Abstract and Applied Analysis

Special Issue
Functional Differential and Difference Equations with Applications

Guest Editors
Josef Diblík, Elena Braverman, István Györi,
Yuriy Rogovchenko, Miroslava Růžičková, and Ağacık Zafer

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Editorial Board

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 986585, 3 pages
doi:10.1155/2012/986585

Editorial

Functional Differential and Difference Equations with Applications

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Received 1 November 2012; Accepted 1 November 2012

This special issue may be viewed as a sequel to Recent Progress in Differential and Difference Equations edited by the four members of the present team and published by the Abstract and Applied Analysis in 2011. Call for papers prepared by the Guest Editors and posted on the journal's web page encouraged the submission of state-of-the-art contributions on a wide spectrum of topics such as asymptotic behavior of solutions, boundedness and periodicity of solutions, nonoscillation and oscillation of solutions, representation of solutions, stability, numerical algorithms, computational aspects, and applications to real-world phenomena. Our invitation was warmly welcomed by the mathematical community—more than one hundred manuscripts addressing important problems in the qualitative theory of functional differential and difference equations were submitted to the editorial office and went through a thorough peer-refereeing process. A thirty-eight carefully selected research articles for this special issue reflect modern trends and advances in functional differential and difference equations. Eighty-seven authors from seventeen countries (Bulgaria, China, Czech Republic, Egypt, Germany, Greece, Korea, Libya, Poland, Saudi Arabia, Slovak Republic, Spain, Thailand, Turkey, Ukraine, United Kingdom, and USA) have contributed to the success of this thematic collection of papers dealing with various classes of delay differential equations, dynamic equations on time scales, partial differential equations, neutral functional differential equations, systems with p-Laplacian, stochastic differential equations, and related topics.

For many decades, the stability problems has attracted attention of researchers working with the qualitative theory of differential, functional differential, and difference equations. In this issue, the reader will find papers addressing stability of nonlinear differential systems with random parameters, robust stability of interval neural networks with discrete and distributed time delays, mean square exponential stability of stochastic-switched systems with interval time-varying delays, global exponential stability of periodic solutions to neural networks and impulsive neural networks with time-varying delays, and stability of impulsive stochastic functional differential systems.

Important directions in the modern qualitative theory of differential equations include periodicity, almost periodicity, oscillation, and nonoscillation of solutions to various classes of equations. This special issue contains papers that are concerned with the existence of periodic solutions to nonlinear dynamic equations on time scales, oscillation of second-order quasilinear neutral functional differential equations, existence of periodic solutions to difference systems with a p-Laplacian, interval oscillation criteria for second-order-mixed nonlinear impulsive differential equations with delay, existence of positive periodic solutions to first-order neutral functional differential equations with periodic delays, existence of almost periodic solutions to parabolic inverse Cauchy problems, bounded oscillation of forced nonlinear neutral differential equations, and existence of periodic solutions to Duffing-type p-Laplacian equations with multiple constant delays. Two papers in this collection deal with the asymptotic behavior of solutions to a class of two-dimensional differential systems with a finite number of nonconstant delays and with the properties of smooth solutions to a class of iterative functional differential equations.

It is for sure that existence of positive solutions is important for many applied problems. In this special issue, the reader can find contributions that address positivity of solutions to nonlinear two-dimensional difference systems with multiple delays, existence of positive bounded solutions to third-order discrete equations, partial difference equations with delays, Neumann boundary value problems for second-order impulsive differential equations in Banach spaces, neutral differential equations, existence and multiplicity of positive solutions to a class of nonlinear discrete fourth-order boundary value problems, and existence of positive monotonic solutions to nonlocal boundary value problems for a class of second-order functional differential equations. Estimates for positive solutions to discrete linear equations with a single delay are also established in one of the papers.

Several articles deal with important aspects of the theory of boundary value and initial value problems providing the analysis of a class of boundary value problems for a system of autonomous second-order linear partial differential equations of parabolic type with a single delay and Dirichlet problems with an indefinite and unbounded potential and concave-convex nonlinearities. In other papers included in this special issue, linear homogeneous partial differential equations with entire solutions represented by Laguerre polynomials are studied, monotone-iterative methods for initial value problems for differential equations with "maxima" and initial time differences are developed, necessary and sufficient conditions for the existence of solutions to a class of discrete second-order boundary value problems are derived, existence of one-signed solutions to some discrete second-order periodic boundary value problems is established, and optimal conditions for the existence and uniqueness of solutions to a class of nonlocal boundary value problems for linear homogeneous second-order functional differential equations with piecewise constant arguments are obtained.

The last but not the least, this issue features a number of publications that report recent progress in the analysis of problems arising in various applications. In particular, dynamics of delayed neural network models consisting of two neurons with inertial coupling were studied, properties of a stochastic delay logistic model under regime switching were explored, and analysis of the permanence and extinction of a single species with contraception and feedback controls was conducted. Other applied problems addressed in this special issue regard Cohen-Grossberg BAM neural networks with time-varying delays, adaptive observer-based fault estimation for stochastic Markovian jumping systems, hematopoiesis models, Lotka-Volterra systems, and finite-time attractivity for diagonally dominant systems with off-diagonal delays.

It is certainly impossible to provide in this short editorial note a more comprehensive description for all articles in the collection. However, the team of the Guest Editors believes that results included in this volume represent significant contemporary trends in the qualitative theory of ordinary, functional, partial, impulsive, dynamic, stochastic differential, and difference equations and in applications. We hope that this special issue will serve as a source of inspiration for researchers working in related areas providing specialists with a wealth of new ideas, techniques, and unsolved problems.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 689319, 17 pages
doi:10.1155/2012/689319

Research Article

Dynamics in a Delayed Neural Network Model of Two Neurons with Inertial Coupling

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Received 27 February 2012; Accepted 26 May 2012

Academic Editor: Yuriy Rogovchenko

A delayed neural network model of two neurons with inertial coupling is dealt with in this paper. The stability is investigated and Hopf bifurcation is demonstrated. Applying the normal form theory and the center manifold argument, we derive the explicit formulas for determining the properties of the bifurcating periodic solutions. An illustrative example is given to demonstrate the effectiveness of the obtained results.

1. Introduction

In recent years, a number of different classes of neural networks with or without delays, including Hopfield networks, cellular neural networks, Cohen-Grossberg neural networks, and bidirectional associate memory neural networks have been active research topic as [1], and substantial efforts have been made in neural network models, for example, Huang et al. [2] studied the global exponential stability and the existence of periodic solution of a class of cellular neural networks with delays, Guo and Huang [3] investigated the Hopf bifurcation natures of a ring of neurons with delays, Yan [4] analyzed the stability and bifurcation of a delayed tri-neuron network model, Hajihosseini et al. [5] made a discussion on the Hopf bifurcation of a delayed recurrent neural network in the frequency domain, and Liao et al. [6] did a theoretical and empirical investigation of a two-neuron system with distributed delays in the frequency domain. Agranovich et al. [7] considered the impulsive control of a hysteresis cellular neural network model. For more information, one can see [8–24]. In 1986 and 1987, Babcock and Westervelt [25, 26] had investigated the stability and dynamics of the following simple neural network model of two neurons with inertial coupling:

$$\begin{aligned}\frac{dx_1}{dt} &= x_3,\\ \frac{dx_2}{dt} &= x_4,\\ \frac{dx_3}{dt} &= -2\xi x_3 - x_1 + A_2 \tanh(x_2),\\ \frac{dx_4}{dt} &= -2\xi x_4 - x_2 + A_1 \tanh(x_1),\end{aligned} \tag{1.1}$$

where x_i $(i = 1,2)$ is the input voltage of the ith neuron, x_j $(j = 3,4)$ denotes the output of the jth neuron, $\xi > 0$ is the damping factor, and A_i $(i = 1,2)$ is the overall gain of the neuron which determines the strength of the nonlinearity. For a more detailed interpretation of the parameters, one can see [25, 26]. In 1997, Lin and Li [27] made a detail discussion on the bifurcation direction of periodic solution for system (1.1).

From applications point of view, considering that there is a time delay (we assume that it is τ) in the response of the output voltages to changes in the input, that is, there exists a feedback delay of the input voltage of the ith neuron to the growth of the output of the jth neuron, then we modify system (1.1) as follows:

$$\begin{aligned}\frac{dx_1}{dt} &= x_3,\\ \frac{dx_2}{dt} &= x_4,\\ \frac{dx_3}{dt} &= -2\xi x_3 - x_1(t-\tau) + A_2 \tanh(x_2(t-\tau)),\\ \frac{dx_4}{dt} &= -2\xi x_4 - x_2(t-\tau) + A_1 \tanh(x_1(t-\tau)).\end{aligned} \tag{1.2}$$

It is well known that the research on the Hopf bifurcation, especially on the stability of bifurcating periodic solutions and direction of Hopf bifurcation is very critical. When delays are incorporated into the network models, stability and Hopf bifurcation analysis become more difficult. To obtain a deep and clear understanding of dynamics of neural network model of two neurons with inertial coupling, we will make a discussion on system (1.2), that is, we study the stability, the local Hopf bifurcation for system (1.2).

The remainder of this paper is organized as follows. In Section 2, we investigate the stability of the equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results.

2. Stability of the Equilibrium and Local Hopf Bifurcations

In this section, we shall study the stability of the equilibrium and the existence of local Hopf bifurcations. For simplification, we only consider the zero equilibrium. One can check that if the following condition:

$$(H1) A_1 A_2 < 1 \tag{2.1}$$

holds, then (1.2) has a unique equilibrium $E(0,0,0,0)$. The linearization of (1.2) at $E(0,0,0,0)$ is given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_3, \\ \frac{dx_2}{dt} &= x_4, \\ \frac{dx_3}{dt} &= -2\xi x_3 - x_1(t-\tau) + A_2 x_2(t-\tau), \\ \frac{dx_4}{dt} &= -2\xi x_4 - x_2(t-\tau) + A_1 x_1(t-\tau), \end{aligned} \tag{2.2}$$

whose characteristic equation takes the form of

$$\det\begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ e^{-\lambda\tau} & -A_2 e^{-\lambda\tau} & \lambda + 2\xi & 0 \\ -A_1 e^{-\lambda\tau} & e^{-\lambda\tau} & 0 & \lambda + 2\xi \end{pmatrix} = 0, \tag{2.3}$$

that is,

$$\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2 + \left(2\lambda^2 + 4\xi\lambda\right)e^{-\lambda\tau} + (1 - A_1A_2)e^{-2\lambda\tau} = 0. \tag{2.4}$$

Multiplying $e^{\lambda\tau}$ on both sides of (2.4), it is easy to obtain

$$\left(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2\right)e^{\lambda\tau} + 2\lambda^2 + 4\xi\lambda + (1 - A_1A_2)e^{-\lambda\tau} = 0. \tag{2.5}$$

In order to investigate the distribution of roots of the transcendental equation (2.5), the following lemma is helpful.

Lemma 2.1 (see [28]). *For the following transcendental equation:*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}\right) = {} & \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ & + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ & + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0, \end{aligned} \tag{2.6}$$

as $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ in the open right half-plane can change, and only a zero appears on or crosses the imaginary axis.

For $\tau = 0$, (2.5) becomes

$$\lambda^4 + 4\xi\lambda^3 + \left(4\xi^2 + 2\right)\lambda^2 + 4\xi\lambda + 1 - A_1A_2 = 0. \tag{2.7}$$

In view of Routh-Hurwitz criteria, we know that all roots of (2.7) have a negative real part if the following condition:

$$(H2)4\xi^2 + A_1A_2 > 0 \tag{2.8}$$

is fulfilled.

For $\omega > 0$, $i\omega$ is a root of (2.5) if and only if

$$\left(\omega^4 - 4\xi\omega^3 i - 4\xi^2\omega^2\right)(\cos\omega\tau + i\sin\omega\tau) - 2\omega^2 + 4\xi\omega i + (1 - A_1A_2)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.9}$$

Separating the real and imaginary parts gives

$$\begin{aligned} &\left(\omega^4 - 4\xi^2\omega^2 + 1 - A_1A_2\right)\cos\omega\tau + 4\xi\omega^3\sin\omega\tau = 2\omega^2, \\ &\left(\omega^4 - 4\xi^2\omega^2 - 1 + A_1A_2\right)\sin\omega\tau - 4\xi\omega^3\cos\omega\tau = -4\xi\omega. \end{aligned} \tag{2.10}$$

Then, we obtain

$$\sin\omega\tau = \frac{4\xi\omega^5 + 16\xi^3\omega^3 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}, \tag{2.11}$$

$$\cos\omega\tau = \frac{2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}. \tag{2.12}$$

In view of $\sin^2\omega\tau + \cos^2\omega\tau = 1$, then, we have

$$\left[4\xi\omega^5 + 16\xi^3\omega^3 - (1 - A_1A_2)\right]^2 + \left[2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)\right]^2 = \left[\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2\right]^2, \tag{2.13}$$

which is equivalent to

$$\omega^{16} + l_1\omega^{14} + l_2\omega^{12} + l_3\omega^{10} + l_4\omega^8 + l_5\omega^6 + l_6\omega^5 + l_7\omega^4 + l_8\omega^3 + l_9 = 0, \tag{2.14}$$

where

$$\begin{aligned}
&l_1 = 16\xi^2,\ l_2 = 96\xi^4 - 4,\ l_3 = 256\xi^6 - 48\xi^2,\\
&l_4 = 256\xi^8 - 64\xi^4 - 128\xi^3 - 2(1 - A_1A_2)^2,\ l_5 = 2(1 - A_1A_2) - 16(1 - A_1A_2)^2\xi^2 - 256\xi^6,\\
&l_6 = 2\xi(1 - A_1A_2),\ l_7 = 16\xi^2(1 - A_1A_2) - 32\xi^4(1 - A_1A_2)^2,\\
&l_8 = 32\xi^3(1 - A_1A_2),\ l_9 = -[(1 - A_1A_2)^4 + (1 - A_1A_2)^2].
\end{aligned}$$

Denote

$$h(\omega) = \omega^{16} + l_1\omega^{14} + l_2\omega^{12} + l_3\omega^{10} + l_4\omega^8 + l_5\omega^6 + l_6\omega^5 + l_7\omega^4 + l_8\omega^3 + l_9. \tag{2.15}$$

Since $l_9 < 0$ and $\lim_{\omega\to+\infty} h(\omega) = +\infty$, then we can conclude that (2.14) has at least one positive root. Without loss of generality, we assume that (2.14) has sixteen positive roots, denoted by ω_k $(k = 1, 2, 3, \ldots, 16)$. Then, by (2.12), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left[\frac{2\omega^6 + 8\xi^2\omega^4 - (1 - A_1A_2)}{\left(\omega^4 + 4\xi^2\omega^2\right)^2 - (1 - A_1A_2)^2}\right] + 2j\pi\right\}, \tag{2.16}$$

where $k = 1, 2, 3, \ldots, 16;\ j = 0, 1, \ldots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (2.4) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3,\ldots,16\}}\left\{\tau_k^{(0)}\right\}. \tag{2.17}$$

The above analysis leads to the following result.

