numerical simulation, it is assumed that $\tau(z(t))$ takes the following form [23]:

$$
\begin{equation*}
\tau(z(t))=\tau_{0}+\tau_{m}-\tau_{m} e^{-z(t)} \tag{83}
\end{equation*}
$$

where $\tau_{m} \in\left(\tau_{0}, \tau_{1}\right)$ satisfying $\tau_{0}+\tau_{m}=\tau_{1}$. Based on Theorem 1, it follows from simple computation that

$$
\begin{equation*}
\lim _{t \rightarrow 0} \tau(z(t))=\tau_{0}, \quad \lim _{t \rightarrow+\infty} \tau(z(t))=\tau_{1} \tag{84}
\end{equation*}
$$

which implies that (H2) holds.
Values of parameters are taken from [24] which are used in Example 1 of [24] and set in appropriate units. $r_{1}=12$, $d=0.2, b=1.2, a_{1}=0.5, a_{2}=2, k_{1}=2, k_{2}=1$, $r_{2}=2, q=0.25$, and $E=4$. According to the given values of parameters, it follows from Theorems 3 and 4 that there exists a unique positive equilibrium $P^{*}(141.5454,5.6252,3.3126)$. Furthermore, it can be verified that $P^{*}$ is globally attractive based on Theorem 8. Responses of model system (2) are indicated in Figure 1, and the phase portrait of model system (2) with different initial values is plotted in Figure 2.


Figure 1: Dynamical responses of model system (2).

## 5. Conclusion

In this paper, a harvested prey predator model is proposed to investigate the effects of density-dependent maturation delay and harvest effort on the dynamics. Conditions which influence positiveness and boundedness of solutions of model system are obtained in Theorems 1 and 2, respectively. Existence of all equilibria of model system and uniqueness of the positive equilibrium are studied in Theorems 3 and 4, respectively. Biological interpretations of the positive equilibrium mean immature prey, mature prey, predator and harvest effort on predator population all exist in the harvested ecosystem. Consequently, we mainly concentrate on dynamical analysis around positive equilibrium in this paper. Local stability analysis in Theorem 6 reveals that local stability of the positive equilibrium loses due to variation of maturation delay and harvest effort level. Furthermore, global stability of the positive equilibrium is discussed by utilizing an iterative technique in Theorem 8, which is utilized to investigate the coexistence and interaction mechanism of harvested prey-predator ecosystem.

Compared with the work done in [24] and the related work in [26], maturation delay for prey population in this paper relates to the density of all population within the harvested ecosystem, which accurately reflects the practical phenomena in the real world [23, 34-36]. Furthermore, it should be noted that dynamics of prey predator model with density-dependent delay for predator population is investigated in [23], while dynamics of harvest effort on population within ecosystem is not considered. Compared with the work done in [23], harvest effort on predator population is introduced, and the effect of harvesting on model dynamics is also investigated in this paper. With the rapid development


Figure 2: Phase portrait of model system (2) with different initial values.
of commercial harvesting on prey predator ecosystem in the real world, the introduction of harvest effort and related qualitative analysis makes the work done in this paper have some new and positive feature.

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## Research Article

# Collaboration Control of Fractional-Order Multiagent Systems with Sampling Delay 

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#### Abstract

Because of the complexity of the practical environments, many distributed multiagent systems cannot be illustrated with the integerorder dynamics and can only be described with the fractional-order dynamics. In this paper, collaboration control problems of continuous-time networked fractional-order multiagent systems via sampled control and sampling delay are investigated. Firstly, the sampled-data control of multiagent systems with fractional-order derivative operator is analyzed in a directed weighted network ignoring sampling delay. Then, the collaborative control of fractional-order multiagent systems with sampled data and sampling delay is studied in a directed and symmetrical network. Many sufficient conditions for reaching consensus with sampled data and sampling delay are obtained. Some numerical simulations are presented to illustrate the utility of our theoretical results.


## 1. Introduction

In recent years, consensus problems in distributed networked multiagent systems have attracted increasing attention of more and more researches including control theory, mathematics, biology, physics, computer science, and robotics. The applications of multiagent systems are extensive, ranging from multiple space-craft alignment, heading direction in flocking behavior, distributed computation, and rendezvous of multiple vehicles. Based on certain quantities of interest, collaboration control problems of agent systems have been studied by many researchers and many important results have been achieved in much literature [1-8].

With the development of digital sensors and controllers, in many cases that the system itself is a continuous process, the synthesis of control law can only use the data sampled at the discrete sampling instants. Compare to continuoustime systems with continuous-time controller, continuoustime systems via sampled control have many advantages, such as flexibility, robustness, and low cost. Therefore, sampled control for continuous-time system is more coincident with applications in our real life. Robots, vehicles, airplanes, satellites, and almost all of the modern artificial products
are controlled by digital controller where continuous signals are transferred into discrete ones. For consensus problems of continuous-time multiagent systems via sampled control, some interesting results about consensus problem for multiagent system have been reported [9-15]. However, in the many real applications, we always want to find how large the sampling period should be chosen to guarantee that the system runs well. This requires us to look for an upper bound of sampling period. Moreover, sampling delay of the system cannot be ignored and sometimes may play a key role in the stability analysis of the networks. Therefore, we will also study the case when sampling delay exists.

The important results of the above literature focus the consensus problems of multiagent systems with integer-order dynamical equation. In the complex environment, many phenomena cannot be explained by the framework of integerorder dynamics, for example, the synchronized motion of agents in complex environments such as macromolecule fluids and porous media [16-18]. Under these circumstances, many dynamic characteristics of natural phenomena can only be described in the dynamics of fractional-order (noninteger order) behavior, for example: flocking movement and food searching by means of the individual secretions
and microbial, submarine underwater robots in the bottom of the sea with a large number of microorganisms and viscous substances, unmanned aerial vehicles running in the complex space environment [19, 20]. Cao et al. [21, 22] studied distribution coordination of multiagent systems with fractional-order dynamics firstly and gave the relationship between the number of individuals and the fractional order in the stable multiagent systems. However, to the best of our knowledge, there are few researches done on the coordination control of fractional-order multiagent systems via sampled data.

In this paper, we investigate the consensus of fractionalorder multiagent systems (FOMAS) with sampled-data control. Because few methods are presented to analyze the fractional-order systems with sampling delay, the problems of the fractional-order systems with sampling delay and sampled data will become more difficult. The main innovation of this paper lies in the study on the distributed coordination of FOMAS with sampled data and sampling delay. The rest of the paper is organized as follows. In Section 2, we recall some basic definitions about fractional calculus. In Section 3, some preliminaries about graph theory, fractional-order coordination model of multiagent systems are shown out. A distributed coordination algorithm for FOMAS with sampled data control is studied in Section 4. Section 5 presents the consensus of FOMAS with sampled data and sampling delay. In Section 6, numerical examples are simulated to verify the theoretical analysis. Conclusions are finally drawn in Section 7.

## 2. Fractional Calculus

Fractional derivatives provide an excellent instrument to describe the memory and hereditary features of various materials and processes. Fractional calculus also appears in the control of dynamical systems, when the controlled system and the controller are described by a fractionalorder differential equation. This is the main advantage of fractional derivatives in comparison with classical integerorder models, in which such effects are in fact neglected. The advantages of fractional-order derivatives become evident in modeling mechanical and electrical characteristics of real materials, as well as in many fields to describe the rheological properties of rocks.

Fractional operator plays an important role in modern science, which is used as a generalization of integration and differentiation with noninteger order fundamental operator ${ }_{a} D_{t}^{p}$, where a and t are the limits of the operation and $p \in R$. The continuous integrodifferential operator is defined as

$$
{ }_{a} D_{t}^{p}= \begin{cases}\frac{d^{p}}{d t^{p}}, & p>0  \tag{1}\\ 1, & p=0 \\ \int_{a}^{t}(d \theta)^{-p}, & p<0\end{cases}
$$

Three definitions most frequently used for the general fractional operators are the Grünwald-Letnikov (GL) definition, the Riemann-Liouville (RL), and the Caputo definition [16-18]. The GL definition is given by

$$
\begin{equation*}
{ }_{a}^{G} D_{t}^{p} f(t)=\lim _{h \rightarrow 0} h^{-p} \sum_{k=0}^{[(t-a) / h]}(-1)^{k}\binom{p}{k} f(t-k h), \tag{2}
\end{equation*}
$$

where [•] means the integer part, and $\binom{p}{k}$ is fractional binomial coefficients. The RL definition is given as

$$
\begin{equation*}
{ }_{a}^{R} D_{t}^{p} f(t)=\frac{1}{\Gamma(n-p)} \frac{d^{n}}{d t^{n}} \int_{a}^{t} \frac{f(\theta)}{(t-\theta)^{1+p-n}} d \theta \tag{3}
\end{equation*}
$$

for $(n-1<p<n)$ and where $\Gamma(\cdot)$ is the Gamma function. The Caputo definition can be written as

$$
\begin{equation*}
{ }_{a}^{C} D_{t}^{p} f(t)=\frac{1}{\Gamma(p-n)} \int_{a}^{t} \frac{f^{(n)}(\theta)}{(t-\theta)^{1+p-n}} d \theta \tag{4}
\end{equation*}
$$

The initial conditions for the fractional order differential equations with the Caputo derivatives are in the same form as that for the integer-order differential equations. In this paper, a simple notation $f^{(p)}$ is used to replace ${ }_{a} D_{t}^{p} f(t)$.

## 3. Problem Statement

Assume that multiagent system consists of $n$ autonomous agents then connected relations between the agents constitute a network topology $\mathscr{G}$. Assume $\mathscr{G}=\{V, E, A\}$ represents a directed weighted graph, in which $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ represents a collection of $n$ nodes, and its set of edges is $E \subseteq V \times V$. The node indexes belong to a finite index set $I=\{1,2, \ldots, n\}$, with adjacency matrix $A=\left[a_{i l}\right] \in R^{n \times n}$ with weighted adjacency elements $a_{i l} \geq 0$. An edge of the weighted diagraph $\mathscr{G}$ is denoted by $e_{i l}=\left(v_{i}, v_{l}\right) \in E$. We assume that the adjacency element $a_{i l}>0$ when $e_{i l} \in E$; otherwise, $a_{i l}=0$. The set of neighbors of a node $i$ is denoted by $N_{i}=\left\{l \in I: a_{i l}>0\right\}$.

Let $\mathscr{G}$ be a weighted digraph without self-loops, that is, let $a_{i i}=0$, and matrix $D=\operatorname{diag}\left\{d_{1}, d_{2}, \ldots, d_{n}\right\}$ be the diagonal matrix with the diagonal elements $d_{i}=\sum_{l=1}^{n} a_{i l}$ representing the sum of the elements in the $i$ th row of matrix $A$. The Laplacian matrix of the weighted digraph $\mathscr{G}$ is defined as $L=$ $D-A$. For two nodes $i$ and $l$, there is subscript set $\left\{l_{1}, l_{2}, \ldots l_{k}\right\}$ satisfying $a_{i l_{1}}>0, a_{l_{1} l_{2}}>0, \ldots, a_{l_{k} l}>0$, and then there is a directed linked path between node $i$ and node $l$ which is used for the information transmission, also we can say that node $i$ can receive the information from node $l$. If node $i$ can find a path to reach any node of the graph, then node $i$ is globally reachable from every other node in the digraph. In this paper, the directed graph and directed symmetrical graph for fractional-order multiagent systems will be considered.

Lemma 1 (see [5]). 0 is a simple eigenvalue of Laplacian matrix $L$, and $X_{0}=C[1,1, \ldots, 1]^{T}$ is corresponding right eigenvector, that is, $L X_{0}=0$, if and only if the digraph $\mathscr{G}=$ ( $V, E, A$ ) has a globally reachable node.

Given that the dynamics of multiagent systems indicated with fractional derivative in the complex environments, the fractional dynamical equations are defined as

$$
\begin{equation*}
x_{i}^{(\alpha)}=u_{i}(t), \quad i=1, \ldots, n \tag{5}
\end{equation*}
$$

where $x_{i}(t) \in R$ and $u_{i}(t) \in R$ represent the $i$ th agent's state and control input, respectively, and $x_{i}^{(\alpha)}$ represents the $\alpha$ ( $\alpha>$ 0 ) order fractional derivative. Assume that the following control protocols are used in multiagent systems:

$$
\begin{equation*}
u_{i}(t)=-\gamma \sum_{l \in N_{i}} a_{i l}\left[x_{i}(t)-x_{l}(t)\right], \quad i=1, \ldots, n \tag{6}
\end{equation*}
$$

where $a_{i l}$ represents the $(i, l)$ elements of adjacency matrix $A, \gamma>0$ is control gain, and $N_{i}$ represents the neighbors collection of the $i$ th agent.

Suppose that for any initial value of the system, the states of autonomous agents meet $\lim _{t \rightarrow \infty}\left(x_{i}(t)-x_{l}(t)\right)=0$, for $i, l \in I$, and then multiagent systems asymptotically reach consensus.

## 4. Sampled Control of FOMAS

Suppose that the sampling period is $h$; then the discrete-time dynamics of multiagent systems with fractional derivative can be rewritten as

$$
\begin{equation*}
x_{i}(k+1)=\alpha x_{i}(k)+h^{\alpha} u_{i}(k), \quad i=1, \ldots, n, \tag{7}
\end{equation*}
$$

we have

$$
\begin{equation*}
X(k+1)=\Psi X(k) \tag{8}
\end{equation*}
$$

where $X(k)=\left[x_{1}(k), x_{2}(k), \ldots, x_{n}(k)\right]^{T}, \Psi=\alpha I_{n}-h^{\alpha} \gamma L$, and $I_{n}$ is a unit matrix with $n$-dimensions. If the norm of matrix $\Psi$ is satisfying $\|\Psi\|<1$, the fractional-order discrete-time multiagent system (8) will asymptotically reach consensus.

Theorem 2. Suppose that multiagent systems are composed of $n$ independent agents, whose connection network topology is directed, and there is a global reachable node. Then fractionalorder multiagent system (7) with sampled data can asymptotically reach consensus, if $\alpha<1$ and

$$
\begin{gather*}
h<\min \left\{\left[\frac{\alpha \operatorname{Re}\left(\lambda_{i}\right)}{\gamma\left|\lambda_{i}^{2}\right|}+\frac{1}{\gamma\left|\lambda_{i}\right|} \sqrt{1-\alpha^{2}\left(\frac{\operatorname{Im}\left(\lambda_{i}\right)}{\left|\lambda_{i}\right|}\right)^{2}}\right]^{1 / \alpha},\right. \\
\quad i=2, \ldots, n\} \tag{9}
\end{gather*}
$$

where $\lambda_{i}$ is the eigenvalue of the Laplacian matrix $L$.
Proof. Since the spectral radius $\rho$ of the matrix $\Psi$ is satisfying $\rho \leq\|\Psi\|$, we will require the spectral radius $\rho<1$. By the definition $\rho(\Psi)=\max \left\{\left|\eta_{i}\right|, i=1, \ldots, n\right\}$ where $\eta_{i}$ is the
characteristic value of the matrix $\Psi$, we should calculate the characteristic value $\eta_{i}$ of the matrix $\Psi$ with $\left|\eta_{i}\right|<1$.

For any matrix $L$, there exists a unitary matrix $P$ satisfying

$$
\begin{equation*}
\Lambda=P L P^{H} \tag{10}
\end{equation*}
$$

where conjugate matrix $P^{H}=P^{-1}, \Lambda$ is an upper triangular matrix, and its diagonal elements $\lambda_{1}, \ldots, \lambda_{n}$ are the eigenvalues of matrix $L$. From $P \Psi P^{H}=\alpha I_{n}-h^{\alpha} \gamma \Lambda$, we can obtain the characteristic values of matrix $\Psi$ being

$$
\begin{equation*}
\eta_{i}=\alpha-h^{\alpha} \gamma \lambda_{i}, \quad i=1, \ldots, n \tag{11}
\end{equation*}
$$

Suppose that the connection network topology of FOMAS is directed, and there is a global reachable node; then $\operatorname{Rank}(L)=$ $n-1$ and $\lambda=0$ is a single eigenvalue of Laplacian matrix $L$. Suppose $\lambda_{1}=0$, we have $\eta_{1}=\alpha-h^{\alpha} \gamma \lambda_{1}=\alpha$. From the stability requirement of the system (7), we obtain $\alpha<1$.

For other characteristic values of Laplacian matrix $L$ with $\lambda_{i} \neq 0(i \neq 1)$, the corresponding characteristic values of matrix $\Psi$ are being $\eta_{i}=\alpha-h^{\alpha} \gamma \lambda_{i}, i=2, \ldots, n$. Let $\lambda=$ $\operatorname{Re}\left(\lambda_{i}\right)+j \operatorname{Im}\left(\lambda_{i}\right)$ (where $j$ is complex number unit), and then the corresponding characteristic values, for $i=2, \ldots, n$, are

$$
\begin{equation*}
\eta_{i}=\alpha-h^{\alpha} \gamma\left(\operatorname{Re}\left(\lambda_{i}\right)+j \operatorname{Im}\left(\lambda_{i}\right)\right) \tag{12}
\end{equation*}
$$

From $\left|\eta_{i}\right|<1$, we can obtain

$$
\begin{equation*}
\alpha^{2}-2 \alpha h^{\alpha} \gamma \operatorname{Re}\left(\lambda_{i}\right)+h^{2 \alpha} \gamma^{2}\left|\lambda_{i}^{2}\right|<1 \tag{13}
\end{equation*}
$$

It has

$$
\begin{equation*}
h^{\alpha}<\frac{\alpha \operatorname{Re}\left(\lambda_{i}\right)}{\gamma\left|\lambda_{i}^{2}\right|}+\frac{1}{\gamma\left|\lambda_{i}\right|} \sqrt{1-\alpha^{2}\left(\frac{\operatorname{Im}\left(\lambda_{i}\right)}{\left|\lambda_{i}\right|}\right)^{2}}, \quad i=2, \ldots, n \tag{14}
\end{equation*}
$$

Then, the condition of the fractional-order multiagent system (7) is obtained. The proof is finished.

Corollary 3. Suppose that multiagent systems are composed of $n$ independent agents, whose connection network topology is directed and symmetrical, and there is a global reachable node. Then FOMAS (7) can asymptotically reach consensus via sampled data, if $\alpha<1$ and

$$
\begin{equation*}
h<\left(\frac{\alpha+1}{\gamma \lambda_{n}}\right)^{1 / \alpha} \tag{15}
\end{equation*}
$$

where $\lambda_{n}$ is the maximum eigenvalue of the Laplacian matrix $L$.

Corollary 4. Suppose multiagent systems are composed of $n$ independent agents, whose connection network topology is directed and symmetrical, and there is a global reachable node. Then multiagent system (7) with $\alpha=1$ can asymptotically reach consensus via sampled data, if

$$
\begin{equation*}
h<\frac{2}{\gamma \lambda_{n}}, \tag{16}
\end{equation*}
$$

where $\lambda_{n}$ is the maximum eigenvalue of the Laplacian matrix $L$.

Remark 5. The system of Corollary 4 forfractional-order $\alpha=1$ becomes the first-order multiagent system. The consensus condition obtained in Corollary 4 is same as that in [9].

## 5. Consensus of FOMAS with Sampled Data and Sampling Delay

In the practical application, the sampled-data transferring will result in the communication delays. The sampling delays will affect the control features of the system and sometimes may play a key role in the stability analysis of the network. In this section, we will study the consensus of multiagent systems with sampled data and sampling delay.

Suppose that land the sampling period is $h$, the sampling delay is $\tau$. The sampled control protocols are used in multiagent systems as follows:
$u_{i}(t)$

$$
= \begin{cases}-\gamma \sum_{l \in N_{i}} a_{i l}\left[x_{i}(k-1)-x_{l}(k-1)\right], & t \in[k h, k h+\tau),  \tag{17}\\ -\gamma \sum_{l \in N_{i}} a_{i l}\left[x_{i}(k)-x_{l}(k)\right], & t \in[k h+\tau, k h+h) .\end{cases}
$$

Based on (7) and (17), the dynamics of FOMAS with sampled data and sampling delay can be rewritten as

$$
\begin{equation*}
\binom{X(k+1)}{X(k)}=\Phi\binom{X(k)}{X(k-1)} \tag{18}
\end{equation*}
$$

where

$$
\begin{gather*}
X(k)=\left[x_{1}(k), x_{2}(k), \ldots, x_{n}(k)\right]^{T} \\
\Phi=\left(\begin{array}{cc}
\alpha I_{n}-\gamma\left(h^{\alpha}-h^{\alpha} \tau\right) L & -\gamma h^{\alpha} \tau L \\
I_{n} & 0
\end{array}\right), \tag{19}
\end{gather*}
$$

and $I_{n}$ is a unit matrix with $n$-dimensions. If the norm of matrix $\Phi$ is satisfying $\|\Phi\|<1$, the discrete-time FOMAS (18) will asymptotically reach consensus.

Lemma 6 (see [23, Hermite-Biehler Theorem]). Suppose the polynomial

$$
\begin{equation*}
p(s)=p_{0}+p_{1} s+\cdots+p_{n} s^{n} . \tag{20}
\end{equation*}
$$

Substituting $s=i \omega$ into the polynomial $p(s)$ yields

$$
\begin{equation*}
p(\omega)=m(\omega)+i n(\omega) \tag{21}
\end{equation*}
$$

Then, the polynomial $p(s)$ is Hurwitz stability if and only if the related pair $m(\omega)$ and $n(\omega)$ is interlaced, and $m(0) n^{\prime}(0)-$ $m^{\prime}(0) n(0)>0$.

Theorem 7. Suppose that multiagent systems are composed of $n$ independent agents, whose connection network topology is directed and symmetrical, and there is a global reachable node.

Then FOMAS (18) with sampled data and sampling delay can asymptotically reach consensus, if $\alpha<1$ and

$$
\begin{gather*}
h<\left[\frac{1}{\gamma \tau \lambda_{n}}\right]^{1 / \alpha}, \quad \tau>\frac{1}{(3+\alpha)}, \\
h<\left[\frac{1+\alpha}{\gamma \lambda_{n}(1-2 \tau)}\right]^{1 / \alpha}, \quad \tau \leq \frac{1}{(3+\alpha)}, \tag{22}
\end{gather*}
$$

where $\lambda_{n}$ is the maximum eigenvalue of the Laplacian matrix $L$.

Proof. In order to prove the asymptotical consensus of discrete-time systems (18), the spectral radius $\rho$ of the matrix $\Phi$ should be satisfied with $\rho(\Phi)<1$. Since $\rho(\Phi)=$ $\max \left\{\left|\nu_{i}\right|, i=1, \ldots, n\right\}$ where $v_{i}$ is the characteristic value of the matrix $\Phi$, we should calculate the characteristic value $\nu_{i}$ of the matrix $\Phi$ with $\left|v_{i}\right|<1$.

Suppose that the network topology of FOMAS is directed and symmetrical, and there is a global reachable node; then $\operatorname{Rank}(L)=n-1$ and $\lambda=0$ is a single eigenvalue of Laplacian matrix $L$. There exists an orthogonal matrix $P$ satisfying $\Lambda=$ $P L P^{T}$ where $P^{T}=P^{-1}, \Lambda$ is a diagonal matrix, and its diagonal elements $\lambda_{1}, \ldots, \lambda_{n}$ are the characteristic values of matrix $L$. Without loss of generality, we suppose $0=\lambda_{1}<\lambda_{2} \leq \cdots \leq$ $\lambda_{n}$. The characteristic equation of matrix $\Phi$ will be calculated as follows:

$$
\begin{align*}
& \operatorname{det}\left(\nu I_{2 n \times 2 n}-\Phi\right) \\
& \quad=\operatorname{det}\left(\begin{array}{cc}
\nu I_{n}-\alpha I_{n}+\gamma\left(h^{\alpha}-h^{\alpha} \tau\right) L & \gamma h^{\alpha} \tau L \\
-I_{n} & \nu I_{n}
\end{array}\right)  \tag{23}\\
& =\operatorname{det}\left(v^{2} I_{n}-\nu\left(\alpha I_{n}-\gamma\left(h^{\alpha}-h^{\alpha} \tau\right) L\right)+\gamma h^{\alpha} \tau L\right) \\
& =\Pi_{i=1}^{n}\left[v^{2}-v \alpha+\left(\nu h^{\alpha}-\nu h^{\alpha} \tau+h^{\alpha} \tau\right) \gamma \lambda_{i}\right] .
\end{align*}
$$

Let $a(\nu)=\nu^{2}-v \alpha+\left(v h^{\alpha}-v h^{\alpha} \tau+h^{\alpha} \tau\right) \gamma \lambda_{i}$. When $\lambda_{1}=0$ and $a(\nu)=\nu^{2}-\nu \alpha$, we can obtain the characteristic values $\nu_{1}=0$ and $\nu_{2}=\alpha$. From the stability requirement of the system (18), we obtain $\alpha<1$.

When $\lambda_{i}>0(i=2, \ldots, n)$, applying the bilinear transform $v=(s+1) /(s-1), a(\nu)$ can be converted into $b(s)$ as follows:

$$
\begin{align*}
b(s)= & \left(1-\alpha+h^{\alpha} \gamma \lambda_{i}\right) s^{2}+2\left(1-h^{\alpha} \gamma \tau \lambda_{i}\right) s  \tag{24}\\
& +\left(1+\alpha-h^{\alpha} \gamma \lambda_{i}+2 h^{\alpha} \tau \gamma \lambda_{i}\right) .
\end{align*}
$$

Let $s=j \omega$, and the we have

$$
\begin{equation*}
b(\omega)=m(\omega)+j n(\omega), \tag{25}
\end{equation*}
$$

where $m(\omega)=-\left(1-\alpha+h^{\alpha} \gamma \lambda_{i}\right) \omega^{2}+\left(1+\alpha-h^{\alpha} \gamma \lambda_{i}+2 h^{\alpha} \tau \gamma \lambda_{i}\right)$ and $n(\omega)=2\left(1-h^{\alpha} \gamma \tau \lambda_{i}\right) \omega$. Applying Lemma 6, we have the following.
(1) By means of $m(0) n^{\prime}(0)-m^{\prime}(0) n(0)=2\left(1+\alpha-h^{\alpha} \gamma \lambda_{i}+\right.$ $\left.2 h^{\alpha} \tau \gamma \lambda_{i}\right)\left(1-h^{\alpha} \gamma \tau \lambda_{i}\right)>0$, we have

$$
\begin{gather*}
1-h^{\alpha} \gamma \tau \lambda_{i}>0  \tag{26}\\
1+\alpha-h^{\alpha} \gamma \lambda_{i}+2 h^{\alpha} \tau \gamma \lambda_{i}>0 \tag{27}
\end{gather*}
$$

Based on (26), it has

$$
\begin{equation*}
h^{\alpha}<\frac{1}{\gamma \tau \lambda_{i}} \tag{28}
\end{equation*}
$$

Discussing (27), if $\tau \geq 0.5$, then (27) comes into existence. If $\tau<0.5$, then it require is

$$
\begin{equation*}
h^{\alpha}<\frac{1+\alpha}{\gamma \lambda_{i}(1-2 \tau)} . \tag{29}
\end{equation*}
$$

(2) When $2\left(1-h^{\alpha} \gamma \tau \lambda_{i}\right) \neq 0$ from (26), the solution of $n(\omega)=0$ is 0 . Since $\alpha<1$ we have $1-\alpha+h^{\alpha} \gamma \lambda_{i}>0$ and $1+\alpha-h^{\alpha} \gamma \lambda_{i}+2 h^{\alpha} \tau \gamma \lambda_{i}>0$ from (27); then the solutions of $m(\omega)=0$ are $\pm \sqrt{\left(1+\alpha-h^{\alpha} \gamma \lambda_{i}+2 h^{\alpha} \tau \gamma \lambda_{i}\right) /\left(1-\alpha+h^{\alpha} \gamma \lambda_{i}\right) .}$ Therefore, the solutions of the related pair $m(\omega)$ and $n(\omega)$ are interlaced.

By comparing the right parts of (28) and (29), we can obtain that when $\tau>1 /(3+\alpha), 1 / \gamma \tau \lambda_{i}<(1+\alpha) / \gamma \lambda_{i}(1-2 \tau)$; otherwise, when $\tau \leq 1 /(3+\alpha), 1 / \gamma \tau \lambda_{i} \geq(1+\alpha) / \gamma \lambda_{i}(1-2 \tau)$. Then, the consensus conditions of the multiagent system are obtained. The proof is finished.

Remark 8. Suppose that the fractional-order $\alpha=1$; we can get the consensus condition $h<1 / \gamma \tau \lambda_{n}$ when $\tau>1 / 4$, and $h<2 / \gamma \lambda_{n}(1-2 \tau)$ when $\tau \leq 1 / 4$, where $\lambda_{n}$ is the maximum eigenvalue of matrix $L$.

Remark 9. Suppose that the communication delay $\tau=0$, we can get $\tau<1 /(3+\alpha)$; therefore, $h<\left((1+\alpha) / \gamma \lambda_{n}\right)^{1 / \alpha}$ where $\lambda_{n}$ is the maximum eigenvalue of matrix $L$. This result is the same as the consensus condition in Corollary 3.

## 6. Simulations

Suppose that the system is composed of four fractional-order dynamical agents (Figure 1). The connection weights between individuals are $a_{21}=a_{12}=0.7, a_{42}=a_{24}=0.8, a_{31}=a_{13}=$ 0.9 , and $a_{14}=a_{41}=1$. Through the network topology of the system, we can get the adjacency matrix

$$
A=\left(\begin{array}{cccc}
0 & 0.7 & 0.9 & 1  \tag{30}\\
0.7 & 0 & 0 & 0.8 \\
0.9 & 0 & 0 & 0 \\
1 & 0.8 & 0 & 0
\end{array}\right)
$$

Suppose that the order of the fractional multiagent dynamics is $\alpha=0.8$ and the system control gain is $\gamma=1$; then we can obtain the relationship between the sampling period and the upper bound of sampling delays (Figure 2) from the conditions in Theorem 7. In order to make the system meet the condition of reaching consensus, we can set the sampling period according to the sampling delay of the system or decide the stable fields of sampling delays by means of the sampling period. Suppose the order of the fractional multiagent dynamics is $\alpha=0.8$ and the control gain $\gamma=1$; then we can obtain that the upper bound of sampling period $h_{\max }=1.0993 \mathrm{~s}$ corresponding delay $\tau=0.2632 \mathrm{~s}$ from Figure 2.


Figure 1: Network topology of the multiagent systems.


Figure 2: Relationship between the communication delay and the upper bound of sampling period.

Simulation 1. Assume the sampling delay of multiagent system is $\tau=0.25 \mathrm{~s}$, we can obtain the upper bound of sampling period is 1.0272 s . In computer simulation, selecting the sampling period $h=0.90 \mathrm{~s}$, the consensus can be asymptotically reached (Figure 3) through fractional-order coordination algorithm.

Simulation 2. Assume the sampled delay of multiagent system is $\tau=0.49 \mathrm{~s}$, the upper bound of sampling period is 0.5054 in Figure 2. In computer simulation, selecting the sampling period $h=0.50 \mathrm{~s}$, the consensus can be asymptotically reached much more slowly with increasing of time delay (Figure 4).

Simulation 3. Assume the sampling delay of multiagent system is $\tau=0.50 \mathrm{~s}$, the upper bound of sampling period is 0.4928 in Figure 2. In computer simulation, selecting the sampling period $h=0.50 \mathrm{~s}$, the movement trajectories of the multiagent systems will be asymptotically diverged and the consensus cannot be reached (Figure 5) through fractionalorder coordination algorithm.

Simulation 4. Let time delay continue increasing, suppose the sampling delay is $\tau=0.80 \mathrm{~s}$, the upper bound of sampling period will be 0.2739 in Figure 2. In computer simulation, selecting the sampling period $h=0.25 \mathrm{~s}$, the movement trajectories of the multiagent systems will be asymptotically converged and the consensus can be reached (Figure 6).


Figure 3: Movement trajectories of the multiagent systems with delay 0.25 s and sampling period 0.90 s .


Figure 4: Movement trajectories of the multiagent systems with delay 0.49 s and sampling period 0.50 s .

Although the sampling period is less than the sampling delay, the consensus can be still achieved under the condition of Theorem 7.

## 7. Conclusions

This paper studies distributed coordination of fractionalorder multiagent system with sampled control and sampling delay. By applying the stability theory of discretetime domain, sampled-data control of FOMAS with directed network topology is investigated, and the upper bound of the sampling period is obtained. Based on the HermiteBiehler Theorem, the collaborative control of fractional-order multiagent systems with sampled data and sampling delay is


Figure 5: Movement trajectories of the multiagent systems with delay 0.50 s and sampling period 0.50 s .


Figure 6: Movement trajectories of the multiagent systems with delay 0.80 s and sampling period 0.25 s .
studied. The relations between sampling delay and sampled period are obtained to ensure the consensus of FOMAS. Research of the robust stability of FOMAS will be carried out in the following work.

## Disclosure

This paper has not been published previously and is not under consideration of publication elsewhere.

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# Robust $H_{\infty}$ Control of Uncertain T-S Fuzzy Time-Delay System: A Delay Decomposition Approach 

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#### Abstract

This paper is concerned with the problem of robust $H_{\infty}$ control for a class of uncertain time-delay fuzzy systems with normbounded parameter uncertainties. By utilizing the instrumental idea of delay decomposition, the decomposed Lyapunov-Krasovskii functional is introduced to uncertain T-S fuzzy system, and some delay-dependent conditions for the existence of robust controller are formulated in the form of linear matrix inequalities (LMIs). When these LMIs are feasible, a controller is presented. A numerical example is given to demonstrate the effectiveness of the proposed method.


## 1. Introduction

It is well known that time delay is built-in features in various nonlinear systems such as tandem mills, remote control systems, long transmission lines in pneumatic systems, and chemical system. The time delay is recognized to be a source of instability and performance deterioration of control systems. Therefore, stability analysis and controller synthesis for time-delay system have been one of the most hot research area in the control community over the past years [1-14].

Fuzzy systems in the form of the Takagi-Sugeno (T-S) model have attracted rapidly growing interest in recent years. It has been shown that the T-S model method is a simple and effective way to represent complex nonlinear systems by a set of simple local linear dynamic systems with their linguistic description [12, 15-19]. Over the past few years, most work has been devoted to analysis and synthesis of T-S fuzzy control systems. See the survey papers $[16,17]$ and the reference citied therein for the most recent advances on this topic. The appeal and superiority of T-S fuzzy models is that the analysis and synthesis of the overall fuzzy systems can be carried out in the Lyapunov-function-based framework. To mention a few, by using LMI, Cao and Frank presented controller design for a class of fuzzy dynamic systems with time delay in both continuous and discrete cases in [20,21]. Wu et al. studied the
model approximation problem and $L_{2}-L_{\infty}$ control problem for nonlinear time-delay systems in [22, 23]. Moreover, great attention from researchers has been drawn to the study of stability analysis and controller design for T-S fuzzy systems with time delays [24-28]. On the other hand, type-2 fuzzy mode are considered in $[29,30]$.

Recently, many scholars studied the stability problem based on the piecewise Lyapunov-Krasovskii functional [3133]. Reference [31] investigated the linear continuous/discrete systems with time-varying delay and divided the variation interval of the time delay into several subintervals. based on this method, [32]addressed the problem of the robust $H_{\infty}$ filtering for singular linear parameter varying (LPV). Reference [33] researched the stability of linear time-invariant systems and divided the delay interval into $N$ subintervals. The simulations show these methods can lead to much less conservative results than those in the existing references.

Motivated by the above observations, in this paper, we will investigate the problem of robust $H_{\infty}$ control of uncertain T-S fuzzy systems with constant delay. Attention is focused on the design of robust $H_{\infty}$ controllers via the parallel distributed compensation scheme such that the closed-loop fuzzy time-delay system is asymptotically stable and the $H_{\infty}$ disturbance attenuation is below a prescribed level.

Based on delay decomposition approach [33], the decomposed Lyapunov-Krasovskii functional is introduced, and some delay-dependent conditions have been obtained. These conditions are formulated in the form of LMIs, and the controller design is cast into a convex optimization problem subject to LMI constraints, which can be readily solved via standard numerical software. Finally, a numerical example is provided to show the effectiveness and less conservatism of the proposed results.

The rest of this paper is organized as follows. In Section 2, the model description and problem are first formulated. The main results for delay-dependent robust $H_{\infty}$ controller are presented in Section 3. Illustrative examples are given in Section 4, and the paper is concluded in Section 5.

Notations. The notations used throughout this paper are fairly standard. The superscript " $T$ " stands for matrix transpose, and the notation $P>0(P \geq 0)$ means that matrix $P$ is real symmetric and positive (or being positive semidefinite). $I$ and 0 are used to denote appropriate dimensions identity matrix and zero matrix, respectively. The notation * in a symmetric always denotes the symmetric block in the matrix. The parameter $\operatorname{diag}\{\cdots\}$ denotes a block-diagonal matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

## 2. System Descriptions and Preliminaries

Consider the uncertain nonlinear system with state delay that is described by the following T-S model with uncertain parameter matrices.

Plant Rule $i$. IF $s_{1}(t)$ is $F_{i 1}$ and $s_{2}(t)$ is $F_{i 2}$ and $\ldots$ and $s_{n}(t)$ is $F_{\text {in }}$ THEN

$$
\begin{gather*}
\dot{x}(t)=\left(A_{i}+\Delta A_{i}\right) x(t)+\left(A_{d i}+\Delta A_{d i}\right) x(t-\tau) \\
+\left(B_{1 i}+\Delta B_{1 i}\right) u(t)+B_{2 i} \omega(t), \\
z(t)=C_{i} x(t),  \tag{1}\\
x(t)=\phi(t), \quad t \in[-\tau, 0], \quad i=1,2, \ldots, r,
\end{gather*}
$$

where $s_{1}(t), s_{2}(t), \ldots, s_{n}(t)$ are the premise variables that are measurable and each $F_{i j}(j=1,2, \ldots, n)$ is fuzzy set. $x(t) \in R^{n}$ is the state vector and $u(t) \in R^{m}$ is the control input vector. $z(t)$ is the output vector. $\omega(t) \in R^{q}$ is the disturbance input vector belongs to $L_{2}[0, \infty) . r$ is the number of IF-THEN rules, $\tau$ is the constant delay in the state. $\phi(t)$ is a vector-valued initial continuous function.

The matrices $\Delta A_{i}, \Delta A_{d i}$, and $\Delta B_{1 i}$ denote the parameters uncertainties, which are assumed of the form

$$
\begin{equation*}
\left[\Delta A_{i}, \Delta A_{d i}, \Delta B_{1 i}\right]=\operatorname{MF}(t)\left[E_{i}, E_{d i}, E_{1 i}\right] \tag{2}
\end{equation*}
$$

where $M, E_{i}, E_{d i}$, and $E_{1 i}$ are known constant matrices and $F(t)$ is an unknown time-varying matrix function satisfying $F^{T}(t) F(t) \leq I$.

For simplicity, introduce the following notations:

$$
\begin{equation*}
\bar{A}_{i}=A_{i}+\Delta A_{i} \quad \bar{A}_{d i}=A_{d i}+\Delta A_{d i} \quad \bar{B}_{1 i}=B_{1 i}+\Delta B_{1 i} \tag{3}
\end{equation*}
$$

By using a center-average defuzzier, product fuzzy inference, and a singleton fuzzifier, the following global T-S fuzzy model can be obtained:

$$
\begin{align*}
& \begin{aligned}
& \dot{x}(t)=\left(\sum _ { i = 1 } ^ { r } \alpha _ { i } ( s ( t ) ) \left[\left(A_{i}+\Delta A_{i}\right) x(t)\right.\right. \\
&+\left(A_{d i}+\Delta A_{d i}\right) x(t-\tau) \\
&\left.\left.+\left(B_{1 i}+\Delta B_{1 i}\right) u(t)+B_{2 i} \omega(t)\right]\right) \\
& \quad \times\left(\sum_{i=1}^{r} \alpha_{i}(s(t))\right)^{-1} \\
&=\sum_{i=1}^{r} \mu_{i}(s(t))\left[\left(A_{i}+\Delta A_{i}\right) x(t)+\left(A_{d i}+\Delta A_{d i}\right) x(t-\tau)\right. \\
&\left.\quad+\left(B_{1 i}+\Delta B_{1 i}\right) u(t)+B_{2 i} \omega(t)\right], \\
& z(t)= \sum_{i=1}^{r} \mu_{i}(s(t)) C_{i} x(t) \\
&= C(t) x(t), x(t) \\
&= \phi(t) \quad t \in[-\tau, 0]
\end{aligned} \\
&
\end{align*}
$$

where $\alpha_{i}(s(t))=\prod_{j=1}^{n} F_{i j}\left(s_{j}(t)\right), \mu_{i}(s(t))=\alpha_{i}(s(t)) /$ $\sum_{i=1}^{r} \alpha_{i}(s(t))$, and $F_{i j}\left(s_{j}(t)\right)$ is the grade of membership of $s_{j}(t)$ in $F_{i j}$, and it is assumed that $\alpha_{i}(s(t)) \geq 0, i=$ $1,2, \ldots, r \sum_{i=1}^{r} \alpha_{i}(s(t))>0$ for all $t$. Therefore, $\mu_{i}(s(t)) \geq 0$ and $\sum_{i=1}^{r} \mu_{i}(s(t))=1$ for all $t$.

In this paper, employing the idea of parallel distributed compensation (PDC), the T-S fuzzy-model-based controller via the PDC can be constructed as follows.

Controller Rule i. IF $s_{1}(t)$ is $F_{i 1}$ and $s_{2}(t)$ is $F_{i 2}$ and $\ldots$ and $s_{n}(t)$ is $F_{\text {in }}$ THEN

$$
\begin{equation*}
u(t)=K_{i} x(t) \tag{5}
\end{equation*}
$$

where $K_{i}(i=1,2, \ldots, r)$ are the controller gains of (5) to be determined.

Then, the overall output of the controller rules is given by

$$
\begin{equation*}
u(t)=\sum_{i=1}^{r} \mu_{i}(s(t)) K_{i} x(t) \tag{6}
\end{equation*}
$$

Substituting (6) into (4), the closed-loop system can be given as

$$
\begin{gather*}
\dot{x}(t)=\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i}(s(t)) \mu_{j}(s(t)) \\
\times\left[\left(\bar{A}_{i}+\bar{B}_{1 i} K_{j}\right) x(t)+\bar{A}_{d i} x(t-\tau)+B_{2 i} \omega(t)\right] \\
z(t)=\sum_{i=1}^{r} \mu_{i}(s(t)) C_{i} x(t) \tag{7}
\end{gather*}
$$

with its compact form

$$
\begin{gather*}
\dot{x}(t)=\left(\bar{A}(t)+\bar{B}_{1}(t) K(t)\right) x(t) \\
+\bar{A}_{d}(t) x(t-\tau)+B_{2}(t) \omega(t),  \tag{8}\\
z(t)=C(t) x(t), \tag{9}
\end{gather*}
$$

where

$$
\begin{align*}
& \bar{A}(t)=A(t)+\Delta A(t)=\sum_{i=1}^{r} \mu_{i}(s(t))\left[A_{i}+\Delta A_{i}(t)\right] \\
& \bar{A}_{d}(t)=A_{d}(t)+\Delta A_{d}(t)=\sum_{i=1}^{r} \mu_{i}(s(t))\left[A_{d i}+\Delta A_{d i}(t)\right] \\
& \bar{B}_{1}(t)=B_{1}(t)+\Delta B_{1}(t)=\sum_{i=1}^{r} \mu_{i}(s(t))\left[B_{1 i}+\Delta B_{1 i}(t)\right] \\
& B_{2}(t)=\sum_{i=1}^{r} \mu_{i}(s(t)) B_{2 i}, \quad C(t)=\sum_{i=1}^{r} \mu_{i}(s(t)) C_{i}, \\
& K(t)=\sum_{i=1}^{r} \mu_{i}(s(t)) K_{i} . \tag{10}
\end{align*}
$$

Before ending this section, we introduce the following definitions and lemmas, which will be used in the derivation of our main results.

Definition 1 ( $H_{\infty}$ performance). Given a scalar $\gamma>0$ and under zero initial condition, the system (1) is said to be asymptotically stable with $\gamma$-disturbance attenuation if the system (4) is asymptotically stable and the output $z(t)$ satisfies

$$
\begin{equation*}
\|z(t)\|_{2} \leq \gamma\|\omega(t)\|_{2} \tag{11}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\int_{0}^{\infty}\left[z^{T}(t) z(t)-\gamma^{2} \omega^{T}(t) \omega(t)\right] d t \leq 0 \tag{12}
\end{equation*}
$$

for all nonzero $\omega(t) \in L_{2}[0, \infty)$.
Lemma 2 (see [34]). For any constant matrix $W \in$ $R^{n \times n}, W=W^{T}>0$, scalar $r>0$, and vector-valued function $\dot{x}:[-r, 0] \rightarrow R^{n}$ such that the following integration is well defined; then

$$
\begin{align*}
& -\int_{t-r}^{t} \dot{x}^{T}(s)(r W) \dot{x}(s) d s \\
& \quad \leq\left(x^{T}(t) \quad x^{T}(t-r)\right)\left(\begin{array}{cc}
-W & W \\
W & -W
\end{array}\right)\binom{x(t)}{x(t-r)} \tag{13}
\end{align*}
$$

Lemma 3 (see [35]). Given a symmetric matrix $M$ and matrices $D, F(t)$, and $E$ of compatible dimensions, then, for $F^{T}(t) F(t) \leq I$, the inequality

$$
\begin{equation*}
M+D F(t) E+(D F(t) E)^{T}<0 \tag{14}
\end{equation*}
$$

holds if and only if there exists a scalar $\varepsilon>0$ such that

$$
\begin{equation*}
M+\varepsilon D D^{T}+\varepsilon^{-1} E^{T} E<0 \tag{15}
\end{equation*}
$$

## 3. Main Results

In this section, some delay-dependent sufficient conditions on the existence of robust $H_{\infty}$ controller for T-S fuzzy system (7) will be presented. A Lyapunov-Krasovskii functional, based on the idea of delay decomposition approach, will be introduced, which can potentially reduce the conservatism of the results.

To this end, we first consider the following nominal closed-loop system:

$$
\begin{gather*}
\dot{x}(t)=\left(A(t)+B_{1}(t) K(t)\right) x(t) \\
+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t),  \tag{16}\\
z(t)=C(t) x(t) . \tag{17}
\end{gather*}
$$

Firstly, the sufficient condition of $H_{\infty}$ performance analysis for the unforced case of system (16) is established in Proposition 4.

Proposition 4. For some prescribed $\gamma>0$ and $\tau>0$, the unforced case of system (16) is asymptotically stable with a guaranteed $H_{\infty}$ performance $\gamma$, if there exist matrices $P>0$, $R_{l}>0$, and $Q_{l}>0(l=1,2, \ldots, N)$ such that the following LMIs hold for $i=1,2, \ldots, r$ :

$$
\Sigma=\left[\begin{array}{ccc}
\Sigma^{(1)} & \Sigma^{(2)} & \Sigma^{(3)}  \tag{18}\\
* & -\gamma^{2} I & \Sigma^{(4)} \\
* & * & \Sigma^{(5)}
\end{array}\right]<0
$$

where

$$
\left.\begin{array}{c}
\Sigma^{(1)}=\left[\begin{array}{cccccc}
\Sigma_{11}^{(1)} & R_{1} & 0 & \cdots & 0 & P A_{d i} \\
* & \Sigma_{22}^{(1)} & R_{2} & \cdots & 0 & 0 \\
* & * & \Sigma_{33}^{(1)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & \Sigma_{N N}^{(1)} & R_{N} \\
* & * & * & \cdots & * & \Sigma_{N+1}^{(1)},
\end{array}\right], \\
\Sigma_{11}^{(1)}=Q_{1}-R_{1}+A_{i}^{T} P+P A_{i}+C_{i}^{T} C_{i}, \\
\Sigma_{22}^{(1)}=Q_{2}-Q_{1}-R_{1}-R_{2}, \\
\Sigma_{33}^{(1)}=Q_{3}-Q_{2}-R_{2}-R_{3}, \\
\vdots \\
\Sigma_{N N}^{(1)}=Q_{N}-Q_{N-1}-R_{N-1}-R_{N}, \\
\Sigma_{N+1}^{(1)} N+1
\end{array}\right]-Q_{N}-R_{N},
$$

$$
\begin{align*}
& \Sigma^{(2)}=\left[\begin{array}{c}
P B_{2 i} \\
0 \\
\vdots \\
0
\end{array}\right], \\
& \Sigma^{(3)}=\left[\begin{array}{cccc}
h A_{i}^{T} R_{1} & h A_{i}^{T} R_{2} & \cdots & h A_{i}^{T} R_{N} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
h A_{d i}^{T} R_{1} & h A_{d i}^{T} R_{2} & \cdots & h A_{d i}^{T} R_{N}
\end{array}\right] \\
& \Sigma^{(4)}=\left[\begin{array}{llll}
h B_{2 i}^{T} R_{1} & h B_{2 i}^{T} R_{2} & \cdots & h B_{2 i}^{T} R_{N}
\end{array}\right], \\
& \Sigma^{(5)}=\operatorname{diag}\left\{-R_{1},-R_{2}, \ldots,-R_{N}\right\} . \tag{19}
\end{align*}
$$

Proof. Choose a Lyapunov-Krasovskii functional candidate as

$$
\begin{gather*}
V(t)=V_{1}(t)+V_{2}(t)+V_{3}(t), \\
V_{1}(t)=x^{T}(t) P x(t), \\
V_{2}(t)=\sum_{l=1}^{N} \int_{t-l h}^{t-(l-1) h} x^{T}(s) Q_{l} x(s) d s  \tag{20}\\
V_{3}(t)=\sum_{l=1}^{N} \int_{-l h}^{-(l-1) h} \int_{t+\theta}^{t} \dot{x}^{T}(s)\left(h R_{l}\right) \dot{x}(s) d s d \theta
\end{gather*}
$$

where $h=\tau / N$ and $N$ is the partitioning number of time delay $\tau$.

Taking the derivative of $V_{i}(t)$, for $i=1,2,3$, with respect to $t$ along the trajectory of unforced case of system (16) yields

$$
\begin{aligned}
\dot{V}_{1}(t)= & \dot{x}^{T}(t) P x(t)+x^{T}(t) P \dot{x}(t) \\
= & \left(A(t) x(t)+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right)^{T} \\
& \times P x(t)+x^{T}(t) P(A(t) x(t) \\
& \left.+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right), \\
& \left.\quad \times Q_{l} x(t-(l-1) h)-x^{T}(t-l h) Q_{l} x(t-l h)\right)
\end{aligned}
$$

$$
\begin{align*}
= & \sum_{l=1}^{N} x^{T}(t-(l-1) h) Q_{l} x(t-(l-1) h) \\
& -\sum_{l=1}^{N} x^{T}(t-l h) Q_{l} x(t-l h) \\
\dot{V}_{3}(t)= & \sum_{l=1}^{N} h \dot{x}^{T}(t)\left(h R_{l}\right) \dot{x}(t) \\
& -\sum_{l=1}^{N} \int_{t-l h}^{t-(l-1) h} \dot{x}^{T}(s)\left(h R_{l}\right) \dot{x}(s) d s . \tag{21}
\end{align*}
$$

By using Lemma 2, we obtain that

$$
\begin{aligned}
& \dot{V}_{3}(t) \\
& \leq \sum_{l=1}^{N} h \dot{x}^{T}(t)\left(h R_{l}\right) \dot{x}(t) \\
& \quad+\sum_{l=1}^{N}\left(x^{T}(t-(l-1) h)\right. \\
& \left.\quad x^{T}(t-l h)\right) \\
& \quad \times\left(\begin{array}{cc}
-R_{l} & R_{l} \\
R_{l} & -R_{l}
\end{array}\right)\binom{x(t-(l-1) h)}{x(t-l h)}
\end{aligned}
$$

$$
=\sum_{l=1}^{N}\left(A(t) x(t)+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right)^{T}
$$

$$
\times\left(h^{2} R_{l}\right)\left(A(t) x(t)+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right)
$$

$$
+\sum_{l=1}^{N}\left(x^{T}(t-(l-1) h) \quad x^{T}(t-l h)\right)
$$

$$
\times\left(\begin{array}{cc}
-R_{l} & R_{l} \\
R_{l} & -R_{l}
\end{array}\right)\binom{x(t-(l-1) h)}{x(t-l h)}
$$

$$
=\left(A(t) x(t)+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right)^{T}
$$

$$
\times\left(\sum_{l=1}^{N} h^{2} R_{l}\right)\left(A(t) x(t)+A_{d}(t) x(t-\tau)+B_{2}(t) \omega(t)\right)
$$

$$
+\sum_{l=1}^{N}\left(x^{T}(t-(l-1) h) \quad x^{T}(t-l h)\right)\left(\begin{array}{cc}
-R_{l} & R_{l} \\
R_{l} & -R_{l}
\end{array}\right)\binom{x(t-(l-1) h)}{x(t-l h)}
$$

$$
=\left[\begin{array}{c}
x(t) \\
x(t-\tau) \\
\omega(t)
\end{array}\right]^{T}
$$

$$
\times\left[\begin{array}{lll}
A^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) A(t) & A^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) A_{d}(t) & A^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) B_{2}(t) \\
A_{d}^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) A(t) & A_{d}^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) A_{d}(t) & A_{d}^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) B_{2}(t) \\
B_{2}^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{j}\right) A(t) & B_{2}^{T}(t)\left(\sum_{j=1}^{N} h^{2} R_{l}\right) A_{d}(t) & B_{2}^{T}(t)\left(\sum_{l=1}^{N} h^{2} R_{l}\right) B_{2}(t)
\end{array}\right]
$$

$$
\times\left[\begin{array}{c}
x(t) \\
x(t-\tau) \\
\omega(t)
\end{array}\right]
$$

$$
+\sum_{l=1}^{N}\left(x^{T}(t-(l-1) h) \quad x^{T}(t-l h)\right)
$$

$$
\times\left(\begin{array}{cc}
-R_{l} & R_{l}  \tag{22}\\
R_{l} & -R_{l}
\end{array}\right)\binom{x(t-(l-1) h)}{x(t-l h)} .
$$

Define the variable $\eta^{T}(t)=\left[x^{T}(t), x^{T}(t-h), \ldots, x^{T}(t-(N-\right.$ 1)h), $x^{T}(t-\tau)$ ] and by simple manipulation, we have

$$
\begin{align*}
\dot{V}(t) & =\dot{V}_{1}(t)+\dot{V}_{2}(t)+\dot{V}_{3}(t)  \tag{23}\\
& \leq \xi^{T}(t)\left(\Pi_{1}+\Pi_{2}+\Pi_{3}\right) \xi(t)
\end{align*}
$$

where

$$
\begin{aligned}
& \Pi_{1} \\
& =\left[\begin{array}{cccc}
A^{T}(t) P+P A(t) & \cdots & P A_{d}(t) & P B_{2}(t) \\
\vdots & \ddots & \vdots & \vdots \\
A_{d}^{T}(t) P & \cdots & 0 & 0 \\
B_{2}^{T}(t) P & \cdots & 0 & 0
\end{array}\right],
\end{aligned}
$$

$$
\begin{align*}
& \Pi_{3} \\
& =\left[\begin{array}{ccccccc}
-R_{1} & R_{1} & 0 & \ldots & 0 & 0 & 0 \\
* & -R_{1}-R_{2} & R_{2} & \ldots & 0 & 0 & 0 \\
* & * & -R_{2}-R_{3} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
* & * & * & \ldots & -R_{N-1}-R_{N} & R_{N} & 0 \\
* & * & * & \ldots & * & -R_{N} & 0 \\
* & * & * & \ldots & * & * & 0
\end{array}\right] \\
& -\left[\begin{array}{c}
\Sigma^{(3)} \\
\Sigma^{(4)}
\end{array}\right]\left(\Sigma^{(5)}\right)^{-1}\left[\begin{array}{c}
\Sigma^{(3)} \\
\Sigma^{(4)}
\end{array}\right]^{T}, \\
& \xi(t)=\left[\begin{array}{ll}
\eta^{T}(t) & \omega^{T}(t)
\end{array}\right]^{T} . \tag{24}
\end{align*}
$$

First, we prove the asymptotic stability of the system in (16). To this end, assume $\omega(\mathrm{t}) \equiv 0$, and thus $\Pi_{1}+\Pi_{2}+\Pi_{3}$ in (23) reads

$$
\begin{equation*}
\widetilde{\Pi}_{1}+\widetilde{\Pi}_{2}+\widetilde{\Pi}_{3} \tag{25}
\end{equation*}
$$

where

$$
\widetilde{\Pi}_{1}=\left[\begin{array}{ccc}
A^{T}(t) P+P A(t) & \cdots & P A_{d}(t) \\
\vdots & \ddots & \vdots \\
* & \cdots & 0
\end{array}\right]
$$

$$
\begin{gather*}
\widetilde{\Pi}_{2}=\left[\begin{array}{ccccc}
Q_{1} & 0 & \cdots & 0 \\
* & Q_{2}-Q_{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & & 0 \\
* & * & \cdots & Q_{N}-Q_{N-1}
\end{array}\right], \\
\widetilde{\Pi}_{3}=\left[\begin{array}{cccccc}
-R_{1} & R_{1} & 0 & \cdots & 0 & 0 \\
* & -R_{1}-R_{2} & R_{2} & \cdots & 0 & 0 \\
* & * & -R_{2}-R_{3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & -R_{N-1}-R_{N} & 0 \\
* & * & * & \cdots & * & -R_{N}
\end{array}\right] \\
 \tag{26}\\
-\left(\Sigma^{(3)}\right)\left(\Sigma^{(5)}\right)^{-1}\left(\Sigma^{(3)}\right)^{T} .
\end{gather*}
$$

From (18), we know that

$$
\begin{equation*}
\widetilde{\Pi}_{1}+\widetilde{\Pi}_{2}+\widetilde{\Pi}_{3}<0 \tag{27}
\end{equation*}
$$

which guarantees $\dot{V}(t)<0$ for all non-zero $\eta(t)$. Thus one can always find a sufficiently small $v>0$ such that $\dot{V}(t)<$ $-v\|x(t)\|^{2}$. The asymptotic stability of the considered system is proved.

Next, assuming that $\omega(t) \neq 0$ and $\phi(t)=0, t \in[-\tau, 0]$, we consider $H_{\infty}$ performance of the system in Definition 1.

Considering $\mu_{i}(s(t)) \geq 0, \sum_{i=1}^{r} \mu_{i}(s(t))=1$, we obtain that

$$
\begin{equation*}
z^{T}(t) z(t)-\gamma^{2} \omega^{T}(t) \omega(t)+\dot{V}(t) \leq \sum_{i=1}^{r} \mu_{i}(s(t)) \xi^{T}(t) \Psi \xi(t), \tag{28}
\end{equation*}
$$

where

$$
\Psi=\left[\begin{array}{cc}
\Sigma^{(1)} & \Sigma^{(2)}  \tag{29}\\
* & -\gamma^{2} I
\end{array}\right]-\left[\begin{array}{c}
\Sigma^{(3)} \\
\Sigma^{(4)}
\end{array}\right]\left[\Sigma^{(5)}\right]^{-1}\left[\begin{array}{c}
\Sigma^{(3)} \\
\Sigma^{(4)}
\end{array}\right]^{T} .
$$

Applying Schur complement, guarantees $\Psi<0$. From (28), we can get that

$$
\begin{equation*}
z^{T}(t) z(t)-\gamma^{2} \omega^{T}(t) \omega(t)+\dot{V}(t) \leq 0 \tag{30}
\end{equation*}
$$

Integrating the preceding inequality from 0 to $\infty$, it is easy to get that

$$
\begin{equation*}
\int_{0}^{\infty} z^{T}(t) z(t) d t \leq \int_{0}^{\infty} \gamma^{2} \omega^{T}(t) \omega(t) d t+V\left(x_{0}\right)-V\left(x_{\infty}\right) \tag{31}
\end{equation*}
$$

Since $V\left(x_{0}\right)=0$ and $V\left(x_{\infty}\right) \geq 0$, we have

$$
\begin{equation*}
\int_{0}^{\infty} z^{T}(t) z(t) d t \leq \int_{0}^{\infty} \gamma^{2} \omega^{T}(t) \omega(t) d t \tag{32}
\end{equation*}
$$

Then, according to Definition 1, the $H_{\infty}$ performance of the system in (16) is established. This completes the proof.

In the following, based on Proposition 4, we design robust state feedback $H_{\infty}$ controller for the system (7).

Theorem 5. For some prescribed $\gamma>0, \tau>0, \delta>0$, and $N$ is a positive integer, if there exist scalar $\varepsilon>0$, matrices $X>0, V_{l}>0(l=2,3, \ldots, N)$, and $Q_{l}>0(l=1,2 \ldots N)$, and appropriate dimension matrices $W_{j}(j=1,2, \ldots, r)$ such that the following LMIs simultaneously hold for $i$, $j=1,2, \ldots, r$ :

$$
\begin{align*}
& \Omega_{i i}<0, \quad(i=1,2, \ldots, r), \\
& \frac{\left(\Omega_{i j}+\Omega_{j i}\right)}{2}<0, \quad(1 \leq i<j \leq r),  \tag{33}\\
& V_{i} R_{i}=I, \quad(i=1,2, \ldots, N),
\end{align*}
$$

where

$$
\begin{aligned}
& \Omega_{i j}=\left[\begin{array}{ccc}
\Upsilon_{i j} & \varepsilon U & T_{i j}^{T} \\
* & -\varepsilon I & 0 \\
* & * & -\varepsilon I
\end{array}\right], \quad \Upsilon_{i j}=\left[\begin{array}{cccc}
\Pi_{i j}^{(1)} & \Pi_{i}^{(2)} & \Pi_{i j}^{(3)} & \Pi_{i}^{(6)} \\
* & -\gamma^{2} I & \Pi_{i}^{(4)} & 0 \\
* & * & \Pi^{(5)} & 0 \\
* & * & * & -I
\end{array}\right], \\
& U=\left[\begin{array}{cccc}
H & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & * & N & 0 \\
* & * & * & 0
\end{array}\right], \quad T_{i j}=\left[\begin{array}{cccc}
S_{i j} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
L_{i j} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \\
& H=\left[\begin{array}{cccccc}
M & 0 & 0 & \cdots & 0 & 0 \\
* & 0 & 0 & \cdots & 0 & 0 \\
* & * & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & 0 & 0 \\
* & * & * & \cdots & * & 0
\end{array}\right], \\
& S_{i j}=\left[\begin{array}{cccccc}
E_{i} X+E_{1 i} W_{j} & 0 & 0 & \cdots & 0 & E_{d i} \\
* & 0 & 0 & \cdots & 0 & 0 \\
* & * & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & 0 & 0 \\
* & * & * & \cdots & * & 0
\end{array}\right], \\
& L_{i j}^{T} \\
& =\left[\begin{array}{cccc}
h X E_{i}^{T}+h W_{j}^{T} E_{1 i}^{T} & h X E_{i}^{T}+h W_{j}^{T} E_{1 i}^{T} & \cdots & h X E_{i}^{T}+h W_{j}^{T} E_{1 i}^{T} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
h E_{d i}^{T} & h E_{d i}^{T} & \cdots & h E_{d i}^{T}
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
& \Pi_{i j}^{(1)}=\left[\begin{array}{cccccc}
\Pi_{11 i j}^{(1)} & \delta I & 0 & \cdots & 0 & A_{d i} \\
* & \Pi_{22}^{(1)} & R_{2} & \cdots & 0 & 0 \\
* & * & \Pi_{33}^{(1)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & \Pi_{N N}^{(1)} & R_{N} \\
* & * & * & \cdots & * & \Pi_{N+1 ~ N+1}^{(1)}
\end{array}\right] \\
& N^{T}=\left[\begin{array}{cccccc}
M^{T} & 0 & 0 & \cdots & 0 & 0 \\
* & M^{T} & 0 & \cdots & 0 & 0 \\
* & * & M^{T} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \cdots & M^{T} & 0 \\
* & * & * & \cdots & * & M^{T}
\end{array}\right], \\
& \Pi_{11 i j}^{(1)}=X Q_{1} X-\delta X+X A_{i}^{T}+W_{j}^{T} B_{1 i}^{T}+A_{i} X+B_{1 i} W_{j}, \\
& \Pi_{22}^{(1)}=Q_{2}-Q_{1}-\delta P-R_{2}, \\
& \Pi_{33}^{(1)}=Q_{3}-Q_{2}-R_{2}-R_{3}, \\
& \Pi_{N N}^{(1)}=Q_{N}-Q_{N-1}-R_{N-1}-R_{N}, \\
& \Pi_{N+1 N+1}^{(1)}=-Q_{N}-R_{N}, \\
& \Pi_{i}^{(4)}=\left[\begin{array}{llll}
h B_{2 i}^{T} & h B_{2 i}^{T} & \cdots & h B_{2 i}^{T}
\end{array}\right], \\
& \Pi^{(5)}=\operatorname{diag}\left\{-\delta^{-1} X,-V_{2}, \ldots,-V_{N}\right\}, \\
& \Pi_{i}^{(2)}=\left[\begin{array}{c}
B_{2 i} \\
0 \\
\vdots \\
0
\end{array}\right], \\
& \Pi_{i j}^{(3)}=\left[\begin{array}{cccc}
h X A_{i}^{T}+h W_{j}^{T} B_{1 i}^{T} & h X A_{i}^{T}+h W_{j}^{T} B_{1 i}^{T} & \cdots & h X A_{i}^{T}+h W_{j}^{T} B_{1 i}^{T} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
h A_{d i}^{T} & h A_{d i}^{T} & \cdots & h A_{d i}^{T}
\end{array}\right] \\
& \Pi_{i}^{(6)}=\left[\begin{array}{c}
X C_{i}^{T} \\
0 \\
\vdots \\
0
\end{array}\right] . \tag{34}
\end{align*}
$$

Then the closed-loop system (16) is asymptotically stable with the $H_{\infty}$ performance index $\gamma$. Moreover, if the above condition is feasible, the gain matrices of a desired controller in the form of (6) can be designed by

$$
\begin{equation*}
K_{j}=W_{j} X^{-1} \tag{35}
\end{equation*}
$$

Proof. The proof of this theorem is divided into two parts. First, we design the state-feedback $H_{\infty}$ controller of the nominal case of closed-loop system (7).

According to Proposition 4, it is easy to know that the $H_{\infty}$ performance requirement of the nominal case of closed-loop system (7) implies

$$
\begin{equation*}
\tilde{\Sigma}<0 \tag{36}
\end{equation*}
$$

where $\widetilde{\Sigma}$ is a matrix derived from (18) by changing the term $A_{i}$ to $A_{i}+B_{1 i} K_{j}$.

Introduce the following matrix variables:

$$
\begin{gather*}
X=P^{-1}, \quad R_{1}=\delta P  \tag{37}\\
V_{2}=R_{2}^{-1}, \ldots, V_{N}=R_{N}^{-1}, \quad K_{j} X=W_{j} .
\end{gather*}
$$

Combined with Schur complement, and pre- and postmultiply (36) by $\operatorname{diag}\left\{X, I, \delta^{-1} X, V_{2}, \ldots, V_{N}, I\right\}$ and its transpose, respectively, we get

$$
\begin{equation*}
Y_{i j}<0 \tag{38}
\end{equation*}
$$

Next, we investigate robust state feedback $H_{\infty}$ controller of the closed-loop system (7).

Replace $A_{i}, \mathrm{~A}_{d i}$, and $B_{i}$ with $A_{i}+\Delta A_{i}, \mathrm{~A}_{d i}+\Delta \mathrm{A}_{d i}$, and $B_{1 i}+\Delta B_{1 i}$, respectively; then we obtain from (2) and (38) that

$$
\begin{equation*}
\Upsilon_{i j}+U F(t) T_{i j}+T_{i j}^{T} F(t) U^{T}<0 \tag{39}
\end{equation*}
$$

By Lemma 3, we can know (39) holds, if and only if the following inequality holds:

$$
\begin{equation*}
\Upsilon_{i j}+\varepsilon U U^{T}+\varepsilon^{-1} T_{i j}^{T} T_{i j}<0 \tag{40}
\end{equation*}
$$

where $\varepsilon$ is a positive scalar.
Applying Schur complement to (40), we have that

$$
E_{i j}=\left[\begin{array}{ccc}
\Upsilon_{i j} & \varepsilon U & T_{i j}^{T}  \tag{41}\\
* & -\varepsilon I & 0 \\
* & * & -\varepsilon I
\end{array}\right]<0
$$

Noting $\mu_{i}(s(t)) \geq 0, \quad \sum_{i=1}^{r} \mu_{i}(s(t))=1$,

$$
\begin{align*}
& \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i}(s(t)) \mu_{j}(s(t)) E_{i j} \\
& \quad=\sum_{i=1}^{r} \mu_{i}^{2}(s(t)) E_{i i}+\sum_{i<j}^{r} \mu_{i}(s(t)) \mu_{j}(s(t))\left(E_{i j}+E_{j i}\right) . \tag{42}
\end{align*}
$$

Therefore, condition (33) can guarantee that condition (41) holds. This completes the proof.

It should be noted that the obtained conditions in Theorem 5 are not strict LMI conditions due to the existence of nonlinear term $X Q_{1} X$ in (33). It cannot be directly solved by standard LMI Toolbox. In the following, we present an approach to solving the condition in Theorem 5.

Introduce additional matrix variable $G>0$ such that

$$
\begin{equation*}
X Q_{1} X \leq G \tag{43}
\end{equation*}
$$

By Schur complement, it follows from (43) that

$$
\left[\begin{array}{cc}
-G & X  \tag{44}\\
X & -Q_{1}^{-1}
\end{array}\right] \leq 0
$$

Then, we readily obtain the following theorem.
Theorem 6. For some prescribed $\gamma>0, \tau>0, \delta>0$, and $N$ is a positive integer, if there exist scalar $\varepsilon>0$, matrices $G>0, P>0, X>0, V_{l}>0(l=2,3, \ldots, N), Q_{l}>0$, $(l=1,2 \ldots N)$, and $\bar{Q}_{1}>0$, and appropriate dimension matrices $W_{j}(j=1,2 \ldots, r)$ such that the following LMIs simultaneously hold for $i, j=1,2, \ldots, r$ :

$$
\begin{gather*}
\hat{\Omega}_{i i}<0, \quad(i=1,2, \ldots, r), \\
\frac{\left(\hat{\Omega}_{i j}+\hat{\Omega}_{j i}\right)}{2}<0, \quad(1 \leq i<j \leq r),  \tag{45}\\
{\left[\begin{array}{cc}
-G & X \\
X & -\bar{Q}_{1}
\end{array}\right] \leq 0,}  \tag{46}\\
Q_{1} \bar{Q}_{1}=I, \quad P X=I, \quad V_{i} R_{i}=I, \quad(i=1,2, \ldots, N), \tag{47}
\end{gather*}
$$

where $\widehat{\Omega}_{i j}$ is a matrix derived from $\Omega_{i j}$ by replacing the term

$$
\begin{equation*}
\Pi_{11 i j}^{(1)}=X Q_{1} X-X+X A_{i}^{T}+W_{j}^{T} B_{1 i}^{T}+A_{i} X+B_{1 i} W_{j} \tag{48}
\end{equation*}
$$

with

$$
\begin{equation*}
\widehat{\Pi}_{11 i j}^{(1)}=G-\delta X+X A_{i}^{T}+W_{j}^{T} B_{1 i}^{T}+A_{i} X+B_{1 i} W_{j} \tag{49}
\end{equation*}
$$

then the closed-loop system (25) is asymptotically stable with the $H_{\infty}$ performance index $\gamma$. Moreover, if the above condition is feasible, the gain matrices of a desired controller in the form of (6) can be designed by

$$
\begin{equation*}
K_{j}=W_{j} X^{-1} \tag{50}
\end{equation*}
$$

Remark 7. Note that the obtained conditions in Theorem 6 are not all in LMI form due to equality constraints, which cannot be solved directly using standard LMI procedures. However, via the result in [35] which has been widely used by many scholars [36, 37], we can solve these nonconvex feasibility problems by formulating them into some sequential optimization problems subject to LMIs constraints.

Now using the approach [35], we suggest the following minimization problem involving LMI conditions instead of
the original nonconvex feasibility problem formulated in Theorem 6.

Problem HSFCD ( $H_{\infty}$ state feedback controller design). Consider the following:

$$
\text { Minimize trace }\left(P X+G \bar{G}+\sum_{j=2}^{N} R_{j} V_{j}\right)
$$

subject to

$$
\text { (45), (46) and }\left[\begin{array}{cc}
P & I  \tag{51}\\
I & X
\end{array}\right] \geq 0,
$$

$$
\left[\begin{array}{cc}
Q_{1} & I \\
I & Q_{1}
\end{array}\right] \geq 0, \quad\left[\begin{array}{cc}
R_{j} & I \\
I & V_{j}
\end{array}\right]_{j=2}^{N} \geq 0 .
$$

When minimize Trace $\left(P X+Q_{1} \bar{Q}_{1}+\sum_{j=2}^{N} R_{j} V_{j}\right)=$ $(N+1) n$, then the conditions in Theorem 6 are solvable. Algorithm 1 in [35] can be easily adapted to solve Problem HSFCD.

## 4. Numerical Example

In this section, we use an example to show the applicability of the results proposed in this paper.

Example 8. Consider the truck trailer system borrowed from [38], which can be represented by the following uncertain time-delay T-S fuzzy model.

Rule 1. If $\theta(t)=x_{2}(t)+a(v \bar{t} / 2 L) x_{1}(t)+(1-a)(v \bar{t} / 2 L) x_{1}(t-\tau)$ is about 0 , then

$$
\begin{gather*}
\dot{x}(t)=\left(A_{1}+\Delta A_{1}\right) x(t)+\left(A_{d 1}+\Delta A_{d 1}\right) x(t-\tau) \\
+\left(B_{u 1}+\Delta B_{u 1}\right) u(t)+B_{w 1} \omega(t)  \tag{52}\\
z(t)=C_{1} x(t)
\end{gather*}
$$

Rule 2. If $\theta(t)=x_{2}(t)+a(v \bar{t} / 2 L) x_{1}(t)+(1-a)(v \bar{t} / 2 L) x_{1}(t-\tau)$ is about $\pi$ or $-\pi$, then

$$
\begin{gather*}
\dot{x}(t)=\left(A_{2}+\Delta A_{2}\right) x(t)+\left(A_{d 2}+\Delta A_{d 2}\right) x(t-\tau) \\
+\left(B_{u 2}+\Delta B_{u 2}\right) u(t)+B_{w 2} \omega(t)  \tag{53}\\
z(t)=C_{2} x(t)
\end{gather*}
$$

where

$$
\begin{gathered}
a=0.7, \quad v=-1, \quad \bar{t}=2.0, \quad L=5.5, \quad \tau=0.5, \\
A_{1}=\left[\begin{array}{ccc}
0.5091 & 0 & 0 \\
-0.5091 & 0 & 0 \\
-0.5091 & -4 & 0
\end{array}\right], \quad A_{d 1}=\left[\begin{array}{ccc}
0.2182 & 0 & 0 \\
-0.2182 & 0 & 0 \\
0.2182 & 0 & 0
\end{array}\right], \\
B_{u 1}=\left[\begin{array}{c}
-1.4286 \\
0 \\
0
\end{array}\right], \quad B_{w 1}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right], \\
A_{2}=\left[\begin{array}{ccc}
0.5091 & 0 & 0 \\
-0.5091 & 0 & 0 \\
-0.8102 & -6.3662 & 0
\end{array}\right], \\
A_{d 2}
\end{gathered}=\left[\begin{array}{ccc}
0.2182 & 0 & 0 \\
-0.2182 & 0 & 0 \\
0.3472 & 0 & 0
\end{array}\right], \quad B_{u 2}=\left[\begin{array}{c}
-1.4286 \\
0 \\
0
\end{array}\right], ~ \$, ~ l
$$

$$
B_{w 1}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],
$$

$$
\left[\Delta A_{i}, \Delta A_{d i}, \Delta B_{u i}\right]=M F(t)\left[E_{i}, E_{d i}, E_{u i}\right]
$$

$$
C_{1}=C_{2}=\left[\begin{array}{lll}
0 & 1 & 0
\end{array}\right],
$$

$$
M=\operatorname{diag}\{0.05,0.05,0.05\}
$$

$$
E_{1}=\left[\begin{array}{ccc}
0.5091 & 0 & 0 \\
-0.5091 & 0 & 0 \\
0.5091 & 0 & 0
\end{array}\right], \quad E_{2}=\left[\begin{array}{ccc}
0.5091 & 0 & 0 \\
-0.5091 & 0 & 0 \\
0.8107 & 0 & 0
\end{array}\right],
$$

$$
E_{d 1}=\left[\begin{array}{ccc}
0.2182 & 0 & 0 \\
-0.2182 & 0 & 0 \\
0.2182 & 0 & 0
\end{array}\right], \quad E_{d 2}=\left[\begin{array}{ccc}
0.2182 & 0 & 0 \\
-0.2182 & 0 & 0 \\
0.3472 & 0 & 0
\end{array}\right],
$$

$$
E_{u 1}=\left[\begin{array}{c}
-0.3571  \tag{54}\\
0 \\
0
\end{array}\right], \quad E_{u 2}=\left[\begin{array}{c}
-0.3571 \\
0 \\
0
\end{array}\right]
$$

For this example, the prescribed $H_{\infty}$ performance level is chosen as $\gamma=0.5$. In order to design a robust $H_{\infty}$ state feedback controller for the given T-S fuzzy model, choose $\delta=1, N=4$. According to Theorem 5, the gain matrix of controller is given as

$$
\begin{align*}
& K_{1}=\left[\begin{array}{lll}
25.6229 & -29.9005 & 8.0667
\end{array}\right],  \tag{55}\\
& K_{2}=\left[\begin{array}{lll}
23.8882 & -31.9243 & 8.1041
\end{array}\right] .
\end{align*}
$$

According to [38], take the membership function as

$$
\begin{gather*}
h_{1}=\left(1-\frac{1}{1+\exp (-3(\theta(t)-0.5 \pi))}\right) \\
\times\left(\frac{1}{1+\exp (-3(\theta(t)+0.5 \pi))}\right),  \tag{56}\\
h_{2}=1-h_{1} .
\end{gather*}
$$



Figure 1: Controlled output $z(t)$ and the disturbance input $w(t)$.