Lemma 2.2. *If* $(H1)$ *and* $(H2)$ *hold, then all roots of* (2.4) *have a negative real part when* $\tau \in [0, \tau_0)$ *and* (2.4) *admits a pair of purely imaginary roots* $\pm\omega_k$ *when* $\tau = \tau_k^{(j)}$ $(k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, \ldots)$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (2.5) near $\tau = \tau_k^{(j)}$, $\alpha(\tau_k^{(j)}) = 0$, and $\omega(\tau_k^{(j)}) = \omega_k$. Due to functional differential equation theory, for every $\tau_k^{(j)}$, $k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, \ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{(j)}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (2.5) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi}{\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2)e^{\lambda\tau}} - \frac{\tau}{\lambda}. \tag{2.18}$$

Noting that

$$\begin{gathered}\left[4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi\right]_{\tau=\tau_k^{(j)}} = K_1 + K_2 i,\\ \left[\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda\left(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2\right)e^{\lambda\tau}\right]_{\tau=\tau_k^{(j)}} = P_1 + P_2 i,\end{gathered} \tag{2.19}$$

where

$$\begin{aligned}K_1 &= 4\xi - 12\xi\omega_k^2\cos\omega_k\tau_k^{(j)} - (18\xi^2\omega_k - 4\omega_k^3)\sin\omega_k\tau_k^{(j)},\\ K_2 &= 4\omega_k + (8\xi^2\omega_k - 4\omega_k^3)\cos\omega_k\tau_k^{(j)} - 12\xi\omega_k^2\sin\omega_k\tau_k^{(j)}.\\ P_1 &= 4\xi\omega_k^4\cos\omega_k\tau_k^{(j)} + [(1 - A_1A_2)\omega_k - \omega_k^5 - 4\xi^2\omega_k^3]\sin\omega_k\tau_k^{(j)}\\ P_2 &= [(1 - A_1A_2)\omega_k - \omega_k^5 - 4\xi^2\omega_k^3]\cos\omega_k\tau_k^{(j)} - 4\xi\omega_k^4\sin\omega_k\tau_k^{(j)},\end{aligned}$$

we derive

$$\begin{aligned}\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1}_{\tau=\tau_k^{(j)}} &= \operatorname{Re}\left\{\frac{(4\lambda^3 + 12\xi\lambda^2 + 8\xi^2\lambda)e^{\lambda\tau} + 4\lambda + 4\xi}{\lambda(1 - A_1A_2)e^{-\lambda\tau} - \lambda(\lambda^4 + 4\xi\lambda^3 + 4\xi^2\lambda^2)e^{\lambda\tau}}\right\}_{\tau=\tau_k^{(j)}}\\ &= \operatorname{Re}\left\{\frac{K_1 + K_2 i}{P_1 + P_2 i}\right\} = \frac{K_1P_1 - K_2P_2}{P_1^2 + P_2^2}.\end{aligned} \tag{2.20}$$

We assume that the following condition holds:

$$(H3) K_1P_1 \neq K_2P_2. \tag{2.21}$$

According to above analysis and the results of Kuang [29] and Hale [30], we have the following.

Theorem 2.3. *If* $(H1)$ *and* $(H2)$ *hold, then the equilibrium* $E(0,0,0,0)$ *of system* (1.2) *is asymptotically stable for* $\tau \in [0,\tau_0)$. *Under the conditions* $(H1)$ *and* $(H2)$, *if the condition* $(H3)$ *holds, then system* (1.2) *undergoes a Hopf bifurcation at the equilibrium* $E(0,0,0,0)$ *when* $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$.

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$. In this section, we shall obtain the explicit formulae for determining the direction, stability, and periods of these periodic solutions bifurcating from the equilibrium $E(0,0,0,0)$ at these critical value of τ, by using techniques from normal form and center manifold theory [31]. Throughout this section, we always assume that system (1.2) undergoes Hopf bifurcation at the equilibrium $E(0,0,0,0)$ for $\tau = \tau_k^{(j)}$, $k = 1,2,3,\ldots,16$; $j = 0,1,2,\ldots$, and then $\pm i\omega_k$ are corresponding purely imaginary roots of the characteristic equation at the equilibrium $E(0,0,0,0)$.

For convenience, let $\overline{x}_i(t) = x_i(\tau t)$ $(i = 1,2,3,4)$ and $\tau = \tau_k^{(j)} + \mu$, where $\tau_k^{(j)}$ is defined by (2.16) and $\mu \in R$, drop the bar for the simplification of notations, then system (2.2) can be written as an FDE in $C = C([-1,0]), R^4)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (x_1(t+\theta), x_2(t+\theta), x_3(t+\theta), x_4(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$L_\mu\phi = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & -2\xi \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{pmatrix} + \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & A_2 & 0 & 0 \\ A_1 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{pmatrix},$$

$$f(\mu, \phi) = \left(\tau_k^{(j)} + \mu\right)\begin{pmatrix} 0 \\ 0 \\ A_2\phi_2^3(-1) + \text{ h.o.t.} \\ A_1\phi_1^3(-1) + \text{ h.o.t.} \end{pmatrix}, \tag{3.2}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))^T \in C$.

From the discussion in Section 2, we know that if $\mu = 0$, then system (3.1) undergoes a Hopf bifurcation at the equilibrium $E(0,0,0,0)$ and the associated characteristic equation of system (3.1) has a pair of simple imaginary roots $\pm\omega_k\tau_k^{(j)}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta,\mu)$, $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{3.3}$$

In fact, we can choose

$$\eta(\theta,\mu) = \left(\tau_k^{(j)}+\mu\right)\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&-2\xi&0\\0&0&0&-2\xi\end{pmatrix}\delta(\theta) - \left(\tau_k^{(j)}+\mu\right)\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&A_2&0&0\\A_1&-1&0&0\end{pmatrix}\delta(\theta+1), \tag{3.4}$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0],R^4)$, define

$$A(\mu)\phi = \begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0,\\ \displaystyle\int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0,\end{cases}$$
$$R\phi = \begin{cases}0, & -1 \le \theta < 0,\\ f(\mu,\phi), & \theta = 0.\end{cases} \tag{3.5}$$

Then, (3.1) is equivalent to the following abstract differential equation:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.6}$$

where $u_t(\theta) = u(t+\theta)$, $\theta \in [-1,0]$. For $\psi \in C([0,1],(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases}-\dfrac{d\psi(s)}{ds}, & s \in (0,1],\\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0.\end{cases} \tag{3.7}$$

For $\phi \in C([-1,0],R^4)$ and $\psi \in C([0,1],(R^4)^*)$, define the following bilinear form:

$$\langle\psi,\phi\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.8}$$

where $\eta(\theta) = \eta(\theta, 0)$, and the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_k\tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_k\tau_k^{(j)}$ and $-i\omega_k\tau_k^{(j)}$, respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta, \gamma)^T e^{i\omega_k\tau_k^{(j)}\theta}, \qquad q^*(s) = D(1, \alpha^*, \beta^*, \gamma^*)e^{i\omega_k\tau_k^{(j)}s}, \tag{3.9}$$

where

$$\alpha = \frac{i\omega_k(i\omega_k + 2\xi) + e^{-i\omega_k\tau_k^{(j)}}}{A_2 e^{-i\omega_k\tau_k^{(j)}}}, \qquad \beta = -i\omega_k, \qquad \gamma = \frac{i\omega_k e^{-i\omega_k\tau_k^{(j)}} - \omega_k^2(i\omega_k + 2\xi)}{A_2 e^{-i\omega_k\tau_k^{(j)}}},$$
$$\alpha^* = \frac{i\omega_k(2\xi - i\omega_k) - e^{-i\omega_k\tau_k^{(j)}}}{A_1 e^{-i\omega_k\tau_k^{(j)}}}, \qquad \beta^* = \frac{1}{2\xi - i\omega_k}, \qquad \gamma^* = \frac{\omega_k^2(i\omega_k + 2\xi) - i\omega_k e^{-i\omega_k\tau_k^{(j)}}}{A_2 e^{-i\omega_k\tau_k^{(j)}}},$$
$$D = \frac{1}{1 + \overline{\alpha}\alpha^* + \overline{\beta}\beta^* + \overline{\gamma}\gamma^* + \tau_k^{(j)}\left[\overline{\gamma}(A_1 - \alpha^*) - \overline{\beta}(A_2\alpha^* + 1)\right]e^{i\omega_k\tau_k^{(j)}}}. \tag{3.10}$$

Furthermore, $\langle q^*(s), q(\theta)\rangle 1$ and $\langle q^*(s), \overline{q}(\theta)\rangle 0$.

Next, we use the same notations as those in Hassard et al. [31] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (3.1), when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t\rangle, \qquad W(t, \theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}, \tag{3.11}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \overline{z}(t), \theta), \tag{3.12}$$

where

$$W(z(t), \overline{z}(t), \theta) = W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{3.13}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3.1), we have

$$\dot{z}(t) = i\omega_k\tau_k^{(j)}z + \overline{q}^*(\theta)f\left(0, W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{zq(\theta)\}\right) \stackrel{\text{def}}{=} i\omega_k\tau_k^{(j)}z + \overline{q}^*(0)f_0. \tag{3.14}$$

That is,

$$\dot{z}(t) = i\omega_k \tau_k^{(j)} z + g(z, \overline{z}), \tag{3.15}$$

where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots . \tag{3.16}$$

Hence, we have

$$\begin{aligned} g(z, \overline{z}) &= \overline{q}^*(0) f_0(z, \overline{z}) = \overline{q}^*(0) f(0, u_t) \\ &= \tau_k^{(j)} \overline{D}\left(1, \overline{\alpha}^*, \overline{\beta}^*, \overline{\gamma}^*\right) \begin{pmatrix} 0 \\ 0 \\ A_2 x_{2t}^3(-1) + \text{h.o.t.} \\ A_1 x_{1t}^3(-1) + \text{h.o.t.} \end{pmatrix} \\ &= \overline{D}\tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \overline{\alpha} e^{-2i\omega_k \tau_k^{(j)}}\right] z^2 \overline{z} + \text{h.o.t}, \end{aligned} \tag{3.17}$$

and we obtain

$$g_{20} = g_{11} = g_{02} = 0, \tag{3.18}$$

$$g_{21} = 2\overline{D}\tau_k^{(j)} \left[3\beta^* e^{-i\omega_k \tau_k^{(j)}} + 3\gamma^* \alpha^2 \overline{\alpha} e^{-2i\omega_k \tau_k^{(j)}}\right]. \tag{3.19}$$

Thus, we derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_k \tau_k^{(j)}} \left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}, \\ \beta_2 &= 2\operatorname{Re}(c_1(0)), \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'\left(\tau_k^{(j)}\right)\right\}}{\omega_k \tau_k^{(j)}}, \end{aligned} \tag{3.20}$$

which determine the quantities of bifurcation periodic solutions of (3.1) on the center manifold at the critical value $\tau = \tau_k^{(j)}$, $(k = 1, 2, 3, \ldots, 16;\ j = 0, 1, 2, 3, \ldots)$. Summarizing the results obtained above leads to the following theorem.

Theorem 3.1. *The periodic solution is forward (backward) if $\mu_2 > 0$ ($\mu_2 < 0$). The bifurcating periodic solutions on the center manifold are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$). The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

4. Numerical Examples

In this section, we present some numerical results of system (1.2) to verify the analytical predictions obtained in the previous section. Let us consider the following system:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_3, \\
\frac{dx_2}{dt} &= x_4, \\
\frac{dx_3}{dt} &= -0.8x_3 - x_1(t-\tau) + 0.2\tanh(x_2(t-\tau)), \\
\frac{dx_4}{dt} &= -0.8x_4 - x_2(t-\tau) + 0.4\tanh(x_1(t-\tau)),
\end{aligned} \tag{4.1}$$

which has an equilibrium $E_0(0,0,0,0)$ and satisfies the conditions indicated in Theorem 2.3. The equilibrium $E_0(0,0,0,0)$ is asymptotically stable for $\tau = 0$. Using the software MATLAB (here take $j = 0$ for example), we derive $\omega_0 \approx 4.1208$, $\tau_0 \approx 0.6801$, $\lambda'(\tau_0) \approx 0.3307 - 3.1524i$, $g_{21} \approx -1.4203 - 4.5518i$. Thus by algorithm (3.20) derived in Section 3, we have $c_1(0) \approx -0.7102 - 2.1609i$, $\mu_2 \approx -2.1476$, $\beta_2 \approx -1.4202$, $T_2 \approx 3.1867$. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the equilibrium $E_0(0,0,0,0)$ is stable when $\tau < \tau_0 \approx 0.6801$. Figures 1(a)–1(j) show that the equilibrium $E_0(0,0,0,0)$ is asymptotically stable when $\tau = 0.65 < \tau_0 \approx 0.6801$. It is observed from Figures 1(a)–1(j) that the input voltage of the i ($i = 1,2$)th neuron and the output of the j ($j = 3,4$)th neuron converge to their steady states in finite time. If we gradually increase the value of τ and keep other parameters fixed, when τ passes through the critical value $\tau_0 \approx 0.6801$, the equilibrium $E_0(0,0,0,0)$ loses its stability and a Hopf bifurcation occurs, that is, the input voltage of the i ($i = 1,2$)th neuron the output of the j ($j = 3,4$)th neuron will keep an oscillary mode near the equilibrium $E_0(0,0,0,0)$. Due to $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0 \approx 0.6801$, and these bifurcating periodic solutions from $E_0(0,0,0,0)$ at $\tau_0 \approx 0.6801$ are stable. Figures 1(j)–2(d) suggest that Hopf bifurcation occurs from the equilibrium $E_0(0,0,0,0)$ when $\tau = 0.8 > \tau_0 \approx 0.6801$.

5. Conclusions

In this paper, we have studied the bifurcation natures of a delayed neural network model of two neurons with inertial coupling. Regarding delay as the bifurcation parameter and analyzing the characteristic equation of the linearized system of the original system at the equilibrium $E_0(0,0,0,0)$, we proposed the conditions to define the parameters for

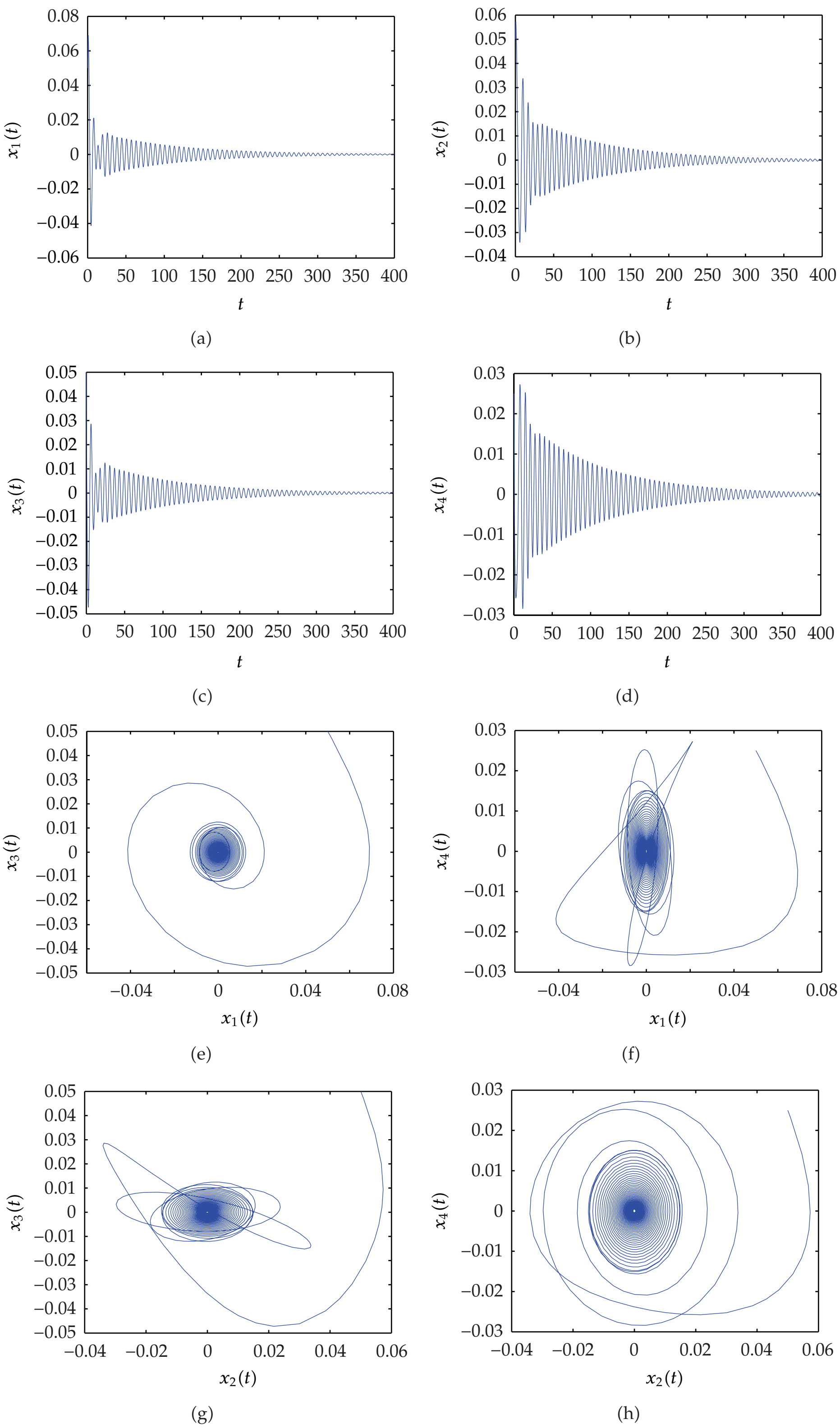

Figure 1: Continued.

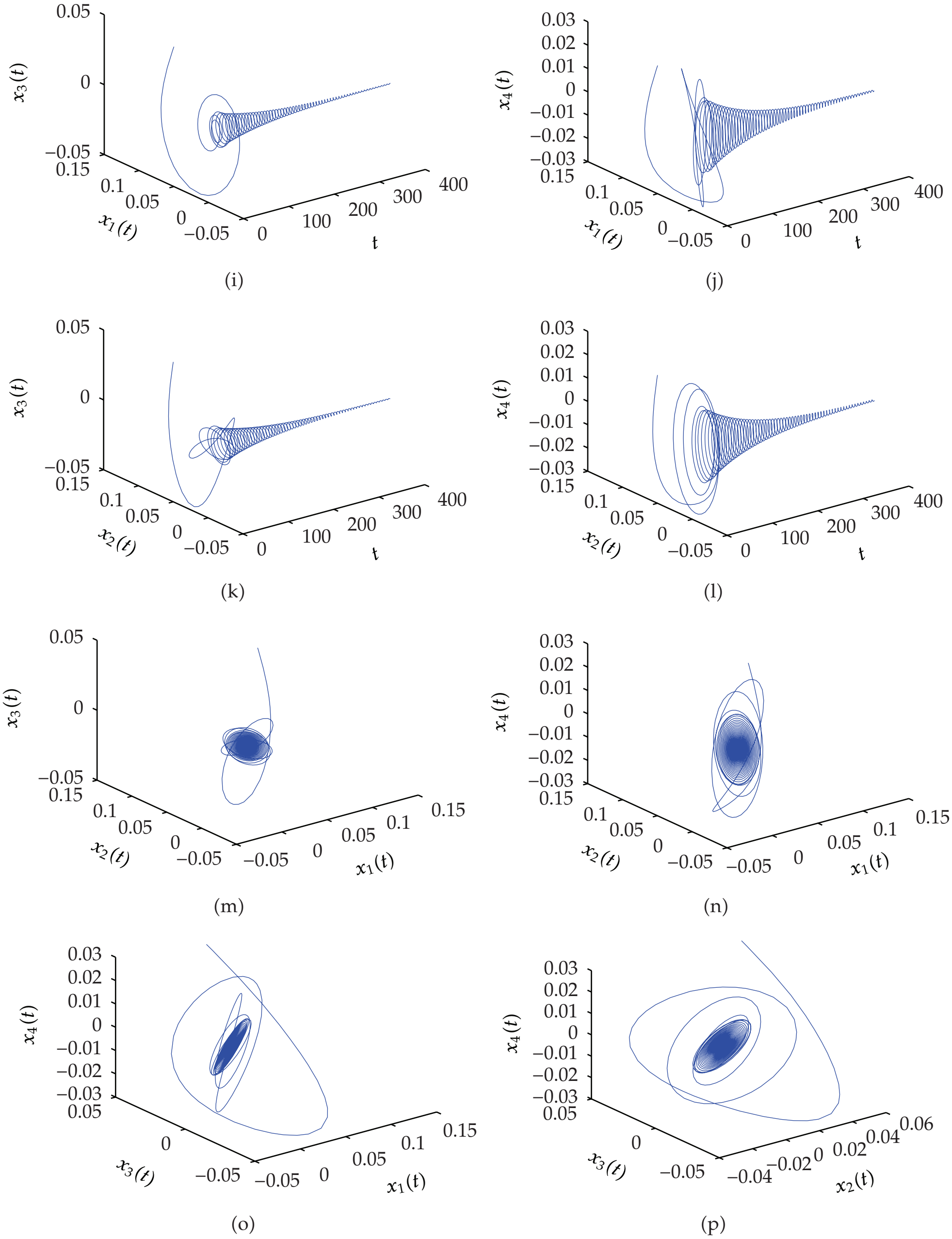

Figure 1: The time histories and phase trajectories of system (4.1) with $\tau = 0.65 < \tau_0 \approx 0.6801$ and the initial value (0.05, 0.05, 0.05, 0.025). The equilibrium $E_0(0, 0, 0, 0)$ is asymptotically stable.

the occurrence of Hopf bifurcation and the oscillatory solutions of the models equations. It is shown that if conditions $(H1)$ and $(H2)$ hold, the equilibrium $E_0(0,0,0,0)$ of system (1.2) is asymptotically stable for all $\tau \in [0,\tau_0)$. Under conditions $(H1)$ and $(H2)$, if condition $(H3)$ is satisfied, as the delay τ increases and crosses a threshold value $\tau_k^{(j)}$, the equilibrium loses its stability and the delayed network model of two

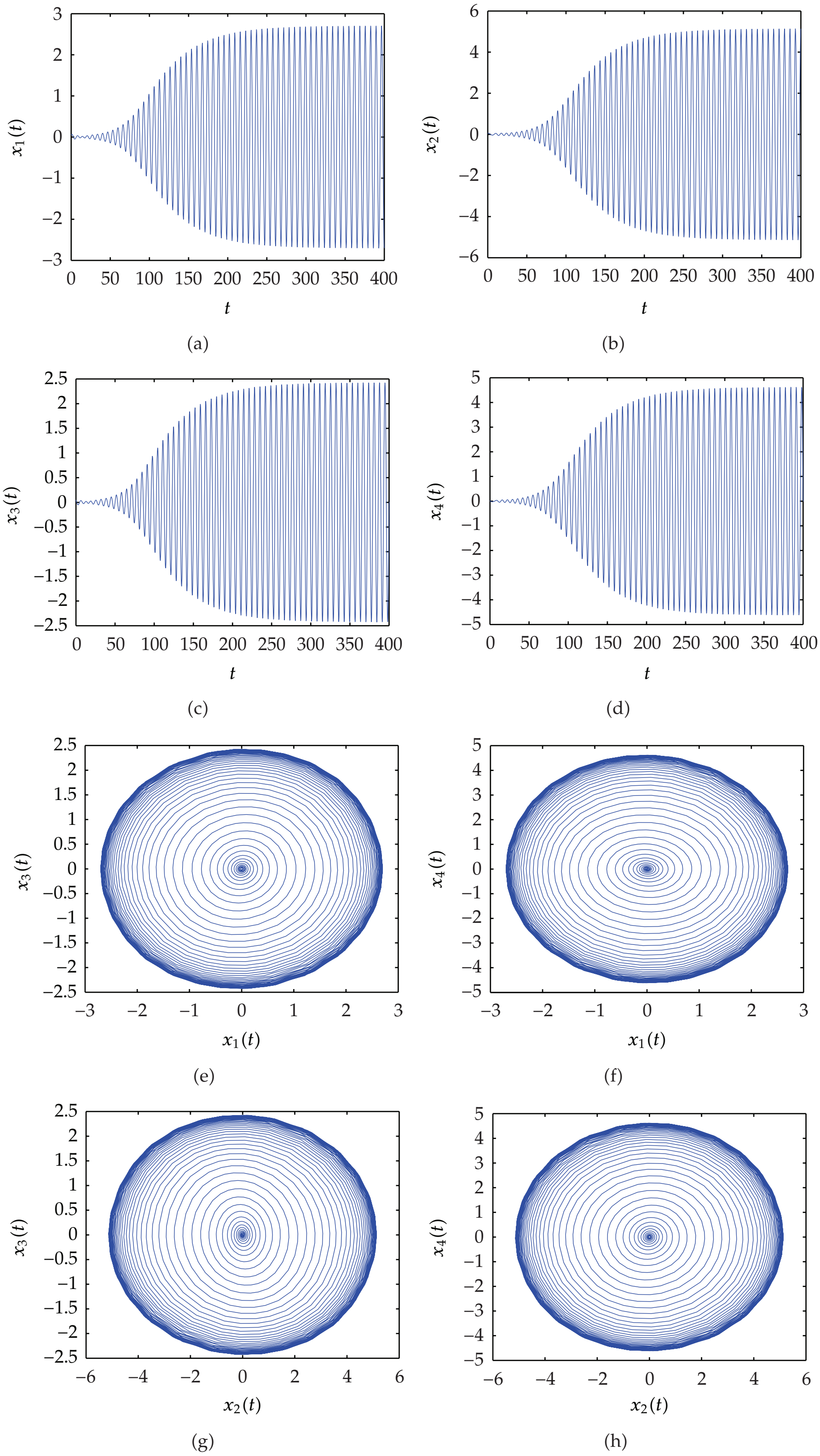

Figure 2: Continued.

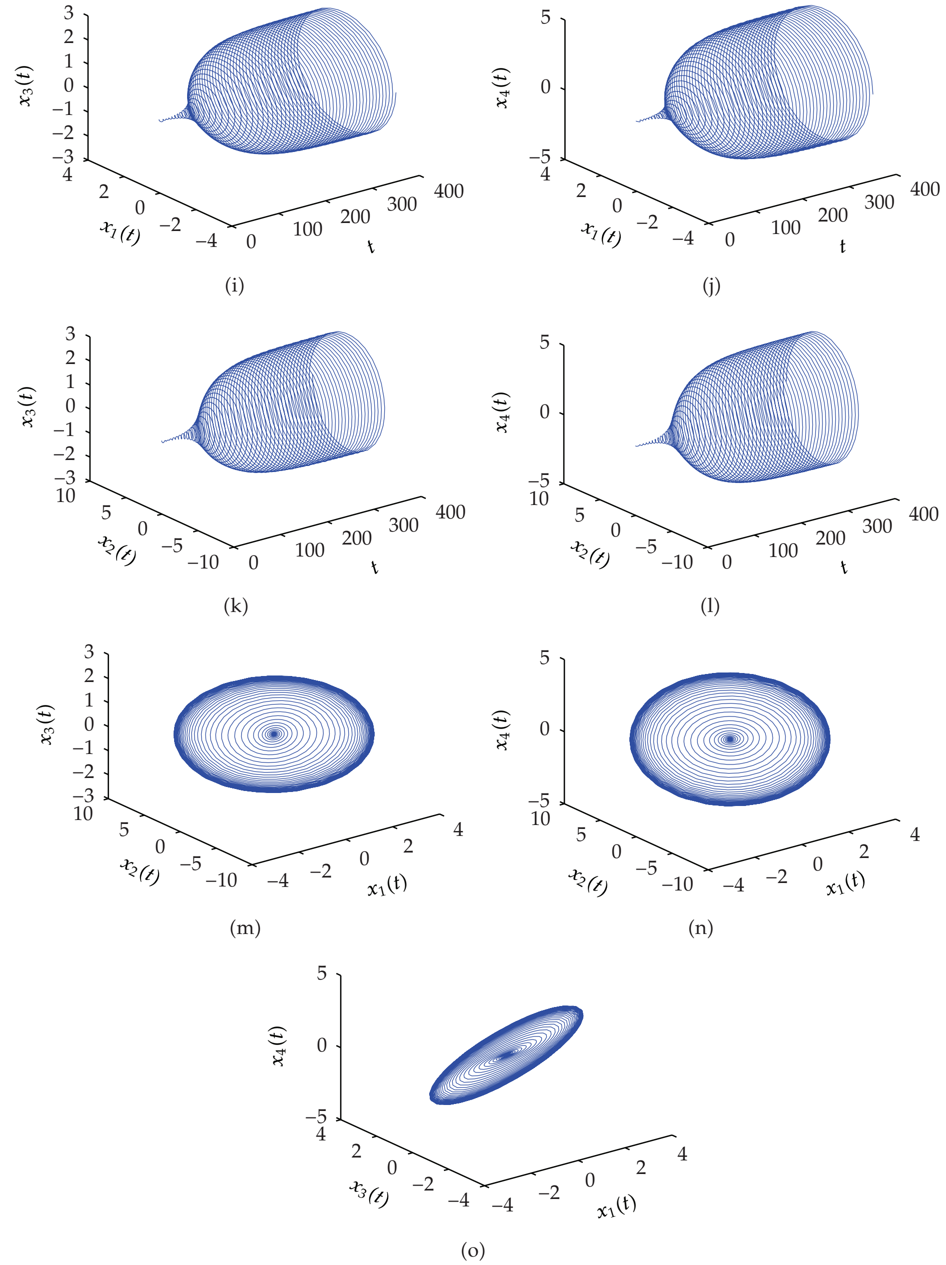

Figure 2: The time histories and phase trajectories of system (4.1) with $\tau = 0.8 > \tau_0 \approx 0.6801$ and the initial value $(0.05, 0.05, 0.05, 0.025)$. Hopf bifurcation occurs from the equilibrium $E_0(0, 0, 0, 0)$.

neurons with inertial coupling enters into a Hopf bifurcation. In addition, using the normal form method and center manifold theorem, explicit formulaes for determining the properties of periodic solutions are worked out. Simulations are included to verify the theoretical findings. The obtained findings are useful in applications of network control.

Acknowledgments

This work is supported by National Natural Science Foundation of China (no. 60902044), Soft Science and Technology Program of Guizhou Province (no. 2011LKC2030), Natural Science and Technology Foundation of Guizhou Province (J[2012]2100) and Doctoral Foundation of Guizhou University of Finance and Economics (2010), Governor Foundation of Guizhou Province (2012), and the Science and Technology Program of Hunan Province (no. 2010FJ6021).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 707319, 17 pages
doi:10.1155/2012/707319

Research Article

Periodic Solutions in Shifts $\delta_{\pm}$ for a Nonlinear Dynamic Equation on Time Scales

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Received 16 March 2012; Accepted 27 June 2012

Academic Editor: Elena Braverman

Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$. We use a fixed point theorem due to Krasnosel'skiĭ to show that nonlinear delay in dynamic equations of the form $x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x^{\Delta}(\delta_{-}(k,t))\delta_{-}^{\Delta}(k,t) + q(t, x(t), x(\delta_{-}(k,t)))$, $t \in \mathbb{T}$, has a periodic solution in shifts $\delta_{\pm}$. We extend and unify periodic differential, difference, h-difference, and q-difference equations and more by a new periodicity concept on time scales.

1. Introduction

The time scales approach unifies differential, difference, h-difference, and q-differences equations and more under dynamic equations on time scales. The theory of dynamic equations on time scales was introduced by Hilger in this Ph.D. thesis in 1988 [1]. The existence problem of periodic solutions is an important topic in qualitative analysis of ordinary differential equations. There are only a few results concerning periodic solutions of dynamic equations on time scales such as in [2, 3]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition

$$\text{“there exists a } \omega > 0 \text{ such that } t \pm \omega \in \mathbb{T} \quad \forall t \in \mathbb{T}.\text{”} \tag{1.1}$$

Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy condition (1.1). Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed

$\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_{\pm}$ which are first defined in [4] and then generalized in [5].

Let $\mathbb{T}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. We are concerned with the existence of periodic solutions in shifts $\delta_{\pm}$ for the nonlinear dynamic equation with a delay function $\delta_{-}(k,t)$:

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x^{\Delta}(\delta_{-}(k,t))\delta_{-}^{\Delta}(k,t) + q(t, x(t), x(\delta_{-}(k,t))), \quad t \in \mathbb{T}, \tag{1.2}$$

where k is fixed if $\mathbb{T} = \mathbb{R}$ and $k \in [P, \infty)_{\mathbb{T}}$ if $\mathbb{T}$ is periodic in shifts $\delta_{\pm}$ with period P.

Kaufmann and Raffoul in [2] used Krasnosel'skiĭ fixed point theorem and showed the existence of a periodic solution of (1.2) and used the contraction mapping principle to show that the periodic solution is unique when $\mathbb{T}$ satisfies condition (1.1). Similar results were obtained concerning (1.2) in [6, 7] in the case $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, respectively. Currently, Adıvar and Raffoul used Lyapunov's direct method to obtain inequalities that lead to stability and instability of delay dynamic equations of (1.2) when $q = 0$ on a time scale having a delay function δ_{-} in [8] and also using the topological degree method and Schaefers fixed point theorem, they deduce the existence of periodic solutions of nonlinear system of integrodynamic equations on periodic time scales in [9].

Hereafter, we use the notation $[a,b]_{\mathbb{T}}$ to indicate the time scale interval $[a,b] \cap \mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and $(a,b)_{\mathbb{T}}$ are similarly defined.