(b)

(c)

(d)

Figure 2: Response of the closed-loop system and controller input.

Let disturbance input $\omega(t)=\sin (2 t) e^{-0.05 t}$ and initial condition $\phi(t)=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{T}, t \in[-0.5,0]$, and simulation time is 100 s .

Figure 1 shows the controlled output $z(t)$ and the disturbance input $\omega(t)$. According to Figure 1 , the resulting output energy of the robust $H_{\infty}$ controller is $\int_{0}^{100} z^{2}(t) d t=0.832$, while the input energy is $\int_{0}^{100} \omega^{2}(t) d t=5$. Simulation result for the ratio of the output energy to the disturbance energy is 0.1664 , and the $l_{2}$-norm is $\sqrt{0.1664}=0.41<\gamma=0.5$ (due to the fact that the state has been stable for a long time, we can regard the value 0.41 as the $l_{2}$-norm).

State response of the closed-loop system and controller input are shown in Figure 2.

The simulation results show that the algorithm proposed in this paper is effective for robust $H_{\infty}$ control problem of the truck trailer system with time delay.

## 5. Conclusion

The problem of robust $H_{\infty}$ controller design has been addressed for a class of T-S fuzzy-model-based systems with constant delay and norm-bounded parameter uncertainty. Based on the Lyapunov-Krasovskii functional approach, a sufficient condition for the existence of the robust $H_{\infty}$ controller, which robustly stabilizes the T-S fuzzy-model-based uncertain systems and guarantees a prescribed level on disturbance attenuation, has been obtained in an LMI form. The given numerical example has shown the effectiveness of the proposed method. In addition, the filtering problems of T-S fuzzy delayed systems by using the delay decomposition approach are also challenging, which could be our further work.

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# Composite Disturbance Observer-Based Control and $H_{\infty}$ Output Tracking Control for Discrete-Time Switched Systems with Time-Varying Delay 

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#### Abstract

This paper considers the problem of $H_{\infty}$ output tracking control for discrete-time switched systems with time-varying delay and external disturbances. The control scheme combining disturbance observer-based control (DOBC) and $H_{\infty}$ control is proposed. The disturbances are assumed to include two parts. One part is generated by an exogenous system, which imposes on system with control inputs in the same channel. The other part is supposed to have the bounded $\mathrm{H}_{2}$ norm. A new disturbance observer is developed to estimate and reject the first case disturbances for switched system with time-varying delay, and the second case disturbances are attenuated by $H_{\infty}$ control scheme. The stability analysis of the closed-loop system is developed by switched Lyapunov function, and a solvable delay-dependent sufficient condition is presented in terms of linear matrix inequalities (LMIs) and cone complement linearization (CCL) methods. A numerical example is given to demonstrate the effectiveness of the proposed composite control scheme.


## 1. Introduction

Switched systems are a special class of hybrid control systems, which are composed of a family of subsystems described by continuous or discrete time dynamics and a switched rule among them [1, 2]. Switched systems can be employed to represent many practical systems, for example, power electronics, embedded systems, chemical processes, computer-controlled systems, automotive industries, and so on. Hence, switched systems have attracted considerable attention during the last decade, and many meaningful results have been reported on stability analysis and controller design for switched systems (see, e.g., [3-14]). By far, there are a number of methodologies on dealing with the stability analysis and control synthesis of switched systems, such as common Lyapunov function [1, 11], multiple Lyapunov function [3, 12], dwell time and average dwell time method $[7,13]$, and switched Lyapunov function $[4,14]$.

On the one hand, time delay always encounters in various engineering systems, such as long transmission lines in pneumatic, hydraulic, and chemical processes, economic and rolling mill systems, to name a few, which usually leads to instability and poor performance of the closedloop system. Recently, many researches have paid more and more attention to stabilization and performance analysis of systems or switched systems with delays [15-20]. On the other hand, disturbances widely exist in systems, which may degrade the closed-loop system performance. Usually, some performance indexes, that is, $H_{\infty}$ and $L_{2}-L_{\infty}\left(l_{2}-l_{\infty}\right)$, are employed to deal with external disturbances. Many results have been developed with these performance indexes [10, $13,18,19,21-25]$. However, disturbances need to satisfy the bounded $\mathrm{H}_{2}$ norm. In reality, many disturbances do not satisfy this condition, for example, constant disturbances or harmonics disturbances. To improve the ability of disturbance attenuation and rejection, disturbance observer-based
control (DOBC) scheme has been studied from 1980s and applied in many control fields [26-37], which can be regarded as an active disturbance rejection method. In this method, disturbances are estimated by disturbance observer (DOB), and the disturbance estimation is employed to feedforward compensation.

The problem of output tracking control is important and numerous results have been developed for kinds of systems in the literature [38-40]. Along with the development of switched system theory, increased attention has been paid to studying output tracking controller design for switched systems. In [41-43], the tracking control problems have been considered for kinds of continuous switched systems. Exponential $l_{2}-l_{\infty}$ output tracking control problem has been solved in [13] for a class of discrete-time switched systems. However, in [13], disturbances are assumed to be the bounded $H_{2}$ norm.

In this paper, a composite control scheme is developed to solve the problem of $H_{\infty}$ output tracking control for discrete-time switched systems with time-varying delay and external disturbances. Here, external disturbances can be divided into two parts. One part is generated by exogenous system, which imposes on system in the same channel with control inputs. The other part satisfies bounded $\mathrm{H}_{2}$ norm. A new disturbance observer is designed to estimate and reject the first case disturbances for switched system with time-varying delay. $H_{\infty}$ control is employed to analyze attenuation performance with respect to the second case disturbances. Hence, a composite control method, consisting of DOBC and $H_{\infty}$ control, is proposed. By resorting to the switched Lyapunov function approach and inspired by [44, 45], some delay-dependent conditions for the problem of $H_{\infty}$ output tracking control are presented. In order to obtain the desired controller and observer gains, a cone complement linearization (CCL) method is used to transform the nonconvex feasibility problem to some sequential optimization problem subject to linear matrix inequalities (LMIs) constraints. Finally, a numerical example is provided to demonstrate the effectiveness of the main result.

## 2. Problem Formulation and Preliminaries

Consider the following discrete-time switched systems with time-varying delays described by

$$
\begin{align*}
& \Sigma_{0}: x(k+1)= A_{\sigma(k)} x(k)+A_{d \sigma(k)} x(k-\tau(k)) \\
&+B_{\sigma(k)}\left(u(k)+d_{1}(k)\right)+B_{1 \sigma(k)} d_{2}(k), \\
& z(k)=C_{\sigma(k)} x(k)+C_{d \sigma(k)} x(k-\tau(k))+D_{\sigma(k)} u(k), \\
& x(s)=\phi(s), \quad s=-\bar{\tau}, \ldots,-1,0, \tag{1}
\end{align*}
$$

where $x(k) \in \mathbb{R}^{n}$ is the states, $z(k) \in \mathbb{R}^{q}$ is the signal to be estimated, and $u(k)$ is the control input. $\sigma(k): \mathbb{Z}^{+} \rightarrow$ $\mathcal{N}=\{1,2, \ldots, N\}$ is the switching signal, which specifies which subsystem will be activated at a certain discrete time
instant and $A_{i}, A_{d i}, B_{i}, B_{1 i}, C_{i}, C_{d i}, D_{i}, i \in \mathcal{N}$, are constant matrices with appropriate dimensions. $\tau(k)$ is a time-varying delay of the system and satisfies

$$
\begin{equation*}
\underline{\tau} \leq \tau(k) \leq \bar{\tau}, \quad \forall k \in \mathbb{Z}^{+} \tag{2}
\end{equation*}
$$

where $\underline{\tau}, \bar{\tau}, \tau(k)$ are nonnegative integer numbers. $\phi(k)$ is the initial condition. $d_{2}(k) \in \mathbb{R}^{p}$ is the external disturbances, which is assumed to belong to $l_{2}[0, \infty) . d_{1}(k)$ is also the external disturbances, which is generated by the exogenous system

$$
\begin{equation*}
\omega(k+1)=W \omega(k), \quad d_{1}(k)=V \omega(k) \tag{3}
\end{equation*}
$$

where $\left(W, B_{i} V\right)$ is observable.
Remark 1. It is clear that $\tau(k)$ is an interval-like time-varying delay. When $\underline{\tau}=0$, it is reduced to $0 \leq \tau(k) \leq \bar{\tau}$, for all $k \in \mathbb{Z}^{+}$, which was recently discussed in [23]. Therefore, this note can be viewed as some extensions of their results.

Remark 2. System $\Sigma_{0}$ contains two kinds of external disturbances: matched external disturbances and mismatched external disturbances. Two different methods are employed to deal with these disturbances. First, a disturbance observer is introduced to estimate matched disturbances; then the estimation values are used to feedforward compensation. Then, a performance index, that is, $H_{\infty}$ performance index, is presented to reject and attenuate mismatched disturbances.

Assume that the reference signal $z_{r}(k)$ is generated by the following system:

$$
\begin{gather*}
x_{r}(k+1)=A_{r} x_{r}(k)+B_{r} r(k),  \tag{4}\\
z_{r}(k)=C_{r} x_{r}(k),
\end{gather*}
$$

where $x_{r}(k)$ is reference states, $r(k) \in l_{2}[0, \infty)$ is reference input, and $A_{r}$ is Hurwitz matrix with an appropriate dimension. Here we are interested in designing a state feedback controller by the following formula:

$$
\begin{equation*}
u(k)=K_{1 \sigma(k)} x(k)-\widehat{d}_{1}(k)+K_{2 \sigma(k)} x_{r}(k) \tag{5}
\end{equation*}
$$

where $\widehat{d}_{1}(k)$ is the estimation of the disturbances $d_{1}(k)$, which is obtained by the following disturbance observer:

$$
\begin{gather*}
\widehat{\omega}(k)=\nu(k)-L x(k), \\
\nu(k+1)=W(v(k)-L x(k)) \\
+L\left(A_{\sigma(k)} x(k)+A_{d \sigma(k)} x(k-\tau(k))\right.  \tag{6}\\
\left.+B_{\sigma(k)}\left(u(k)+\widehat{d}_{1}(k)\right)\right), \\
\hat{d}_{1}(k)=V \widehat{\omega}(k) . \tag{7}
\end{gather*}
$$

Remark 3. In this paper, we assume that the switching signal $\sigma(k)$ is not known a priori but its instantaneous value is available in real time [4]. Here we only consider the case of synchronous switching; that is, the controller switches just as the system $\Sigma_{0}$ does.

Defining $e_{\omega}(k)=\omega(k)-\widehat{\omega}(k)$ yields

$$
\begin{equation*}
e_{\omega}(k+1)=\left(W+L B_{\sigma(k)} V\right) e_{\omega}(k)+\bar{B}_{1 \sigma(k)} d_{2}(k), \tag{8}
\end{equation*}
$$

where $\bar{B}_{1 \sigma(k)}=L B_{1 \sigma(k)}$.
Applying controller (5) to system $\Sigma_{0}$ and combining (3) and (7), we obtain

$$
\begin{align*}
x(k+1)= & \bar{A}_{\sigma(k)} x(k)+A_{d \sigma(k)} x(k-\tau(k)) \\
& +B_{\sigma(k)}\left(d_{1}(k)-\widehat{d}_{1}(k)\right)+B_{1 \sigma(k)} d_{2}(k) \\
& +\bar{B}_{\sigma(k)} x_{r}(k) \\
= & \bar{A}_{\sigma(k)} x(k)+A_{d \sigma(k)} x(k-\tau(k))+\widehat{B}_{\sigma(k)} e_{\omega}(k) \\
& +B_{1 \sigma(k)} d_{2}(k)+\bar{B}_{\sigma(k)} x_{r}(k), \tag{9}
\end{align*}
$$

where

$$
\begin{align*}
& \bar{A}_{\sigma}(k)=A_{\sigma(k)}+B_{\sigma(k)} K_{1 \sigma(k)}, \quad \widehat{B}_{\sigma(k)}=B_{\sigma(k)} V \\
& \bar{B}_{\sigma(k)}=B_{\sigma(k)} K_{2 \sigma(k)}, \quad e_{\omega}(k)=\omega(k)-\widehat{\omega}(k) \tag{10}
\end{align*}
$$

Let

$$
\begin{equation*}
e(k)=z(k)-z_{r}(k) \tag{11}
\end{equation*}
$$

then we have the following augmented switched system:

$$
\begin{align*}
& \Sigma_{e}: \zeta(k+1)= \mathscr{A}_{\sigma(k)} \zeta(k)+\mathscr{A}_{d \sigma(k)} \zeta(k-\tau(k)) \\
&+\mathscr{B}_{\sigma(k)} e_{\omega}(k)+\mathscr{B}_{1 \sigma(k)} v(k), \\
& e_{\omega}(k+1)=\left(W+L B_{\sigma(k)} V\right) e_{\omega}(k)+\breve{\mathscr{B}}_{1 \sigma(k)} v(k),  \tag{12}\\
& e(k)=\mathscr{C}_{\sigma(k)} \zeta(k)+\mathscr{C}_{d \sigma(k)} \zeta(k-\tau(k)),
\end{align*}
$$

where

$$
\left.\begin{array}{c}
\mathscr{A}_{\sigma(k)}=\left[\begin{array}{cc}
\bar{A}_{\sigma(k)} & \bar{B}_{\sigma(k)} \\
0 & A_{r}
\end{array}\right], \quad \mathscr{A}_{d \sigma(k)}=\left[\begin{array}{cc}
A_{d \sigma(k)} & 0 \\
0 & 0
\end{array}\right], \\
\mathscr{B}_{\sigma(k)}=\left[\begin{array}{c}
\widehat{B}_{\sigma(k)} \\
0
\end{array}\right], \quad \mathscr{B}_{1 \sigma(k)}=\left[\begin{array}{cc}
B_{1 \sigma(k)} & 0 \\
0 & B_{r}
\end{array}\right], \\
\zeta(k)=\left[\begin{array}{c}
x(k) \\
x_{r}(k),
\end{array}\right] \in \mathbb{R}^{m}, \quad v(k)=\left[\begin{array}{c}
d_{2}(k) \\
r(k)
\end{array}\right], \\
\mathscr{C}_{\sigma(k)}=\left[\begin{array}{ll}
C_{\sigma(k)}+D_{\sigma(k)} K_{1 \sigma(k)} \quad D_{\sigma(k)} K_{2 \sigma(k)}-C_{r}
\end{array}\right], \\
\mathscr{C}_{d \sigma(k)}=\left[\begin{array}{ll}
C_{d \sigma(k)} & 0
\end{array}\right], \quad \breve{\mathscr{B}}_{1 \sigma(k)}=\left[\bar{B}_{1 \sigma(k)}\right.  \tag{13}\\
0
\end{array}\right] .
$$

By defining

$$
\begin{gather*}
\widehat{\mathscr{A}}_{\sigma(k)}=\left[\begin{array}{cc}
A_{\sigma(k)} & 0 \\
0 & A_{r}
\end{array}\right], \quad \widehat{\mathscr{B}}_{\sigma(k)}=\left[\begin{array}{c}
B_{\sigma(k)} \\
0
\end{array}\right], \\
K_{\sigma(k)}=\left[\begin{array}{ll}
K_{1 \sigma(k)} & K_{2 \sigma(k)}
\end{array}\right]  \tag{14}\\
\widehat{\mathscr{C}}_{\sigma(k)}=\left[\begin{array}{ll}
C_{\sigma(k)} & -C_{r}
\end{array}\right]
\end{gather*}
$$

then we have

$$
\begin{align*}
\mathscr{A}_{\sigma(k)} & =\widehat{\mathscr{A}}_{\sigma(k)}+\widehat{\mathscr{B}}_{\sigma(k)} K_{\sigma(k)} \\
\mathscr{C}_{\sigma(k)} & =\widehat{\mathscr{C}}_{\sigma(k)}+D_{\sigma(k)} K_{\sigma(k)} \tag{15}
\end{align*}
$$

and the controller can be rewritten as

$$
\begin{equation*}
u(k)=K_{\sigma(k)} \zeta(k)-\widehat{d}_{1}(k) \tag{16}
\end{equation*}
$$

In order to prepare for a precise formulation, we introduce the following definition.

Definition 4 ( $H_{\infty}$ output tracking control problem). Consider system $\Sigma_{e}$. Given a prescribed $\gamma>0$, if there exists composite controller (5) such that the following two conditions are satisfied:
(R1) system $\Sigma_{e}$ is asymptotically stable when $v(k)=0$, for all $k \geq 0$;
(R2) under the zero-initial condition, the following inequality holds:

$$
\begin{equation*}
\|e(k)\|_{2}<\gamma\|v(k)\|_{2} \tag{17}
\end{equation*}
$$

for any nonzero $v(k) \in l_{2}[0, \infty)$, then system $\Sigma_{e}$ is asymptotically stable with an $H_{\infty}$ performance index $\gamma$.

## 3. Main Results

3.1. $H_{\infty}$ Output Tracking Performance Analysis. In this subsection, we focus on developing delay-dependent condition to solve the $H_{\infty}$ output tracking control problem formulated in previous section.

Theorem 5 (consider system $\Sigma_{e}$ ). For scalars $\underline{\tau}>0, \bar{\tau}>$ $0, \gamma>0$, the error system $\Sigma_{e}$ is asymptotically stable with an $H_{\infty}$ performance index, if there exist matrices $M_{1}, M_{2}, N_{1}, N_{2}$, $T_{1}, T_{2}, K_{1 i}, K_{2 i}, P_{i}>0, Q_{1}>0, Q_{2}>0, Q_{3}>0, R_{1}>0, R_{2}>$ $0, S>0, i \in \mathcal{N}$, such that the following inequality holds:

$$
\Phi_{i}=\left[\begin{array}{ccccccccccccc}
\Phi_{11 i} & \Phi_{12} & 0 & 0 & M_{1}^{T} & -T_{1}^{T} & \mathscr{A}_{i}^{T} & \Phi_{18 i} & \Phi_{19 i} & \Phi_{1,10} & \Phi_{1,11} & \Phi_{1,12} & 0  \tag{18}\\
* & \Phi_{22 i} & 0 & 0 & M_{2}^{T} & -T_{2}^{T} & \mathscr{A}_{d i}^{T} & \mathscr{A}_{d i}^{T} & \mathscr{A}_{d i}^{T} & \Phi_{2,10} & \Phi_{2,11} & \Phi_{2,12} & 0 \\
* & * & -S & 0 & 0 & 0 & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & 0 & 0 & 0 & \Phi_{3,13 i} \\
* & * & * & -\gamma^{2} I & 0 & 0 & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & 0 & 0 & 0 & \mathscr{B}_{1 i}^{T} \\
* & * & * & * & -Q_{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -P_{l}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Phi_{88} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \Phi_{99} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \Phi_{10,10} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \Phi_{11,11} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \Phi_{12,12} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -S^{-1}
\end{array}\right]<0,
$$

where

$$
\begin{gather*}
\Phi_{11 i}=-P_{i}+Q_{1}+Q_{2}+(\bar{\tau}-\underline{\tau}+1) Q_{3} \\
+N_{1}+N_{1}^{T}+\mathscr{C}_{i}^{T} \mathscr{C}_{i}+\mathscr{C}_{i}^{T} \mathscr{C}_{d i}, \\
\Phi_{12}=-N_{1}^{T}+N_{2}+T_{1}^{T}-M_{1}^{T}, \quad \Phi_{18 i}=\Phi_{19 i}=\left(\mathscr{A}_{i}-I\right)^{T}, \\
\Phi_{1,10}=\bar{\tau} N_{1}^{T}, \quad \Phi_{1,11}=(\bar{\tau}-\underline{\tau}) T_{1}^{T}, \\
\Phi_{1,13}=(\bar{\tau}-\underline{\tau}) M_{1}^{T}, \\
\Phi_{22 i}=-Q_{3}-N_{2}-N_{2}^{T}+T_{2}+T_{2}^{T}-M_{2}-M_{2}^{T}+\mathscr{C}_{d i}^{T} \mathscr{C}_{d i}, \\
\Phi_{2,10}=\bar{\tau} N_{2}^{T}, \quad \Phi_{2,11}=(\bar{\tau}-\underline{\tau}) T_{2}^{T}, \\
\Phi_{2,12}=(\bar{\tau}-\underline{\tau}) M_{2}^{T}, \quad \Phi_{3,13 i}=\left(W+L B_{i} V\right)^{T}, \\
\Phi_{88}=-\left(\bar{\tau} R_{1}\right)^{-1}, \quad \Phi_{99}=-\left((\bar{\tau}-\underline{\tau}) R_{2}\right)^{-1}, \\
\Phi_{10,10}=-\bar{\tau} R_{1}, \\
\Phi_{11,11}=-(\bar{\tau}-\underline{\tau})\left(R_{1}+R_{2}\right), \quad \Phi_{12,12}=-(\bar{\tau}-\underline{\tau}) R_{2} . \tag{19}
\end{gather*}
$$

Proof. Choose a Lyapunov-Krasovskii functional candidate as

$$
\begin{equation*}
V(k)=V_{1}(k)+V_{2}(k)+V_{3}(k), \tag{20}
\end{equation*}
$$

where

$$
\begin{aligned}
V_{1}(k) & =\zeta^{T}(k) P_{\sigma(k)} \zeta(k)+e_{\omega}^{T}(k) S e_{\omega}(k), \\
V_{2}(k)= & \sum_{\theta=k-\underline{\tau}}^{k-1} \zeta^{T}(\theta) Q_{1} \zeta(\theta)+\sum_{\theta=k-\bar{\tau}}^{k-1} \zeta^{T}(\theta) Q_{2} \zeta(\theta) \\
& +\sum_{\theta=k-\tau(k)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta)
\end{aligned}
$$

$$
\begin{align*}
V_{3}(k)= & \sum_{i=-\bar{\tau}}^{-1} \sum_{\theta=k+i}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta) \\
& +\sum_{i=-\bar{\tau}+1}^{-\tau} \sum_{\theta=k+i-1}^{k-1} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& +\sum_{i=-\bar{\tau}+2}^{-\frac{\tau}{\tau}+1} \sum_{\theta=k+i-1}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta), \tag{21}
\end{align*}
$$

and $\rho(\theta)=\zeta(\theta+1)-\zeta(\theta)$.
Without loss of generality, we assume that $\sigma(k+1)=$ $l, \sigma(k)=i$, for all $i, l \in \mathcal{N}$. Then taking the forward difference yields

$$
\begin{align*}
\Delta V_{1}(k)= & V_{1}(k+1)-V_{1}(k) \\
= & \zeta^{T}(k+1) P_{l} \zeta(k+1)-\zeta^{T}(k) P_{i} \zeta(k) \\
& +e_{\omega}^{T}(k+1) S e_{\omega}(k+1)-e_{\omega}^{T}(k) S e_{\omega}(k)  \tag{22}\\
= & \xi^{T}(k) \mathscr{A}_{i}^{T} P_{l} \mathscr{A}_{i} \xi(k)-\zeta^{T}(k) P_{i} \zeta(k) \\
& +\xi^{T}(k) \mathscr{B}_{i}^{T} S \mathscr{B}_{i} \xi(k)-e_{\omega}^{T}(k) S e_{\omega}(k),
\end{align*}
$$

where

$$
\begin{gather*}
\xi^{T}(k)=\left[\begin{array}{llll}
\zeta^{T}(k) & \zeta^{T}(k-\tau(k)) & e_{\omega}^{T}(k) & v^{T}(k)
\end{array}\right] \\
\mathscr{A}_{i}=\left[\begin{array}{llll}
\mathscr{A}_{i} & \mathscr{A}_{d i} & \mathscr{B}_{i} & \mathscr{B}_{1 i}
\end{array}\right]  \tag{23}\\
\mathscr{B}_{i}=\left[\begin{array}{lll}
0 & 0 & \left(W+L B_{i} V\right) \\
\mathscr{B}_{1 i}
\end{array}\right] .
\end{gather*}
$$

## Direct computation gives

$$
\begin{align*}
\Delta V_{2}(k)= & \sum_{\theta=k+1-\underline{\tau}}^{k} \zeta^{T}(\theta) Q_{1} \zeta(\theta)-\sum_{\theta=k-\underline{\tau}}^{k-1} \zeta^{T}(\theta) Q_{1} \zeta(\theta) \\
& +\sum_{\theta=k+1-\bar{\tau}}^{k} \zeta^{T}(\theta) Q_{2} \zeta(\theta)-\sum_{\theta=k-\bar{\tau}}^{k-1} \zeta^{T}(\theta) Q_{2} \zeta(\theta) \\
& +\sum_{\theta=k+1-\tau(k+1)}^{k} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
& -\sum_{\theta=k-\tau(k)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
= & \zeta^{T}(k)\left(Q_{1}+Q_{2}+Q_{3}\right) \zeta(k) \\
& -\zeta^{T}(k-\underline{\tau}) Q_{1} \zeta(k-\underline{\tau}) \\
& -\zeta^{T}(k-\bar{\tau}) Q_{2} \zeta(k-\bar{\tau}) \\
& -\zeta^{T}(k-\tau(k)) Q_{3} \zeta(k-\tau(k)) \\
& +\sum_{\theta=k+1-\tau(k+1)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
& -\sum_{\theta=k+1-\tau(k)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta) . \tag{24}
\end{align*}
$$

Note that

$$
\begin{align*}
& \sum_{\theta=k+1-\tau(k+1)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
& \quad=\sum_{\theta=k+1-\underline{\tau}}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta)+\sum_{\theta=k+1-\tau(k+1)}^{k-\tau} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
& \quad \leq \sum_{\theta=k+1-\tau(k)}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta)+\sum_{\theta=k+1-\bar{\tau}}^{k-\tau} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \tag{25}
\end{align*}
$$

Hence we obtain

$$
\begin{align*}
\Delta V_{2}(k) \leq & \zeta^{T}(k)\left(Q_{1}+Q_{2}+Q_{3}\right) \zeta(k) \\
& -\zeta^{T}(k-\underline{\tau}) Q_{1} \zeta(k-\underline{\tau}) \\
& -\zeta^{T}(k-\bar{\tau}) Q_{2} \zeta(k-\bar{\tau})  \tag{26}\\
& -\zeta^{T}(k-\tau(k)) Q_{3} \zeta(k-\tau(k)) \\
& +\sum_{\theta=k+1-\bar{\tau}}^{k-\tau} \zeta^{T}(\theta) Q_{3} \zeta(\theta)
\end{align*}
$$

After some manipulations, the following inequality is satisfied:

$$
\begin{align*}
\Delta V_{3}(k)= & \sum_{i=-\bar{\tau}}^{-1} \sum_{\theta=k+1+i}^{k} \rho^{T}(\theta) R_{1} \rho(\theta) \\
& -\sum_{i=-\bar{\tau}}^{-1} \sum_{\theta=k+i}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta) \\
& +\sum_{i=-\bar{\tau}+1}^{-\tau} \sum_{\theta=k+i}^{k} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& -\sum_{i=-\bar{\tau}+1}^{-\tau} \sum_{\theta=k+i-1}^{k-1} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& +\sum_{i=-\bar{\tau}+2}^{-\tau+1} \sum_{\theta=k+i}^{k} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
& -\sum_{i=-\bar{\tau}+2}^{-\tau+1} \sum_{\theta=k+i-1}^{k-1} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \\
= & \bar{\tau} \rho^{T}(k) R_{1} \rho(k)-\sum_{\theta=k-\bar{\tau}}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta) \\
& +(\bar{\tau}-\underline{\tau}) \rho^{T}(k) R_{2} \rho(k)-\sum_{\theta=k-\bar{\tau}}^{k-\tau-1} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& +(\bar{\tau}-\underline{\tau}) \zeta^{T}(k) Q_{3} \zeta(k)-\sum_{\theta=k+1-\bar{\tau}}^{k-\tau} \zeta^{T}(\theta) Q_{3} \zeta(\theta) \tag{27}
\end{align*}
$$

Observe that the following equalities hold naturally:

$$
\begin{align*}
& 2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \quad \times\left(\zeta(k)-\zeta(k-\tau(k))-\sum_{\theta=k-\tau(k)}^{k-1} \rho(\theta)\right)=0 \\
& 2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}\left(k-d_{j}(k)\right) T_{2}^{T}\right) \\
& \quad \times\left(\zeta(k-\tau(k))-\zeta(k-\bar{\tau})-\sum_{\theta=k-\bar{\tau}}^{k-1-\tau(k)} \rho(\theta)\right)=0  \tag{28}\\
& 2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \quad \times\left(\zeta(k-\underline{\tau})-\zeta(k-\tau(k))-\sum_{\theta=k-\tau(k)}^{k-1-\underline{\tau}} \rho(\theta)\right)=0
\end{align*}
$$

Then

$$
\begin{aligned}
& -\sum_{\theta=k-\bar{\tau}}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta)-\sum_{\theta=k-\bar{\tau}}^{k-\tau-1} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& =-\sum_{\theta=k-\bar{\tau}}^{k-\tau(k)-1} \rho^{T}(\theta)\left(R_{1}+R_{2}\right) \rho(\theta) \\
& -\sum_{\theta=k-\tau(k)}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta)-\sum_{\theta=k-\tau(k)}^{k-\tau-1} \rho^{T}(\theta) R_{2} \rho(\theta) \\
& +2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \times\left(\zeta(k)-\zeta(k-\tau(k))-\sum_{\theta=k-\tau(k)}^{k-1} \rho(\theta)\right) \\
& +2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times\left(\zeta(k-\tau(k))-\zeta(k-\bar{\tau})-\sum_{\theta=k-\bar{\tau}}^{k-1-\tau(k)} \rho(\theta)\right) \\
& +2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \times\left(\zeta(k-\underline{\tau})-\zeta(k-\tau(k))-\sum_{\theta=k-\tau(k)}^{k-1-\underline{\tau}} \rho(\theta)\right) \\
& =2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \times(\zeta(k)-\zeta(k-\tau(k))) \\
& +2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times(\zeta(k-\tau(k))-\zeta(k-\bar{\tau})) \\
& +2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \times(\zeta(k-\underline{\tau})-\zeta(k-\tau(k))) \\
& -2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times \sum_{\theta=k-\bar{\tau}}^{k-1-\tau(k)} \rho(\theta)-\sum_{\theta=k-\bar{\tau}}^{k-\tau(k)-1} \rho^{T}(\theta)\left(R_{1}+R_{2}\right) \rho(\theta) \\
& -2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \times \sum_{\theta=k-\tau(k)}^{k-1} \rho(\theta)-\sum_{\theta=k-\tau(k)}^{k-1} \rho^{T}(\theta) R_{1} \rho(\theta) \\
& -2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \times \sum_{\theta=k-\tau(k)}^{k-1-\underline{\tau}} \rho(\theta)-\sum_{\theta=k-\tau(k)}^{k-\tau-1} \rho^{T}(\theta) R_{2} \rho(\theta)
\end{aligned}
$$

$$
\begin{align*}
& =2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \times(\zeta(k)-\zeta(k-\tau(k))) \\
& +2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times(\zeta(k-\tau(k))-\zeta(k-\bar{\tau})) \\
& +2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \times(\zeta(k-\underline{\tau})-\zeta(k-\tau(k))) \\
& +\tau(k)\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) R_{1}^{-1} \\
& \times\left(N_{1} \zeta(k)+N_{2} \zeta(k-\tau(k))\right) \\
& +(\bar{\tau}-\tau(k))\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times\left(R_{1}+R_{2}\right)^{-1}\left(T_{1} \zeta(k)+T_{2} \zeta(k-\tau(k))\right) \\
& +(\tau(k)-\underline{\tau})\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) R_{2}^{-1} \\
& \times\left(M_{1} \zeta(k)+M_{2} \zeta(k-\tau(k))\right) \\
& -\sum_{\theta=k-\tau(k)}^{k-1} \eta^{T}(k) N^{T} R_{1}^{-1} N \eta(k) \\
& -\sum_{\theta=k-\bar{\tau}}^{k-1-\tau(k)} \eta^{T}(k) T^{T}\left(R_{1}+R_{2}\right)^{-1} T \eta(k) \\
& -\sum_{\theta=k-\tau(k)}^{k-\underline{\tau}-1} \eta^{T}(k) M^{T} R_{2}^{-1} M \eta(k) \\
& \leq 2\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) \\
& \times(\zeta(k)-\zeta(k-\tau(k))) \\
& +2\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times(\zeta(k-\tau(k))-\zeta(k-\bar{\tau})) \\
& +2\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) \\
& \times(\zeta(k-\underline{\tau})-\zeta(k-\tau(k))) \\
& +\bar{\tau}\left(\zeta^{T}(k) N_{1}^{T}+\zeta^{T}(k-\tau(k)) N_{2}^{T}\right) R_{1}^{-1} \\
& \times\left(N_{1} \zeta(k)+N_{2} \zeta(k-\tau(k))\right) \\
& +(\bar{\tau}-\underline{\tau})\left(\zeta^{T}(k) T_{1}^{T}+\zeta^{T}(k-\tau(k)) T_{2}^{T}\right) \\
& \times\left(R_{1}+R_{2}\right)^{-1}\left(T_{1} \zeta(k)+T_{2} \zeta(k-\tau(k))\right) \\
& +(\bar{\tau}-\underline{\tau})\left(\zeta^{T}(k) M_{1}^{T}+\zeta^{T}(k-\tau(k)) M_{2}^{T}\right) R_{2}^{-1} \\
& \times\left(M_{1} \zeta(k)+M_{2} \zeta(k-\tau(k))\right), \tag{29}
\end{align*}
$$

where

$$
\begin{gather*}
\eta^{T}(k)=\left[\begin{array}{lll}
\zeta^{T}(k) & \zeta^{T}(k-\tau(k)) & \rho^{T}(\theta)
\end{array}\right], \\
N=\left[\begin{array}{lll}
N_{1} & N_{2} & R_{1}
\end{array}\right],  \tag{30}\\
M=\left[\begin{array}{llll}
M_{1} & M_{2} & R_{2}
\end{array}\right], \quad T=\left[\begin{array}{lll}
T_{1} & T_{2} & R_{1}+R_{2}
\end{array}\right] .
\end{gather*}
$$

From (22)-(29), and by some manipulations, we obtain

$$
\begin{align*}
\Delta V(k) \leq & \lambda^{T}(k) \Phi_{1 i} \lambda(k)+\xi^{T}(k) \Phi_{2 i} \xi(k)+\bar{\eta}^{T}(k) \Phi_{3} \bar{\eta}(k) \\
& -\bar{\eta}^{T}(k)\left[\begin{array}{c}
\mathscr{C}_{i}^{T} \\
\mathscr{C}_{d i}^{T}
\end{array}\right]\left[\begin{array}{ll}
\mathscr{C}_{i} & \mathscr{C}_{d i}
\end{array}\right] \bar{\eta}(k), \tag{31}
\end{align*}
$$

where

$$
\begin{align*}
& \lambda^{T}(k) \\
& =\left[\zeta^{T}(k) \zeta^{T}(k-\tau(k)) e_{\omega}^{T}(k) v^{T}(k) \zeta^{T}(k-\underline{\tau}) \zeta^{T}(k-\bar{\tau})\right], \\
& \bar{\eta}^{T}(k)=\left[\zeta^{T}(k) \zeta^{T}(k-\tau(k))\right], \\
& \Phi_{2 i}=\left[\mathscr{A}_{i}^{T} P_{l} \mathscr{A}_{i}+\mathscr{B}_{i}^{T} S \mathscr{B}_{i}+\overline{\mathscr{A}}_{i}^{T}\left(\bar{\tau} R_{1}+(\bar{\tau}-\underline{\tau}) R_{2}\right) \overline{\mathscr{A}}_{i}\right], \\
& \overline{\mathscr{A}}_{i}=\left[\mathscr{A}_{i}-\bar{I}\right], \quad \bar{I}=\left[\begin{array}{llll}
I & 0 & 0 & 0
\end{array}\right], \\
& \Phi_{3}=\bar{\tau}\left[\begin{array}{ll}
N_{1} & N_{2}
\end{array}\right]^{T} R_{1}^{-1}\left[\begin{array}{ll}
N_{1} & N_{2}
\end{array}\right]+(\bar{\tau}-\underline{\tau})\left[\begin{array}{ll}
T_{1} & T_{2}
\end{array}\right]^{T} \\
& \times\left(R_{1}+R_{2}\right)^{-1}\left[\begin{array}{ll}
T_{1} & T_{2}
\end{array}\right] \\
& +(\bar{\tau}-\underline{\tau})\left[\begin{array}{ll}
M_{1} & M_{2}
\end{array}\right]^{T} R_{2}^{-1}\left[\begin{array}{ll}
M_{1} & M_{2}
\end{array}\right], \\
& \Phi_{1 i}=\left[\begin{array}{cccccc}
\Phi_{11 i} & \Phi_{12} & 0 & 0 & M_{1}^{T} & -T_{1}^{T} \\
* & \Phi_{22 i} & 0 & 0 & M_{2}^{T} & -T_{2}^{T} \\
* & * & -S & 0 & 0 & 0 \\
* & * & * & -\gamma^{2} I & 0 & 0 \\
* & * & * & * & -Q_{1} & 0 \\
* & * & * & * & * & -Q_{2}
\end{array}\right] . \tag{32}
\end{align*}
$$

Now, we develop the conclusion from two aspects. We first establish the asymptotic stability of system $\Sigma_{e}$ under the condition of zero disturbances. In fact, when $v(k)=0$, it is verified that

$$
\begin{align*}
\Delta V(k) \leq & \widetilde{\lambda}^{T}(k) \widetilde{\Phi}_{1 i} \widetilde{\lambda}(k)+\xi^{T}(k) \widetilde{\Phi}_{2 i} \xi(k) \\
& +\sum_{j=1}^{m} \bar{\eta}_{j}^{T}(k) \Phi_{3 j} \bar{\eta}_{j}(k)-\zeta^{T}(k) \mathscr{G}_{i}^{T} \mathscr{G}_{i} \zeta(k), \tag{33}
\end{align*}
$$

where

$$
\begin{align*}
& \tilde{\lambda}^{T}(k) \\
& =\left[\begin{array}{llll}
\zeta^{T}(k) & \zeta^{T}(k-\tau(k)) & e_{\omega}^{T}(k) & \zeta^{T}(k-\underline{\tau}) \\
\zeta^{T}(k-\bar{\tau})
\end{array}\right], \\
& \widetilde{\Phi}_{2 i}=\left[\widetilde{\mathscr{A}_{i}^{T}} P_{l} \widetilde{\mathscr{A}}_{i}+\widehat{\mathscr{B}}_{i}^{T} S \widehat{\mathscr{B}}_{i}+\widehat{\mathscr{A}_{i}^{T}}\left(\bar{\tau} R_{1}+(\bar{\tau}-\tau) R_{2}\right) \widehat{\mathscr{A}}_{i}\right], \\
& \widetilde{\mathscr{A}_{i}}=\left[\begin{array}{lll}
\mathscr{A}_{i} & \mathscr{A}_{d i} & \mathscr{B}
\end{array}\right], \quad \widehat{\mathscr{A}_{i}}=\left[\widetilde{\mathscr{A}}_{i}-\widetilde{I}\right], \quad \widetilde{I}=\left[\begin{array}{lll}
1 & 0 & 0
\end{array}\right], \\
& \widehat{\mathscr{B}}=\left[\begin{array}{lll}
0 & 0 & W+L B_{i} V
\end{array}\right], \\
& \widetilde{\Phi}_{1 i}=\left[\begin{array}{ccccc}
\Phi_{11 i} & \Phi_{12} & 0 & M_{1}^{T} & -T_{1}^{T} \\
* & \Phi_{22 i} & 0 & M_{2}^{T} & -T_{2}^{T} \\
* & * & -S & 0 & 0 \\
* & * & * & -Q_{1} & 0 \\
* & * & * & * & -Q_{2}
\end{array}\right] . \tag{34}
\end{align*}
$$

Applying Schur complement formula, we obtain $\Delta V(k)<0$ if (18) is true. Therefore, it is easy to see that the error system $\Sigma_{e}$ is asymptotically stable by the Lyapunov-Krasovskii stability theorem.

Next, we will present that under the zero-initial condition, the time-delay system $\Sigma_{e}$ satisfies (17) for all nonzero $v(k) \in l_{2}[0, \infty)$. To this end, we introduce

$$
\begin{equation*}
J_{N}=\sum_{k=0}^{N}\left(e^{T}(k) e(k)-\gamma^{2} v^{T}(k) v(k)+\Delta V(k)\right)-V(N+1) . \tag{35}
\end{equation*}
$$

Then, by the Schur complement formula, it easily follows from (18) and (31) that

$$
\begin{equation*}
J_{N} \leq \sum_{k=0}^{\infty}\left(e^{T}(k) e(k)-\gamma^{2} v^{T}(k) v(k)+\Delta V(k)\right)<0 \tag{36}
\end{equation*}
$$

which implies that $\|e(k)\|_{2}<\gamma\|v(k)\|_{2}$ holds under the zeroinitial condition. This completes the proof.

Remark 6. It is obvious that the condition in (18) is not an LMI with respect to the parameters $P_{1}, \ldots, P_{N}, R_{1}, R_{2}, S$. In order to solve the controller in the form of (5), we will cast the $H_{\infty}$ output tracking control problem into an LMI framework.
3.2. $H_{\infty}$ Output Tracking Controller Design. In this subsection, we try to obtain a solvable condition for the problem of $H_{\infty}$ output tracking controller design using CCL method.

Define

$$
\begin{gathered}
X_{i}=P_{i}^{-1}, \quad \bar{R}_{1}=R_{1}^{-1}, \quad \bar{R}_{2}=R_{2}^{-1} \\
\bar{K}_{i}=K_{i} X_{i}, \quad \bar{L}=S L \\
\bar{Q}_{1 i}=X_{i} Q_{1} X_{i}, \quad \bar{Q}_{2 i}=X_{i} Q_{2} X_{i}, \quad \bar{Q}_{3 i}=X_{i} Q_{3} X_{i} \\
\bar{M}_{1}=X_{i} M_{1} X_{i}, \quad \bar{M}_{2}=X_{i} M_{2} X_{i}, \quad \bar{N}_{1}=X_{i} N_{1} X_{i}
\end{gathered}
$$

$$
\begin{gather*}
\bar{N}_{2}=X_{i} N_{2} X_{i}, \quad \bar{T}_{1}=X_{i} T_{1} X_{i}, \\
\bar{T}_{2}=X_{i} T_{2} X_{i}, \\
\Gamma=\operatorname{diag}\left\{X_{i}, X_{i}, I, I, X_{i}, X_{i}, I, I, I, X_{i}, X_{i}, X_{i}, S\right\} . \tag{37}
\end{gather*}
$$

Pre- and postmultiplying $\Gamma$ and $\Gamma^{T}$ on (18), and using Schur complement formula, we obtain

$$
\bar{\Phi}_{i}=\left[\begin{array}{cccccccccccccc}
\bar{\Phi}_{11 i} & \bar{\Phi}_{12} & 0 & 0 & \bar{M}_{1}^{T} & -\bar{T}_{1}^{T} & \bar{\Phi}_{17 i} & \bar{\Phi}_{18 i} & \bar{\Phi}_{19 i} & \bar{\Phi}_{1,10} & \bar{\Phi}_{1,11} & \bar{\Phi}_{1,12} & 0 & \bar{\Phi}_{1,14 i}  \tag{38}\\
* & \bar{\Phi}_{22 i} & 0 & 0 & \bar{M}_{2}^{T} & -\bar{T}_{2}^{T} & X_{i} \mathscr{A}_{d i}^{T} & X_{i} \mathscr{A}_{d i}^{T} & X_{i} \mathscr{A}_{d i}^{T} & \bar{\Phi}_{2,10} & \bar{\Phi}_{2,11} & \bar{\Phi}_{2,12} & 0 & \bar{\Phi}_{2,14 i} \\
* & * & -S & 0 & 0 & 0 & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & 0 & 0 & 0 & \bar{\Phi}_{3,13 i} & 0 \\
* & * & * & -\gamma^{2} I & 0 & 0 & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & 0 & 0 & 0 & \Phi_{4,13 i} & 0 \\
* & * & * & * & -\bar{Q}_{1 i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{Q}_{2 i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -X_{l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Phi_{88} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \bar{\Phi}_{99} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \bar{\Phi}_{10,10 i} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \Phi_{11,11 i} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \Phi_{12,12 i} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -S & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -I
\end{array}\right]<0,
$$

where

$$
\begin{gather*}
\bar{\Phi}_{11 i}=-X_{i}+\bar{Q}_{1 i}+\bar{Q}_{2 i}+(\bar{\tau}-\underline{\tau}+1) \bar{Q}_{3 i}+\bar{N}_{1}+\bar{N}_{1}^{T}, \\
\bar{\Phi}_{12}=-\bar{N}_{1}^{T}+\bar{N}_{2}+\bar{T}_{1}^{T}-\bar{M}_{1}^{T},  \tag{39}\\
\bar{\Phi}_{17 i}=X_{i} \widehat{\mathscr{A}}_{i}^{T}+\bar{K}_{i}^{T} \widehat{\mathscr{B}}_{i}^{T}, \\
\bar{\Phi}_{18 i}=\Phi_{19 i}=\bar{\Phi}_{17 i}-X_{i}, \\
\bar{\Phi}_{1,10}=\bar{\tau} \bar{N}_{1}^{T}, \quad \bar{\Phi}_{1,11}=(\bar{\tau}-\underline{\tau}) \bar{T}_{1}^{T}, \\
\bar{\Phi}_{1,13}=(\bar{\tau}-\underline{\tau}) \bar{M}_{1}^{T}, \quad \bar{\Phi}_{1,14 i}=X_{i} \widehat{\mathscr{C}}_{i}^{T}+\bar{K}_{i}^{T} D_{i}^{T} \\
\bar{\Phi}_{22 i}=-\bar{Q}_{3 i}-\bar{N}_{2}-\bar{N}_{2}^{T}+\bar{T}_{2}+\bar{T}_{2}^{T}-\bar{M}_{2}-\bar{M}_{2}^{T} \\
\bar{\Phi}_{2,14 i}=X_{i} \mathscr{C}_{d i}^{T}, \quad \bar{\Phi}_{2,10}=\bar{\tau} \bar{N}_{2}^{T} \\
\bar{\Phi}_{2,11}=(\bar{\tau}-\underline{\tau}) \bar{T}_{2}^{T}, \quad \bar{\Phi}_{2,12}=(\bar{\tau}-\underline{\tau}) \bar{M}_{2}^{T} \\
\bar{\Phi}_{3,13 i}=W^{T} S+\left(\bar{L} B_{i} V\right)^{T}, \\
\bar{\Phi}_{88}=-\frac{1}{\bar{\tau} \bar{R}_{1}}, \quad \bar{\Phi}_{99}=-\frac{1}{(\bar{\tau}-\underline{\tau}) \bar{R}_{2}} \\
\bar{\Phi}_{10,10 i}=-\bar{\tau} X_{i} R_{1} X_{i}, \quad \bar{\Phi}_{4,13 i}=\breve{\mathscr{B}}_{1 i}^{T} S
\end{gather*}
$$

$$
\begin{gathered}
\bar{\Phi}_{11,11 i}=-(\bar{\tau}-\underline{\tau}) X_{i}\left(R_{1}+R_{2}\right) X_{i} \\
\bar{\Phi}_{12,12 i}=-(\bar{\tau}-\underline{\tau}) X_{i} R_{2} X_{i}
\end{gathered}
$$

Theorem 7 (consider the system $\Sigma_{e}$ ). For scalars $\underline{\tau}>$ $0, \bar{\tau}>0, \gamma>0$, the error system $\Sigma_{e}$ is asymptotically stable with an $H_{\infty}$ performance index, if there exist matrices $\bar{M}_{1}, \bar{M}_{2}, \bar{N}_{1}, \bar{N}_{2}, \bar{T}_{1}, \bar{T}_{2}, \bar{K}_{1 i}, \bar{K}_{2 i}, X_{i}>0, \bar{Q}_{1}>0, \bar{Q}_{2}>$ $0, \bar{Q}_{3}>0, \bar{R}_{1}>0, \bar{R}_{2}>0, R_{1}>0, R_{2}>0, S>0, i \in \mathcal{N}$, such that inequality (38) holds.

It is clear that (38) is a nonlinear matrix inequality due to the existence of terms $\bar{R}_{1}, \bar{R}_{2}, X_{i} R_{1} X_{i}$ and $X_{i} R_{2} X_{i}$. In the sequel, the CCL method is resorted to solve the desired controller gains and observer gain.

Introduce two new variables $S_{1 i}$ and $S_{2 i}$ such that $X_{i} R_{1} X_{i} \geq S_{1 i}$ and $X_{i} R_{2} X_{i} \geq S_{2 i}$, then we obtain the following results.

Theorem 8 (Consider system $\Sigma_{e}$ ). For scalars $\underline{\tau}>0, \bar{\tau}>$ $0, \gamma>0$, the error system $\Sigma_{e}$ is asymptotically stable with an $H_{\infty}$ performance index, if there exist matrices $\bar{M}_{1}, \bar{M}_{2}, \bar{N}_{1}, \bar{N}_{2}, \bar{T}_{1}, \bar{T}_{2}, \bar{K}_{1 i}, \bar{K}_{2 i}, X_{i}>0, \bar{Q}_{1}>0, \bar{Q}_{2}>$ $0, \bar{Q}_{3}>0, \bar{R}_{1}>0, \bar{R}_{2}>0, R_{1}>0, R_{2}>0, S>$ $0, S_{1 i}, S_{2 i}, i \in \mathcal{N}$, such that the following inequalities hold

$$
\widehat{\Phi}_{i}=\left[\begin{array}{cccccccccccccc}
\bar{\Phi}_{11 i} & \bar{\Phi}_{12} & 0 & 0 & \bar{M}_{1}^{T} & -\bar{T}_{1}^{T} & \bar{\Phi}_{17 i} & \bar{\Phi}_{18 i} & \bar{\Phi}_{19 i} & \bar{\Phi}_{1,10} & \bar{\Phi}_{1,11} & \bar{\Phi}_{1,12} & 0 & \bar{\Phi}_{1,14 i} \\
* & \bar{\Phi}_{22 i} & 0 & 0 & \bar{M}_{2}^{T} & -\bar{T}_{2}^{T} & X_{i} \mathscr{A}_{d i}^{T} & X_{i} \mathscr{A}_{d i}^{T} & X_{i} \mathscr{A}_{d i}^{T} & \bar{\Phi}_{2,10} & \bar{\Phi}_{2,11} & \bar{\Phi}_{2,12} & 0 & \bar{\Phi}_{2,14 i} \\
* & * & -S & 0 & 0 & 0 & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & \mathscr{B}_{i}^{T} & 0 & 0 & 0 & \bar{\Phi}_{3,13 i} & 0  \tag{42}\\
* & * & * & -\gamma^{2} I & 0 & 0 & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & \mathscr{B}_{1 i}^{T} & 0 & 0 & 0 & \bar{\Phi}_{4,13 i} & 0 \\
* & * & * & * & -\bar{Q}_{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{Q}_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -X_{l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Phi_{88} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \bar{\Phi}_{99} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \widehat{\Phi}_{10,10 i} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \widehat{\Phi}_{11,11 i} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \Phi_{12,12 i} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -S & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -I
\end{array}\right]
$$

where $\widehat{\Phi}_{10,10 i}=-\bar{\tau} S_{1 i}, \widehat{\Phi}_{11,11 i}=-(\bar{\tau}-\underline{\tau})\left(S_{1 i}+S_{2 i}\right), \widehat{\Phi}_{12,12 i}=$ $-(\bar{\tau}-\underline{\tau}) S_{2 i}$.

From (41) and (42), we know that the conditions in Theorem 8 are not strict linear matrix inequalities. By the assistance of the CCL method [46], the nonconvex feasibility problem formulated by (40)-(42) can be transformed into the following nonlinear minimization problem:

Minimize $\operatorname{Tr}\left(\sum_{i=1}^{N}\left(X_{i} H_{i}+S_{1 i} J_{1 i}+S_{2 i} J_{2 i}\right)+\bar{R}_{1} G_{1}+\bar{R}_{2} G_{2}\right)$
Subject to (40) and

$$
\begin{array}{ll}
{\left[\begin{array}{cc}
G_{1} & H_{i} \\
H_{i} & J_{1 i}
\end{array}\right] \geq I,} & {\left[\begin{array}{cc}
G_{2} & H_{i} \\
H_{i} & J_{2 i}
\end{array}\right] \geq I,} \\
{\left[\begin{array}{cc}
S_{1 i} & I \\
I & J_{1 i}
\end{array}\right] \geq 0,} & {\left[\begin{array}{cc}
S_{2 i} & I \\
I & J_{2 i}
\end{array}\right] \geq 0,} \\
{\left[\begin{array}{cc}
\bar{R}_{1} & I \\
I & G_{1}
\end{array}\right] \geq 0,} & {\left[\begin{array}{cc}
\bar{R}_{2} & I \\
I & G_{2}
\end{array}\right] \geq 0,} \\
{\left[\begin{array}{cc}
X_{i} & I \\
I & H_{i}
\end{array}\right] \geq 0,} & \tag{43}
\end{array}
$$

If the solution of the above minimization problem is ( $6 \mathrm{~N}+$ 4) $m$, then system $\Sigma_{e}$ is asymptotically stable with an $H_{\infty}$ performance index via controller (5) and observer (6) with gains $K_{i}=\bar{K}_{i} X_{i}^{-1}$ and $L=S^{-1} \bar{L}$, respectively.

## 4. A Numerical Example

Now, we provide an example to show the effectiveness of the main result in this paper.

Consider discrete-time switched system with parameters as follows:

$$
\begin{gather*}
A_{1}=\left[\begin{array}{cc}
0.2 & 0 \\
0 & -0.035
\end{array}\right], \quad A_{d 1}=\left[\begin{array}{cc}
0.02 & 0 \\
0.01 & -0.03
\end{array}\right] \\
B_{1}=\left[\begin{array}{l}
8 \\
7
\end{array}\right], \\
B_{11}=\left[\begin{array}{c}
-0.03 \\
0.05
\end{array}\right], \quad C_{1}=\left[\begin{array}{ll}
1.2 & 0.8
\end{array}\right], \\
C_{d 1}=\left[\begin{array}{ll}
1 & -0.6
\end{array}\right], \quad D_{1}=16, \\
A_{2}=\left[\begin{array}{cc}
-0.3 & 0 \\
0.01 & -0.015
\end{array}\right], \quad A_{d 2}=\left[\begin{array}{cc}
0.01 & -0.01 \\
0.02 & -0.05
\end{array}\right], \\
B_{2}=\left[\begin{array}{c}
10 \\
10
\end{array}\right], \quad B_{12}=\left[\begin{array}{c}
-0.01 \\
0.07
\end{array}\right], \quad C_{2}=\left[\begin{array}{ll}
0.6 & 1.6
\end{array}\right], \\
C_{d 2}=\left[\begin{array}{ll}
0.8 & -0.4
\end{array}\right], \quad D_{2}=14 . \tag{44}
\end{gather*}
$$

The reference model is given by the following parameters:

$$
\begin{gather*}
A_{r}=\left[\begin{array}{cc}
0.5 & -1 \\
0.3 & -0.035
\end{array}\right], \quad B_{r}=\left[\begin{array}{l}
0.2 \\
0.8
\end{array}\right],  \tag{45}\\
C_{r}=\left[\begin{array}{ll}
0.5 & 0.5
\end{array}\right] .
\end{gather*}
$$

The disturbance model is presented by the following parameters:

$$
W=\left[\begin{array}{cc}
0.8776 & 0.4794  \tag{46}\\
-0.4794 & 0.8776
\end{array}\right], \quad V=\left[\begin{array}{cc}
10 & 0
\end{array}\right]
$$

Set $\underline{\tau}=1$ and $\bar{\tau}=4$. Here, we suppose the disturbance attenuation level $\gamma=0.8$. Then, using Matlab Control Toolbox to solve Theorem 8, we obtain controller gains and observer gain as follows:

$$
\begin{gather*}
K_{11}=\left[\begin{array}{ll}
-0.0477 & -0.0308
\end{array}\right], \quad K_{21}=\left[\begin{array}{ll}
-0.0011 & 0.0432
\end{array}\right], \\
L=\left[\begin{array}{cc}
-0.0012 & -0.0004 \\
0.0005 & 0.0002
\end{array}\right], \\
K_{12}=\left[\begin{array}{lll}
-0.0159 & -0.0103
\end{array}\right], \quad K_{22}=\left[\begin{array}{ll}
-0.0004 & 0.0144
\end{array}\right] . \tag{47}
\end{gather*}
$$

Suppose the switching sequence as $121212 \cdots$. The initial value of the states is chosen as $\phi(s)=\left[-0.2 e^{-s} 0.1 e^{-s}\right]^{T}$, and the reference model of initial condition is selected as $x_{r}=[0.1-0.2]^{T}$. In the sequel, two kinds of reference inputs, that is, step reference input and sinusoidal reference input, are considered to demonstrate the effectiveness of the proposed method.

Case 1 (step reference input). Let

$$
\begin{gather*}
r(k)= \begin{cases}10, & 80 \leq k<150, \\
0, & \text { else } k\end{cases} \\
d_{2}(k)= \begin{cases}\frac{30}{k+1}, & 80 \leq k<150 \\
0, & \text { else } k\end{cases} \tag{48}
\end{gather*}
$$

Curves of $z_{k}$ and $z_{r}$ are depicted in Figure 1 under input signal (48). From Figure 1, we can see that the system output can effectively track the reference model output in presence of matched disturbances and mismatched disturbances, which demonstrates the effectiveness of the proposed method. In order to evaluate the effectiveness of the DOB, curves of disturbances and disturbances estimation are shown in Figure 2. It illustrates that DOB can effectively estimate disturbances.

Case 2 (sinusoidal reference input). Let

$$
\begin{align*}
& r(k)= \begin{cases}10 \sin (k), & 80 \leq k<150, \\
0, & \text { else } k,\end{cases} \\
& d_{2}(k)= \begin{cases}\frac{30}{k+1}, & 80 \leq k<150, \\
0, & \text { else } k .\end{cases} \tag{49}
\end{align*}
$$

Figure 3 presents the curves of $z_{k}$ and $z_{r}$ under input signal (49), which demonstrates that the proposed method obtains good tracking performance in spite of matched disturbances and mismatched disturbances. Figure 4 shows the disturbances estimation results, which depicts that the DOB can effectively estimate disturbances.


Figure 1: Curves of $z(k)$ and $z_{r}(k)$ under Case 1.


Figure 2: Curves of disturbances and disturbances estimation under Case 1.

Remark 9. In this paper, a composite tracking controller is designed for a class of discrete-time switched systems with time-varying delay. In reality, many physical systems can be modelled as switched system, for example, flight control systems [47, 48] and inverted-pendulum system [49]. Take flight control system; for example, the flight control system can be modelled as switched system corresponding to finite operating points within the flight envelope, where $d_{2}(t)$ is denoted as wind gust and $d_{1}(t)$ is regarded as unknown harmonic disturbances. In order to better serve engineering, we will pay attention to studying a tracking controller design for a practical system based on our method in future.

## 5. Conclusions

The problem of $H_{\infty}$ output tracking control for discretetime switched systems subject to time-varying delay and disturbances has been studied. $H_{\infty}$ control has achieved the attenuation performance with respect to norm bounded disturbances. DOBC has been employed to reject the disturbances with some known information. In this paper, a


Figure 3: Curves of $z(k)$ and $z_{r}(k)$ under Case 2.


Figure 4: Curves of disturbances and disturbances estimation under Case 2.
composite control scheme, that is, consisting of $H_{\infty}$ control method and DOBC technique, has been proposed, which can effectively attenuate and reject the external disturbances. A numerical example has been provided to show the effectiveness of the proposed algorithm.

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## Research Article

# $H_{\infty}$ Control for Networked Control Systems with Time Delays and Packet Dropouts 

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#### Abstract

This paper is concerned with the $H_{\infty}$ control issue for a class of networked control systems (NCSs) with packet dropouts and time-varying delays. Firstly, the addressed NCS is modeled as a Markovian discrete-time switched system with two subsystems; by using the average dwell time method, a sufficient condition is obtained for the mean square exponential stability of the closed-loop NCS with a desired $H_{\infty}$ disturbance attenuation level. Then, the desired $H_{\infty}$ controller is obtained by solving a set of linear matrix inequalities (LMIs). Finally, a numerical example is given to illustrate the effectiveness of the proposed method.


## 1. Introduction

Networked control systems (NCSs) are distributed systems in which communication between sensors, actuators, and controllers is supported by a shared real-time network. Compared with conventional point-to-point system connection, this new network-based control scheme reduces system wiring and has low cost, high reliability, information sharing, and remote control [1,2]. Nevertheless, the introduction of communication networks also brings some new problems and challenges, such as time-delay, packet dropout, quantization, and band-limited channel [3-8], which all might be potential sources of poor performance, even of instability.

Random delay and packet dropout in NCS are two major causes for the deterioration of system stability; various approaches have been developed for the NCS with random communication delays and packet dropout in [9-17]. The time delay occurs in various physical, industrial, and engineering systems and is a source of poor performance and instability of systems. In $[9,10]$, the uncertainties of the delays are transformed into those of the system models with uncertain parameters. The delay is limited to take finite values during a sampling period, and the NCS is ultimately modeled as a discrete-time switched system with a finite number of subsystems [11, 12]. In [13-15], the delay is assumed to be random
and follows some specific distribution laws, which may not be exactly known prior in practice. And in some literature the delay is separated into a nominal part and an uncertain part; in this way, the NCS is represented as an uncertain system with norm-bounded uncertainties or polytopic uncertainties. Another important issue in NCS control problem is packet dropout; most of the NCS models are presented by using the Bernoulli random binary distributed sequence methods or the Markov chain. For NCSs, let the binary-valued function denote the data transmission status from sensor to controller and controller to actuator, respectively, where 1 means successful packet communication and 0 is the case of packet dropout [16]. Reference [17] proposes an iterative method to model NCSs with bounded packet dropout as MJLSs with partly unknown transition probabilities.

On the other hand, in view of abrupt variation in the structures, such as component failures, sudden environmental disturbance, and abrupt variations of the operating points of NCSs, it is more appropriate to model such class of systems as a special class of stochastic hybrid systems with finite operation modes. And packet dropout (time-delay) of the next sampling moment may have a close relation to the previous moment, so it is reasonable to model NCS as the Markov switched system. The mean square stabilization of a class of Markovian NCS is studied in [18], and the average
dwell time (ADT) approach is applied to investigate the stability of the NCS in [19]. However, to the best of our knowledge, the problems of mean square exponential stability and control for the NCS have not been fully investigated to date. This motivates the present study.

With the motivation of the above reasons, we consider the mean square exponential $H_{\infty}$ performance for NCS with random delay and packet dropout. The main contribution can be summarized as follows: (i) an NCS model with random delay and packet dropout is proposed firstly; the packet dropout process is modeled as a finite state Markov chain and the resulting closed-loop system is a Markovian switching system; (ii) the parameter-dependent Lyapunov function is applied for stability analysis and control synthesis, and sufficient conditions for the robustly mean square exponential stability of the closed-loop system are given by using the ADT method [20], where the convergence of the Markov chain is utilized; and (iii) a state feedback controller is designed by using a cone complementary linearization approach to ensure that the closed-loop system is mean square exponentially stable and achieves the disturbance attenuation level.

The paper is organized as follows. In Section 2, the NCS with packet dropouts and time-varying delays is modeled as a class of the Markovian discrete-time switched system with two subsystems. The mean square exponential stability of the closed-loop NCS with a desired $H_{\infty}$ disturbance attenuation level is developed in Section 3 and the desired $H_{\infty}$ controller is formulated in a set of LMIs. A numerical example is provided in Section 4. Finally, Section 5 concludes this paper.

Notation 1. Throughout the paper, the superscript " -1 " and " $T$ " stand for the inverse and transpose of a matrix, respectively; $R^{n}$ denotes the $n$-dimensional Euclidean space and the notation $P>0$ means that $P$ is a real symmetric positive definite matrix. $E\{x\}$ is the expectation of the stochastic variable $x . I$ and 0 represent identity matrix and zero matrix with appropriate dimensions in different places. In symmetric block matrices or complex matrix expressions, we use an asterisk $*$ to represent a term that is induced by symmetry and $\operatorname{diag}\{\cdots\}$ stands for a block diagonal matrix. $\|\cdot\|$ refers to the Euclidean norm for vectors and induced 2-norm for matrix. $L_{2}\left[k_{0}, \infty\right)$ stands for the space of square integrable functions on $\left[k_{0}, \infty\right)$.

## 2. Model of Networked Control System

Consider the following system:

$$
\begin{gather*}
\dot{x}(t)=A_{p} x(t)+B_{p} u(t)+f(x, t)+H_{p} w(t),  \tag{1}\\
z(t)=C x(t)
\end{gather*}
$$

where $x(t) \in R^{n}, u(t) \in R^{m}$, and $z(t) \in R^{P}$ are the state vector, control input vector, and controlled output vector, respectively, and $w(t) \in R^{d}$ is the exogenous disturbance signal belonging to $L_{2}[0, \infty) . A_{p}, B_{p}, H_{p}$, and $C$ are known
real matrices with appropriate dimensions. $f: \Omega \times\left[t_{0}, \infty\right) \rightarrow$ $R^{n}\left(\Omega \subset R^{n}\right)$ is the nonlinear function vector, and $f\left(0, t_{0}\right)=0$. $f$ satisfies the local Lipschitz condition, that is,

$$
\begin{gather*}
\left\|f\left(x_{1}, t\right)-f\left(x_{2}, t\right)\right\|_{2} \leq \alpha\left\|x_{1}-x_{2}\right\|_{2}  \tag{2}\\
\forall x_{1}, x_{2} \in \Omega \subset R^{n}, \forall t \in\left[t_{0}, \infty\right)
\end{gather*}
$$

where $\alpha>0$ is a known constant.
In the considered NCS, time delays exist in both channels from sensor to controller and from controller to actuator. Sensor-to-controller delay and controller-to-actuator delay are denoted by $\tau^{\text {sc }}$ and $\tau^{\mathrm{ca}}$, respectively. The assumptions in the above NCS are as follows:
(1) the discrete-time state-feedback controller and the actuator are event driven; the sensor is time-driven with sampling period $T$,
(2) the network-induced delay $\tau_{k} \triangleq \tau_{k}^{\mathrm{sc}}+\tau_{k}^{\mathrm{ca}}$ satisfies $0<$ $\tau_{\text {min }} \leq \tau_{k} \leq \tau_{\text {max }}<T$,
(3) the zero-order hold device does not update the output value until the new value arrives.

The output value of the discrete-time state-feedback controller corresponding to $x(k)$ is denoted by

$$
\begin{equation*}
u(k)=K x(k) \tag{3}
\end{equation*}
$$

Consider the plant input:
$u(k)$
$= \begin{cases}\widehat{u}(k) & \text { if } \widehat{u}(k) \text { and } x(k) \text { is successfully transmitted, } \\ u(k-1) & \text { if } \widehat{u}(k) \text { or } x(k) \text { is lost during transmission. }\end{cases}$

Discretizing system (1) in one period, we can obtain the discrete state equation of the NCS:

$$
\begin{align*}
x(k+1)= & A x(k)+B_{0}\left(\tau_{k}\right) u(k) \\
& +B_{1}\left(\tau_{k}\right) u(k-1)+\tilde{f}(x, k)+H w(k), \tag{5}
\end{align*}
$$

where $A=e^{A_{p} T}, B_{0}\left(\tau_{k}\right)=\int_{0}^{T-\tau_{k}} e^{A_{p} s} d s B_{p}, B_{1}\left(\tau_{k}\right)=$ $\int_{T-\tau_{k}}^{T} e^{A_{p} s} d s B_{p}, H=\int_{0}^{T} e^{A_{p} s} d s H_{p}$, and $\tilde{f}(x, k)=$ $\int_{0}^{T} e^{A_{p} s} d s f(x, k)$.

By using the Jordan form of the matrix $A_{p}, B_{0}\left(\tau_{k}\right)$ is rewritten as [21]

$$
\begin{equation*}
B_{0}\left(\tau_{k}\right)=F_{0}+\sum_{i=1}^{v} \eta_{i}\left(\tau_{k}\right) F_{i} \tag{6}
\end{equation*}
$$

with $v \leq n$, where $F_{0}$ and $F_{i}$ are constant matrices, $\eta_{i}\left(\tau_{k}\right)=$ $e^{a\left(T-\tau_{k}\right)} \cos \left(b\left(T-\tau_{k}\right)\right)$ and the eigenvalue of $A$ is $\lambda=a+i b$.

Then, $\left\{B_{0}\left(\tau_{k}\right) \mid k \in N\right\}$ is a subset of $\operatorname{co}(F)$ with

$$
\begin{align*}
F & =\left\{F_{0}+\sum_{i=1}^{v} \widetilde{\tilde{\eta}}_{i} F_{i} \mid \widetilde{\eta}_{i}=\bar{\eta}_{i}, \underline{\eta}_{i}, i=1,2, \ldots, v\right\}  \tag{7}\\
& =\left\{\widetilde{F}_{i} \mid i=1,2, \ldots, 2^{v}\right\},
\end{align*}
$$

where $\bar{\eta}_{i}=\max \eta_{i}\left(\tau_{k}\right), \underline{\eta}_{i}=\min \eta_{i}\left(\tau_{k}\right), F$ is the set of vertices, and $\operatorname{co}(\cdot)$ denotes the convex hull. Thus we obtain

$$
\begin{equation*}
B_{0}\left(\tau_{k}\right)=\sum_{i=1}^{2^{v}} \xi_{i}(k) \widetilde{F}_{i} \tag{8}
\end{equation*}
$$

with $\sum_{i=1}^{2^{v}} \xi_{i}(k)=1, \xi_{i}(k) \in[0,1]$.
Defining an augmented vector $\bar{x}(k)=$ $\left[x^{T}(k) u^{T}(k-1)\right]^{T}$, during each sampling period, two cases may arise, which can be listed as follows.
$S_{1}$ : no packet dropout happens; (5) can be written as

$$
\begin{gather*}
\bar{x}(k+1)=\bar{A}_{1}(k) \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k), \\
z(k)=\bar{C} \bar{x}(k), \tag{9}
\end{gather*}
$$

where

$$
\begin{gather*}
\bar{A}_{1}(k)=\left[\begin{array}{cc}
A+B_{0}\left(\tau_{k}\right) K & B_{1}\left(\tau_{k}\right) \\
K & 0
\end{array}\right] \\
=\left[\begin{array}{cc}
A & B-B_{0}\left(\tau_{k}\right) \\
0 & 0
\end{array}\right]+\left[\begin{array}{c}
B_{0}\left(\tau_{k}\right) \\
I
\end{array}\right]\left[\begin{array}{ll}
K & 0
\end{array}\right], \\
B=\int_{0}^{T} e^{A_{p} s} d s B_{p}, \quad B_{1}\left(\tau_{k}\right)=B-B_{0}\left(\tau_{k}\right), \\
\bar{f}(x, k)=\left[\begin{array}{c}
\tilde{f}(x, k) \\
0
\end{array}\right], \quad \bar{H}=\left[\begin{array}{c}
H \\
0
\end{array}\right], \quad \bar{C}=\left[\begin{array}{ll}
C & 0
\end{array}\right] . \tag{10}
\end{gather*}
$$

Substituting (8) into (9) gives rise to

$$
\begin{equation*}
\bar{A}_{1}(k)=\sum_{i=1}^{2^{v}} \xi_{i}(k) A_{i} \tag{11}
\end{equation*}
$$

where

$$
\begin{gather*}
A_{i}=\widetilde{A}_{i}+\widetilde{B}_{i}\left[\begin{array}{ll}
K & 0
\end{array}\right]=\widetilde{A}_{i}+\widetilde{B}_{i} \bar{K} \\
\widetilde{A}_{i}=\left[\begin{array}{cc}
A & B-\widetilde{F}_{i} \\
0 & 0
\end{array}\right], \quad \widetilde{B}_{i}=\left[\begin{array}{c}
\widetilde{F}_{i} \\
I
\end{array}\right], \quad \bar{K}=\left[\begin{array}{ll}
K & 0
\end{array}\right] . \tag{12}
\end{gather*}
$$

$S_{2}$ : packet dropout happens; (5) can be written as

$$
\begin{align*}
\bar{x}(k+1) & =\bar{A}_{2}(k) \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k), \\
z(k) & =\bar{C} \bar{x}(k), \tag{13}
\end{align*}
$$

where $\bar{A}_{2}=\left[\begin{array}{cc}A & B \\ 0 & I\end{array}\right]$.
From (2)-(9), the nonlinear uncertainty $\bar{f}(x, k)$ satisfies

$$
\begin{equation*}
\bar{f}^{T}(x, k) \bar{f}(x, k)=\tilde{f}^{T}(x, k) \tilde{f}(x, k) \leq \bar{x}^{T}(k) U^{T} U \bar{x}(k) \tag{14}
\end{equation*}
$$

where $U$ is a known constant positive-definite matrix.
By the above analysis and assumptions, we can see that networked control system can be described by the following switched system with two modes:

$$
\begin{gather*}
\bar{x}(k+1)=\bar{A}_{\sigma(k)}(k) \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k), \\
z(k)=\bar{C} \bar{x}(k), \tag{15}
\end{gather*}
$$

where $\sigma(k)$ is called a switching signal. $\sigma(k)=1$ represents no packet dropout, while $\sigma(k)=2$ implies packet dropout. The switching characteristics between the two modes are often assumed as the Markov chain, and $\pi_{r l}$ is transition probability from mode $r$ to $l, r, l=1,2$; therefore, $\sigma(k)$ of the Markov chain has ergodicity and satisfied the following condition:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \pi_{r l}^{(n)}=\pi_{l}, \quad r, l=1,2 \tag{16}
\end{equation*}
$$

where $\pi_{l}$ is the limitation of state $l$. So $\left\{\pi_{1}, \pi_{2}\right\}$ is the stationary distribution of the Markov chain.

For an arbitrary switching sequence $\sigma(k)$ and any given integer $k>0$, let $k_{0}$ imply the initial time, and $k_{0}<k_{1}<$ $k_{2}<\cdots k_{q}<\cdots<k, q \geq 1$ represent the switching instants. Denote $T^{1}\left[k_{0}, k\right)$ as the all sequence of the time period in which subsystem 1 is active during the time interval $\left[k_{0}, k\right)$. Similarly, $T^{2}\left[k_{0}, k\right)$ represents the all period sequence that subsystem 2 is active during the time interval $\left[k_{0}, k\right)$.

Lemma 1 (Schur complement [22]). For a given matrix $S=$ $\left[\begin{array}{ll}S_{11} & S_{12} \\ S_{12}^{T} & S_{22}\end{array}\right]$, where $S_{11}, S_{22}$ are square matrices, the following conditions are equivalent:
(1) $S<0$;
(2) $S_{11}<0, S_{22}-S_{12}^{T} S_{11}^{-1} S_{12}<0$;
(3) $S_{22}<0, S_{11}-S_{12} S_{22}^{-1} S_{12}^{T}<0$.