In Section 2, we will state some facts about the exponential function on time scales, the new periodicity concept for time scales, and some important theorems which will be needed to show the existence of a periodic solution in shifts $\delta_{\pm}$. In Section 3, we will give some lemmas about the exponential function and the graininess function with shift operators. Finally, we present our main result in Section 4 by using Krasnosel'skiĭ fixed point theorem and give an example.

2. Preliminaries

In this section, we mention some definitions, lemmas, and theorems from calculus on time scales which can be found in [10, 11]. Next, we state some definitions, lemmas, and theorems about the shift operators and the new periodicity concept for time scales which can be found in [12].

Definition 2.1 (see [10]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^{+}$ is given by $\mathcal{R}^{+} = \{\varphi \in \mathcal{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $\varphi \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_{\varphi}(t,s) = \exp\left(\int_{s}^{t} \zeta_{\mu(r)}(\varphi(r))\Delta r\right), \tag{2.1}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(\varphi(r)) := \begin{cases} \dfrac{1}{\mu(r)}\operatorname{Log}(1+\mu(r)\varphi(r)), & \text{if } \mu(r) > 0, \\ \varphi(r), & \text{if } \mu(r) = 0. \end{cases} \tag{2.2}$$

Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma [10, Theorem 2.36].

Lemma 2.2 (see [10]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $1/e_p(t,s) = e_{\ominus}(t,s)$, *where,* $\ominus p(t) = -(p(t))/(1+\mu(t)p(t))$;

(iv) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$;

(viii) $(1/e_p(\cdot,s))^{\Delta} = -p(t)/e_p^{\sigma}(\cdot,s)$.

The following definitions, lemmas, corollaries, and examples are about the shift operators and new periodicity concept for time scales which can be found in [12].

Definition 2.3 (see [12]). Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_{\pm} : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties.

(P.1) The function $\delta_{\pm}$ are strictly increasing with respect to their second arguments, that is, if

$$(T_0,t),(T_0,u) \in \mathfrak{D}_{\pm} := \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\mp}(s,t) \in \mathbb{T}^*\}, \tag{2.3}$$

then

$$T_0 \le t < u \quad \text{implies} \quad \delta_{\pm}(T_0,t) < \delta_{\pm}(T_0,u). \tag{2.4}$$

(P.2) If $(T_1,u),(T_2,u) \in \mathfrak{D}$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in D_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$.

(P.3) If $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$ holds.

(P.4) If $(s,t) \in D_{\pm}$, then $(s,\delta_{\pm}(s,t)) \in D_{\mp}$ and $\delta_{\mp}(s,\delta_{\pm}(s,t)) = t$, respectively.

(P.5) If $(s,t) \in D_{\pm}$ and $(u,\delta_{\pm}(s,t)) \in D_{\pm}$, then $(s,\delta_{\mp}(u,t)) \in D_{\pm}$ and $\delta_{\mp}(u,\delta_{\pm}(s,t)) = \delta_{\pm}(s,\delta_{\mp}(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0,\infty)_{\mathbb{T}}$ in $\delta_\pm(s,t)$ is called the shift size. The values $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathfrak{D}_\pm$ are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Example 2.4 (see [12]).

(i) $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{R}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.

(ii) $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{Z}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.

(iii) $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $t_0 = 1$, $\mathbb{T}^* = q^{\mathbb{Z}}$, $\delta_-(s,t) = t/s$ and $\delta_+(s,t) = ts$.

(iv) $\mathbb{T} = \mathbb{N}^{1/2}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{N}^{1/2}$, $\delta_-(s,t) = \sqrt{t^2 - s^2}$ and $\delta_+(s,t) = \sqrt{t^2 + s^2}$.

Definition 2.5 (periodicity in shifts [12]). Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shift $\delta_\pm$ if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\left\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in D_\pm, \quad \forall t \in \mathbb{T}^*\right\} \neq t_0, \tag{2.5}$$

then P is called the period of the time scale $\mathbb{T}$.

Example 2.6 (see [12]). The following time scales are not periodic in the sense of condition (1.1) but periodic with respect to the notion of shift operators given in Definition 2.5:

(i) $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}, \delta_\pm(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2, & t>0; \\ \pm P, & t=0; \\ -(\sqrt{-t} \pm \sqrt{P})^2, & t<0; \end{cases}$, $P = 1, t_0 = 0$,

(ii) $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q, t_0 = 1$,

(iii) $\mathbb{T}_3 = \overline{\cup_{n\in\mathbb{Z}}[2^{2n}, 2^{2n+1}]}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$,

(iv) $\mathbb{T}_4 = \{q^n/(1+q^n) : q > 1 \text{ is constant and } n \in \mathbb{Z}\} \cup \{0,1\}$,

$$\delta_\pm(P,t) = \frac{q^{((\ln(t/(1-t)) \pm \ln(P/(1-P)))/\ln q)}}{1 + q^{((\ln(t/(1-t)) \pm \ln(P/(1-P)))/\ln q)}}, \quad P = \frac{q}{1-q}. \tag{2.6}$$

Notice that the time scale $\mathbb{T}_4$ in Example 2.6 is bounded above and below and $\mathbb{T}_4^* = \{q^n/(1+q^n) : q > 1$ is constant and $n \in \mathbb{Z}\}$.

Remark 2.7 (see [12]). Let $\mathbb{T}$ be a time scale, that is, periodic in shifts with the period P. Thus, by $(P.4)$ of Definition 2.3 the mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, by $(P.1)$ of Definition 2.3 shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) = \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $(\delta_+^P)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.

We assume that $\mathbb{T}$ is a periodic time scale in shift $\delta_\pm$ with period P. The operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ given by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. That is, $(\delta_\pm^P \circ \sigma)(t) = (\sigma \circ \delta_\pm^P)(t)$ for all $t \in \mathbb{T}^*$.

Lemma 2.8 (see [12]). *The mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ preserves the structure of the points in $\mathbb{T}^*$. That is,*

$$\sigma(t) = t \quad \text{implies } \sigma(\delta_+(P,t)) = \delta_+(P,t) \text{ and } \sigma(t) > t \quad \text{implies } \sigma(\delta_+(P,t)) > \delta_+(P,t). \tag{2.7}$$

Corollary 2.9 (see [12]). *$\delta_+(P,\sigma(t)) = \sigma(\delta_+(P,t))$ and $\delta_-(P,\sigma(t)) = \sigma(\delta_-(P,t))$ for all $t \in \mathbb{T}^*$.*

Definition 2.10 (periodic function in shift $\delta_\pm$ [12]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in D_\pm, \qquad f\left(\delta_\pm^T(t)\right) = f(t) \quad \forall t \in \mathbb{T}^*, \tag{2.8}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (5) holds is called the period of f.

Definition 2.11 (Δ-periodic function in shifts $\delta_\pm$ [12]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that

$$\begin{gathered} (T,t) \in D_\pm \quad \forall t \in \mathbb{T}^*, \\ \text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with } rd\text{-continuous derivatives,} \\ f\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T} = f(t) \quad \forall t \in \mathbb{T}^*, \end{gathered} \tag{2.9}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (2.9) hold is called the period of f.

Notice that Definitions 2.10 and 2.11 give the classic periodicity definition on time scales whenever $\delta_\pm^T := t \pm T$ are the shifts satisfying the assumptions of Definitions 2.10 and 2.11.

Now, we give two theorems concerning the composition of two functions. The first theorem is the chain rule on time scales [10, Theorem 1.93].

Theorem 2.12 (chain rule [10]). *Assume that $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\widetilde{\Delta}}$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \upsilon)^\Delta = \left(w^{\widetilde{\Delta}} \circ \nu\right)\nu^\Delta. \tag{2.10}$$

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$. If one takes $\nu(t) = \delta_\pm(T,t)$, then one has $\nu(\mathbb{T}) = \mathbb{T}$ and $[f(\nu(t))]^\Delta = (f^\Delta \circ \nu(t))\nu^\Delta(t)$.

The second theorem is the substitution rule on periodic time scales in shifts $\delta_\pm$ which can be found in [12].

Theorem 2.13 (see [12]). *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_{\pm}$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} f(s)\Delta s. \tag{2.11}$$

This work is mainly based on the following theorem [13].

Theorem 2.14 (Krasnosel'skiĭ). *Let M be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that A and B map M into $\mathbb{B}$ such that*

(i) *$x, y \in M$ imply $Ax + By \in M$,*

(ii) *A is completely continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in M$ with $z = Az + Bz$.

3. Some Lemmas

In this section, we show some interesting properties of the exponential functions $e_p(t, t_0)$ and shift operators on time scales.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P and the shift $\delta_{\pm}^{T}$ is Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Then the graininess function $\mu : \mathbb{T} \to [0, \infty)$ satisfies*

$$\mu\left(\delta_{\pm}^{T}(t)\right) = \delta_{\pm}^{\Delta T}(t)\mu(t). \tag{3.1}$$

Proof. Since $\delta_{\pm}^{T}$ is Δ-differentiable at t we can use Theorem 1.16 (iv) in [10]. Then we have

$$\mu(t)\delta_{\pm}^{\Delta T}(t) = \delta_{\pm}^{T}(\sigma(t)) - \delta_{\pm}^{T}(t). \tag{3.2}$$

Then by using Corollary 2.9 we have

$$\begin{aligned} \mu(t)\delta_{\pm}^{\Delta T}(t) &= \sigma\left(\delta_{\pm}^{T}(t)\right) - \delta_{\pm}^{T}(t) \\ &= \mu\left(\delta_{\pm}^{T}(t)\right). \end{aligned} \tag{3.3}$$

Thus, the proof is complete. □

Lemma 3.2. *Let $\mathbb{T}$ be a time scale, that is, periodic in shifts $\delta_{\pm}$ with the period P and the shift $\delta_{\pm}^{T}$ is Δ-differentiable on $t \in \mathbb{T}^*$, where $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_{\pm}$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Then,*

$$e_p\left(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(t_0)\right) = e_p(t, t_0) \quad for\ t, t_0 \in \mathbb{T}^*. \tag{3.4}$$

Proof. Assume that $\mu(\tau) \neq 0$. Set $f(\tau) = (1/\mu(\tau))\mathrm{Log}(1 + p(\tau)\mu(\tau))$. Using Lemma 3.1 and Δ-periodicity of p in shifts $\delta_\pm$ we get

$$\begin{aligned}
f\left(\delta_\pm^T(\tau)\right)\delta_\pm^{\Delta T}(\tau) &= \frac{\delta_\pm^{\Delta T}(\tau)}{\mu(\delta_\pm^T(\tau))}\mathrm{Log}\left(1 + p\left(\delta_\pm^T(\tau)\right)\mu\left(\delta_\pm^T(\tau)\right)\right) \\
&= \frac{\delta_\pm^{\Delta T}(\tau)}{\mu(\delta_\pm^T(\tau))}\mathrm{Log}\left(1 + p\left(\delta_\pm^T(\tau)\right)\delta_\pm^{\Delta T}\frac{1}{\delta_\pm^{\Delta T}}\mu\left(\delta_\pm^T(\tau)\right)\right) \\
&= \frac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau)) \\
&= f(\tau).
\end{aligned} \tag{3.5}$$

Thus, f is Δ−periodic in shifts $\delta_\pm$ with the period T. By using Theorem 2.13 we have

$$\begin{aligned}
e_p\left(\delta_\pm^T(t), \delta_\pm^T(t_0)\right) &= \begin{cases} \exp\left(\int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} \frac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{for } \mu(\tau) \neq 0, \\ \exp\left(\int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} p(\tau)\Delta\tau\right), & \text{for } \mu(\tau) = 0, \end{cases} \\
&= \begin{cases} \exp\left(\int_{t_0}^{t} \frac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{for } \mu(\tau) \neq 0, \\ \exp\left(\int_{t_0}^{t} p(\tau)\Delta\tau\right), & \text{for } \mu(\tau) = 0, \end{cases} \\
&= e_p(t, t_0).
\end{aligned} \tag{3.6}$$

The proof is complete. □

Lemma 3.3. *Let $\mathbb{T}$ be a time scale, that is, periodic in shifts $\delta_\pm$ with the period P and the shift $\delta_\pm^T$ is Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Then*

$$e_p\left(\delta_\pm^T(t), \sigma\left(\delta_\pm^T(s)\right)\right) = e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(t)p(t)} \quad \text{for } t, s \in \mathbb{T}^*. \tag{3.7}$$

Proof. From Corollary 2.9, we know $\sigma(\delta_\pm^T(s)) = \delta_\pm^T(\sigma(s))$. By Lemmas 3.2 and 2.2 we obtain

$$e_p\left(\delta_\pm^T(t), \sigma\left(\delta_\pm^T(s)\right)\right) = e_p\left(\delta_\pm^T(t), \delta_\pm^T(\sigma(s))\right) = e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(t)p(t)}. \tag{3.8}$$

The proof is complete. □

4. Main Result

We will state and prove our main result in this section. We define

$$P_T = \left\{ x \in C(\mathbb{T}, \mathbb{R}) : x\left(\delta_+^T(t)\right) = x(t) \right\}, \tag{4.1}$$

where $C(\mathbb{T}, \mathbb{R})$ is the space of all real valued continuous functions. Endowed with the norm

$$\|x\| = \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} |x(t)|, \tag{4.2}$$

P_T is a Banach space.

Lemma 4.1. *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

Proof. Since $x \in P_T$, then $x(\delta_+^T(t_0)) = x(t_0)$, and by Corollary 2.9, we have $x(\sigma(\delta_+^T(t_0))) = x(\sigma(t_0))$. For all $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}, |x(\sigma(t))| \le \|x\|$. Hence $\|x^\sigma\| \le \|x\|$. Since $x \in C(\mathbb{T}, \mathbb{R})$, there exists $t_1 \in [t_0, \delta_+^T(t_0)]$ such that $\|x\| = |x(t_1)|$. If t_1 is left scattered, then $\sigma(\rho(t_1)) = t_1$. And so, $\|x^\sigma\| \ge |x^\sigma(\rho(t_1))| = x(t_1) = \|x\|$. Thus, we have $\|x^\sigma\| = \|x\|$. If t_1 is dense, $\sigma(t_1) = t_1$ and $\|x^\sigma\| = \|x\|$.

Assume that t_1 is left dense and right scattered. Note that if $t_1 = t_0$ then we work $t_1 = \delta_+^T(t_0)$. Fix $\epsilon > 0$ and consider a sequence $\{a_n\}$ such that $a_n \uparrow t_1$. Note that $\sigma(a_n) \le t_1$ for all n. By the continuity of x, there exists N such that for all $n > N$, $|x(t_1) - x^\sigma(a_n)| < \epsilon$. This implies that $\|x\| - \epsilon \le \|x^\sigma\|$. Since $\epsilon > 0$ was arbitrary, then $\|x\| = \|x^\sigma\|$ and the proof is complete. □

In this paper we assume that $a(t) \in \mathcal{R}^+$ is a continuous function with $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a\left(\delta_+^T(t)\right)\delta_+^{\Delta T}(t) = a(t), \quad b\left(\delta_+^T(t)\right) = b(t), \tag{4.3}$$

where $b^\Delta(t)$ is continuous. We further assume that $q(t, x, y)$ is continuous and periodic with $\delta_\pm$ in t and Lipschitz continuous in x and y. That is,

$$q\left(\delta_+^T(t), x, y\right)\delta_+^{\Delta T}(t) = q(t, x, y), \tag{4.4}$$

and there are some positive constants L and E such that

$$\left|q(t, x, y) - q(t, z, w)\right| \le L\|x - z\| + E\left\|y - w\right\|. \tag{4.5}$$

Lemma 4.2. *Suppose that* (4.3)–(4.5) *hold. If* $x(t) \in P_T$*, then* $x(t)$ *is a solution of* (1.2) *if and only if*

$$x(t) = b(t)x(\delta_-(k,t)) + \frac{1}{1 - e_{\ominus a(t)}(t, \delta_-^T(t))} \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s, \tag{4.6}$$

where

$$r(s) = a(s)b^\sigma(s) + b^\Delta(s). \tag{4.7}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.2). We can rewrite (1.2) as

$$x^\Delta(t) + a(t)x^\sigma(t) = b(t)x^\Delta(\delta_-(k,t))\delta_-^\Delta(k,t) + q(t, x(t), x(\delta_-(k,t))). \tag{4.8}$$

Multiply both sides of the above equation by $e_{a(t)}(t, t_0)$ and then integrate from $\delta_-^T(t)$ to t to obtain

$$\int_{\delta_-^T(t)}^{t} \left[x(s)e_{a(s)}(s,t_0)\right]^\Delta \Delta s = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t_0)\Delta s. \tag{4.9}$$

We arrive at

$$x(t)\left[e_{a(t)}(t,t_0) - e_{a(t)}\left(\delta_-^T(t), t_0\right)\right] = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t_0)\Delta s. \tag{4.10}$$

Dividing both sides of the above equation by $e_{a(t)}(t, t_0)$ and using $x(\delta_+^T(t)) = x(t)$ and Lemma 2.2, we have

$$x(t)\left(1 - e_{a(t)}\left(\delta_-^T(t), t\right)\right) = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t)\Delta s. \tag{4.11}$$

Now, we consider the first term of the integral on the right-hand side of (4.11)

$$\int_{\delta_-^T(t)}^{t} b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s)e_{a(s)}(s,t)\Delta s. \tag{4.12}$$

Using integration by parts from rule [10] we obtain

$$\begin{aligned}&\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s\\&\quad=\int_{\delta_-^T(t)}^{t}\left[b(s)e_{a(s)}(s,t)x(\delta_-(k,s))\right]^{\Delta}\Delta s-\int_{\delta_-^T(t)}^{t}\left[b(s)e_{a(s)}(s,t)\right]_s^{\Delta}x^{\sigma}(\delta_-(k,s))\Delta s\end{aligned}\tag{4.13}$$

$$\begin{aligned}&\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s\\&\quad=b(t)e_{a(t)}(t,t)x(\delta_-(k,t))-b\left(\delta_-^T(t)\right)e_{a(s)}\left(\delta_-^T(t),t\right)x\left(\delta_-\left(k,\delta_-^T(t)\right)\right)\\&\qquad-\int_{\delta_-^T(t)}^{t}\left[b^{\sigma}(s)a(s)e_{a(s)}(s,t)+b^{\Delta}(s)e_{a(s)}(s,t)\right]x^{\sigma}(\delta_-(k,s))\Delta s.\end{aligned}\tag{4.14}$$

Since $b(\delta_-^T(t))=b(t)$ and $x(\delta_-^T(t))=x(t)$, the above equality reduces to

$$\begin{aligned}&\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s\\&\quad=b(t)x(\delta_-(k,t))\left(1-e_{a(s)}\left(\delta_-^T(t),t\right)\right)\\&\qquad-\int_{\delta_-^T(t)}^{t}\left[a(s)b^{\sigma}(s)+b^{\Delta}(s)\right]x^{\sigma}(\delta_-(k,s))e_{a(s)}(s,t)\Delta s.\end{aligned}\tag{4.15}$$

Substituting (4.15) into (4.11) we get

$$\begin{aligned}x(t)&=b(t)x(\delta_-(k,t))\\&\quad+\frac{1}{1-e_{\ominus a(t)}(t,\delta_-^T(t))}\\&\quad\times\int_{\delta_-^T(t)}^{t}\left[-r(s)x^{\sigma}(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s.\end{aligned}\tag{4.16}$$

Thus the proof is complete. □

Define the mapping $H:P_T\to P_T$ by

$$\begin{aligned}Hx(t)&:=b(t)x(\delta_-(k,t))\\&\quad+\frac{1}{\left(1-e_{\ominus a(t)}(t,\delta_-^T(t))\right)}\\&\quad\times\int_{\delta_-^T(t)}^{t}\left[-r(s)x^{\sigma}(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s.\end{aligned}\tag{4.17}$$

To apply Theorem 2.14 we need to construct two mappings: one map is a contraction and the other map is compact and continuous. We express (4.17) as

$$Hx(t) = Bx(t) + Ax(t), \tag{4.18}$$

where A, B are given

$$Bx(t) = b(t)x(\delta_-(k,t)), \tag{4.19}$$

$$\begin{aligned} Ax(t) = {} & \frac{1}{1 - e_{\ominus a(t)}\left(t, \delta_-^T(t)\right)} \\ & \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s, \end{aligned} \tag{4.20}$$

and $r(s)$ is defined in (4.7).

Lemma 4.3. *Suppose that* (4.3)–(4.5) *hold. Then* $A : P_T \to P_T$*, as defined by* (4.20)*, is compact and continuous.*

Proof. We show that $A : P_T \to P_T$. Evaluate (4.20) at $\delta_+^T(t)$,

$$\begin{aligned} Ax\left(\delta_+^T(t)\right) = {} & \frac{1}{1 - e_{\ominus a(t)}\left(\delta_+^T(t), \delta_-^T\left(\delta_+^T(t)\right)\right)} \\ & \times \int_{\delta_-^T\left(\delta_+^T(t)\right)}^{\delta_+^T(t)} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}\left(\delta_+^T(t), s\right)\Delta s \\ = {} & \frac{1}{1 - e_{p(t)}\left(\delta_+^T(t), t\right)} \\ & \times \int_{\delta_+^T\left(\delta_-^T(t)\right)}^{\delta_+^T(t)} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}\left(\delta_+^T(t), s\right)\Delta s. \end{aligned} \tag{4.21}$$

Now, since (4.3) and Corollary 2.9 hold, then we have

$$\begin{aligned} r\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) &= a\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) b^\sigma\left(\delta_+^T(s)\right) + b^\Delta\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) \\ &= a(s)b^\sigma(s) + b^\Delta(s) = r(s). \end{aligned} \tag{4.22}$$

That is, $r(s)$ is Δ-periodic in $\delta_\pm$ with period T. Using the periodicity of $r,\ x,\ q$, and Lemma 3.2 we get

$$\begin{aligned}
&\left[-r\left(\delta_+^T(s)\right)x^\sigma\left(\delta_-\left(k,\delta_+^T(s)\right)\right)+q\left(\delta_+^T(s),x\left(\delta_+^T(s)\right),x\left(\delta_-\left(k,\delta_+^T(s)\right)\right)\right)\right]\delta_+^{T\Delta}(s)\\
&\qquad\times e_{\ominus a(s)}\left(\delta_+^T(t),\delta_+^T(s)\right)\\
&\quad=\left[-r(s)x^\sigma(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s).
\end{aligned}\tag{4.23}$$

That is, inside the integral of (4.21) is Δ-periodic in $\delta_\pm$ with period T. By Theorem 2.13 and Lemma 3.2 we have

$$\begin{aligned}
Ax\left(\delta_+^T(t)\right)&=\frac{1}{1-e_{\ominus a(t)}\left(t,\delta_-^T(t)\right)}\\
&\qquad\times\int_{\delta_-^T(t)}^{t}\left[-r(s)x^\sigma(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s\\
&=Ax(t).
\end{aligned}\tag{4.24}$$

That is, $A:P_T\to P_T$.

To see that A is continuous, we let $\varphi,\ \psi\in P_T$ with $\|\varphi\|\le C$ and $\|\psi\|\le C$ and define

$$\begin{gathered}
\eta:=\max_{t\in\left[t_0,\delta_+^T(t_0)\right]_{\mathbb{T}}}\left|\left(1-e_{\ominus a(t)}\left(t,\delta_-^T(t)\right)\right)^{-1}\right|,\qquad \gamma:=\max_{u\in\left[\delta_-^T(t),t\right]_{\mathbb{T}}}e_{\ominus a(t)}(t,u),\\
\beta:=\max_{t\in\left[t_0,\delta_+^T(t_0)\right]_{\mathbb{T}}}|r(t)|.
\end{gathered}\tag{4.25}$$

Given that $\epsilon>0$, take $\delta=\epsilon/M$ such that $\|\varphi-\psi\|<\delta$. By making use of the Lipschitz inequality (4.5) in (4.20), we get

$$\begin{aligned}
\|A\varphi-A\psi\|&\le\gamma\eta\int_{\delta_-^T(t)}^{t}\beta\|\varphi-\psi\|+L\|\varphi-\psi\|+E\|\varphi-\psi\|\Delta s\\
&=\eta\gamma[\beta+L+E]\left(t_0-\delta_-^T(t_0)\right)\|\varphi-\psi\|\\
&\le M\|\varphi-\psi\|<\epsilon,
\end{aligned}\tag{4.26}$$

where L, E are given by (4.5) and $M=\eta\gamma[\beta+L+E](t_0-\delta_-^T(t_0))$. This proves that A is continuous.

We need to show that A is compact. Consider the sequence of periodic functions in $\delta_{\pm}\{\varphi_n\} \subset P_T$ and assume that the sequence is uniformly bounded. Let $R > 0$ be such that $\|\varphi_n\| \le R$, for all $n \in \mathbb{N}$. In view of (4.5) we arrive at

$$\begin{aligned} |q(t,x,y)| &= |q(t,x,y) - q(t,0,0) + q(t,0,0)| \\ &\le |q(t,x,y) - q(t,0,0)| + |q(t,0,0)| \\ &\le L\|x\| + E\|y\| + \alpha, \end{aligned} \tag{4.27}$$

where $\alpha := \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|q(t,0,0)|$. Hence,

$$\begin{aligned} |A\varphi_n| &= \left| \frac{1}{1-e_{\ominus a(t)}(t,\delta_-^T(t))} \int_{\delta_-^T(t)}^{t} \left[-r(s)\varphi_n^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s \right| \\ &\le \eta\gamma\left[(\beta+L+E)\|\varphi_n\| + \alpha\right]\left(t_0 - \delta_-^T(t_0)\right) \\ &\le \eta\gamma\left[(\beta+L+E)R + \alpha\right]\left(t_0 - \delta_-^T(t_0)\right) := D. \end{aligned} \tag{4.28}$$

Thus, the sequence $\{A\varphi_n\}$ is uniformly bounded. If we find the derivative of $A\varphi_n$, we have

$$\begin{aligned} (A\varphi_n)^{\Delta}(t) = {}& a(t)A\varphi_n(t) \\ &+ \frac{-a(t)+\ominus a(t)}{1-e_{\ominus a(t)}(\sigma(t),\delta_-^T(\sigma(t)))} \\ &\times \int_{\delta_-^T(t)}^{t} \left[-r(s)\varphi_n^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s \\ &+ \frac{1}{1+\mu(t)a(t)}\left[-r(t)\varphi^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))\right]. \end{aligned} \tag{4.29}$$

Consequently,

$$\left|(A\varphi_n)^{\Delta}(t)\right| \le D\|a\| + \left[(\beta+E+L)R + \alpha\right]\left[2\|a\|\gamma\eta\left(t_0 - \delta_-^T(t_0)\right)\right] := F. \tag{4.30}$$

for all n. That is, $|(A\varphi_n)^{\Delta}(t)| \le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equicontinuous. The Arzela-Ascoli theorem implies that $\{A\varphi_{n_k}\}$ uniformly converges to a continuous T-periodic function φ^* in $\delta_{\pm}$. Thus A is compact. □

Lemma 4.4. *Let B be defined by* (4.19) *and*

$$\|b(t)\| \le \xi < 1. \tag{4.31}$$

Then $B : P_T \to P_T$ is a contraction.

Proof. Trivially, $B : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}
\|B\varphi - B\psi\| &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \left|B\varphi(t) - B\psi(t)\right| \\
&= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \left\{|b(t)|\left|\varphi(\delta_-(k,s)) - \psi(\delta_-(k,s))\right|\right\} \\
&\le \xi\|\varphi - \psi\|.
\end{aligned} \tag{4.32}$$

Hence B defines a contraction mapping with contraction constant ξ. □

Theorem 4.5. *Let $\alpha := max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|q(t,0,0)|$. Let β, η, and γ be given by* (4.39)*. Suppose that* (4.3)–(4.5) *and* (4.31) *hold and that there is a positive constant G such that all solutions $x(t)$ of* (1.2)*, $x \in P_T$, satisfy $|x(t)| \le G$, the inequality*

$$\left\{\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right\}G + \gamma\eta\alpha\left(t_0 - \delta_-^T(t_0)\right) \le G \tag{4.33}$$

holds. Then (1.2) *has a T-periodic solution in $\delta_\pm$.*

Proof. Define $M := \{x \in P_T : \|x\| \le G\}$. Then Lemma 4.3 implies that $A : P_T \to P_T$ is compact and continuous. Also, from Lemma 4.4, the mapping $B : P_T \to P_T$ is contraction.

We need to show that if $\varphi, \psi \in M$, we have $\|A\varphi - B\psi\| \le G$. Let $\varphi, \psi \in M$ with $\|\varphi\|,\ \|\psi\| \le G$. From (4.19) and (4.20) and the fact that $|q(t,x,y)| \le L\|x\| + E\|y\| + \alpha$, we have

$$\begin{aligned}
\|A\varphi + B\psi\| &\le \gamma\eta\int_{\delta_-^T(t)}^{t} [L\|\varphi\| + E\|\varphi\| + \beta\|\varphi\| + \alpha]\,\Delta s + \xi\|\psi\| \\
&\le \left\{\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right\}G + \gamma\eta\alpha\left(t_0 - \delta_-^T(t_0)\right) \\
&\le G.
\end{aligned} \tag{4.34}$$

We see that all the conditions of Krasnosel'skiĭ theorem are satisfied on the set M. Thus there exists a fixed point z in M such that $z = Bz + Az$. By Lemma 4.2, this fixed point is a solution of (2) has a T-periodic solution in $\delta_\pm$. □

Theorem 4.6. *Suppose that* (4.3)–(4.5) *and* (4.31) *hold. Let β, η, and γ be given by* (4.39)*. If*

$$\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right) \le 1, \tag{4.35}$$

then (1.2) *has a unique T-periodic solution in $\delta_\pm$.*

Proof. Let the mapping H be given by (4.17). For $\varphi, \psi \in P_T$ we have

$$\begin{aligned} \|H\varphi - H\psi\| &\leq \xi\|\varphi - \psi\| + \gamma\eta \int_{\delta_-^T(t)}^{t} [L\|\varphi - \psi\| + E\|\varphi - \psi\| + \beta\|\varphi - \psi\|]\Delta s \\ &\leq \left[\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right]\|\varphi - \psi\|. \end{aligned} \tag{4.36}$$

This completes the proof. □

Example 4.7. Let $\mathbb{T} = \{2^n\}_{n\in N_0} \cup \{1/4, 1/2\}$ be a periodic time scale in shift $\delta_\pm(P,t) = P^{\pm 1}t$ with period $P = 2$. We consider the dynamic equation (1.2) with $a(t) = 1/5t$, $b(t) = (1/500)(-1)^{\ln t/\ln q}$ and $q(t,x,y) = (\sin x + \arctan x + 1)/1000t$.

The operators $\delta_-(s,t) = t/s$ and $\delta_+(s,t) = st$ are backward and forward shift operators for $(s,t) \in D_\pm$. Here $\mathbb{T}^* = \mathbb{T}$, the initial point $t_0 = 1$ and $\delta_-(k,t) = t/k$ for $k \in [2,\infty)_\mathbb{T}$. If we consider conditions (4.3)-(4.4) we find $T = 4$. Then $a(t), b(t)$ satisfy condition (4.3), $a(t) \in \mathcal{R}^+$ and $q(t,x,y)$ satisfies the condition (4.4) for all $t \in \mathbb{T}$. Also, $q(t,x,y)$ is Lipschitz continuous in x and y for $L = E = 1/250$. Since $\|b(t)\| = \|(1/500)(-1)^{\ln t/\ln q}\| = (1/500) = \xi < 1$, then the condition (4.31) holds.

If we compute η, γ, and β, we have

$$\eta = \max_{t\in\left[t_0, \delta_+^T(t_0)\right]_\mathbb{T}} \left|\left(1 - e_{\ominus a(t)}\left(t, \delta_-^T(t)\right)\right)^{-1}\right| \cong 3,45, \qquad \alpha = \max_{t\in\left[t_0, \delta_+^T(t_0)\right]_\mathbb{T}} |q(t,0,0)| = \frac{1}{250} \tag{4.37}$$

$$\gamma = \max_{u\in\left[\delta_-^T(t), t\right]_\mathbb{T}} e_{\ominus a(t)}(t,u) \cong 1,5, \qquad \beta = \max_{t\in\left[t_0, \delta_+^T(t_0)\right]_\mathbb{T}} |r(t)| = \frac{11}{2500}. \tag{4.38}$$

If we take $G = 1$, then inequality (4.33) satisfies.