Lemma 2 (see [23]). The stochastic stability in discrete time implies the stochastic stability in continuous time.

Definition 3 (see [24]). The closed-loop system (15) is mean square exponentially stable with $w(k)=0$, if there exists $\delta>$ $0,0<\beta<1$, such that

$$
\begin{equation*}
E\left\{\|\bar{x}(k)\|^{2}\right\}<\delta \beta^{k-k_{0}} E\left\{\left\|\bar{x}\left(k_{0}\right)\right\|^{2}\right\} \tag{17}
\end{equation*}
$$

for all initial condition $\left(\bar{x}\left(k_{0}\right), \sigma\left(k_{0}\right)\right)$.
Definition 4 (see [25]). For any $k>k_{0} \geq 0$, let $N_{\sigma}\left[k_{0}, k\right)$ denote the total number of the switching of $\sigma(k)$ during the interval $\left[k_{0}, k\right)$. If

$$
\begin{equation*}
N_{\sigma}\left[k_{0}, k\right) \leq N_{0}+\frac{k-k_{0}}{T_{a}} \tag{18}
\end{equation*}
$$

holds for a given $N_{0} \geq 0, T_{a}>0$, then the constant $T_{a}$ is called the average dwell time and $N_{0}$ is the chatter bound. For simplicity, we choose $N_{0}=0$ without loss of generality.

Definition 5 (see [20]). Given scalars $\gamma>0$ and $0<\lambda<1$, the closed-loop system (15) is robustly exponentially stable with an exponential $H_{\infty}$ performance $\gamma$ if the following conditions are satisfied:
(a) the closed-loop system (15) with $\omega(k) \equiv 0$ is exponentially stable;
(b) under the zero-initial condition, it holds that

$$
\begin{align*}
& \sum_{k=k_{0}}^{\infty} E\left\{\lambda^{k} z^{T}(k) z(k)\right\} \\
& \quad<\gamma^{2} \sum_{k=k_{0}}^{\infty} E\left\{w^{T}(k) w(k)\right\}, \quad \forall w(k) \in L_{2}\left[k_{0}, \infty\right) \tag{19}
\end{align*}
$$

## 3. Main Results

The following theorems present a sufficient condition for the mean square stability of the considered system and the $H_{\infty}$ controller design method.
3.1. Stability Analysis. In this subsection, sufficient conditions for the existence of mean square exponential stability of system (15) with $\omega(k) \equiv 0$ are given in the following theorem.

Theorem 6. System (15) is mean square exponentially stable with a decay rate $\lambda^{\rho}$, if there exist positive definite matrices $P_{i}$, $Q$, scalars $\mu \geq 1, \lambda_{1}$, and $\lambda_{2}$, such that

$$
\begin{align*}
& {\left[\begin{array}{cc}
A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) A_{i}-\lambda_{1} P_{i}+U^{T} U & A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) \\
* & \left(\pi_{11} P_{j}+\pi_{12} Q\right)-I
\end{array}\right]} \\
& \quad<0 \tag{20}
\end{align*}
$$

$$
\left[\begin{array}{cc}
\bar{A}_{2}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) \overline{A_{2}}-\lambda_{2} Q+U^{T} U & \bar{A}_{2}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) \\
* & \left(\pi_{21} P_{j}+\pi_{22} Q\right)-I
\end{array}\right]
$$

$$
\begin{equation*}
<0 \quad i, j=1,2,3, \ldots, 2^{v} \tag{21}
\end{equation*}
$$

$$
\begin{gather*}
\frac{1}{\mu} Q \leq P_{i} \leq \mu \mathrm{Q},  \tag{22}\\
0<\lambda<1,  \tag{23}\\
\max \left\{\pi_{12}, \pi_{21}\right\}<-\frac{\ln \lambda}{\ln \mu}, \tag{24}
\end{gather*}
$$

where $\lambda=\lambda_{1}^{\pi_{21} /\left(\pi_{12}+\pi_{21}\right)} \lambda_{2}^{\pi_{12} /\left(\pi_{12}+\pi_{21}\right)}, \rho=1+\max \left\{\pi_{12}, \pi_{21}\right\}$. $(\ln \mu / \ln \lambda)$.

Proof. For the system (15), define the following Lyapunov function:

$$
\begin{equation*}
V_{\sigma(k)}(\bar{x}(k), \xi(k))=\bar{x}^{T}(k) \widetilde{P}_{\sigma(k)} \bar{x}(k), \tag{25}
\end{equation*}
$$

where

$$
\begin{gather*}
\widetilde{P}_{\sigma(k)}=\sum_{i=1}^{2^{v}} \xi_{i}(k) P_{i}, \quad \text { for } \sigma(k)=1  \tag{26}\\
\widetilde{P}_{\sigma(k)}=Q, \quad \text { for } \sigma(k)=2
\end{gather*}
$$

For subsystem 1, it follows from (15) that

$$
\begin{align*}
& \Delta V_{1} {[\bar{x}(k+1)] } \\
&= E\left[V_{1}(\bar{x}(k+1), \xi(k+1))\right]-\lambda_{1} V_{1}(\bar{x}(k), \xi(k)) \\
&= \bar{x}^{T}(k+1)\left(\pi_{11} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{12} Q\right) \bar{x}(k+1) \\
& \quad \lambda_{1} \bar{x}^{T}(k)\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) P_{i}\right) \bar{x}(k) \\
& \leq\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) A_{i} \bar{x}(k)+\bar{f}(x, k)\right)^{T} \\
& \quad\left(\pi_{11} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{12} Q\right) \\
& \quad\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) A_{i} \bar{x}(k)+\bar{f}(x, k)\right) \\
& \quad-\lambda_{1} \bar{x}^{T}(k)\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) P_{i}\right) \bar{x}(k) \\
&+\bar{x}^{T}(k) U^{T} U \bar{x}(k)-\bar{f}^{T}(x, k) \bar{f}(x, k) \\
&=\sum_{i=1}^{2^{v}} \xi_{i}(k) \sum_{j=1}^{2^{v}} \xi_{j}(k+1)\left[\bar{x}_{f}(x)\right]^{T} \Theta[\bar{x}(k)]  \tag{27}\\
&f(x, k)]
\end{align*}
$$

where

$$
\Theta=\left[\begin{array}{cc}
A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) A_{i}-\lambda_{1} P_{i}+U^{T} U & A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right)  \tag{28}\\
* & \left(\pi_{11} P_{j}+\pi_{12} Q\right)-I
\end{array}\right] .
$$

From inequality (20), one obtains

$$
\begin{equation*}
E\left[V_{1}(\bar{x}(k+1), \xi(k+1))\right]<\lambda_{1} V_{1}(\bar{x}(k), \xi(k)) \tag{29}
\end{equation*}
$$

In the same way, for subsystem 2, we obtain

$$
\begin{aligned}
\Delta V_{2} & {[\bar{x}(k+1)] } \\
& =E\left[V_{2}(\bar{x}(k+1), \xi(k+1))\right]-\lambda_{2} V_{2}(\bar{x}(k), \xi(k)) \\
& =\bar{x}^{T}(k+1)\left(\pi_{21} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{22} Q\right) \bar{x}(k+1) \\
& -\lambda_{2} \bar{x}^{T}(k) Q \bar{x}(k)
\end{aligned}
$$

$$
\begin{align*}
\leq & \left(\overline{A_{2}} \bar{x}(k)+\bar{f}(x, k)\right)^{T} \\
& \times\left(\pi_{21} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{22} Q\right) \\
& \cdot\left(\overline{A_{2}} \bar{x}(k)+\bar{f}(x, k)\right)-\lambda_{2} \bar{x}^{T}(k) Q \bar{x}(k) \\
& +\bar{x}^{T}(k) U^{T} U \bar{x}(k)-\bar{f}^{T}(x, k) \bar{f}(x, k) ; \tag{30}
\end{align*}
$$

then

$$
\begin{equation*}
E\left[V_{2}(\bar{x}(k+1), \xi(k+1))\right]<\lambda_{2} V_{2}(\bar{x}(k), \xi(k)) . \tag{31}
\end{equation*}
$$

Considering the condition (22), we get that

$$
\begin{align*}
& V_{1}(\bar{x}(k), \xi(k)) \leq \mu V_{2}\left(\bar{x}\left(k^{-}\right), \xi\left(k^{-}\right)\right)  \tag{32}\\
& V_{2}(\bar{x}(k), \xi(k)) \leq \mu V_{1}\left(\bar{x}\left(k^{-}\right), \xi\left(k^{-}\right)\right) .
\end{align*}
$$

Then for $k_{q}<k<k_{q+1}$, we get

$$
\begin{align*}
E & {\left[V_{\sigma(k)}(\bar{x}(k), \xi(k))\right] } \\
& <E\left[\mu \lambda_{\sigma\left(k_{q}\right)}^{\left(k-k_{q}\right)} V_{\sigma\left(k_{q-1}\right)}\left(\bar{x}\left(k_{q}\right), \xi\left(k_{q}\right)\right)\right] \\
& <E\left[\mu \lambda_{\sigma\left(k_{q}\right)}^{\left(k-k_{q}\right)} \lambda_{\sigma\left(k_{q-1}\right)}^{\left(k_{q}-k_{q-1}\right)} V_{\sigma\left(k_{q-1}\right)}\left(\bar{x}\left(k_{q-1}\right), \xi\left(k_{q-1}\right)\right)\right] \\
& \vdots \\
& <E\left[\mu^{N_{\sigma}\left[k_{0}, k\right)} \lambda_{1}^{T^{1}\left[k_{0}, k\right)} \lambda_{2}^{T^{2}\left[k_{0}, k\right)} V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] . \tag{33}
\end{align*}
$$

Note that the Markov chain is stationary (16); then

$$
\begin{align*}
& E\left[T^{1}\left[k_{0}, k\right)\right]=\pi_{1}\left(k-k_{0}\right),  \tag{34}\\
& E\left[T^{2}\left[k_{0}, k\right)\right]=\pi_{2}\left(k-k_{0}\right) .
\end{align*}
$$

Therefore, we can obtain that

$$
\begin{align*}
& E[ \left.V_{\sigma(k)}(\bar{x}(k), \xi(k))\right] \\
&< E\left[\mu^{N_{\sigma}\left[k_{0}, k\right)} \lambda_{1}^{\pi_{1}\left(k-k_{0}\right)} \lambda_{2}^{\pi_{2}\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
&=E {\left[\mu^{N_{\sigma}\left[k_{0}, k\right)} \lambda_{1}^{\left(\pi_{21} /\left(\pi_{12}+\pi_{21}\right)\right)\left(k-k_{0}\right)}\right.} \\
&\left.\times \lambda_{2}^{\left(\pi_{12} /\left(\pi_{12}+\pi_{21}\right)\right)\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
&=E\left[\mu^{N_{\sigma}\left[k_{0}, k\right)} \lambda^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
&=E\left[\lambda^{N_{\sigma}\left[k_{0}, k\right)(\ln \mu / \ln \lambda)} \lambda^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
&=E\left[\left(\lambda^{\left.\left.1+\left(N_{\sigma}\left[k_{0}, k\right) /\left(k-k_{0}\right)\right) \cdot(\ln \mu / \ln \lambda)\right)^{\left(k-k_{0}\right)}\right]}\right.\right. \\
& \times E\left[V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] . \tag{35}
\end{align*}
$$

From Definition 4 we have that

$$
\begin{equation*}
\frac{k-k_{0}}{N_{\sigma}\left[k_{0}, k\right)} \geq T_{a} . \tag{36}
\end{equation*}
$$

And from [24], we can get $1 / E\left(T_{a}\right) \leq \max \left\{\pi_{12}, \pi_{21}\right\}$; then combining (23) and (24), we can know that

$$
\begin{equation*}
0<\lambda^{1+\left(N_{\sigma}\left[k_{0}, k\right) /\left(k-k_{0}\right)\right) \cdot(\ln \mu / \ln \lambda)}<1, \tag{37}
\end{equation*}
$$

which ensure the convergence of $E\left[V_{\sigma(k)}(\bar{x}(k), \xi(k))\right]$.
In this case,

$$
\begin{align*}
E & {\left[V_{\sigma(k)}(\bar{x}(k), \xi(k))\right] } \\
& <E\left[\left(\lambda^{1+\left(1 / T_{a}\right) \cdot(\ln \mu / \ln \lambda)}\right)^{\left(k-k_{0}\right)}\right] \\
& \times E\left[V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
= & \lambda^{\left(k-k_{0}\right)\left[1+\left(1 / E\left(T_{a}\right)\right) \cdot(\ln \mu / \ln \lambda)\right]} E\left[V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] \\
& \leq\left(\lambda^{\rho}\right)^{\left(k-k_{0}\right)} E\left[V_{\sigma\left(k_{0}\right)}\left(\bar{x}\left(k_{0}\right), \xi\left(k_{0}\right)\right)\right] . \tag{38}
\end{align*}
$$

Furthermore

$$
\begin{align*}
E\left[a\|\bar{x}(k)\|^{2}\right] & \leq E\left[V_{\sigma(k)}(\bar{x}(k), \xi(k))\right] \\
& <\left(\lambda^{\rho}\right)^{\left(k-k_{0}\right)} E\left[b\left\|\bar{x}\left(k_{0}\right)\right\|^{2}\right] \tag{39}
\end{align*}
$$

Then

$$
\begin{equation*}
E\left[\|\bar{x}(k)\|^{2}\right] \leq \frac{b}{a}\left(\lambda^{\rho}\right)^{\left(k-k_{0}\right)} E\left[\left\|\bar{x}\left(k_{0}\right)\right\|^{2}\right] \tag{40}
\end{equation*}
$$

Therefore, by Definition 3, system (15) is mean square exponentially stable.

Remark 7. From Lemma 2, we know that system (1) is also mean square exponentially stable.
3.2. $H_{\infty}$ Performance Analysis and Controller Design. In this subsection, we are in the position to prove the main result. The $H_{\mathbf{\infty}}$ controller design method is given in the following theorem.

Theorem 8. For given scalars $\lambda_{1}, \lambda_{2}, \gamma$, and $\mu \geq 1$, if there exist positive definite matrices $P_{i}, X_{i}, S, Q$, and matrix $\bar{K}=\left[\begin{array}{ll}K & 0\end{array}\right]$ of appropriate dimensions, $i, j=1,2,3, \ldots, 2^{v}$, such that (22)(24) and the following inequalities:

$$
\left[\begin{array}{ccccccc}
-\lambda_{1} P_{i} & 0 & 0 & \Gamma_{14} & \Gamma_{15} & \bar{C}^{T} & U^{T}  \tag{41}\\
* & -I & 0 & \sqrt{\pi_{11}} I & \sqrt{\pi_{12}} I & 0 & 0 \\
* & * & -\gamma^{2} I & \sqrt{\pi_{11}} \bar{H}^{T} & \sqrt{\pi_{12}} \bar{H}^{T} & 0 & 0 \\
* & * & * & -X_{j} & 0 & 0 & 0 \\
* & * & * & * & -S & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & -I
\end{array}\right]<0,
$$

where $\Gamma_{14}=\sqrt{\pi_{11}} \widetilde{A}_{i}^{T}+\sqrt{\pi_{11}} \bar{K}^{T} \widetilde{B}_{i}^{T}, \Gamma_{15}=\sqrt{\pi_{12}} \widetilde{A}_{i}^{T}+$ $\sqrt{\pi_{12}} \bar{K}^{T} \widetilde{B}_{i}^{T}$,

$$
\left.\begin{array}{ccccccc}
-\lambda_{2} Q & 0 & 0 & \sqrt{\pi_{21}} \bar{A}_{2}^{T} & \sqrt{\pi_{22}} \bar{A}_{2}^{T} & \bar{C}^{T} & U^{T}  \tag{42}\\
* & -I & 0 & \sqrt{\pi_{21}} I & \sqrt{\pi_{22}} I & 0 & 0 \\
* & * & -\gamma^{2} I & \sqrt{\pi_{21}} \bar{H}^{T} & \sqrt{\pi_{22}} \bar{H}^{T} & 0 & 0 \\
* & * & * & -X_{j} & 0 & 0 & 0 \\
* & * & * & * & -S & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & -I
\end{array}\right]<0,
$$

hold, then system (15) with the controller gain matrix $\bar{K}$ has robustly mean square exponential stability with $H_{\infty}$ disturbance attenuation level $\gamma$.

Proof. It is easy to obtain that (20) and (21) can be deduced from (41) and (42), respectively. Then from Theorem 6, it can be verified that closed-loop system (15) is mean square exponentially stable with $w(k)=0$.

For the nonzero $w(k)$, using the same Lyapunov function candidates as in Theorem 6, the following relations can be obtained:

$$
\begin{aligned}
& \Delta V_{1}[\bar{x}(k+1)] \\
& =E\left[V_{1}(\bar{x}(k+1), \xi(k+1))\right]-\lambda_{1} V_{1}(\bar{x}(k), \xi(k)) \\
& \leq\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) A_{i} \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k)\right)^{T} \\
& \times\left(\pi_{11} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{12} Q\right) \\
& \cdot\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) A_{i} \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k)\right) \\
& -\lambda_{1} \bar{x}^{T}(k)\left(\sum_{i=1}^{2^{v}} \xi_{i}(k) P_{i}\right) \bar{x}(k) \\
& +\bar{x}^{T}(k) U^{T} U \bar{x}(k)-\bar{f}^{T}(x, k) \bar{f}(x, k), \\
& \Delta V_{2}[\bar{x}(k+1)] \\
& =E\left[V_{2}(\bar{x}(k+1), \xi(k+1))\right]-\lambda_{2} V_{2}(\bar{x}(k), \xi(k)) \\
& \leq\left(\overline{A_{2}} \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k)\right)^{T} \\
& \times\left(\pi_{21} \sum_{j=1}^{2^{v}} \xi_{j}(k+1) P_{j}+\pi_{22} Q\right) \\
& \cdot\left(\overline{A_{2}} \bar{x}(k)+\bar{f}(x, k)+\bar{H} w(k)\right) \\
& -\lambda_{2} \bar{x}^{T}(k) Q \bar{x}(k)+\bar{x}^{T}(k) U^{T} U \bar{x}(k) \\
& -\bar{f}^{T}(x, k) \bar{f}(x, k) \text {. }
\end{aligned}
$$

From inequalities (43), we have

$$
\begin{align*}
& \Delta V_{1}[\bar{x}(k+1)]+z^{T}(k) z(k)-\gamma^{2} w^{T}(k) w(k) \\
& \quad \leq \sum_{i=1}^{2^{v}} \xi_{i}(k) \sum_{j=1}^{2^{v}} \xi_{j}(k+1) \eta^{T}(k) \Xi_{1} \eta(k),  \tag{44}\\
& \Delta V_{2}[\bar{x}(k+1)]+z^{T}(k) z(k)-\gamma^{2} w^{T}(k) w(k) \\
& \quad \leq \sum_{j=1}^{2^{v}} \xi_{j}(k+1) \eta^{T}(k) \Xi_{2} \eta(k), \tag{45}
\end{align*}
$$

where

$$
\begin{gather*}
\eta(k)=\left[\begin{array}{lc}
\bar{x}^{T}(k) \bar{f}^{T}(x, k) & w^{T}(k)
\end{array}\right]^{T}, \\
\Xi_{1}=\left[\begin{array}{ccc}
\psi_{1} & A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) & A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) \bar{H} \\
* & \pi_{11} P_{j}+\pi_{12} Q-I & \left(\pi_{11} P_{j}+\pi_{12} Q\right) \bar{H} \\
* & * & \varphi_{1}
\end{array}\right], \\
\Xi_{2}=\left[\begin{array}{ccc}
\psi_{2} & \bar{A}_{2}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) & \bar{A}_{2}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) \bar{H} \\
* & \pi_{21} P_{j}+\pi_{22} Q-I & \left(\pi_{21} P_{j}+\pi_{22} Q\right) \bar{H} \\
* & * & \varphi_{2}
\end{array}\right], \\
\psi_{1}=A_{i}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) A_{i}-\lambda_{1} P_{i}+\bar{C}^{T} \bar{C}+U^{T} U, \\
\psi_{2}=\bar{A}_{2}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) \overline{A_{2}}-\lambda_{2} Q+\bar{C}^{T} \bar{C}+U^{T} U, \\
\varphi_{1}=\bar{H}^{T}\left(\pi_{11} P_{j}+\pi_{12} Q\right) \bar{H}-\gamma^{2} I, \\
\varphi_{2}=\bar{H}^{T}\left(\pi_{21} P_{j}+\pi_{22} Q\right) \bar{H}-\gamma^{2} I . \tag{46}
\end{gather*}
$$

In terms of the Schur complement, we obtain

$$
\left[\begin{array}{ccccc}
-\lambda_{1} P_{i}+\bar{C}^{T} \bar{C}+\bar{U}^{T} \bar{U} & 0 & 0 & \sqrt{\pi_{11}} A_{i}^{T} & \sqrt{\pi_{12}} A_{i}^{T}  \tag{47}\\
* & -I & 0 & \sqrt{\pi_{11}} I & \sqrt{\pi_{12}} I \\
* & * & -\gamma^{2} I & \sqrt{\pi_{11}} \bar{H}^{T} & \sqrt{\pi_{12}} \bar{H}^{T} \\
* & * & * & -P_{j}^{-1} & 0 \\
* & * & * & * & -Q^{-1}
\end{array}\right]<0,
$$

where $A_{i}=\widetilde{A}_{i}+\widetilde{B}_{i} \bar{K}$,

$$
\left[\begin{array}{ccccc}
-\lambda_{2} Q+\bar{C}^{T} \bar{C}+\bar{U}^{T} \bar{U} & 0 & 0 & \sqrt{\pi_{21}} \bar{A}_{2}^{T} & \sqrt{\pi_{22}} \bar{A}_{2}^{T}  \tag{48}\\
* & -I & 0 & \sqrt{\pi_{21}} I & \sqrt{\pi_{22}} I \\
* & * & -\gamma^{2} I & \sqrt{\pi_{21}} \bar{H}^{T} & \sqrt{\pi_{22}} \bar{H}^{T} \\
* & * & * & -P_{j}^{-1} & 0 \\
* & * & * & * & -Q^{-1}
\end{array}\right]<0 .
$$

In light of Lemma 1, if equalities (47) and (48) hold, then combining (44) and (45), we have that

$$
\begin{align*}
& E\left[V_{1}(\bar{x}(k+1), \xi(k+1))\right]<\lambda_{1} V_{1}(\bar{x}(k), \xi(k))-J(k), \\
& E\left[V_{2}(\bar{x}(k+1), \xi(k+1))\right]<\lambda_{2} V_{2}(\bar{x}(k), \xi(k))-J(k), \tag{49}
\end{align*}
$$

where $J(k)=z^{T}(k) z(k)-\gamma^{2} w^{T}(k) w(k)$.

Combining (22) and (49), it can be seen that

$$
\begin{align*}
& E\left[V_{\sigma(k)}(\bar{x}(k), \xi(k))\right] \\
& \quad<E\left[\mu \lambda_{\sigma\left(k_{q}\right)}^{k-k_{q}} V_{\sigma\left(k_{q-1}\right)}\left(k_{q-1}\right)-\sum_{s=k_{q}}^{k-1} \lambda_{\sigma\left(k_{q}\right)}^{k-s-1} J(s)\right] \\
& \quad<E\left\{\mu \lambda_{\sigma\left(k_{q}\right)}^{k-k_{q}}\left[\lambda_{k_{q-1}}^{k_{q}-k_{q-1}} V_{\sigma\left(k_{q-1}\right)}\left(k_{q-1}\right)-\sum_{s=k_{q-1}}^{k_{q}-1} \lambda_{\sigma\left(k_{q-1}\right)}^{k_{q}-s-1} J(s)\right]\right. \\
& \begin{aligned}
& \vdots\left.\quad-\sum_{s=k_{q}}^{k-1} \lambda_{\sigma\left(k_{q}\right)}^{k-s-1} J(s)\right\} \\
&<E\left[\mu^{N_{\sigma}\left[k_{0}, k\right)} \lambda_{1}^{T^{1}\left[k_{0}, k\right)} \lambda_{2}^{T^{2}\left[k_{0}, k\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right)\right. \\
&\left.\quad-\sum_{s=k_{0}}^{k-1} \mu^{N_{\sigma}[s, k)} \lambda_{1}^{T_{1}^{1}[s, k-1)} \lambda_{2}^{T^{2}[s, k-1)} J(s)\right] .
\end{aligned}
\end{align*}
$$

Since $V_{\sigma(k)}>0$ and the zero-initial state assumption, it can be seen that

$$
\begin{equation*}
E\left[\sum_{s=k_{0}}^{k-1} \mu^{N_{\sigma}[s, k)} \lambda_{1}^{T^{1}[s, k-1)} \lambda_{2}^{T^{2}[s, k-1)} J(s)\right]<0 \tag{51}
\end{equation*}
$$

From (34), (51) can be written as

$$
\begin{equation*}
E\left[\sum_{s=k_{0}}^{k-1} \mu^{N_{\sigma}[s, k)} \lambda^{k-1-s} J(s)\right]<0 \tag{52}
\end{equation*}
$$

Multiplying both sides of inequality (52) by $-N_{\sigma}[0, k)$, we can obtain

$$
\begin{align*}
& E\left[\mu^{-N_{\sigma}[0, k)} \sum_{s=k_{0}}^{k-1} \mu^{N_{\sigma}[s, k)} \lambda^{(k-1-s)} z^{T}(s) z(s)\right]  \tag{53}\\
& \quad<E\left[\mu^{-N_{\sigma}[0, k)} \sum_{s=k_{0}}^{k-1} \mu^{N_{\sigma}[s, k)} \lambda^{(k-1-s)} \gamma^{2} w^{T}(s) w(s)\right]
\end{align*}
$$

which is equivalent to

$$
\begin{aligned}
& E\left[\sum_{s=k_{0}}^{k-1} \mu^{-N_{\sigma}[0, s)} \lambda^{(k-1-s)} z^{T}(s) z(s)\right] \\
& \quad<E\left[\sum_{s=k_{0}}^{k-1} \mu^{-N_{\sigma}[0, s)} \lambda^{(k-1-s)} \gamma^{2} w^{T}(s) w(s)\right] .
\end{aligned}
$$

Then, from Definition 4 and (24)

$$
\begin{equation*}
N_{\sigma}[0, s) \leq \frac{s}{T_{a}}<s \cdot \max \left\{\pi_{12}, \pi_{21}\right\}<s \cdot\left(-\frac{\ln \lambda}{\ln \mu}\right) \tag{55}
\end{equation*}
$$

we have

$$
\begin{align*}
& E {\left[\sum_{s=k_{0}}^{k-1} \mu^{-N_{\sigma}[0, s)} \lambda^{(k-1-s)} z^{T}(s) z(s)\right] } \\
&>E\left[\sum_{s=k_{0}}^{k-1} \mu^{s \cdot \ln \lambda / \ln \mu} \lambda^{(k-1-s)} z^{T}(s) z(s)\right] \\
&=E\left[\sum_{s=k_{0}}^{k-1} \lambda^{(k-1)} z^{T}(s) z(s)\right]  \tag{56}\\
& E\left[\sum_{s=k_{0}}^{k-1} \mu^{-N_{\sigma}[0, s)} \lambda^{(k-1-s)} \gamma^{2} w^{T}(s) w(s)\right] \\
&<E\left[\sum_{s=k_{0}}^{k-1} \lambda^{(k-1-s)} \gamma^{2} w^{T}(s) w(s)\right]
\end{align*}
$$

Therefore

$$
\begin{align*}
& E\left[\sum_{s=k_{0}}^{k-1} \lambda^{(k-1)} z^{T}(s) z(s)\right] \\
& \quad<E\left[\sum_{s=k_{0}}^{k-1} \lambda^{(k-1-s)} \gamma^{2} w^{T}(s) w(s)\right] \tag{57}
\end{align*}
$$

which implies that

$$
\begin{align*}
E & {\left[\sum_{s=k_{0}}^{\infty} z^{T}(s) z(s) \sum_{k=s+1}^{\infty} \lambda^{(k-1)}\right] } \\
& <E\left[\sum_{s=k_{0}}^{\infty} \gamma^{2} w^{T}(s) w(s) \sum_{k=s+1}^{\infty} \lambda^{(k-1-s)}\right] . \tag{58}
\end{align*}
$$

Then

$$
\begin{equation*}
E\left[\sum_{s=k_{0}}^{\infty} \lambda^{s} z^{T}(s) z(s)\right]<E\left[\sum_{s=k_{0}}^{\infty} \gamma^{2} w^{T}(s) w(s)\right] \tag{59}
\end{equation*}
$$

By Definition 5, system (15) has an exponential $H_{\infty}$ performance $\gamma$. This completes the proof.

Remark 9. It should be pointed out that the conditions proposed in Theorem 8 are not standard LMIs. In this paper, it is suggested to use the cone complementarity linearization (CCL) algorithm to solve this problem [26]; a nonlinear constraint can be converted to a linear optimization problem with a rank constraint.

Remark 10. In this paper, the mean square exponential $H_{\infty}$ performance of the system (15) can be guaranteed, which means the noise attenuation performance is different when the decay degree of the system is different, and the decay degree has a close relation with the elements of the transition probabilities. Note that the scalar $\lambda$ in the sequel symbolizes the decreasing rate of the Lyapunov function to be constructed for each subsystem from Theorem 6. Then, if $\lambda \rightarrow 1$, the evaluated performance index will approach the normal $H_{\infty}$ performance for the whole time domain.

## 4. Numerical Example

In this section, we present an example to illustrate the effectiveness of the proposed approach. Consider the following system:

$$
\begin{align*}
\dot{x}(t)= & {\left[\begin{array}{cc}
-1 & 1 \\
0 & -0.1
\end{array}\right] x(t)+\left[\begin{array}{c}
0 \\
0.1
\end{array}\right] u(t) } \\
& +\left[\begin{array}{l}
0.06 x_{1} \sin x_{1} \\
0.01 x_{2} \cos x_{2}
\end{array}\right]+\left[\begin{array}{l}
0.05 \\
0.01
\end{array}\right] w(t),  \tag{60}\\
& z(t)=\left[\begin{array}{ll}
0.1 & 0.5
\end{array}\right] x(t) .
\end{align*}
$$

Let the sampling period be $T=0.3 \mathrm{~s}$, and $0 \leq \tau_{k} \leq 0.1 \mathrm{~s}$. Assume that the transition probability matrix of stochastic switching signals is given as $P=\left[\begin{array}{ccc}0.8 & 0.2 \\ 0.6 & 0.4\end{array}\right]$; the corresponding matrices are given by

$$
\begin{align*}
& \widetilde{A}_{1}=\left[\begin{array}{ccc}
0.5488 & 0.2219 & 0 \\
0 & 0.9704 & 0 \\
0 & 0 & 0
\end{array}\right], \\
& \widetilde{A}_{2}=\left[\begin{array}{ccc}
0.5488 & 0.2219 & 0.1 \\
0 & 0.9704 & 0 \\
0 & 0 & 0
\end{array}\right] \text {, } \\
& \widetilde{A}_{3}=\left[\begin{array}{ccc}
0.5488 & 0.2219 & -0.0980 \\
0 & 0.9704 & 0.0098 \\
0 & 0 & 0
\end{array}\right] \text {, } \\
& \widetilde{A}_{4}=\left[\begin{array}{ccc}
0.5488 & 0.2219 & 0.0020 \\
0 & 0.9704 & 0.0098 \\
0 & 0 & 0
\end{array}\right] \text {, } \\
& \bar{A}_{2}=\left[\begin{array}{ccc}
0.5488 & 0.2219 & 0.0037 \\
0 & 0.9704 & 0.0296 \\
0 & 0 & 1
\end{array}\right], \quad \widetilde{B}_{1}=\left[\begin{array}{c}
0.0037 \\
0.0296 \\
1
\end{array}\right] \text {, } \\
& \widetilde{B}_{2}=\left[\begin{array}{c}
-0.0963 \\
0.0296 \\
1
\end{array}\right], \quad \widetilde{B}_{3}=\left[\begin{array}{c}
0.1017 \\
0.0198 \\
1
\end{array}\right], \\
& \widetilde{B}_{4}=\left[\begin{array}{c}
0.1017 \\
0.0198 \\
1
\end{array}\right], \quad \bar{H}=\left[\begin{array}{c}
0.0116 \\
0.0030 \\
0
\end{array}\right], \\
& U=\operatorname{diag}\{0.2,0.2,0\}, \quad \bar{C}=\left[\begin{array}{lll}
0.1 & 0.5 & 0
\end{array}\right] . \tag{61}
\end{align*}
$$



Figure 1: State trajectories of the closed-loop system.

For subsystem 2 without state feedback, $\bar{A}_{2}$ is an unstable matrix, and $\lambda_{2}>1$. By Theorem 6 , we can get $0<\lambda_{1}<1$. Take $\lambda_{1}=0.45, \lambda_{2}=2$; then $\lambda=0.6533<1$, which satisfies the condition (16). It is assumed that $\gamma=2$; solving LMIs (41) and (42) in Theorem 8, we get the following solutions:

$$
\begin{align*}
X_{1} & =\left[\begin{array}{ccc}
2.3889 & -0.1809 & 0.9960 \\
-0.1809 & 0.0703 & -0.8737 \\
0.9960 & -0.8737 & 40.8886
\end{array}\right], \\
X_{2} & =\left[\begin{array}{ccc}
6.4614 & -0.1005 & -31.2945 \\
-0.1005 & 0.0663 & -1.2282 \\
-31.2945 & -1.2282 & 275.1873
\end{array}\right], \\
X_{3} & =\left[\begin{array}{ccc}
7.8714 & -0.3703 & 43.1660 \\
-0.3703 & 0.0723 & -1.6709 \\
43.1660 & -1.6709 & 343.6616
\end{array}\right],  \tag{62}\\
X_{4} & =\left[\begin{array}{ccc}
2.4000 & -0.1227 & 4.0524 \\
-0.1227 & 0.2325 & -11.7220 \\
4.0524 & -11.7220 & 145.1491
\end{array}\right], \\
S & =\left[\begin{array}{ccc}
9.7289 & -2.4859 & 9.0787 \\
-2.4859 & 0.7924 & -4.3508 \\
9.0787 & -4.3508 & 85.3273
\end{array}\right],
\end{align*}
$$

Then the controller gain can be obtained:

$$
K=\left[\begin{array}{ll}
-0.8405 & -14.2332 \tag{63}
\end{array}\right] .
$$

The state trajectories of the NCS and the corresponding switching signal are shown in Figures 1 and 2, respectively, where the initial condition $x_{0}=\left[\begin{array}{ll}-1 & 1\end{array}\right]^{T}$ and $w(k)=$ $0.05 \exp (-0.01 k)$.

From simulation results, it can be seen that the NCS is robustly mean square exponentially stable and the $H_{\infty}$ disturbance attenuation level $\gamma=2$.


Figure 2: The stochastic switching signal.

## 5. Conclusions

In this paper, a discrete-time switched system with two subsystems has been presented to model the NCS with time delay and packet dropout. A new approach by using the average dwell time method is proposed to study the robust stabilization and $H_{\infty}$ control of the addressed NCS. Finally, a numerical example has been given to demonstrate the effectiveness of the proposed method.

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## Research Article

# Enhanced Disturbance-Observer-Based Control for a Class of Time-Delay System with Uncertain Sinusoidal Disturbances 

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#### Abstract

This paper is concerned with disturbance-observer-based control (DOBC) for a class of time-delay systems with uncertain sinusoidal disturbances. The disturbances are decomposed as precise and uncertain parts using nonlinear disturbance observer (DO) after appropriate coordinate transformation. And then the two parts can be compensated by corresponding controller, respectively, such that the classic DOBC method is extended to uncertain disturbance rejection. One novel feature of the proposed method is that even if the precise disturbance parameters are inaccessible, the merits of DOBC can be inherited. By integrating the disturbance observers with feedback control laws with time delay, the disturbances can be rejected and the desired dynamic performances can be guaranteed. Finally, simulations for a flight control system are given to demonstrate the effectiveness of the results.


## 1. Introduction

Dynamical systems with time delays [1-3] widely exist in many systems, such as hydraulic processes, chemical systems, and temperature processes. In addition, the presence of exogenous disturbances is inevitable in engineering control systems; a complex system may suffer various of disturbances due to inherent physical property including sensor measurement noise, control error, and structural vibrations. As the phenomenons mentioned above are often primary sources of instability and performance degradation, it is an impendence thing to design control strategy for time-delay systems characteristic with antidisturbance performance. Some researchers [4-6] have contributed on this subject recently, in $[4,5]$ the $H_{\infty}$ control is adopted to attenuate the influences from disturbances to a desired level, for systems with the bounded disturbances. In [6] a reduced-order observer is structured for the estimation of the modeled disturbance; simultaneously, $H_{\infty}$ scheme can attenuate norm bounded signals. Due to the increasing complexity of the controlled plants and environment, it makes the higher demand for system accuracy, reliability, and real-time performance.

Disturbance-observer-based control (DOBC) is a prevalent anti-disturbance control strategy, which has a simple
structure and is easily implemental in engineering (see surveys [7] and references therein). If the priori characteristics of disturbance to be estimated can be obtained, DOBC can be implemented where the disturbance compensation dynamic property within a composite system can be analyzed [811]. Originating from [9], a hierarchical control strategy is established in $[6,10,11]$ aiming at multiple disturbances in multiinput multioutput (MIMO) nonlinear system; the outcome shows that the strategy has high precision together with strong robustness. The literature mentioned above shows that the DOBC is feasible for more complex structure and can avoid heavy computation, such as resolve of partial differential equations (PDEs) compared with output regulation theory. However, the main limitation of the classical DOBC is that the precise characteristic parameters of disturbance must be available. Moreover, failure in modeling for disturbance accurately may lead to severe deterioration of closed loop system performance, even to instability. It has not been reported that DOBC is presented for time-delay systems subject to uncertain disturbances.

In DOBC [9-11, 13], the disturbance is seen as extended state, correspondingly an extended state observer; that is, disturbance observer (DO) can be constructed to estimate the disturbance. Once we have no access of the precise
disturbance model, no effective observer can be constructed directly to estimate the disturbance as the matching condition [12] is not satisfied. It still remains challenging work to extend the DOBC to the general case, in which the disturbance dynamic model has parametric uncertainty The aim of this paper is to provide a novel approach to estimate and reject the uncertain disturbances, such that the merits of DOBC can be inherited. We first construct an auxiliary observer and then decompose the disturbances into a known precise function, an uncertain nonlinear function, and a decaying vector defined by the auxiliary observer. Corresponding disturbance rejection strategy can be implemented to deal with the uncertain disturbance after the sophisticated design with lower conservativeness compared with the literature mentioned above.

The organization of the problem is given below. Section 2 gives the problem formulation. In Section 3, the formulation for the uncertain disturbance estimation with time delay is introduced. In Section 4, by using the auxiliary vector, DOBC combined with adaptive controller is designed to reject the disturbance and globally stabilize the closed-loop systems. In Section 5, the proposed method is applied to an A4D aircraft model; simulations show the effectiveness of the proposed approaches. Section 6 provides conclusions.

## 2. Formulation of the Problem

The following continuous time-delay system with uncertain perturbation is considered:
$\dot{x}(t)=A x(t)+F f(x(t), t)+A_{d} x(t-\tau)+B[u(t)+d(t)]$,
where $x(t) \in R^{n}, u(t) \in R^{m}$ are the state and the control input, respectively. $A \in R^{n \times n}, B \in R^{n \times m}$, and $A_{d} \in R^{n \times n}$ are the coefficient matrices, satisfying $\operatorname{rank}(B)=m . F$ is the corresponding weighting matrix, $f(x(t), t)$ is nonlinear function which is supposed to satisfy bounded conditions described as Assumption 1. $d(t)$ is a vector of sinusoidal disturbance and $\tau$ is the delay time. Such a model can also represent a wider class of time-delay system compared with papers $[6,9,14]$.

Assumption 1. For any $x_{j}(t) \in R^{n}, j=1,2$ nonlinear functions $f(x, t)$ satisfy

$$
\begin{gather*}
f(0, t)=0, \\
\left\|f\left(x_{1}(t), t\right)-f\left(x_{2}(t), t\right)\right\| \leq\left\|U\left(x_{1}(t)-x_{2}(t)\right)\right\|, \tag{2}
\end{gather*}
$$

where $U$ is the given constant weighting matrix.
Similar to the output regulation theory, DOBC strategy [9, 15], each unknown external disturbance $d_{i}(i=$ $1,2, \ldots, m$ ) is supposed to be generated by an exogenous system described by

$$
\begin{align*}
& \dot{w}_{i}=\Gamma_{i} w_{i},  \tag{3}\\
& d_{i}=V_{i} w_{i}
\end{align*}
$$

where $\left(\Gamma_{i}, V_{i}\right)$ is uniformly observable. To show the main ideology of our paper, suppose $w_{i} \in R^{2}$ and the linear uncertain matrix $\Gamma_{i} \in R^{2 \times 2}$. For sake of simplicity, $\left(\Gamma_{i}, V_{i}\right)$ has observable canonical form, which can be expressed as follows:

$$
\Gamma_{i}=\left[\begin{array}{cc}
0 & 1  \tag{4}\\
-W_{i} & 0
\end{array}\right], \quad V_{i}=\left[\begin{array}{cc}
1 & 0
\end{array}\right], \quad i=1, \ldots, m
$$

where $W_{i}$ is parameter characteristics related to disturbance frequency; different from the present work, we consider $W_{i}$ to be uncertain constant values, for the sake of simplicity, denote that

$$
W=\left[\begin{array}{c}
W_{1}  \tag{5}\\
\vdots \\
W_{n}
\end{array}\right]
$$

$$
\Theta=\left[\begin{array}{c}
\Theta_{1} \\
\vdots \\
\Theta_{n}
\end{array}\right]
$$

$$
\Xi=\left[\begin{array}{c}
\Xi_{1} \\
\vdots \\
\Xi_{n}
\end{array}\right]
$$

where $\Theta_{i}$ and $\Xi_{i}$ represent precise and unknown part of $W_{i}$, respectively, that is, $W_{i}=\Theta_{i}+\Xi_{i}$. In application, many kinds of disturbances in engineering can be described by this model, for example, the control of aircraft control [9], magnetic bearing control [16], robotic systems [14], and so forth.

In the conventional DOBC strategy [8, 9, 11, 14, 17], the $\Gamma_{i}$ in disturbance must be known in advance. This condition is so strict for reason that the disturbances acting on a system are difficult to be modeled precisely in general. Up to now, there is no related method discussing the uncertain disturbances estimation problem subject to time delay. This is the major hurdle that mostly impedes the further research and application in DOBC and other disturbance rejection research.

In this paper, we will derive the relation between the uncertain parameters $\Gamma$ and $d(t)$, according to which the exogenous disturbance may be expressed as nonlinear functions including precise part and uncertain part. The control problem considered will be solved by means of DOBC combined with adaptive control (DOBC + adaptive) such that the proposed controller can achieve arbitrary disturbance attenuation.

## 3. Nonlinear Disturbance Observer

The disturbance parameters are inaccessible in this state timedelay system (1), so it is difficult to construct the disturbance observer with traditional ways directly as in [9, 14]. In this section, we first design the auxiliary observer for nonlinear vector $\xi$ with time delay. After an appropriate coordinate transformation, the disturbance $d$ may be formulated as a parametric uncertain function. According to (2), $d$ in (1) can be expressed as follows:

$$
\begin{align*}
& \dot{w}=\Gamma w, \\
& d=V w, \tag{6}
\end{align*}
$$

where $w \in R^{2 m}, \Gamma \in R^{2 m \times 2 m}$, and

$$
\Gamma=\left[\begin{array}{ccc}
\Gamma_{1} & 0 & 0  \tag{7}\\
0 & \ddots & 0 \\
0 & 0 & \Gamma_{n}
\end{array}\right], \quad V=\left[\begin{array}{ccc}
V_{1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & V_{n}
\end{array}\right]
$$

In this section, we suppose that $f(x(t), t)$ is given and Assumption 1 holds. When all states of the system are available, it is unnecessary to estimate the states, then only the estimation of the disturbance need to be concerned. Construct an auxiliary MIMO nonlinear system as follows:

$$
\begin{gather*}
\xi=\nu(t)+\psi, \\
\dot{\nu}=G(\nu(t)+\psi) \\
-L \check{B}\left(A x+F f(x(t), t)+A_{d} x(t-\tau)+B u(t)\right),  \tag{8}\\
\psi=L \check{B} x(t),
\end{gather*}
$$

where

$$
\xi=\left[\begin{array}{c}
\xi_{1}  \tag{9}\\
\xi_{2} \\
\vdots \\
\xi_{n}
\end{array}\right], \quad \xi_{i}=\left[\begin{array}{c}
\xi_{i 1} \\
\xi_{i 2}
\end{array}\right]
$$

$G$ and $L$ are given constant matrices in form of

$$
G=\left[\begin{array}{ccc}
G_{1} & 0 & 0  \tag{10}\\
0 & \ddots & 0 \\
0 & 0 & G_{n}
\end{array}\right], \quad L=\left[\begin{array}{ccc}
L_{1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & L_{n}
\end{array}\right]
$$

where

$$
L_{i}=\left[\begin{array}{l}
0  \tag{11}\\
1
\end{array}\right], \quad G_{i}=\left[\begin{array}{cc}
0 & 1 \\
-g_{i 1} & -g_{i 2}
\end{array}\right]
$$

$G_{i}$ is Hurwitz by selection of $g_{i 1}$ and $g_{i 2}$. Considering $\operatorname{rank}(B)=m$, there exists pseudoinverse $\check{B}$ such that $\check{B} B=I$, so system (8) can be transformed as

$$
\begin{equation*}
\dot{\xi}=G \xi+L d \tag{12}
\end{equation*}
$$

Comparing (6) with (12) yields

$$
\left[\begin{array}{c}
\dot{\xi}  \tag{13}\\
\dot{w}
\end{array}\right]=\left[\begin{array}{cc}
G & L V \\
0 & \Gamma
\end{array}\right]\left[\begin{array}{c}
\xi \\
w
\end{array}\right] .
$$

Lemma 2. For system (6), if $G_{i}$ and $L_{i}$ have form of (11) and guarantee

$$
\begin{equation*}
\left(g_{i 1}-W_{i}\right)^{2}+g_{i 2}^{2} W_{i} \neq 0 \tag{14}
\end{equation*}
$$

in global region of $W_{i}$, then there exists an invertible constant matrix $\Pi_{i}$ such that

$$
\begin{equation*}
\dot{\xi}_{i}+\Pi_{i} \dot{w}_{i}=G_{i}\left(\xi_{i}+\Pi_{i} w_{i}\right) \tag{15}
\end{equation*}
$$

Proof. Considering an invertible matrix

$$
P_{i}^{-1}=\left[\begin{array}{cc}
I^{2 \times 2} & \Pi_{i}  \tag{16}\\
0 & I^{2 \times 2}
\end{array}\right]
$$

where

$$
\Pi_{i}=\frac{\left[\begin{array}{cc}
-\left(g_{i 1}-W_{i}\right) & g_{i 2}  \tag{17}\\
-W_{i} g_{i 2} & -\left(g_{i 1}-W_{i}\right)
\end{array}\right]}{\left(g_{i 1}-W_{i}\right)^{2}+g_{i 2}^{2} W_{i}}
$$

it is obvious that if (14) is satisfied, then $\Pi_{i}$ is invertible in global region of $W_{i}$; furthermore it can be derived that

$$
\Pi_{i}^{-1}=\left[\begin{array}{cc}
-g_{i 1}+W_{i} & -g_{i 2}  \tag{18}\\
W_{i} g_{i 2} & -g_{i 1}+W_{i}
\end{array}\right]
$$

According to (13), notice that

$$
\left[\begin{array}{c}
\dot{\xi}_{i}  \tag{19}\\
\dot{w}_{i}
\end{array}\right]=\left[\begin{array}{cc}
G_{i} & V_{i} L_{i} \\
0 & \Gamma_{i}
\end{array}\right]\left[\begin{array}{c}
\xi_{i} \\
w_{i}
\end{array}\right]
$$

we can define the following coordinate transformation

$$
\left[\frac{\bar{\xi}_{i}}{w_{i}}\right]=P_{i}^{-1}\left[\begin{array}{c}
\xi_{i}  \tag{20}\\
w_{i}
\end{array}\right]
$$

Combining (16) with (20) yields

$$
\left[\begin{array}{c}
\dot{\bar{\xi}}_{i}  \tag{21}\\
\frac{\bar{w}_{i}}{i}
\end{array}\right]=P_{i}^{-1}\left[\begin{array}{cc}
G_{i} & V_{i} L_{i} \\
0 & \Gamma_{i}
\end{array}\right] P_{i}\left[\begin{array}{l}
\bar{\xi}_{i} \\
\bar{w}_{i}
\end{array}\right] .
$$

After calculation, it can be verified that

$$
P_{i}^{-1}\left[\begin{array}{cc}
G_{i} & L_{i} V_{i}  \tag{22}\\
0 & \Gamma_{i}
\end{array}\right] P_{i}=\left[\begin{array}{cc}
G_{i} & 0 \\
0 & \Gamma_{i}
\end{array}\right]
$$

Thus (15) can be got directly following (21) and (22).
Based on Lemma 2, we can give another form of $d_{i}$ as

$$
\begin{equation*}
d_{i}(t)=V_{i} w_{i}(t)=-\bar{V}_{i}(t) \xi_{i}(t)+\bar{V}_{i}(t) \bar{\xi}_{i}(t) \tag{23}
\end{equation*}
$$

where

$$
\begin{gather*}
w_{i}(t)=-\Pi_{i}^{-1} \xi_{i}(t)+\Pi_{i}^{-1} \bar{\xi}_{i}(t), \quad \bar{V}_{i}=V_{i} \Pi_{i}^{-1} \\
\bar{\xi}_{i}=\xi_{i}+\Pi_{i} w_{i} \tag{24}
\end{gather*}
$$

and satisfy

$$
\begin{equation*}
\dot{\bar{\xi}}_{i}=G_{i} \bar{\xi}_{i} \tag{25}
\end{equation*}
$$

Similarly, $d$ can be rewritten as

$$
\begin{equation*}
d=\bar{V} \xi+\bar{V} \bar{\xi} \tag{26}
\end{equation*}
$$

where

$$
\bar{V}=\left[\begin{array}{ccc}
V_{1} \Pi_{1}^{-1} & 0 & 0  \tag{27}\\
0 & \ddots & 0 \\
0 & 0 & V_{m} \Pi_{m}^{-1}
\end{array}\right], \quad \bar{\xi}=\left[\begin{array}{c}
\bar{\xi}_{1} \\
\vdots \\
\bar{\xi}_{m}
\end{array}\right]
$$

according to (10) and (25), $\bar{\xi}$ satisfies

$$
\begin{equation*}
\dot{\bar{\xi}}=G \bar{\xi} \tag{28}
\end{equation*}
$$

From Lemma 2, we have given another form of $d$ through auxiliary observer $\xi$, and can construct observer of $d$ as

$$
\begin{equation*}
\widehat{d}=\bar{V} \xi \tag{29}
\end{equation*}
$$

thus the proposed method will exhibit classic DOBC property.

Notice that $\bar{V}$ in observer $\hat{d}$ cannot be implemented directly as the $W_{i}$ is uncertain. To show it clearly, we divide $\widehat{d}$ into two parts. One part is a precise value that can be predicted and the other is uncertain constant parameter multiplied by a known nonlinear term. For the sake of simplicity, denote

$$
\begin{align*}
\bar{V} \xi= & {\left[\begin{array}{ccc}
V_{1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & V_{n}
\end{array}\right]\left[\begin{array}{c}
-\Pi_{1}^{-1} \xi_{1} \\
-\Pi_{2}^{-1} \xi_{2} \\
\vdots \\
-\Pi_{n}^{-1} \xi_{n}
\end{array}\right] } \\
= & {\left[\begin{array}{c}
\left(-g_{11}+\Theta_{11}\right) \xi_{11}+\left(\Theta_{12}-g_{12}\right) \xi_{12} \\
\vdots \\
\left(-g_{m 1}+\Theta_{m 1}\right) \xi_{m 1}+\left(\Theta_{m 2}-g_{m 2}\right) \xi_{n 2}
\end{array}\right] }  \tag{30}\\
& +\left[\begin{array}{ccc}
\xi_{11} & \ddots & \\
& & \xi_{m 1}
\end{array}\right]\left[\begin{array}{c}
\Xi_{1} \\
\vdots \\
\Xi_{n}
\end{array}\right] .
\end{align*}
$$

A notable property of (29) is that uncertain sinusoidal $d$ can be expressed in form of parametric uncertainty. So, we need not estimate the upper bounds of $d$ as in [18-21].

## 4. DOBC with Stability Analysis

After substituting (26) into system (1), we have

$$
\begin{align*}
\dot{x}(t)= & A x(t)+F f(x(t), t)+A_{d} x(t-\tau) \\
& +B[u(t)+\bar{V} \xi+\bar{V} \bar{\xi}] . \tag{31}
\end{align*}
$$

For the plants with known nonlinearity, the DOBC+adaptive strategy can be designed by using the separation principle as follows:

$$
\begin{equation*}
u=K x+u_{o}+u_{a}, \tag{32}
\end{equation*}
$$

where $K$ is the conventional control gain for stabilization, $u_{o}$ and $u_{a}$ are used to reject and attenuate the disturbances known and uncertain parts, respectively, and according to (30) we select

$$
u_{o}=-\left[\begin{array}{c}
\left(-g_{11}+\Theta_{11}\right) \xi_{11}+\left(\Theta_{12}-g_{12}\right) \xi_{12}  \tag{33}\\
\vdots \\
\left(-g_{m 1}+\Theta_{m 1}\right) \xi_{m 1}+\left(\Theta_{m 2}-g_{m 2}\right) \xi_{m 2}
\end{array}\right]
$$

Similar to [22], adaptive controller $u_{a}$ is used to compensate unknown part of disturbance $d$ which satisfies

$$
\begin{equation*}
u_{a}=-\bar{\Theta} \widehat{\Xi} \tag{34}
\end{equation*}
$$

where

$$
\bar{\Theta}=\left[\begin{array}{lll}
\xi_{11} & &  \tag{35}\\
& \ddots & \\
& & \xi_{n 1}
\end{array}\right]
$$

and $\widehat{\Xi}$ is estimation of $\Xi$. At last the dynamic system (31) may be rewritten as follows:

$$
\begin{align*}
\dot{x}(t)= & A x(t)+F f(x(t), t)+A_{d} x(t-d) \\
& +B[K x+\bar{\Theta}(\Xi-\widehat{\Xi})+\bar{V} \bar{\xi}] . \tag{36}
\end{align*}
$$

At this stage, our objective is to find $K$ such that the closed-loop system (31) with $u=K x$ is asymptotically stable. For the sake of simplifying descriptions, we denote $\operatorname{sym}(M):=M+M^{T}$ and

$$
\begin{equation*}
N_{1}=P_{1}(A+B K)+(A+B K)^{T} P_{1} \tag{37}
\end{equation*}
$$

Theorem 3. For given $\lambda>0$, if (14) can be guaranteed and there exist $Q_{1}>0, P_{1}>0, E=S^{-1}>0$ and $R_{1}$ satisfying

$$
\Omega=\left[\begin{array}{ccccc}
\operatorname{Sym}\left(\mathrm{AQ}_{1}+\mathrm{BR}_{1}\right) & A_{d} E & F_{1} & \mathrm{Q}_{1} & U Q_{1}  \tag{38}\\
* & -E & 0 & 0 & 0 \\
* & * & -\frac{1}{\lambda_{1}^{2}} I & 0 & 0 \\
* & * & * & -E & 0 \\
* & * & * & * & *-\frac{1}{\lambda_{1}^{2}} I
\end{array}\right]<0,
$$

then under DOBC law (32) and adaptive dynamic

$$
\begin{equation*}
\dot{\widehat{\Xi}}=\gamma_{1} \bar{\Theta}^{T} B^{T} P x, \quad\left(\gamma_{1}>0\right) \tag{39}
\end{equation*}
$$

the closed-loop system (36) with gain $K=R_{1} Q_{1}^{-1}$ is asymptotically stable.

Proof. Denote $V(x(t), \bar{\xi}(t), \widetilde{\Xi}(t), t)=V_{1}(x(t), t)+V_{2}(\bar{\xi}(t), t)+$ $V_{3}(\tilde{\Xi}(t), t)$, where

$$
\begin{align*}
& V_{1}(x(t), t)= x(t)^{T} P x(t) \\
&+\frac{1}{\lambda^{2}} \int_{0}^{t}\left(\|U x(\sigma)\|^{2}-\|f(x(\sigma), \sigma)\|^{2}\right) d \sigma \\
&+\int_{t-\tau}^{t} x^{T}(\sigma) S x(\sigma) d \sigma \\
& V_{2}(\bar{\xi}(t), t)=\gamma_{2} \bar{\xi}(t)^{T} P_{2} \bar{\xi}(t), \quad\left(\gamma_{2}>0\right) \\
& V_{3}(\Xi(t), t)=\gamma_{1}^{-1} \widetilde{\Xi}^{T} \widetilde{\Xi} . \tag{40}
\end{align*}
$$

Along with the trajectories of (36) and (39), firstly it can be verified that

$$
\begin{align*}
\dot{V}_{1}(x & (t), t)+\dot{V}_{3}(\widetilde{\Xi}, t) \\
= & \dot{x}(t)^{T} P x(t)+x(t)^{T} P \dot{x}(t) \\
& +\frac{1}{\lambda^{2}}\left(\|U x(t)\|^{2}-\|f(x(t), t)\|^{2}\right) \\
& +x^{T}(t) S x(t)-x^{T}(t-\tau) S x(t-\tau)+2 x^{T}(t) P B \bar{V} \bar{\xi} \\
\leq & \left(A x(t)+F f(x(t), t)+A_{d} x(t-\tau)\right)^{T} P x(t) \\
& +x^{T}(t) P\left(A x(t)+F f(x(t), t)+A_{d} x(t-\tau)\right) \\
& +\frac{1}{\lambda^{2}} x^{T}(t) U^{T} U x(t)-\frac{1}{\lambda^{2}} f^{T}(x(t), t) f(x(t), t) \\
& +x^{T}(t) S x(t)-x^{T}(t-\tau) S x(t-\tau)+2 x^{T} P B \bar{V} \bar{\xi} \\
= & {\left[\begin{array}{c}
x(t) \\
x(t-\tau) \\
f(x(t), t)
\end{array}\right]^{T} \begin{array}{c}
x(t) \\
\Omega_{1}\left[\begin{array}{c}
x(t-\tau) \\
f(x(t), t)
\end{array}\right]+2 x^{T} P B \bar{V} \bar{\xi},
\end{array} } \tag{41}
\end{align*}
$$

where

$$
\Omega_{1}=\left[\begin{array}{ccc}
N_{1}+S+\frac{1}{\lambda_{1}^{2}} U_{1}^{T} U_{1} & P A_{d} & P F_{1}  \tag{42}\\
* & -S & 0 \\
* & * & -\frac{1}{\lambda_{1}^{2}} I
\end{array}\right]
$$

Premultiplied and postmultiplied simultaneously by diag $\{Q, I, I\}, \Omega_{1}<0$ is equivalent to $\Omega_{2}<0$, where

$$
\Omega_{2}=\left[\begin{array}{ccc}
\operatorname{Sym}\left(A Q_{1}+B R_{1}\right)+Q_{1} S Q_{1}+\frac{1}{\lambda_{1}^{2}} Q_{1} U_{1}^{T} U_{1} Q_{1} & A_{d} & F_{1}  \tag{43}\\
* & -S & 0 \\
* & * & -\frac{1}{\lambda_{1}^{2}} I
\end{array}\right]
$$

Based on Schur complement, it can be seen that $\Omega_{2}<0$ is equivalent to $\Omega_{3}<0$ and

$$
\Omega_{3}=\left[\begin{array}{ccccc}
\operatorname{Sym}\left(A Q_{1}+B R_{1}\right) & A_{d} E & F_{1} & Q_{1} & U_{1} Q_{1}  \tag{44}\\
* & -S & 0 & 0 & 0 \\
* & * & -\frac{1}{\lambda_{1}^{2}} I & 0 & 0 \\
* & * & * & -S^{-1} & 0 \\
* & * & * & * & *-\frac{1}{\lambda_{1}^{2}} I
\end{array}\right] .
$$

Premultiplied and postmultiplied simultaneously by diag $\{I, E, I, I, I\}, \Omega_{3}<0$ is equivalent to $\Omega<0$. That is to say
that if (38) upholds, there exists constant $\alpha_{1}>0$ such that

$$
\dot{V}_{1}(x(t), t)+\dot{V}_{3}(\widetilde{\Xi}, t)<-\alpha_{1}\left\|\left[\begin{array}{c}
x(t)  \tag{45}\\
x(t-\tau) \\
f(x(t), t)
\end{array}\right]\right\|^{2}+2 x^{T} P B \bar{V} \bar{\xi}
$$

Together with the definition of $G$ in (10), we can find that $\alpha_{2}>$ 0 satisfies

$$
\begin{equation*}
G^{T} P_{2}+P_{2} G+\alpha_{2}<0 \tag{46}
\end{equation*}
$$

Furthermore, there exists $\gamma_{3}>0$ depending on $P_{1}$ such that for any $x$ and $\bar{\xi}$

$$
\begin{equation*}
2 x^{T} P_{1} B \bar{V} \bar{\xi} \leq \gamma_{3}\|x\|\|\bar{\xi}\| \tag{47}
\end{equation*}
$$

After substituting (45), (46), and (49), derivative along Lyapunov function candidate is given by

$$
\begin{align*}
\dot{V}(x, \bar{\xi}, \widetilde{\Xi}) & \leq-\alpha_{1}\left\|\left[\begin{array}{c}
x(t) \\
x(t-\tau) \\
f(x(t), t)
\end{array}\right]\right\|^{2}+\gamma_{3}\|x\|\|\bar{\xi}\|-\alpha_{2} \gamma_{2}\|\bar{\xi}\|^{2} \\
& \leq-\alpha_{1}\|[x(t)]\|^{2}+\gamma_{3}\|x\|\|\bar{\xi}\|-\alpha_{2} \gamma_{2}\|\bar{\xi}\|^{2} . \tag{48}
\end{align*}
$$

The right part of the above inequality can be regarded as a polynomial with respect to two variables $\|x\|$ and $\|\bar{\xi}\|$. Thus for all $\|x\|$ and $\|\bar{\xi}\|, \dot{V}(x, \bar{\xi}, \widetilde{\Xi}) \leq 0$ holds if there exists a group of parameters $\gamma_{i}(i=2,3)$ satisfying

$$
\begin{equation*}
2 \sqrt{\alpha_{1} \alpha_{2} \gamma_{2}} \geq \gamma_{3} \tag{49}
\end{equation*}
$$

The disturbance-observer-based control design procedure can be summarized as follows.

Step 1. Select weighting matrices $G$ and $L$ with form of (10) and (11), apply $G$ and $L$ into (8) to calculate auxiliary vector.

Step 2. According to (26), give another form of disturbance represented by auxiliary vector.

Step 3. Design time-delay feedback controller $K X$ based on Theorem 3, apply controller into (32), then DOBC + adaptive control can be realized.

## 5. Simulation

To show the efficiency of the proposed scheme, let us consider the continuous time-delay models under the proposed DOBC + adaptive scheme. The longitudinal dynamics of A4D aircraft at a flight condition of 15000 ft altitude and 0.9 Mach can be given by (1). The meaning and significance of the parameters are the same as in [6,9], where $x_{1}(t)$ is the forward velocity $(f t / s), x_{2}(t)$ is the angle of attack $(\mathrm{rad}), x_{3}(t)$ is the pitching velocity $(\mathrm{rad} / \mathrm{s}), x_{4}(t)$ is the pitching angle ( rad ), and
$u(t)$ is elevator deflection (deg) and coefficient matrices:

$$
\begin{gather*}
A=\left[\begin{array}{cccc}
-0.0605 & 32.37 & 0 & 32.2 \\
-0.00014 & -1.475 & 1 & 0 \\
-0.0111 & -34.72 & -2.793 & 0 \\
0 & 0 & 1 & 0
\end{array}\right], \\
B=\left[\begin{array}{c}
0 \\
-1.1064 \\
-33.8 \\
0
\end{array}\right], \\
A_{d}=\left[\begin{array}{ccc}
0.09 & 0.03 & 0.03 \\
0.002 & 0.04 & 0.07 \\
-0.0034 \\
-0.004 & -0.04 & 0.03 \\
-0.001 \\
-0.0004 & -0.014 & 0.008 \\
0.03
\end{array}\right],  \tag{50}\\
F=\left[\begin{array}{c}
0 \\
0 \\
50 \\
0
\end{array}\right] .
\end{gather*}
$$

Similar to [9], it is supposed that nonlinearity and/or uncertainty $f(x(t), t)=\sin (2 \pi 5 t) x_{2}(t)$, state delay time $\tau=2$, and set $U=\operatorname{diag}\{0100\}$ guarantee $\|f(x(t), t)\| \leq\|U f(x(t), t)\|$. Paper $[9,14]$ pointed out that if the frequency is perturbed, the pure DOBC approach will be unavailable because the disturbances cannot be rejected accurately. In order to investigate further, it has been considered that uncertainties exist in such an exogenous model for the disturbance in (3). That is,

$$
\begin{equation*}
d=25[\sin ((1+\Xi) t)], \tag{51}
\end{equation*}
$$

where frequency perturbations $\Xi_{11}=\Xi_{21}=4$. Set

$$
G=\left[\begin{array}{cc}
0 & 1  \tag{52}\\
-4 & -4
\end{array}\right], \quad \gamma_{1}=10000
$$

It is noted that the selection of $\lambda$ is tradeoff and we select $\lambda=$ 0.8 , and the initial value of the disturbance is 0 . When the full states can be measured, applying the approach in Theorem 3, when DOBC law is applied in system (31), the corresponding parameter in (32) can be gotten that

$$
K=\left[\begin{array}{llll}
9.9316 & 530.5033 & 1.7402 & 552.7417 \tag{53}
\end{array}\right] .
$$

Figure 1 plots the estimation error of the uncertain disturbances with traditional DO $[9,11,14]$. The results show that if system suffers uncertain disturbance, it may bring large disturbance estimation error, and the control performance is deteriorated (Figure 2). Figures 3 and 4 demonstrate the system performance using the proposed DOBC + adaptive schemes, obviously the system output converges to zero with sufficiently small steady error. The results show that although there exists uncertainty in the disturbance parameters, the disturbance rejection performance is improved and enhanced system responses can be achieved.

## 6. Conclusion

The DOBC strategy is extended to the uncertain disturbance rejection problem by combining with adaptive control. We


Figure 1: Estimation error using traditional DO.


Figure 2: System output using traditional DOBC.
first construct the auxiliary observer for $d(t)$ with uncertain parameters, and then the exogenous disturbance may be divided into two parts. One part is a precise term which can be compensated by a feed-forward controller, and the other can be expressed as uncertain constant parameter multiplied by a known nonlinear term; an adaptive controller is adopted to compensate the effect of the second part. Simulations on an aircraft model demonstrate the advantages of the proposed scheme. However, if there exist multiple disturbances and unmodeled dynamics in the system as in $[10,11]$, the situation turns to be more complicated, and further research is required in the future.


Figure 3: Estimation error using DO proposed in this paper.


Figure 4: System output using DOBC + adaptive.

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## Research Article

# Sliding Mode Control Based on Observer for a Class of State-Delayed Switched Systems with Uncertain Perturbation 

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#### Abstract

This paper is concerned with a state observer-based sliding mode control design methodology for a class of continuous-time statedelayed switched systems with unmeasurable states and nonlinear uncertainties. The advantages of the proposed scheme mainly lie in which it eliminates the need for state variables to be full accessible and parameter uncertainties to be satisfied with the matching condition. Firstly, a state observer is constructed, and a sliding surface is designed. By matrix transformation techniques, combined with Lyapunov function and sliding surface function, a sufficient condition is given to ensure asymptotic stability of the overall closed-loop systems composed of the observer dynamics and the estimation error dynamics. Then, reachability of sliding surface is investigated. At last, an illustrative numerical example is presented to prove feasibility of the proposed approaches.


## 1. Introduction

A switched system consists of a finite number of subsystems described by a class of differential or difference equations and a logical law that is used to orchestrate switching between these subsystems [1]. As increasing demand, this theory is widely developed, which means that many working systems can be modeled as switched systems, such as automated highway systems [2].

It is well known that different switching strategies produce different systems stability and performance, and the systems states cannot be directly measured. Accordingly, choosing a suitable switching law that stabilizes switched systems and designing an observer become an important problem [3]. Various methods of observer design have been successfully proposed, such as algebraic transfer function and singular-value decomposition. However, designing state observer turns out to be much more difficult when the system is nonlinear [4] and uncertain [5]. In [6], this problem of state observer is considered for discrete time delay switched systems with Lipschitz's nonlinearity and random switching law.

Since 1916, sliding mode control (SMC) has been proven to be an effective robust control strategy for hybrid or uncertain systems [7]. SMC belongs to variable structure
control (VSC), which utilizes a discontinuous control to force the state trajectories of the system to some specific sliding surfaces on which the system acquires desired properties such as decay speed, disturbance rejection capability, and robustness. Developments on SMC involve uncertain systems $[8,9]$, time-delay systems [10, 11], fuzzy systems [12, 13], and Markovian jump systems [14-18].

Full state feedback is not always available and measurable in many practical systems. In Wu's work [19] a new robust stability condition based on observer is proposed for a class of uncertain nonlinear neutral delay systems by using the sliding mode control theory combined with reaching law technique. In He's paper [20], a sliding mode control strategy of uncertain switched linear systems based on observer is presented. The average dwell time is introduced and the matching condition of parameter uncertainties needs not to be satisfied. As the paper $[19,20]$ improved, this paper presents a new observer design for a class of uncertain nonlinear state-delayed switched systems by using the sliding mode control theory.

In the present paper, the motivation of our work is to introduce an SMC scheme based on observer for a class of switched systems that are uncertain nonlinear state-delayed systems with unmeasurable state, unknown nonlinear function, and mismatching parameter uncertainties. This is a new
problem in SMC and switched systems research areas. In this work, the sliding mode observer for each subsystem is designed to estimate full state. By matrix transformation techniques, a sufficient condition is proposed to ensure Lyapunov asymptotic stability of the overall closed-loop switched system. In addition, the derived SMC law is provided to guarantee reachability of the designed sliding surface. Finally, an example is given to prove the feasibility of the proposed approaches.

## 2. Problem Formulation and Preliminaries

2.1. Switched Uncertain Nonlinear State-Delayed Systems. In this work, consider the following uncertain nonlinear statedelayed switched systems:

$$
\begin{align*}
& \dot{x}(t)= {\left[A_{\sigma(t)}+\Delta A_{\sigma(t)}(t)\right] x(t) } \\
&+\left[A_{d \sigma(t)}+\Delta A_{d \sigma(t)}(t)\right] x(t-d) \\
&+B\left[u_{\sigma(t)}(t)+f_{\sigma(t)}(x(t), t)\right],  \tag{1}\\
& y(t)=C x(t), \\
& x(t)=\varphi(t), \quad t \in[-d, 0],
\end{align*}
$$

where $x(t) \in R^{n}$ denotes the vector of continuous-time state variables; $u_{\sigma(t)}(t) \in R^{m}$ denotes the vector of control inputs; $y(t) \in R^{p}$ denotes the system outputs; $f_{\sigma(t)}(x(t), t) \in R^{m}$ denotes an unknown nonlinearity; $A_{\sigma(t)}, A_{d \sigma(t)}, B$, and $C$ are known real constant matrixes of corresponding dimension and the matrix $B$ is of full column rank for arbitrary $i \in \mathbf{I}$; $\Delta A_{\sigma(t)}(t)$ and $\Delta A_{d \sigma(t)}(t)$ are unknown time-varying system parameter uncertainties; $\sigma(t):[0, \infty) \rightarrow \mathbf{I}\{1, \ldots, N\}$ is switching signal which is assumed to be a piecewise constant function of time $t ; k$ denotes the total number of switching modes. In this paper, the notations $t_{i_{r} \text { in }}$ and $t_{i_{r} \text { out }}$ are used to denote the time at which, for the $r$ th, the $i$ th subsystem is switched in and out, respectively, that is, $\sigma\left(t_{i_{r} \text { in }}^{+}\right)=\sigma\left(t_{i_{r} \text { out }}^{-}\right)=$ $i$. With these notations, $i$ is used to replace $\sigma(t)$ for $t_{i_{r} \text { in }} \leq$ $t<t_{i_{r} \text { out }}$. $d$ is a known constant delay; $\varphi(t)$ denotes a differentiable vector-valued initial function on $[-d, 0]$.

The following assumptions are useful for the development of our work.

Assumption 1. Each of the subsystems is completely observable.

Assumption 2. The uncertain parameters are of the form

$$
\left(\Delta A_{i}(t) \Delta A_{d i}(t)\right)=E_{i} F_{i}(t)\left(\begin{array}{ll}
H_{i} & H_{d i} \tag{2}
\end{array}\right)
$$

$\Delta A_{i}(t)$ and $\Delta A_{d i}(t)$ are composed by $E_{i}, H_{i}, H_{d i}$, and $F_{i}(t)$, where $E_{i}, H_{i}$, and $H_{d i}$ are constant matrices and $F_{i}(t)$ is a timevarying matrix function satisfying

$$
\begin{equation*}
F_{i}^{T}(t) F_{i}(t) \leq I \tag{3}
\end{equation*}
$$

Remark 1. Obviously, by comparison with Assumption 2, the matching condition that the following function must be satisfied is a stronger condition:

$$
\begin{equation*}
\left(\Delta A_{i}(t) \quad \Delta A_{d i}(t)\right)=B\left(M_{i}(t) \quad M_{d i}(t)\right) . \tag{4}
\end{equation*}
$$

Assumption 3. There exists a known scalar function $\rho(t, y)$ such that the nonlinearities $f_{i}(x(t), t)$, for arbitrary $i \in \mathbf{I}$, satisfy

$$
\begin{equation*}
\left\|f_{i}(x(t), t)\right\| \leq \rho_{i}(t, y) \tag{5}
\end{equation*}
$$

2.2. Sliding Mode Control. According to the sliding mode control theory, a sliding mode function is chosen as follows:

$$
\begin{equation*}
S(x, t)=B^{T} X x(t) \tag{6}
\end{equation*}
$$

where $X \in R^{n \times n}$ is a positive definite matrix that will be designed.

We know that an ideal sliding mode within the stateestimate space exists if there exists a finite time $t_{s}$, such that

$$
\begin{equation*}
S(t)=0, \quad \dot{S}(t)=0, \quad t \geq t_{s} \quad \text { or } \quad S^{T} \dot{S}(t)<0 \tag{7}
\end{equation*}
$$

The second function of (7) will be used to prove the accessibility condition. And in a conventional SMC design, a class of equivalent control laws and some switching control terms can be chosen to undertake that the accessibility condition $S^{T} \dot{S}<0$ is satisfied.

Then, from

$$
\begin{align*}
& \dot{S}= \frac{\partial S}{\partial x} \frac{\partial x}{\partial t} \\
&=B^{T} X \dot{x}(t) \\
& B^{T} X\left\{\left[A_{i}+\Delta A_{i}(t)\right] x(t)+\left[A_{d i}+\Delta A_{d i}(t)\right] x(t-d)\right.  \tag{8}\\
&\left.+B\left[u_{i}(t)+f_{i}(x(t), t)\right]\right\}
\end{align*}
$$

we get

$$
\begin{align*}
B^{T} X\{ & {\left[A_{i}+\Delta A_{i}(t)\right] x(t)+\left[A_{d i}+\Delta A_{d i}(t)\right] x(t-d) } \\
& \left.+B\left[u_{i}(t)+f_{i}(x(t), t)\right]\right\}=0 \tag{9}
\end{align*}
$$

The solution $u_{i}(t)$ of (7), namely, sliding mode control law, can be broken down into two parts: equivalent control law $u_{\text {eqi }}(t)$ and switching control term $u_{c i}(t)$, where the equivalent control $u_{\text {eqi }}(t)$ is designed for a class of certain systems without nonlinear disturbance and the switching control $u_{c i}(t)$ is robust control of nonlinear uncertain systems.

So, $u_{\text {eqi }}(t)$ and $u_{c i}(t)$ are designed as the following functions (10) and (11), respectively:

$$
\begin{gather*}
u_{\mathrm{eq} i}(t)=-\left(B^{T} X B\right)^{-1} B^{T} X\left(A_{i} x(t)+A_{d i} x(t-d)\right),  \tag{10}\\
u_{c i}(t)=-\left[\left(\rho_{i}(t, y)+\gamma_{1 i}\right)+\gamma_{2 i}\right] \operatorname{sgn}(S(t)), \tag{11}
\end{gather*}
$$

where, $\gamma_{1 i}, \gamma_{2 i}$ are real positive scalars to be specified and function (10) is the solution of the following equation:

$$
\begin{equation*}
B^{T} X\left[A_{i} x(t)+A_{d i} x(t-d)+B u_{\mathrm{eq} i}(t)\right]=0 \tag{12}
\end{equation*}
$$

Remark 2. It should be noticed that the sliding surface function defined in (6) does not switch with switching single (so $B$ not $B_{i}$ and $X$ not $X_{i}$ are designed). It means that there is a unique nonswitched sliding surface in order to avoid repetitive jumps of the state trajectories between sliding surfaces leading to instability and chattering.
2.3. Some Lemmas. For further analysis, some lemmas are given that are useful for stability analysis of the sliding mode dynamics and the development of other theorems.

Lemma 3 (see [21]). Let $E, F$, and $H$ be real matrices of appropriate dimensions, with $F^{T} F \leq I$; then one has that for any scalar $\varepsilon>0$,

$$
\begin{equation*}
E F H+H^{T} F^{T} E^{T} \leq \varepsilon^{-1} E E^{T}+\varepsilon H^{T} H \tag{13}
\end{equation*}
$$

Lemma 4 (Schur complement). Let $S=\left[\begin{array}{ll}S_{11} & S_{12} \\ S_{21} & S_{22}\end{array}\right]$ be symmetrical matrix; then the following three functions are equivalent:

$$
\begin{array}{ll} 
& S<0 \\
S_{11}<0, & S_{22}-S_{12}^{T} S_{11}^{-1} S_{12}<0  \tag{14}\\
S_{22}<0, & S_{11}-S_{12} S_{22}^{-1} S_{12}^{T}<0
\end{array}
$$

Lemma 5 (see [22]). Given matrixes $G \in R^{n \times n}$ and $U \in R^{n \times m}$. Assume that $U$ hasfull rank $m<n$ and $G=G^{T}$. Then, $\delta U U^{T}{ }_{-}$ $G>0$ for some scalar $\delta$ if and only if $\widetilde{U}^{T} G \widetilde{U}<0$ where $\widetilde{U}$ is any matrix whose columns from basis of the null space of $U^{T}$.

When the SMC is designed, we always suppose that all of the system states are available. But, this assumption is hardly satisfied on the practical viewpoint. Hence, in the next chapter, the sliding mode control-based observer will be designed.

## 3. Observer-Based Sliding Mode Control

In this section, a sliding mode observer is designed to provide the estimate of state vector, and then a sliding mode controller is synthesized based on state estimates. Furthermore, by applying the sliding mode control and the multiple Lyapunov function technique, a sufficient condition is given to ensure the asymptotic stability of the overall closed-loop statedelayed system. Finally, we guarantee that the sliding modes within both the state estimate space and state estimation errors pace are attained, respectively.
3.1. Sliding Mode Observer Design. The sliding mode observer to be designed, for state-delayed system (1)-(4), has the form of (6), which can be given by

$$
\begin{align*}
\dot{\hat{x}}(t)= & A_{i} \hat{x}(t)+A_{d i} \widehat{x}(t-d)+B u_{i}(t)  \tag{15}\\
& +B u_{e i}(t)+L_{i}(y(t)-C \hat{x}(t)),
\end{align*}
$$

where $\widehat{x}(t)$ denotes the estimate of system state $x(t)$ and $L_{i} \in$ $R^{n \times p}$ is the observer feedback matrix to be computed later. $u_{e i}$ is a robust control term used to eliminate impact of nonlinear $f_{i}(x(t), t)$, given by

$$
\begin{equation*}
u_{e i}(t)=\left(\rho_{i}(t, y)+\gamma_{1 i}\right) \operatorname{sgn}\left(S_{e}(t)\right) . \tag{16}
\end{equation*}
$$

The corresponding state-estimation error dynamics is given by

$$
\begin{align*}
\dot{e}(t)= & \left(A_{i}-L_{i} C+\Delta A_{i}(t)\right) e(t) \\
& +\left(A_{d i}-\Delta A_{d i}(t)\right) e(t-d)+\Delta A_{i}(t) \hat{x}(t)  \tag{17}\\
& +\Delta A_{d i}(t) \hat{x}(t-d)-B_{i}\left(u_{e i}(t)-f_{i}(x(t), t)\right) .
\end{align*}
$$

According to the sliding mode control theory and (6), $S_{e}(e(t), t)=B^{T} X e(t)$.

In (6), it is assumed that matrix $X$ satisfies

$$
\begin{equation*}
B^{T} X=N C \tag{18}
\end{equation*}
$$

We obtain

$$
\begin{align*}
S_{e}(e(t), t) & =N C e(t)=N C(x(t)-\widehat{x}(t)) \\
& =N(y(t)-C \widehat{x}(t)) . \tag{19}
\end{align*}
$$

This estimation error dynamics corresponds to a nonlinear uncertain state-delayed system, which is dependent on the observer feedback matrix $L_{i}$ and state estimates $\widehat{x}(t), \widehat{x}(t-d)$. This means that stability analysis of the error dynamics (17) is dependent on the observer dynamics. So, when designing the sliding mode observer, the overall closed-loop system composed of (15) and (17) must be considered to guarantee system stability and accessibility of both the sliding surface $S_{e}(e(t), t)=B^{T} X e(t)=N(y(t)-C \hat{x}(t))=0$ in state-estimation error space and the sliding surface $\widehat{S}(\widehat{x}(t), t)=B^{T} X \widehat{x}(t)$ $=0$ in the state estimate space. Namely, as (7), the following function should be satisfied:

$$
\begin{equation*}
\widehat{S}(t)=0, \quad \dot{\hat{S}}(t)=0, \quad t \geq t_{s} . \tag{20}
\end{equation*}
$$

Hence, as functions (10) and (11), equivalent control law and switching control term are designed as follows, respectively:

$$
\begin{gather*}
u_{\mathrm{eq} i}(t)=-\left(B^{T} X B\right)^{-1} B^{T} X\left(A_{i} \widehat{x}(t)+A_{d i} \widehat{x}(t-d)\right), \\
u_{c i}(t)=-\left[\left(\rho_{i}(t, y)+\gamma_{1 i}\right)+\gamma_{2 i}\right] \operatorname{sgn}(\widehat{S}(t)), \tag{21}
\end{gather*}
$$

where $\gamma_{1 i}$ and $\gamma_{2 i}$ are positive design constants, which should be chosen suitably because approaching rate and chattering of sliding mode face can be influenced by it.

Hence, we obtain the following sliding mode controller:

$$
\begin{equation*}
u_{i}(t)=u_{\mathrm{eq} i}(t)+u_{c i}(t) \tag{22}
\end{equation*}
$$

3.2. Stability of Closed-Loop System with Observer. Summing up, considering the design of sliding mode observer and synthesis of sliding mode controller, a sufficient condition for asymptotic stability of the overall closed-loop system can be given as follows.

Theorem 6. If there exist matrices $Y_{i}, N, X>0, Q_{1 i}>0$, and $Q_{2 i}>0$, and scalars $\delta_{1 i}>0$, and $\delta_{2 i}>0$ satisfying

$$
\left(\begin{array}{ccccc}
\Pi_{1 i} & X A_{d i}+\delta_{1 i} H_{i}^{T} H_{d i} & C^{T} Y_{i}^{T}+\delta_{1 i} H_{i}^{T} H_{i} & \delta_{1 i} H_{i}^{T} H_{d i} & X E \\
A_{d i}^{T} X+\delta_{1 i} H_{d i}^{T} H_{i} & \Pi_{2 i} & \delta_{1 i} H_{d i}^{T} H_{i} & \delta_{1 i} H_{d i}^{T} H_{d i} & 0  \tag{24}\\
Y_{i} C+\delta_{1 i} H_{i}^{T} H_{i} & \delta_{1 i} H_{i}^{T} H_{d i} & \Pi_{3 i} & X A_{d i}+\delta_{1 i} H_{i}^{T} H_{d i} & 0 \\
\delta_{1 i} H_{d i}^{T} H_{i} & \delta_{1 i} H_{d i}^{T} H_{d i} & A_{d i}^{T} X+\delta_{1 i} H_{d i}^{T} H_{i} & \Pi_{4 i} & 0 \\
E^{T} X & 0 & 0 & 0 & -\delta_{1 i} I
\end{array}\right) \leq 0,
$$

where
$\Pi_{1 i}=X A_{i}+A_{i}^{T} X-Y_{i} C-C^{T} Y_{i}^{T}+Q_{1 i}+\delta_{1 i} H_{i}^{T} H_{i}-\delta_{2 i} B B^{T}$,
$\Pi_{2 i}=-Q_{1 i}+\delta_{1 i} H_{d i}^{T} H_{d i}-\delta_{2 i} B B^{T}$,
$\Pi_{3 i}=A_{i}^{T} X+X A_{i}+Q_{2 i}+\delta_{1 i} H_{i}^{T} H_{i}-\delta_{2 i} B B^{T}$,
$\Pi_{4 i}=-Q_{2 i}+\delta_{1 i} H_{d i}^{T} H_{d i}-\delta_{2 i} B B^{T}$,
then the sliding mode control law (21)-(22) guarantees that the combined closed-loop switched system is asymptotically stable for the switching signal satisfies

$$
\begin{equation*}
\sigma(t)=i=\arg \left\{\min \left(V_{i}(t)\right)\right\}, \quad i \in \mathbf{I} . \tag{26}
\end{equation*}
$$

Furthermore, an observer feedback matrix is given by $L_{i}=$ $X^{-1} Y_{i}$.

Proof. Firstly, using the method of matrix transformation [23] function (15) is transformed, and the transformation matrix and the associated vector $\widehat{z}(t)$ are defined as

$$
\begin{equation*}
\widehat{z}(t)=\binom{\widehat{z}_{1}(t)}{\widehat{z}_{2}(t)}=W \widehat{x}(t), \tag{27}
\end{equation*}
$$

where $\widehat{z}_{1}(t) \in R^{n-m}, \widehat{z}_{2}(t) \in R^{m}, \widetilde{B}$ is an orthogonal complement of $B$, which means $\widetilde{B}^{T} B=0$. So $B$ is designed to a nonsingular matrix. It is easily known that $W^{-1}=\left(X^{-1} \widetilde{B}\right.$ and $\widehat{z}_{2}(t)=\left(B^{T} X B\right)^{-1} \widehat{\mathcal{s}}(t)$ due to the following function:

$$
\begin{align*}
& \binom{\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T}}{\left(B^{T} X B\right)^{-1} B^{T} X}\left(\begin{array}{ll}
X^{-1} \widetilde{B} & B
\end{array}\right) \\
& =\left(\begin{array}{ll}
\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} X^{-1} \widetilde{B} & \left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} B \\
\left(B^{T} X B\right)^{-1} B^{T} X X^{-1} \widetilde{B} & \left(B^{T} X B\right)^{-1} B^{T} X B
\end{array}\right)  \tag{28}\\
& =\left(\begin{array}{ll}
I & 0 \\
0 & I
\end{array}\right)=I .
\end{align*}
$$

Because of the sliding surface $\widehat{s}(x, t)=0$, we obtain $\widehat{z}_{2}(t)=0$. So $(n-m)$ reduced-order sliding mode dynamic
system as follows:

$$
\begin{align*}
\dot{z}_{1}= & \left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T}\left(A_{i}-L_{i} C\right) X^{-1} \widetilde{B} \widehat{z}_{1}(t) \\
& +\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} A_{d i} X^{-1} \widetilde{B} \widehat{z}_{1}(t-d)  \tag{29}\\
& +\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} L_{i} C X^{-1} \widetilde{B} z_{1}(t)
\end{align*}
$$

Similar to the previous method, function (17) is transformed as

$$
\begin{align*}
\dot{e}_{z 1}(t)= & \left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T}\left(A_{i}-L_{i} C+\Delta A_{i}(t)\right) X^{-1} \widetilde{B} e_{z 1}(t) \\
& +\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} \Delta A_{i}(t) X^{-1} \widetilde{B} \widehat{v}_{1}(t) \\
& +\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T}\left(A_{d i}+\Delta A_{d i}(t)\right) X^{-1} \widetilde{B} e_{z 1}(t-d) \\
& +\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right)^{-1} \widetilde{B}^{T} \Delta A_{d i}(t) X^{-1} \widetilde{B} \widehat{v}_{1}(t-d) . \tag{30}
\end{align*}
$$

Then, the Lyapunov functional is chosen as

$$
\begin{align*}
V_{i}(t)= & e_{z 1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) e_{z 1}(t) \\
& +\int_{t-d}^{t} e_{z 1}^{T}(\tau)\left(\widetilde{B}^{T} \bar{Q}_{1 i} \widetilde{B}\right) e_{z 1}(\tau) d \tau \\
& +\widehat{z}_{1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) \widehat{z}_{1}(t) \\
& +\int_{t-d}^{t} \widehat{z}_{1}^{T}(\tau)\left(\widetilde{B}^{T} \bar{Q}_{2 i} \widetilde{B}\right) \widehat{z}_{1}(\tau) d \tau \\
\dot{V}_{i}(t)= & \dot{e}_{z 1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) e_{z 1}(t)+e_{z 1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) \dot{e}_{z 1}(t) \\
& +\left(e_{z 1}^{T}(t)\left(\widetilde{B}^{T} \bar{Q}_{1 i} \widetilde{B}\right) e_{z 1}(t)\right)_{t-d}^{t} \\
& +\dot{\widehat{z}}_{1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) \widehat{z}_{1}(t)+\widehat{z}_{1}^{T}(t)\left(\widetilde{B}^{T} X^{-1} \widetilde{B}\right) \dot{\widehat{z}}_{1}(t) \\
& +\left(\widehat{z}_{1}^{T}(t)\left(\widetilde{B}^{T} \bar{Q}_{2 i} \widetilde{B}\right) \widehat{z}_{1}(t)\right)_{t-d}^{t} . \tag{31}
\end{align*}
$$

Taking the time derivative along the state trajectories of (15) and (17), and considering (21)-(22), it follows that

$$
\begin{equation*}
\dot{V}_{i}(t)=Z^{T}(t) G_{i}(t) Z(t) \tag{32}
\end{equation*}
$$

where

$$
\begin{align*}
& G_{i}(t)=\left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right)^{T}\left(G_{1 i}+G_{2 i}(t)\right) \\
& \times\left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right), \\
& Z(t)=\left(\begin{array}{c}
e_{z 1}(t) \\
e_{z 1}(t-d) \\
\widehat{z}_{1}(t) \\
\widehat{z}_{1}(t-d)
\end{array}\right), \\
& G_{1 i}=\left(\begin{array}{cccc}
\bar{\Pi}_{1 i} & X A_{d i} & C^{T} L_{i}^{T} X & 0 \\
A_{d i}^{T} X & -Q_{1 i} & 0 & 0 \\
X L_{i} C & 0 & \bar{\Pi}_{2 i} & X A_{d i} \\
0 & 0 & A_{d i}^{T} X & -Q_{2 i}
\end{array}\right), \\
& G_{2 i}(t)=\left(\begin{array}{c}
X E \\
0 \\
0 \\
0
\end{array}\right) F_{i}(t)\left(\begin{array}{llll}
H_{i} & H_{d i} & H_{i} & H_{d i}
\end{array}\right) \\
& +\left(\left(\begin{array}{c}
X E \\
0 \\
0 \\
0
\end{array}\right) F_{i}(t)\left(\begin{array}{llll}
H_{i} & H_{d i} & H_{i} & H_{d i}
\end{array}\right)\right)^{T}, \\
& \bar{\Pi}_{1 i}=\left(A_{i}-L_{i} C\right)^{T} X+X\left(A_{i}-L_{i} C\right)+Q_{1 i}, \\
& \bar{\Pi}_{2 i}=A_{i}^{T} X+X A_{i}+Q_{2 i}, \\
& Q_{1 i}=X \bar{Q}_{1 i} X, \quad Q_{2 i}=X \bar{Q}_{2 i} X . \tag{33}
\end{align*}
$$

From (32), it is easily seen that $\dot{V}_{i}(t)<0$ if $G_{i}(t)<0$ for $Z \neq 0$. So, we just prove the inequality $G_{i}(t)<0$ is satisfied.

By Lemma 3, the previous matrix inequality holds for $F_{i}(t)$ satisfying $F_{i}^{T}(t) F_{i}(t) \leq I$ if there exists a constant $\delta_{1 i}>0$ such that

$$
\begin{align*}
G_{2 i}(t) \leq \bar{G}_{2 i}= & \delta^{-1}\left(\begin{array}{c}
X E \\
0 \\
0 \\
0
\end{array}\right)\left(\begin{array}{llll}
E^{T} X & 0 & 0 & 0
\end{array}\right) \\
& +\delta\left(\begin{array}{c}
H_{i}^{T} \\
H_{d i}^{T} \\
H_{i}^{T} \\
H_{d i}^{T}
\end{array}\right)\left(\begin{array}{llll}
H_{i} & H_{d i} & H_{i} & H_{d i}
\end{array}\right) . \tag{34}
\end{align*}
$$

So, $G_{i}<0$ only if

$$
\begin{align*}
& \left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right)^{T}\left(G_{1 i}+\bar{G}_{2 i}\right)  \tag{35}\\
& \quad \times\left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right)<0
\end{align*}
$$

By Lemma 4, inequality (35) is equivalent to the following inequality (36):

$$
\begin{align*}
& \left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right)^{T} \\
& \times G_{3 i}\left(\begin{array}{cccc}
X^{-1} \widetilde{B} & 0 & 0 & 0 \\
0 & X^{-1} \widetilde{B} & 0 & 0 \\
0 & 0 & X^{-1} \widetilde{B} & 0 \\
0 & 0 & 0 & X^{-1} \widetilde{B}
\end{array}\right)<0 \tag{36}
\end{align*}
$$

where

$$
G_{3 i}=\left(\begin{array}{ccccc}
\Pi_{1 i} & X A_{d i}+\delta_{1 i} H_{i}^{T} H_{d i} & C^{T} Y_{i}^{T}+\delta_{1 i} H_{i}^{T} H_{i} & \delta_{1 i} H_{i}^{T} H_{d i} & X E  \tag{37}\\
A_{d i}^{T} X+\delta_{1 i} H_{d i}^{T} H_{i} & \Pi_{2 i} & \delta_{1 i} H_{d i}^{T} H_{i} & \delta_{1 i} H_{d i}^{T} H_{d i} & 0 \\
Y_{i} C+\delta_{1 i} H_{i}^{T} H_{i} & \delta_{1 i} H_{i}^{T} H_{d i} & \Pi_{3 i} & X A_{d i}+\delta_{1 i} H_{i}^{T} H_{d i} & 0 \\
\delta_{1 i} H_{d i}^{T} H_{i} & \delta_{1 i} H_{d i}^{T} H_{d i} & A_{d i}^{T} X+\delta_{1 i} H_{d i}^{T} H_{i} & \Pi_{4 i} & 0 \\
E^{T} X & 0 & 0 & 0 & -\delta_{1 i} I
\end{array}\right) .
$$

By Lemma 5 and given $L_{i}=X^{-1} Y_{i}$, LMI (23) is obtained. That means $\dot{V}_{i}(t)<0$ only if there exist matrices $Y_{i}, N, X>0$, $Q_{1 i}>0$, and $Q_{2 i}>0$ and scalars $\delta_{1 i}>0$, and $\delta_{2 i}>0$ satisfying LMI (23).

Remark 7. As a result of the given Theorem 6, the observerbased SMC stable problem becomes a linear matrix inequality feasibility problem. Consider the linear equality condition
$B^{T} X=N C$, where $N, X>0$ satisfies LMI (23), which can be equivalently converted to

$$
\begin{equation*}
\operatorname{tr}\left(\left(B^{T} X-N C\right)^{T}\left(B^{T} X-N C\right)\right)=0 \tag{38}
\end{equation*}
$$

Introduce the condition

$$
\begin{equation*}
\left(B^{T} X-N C\right)^{T}\left(B^{T} X-N C\right) \leq \beta I \tag{39}
\end{equation*}
$$

and Lemma 4 gives

$$
\left(\begin{array}{cc}
-\sqrt{\beta} I & X B-C^{T} N^{T}  \tag{40}\\
B^{T} X-N C & -\sqrt{\beta} I
\end{array}\right) \leq 0 .
$$

Hence, it is now changed to a problem in which a global solution of the following minimization is found:

$$
\begin{equation*}
\min \beta \text { subject to }(23) \text { and }(40) . \tag{41}
\end{equation*}
$$

So, it is a minimization problem relating linear objective and LMI constraints; it has an infimum and when infimum equals zero, the observed-based SMC problem is solvable.
3.3. Accessibility Condition. Finally, we prove the accessibility of sliding surfaces $\widehat{S}(\widehat{x}(t), t)=0$ in the state-estimate space and $S_{e}(e(t), t)=0$ in estimation error space. So, Theorem 8 is given.

Theorem 8. If there exist matrices $Y_{i}, N, X>0, Q_{1 i}>0, Q_{2 i}>$ 0 and scalars $\delta_{1 i}>0$, and $\delta_{2 i}>0$ satisfying (23) with (24), and observer feedback matrix $L_{i}=X^{-1} Y_{i}$, then the sliding mode control law (26) with (23) and (24) guarantees that the sliding motion is attained on the sliding surfaces $\widehat{S}(\widehat{x}(t), t)=0$ and $S_{e}(e(t), t)=0$, respectively.

Proof. As (7), $\widehat{S}(\widehat{x}(t), t)=0$ and $S_{e}(e(t), t)=0$ can be replaced by $\widehat{S}^{T} \dot{\hat{S}}(t)<0$ and $S_{e}^{T} \dot{S}_{e}(t)<0$.

We know $\hat{S}^{T} \dot{\hat{S}}(t)<0$ is the same as $\widehat{S}^{T}\left(B^{T} X B\right)^{-1} \dot{\hat{S}}(t)<$ 0 , so $\widehat{S}^{T} \dot{\hat{S}}(t)<0$ and $S_{e}^{T} \dot{S}_{e}(t)<0$ can be replaced by $\hat{S}^{T}\left(B^{T} X B\right)^{-1} \dot{\hat{S}}(t)^{-1}<0$ and $S_{e}^{T}\left(B^{T} X B\right)^{-1} \dot{S}_{e}(t)<0$ to make proving simpler:

$$
\begin{aligned}
& \widehat{S}^{T}(t)\left(B^{T} X B\right)^{-1} \dot{\hat{S}}(t) \\
& =\widehat{S}^{T}(t)\left(B^{T} X B\right)^{-1} B^{T} X\left[A_{i} \hat{x}(t)+A_{d i} \widehat{x}(t-d)+B u_{i}(t)\right. \\
& \left.+B u_{e i}(t)+L_{i}(y(t)-C \hat{x}(t))\right] \\
& =\widehat{S}^{T}(t)\left(B^{T} X B\right)^{-1} B^{T} X \\
& \times\left(A_{i} \hat{x}(t)+A_{d i} \widehat{x}(t-d)+L_{i} C e(t)+B u_{\mathrm{eq} i}\right) \\
& -\widehat{S}^{T}(t)\left(u_{c i}+u_{e i}\right) \\
& \leq\|\widehat{S}(t)\|\left[\left\|\left(B^{T} X B\right)^{-1}-I\right\|\right. \\
& \times\left(\left\|B^{T} X A_{i}\right\|\|\widehat{x}(t)\|+\left\|B^{T} X A_{d i}\right\|\|\widehat{x}(t-d)\|\right) \\
& \left.+\left\|\left(B^{T} X B\right)^{-1}\right\|\left\|B^{T} X L_{i} C\right\|\|e(t)\|\right] \\
& -\|\widehat{S}(t)\|\left[\left(\rho_{i}(t, y)+\gamma_{1 i}\right)+\gamma_{2 i}\right] \\
& +\|\widehat{S}(t)\|\left(\rho_{i}(t, y)+\gamma_{1 i}\right) \\
& \leq\|\widehat{S}(t)\|\left[\left(-\gamma_{2 i}+\alpha_{1 i} \alpha_{2 i}\right)\|\widehat{x}(t)\|\right. \\
& \left.+\alpha_{1 i} \alpha_{3 i}\|\widehat{x}(t-d)\|+\alpha_{4 i} \alpha_{5 i}\|e(t)\|\right],
\end{aligned}
$$

$$
\begin{align*}
& S_{e}^{T}(t)\left(B^{T} X B\right)^{-1} \dot{S}_{e}(t) \\
& =S_{e}^{T}(t)\left(B^{T} X B\right)^{-1} B^{T} X\left[\left(A_{i}-L_{i} C+\Delta A_{i}(t)\right) e(t)\right. \\
& +\left(A_{d i}+\Delta A_{d i}(t)\right) e(t-d) \\
& -B\left(u_{e i}(t)-f_{i}(x(t), t)\right) \\
& \left.+\Delta A_{i} \widehat{x}(t)+\Delta A_{d i} \widehat{x}(t-d)\right] \\
& =-S_{e}^{T}(t)\left(u_{e i}(t)-f_{i}(x(t), t)\right) \\
& +S_{e}^{T}(t)\left(B^{T} X B\right)^{-1} B^{T} X\left[\left(A_{i}-L_{i} C+\Delta A_{i}(t)\right) e(t)\right. \\
& +\left(A_{d i}+\Delta A_{d i}(t)\right) e(t-d) \\
& \left.+\Delta A_{i} \hat{x}(t)+\Delta A_{d i} \hat{x}(t-d)\right] \\
& \leq\left\|S_{e}(t)\right\| \rho_{i}(t, y)+\left\|S_{e}(t)\right\|\left\|\left(B^{T} X B\right)^{-1}\right\| \\
& \times\left[\left\|B^{T} X\left(A_{i}-L_{i} C\right)\right\|\|e(t)\|+\left\|B^{T} X E\right\|\left\|H_{i}\right\|\|e(t)\|\right. \\
& +\left\|B^{T} X A_{d i}\right\|\|e(t-d)\|+\left\|B^{T} X E\right\|\left\|H_{d i}\right\|\|e(t-d)\| \\
& \left.+\left\|B^{T} X E\right\|\left\|H_{i}\right\|\|\widehat{x}(t)\|+\left\|B^{T} X E\right\|\left\|H_{d i}\right\|\|\hat{x}(t-d)\|\right] \\
& \leq\left\|S_{e}(t)\right\|\left[-\gamma_{1 i}+\alpha_{4 i}\left(\beta_{1 i}-\beta_{2 i}\right)\|e(t)\|\right. \\
& +\left(\alpha_{3 i}+\beta_{3 i}\right)\|e(t-d)\|+\beta_{2 i}\|\widehat{x}(t)\| \\
& \left.+\beta_{3 i}\|\widehat{x}(t-d)\|\right] \tag{42}
\end{align*}
$$

with $\alpha_{1 i}=\left\|\left(B^{T} X B\right)^{-1}-I\right\|, \alpha_{2 i}=\left\|B^{T} X A_{i}\right\|, \alpha_{3 i}=\left\|B^{T} X A_{d i}\right\|$, $\alpha_{4 i}=\left\|\left(B^{T} X B\right)^{-1}\right\|, \alpha_{5 i}=\left\|B^{T} X L_{i} C\right\|, \beta_{1 i}=\| B^{T} X\left(A_{i}-\right.$ $\left.L_{i} C\right)\left\|, \beta_{2 i}=\right\| B^{T} X E\| \|\left\|H_{i}\right\|, \beta_{3 i}=\left\|B^{T} X E\right\|\left\|\mid H_{d i}\right\|$.

In the state space composed of state estimate vector and estimation error vector, we define the following domain:

$$
\Omega_{\delta i}=\left\{\begin{array}{c}
Z \in R^{4 n}: \alpha_{1 i} \alpha_{2 i}\|\widehat{x}(t)\|+\alpha_{1 i} \alpha_{3 i}\|\widehat{x}(t-d)\|  \tag{43}\\
+\alpha_{4 i} \alpha_{5 i}\|e(t)\|<\gamma_{2 i}-\delta, \\
\alpha_{4 i}\left(\beta_{1 i}-\beta_{2 i}\right)\|e(t)\|+\beta_{2 i}\|\widehat{x}(t)\| \\
+\left(\alpha_{3 i}+\beta_{3 i}\right)\|e(t-d)\| \\
+\beta_{3 i}\|\widehat{x}(t-d)\|<\gamma_{1 i}-\delta
\end{array}\right\}
$$

with constant $\delta$ satisfying $0<\delta<\min \left\{\gamma_{1 i}, \gamma_{2 i}\right\}$, so from (42), in the domain $\Omega_{\delta i}$, we have $\widehat{S}^{T} \dot{\hat{S}}(t)<0$ and $S_{e}^{T} \dot{S}_{e}(t)<0$.

So, the sliding surfaces $\widehat{S}(\widehat{x}(t), t)=0$ in the state-estimate space and $S_{e}(e(t), t)=0$ in estimation error space can be accessed.

Remark 9. Only when the state trajectories enter the $\Omega_{\delta i}$, could $\widehat{S}(t)=0$ and $S_{e}(t)=0$ happen. The region in which sliding motion takes place is usually referred to as sliding patch [24]. In the present scheme, it can be seen that size of the sliding patch depends on design constants $\gamma_{1 i}, \gamma_{2 i}$.

## 4. Simulation

In this section, the following switch system will be used to illustrate the proposed sliding mode control scheme-based observer is feasible:

$$
\begin{align*}
& \dot{x}(t)= {\left[A_{\sigma(t)}+\Delta A_{\sigma(t)}(t)\right] x(t) } \\
&+\left[A_{d \sigma(t)}+\Delta A_{d \sigma(t)}(t)\right] x(t-d) \\
&+B\left[u_{\sigma(t)}(t)+f_{\sigma(t)}(x(t), t)\right]  \tag{44}\\
& y(t)=C x(t)
\end{align*}
$$

where

$$
\begin{align*}
& A_{1}=\left(\begin{array}{ccc}
-2 & 4 & 0 \\
-7 & -1.5 & 0 \\
0 & 1 & -1
\end{array}\right), \quad A_{2}=\left(\begin{array}{ccc}
-1.2 & 0.9 & 0 \\
2 & -2 & 0 \\
0 & 1.4 & -1.3
\end{array}\right), \\
& A_{d 1}=\left(\begin{array}{ccc}
-0.3 & 0.1 & 0.1 \\
0.1 & 0.1 & 0 \\
0 & 0.1 & 0.2
\end{array}\right), \quad A_{d 2}=\left(\begin{array}{ccc}
0.1 & 0.2 & 0.3 \\
0.12 & 0.09 & 0 \\
0 & 0.1 & 0.2
\end{array}\right) \text {, } \\
& B=\left(\begin{array}{ll}
2 & 1 \\
1 & 2 \\
1 & 1
\end{array}\right), \quad C=\left(\begin{array}{lll}
1 & 3 & 2 \\
2 & 2 & 0
\end{array}\right), \\
& E_{1}=E_{2}=\left(\begin{array}{ccc}
0.1 & 0.2 & 0.2 \\
0 & 0.2 & 0.1 \\
0 & 0 & 0.3
\end{array}\right) \text {, } \\
& H_{1}=H_{2}=\left(\begin{array}{ccc}
0.1 & 0.2 & 0.1 \\
0 & 0.2 & 0.3 \\
0.2 & 0.2 & 0.1
\end{array}\right), \\
& H_{d 1}=H_{d 2}=\left(\begin{array}{ccc}
0.2 & 0 & 0.1 \\
0 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.2
\end{array}\right), \\
& f_{1}(x(t), t)=\binom{0.4 \sin \left(x_{1}(t)\right)}{0.2 \cos \left(x_{2}(t)\right)}, \\
& f_{2}(x(t), t)=\binom{0.3 \sin \left(x_{2}(t)\right)}{0.3 \cos \left(x_{3}(t)\right)}, \\
& F_{1}(t)=\left(\begin{array}{ccc}
\sin (t) & 0 & 0 \\
0 & \sin (t) & 0 \\
0 & 0 & \cos (t)
\end{array}\right), \\
& F_{2}(t)=\left(\begin{array}{ccc}
\cos (t) & 0 & 0 \\
0 & \sin (t) & 0 \\
0 & 0 & 1
\end{array}\right), \\
& d(t)=2 \sin (t)+1, \quad d=3, \\
& x(t)=\left(\begin{array}{lll}
1 & -1 & 0
\end{array}\right)^{T}, \quad t \in[-2,0] \text {, } \\
& x(0)=\left(\begin{array}{lll}
1 & -1 & 0.5
\end{array}\right)^{T} . \tag{45}
\end{align*}
$$

Then, the linear inequalities (23) and (24) with (41) are solved, and we obtain:

$$
\begin{align*}
& X=\left(\begin{array}{ccc}
2.8965 & -0.8585 & -5.1634 \\
-0.8585 & 1.7945 & 1.2445 \\
-5.1634 & 1.2445 & 10.6332
\end{array}\right), \\
& N=\left(\begin{array}{ll}
0.7754 & -0.5022 \\
3.9794 & -3.9817
\end{array}\right), \\
& Y_{1}=\left(\begin{array}{cc}
-4.3062 & 0.7478 \\
-3.3264 & 0.1157 \\
-1.9617 & -1.3973
\end{array}\right) \text {, } \\
& Y_{2}=\left(\begin{array}{cc}
-0.6094 & 0.4116 \\
0.5758 & -0.6553 \\
-0.0320 & -0.1995
\end{array}\right) \text {, } \\
& L_{1}=\left(\begin{array}{cc}
-16.3887 & 0.3210 \\
-4.4045 & 0.2188 \\
-7.6272 & -0.0012
\end{array}\right) \text {, } \\
& L_{2}=\left(\begin{array}{cc}
-1.5303 & 0.6209 \\
0.1156 & -0.2876 \\
-0.7596 & 0.3164
\end{array}\right) \text {, } \\
& \widehat{S}(t)=B^{T} X \widehat{x}(t)=\left(\begin{array}{ccc}
0.0091 & 0.0038 & 0.0017 \\
-0.0090 & 0.0060 & 0.0145
\end{array}\right) \widehat{x}(t), \\
& u_{i}(t)=u_{\mathrm{eqi}}(t)+u_{c i}(t), \\
& u_{e i}(t)=\left(\rho_{i}(t, y)+\gamma_{1 i}\right) \operatorname{sgn}\left(S_{e}(t)\right), \\
& u_{c i}(t)=-\left[\left(\rho_{i}(t, y)+\gamma_{1 i}\right)+\gamma_{2 i}\right] \operatorname{sgn}(\widehat{S}(t)), \\
& u_{\mathrm{eqi}}(t)=-\left(B^{T} X B\right)^{-1} B^{T} X\left(A_{i} \widehat{x}(t)+A_{d i} \widehat{x}(t-d)\right), \\
& u_{\text {eq } 1}(t)=\left(\begin{array}{ccc}
2.000 & -3.0000 & -1.0000 \\
1.0000 & 2.1667 & 1.0000
\end{array}\right) \widehat{x}(t) \\
& +\left(\begin{array}{ccc}
-0.3000 & 0.0000 & 0.1000 \\
0.1667 & -0.0667 & 0.1333
\end{array}\right) \widehat{x}(t-d), \\
& u_{\mathrm{eq} 2}(t)=\left(\begin{array}{ccc}
1.2000 & 0.5000 & -1.3000 \\
-1.4667 & -0.1333 & 1.3000
\end{array}\right) \widehat{x}(t) \\
& +\left(\begin{array}{ccc}
-0.1000 & -0.1000 & -0.1000 \\
0.0267 & 0.0033 & -0.0000
\end{array}\right) \hat{x}(t-d) . \tag{46}
\end{align*}
$$

Let $\gamma_{11}=\gamma_{12}=60, \gamma_{21}=\gamma_{22}=40, \widehat{x}(t)=\left(\begin{array}{lll}0 & 0 & 0\end{array}\right)^{T}, t \in$ $[-2,0], \widehat{x}(0)=\left(\begin{array}{lll}0 & 0 & 0\end{array}\right)^{T}$. To restrain control signals from quiver, $\operatorname{sgn}(\widehat{S}(t))$ is replaced by $\widehat{S}(t) /(\|\widehat{S}(t)\|+0.01)$, and $S_{e}(t)$ is replaced by $S_{e}(t) /\left(\left\|S_{e}(t)\right\|+0.01\right) . \rho_{i}(t, y)$ is very tiny, compared with $\gamma_{1 i}$, so it is ignored.

Figures 1-4 testify the simulation results. Figures 1 and 2 show the time evolution of states and confirm asymptotic stability of the respective switched system. According to Figure 1, we know that the opened-loop switched system is instability and oscillation. Nevertheless, Figure 2 shows that the closed-loop switched system is stability. So it can be


Figure 1: State response of the opened-loop switched system.


Figure 2: State response of the closed-loop switched system.


Figure 3: State response of the observer system.


Figure 4: Sliding mode variables.
seen that state response of the closed-loop switched system is clearly superior to the opened-loop switched system.

And we check that, from Figure 3, the present observer based on sliding mode control scheme effectively eliminated effects of parameter uncertainties and nonlinearities and guaranteed asymptotic stability of the closed-loop system. Figure 4 shows that the sliding surface is accessibility but continue chattering. The chattering of the sliding mode variable $\widehat{S}(t)$ is treated as reduced-order compensation of dynamic system.

## 5. Conclusion

In this paper, an observer-based sliding mode control strategy is presented for a class of uncertain nonlinear state-delayed switched systems. Through this work, state-delayed switched system with immeasurable states and nonlinear uncertainties get better performance. Not only are the parameter uncertainties not needed to satisfy matching condition, but also the availabilities of all system states are no longer required. Meanwhile, a unique nonswitched sliding surface is designed in order to avoid repetitive jumps of the state trajectories between sliding surfaces leading to instability and chattering.

The weakness is that only the switching signal satisfies $\sigma(t)=i=\arg \left\{\min \left(V_{i}(t)\right)\right\}$, can the switched system be asymptotically stable. It is expected that the results developed in the paper can be extended to common cases that the underlying systems are involved in Markovian switching signal [25-27] or arbitrary switching signal.

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## Research Article

# $H_{\infty}$ Control for Flexible Spacecraft with Time-Varying Input Delay 

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#### Abstract

This paper is concerned with $H_{\infty}$ control problem for flexible spacecraft with disturbance and time-varying control input delay. By constructing an augmented Lyapunov functional with slack variables, a new delay-dependent state feedback controller is obtained in terms of linear inequality matrix. These slack variables can make the design more flexible, and the resultant design also can guarantee the asymptotic stability and $H_{\infty}$ attenuation level of closed-loop system. The effectiveness of the proposed design method is illustrated via a numerical example.


## 1. Introduction

High-precision attitude control for flexible spacecrafts is a difficult problem in communication, navigation and remote sensing, and so forth. It is because modern spacecrafts often employs large, complex, and lightweight structures such as solar arrays in order to achieve the increased functionality at a reduced launch cost and also provide sustainable energy during space flight [1-4]. Consequently, the complex space structure may lead to the decreased rigidity and low frequency elastic modes. The dynamic model of a flexible spacecraft usually includes the interaction between the rigid and elastic modes $[5,6]$. During the control of the rigid body attitude, the unwanted excitation of the flexible modes, together with other external disturbances, measurement, and actuator error, may degrade the performance of attitude control systems (ACSs). Meanwhile, the spacecraft commonly operates in the presence of various disturbances, including gravitational torque, aerodynamic torque, radiation torque, and other environmental and nonenvironmental torques. The problem of disturbance rejection is particularly pronounced in the case of low-earth-orbiting satellites that operate in altitude ranges where their dynamics are substantially affected by most of the disturbances mentioned above [ 7,8 ]. In the face of disturbance and uncertainty, $H_{\infty}$ methods are ideally
suited for yielding a good performance of flexible spacecrafts. $H_{\infty}$ control has been used in attitude control systems design in $[9,10]$ where external disturbance and model uncertainty are considered. An $H_{\infty}$ multiobjective controller based on the linear matrix inequality (LMI) framework is designed for flexible spacecraft in [11].

On the other hand, in recent years, several studies related to control of flexible spacecraft attitude system with input saturation have been done in $[12,13]$. However, the input delay often exists in flexible spacecraft due to the physical structure and energy consumption of the actuators. Although it is not the most important factor to affect the attitude control, it still leads to substantial performance deterioration and even to instability of the system [14, 15]. Hence, stabilization algorithms for such systems that explicitly take input time delay into account are practical interest. Up to now, the issue of control problems for flexible spacecraft subject to both disturbance and input time delay has not been fully investigated and remains to be open and challenging.

In control system design, it is usually desirable to design the control systems which not only is robustly stable but also guarantees an adequate level of performance. One approach to this problem is the so-called guaranteed cost control approach.

Motivated by the preceding discussion, in this paper, we consider $H_{\infty}$ control problem for flexible spacecraft subject to both disturbance and input time-varying delay. By constructing an augmented Lyapunov functional with slack variables, a new delay-dependent state feedback controller is obtained in terms of linear inequality matrix. These slack variables can make the controller design more flexible and be extended to the systems without time delay. The resultant design also can guarantee the asymptotic stability and $H_{\infty}$ performances of closed-loop system. Finally, a numerical example is shown to demonstrate the good performance of our method.

Notation. Throughout this paper, $R^{n}$ denotes the $n$-dimensional Euclidean space; the space of square-integrable vector functions over $[0, \infty)$ is denoted by $l_{2}[0, \infty)$; the superscripts " $T$ " and " -1 " stand for matrix transposition and matrix inverse, respectively; $P>(\geq 0)$ means that $P$ is a real symmetric and a positive definite (semidefinite). In symmetric block matrices or complex matrix expressions, $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix, and $*$ represents a term that is induced by symmetry. For a vector $v(t)$, its norm is given by $\|v(t)\|_{2}^{2}=\int_{0}^{\infty} v^{\top}(t) v(t) d t$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for related algebraic operations.

## 2. Problem Formulation and Preliminaries

To simplify the problem, only single-axis rotation is considered. We can obtain the single-axis model derived from the nonlinear attitude dynamics of the flexible spacecraft (see also $[1,7])$. It is assumed that this model includes one rigid body and one flexible appendage, and the relative elastic spacecraft model is described as follows:

$$
\begin{gather*}
J \ddot{\theta}+F \ddot{\eta}=u(t-\tau(t)),  \tag{1}\\
\ddot{\eta}+2 \xi \omega \dot{\eta}+\omega^{2} \eta+F^{\top} \ddot{\theta}=0,
\end{gather*}
$$

where $\theta$ is the attitude angle, $J$ is the moment of inertia of the spacecraft, $F$ is the rigid-elastic coupling matrix, $u(t-\tau(t))$ is the control torque generated by the reaction wheels that are installed in the flexible spacecraft, where $\tau(t)$ satisfies $0 \leq$ $\tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \mu \leq 1, \eta$ is the flexible modal coordinate, $\xi$ is the damping ratio, and $\omega$ is the modal frequency. Since the vibration energy is concentrated in low frequency modes in a flexible structure, its reduced order model can be obtained by modal truncation. In this paper, only the first two bending modes are taken into account. Then, we can get

$$
\begin{equation*}
\left(J-F F^{\top}\right) \ddot{\theta}=F\left(2 \xi \omega \dot{\eta}+\omega^{2} \eta\right)+u(t-\tau(t)) . \tag{2}
\end{equation*}
$$

We consider that $F\left(2 \xi \omega \dot{\eta}+\omega^{2} \eta\right)$ as the disturbance due to the elastic vibration of the flexible appendages. Denote $x(t)=$ $[\theta(t) \dot{\theta}(t)]^{\top}$, then (2) can be transformed into the state-space form

$$
\begin{gather*}
\dot{x}(t)=A x(t)+B u(t-\tau(t))+B d(t), \\
z(t)=C x(t), \tag{3}
\end{gather*}
$$

where

$$
\begin{gather*}
A=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad B=\left[\begin{array}{c}
0 \\
\left(J-F F^{\top}\right)^{-1}
\end{array}\right],  \tag{4}\\
C=\left[\begin{array}{ll}
I & 0
\end{array}\right] .
\end{gather*}
$$

$d(t)=F\left(2 \xi \omega \dot{\eta}+\omega^{2} \eta\right)$ is the disturbance from the flexible appendages and belongs to $l_{2}[0, \infty)$ and $\|d(t)\| \leq \delta_{d}$. For system (3), the following control law is employed to deal with the problem of $H_{\infty}$ control via state feedback

$$
\begin{equation*}
u(t)=K x(t) \tag{5}
\end{equation*}
$$

Then, with the control law (5), the system (3) can be expressed as follows:

$$
\begin{gather*}
\dot{x}(t)=A x(t)+B K x(t-\tau(t))+B d(t), \\
z(t)=C x(t) . \tag{6}
\end{gather*}
$$

Before stating our main results, the following lemmas are first presented, which will be used in the proof of our result.

Lemma 1 (see [16]). For any matrix $M>0$, scalars $b>a$ and $c<d \leq 0$, if there exists a Lebesgue vector function $\omega(s)$, then the following inequalities hold:

$$
\begin{gather*}
-\int_{a}^{b} \omega^{\top}(s) M \omega(s) d s \leq-\frac{1}{b-a} \widetilde{\omega}^{\top}(s) M \widetilde{\omega}(s) \\
-\int_{c}^{d} \int_{t+\theta}^{t} \omega^{\top}(s) M \omega(s) d s d \theta \leq-\frac{2}{c^{2}-d^{2}} \bar{\omega}^{\top}(s) M \bar{\omega}(s), \tag{7}
\end{gather*}
$$

where $\widetilde{\omega}(s)=\int_{a}^{b} \omega^{\top}(s) d s, \bar{\omega}(s)=\int_{c}^{d} \int_{t+\theta}^{t} \omega(s) d s d \theta$.

## 3. Main Result

First of all, a new version of delay-dependent bounded real lemma for the system (6) is proposed in this section.

Theorem 2. Given scalars $\gamma>0, \mu \leq 1$. For any delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq \tau$, the system (6) is asymptotically stable and satisfies $\|z(t)\|_{2}<\gamma\|d(t)\|_{2}$ for any nonzero $d(t) \in$ $l_{2}[0, \infty)$ under the zero initial condition if there exist matrices $P_{1}>0, Q_{1}>0, Q_{2}>0, R_{1}>0, R_{2}>0, P_{2}$, and $P_{3}$ such that the following inequality holds:

$$
\left[\begin{array}{cc}
\Omega+\bar{\Omega} & Y  \tag{8}\\
Y^{\top} & -\gamma^{2} I
\end{array}\right]<0
$$

where

$$
\begin{gathered}
\Omega=\left[\begin{array}{ccccc}
\Omega_{11} & \Omega_{12} & \Omega_{13} & 0 & \frac{1}{\tau} R_{2} \\
* & \Omega_{22} & P_{3}^{\top} B K & 0 & 0 \\
* & * & \Omega_{33} & \frac{1}{\tau} R_{1} & 0 \\
* & * & * & -Q_{1}-\frac{1}{\tau} R_{1} & 0 \\
* & * & * & * & -\frac{2}{\tau^{2}} R_{2}
\end{array}\right], \\
Y=\left[\begin{array}{c}
P_{2}^{\top} B \\
P_{3}^{\top} B \\
0 \\
0 \\
0
\end{array}\right], \\
\Omega_{11}=P_{2}^{\top} A+A^{\top} P_{2}+Q_{1}+Q_{2}-\frac{1}{\tau} R_{1}-2 R_{2}, \\
\Omega_{12}=P_{1}-P_{2}^{\top}+A^{\top} P_{3}, \\
\Omega_{13}=P_{2}^{\top} B K+\frac{1}{\tau} R_{1}, \\
\Omega_{22}=-P_{3}-P_{3}^{\top}+\tau R_{1}+\frac{\tau^{2}}{2} R_{2}, \\
\Omega_{33}=-(1-\mu) Q_{2}-\frac{2}{\tau} R_{1} .
\end{gathered}
$$

Proof. The first step is to analyze the asymptotic stability of the system (6). Consider the system (6) in the absence of $d(t)$, that is,

$$
\begin{equation*}
\dot{x}(t)=A x(t)+B K x(t-\tau(t)) \tag{10}
\end{equation*}
$$

Choose the following Lyapunov-Krasovskii functional:

$$
\begin{align*}
V(t)= & \xi^{\top}(t) E P \xi(t)+\int_{t-\tau}^{t} x^{\top}(s) Q_{1} x(s) d s \\
& +\int_{t-\tau(t)}^{t} x^{\top}(s) Q_{2} x(s) d s \\
& +\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^{\top}(s) R_{1} \dot{x}(s) d s d \theta  \tag{11}\\
& +\int_{-\tau}^{0} \int_{\theta}^{0} \int_{t+v}^{t} \dot{x}^{\top}(s) R_{2} \dot{x}(s) d s d \theta d v
\end{align*}
$$

where

$$
\begin{gather*}
E=\left[\begin{array}{ll}
I & 0 \\
0 & 0
\end{array}\right], \quad P=\left[\begin{array}{cc}
P_{1} & 0 \\
P_{2} & P_{3}
\end{array}\right], \quad \xi(t)=\left[\begin{array}{c}
x(t) \\
\dot{x}(t)
\end{array}\right],  \tag{12}\\
P_{1}>0, \quad Q_{i}>0, \quad R_{i}>0, \quad i=1,2 .
\end{gather*}
$$

Then, along the solution of the system in (10), the time derivative of $V(t)$ is given by
$\dot{V}(t)$

$$
\begin{align*}
= & 2 \xi^{\top}(t) P^{\top}\left[\begin{array}{c}
\dot{x}(t) \\
-\dot{x}(t)+A x(t)+B K x(t-\tau(t))
\end{array}\right] \\
& +x^{\top}(t)\left(Q_{1}+Q_{2}\right) x(t) \\
& -x^{\top}(t-\tau) Q_{1} x(t-\tau) \\
& -(1-\mu) x^{\top}(t-\tau(t)) Q_{2} x(t-\tau(t)) \\
& +\dot{x}^{\top}(t)\left(\tau R_{1}+\frac{\tau^{2}}{2} R_{2}\right) \dot{x}(t) \\
& -\int_{t-\tau}^{t} \dot{x}^{\top}(s) R_{1} \dot{x}(s) d s-\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^{\top}(s) R_{2} \dot{x}(s) d s d \theta \tag{13}
\end{align*}
$$

From Lemma 1, It is easily shown that

$$
\begin{align*}
&-\int_{t-\tau}^{t} \dot{x}^{\top}(s) R_{1} \dot{x}(s) d s \\
&=-\int_{t-\tau}^{t-d(t)} \dot{x}^{\tau}(s) R_{1} \dot{x}(s) d s \\
&-\int_{t-d(t)}^{t} \dot{x}^{\tau}(s) R_{1} \dot{x}(s) d s  \tag{14}\\
& \quad \leq\left[\begin{array}{lll}
x^{\top}(t) & x^{\top}(t-d(t)) & x^{\top}(t-\tau)
\end{array}\right] \\
& \times\left[\begin{array}{ccc}
-\frac{R_{1}}{\tau} & \frac{R_{1}}{\tau} & 0 \\
* & -\frac{2 R_{1}}{\tau} & \frac{R_{1}}{\tau} \\
* & * & -\frac{R_{1}}{\tau}
\end{array}\right]\left[\begin{array}{c}
x(t) \\
x(t-d(t)) \\
x(t-\tau)
\end{array}\right]
\end{align*}
$$

Similarly, the following inequality is also true:

$$
\begin{align*}
&-\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^{\top}(s) R_{2} \dot{x}(s) d s d \theta \\
& \leq-\frac{2}{h^{2}}\left(\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(s) d s d \theta\right)^{\top} R_{2} \\
& \times\left(\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(s) d s d \theta\right)  \tag{15}\\
&= {\left[x^{\top}(t) \int_{t-\tau}^{t} x^{\top}(s) d s\right]\left[\begin{array}{cc}
-2 R_{2} & \frac{2}{\tau} R_{2} \\
* & -\frac{2}{\tau^{2}} R_{2}
\end{array}\right] } \\
& \times\left[\int_{t-\tau}^{t} x(t)\right. \\
&x(s) d s]
\end{align*}
$$

Substituting (14) and (15) into (13) gives

$$
\begin{equation*}
\dot{V}(t) \leq \eta^{\top}(t) \Omega \eta(t), \tag{16}
\end{equation*}
$$

where $\eta(t)=\left[\begin{array}{llll}x(t) & \dot{x}(t) & x(t-\tau(t)) & x(t-\tau)\end{array} \int_{t-\tau}^{t} x(s) d s\right]^{\top}$ and $\Omega$ is defined in (8). Applying the Schur complement to (8) gives $\Omega<0$, which implies $\dot{V}(t)<0$. Hence, the system (6) is asymptotically stable. Next, we will establish the $H_{\infty}$ performance of the uncertain delay system (6) under zero initial condition. Let

$$
\begin{equation*}
J(t)=\int_{0}^{t}\left[z^{\top}(s) z(s)-\gamma^{2} d^{\top}(s) d(s)\right] d s \tag{17}
\end{equation*}
$$

It can be shown that for any nonzero $d(t) \in L_{2}[0, \infty)$ and $t>0$,

$$
\begin{equation*}
J(t) \leq \int_{0}^{t}\left[z^{\top}(s) z(s)-\gamma^{2} d^{\top}(s) d(s)+\dot{V}(s)\right] d s \tag{18}
\end{equation*}
$$

It is noted that

$$
\begin{align*}
z^{\top}(s) & z(s)-\gamma^{2} d^{\top}(s) d(s) \\
& =\phi^{\top}(t) \operatorname{diag}\left\{C^{\top} C, 0,0,0,0,-\gamma^{2} I\right\} \phi(t) \tag{19}
\end{align*}
$$

where $\phi(t)=[\eta(t), d(t)]$ and the time derivative of $\dot{V}\left(x_{s}\right)$ along the solution of (6) gives

$$
\dot{V}(s) \leq \phi^{\top}(t)\left[\begin{array}{cc}
\Omega & Y  \tag{20}\\
Y^{\top} & 0
\end{array}\right] \phi(t)
$$

Hence, $J(t)<0$ follows from (8), (19) and (20), which implies that $\|z(t)\|_{2}<\gamma\|d(t)\|_{2}$ holds for any nonzero $d(t) \in$ $L_{2}[0, \infty)$.

From the proof procedure of Theorem 2, one can see that a new Lyapunov-Krasovskii functional is constructed by employing slack variables $P_{2}$ and $P_{3}$. It is worth pointing out that the matrices $P_{2}$ and $P_{3}$ are useless for reducing the conservatism of stability conditions by using the equivalence idea in $[17,18]$. However, they separates the Lyapunov function matrix $P_{1}>0$ from system matrices $A$ and $B$, that is, there are no terms containing the product of $P_{1}$ and any of them, which is useful for the design of $H_{\infty}$ controller later on.

On the basis of Theorem 2, we will present a design method of $H_{\infty}$ stabilizing controllers in the following.

Theorem 3. Given scalars $\gamma>0, \mu \leq 1$ and $\varepsilon \neq 0$. For any delay $\tau(t)$ satisfying $0 \leq \tau(t) \leq \tau$, the system (6) is asymptotically stable and satisfies $\|z(t)\|_{2}<\gamma\|d(t)\|_{2}$ for any nonzero $d(t) \in l_{2}[0, \infty)$ under the zero initial condition if there exist matrices $\bar{P}_{1}>0, \bar{Q}_{1}>0, \bar{Q}_{2}>0, \bar{R}_{1}>0, \bar{R}_{2}>0, F$, and invertible matrix $\bar{P}_{2}$ such that the following inequality holds:

$$
\left[\begin{array}{cc}
\Pi & \bar{Y}  \tag{21}\\
\bar{Y}^{\top} & -\gamma^{2} I
\end{array}\right]<0
$$

where

$$
\begin{gather*}
\Pi=\left[\begin{array}{cccccc}
\Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & \frac{1}{\tau} \bar{R}_{2} & \bar{P}_{2}^{\top} C^{\top} \\
* & \Pi_{22} & \varepsilon B F & 0 & 0 & 0 \\
* & * & \Pi_{33} & \frac{1}{\tau} \bar{R}_{1} & 0 & 0 \\
* & * & * & -\bar{Q}_{1}-\frac{1}{\tau} \bar{R}_{1} & 0 & 0 \\
* & * & * & * & -\frac{2}{\tau^{2}} \bar{R}_{2} & 0 \\
* & * & * & * & * & -I
\end{array}\right], \\
\bar{Y}=\left[\begin{array}{c}
B \\
\varepsilon B \\
0 \\
0 \\
0
\end{array}\right],  \tag{22}\\
\Pi_{11}=A \bar{P}_{2}+\bar{P}_{2}^{\top} A^{\top}+\bar{Q}_{1}+\bar{Q}_{2}-\frac{1}{\tau} \bar{R}_{1}-2 \bar{R}_{2}, \\
\Pi_{12}=\bar{P}_{1}-\bar{P}_{2}+\varepsilon \bar{P}_{2}^{\top} A^{\top}, \\
\Pi_{13}=B F+\frac{1}{\tau} \bar{R}_{1}, \\
\Pi_{22}=-\varepsilon \bar{P}_{2}-\varepsilon \bar{P}_{2}^{\top}+\tau \bar{R}_{1}+\frac{\tau^{2}}{2} \bar{R}_{2}, \\
\Pi_{3}=-(1-\mu) \bar{Q}_{2}-\frac{2}{\tau} \bar{R}_{1} .
\end{gather*}
$$

Moreover, the feedback gain matrices $K$ are given by

$$
\begin{equation*}
K=F \bar{P}_{2}^{-1} \tag{23}
\end{equation*}
$$

Proof. Define some matrices as follows:

$$
\begin{gather*}
P_{2}=\bar{P}_{2}^{-1}, \quad P_{3}=\frac{1}{\varepsilon} P_{2}^{-1}, \quad P_{1}=P_{2}^{\top} \bar{P}_{1} P_{2} \\
Q_{1}=P_{2}^{\top} \bar{Q}_{1} P_{2}, \quad Q_{2}=P_{2}^{\top} \bar{Q}_{2} P_{2}  \tag{24}\\
R_{1}=P_{2}^{\top} \bar{R}_{1} P_{2}, \quad R_{2}=P_{2}^{\top} \bar{R}_{2} P_{2} .
\end{gather*}
$$

Then, premultiplying (21) by $\operatorname{diag}\left\{P_{2}^{\top}, P_{2}^{\top}, P_{2}^{\top}, P_{2}^{\top}, P_{2}^{\top}\right.$, $I, I\}$ and postmultiplying by $\operatorname{diag}\left\{P_{2}, P_{2}, P_{2}, P_{2}, P_{2}, I, I\right\}$, we can get the following inequality:

$$
\left[\begin{array}{ccc}
\Omega & W & Y  \tag{25}\\
W^{\top} & -I & 0 \\
Y^{\top} & 0 & -\gamma^{2} I
\end{array}\right]<0
$$

where $W^{\top}=\left[\begin{array}{lllll}C & 0 & 0 & 0 & 0\end{array}\right]$. Using Schur complement, from (25), it is clear that (8) holds. As a result, the closed-loop system (6) is asymptotically stable and satisfies $\|z(t)\|_{2}<\gamma\|d(t)\|_{2}$. The proof is thus completed.

Comparing with the traditional controller design method, the matrix $\bar{P}_{2}$ is invertible matrix instead of positive definite matrix, which make the design more flexible. Moreover, this method also can be extended to the systems without time delay.


Figure 1: Spacecraft with flexible appendages.


Figure 2: The responses of pitch attitude $\theta$.

## 4. Numerical Examples

In this section, we consider flexible spacecraft including one rigid body and one flexible appendage depicted in Figure 1. Numerical application of the proposed control schemes to the attitude control of such system is presented using MATLAB/SIMULINK software. In this paper, we only consider the attitude in the pitch channel. Four bending modes are considered for the practical spacecraft model at $\omega_{1}=3.17 \mathrm{rad} / \mathrm{s}$ and $\omega_{2}=7.18 \mathrm{rad} / \mathrm{s}$ with damping $\xi=$ 0.001 and $\xi=0.0015$. We suppose that $F=\left[\begin{array}{ll}F_{1} & F_{2}\end{array}\right]$, where the coupling coefficients of the first two bending modes are $F_{1}=1.27814, F_{2}=0.91756$, and $J=35.72 \mathrm{kgm}^{2}$ which is the nominal principal moment of inertia of pitch axis. The flexible spacecraft is supposed to move in a circular orbit with the altitude of 500 km , then the orbit rate $n=0.0011 \mathrm{rad} / \mathrm{s}$. The initial pitch attitude be of the spacecraft are $\theta(0)=$ 0.08 rad and $\dot{\theta}(0)=0.001 \mathrm{rad} / \mathrm{s}$. And $H_{\infty}$ performance index is supposed to $\gamma=0.1$, and time delay satisfies $\tau=0.3$ and $u=0.1$. The tuning parameter is chosen as $\varepsilon=1$. The response of pitch attitude $\theta, \dot{\theta}$, and the control are shown in Figures 2, 3, and 4 , respectively. From these figures, we can see that our proposed method has a good performance under disturbance and input time delay.


Figure 3: The responses of pitch attitude $\dot{\theta}$.


Figure 4: The responses of control $u(t-\tau(t))$.

## 5. Conclusion

In this paper, $H_{\infty}$ control problem for flexible spacecraft with disturbance and input time-varying delay has been investigated. The LMI-based condition has been formulated for the existence of the admissible controller, which ensures that the closed-loop system is asymptotically stable with a $H_{\infty}$ disturbance attenuation level. Further improvement in precision attitude control for flexible spacecrafts will be considered in our future work.

## Conflict of Interests

The authors of this paper do not have a direct financial relation with the commercial identity mentioned in this paper. This does not lead to a conflict of interests to any of the authors.

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# A Novel Approach to $H_{\infty}$ Control Design for Linear Neutral Time-Delay Systems 

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#### Abstract

This paper is concerned with the problem of $H_{\infty}$ control of linear neutral systems with time-varying delay. Firstly, by applying a novel Lyapunov-Krasovskii functional which is constructed with the idea of delay partitioning approach, appropriate free-weighting matrices, an improved delay-dependent bounded real lemma (BRL) for neutral systems is established. By using the obtained BRL, a delay-dependent sufficient condition for the existence of a state-feedback controller, which ensures asymptotic stability and a prescribed $H_{\infty}$ performance level of the corresponding closed-loop system, is formulated in terms of linear matrix inequalities. Some numerical examples are given to illustrate the effectiveness of the proposed design method.


## 1. Introduction

A neutral time-delay system contains delays both in its states and in its derivatives of states, which occurs in various dynamic systems, such as economical systems, biological systems, metallurgical processing systems, nuclear reactor, power systems, and long-transmission lines in pneumatic and hydraulic systems [1-10]. It has been recognized that time delays can degrade system performance and even result in instability [3-8]. Therefore, researchers have paid considerable attention to the problems of analysis and synthesis for time-delay systems in the last decades (e.g., [1-16]).

In practical applications, however, it is desirable to design a controller such that the closed-loop system is not only stable but also possesses an adequate level of performance $[6,7,10$, 11]. One approach to cope with this problem is the so-called $H_{\infty}$ control approach. The main objective of the $H_{\infty}$ control is to obtain a controller such that the resulted closed system allows a maximum delay size for a fixed $H_{\infty}$ performance bound or achieves a minimum $H_{\infty}$ performance bound for a fixed delay size [10-12]. The conservatism in the $H_{\infty}$ control is hence measured by the allowable delay size or performance
bound obtained. Recently, many results on $H_{\infty}$ control of neutral systems have appeared in the literature; see [4, 10$12,16,17]$ and the references therein.

In general, the existing literature for time-delay systems can be roughly divided into two types: delay-independent results [8-10] and delay-dependent ones [16-27]. The former is irrelevant to the delay size while the latter includes the information of delay size. Obviously, it has been recognized that delay-dependent results are generally less conservative than delay-independent ones, particularly when the delay size is time varying [20-24, 28, 29]. In order to further reduce the conservatism, some improved delay-dependent stability conditions are derived by introducing free-weighting matrices in [19]. In fact, Wu et al. and He et al. [20, 28] have proposed some effective methods for dealing with time-delay systems, which employ free-weighting matrices to express the relationships between the terms in the Leibinz-Newton formula. The method therein reduces the conservativeness of methods involving a fixed model transformation. Additional studies can be found in $[8,16,20,25,28,30]$ and references cited therein.

On the other hand, some other efforts on improving the delay-dependent conditions were made through introducing new Lyapunov-Krasovskii functional. To mention a few, new classes of Lyapunov functional and augmented Lyapunov functional were introduced to study the delay-dependent stability for systems with time-varying delay in [28], which is shown to possess less conservatism than the existing ones. In [21], based on a novel fuzzy Lyapunov-Krasovskii functional which is constructed using a delay partitioning method, a delay-dependent criterion is developed for the stability analysis of fuzzy time-varying state delay systems. In [24], a less conservative delay-dependent robust $H_{\infty}$ control is proposed for uncertain linear systems with a state-delay based on a new Lyapunov-Krasovskii functional. A new criterion of asymptotic stability is derived in [26] by introducing a novel Lyapunov functional with the idea of partitioning the lower bound of the time-varying delay. Recently, there is enormous growth of interest in using the delay partitioning technique to deal with time-delay systems; see, for example [21-23, 26, 3133]. The basic idea of this approach is to evenly partition time delay into several components. By constructing a LyapunovKrasovskii functional (LKF) with every delay component, one can obtain a less conservative stability condition as discussed by F. Gouaisbaut and Peaucelle [22] and Wu et al. [21, 29]. More results can be found in articles [17, 21-27, 2933] and the references therein.

In this context, motivated by Wu et al. [21] and Zhang and Li [23], we will study the delay-dependent $H_{\infty}$ control problem of a class of neutral time-delay systems based on a delay partitioning technique. The remainder of the paper is organized as follows. Section 2 gives problem formulation and a necessary lemma. In Section 3, dividing the delay interval into multiple segments, using the Lyapunov functional technology combined with matrix inequality technology, a new delay-dependent bounded real lemma is proposed. Based on the BRL, a condition for the existence of a state-feedback $H_{\infty}$ controller is introduced in terms of linear matrix inequalities. Numerical examples are given in Section 4, followed by the conclusions, which are presented in Section 5.

Notation. Throughout this paper, " $T$ " stands for matrix transposition. " $I$ " denotes the identity matrix of appropriate dimensions. " $P>0$ " means that $P$ is positive definite. "*" represents the elements below the main diagonal of a symmetric matrix.

## 2. Problem Formulations

Consider a class of linear time-varying discrete neutral system:

$$
\begin{gather*}
\dot{x}(t)-C_{d} \dot{x}(t-\tau)=A x(t)+A_{d} x(t-d(t)) \\
\\
\quad+B w(t)+B_{u} u(t), \quad t>0,  \tag{1}\\
z(t)=C x(t)+C_{u} u(t)+D w(t) \\
x(t)=\phi(t), \quad t \in[-r, 0]
\end{gather*}
$$

where $x(t)$ is the state vector; the matrices $A, A_{d}, B, C, C_{d}$, and $D$ are known constant matrices of appropriate dimensions,
and the eigenvalue of the matrix $C_{d}, \rho\left(C_{d}\right)$ satisfies $\rho\left(C_{d}\right)<1$. $w(t)$ is the disturbance input that belongs to $L_{2}[0, \infty) . z(t)$ is the controlled output, and $u(t)$ is the controlled input. $\phi(t)(t \in[-r, 0])$ is the system's initial function which is continuous differentiable on $[-r, 0]$. The scalar $\tau$ is a positive constant time delay. Time delay $d(t)$ is a continuously differentiable function, satisfying the following conditions:

$$
\begin{gather*}
0 \leq d(t) \leq h \\
\dot{d}(t) \leq \mu \tag{2}
\end{gather*}
$$

where $h$, and $\mu$ are known positive real constants, and it is assumed that $r=\max \{h, \tau\}$. In this paper, we are interested in designing a memoryless state-feedback controller

$$
\begin{equation*}
u(t)=K x(t) \tag{3}
\end{equation*}
$$

where $K$ is a constant matrix, such that for a given scalar $\gamma$, the following requirements are satisfied:
(I) the corresponding closed-loop system is asymptotically stable when $w(t)=0$;
(II) under zero initial condition (i.e., $x(t)=0(t \in[-r$, $0])$ ), the corresponding closed-loop system satisfies

$$
\begin{equation*}
\|z\|_{2} \leq \gamma\|w\|_{2}, \quad \forall w \in L_{2}[0, \infty) \tag{4}
\end{equation*}
$$

where $\gamma>0$ is a prescribed scalar.
In obtaining the main results of this paper, the following lemma plays an important role.

Lemma 1 (Schur complement). For the symmetrical matrix $L=\left[\begin{array}{ll}L_{11} & L_{12} \\ L_{12}^{T} & L_{22}\end{array}\right]$, the followings are equivalent:
(1) $L<0$,
(2) $L_{11}<0, \quad L_{22}-L_{12}^{T} L_{11}^{-1} L_{12}<0$,
(3) $L_{22}<0, \quad L_{11}-L_{12} L_{22}^{-1} L_{12}^{T}<0$.

## 3. Main Results

In this section, we discuss the problem of $H_{\infty}$ performance and state-feedback $H_{\infty}$ controller design of system (1).
3.1. $H_{\infty}$ Performance Analysis. In the following theorem, we present a new version of delay-dependent bounded real lemma for neutral system (1) with $u(t) \equiv 0$; that is, we consider the following system:

$$
\begin{gather*}
\dot{x}(t)-C_{d} \dot{x}(t-\tau)=A x(t)+A_{d} x(t-d(t))+B w(t), \\
z(t)=C x(t)+D w(t) . \tag{6}
\end{gather*}
$$

Theorem 2. For given positive scalars $h>0$ and $\mu>0$, the neutral system (6) with delay restrictions (2) is asymptotically stable and satisfies $\|z\|_{2} \leq \gamma\|w\|_{2}$ for any nonzero $w(t) \in$ $L_{2}[0, \infty)$ under the zero initial condition if there exist matrices
$P=P^{T}>0, Q_{i}=Q_{i}^{T} \geq 0(i=1, \ldots, 4), R=R^{T}>0, Z=$ $Z^{T}>0$,

$$
X=\left[\begin{array}{cccccc}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16}  \tag{7}\\
* & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\
* & * & X_{33} & X_{34} & X_{35} & X_{36} \\
* & * & * & X_{44} & X_{45} & X_{46} \\
* & * & * & * & X_{55} & X_{56} \\
* & * & * & * & * & X_{66}
\end{array}\right] \geq 0,
$$

and free-weighting matrices $N_{i}(i=1, \ldots, 6)$ with appropriate dimensions, such that the following LMIs hold:

$$
\begin{gather*}
\Phi=\left[\begin{array}{ccccccccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} & A^{T} H & 0 & C^{T} \\
* & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} & 0 & 0 & 0 \\
* & * & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} & A_{d}^{T} H & 0 & 0 \\
* & * & * & \Phi_{44} & \Phi_{45} & \Phi_{46} & 0 & 0 & 0 \\
* & * & * & * & \Phi_{55} & \Phi_{56} & 0 & 0 & 0 \\
* & * & * & * & * & \Phi_{66} & C_{d}^{T} H & 0 & 0 \\
* & * & * & * & * & * & -H & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^{2} I & D^{T} \\
* & * & * & * & * & * & * & * & -I
\end{array}\right] \\
<0,  \tag{8}\\
\Psi=\left[\begin{array}{cccccccccc}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & N_{1} \\
* & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & N_{2} \\
* & * & X_{33} & X_{34} & X_{35} & X_{36} & N_{3} \\
* & * & * & X_{44} & X_{45} & X_{46} & N_{4} \\
* & * & * & * & X_{55} & X_{56} & N_{5} \\
* & * & * & * & * & X_{66} & N_{6} \\
* & * & * & * & * & * & Z
\end{array}\right] \geq 0, \tag{9}
\end{gather*}
$$

where

$$
\begin{gathered}
\Phi_{11}=P A+A^{T} P+Q_{1}+N_{1}+N_{1}^{T}+h X_{11}, \\
\Phi_{12}=N_{2}^{T}+h X_{12}, \quad \Phi_{13}=P A_{d}-N_{1}+N_{3}^{T}+h X_{13}, \\
\Phi_{14}=N_{4}^{T}+h X_{14}, \quad \Phi_{15}=N_{5}^{T}+h X_{15}, \\
\Phi_{16}=P C_{d}+N_{6}^{T}+h X_{16}, \\
\Phi_{22}=-\left(1-\frac{\mu}{2}\right) Q_{1}+Q_{2}+h X_{22} \\
\Phi_{23}=-N_{2}+h X_{23}, \quad \Phi_{24}=h X_{24}, \\
\Phi_{25}=h X_{25}, \quad \Phi_{26}=h X_{26}, \\
\Phi_{33}=-(1-\mu) Q_{2}+Q_{3}-N_{3}-N_{3}^{T}+h X_{33}, \\
\Phi_{34}=-N_{4}^{T}+h X_{34}, \quad \Phi_{35}=-N_{5}^{T}+h X_{35} \\
\Phi_{36}=-N_{6}^{T}+h X_{36}, \\
\Phi_{44}=-\left(1-\frac{\mu}{2}\right) Q_{3}+Q_{4}+h X_{44},
\end{gathered}
$$

$$
\begin{array}{cc}
\Phi_{45}=h X_{45}, & \Phi_{46}=h X_{46} \\
\Phi_{55}=-Q_{4}+h X_{55}, & \Phi_{56}=h X_{56} \\
\Phi_{66}=-R+h X_{66}, & H=R+h Z \tag{10}
\end{array}
$$

Proof. Under the condition of the theorem, we first show the asymptotic stability of system (6). To this end, we consider system (6) with $w(t)=0$, that is,

$$
\begin{equation*}
\dot{x}(t)-C_{d} \dot{x}(t-\tau)=A x(t)+A_{d} x(t-d(t)) . \tag{11}
\end{equation*}
$$

Inspired by the works of [23], we divided the delay interval $[0, h]$ into $[0, d(t) / 2],[d(t) / 2, d(t)],[d(t),(d(t)+h) / 2]$, and $[(d(t)+h) / 2, h]$ for system (6). Corresponding to such a division, the following Lyapunov-Krasovskii functional is chosen for this system:

$$
\begin{align*}
V\left(t, x_{t}\right)= & x^{T}(t) P x(t)+\int_{t-d(t) / 2}^{t} x^{T}(s) Q_{1} x(s) d s \\
& +\int_{t-d(t)}^{t-d(t) / 2} x^{T}(s) Q_{2} x(s) d s \\
& +\int_{t-(d(t)+h) / 2}^{t-d(t)} x^{T}(s) Q_{3} x(s) d s  \tag{12}\\
& +\int_{t-h}^{t-(d(t)+h) / 2} x^{T}(s) Q_{4} x(s) d s \\
& +\int_{t-\tau}^{t} \dot{x}^{T}(s) R \dot{x}(s) d s \\
& +\int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(s) Z \dot{x}(s) d s d \theta
\end{align*}
$$

where $P=P^{T}>0, Q_{i}=Q_{i}^{T} \geq 0(i=1, \ldots, 4), R=R^{T}>0$, and $Z=Z^{T}>0$ are matrices to be determined. By using the Leibniz-Newton formula, one has

$$
\begin{equation*}
x(t-d(t))=x(t)-\int_{t-d(t)}^{t} \dot{x}(s) d s \tag{13}
\end{equation*}
$$

Since (13) can be rewritten as $x(t)-x(t-d(t))-\int_{t-d(t)}^{t} \dot{x}(s) d s=$ 0 . Due to this relation, one can introduce some appropriate dimensional matrices $N_{i}(i=1, \ldots, 6)$, such that

$$
\begin{align*}
& 2\left[x^{T}(t) N_{1}+x^{T}\left(t-\frac{d(t)}{2}\right) N_{2}+x^{T}(t-d(t)) N_{3}\right. \\
& \quad+x^{T}\left(t-\frac{d(t)+h}{2}\right) N_{4}+x^{T}(t-h) N_{5} \\
& \left.\quad+\dot{x}^{T}(t-\tau) N_{6}\right]  \tag{14}\\
& \times\left[x(t)-x(t-d(t))-\int_{t-d(t)}^{t} \dot{x}(s) d s\right]=0
\end{align*}
$$

Moreover, it follows from (2) that for any appropriate dimensional matrix $X \geq 0$,

$$
\begin{equation*}
h \eta^{T}(t) X \eta(t)-\int_{t-d(t)}^{t} \eta^{T}(t) X \eta(t) d s \geq 0 \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta(t)=\left[x^{T}(t) x^{T}\left(t-\frac{d(t)}{2}\right) x^{T}(t-d(t)) x^{T}\left(t-\frac{d(t)+h}{2}\right) x^{T}(t-h) \dot{x}^{T}(t-\tau)\right]^{T} . \tag{16}
\end{equation*}
$$

According to (14) and (15), and taking the time derivative of the Lyapunov-Krasovskii functional candidate (12) gives that

$$
\begin{aligned}
& \dot{V}\left(t, x_{t}\right) \\
& =2 x^{T}(t) P \dot{x}(t)+x^{T}(t) Q_{1} x(t) \\
& -\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)}{2}\right) Q_{1} x\left(t-\frac{d(t)}{2}\right) \\
& +\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)}{2}\right) Q_{2} x\left(t-\frac{d(t)}{2}\right) \\
& -(1-\dot{d}(t)) x^{T}(t-d(t)) Q_{2} x(t-d(t)) \\
& +\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)}{2}\right) Q_{2} x\left(t-\frac{d(t)}{2}\right) \\
& -(1-\dot{d}(t)) x^{T}(t-d(t)) Q_{2} x(t-d(t)) \\
& +(1-\dot{d}(t)) x^{T}(t-d(t)) Q_{3} x(t-d(t)) \\
& -\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)+h}{2}\right) Q_{3} \\
& \times x\left(t-\frac{d(t)+h}{2}\right) \\
& +\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)+h}{2}\right) Q_{4} \\
& \times x\left(t-\frac{d(t)+h}{2}\right)-x^{T}(t-h) Q_{4} x(t-h) \\
& +\dot{x}^{T}(t) R \dot{x}(t)-\dot{x}^{T}(t-\tau) R \dot{x}(t-\tau) \\
& +h \dot{x}^{T}(t) Z \dot{x}(t)-\int_{t-h}^{t} \dot{x}^{T}(s) Z \dot{x}(s) d s \\
& \leq 2 x^{T}(t) P\left[A x(t)+A_{d} x(t-d(t))+C_{d} \dot{x}(t-\tau)\right] \\
& +x^{T}(t) Q_{1} x(t) \\
& -\left(1-\frac{\mu}{2}\right) x^{T}\left(t-\frac{d(t)}{2}\right) Q_{1} x\left(t-\frac{d(t)}{2}\right) \\
& +\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)}{2}\right) Q_{2} x\left(t-\frac{d(t)}{2}\right) \\
& -(1-\mu) x^{T}(t-d(t)) Q_{2} x(t-d(t)) \\
& +(1-\dot{d}(t)) x^{T}(t-d(t)) Q_{3} x(t-d(t))
\end{aligned}
$$

$$
\begin{align*}
& -\left(1-\frac{\mu}{2}\right) x^{T}\left(t-\frac{d(t)+h}{2}\right) Q_{3} x\left(t-\frac{d(t)+h}{2}\right) \\
& +\left(1-\frac{\dot{d}(t)}{2}\right) x^{T}\left(t-\frac{d(t)+h}{2}\right) Q_{4} \\
& \times x\left(t-\frac{d(t)+h}{2}\right)-x^{T}(t-h) Q_{4} x(t-h) \\
& +\dot{x}^{T}(t) R \dot{x}(t)-\dot{x}^{T}(t-\tau) R \dot{x}(t-\tau) \\
& +h \dot{x}^{T}(t) Z \dot{x}(t)-\int_{t-d(t)}^{t} \dot{x}^{T}(s) Z \dot{x}(s) d s \\
& +2\left[x^{T}(t) N_{1}+x^{T}\left(t-\frac{d(t)}{2}\right) N_{2}\right. \\
& +x^{T}(t-d(t)) N_{3}+x^{T}\left(t-\frac{d(t)+h}{2}\right) N_{4} \\
& \left.+x^{T}(t-h) N_{5}+\dot{x}^{T}(t-\tau) N_{6}\right] \\
& \times\left[x(t)-x(t-d(t))-\int_{t-d(t)}^{t} \dot{x}(s) d s\right] \\
& +h \eta^{T}(t) X \eta(t)-\int_{t-d(t)}^{t} \eta^{T}(t) X \eta(t) d s \\
& =\eta^{T}(t) \Xi \eta(t)-\int_{t-d(t)}^{t} \xi^{T}(t, s) \Psi \xi(t, s) d s \\
& -\frac{\dot{d}(t)}{2} x^{T}\left(t-\frac{d(t)}{2}\right) Q_{2} x\left(t-\frac{d(t)}{2}\right) \\
& -\dot{d}(t) x^{T}(t-d(t)) Q_{3} x(t-d(t)) \\
& -\frac{\dot{d}(t)}{2} x^{T}\left(t-\frac{d(t)+h}{2}\right) Q_{4} x\left(t-\frac{d(t)+h}{2}\right), \tag{17}
\end{align*}
$$

where

$$
\begin{gather*}
\xi(t)=\left[\begin{array}{lllll}
\eta^{T}(t) & \dot{x}^{T}(s)
\end{array}\right]^{T}, \\
\Xi=\left[\begin{array}{cccccc}
\Phi_{11}+A^{T} H A & \Phi_{12} & \Phi_{13}+A^{T} H A_{d} & \Phi_{14} & \Phi_{15} & \Phi_{16}+A^{T} H C_{d} \\
* & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} \\
* & * & \Phi_{33}+A_{d}^{T} H A_{d} & \Phi_{34} & \Phi_{35} & \Phi_{36}+A_{d}^{T} H C_{d} \\
* & * & * & \Phi_{44} & \Phi_{45} & \Phi_{46} \\
* & * & * & * & \Phi_{55} & \Phi_{56} \\
* & * & * & * & * & \Phi_{66}+C_{d}^{T} H C_{d}
\end{array}\right] \tag{18}
\end{gather*}
$$

and $\Psi$ is denoted in (9). The last three items of (17) are not more than zero since $Q_{i} \geq 0(i=2, \ldots, 4)$. Therefore, if
$\Xi<0$ and $\Psi \geq 0$, there exists a positive scalar $\varepsilon$ such that $\dot{V}\left(t, x_{t}\right) \leq-\varepsilon\|x(t)\|^{2}$, which guarantees system (6) is asymptotically stable. By Lemma 1, we can conclude that if the matrix inequality (8) is feasible then the inequality $\Xi<0$ is feasible. This implies that system (6) is asymptotically stable if LMIs (8) and (9) are feasible.