Let $x(t) \in P_T$. We show that $\|x(t)\| \leq G = 1$. Integrate (1.2) from 1 to 4, we get

$$x(4) - x(1) = \int_1^4 \left[-a(t)x^\sigma(t) + b(t)x^\Delta\left(\frac{t}{k}\right)\frac{1}{k} + q\left(t, x(t), x\left(\frac{t}{k}\right)\right)\right]\Delta t. \tag{4.39}$$

Since $x(t) \in P_T$, then $x(4) = x(1)$ and so after integration by parts (23) becomes

$$\int_1^4 a(t)x^\sigma(t)\Delta t = \int_1^4 q\left(t, x(t), x\left(\frac{t}{k}\right)\right) - b^\Delta(t)x\left(\frac{t}{k}\right)\Delta t. \tag{4.40}$$

Claim 1. There exist $t^* \in [1,4]_\mathbb{T}$ such that $3a(t^*)x^\sigma(t^*) \leq \int_1^4 a(t)x^\sigma(t)\Delta t$.

Suppose that the claim is false. Define $S := \int_1^4 a(t)x^\sigma(t)\Delta t$. Then there exists $\epsilon > 0$ such that

$$3a(t)x^\sigma(t) > S + \epsilon, \tag{4.41}$$

for all $t \in [1,4]_{\mathbb{T}}$. So,

$$S = \int_1^4 a(t)x^\sigma(t)\Delta t > \frac{1}{3}\int_1^4 (S+\epsilon)\Delta T = S+\epsilon. \tag{4.42}$$

That is, $S > S+\epsilon$, a contradiction.

As a consequence of the claim, we have

$$\begin{aligned}
3|a(t^*)||x^\sigma(t^*)| &\le \int_1^4 \left|q\left(t, x(t), x\left(\frac{t}{k}\right)\right)\right| + \left|b^\Delta(t)x\left(\frac{t}{k}\right)\right|\Delta t \\
&\le \int_1^4 [(L+E)\|x\| + \alpha + \delta\|x\|]\Delta t \\
&= 3\left[\left(\frac{2}{250}+\frac{1}{250}\right)\|x\| + \frac{1}{250}\right] = 3\left[\frac{3}{250}\|x\| + \frac{1}{250}\right],
\end{aligned} \tag{4.43}$$

where $\delta = \max_{[1,4]_{\mathbb{T}}}|b^\Delta(t)| = 1/250$.

So, $|a(t^*)||x^\sigma(t^*)| \le (3/250)\|x\| + (1/250)$, which implies $|x^\sigma(t^*)| \le 20[(3/250)\|x\| + (1/250)]$. Since for all $t \in [1,4]_{\mathbb{T}}$,

$$x^\sigma(t) = x^\sigma(t^*) + \int_{t^*}^t x^\Delta(\sigma(s))\Delta s, \tag{4.44}$$

we have

$$|x^\sigma(t)| \le |x^\sigma(t^*)| + \int_1^t \left|x^\Delta(\sigma(s))\right|\Delta s \le 20\left[\frac{3}{250}\|x\| + \frac{1}{250}\right] + 3\left\|x^\Delta\right\|. \tag{4.45}$$

This implies that

$$\|x\| \le \frac{2}{19} + \frac{1500}{19}\left\|x^\Delta\right\|. \tag{4.46}$$

Taking the norm in (1.2) yields

$$\left\|x^\Delta\right\| \le \frac{(\|a\| + L + E)\|x\| + \alpha}{1-\|b\|} = \frac{(1/5 + 2/250)\|x\| + 1/250}{1-1/500} = \frac{104\|x\| + 2}{250.499}. \tag{4.47}$$

Substitution of (4.47) into (4.46) yields that for all $x(t) \in P_T$, $\|x(t)\| \le G = 1$. Then by Theorem 4.5, (1.2) has a 4-periodic solution in shifts $\delta_\pm$.

In this example, if we take $q(t,x,y) = (\sin x + \arctan x)/1000t$, we have

$$\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right) = \frac{1}{500} + \frac{96255}{2.10^6} < 1. \tag{4.48}$$

So, all the conditions of Theorem 4.6 are satisfied. Therefore, (1.2) has a unique 4-periodic solution in shifts $\delta_{\pm}$.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 240378, 57 pages
doi:10.1155/2012/240378

Research Article

Positive Solutions and Iterative Approximations for a Nonlinear Two-Dimensional Difference System with Multiple Delays

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Received 16 March 2012; Accepted 30 May 2012

Academic Editor: Josef Diblík

This paper studies the nonlinear two-dimensional difference system with multiple delays $\Delta(x_n + p_{1n}x_{n-\tau_1}) + f_1(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}}) = q_{1n}, \Delta(y_n + p_{2n}y_{n-\tau_2}) + f_2(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}}) = q_{2n}, n \geq n_0$. Using the Banach fixed point theorem and a few new analysis techniques, we show the existence of uncountably many bounded positive solutions for the system, suggest Mann iterative algorithms with errors, and discuss the error estimates between the positive solutions and iterative sequences generated by the Mann iterative algorithms. Examples to illustrates the results are included.

1. Introduction and Preliminaries

In recent years, there has been an increasing interest in the study of oscillation, nonoscillation, asymptotic behavior, existence and multiplicity of solutions, positive solutions, nonoscillatory solutions, and periodic solutions, respectively, for various difference equations and systems, see for example, [1–34] and the references cited therein.

Jiang and Tang [18] and Graef and Thandapani [14] studied the oscillation of the linear two-dimensional difference system

$$\begin{aligned} \Delta x_n &= p_n y_n, \\ \Delta y_{n-1} &= -q_n x_n, \quad n \geq n_0, \end{aligned} \tag{1.1}$$

and nonlinear two-dimensional difference system

$$\begin{aligned} \Delta x_n &= b_n g(y_n), \\ \Delta y_{n-1} &= -a_n f(x_n), \quad n \geq n_0, \end{aligned} \tag{1.2}$$

where $n_0 \in \mathbb{N}$, $\{p_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, $\{a_n\}_{n\geq n_0}$, and $\{b_n\}_{n\geq n_0}$ are nonnegative sequences, $f, g \in C(\mathbb{R}, \mathbb{R})$ with $uf(u) > 0$ and $ug(u) > 0$ for all $u \neq 0$. Jiang and Tang [19] also gave some necessary and sufficient conditions for all solutions of System (1.2) to be oscillatory. Agarwal et al. [2] discussed the two-dimensional nonlinear difference system of the form

$$\begin{aligned} \Delta x_n &= a_n f(y_n), \\ \Delta y_{n-1} &= b_n g(x_n), \quad n \geq n_0, \end{aligned} \tag{1.3}$$

where $n_0 \in \mathbb{N}$, $\{a_n\}_{n\geq n_0}$ and $\{b_n\}_{n\geq n_0}$ are positive sequences, $f, g \in C(\mathbb{R}, \mathbb{R})$ are increasing with $uf(u) > 0$ and $ug(u) > 0$ for all $u \neq 0$, and they provided a classification scheme of positive solutions for System (1.3) and established conditions for the existence of solutions with designated asymptotic behavior. Li [21] introduced the two-dimensional nonlinear difference system:

$$\begin{aligned} \Delta x_n &= a_n f(y_n), \\ \Delta y_n &= -b_n g(x_n), \quad n \geq n_0, \end{aligned} \tag{1.4}$$

where $n_0 \in \mathbb{N}$, $\{a_n\}_{n\geq n_0}$ and $\{b_n\}_{n\geq n_0}$ are nonnegative sequences, $f, g \in C(\mathbb{R}, \mathbb{R})$ with $uf(u) > 0$, and $ug(u) > 0$ for all $u \neq 0$, he showed both classification schemes for nonoscillatory solutions of System (1.4) and gave necessary and sufficient conditions for the existence of these solutions. Huo and Li [15, 16] considered the nonlinear two-dimensional difference system

$$\begin{aligned} \Delta x_n &= b_n g(y_n), \\ \Delta y_n &= -a_n f(x_n) + r_n, \quad n \geq n_0, \end{aligned} \tag{1.5}$$

and the Emden-Fowler difference system

$$\begin{aligned} \Delta x_n &= b_n g(y_n), \\ \Delta y_{n-1} &= -a_n f(x_n) + r_n, \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $n_0 \in \mathbb{N}$, $\{a_n\}_{n\geq n_0}$ is a real sequence, $\{b_n\}_{n\geq n_0}$ is a nonnegative sequence, $\sum_{i=1}^{\infty} |r_i| < +\infty$, $f, g \in C(\mathbb{R}, \mathbb{R})$ with $uf(u) > 0$, and $ug(u) > 0$ for all $u \neq 0$, they proved some oscillation results for Systems (1.5) and (1.6). Jiang and Li [17] investigated the nonlinear two-dimensional difference system:

$$\begin{aligned} \Delta x_n &= a_n g(y_n), \\ \Delta y_{n-1} &= f(n, x_n), \quad n \geq n_0, \end{aligned} \tag{1.7}$$

where $n_0 \in \mathbb{N}$, $\{a_n\}_{n\geq n_0}$ is a nonnegative sequence, $f \in C(\mathbb{N}_{n_0} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$ with $uf(n,u) > 0$ for all $n \in \mathbb{N}_{n_0}$, and $u \neq 0$ and $ug(u) > 0$ for all $u \neq 0$, they obtained some necessary and sufficient conditions for all solutions of System (1.7) to be oscillatory. Thandapani and Kumar [31] studied the oscillation for the nonlinear two-dimensional difference system of the neutral type

$$\begin{aligned} \Delta(x_n - a_n x_{\sigma_n}) &= p_n g(y_n), \\ \Delta y_n &= \delta q_n f(x_{\tau_n}), \quad n \geq n_0, \end{aligned} \tag{1.8}$$

where $\delta = \pm 1$, $n_0 \in \mathbb{N}$, $\{a_n\}_{n\geq n_0}$ is a positive sequence, $\{p_n\}_{n\geq n_0}$ and $\{q_n\}_{n\geq n_0}$ are nonnegative sequences with $\sum_{i=n_0}^{\infty} p_i = +\infty$, $\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\tau_n = +\infty$, $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing with $uf(u) > 0$ and $ug(u) > 0$ for all $u \neq 0$. Wu and Liu [33] established the existence and multiplicity of periodic solutions for the first-order neutral difference system

$$\begin{aligned} \Delta(x_n - cx_{n-\tau}) &= a_{1n} g_1(x_n) x_n - \lambda b_{1n} f_1(x_{n-\tau_{1n}}, y_{n-\rho_{1n}}), \\ \Delta(y_n - cy_{n-\tau}) &= a_{2n} g_2(y_n) y_n - \mu b_{2n} f_2(x_{n-\tau_{2n}}, y_{n-\rho_{2n}}), \end{aligned} \tag{1.9}$$

where $\tau \in \mathbb{N}$, $\lambda, \mu \in \mathbb{R}^+ \setminus \{0\}$, $c \in \mathbb{R}$ with $|c| \neq 1$, $\{a_{1n}\}$, $\{a_{2n}\}$, $\{b_{1n}\}$, and $\{b_{2n}\}$ are positive T-periodic sequences, $\{\tau_{1n}\}$, $\{\tau_{2n}\}$, $\{\rho_{1n}\}$, and $\{\rho_{2n}\}$ are positive T-periodic integer sequences. Tang [30] proved the existence results of a bounded nonoscillatory solution for the second-order linear delay difference equation

$$\Delta^2 x_n = p_n x_{n-k}, \quad n \geq 0, \tag{1.10}$$

where $k \in \mathbb{N}$ and $\{p_n\}_{n\geq 0}$ is a nonnegative sequence. Cheng [20] utilized the Banach fixed point theorem to discuss the existence of a nonoscillatory solution for the second-order neutral delay difference equation with positive and negative coefficients:

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0, \tag{1.11}$$

where $m, k, l \in \mathbb{N}$, $p \in \mathbb{R} \setminus \{-1\}$, $\{p_n\}_{n\geq 0}$ and $\{q_n\}_{n\geq 0}$ are nonnegative sequences with $\sum_{n=n_0}^{\infty} n \max\{p_n, q_n\} < +\infty$. M. Migda and J. Migda [29] obtained the asymptotic behavior of the second-order neutral difference equation:

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1, \tag{1.12}$$

where $p \in \mathbb{R}$, $k \in \mathbb{N}_0$, and $f \in C(\mathbb{N} \times \mathbb{R}, \mathbb{R})$. Liu et al. [27] studied the global existence of uncountably many bounded nonoscillatory solutions for the second-order nonlinear neutral delay difference equation:

$$\Delta(a_n \Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \tag{1.13}$$

where $b \in \mathbb{R}$, $\tau, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ is a positive sequence, $\{c_n\}_{n\in\mathbb{N}_{n_0}}$ is a real sequence, $\bigcup_{l=1}^{k}\{d_{ln}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$ with $\lim_{n\to\infty}(n-d_{ln}) = +\infty$ for $l \in \{1,2,\ldots,k\}$, and $f : \mathbb{N}_{n_0} \times \mathbb{R}^k \to \mathbb{R}$ is a mapping.

The purpose of this paper is to study the below nonlinear two-dimensional difference system with multiple delays

$$\begin{gathered}\Delta\left(x_n + p_{1n}x_{n-\tau_1}\right) + f_1\left(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}}\right) = q_{1n}, \\ \Delta\left(y_n + p_{2n}y_{n-\tau_2}\right) + f_2\left(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}}\right) = q_{2n}, \quad n \geq n_0,\end{gathered} \tag{1.14}$$

where $h, k, \tau_i \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{p_{in}\}_{n\in\mathbb{N}_{n_0}}, \{q_{in}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, and $f_i \in C(\mathbb{N}_{n_0} \times \mathbb{R}^{h+k}, \mathbb{R})$ for $i \in \Lambda_2$, $\{a_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{c_{ln}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{Z}$ with $\lim_{n\to\infty} a_{ln} = \lim_{n\to\infty} c_{ln} = +\infty$ for $l \in \Lambda_h$ and $\{b_{jn}\}_{n\in\mathbb{N}_{n_0}}$, $\{d_{jn}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{Z}$ with $\lim_{n\to\infty} b_{jn} = \lim_{n\to\infty} d_{jn} = +\infty$ for $j \in \Lambda_k$. It is easy to see that the system (1.14) includes Systems (1.1)–(1.9), (1.10)–(1.13), and a lot of the first- and second-order nonlinear, half-linear, and quasilinear difference equations as special cases. Using the Banach fixed point theorem and some analysis techniques, we prove the existence of uncountably many bounded positive solutions for System (1.14), establish their iterative approximations, and discuss the error estimates between the positive solutions and iterative approximations. Our results sharp, and improve [14, Theorem 1] and [27, Theorems 2.1–2.7]. To illustrate our results, fifteen examples are also included.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ stand for the sets of all integers, positive integers, and nonnegative integers, respectively,

$$\begin{gathered}\mathbb{N}_{n_0} = \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\}, \quad \Lambda_n = \{1, \ldots, n\}, \quad \forall n \in \mathbb{N}, \\ \alpha = \inf\left\{n_0 - \tau_1, n_0 - \tau_2, a_{ln}, c_{ln}, b_{jn}, d_{jn} : n \in \mathbb{N}_{n_0},\ l \in \Lambda_h,\ j \in \Lambda_k\right\}, \\ \mathbb{Z}_\alpha = \{n : n \in \mathbb{Z} \text{ with } n \geq \alpha\},\end{gathered} \tag{1.15}$$

l_α^∞ denotes the Banach space of all bounded sequences on $\mathbb{Z}_\alpha$ with norm

$$\|x\| = \sup_{n\in\mathbb{Z}_\alpha}|x_n| \quad \text{for } x = \{x_n\}_{n\in\mathbb{Z}_\alpha} \in l_\alpha^\infty. \tag{1.16}$$

Let

$$\begin{aligned}&A(N, M) \\ &\quad = \left\{(x, y) = \left(\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}\right) \in l_\alpha^\infty \times l_\alpha^\infty : N \leq x_n \leq M,\ N \leq y_n \leq M,\ n \in \mathbb{Z}_\alpha\right\}, \\ &A(N, M, N_0, M_0) \\ &\quad = \left\{(x, y) = \left(\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}\right) \in l_\alpha^\infty \times l_\alpha^\infty : N \leq x_n \leq M,\ N_0 \leq y_n \leq M_0,\ n \in \mathbb{Z}_\alpha\right\},\end{aligned} \tag{1.17}$$

for any $M > N > 0$ and $M_0 > N_0 > 0$. It is easy to see that $A(N, M)$ and $A(N, M, N_0, M_0)$ are bounded closed and convex subsets of the Banach space $l_\alpha^\infty \times l_\alpha^\infty$ with norm

$$\|(x, y)\|_1 = \max\{\|x\|, \|y\|\} \quad \text{for } (x, y) \in l_\alpha^\infty \times l_\alpha^\infty. \tag{1.18}$$

By a solution of System (1.14), we mean a sequence $(\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha})$ with a positive integer $T \geq n_0 + \tau_1 + \tau_2 + |\alpha|$ such that System (1.14) is satisfied for all $n \geq T$. A solution $(\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha})$ of System (1.14) is said to be positive if both components are positive.