Next, we shall establish the $H_{\infty}$ performance of system (6) under zero initial condition. To this end, we introduce

$$
\begin{equation*}
J_{\gamma}=\int_{0}^{t}\left(z^{T}(\theta) z(\theta)-\gamma^{2} w^{T}(\theta) w(\theta)\right) d \theta \tag{19}
\end{equation*}
$$

where $t>0$.
By combining (17) with those results analyzed above, now it is interesting to note that

$$
\begin{equation*}
\dot{V}\left(t, x_{t}\right) \leq \eta^{T}(t) \Xi \eta(t)-\int_{t-d(t)}^{t} \xi^{T}(t, s) \Psi \xi(t, s) d s \tag{20}
\end{equation*}
$$

Considering zero initial condition, it is easy to see that for any nonzero $w \in L_{2}[0, \infty)$ and $t>0$, the following expression holds.

$$
\begin{align*}
J_{\gamma}= & \int_{0}^{t}\left(z^{T}(\theta) z(\theta)-\gamma^{2} w^{T}(\theta) w(\theta)+\dot{V}(\theta)\right) d \theta-V(t) \\
\leq & \int_{0}^{t}\left(z^{T}(\theta) z(\theta)-\gamma^{2} w^{T}(\theta) w(\theta)+\dot{V}(\theta)\right) d \theta \\
= & \int_{0}^{t} \bar{\eta}^{T}(\theta) \Xi^{\prime} \bar{\eta}(\theta) d \theta \\
& -\int_{0}^{t} \int_{\theta-d(\theta)}^{\theta} \xi^{T}(\theta, s) \Psi \xi(\theta, s) d s d \theta \tag{21}
\end{align*}
$$

where

By carrying out some algebraic manipulations, the aforementioned matrix inequality (22) can be rewritten as follows:

$$
\left[\begin{array}{cccccccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} & A^{T} H & 0 \\
* & \Phi_{22} \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} & 0 & 0 \\
* & * & \Phi_{33} & \Phi_{34} & \Phi_{35} \Phi_{36} A_{d}^{T} H & 0 \\
* & * & * & \Phi_{44} & \Phi_{45} \Phi_{46} & 0 & 0 \\
* & * & * & * & \Phi_{55} & \Phi_{56} & 0 & 0 \\
* & * & * & * & * & \Phi_{66} & C_{d}^{T} H & 0 \\
* & * & * & * & * & * & -H & 0 \\
* & * & * & * & * & * & * & -\gamma^{2} I
\end{array}\right]
$$

$$
+\left[\begin{array}{c}
C^{T}  \tag{23}\\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
D^{T}
\end{array}\right] \times\left[\begin{array}{llllllll}
C & 0 & 0 & 0 & 0 & 0 & 0 & D
\end{array}\right]<0
$$

By Schur complement (Lemma 1) and some matrices primary manipulations, it is easy to see that the abovementioned matrix inequality (23) is equivalent to (8).

Combining (8) and (9), we have $\bar{\eta}^{T}(t) \Xi^{\prime} \bar{\eta}(t) \leq 0$ for any $\bar{\eta}(t)$. Therefore, the following expression holds for any $t>0$ :

$$
\begin{equation*}
J_{\gamma}=\int_{0}^{t}\left(z^{T}(\theta) z(\theta)-\gamma^{2} w^{T}(\theta) w(\theta)\right) d \theta \leq 0 \tag{24}
\end{equation*}
$$

By letting $t \rightarrow \infty$, we lead to

$$
\begin{equation*}
\int_{0}^{\infty} z^{T}(\theta) z(\theta) d \theta \leq \gamma^{2} \int_{0}^{\infty} w^{T}(\theta) w(\theta) d \theta \tag{25}
\end{equation*}
$$

And hence, (4) is satisfied for any nonzero $w(t) \in L_{2}[0, \infty)$.
The proof is thus completed.
Remark 3. Theorem 2 presents an improved bounded real lemma for linear neutral system with time-varying delay by defining a novel Lyapunov-Krasovskii functional. The merit of the proposed BRL lies in its reduced conservatism, which is based on a time-delay fractioning approach.

Remark 4. About the delay partitioning technique, the classical approach is to represent the time delay as two parts: constant part and time-varying part, and then a LyapunovKrasovskii functional is introduced by applying the idea of delay partitioning to the constant part. However, in this brief, we partition the whole time-varying delay interval into multiparts and a novel Lyapunov-Krasovskii functional (LKF) is constructed with every delay component. Then, we
obtain a time-varying LKF since it is dependent on the timevarying delay, which constitutes the major difference from most existing results in the literature.
3.2. $H_{\infty}$ Control of Neutral Time-Delay Systems. Now, we are in a position to state the $H_{\infty}$ control result based on the BRL derived in the previous section. A sufficient condition under which there exists a memoryless state-feedback $H_{\infty}$ controller for the neutral system (1) is given in Theorem 5.

Theorem 5. For given positive scalars $\mu$ and $\lambda$, there exists a state-feedback controller in the form of (3) such that the resulting closed-loop system satisfies the requirements (I) and
(II) if there exist matrices $\widetilde{P}>0, \widetilde{Q}_{i}>0(i=1, \ldots, 4), \widetilde{R}>0$, $\widetilde{Z}>0$,

$$
\widetilde{X}=\left[\begin{array}{cccccc}
\widetilde{X}_{11} & \widetilde{X}_{12} & \widetilde{X}_{13} & \widetilde{X}_{14} & \widetilde{X}_{15} & \widetilde{X}_{16}  \tag{26}\\
* & \widetilde{X}_{22} & \widetilde{X}_{23} & \widetilde{X}_{24} & \widetilde{X}_{25} & \widetilde{X}_{26} \\
* & * & \widetilde{X}_{33} & \widetilde{X}_{34} & \widetilde{X}_{35} & \widetilde{X}_{36} \\
* & * & * & \widetilde{X}_{44} & \widetilde{X}_{45} & \widetilde{X}_{46} \\
* & * & * & * & \widetilde{X}_{55} & \widetilde{X}_{56} \\
* & * & * & * & * & \widetilde{X}_{66}
\end{array}\right] \geq 0,
$$

free-weighting matrices $\widetilde{N}_{i}(i=1, \ldots, 6)$ of appropriate dimensions, and a scalar $\gamma>0$, such that the following matrix inequalities hold:

$$
\begin{gather*}
{\left[\begin{array}{ccccccccc}
\widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} & \widetilde{\Phi}_{13} & \widetilde{\Phi}_{14} & \widetilde{\Phi}_{15} & \widetilde{\Phi}_{16} & \lambda \widetilde{P} B^{T}+\lambda Y^{T} B_{u}^{T} & 0 & \widetilde{P} C^{T}+Y^{T} C_{u}^{T} \\
* & \widetilde{\Phi}_{22} & \widetilde{\Phi}_{23} & \widetilde{\Phi}_{24} & \widetilde{\Phi}_{25} & \widetilde{\Phi}_{26} & 0 & 0 & 0 \\
* & * & \widetilde{\Phi}_{33} & \widetilde{\Phi}_{34} & \widetilde{\Phi}_{35} & \widetilde{\Phi}_{36} & \lambda \widetilde{P}_{d}^{T} & 0 & 0 \\
* & * & * & \widetilde{\Phi}_{44} & \Phi_{45} & \widetilde{\Phi}_{46} & 0 & 0 & 0 \\
* & * & * & * & \widetilde{\Phi}_{55} & \widetilde{\Phi}_{56} & 0 & 0 & 0 \\
* & * & * & * & * & \widetilde{\Phi}_{66} & \lambda \widetilde{P} C_{d}^{T} & 0 & 0 \\
* & * & * & * & * & * & -\lambda & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^{2} I & D^{T} \\
* & * & * & * & * & * & * & * & -I
\end{array}\right]<0,}  \tag{27}\\
 \tag{28}\\
\\
\\
\\
\\
\end{gather*}
$$

where

$$
\begin{gather*}
\widetilde{\Phi}_{11}=B_{u} Y+Y^{T} B_{u}^{T}+\widetilde{P} B^{T}+B \widetilde{P}+\widetilde{Q}_{1}+\widetilde{N}_{1}+\widetilde{N}_{1}^{T}+h \widetilde{X}_{11}, \\
\widetilde{\Phi}_{12}=\widetilde{N}_{2}^{T}+h \widetilde{X}_{12} \\
\widetilde{\Phi}_{13}=A_{d} \widetilde{P}-\widetilde{N}_{1}+\widetilde{N}_{3}^{T}+h \widetilde{X}_{13} \\
\widetilde{\Phi}_{14}=\widetilde{N}_{4}^{T}+h \widetilde{X}_{14}, \quad \widetilde{\Phi}_{15}=\widetilde{N}_{5}^{T}+h \widetilde{X}_{15} \\
\widetilde{\Phi}_{16}=C_{d} \widetilde{P}+\widetilde{N}_{6}^{T}+h \widetilde{X}_{16} \\
\widetilde{\Phi}_{22}=-\left(1-\frac{\mu}{2}\right) \widetilde{Q}_{1}+\widetilde{Q}_{2}+h \widetilde{X}_{22}  \tag{29}\\
\widetilde{\Phi}_{24}=h \widetilde{X}_{24}, \quad \widetilde{\Phi}_{25}=h \widetilde{N}_{25}+h \widetilde{X}_{23}, \quad \widetilde{\Phi}_{26}=h \bar{X}_{26} \tag{30}
\end{gather*}
$$

$$
\begin{gathered}
\widetilde{\Phi}_{33}=-(1-\mu) \widetilde{Q}_{2}+\widetilde{Q}_{3}-\widetilde{N}_{3}-\widetilde{N}_{3}^{T}+h \widetilde{X}_{33} \\
\widetilde{\Phi}_{34}=-\widetilde{N}_{4}+h \widetilde{X}_{34}, \quad \widetilde{\Phi}_{35}=-\widetilde{N}_{5}+h \widetilde{X}_{35} \\
\widetilde{\Phi}_{36}=-\widetilde{N}_{6}+h \widetilde{X}_{36} \\
\widetilde{\Phi}_{44}=\left(\frac{\mu}{2}-1\right) \widetilde{Q}_{3}+\widetilde{Q}_{4}+h \widetilde{X}_{44} \\
\widetilde{\Phi}_{45}=h \widetilde{X}_{45}, \quad \widetilde{\Phi}_{46}=h \widetilde{X}_{46} \\
\widetilde{\Phi}_{55}=-\widetilde{Q}_{4}+h \widetilde{X}_{55}, \quad \widetilde{\Phi}_{56}=h \widetilde{X}_{56} \\
\widetilde{\Phi}_{66}=-\widetilde{R}+h \widetilde{X}_{66}
\end{gathered}
$$

In this case, an $H_{\infty}$ state-feedback controller can be chosen as

$$
u(t)=Y \widetilde{P}^{-1} x(t)
$$

Proof. Assume that $u(t)=K x(t) ; B_{u} u(t)$ and $C_{u} u(t)$ in (1) are replaced by $B_{u} K x(t)$ and $C_{u} K x(t)$, respectively. Define $A_{k} \doteq$
$A+B_{u} K$ and $C_{k} \doteq C+C_{u} K$. Taking this into account, the condition in Theorem 2 is replaced by

$$
\left[\begin{array}{ccccccccc}
\Phi_{11}^{\prime} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} & A_{k}^{T} H & 0 & C_{k}^{T}  \tag{31}\\
* & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} & 0 & 0 & 0 \\
* & * & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} & A_{d}^{T} H & 0 & 0 \\
* & * & * & \Phi_{44} & \Phi_{45} & \Phi_{46} & 0 & 0 & 0 \\
* & * & * & * & \Phi_{55} & \Phi_{56} & 0 & 0 & 0 \\
* & * & * & * & * & \Phi_{66} & C_{d}^{T} H & 0 & 0 \\
* & * & * & * & * & * & -H & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^{2} I & D^{T} \\
* & * & * & * & * & * & * & * & -I
\end{array}\right]<0,
$$

where

$$
\begin{equation*}
\Phi_{11}^{\prime}=P A_{k}+A_{k}^{T} P+Q_{1}+N_{1}+N_{1}^{T}+h X_{11} \tag{32}
\end{equation*}
$$

Pre- and postmultiplying (31) by $\operatorname{diag}\left\{P^{-T}, P^{-T}, P^{-T}\right.$, $\left.P^{-T}, P^{-T}, P^{-T}, P^{-T}, I, I\right\}$ and $\operatorname{diag}\left\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}\right.$, $\left.P^{-1}, P^{-1}, I, I\right\}$, respectively, and introducing an additional constraint $H=\lambda P$, then one can obtain an equivalent expression of (31) as follows:

$$
\left[\begin{array}{ccccccccc}
\Phi_{11}^{\prime} & \bar{\Phi}_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \bar{\Phi}_{16} & \lambda P^{-T} B^{T} & & P^{-T} C^{T}  \tag{33}\\
* & \bar{\Phi}_{22} & \bar{\Phi}_{23} & \bar{\Phi}_{24} & \bar{\Phi}_{25} & \bar{\Phi}_{26} & 0 & 0 & K^{T} B_{u}^{T}
\end{array}\right)
$$

where

$$
\begin{gathered}
\bar{\Phi}_{11}^{\prime}=B_{u} K P^{-1}+P^{-T} K^{T} B_{u}^{T}+P^{-T} B^{T}+B P^{-1} \\
+\bar{Q}_{1}+\bar{N}_{1}+\bar{N}_{1}^{T}+h \bar{X}_{11} \\
\bar{\Phi}_{12}=\bar{N}_{2}^{T}+h \bar{X}_{12} \\
\bar{\Phi}_{13}=A_{d} P^{-1}-\bar{N}_{1}+\bar{N}_{3}^{T}+h \bar{X}_{13} \\
\bar{\Phi}_{14}=\bar{N}_{4}^{T}+h \bar{X}_{14}, \quad \bar{\Phi}_{15}=\bar{N}_{5}^{T}+h \bar{X}_{15} \\
\bar{\Phi}_{16}=C_{d} P^{-1}+\bar{N}_{6}^{T}+h \bar{X}_{16} \\
\bar{\Phi}_{22}=-\left(1-\frac{\mu}{2}\right) \bar{Q}_{1}+\bar{Q}_{2}+h \bar{X}_{22} \\
\bar{\Phi}_{23}=-\bar{N}_{2}+h \bar{X}_{23}, \quad \bar{\Phi}_{24}=h \bar{X}_{24} \\
\bar{\Phi}_{25}=h \bar{X}_{25}, \quad \bar{\Phi}_{26}=h \bar{X}_{26} \\
\bar{\Phi}_{33}=-(1-\mu) \bar{Q}_{2}+\bar{Q}_{3}-\bar{N}_{3}-\bar{N}_{3}^{T}+h \bar{X}_{33} \\
\bar{\Phi}_{34}=-\bar{N}_{4}+h \bar{X}_{34}, \quad \bar{\Phi}_{35}=-\bar{N}_{5}+h \bar{X}_{35} \\
\bar{\Phi}_{36}=-\bar{N}_{6}+h \bar{X}_{36} \\
\bar{\Phi}_{44}=-\left(1-\frac{\mu}{2}\right) \bar{Q}_{3}+\bar{Q}_{4}+h \bar{X}_{44} \\
\bar{\Phi}_{45}=h \bar{X}_{45}, \quad \bar{\Phi}_{46}=h \bar{X}_{46}
\end{gathered}
$$

$$
\begin{gather*}
\bar{\Phi}_{55}=-\bar{Q}_{4}+h \bar{X}_{55}, \quad \bar{\Phi}_{56}=h \bar{X}_{56} \\
\bar{\Phi}_{66}=-\bar{R}+h \bar{X}_{66}, \\
\bar{Q}_{i}=P^{-T} Q_{i} P^{-1} \quad(i=1, \ldots, 4), \\
\bar{N}_{i}=P^{-T} N_{i} P^{-1} \quad(i=1, \ldots, 6) \\
\bar{R}=P^{-T} R P^{-1}, \\
\bar{X}_{i j}=P^{-T} X_{i j} P^{-1} \quad(i=1, \ldots, 6, j=1, \ldots, 6), \\
\bar{Z}=P^{-T} Z P^{-1} \tag{34}
\end{gather*}
$$

Introducing change of those above-mentioned variables such that

$$
\begin{gather*}
\widetilde{P}=P^{-T}, \quad Y=K \widetilde{P}, \quad \widetilde{Q}_{i}=\bar{Q}_{i} \quad(i=1, \ldots, 4), \\
\widetilde{N}_{i}=\bar{N}_{i} \quad(i=1, \ldots, 6) \\
\widetilde{X}_{i j}=\bar{X}_{i j} \quad(i=1, \ldots, 6, j=1, \ldots, 6)  \tag{35}\\
\widetilde{R}=\bar{R}, \quad \widetilde{Z}=\bar{Z}
\end{gather*}
$$

thus we can obtain (27).
The proof is thus completed.
Remark 6. It should be pointed out that Theorem 5 gives a sufficient condition for the existence of a sate feedback
compensation controller with the $H_{\infty}$ performance bound $\gamma$ in the form of (3) for system (1), which guarantees the closedloop system to be asymptotically stable.

Remark 7. Since conditions (27) and (28) in Theorem 5 are in the LMI forms, for a given scalar $\gamma>0$, the solutions can be easily obtained using LMI Toolbox. The problem is then how to find the optimal values of $\gamma$. A feasible optimizing approach is given in Corollary 8 as follows, which can be completed using the Matlab command mincx.

Corollary 8. A suboptimal $H_{\infty}$ controller in the form of (3) for the neutral time-delay system (1) can be found by solving the following optimization problem:

$$
\begin{array}{ll}
\min & \gamma  \tag{36}\\
\text { s.t. } & (27),(28) .
\end{array}
$$

## 4. Numerical Examples and Discussions

This section presents some examples to illustrate the effectiveness of the methods described above.

Example 9. Consider the neutral system

$$
\begin{gather*}
\dot{x}(t)-C_{d} \dot{x}(t-\tau)=A x(t)+A_{d} x(t-d(t))+B w(t), \\
z(t)=C x(t)+D w(t) \tag{37}
\end{gather*}
$$

with

$$
\begin{gather*}
A=\left[\begin{array}{cc}
-1.7 & 0 \\
0 & -0.9
\end{array}\right], \quad B=\left[\begin{array}{c}
0.5 \\
1
\end{array}\right] \\
A_{d}=\left[\begin{array}{cc}
-1 & 0 \\
-1 & -1
\end{array}\right], \quad C_{d}=\left[\begin{array}{cc}
0.1 & 0 \\
0 & 0.1
\end{array}\right]  \tag{38}\\
C=\left[\begin{array}{ll}
1 & 0
\end{array}\right], \quad D=0.1
\end{gather*}
$$

By Theorem 2, the $H_{\infty}$ index $\gamma$ is listed in Table 1 for various values of $\mu$ and $h$. It is clear that our results presented in this paper are feasible.

Example 10. Consider the neutral system

$$
\begin{gather*}
\dot{x}(t)-C_{d} \dot{x}(t-\tau)=A x(t)+A_{d} x(t-d(t)) \\
 \tag{39}\\
+B w(t)+B_{u} u(t), \\
z(t)=C x(t)+C_{u} u(t)+D w(t),
\end{gather*}
$$

with

$$
\begin{gather*}
A=\left[\begin{array}{cc}
-1 & -0.2 \\
0 & -1
\end{array}\right], \quad A_{d}=\left[\begin{array}{cc}
1 & 0 \\
0 & 1.2
\end{array}\right], \\
C_{d}=\left[\begin{array}{cc}
0.2 & 0 \\
0 & 0.1
\end{array}\right], \quad B_{u}=\left[\begin{array}{l}
1 \\
1
\end{array}\right], \quad B=\left[\begin{array}{l}
1 \\
1
\end{array}\right],  \tag{40}\\
C=\left[\begin{array}{ll}
0 & 0.5
\end{array}\right], \quad C_{u}=0.5, \quad D=0.2 .
\end{gather*}
$$

TABLE 1: $H_{\infty}$ index $\gamma$ for various values of $\mu$ and $h$.

| $h$ | 0 | 0.5 | 1 | $1.97 / 1.57 / 1.31$ |
| :--- | :---: | :---: | :---: | :---: |
| $\mu=0$ | 0.2859 | 0.3205 | 0.4678 | 0.6693 |
| $\mu=0.5$ | 0.2859 | 0.3205 | 0.5158 | 0.7722 |
| $\mu=0.9$ | 0.2859 | 0.3205 | 0.6010 | 0.9340 |

For $\mu=0.5, h=0.8, x(t)=\left[\begin{array}{ll}1 & -1\end{array}\right]^{T}$, a minimum of $\gamma=$ 0.8 with a corresponding gain $K=\left[\begin{array}{ll}0.9646 & -12.1649\end{array}\right]$ was obtained with Theorem 5, which implies that the proposed method is effective and feasible.

## 5. Conclusions

In this contribution, the $H_{\infty}$ performance for linear neutral system with time-varying delay is discussed and a new bounded real lemma is presented by introducing a novel Lyapunov-Krasovskii functional. Based on the BRL, an approach to design memoryless state-feedback $H_{\infty}$ controller using LMI technique for linear neutral system with timevarying delay is proposed, which can be solved readily by using existing LMI optimization techniques. The numerical example simulation results demonstrate that the method is feasible and effective. Therefore, how to further reduce the conservatism constitutes is an important problem for future investigation.

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# A Simplified Descriptor System Approach to Delay-Dependent Stability and Robust Performance Analysis for Discrete-Time Systems with Time Delays 

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A simplified descriptor system approach is proposed for discrete-time systems with delays in terms of linear matrix inequalities. In comparison with the results obtained by combining the descriptor system approach with recently developed bounding technique, our approach can remove the redundant matrix variables while not reducing the conservatism. It is shown that the bounding technique is unnecessary in the derivation of our results. Via the proposed method, delay-dependent results on quadratic cost and $H_{\infty}$ performance analysis are also presented.

## 1. Introduction

In the past decades, considerable attention has been paid to the problems of stability analysis and control synthesis of time-delay systems. Many methodologies have been proposed, and a large number of results have been established (see, e.g., [1, 2] and the references therein). All these results can be generally divided into two categories: delayindependent stability conditions $[3,4]$ and delay-dependent stability conditions [5-11]. The delay-independent stability condition does not take the delay size into consideration and thus is often conservative especially for systems with small delays, while the delay-dependent stability condition makes fully use of the delay information and thus is less conservative than the delay-independent one. Very recently, in order to provide less conservative delay-dependent stability criteria, a descriptor system approach was proposed in [12, 13], while a new bounding technique has been presented in [14] (also called Moon's inequality). By combining the descriptor system approach with the bounding technique, novel delaydependent sufficient conditions for the existence of a memoryless feedback guaranteed cost controller are derived for a class of discrete-time systems with delays in $[6,7]$.

Although the descriptor system approach proposed in [12, 13] is powerful to deal with the stability analysis of time-delay
systems, there are too many matrix variables introduced. In [15], a simplified but equivalent descriptor system approach to delay-dependent stability analysis was established for the continuous-time systems with delays. It is shown in [15] that the bounding technique in [14] is not necessary when deriving the delay-dependent stability results. It should be pointed out that the result in [15] is only applicable to continuoustime systems with delays. In this paper, we focus our attention upon deriving a simplified descriptor system approach to delay-dependent stability analysis in the context of discretetime systems with delays. It is shown that the results derived by our approach are also equivalent to those obtained in $[6,7]$ but with fewer variables to be determined. It is also proved that, for discrete-time systems, the bounding technique in [14] will introduce some redundant variables and thus is unnecessary. Via the proposed method, delay-dependent results on quadratic cost and $H_{\infty}$ performance analysis are also presented. It is worth mentioning that through the approach proposed in this paper, the delay-dependent guaranteed cost control conditions in $[6,7]$ obtained by the descriptor system approach and the bounding technique can also be simplified.

Notations. Throughout this paper, for real symmetric matrices $X$ and $Y$, the notation $X \geq Y$ (resp., $X>Y$ ) means
that the matrix $X-Y$ is positive semidefinite (resp., positive definite). The superscript " $T$ " represents the transpose. $I$ is an identity matrix with appropriate dimension. diag (•) denotes a diagonal matrix. $l_{2}[0, \infty)$ refers to the space of square summable infinite vector sequences. In symmetric block matrices, we use an asterisk "*" to represent a term that is induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

## 2. Main Results

In order to introduce the simplified descriptor system approach, we consider the following discrete time-delay system

$$
\begin{align*}
\left(\Sigma_{a}\right): x(k+1) & =\sum_{i=0}^{2} A_{i} x\left(k-d_{i}(k)\right),  \tag{1}\\
x(k) & =\phi(k), \quad \forall k \in[-h, 0]
\end{align*}
$$

where $x(k) \in \mathbb{R}^{n}$ is the state, $d_{0}(k)=0, \phi(k)$ is the initial condition, the scalar $h>0$ is an upper bound on the time delays $d_{i}(k), i=1,2$, and $A_{i}, i=0,1,2$, are known real constant matrices.

Throughout this paper, we make the following assumption.

Assumption 1. $d_{i}(k)$ are unknown but satisfy for all $k \in \mathbb{Z}^{+}$

$$
\begin{equation*}
0<d_{i}(k) \leq d_{i}, \quad i=1,2 \tag{2}
\end{equation*}
$$

Now, we are in a position to present the main result of this paper.

Theorem 2. Under Assumption 1, the time-delay system $\left(\Sigma_{a}\right)$ is asymptotically stable for all $d_{i}(k), i=1,2$, satisfying (2) if there exist matrices $P>0, P_{i}, R_{i}, S_{i}$, and $L_{i}, i=1,2$, such that the following LMI holds:

$$
\left[\begin{array}{ccccc}
\Omega & G^{T} & {\left[\begin{array}{c}
0 \\
A_{1}
\end{array}\right]-L_{1}^{T}} & G^{T}\left[\begin{array}{c}
0 \\
A_{2}
\end{array}\right]-L_{2}^{T} & -d_{1} L_{1}^{T}  \tag{3}\\
* & -d_{2} L_{2}^{T} \\
* & * & 0 & 0 & 0 \\
* & * & -R_{2} & 0 & 0 \\
* & * & * & -d_{1} S_{1} & 0 \\
* & * & * & -d_{2} S_{2}
\end{array}\right]<0
$$

where

$$
\begin{aligned}
G= & {\left[\begin{array}{cc}
P & 0 \\
P_{1} & P_{2}
\end{array}\right] } \\
\Omega= & G^{T}\left[\begin{array}{cc}
0 & I \\
A_{0}-I & -I
\end{array}\right]+\left[\begin{array}{cc}
0 & I \\
A_{0}-I & -I
\end{array}\right]^{T} G+\sum_{i=1}^{2}\left[\begin{array}{c}
L_{i} \\
0
\end{array}\right] \\
& +\sum_{i=1}^{2}\left[\begin{array}{c}
L_{i} \\
0
\end{array}\right]^{T}+\left[\begin{array}{cc}
\sum_{i=1}^{2} R_{i} & 0 \\
0 & P+\sum_{i=1}^{2} d_{i} S_{i}
\end{array}\right]
\end{aligned}
$$

Proof. For all $d_{i}(k), i=1,2$, satisfying (2), it can be verified that (3) implies that
$\Theta(k)$
$:=\left[\begin{array}{ccccc}\Omega & G^{T}\left[\begin{array}{c}0 \\ A_{1}\end{array}\right]-L_{1}^{T} & G^{T}\left[\begin{array}{c}0 \\ A_{2}\end{array}\right]-L_{2}^{T} & -d_{1}(k) L_{1}^{T} & -d_{2}(k) L_{2}^{T} \\ * & -R_{1} & 0 & 0 & 0 \\ * & * & -R_{2} & 0 & 0 \\ * & * & * & -d_{1}(k) S_{1} & 0 \\ * & * & * & * & -d_{2}(k) S_{2}\end{array}\right]$
$<0$.

Let

$$
\begin{equation*}
y(k)=x(k+1)-x(k) . \tag{6}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
x\left(k-d_{i}(k)\right)=x(k)-\sum_{l=k-d_{i}(k)}^{k-1} y(l) \tag{7}
\end{equation*}
$$

Then, the system $\left(\Sigma_{a}\right)$ can be transformed into an equivalent descriptor form

$$
\begin{gather*}
x(k+1)=x(k)+y(k), \\
0=-y(k)+\left(\sum_{i=0}^{2} A_{i}-I\right) x(k)-\sum_{i=1}^{2} A_{i}\left(\sum_{l=k-d_{i}(k)}^{k-1} y(l)\right) . \tag{8}
\end{gather*}
$$

Now, choose a Lyapunov functional candidate as

$$
\begin{equation*}
V(k)=V_{1}(k)+V_{2}(k)+V_{3}(k) \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& V_{1}(k)=x^{T}(k) P x(k), \\
& V_{2}(k)=\sum_{i=1}^{2}\left\{\sum_{l=k-d_{i}(k)}^{k-1} x^{T}(l) R_{i} x(l)\right\},  \tag{10}\\
& V_{3}(k)=\sum_{i=1}^{2}\left\{\sum_{\theta=-d_{i}(k)+1}^{0} \sum_{l=k-1+\theta}^{k-1} y^{T}(l) S_{i} y(l)\right\} .
\end{align*}
$$

## Then,

$$
\begin{aligned}
& \Delta V_{1}(k) \\
& =V_{1}(k+1)-V_{1}(k) \\
& =2 x^{T}(k) P y(k)+y^{T}(k) P y(k) \\
& =2 \bar{x}^{T}(k) G^{T}\left[\begin{array}{c}
y(k) \\
0
\end{array}\right]+y^{T}(k) P y(k) \\
& =2 \bar{x}^{T}(k) G^{T} \\
& \times\left[-y(k)+\left(\sum_{i=0}^{2} A_{i}-I\right) x(k)-\sum_{i=1}^{2} A_{i}\left(\sum_{l=k-d_{i}(k)}^{k-1} y(l)\right)\right] \\
& +y^{T}(k) P y(k) \\
& =2 \bar{x}^{T}(k) G^{T}\left[\begin{array}{ccc}
0 & I \\
\sum_{i=0}^{2} A_{i}-I & -I
\end{array}\right] \bar{x}(k) \\
& +\bar{x}^{T}(k)\left[\begin{array}{ll}
0 & 0 \\
0 & P
\end{array}\right] \bar{x}(k) \\
& -2 \bar{x}^{T}(k) \sum_{i=1}^{2} G^{T}\left[\begin{array}{c}
0 \\
A_{i}
\end{array}\right] \sum_{l=k-d_{i}(k)}^{k-1} y(l) \\
& =2 \bar{x}^{T}(k) G^{T}\left[\begin{array}{ccc}
0 & I \\
\sum_{i=0}^{2} A_{i}-I & -I
\end{array}\right] \bar{x}(k) \\
& +\bar{x}^{T}(k)\left[\begin{array}{ll}
0 & 0 \\
0 & P
\end{array}\right] \bar{x}(k) \\
& +2 \bar{x}^{T}(k) \sum_{i=1}^{2}\left(L_{i}^{T}-G^{T}\left[\begin{array}{c}
0 \\
A_{i}
\end{array}\right]\right) \sum_{l=k-d_{i}(k)}^{k-1} y(l) \\
& -2 \bar{x}^{T}(k) \sum_{i=1}^{2} L_{i}^{T} \sum_{l=k-d_{i}(k)}^{k-1} y(l) \\
& =2 \bar{x}^{T}(k) G^{T}\left[\begin{array}{cc}
0 & I \\
\sum_{i=0}^{2} A_{i}-I & -I
\end{array}\right] \bar{x}(k)+\bar{x}^{T}(k)\left[\begin{array}{ll}
0 & 0 \\
0 & P
\end{array}\right] \bar{x}(k) \\
& +2 \bar{x}^{T}(k) \sum_{i=1}^{2}\left(L_{i}^{T}-G^{T}\left[\begin{array}{c}
0 \\
A_{i}
\end{array}\right]\right)\left[x(k)-x\left(k-d_{i}(k)\right)\right] \\
& -2 \bar{x}^{T}(k) \sum_{i=1}^{2} L_{i}^{T} \sum_{l=k-d_{i}(k)}^{k-1} y(l) \\
& =2 \bar{x}^{T}(k)\left(G^{T}\left[\begin{array}{cc}
0 & I \\
A_{0}-I & -I
\end{array}\right]+\sum_{i=1}^{2}\left[\begin{array}{c}
L_{i} \\
0
\end{array}\right]^{T}\right) \bar{x}(k) \\
& +\bar{x}^{T}(k)\left[\begin{array}{ll}
0 & 0 \\
0 & P
\end{array}\right] \bar{x}(k)
\end{aligned}
$$

$$
\begin{align*}
& +2 \bar{x}^{T}(k) \sum_{i=1}^{2}\left(G^{T}\left[\begin{array}{c}
0 \\
A_{i}
\end{array}\right]-L_{i}^{T}\right) x\left(k-d_{i}(k)\right) \\
& -2 \bar{x}^{T}(k) \sum_{i=1}^{2} L_{i}^{T} \sum_{l=k-d_{i}(k)}^{k-1} y(l) \tag{11}
\end{align*}
$$

where $\bar{x}(k)=\left[\begin{array}{ll}x^{T}(k) & y^{T}(k)\end{array}\right]^{T}$.
Furthermore, from (11), we obtain

$$
\begin{align*}
& \Delta V_{1}(k) \\
& =\frac{1}{d_{1}(k) d_{2}(k)} \\
& \quad \times \sum_{\alpha_{2}=k-d_{2}(k)}^{k-1} \sum_{\alpha_{1}=k-d_{1}(k)}^{k-1}\left[2 \bar{x}^{T}(k)\right. \\
& \\
& \\
& \\
& \\
& \\
&  \tag{12}\\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
& \\
&
\end{align*}
$$

After some manipulations, we get

$$
\begin{aligned}
\Delta V_{2}(k) & +\Delta V_{3}(k) \\
\leq \sum_{i=1}^{2}[ & x^{T}(k) R_{i} x(k)+d_{i} y^{T}(k) S_{i} y(k) \\
& -x^{T}\left(k-d_{i}(k)\right) R_{i} x\left(k-d_{i}(k)\right) \\
& \left.\quad-\sum_{l=k-d_{i}(k)}^{k-1} y^{T}(l) S_{i} y(l)\right] \\
= & \frac{1}{d_{1}(k) d_{2}(k)}
\end{aligned}
$$

$$
\times \sum_{\alpha_{2}=k-d_{2}(k)}^{k-1} \sum_{\alpha_{1}=k-d_{1}(k)}^{k-1}\left\{\bar{x}^{T}(k)\right.
$$

$$
\times\left[\begin{array}{cc}
\sum_{i=1}^{2} R_{i} & 0 \\
0 & \sum_{i=1}^{2} d_{i} S_{i}
\end{array}\right] \bar{x}(k)
$$

$$
-d_{2}(k) y^{T}\left(\alpha_{2}\right) S_{2} y\left(\alpha_{2}\right)
$$

$$
-d_{1}(k) y^{T}\left(\alpha_{1}\right) S_{1} y\left(\alpha_{1}\right)
$$

$$
-\sum_{i=1}^{2} x^{T}\left(k-d_{i}(k)\right)
$$

$$
\begin{equation*}
\left.\times R_{i} x\left(k-d_{i}(k)\right)\right\} \tag{13}
\end{equation*}
$$

Combining (12) with (13) yields
$\Delta V(k)$

$$
\begin{align*}
\leq & \frac{1}{d_{1}(k) d_{2}(k)} \\
& \times \sum_{\alpha_{2}=k-d_{2}(k)}^{k-1} \sum_{\alpha_{1}=k-d_{1}(k)}^{k-1} \eta^{T}\left(k, \alpha_{1}, \alpha_{2}\right) \Theta(k) \eta\left(k, \alpha_{1}, \alpha_{2}\right) \tag{14}
\end{align*}
$$

where $\Theta(k)$ is given in (5) and

$$
\begin{align*}
& \eta\left(k, \alpha_{1}, \alpha_{2}\right) \\
& =\left[\bar{x}^{T}(k) x^{T}\left(k-d_{1}(k)\right) x^{T}\left(k-d_{2}(k)\right) y^{T}\left(\alpha_{1}\right) y^{T}\left(\alpha_{2}\right)\right]^{T} \tag{15}
\end{align*}
$$

Therefore, the time-delay system $\left(\Sigma_{a}\right)$ is asymptotically stable for all $d_{i}(k), i=1,2$, satisfying (2) by the Lyapunov stability theory. This completes the proof.

Remark 3. It is noted that only two time delays are considered for the sake of simplicity. However, the results in Theorem 2 can be extended to the case of multiple delays. The simplified approach in Theorem 2 can also be used to tackle with the discrete time-delay systems with uncertainties, such as normbounded parameter uncertainties and linear fractional uncertainties.

Remark 4. Note that the delays considered here satisfy (2). From the proof of Theorem 2, the delay-dependent results in this paper can be extended to the case of interval delays (see

Table 1: the maximum delay bound of $d_{1}$.

| References | $[18]$ | $[17]$ | $[6]$ | Theorem 2 |
| :--- | :---: | :---: | :---: | :---: |
| $d_{1}$ | - | 12 | 16 | 18 |

[16] for more details), where the delays vary between a lower bound (may be not zero) and an upper bound.

By the method proposed in Theorem 2, the quadratic cost analysis result derived by using the descriptor system approach, together with the inequality in [14] as shown in $[6,7]$, can also be simplified. To make it clear, introduce the following quadratic cost function

$$
\begin{equation*}
J=\sum_{k=0}^{\infty} x^{T}(k) Q x(k) . \tag{16}
\end{equation*}
$$

Then, by Theorem 2, we have the following result.
Theorem 5. There exist matrices $P>0, P_{i}, R_{i}, S_{i}$, and $L_{i}, i=$ 1,2 , such that the following LMI holds:

$$
\left[\begin{array}{ccccc}
\Omega_{1} & G^{T} & {\left[\begin{array}{c}
0 \\
A_{1}
\end{array}\right]-L_{1}^{T}} & G^{T} & {\left[\begin{array}{c}
0 \\
A_{2}
\end{array}\right]-L_{2}^{T}}  \tag{17}\\
* & -d_{1} L_{1}^{T} & -d_{2} L_{2}^{T} \\
* & * & 0 & 0 & 0 \\
* & * & -R_{2} & 0 & 0 \\
* & * & * & -d_{1} S_{1} & 0 \\
* & * & * & -d_{2} S_{2}
\end{array}\right]<0
$$

where $\Omega_{1}=\Omega+\operatorname{diag}(Q, 0)$, with $G$ and $\Omega$ being defined in (4), then the system $\left(\Sigma_{a}\right)$ is asymptotically stable, and the cost function in (16) satisfies

$$
\begin{align*}
J \leq & J_{0} \\
= & x^{T}(0) P x(0) \\
& +\sum_{i=1}^{2}\left\{\sum_{l=-d_{i}}^{-1} x^{T}(l) R_{i} x(l)+\sum_{\theta=-d_{i}+1}^{0} \sum_{l=-1+\theta}^{-1} y^{T}(l) S_{i} y(l)\right\}, \tag{18}
\end{align*}
$$

where $y(l)=x(l+1)-x(l)$.
In the next, via the method proposed in Theorem 2, we will present the $H_{\infty}$ performance analysis result.

Consider the following time-delay system:

$$
\begin{align*}
\left(\Sigma_{b}\right): x(k+1) & =\sum_{i=0}^{2} A_{i} x\left(k-d_{i}(k)\right)+B \omega(k),  \tag{19}\\
z(k) & =C x(k)+D \omega(k),
\end{align*}
$$

where $z(k) \in \mathbb{R}^{p}$ is the output and $\omega(k) \in \mathbb{R}^{q}$ is the disturbance signal which is assumed to be in $l_{2}[0, \infty)$.

Then, the following delay-dependent result on $H_{\infty}$ performance analysis can be obtained by Theorem 2.

Theorem 6. Given a scalar $\gamma>0$. Then, under Assumption 1, the time-delay system $\left(\Sigma_{b}\right)$ :
(i) is asymptotically stable with $\omega(k)=0$,
(ii) satisfies

$$
\begin{equation*}
\|z\|_{2}<\gamma\|\omega\|_{2}, \tag{20}
\end{equation*}
$$

under zero-initial condition for all nonzero $\omega \in l_{2}[0, \infty)$ if there exist matrices $P>0, P_{i}, R_{i}>0, S_{i}>0, L_{i}, i=1,2$, such that the following LMI holds:

$$
\left[\begin{array} { c c c c c c } 
{ \Omega _ { 2 } } & { G ^ { T } [ \begin{array} { c } 
{ 0 } \\
{ A _ { 1 } }
\end{array} ] - L _ { 1 } ^ { T } } & { G ^ { T } [ \begin{array} { c } 
{ 0 } \\
{ A _ { 2 } }
\end{array} ] - L _ { 2 } ^ { T } } & { - d _ { 1 } L _ { 1 } ^ { T } } & { - d _ { 2 } L _ { 2 } ^ { T } } & { G ^ { T } [ \begin{array} { c } 
{ 0 } \\
{ B }
\end{array} ] }
\end{array} [ \begin{array} { c } 
{ C ^ { T } }  \tag{21}\\
{ 0 }
\end{array} ] [ \begin{array} { c } 
{ * }
\end{array} ] \left[\begin{array}{l}
0 \\
*
\end{array}\right.\right.
$$

where $\Omega_{2}=\Omega+\operatorname{diag}\left(C^{T} C, 0\right)$, with $G$ and $\Omega$ being defined in (4).

## 3. A Numerical Example

In this section, we present a numerical example to the effectiveness of the proposed algorithm. In order to show the comparison, we choose $A_{2}=0$ and $d_{2}=0$.

Example 7. Consider the system $\left(\Sigma_{a}\right)$ with

$$
A_{0}=\left[\begin{array}{cc}
0.8 & 0  \tag{22}\\
0 & 0.97
\end{array}\right], \quad A_{1}=\left[\begin{array}{cc}
-0.1 & 0 \\
-0.1 & -0.1
\end{array}\right] .
$$

Based on Theorem 2, we seek the maximum value of $d_{1}$. Compared with three methods, which are in [6, 17, 18], respectively; we can illustrate the advantage of the proposed algorithm in this paper. Table 1 presents the result of comparison.

## 4. Conclusions

In this paper, we have proposed a simplified delay-dependent stability condition for discrete-time systems with delays. The given condition has fewer variables compared with those established using the descriptor system approach with Moon's bounding technique. It has been shown that Moon's bounding technique is unnecessary when deriving the delay-dependent stability conditions. By the proposed method in this paper, the delay-dependent results on quadratic cost and $H_{\infty}$ performance analysis have also been provided.

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## Research Article

# Stability and $l_{1}$-Gain Analysis for Positive 2D Systems with State Delays in the Roesser Model 

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#### Abstract

This paper considers the problem of delay-dependent stability and $l_{1}$-gain analysis for positive 2D systems with state delays described by the Roesser model. Firstly, the copositive-type Lyapunov function method is used to establish the sufficient conditions for the addressed positive 2 D system to be asymptotically stable. Then, $l_{1}$-gain performance for the system is also analyzed. All the obtained results are formulated in the form of linear matrix inequalities (LMIs) which are computationally tractable. Finally, an illustrative example is given to verify the effectiveness of the proposed results.


## 1. Introduction

2D systems exist in many practical applications, such as circuits analysis, digital image processing, signal filtering, and thermal power engineering [1-4]. Thus the analysis and synthesis of 2D systems are interesting and challenging problems, and they have received considerable attention; for example, 2D state-space realization theory was researched in [5], the stability and 2D optimal control theory was studied in $[6,7]$, and $H_{\infty}$ control and filtering problem for 2D systems were addressed in [8-11]. In addition, linear repetitive processes, a distinct class of 2D systems, have also been investigated. For example, the quasi-sliding mode control problem for linear repetitive processes with unknown input disturbance was solved in [12].

The most popular models of two-dimensional (2D) linear systems were introduced by Roesser [13], Fornasini and Marchesini [5, 14], and Kurek [15]. These models have been extended to positive systems in [16-19]. A positive system means that its state and output are nonnegative whenever the initial condition and input are nonnegative [19-21]. Positive 2 D systems are needed in many cases such as the wave equation in fluid dynamics and the heat equation which describes the temperature (using thermodynamic temperature scale) in a given region over time and the Poisson's
equation. These facts stimulate the research on 2D positive discrete systems. Reference [22] investigated the choice of the forms of Lyapunov functions for positive 2D Roesser model. The problem of stability analysis for 2D positive systems has been investigated in [17, 23-25]. It should be noted that although positive 2D systems have been discussed in control engineering and mathematics literature recently, there are still many questions which deserve further investigation.

On the other hand, the reaction of real-world systems to exogenous signals is never instantaneous and, always infected by certain time delays. For general systems, even nominal stable systems when were affected by delays may inherit very complex behaviors such as oscillations, instability, and bad performance [26], and delayed systems have attracted many researchers' attention [27-32]. The reachability, minimum energy control, and realization problem for positive 2D discrete-time systems with delays has been analyzed in [18, 33]. And the stability analysis for 2D positive delayed systems has been investigated in [34-36]. In addition, perturbations and uncertainties widely exist in the practical systems. In some cases, the perturbations and unmodeled errors can be merged into disturbances, which can be supposed to be bounded in the appropriate norms. It is important and necessary to establish a criterion evaluating the disturbance attenuation performance for the positive 2D discrete-time
systems. However, to the best of our knowledge, there has been no literature considering the disturbance attenuation performance for positive 2D systems, which motivates the present study.

In this paper, we will study the problem of delaydependent stability and $l_{1}$-gain analysis for positive 2 D linear systems with delays. The main theoretical contributions of this paper are as follows (1) We use $l_{1}$-gain to evaluate the disturbance attenuation performance of positive 2D linear systems. This important performance is firstly considered for positive 2D systems, and a delay-dependent stability criterion of these systems with state delays is developed. (2) Copositive-type Lyapunov function method is firstly used to analyze delay-dependent stability and $l_{1}$-gain performance for positive 2D linear systems. (3) It is significant to characterize conditions under which the positive 2D delayed system is asymptotically stable. All the developed results are expressed in terms of feasibility testing of LMIs which is computationally tractable.

The paper is organized as follows. In Section 2, problem statement and some definitions concerning the positive 2 D linear systems with delays are given. In Section 3, some theorems concerning the delay-dependent stability and $l_{1}-$ gain analysis of positive 2D linear systems are presented. In Section 4, a numerical example is given to illustrate the effectiveness of the proposed results. Finally, concluding remarks are provided in Section 5.

Notations. In this paper, the superscript " $T$ " denotes the transpose. The notation $X>Y(X \geq Y)$ means that matrix $X-Y$ is positive definite (positive semidefinite, resp.). $A \succeq$ $0(\preceq 0)$ means that all entries of matrix $A$ are nonnegative (nonpositive). $A \succ 0(<0)$ means that all entries of matrix $A$ are positive (negative). $R^{n \times m}$ denotes the set of $n \times m$ real matrices. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $R_{+}^{n \times m}, R_{+}^{n}$ denotes the set of vectors with nonnegative entries, and the set of nonnegative integers will be denoted by $Z_{+}$. The $n \times n$ identity matrix will be denoted by $I_{n}$. The $l_{1}$ norm of a 2D signal $w(i, j)=$ $\left[w_{1}(i, j), w_{2}(i, j), \ldots, w_{m}(i, j)\right]^{T}$ is given by

$$
\begin{equation*}
\|w(i, j)\|_{1}=\sum_{k=1}^{m} w_{k}(i, j) \tag{1}
\end{equation*}
$$

And we say $w(i, j) \in l_{1}$, if $\|w(i, j)\|_{1}<\infty$.

## 2. Problem Formulation and Preliminaries

Consider the positive 2D Roesser model with state delays [25]:

$$
\begin{align*}
{\left[\begin{array}{l}
x^{h}(i+1, j) \\
x^{v}(i, j+1)
\end{array}\right]=} & A\left[\begin{array}{l}
x^{h}(i, j) \\
x^{v}(i, j)
\end{array}\right]+A_{d}\left[\begin{array}{l}
x^{h}\left(i-d_{h}(i), j\right) \\
x^{v}\left(i, j-d_{v}(j)\right)
\end{array}\right] \\
& +B w(i, j), \tag{1a}
\end{align*}
$$

$$
\begin{equation*}
z(i, j)=H x(i, j)+L w(i, j) \tag{1b}
\end{equation*}
$$

where $i$ and $j$ are integers in $Z_{+}, x^{h}(i, j)$ is the horizontal state in $R_{+}^{n_{1}}, x^{v}(i, j)$ is the vertical state in $R_{+}^{n_{2}}, x(i, j)$ is the whole state in $R_{+}^{n}, w(i, j) \in R_{+}^{m_{1}}$ is the $l_{1}$ norm bounded disturbance input, $z(i, j) \in R_{+}^{m_{2}}$ is the controlled output, and $A, A_{d}, B, H$, $L \succeq 0$ are system matrices with compatible dimensions. The matrices are

$$
A=\left[\begin{array}{ll}
A_{11} & A_{12}  \tag{2}\\
A_{21} & A_{22}
\end{array}\right], \quad A_{d}=\left[\begin{array}{ll}
A_{d 11} & A_{d 12} \\
A_{d 21} & A_{d 22}
\end{array}\right], \quad B=\left[\begin{array}{l}
B_{1} \\
B_{2}
\end{array}\right] .
$$

$d_{h}(i)$ and $d_{v}(j)$ are delays along horizontal and vertical directions, respectively. We assume that $d_{h}(i)$ and $d_{v}(j)$ satisfy

$$
\begin{equation*}
d_{h L} \leq d_{h}(i) \leq d_{h H}, \quad d_{v L} \leq d_{v}(j) \leq d_{v H} \tag{3}
\end{equation*}
$$

where $d_{h L}, d_{h H}$ and $d_{v L}, d_{v H}$ denote the lower and upper delay bounds along horizontal and vertical directions, respectively. The boundary conditions are defined by

$$
\begin{gather*}
x^{h}(i, j)=h_{i j}, \quad \forall 0 \leq j \leq z_{1},-d_{h H} \leq i \leq 0 \\
x^{h}(i, j)=0, \quad \forall j>z_{1},-d_{h H} \leq i \leq 0 \\
x^{v}(i, j)=v_{i j}, \quad \forall 0 \leq i \leq z_{2},-d_{v H} \leq j \leq 0  \tag{4}\\
x^{v}(i, j)=0, \quad \forall i>z_{2},-d_{v H} \leq j \leq 0 \\
h_{00}=v_{00}
\end{gather*}
$$

where $z_{1}<\infty$ and $z_{2}<\infty$ are positive integers, $h_{i j} \in R_{+}^{n_{1}}$ and $v_{i j} \in R_{+}^{n_{2}}$ are given vectors.

Definition 1. The 2D positive system (1a) and (1b) with $w(i, j)=0$ is said to be asymptotically stable if $\lim _{l \rightarrow \infty} X_{l}=0$ for all bounded boundary conditions (4), where

$$
\begin{equation*}
X_{l}=\sup \{\|x(i, j)\|: i+j=l, i, j \geq 1\} \tag{5}
\end{equation*}
$$

Definition 2. For $\gamma>0$, the system (1a) and (1b) is said to be asymptotically stable with the $l_{1}$-gain index $\gamma$, if the following conditions hold.
(1) The system (1a) and (1b) with $w(i, j)=0$ is asymptotically stable.
(2) Under zero boundary conditions, that is, $h_{i j}=0, v_{i j}=$ 0 in (4), it holds that

$$
\begin{equation*}
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|z(i, j)\|_{1}<\gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|w(i, j)\|_{1}, \quad \forall 0 \neq w(i, j) \in l_{1} \tag{6}
\end{equation*}
$$

Remark 3. From (6), we see that $\gamma$ can characterize the disturbance attenuation performance of the system (1a) and (1b). The smaller the $\gamma$ is, the better the disturbance attenuation performance is.

## 3. Main Results

3.1. Stability Analysis. In this subsection, we focus on the problem of delay-dependent asymptotically stability analysis for the positive 2D discrete linear systems with state delays.

Theorem 4. For given positive constants $d_{h L}, d_{h H}, d_{v L}, d_{v H}$, the positive $2 D$ system ( $1 a$ ) and ( $1 b$ ) with $w(i, j)=0$ is asymptotically stable if there exist vectors $p, q, \varsigma_{1}, \varsigma_{2}, \zeta \in R_{+}^{n}$, such that

$$
\begin{gather*}
\Phi=\operatorname{diag}\left\{\Phi_{1}, \Phi_{2}, \ldots, \Phi_{n}, \Phi_{1}^{\prime}, \Phi_{2}^{\prime}, \ldots, \Phi_{n}^{\prime}, \Phi_{1}^{\prime \prime}, \Phi_{2}^{\prime \prime}, \ldots, \Phi_{n}^{\prime \prime}\right. \\
\left.\Phi_{1}^{\prime \prime \prime}, \Phi_{2}^{\prime \prime \prime}, \ldots, \Phi_{n}^{\prime \prime \prime}\right\}<0 \tag{7}
\end{gather*}
$$

where

$$
\begin{align*}
& \Phi_{k}=\left\{\begin{array}{l}
\left(\begin{array}{l}
\left.a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{h H}-d_{h L}\right) E_{k}\right) q \\
+E_{k} \zeta+\left(d_{h H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{h H} E_{k}\right) \varsigma_{2} \quad 1 \leq k \leq n_{1}, \\
+d_{h H}^{2} E_{k} \varsigma_{1}, \\
\left(a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{v H}-d_{v L}\right) E_{k}\right) q \\
+E_{k} \zeta+\left(d_{v H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{v H} E_{k}\right) \varsigma_{2} \\
+n_{1}+1 \leq k \leq n,
\end{array}\right. \\
\Phi_{v H}^{\prime}= \begin{cases}E_{k} \varsigma_{1}, \\
a_{d k}^{T} p+\left(a_{d k}^{T}-E_{k}\right) q+d_{h H}^{2} a_{d k}^{T} \varsigma_{2}, & 1 \leq k \leq n_{1}, \\
a_{d k}^{T} p+\left(a_{d k}^{T}-E_{k}\right) q+d_{v H}^{2} a_{d k}^{T} \varsigma_{2}, & n_{1}+1 \leq k \leq n,\end{cases} \\
\Phi_{k}^{\prime \prime}= \begin{cases}-E_{k} \zeta+d_{h H} E_{k}\left(\varsigma_{2}-\varsigma_{1}\right), & 1 \leq k \leq n_{1}, \\
-E_{k} \zeta+d_{v H} E_{k}\left(\varsigma_{2}-\varsigma_{1}\right), & n_{1}+1 \leq k \leq n,\end{cases} \\
\Phi_{k}^{\prime \prime \prime}= \begin{cases}-d_{h H} E_{k} \varsigma_{1}, & 1 \leq k \leq n_{1}, \\
-d_{v H} E_{k} \varsigma_{1}, & n_{1}+1 \leq k \leq n,\end{cases}
\end{array}\right.
\end{align*}
$$

with $k \in \underline{n}=\{1,2, \ldots, n\}, E_{k}=[\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k}$, and $a_{k}\left(a_{d k}\right)$ represents the $k$ th column vector of matrix $A\left(A_{d}\right)$.

Proof. Choose the following copositive Lyapunov-Krasovskii functional candidate:

$$
\begin{equation*}
V(i, j)=V^{h}(i, j)+V^{v}(i, j) \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& V^{h}(i, j)=\sum_{k=1}^{5} V_{k}^{h}(i, j), \\
& V_{1}^{h}(i, j)=x^{h T}(i, j) p^{h}, \\
& V_{2}^{h}(i, j)=\sum_{r=i-d_{h}(i)}^{i} x^{h T}(r, j) q^{h}, \\
& V_{3}^{h}(i, j)=\sum_{r=i-d_{h H}}^{i-1} x^{h T}(r, j) \zeta^{h}, \\
& V_{4}^{h}(i, j)=\sum_{s=-d_{h H^{+}}}^{-d_{h L}} \sum_{r=i+s}^{i-1} x^{h T}(r, j) q^{h}, \\
& V_{5}^{h}(i, j)=d_{h H} \sum_{s=-d_{h H}}^{-1} \sum_{r=i+s}^{i-1} \eta^{h T}(r, j) \varsigma^{h}, \\
& V^{v}(i, j)=\sum_{k=1}^{5} V_{k}^{v}(i, j), \\
& V_{1}^{v}(i, j)=x^{v T}(i, j) p^{v},  \tag{10}\\
& V_{2}^{v}(i, j)=\sum_{s=j-d_{v}(j)}^{j} x^{v T}(i, s) q^{v}, \\
& V_{3}^{v}(i, j)=\sum_{t=j-d_{v H}}^{j-1} x^{v T}(i, t) \zeta^{v}, \\
& V_{4}^{v}(i, j)=\sum_{s=-d_{v H}+1}^{-d_{v L}} \sum_{t=j+s}^{j-1} x^{v T}(i, t) q^{v}, \\
& V_{5}^{v}(i, j)=d_{v H} \sum_{s=-d v_{H}}^{-1} \sum_{t=j+s}^{j-1} \eta^{v T}(i, t) \varsigma^{v}, \\
& \eta^{h}(r, j)=\left[\begin{array}{ll}
x^{h T}(r, j) & \delta^{h T}(r, j)
\end{array}\right]^{T}, \\
& \eta^{v}(i, t)=\left[\begin{array}{ll}
x^{v T}(i, t) & \delta^{v T}(i, t)
\end{array}\right]^{T}, \\
& \delta^{h}(r, j)=x^{h}(r+1, j)-x^{h}(r, j) \text {, } \\
& \delta^{v}(i, t)=x^{v}(i, t+1)-x^{v}(i, t),
\end{align*}
$$

with $p^{h}, q^{h}, \zeta^{h}, \varsigma_{1}^{h}$, and $\varsigma_{2}^{h} \in R_{+}^{n_{1}}, p^{v}, q^{v}, \zeta^{v}, \varsigma_{1}^{v}$, and $\varsigma_{2}^{v} \in R_{+}^{n_{2}}$, $\varsigma^{h}=\left[\begin{array}{ll}\varsigma_{1}^{h} & \varsigma_{2}^{h}\end{array}\right]^{T} \in R_{+}^{2 n_{1}}$, and $\varsigma^{v}=\left[\begin{array}{ll}\varsigma_{1}^{v} & \varsigma_{2}^{v}\end{array}\right]^{T} \in R_{+}^{2 n_{2}}$.

Along the trajectory of the system (1a) and (1b), we have

$$
\begin{align*}
\Delta V(i, j) & =V^{h}(i+1, j)-V^{h}(i, j)+V^{v}(i, j+1)-V^{v}(i, j) \\
& =\sum_{k=1}^{5} \Delta V_{k}^{h}(i, j)+\sum_{k=1}^{5} \Delta V_{k}^{v}(i, j) \tag{11}
\end{align*}
$$

where

$$
\begin{aligned}
& \Delta V_{1}^{h}(i, j)=x^{h T}(i+1, j) p^{h}-x^{h T}(i, j) p^{h}, \\
& \Delta V_{2}^{h}(i, j)=\sum_{r=i+1-d_{h}(i+1)}^{i+1} x^{h T}(r, j) q^{h} \\
& -\sum_{r=i-d_{h}(i)}^{i} x^{h T}(r, j) q^{h} \\
& =x^{h T}(i+1, j) q^{h}-x^{h T}\left(i-d_{h}(i), j\right) q^{h} \\
& +\sum_{r=i+1-d_{h}(i+1)}^{i} x^{h T}(r, j) q^{h} \\
& -\sum_{r=i+1-d_{h}(i)}^{i} x^{h T}(r, j) q^{h} \\
& \leq x^{h T}(i+1, j) q^{h}-x^{h T}\left(i-d_{h}(i), j\right) q^{h} \\
& +\sum_{r=i+1-d_{h H}}^{i} x^{h T}(r, j) q^{h} \\
& -\sum_{r=i+1-d_{h L}}^{i} x^{h T}(r, j) q^{h} \\
& =x^{h T}(i+1, j) q^{h}-x^{h T}\left(i-d_{h}(i), j\right) q^{h} \\
& +\sum_{r=i+1-d_{h H}}^{r=i-d_{h L}} x^{h T}(r, j) q^{h}, \\
& \Delta V_{3}^{h}(i, j)=\sum_{r=i+1-d_{h H}}^{i} x^{h T}(r, j) \zeta^{h}-\sum_{r=i-d_{h H}}^{i-1} x^{h T}(r, j) \zeta^{h} \\
& =x^{h T}(i, j) \zeta^{h}-x^{h T}\left(i-d_{h H}, j\right) \zeta^{h}, \\
& \Delta V_{4}^{h}(i, j)=\sum_{s=-d_{h H}+1}^{-d_{h L}} \sum_{r=i+1+s}^{i} x^{h T}(r, j) q^{h} \\
& -\sum_{s=-d_{h H}+1}^{-d_{h L}} \sum_{r=i+s}^{i-1} x^{h T}(r, j) q^{h} \\
& =\sum_{s=-d_{h H^{+}}+}^{-d_{h L}}\left[x^{h T}(i, j) q^{h}-x^{h T}(i+s, j) q^{h}\right] \\
& =\left(d_{h H}-d_{h L}\right) x^{h T}(i, j) q^{h} \\
& -\sum_{r=i-d_{h H^{+}}}^{i-d_{h L}} x^{h T}(r, j) q^{h}, \\
& \Delta V_{5}^{h}(i, j) \\
& =d_{h H} \sum_{s=-d_{h H}}^{-1} \sum_{r=i+1+s}^{i} \eta^{h T}(r, j) \varsigma^{h}
\end{aligned}
$$

$$
\begin{aligned}
& -d_{h H} \sum_{s=-d_{h H}}^{-1} \sum_{r=i+s}^{i-1} \eta^{h T}(r, j) \varsigma^{h} \\
& =d_{h H} \sum_{s=-d_{h H}}^{-1}\left(\eta^{h T}(i, j) \varsigma^{h}-\eta^{h T}(i+s, j) \varsigma^{h}\right) \\
& =d_{h H}^{2} \eta^{h T}(i, j) \varsigma^{h}-d_{h H} \sum_{r=i-d_{h H}}^{i-1} \eta^{h T}(r, j) \varsigma^{h} \\
& =d_{h H}^{2}\left[x^{h T}(i, j) x^{h T}(i+1, j)-x^{h T}(i, j)\right]\left[\begin{array}{l}
\varsigma_{1}^{h} \\
\varsigma_{2}^{h}
\end{array}\right] \\
& -d_{h H}\left[\sum_{r=i-d_{h H}}^{i-1} x^{h T}(i, j) x^{h T}(i, j)-x^{h T}\left(i-d_{h H}, j\right)\right] \\
& \times\left[\begin{array}{l}
\zeta_{1}^{h} \\
\zeta_{2}^{h}
\end{array}\right], \\
& \Delta V_{1}^{v}(i, j)=x^{v T}(i, j+1) p^{v}-x^{v T}(i, j) p^{v}, \\
& \Delta V_{2}^{v}(i, j)=\sum_{s=j+1-d_{v}(j+1)}^{j+1} x^{v T}(i, s) q^{v} \\
& -\sum_{s=j-d_{v}(j)}^{j} x^{v T}(i, s) q^{v} \\
& =x^{v T}(i, j+1) q^{v}-x^{v T}\left(i, j-d_{v}(j)\right) q^{v} \\
& +\sum_{s=j+1-d_{v}(j+1)}^{j} x^{v T}(i, s) q^{v} \\
& -\sum_{s=j+1-d_{v}(j)}^{j} x^{v T}(i, s) q^{v} \\
& \leq x^{v T}(i, j+1) q^{v}-x^{v T}\left(i, j-d_{v}(j)\right) q^{v} \\
& +\sum_{s=j+1-d_{v H}}^{j} x^{v T}(i, s) q^{v} \\
& -\sum_{s=j+1-d_{v L}}^{j} x^{v T}(i, s) q^{v} \\
& =x^{v T}(i, j+1) q^{v}-x^{v T}\left(i, j-d_{v}(j)\right) q^{v} \\
& +\sum_{t=j+1-d_{v H}}^{j-d_{v L}} x^{v T}(i, t) q^{v},
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_{3}^{v}(i, j)= & \sum_{s=j+1-d_{v H}}^{j} x^{h T}(i, s) \zeta^{v} \\
& -\sum_{s=j-d_{v H}}^{j-1} x^{h T}(i, a) \zeta^{v} \\
= & x^{v T}(i, j) \zeta^{v}-x^{v T}\left(i, j-d_{v H}\right) \zeta^{v}, \\
\Delta V_{4}^{v}(i, j)= & \sum_{s=-d_{v H}+1}^{-d_{v L}} \sum_{t=j+1+s}^{j} x^{v T}(i, t) q^{v} \\
& -\sum_{s=-d_{v H}+1}^{-d_{v L}} \sum_{t=j+s}^{j-1} x^{v T}(i, t) q^{v} \\
= & \sum_{s=-d_{v H}+1}^{-d_{v L}}\left[x^{v T}(i, j) q^{v}-x^{v T}(i, j+s) q^{v}\right] \\
= & \left(d_{v H}-d_{v L}\right) x^{h T}(i, j) q^{v} \\
& -\sum_{t=j-d_{v H}+1}^{j-d_{v L}} x^{v T}(i, t) q^{v},
\end{aligned}
$$

$$
\Delta V_{5}^{v}(i, j)
$$

$$
=d_{v H} \sum_{s=-d v_{H}}^{-1} \sum_{t=j+1+s}^{j} \eta^{v T}(i, t) \varsigma^{v}
$$

$$
-d_{v H} \sum_{s=-d v_{H}}^{-1} \sum_{t=j+s}^{j-1} \eta^{v T}(i, t) \varsigma^{v}
$$

$$
=d_{v H} \sum_{s=-d_{v H}}^{-1}\left(\eta^{v T}(i, j) \varsigma^{v}-\eta^{v T}(i, j+s) \varsigma^{v}\right)
$$

$$
=d_{v H}^{2} \eta^{v T}(i, j) \varsigma^{v}-d_{v H} \sum_{t=j-d_{v H}}^{j-1} \eta^{v T}(i, t) \varsigma^{v}
$$

$$
=d_{v H}^{2}\left[x^{\nu T}(i, j) \quad x^{\nu T}(i, j+1)-x^{\nu T}(i, j)\right]
$$

$$
\times\left[\begin{array}{l}
\varsigma_{1}^{v} \\
\varsigma_{2}^{v}
\end{array}\right]
$$

$$
-d_{v H}\left[\sum_{t=j-d_{v H}}^{j-1} x^{v T}(i, j) x^{v T}(i, j)-x^{v T}\left(i, j-d_{v H}\right)\right]
$$

$$
\times\left[\begin{array}{c}
\varsigma_{1}^{v}  \tag{12}\\
\varsigma_{2}^{v}
\end{array}\right] .
$$

Substitute the previously mentioned formulations into (11), and take

$$
\left.\begin{array}{c}
p=\left[\begin{array}{l}
p^{h} \\
p^{v}
\end{array}\right], \quad q=\left[\begin{array}{l}
q^{h} \\
q^{v}
\end{array}\right], \quad \zeta=\left[\begin{array}{l}
\zeta^{h} \\
\zeta^{v}
\end{array}\right], \\
\varsigma_{1}=\left[\begin{array}{c}
\varsigma_{1}^{h} \\
\varsigma_{1}^{h}
\end{array}\right], \quad \varsigma_{2}=\left[\begin{array}{l}
\varsigma_{2}^{h} \\
\varsigma_{2}^{h}
\end{array}\right], \\
D_{H}=\left[\begin{array}{cc}
d_{h H} I_{n_{1}} & 0 \\
0 & d_{v H} I_{n_{2}}
\end{array}\right], \quad D_{L}=\left[\begin{array}{cc}
d_{h L} I_{n_{1}} & 0 \\
0 & d_{v L} I_{n_{2}}
\end{array}\right], \\
x(i, j)=\left[\begin{array}{ll}
x^{h T}(i, j) & x^{v T}(i, j)
\end{array}\right]^{T}, \\
x_{d}(i, j)=\left[\begin{array}{ll}
x^{h T}\left(i-d_{h}(i), j\right) & x^{v T}\left(i, j-d_{v}(j)\right)
\end{array}\right]^{T}, \\
x_{H}(i, j)=\left[\begin{array}{ll}
x^{h T}\left(i-d_{h H}, j\right) & x^{v T}\left(i, j-d_{v H}\right)
\end{array}\right]^{T}, \\
x_{s}(i, j)=\left[\begin{array}{c}
\sum_{r=i-d_{h H}+1}^{i-1} x^{h T}(r, j)
\end{array} \sum_{t=j-d_{v H+1}}^{j-1} x^{v T}(i, t)\right. \tag{13}
\end{array}\right]^{T} .
$$

Then we have

$$
\begin{align*}
\Delta V(i, j)= & x^{T}(i, j)\left\{\left(A^{T}-I_{n}\right) p+\left(A^{T}+D_{H}-D_{L}\right) q\right. \\
& \left.+\zeta+D_{H}^{2}\left(\left(A^{T}-I_{n}\right) \varsigma_{2}+\varsigma_{1}\right)-D_{H} \varsigma_{2}\right\} \\
& +x_{d}^{T}(i, j)\left\{A_{d}^{T} p+\left(A_{d}^{T}-I_{n}\right) q+D_{H}^{2} A_{d}^{T} \varsigma_{2}\right\} \\
& +x_{H}^{T}(i, j)\left\{-\zeta+D_{H}\left(\varsigma_{2}-\varsigma_{1}\right)\right\} \\
& +x_{s}^{T}(i, j)\left\{-D_{H} \varsigma_{1}\right\} . \tag{14}
\end{align*}
$$

If condition (7) holds, one obtains

$$
\begin{gather*}
\left(A^{T}-I_{n}\right) p+\left(A^{T}+D_{H}-D_{L}\right) q+\zeta \\
+D_{H}^{2}\left(\left(A^{T}-I_{n}\right) \varsigma_{2}+\varsigma_{1}\right)-D_{H} \varsigma_{2} \prec 0 \\
A_{d}^{T} p+\left(A_{d}^{T}-I_{n}\right) q+D_{H}^{2} A_{d}^{T} \varsigma_{2} \prec 0  \tag{15}\\
-\zeta+D_{H}\left(\varsigma_{2}-\varsigma_{1}\right) \prec 0 \\
\quad-D_{H} \varsigma_{1} \prec 0
\end{gather*}
$$

It follows that $\Delta V(i, j)<0$, which means that

$$
\begin{equation*}
V^{h}(i+1, j)+V^{v}(i, j+1)<V^{h}(i, j)+V^{v}(i, j) \tag{16}
\end{equation*}
$$

Summing up both sides of (16) from $D$ to 0 with respect to $i$ and from 0 to $D$ with respect to $j$, for any nonnegative integer
$D \geq \max \left(z_{1}, z_{2}\right)$, one gets

$$
\begin{align*}
V^{h}(1, D) & +V^{v}(0, D+1)+V^{h}(2, D-1) \\
& +V^{v}(1, D)+\cdots+V^{h}(D+1,0)+V^{v}(D, 1) \\
= & \sum_{i+j=D+1} V^{h}(i, j)+\sum_{i+j=D+1} V^{v}(i, j) \\
= & \sum_{i+j=D+1} V(i, j)  \tag{17}\\
\leq & V^{h}(0, D)+V^{v}(0, D)+V^{h}(1, D-1) \\
& +V^{v}(1, D-1)+\cdots+V^{h}(D, 0)+V^{v}(D, 0) \\
= & \sum_{i+j=D} V(i, j) .
\end{align*}
$$

Then from (9), we can conclude that

$$
\begin{equation*}
\lim _{i+j \rightarrow \infty} x(i, j)=0 \tag{18}
\end{equation*}
$$

which implies that the system (1a) and (1b) with $w(i, j)=0$ is asymptotically stable.