Lemma 1.1 (see [35]). *Let $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, and $\{D_n\}_{n\geq 0}$ be four nonnegative real sequences satisfying the inequality*

$$A_{n+1} \leq (1 - D_n)A_n + D_n B_n + C_n, \quad \forall n \geq 0, \tag{1.19}$$

where $\{D_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} D_n = +\infty$, $\lim_{n\to\infty} B_n = 0$, and $\sum_{n=0}^{\infty} C_n < +\infty$. Then $\lim_{n\to\infty} A_n = 0$.

Lemma 1.2 (see [36]). *Let $\tau \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, and $\{B_n\}_{n\in\mathbb{N}_{n_0}}$ be a nonnegative sequence. Then*

$$\sum_{i=0}^{\infty} \sum_{n=n_0+i\tau}^{\infty} B_n < +\infty \Longleftrightarrow \sum_{n=n_0}^{\infty} n B_n < +\infty. \tag{1.20}$$

2. Main Results

In this section, we investigate the existence of uncountably many bounded positive solutions for System (1.14) and their iterative approximations and the error estimates between the positive solutions and iterative approximations.

Theorem 2.1. *Assume that there exist constants $M, N \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$, and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$|f_i(n, u_1, \ldots, u_h, v_1, \ldots, v_k)| \leq r_{in}, \quad \forall (n, u_l, v_j) \in \mathbb{N}_{n_1} \times [N, M]^2,\ l \in \Lambda_h,\ j \in \Lambda_k,\ i \in \Lambda_2, \tag{2.1}$$

$$\begin{aligned} &|f_i(n, u_1, \ldots, u_h, v_1, \ldots, v_k) - f_i(n, w_1, \ldots, w_h, z_1, \ldots, z_k)| \\ &\quad \leq t_{in} \max\{|u_l - w_l|, |v_j - z_j| : l \in \Lambda_h, j \in \Lambda_k\}, \\ &\qquad \forall (n, u_l, w_l, v_j, z_j) \in \mathbb{N}_{n_1} \times [N, M]^4,\ l \in \Lambda_h,\ j \in \Lambda_k,\ i \in \Lambda_2, \end{aligned} \tag{2.2}$$

$$\sum_{n=n_0}^{\infty} n \max\{r_{in}, t_{in}, |q_{in}| : i \in \Lambda_2\} < +\infty, \tag{2.3}$$

$$N < M, \qquad p_{in} = -1, \quad \forall n \geq n_1,\ i \in \Lambda_2. \tag{2.4}$$

Then,

(a) *for each* $L \in (N, M)$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by the schemes*

$$x_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)x_n^\mu \\ \quad +\beta_\mu\left\{L-\displaystyle\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right)-q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)x_{T_L}^\mu \\ \quad +\beta_\mu\left\{L-\displaystyle\sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_1}^{\infty}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right)-q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L, \end{cases} \tag{2.5}$$

$$y_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)y_n^\mu \\ \quad +\beta_\mu\left\{L-\displaystyle\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right)-q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)y_{T_L}^\mu \\ \quad +\beta_\mu\left\{L-\displaystyle\sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_2}^{\infty}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right)-q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L \end{cases} \tag{2.6}$$

converges to a bounded positive solution $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate:*

$$\left\|\left(x^{\mu+1}, y^{\mu+1}\right)-(x, y)\right\|_1 \leq (1-\beta_\mu(1-\theta_L))\|(x^\mu, y^\mu)-(x, y)\|_1 + 2M\gamma_\mu, \quad \forall \mu \geq 0, \tag{2.7}$$

where $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$ *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N,M)$ *are arbitrary sequences with*

$$\sum_{\mu=0}^{\infty}\beta_\mu = +\infty, \tag{2.8}$$

$$\sum_{\mu=0}^{\infty}\gamma_\mu < +\infty \text{ or there exists a sequence } \{\eta_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,+\infty) \text{ satisfying}$$
$$\gamma_\mu = \beta_\mu\eta_\mu, \quad \forall\mu \in \mathbb{N}_0, \quad \lim_{\mu\to\infty}\eta_\mu = 0, \tag{2.9}$$

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N,M)$.

Proof. (a) Let $L \in (N,M)$. Now we construct a contraction mapping $S_L : A(N,M) \to A(N,M)$ and prove that its fixed point is a bounded positive solution of System (1.14). It follows from (2.3) and (2.4) that there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1,\tau_2\}+n_1+|\alpha|$ satisfying

$$\theta_L = \sum_{m=T_L}^{\infty}(1+m)\max\{t_{im} : i \in \Lambda_2\}, \tag{2.10}$$

$$\sum_{m=T_L}^{\infty}(1+m)\max\{r_{im} + |q_{im}| : i \in \Lambda_2\} < \min\{M-L, L-N\}. \tag{2.11}$$

Define three mappings $S_L : A(N,M) \to l_\alpha^\infty \times l_\alpha^\infty$, $S_{1L}, S_{2L} : A(N,M) \to l_\alpha^\infty$ by

$$S_{1L}(x_n,y_n) = \begin{cases} L - \displaystyle\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left[f_1(m, x_{a_{1m}},\ldots,x_{a_{hm}}, y_{b_{1m}},\ldots,y_{b_{km}}) - q_{1m}\right], & n \geq T_L, \\ S_{1L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.12}$$

$$S_{2L}(x_n,y_n) = \begin{cases} L - \displaystyle\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}\left[f_2(m, x_{c_{1m}},\ldots,x_{c_{hm}}, y_{d_{1m}},\ldots,y_{d_{km}}) - q_{2m}\right], & n \geq T_L, \\ S_{2L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.13}$$

$$S_L(x_n,y_n) = (S_{1L}(x_n,y_n), S_{2L}(x_n,y_n)), \quad n \in \mathbb{Z}_\alpha, \tag{2.14}$$

for all $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$. It follows from (2.1), (2.3), (2.12), (2.13), and Lemma 1.2 that S_{1L} and S_{2L} are well defined.

In view of (2.1), (2.2), and (2.10)–(2.14), we know that for any $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$ and $(u,v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha}, \{v_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$,

$$
\begin{aligned}
&\left\|S_L(x,y) - S_L(u,v)\right\|_1 \\
&\quad = \max\left\{\left\|S_{iL}(x,y) - S_{iL}(u,v)\right\| : i \in \Lambda_2\right\} \\
&\quad = \max\left\{\sup_{n\ge T_L}\left|S_{iL}(x_n,y_n) - S_{iL}(u_n,v_n)\right| : i \in \Lambda_2\right\} \\
&\quad \le \max\left\{\sup_{n\ge T_L}\left\{\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left|f_1(m,x_{a_{1m}},\dots,x_{a_{hm}},y_{b_{1m}},\dots,y_{b_{km}})\right.\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. - f_1(m,u_{a_{1m}},\dots,u_{a_{hm}},v_{b_{1m}},\dots,v_{b_{km}})\right|\right\}, \\
&\qquad\qquad \sup_{n\ge T_L}\left\{\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}\left|f_2(m,x_{c_{1m}},\dots,x_{c_{hm}},y_{db_{1m}},\dots,y_{d_{km}})\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left.\left. - f_2(m,u_{c_{1m}},\dots,u_{c_{hm}},v_{d_{1m}},\dots,v_{d_{km}})\right|\right\}\right\} \\
&\quad \le \max\left\{\sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_1}^{\infty} t_{1m}\max\left\{\left|x_{a_{im}} - u_{a_{im}}\right|, \left|y_{b_{jm}} - v_{b_{jm}}\right| : i\in\Lambda_h, j\in\Lambda_k\right\},\right. \\
&\qquad\qquad \left.\sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_2}^{\infty} t_{2m}\max\left\{\left|x_{c_{im}} - u_{c_{im}}\right|, \left|y_{d_{jm}} - v_{d_{jm}}\right| : i\in\Lambda_h, j\in\Lambda_k\right\}\right\} \\
&\quad \le \max\left\{\sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_1}^{\infty} t_{1m}, \sum_{l=0}^{\infty}\sum_{m=T_L+(l+1)\tau_2}^{\infty} t_{2m}\right\}\max\{\|x-u\|, \|y-v\|\} \\
&\quad \le \max\left\{\sum_{m=T_L}^{\infty}\left(1+\frac{m}{\tau_1}\right)t_{1m}, \sum_{m=T_L}^{\infty}\left(1+\frac{m}{\tau_2}\right)t_{2m}\right\}\left\|(x,y) - (u,v)\right\|_1 \\
&\quad \le \left(\sum_{m=T_L}^{\infty}(1+m)\max\{t_{1m},t_{2m}\}\right)\left\|(x,y) - (u,v)\right\|_1 \\
&\quad = \theta_L\left\|(x,y) - (u,v)\right\|_1, \\
&\max\left\{\left|S_{1L}(x_n,y_n) - L\right|, \left|S_{2L}(x_n,y_n) - L\right|\right\} \\
&\quad \le \max\left\{\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left|f_1(m,x_{a_{1m}},\dots,x_{a_{hm}},y_{b_{1m}},\dots,y_{b_{km}}) - q_{1m}\right|,\right. \\
&\qquad\qquad \left.\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}\left|f_2(m,x_{c_{1m}},\dots,x_{c_{hm}},y_{d_{1m}},\dots,y_{d_{km}}) - q_{2m}\right|\right\}
\end{aligned}
$$

$$\begin{aligned}
&\le \max\left\{\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}(r_{1m}+|q_{1m}|),\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}(r_{2m}+|q_{2m}|)\right\}\\
&\le \max\left\{\sum_{m=T_L}^{\infty}\left(1+\frac{m}{\tau_1}\right)(r_{1m}+|q_{1m}|),\sum_{m=T_L}^{\infty}\left(1+\frac{m}{\tau_2}\right)(r_{2m}+|q_{2m}|)\right\}\\
&\le \sum_{m=T_L}^{\infty}(1+m)\max\{r_{im}+|q_{im}| : i\in\Lambda_2\}\\
&< \min\{M-L,L-N\},\quad \forall n\ge T_L,
\end{aligned} \tag{2.15}$$

which give that

$$\|S_L(x,y)-S_L(u,v)\|_1 \le \theta_L\|(x,y)-(u,v)\|_1,\quad \forall(x,y),(u,v)\in A(N,M), \tag{2.16}$$

$$N\le S_{1L}(x_n,y_n)\le M,\quad N\le S_{2L}(x_n,y_n)\le M,\quad \forall\left(\{x_n\}_{n\in\mathbb{Z}_\alpha},\{y_n\}_{n\in\mathbb{Z}_\alpha}\right)\in A(N,M),\ n\ge T_L. \tag{2.17}$$

Observe that $\theta_L\in(0,1)$. It is easy to see that (2.12)–(2.17) give that $S_L : A(N,M)\to A(N,M)$ is a contraction. Consequently, S_L possesses a unique fixed point $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha},\{y_n\}_{n\in\mathbb{Z}_\alpha})\in A(N,M)$, which implies that

$$\begin{aligned}
x_n &= L-\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left[f_1(m,x_{a_{1m}},\ldots,x_{a_{hm}},y_{b_{1m}},\ldots,y_{b_{km}})-q_{1m}\right],\quad n\ge T_L,\\
y_n &= L-\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_2}^{\infty}\left[f_2(m,x_{c_{1m}},\ldots,x_{c_{hm}},y_{d_{1m}},\ldots,y_{d_{km}})-q_{2m}\right],\quad n\ge T_L,
\end{aligned} \tag{2.18}$$

which yield that

$$\begin{aligned}
x_n-x_{n-\tau_1} &= L-\sum_{l=0}^{\infty}\sum_{m=n+(l+1)\tau_1}^{\infty}\left[f_1(m,x_{a_{1m}},\ldots,x_{a_{hm}},y_{b_{1m}},\ldots,y_{b_{km}})-q_{1m}\right]\\
&\quad -L+\sum_{l=0}^{\infty}\sum_{m=n+l\tau_1}^{\infty}\left[f_1(m,x_{a_{1m}},\ldots,x_{a_{hm}},y_{b_{1m}},\ldots,y_{b_{km}})-q_{1m}\right]\\
&= \sum_{m=n}^{\infty}\left[f_1(m,x_{a_{1m}},\ldots,x_{a_{hm}},y_{b_{1m}},\ldots,y_{b_{km}})-q_{1m}\right],\quad \forall n\ge T_L+\tau_1,
\end{aligned}$$

$$\begin{aligned} y_n - y_{n-\tau_2} &= L - \sum_{l=0}^{\infty} \sum_{m=n+(l+1)\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m}\right] \\ &\quad - L + \sum_{l=0}^{\infty} \sum_{m=n+l\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m}\right] \\ &= \sum_{m=n}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m}\right], \quad \forall n \geq T_L + \tau_2, \end{aligned} \tag{2.19}$$

which give that

$$\begin{aligned} \Delta\left(x_n - x_{n-\tau_1}\right) &= -f_1\left(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}}\right) + q_{1n}, \quad \forall n \geq T_L + \tau_1, \\ \Delta\left(y_n - y_{n-\tau_2}\right) &= -f_2\left(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}}\right) + q_{2n}, \quad \forall n \geq T_L + \tau_2, \end{aligned} \tag{2.20}$$

that is, (x, y) is a bounded positive solution of System (1.14) in $A(N, M)$.

In light of (2.5), (2.6), and (2.11)–(2.16), we arrive at

$$\begin{aligned} &\left\|\left(x^{\mu+1}, y^{\mu+1}\right) - (x, y)\right\|_1 \\ &= \max\left\{\sup_{n \geq T_L}\left|x_n^{\mu+1} - x_n\right|, \sup_{n \geq T_L}\left|y_n^{\mu+1} - y_n\right|\right\} \\ &\leq \max\left\{\sup_{n \geq T_L}\left[\left(1 - \beta_\mu - \gamma_\mu\right)\left|x_n^\mu - x_n\right|\right.\right. \\ &\qquad + \beta_\mu\left|L - \sum_{l=0}^{\infty} \sum_{m=n+(l+1)\tau_1}^{\infty}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m}\right] - x_n\right| \\ &\qquad \left. + \gamma_\mu\left|\delta_{1n}^\mu - x_n\right|\right], \\ &\qquad \sup_{n \geq T_L}\left[\left(1 - \beta_\mu - \gamma_\mu\right)\left|y_n^\mu - y_n\right|\right. \\ &\qquad + \beta_\mu\left|L - \sum_{l=0}^{\infty} \sum_{m=n+(l+1)\tau_2}^{\infty}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m}\right] - y_n\right| \\ &\qquad \left.\left. + \gamma_\mu\left|\delta_{2n}^\mu - y_n\right|\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \max\Bigg\{\sup_{n\geq T_L}\Big[(1-\beta_\mu-\gamma_\mu)\Big|x_n^\mu-x_n\Big|+\beta_\mu\Big|S_{1L}\Big(x_n^\mu,y_n^\mu\Big)-S_{1L}(x_n,y_n)\Big|+2M\gamma_\mu\Big],\\
&\qquad\sup_{n\geq T_L}\Big[(1-\beta_\mu-\gamma_\mu)\Big|y_n^\mu-y_n\Big|+\beta_\mu\Big|S_{2L}\Big(x_n^\mu,y_n^\mu\Big)-S_{2L}(x_n,y_n)\Big|+2M\gamma_\mu\Big]\Bigg\}\\
&\leq \max\{(1-\beta_\mu-\gamma_\mu)\|x^\mu-x\|+\theta_L\beta_\mu\|(x^\mu,y^\mu)-(x,y)\|_1+2M\gamma_\mu,\\
&\qquad(1-\beta_\mu-\gamma_\mu)\|y^\mu-y\|+\theta_L\beta_\mu\|(x^\mu,y^\mu)-(x,y)\|_1+2M\gamma_\mu\}\\
&\leq (1-\beta_\mu(1-\theta_L))\|(x^\mu,y^\mu)-(x,y)\|_1+2M\gamma_\mu,\quad \forall\mu\geq 0,
\end{aligned}
\tag{2.21}
$$

that is, (2.7) holds. It follows from (2.7)–(2.9) and Lemma 1.1 that $\lim_{\mu\to\infty}\|(x^\mu,y^\mu)-(x,y)\|_1 = 0$.