This completes the proof.
When $d_{h H}=d_{h L}=d_{h}$, and $d_{v H}=d_{v L}=d_{v}$, the system (1a) and (1b) with $w(i, j)=0$ is reduced to the following system:

$$
\left[\begin{array}{l}
x^{h}(i+1, j)  \tag{19}\\
x^{v}(i, j+1)
\end{array}\right]=A\left[\begin{array}{l}
x^{h}(i, j) \\
x^{v}(i, j)
\end{array}\right]+A_{d}\left[\begin{array}{l}
x^{h}\left(i-d_{h}, j\right) \\
x^{v}\left(i, j-d_{v}\right)
\end{array}\right],
$$

where $d_{h}$ and $d_{v}$ are constant delays along horizontal and vertical directions, respectively, and the boundary conditions are defined in (4). Then we can get the following result.

Corollary 5. For given positive constants $d_{h}$ and $d_{v}$, the positive $2 D$ system (19) is asymptotically stable if there exist vectors $p$ and $q \in R_{+}^{n}$, such that

$$
\begin{equation*}
\widetilde{\Phi}=\operatorname{diag}\left\{\Phi_{1}, \Phi_{2}, \ldots, \Phi_{n}, \Phi_{1}^{\prime}, \Phi_{2}^{\prime}, \ldots, \Phi_{n}^{\prime}\right\}<0 \tag{20}
\end{equation*}
$$

where

$$
\begin{align*}
& \widetilde{\Phi}_{k}=\left(a_{k}^{T}-E_{k}\right) p+a_{k}^{T} q \\
& \widetilde{\Phi}_{k}^{\prime}=a_{d k}^{T} p+\left(a_{d k}^{T}-E_{k}\right) q \tag{21}
\end{align*}
$$

with $k \in \underline{K}=\{1,2, \ldots, n\}, E_{k}=[\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k}], p$, $q \in R_{+}^{n}$, and $a_{k}\left(a_{d k}\right)$ represents the $k$ th column vector of matrix $A\left(A_{d}\right)$.

Proof. Choose the following copositive Lyapunov-Krasovskii functional candidate for the system (19):

$$
\begin{equation*}
V(i, j)=V^{h}(i, j)+V^{v}(i, j) \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
& V^{h}(i, j)=\sum_{k=1}^{2} V_{k}^{h}(i, j), \quad V_{1}^{h}(i, j)=x^{h T}(i, j) p^{h}, \\
& V_{2}^{h}(i, j)=\sum_{r=i-d_{h}}^{i} x^{h T}(r, j) q^{h}, \\
& V^{v}(i, j)=\sum_{k=1}^{2} V_{k}^{v}(i, j), \quad V_{1}^{v}(i, j)=x^{v T}(i, j) p^{v},  \tag{23}\\
& V_{2}^{v}(i, j)=\sum_{s=j-d_{v}}^{j} x^{v T}(i, s) q^{v},
\end{align*}
$$

with $p^{h}, q^{h} \in R_{+}^{n_{1}}, p^{v}, q^{v} \in R_{+}^{n_{2}}$. Then following the proof line of Theorem 4, the corollary can be obtained.
3.2. $l_{1}$-Gain Analysis. The following theorem establishes sufficient condition of the asymptotical stability with $l_{1}$-gain performance for the system (1a) and (1b).

Theorem 6. For given positive constants $d_{h L}, d_{h H}, d_{v L}, d_{v H}$, and $\gamma$, the positive $2 D$ system ( $1 a$ ) and ( $1 b$ ) is asymptotically stable with the $l_{1}$-gain index $\gamma$ if there exist vectors $p \in R_{+}^{n}$, $q \in R_{+}^{n}, \varsigma_{1} \in R_{+}^{n}, \varsigma_{2} \in R_{+}^{n}$, and $\zeta \in R_{+}^{n}$, such that

$$
\begin{gather*}
\bar{\Phi}=\operatorname{diag}\left\{\bar{\Phi}_{1}, \bar{\Phi}_{2}, \ldots, \bar{\Phi}_{n}, \Phi_{1}^{\prime}, \Phi_{2}^{\prime}, \ldots, \Phi_{n}^{\prime}, \Phi_{1}^{\prime \prime}, \Phi_{2}^{\prime \prime}, \ldots, \Phi_{n}^{\prime \prime},\right. \\
\left.\Phi_{1}^{\prime \prime \prime}, \Phi_{2}^{\prime \prime \prime}, \ldots, \Phi_{n}^{\prime \prime \prime}, \mathrm{T}_{1}, \mathrm{~T}_{2}, \ldots, \mathrm{~T}_{m}\right\}<0 \tag{24}
\end{gather*}
$$

where $\Phi_{k}^{\prime}$, $\Phi_{k}^{\prime \prime}$, and $\Phi_{k}^{\prime \prime \prime}$ are denoted as in Theorem 4, and

$$
\begin{align*}
& \bar{\Phi}_{k} \\
& = \begin{cases}\left(a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{h H}-d_{h L}\right) E_{k}\right) q+E_{k} \zeta \\
+\left(d_{h H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{h H} E_{k}\right) \varsigma_{2} & 1 \leq k \leq n_{1}, \\
+d_{h H}^{2} E_{k} \varsigma_{1}+\left\|h_{k}\right\|_{1}, \\
\left(a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{v H}-d_{v L}\right) E_{k}\right) q+E_{k} \zeta \\
+\left(d_{v H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{v H} E_{k}\right) \varsigma_{2} & n_{1}+1 \leq k \leq n, \\
+d_{v H}^{2} E_{k} \varsigma_{1}+\left\|h_{k}\right\|_{1},\end{cases}  \tag{25}\\
& T_{\varepsilon}=b_{\varepsilon}^{T} p+b_{\varepsilon}^{T} q+b_{\varepsilon}^{T} D_{h H}^{2} \varsigma_{2}+\left\|l_{\varepsilon}\right\|_{1}-\gamma, \\
& D_{H}=\operatorname{diag}\left\{d_{h H} I_{n_{1}}, d_{v H} I_{n_{2}}\right\},
\end{align*}
$$

with $k \in \underline{n}=\{1,2, \ldots, n\}, \varepsilon \in \underline{m}=\{1,2, \ldots, m\}$, $E_{k}=[\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k}], a_{k}, a_{d k}, b_{k}$, and $h_{k}$ represent the $k$ th column vector of matrices $A, A_{d}, B$, and $H$, respectively, and $l_{\varepsilon}$ represents the eth column vector of matrix $L$.

Proof. It is an obvious fact that (24) implies the following inequality:

$$
\begin{gather*}
\Psi=\operatorname{diag}\left\{\Psi_{1}, \Psi_{2}, \ldots, \Psi_{n}, \Psi_{1}^{\prime}, \Psi_{2}^{\prime}, \ldots, \Psi_{n}^{\prime}, \Psi_{1}^{\prime \prime}, \Psi_{2}^{\prime \prime}, \ldots,\right. \\
\left.\Psi_{n}^{\prime \prime}, \Psi_{1}^{\prime \prime \prime}, \Psi_{2}^{\prime \prime \prime}, \ldots, \Psi_{n}^{\prime \prime \prime}\right\}<0 \tag{26}
\end{gather*}
$$

where

$$
\left.\begin{array}{l}
\left(\begin{array}{l}
\left(a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{h H}-d_{h L}\right) E_{k}\right) q \\
+\zeta_{k}+\left(d_{h H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{h H} E_{k}\right) \varsigma_{2} \\
+d_{h H}^{2} E_{k} \varsigma_{1}, \\
\left(a_{k}^{T}-E_{k}\right) p+\left(a_{k}^{T}+\left(d_{v H}-d_{v L}\right) E_{k}\right) q \\
+\zeta_{k}+\left(d_{v H}^{2}\left(a_{k}^{T}-E_{k}\right)-d_{v H} E_{k}\right) \varsigma_{2}, \\
+d_{v H}^{2} E_{k} \varsigma_{1},
\end{array}\right. \\
\Psi_{1}+1 \leq k \leq n,
\end{array}\right\}
$$

with $k \in \underline{n}=\{1,2, \ldots, n\}, E_{k}=[\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k}$, and $a_{k}\left(a_{d k}\right)$ represents the $k$ th column vector of matrix $A\left(A_{d}\right)$.

By Theorem 4, we can obtain that the system (1a) and (1b) with $w(i, j)=0$ is asymptotically stable. Now we are in a position to prove that the system (1a) and (1b) has a prescribed $l_{1}$-gain index $\gamma$ for any nonzero $w(i, j) \in l_{1}$. To establish the $l_{1}$-gain performance, we choose the same copositive Lyapunov-Krasovskii functional candidate as in (9) for the system (1a) and (1b). Following the proof line of Theorem 4, we can get that

$$
\begin{align*}
\Delta V(i, j)+ & \|z(i, j)\|_{1}-\gamma\|w(i, j)\|_{1} \\
= & x^{T}(i, j)\left\{\left(A^{T}-I_{n}\right) p+\left(A^{T}+D_{H}-D_{L}\right) q\right. \\
& \left.+\zeta+D_{H}^{2}\left(\left(A^{T}-I_{n}\right) \varsigma_{2}+\varsigma_{1}\right)-D_{H} \varsigma_{2}\right\} \\
& +x_{d}^{T}(i, j)\left\{A_{d}^{T} p+\left(A_{d}^{T}-I_{n}\right) q+D_{H}^{2} A_{d}^{T} \varsigma_{2}\right\} \\
& +x_{H}^{T}(i, j)\left\{-\zeta+D_{H}\left(\varsigma_{2}-\varsigma_{1}\right)\right\} \\
& +x_{s}^{T}(i, j)\left\{-D_{H} \varsigma_{1}\right\} \\
& +w^{T}(i, j)\left\{B^{T}\left(p+q+D_{H}^{2} \varsigma_{2}\right)\right\} \\
& +\|H x(i, j)\|_{1}+\|L w(i, j)\|_{1}-\gamma\|w(i, j)\|_{1} \tag{28}
\end{align*}
$$

According to the definition of $l_{1}$ norm, one obtains

$$
\begin{align*}
& H x(i, j)= {\left[\begin{array}{cccccccc}
h_{1,1} & h_{1,2} & \cdots & h_{1, n_{1}} & h_{1, n_{1}+1} & h_{1, n_{1}+2} & \cdots & h_{1, n} \\
h_{2,1} & h_{2,2} & \cdots & h_{2, n_{1}} & h_{2, n_{1}+1} & h_{2, n_{1}+2} & \cdots & h_{2, n} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
h_{p, 1} & h_{p, 2} & \cdots & h_{p, n_{1}} & h_{p, n_{1}+1} & h_{p, n_{1}+2} & \cdots & h_{p, n}
\end{array}\right]\left[\begin{array}{c}
x_{1}^{h}(i, j) \\
\vdots \\
x_{n_{1}}^{h}(i, j) \\
x_{1}^{v}(i, j) \\
\vdots \\
x_{n_{2}}^{v}(i, j)
\end{array}\right] } \\
&= {\left[\begin{array}{c}
h_{1,1} x_{1}^{h}(i, j)+\cdots+h_{1, n_{1}} x_{n_{1}}^{h}(i, j)+h_{1, n_{1}+1} x_{1}^{v}(i, j)+\cdots+h_{1, n} x_{n_{2}}^{v}(i, j) \\
h_{2,1} x_{1}^{h}(i, j)+\cdots+h_{2, n_{1}} x_{n_{1}}^{h}(i, j)+h_{2, n_{1}+1} x_{1}^{v}(i, j)+\cdots+h_{2, n} x_{n_{2}}^{v}(i, j) \\
\vdots H x(i, j) \|_{1}= \\
h_{p, 1} x_{1}^{h}(i, j)+\cdots+h_{p, n_{1}} x_{n_{1}}^{h}(i, j)+h_{p, n_{1}+1} x_{1}^{v}(i, j)+\cdots+h_{p, n} x_{n_{2}}^{v}(i, j)
\end{array}\right], }  \tag{29}\\
& h_{k, 1} x_{1}^{h}(i, j)+\cdots+h_{k, n_{1}} x_{n_{1}}^{h}(i, j)+h_{k, n_{1}+1} x_{1}^{v}(i, j)+\cdots+h_{k, n} x_{n_{2}}^{v}(i, j) \\
&=\left(\sum_{k=1}^{p} h_{k, 1}\right) x_{1}^{h}(i, j)+\cdots+\left(\sum_{k=1}^{p} h_{k, n_{1}}\right) x_{n_{1}}^{h}(i, j)+\left(\sum_{k=1}^{p} h_{k, n_{1}+1}\right) x_{1}^{v}(i, j) \\
&+\cdots+\left(\sum_{k=1}^{p} h_{k, n}\right) x_{n_{2}}^{v}(i, j) \\
&= x^{T}(i, j)\left[\left\|h_{1}\right\|_{1}\left\|h_{2}\right\|_{1} \cdots\left\|h_{n}\right\|_{1}\right]^{T},
\end{align*}
$$

where $h_{k}$ represents the $k$ th column vector and $h_{i, j}$ represents the entry located at $(i, j)$ of matrix $H$. Then, similarly

$$
\begin{align*}
& \|L w(i, j)\|_{1}=w^{T}(i, j)\left[\left\|l_{1}\right\|_{1} \quad\left\|l_{2}\right\|_{1} \cdots \quad\left\|l_{m}\right\|_{1}\right]^{T},  \tag{30}\\
& \gamma\|w(i, j)\|_{1}=w^{T}(i, j)\left[\begin{array}{llll}
\gamma & \gamma & \cdots & \gamma
\end{array}\right]^{T},
\end{align*}
$$

where $l_{k}$ represents the $k$ th column vector of matrix $L$.
Substituting (29)-(30) into (28) leads to

$$
\left.\begin{array}{rl}
\Delta V(i, j)+\|z(i, j)\|_{1}-\gamma\|w(i, j)\|_{1} \\
=x^{T}(i, j)\left\{\left(A^{T}-I_{n}\right) p+\left(A^{T}+D_{H}-D_{L}\right) q+\zeta\right. \\
& +D_{H}^{2}\left(\left(A^{T}-I_{n}\right) \varsigma_{2}+\varsigma_{1}\right)-D_{H} \varsigma_{2} \\
& \left.+\left[\left\|h_{1}\right\|_{1}\left\|h_{2}\right\|_{1} \cdots\left\|h_{n}\right\|_{1}\right]^{T}\right\}
\end{array}\right\} \begin{aligned}
+ & x_{d}^{T}(i, j)\left\{A_{d}^{T} p+\left(A_{d}^{T}-I_{n}\right) q+D_{H}^{2} A_{d}^{T} \varsigma_{2}\right\} \\
+ & x_{H}^{T}(i, j)\left\{-\zeta+D_{H}\left(\varsigma_{2}-\varsigma_{1}\right)\right\} \\
+ & x_{s}^{T}(i, j)\left\{-D_{H} \varsigma_{1}\right\} \\
+ & w^{T}(i, j)\left\{B^{T}\left(p+q+D_{H}^{2} \varsigma_{2}\right)\right. \\
& +\left[\left\|l_{1}\right\|_{1}\left\|l_{2}\right\|_{1} \cdots\left\|l_{m}\right\|_{1}\right]^{T} \\
& \left.\quad[\gamma \gamma \cdots \gamma]^{T}\right\} .
\end{aligned}
$$

If condition (24) holds, we have

$$
\begin{align*}
V^{h}(i & +1, j)-V^{h}(i, j)+V^{v}(i, j+1)-V^{v}(i, j)  \tag{32}\\
& +\|z(i, j)\|_{1}-\gamma\|w(i, j)\|_{1}<0
\end{align*}
$$

We know that

$$
\begin{equation*}
\Delta V(i, j)=V^{h}(i+1, j)-V^{h}(i, j)+V^{v}(i, j+1)-V^{v}(i, j) \tag{33}
\end{equation*}
$$

For any positive scalars $k_{h}$, and $k_{v} \in Z_{+}$, it can be verified that

$$
\begin{aligned}
\sum_{i=0}^{k_{h}} \sum_{j=0}^{k_{v}} \Delta V(i, j)= & \sum_{i=0}^{k_{h}} \sum_{j=0}^{k_{v}}\left(V^{h}(i+1, j)-V^{h}(i, j)\right) \\
& +\sum_{i=0}^{k_{h}} \sum_{j=0}^{k_{v}}\left(V^{v}(i, j+1)-V^{v}(i, j)\right) \\
= & \sum_{j=0}^{k_{v}}\left(V^{h T}\left(k_{h}+1, j\right)-V^{h T}(0, j)\right) \\
& +\sum_{i=0}^{k_{h}}\left(V^{v T}\left(i, k_{v}+1\right)-V^{v T}(i, 0)\right)
\end{aligned}
$$

When $k_{h}$ and $k_{v}=\infty$, we have

$$
\begin{equation*}
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\left(\|z(i, j)\|_{1}-\gamma\|w(i, j)\|_{1}\right)<\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V(i, j) \tag{35}
\end{equation*}
$$

The existence of solution for LMI (24) implies that the positive 2D system (1a) and (1b) is asymptotically stable. Together with the zero boundary conditions, one can get

$$
\begin{equation*}
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V(i, j)=0 \tag{36}
\end{equation*}
$$

Applying (36) to (35), one has

$$
\begin{equation*}
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|z(i, j)\|_{1}<\gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|w(i, j)\|_{1} \tag{37}
\end{equation*}
$$

By Definition 2, the positive 2D system (1a) and (1b) is asymptotically stable and has the $l_{1}$-gain index $\gamma$.

This completes the proof.
Remark 7. In Theorem 6, the disturbance attenuation performance of positive 2D linear systems is analyzed, and sufficient conditions for the existence of $l_{1}$-gain performance for positive 2D system to (1a) and (1b) are proposed in terms of LMIs which are computationally tractable. This is also the major contribution of our paper.

## 4. Numerical Example

Consider the positive 2D system with delays in the Roesser model (1a) and (1b), where

$$
\begin{gather*}
A=\left[\begin{array}{cccc}
0.10 & 0.20 & \vdots & 0.2 \\
0.00 & 0.30 & \vdots & 0.10 \\
\cdots & \cdots & \cdots & \cdots \\
0.00 & 0.10 & \vdots & 0.40
\end{array}\right], \\
A_{d}=\left[\begin{array}{cccc}
0.10 & 0.01 & \vdots & 0.05 \\
0.10 & 0.02 & \vdots & 0.05 \\
\cdots & \cdots & \cdots & \cdots \\
0.03 & 0.12 & \vdots & 0.03
\end{array}\right], \\
B=\left[\begin{array}{c}
0.2 \\
0.1 \\
0.1
\end{array}\right], \\
H=\left[\begin{array}{lll}
0.1 & 0 & 0.2
\end{array}\right], \quad L=0.1, \\
d_{h}(i)=4+2 \sin \left(\frac{\pi i}{2}\right),  \tag{38}\\
d_{v}(j)=5+2 \sin \left(\frac{\pi j}{2}\right), \\
w(i, j)=e^{-(i+0.5 j)},
\end{gather*}
$$



Figure 1: State response of $x_{1}^{h}(i, j)$.
where state dimensions are $n_{h}=2$ and $n_{v}=1$. The boundary conditions are given by

$$
\begin{gather*}
x^{h}(i, j)=\left[\begin{array}{c}
0.1 \\
0.1
\end{array}\right], \quad \forall 0 \leq j \leq 52,-d_{h H} \leq i \leq 0  \tag{39}\\
x^{v}(i, j)=0.1, \quad \forall 0 \leq i \leq 52,-d_{v H} \leq j \leq 0
\end{gather*}
$$

In this example, we can get $d_{h L}=2, d_{h H}=6, d_{v L}=3$, and $d_{v H}=7$. Given $\gamma=4.5$, then by using the LMI Control Toolbox [37] to solve the inequalities in Theorem 6, we can get the following solutions:

$$
\begin{align*}
& p=\left[\begin{array}{lll}
1.7310 & 1.6842 & 1.7364
\end{array}\right]^{T}, \\
& q=\left[\begin{array}{lll}
3.2077 & 2.8768 & 3.3026
\end{array}\right]^{T}, \\
& \zeta=\left[\begin{array}{lll}
1.0334 & 1.1075 & 0.9957
\end{array}\right]^{T},  \tag{40}\\
& \varsigma_{1}=\left[\begin{array}{lll}
0.3675 & 0.4042 & 0.3552
\end{array}\right]^{T}, \\
& \varsigma_{2}=\left[\begin{array}{lll}
0.0385 & 0.0489 & 0.0330
\end{array}\right]^{T} .
\end{align*}
$$

Figures 1, 2, and 3 show the state responses of the system; it can be seen that the corresponding positive 2D system is asymptotically stable. Furthermore, by computing, under zero boundary conditions, we have $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|z(i, j)\|_{1}=$ $4.0206 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\|w(i, j)\|_{1}=1.0977$. It is obvious that the prescribed $l_{1}$-gain performance level $\gamma=4.5$ is satisfied.

## 5. Conclusions

This paper has addressed the delay-dependent stability analysis with $l_{1}$-gain performance for positive 2 D systems with state delays in the Roesser model. A sufficient condition for the existence of the delay-dependent asymptotic stability of positive 2D linear systems with time delays has been established. Copositive-type Lyapunov function method has been used to get a computationally tractable LMI-based sufficient


Figure 2: State response of $x_{2}^{h}(i, j)$.


Figure 3: State response of $x^{v}(i, j)$.
criterion which ensures that the system is asymptotically stable and has a prescribed $l_{1}$-gain performance. A numerical example has been given to illustrate the efficiency of the results. Furthermore, our future work will be devoted to the $l_{1}$-gain control problem for positive 2D systems with delays.

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# Robust Reliable Control of Uncertain Discrete Impulsive Switched Systems with State Delays 

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#### Abstract

This paper is concerned with the problem of robust reliable control for a class of uncertain discrete impulsive switched systems with state delays, where the actuators are subjected to failures. The parameter uncertainties are assumed to be norm-bounded, and the average dwell time approach is utilized for the stability analysis and controller design. Firstly, an exponential stability criterion is established in terms of linear matrix inequalities (LMIs). Then, a state feedback controller is constructed for the underlying system such that the resulting closed-loop system is exponentially stable. A numerical example is given to illustrate the effectiveness of the proposed method.


## 1. Introduction

Switched systems are a class of dynamical systems comprised of several continuous-time or discrete-time subsystems and a rule that orchestrates the switching among different subsystems. These systems have attracted considerable attention because of their applicability and significance in various areas, such as power electronics, embedded systems, chemical processes, and computer-controlled systems [1, 2]. Many works in the field of stability analysis and control synthesis for switched systems have appeared (see [3-11] and references cited therein). However, in the real world, they may not cover all the practical cases. People found that many systems are affected not only by switching among different subsystems, but also impulsive jumps at the switching instants. This kind of systems is named after impulsive switched systems, which have numerous applications in many fields, such as mechanical systems, automotive industry, aircraft, air traffic control, networked control, chaotic-based secure communication, quality of service in the internet, and video coding [12].

Impulsive switched systems have received a considerable research attention for more than one decade. The problems of stability, controllability, and observability for impulsive
switched systems have been successfully investigated, and a rich body of the literature has been available [13-17]. In [13], the authors established the necessary and sufficient conditions for controllability and controlled observability with respect to a given switching time sequence. Some results on the stability analysis and stabilization were developed in [14-17]. Because time-delay exists widely in practical environment and often causes undesirable performance, it is necessary and significant to study time delayed systems. Recently, such systems have stirred a great deal of research attention [18-22]. So far, many stability conditions of impulsive switched systems with state delays have been obtained in [23-26].

On the other hand, it is inevitable that the actuators will be subjected to failures in a real environment. A control system is said to be reliable if it retains certain properties when there exist failures. When failure occurs, the conventional controller will become conservative and may not satisfy certain control performance indexes. In this case, reliable control is a kind of effective control approach to improve system reliability. Recently, several approaches for designing reliable controllers have been proposed, and some of them have been used to research the problem of reliable control for switched systems [27-33]. In [27], a design methodology of the robust
reliable control for switched nonlinear systems with time delays was presented. In [32], $L_{\infty}$ reliable control problem for a class of continuous impulsive switched systems was researched, and a state feedback controller was constructed to restrain the outputs of the faulty actuators as well as disturbance inputs below a specified level. However, to the best of our knowledge, the existing results of the reliable control for impulsive switched systems are in the continuoustime framework, such topic on discrete impulsive switched systems has not been fully investigated, which motivates our present study.

In this paper, we will focus our interest on robust reliable control problem for a class of uncertain discrete impulsive switched systems with state delays. The dwell time approach is utilized for the stability analysis and controller design. The main contributions of this paper can be summarized as follows: (i) stability and reliability of discrete impulsive switched systems in the presence of actuators failures are first considered; (ii) a state feedback design methodology is proposed to achieve the exponential stability and reliability for the underlying systems.

The remainder of the paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, based on the dwell time approach, an exponential stability criterion is established in terms of LMIs. Then a delay-dependent sufficient condition for the existence of a robust reliable controller is derived in terms of a set of matrix inequalities. Section 4 gives a numerical example to illustrate the effectiveness of the proposed approach. Concluding remarks are given in Section 5.

Notations. Throughout this paper, the superscript " $T$ " denotes the transpose, and the notation $X \geq Y(X>Y)$ means that matrix $X-Y$ is a positive semidefinite (positive definite, resp.). $\|\cdot\|$ denotes the Euclidean norm. I represents identity matrix with appropriate dimension; $\operatorname{diag}\left\{a_{i}\right\}$ denotes diagonal matrix with the diagonal elements $a_{i}, i=1,2, \ldots, n . X^{-1}$ denotes the inverse of $X$. The asterisk $*$ in a matrix is used to denote a term that is induced by symmetry. The set of all positive integers is represented by $Z^{+}$.

## 2. Problem Formulation and Preliminaries

Consider the following uncertain discrete impulsive switched systems with state delays:

$$
\begin{gather*}
x(k+1)=\widehat{A}_{\sigma(k)} x(k)+\widehat{A}_{d \sigma(k)} x(k-d)+B_{\sigma(k)} u^{f}(k),  \tag{1}\\
\quad k \neq k_{b}-1, b \in Z^{+}, \\
x(k+1)=E_{\sigma(k+1) \sigma(k)} x(k), \quad k=k_{b}-1, b \in Z^{+},  \tag{2}\\
x\left(k_{0}+\theta\right)=\phi(\theta), \quad \theta=[-d, 0], \tag{3}
\end{gather*}
$$

where $x(k) \in R^{n}$ is the state vector. $u^{f}(k) \in R^{p}$ is the control input of actuator fault; $\phi(\theta)$ is a discrete vectorvalued initial function. $d$ is discrete time delay. $\sigma(k)$ is a switching signal which takes its values in the finite set $\underline{N}:=$ $\{1, \ldots, N\}$, corresponding to it is the switching sequence
$\mathcal{\vartheta}=\left\{\left(k_{0}, \sigma\left(k_{0}\right)\right),\left(k_{1}, \sigma\left(k_{1}\right)\right), \ldots,\left(k_{b}, \sigma\left(k_{b}\right)\right), \ldots\right\}$, where $k_{0}$ is the initial time and $k_{b}\left(b \in Z^{+}\right)$denotes the $b$ th switching instant. Moreover, $\sigma(k)=i \in \underline{N}$ means that the $i$ th subsystem is activated. $\sigma(k-1)=j$ and $\sigma(k)=i(i \neq j)$ indicate that $k$ is a switching instant at which the system is switched from the $j$ th subsystem to the $i$ th subsystem. $N$ denotes the number of subsystems. Note that there exists an impulsive jump described by (2) at the switching instant $k_{b}\left(b \in Z^{+}\right)$.

Remark 1. The impulsive jump at the switching instant $k_{b}$ is represented by $E_{\sigma\left(k_{b}\right) \sigma\left(k_{b}-1\right)}$. The matrix $E_{i j}(i, j \in \underline{N})$ is also used in [34]. Moreover, $E_{i j}$ is a certain real-valued matrix with appropriate dimension and means that the impulse is only determined by the subsystems activated before and after the specific switching instant $k_{b}$.

For each $i \in \underline{N}, \widehat{A}_{i} \widehat{A}_{d i}$ are uncertain real-valued matrices with appropriate dimensions and satisfy

$$
\left[\begin{array}{ll}
\widehat{A}_{i} & A_{d i}
\end{array}\right]=\left[\begin{array}{ll}
A_{i} & A_{d i}
\end{array}\right]+H_{i} F_{i}(k)\left[\begin{array}{ll}
M_{1 i} & M_{2 i} \tag{4}
\end{array}\right],
$$

where $A_{i}, A_{d i}, H_{i}, M_{1 i}$, and $M_{2 i}(i \in \underline{N})$ are known real constant matrices with appropriate dimensions. $F_{i}(k)$ are unknown and possibly time-varying matrices with Lebesgue measurable elements and satisfy

$$
\begin{equation*}
F_{i}^{T}(k) F_{i}(k) \leq I . \tag{5}
\end{equation*}
$$

The control input of actuator fault $u^{f}(k)$ can be described as

$$
\begin{equation*}
u^{f}(k)=\Omega_{\sigma(k)} u(k), \tag{6}
\end{equation*}
$$

where $u(k)=K_{\sigma(k)} x(k)$ is the control input to be designed, $\Omega_{i}(i \in \underline{N})$ are the actuator fault matrices with the following form:

$$
\begin{equation*}
\Omega_{i}=\operatorname{diag}\left\{\omega_{i 1}, \omega_{i 2}, \ldots, \omega_{i l}, \ldots, \omega_{i p}\right\} \tag{7}
\end{equation*}
$$

where $0 \leq \omega_{L i k} \leq \omega_{i k} \leq \omega_{H i k}, \omega_{H i k} \leq 1$.
For simplicity, we define

$$
\begin{gather*}
\Omega_{10}=\operatorname{diag}\left\{\widetilde{\omega}_{i 1}, \widetilde{\omega}_{i 2}, \ldots, \widetilde{\omega}_{i i}, \ldots, \widetilde{\omega}_{i p}\right\}, \\
\widetilde{\omega}_{i k}=\frac{1}{2}\left(\omega_{L i k}+\omega_{H i k}\right), \\
\Xi_{i}^{2}=\operatorname{diag}\left\{\xi_{i 1}, \xi_{i 2}, \ldots, \xi_{i i}, \ldots, \xi_{i p}\right\}, \\
\xi_{i k}=\frac{\omega_{H i k}-\omega_{L i k}}{\omega_{H i k}+\omega_{L i k}},  \tag{8}\\
\Theta_{i}=\operatorname{diag}\left\{\Theta_{i 1}, \Theta_{i 2}, \ldots, \Theta_{i i}, \ldots, \Theta_{i p}\right\}, \\
\Theta_{i k}=\frac{\omega_{i k}-\widetilde{\omega}_{i k}}{\widetilde{\omega}_{i k}} .
\end{gather*}
$$

Thus, we have

$$
\begin{equation*}
\Omega_{i}=\Omega_{i 0}\left(I+\Theta_{i}\right), \quad\left|\Theta_{i}\right| \leq \Xi_{i}^{2} \leq I, \tag{9}
\end{equation*}
$$

where $\left|\Theta_{i}\right|=\operatorname{diag}\left\{\left|\Theta_{i 1}\right|,\left|\Theta_{i 2}\right|, \ldots,\left|\Theta_{i i}\right|, \ldots,\left|\Theta_{i p}\right|\right\}$.
Before ending this section, we introduce the following definitions and lemmas.

Definition 2 (see [34]). Let $N_{\sigma}\left(k_{0}, k\right)$ denote the switching number of $\sigma(k)$ during the interval $\left[k_{0}, k\right)$. If there exist $N_{0} \geq$ 0 and $\tau_{a} \geq 0$ such that

$$
\begin{equation*}
N_{\sigma}\left(k_{0}, k\right) \leq N_{0}+\frac{k-k_{0}}{\tau_{a}}, \quad \forall k \geq k_{0} \tag{10}
\end{equation*}
$$

then $\tau_{a}$ and $N_{0}$ are called the average dwell time and the chatter bound, respectively.

Remark 3. In this paper, the average dwell time method is used to restrict the switching number during a time interval such that the stability of system (1), (2), and (3) can be guaranteed.

Definition 4 (see [35]). The system (1), (2), and (3) is said to be exponentially stable if its solution satisfies

$$
\begin{equation*}
\|x(k)\| \leq \eta\left\|x\left(k_{0}\right)\right\|_{h} \rho^{-\left(k-k_{0}\right)}, \quad \forall k \geq k_{0} \tag{11}
\end{equation*}
$$

for any initial condition $x\left(k_{0}+\theta\right), \theta=[-d, 0]$, where $\eta>0$ and $\rho>1$ is the decay rate, $\left\|x\left(k_{0}\right)\right\|_{h}=\max _{k_{0}-d \leq k \leq k_{0}}\|x(k)\|$.

Lemma 5 (see [35]). For a given matrix $S=\left[\begin{array}{ll}S_{11} & S_{12} \\ S_{12}^{T} & S_{22}\end{array}\right]$, where $S_{11}, S_{22}$ are square matrices, then the following conditions are equivalent:
(i) $S<0$,
(ii) $S_{11}<0, S_{22}-S_{12}^{T} S_{11}^{-1} S_{12}<0$,
(iii) $S_{22}<0, S_{11}-S_{12} S_{22}^{-1} S_{12}^{T}<0$.

Lemma 6 (see [36]). Let $U, V, W$, and $X$ be real matrices of appropriate dimensions with $X$ satisfying $X=X^{T}$, then for all $V^{T} V \leq I, X+U V W+W^{T} V^{T} U^{T}<0$, if and only if there exists a scalar $\varepsilon$ such that $X+\varepsilon U U^{T}+\varepsilon^{-1} W^{T} W<0$.

Lemma 7 (see [37]). For matrices $Q_{1}, Q_{2}$ with appropriate dimensions, there exists a positive scalar $\varepsilon$ such that

$$
\begin{equation*}
Q_{1} \Sigma Q_{2}+Q_{2}^{T} \Sigma^{T} Q_{1}^{T} \leq \varepsilon Q_{1} U Q_{1}^{T}+\varepsilon^{-1} Q_{2}^{T} U Q_{2} \tag{12}
\end{equation*}
$$

holds, where $\Sigma$ is a diagonal matrix and $U$ is a known real-value matrix satisfying $|\Sigma| \leq U$.

## 3. Main Results

3.1. Stability Analysis. In this subsection, we consider the exponential stability of the following uncertain discrete impulsive switched systems with state delays:

$$
\begin{gather*}
x(k+1)=\widehat{A}_{\sigma(k)} x(k)+\widehat{A}_{d \sigma(k)} x(k-d), \\
k \neq k_{b}-1, b \in Z^{+},  \tag{13}\\
x(k+1)=E_{\sigma(k+1) \sigma(k)} x(k), \quad k=k_{b}-1, b \in Z^{+},  \tag{14}\\
x\left(k_{0}+\theta\right)=\phi(\theta), \quad \theta=[-d, 0] . \tag{15}
\end{gather*}
$$

Theorem 8. Consider system (13), (14), and (15), for given positive scalars $d, 0<\alpha<1$, if there exist positive
definite symmetric matrices $X_{i}, N_{i}(i \in \underline{N})$ with appropriate dimensions and positive scalars $\varepsilon_{i}$ such that

$$
\left[\begin{array}{ccccc}
-\alpha X_{i} & 0 & X_{i} A_{i}^{T} & X_{i} & X_{i} M_{1 i}^{T}  \tag{16}\\
* & -\alpha^{d} N_{i} & N_{i} A_{d i}^{T} & 0 & N_{i} M_{2 i}^{T} \\
* & * & -X_{i}+\varepsilon_{i} H_{i} H_{i}^{T} & 0 & 0 \\
* & * & * & -N_{i} & 0 \\
* & * & * & * & -\varepsilon_{i} I
\end{array}\right]<0
$$

Then, under the following average dwell time scheme:

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=-\frac{\ln \mu}{\ln \alpha}+1 \tag{17}
\end{equation*}
$$

the system is exponentially stable, where $\mu \geq 1$ satisfies

$$
\begin{gather*}
{\left[\begin{array}{ccc}
-\mu X_{i} & X_{i} E_{j i}^{T} & X_{i} \\
* & -X_{j} & 0 \\
* & * & -N_{j}
\end{array}\right]<0,}  \tag{18}\\
\alpha N_{i} \leq \mu N_{j}, \quad \forall i, j \in \underline{N}, i \neq j .
\end{gather*}
$$

Proof. Choose the following piecewise Lyapunov function candidate for system (13), (14), and (15):

$$
\begin{equation*}
V(k)=V_{\sigma(k)}(k), \tag{19}
\end{equation*}
$$

and the form of each $V_{\sigma(k)}(k)$ is given by

$$
\begin{equation*}
V_{\sigma(k)}(k)=V_{1 \sigma(k)}(k)+V_{2 \sigma(k)}(k), \tag{20}
\end{equation*}
$$

where

$$
\begin{gather*}
V_{1 \sigma(k)}(k)=x^{T}(k) P_{\sigma(k)} x(k) \\
V_{2 \sigma(k)}(k)=\sum_{r=k-d}^{k-1} x^{T}(r) R_{\sigma(k)} x(r) \alpha^{k-r-1} \tag{21}
\end{gather*}
$$

Let $k_{1}, \ldots, k_{b}$ denote the switching instants during the interval $\left[k_{0}, k\right)$. Without loss of generality, assume that the $i$ th subsystem is activated at the switching instant $k_{b-1}$, and the $j$ th subsystem is activated at the switching instant $k_{b}$.

When $k \in\left[k_{b-1}, k_{b}-1\right), b \in Z^{+}, \sigma(k)=\sigma(k+1)=i$ ( $i \in \underline{N}$ ), along the trajectory of system (13), (14), and (15), we have

$$
\begin{align*}
V_{i}(x(k & +1))-\alpha V_{i}(x(k)) \\
= & x^{T}(k+1) P_{i} x(k+1) \\
& +\sum_{r=k+1-d}^{k} x^{T}(r) R_{i} x(r) \alpha^{k-r}-\alpha x^{T}(k) P_{i} x(k)  \tag{22}\\
& -\sum_{r=k-d}^{k-1} x^{T}(r) R_{i} x(r) \alpha^{k-r} .
\end{align*}
$$

Thus,

$$
\begin{equation*}
V_{i}(x(k+1))-\alpha V_{i}(x(k))=X^{T}(k) \varphi_{i} X(k) \tag{23}
\end{equation*}
$$

where

$$
\begin{gather*}
\varphi_{i}=\left(\begin{array}{cc}
R_{i}-\alpha P_{i} & 0 \\
0 & -\alpha^{d} R_{i}
\end{array}\right)+\binom{\widehat{A}_{i}^{T}}{\widehat{A}_{d i}^{T}} P_{i}\left(\begin{array}{ll}
\widehat{A}_{i} & \widehat{A}_{d i}
\end{array}\right),  \tag{24}\\
X(k)=\left[\begin{array}{ll}
x^{T}(k) & x^{T}(k-d)
\end{array}\right]^{T} .
\end{gather*}
$$

Thus, if the following inequality holds:

$$
\left(\begin{array}{cc}
R_{i}-\alpha P_{i} & 0  \tag{25}\\
0 & -\alpha^{d} R_{i}
\end{array}\right)+\binom{\widehat{A}_{i}^{T}}{\widehat{A}_{d i}^{T}} P_{i}\left(\begin{array}{ll}
\widehat{A}_{i} & \widehat{A}_{d i}
\end{array}\right)<0,
$$

then we have

$$
\begin{equation*}
V_{i}(x(k+1))<\alpha V_{i}(x(k)) . \tag{26}
\end{equation*}
$$

Using $\operatorname{diag}\left\{P_{i}^{-1}, R_{i}^{-1}\right\}$ to pre- and postmultiply the left term of (25) and applying Lemma 5, we can obtain that (25) is equivalent to the following inequality:

$$
\left(\begin{array}{cccc}
-\alpha P_{i}^{-1} & 0 & P_{i}^{-1} \widehat{A}_{i}^{T} & P_{i}^{-1}  \tag{27}\\
* & -\alpha^{d} R_{i}^{-1} & R_{i}^{-1} \widehat{A}_{d i}^{T} & 0 \\
* & * & -P_{i}^{-1} & 0 \\
* & * & * & -R_{i}^{-1}
\end{array}\right)<0
$$

Denote that $X_{i}=P_{i}^{-1}, N_{i}=R_{i}^{-1}$, then substituting (4) into (27) and applying Lemma 6, we can obtain that (16) and (27) are equivalent.

When $k=k_{b}-1, \sigma(k+1)=\sigma\left(k_{b}\right)=j, \sigma(k)=\sigma\left(k_{b}-1\right)=$ $i, i \neq j$, along the trajectory of system (13), (14), and (15), we have

$$
\begin{aligned}
V_{j}\left(x\left(k_{b}\right)\right)=x^{T}\left(k_{b}\right) P_{j} x\left(k_{b}\right) & +\sum_{r=k_{b}-d}^{k_{b}-1} x^{T}(r) R_{j} x(r) \alpha^{k_{b}-r-1}, \\
V_{i}\left(x\left(k_{b}-1\right)\right)= & x^{T}\left(k_{b}-1\right) P_{i} x\left(k_{b}-1\right) \\
& +\sum_{r=k_{b}-1-d}^{k_{b}-2} x^{T}(r) R_{i} x(r) \alpha^{k_{b}-r-2},
\end{aligned}
$$

$$
\begin{aligned}
V_{j}(x & \left.\left(k_{b}\right)\right)-\mu V_{i}\left(x\left(k_{b}-1\right)\right) \\
& =x^{T}\left(k_{b}-1\right)\left(E_{j i}^{T} P_{j} E_{j i}-\mu P_{i}\right) x\left(k_{b}-1\right)
\end{aligned}
$$

$$
+\sum_{r=k_{b}-d}^{k_{b}-1} x^{T}(r) R_{j} x(r) \alpha^{k_{b}-r-1}
$$

$$
-\mu \sum_{r=k_{b}-1-d}^{k_{b}-2} x^{T}(r) R_{i} x(r) \alpha^{k_{b}-r-2}
$$

$$
=x^{T}\left(k_{b}-1\right)\left(E_{j i}^{T} P_{j} E_{j i}-\mu P_{i}+R_{j}\right) x\left(k_{b}-1\right)
$$

$$
-\mu x^{T}\left(k_{b}-1-d\right) R_{i} x\left(k_{b}-1-d\right) \alpha^{d-2}
$$

$$
\begin{equation*}
+\sum_{r=k_{b}+1-d}^{k_{b}-2} \alpha^{k_{b}-r-2} x^{T}(r)\left(\alpha R_{j}-\mu R_{i}\right) x(r) . \tag{28}
\end{equation*}
$$

From (18), we can get the following inequalities for all $i, j \in$ N, $i \neq j$ :

$$
\begin{gather*}
E_{j i}^{T} P_{j} E_{j i}-\mu P_{i}+R_{j}<0,  \tag{29}\\
\alpha R_{j}-\mu R_{i} \leq 0 .
\end{gather*}
$$

Then, it is not difficult to get

$$
\begin{equation*}
V_{j}\left(x\left(k_{b}\right)\right)<\mu V_{i}\left(x\left(k_{b}-1\right)\right), \quad i \neq j . \tag{30}
\end{equation*}
$$

Thus, for $k \in\left[k_{b}, k_{b+1}\right)$, we have

$$
\begin{align*}
V_{\sigma(k)}(x(k)) & <\alpha^{k-k_{b}} V_{\sigma\left(k_{b}\right)}\left(x\left(k_{b}\right)\right) \\
& <\mu \alpha^{k-k_{b}} V_{\sigma\left(k_{b}-1\right)}\left(x\left(k_{b}-1\right)\right) . \tag{31}
\end{align*}
$$

Repeating the above manipulation, one has that

$$
\begin{align*}
V_{\sigma(k)} & (x(k)) \\
& <\alpha^{k-k_{b}} V_{\sigma\left(k_{b}\right)}\left(x\left(k_{b}\right)\right) \\
& <\mu \alpha^{k-k_{b}} V_{\sigma\left(k_{b}-1\right)}\left(x\left(k_{b}-1\right)\right) \\
& \leq \mu \alpha^{k-k_{b}} \alpha^{k_{b}-1-k_{b-1}} V_{\sigma\left(k_{b-1}\right)}\left(x\left(k_{b-1}\right)\right)  \tag{32}\\
& =\mu \alpha^{k-k_{b-1}-1} V_{\sigma\left(k_{b-1}\right)}\left(x\left(k_{b-1}\right)\right) \\
& <\mu^{2} \alpha^{k-k_{b-1}-1} V_{\sigma\left(k_{b-1}-1\right)}\left(x\left(k_{b-1}-1\right)\right) \\
& <\cdots \\
& <\mu^{b} \alpha^{k-k_{0}-b} V_{\sigma\left(k_{0}\right)}\left(x\left(k_{0}\right)\right) .
\end{align*}
$$

From Definition 2, we know that $b=N_{\sigma}\left(k_{0}, k\right)$, then

$$
\begin{equation*}
b \leq N_{0}+\frac{k-k_{0}}{\tau_{a}} \tag{33}
\end{equation*}
$$

It follows that

$$
\begin{align*}
V_{\sigma(k)} & (x(k)) \\
& <\mu^{b} \alpha^{k-k_{0}-b} V_{\sigma\left(k_{0}\right)}\left(x\left(k_{0}\right)\right) \\
& \leq\left(\mu \alpha^{-1}\right)^{N_{0}+\left(k-k_{0}\right) / \tau_{a}} \alpha^{k-k_{0}} V_{\sigma\left(k_{0}\right)}\left(x\left(k_{0}\right)\right) \\
& =\left(\mu \alpha^{-1}\right)^{N_{0}} e^{\left(\left(k-k_{0}\right) / \tau_{a}\right)(\ln \mu-\ln \alpha)} e^{\left(k-k_{0}\right) \ln \alpha} V_{\sigma\left(k_{0}\right)}\left(x\left(k_{0}\right)\right) \\
& =\left(\mu \alpha^{-1}\right)^{N_{0}} e^{\left((\ln \mu-\ln \alpha) / \tau_{a}+\ln \alpha\right)\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(x\left(k_{0}\right)\right), \tag{34}
\end{align*}
$$

that is,

$$
\begin{equation*}
\|x(k)\|<\eta\left\|x\left(k_{0}\right)\right\|_{h} \rho^{-\left(k-k_{0}\right)}, \quad \forall k \geq k_{0} \tag{35}
\end{equation*}
$$

where

$$
\begin{gather*}
\eta=\sqrt{\frac{\max _{i \in \underline{N}}\left\{\lambda_{\max }\left(X_{i}^{-1}\right)+d \lambda_{\max }\left(N_{i}^{-1}\right)\right\}}{\min _{i \in \underline{N}} \lambda_{\min }\left(X_{i}^{-1}\right)}}\left(\mu \alpha^{-1}\right)^{N_{0} / 2} \\
\rho=e^{-\left((\ln \mu-\ln \alpha) / \tau_{a}+\ln \alpha\right) / 2}, \quad\left\|x\left(k_{0}\right)\right\|_{h}=\max _{k_{0}-d \leq k \leq k_{0}}\|x(k)\| . \tag{36}
\end{gather*}
$$

Then under the average dwell time scheme (17), it is easy to get that $\rho>1$, which implies that the system (13), (14), and (15) is exponentially stable.

This completes the proof.
Remark 9. When $\mu=1$, conditions (18) can be reduced to the following inequalities:

$$
\begin{align*}
& {\left[\begin{array}{ccc}
-X_{i} & X_{i} E_{j i}^{T} & X_{i} \\
* & -X_{j} & 0 \\
* & * & -N_{j}
\end{array}\right]<0,}  \tag{37}\\
& \alpha N_{i} \leq N_{j}, \quad \forall i, j \in \underline{N}, i \neq j,
\end{align*}
$$

then $\tau_{a}^{*}=1$.
Remark 10. It should be noted that some stability results of discrete delayed systems with and without impulsive jumps have been obtained by using standard Lyapunov-Krasovskii function approach (see [5, 7, 38]). In this paper, these stability criteria are extended to discrete impulsive switched delayed system (1), (2), and (3). However, due to that there exist impulsive jumps described by (2) at the switching instants, the criterion in Theorem 8 is different from the existing ones. The result is essential for designing the reliable controller for system (1), (2), and (3).
3.2. Robust Reliable Control. In this subsection, we are interested in designing a state feedback controller such that the resulting closed-loop system is exponentially stable.

For system (1), (2), and (3), under switching controller $u(k)=K_{\sigma(k)} x(k)$, the corresponding closed-loop system is given by

$$
\begin{align*}
x(k+1)= & \left(\widehat{A}_{\sigma(k)}+B_{\sigma(k)} \Omega_{\sigma(k)} K_{\sigma(k)}\right) x(k) \\
& +\widehat{A}_{d \sigma(k)} x(k-d), \quad k \neq k_{b}-1, b \in Z^{+}  \tag{38}\\
x(k+1)= & E_{\sigma(k+1) \sigma(k)} x(k), \quad k=k_{b}-1, b \in Z^{+}  \tag{39}\\
& x\left(k_{0}+\theta\right)=\phi(\theta), \quad \theta=[-d, 0] . \tag{40}
\end{align*}
$$

Theorem 11. Consider the system (1), (2), and (3), for given positive scalars $d$ and $\alpha<1$; suppose there exist positive definite symmetric matrices $X_{i}, N_{i}$, any matrices $W_{i}$ with appropriate dimensions, and positive scalars $\varepsilon_{i}, \gamma_{i}, i \in \underline{N}$, such that

$$
\left(\begin{array}{cccccc}
-\alpha X_{i} & 0 & X_{i} A_{i}^{T}+W_{i}^{T} \Omega_{i 0}^{T} B_{i}^{T} & X_{i} & X_{i} M_{1 i}^{T} & W_{i}^{T} \\
* & -\alpha^{d} N_{i} & N_{i} A_{d i}^{T} & 0 & N_{i} M_{2 i}^{T} & 0  \tag{41}\\
* & * & -X_{i}+\varepsilon_{i} H_{i} H_{i}^{T}+\gamma_{i} B_{i} \Omega_{i 0} \Xi_{i}^{2} \Omega_{i 0}^{T} A_{i}^{T} & 0 & 0 & 0 \\
* & * & * & & & \\
* & * & * & * & 0 & 0 \\
* & * & * & * & * & -\gamma_{i}\left(\Xi_{i}^{2}\right)^{-1}
\end{array}\right)
$$

Then, under the reliable controller

$$
\begin{equation*}
u(k)=K_{\sigma(k)} x(k), \quad K_{i}=W_{i} X_{i}^{-1} \quad(i \in \underline{N}) \tag{42}
\end{equation*}
$$

and the average dwell time scheme (17) with $\mu$ satisfying (18), the corresponding closed-loop system (38), (39), and (40) is exponentially stable.

Proof. From Theorem 8, we know that system (38), (39), and (40) is exponentially stable if (18) and the following inequality hold:

$$
\left[\begin{array}{ccccc}
-\alpha X_{i} & 0 & X_{i} \widetilde{A}_{i}^{T} & X_{i} & X_{i} M_{1 i}^{T}  \tag{43}\\
* & -\alpha^{d} N_{i} & N_{i} A_{d i}^{T} & 0 & N_{i} M_{2 i}^{T} \\
* & * & -X_{i}+\varepsilon_{i} H_{i} H_{i}^{T} & 0 & 0 \\
* & * & * & -N_{i} & 0 \\
* & * & * & * & -\varepsilon_{i} I
\end{array}\right]<0
$$

where $\widetilde{A}_{i}=A_{i}+B_{i} \Omega_{i} K_{i}, \Omega_{i}=\Omega_{i 0}\left(I+\Theta_{i}\right)$, and $\left|\Theta_{i}\right| \leq \Xi_{i}^{2} \leq I$; it can be obtained that (43) can be rewritten as the following inequality:

$$
\begin{gather*}
\left(\begin{array}{ccccc}
-\alpha X_{i} & 0 & X_{i} A_{i}^{T}+\left(K_{i} X_{i}\right)^{T} \Omega_{i 0}^{T} B_{i}^{T} & X_{i} & X_{i} M_{1 i}^{T} \\
* & -\alpha^{d} N_{i} & N_{i} A_{d i}^{T} & 0 & N_{i} M_{2 i}^{T} \\
* & * & -X_{i}+\varepsilon_{i} H_{i} H_{i}^{T} & 0 & 0 \\
* & * & * & -N_{i} & 0 \\
* & * & * & -\varepsilon_{i} I
\end{array}\right) \\
\quad+\left(\begin{array}{c}
0 \\
0 \\
B_{i} \Omega_{i 0} \\
0 \\
0
\end{array}\right) \Theta_{i}\left(\begin{array}{c}
K_{i} X_{i} \\
0 \\
0 \\
0 \\
0
\end{array}\right)+\left(\begin{array}{c}
K_{i} X_{i} \\
0 \\
0 \\
0 \\
0
\end{array}\right) \Theta_{i}^{T}\left(\begin{array}{c}
0 \\
0 \\
B_{i} \Omega_{i 0} \\
0 \\
0
\end{array}\right)^{T}<0 . \tag{44}
\end{gather*}
$$

Denote that $W_{i}=K_{i} X_{i}$, then according to Lemmas 5 and 7, we can easily get that (44) holds if (41) is satisfied, that is to say, (41) guarantees that (43) is tenable.

This completes the proof.
Remark 12. In Theorem 11, a reliable controller design method is proposed for discrete impulsive switched delayed system (1), (2), and (3) with actuator fault. It is noted that a kind of matrix $\Omega_{i}(i \in \underline{N})$, which is successfully adopted in [27, 28], is introduced to describe all the situations that may be encountered in the actuator.

Remark 13. It should be noted that $\alpha$ plays a key role in obtaining the infimum of the average dwell time $\tau_{a}$. From Theorem 11 , it is easy to see that a larger $\alpha$ will be favorable to the solvability of inequality (41), which leads to a larger value for the average dwell time $\tau_{a}$. Considering these, we can first select a larger $\alpha$ to guarantee the feasible solution of inequality (41) and then decrease $\alpha$ to obtain the suitable infimum of the average dwell time $\tau_{a}$.

The detailed procedure of controller design can be given in the following algorithm.

Algorithm 14. We have the following.
Step 1. Given the system matrices and positive constants $\varepsilon_{i}$, $\gamma_{i}$, and $0<\alpha<1$, by solving the LMI (41), we can get the solutions of the matrices $W_{i}, X_{i}$, and $N_{i}$. Then the controller gain matrices can be obtained by (42).
Step 2. Substitute matrices $X_{i}$ and $N_{i}$ into (18), then solving (18), we can find the infimum of $\mu$.

Step 3. Then the average dwell time $\tau_{a}$ can be obtained by (17).

## 4. Numerical Example

In this section, we present an example to illustrate the effectiveness of the proposed approach. Consider system (1), (2), and (3) with parameters as follows:

$$
\begin{array}{ll}
A_{1}=\left[\begin{array}{cc}
2 & -5 \\
1 & -1.5
\end{array}\right], & A_{d 1}=\left[\begin{array}{cc}
-0.4 & 0 \\
-0.1 & -0.1
\end{array}\right], \\
B_{1}=\left[\begin{array}{cc}
-0.4 & 0 \\
-0.1 & -0.1
\end{array}\right], & H_{1}=\left[\begin{array}{cc}
0.1 & 0 \\
0.1 & 0.1
\end{array}\right],
\end{array}
$$

$$
\begin{align*}
& M_{11}=\left[\begin{array}{cc}
0.2 & -0.3 \\
0 & -0.2
\end{array}\right], \quad M_{21}=\left[\begin{array}{cc}
0.1 & 0 \\
0.1 & 0.22
\end{array}\right], \\
& F_{1}=\left[\begin{array}{cc}
\sin (0.5 \pi k) & 0 \\
0 & \sin (0.2 \pi k)
\end{array}\right], \\
& A_{2}=\left[\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right], \quad A_{d 2}=\left[\begin{array}{cc}
-0.2 & 0 \\
-0.4 & 0.3
\end{array}\right], \\
& B_{2}=\left[\begin{array}{cc}
-0.2 & 0 \\
-0.4 & 0.3
\end{array}\right], \quad H_{2}=\left[\begin{array}{ll}
0.2 & 0.1 \\
0.1 & 0.3
\end{array}\right], \\
& M_{12}=\left[\begin{array}{cc}
0.2 & 0 \\
0.2 & 0.1
\end{array}\right], \quad M_{22}=\left[\begin{array}{cc}
0.2 & 0 \\
0 & 0.2
\end{array}\right], \\
& F_{2}=\left[\begin{array}{cc}
\sin (0.5 \pi k) & 0 \\
0 & \sin (0.2 \pi k)
\end{array}\right], \\
& E_{12}=\left[\begin{array}{cc}
3.5 & 0 \\
0 & 3.6
\end{array}\right],
\end{align*} E_{21}=\left[\begin{array}{ll}
3 & 0  \tag{45}\\
0 & 4
\end{array}\right] . ~ \$
$$

The fault matrices $\Omega_{i}=\operatorname{diag}\left\{\omega_{i 1}, \omega_{i 2}\right\}(i=1,2)$, where

$$
\begin{array}{ll}
0.4 \leq \omega_{11} \leq 0.5, & 0.5 \leq \omega_{12} \leq 0.6 \\
0.5 \leq \omega_{21} \leq 0.6, & 0.4 \leq \omega_{22} \leq 0.5 \tag{46}
\end{array}
$$

Then we can obtain

$$
\begin{array}{ll}
\Omega_{10}=\left[\begin{array}{cc}
0.55 & 0 \\
0 & 0.45
\end{array}\right], & \Xi_{1}^{2}=\left[\begin{array}{cc}
\frac{1}{11} & 0 \\
0 & \frac{1}{9}
\end{array}\right], \\
\Omega_{20}=\left[\begin{array}{cc}
0.45 & 0 \\
0 & 0.55
\end{array}\right], & \Xi_{2}^{2}=\left[\begin{array}{cc}
\frac{1}{9} & 0 \\
0 & \frac{1}{11}
\end{array}\right] . \tag{47}
\end{array}
$$

Given $\alpha=0.7, \varepsilon_{1}=\varepsilon_{2}=0.1, \gamma_{1}=0.3, \gamma_{2}=0.3$, then solving the matrix inequality (41) in Theorem 11, we get

$$
\begin{aligned}
X_{1} & =\left[\begin{array}{ll}
0.0095 & 0.0046 \\
0.0046 & 0.0058
\end{array}\right], \\
N_{1} & =\left[\begin{array}{ll}
0.0203 & 0.0116 \\
0.0116 & 0.0573
\end{array}\right], \\
W_{1} & =\left[\begin{array}{cc}
-0.0192 & -0.0902 \\
0.0657 & 0.0170
\end{array}\right], \\
X_{2} & =\left[\begin{array}{cc}
0.0106 & 0.0098 \\
0.0098 & 0.0445
\end{array}\right], \\
N_{2} & =\left[\begin{array}{cc}
0.0528 & 0.0383 \\
0.0383 & 0.1505
\end{array}\right], \\
W_{2} & =\left[\begin{array}{cc}
0.1124 & 0.0990 \\
-0.0113 & 0.1886
\end{array}\right]
\end{aligned}
$$



Figure 1: Switching signal.


Figure 2: State trajectories of the closed-loop system.

Then from (42), the controller gain matrices can be obtained

$$
\begin{align*}
& K_{1}=\left[\begin{array}{cc}
8.9155 & -22.5632 \\
8.9651 & -4.1574
\end{array}\right]  \tag{49}\\
& K_{2}=\left[\begin{array}{cc}
10.6963 & -0.1343 \\
-6.2374 & 5.6124
\end{array}\right]
\end{align*}
$$

According to conditions (18), we can get $\mu=11.5633$. From (17), it can be obtained that $\tau_{a}^{*}=7.863$. Choosing $\tau_{a}=$ 8, the simulation results are shown in Figures 1 and 2, where the initial value $x(0)=\left[\begin{array}{ll}3 & 4\end{array}\right]^{T}, x(\theta)=0$, and $\theta \in[-d, 0)$. Figure 1 depicts the switching signal, and the state trajectories of the closed-loop system are shown in Figure 2.

From Figures 1 and 2, it can be observed that the designed controller can guarantee the asymptotic stability of the closed-loop system. This demonstrates the effectiveness of the proposed method.

## 5. Conclusions

This paper has investigated the problem of robust reliable control for a class of uncertain discrete impulsive switched systems with state delays. By employing the average dwell time approach, an exponential stability criterion has been proposed in terms of a set of LMIs. On the basis of the obtained stability criterion, the robust reliable controller has been designed. An illustrative example has also been given to illustrate the applicability of the proposed approach.

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## Research Article

# Robust Filtering for Networked Stochastic Systems Subject to Sensor Nonlinearity 

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#### Abstract

The problem of network-based robust filtering for stochastic systems with sensor nonlinearity is investigated in this paper. In the network environment, the effects of the sensor saturation, output quantization, and network-induced delay are taken into simultaneous consideration, and the output measurements received in the filter side are incomplete. The random delays are modeled as a linear function of the stochastic variable described by a Bernoulli random binary distribution. The derived criteria for performance analysis of the filtering-error system and filter design are proposed which can be solved by using convex optimization method. Numerical examples show the effectiveness of the design method.


## 1. Introduction

In recent years, networked control systems (NCSs) have been extensively investigated due to thier broad applications in industrial engineering [1]. NCSs hold a few excellent advantages such as reduction of costs of cables and power, simplification of the installation and maintenance of the whole system, and increase of the reliability [2]. However, the insertion of the communication channels also arises some unexpected phenomenon in NCSs such as signals quantization [1], intermittent data packet losses, and the signal-transmission delay $[3,4]$. These phenomena emerging in NCSs are known to be the main causes for the performance deterioration or even the instability of the controlled networked system. Over the past few years, intensive research interest has been reported in a wealth of the literature focusing on the control and filtering problems of NCSs involved with networked-induced time delay, packet losses, and signal quantization (see, e.g., [1] and the references therein).

Stochastic phenomenon frequently exhibits in many branches of science and engineering applications [5-10]. In the past few years, increasing research interests have recently
been paid to the study of control and filtering problems for various continuous-time or discrete-time stochastic systems [5]. For instance, the $H_{\infty}$ nonlinear filtering problem has been investigated for discrete-time stochastic systems subject to signal quantization in [2]. The design problem of state estimation and stabilization for a nonlinear networked control systems has been addressed in [11], while the $H_{\infty}$ output feedback control problem has been considered in [12].

In practical physical systems, sensors and actuators cannot always provide unlimited amplitude signal mainly due to the physical or safety constraints [13]. The phenomenon of sensor and/or actuator saturation can yield significant limitations on various aspects of sensor and/or actuator performance, for example, the range limitations that results in the nonlinear characteristic of sensors and/or actuator [13, 14]. For actuator saturation, a great deal of attention has been focused on the control and filtering problems for various types of systems [15]. In particular, the control problem has been investigated for continuous-time linear delay systems subject to quantization and saturation in [16], where both quantized state and quantized input are taken into consideration.


Figure 1: The structure of networked filtering systems.

It should be pointed out that, if we consider the filtering problem for stochastic systems in a realistic networked environment, the effects of sensor saturation, sensor quantization, and random communication delay always exhibit simultaneously. However, in networked environments, the sensor saturation may occur to be involved with state-dependent disturbance, and it may result from random sensor failures leading to intermittent saturation, sensor aging resulting in changeable saturation level, repairs of partial components, changes in the interconnections of subsystems, and so forth. Therefore, when investigating the filtering problems of NCSs with a stochastic plant, the model under consideration should be more comprehensive to reflect the realities such as the the state-dependent stochastic disturbances, the coupling effects of sensor saturation, output quantization, and networkedinduced transmission delay. Unfortunately, however, to the best of the authors' knowledge, the $H_{\infty}$ filtering problem on stochastic systems subject to sensor saturation, quantization, and random communication delay has not been investigated and remains to be important and challenging. This motivates our current work.

In this paper, we are concerned with the filter design problem for discrete-time networked stochastic systems subject to output saturation, quantization, and random communication delay. The networked-induced communication delay phenomena are modeled by a Bernoulli random binary distributed white sequence with a known conditional probability. In this network setting, the effects of sensor saturation, output quantization, and communication delay in the digital communication channel exhibit simultaneously, and the signal received in the filter side is imperfect. The objective is to analyze and design a robust filter such that the asymptotic estimates of system states are obtained by employing the incomplete output measurements. Moreover, sufficient conditions will be proposed such that the derived filtering error system is robustly stochastically stable with a prescribed disturbance attenuation level. Finally, a numerical example is provided to illustrate the effectiveness of the proposed filtering design approach.

Throughout the paper, $\mathbb{E}\{\cdot\}$ is the mathematical expectation. $\|\cdot\|$ denotes the Euclidean norm of a vector. Given a symmetric matrix $A$, the notation $A>0(<0)$ denotes a positive definite matrix (negative definite, resp.). $I_{n}$ denotes an identity matrix with dimension $n$.

## 2. Problem Description

We consider the following discrete-time stochastic system with state-dependent disturbance:

$$
\begin{gather*}
x(k+1)=A x(k)+B v(k)+[E x(k)+G v(k)] w(k) \\
y(k)=C x(k) \\
y_{\phi}(k)=\phi(y(k))  \tag{1}\\
y_{q}(k)=q\left(y_{\phi}(k)\right) \\
z(k)=L x(k)
\end{gather*}
$$

where $x(k) \in \mathbb{R}^{n}$ is the state; $y(k) \in \mathbb{R}^{p}$ is the output; the saturation function $\phi(\cdot)$ is defined as in (3); $y_{q}(k) \in \mathbb{R}^{p}$ is the quantized output, and $q(\cdot)$ is the logarithmic quantizer defined in (6)-(7); $z(k) \in \mathbb{R}^{r}$ is the state combination to be estimated; $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, E \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$, and $L \in \mathbb{R}^{r \times n}$ are known constant matrices.

In plant (1), w(k) is a standard one-dimensional random process on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where $\Omega$ is the sample space, $\mathscr{F}$ is the $\sigma$-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$. The sequence of $w(k)$ is generated by $(w(k))_{k \in \mathbb{N}}$ where $\mathbb{N}$ denotes the set of natural numbers, and it satisfies that $\mathbb{E}\{w(k)\}=0, \mathbb{E}\left\{w(k)^{2}\right\}=$ $1, \mathbb{E}\{w(i) w(j)\}=0$ for $i \neq j$.

Besides, it is assumed that the exogenous disturbance $v(k) \in \mathbb{R}^{m}$ belongs to $\mathscr{L}_{E_{2}}\left([0, \infty) ; \mathbb{R}^{m}\right)$, where $\mathscr{L}_{E_{2}}\left([0, \infty) ; \mathbb{R}^{m}\right)$ denotes the space of $k$-dimensional nonanticipatory square-integrable process $\varphi(\cdot)=(\varphi(k))_{k \in \mathbb{N}}$ on $\mathbb{N}$ with respect to $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$, and $\varphi(\cdot)$ satisfies

$$
\begin{equation*}
\|\varphi\|_{E_{2}}^{2}=\mathbb{E}\left\{\sum_{k=0}^{\infty}\|\varphi(k)\|^{2}\right\}=\sum_{k=0}^{\infty} \mathbb{E}\left\{\|\varphi(k)\|^{2}\right\}<\infty . \tag{2}
\end{equation*}
$$

Remark 1. As seen in plant (1), the phenomena of sensor quantization and saturation are taken into consideration simultaneously, which is one of the main contributions of this paper. Although there have been much of the literature devoted to quantized filtering, few of which has considered the effect of sensor saturation.

The structure of the quantized filtering system is illustrated in Figure 1.

We denote $y(k)$ as $y_{k}$ for simplicity in the following discussion. It is assumed that the saturation function $\phi(\cdot)$ : $\mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ in (1) belongs to [ $K_{1}, K_{2}$ ] for some given diagonal matrices $K_{1} \in \mathbb{R}^{p \times p}, K_{2} \in \mathbb{R}^{p \times p}$ with $K_{1} \geq 0, K_{2} \geq 0$ and $K_{2}>K_{1}$, and $\phi(\cdot)$ satisfies the following sector condition:

$$
\begin{equation*}
\left(\phi\left(y_{k}\right)-K_{1} y_{k}\right)^{T}\left(\phi\left(y_{k}\right)-K_{2} y_{k}\right) \leq 0, \quad \forall y_{k} \in \mathbb{R}^{q} \tag{3}
\end{equation*}
$$

In the light of (3), the nonlinear function $\phi\left(y_{k}\right)$ can be decomposed into a linear and a nonlinear part as follows:

$$
\begin{equation*}
\phi\left(y_{k}\right)=\phi_{s}\left(y_{k}\right)+K_{1} y_{k} \tag{4}
\end{equation*}
$$

and the nonlinearity $\phi_{s}\left(y_{k}\right)$ satisfies $\phi_{s}\left(y_{k}\right) \in \Phi_{s}$, where the set $\Phi_{s}$ is defined as

$$
\begin{equation*}
\Phi_{s} \triangleq\left\{\phi_{s}: \phi_{s}^{T}\left(y_{k}\right)\left(\phi_{s}\left(y_{k}\right)-K y_{k}\right) \leq 0\right\} \tag{5}
\end{equation*}
$$

and $K \triangleq K_{2}-K_{1}$.
In this paper, we employ the logarithmic quantizer for system (1) which is described as follows:

$$
\begin{equation*}
q(\cdot)=\left[q_{1}(\cdot), q_{2}(\cdot), \ldots, q_{p}(\cdot)\right]^{T} \tag{6}
\end{equation*}
$$

and $q_{i}(\cdot)$ is defined as follows:

$$
\begin{align*}
& q_{i}\left(\phi\left(y_{i}(k)\right)\right) \\
& \quad=\left\{\begin{aligned}
& \eta_{i}^{(j)} \\
& \text { if } \frac{1}{1+\delta_{i}} \eta_{i}^{(j)}<\phi\left(y_{i}(k)\right) \\
& \leq \frac{1}{1-\delta_{i}} \eta_{i}^{(j)}, \\
& \quad \phi\left(y_{i}(k)\right)>0, \\
& 0 \quad \\
& \text { if } \phi\left(y_{i}(k)\right)=0, \\
&-q_{i}\left(-\phi\left(y_{i}(k)\right)\right) \\
& \text { if } \phi\left(y_{i}(k)\right)<0, i=1,2, \ldots, p ; \\
& \quad j= \pm 1, \pm 2, \ldots,
\end{aligned}\right.
\end{align*}
$$

where $\delta_{i}=\left(1-\rho_{i}\right) /\left(1+\rho_{i}\right)$ are the quantizer parameters.
In fact, the logarithmic quantizer (7) can be characterized by the following form:

$$
\begin{equation*}
q\left(\phi\left(y_{k}\right)\right)=\left(I_{p}+\Lambda(k)\right) \phi\left(y_{k}\right) \tag{8}
\end{equation*}
$$

where

$$
\begin{gather*}
\Lambda(k)=\operatorname{diag}\left\{\Lambda_{1}(k), \Lambda_{2}(k), \ldots, \Lambda_{p}(k)\right\}, \\
\Lambda_{j}(k) \in\left[-\eta_{j}, \eta_{j}\right], \quad j=1, \ldots, p \tag{9}
\end{gather*}
$$

In this filtering problem, the measured output received in the filter side is involved with the effects of quantization and communication delay, and it is described by

$$
\begin{equation*}
y_{f}(k)=\left(1-\theta_{k}\right) q\left(\phi\left(y_{k}\right)\right)+\theta_{k} q\left(\phi\left(y_{k-1}\right)\right)+D v(k), \tag{10}
\end{equation*}
$$

with $D \in \mathbb{R}^{p \times m}$, and the stochastic variable $\theta_{k} \in \mathbb{R}$ is a Bernoulli distributed white sequence with the probability distribution as follows:

$$
\begin{gather*}
\operatorname{Prob}\left\{\theta_{k}=1\right\}=\mathbb{E}\left\{\theta_{k}\right\}=\bar{\theta}, \\
\operatorname{Prob}\left\{\theta_{k}=0\right\}=1-\mathbb{E}\left\{\theta_{k}\right\}=1-\bar{\theta},  \tag{11}\\
\operatorname{var}\left\{\theta_{k}\right\}=\mathbb{E}\left\{\left(\theta_{k}-\bar{\theta}\right)^{2}\right\}=(1-\bar{\theta}) \bar{\theta}=\theta_{1}^{2},
\end{gather*}
$$

where $0 \leq \bar{\theta}<1$ is a known positive constant to denote the probability that the packet will be transmitted successfully from sensor to filter, and $0 \leq \theta_{1}<1$ denotes the variance of $\theta_{k}$.

For simplicity, we denote $x_{k}:=x(k), f_{k}:=f\left(k, x_{k}\right), \omega_{k}:=$ $\omega(k), v_{k}:=v(k), \Lambda(k):=\Lambda_{k}$ in the following discussion. In the light of (4), (10) can be written as

$$
\begin{align*}
y_{f}(k)= & \left(1-\theta_{k}\right)\left(I_{p}+\Delta_{k}\right)\left(\phi_{s}\left(y_{k}\right)+K_{1} C x_{k}\right) \\
& +\theta_{k}\left(I_{p}+\Delta_{k}\right)\left(\phi_{s}\left(y_{k-1}\right)+K_{1} C x_{k-1}\right)+D v_{k} \tag{12}
\end{align*}
$$

The main objective of this paper is to address the filtering problem for stochastic system (1) by employing the incomplete measurements $y_{f}(k)$. To this end, we consider the following filter of full order $n$ :

$$
\begin{gather*}
\widehat{x}(k+1)=\widehat{A} \hat{x}(k)+\widehat{B} y_{f}(k), \quad \widehat{x}(k)=0,  \tag{13}\\
\widehat{z}(k)=L \widehat{x}(k),
\end{gather*}
$$

where $\widehat{x}(k) \in \mathbb{R}^{p}$ is the state of the filter, and $\widehat{z}(k) \in \mathbb{R}^{r}$ is the estimated signal; $\widehat{A} \in \mathbb{R}^{n \times n}$ and $\widehat{B} \in \mathbb{R}^{n \times m}$ are filter gains to be designed.

We define the following error variables:

$$
\begin{equation*}
e_{x}(k) \triangleq x(k)-\widehat{x}(k) . \tag{14}
\end{equation*}
$$

Subtracting (13) from (1) and considering the imperfect output measurements (12), we obtain the filtering error dynamics as follows:

$$
\begin{align*}
e_{x}(k+1)= & {\left[A-\widehat{A}-\left(I_{p}+\Delta_{k}\right) \widehat{B} K_{1} C\right] x_{k} } \\
& +\widehat{A} e_{x}(k)-\widehat{B}\left(I_{p}+\Delta_{k}\right) \phi_{s}\left(y_{k}\right) \\
& +\left[E x_{k}+G v_{k}\right] \omega_{k}-\widehat{B} D v_{k} \\
& +\widehat{B}\left(\theta_{k}-\bar{\theta}\right)\left(I_{p}+\Delta_{k}\right)\left[\phi_{s}\left(y_{k}\right)+K_{1} C x_{k}\right] \\
& +\widehat{B} \bar{\theta}\left(I_{p}+\Delta_{k}\right)\left[\phi_{s}\left(y_{k}\right)+K_{1} C x_{k}\right] \\
& -\widehat{B}\left(\theta_{k}-\bar{\theta}\right)\left(I_{p}+\Delta_{k-1}\right)\left[\phi_{s}\left(y_{k-1}\right)+K_{1} C x_{k-1}\right] \\
& -\widehat{B} \bar{\theta}\left(I_{p}+\Delta_{k-1}\right)\left[\phi_{s}\left(y_{k-1}\right)+K_{1} C x_{k-1}\right] . \tag{15}
\end{align*}
$$

We define the following error variables:

$$
\begin{equation*}
e(k) \triangleq\left[x_{k}^{T}, e_{x}^{T}(k)\right]^{T}, \quad e_{z}(k) \triangleq z(k)-\widehat{z}(k) \tag{16}
\end{equation*}
$$

Combining the error dynamics (15) with the plant (1), we obtain the following augmented filtering error dynamics:

$$
\begin{align*}
e(k+1)= & \left(\bar{A}_{1}+\Delta \bar{A}_{1}\right) e_{k}+\left(\bar{H}_{1}+\Delta \bar{H}_{1}\right) \phi_{s}\left(y_{k}\right)+\bar{B}_{1} v_{k} \\
& +\left(\theta_{k}-\bar{\theta}\right) \\
& \times\left\{\left(\bar{A}_{2}+\Delta \bar{A}_{2}\right) e_{k}+\left(\bar{H}_{2}+\Delta \bar{H}_{2}\right) \phi_{s}\left(y_{k}\right)\right\} \\
& +\left(\theta_{k}-\bar{\theta}\right) \\
& \times\left\{\left(\bar{A}_{3}+\Delta \bar{A}_{3}\right) e_{k-1}+\left(\bar{H}_{3}+\Delta \bar{H}_{3}\right) \phi_{s}\left(y_{k-1}\right)\right\} \\
& +\left(\bar{H}_{4}+\Delta \bar{H}_{4}\right) \phi\left(y_{k-1}\right)+\left(\bar{A}_{4}+\Delta \bar{A}_{4}\right) e_{k-1} \\
& +\left(\bar{E} e_{k}+\bar{G} v_{k}\right) \omega_{k}, \\
& e_{z}(k)=\bar{L} e(k), \tag{17}
\end{align*}
$$

where

$$
\begin{align*}
& \bar{A}_{1}=\left[\begin{array}{cc}
A, & 0 \\
A-\widehat{A}-\bar{\theta} \widehat{B} K_{1} C, & \widehat{A}
\end{array}\right], \quad \bar{A}_{2}=\left[\begin{array}{cc}
0, & 0 \\
\widehat{B} K_{1} C, & 0
\end{array}\right], \\
& \bar{A}_{3}=\left[\begin{array}{cc}
0, & 0 \\
-\widehat{B} K_{1} C, & 0
\end{array}\right], \quad \bar{A}_{4}=\left[\begin{array}{cc}
0, & 0 \\
-\bar{\theta} \widehat{B} K_{1} C, & 0
\end{array}\right], \\
& \bar{B}_{1}=\left[\begin{array}{c}
B \\
\widehat{B} D
\end{array}\right], \quad \bar{B}_{3}=\left[\begin{array}{c}
0 \\
-\widehat{B} D
\end{array}\right], \quad \bar{B}_{4}=\left[\begin{array}{c}
0 \\
-\bar{\theta} \widehat{B} D
\end{array}\right], \\
& \bar{H}_{1}=\left[\begin{array}{c}
0 \\
(\bar{\theta}-1) \widehat{B}
\end{array}\right], \quad \bar{H}_{2}=\left[\begin{array}{l}
0 \\
\widehat{B}
\end{array}\right], \\
& \bar{H}_{3}=\left[\begin{array}{c}
0 \\
-\widehat{B}
\end{array}\right], \quad \bar{H}_{4}=\left[\begin{array}{c}
0 \\
-\bar{\theta} \widehat{B}
\end{array}\right], \\
& \Delta \bar{A}_{1}=\left[\begin{array}{cc}
A, & 0 \\
(\bar{\theta}-1) \widehat{B} \Lambda_{k} K_{1} C, & 0
\end{array}\right], \quad \Delta \bar{A}_{2}=\left[\begin{array}{cc}
0, & 0 \\
\widehat{B} \Lambda_{k} K_{1} C, & 0
\end{array}\right], \\
& \Delta \bar{A}_{3}=\left[\begin{array}{cc}
0, & 0 \\
-\widehat{B} \Lambda_{k} K_{1} C, & 0
\end{array}\right], \quad \Delta \bar{A}_{4}=\left[\begin{array}{cc}
0, & 0 \\
-\bar{\theta} \widehat{B} \Lambda_{k} K_{1} C, & 0
\end{array}\right], \\
& \Delta \bar{B}_{3}=\left[\begin{array}{c}
0 \\
-\widehat{B} \Lambda_{k-1} D
\end{array}\right], \quad \Delta \bar{B}_{4}=\left[\begin{array}{c}
0 \\
-\bar{\theta} \widehat{B} \Lambda_{k-1} D
\end{array}\right], \\
& \Delta \bar{H}_{1}=\left[\begin{array}{c}
0 \\
(\bar{\theta}-1) \widehat{B} \Lambda_{k}
\end{array}\right], \quad \Delta \bar{H}_{2}=\left[\begin{array}{c}
0 \\
\widehat{B} \Lambda_{k}
\end{array}\right], \\
& \Delta \bar{H}_{3}=\left[\begin{array}{c}
0 \\
-\widehat{B} \Lambda_{k-1}
\end{array}\right], \quad \Delta \bar{H}_{4}=\left[\begin{array}{c}
0 \\
-\bar{\theta} \widehat{B} \Lambda_{k-1}
\end{array}\right], \\
& \bar{E}=\left[\begin{array}{ll}
E & 0 \\
E & 0
\end{array}\right], \quad \bar{G}=\left[\begin{array}{c}
G \\
G
\end{array}\right], \\
& \bar{L}=[0, L] \text {. } \tag{18}
\end{align*}
$$

For simplicity, we denote

$$
\begin{array}{ll}
\widetilde{A}_{1}:=\bar{A}_{1}+\Delta \bar{A}_{1}(k), & \widetilde{A}_{2}:=\bar{A}_{2}+\Delta \bar{A}_{2}(k) \\
\widetilde{A}_{3}:=\bar{A}_{3}+\Delta \bar{A}_{3}(k), & \widetilde{A}_{4}:=\bar{A}_{4}+\Delta \bar{A}_{4}(k) \\
\widetilde{B}_{3}:=\bar{B}_{3}+\Delta \bar{B}_{3}(k), & \widetilde{B}_{4}:=\bar{B}_{4}+\Delta \bar{B}_{4}(k)  \tag{19}\\
\widetilde{H}_{1}:=\bar{H}_{1}+\Delta \bar{H}_{1}(k), & \widetilde{H}_{2}:=\bar{H}_{2}+\Delta \bar{H}_{2}(k) \\
\widetilde{H}_{3}:=\bar{H}_{3}+\Delta \bar{H}_{3}(k), & \widetilde{H}_{4}:=\bar{H}_{4}+\Delta \bar{H}_{4}(k),
\end{array}
$$

in the following discussion.
System (17) contains sector-bounded uncertainty $\Lambda_{k}$ and stochastic parameters $\theta_{k}$, and thus it is an uncertain stochastic parameter system where the uncertainties resulted from the quantization error. Due to this fact, it is required to introduce the notion of stochastic stability before proceeding with the subsequent analysis.

Before formulating the problem to be investigated, we introduce the following definition and lemma.

Definition 2 (see [17]). The stochastic system (17) under $v(k)=0$ is said to be stochastically stable if there exists a scalar $\beta>0$ such that

$$
\begin{equation*}
\mathbb{E}\left\{\sum_{k=0}^{\infty}\|x(k)\|^{2}\right\} \leq \beta \mathbb{E}\left\{\|x(0)\|^{2}\right\} \tag{20}
\end{equation*}
$$

Lemma 3 (see [18]). For any real vectors $a, b$ and matrix $R>0$ of compatible dimensions, the following inequality holds:

$$
\begin{equation*}
a^{T} b+b^{T} a \leq a^{T} R a+b^{T} R^{-1} b \tag{21}
\end{equation*}
$$

In the sequel, the main objective of this paper is as follows.
$H_{\infty}$ Filtering Problem. Given a disturbance attenuation level $\gamma>0$, the parameters $\widehat{A}$ and $\widehat{B}$ of filter (13) are designed such that (i) the resulting filtering error system is stochastically stable for $v(k)=0$, and (ii) for any function $\phi(\cdot) \in\left[K_{1}, K_{2}\right]$, $\|z-\widehat{z}\|_{E_{2}}<\gamma\left\|v_{k}\right\|_{E_{2}}$ holds under zero initial conditions for all $\widetilde{v}(k) \in \mathscr{L}_{E_{2}}\left([0, \infty) ; \mathbb{R}^{m}\right)$.

## 3. Filtering Performance Analysis

In this section, we shall focus on the $H_{\infty}$ performance, that is, presenting sufficient conditions under which the $H_{\infty}$ performance index is achieved for a given filter.

Theorem 4. If there exist positive and definite matrices $\bar{P} \in$ $\mathbb{R}^{n \times n}$, such that the following matrix inequality constraint holds:

$$
\Gamma \triangleq\left[\begin{array}{ccccccc}
-\bar{P}+\bar{L}^{T} \bar{L} & 0 & \bar{C}^{T} K^{T} & 0 & 0 & \widetilde{A}_{1}^{T} & \widetilde{A}_{2}^{T} \\
* & -\gamma^{2} I_{m} & 0 & 0 & 0 & \bar{B}_{1}^{T} & 0 \\
* & * & -2 I_{p} & 0 & 0 & -\widetilde{H}_{1}^{T} & \widetilde{H}_{2}^{T} \\
* & * & * & -\bar{Q} \bar{C}^{T} K^{T} & \widetilde{A}_{4}^{T} & \widetilde{A}_{3}^{T} \\
* & * & * & * & -2 I_{p} & \widetilde{H}_{4}^{T} & \widetilde{H}_{3}^{T} \\
* & * & * & * & * & -\bar{P}^{-1} & 0 \\
* & * & * & * & * & * & -\theta_{1}^{2} \bar{P}^{-1}
\end{array}\right]
$$

$$
\begin{equation*}
<0 \tag{22}
\end{equation*}
$$

then the filtering error system (17) is stochastically stable.
Proof. Consider the filtering error system (17) with $v(k)=0$, select the stochastic Lyapunov functional candidate as

$$
\begin{equation*}
V(e(k), k)=e^{T}(k) \bar{P} e(k)+e_{k-1}^{T} \bar{Q} e_{k-1} \tag{23}
\end{equation*}
$$

with $\bar{P}=\left[\begin{array}{cc}P_{1} & 0 \\ 0 & P_{2}\end{array}\right]>0, \bar{Q}=\left[\begin{array}{cc}Q_{1} & 0 \\ 0 & Q_{2}\end{array}\right]>0$. It follows from (17) that

$$
\begin{align*}
\Delta V(k)= & \mathbb{E}\left\{V(e(k+1), k+1) \mid x_{k}, e_{k}\right\}-V(e(k), k) \\
= & {\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right]^{T} } \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& +\left(\bar{E} e_{k}\right)^{T} \bar{P} \bar{E} e_{k}-e_{k}^{T} \bar{Q} e_{k}-e_{k-1}^{T} \bar{Q} e_{k-1} \\
& +\mathbb{E}\left\{\left(\theta_{k}-\bar{\theta}\right)^{2}\right\} \\
& \times\left[\widetilde{A}_{2} e_{k}+\widetilde{H}_{2} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{3} e_{k-1}+\widetilde{H}_{3} \phi\left(y_{k-1}\right)\right]^{T} \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \tag{24}
\end{align*}
$$

Notice that $\mathbb{E}\left\{\left(\theta_{k}-\bar{\theta}\right)^{2}\right\}=\theta_{1}^{2}$, and thus we have

$$
\begin{align*}
\Delta V(k)= & {\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right]^{T} } \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& +\left(\bar{E} e_{k}\right)^{T} \bar{P}\left(\bar{E} e_{k}\right)-e_{k}^{T} \bar{Q} e_{k}-e_{k-1}^{T} \bar{Q} e_{k-1} \\
& +\theta_{1}^{2}\left[\widetilde{A}_{2} e_{k}+\widetilde{H}_{2} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{3} e_{k-1}+\widetilde{H}_{3} \phi\left(y_{k-1}\right)\right]^{T} \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \tag{25}
\end{align*}
$$

In fact, for the saturation function $\phi_{s}\left(y_{k}\right)$, from (5) we have that

$$
\begin{gather*}
-2 \phi_{s}^{T}\left(y_{k}\right) \phi_{s}\left(y_{k}\right)+2 \phi_{s}^{T}\left(y_{k}\right) K y_{k}>0 \\
-2 \phi_{s}^{T}\left(y_{k-1}\right) \phi_{s}\left(y_{k-1}\right)+2 \phi_{s}^{T}\left(y_{k-1}\right) K y_{k-1}>0 \tag{26}
\end{gather*}
$$

which implies that

$$
\begin{gather*}
-2 \phi_{s}^{T}\left(y_{k}\right) \phi_{s}\left(y_{k}\right)+2 \phi_{s}^{T}\left(y_{k}\right) K \bar{C} e_{k}>0 \\
-2 \phi_{s}^{T}\left(y_{k-1}\right) \phi_{s}\left(y_{k-1}\right)+2 \phi_{s}^{T}\left(y_{k-1}\right) K \bar{C} e_{k-1}>0 \tag{27}
\end{gather*}
$$

with $\bar{C} \triangleq[C, 0]$.

In the light of (26), (27), and (24), one can obtain

$$
\begin{align*}
\Delta V(k)= & {\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right]^{T} } \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& +\left(\bar{E} e_{k}\right)^{T} \bar{P}\left(\bar{E} e_{k}\right)-e_{k}^{T} \bar{Q} e_{k}-e_{k-1}^{T} \bar{Q} e_{k-1} \\
& +\theta_{1}^{2}\left[\widetilde{A}_{2} e_{k}+\widetilde{H}_{2} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{3} e_{k-1}+\widetilde{H}_{3} \phi\left(y_{k-1}\right)\right]^{T} \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& -2 \phi_{s}^{T}\left(y_{k}\right) \phi_{s}\left(y_{k}\right)+e^{T}(k) \bar{C}^{T} K^{T} \phi_{s}\left(y_{k}\right) \\
& +\phi_{s}^{T}\left(y_{k}\right) K \bar{C} e(k)-2 \phi_{s}^{T}\left(y_{k-1}\right) \phi_{s}\left(y_{k-1}\right) \\
& +e_{k-1}^{T} \bar{C}^{T} K^{T} \phi_{s}\left(y_{k-1}\right)+\phi_{s}^{T}\left(y_{k-1}\right) K \bar{C} e_{k-1} . \tag{28}
\end{align*}
$$

The following proof is divided into the following two parts: (i) a proof that the filtering error system (17) is stochastically stable with $v(k)=0$, and (ii) a proof that $\left\|e_{z}(k)\right\|_{E_{2}}<\gamma\left\|v_{k}\right\|_{E_{2}}$.
(i) Firstly, we establish the stochastic stability of the filtering error system (17) under the condition (22). It follows from (28) with $v_{k}=0$ that

$$
\begin{align*}
\Delta V(k)= & {\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right]^{T} } \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& +e_{k}^{T} \bar{E}^{T} \overline{P E} e_{k}-e_{k}^{T} \bar{Q} e_{k}-e_{k-1}^{T} \bar{Q} e_{k-1} \\
& +\theta_{1}^{2}\left[\widetilde{A}_{2} e_{k}+\widetilde{H}_{2} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{3} e_{k-1}+\widetilde{H}_{3} \phi\left(y_{k-1}\right)\right]^{T} \\
& \times \bar{P}\left[\widetilde{A}_{1} e_{k}+\widetilde{H}_{1} \phi_{s}\left(y_{k}\right)+\widetilde{A}_{4} e_{k-1}+\widetilde{H}_{4} \phi\left(y_{k-1}\right)\right] \\
& -2 \phi_{s}^{T}\left(y_{k}\right) \phi_{s}\left(y_{k}\right)+e^{T}(k) \bar{C}^{T} K^{T} \phi_{s}\left(y_{k}\right) \\
& +\phi_{s}^{T}\left(y_{k}\right) K \bar{C} e(k)-2 \phi_{s}^{T}\left(y_{k-1}\right) \phi_{s}\left(y_{k-1}\right) \\
& +e_{k-1}^{T} \bar{C}^{T} K^{T} \phi_{s}\left(y_{k-1}\right)+\phi_{s}^{T}\left(y_{k-1}\right) K \bar{C} e_{k-1} \\
\leq & \xi^{T}(k) \Pi \xi(k), \tag{29}
\end{align*}
$$

where

$$
\begin{align*}
& \xi(k) \triangleq\left[e^{T}(k), \phi_{s}^{T}\left(y_{k}\right), e^{T}(k-1), \phi_{s}^{T}\left(y_{k-1}\right)\right]^{T}, \\
& \Pi \triangleq\left[\begin{array}{llll}
\widetilde{A}_{1} & \widetilde{H}_{1} & \widetilde{A}_{4} & \widetilde{H}_{4} \\
\widetilde{A}_{2} & \widetilde{H}_{2} & \widetilde{A}_{3} & \widetilde{H}_{3}
\end{array}\right]^{T}\left[\begin{array}{cc}
\bar{P} & 0 \\
0 & \theta_{1}^{2} \bar{P}
\end{array}\right]\left[\begin{array}{cccc}
\widetilde{A}_{1} & \widetilde{H}_{1} & \widetilde{A}_{4} & \widetilde{H}_{4} \\
\widetilde{A}_{2} & \widetilde{H}_{2} & \widetilde{A}_{3} & \widetilde{H}_{3}
\end{array}\right] \\
&+\left[\begin{array}{cccc}
-\bar{P}+\bar{L}^{T} \bar{L} & \bar{C}^{T} \bar{K}^{T} & 0 & 0 \\
* & -2 I_{p} & 0 & 0 \\
* & * & -Q & \bar{C}^{T} K^{T} \\
* & * & * & -2 I_{p}
\end{array}\right] . \tag{30}
\end{align*}
$$

It is obvious that if $\Pi<0$, one can obtain that $\Delta V(k)<$ 0 . Therefore, it follows from Kolmanovskii and Myshkis [19] that the filtering error system (17) with $v(k)=0$ is stochastically stable. On the other hand, by means of Schur's complement, $\Pi<0$ is equivalent to the following matrix condition:

$$
\left[\begin{array}{cccccc}
-\bar{P}+\bar{L}^{T} \bar{L} & \bar{C}^{T} \bar{K}^{T} & 0 & 0 & \widetilde{A}_{1}^{T} & \widetilde{A}_{2}^{T}  \tag{31}\\
* & -2 I_{p} & 0 & 0 & \widetilde{H}_{1}^{T} & \widetilde{H}_{2}^{T} \\
* & * & -Q & \bar{C}^{T} K^{T} & \widetilde{A}_{4}^{T} & \widetilde{A}_{3}^{T} \\
* & * & * & -2 I_{p} & \widetilde{H}_{4}^{T} & \widetilde{H}_{3}^{T} \\
* & * & * & * & -\bar{P}^{-1} & 0 \\
* & * & * & * & * & -\bar{\theta}^{2} \bar{P}^{-1}
\end{array}\right]<0 .
$$

Notice that the condition (22) can imply (31). This means that if there exist positive and definite matrices $\bar{P}$, such that matrix condition (22) holds, then system (17) is stochastically stable.
(ii) Next, the objective should be devoted to prove that the filtering error system (17) satisfies

$$
\begin{equation*}
\left\|e_{z}(k)\right\|_{E_{2}}<\gamma\left\|v_{k}\right\|_{E_{2}} \tag{32}
\end{equation*}
$$

for all nonzero $v(k) \in \mathscr{L}_{E_{2}}\left([0, \infty) ; \mathbb{R}^{m}\right)$.
In fact, under zero initial conditions, it is shown that

$$
\begin{equation*}
\mathbb{E}\{V(e(k), t)\}=\mathbb{E}\left\{\sum_{k=1}^{N} \Delta V(k)\right\} . \tag{33}
\end{equation*}
$$

We define the following performance index function:

$$
\begin{equation*}
J(k)=\mathbb{E}\left\{\sum_{k=1}^{N}\left[e_{z}^{T}(k) e_{z}(k)-\gamma^{2} v_{k}^{T} v_{k}\right]\right\} \tag{34}
\end{equation*}
$$

for any integer $N$.
It follows from (34) that

$$
\begin{align*}
J(k)= & \mathbb{E}\left\{\sum_{k=1}^{N}\left[e_{z}^{T}(k) e_{z}(k)-\gamma^{2} v_{k}^{T} v_{k}+\Delta V(k)\right]\right\} \\
& -\mathbb{E}\{V(e(k), N)\}  \tag{35}\\
& \leq \mathbb{E}\left\{\sum_{k=1}^{N} \zeta^{T}(k) \Xi \zeta(k)\right\}
\end{align*}
$$

where

$$
\begin{align*}
\zeta(k) \triangleq & {\left[e^{T}(k), v^{T}(k), \phi_{s}^{T}\left(y_{k}\right), e^{T}(k-1), \phi_{s}^{T}\left(y_{k-1}\right)\right]^{T} } \\
\Xi \triangleq & {\left[\begin{array}{ccccc}
\widetilde{A}_{1} & \widetilde{B}_{1} & \widetilde{H}_{1} & \widetilde{A}_{4} & \widetilde{H}_{4} \\
\widetilde{A}_{2} & 0 & \widetilde{H}_{2} & \widetilde{A}_{3} & \widetilde{H}_{3}
\end{array}\right]^{T}\left[\begin{array}{cc}
\bar{P} & 0 \\
0 & \theta_{1}^{2} \bar{P}
\end{array}\right] } \\
& \times\left[\begin{array}{cccccc}
\widetilde{A}_{1} & \widetilde{B}_{1} & \widetilde{H}_{1} & \widetilde{A}_{4} & \widetilde{H}_{4} \\
\widetilde{A}_{2} & 0 & \widetilde{H}_{2} & \widetilde{A}_{3} & \widetilde{H}_{3}
\end{array}\right] \\
& +\left[\begin{array}{cccccc}
-\bar{P}+\bar{L}^{T} \bar{L} & 0 & \bar{C}^{T} \bar{K}^{T} & 0 & 0 \\
* & -\gamma^{2} I_{m} & 0 & 0 & 0 \\
* & * & -2 I_{p} & 0 & 0 \\
* & * & * & -Q & \bar{C}^{T} K^{T} \\
* & * & * & * & -2 I_{p}
\end{array}\right] \tag{36}
\end{align*}
$$

From (35), it can be shown that if $\Xi<0$ holds, then $J(k)<0$, which implies that (32) holds for any nonzero $v(k) \in \mathscr{L}_{E_{2}}\left([0, \infty) ; \mathbb{R}^{m}\right)$. Furthermore, it is easy to see that $\Xi<0$ is equivalent to the condition (22). This means that the condition (22) in Theorem 4 can imply $\left\|e_{z}(k)\right\|_{E_{2}}<\gamma\|v(k)\|_{E_{2}}$. This completes the proof.

## 4. $H_{\infty}$ Filter Design

In this section, the attention should be paid on coping with the addressed filter design problem for the discrete-time stochastic system (1) based on Theorem 4.

Lemma 5 (see [18]). Let $Q, R$, and $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F^{T}(t) F(t)<I$. Then, for any scalar $\varepsilon>0$,

$$
\begin{equation*}
Q F(t) R+R^{T} F^{T}(t) Q^{T} \leq \varepsilon Q Q^{T}+\varepsilon^{-1} R^{T} R \tag{37}
\end{equation*}
$$

The following theorem provides the sufficient LMI condition for the existence of the proposed robust filter (13).

Theorem 6. Consider the discrete-time stochastic system (1), for a given disturbance level $\gamma>0$, if there exist positive and definite matrices $X, Y \in \mathbb{R}^{n \times n}$, and matrices $W \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}^{n \times m}$, and positive scalars $\varepsilon_{1}, \varepsilon_{2}$, and $\varepsilon_{3}$ such that the following LMI holds:

$$
\left[\begin{array}{ccc}
\Phi_{11} & \Phi_{12} & \Phi_{13}  \tag{38}\\
* & \Phi_{22} & \Phi_{23} \\
* & * & \Phi_{33}
\end{array}\right]<0
$$

with

$$
\begin{aligned}
& \Phi_{11}=\left[\begin{array}{ccccccc}
\widetilde{\Gamma}_{11} & 0 & 0 & C^{T} K^{T} & 0 & 0 & 0 \\
* & \widetilde{\Gamma}_{22} & 0 & 0 & 0 & 0 & 0 \\
* & * & \widetilde{\Gamma}_{33} & 0 & 0 & 0 & 0 \\
* & * & * & \widetilde{\Gamma}_{44} & 0 & 0 & 0 \\
* & * & * & * & -Q_{1} & 0 & C^{T} K^{T} \\
* & * & * & * & * & -Q_{2} & 0 \\
* & * & * & * & * & * & -2 I_{p}
\end{array}\right], \\
& \Phi_{12}=\left[\begin{array}{cccc}
A^{T} X & \widetilde{\Gamma}_{1,9} & 0 & \widetilde{\Gamma}_{1,11} \\
0 & \widehat{A}^{T} Y & 0 & 0 \\
0 & \widetilde{\Gamma}_{3,9} & 0 & D^{T} U^{T} \\
0 & \widetilde{\Gamma}_{4,9} & 0 & U^{T} \\
0 & \widetilde{\Gamma}_{5,9} & 0 & \widetilde{\Gamma}_{5,11} \\
0 & 0 & 0 & 0 \\
0 & -\bar{\theta} U^{T} & 0 & -U^{T}
\end{array}\right], \\
& \Phi_{13}=\left[\begin{array}{ccccc}
0 & 0 & 0 & E^{T} X & E^{T} Y \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G^{T} X & G^{T} Y \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right], \\
& \Phi_{22}=\operatorname{diag}\left\{-X,-Y,-\theta_{1}^{2} X,-\theta_{1}^{2} Y\right\} \text {, } \\
& \Phi_{23}=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
\widetilde{\Gamma}_{9,12} & \widetilde{\Gamma}_{9,13} & \widetilde{\Gamma}_{9,14} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
U & U & U & 0 & 0
\end{array}\right], \\
& \Phi_{33}=\operatorname{diag}\left\{-\varepsilon_{1} I_{p},-\varepsilon_{2} I_{p},-\varepsilon_{3} I_{p},-X,-Y\right\}, \\
& \widetilde{\Gamma}_{11}=-\bar{P}+\bar{L}^{T} \bar{L}+\varepsilon_{1} \bar{C}^{T} K_{1}^{T} \Delta^{T} \Delta K_{1} \bar{C}, \\
& \widetilde{\Gamma}_{1,9}=A^{T} Y-\widehat{A}^{T} Y+(\bar{\theta}-1) C^{T} K_{1}^{T} \widehat{B}^{T} Y, \\
& \widetilde{\Gamma}_{1,11}=A^{T} Y-W^{T}+(\bar{\theta}-1) C^{T} K_{1}^{T} U^{T}, \\
& \tilde{\Gamma}_{22}=-Y+L^{T} L, \\
& \widetilde{\Gamma}_{33}=-\gamma^{2} I_{r}+\varepsilon_{2} D^{T} \Delta \Delta D, \\
& \widetilde{\Gamma}_{3,9}=(\bar{\theta}-1) D^{T} U^{T}, \\
& \widetilde{\Gamma}_{4,4}=-2 I_{p}+\varepsilon_{3} \Delta \Delta, \\
& \widetilde{\Gamma}_{4,9}=(\bar{\theta}-1) U^{T},
\end{aligned}
$$



Figure 2: $x_{1}(k)$ and $\widehat{x}_{1}(k)$.

$$
\begin{gather*}
\widetilde{\Gamma}_{5,9}=-\bar{\theta} C^{T} K_{1}^{T} U^{T}, \\
\widetilde{\Gamma}_{5,11}=-C^{T} K_{1}^{T} Y, \\
\widetilde{\Gamma}_{9,12}=\widetilde{\Gamma}_{9,13}=\widetilde{\Gamma}_{9,14}=-\bar{\theta} U, \tag{39}
\end{gather*}
$$

then the $H_{\infty}$ filtering problem is solved by the filter (13). Furthermore, the filter gains are given by

$$
\begin{equation*}
\widehat{A}=Y^{-1} W, \quad \widehat{B}=Y^{-1} U \tag{40}
\end{equation*}
$$

Proof. The matrix condition (22) in Theorem 4 can be rewritten as

$$
\left[\begin{array}{cccccccc}
\Gamma_{11} & 0 & \bar{C}^{T} K^{T} & 0 & 0 & 0 & \bar{A}_{1}^{T} & \bar{A}_{2}^{T} \\
* & \Gamma_{22} & 0 & 0 & 0 & 0 & \bar{B}_{1}^{T} & 0 \\
* & * & -2 I_{p} & 0 & 0 & 0 & -\bar{H}_{1}^{T} & \bar{H}_{2}^{T} \\
* & * & * & -\bar{Q} & 0 & \bar{C}^{T} K^{T} & \bar{A}_{4}^{T} & A_{3}^{T} \\
* & * & * & * & -\gamma^{2} I_{m} & 0 & \bar{B}_{4}^{T} & \bar{B}_{3}^{T} \\
* & * & * & * & * & -2 I_{p} & \bar{H}_{4}^{T} & \bar{H}_{3}^{T}  \tag{41}\\
* & * & * & * & * & * & -\bar{P}^{-1} & 0 \\
* & * & * & * & * & * & * & -\theta_{1}^{2} \bar{P}^{1} \\
& +M_{1} \Lambda_{k} N_{1}+N_{1}^{T} \Lambda_{k} M_{1}^{T}+M_{2} \Lambda_{k} N_{2} & \\
& +N_{2}^{T} \Lambda_{k} M_{2}^{T}+M_{1} \Lambda_{k} N_{3}+N_{3}^{T} \Lambda_{k} M_{1}^{T}<0,
\end{array}\right.
$$

where

$$
\begin{gather*}
M_{1}=\left[0,0,0,0,0,0, \bar{H}_{1}^{T}, \bar{H}_{2}^{T}\right]^{T}, \\
N_{1}=\left[K_{1} \bar{C}, 0,0, K_{1} \bar{C}, 0,0,0,0\right],  \tag{42}\\
N_{2}=[0, D, 0,0, D, 0,0,0]^{T}, \\
N_{3}=[0,0, I, 0,0, I, 0,0]^{T} .
\end{gather*}
$$

In the light of Lemma 5, it is shown that the condition (41) holds if the following condition holds:

$$
\left[\begin{array}{ccccccccccc}
\Gamma_{11} & 0 & \bar{C}^{T} K^{T} & 0 & 0 & 0 & \bar{A}_{1}^{T} & \bar{A}_{2}^{T} & 0 & 0 & 0  \tag{43}\\
* & \Gamma_{22} & 0 & 0 & 0 & 0 & \bar{B}_{1}^{T} & 0 & 0 & 0 & 0 \\
* & * & -2 I_{p} & 0 & 0 & 0 & -\bar{H}_{1}^{T} & \bar{H}_{2}^{T} & 0 & 0 & 0 \\
* & * & * & -Q & 0 & \bar{C}^{T} K^{T} & \bar{A}_{4}^{T} & \bar{A}_{3}^{T} & 0 & 0 & 0 \\
* & * & * & * & -\gamma^{2} I_{m} & 0 & \bar{B}_{4}^{T} & \bar{B}_{3}^{T} & 0 & 0 & 0 \\
* & * & * & * & * & -2 I_{p} & \bar{H}_{4}^{T} & \bar{H}_{3}^{T} & 0 & 0 & 0 \\
* & * & * & * & * & * & -\bar{P}^{-1} & 0 & (\bar{\theta}-1) \bar{H}_{2} & (\bar{\theta}-1) \bar{H}_{2} & (\bar{\theta}-1) \bar{H}_{2} \\
* & * & * & * & * & * & * & -\theta_{1}^{2} \bar{P}^{1} & \bar{H}_{2} & \bar{H}_{2} & \bar{H}_{2} \\
* & * & * & * & * & * & * & * & -\varepsilon_{1} I_{m} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\varepsilon_{2} I_{m} & 0 \\
* & * & * & * & * & * & * & * & * & * & -\varepsilon_{3} I_{m}
\end{array}\right]<0 .
$$



Figure 3: $x_{2}(k)$ and $\widehat{x}_{2}(k)$.

Multiply $\operatorname{diag}\left\{I, I, I, I, I, I, P_{1}, P_{1}, I, I, I\right\}$ and its transpose on the left side and the right side of (43), respectively, and let $W=Y \widehat{A}, U=Y \widehat{B}$, one can obtain the condition (38). This means that if there exist scalars $\varepsilon_{1}>0, \varepsilon_{2}>0, \varepsilon_{3}>0$ such that the LMI condition (38) holds, then the error system (17) is stochastically stable, and the $H_{\infty}$ performance (32) is guaranteed. This completes the proof.

Remark 7. The LMI condition (38) of Theorem 6 is not conservative, since the system matrix $A$ has been supposed to be stable.

## 5. Simulation

We consider the system (1) with the following data: $n=3$, $m=2$, and $p=2, r=2$. For the logarithmic quantizer (7), the quantizer densities are chosen as $\rho_{1}=0.6667$ and $\rho_{2}=$ 0.7391. The initial quantizer points are chosen as $\eta_{1}^{(0)}=40$ and $\eta_{2}^{(0)}=40$. It can be calculated that $\delta_{1}=0.2$ and $\delta_{2}=0.15$. The random communication delay parameters are selected as $\bar{\theta}=0.6$, and $\theta_{1}=\sqrt{\bar{\theta}(1-\bar{\theta})}=0.4899$. The saturation parameter matrices are selected as follows:

$$
K_{1}=\left[\begin{array}{cc}
0.6 & 0  \tag{44}\\
0 & 0.7
\end{array}\right], \quad K_{2}=\left[\begin{array}{cc}
0.8 & 0 \\
0 & 0.8
\end{array}\right]
$$

and $\phi\left(y_{k}\right)=\left(K_{1}+K_{2}\right) / 2 y_{k}+\left(K_{2}-K_{1}\right) / 2 \sin \left(y_{k}\right)$.

Besides, the model parameters are given as follows:

$$
\begin{gather*}
A=\left[\begin{array}{ccc}
-0.3 & 0 & 0.01 \\
-0.59 & -0.24 & 0.02 \\
0.1 & -0.06 & -0.68
\end{array}\right], \quad B=\left[\begin{array}{c}
-0.202 \\
0.383 \\
0.139
\end{array}\right], \\
C=\left[\begin{array}{ccc}
-0.2 & -0.1 & -0.2 \\
0.5 & 0.2 & 0.21
\end{array}\right], \quad D=\left[\begin{array}{c}
0.1 \\
0.47
\end{array}\right] \\
E=\left[\begin{array}{ccc}
-0.12 & -0.11 & 0.38 \\
0.11 & 0.64 & -0.18 \\
-0.31 & -0.63 & -0.6
\end{array}\right]  \tag{45}\\
G=\left[\begin{array}{c}
-0.13 \\
0.11 \\
0.051
\end{array}\right], \quad L=\left[\begin{array}{ccc}
0.1 & 0.09 & 0.1 \\
0.05 & 0.05 & 0.05
\end{array}\right]
\end{gather*}
$$

Without loss of generality, we assume that the noises $v(k)$ in system (1) have the following form:

$$
\begin{equation*}
v(k)=\frac{1}{0.1+k^{2}}, \tag{46}
\end{equation*}
$$

and it can be checked that $v(k)$ satisfies the constraint (2). Solving the LMI condition (38), one can obtain the following solutions:

$$
\begin{gather*}
X=\left[\begin{array}{ccc}
3.6195 & 3.3264 & 2.2784 \\
3.3264 & 10.8295 & 2.5230 \\
2.2784 & 2.5230 & 7.4548
\end{array}\right], \\
Y=\left[\begin{array}{lll}
4.3244 & 1.5752 & 1.6206 \\
1.5752 & 2.3707 & 0.2978 \\
1.6206 & 0.2978 & 2.3496
\end{array}\right],  \tag{47}\\
W=\left[\begin{array}{ccc}
-1.3882 & -0.4611 & -0.7446 \\
-1.1207 & -0.0697 & -0.4972 \\
-0.4623 & -0.1005 & -1.4395
\end{array}\right], \\
U=\left[\begin{array}{ccc}
0.0118 & 0.0050 \\
0.0469 & 0.1449 \\
0.0390 & 0.0963
\end{array}\right] .
\end{gather*}
$$

The filter gain are then calculated as follows:

$$
\begin{gather*}
\widehat{A}=\left[\begin{array}{ccc}
-0.1855 & -0.1492 & 0.1803 \\
-0.3463 & 0.0632 & -0.2408 \\
-0.0249 & 0.0521 & -0.7065
\end{array}\right], \\
\widehat{B}=\left[\begin{array}{cc}
-0.0165 & -0.0580 \\
0.0277 & 0.0909 \\
0.0245 & 0.0695
\end{array}\right] \tag{48}
\end{gather*}
$$

The initial condition is selected as $x(0)=\left[\begin{array}{lll}1.5 & 0 & -1\end{array}\right]^{T}$, $e(0)=\left[\begin{array}{ccc}-0.5 & 1 & 1.5\end{array}\right]^{T}$, and the quantizer parameter $\eta_{0}$ is selected as $\eta_{0}=50$. The trajectories of plant states $x_{k}$ and its estimates are shown in Figures 2, 3, and 4; the comparisons of the unquantized saturated outputs $\phi\left(y_{k}\right)$ and quantized saturated outputs $q\left(\phi\left(y_{k}\right)\right)$ are shown in Figures 5 and 6; the trajectory of the error estimation signal $e_{z}(k)=z(k)-\widehat{z}(k)$ is illustrated in Figures 7 and 8. It can be seen that the obtained state estimation is desirable.


Figure 4: $x_{3}(k)$ and $\widehat{x}_{3}(k)$.


Figure 5: $\phi\left(y_{1}(k)\right)$ and quantized $\phi\left(y_{1}(k)\right)$.

## 6. Conclusion

In this paper, the $H_{\infty}$ filtering problem has been investigated for stochastic systems subject to sensor saturation over limited capacity channel. The plant under consideration is a class of stochastic systems with random noise depending on state and external disturbance. In this setting, the effects of sensor quantization, output logarithmic quantization, and networked-induced communication delay are taken into account simultaneously. The phenomenon of random communication delay is described by a Bernoulli type stochastic variable. Subsequently, the $H_{\infty}$ filter is designed for the


Figure 6: $\phi\left(y_{2}(k)\right)$ and quantized $\phi\left(y_{2}(k)\right)$.


Figure 7: $z_{1}(k)$ and $\widehat{z}_{1}(k)$.
considered plant by employing the quantized output measurements, and sufficient conditions are established for the existence of the proposed filter.

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Figure 8: $z_{2}(k)$ and $\widehat{z}_{2}(k)$.

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## Research Article

# Stability Analysis for Delayed Neural Networks: Reciprocally Convex Approach 

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#### Abstract

This paper is concerned with global stability analysis for a class of continuous neural networks with time-varying delay. The lower and upper bounds of the delay and the upper bound of its first derivative are assumed to be known. By introducing a novel Lyapunov-Krasovskii functional, some delay-dependent stability criteria are derived in terms of linear matrix inequality, which guarantee the considered neural networks to be globally stable. When estimating the derivative of the LKF, instead of applying Jensen's inequality directly, a substep is taken, and a slack variable is introduced by reciprocally convex combination approach, and as a result, conservatism reduction is proved to be more obvious than the available literature. Numerical examples are given to demonstrate the effectiveness and merits of the proposed method.


## 1. Introduction

In recent years, important application in the fields of pattern recognition, signal processing, optimization and associative memories, and so forth has made neural networks (NNs) the eye of attention. Stability analysis of NNs with time-varying delay, as a result, has received much attention ever since, because time delay is frequently encountered in NNs, owing to finite switching speed of amplifiers in the communication and response of neurons, and could cause instability and oscillations in the system. Both delay-independent and delaydependent stability criteria have been brought up. While delay-independent criteria tend to be more conservative, more attention is given to delay-dependent criteria, because they could make use of the length of the delay.

Global stability of various recurrent neural networks has been investigated in [1-4]. In stability analysis of neural networks, the qualitative properties primarily concerned are uniqueness, global asymptotic stability, and global exponential stability of their equilibria. When the system with time
delay is described by the following dynamic equation:

$$
\begin{equation*}
\dot{x}(t)=-C x(t)+A f(x(t))+B f(x(t-h))+u \tag{1}
\end{equation*}
$$

exponential and asymptotic stability analysis has been done by [5-10]. The stability criteria in [5, 9, 10] are delay independent, while those in $[6,8,11]$ are delay dependent, and [12] includes both. Also, [8-10] adopted the delay-partitioning approach. After some changes are made on system description, such as setting $A=0$, adding a new term $D \int_{t-d}^{t} f(x(s)) d s$, and introducing a new activation function $g(x(t-h))$ to substitute for $f(x(t-h))$, asymptotic stability criteria are derived in [13-17], and uniqueness analysis is done in $[5,18]$.

Various methods have been proposed to reduce conservatism when deriving stability criteria. For example, the freeweighting matrix approach noticed in $[6,7,19-22]$ is proved to be very effective since bounding techniques on some crossproduct terms are avoided. Stability analyses of NNs with multiple and single time-varying delays are done by [17, 23], respectively. Moreover, a new free-weighting matrix
approach is brought up in [24] to estimate the derivative of Lyapunov functional without missing any negative quadratic terms, and thus, improved delay-dependent stability criteria are established. Along with free-weighting matrix approach, $[14,25,26]$ adopted the delay partitioning idea to solve delay-dependent stability problem, and the proposed methods have significantly reduced conservatism.

When deriving stability criteria for delayed systems, two kinds of approaches are usually used, namely, Lyapunov function approaches [27-29] and Lyapunov-Krasovskii functional (LKF) approaches [17, 23, 24, 30-33]. The former makes no restriction on the derivative of time delay and usually gives a simpler stability criterion or delay-independent criterion while the latter, expressed in the form of LMI, takes the derivative of time delay into account, gives a delay-dependent criterion, and thus can be less conservative since LKF makes use of more information about the system. Discretized LKF method developed in [34] is another method for stability analysis, and [13] made some necessary adjustments to make the method compatible with robust stability problem for NNs with uncertain delays.

In addition, the range of time-varying delay for NNs is mostly considered to have a lower bound of zero, as seen in $[7,11,23,24]$, while in practice it may not be restricted to 0 , and setting $h_{1}$ to zero would result in increased conservatism. It is the same with the derivative of time delay. In many papers, time delay of NNs is either constant or unknown, as noticed in $[14,25,26]$, while an upper or lower bound could be assumed.

In this paper, the stability problem for continuous NNs with time-varying delay is taken into consideration. A novel LKF is brought up, and changes are made to deal with different cases concerning time delay and its derivative. When estimating the derivative of LKF, instead of applying Jensen's inequality directly, a substep is taken, and a slack variable is introduced, and consequently, conservatism reduction is proved to be more obvious than existing results. Numerical examples are given, and analysis is made to demonstrate the effectiveness and merits of the proposed method.

In Section 1, a brief introduction is presented, and some notations are defined. In Section 2, the stability problem is formulated, and some preliminaries are given. In Section 3, new criteria in the form of one theorem and three corollaries for NNs with time-varying delay are presented. In Section 4, numerical examples are presented, along with results from the other literature. The paper is concluded in Section 5.

Notations. The notations used throughout the paper are standard. The superscript " $T$ " stands for matrix transposition; $R^{n}$ denotes the $n$-dimensional Euclidean space; the notation $P>0$ means that $P$ is a real positive definite; $I$ and 0 represent the identity matrix and a zero matrix, respectively; $\operatorname{diag}(\cdots)$ stands for a block-diagonal matrix; $\lambda_{\text {min }}(P)$ $\left(\lambda_{\max }(P)\right)$ denotes the minimum (maximum) eigenvalue of symmetric matrix $P ;\|\cdot\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices, we use an asterisk $(*)$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not
explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Model Descriptions and Preliminaries

The dynamic behavior of a continuous-time neural networks with time delay can be described by the following state equation:

$$
\begin{equation*}
\frac{d y(t)}{d t}=-C y(t)+A g(y(t))+B g(y(t-h(t)))+u \tag{2}
\end{equation*}
$$

where $y(t)$ is the state vector of the neural network; $C$ is a positive matrix; $A$ and $B$ are the connection weight and the delayed connection weight matrices, respectively. $g(y(t))$ represents the activation function vector of neurons, and $u$ is a constant external bias vector. $h(t)$ denotes axonal signal transmission delay, which is nonnegative, bounded, and has $0<h_{1}<h<h_{2}, 0 \leq \dot{h} \leq \gamma$, and will be written as $h$ for short throughout the paper. The initial conditions associated with system (2) are of the form

$$
\begin{equation*}
y_{i}(\theta)=\varphi_{i}(\theta), \quad \theta \in\left[-h_{2}, 0\right], i=1,2, \ldots, n \tag{3}
\end{equation*}
$$

where $\varphi_{i}(\theta) \in C\left(\left[-h_{2}, 0\right], R^{n}\right)$ is the Banach space of continuous functions mapping interval $\left[-h_{2}, 0\right]$ into $R^{n}$.

The following assumptions are made on system (2) throughout this paper.
(H1) The activation functions $g(y(t))$ are bounded and monotonically nondecreasing on $R$.
(H2) The activation functions $g(y(t))$ satisfy

$$
\begin{equation*}
l_{m, i} \leq \frac{g_{i}\left(\omega_{1}\right)-g_{i}\left(\omega_{2}\right)}{\omega_{1}-\omega_{2}} \leq l_{p, i}, \quad \omega_{1} \neq \omega_{2} \tag{4}
\end{equation*}
$$

It is known that bounded activation functions always guarantee the existence of an equilibrium point for model (2). For convenience of exposition, in the following, we will shift the equilibrium point $y^{*}$ of model (2) to the origin. The transformation $x(t)=y(t)-y^{*}$ puts system (2) into the following form:

$$
\begin{equation*}
\frac{d x(t)}{d t}=-C x(t)+A f(x(t))+B f(x(t-h)) \tag{5}
\end{equation*}
$$

where $x(t)$ is the state vector of the transformed system, and $f_{i}\left(x_{i}\right)=g_{i}\left(x_{i}+y_{i}^{*}\right)-g_{i}\left(y_{i}^{*}\right)$ with $f_{i}(0)=0, x_{i}(\theta)=\phi_{i}(\theta)$, $\theta \in\left[-h_{2}, 0\right],(i=1, \ldots, n)$. Obviously, the equilibrium point $y^{*}$ of system (2) is globally stable if and only if the origin of system (5) is globally stable. Assume that $f_{i}(0)=0$, then the functions $f_{i}(\cdot)$ satisfy

$$
\begin{align*}
& f_{\mathrm{i}}\left(\omega_{i}\right) \leq l_{p, i} \omega_{i} \\
& f_{i}\left(\omega_{i}\right) \geq l_{m, i} \omega_{i}, \quad \forall \omega_{i} \in R \tag{6}
\end{align*}
$$

Rewriting (6), we can get

$$
\begin{gather*}
L_{p} \omega-f(\omega) \geq 0, \quad f(\omega)-L_{m} \omega \geq 0 \\
{\left[L_{p} \omega-f(\omega)\right]^{T} S\left[f(\omega)-L_{m} \omega\right] \geq 0, \quad S \geq 0} \tag{7}
\end{gather*}
$$

where $L_{p} \triangleq \operatorname{diag}\left(l_{p, 1}, l_{p, 2}, \ldots, l_{p, n}\right)$ and $L_{m} \triangleq \operatorname{diag}\left(l_{m, 1}, l_{m, 2}\right.$, $\left.\ldots, l_{m, n}\right)$.

Lemma 1 (see [35]). For positive definite $M \in R^{n \times n}$, scalars $r_{1}$ and $r_{2}$, and a vector function $w \in R^{n}$, the following inequality holds:

$$
\begin{align*}
& \left(\int_{r_{1}}^{r_{2}} w^{T}(s) d s\right) M\left(\int_{r_{1}}^{r_{2}} w(s) d s\right) \\
& \quad \leq\left(r_{2}-r_{1}\right) \int_{r_{1}}^{r_{2}} w^{T}(s) M w(s) d s \tag{8}
\end{align*}
$$

## 3. Main Results

In this section, some delay-dependent sufficient conditions of the global stability for the neural networks with time-varying delay in (5) are derived. First, we consider the case where the upper bound of $\dot{h}$ is known, and correspondingly, the global stability condition is given as follows.

Theorem 2. Suppose that in reference system (5), the time delay $h$ satisfies $0<h_{1}<h<h_{2}, 0 \leq \dot{h} \leq \gamma$. Under the condition given in (6), if there exist matrices $P>0, Q_{1}>0$, $Q_{2}>0, R_{1}>0, R_{2}>0, K_{1}>0, K_{2}>0, S_{i}>0, i=1, \ldots, 4$, $S_{1} \triangleq \operatorname{diag}\left(s_{1,1}, s_{1,2}, \ldots, s_{1, n}\right), S_{2} \triangleq \operatorname{diag}\left(s_{2,1}, s_{2,2}, \ldots, s_{2, n}\right)$, and $T$ such that

$$
\begin{align*}
& \Theta \triangleq \Upsilon_{1}+\Upsilon_{2}+\Upsilon_{3}+\Upsilon_{4}<0 \\
& \Psi \triangleq\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \geq 0 \tag{9}
\end{align*}
$$

where $\Upsilon_{1}, \Upsilon_{2}, \Upsilon_{3}$, and $\Upsilon_{4}$ are defined as

$$
\begin{aligned}
\Upsilon_{1} \triangleq & D_{x}^{T} P D_{\dot{x}}+D_{\dot{x}}^{T} P D_{x}+D_{x}^{T} Q_{1} D_{x}-D_{x h_{1}}^{T} Q_{1} D_{x h_{1}} \\
& +D_{x}^{T} Q_{2} D_{x}-D_{x h_{2}}^{T} Q_{2} D_{x h_{2}}, \\
\Upsilon_{2} \triangleq & D_{x}^{T} S_{3} D_{x}-(1-\gamma) D_{x h}^{T} S_{3} D_{x h}+D_{f}^{T} S_{4} D_{f} \\
& -(1-\gamma) D_{f h}^{T} S_{4} D_{f h}+2 D_{f}^{T} S_{1} D_{\dot{x}}-2 D_{x}^{T} L_{m}^{T} S_{1} D_{\dot{x}} \\
& -2 D_{f}^{T} S_{2} D_{\dot{x}}+2 D_{x}^{T} L_{p}^{T} S_{2} D_{\dot{x}}, \\
\Upsilon_{3} \triangleq & D_{f h}^{T} K_{2} L_{p} D_{x h}-D_{f h}^{T} K_{2} D_{f h}-D_{x h}^{T} L_{m}^{T} K_{2} L_{p} D_{x h} \\
& +D_{x h}^{T} L_{m}^{T} K_{2} D_{f h}+D_{f}^{T} K_{1} L_{p} D_{x}-D_{f}^{T} K_{1} D_{f} \\
& -D_{x}^{T} L_{m}^{T} K_{1} L_{p} D_{x}+D_{x}^{T} L_{m}^{T} K_{1} D_{f}, \\
\Upsilon_{4} \triangleq & h_{2}^{2} D_{\dot{x}}^{T} R_{1} D_{\dot{x}}-\left(D_{x}-D_{x h_{2}}\right)^{T} R_{1}\left(D_{x}-D_{x h_{2}}\right) \\
& +\left(h_{2}-h_{1}\right)^{2} D_{\dot{x}}^{T} R_{2} D_{\dot{x}}-\left(D_{x h}-D_{x h_{1}}\right)^{T} R_{2}\left(D_{x h}-D_{x h_{1}}\right) \\
& -\left(D_{x h_{2}}-D_{x h}\right)^{T} R_{2}\left(D_{x h_{2}}-D_{x h}\right) \\
& -\left(D_{x h}-D_{x h_{2}}\right)^{T} T^{T}\left(D_{x h_{1}}-D_{x h}\right) \\
& -\left(D_{x h_{1}}-D_{x h}\right)^{T} T\left(D_{x h}-D_{x h_{2}}\right)
\end{aligned}
$$

where $D_{\dot{x}}, D_{x}, D_{x h}, D_{x h_{1}}, D_{x h_{2}}, D_{f}$, and $D_{f h}$ are defined as

$$
\begin{align*}
& D_{f h} \triangleq\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & I
\end{array}\right], \quad D_{x} \triangleq\left[\begin{array}{llllll}
I & 0 & 0 & 0 & 0 & 0
\end{array}\right], \\
& D_{x h} \triangleq\left[\begin{array}{llllll}
0 & I & 0 & 0 & 0 & 0
\end{array}\right], \\
& D_{x h_{2}} \triangleq\left[\begin{array}{llllll}
0 & 0 & 0 & I & 0 & 0
\end{array}\right], \\
&  \tag{11}\\
& \\
& \\
& D_{\dot{x}} \triangleq\left[\begin{array}{llllll}
-C & D_{f} \triangleq\left[\begin{array}{lllllll}
0 & 0 & I & 0 & 0 & 0
\end{array}\right], \\
0 & 0 & 0 & 0 & I & 0
\end{array}\right],
\end{align*}
$$

then system (5) is globally stable. Moreover,

$$
\begin{equation*}
\|x(t)\| \leq \sqrt{\frac{\Delta}{\lambda_{\min }(P)}}\|\phi\| \tag{12}
\end{equation*}
$$

where $\Delta$ is defined as

$$
\begin{align*}
\Delta \triangleq & 2\left(\lambda_{\max }\left(S_{2} L_{p}\right)-\lambda_{\min }\left(S_{2} L_{m}\right)\right) \\
& +3\left[h_{2}^{3} \lambda_{\max }\left(R_{1}\right)+\left(h_{2}-h_{1}\right)^{3} \lambda_{\max }\left(R_{2}\right)\right] \\
& \times\left(\lambda_{\max }\left(C^{T} C\right)+\lambda_{\max }\left(A^{T} A\right) \lambda_{\max }\left(L_{p}^{2}\right)\right. \\
& \left.+\lambda_{\max }\left(B^{T} B\right) \lambda_{\max }\left(L_{p}^{2}\right)\right)+\lambda_{\max }(P)+h_{2} \lambda_{\max }\left(S_{3}\right) \\
& +h_{2} \lambda_{\max }\left(S_{4}\right) \lambda_{\max }\left(L_{p}^{2}\right)+h_{1} \lambda_{\max }\left(Q_{1}\right) \\
& +h_{2} \lambda_{\max }\left(Q_{2}\right)+2\left(\lambda_{\max }\left(S_{1} L_{p}\right)-\lambda_{\min }\left(S_{1} L_{m}\right)\right) . \tag{13}
\end{align*}
$$

Proof. We choose an LKF as

$$
\begin{equation*}
V(t) \triangleq V_{1}(t)+V_{2}(t)+V_{3}(t) \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
V_{1}(t) \triangleq & x^{T}(t) P x(t)+2 \sum_{i=1}^{n} s_{1, i} \int_{0}^{x_{i}(t)}\left(f_{i}(s)-l_{m, i} x_{i}\right) d s \\
& +2 \sum_{i=1}^{n} s_{2, i} \int_{0}^{x_{i}(t)}\left(l_{p, i} x_{i}-f_{i}(s)\right) d s \\
V_{2}(t) \triangleq & \int_{t-h}^{t}\left[x^{T}(w) S_{3} x(w)+f^{T}(x(w)) S_{4} f(x(w))\right] d w \\
& +\int_{t-h_{1}}^{t} x^{T}(w) Q_{1} x(w) d w \\
& +\int_{t-h_{2}}^{t} x^{T}(w) Q_{2} x(w) d w \\
V_{3}(t) \triangleq & h_{2} \int_{-h_{2}}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{1} \dot{x}(w) d w d \theta \\
& +\left(h_{2}-h_{1}\right) \int_{-h_{2}}^{-h_{1}} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w d \theta \\
Z(w) \triangleq & {\left[x^{T}(w) x^{T}(w-h) x^{T}\left(w-h_{1}\right)\right.} \\
& \left.x^{T}\left(w-h_{2}\right) f^{T}(x(w)) f^{T}(x(w-h))\right]^{T} \tag{15}
\end{align*}
$$

The derivatives of $V_{i}(t), i=1,2,3$, are given, respectively, by

$$
\begin{aligned}
\dot{V}_{1}(t)= & x^{T}(t) P \dot{x}(t)+\dot{x}^{T}(t) P x(t) \\
& +2\left[S_{1}\left(f(x)-L_{m} x(t)\right)\right]^{T} \dot{x}(t) \\
& +2\left[S_{2}\left(L_{p} x(t)-f(x)\right)\right]^{T} \dot{x}(t) \\
= & Z^{T}(t)\left(D_{x}^{T} P D_{\dot{x}}+D_{\dot{x}}^{T} P D_{x}+2 D_{f}^{T} S_{1} D_{\dot{x}}\right. \\
& \quad-2 D_{x}^{T} L_{m}^{T} S_{1} D_{\dot{x}}+2 D_{x}^{T} L_{p}^{T} S_{2} D_{\dot{x}} \\
& \left.\quad-2 D_{f}^{T} S_{2} D_{\dot{x}}\right) Z(t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_{2}(t) \leq & x^{T}(t) S_{3} x(t)+f^{T}(x(t)) S_{4} f(x(t)) \\
& +x^{T}(t) Q_{1} x(t)+x^{T}(t) Q_{2} x(t) \\
& -(1-\gamma) f^{T}(x(t-h)) S_{4} f(x(t-h)) \\
& -(1-\gamma) x^{T}(t-h) S_{3} x(t-h) \\
& \quad-x^{T}\left(t-h_{1}\right) Q_{1} x\left(t-h_{1}\right)-x^{T}\left(t-h_{2}\right) Q_{2} x\left(t-h_{2}\right) \\
= & Z^{T}(t)\left[D_{x}^{T} S_{3} D_{x}-(1-\gamma) D_{x h}^{T} S_{3} D_{x h}\right. \\
& +D_{f}^{T} S_{4} D_{f}-(1-\gamma) D_{f h}^{T} S_{4} D_{f h} \\
& +D_{x}^{T} Q_{1} D_{x}-D_{x h_{1}}^{T} Q_{1} D_{x h_{1}} \\
& \left.+D_{x}^{T} Q_{2} D_{x}-D_{x h_{2}}^{T} Q_{2} D_{x h_{2}}\right] Z(t),
\end{aligned}
$$

$$
\dot{V}_{3}(t)=\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t)
$$

$$
-\left(h_{2}-h_{1}\right) \int_{t-h_{2}}^{t-h_{1}} \dot{x}^{T}(w) R_{1} \dot{x}(w) d w
$$

$$
+h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)-h_{2} \int_{t-h_{2}}^{t} \dot{x}^{T}(w) R_{1} \dot{x}(w) d w
$$

(16)

By Lemma 1, we can get

$$
\begin{aligned}
\dot{V}_{3}(t) \leq & h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)+\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t) \\
& -\left(x(t)-x\left(t-h_{2}\right)\right)^{T} R_{1}\left(x(t)-x\left(t-h_{2}\right)\right) \\
& -\left(h_{2}-h_{1}\right)\left(\int_{t-h_{2}}^{t-h} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w\right. \\
& \left.+\int_{t-h_{2}}^{t-h} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w\right) \\
\leq & h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)+\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t) \\
& -\left(x(t)-x\left(t-h_{2}\right)\right)^{T} R_{1}\left(x(t)-x\left(t-h_{2}\right)\right) \\
& -\left[\frac{h_{2}-h_{1}}{h_{2}-h}\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \times R_{2}\left(x(t-h)-x\left(t-h_{2}\right)\right) \\
& +\frac{h_{2}-h_{1}}{h-h_{1}}\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} \\
& \left.\times R_{2}\left(x\left(t-h_{1}\right)-x(t-h)\right)\right] \\
& =h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)+\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t) \\
& -\left(x(t)-x\left(t-h_{2}\right)\right)^{T} R_{1}\left(x(t)-x\left(t-h_{2}\right)\right) \\
& -\frac{h-h_{1}}{h_{2}-h}\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T} \\
& \times R_{2}\left(x(t-h)-x\left(t-h_{2}\right)\right) \\
& -\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T} \\
& \times R_{2}\left(x(t-h)-x\left(t-h_{2}\right)\right) \\
& -\frac{h_{2}-h}{h-h_{1}}\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} \\
& \times R_{2}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
& -\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} \\
& \times R_{2}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
& =h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)+\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t) \\
& -\left(x(t)-x\left(t-h_{2}\right)\right)^{T} R_{1}\left(x(t)-x\left(t-h_{2}\right)\right) \\
& -\left[\begin{array}{c}
\sqrt{\frac{h_{2}-h}{h-h_{1}}}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \times\left[\begin{array}{c}
\sqrt{\frac{h_{2}-h}{h-h_{1}}}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right] \\
& -\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \times\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right] \\
& \leq h_{2}^{2} \dot{x}^{T}(t) R_{1} \dot{x}(t)+\left(h_{2}-h_{1}\right)^{2} \dot{x}^{T}(t) R_{2} \dot{x}(t) \\
& -\left(x(t)-x\left(t-h_{2}\right)\right)^{T} R_{1}\left(x(t)-x\left(t-h_{2}\right)\right) \\
& -\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \times\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
\leq Z^{T}(t)[ & h_{2}^{2} D_{\dot{x}}^{T} R_{1} D_{\dot{x}}+\left(h_{2}-h_{1}\right)^{2} D_{\dot{x}}^{T} R_{2} D_{\dot{x}} \\
& -\left(D_{x}-D_{x h_{2}}\right)^{T} R_{1}\left(D_{x}-D_{x h_{2}}\right) \\
& -\left[\begin{array}{c}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \left.\times\left[\begin{array}{c}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}
\end{array}\right]\right] Z(t) \tag{17}
\end{align*}
$$

By (16) and (17), we have

$$
\begin{align*}
\dot{V}(t) \leq Z^{T}(t)[ & D_{x}^{T} P D_{\dot{x}}+D_{\dot{x}}^{T} P D_{x}+2 D_{f}^{T} S_{1} D_{\dot{x}} \\
& -D_{x}^{T} L_{m}^{T} S_{1} D_{\dot{x}}+2 D_{x}^{T} L_{p}^{T} S_{2} D_{\dot{x}}-D_{f}^{T} S_{2} D_{\dot{x}} \\
& +D_{x}^{T} S_{3} D_{x}-(1-\gamma) D_{x h}^{T} S_{3} D_{x h}+D_{f}^{T} S_{4} D_{f} \\
& -(1-\gamma) D_{f h}^{T} S_{4} D_{f h}+D_{x}^{T} Q_{1} D_{x} \\
& -D_{x h_{1}}^{T} Q_{1} D_{x h_{1}}+D_{x}^{T} Q_{2} D_{x}-D_{x h_{2}}^{T} Q_{2} D_{x h_{2}} \\
& +h_{2}^{2} D_{\dot{x}}^{T} R_{1} D_{\dot{x}}+\left(h_{2}-h_{1}\right)^{2} D_{\dot{x}}^{T} R_{2} D_{\dot{x}} \\
& -\left(D_{x}-D_{x h_{2}}\right)^{T} R_{1}\left(D_{x}-D_{x h_{2}}\right) \\
& -\left[\begin{array}{l}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \left.\times\left[\begin{array}{l}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}
\end{array}\right]\right] Z(t) \\
= & Z^{T}(t) \Omega Z(t), \tag{18}
\end{align*}
$$

where $\Omega$ is defined as

$$
\left.\begin{array}{rl}
\Omega \triangleq & D_{x}^{T} P D_{\dot{x}}+D_{\dot{x}}^{T} P D_{x}+2 D_{f}^{T} S_{1} D_{\dot{x}} \\
& -D_{x}^{T} L_{m}^{T} S_{1} D_{\dot{x}}+2 D_{x}^{T} L_{p}^{T} S_{2} D_{\dot{x}}-D_{f}^{T} S_{2} D_{\dot{x}} \\
& +h_{2}^{2} D_{\dot{x}}^{T} R_{1} D_{\dot{x}}+\left(h_{2}-h_{1}\right)^{2} D_{\dot{x}}^{T} R_{2} D_{\dot{x}} \\
& -\left(D_{x}-D_{x h_{2}}\right)^{T} R_{1}\left(D_{x}-D_{x h_{2}}\right) \\
& +D_{x}^{T} S_{3} D_{x}-(1-\gamma) D_{x h}^{T} S_{3} D_{x h}+D_{f}^{T} S_{4} D_{f} \\
& -(1-\gamma) D_{f h}^{T} S_{4} D_{f h}+D_{x}^{T} Q_{1} D_{x}-D_{x h_{1}}^{T} Q_{1} D_{x h_{1}} \\
& +D_{x}^{T} Q_{2} D_{x}-D_{x h_{2}}^{T} Q_{2} D_{x h_{2}} \\
& -\left[\begin{array}{l}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}^{T}
\end{array}\right]^{R_{2}} \begin{array}{c}
T \\
*
\end{array} R_{2}
\end{array}\right]\left[\begin{array}{l}
D_{x h_{1}}-D_{x h} \\
D_{x h}-D_{x h_{2}}
\end{array}\right] .
$$

Then by (7), (18), and (19), we have

$$
\begin{align*}
\dot{V}(t) \leq & Z^{T}(t) \Omega Z(t)+ \\
& \left(Z^{T}(t)\left(L_{p} D_{x}-D_{f}\right)^{T}\right. \\
& \left.\times S_{1}\left(D_{f}-L_{m} D_{x}\right) Z(t)\right) \\
& +Z^{T}(t)\left(L_{p} D_{x h}-D_{f h}\right)^{T} S_{2}\left(D_{f h}-L_{m} D_{x h}\right) Z(t)  \tag{20}\\
= & Z^{T}(t) \Theta Z(t) .
\end{align*}
$$

It is clear that if (9) and (20) hold, then, for any $Z(t) \neq 0$, we have $\dot{V}(t)<0$. It follows that

$$
\begin{equation*}
V(t) \leq V(0) \tag{21}
\end{equation*}
$$

From (15), we get

$$
\begin{align*}
& V(0)= x^{T}(0) P x(0)+2 \sum_{i=1}^{n} \int_{0}^{x_{i}(0)}\left[s_{1, i}\left(f_{i}(w)-L_{m, i} x_{i}\right)\right. \\
&\left.+s_{2, i}\left(L_{p, i} x_{i}-f_{i}(w)\right)\right] d w \\
&+h_{2} \int_{-h_{2}}^{0} \int_{\theta}^{0} \dot{x}^{T}(w) R_{1} \dot{x}(w) d w d \theta \\
&+\left(h_{2}-h_{1}\right) \int_{-h_{1}}^{-h_{2}} \int_{\theta}^{0} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w d \theta \\
&+\int_{-h_{1}}^{0}\left[x^{T}(w) Q_{1} x(w)\right] d w \\
&+\int_{-h_{2}}^{0}\left[x^{T}(w) Q_{2} x(w)\right] d w \\
&+\int_{-h}^{0}\left[x^{T}(w) S_{3} x(w)+f^{T}(x(w)) S_{4} f(x(w))\right] d w \\
& \leq {\left[\lambda_{\max }(P)+h_{2} \lambda_{\max }\left(S_{3}\right)+h_{2} \lambda_{\max }\left(S_{4}\right) \lambda_{\max }\left(L_{p}^{2}\right)\right.} \\
&+h_{1} \lambda_{\max }\left(Q_{1}\right)+h_{2} \lambda_{\max }\left(Q_{2}\right) \\
&+2\left(\lambda_{\max }\left(S_{1} L_{p}\right)-\lambda_{\min }\left(S_{1} L_{m}\right)\right. \\
&\left.\left.+\lambda_{\max }\left(S_{2} L_{p}\right)-\lambda_{\min }\left(S_{2} L_{m}\right)\right)\right]\|\phi\|^{2} \\
&+\lambda_{\max }\left(R_{2}\right)\left(h_{1}-h_{2}\right) \int_{-h_{1}}^{-h_{2}} \int_{\theta}^{0} \dot{x}^{T}(w) \dot{x}(w) d w d \theta \\
&+ \lambda_{\max }\left(R_{1}\right) h_{2} \int_{-h_{2}}^{0} \int_{\theta}^{0} \dot{x}^{T}(w) \dot{x}(w) d w d \theta .  \tag{22}\\
&
\end{align*}
$$

By a similar method in [9], we have

$$
\begin{gather*}
\dot{x}^{T}(w) \dot{x}(w) \leq 3\left[\lambda_{\max }\left(C^{T} C\right)+\lambda_{\max }\left(A^{T} A\right) \lambda_{\max }\left(L_{p}^{2}\right)\right. \\
\left.+\lambda_{\max }\left(B^{T} B\right) \lambda_{\max }\left(L_{p}^{2}\right)\right]\|\phi\|^{2} \tag{23}
\end{gather*}
$$

Thus, $V(0) \leq \Delta\|\phi\|^{2}$, where $\Delta$ is defined in (13).

Moreover,

$$
\begin{equation*}
V(t) \geq x^{T}(t) P x(t) \geq \lambda_{\min }(P)\|x(t)\|^{2} . \tag{24}
\end{equation*}
$$

Therefore, we have

$$
\begin{equation*}
\|x(t)\| \leq \sqrt{\frac{\Delta}{\lambda_{\min }(P)}}\|\phi\| \tag{25}
\end{equation*}
$$

which shows that system (5) is globally stable. This completes the proof.

Remark 3. Theorem 2 presents a stability criterion for the delayed neural network. When coping with $\left(h_{2}-\right.$ $\left.h_{1}\right) \int_{-h_{2}}^{-h_{1}} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w d \theta$, instead of using Jensen's inequality directly, we use a substep which can make the method less conservative, which can be noticed as reciprocally convex combination approach in [36]. It follows that

$$
\begin{aligned}
& -\frac{h_{2}-h_{1}}{h_{2}-h}\left(\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T} R_{2}\right. \\
& \left.\times\left(x(t-h)-x\left(t-h_{2}\right)\right)\right) \\
& -\frac{h_{2}-h_{1}}{h-h_{1}}\left(\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} R_{2}\right. \\
& \left.\times\left(x\left(t-h_{1}\right)-x(t-h)\right)\right) \\
& =-\frac{h-h_{1}}{h_{2}-h}\left(\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T} R_{2}\right. \\
& \left.\times\left(x(t-h)-x\left(t-h_{2}\right)\right)\right) \\
& -\left(\left(x(t-h)-x\left(t-h_{2}\right)\right)^{T} R_{2}\right. \\
& \left.\times\left(x(t-h)-x\left(t-h_{2}\right)\right)\right) \\
& -\frac{h_{2}-h}{h-h_{1}}\left(\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} R_{2}\right. \\
& \left.\times\left(x\left(t-h_{1}\right)-x(t-h)\right)\right) \\
& -\left(\left(x\left(t-h_{1}\right)-x(t-h)\right)^{T} R_{2}\right. \\
& \left.\times\left(x\left(t-h_{1}\right)-x(t-h)\right)\right) \\
& =-\left[\begin{array}{c}
\sqrt{\frac{h_{2}-h}{h-h_{1}}}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
& \times\left[\begin{array}{c}
\sqrt{\frac{h_{2}-h}{h-h_{1}}}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right] \\
& -\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
& \times\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right] \\
\leq & -\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \\
\times & {\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right] . } \tag{26}
\end{align*}
$$

If no substep is taken, it will follow that

$$
\begin{align*}
&-\left(h_{2}-h_{1}\right) \int_{t-h_{2}}^{t-h_{1}} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w \\
& \leq- {\left[x\left(t-h_{1}\right)-x(t-h)+x(t-h)\right.} \\
&\left.-x\left(t-h_{2}\right)\right]^{T} R_{2} \\
& \times {\left[x\left(t-h_{1}\right)-x(t-h)+x(t-h)\right.} \\
&\left.-x\left(t-h_{2}\right)\right] \\
&=- {\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right]^{T}\left[\begin{array}{ll}
R_{2} & R_{2} \\
R_{2} & R_{2}
\end{array}\right] } \\
& \times \times\left[\begin{array}{l}
x\left(t-h_{1}\right)-x(t-h) \\
x(t-h)-x\left(t-h_{2}\right)
\end{array}\right] . \tag{27}
\end{align*}
$$

We can see that compared to (27), (26) is relatively free since $T$ could be more than nonnegative, and consequently, its LMI could suffer bigger delay. For the same reason, if the middle term could be found between

$$
\left.\begin{array}{l}
-\left[\begin{array}{c}
\sqrt{\frac{h_{2}-h}{h-h_{1}}}\left(x\left(t-h_{1}\right)-x(t-h)\right) \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right]^{T}\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right]  \tag{28}\\
\quad \times\left[\begin{array}{c}
\frac{h_{2}-h}{h-h_{1}} \\
\hline
\end{array}\left(x\left(t-h_{1}\right)-x(t-h)\right)\right. \\
-\sqrt{\frac{h-h_{1}}{h_{2}-h}}\left(x(t-h)-x\left(t-h_{2}\right)\right)
\end{array}\right]
$$

and 0 , conservatism would be further reduced.
Remark 4. Based on Theorem 2, we can determine the maximum admissible delay $h_{1}$ and $h_{2}$ at a known upper bound of $\dot{h}$. Moreover, the relationship between $f(x)$ and $x$ could be specified using $L_{m}$ and $L_{p}$. As to the case when $\dot{h}$ is unknown, we can refer to Corollary 5, where some changes are made on the LKF.

Corollary 5. Suppose that the time delay $h$ in reference system (5) satisfies $0<h_{1}<h<h_{2}$. Under the condition given by (6) and (7), if there exist matrices $P>0, Q_{1}>0, Q_{2}>0, R_{1}>0$, $R_{2}>0, K_{1}>0, K_{2}>0, S_{1} \triangleq \operatorname{diag}\left(s_{1,1}, s_{1,2}, \ldots, s_{1, n}\right)>0$,
$S_{2} \triangleq \operatorname{diag}\left(s_{2,1}, s_{2,2}, \ldots, s_{2, n}\right)>0$, and $T$ such that the following matrix inequalities hold:

$$
\begin{gather*}
\widetilde{\Theta} \triangleq \widetilde{\Upsilon}_{1}+\widetilde{\Upsilon}_{2}+\widetilde{\Upsilon}_{3}+\widetilde{\Upsilon}_{4}<0 \\
\widetilde{\Psi} \triangleq\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \geq 0 \tag{29}
\end{gather*}
$$

where $\tilde{\Upsilon}_{1}, \tilde{\Upsilon}_{2}, \widetilde{\Upsilon}_{3}$, and $\tilde{\Upsilon}_{4}$ are defined as

$$
\begin{align*}
\widetilde{\Upsilon}_{1} \triangleq & D_{x}^{T} P D_{\dot{x}}+D_{\dot{x}}^{T} P D_{x}+D_{x}^{T} Q_{1} D_{x} \\
& -D_{x h_{1}}^{T} Q_{1} D_{x h_{1}}+D_{x}^{T} Q_{2} D_{x}-D_{x h_{2}}^{T} Q_{2} D_{x h_{2}}, \\
\widetilde{\Upsilon}_{2} \triangleq & 2 D_{f}^{T} S_{1} D_{\dot{x}}-2 D_{x}^{T} L_{m}^{T} S_{1} D_{\dot{x}} \\
& -2 D_{f}^{T} S_{2} D_{\dot{x}}+2 D_{x}^{T} L_{p}^{T} S_{2} D_{\dot{x}}, \\
\widetilde{\Upsilon}_{3} \triangleq & D_{f h}^{T} K_{2} L_{p} D_{x h}-D_{f h}^{T} K_{2} D_{f h}-D_{x h}^{T} L_{m}^{T} K_{2} L_{p} D_{x h} \\
& +D_{x h}^{T} L_{m}^{T} K_{2} D_{f h}+D_{f}^{T} K_{1} L_{p} D_{x}-D_{f}^{T} K_{1} D_{f} \\
& -D_{x}^{T} L_{m}^{T} K_{1} L_{p} D_{x}+D_{x}^{T} L_{m}^{T} K_{1} D_{f}, \\
\widetilde{\Upsilon}_{4} \triangleq & h_{2}^{2} D_{\dot{x}}^{T} R_{1} D_{\dot{x}}-\left(D_{x}-D_{x h_{2}}\right)^{T} R_{1}\left(D_{x}-D_{x h_{2}}\right) \\
& +\left(h_{2}-h_{1}\right)^{2} D_{\dot{x}}^{T} R_{2} D_{\dot{x}} \\
& -\left(D_{x h}-D_{x h_{1}}\right)^{T} R_{2}\left(D_{x h}-D_{x h_{1}}\right) \\
& -\left(D_{x h_{2}}-D_{x h}\right)^{T} R_{2}\left(D_{x h_{2}}-D_{x h}\right) \\
& -\left(D_{x h}-D_{x h_{2}}\right)^{T} T^{T}\left(D_{x h_{1}}-D_{x h}\right) \\
& -\left(D_{x h_{1}}-D_{x h}\right)^{T} T\left(D_{x h}-D_{x h_{2}}\right) \tag{30}
\end{align*}
$$

where $D_{\dot{x}}, D_{x}, D_{x h}, D_{x h_{1}}, D_{x h_{2}}, D_{f}$, and $D_{f h}$ are defined in (11), then system (5) is globally stable. Moreover,

$$
\begin{equation*}
\|x(t)\| \leq \sqrt{\frac{\tilde{\Delta}}{\lambda_{\min }(P)}}\|\phi\| \tag{31}
\end{equation*}
$$

where $\widetilde{\Delta}$ is defined as

$$
\begin{align*}
& \widetilde{\Delta} \triangleq \lambda_{\max }(P)+h_{1} \lambda_{\max }\left(Q_{1}\right)+h_{2} \lambda_{\max }\left(Q_{2}\right) \\
&+2\left(\lambda_{\max }\left(S_{1} L_{p}\right)-\lambda_{\min }\left(S_{1} L_{m}\right)\right. \\
&\left.+\lambda_{\max }\left(S_{2} L_{p}\right)-\lambda_{\min }\left(S_{2} L_{m}\right)\right) \\
&+ 3\left[h_{2}^{3} \lambda_{\max }\left(R_{1}\right)+\left(h_{2}-h_{1}\right)^{3} \lambda_{\max }\left(R_{2}\right)\right]  \tag{32}\\
& \times {\left[\lambda_{\max }\left(C^{T} C\right)+\lambda_{\max }\left(A^{T} A\right) \lambda_{\max }\left(L_{p}^{2}\right)\right.} \\
&\left.+\lambda_{\max }\left(B^{T} B\right) \lambda_{\max }\left(L_{p}^{2}\right)\right] .
\end{align*}
$$

Proof. We choose an LKF as

$$
\begin{equation*}
\widetilde{V}(t) \triangleq \widetilde{V}_{1}(t)+\widetilde{V}_{2}(t)+\widetilde{V}_{3}(t) \tag{33}
\end{equation*}
$$

where

$$
\begin{align*}
& \widetilde{V}_{1}(t) \triangleq x^{T}(t) P x(t)+2 \sum_{i=1}^{n} s_{1, i} \int_{0}^{x_{i}(t)}\left(f_{i}(s)-l_{m, i} x_{i}\right) d s \\
&+2 \sum_{i=1}^{n} s_{2, i} \int_{0}^{x_{i}(t)}\left(l_{p, i} x_{i}-f_{i}(s)\right) d s \\
& \widetilde{V}_{2}(t) \triangleq \int_{t-h_{1}}^{t} x^{T}(w) Q_{1} x(w) d w  \tag{34}\\
&+\int_{t-h_{2}}^{t} x^{T}(w) Q_{2} x(w) d w \\
& \widetilde{V}_{3}(t) \triangleq h_{2} \int_{-h_{2}}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{1} \dot{x}(w) d w d \theta \\
&+\left(h_{2}-h_{1}\right) \int_{-h_{2}}^{-h_{1}} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w d \theta .
\end{align*}
$$

Since the result can be obtained directly from Theorem 2, the rest of the proof for Corollary 5 is omitted.

Remark 6. Corollary 5 presents the stability criterion when the upper bound of $\dot{h}$ is unknown. Since $\gamma$ is unknown and under current conditions, it cannot be estimated or substituted, the first term in $\widetilde{V}_{2}(t)$ of $\int_{-h}^{0}\left[x^{T}(w) S_{3} x(w)+\right.$ $\left.f^{T}(x(w)) S_{4} f(x(w))\right] d w$ should be changed or eliminated from the LKF. In Corollary 5, the term is eliminated because the other two terms in $\widetilde{V}_{2}(t)$ can serve the same function, and it is unnecessary to keep any extra $x(t-h)$ or $f(x(t-h))$. The rest of the terms were reserved because they will not generate any $\dot{h}$-related terms when estimating the derivative of the LKF.

Corollary 7. Suppose that the time delay $h$ in reference system (5) satisfies $0<h<h_{2}$ and $0 \leq \dot{h} \leq \gamma$. Under the condition given by (6) and (7), if there exist matrices $P>0, Q_{2}>0$, $R_{1}>0, R_{2}>0, K_{1}>0, K_{2}>0, S_{i}>0, i=1, \ldots, 4$, $S_{1} \triangleq \operatorname{diag}\left(s_{1,1}, s_{1,2}, \ldots, s_{1, n}\right), S_{2} \triangleq \operatorname{diag}\left(s_{2,1}, s_{2,2}, \ldots, s_{2, n}\right)$, and $T$ such that the following matrix inequalities hold:

$$
\begin{align*}
& \widehat{\Theta} \triangleq \widehat{\Upsilon}_{1}+\widehat{\Upsilon}_{2}+\widehat{\Upsilon}_{3}+\widehat{\Upsilon}_{4}<0 \\
& \widehat{\Psi} \triangleq\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \geq 0 \tag{35}
\end{align*}
$$

where $\widehat{\Upsilon}_{1}, \widehat{\Upsilon}_{2}, \widehat{\Upsilon}_{3}$, and $\widehat{\Upsilon}_{4}$ are defined as

$$
\begin{aligned}
\widehat{Y}_{1} \triangleq & \widehat{D}_{x}^{T} P \widehat{D}_{\dot{x}}+\widehat{D}_{\dot{x}}^{T} P \widehat{D}_{x}+\widehat{D}_{x}^{T} Q_{2} \widehat{D}_{x}-\widehat{D}_{x h_{2}}^{T} Q_{2} \widehat{D}_{x h_{2}} \\
\widehat{\Upsilon}_{2} \triangleq & \widehat{D}_{x}^{T} S_{3} \widehat{D}_{x}-(1-\gamma) \widehat{D}_{x h}^{T} S_{3} \widehat{D}_{x h}+\widehat{D}_{f}^{T} S_{4} \widehat{D}_{f} \\
& -(1-\gamma) \widehat{D}_{f h}^{T} S_{4} \widehat{D}_{f h}+2 \widehat{D}_{f}^{T} S_{1} \widehat{D}_{\dot{x}}-2 \widehat{D}_{x}^{T} L_{m}^{T} S_{1} \widehat{D}_{\dot{x}} \\
& -2 \widehat{D}_{f}^{T} S_{2} \widehat{D}_{\dot{x}}+2 \widehat{D}_{x}^{T} L_{p}^{T} S_{2} \widehat{D}_{\dot{x}},
\end{aligned}
$$

$$
\begin{align*}
& \widehat{\Upsilon}_{3} \triangleq \widehat{D}_{f h}^{T} K_{2} L_{p} \widehat{D}_{x h}-\widehat{D}_{f h}^{T} K_{2} \widehat{D}_{f h}-\widehat{D}_{x h}^{T} L_{m}^{T} K_{2} L_{p} \widehat{D}_{x h} \\
&+\widehat{D}_{x h}^{T} L_{m}^{T} K_{2} \widehat{D}_{f h}+\widehat{D}_{f}^{T} K_{1} L_{p} \widehat{D}_{x}-\widehat{D}_{f}^{T} K_{1} \widehat{D}_{f} \\
&-\widehat{D}_{x}^{T} L_{m}^{T} K_{1} L_{p} \widehat{D}_{x}+\widehat{D}_{x}^{T} L_{m}^{T} K_{1} \widehat{D}_{f}, \\
& \widehat{\Upsilon}_{4} \triangleq h_{2}^{2} \widehat{D}_{\dot{x}}^{T} R_{1} \widehat{D}_{\dot{x}}-\left(\widehat{D}_{x}-\widehat{D}_{x h_{2}}\right)^{T} R_{1}\left(\widehat{D}_{x}-\widehat{D}_{x h_{2}}\right) \\
&+h_{2}^{2} \widehat{D}_{\dot{x}}^{T} R_{2} \widehat{D}_{\dot{x}}-\left(\widehat{D}_{x h}-\widehat{D}_{x}\right)^{T} R_{2}\left(\widehat{D}_{x h}-\widehat{D}_{x}\right) \\
&-\left(\widehat{D}_{x h_{2}}-\widehat{D}_{x h}\right)^{T} R_{2}\left(\widehat{D}_{x h_{2}}-\widehat{D}_{x h}\right) \\
&-\left(\widehat{D}_{x h}-\widehat{D}_{x h_{2}}\right)^{T} T^{T}\left(\widehat{D}_{x}-\widehat{D}_{x h}\right) \\
&-\left(\widehat{D}_{x}-\widehat{D}_{x h}\right)^{T} T\left(\widehat{D}_{x h}-\widehat{D}_{x h_{2}}\right), \tag{36}
\end{align*}
$$

where $\widehat{D}_{\dot{x}}, \widehat{D}_{x}, \widehat{D}_{x h}, \widehat{D}_{x h_{2}}, \widehat{D}_{f}$, and $\widehat{D}_{f h}$ are defined as

$$
\begin{align*}
\widehat{D}_{\dot{x}} \triangleq\left[\begin{array}{llll}
-C & 0 & 0 & A
\end{array}\right], \\
\widehat{D}_{x} \triangleq\left[\begin{array}{lllll}
1 & 0 & 0 & 0 & 0
\end{array}\right], \\
\widehat{D}_{x h} \triangleq\left[\begin{array}{lllll}
0 & I & 0 & 0 & 0
\end{array}\right], \\
\widehat{D}_{x h_{2}} \triangleq\left[\begin{array}{lllll}
0 & 0 & I & 0 & 0
\end{array}\right],  \tag{37}\\
\widehat{D}_{f} \triangleq\left[\begin{array}{lllll}
0 & 0 & 0 & I & 0
\end{array}\right], \\
\widehat{D}_{f h} \triangleq\left[\begin{array}{lllll}
0 & 0 & 0 & 0 & I
\end{array}\right],
\end{align*}
$$

then system (5) is globally stable. Moreover,

$$
\begin{equation*}
\|x(t)\| \leq \sqrt{\frac{\widehat{\Delta}}{\lambda_{\min }(P)}}\|\phi\|, \tag{38}
\end{equation*}
$$

where $\widehat{\Delta}$ is defined as

$$
\begin{align*}
\widehat{\Delta} \triangleq \lambda_{\max }(P)+h_{2} \lambda_{\max }\left(S_{3}\right) & +h_{2} \lambda_{\max }\left(S_{4}\right) \lambda_{\max }\left(L_{p}^{2}\right) \\
& +h_{2} \lambda_{\max }\left(Q_{2}\right)+2\left(\lambda_{\max }\left(S_{1} L_{p}\right)-\lambda_{\min }\left(S_{1} L_{m}\right)\right. \\
& \left.+\lambda_{\max }\left(S_{2} L_{p}\right)-\lambda_{\min }\left(S_{2} L_{m}\right)\right) \\
+3 h_{2}^{3} \lambda_{\max }\left(R_{1}+R_{2}\right) & \left(\lambda_{\max }\left(C^{T} C\right)\right. \\
& +\lambda_{\max }\left(A^{T} A\right) \lambda_{\max }\left(L_{p}^{2}\right) \\
& \left.+\lambda_{\max }\left(B^{T} B\right) \lambda_{\max }\left(L_{p}^{2}\right)\right) . \tag{39}
\end{align*}
$$

Proof. We choose an LKF as

$$
\begin{equation*}
\widehat{V}(t) \triangleq \widehat{V}_{1}(t)+\widehat{V}_{2}(t)+\widehat{V}_{3}(t) \tag{40}
\end{equation*}
$$

where

$$
\begin{align*}
& \widehat{V}_{1}(t) \triangleq x^{T}(t) P x(t)+2 \sum_{i=1}^{n} s_{1, i} \int_{0}^{x_{i}(t)}\left(f_{i}(s)-l_{m, i} x_{i}\right) d s \\
&+2 \sum_{i=1}^{n} s_{2, i} \int_{0}^{x_{i}(t)}\left(l_{p, i} x_{i}-f_{i}(s)\right) d s, \\
& \widehat{V}_{2}(t) \triangleq \int_{t-h}^{t}\left[x(w)^{T} S_{3} x(w)+f^{T}(x(w)) S_{4} f(x(w))\right] d w \\
&+\int_{t-h_{2}}^{t} x^{T}(w) Q_{2} x(w) d w, \\
& \widehat{V}_{3}(t) \triangleq h_{2} \int_{-h_{2}}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(w)\left(R_{1}+R_{2}\right) \dot{x}(w) d w d \theta . \tag{41}
\end{align*}
$$

Since the result can be obtained directly from Theorem 2, the rest of the proof for Corollary 7 is omitted.

Remark 8. Corollary 7 presents the stability criterion when the lower bound $h_{1}$ is zero. If $h_{1}$ is zero, the second term in $\widehat{V}_{2}(t)$ of $\int_{t-h_{1}}^{t} x^{T}(w) Q_{1} x(w) d w$ should be changed or eliminated from the LKF. In Corollary 7, the term is eliminated because there is no need to introduce an extra variable $Q_{1}$, while $P$ and other matrices can serve the same function. Moreover, $\dot{x}^{T}(w) R_{1} \dot{x}(w) d w d \theta$ and $\left(h_{2}-\right.$ $\left.h_{1}\right) \int_{-h_{2}}^{-h_{1}} \int_{t+\theta}^{t} \dot{x}^{T}(w) R_{2} \dot{x}(w) d w d \theta$ in $\widehat{V}_{3}(t)$ can be merged because when $h_{1}$ is zero, they have the same form. But $R_{1}$ and $R_{2}$ can be reserved, because when estimating the upper bound of the LKF, it is still useful to introduce a $T$ like Theorem 2.

Corollary 9. Suppose that the time delay $h$ in reference system (5) satisfies $0<h<h_{2}$. Under the condition given by (6) and (7), if there exist matrices $P>0, Q_{2}>0, R_{1}>0, R_{2}>0$, $K_{1}>0, K_{2}>0, S_{i}>0, i=1,2, S_{i}>0, i=1, \ldots, 4$, $S_{1} \triangleq \operatorname{diag}\left(s_{1,1}, s_{1,2}, \ldots, s_{1, n}\right), S_{2} \triangleq \operatorname{diag}\left(s_{2,1}, s_{2,2}, \ldots, s_{2, n}\right)$, and $T$ such that the following matrix inequalities hold:

$$
\begin{gather*}
\bar{\Theta} \triangleq \bar{\Upsilon}_{1}+\bar{\Upsilon}_{2}+\bar{\Upsilon}_{3}+\bar{\Upsilon}_{4}<0 \\
\bar{\Psi} \triangleq\left[\begin{array}{cc}
R_{2} & T \\
* & R_{2}
\end{array}\right] \geq 0 \tag{42}
\end{gather*}
$$

where $\bar{\Upsilon}_{1}, \bar{\Upsilon}_{2}, \bar{\Upsilon}_{3}$, and $\bar{\Upsilon}_{4}$ are defined as

$$
\begin{array}{rl}
\bar{\Upsilon}_{1} \triangleq & \widehat{D}_{x}^{T} P \widehat{D}_{\dot{x}}+\widehat{D}_{\dot{x}}^{T} P \widehat{D}_{x}+\widehat{D}_{x}^{T} Q_{2} \widehat{D}_{x}-\widehat{D}_{x h_{2}}^{T} Q_{2} \widehat{D}_{x h_{2}}, \\
\bar{Y}_{2} & 2 \widehat{D}_{f}^{T} S_{1} \widehat{D}_{\dot{x}}-2 \widehat{D}_{x}^{T} L_{m}^{T} S_{1} \widehat{D}_{\dot{x}} \\
& -2 \widehat{D}_{f}^{T} S_{2} \widehat{D}_{\dot{x}}+2 \widehat{D}_{x}^{T} L_{p}^{T} S_{2} \widehat{D}_{\dot{x}}, \\
\bar{Y}_{3} \triangleq & \widehat{D}_{f h}^{T} K_{2} L_{p} \widehat{D}_{x h}-\widehat{D}_{f h}^{T} K_{2} \widehat{D}_{f h}-\widehat{D}_{x h}^{T} L_{m}^{T} K_{2} L_{p} \widehat{D}_{x h} \\
& +\widehat{D}_{x x}^{T} L_{m}^{T} K_{2} \widehat{D}_{f h}+\widehat{D}_{f}^{T} K_{1} L_{p} \widehat{D}_{x}-\widehat{D}_{f}^{T} K_{1} \widehat{D}_{f} \\
& -\widehat{D}_{x}^{T} L_{m}^{T} K_{1} L_{p} \widehat{D}_{x}+\widehat{D}_{x}^{T} L_{m}^{T} K_{1} \widehat{D}_{f},
\end{array}
$$

$$
\begin{align*}
\bar{Y}_{4} \triangleq & h_{2}^{2} \widehat{D}_{\dot{x}}^{T} R_{1} \widehat{D}_{\dot{x}}-\left(\widehat{D}_{x}-\widehat{D}_{x h_{2}}\right)^{T} R_{1}\left(\widehat{D}_{x}-\widehat{D}_{x h_{2}}\right) \\
& +h_{2}^{2} \widehat{D}_{\dot{x}}^{T} R_{2} \widehat{D}_{\dot{x}}-\left(\widehat{D}_{x h}-\widehat{D}_{x}\right)^{T} R_{2}\left(\widehat{D}_{x h}-\widehat{D}_{x}\right) \\
& -\left(\widehat{D}_{x h_{2}}-\widehat{D}_{x h}\right)^{T} R_{2}\left(\widehat{D}_{x h_{2}}-\widehat{D}_{x h}\right) \\
& -\left(\widehat{D}_{x h}-\widehat{D}_{x h_{2}}\right)^{T} T^{T}\left(\widehat{D}_{x}-\widehat{D}_{x h}\right) \\
& -\left(\widehat{D}_{x}-\widehat{D}_{x h}\right)^{T} T\left(\widehat{D}_{x h}-\widehat{D}_{x h_{2}}\right), \tag{43}
\end{align*}
$$

where $\widehat{D}_{\dot{x}}, \widehat{D}_{x}, \widehat{D}_{x h}, \widehat{D}_{x h_{2}}, \widehat{D}_{f}$, and $\widehat{D}_{f h}$ are defined in (37), then system (5) is globally stable. Moreover,

$$
\begin{equation*}
\|x(t)\| \leq \sqrt{\frac{\bar{\Delta}}{\lambda_{\min }(P)}}\|\phi\|, \tag{44}
\end{equation*}
$$

where $\bar{\Delta}$ is defined as

$$
\begin{align*}
& \bar{\Delta} \triangleq 2( \lambda_{\max }\left(S_{1} L_{p}\right)-\lambda_{\min }\left(S_{1} L_{m}\right) \\
&\left.+\lambda_{\max }\left(S_{2} L_{p}\right)-\lambda_{\min }\left(S_{2} L_{m}\right)\right) \\
&+\lambda_{\max }(P)+h_{2} \lambda_{\max }\left(Q_{2}\right) \\
&+3 h_{2}^{3} \lambda_{\max }\left(R_{1}+R_{2}\right) \\
&\left(\lambda_{\max }\left(C^{T} C\right)\right. \\
&+\lambda_{\max }\left(A^{T} A\right) \lambda_{\max }\left(L_{p}^{2}\right)  \tag{45}\\
&\left.+\lambda_{\max }\left(B^{T} B\right) \lambda_{\max }\left(L_{p}^{2}\right)\right) .
\end{align*}
$$

Proof. We choose an LKF as

$$
\begin{equation*}
\bar{V}(t) \triangleq \bar{V}_{1}(t)+\bar{V}_{2}(t)+\bar{V}_{3}(t), \tag{46}
\end{equation*}
$$

where

$$
\begin{align*}
& \bar{V}_{1}(t) \triangleq x^{T}(t) P x(t)+2 \sum_{i=1}^{n} s_{1, i} \int_{0}^{x_{i}(t)}\left(f_{i}(s)-l_{m i} x_{i}\right) d s \\
&+2 \sum_{i=1}^{n} s_{2, i} \int_{0}^{x_{i}(t)}\left(l_{p, i} x_{i}-f_{i}(s)\right) d s, \\
& \bar{V}_{2}(t) \triangleq \int_{t-h_{2}}^{t} x^{T}(w) Q_{2} x(w) d w, \\
& \bar{V}_{3}(t) \triangleq h_{2} \int_{-h_{2}}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(w)\left(R_{1}+R_{2}\right) \dot{x}(w) d w d \theta . \tag{47}
\end{align*}
$$

Since the result can be obtained directly from Theorem 2, the rest of the proof for Corollary 9 is omitted.

Remark 10. Corollary 9 presents the stability criterion when $h_{1}$ is zero and the upper bound of time delay's derivative, or $\gamma$, is unknown. If $h_{1}$ is zero and $\gamma$ is unknown, the first and second terms of $\int_{-h}^{0}\left[x^{T}(w) S_{3} x(w)+f^{T}(x(w)) S_{4} f(x(w))\right] d w$
and $\int_{t-h_{1}}^{t} x^{T}(w) Q_{1} x(w) d w$ in $\bar{V}_{2}(t)$ should be eliminated from the LKF. There is no need to introduce an extra variable $Q_{1}$ or keep any $x(t-h)$ or $f(x(t-h))$, and the reason is the same with Corollaries 5 and 7. Moreover, the terms in $\bar{V}_{3}(t)$ can be merged, while $R_{1}$ and $R_{2}$ should be reserved for the same reason with Corollary 7.

## 4. Numerical Examples

In this section, examples are provided to demonstrate the advantages of the proposed stability criteria.

Example 11. Consider the delayed neural network in (5) with the following parameters, which has also been investigated by [13, 39]:

$$
\begin{gather*}
C=\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right], \quad A=\left[\begin{array}{cc}
1 & 1 \\
-1 & -1
\end{array}\right], \\
B=\left[\begin{array}{cc}
0.88 & 1 \\
1 & 1
\end{array}\right], \quad L_{p}=\left[\begin{array}{cc}
0.4 & 0 \\
0 & 0.8
\end{array}\right],  \tag{48}\\
L_{m}=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .
\end{gather*}
$$

Our objective is to find the allowable maximum time delay $h_{2}$ such that the system is stable under different $h_{1}$ and $\dot{h}$. The simulation results from the available literature are shown in Table 1, along with results from Theorem 2 and Corollaries 5, 7 , and 9. It is clear that the conservatism reduction proves to be more obvious than those in [23, 24, 37, 38].

In addition, when $C, A, B$, and $L_{p}$ take the same values as (48), and $L_{m}$ takes a different value as

$$
L_{m}=\left[\begin{array}{cc}
0.1 & 0  \tag{49}\\
0 & 0.2
\end{array}\right]
$$

results under the same system with (49) are shown in Table 2. Since $L_{m}$ is specified in (49), allowable maximum time delay $h_{2}$ is expected to be different from those of (48). As shown in Table 2, with $h_{1}$ getting bigger, the difference between $h_{2}$ and $h_{1}$ becomes smaller, to ensure stability of (49). Moreover, allowable maximum $h_{2}$ of (49) is apparently smaller than their counterparts of (48) because $L_{m}$ in (49) is a positive definite, while its counterpart in (48) is zero, which means that $f(x(t))$ in (49) is more closely related to $x(t)$ than (48).

Example 12. Consider the delayed system in (5) with

$$
\begin{gather*}
C=\left[\begin{array}{ccc}
1 & -1.5 & 0 \\
-0.6 & -0.2 & 1 \\
0 & -1 & 1
\end{array}\right], \quad A=\left[\begin{array}{ccc}
0.2 & 0.1 & 0.1 \\
-0.1 & 0.1 & 0 \\
0 & 0.1 & -0.5
\end{array}\right], \\
L_{p}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right], \quad B=\left[\begin{array}{ccc}
-0.04 & 0.2 & 0.038 \\
-0.03 & 0.1 & 0.024 \\
0.01 & -0.1 & -0.014
\end{array}\right],  \tag{50}\\
L_{m}=\left[\begin{array}{ccc}
0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.1
\end{array}\right] .
\end{gather*}
$$

Table 1: Allowable maximum time delay $h_{2}$ under different $h_{1}$ and $\gamma$ in this paper along with results from other papers for comparison.

| Methods | $h_{1}$ | $\gamma=0.8$ | $\gamma=0.9$ | Unknown $\gamma$ |
| :--- | :---: | :---: | :---: | :---: |
| [37] | $h_{1}=0$ | $h_{2}=2.3534$ | $h_{2}=1.605$ | $h_{2}=1.5103$ |
| [37] | $h_{1}=1$ | $h_{2}=3.2575$ | $h_{2}=2.4769$ | $h_{2}=2.3606$ |
| [23] | $h_{1}=2$ | $h_{2}=4.2552$ | $h_{2}=3.4769$ | $h_{2}=3.3606$ |
| [24] | $h_{1}=0$ | $h_{2}=1.2281$ | $h_{2}=0.1493$ | $h_{2}=0.8298$ |
| [38] | $h_{1}=0$ | $h_{2}=1.6831$ | $h_{2}=1.9631$ | $h_{2}=1.088$ |
|  | $h_{1}=0$ | $h_{2}=2.8854$ | $h_{2}=4.881$ (Corollary 7) | $h_{2}=1.781$ |
|  | $h_{1}=0$ | $h_{2}=5.088$ (Corollary 7) | $h_{2}=4.751$ (Theorem 2) | $h_{2}=4.066$ (Theorem 2) |

Table 2: Allowable maximum time delay $h_{2}$ of (49) under different $h_{1}$ and $\gamma$.

| Theorem 2 | $\gamma=0.3$ | $\gamma=0.6$ | $\gamma=0.9$ |
| :--- | :---: | :---: | :---: |
| $h_{1}=2$ | $h_{2}=3.836$ | $h_{2}=3.664$ | $h_{2}=4.135$ |
| $h_{1}=2.5$ | $h_{2}=4.152$ | $h_{2}=4.21$ | $h_{2}=3.904$ |
| $h_{1}=3$ | $h_{2}=3.355$ | $h_{2}=3.687$ | $h_{2}=3.908$ |

Table 3: Allowable maximum time delay $h_{2}$ of (50) under different $h_{1}$ and $\gamma$.

| Theorem 2 | $\gamma=0$ | $\gamma=0.3$ | $\gamma=0.5$ | $\gamma=0.8$ | $\gamma=1$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $h_{1}=0.1$ | $h_{2}=1.064$ | $h_{2}=1.019$ | $h_{2}=1.044$ | $h_{2}=1.075$ | $h_{2}=1.062$ |
| $h_{1}=0.2$ | $h_{2}=0.992$ | $h_{2}=1.014$ | $h_{2}=1.021$ | $h_{2}=1.044$ | $h_{2}=1.076$ |
| $h_{1}=0.3$ | $h_{2}=0.987$ | $h_{2}=0.988$ | $h_{2}=1.008$ | $h_{2}=1.0151$ | $h_{2}=1.0171$ |
| $h_{1}=0.9$ | $h_{2}=0.995$ | $h_{2}=1.002$ | $h_{2}=1.008$ | $h_{2}=1.015$ | $h_{2}=1.011$ |
| Theorem 2 | $\gamma=1.2$ | $\gamma=1.5$ | $\gamma=1.8$ | $\gamma=2$ | $\gamma=5$ |
| $h_{1}=0.1$ | $h_{2}=1.072$ | $h_{2}=1.0811$ | $h_{2}=1.103$ | $h_{2}=1.101$ | $h_{2}=1.114$ |
| $h_{1}=0.2$ | $h_{2}=1.041$ | $h_{2}=1.055$ | $h_{2}=1.062$ | $h_{2}=1.054$ | $h_{2}=1.088$ |
| $h_{1}=0.3$ | $h_{2}=1.013$ | $h_{2}=1.036$ | $h_{2}=1.033$ | $h_{2}=1.036$ | $h_{2}=1.048$ |
| $h_{1}=0.9$ | $h_{2}=1.013$ |  | $h_{2}=1.017$ | $h_{2}=1.004$ |  |

TAble 4: Allowable maximum time delay $h_{2}$ of (50) under different $h_{1}$ and $\gamma$.

| Corollary 5 | $h_{1}=0.1$ | $h_{1}=0.15$ | $h_{1}=0.2$ | $h_{1}=0.25$ | $h_{1}=0.3$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Unknown $\gamma$ | $h_{2}=1.022$ | $h_{2}=1.015$ | $h_{2}=0.981$ | $h_{2}=0.972$ | $h_{2}=0.971$ |
| Corollary 5 | $h_{1}=0.5$ | $h_{1}=0.9$ |  |  |  |
| Unknown $\gamma$ | $h_{2}=0.983$ | $h_{2}=0.995$ | $\gamma=0.3$ | $\gamma=0.5$ | $\gamma=0.8$ |
| Corollary 7 | $\gamma=0$ | $h_{2}=1.107$ | $h_{2}=1.111$ | $h_{2}=1.1191$ | $\gamma=1$ |
| $h_{1}=0$ | $h_{2}=1.097$ | $\gamma=1.5$ | $\gamma=1.8$ | $\gamma=2$ | $h_{2}=1.121$ |
| Corollary 7 | $\gamma=1.2$ | $h_{2}=1.132$ | $h_{2}=1.142$ | $h_{2}=1.1272$ | $\gamma=5$ |
| $h_{1}=0$ | $h_{2}=1.133$ | Unknown $\gamma$ |  |  |  |
| Corollary 9 |  |  |  | $h_{2}=1.1471$ |  |
| $h_{1}=0$ |  |  |  |  |  |

This example is used to demonstrate the effectiveness of Theorem 2 in Table 3 and Corollaries 5, 7, and 9 in Table 4. Both $L_{p}$ and $L_{m}$ have taken positive definite values, and allowable maximum time delay $h_{2}$ under different $h_{1}$ and $\gamma$ is presented in Tables 3 and 4.

As seen in Table 3, values of $h_{2}$ change periodically with different $\gamma$ under the same $h_{1}$, but the difference between $h_{2}$ and $h_{1}$ decreases readily with increasing $h_{1}$ under the same $\gamma$. It can be expected that whatever value $\gamma$ takes, when $h_{1}$ is big enough, $h_{1}$ and $h_{2}$ would converge to a single point.

Moreover, it can be seen in Table 3 that the point is the same with the convergence point under $\gamma=0$, which is also the allowable maximum constant time delay.

Table 4 is used to demonstrate the effectiveness of Corollaries 5,7 , and 9 . Allowable maximum time delay is presented under unknown $\gamma$ for Corollary 5, $h_{1}=0$ for Corollary 7, and both unknown $\gamma$ and $h_{1}=0$ for Corollary 9. It can be seen that when $\gamma$ is unknown, $h_{2}$ is apparently smaller than otherwise. Moreover, sums of $h_{1}$ and $h_{2}$ are about the same with those under $h_{1}=0.1$ but smaller than those under
$h_{1} \geq 0.2$, which signifies that $h_{1}$ and $h_{2}$ have a near-linear relationship under $0 \leq h_{1} \leq 0.1$.

## 5. Conclusion

This paper has investigated the global stability of the neural networks with time-varying delay. By introducing a novel LKF, delay-dependent global stability criteria have been obtained. By reciprocally convex combination approach, a substep is taken, and a slack variable is introduced to estimate the derivative of LKF, and as a result, the proposed method is expected to be less conservative than the available literature. The proposed criteria have been formulated in terms of linear matrix inequalities and, thus, can be readily solved by standard computing software. Numerical examples are given, and analysis is made under different ranges and derivatives of time delay. The conservatism reduction has been proved to be more obvious than existing results.

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## Research Article

# Improving the Performance Metric of Wireless Sensor Networks with Clustering Markov Chain Model and Multilevel Fusion 

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#### Abstract

The paper proposes a performance metric evaluation for a distributed detection wireless sensor network with respect to IEEE 802.15.4 standard. A distributed detection scheme is considered with presence of the fusion node and organized sensors into the clustering and non-clustering networks. Sensors are distributed in clusters uniformly and nonuniformly and network has multilevel fusion centers. Fusion centers act as heads of clusters for decision making based on majority-like received signal strength (RSS) with comparison the optimized value of the common threshold. IEEE 802.15.4 Markov chain model derived the performance metric of proposed network architecture with MAC, PHY cross-layer parameters, and Channel State Information (CSI) specifications while it is including Path-loss, Modulation, Channel coding and Rayleigh fading. Simulation results represent significant enhancement on performance of network in terms of reliability, packet failure, average delay, power consumption, and throughput.


## 1. Introduction

In the recent years, employments of wireless sensor networks (WSNs) have increased in many aspects of modern lifestyle. Those applications have motivated the researchers around the world to attempt into this field and investigate Quality of Service (QoS) and improve performance and efficiency of network. Usually, wireless sensor networks are supposed to be in harsh environments; consequently, performance metric evaluation at the real situation is difficult, where human intervention for evaluating process, even maintenance, repair, or fix purposes are in jeopardy. Hence, performance evaluation based on the mathematical model of network and simulation is highly considered. Sometimes controlling a process in the large scale needs sensing a unique phenomenon of interest with several sensors. An actuator reacts precisely in relation to decision which is made based on received signals from sensors. Fusion of multiple sensing signals makes a decision more accurate than just one sensor and consequently increases system efficiency.

To address problem, a novel performance evaluation framework would be proposed. Mathematical model framework of a decentralized distributed detection is studied in cluster-based network with a Markov chain model for IEEE 802.15.4 Medium Access Control (MAC) with respect to CSMA/CA mechanism interplay by physical layer and channel state information. The framework investigates appropriated strategies by configuration of wireless sensor nodes based on the optimal tuning of IEEE 802.15.4 MAC and PHY layer key parameters [1]. Head node of each cluster is called Fusion Center (FC). Decision making at fusion node performs with respect to majority-like reception of RSS with Maximum-Likelihood Test.

Performance metric is evaluated for a clustering network topology with respect to a Markov chain model for CSMA/CA medium access control which proposed in [2] for a single node. Model describes a generalized analytical of the slotted CSMA/CA mechanism of beacon-enabled IEEE 802.15.4 with retry limits for each packet transmission. Behavior of the Markov model proposed at [2] is
describing CSMA/CA algorithm for a single node within star network with $N$ sensor nodes whereas our attempt updates performance metric equations with clustering topology and is accompanied by FCs. Model in [2] is only considered to packet collision probability as case of loss. Nevertheless, physical layer and channel state are provoking factors to loss indeed [3]. Therefore, physical-layer and CSI specification such as modulation and channel coding are utilized through the equations as a probability that denotes with $\left(P_{\text {csi }}\right)$. Network is supposed to be high data rate generation for assessment of performance. Simulation is carried out to represent probability of decision error at FC $\left(P_{e}\right)$ in a clustered network with significant enhancement on performance metric in terms of reliability $\left(R_{c}\right)$, packet failure $\left(P_{f}\right)$, average delay $\left(\mathbb{E}_{c}^{(a v)}\right)$, power consumption with considering different operation modes, idle $\left(P_{i}\right)$, sensing $\left(P_{\mathrm{sc}}\right)$, transmission $\left(P_{t}\right)$ and receiving $\left(P_{r}\right)$, and also Network aggregation throughput $\left(S_{c}\right)$.

## 2. Related Work

In the literature, for instance see [4] and the references therein, wireless sensor network is studied with a small amount of sensors and low signal to noise ratio (SNR), distributed detection, and decision making fusion rules carried out on multi-bit knowledge of local detecting sensors with Monte-Carlo simulation methods. The performance of proposed decision fusion rules is integrated with parameters such as channel Rayleigh fading and adaptive Gaussian noise. In $[5,6]$, the authors with respect to similar field of efforts in [7], proposed a simulation-based analysis impact of data fusion mechanisms in a Zigbee sensor network. It is used to monitor a particular constant binary phenomenon and evaluated performance indicators of interest, for example, Bit Error Rate (BER) and networking oriented (delay and aggregate throughput). In [8, 9] a distributed detection (DD) system is considered for multiple sensors/detectors work, collaboratively and the fusion center is responsible for the final decision-making task based on information gathered from local sensors; moreover, the integration of wireless channel conditions in algorithm design is also taken into the account (also see $[10,11]$ ). In [12], an important channel dynamic is well defined; their studies are represented by the behavior of a real link impact in low-power wireless networks. In particular, there is a large transitional region in wireless link quality which is characterized by significant levels of unreliability and asymmetry, significantly impacting on performance of higher-layer protocols. In [3], the authors used the first way to better understand IEEE 802.15.4 standard. Indeed, they provided a comprehensive model, able more faithfully to mimic the functionalities of this standard at the PHY and MAC layers. They have proposed a combination of two relevant models for the two layers. The PHY layer behavior is reproduced by a mathematical framework, which is based on radio and channel models, in order to quantify link reliability. In $[2,13,14]$ the authors proposed a generalized analysis of the IEEE 802.15.4 medium access control (MAC) protocol with focus on CSMA/CA algorithm in terms of reliability, delay, and energy consumption (for more see $[15,16]$ ). The rest of
this paper is as follows. In Section 3, we will describe the analytical framework to evaluate performance metric. This section consists of the several subsections. In Section 4 we will represent simulation results and finally Section 5 would conclud the paper.

## 3. Problem Framework

In this section, we investigate the problem of decentralized distribution detection particularly when the sensor nodes detect a constant binary phenomenon. Sensing data packages and forwards to access point (AP) through intermediate fusion center (FC). Decision making fusion rule performs at FC with majority-like signal power level reception compared to an optimized threshold. Two ideal and noisy (nonideal) channels assume and channel state information (CSI) considers with its impacts on decision-making fusion rule Probability of decision error measures at FC versus signal to noise ratio with modulation and channel coding influences. Sensor nodes distribution at each cluster is supposed to be uniform and nonuniform.

The rest of section is organized as Sections 3.1 and 3.2 depict sensing model and distributed detection in Parallel Fusion Architecture, respectively and Section 3.3 describes distributed detection in clustered Sensor Networks. Section 3.4 comprises communication channel state information such as The Rayleigh fading, path-loss and modulation, and channel coding. Section 3.5 describes medium access control role on clustered network and its performance metric equations with presence FC and impacts of CSI.
3.1. The Sensing Model. According to the stochastic geometry of sensing model, distribution of the nodes over the observing region A can be modelled by a homogeneous Poisson point process (PPP) with intensity $\rho$. Sensing model is a isotropic signal source model for detecting phenomena of interest (PoI) with path loss factor $\alpha$ depends on distance of sensor from PoI and type of signal (chemical contamination, sound, radioactive radiation, etc.) [17]. Here, we assume $\alpha$ is equal to 1 and sensor distance from PoI is $d=1$ meter. Due to sensors are integrated with transmitters as a element of a WSN, thus, the received detection signal strength to sensor with a distance $d$ away from the PoI is given by:

$$
\begin{gather*}
S(d)=\frac{S_{0}}{d^{\alpha}}  \tag{1}\\
\mathbb{P}\left\{N_{t}=n_{t}\right\}=\frac{\lambda_{t}^{n_{t}} \exp \left(-\lambda_{t}\right)}{n_{t}!}, \quad n_{t} \geq 0 \tag{2}
\end{gather*}
$$

where $N_{t}$ is a Poisson r.v. with mean $\lambda_{t}=\mathbb{E}\left\{N_{t}\right\}=\rho|\mathrm{A}|$, whereas $\rho$ is intensity of distribution nodes over observing a finite region of phenomenon with size $|\mathrm{A}|$. We suppose the nodes sensing periodically independent condition whether PoI is absent or present. Particularly, while the PoI is present, observations are not similar between nodes belong into the


Figure 1: Parallel fusion architecture.
same group of sensors. In this case, observation independently remarks at each sensor node after proper sampling and processing is given by

$$
y_{n}= \begin{cases}z_{n}, & \text { when PoI is absent }  \tag{3}\\ \sqrt{S\left(d_{n}\right)}+z_{n}, & \text { when PoI is present }\end{cases}
$$

where $n=1,2, \ldots, N_{t}, z_{n}$ is an independent observation Gaussian distribution noise with zero-mean and variance $\sigma_{z}^{2}$. $S\left(d_{n}\right)$ is the received signal strength at the $n$th node with a distance $d_{n}$ far from the PoI given by (1). Thus, problem status could be defined as follows:

$$
H= \begin{cases}H_{0}: & \text { absent PoI with probability } p_{0}  \tag{4}\\ H_{1}: & \text { present PoI with probabilty } 1-p_{0}\end{cases}
$$

Information is gathered from observers of PoI, located in center region $A$ (environment of observed PoI); hence, equal probability is assumed in term of present or absent PoI, where $p_{0}=\mathbb{P}\left\{H=H_{0}\right\}, \mathbb{P}\{\cdot\}$ being the probability of a given PoI.
3.2. Distributed Detection in Parallel Fusion Architecture. Sensor nodes are organized within Parallel Fusion Architecture (PFA) which is represented on Figure 1. Each sensor independently detects the event under observation and generates information and sends to FC through an ideal communication link. Information could be sequence of bits as symbol of present or absent PoI. According to (2) and (3),
sensors send 1 bit unit information to FC for decision making. A basic equation derived for received sensor observation signal at the FC from the $n$th sensor node is given by:

$$
\begin{equation*}
r_{n}=c_{E}+w_{n} \tag{5}
\end{equation*}
$$

where $c_{E}=\sqrt{a E_{b}} u_{n}$ and $w_{n}$ is a channel noise modeled zero-mean Gaussian distribution with variance $N_{0} / 2$ and across the nodes there is independent identical distribution (i.i.d). $E_{b}$ is transmission energy per bit and $a$ is up-link path loss coefficient between sensor node and FC. Assume $a$ is identical for all nodes. The $u_{n}$ is quantized local decision for observation of an event and characterized with two levels of unit function as follow:

$$
u_{n}= \begin{cases}+1: & \text { when } \widetilde{H}\left(y_{n}\right)=H_{1}  \tag{6}\\ -1: & \text { when } \widetilde{H}\left(y_{n}\right)=H_{0}\end{cases}
$$

whereas $\widetilde{H}\left(y_{n}\right)$ is the decision that made at the $n$th node [17]. The FC would be synchronized with whole nodes in the region A because of FC sends a beacon periodically when we want to retrieve observation data. All nodes exactly trigger and send observing data to corresponding fusion node at region A . With hypothesis ideal communication channels, decision is made at FC with Likelihood Raito Test (LRT) level of received signal by comparison an optimized common threshold value which denotes by $\xi$. Threshold level could be adapted and trained during detection period according to level of transmission signal power.


Figure 2: Block diagram of a clustered sensors network.
3.2.1. LRT with Neyman-Pearson Hypothesis Testing. Here, observing signal received to fusion node might be affected by many factors in an unforeseen manner, hence, the decisionmaking would be doing necessarily statistical. This formulates with a decision rule based on optimality criterion. Normally, optimal criteria are using three major methods, the Bayes risk criterion, the min-max criterion, and the Neyman-Pearson (NP) criterion. LRT is performed regarding NP criterion. Under NP criterion, the optimal decision rule derives from an LRT choosen based on the null and alternative hypotheses conditional probabilities:

$$
\begin{equation*}
\frac{\mathbb{P}\left\{\mathbf{r} \mid H_{1}\right\}}{\mathbb{P}\left\{\mathbf{r} \mid H_{0}\right\}} \underset{H_{0}}{\mathrm{H}_{1}} \xi, \tag{7}
\end{equation*}
$$

whereas data vector $\mathbf{r}$ is given under the alternative as $\mathbb{P}\left\{\mathbf{r} \mid H_{1}\right\}$ and data vector $\mathbf{r}$ under the null hypothesis as $\mathbb{P}\left\{\mathbf{r} \mid H_{0}\right\}$. FC decision performs based on the $N_{t}$ received observations of nodes. The vector $\mathbf{r}$ denotes as a gain of received signal in ideal Binary Symmetric Channels (BSCs). This is corresponding to $N_{t}$ specified in (5). Nevertheless, the $\xi$ for simplicity is adapted with $\sqrt{\mathrm{SNR}} / 2$ where $\mathrm{SNR}=$ $a E_{b} / N_{0}$ is received signal energy per bit per noise power spectral density, can be expressed using signal to noise (SNR), to FC from each sensor node through communication
channel. The received signals vector from $N_{t}$ sensor nodes is considered as follow:

$$
\begin{equation*}
\mathbf{r}=\left[r_{1}, \ldots, r_{i}\right]^{T}, \quad i=\left(1, \ldots, N_{t}\right) \tag{8}
\end{equation*}
$$

With the Bayesian approach, a priori probabilities of the absent or present hypothesis PoI are $\mathbb{P}\left\{H_{0}\right\}$ and $\mathbb{P}\left\{H_{1}\right\}$ at fusion center, respectively. Probability of decision error is defined at fusion center as follow:

$$
\begin{equation*}
P_{e}=\mathbb{P}\left\{\widehat{H}=H_{1} \mid H_{0}\right\} \mathbb{P}\left\{H_{0}\right\}+\mathbb{P}\left\{\widehat{H}=H_{0} \mid H_{1}\right\} \mathbb{P}\left\{H_{1}\right\} . \tag{9}
\end{equation*}
$$

3.3. Distributed Detection in Clustered Sensor Networks. A network with $n$ sensors observes a common binary phenomenon whose status is defined at (4) with $p_{0}=\mathbb{P}\{H=$ $\left.H_{0}\right\}, \mathbb{P}\{\cdot\}$ denotes the probability of given PoI. The $n$ sensors might be organized into several clusters whereas number of cluster is $n_{c}<n$ sensor nodes. Sensors belong to a cluster working as a RFD (Reduce Function Device) just communicates with corresponding FC which is a FFD (Full Function Device). Each cluster with collection of sensors is a PFA represented in Section 3.2 and Figure 2 shows $n_{c}$ cluster-based architecture [6]. The sensors are distributed in each cluster uniformly or nonuniformly. Initially, the channel between the sensors and fusion center is supposed to be an ideal communication link such as a Binary Symmetric

Channels (BSCs) with probability $p$ cross-over, memoryless communication. To continue, wireless channel also would be a non-ideal with respect to CSI specification.
3.3.1. Data-Fusion Model. Decision is made at fusion node and carries out with majority-like mechanism. In some literature this method is called consensus flooding or voting mechanism. Basically, this mechanism is based on majority similar received signal from sensors on the same cluster and event under observe in precise time. According to Figure 2 two-level fusion is shown; in first level, each cluster contains $d_{c}$ distributed sensors uniformly and $n_{c}$ is number of clusters, thus, $n=d_{c} \times n_{c}$ is number of all sensors in network. $k=\left[d_{c} / 2\right]+1$ is acceptable floor of majority-like for first level of fusion. In second level, decision-making is performed at access Point (AP) similarly with assuming FCs as $n_{c}$ sensors. Obviously, AP accepts mechanism with at least $k_{f}=$ $\left[n_{c} / 2\right]+1$ majority-likes. Non-uniform distribution of sensors is defined as unequal number of sensors for each cluster. It denotes clusters size vector by $D \triangleq\left\{d_{c}^{(1)}, d_{c}^{(2)}, \ldots, d_{c}^{\left(n_{c}\right)}\right\}$, where $d_{c}^{(i)}$ is the number sensors in the $i$ th cluster $(i=1,2$, $\ldots, n_{c}$ ) and $\sum_{i=1}^{n_{c}} d_{c}^{(i)}=N$. The probability of decision error in a generic scenario with non-uniform clustering can be evaluated as below:

$$
\begin{align*}
P_{e}= & p_{0} \sum_{i=k_{f}}^{n_{c}} \sum_{j=1}^{\left.n_{c}\right)} \prod_{\ell=1}^{n_{c}}\left\{c_{i, j}(\ell) p_{\ell}^{100}+\left(1-c_{i, j}(\ell)\right)\left(1-p_{\ell}^{100}\right)\right\} \\
& +\left(1-p_{0}\right) \sum_{i=0}^{k_{f}-1} \sum_{j=1}^{n_{i}^{n}} \prod_{\ell=1}^{n_{c}}\left\{c_{i, j}(\ell) p_{\ell}^{111}\right. \\
& \left.+\left(1-c_{i, j}(\ell)\right)\left(1-p_{\ell}^{111}\right)\right\}, \tag{10}
\end{align*}
$$

where $p_{\ell}^{1 \mid 1} \triangleq\left\{p_{1}^{1 \mid 1}, p_{2}^{1 \mid 1}, \ldots, p_{n_{c}}^{1 \mid 1}\right\}$ represents probability of success and $p_{\ell}^{1 \mid 0} \triangleq\left\{p_{1}^{1 \mid 0}, p_{2}^{1 \mid 0}, \ldots, p_{n_{c}}^{1 \mid 0}\right\}$ represents probability of failure decides at FC [9]. $\mathbf{c}_{i, j}=\left(c_{i, j}(1), \ldots, c_{i, j}\left(n_{c}\right)\right)$ is a vector which designates the $j$ th configuration of the decisions from the first-level FCs in a case with $i, 1 s$ and $n_{c}-i, 0 s[5,12]$. On the other words, $c_{i, j}$ can be represented by $\operatorname{string}(i, j, \ell)=$ 1 if there is a success, corresponding to a decision, at $\ell$ th FC or AP, in favor of $H_{1}$, whereas it is 0 if there is a failure, corresponding to a decision, at $\ell$ FC or AP in favor of $H_{0}$. string $(i, j, \ell)$ could be an auxiliary binary function used to distinguish, in the repeated trials formula, between a success and a failure [12, 18]. For example, possible configuration for $n_{c}=4$ clusters is illustrated in Table 1.
3.4. Communication Channel State Information. In this section, channel rules will be explained in interplaying with decision-making at fusion. Generated packet bits from detected event sequentially, bit to bit would be sent to fusion node through a communication channel. The impact of channel condition or channel state information (CSI) is significant on decision which would be made at fusion node. In addition to sensor observation quality, probability

TABLE 1: Possible configuration of $\mathbf{c}_{i, j}$ for $n_{c}=4$ clusters.

| $i$ | $j$ | $c_{i, j}$ |
| :--- | :---: | :---: |
| 0 | 1 | 0000 |
|  | 1 | 1000 |
| 1 | 2 | 0100 |
|  | 3 | 0010 |
|  | 4 | 0001 |
|  | 1 | 1100 |
| 2 | 2 | 1010 |
|  | 3 | 1001 |
|  | 4 | 0101 |
|  | 5 | 0110 |
|  | 6 | 0011 |
|  | 1 | 1110 |
| 3 | 2 | 1011 |
|  | 3 | 0111 |
|  | 4 | 1101 |
| 4 | 1 | 1111 |

of decision error $\left(P_{e}\right)$ at FC completely is related to channel condition and Received Signal Strength Indication (RSSI). Therefore, new element is taken into the account as CSI probability of channel which is denoted by $P_{\mathrm{csi}}$. Impact of $P_{\mathrm{csi}}$ will investigate decision-making accuracy. Here, the sensor network is modeled with no interference impact (orthogonal transmission) because of an exact scheduling between the sensors and fusion node or AP. A beacon message transmits periodically for synchronization to each sensor node when FC and AP are ready for PoI sample reception.
3.4.1. The Rayleigh Fading. Equation (5) with Rayleigh fading is given by:

$$
\begin{equation*}
r_{i}=f_{i}\left(2 c_{i}-1\right) \sqrt{E_{c}}+w_{i}, \quad i=1, \ldots, N+L \tag{11}
\end{equation*}
$$

where $f_{i}$ is a random variable with Rayleigh distribution which is perfectly coherent demodulation and $c_{i} \in\{0,1\}$ is the symbol transmitted from a sensor, $c_{i}$ is an information bit from sensor nodes [9]. The total number of transmission in sensor network is $N+L$ whereas, $N$ is number of sensors and $L$ is bits according to the parity-check equations of the Hamming code. The $E_{c}$ is the energy per coded bit whereas $E_{c} \triangleq R_{c} E_{b}$. $E_{b}$ denotes the energy per bit information and $R_{c}=1 / M$ being code rate that interpreted as a system embedding a repetition code at each sensor when $M$ is consecutive and independent observations of the same phenomenon for a sensor network with multiple observations [18]. A systematic block channel code hypothesizing that each sensor makes a single observation, by using Hamming systematic block code, generates parity bits and sends them to the FC or AP. For $N=k=4$ observer sensors generate $L=n-k=3$ bits according to the parity-check equations. It remarks $(n, k)=(7,4)$ systematic Hamming code [8].

The total number of transmission acts in the proposed sensor network is $N+L . R_{c}$ is computed in this distributed coded scheme $R_{c}=N /(N+L)=4 / 7$. Bit Error Rate (BER) with QPSK modulation at fusion node for Rayleigh fading channel is given by:

$$
\begin{equation*}
p^{\text {Rayleigh }}=\frac{1}{2}\left[1-\sqrt{\frac{R_{c} \gamma_{b}}{1+R_{c} \gamma_{b}}}\right], \tag{12}
\end{equation*}
$$

where $\gamma_{b} \triangleq E_{b} / N_{0}$ is SNR received at Fusion node or AP [9].
3.4.2. Pathloss. According to channel model distance (d) between transmitter and receiver ( FC or AP), the received power $P_{r}$ in dB is as follow:

$$
\begin{gather*}
P_{r}(d)=P_{t}-P L\left(d_{0}\right)-10 \eta \log _{10}\left(\frac{d}{d_{0}}\right)+N(0, \sigma)  \tag{13}\\
P L\left(d_{0}\right)=20 * \log _{10}(f) \tag{14}
\end{gather*}
$$

where $P_{t}$ is the output power, $\eta$ is the pathloss exponent which takes the rate of signal attenuation based on different environment obtains with empirical measurement [12]. $N(0, \sigma)$ is a Gaussian random variable with mean 0 and variance $\sigma$ (standard deviation due to multipath shadowing effects). $P L\left(d_{0}\right)$ is power attenuation at source with distance $d_{0}$ with frequency $f=v / \lambda, v$ is velocity light and $\lambda$ is wavelength. Equation (13) is an isotropic transmission. SNR in $\mathrm{dB}\left(\gamma_{\mathrm{dB}}\right)$ as a function of distance (meter) is:

$$
\begin{equation*}
\gamma(d)=P_{r}(d)-P_{n}, \tag{15}
\end{equation*}
$$

where $P_{n}$ is noise floor, more details see [12]. With substitute consequently,

$$
\begin{equation*}
\gamma_{\mathrm{dB}}(d)=P_{t}-P L\left(d_{0}\right)-10 \eta \log _{10}\left(\frac{d}{d_{0}}\right)-N(0, \sigma)-P_{n} . \tag{16}
\end{equation*}
$$

3.4.3. Modulation and Channel Coding. The QPSK Modulation and NRZ (non-return zero) channel coding impact, respectively, are:

$$
\begin{equation*}
p_{b}=Q\left(\sqrt{2 \gamma(d) \frac{B_{N}}{R}}\right) \tag{17}
\end{equation*}
$$

where $\gamma_{d}=10^{\gamma_{\mathrm{dB}} / 10}$ and $B_{N}$ is noise bandwidth and $R$ is bit data rate with channel coding given by,

$$
\begin{equation*}
P_{c \mathrm{csi}}=\left(1-p_{b}\right)^{8 e}\left(1-p_{b}\right)^{8(b-e)} \tag{18}
\end{equation*}
$$

where $e$ is Preamble length, $b$ is frame length, for more details see [12]. Rewriting (12) with channel state probability for QPSK modulation and NRZ channel coding we get

$$
\begin{equation*}
P_{\mathrm{csi}}^{\text {Rayleigh }}=\frac{1}{2}\left[1-\sqrt{\frac{R_{c} \gamma_{b}(d)}{1+R_{c} \gamma_{b}(d)}}\right] . \tag{19}
\end{equation*}
$$

Probability of decision error $P_{e}$ at Fusion or AP given in [6, 9] and updated with $P_{\text {csi }}^{\text {Rayleigh }}$ is

$$
\begin{align*}
P_{e}= & p_{0} \sum_{i=k_{f}}^{n_{c}} \sum_{j=1}^{\binom{n}{i}} \prod_{\ell=1}^{n_{c}}\left\{c_{i, j}(\ell) q^{1 \mid 0}+\left(1-c_{i, j}(\ell)\right)\left(1-q^{1 \mid 0}\right)\right\} \\
& +\left(1-p_{0}\right) \sum_{i=0}^{k_{f}-1} \sum_{j=1}^{\binom{n}{i}} \prod_{\ell=1}^{n_{c}}\left\{c_{i, j}(\ell) q^{1 \mid 1}\right. \\
& \left.+\left(1-c_{i, j}(\ell)\right)\left(1-q^{1 \mid 1}\right)\right\}, \tag{20}
\end{align*}
$$

where $q^{1 \mid 0}=p_{\ell, \text { csi }}^{1 \mid 0, \text { Rayleigh }}, q^{1 \mid 1}=p_{\ell, \text { csi }}^{1 \mid 1, \text { Rayleigh }}$, and $\ell=$ $\left\{1, \ldots, n_{c}\right\}$.
3.5. Medium Access Control Role on Clustered Network. Basically, Markov chain and performance metric expression proposed in $[2,14]$ are considered with fusion and clustered network (also see [19]). Three major parameters which reformed into scenario are the probability of a node attempts a first carrier sensing (CCA1) in randomly chosen time slot is denoted with $t$ and given by

$$
\begin{equation*}
\tau=\left(\frac{1-x^{m+1}}{1-x}\right)\left(\frac{1-y^{n+1}}{1-y}\right) \widetilde{b}_{0,0,0} \tag{21}
\end{equation*}
$$

where approximation of state probability is

$$
\begin{align*}
\widetilde{b}_{0,0,0} \approx & \frac{W_{0}}{2}(1+2 x)(1+y)+L_{s}\left(1-x^{2}\right)(1+y) \\
& +K_{0}\left(\left(P_{c}\left(1-x^{2}\right)\right)^{2}\left(\left(P_{c}\left(1-x^{2}\right)\right)^{n-1}+1\right)+1\right)^{-1} \tag{22}
\end{align*}
$$

and, $P_{c}$, probability of transmitted packet encounter collision when $N$ is number of whole nodes, is given by

$$
\begin{equation*}
P_{c}=1-(1-\tau)^{N-1} \tag{23}
\end{equation*}
$$

also, $K_{0}=L_{0} q_{0} /\left(1-q_{0}\right)$ whereas, $L_{0}$ is the idle state length without generating packets and, $q_{0}$ is the probability of going back to the idle state. Consider

$$
\begin{gather*}
x=\alpha+(1-\alpha) \beta  \tag{24}\\
y=P_{c}\left(1-x^{m+1}\right) \tag{25}
\end{gather*}
$$

The busy channel probabilities (CCA1) and (CCA2) are $\alpha, \beta$, respectively, given as follows:

$$
\begin{equation*}
\alpha=\alpha_{1}+\alpha_{2} \tag{26}
\end{equation*}
$$

where

$$
\begin{gather*}
\alpha_{1}=L\left(1-(1-\tau)^{N-1}\right)(1-\alpha)(1-\beta) \\
\alpha_{2}=L_{\mathrm{ack}} \frac{N \tau(1-\tau)^{N-1}}{1-(1-\tau)^{N}}\left(1-(1-\tau)^{N-1}\right)(1-\alpha)(1-\beta), \tag{27}
\end{gather*}
$$

with

$$
\begin{equation*}
\beta=\frac{1-(1-\tau)^{N-1}+N \tau(1-\tau)^{N-1}}{2-(1-\tau)^{N}+N \tau(1-\tau)^{N-1}} . \tag{28}
\end{equation*}
$$

While the Markov chain just declared the probability of collision $P_{c}$ as cause of loss, we bring $P_{c s i}$ which is derived in (18), into the account as another possibility of loss due to different SNR, modulation, and channel coding. Probability of failure is defined as

$$
\begin{equation*}
P_{\text {fail }}=1-\left(1-P_{c}\right)\left(1-P_{\mathrm{csi}}\right) \tag{29}
\end{equation*}
$$

where $P_{c}$ is given in (23) as probability of packet collision. Cluster network could be modeled with binominal random variable with independent $j$ th clusters $D_{c}^{(j)}$, where $j=$ $1, \ldots, n_{c}, d_{c}^{(j)}$ is referring to cluster size, denotes a probability $p_{\text {mac }}\left(d_{c}^{(j)}\right)$ corresponding to $j$ th cluster. Performance metric expression that has been extracted from Markov model could be updated according to our assumptions:

$$
\begin{equation*}
P(\mathfrak{J})=\sum_{i_{1}=0}^{d_{c}^{(1)}} \cdots \sum_{i_{n_{c}}=0}^{d^{\left(n_{c}\right)}} \mathbb{P}\left\{D^{(1)}=i_{1}\right\} \cdots \mathbb{P}\left\{D^{\left(n_{c}\right)}=i_{n_{c}}\right\}, \tag{30}
\end{equation*}
$$

where $\mathfrak{J}$ denotes possible variable which could be computed by

$$
\begin{equation*}
\mathbb{P}\left\{D^{(\ell)}=i_{\ell}\right\}=\binom{d_{c}^{(\ell)}}{i_{\ell}}\left[p_{\mathrm{mac}}\left(d_{c}^{(\ell)}\right)\right]^{i_{\ell}}\left[1-p_{\mathrm{mac}}\left(d_{c}^{(\ell)}\right)\right]^{d_{c}^{(\ell)}-i_{\ell}} \tag{31}
\end{equation*}
$$

Using Markov chain performance metric equations, we will be obtaining the following.
3.5.1. Reliability. The probability of successful delivery of packets $\mathbf{R}$ as a clustering topology network, regarding reliability in [2], (31), and (30) redefining the probability of successful delivery of packets majority sensors per cluster which satisfy majority-like fusion strategy, is:

$$
\begin{equation*}
P_{R}^{i_{\ell}}=\sum_{i_{\ell}=\chi}^{d_{c}^{(\ell)}} \prod_{\ell=1}^{n_{c}}\binom{d_{c}^{(\ell)}}{i_{\ell}}\left[\mathbf{R}\left(d_{c}^{(\ell)}\right)\right]^{i_{\ell}}\left[1-\mathbf{R}\left(d_{c}^{(\ell)}\right)\right]^{d_{c}^{(\ell)}-i_{\ell}} \tag{32}
\end{equation*}
$$

where $\chi=\left\lfloor d_{c}^{(\ell)} / 2\right\rfloor+1, \ell=\left\{1, \ldots, n_{c}\right\}$. Two-level fusion at FC and AP, $P_{R}^{i_{e}}$ is given as the probability of successful delivery distributed sensors in first level fusion; the probability of successful delivery FC to AP has similarity by assuming as a cluster with $n_{c}$ sensors for second level fusion. Hence, reliability equation for both levels of fusion at FC and AP is remarked with $\mathbf{R}_{c}$ given by,

$$
\begin{equation*}
\mathbf{R}_{c}=P_{R}^{i_{e}} \cdot P_{R}^{(f c)} \tag{33}
\end{equation*}
$$

where $P_{R}^{(f c)}$ is obtained from (30) and (31) with $d_{c}^{(f c)}=n_{c}$ for second level.
3.5.2. Average Delay. It is noted that communication delays can deteriorate the performance of the network and even can destabilize the systems when they are not considered in the design (see [20, 21]). Therefore, the average delay for clustering with two-level fusion is defined as average delay of successfully received packet as the time interval from the instant the packet is at the head of its MAC queue and ready to be transmitted, until the transmission is successful and the ACK is received from both level of fusion nodes, respectively. According to [2], in framework except the constants (frame length, Ack length, etc.), MAC parameters have only two terms, $\operatorname{Pr}\left(A_{j} \mid A_{t}\right)$ and $\widetilde{P}\left(B_{i} \mid B_{t}\right)$ that could be computed based on (30) and (31). However, initially $\alpha, \beta, \tau$ should be calculated with respect to a given topology at clusters and also $x, y$, and $P_{c}$ with term (29). Obviously, MAC parameters are similar for all equations with optimal tune. Framework concerning majority-like mechanism should be taken into account when encountered with $N$ number sensors in original Markov chain equations that are replaced by $\left\lfloor d_{c} / 2\right\rfloor+$ $1, \ldots, d_{c}$ for each cluster by corresponding sensors. So far, average delay is described for first level fusion of each cluster separately. For second fusion level, it is acting as a cluster with $n_{c}$ sensors. Average delay of whole network is proposed by:

$$
\begin{equation*}
\mathbb{E}_{c}^{(\mathrm{av})}[\widetilde{D}]=\frac{\mathbb{E}_{\max }+\mathbb{E}_{\min }}{2}+\mathbb{E}^{(f c)}[\widetilde{D}] \tag{34}
\end{equation*}
$$

where, $\mathbb{E}_{\max }=\operatorname{Max}\left\{\mathbb{E}_{c}^{(1)}[\widetilde{D}], \ldots, \mathbb{E}_{c}^{\left(n_{c}\right)}[\widetilde{D}]\right\}$ and $\mathbb{E}_{\min }=$ $\operatorname{Min}\left\{\mathbb{E}_{c}^{(1)}[\widetilde{D}], \ldots, \mathbb{E}_{c}^{\left(n_{c}\right)}[\widetilde{D}]\right\}$, first term is average delay for which packets arriving for first level fusion at FC clusters head, and second term for which packets arriving second level fusion at AP. Because of synchronized network, transmission happens at the same time and concurrently; hence, Max and Min are computed regarding to cluster size and parameters.
3.5.3. Network Aggregate Throughput. Network aggregate throughput would be computed for minimum effective number of nodes each cluster network with two-level fusion and data rate $g(\mathrm{bps})$ is given by:

$$
\begin{equation*}
S_{c}=g \cdot A \cdot L_{s} \cdot \hbar \cdot \mathbf{R}_{c} \tag{35}
\end{equation*}
$$

where $\hbar=\sum_{\ell=1}^{n_{c}}\left\lfloor d_{c}^{(\ell)} / 2\right\rfloor+1, \mathbf{R}_{c}$ is computed at (33) and $A=$ $80 \mathrm{bit} / 0.32 \mathrm{~ms}$ is a normalization constant to convert to bps.
3.5.4. Average Power Consumption. The average power consumption equations are proposed in $[2,14]$ taken into consideration by the clustering framework with two-level fusion. Constant values given in Table 2 are used for first level fusion; however, for second level fusion they are valid except $P_{i} \approx 0$ because of assuming fusion center does not have ideal state at second level, also hypothesis $P_{\text {sc }}$ sensing power constant at sensor is corresponding with power of decision-making at fusion node and assumed same computation term.

## 4. Simulation Results

This section represents the results of simulation based on problem framework. Basically, simulations are figured out

Table 2: Power consumption of different operation modes.

| Operation mode | Power consumption |
| :--- | :---: |
| $P_{i}$ | 0.657 mW |
| $P_{\mathrm{sc}}$ | 35.46 mW |
| $P_{t}$ | 31.32 mW |
| $P_{r}$ | 35.46 mW |



Figure 3: Probability of decision error as a function of SNR, $n=32$ sensors with AWGN.
with " 32 " nodes as detector of an event of interest, each node generates high traffic data rates. Performance metric is evaluated with probability of decision error and developed equations of Markov model. Rest of section is organized into two subsections based on those evaluations.
4.1. Probability of Decision Error. Simulation results shows for evaluating probability of decision error in fusion center based on described framework. Probability of decision error is considered at fusion node with respect to clustering topology as long as presence of uniform and non-uniform distributions of " 32 " sensors. Three non-uniform distributions 12.8.8.4, 16.8.4.4, and 25.5 .2 are versus uniform distribution 8.8.8.8. Non-clustering by " 32 " sensors are shown as a proof of comparison in Figure 3. It represents the probability of decision error for non-clustering topology which looks like a star network with coordinator acting as fusion node. Detection sequences just effect with Additive White Gaussian Noise (AWGN) communication channel with OQPSK modulation format.

Basically, an increment of SNR has improvement on decision. According to various sensors distribution, Figure 3 is shown that non-clustering is worst case with respect to our scenario; the decision is made at fusion based on vector received signals on majority-like strategy. Hence, in case of non-clustering at least 17 sensors similar to record as correct decision should be received but for clustering this limitation reduces to $\left[d_{c} / 2\right]+1$. Number of sensors at each cluster for example in 16.8.4.4 design by 4 clusters have $16,8,4,4$ sensors at each cluster; therefore, fusion node at head of clusters


Figure 4: Probability of decision error versus SNR, $n=32$ sensors with $P_{\text {csi }}$.

Table 3: Parameters value for physical layer.

| Parameter | Value |
| :--- | :---: |
| Minimum distance | 1 meter |
| Maximum distance | 40 meters |
| Frame length | 808 bits |
| Power $T x$ | 3 dB m |
| pramble length | 40 bits |
| Noise figure | -123 dB m |
| Noise | -5 dB m |
| Band width | 30 kHz |
| Signal frequency $(f)$ | 2450 MHz |
| Path loss exponent | 4 |
| Shadowing standard deviation | 4 |

should be evaluated 9, 5, 3, 3 signals similarly which have same level for corresponding clusters. However, in second level decision-making at AP should be outcome of decision on first level satisfies with 3 similar signals received form 4 fusion nodes.

Figure 4 shows the probability of decision error with presence of $P_{\mathrm{csi}}$ and fading effect. Impact of $P_{\mathrm{csi}}$ and fading effect are measured by attenuation on level of signal to change probability in order to increscent decision error due to channel influence. According to literatures of Monte Carlo simulation of corresponding expressions in given framework has confirmed our simulation. MAC and PHY parameters values used for $P_{\text {csi }}$ are shown in Table 3.
4.2. Performance Metric Evaluation Based on Markov Chain. Impact of MAC appraised on proposed framework. Simulation of performance metric equations is carried out with MAC and PHY-layers parameters denoted in Tables 3 and 4.
4.2.1. Reliability. Reliability is obtained for non-uniform and uniform topology which is supposed to be with different

Table 4: Parameters value for MAC layer.

| Parameter | Value |
| :--- | :---: |
| MacMaxFrameRetries $(n)$ | 3 |
| MacMaxCSMABackoffs $(m)$ | 4 |
| MacMinBE $\left(m_{0}\right)$ | 3 |
| MacMaxBE $\left(m_{b}\right)$ | 5 |
| $L$ | 1016 bits |
| $L_{\text {ack }}$ | 88 bits |
| $t_{\text {ack }}$ | $222 e-9$ seconds |
| $t_{\text {IFS }}$ | $640 e-6$ seconds |
| $t_{m, \text { ack }}$ | $200 e-9$ seconds |
| aUnitBackoffPeriod | $320 e-6$ seconds |
| macACKWaitDuration | $1920 e-6$ seconds |
| aTurnaroundTime | $192 e-6$ seconds |
| $S_{b}$ | $128 e-6$ seconds |
| $L_{0}$ | $10 e-12$ |
| $W_{0}$ | $2^{\text {macMinBE }}$ |
| $q_{0}$ | $10 e-12$ |



Figure 5: Reliability, $E_{b} / N_{0}=3,12[\mathrm{~dB}]$, OQPSK.
number of sensors at each clusters. Model is evaluated in high data rate generation. Three non-uniformly distribution 16.8.4.4, 25.5.2, and 12.8 .8 .4 at each cluster and uniformly 8.8.8.8 distribution sensors are compared by non-clustering which is similar to a star topology that originally was assumed in Markov chain model. Result shows a significant improvement in reliability in clustering topology even in two-level fusion. However, in clustering based topologies balance of sensors distribution (uniform) in clusters are more reliable than unbalances (non-uniform). Figure 5 represents the reliability of system with signal to noise ratio equal to 12 dB in solid line by comparison with 3 dB in dots line. Reliability is enhanced in order to increment signal to noise ratio. Direct relation between probability of success packet reception or reliability, with probability of packet failure shown in Figure 6, has consequent improvement on reliability. Increasing SNR from 3 dB to 12 dB causes less failure packet reception at fusion node. Result represents in


Figure 6: Failure probability, $E_{b} / N_{0}=3,12$ [dB], OQPSK.


Figure 7: Average delay for two-level decision.
probability of decision error $P_{e}$ also proving this improvement at FC. However, packet failure is increased versus packet generation rate; therefore, we can expect high packet generation rate more effective than increment of SNR ratio in failure term.
4.2.2. Average Delay. Measurement of average delay is explained in framework. Simulation performs with high traffic regime with two SNRs 3 dB and 12 dB , see Figure 7. Important issue here is synchronizing between nodes by specifying a time slot from FC to nodes for retrieve data. Obviously, this time slot is corresponding to size of each cluster, therefor, time slot for cluster with 8 sensors is four


Figure 8: Average power consumption for two-level decision.


Figure 9: Throughput, $E_{b} / N_{0}=3,12[\mathrm{~dB}]$, OQPSK.
times greater than time slot for cluster with 2 sensors because of preventing collision in each cluster during transmitting and each node of cluster has its own time slot to send. Clusters are independent from each other and transmit in their appropriated bandwidth. IEEE 802.15 .4 has 16 channels in 2.4 GHz , based on simulation with maximum 4 clusters there is not any constraint in bandwidth scheduling; hence, each cluster works in a unique bandwidth. Slotted Markov chain model specification satisfies the condition. Non-clustering topology has more average delay. That delay is imaginable because time slot scheduling scenario for " 32 " nodes need longer time slot length. Uniformed distribution 8.8.8.8 needs a time slot with 8 portions at each cluster for retrieving data
process. Clusters that have more nodes need at least a time slot longer than with 8 portions. Basically, effect of number sensors on $\alpha$ and $\beta$ and $\tau$ are important exact contribution of less sensors causes increasing probability of access channel and directly reduces delays.
4.2.3. Average Power Consumption. An increment of mean power consumption with higher data generation rate obviously is illustrated in Figure 8. Basically, data transmission consumes more power rather than computational matter in sensor module. Nevertheless, number of sensors and fusion level have critical roles to achieve power consumption. Topology without clustering " 32 " sensors contribute in decision-making in fusion node coordinates at least half plus one received bits stream signal. Average power is increased with 12 dB signal to noise ratio due to transmission power consumption, $P_{s}$. While power consumption is a very critical issue to wireless sensor network, increment of power consumption is unwilling with respect to restriction on battery capability. On the other view, preciseness of packet receipt sometimes has privilege to power consumption.
4.2.4. Network Aggregate Throughput. Network aggregate throughput is shown in Figure 9 as function of data generation rate with two SNR ratios. Throughput relation with reliability is explained in framework description. All issues represent improvement in higher signal to noise ratio. Throughput reduction happens when data generation rate is up to 900 bits per seconds in each node.

## 5. Conclusion

This paper considered a distributed detection in cluster sensor network with fusion node as a decision maker head of each cluster. We utilized a Markov chain model for evaluation network performance. Generally speaking, network clusterbased topology with data fusion has better performance with aim of data accuracy. Presence of clustering with balance distribution of sensors is acting more efficiently than nonuniform clustering with more number of distributed sensors. Number of distribution sensors directly impacts average delay in clusters; hence, a topology should be selected for less delay achievement. Throughput has better outcome in cluster-based with balance distribution sensors. Power consumption has been acting better in uniformly distributed topology instead of non-uniformly as well as clustering and non-clustering. The main reason of this difference between sensor arrangements is scheduling and timing issue on network. Those issues influence directly on average delay and power consumption. However, it can affect packet failure and also reliability of system.

Based on the results in the paper, interesting future research may be prospective as follows:
(1) optimized sensor arrangement in cluster and network state estimation could be considered;
(2) fault detection and time delays in the network with Markovian jump systems under partially known
transition probabilities can be studied in the framework of this paper (see for instance [22-25]);
(3) the approach, presented in this work, can also be extended to complex networks with constrained information exchange, and a partial knowledge of the state variables (see [26, 27]).

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