(b) Let $L_1, L_2 \in (N, M)$, and $L_1 \neq L_2$. As in the proof of (a), we similarly infer that for each $i \in \Lambda_2$, there exist a constant $\theta_{L_i} \in (0,1)$, a positive integer $T_{L_i} \geq \max\{\tau_1,\tau_2\}+n_1+|\alpha|$, and mappings $S_{L_i}, S_{1L_i}, S_{2L_i}$ satisfying (2.10)–(2.14), where L, θ_L and T_L are replaced by L_i, θ_{L_i} and T_{L_i}, respectively, and the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $(u,v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha},\{v_n\}_{n\in\mathbb{Z}_\alpha})$, $(w,z) = (\{w_n\}_{n\in\mathbb{Z}_\alpha},\{z_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$, respectively, (u,v) and (w,z) are bounded positive solutions of System (1.14) in $A(N,M)$. In order to show that System (1.14) possesses uncountably many bounded positive solutions in $A(N,M)$, we prove only that $(u,v)\neq(w,z)$. By means of (2.10) and (2.12)–(2.16), we conclude that

$$
\begin{aligned}
&\|(u,v)-(w,z)\|_1\\
&=\max\{\|u-w\|,\|v-z\|\}\\
&\geq \max\left\{\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|u_n-w_n|,\ \sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|v_n-z_n|\right\}\\
&\geq \max\Bigg\{|L_1-L_2|-\sum_{l=0}^{\infty}\sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_1}^{\infty}\Big|f_1(m,u_{a_{1m}},\dots,u_{a_{hm}},v_{b_{1m}},\dots,v_{b_{km}})\\
&\qquad\qquad -f_1(m,w_{a_{1m}},\dots,w_{a_{hm}},z_{b_{1m}},\dots,z_{b_{km}})\Big|,\\
&\qquad |L_1-L_2|-\sum_{l=0}^{\infty}\sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_2}^{\infty}\Big|f_2(m,u_{c_{1m}},\dots,u_{c_{hm}},v_{d_{1m}},\dots,v_{d_{km}})\\
&\qquad\qquad -f_2(m,w_{c_{1m}},\dots,w_{c_{hm}},z_{d_{1m}},\dots,z_{d_{km}})\Big|\Bigg\}\\
&\geq\max\Bigg\{|L_1-L_2|-\sum_{l=0}^{\infty}\sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_1}^{\infty}t_{1m}\max\Big\{|u_{a_{im}}-w_{a_{im}}|,\Big|v_{b_{jm}}-z_{b_{jm}}\Big|:i\in\Lambda_h,j\in\Lambda_K\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\qquad |L_1 - L_2| - \sum_{l=0}^{\infty} \sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_2}^{\infty} t_{2m} \\
&\qquad \times \max\left\{|u_{c_{im}} - w_{c_{im}}|, \left|v_{d_{jm}} - z_{d_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_K\right\}\Bigg\} \\
&\geq \max\Bigg\{|L_1 - L_2| - \sum_{l=0}^{\infty} \sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_1}^{\infty} t_{1m} \max\{\|u - w\|, \|v - z\|\}, \\
&\qquad |L_1 - L_2| - \sum_{l=0}^{\infty} \sum_{m=\max\{T_{L_1},T_{L_2}\}+(l+1)\tau_2}^{\infty} t_{2m} \max\{\|u - w\|, \|v - z\|\}\Bigg\} \\
&\geq |L_1 - L_2| - \left(\sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} (1+m) \max\{t_{im} : i \in \Lambda_2\}\right) \|(u,v) - (w,z)\|_1 \\
&\geq |L_1 - L_2| - \max\{\theta_{L_1}, \theta_{L_2}\} \|(u,v) - (w,z)\|_1,
\end{aligned}
\tag{2.22}
$$

which implies that

$$
\|(u,v) - (w,z)\|_1 \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_{L_1}, \theta_{L_2}\}} > 0, \tag{2.23}
$$

that is, $(u,v) \neq (w,z)$. This completes the proof. □

Theorem 2.2. *Assume that there exist constants $M, N \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$, and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2),

$$
\sum_{n=n_0}^{\infty} \max\{r_{in}, t_{in}, |q_{in}| : i \in \Lambda_2\} < +\infty, \tag{2.24}
$$

$$
N < M, \qquad p_{in} = 1, \quad \forall n \geq n_1, \ i \in \Lambda_2. \tag{2.25}
$$

Then,

(a) *for each* $L \in (N, M)$*, there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$*, the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by the schemes*

$$x_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)x_n^\mu \\ \quad +\beta_\mu\left\{L + \displaystyle\sum_{l=1}^{\infty}\sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)x_{T_L}^\mu \\ \quad +\beta_\mu\left\{L + \displaystyle\sum_{l=1}^{\infty}\sum_{m=T_L+(2l-1)\tau_1}^{T+2l\tau_1-1}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L, \end{cases} \tag{2.26}$$

$$y_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)y_n^\mu \\ \quad +\beta_\mu\left\{L + \displaystyle\sum_{l=1}^{\infty}\sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)y_{T_L}^\mu \\ \quad +\beta_\mu\left\{L + \displaystyle\sum_{l=1}^{\infty}\sum_{m=T_L+(2l-1)\tau_2}^{T_L+2l\tau_2-1}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L \end{cases} \tag{2.27}$$

converges to a bounded positive solution $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7)*, where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$*, and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9)*,*

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$*.*

Proof. (a) Let $L \in (N, M)$. Now we construct a contraction mapping $S_L : A(N, M) \to A(N, M)$ and prove that its fixed point is a bounded positive solution of System (1.14). Note that (2.24) and (2.25) guarantee that there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ satisfying

$$\theta_L = \sum_{m=T_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\}, \tag{2.28}$$

$$\sum_{m=T_L}^{\infty} \max\{r_{im} + |q_{im}| : i \in \Lambda_2\} < \min\{M - L, L - N\}. \tag{2.29}$$

Define three mappings $S_L : A(N,M) \to l_\alpha^\infty \times l_\alpha^\infty$, $S_{1L}, S_{2L} : A(N,M) \to l_\alpha^\infty$ by (2.14),

$$S_{1L}(x_n, y_n) = \begin{cases} L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m} \right], & n \geq T_L, \\ S_{1L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.30}$$

$$S_{2L}(x_n, y_n) = \begin{cases} L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1} \left[f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}) - q_{2m} \right], & n \geq T_L, \\ S_{2L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.31}$$

for all $(x, y) = (\{x_n\}_{n \in \mathbb{Z}_\alpha}, \{y_n\}_{n \in \mathbb{Z}_\alpha}) \in A(N,M)$.

Using (2.1), (2.2), (2.14), and (2.28)–(2.31), we deduce that for any $(x, y) = (\{x_n\}_{n \in \mathbb{Z}_\alpha}, \{y_n\}_{n \in \mathbb{Z}_\alpha})$ and $(u, v) = (\{u_n\}_{n \in \mathbb{Z}_\alpha}, \{v_n\}_{n \in \mathbb{Z}_\alpha}) \in A(N,M)$,

$$\begin{aligned}
&\left\| S_L(x,y) - S_L(u,v) \right\|_1 \\
&\quad = \max\left\{ \left\| S_{iL}(x,y) - S_{iL}(u,v) \right\| : i \in \Lambda_2 \right\} \\
&\quad = \max\left\{ \sup_{n \geq T_L} \left| S_{iL}(x_n, y_n) - S_{iL}(u_n, v_n) \right| : i \in \Lambda_2 \right\} \\
&\quad \leq \max\Bigg\{ \sup_{n \geq T_L} \Bigg\{ \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{m=n+2l\tau_1-1} \Big| f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) \\
&\qquad\qquad - f_1\,(m, u_{a_{1m}}, \ldots, u_{a_{hm}}, v_{b_{1m}}, \ldots, v_{b_{km}}) \Big| \Bigg\}, \\
&\qquad \sup_{n \geq T_L} \Bigg\{ \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{m=n+2l\tau_2-1} \Big| f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}) \\
&\qquad\qquad - f_2\,(m, u_{c_{1m}}, \ldots, u_{c_{hm}}, v_{d_{1m}}, \ldots, v_{d_{km}}) \Big| \Bigg\} \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max\left\{\sum_{m=T_L+\tau_1}^{\infty} t_{1m} \max\left\{|x_{a_{im}} - u_{a_{im}}|, \left|y_{b_{jm}} - v_{b_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_k\right\},\right. \\
&\qquad \left.\sum_{m=T_L+\tau_2}^{\infty} t_{2m} \max\left\{|x_{c_{im}} - u_{c_{im}}|, \left|y_{d_{jm}} - v_{d_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_k\right\}\right\} \\
&\leq \max\left\{\sum_{m=T_L+\tau_1}^{\infty} t_{1m}, \sum_{m=T_L+\tau_2}^{\infty} t_{2m}\right\} \max\{\|x-u\|, \|y-v\|\} \\
&\leq \left(\sum_{m=T_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\}\right)\|(x,y)-(u,v)\|_1 \\
&= \theta_L \|(x,y)-(u,v)\|_1, \\
\max&\{\left|S_{1L}(x_n,y_n) - L\right|, \left|S_{2L}(x_n,y_n) - L\right|\} \\
&\leq \max\left\{\sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left|f_1(m, x_{a_{1m}},\dots,x_{a_{hm}}, y_{b_{1m}},\dots,y_{b_{km}}) - q_{1m}\right|,\right. \\
&\qquad \left.\sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1} \left|f_2(m, x_{c_{1m}},\dots,x_{c_{hm}}, y_{d_{1m}},\dots,y_{d_{km}}) - q_{2m}\right|\right\} \\
&\leq \max\left\{\sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left(r_{1m} + |q_{1m}|\right), \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1} \left(r_{2m} + |q_{2m}|\right)\right\} \\
&\leq \max\left\{\sum_{m=T_L+\tau_1}^{\infty} \left(r_{1m} + |q_{1m}|\right), \sum_{m=T_L+\tau_2}^{\infty} \left(r_{2m} + |q_{2m}|\right)\right\} \\
&\leq \sum_{m=T_L}^{\infty} \max\{r_{im} + |q_{im}| : i \in \Lambda_2\} \\
&< \min\{M-L, L-N\}, \quad \forall n \geq T_L,
\end{aligned}
\tag{2.32}
$$

which imply (2.16) and (2.17) and which ensure that S_L is a self-mapping from $A(N,M)$ into itself and is a contraction by $\theta_L \in (0,1)$. The Banach fixed point theorem means that S_L has a unique fixed point $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$, that is,

$$
\begin{aligned}
x_n &= L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left[f_1(m, x_{a_{1m}},\dots,x_{a_{hm}}, y_{b_{1m}},\dots,y_{b_{km}}) - q_{1m}\right], \quad n \geq T_L, \\
y_n &= L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1} \left[f_2(m, x_{c_{1m}},\dots,x_{c_{hm}}, y_{d_{1m}},\dots,y_{d_{km}}) - q_{2m}\right], \quad n \geq T_L,
\end{aligned}
\tag{2.33}
$$

which reveal that

$$\begin{aligned}
x_n + x_{n-\tau_1} &= L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right] \\
&\quad + L + \sum_{l=1}^{\infty} \sum_{m=n+2(l-1)\tau_1}^{n+(2l-1)\tau_1-1} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right] \\
&= 2L + \sum_{m=n}^{\infty} \left[f_1\left(m, x_{\mathrm{a}_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right], \quad \forall n \geq T_L + \tau_1, \\
y_n + y_{n-\tau_2} &= L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m} \right] \\
&\quad + L + \sum_{l=1}^{\infty} \sum_{m=n+2(l-1)\tau_2}^{n+(2l-1)\tau_2-1} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m} \right] \\
&= 2L + \sum_{m=n}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m} \right], \quad \forall n \geq T_L + \tau_2,
\end{aligned} \tag{2.34}$$

which yield that

$$\begin{aligned}
\Delta\left(x_n - x_{n-\tau_1}\right) &= -f_1\left(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}}\right) + q_{1n}, \quad \forall n \geq T_L + \tau_1, \\
\Delta\left(y_n - y_{n-\tau_2}\right) &= -f_2\left(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}}\right) + q_{2n}, \quad \forall n \geq T_L + \tau_2,
\end{aligned} \tag{2.35}$$

that is, (x, y) is a bounded positive solution of System (1.14) in $A(N, M)$.

It follows from (2.14), (2.16), (2.26)–(2.28), (2.30), and (2.31) that

$$\begin{aligned}
&\left\| \left(x^{\mu+1}, y^{\mu+1}\right) - (x, y) \right\|_1 \\
&= \max\left\{ \sup_{n \geq T_L} \left| x_n^{\mu+1} - x_n \right|, \sup_{n \geq T_L} \left| y_n^{\mu+1} - y_n \right| \right\} \\
&\leq \max\left\{ \sup_{n \geq T_L} \left[\left(1 - \beta_\mu - \gamma_\mu\right) \left| x_n^\mu - x_n \right| \right. \right. \\
&\qquad + \beta_\mu \left| L + \sum_{l=1}^{\infty} \sum_{m=n+(2l-1)\tau_1}^{n+2l\tau_1-1} \left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m} \right] - x_n \right| \\
&\qquad \left. + \gamma_\mu \left| \delta_{1n}^\mu - x_n \right| \right],
\end{aligned}$$

$$
\begin{aligned}
&\sup_{n\ge T_L}\Bigg[(1-\beta_\mu-\gamma_\mu)\left|y_n^\mu-y_n\right| \\
&\qquad+\beta_\mu\left|L+\sum_{l=1}^{\infty}\sum_{m=n+(2l-1)\tau_2}^{n+2l\tau_2-1}\left[f_2\left(m,x_{c_{1m}}^\mu,\ldots,x_{c_{hm}}^\mu,y_{d_{1m}}^\mu,\ldots,y_{d_{km}}^\mu\right)-q_{2m}\right]-y_n\right| \\
&\qquad+\gamma_\mu\left|\delta_{2n}^\mu-y_n\right|\Bigg]\Bigg\} \\
&\le\max\Bigg\{\sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\left|x_n^\mu-x_n\right|+\beta_\mu\left|S_{1L}\left(x_n^\mu,y_n^\mu\right)-S_{1L}(x_n,y_n)\right|+2M\gamma_\mu\right], \\
&\qquad\sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\left|y_n^\mu-y_n\right|+\beta_\mu\left|S_{2L}\left(x_n^\mu,y_n^\mu\right)-S_{2L}(x_n,y_n)\right|+2M\gamma_\mu\right]\Bigg\} \\
&\le(1-\beta_\mu(1-\theta_L))\left\|(x^\mu,y^\mu)-(x,y)\right\|_1+2M\gamma_\mu,\quad\forall\mu\ge0,
\end{aligned}
\tag{2.36}
$$

which implies (2.7). Thus Lemma 1.1 and (2.7)–(2.9) mean that $\lim_{\mu\to\infty}\|(x^\mu,y^\mu)-(x,y)\|_1=0$.

(b) Let $L_1,L_2\in(N,M)$ and $L_1\ne L_2$. As in the proof of (a), we conclude that for each $i\in\Lambda_2$, there exist a constant $\theta_{L_i}\in(0,1)$, a positive integer $T_{L_i}\ge\max\{\tau_1,\tau_2\}+n_1+|\alpha|$, and mappings S_{L_i}, S_{1L_i}, S_{2L_i} satisfying (2.14) and (2.28)–(2.31), where L, θ_L, and T_L are replaced by L_i, θ_{L_i}, and T_{L_i}, respectively, and the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $(u,v)=(\{u_n\}_{n\in\mathbb{Z}_\alpha},\{v_n\}_{n\in\mathbb{Z}_\alpha})$, $(w,z)=(\{w_n\}_{n\in\mathbb{Z}_\alpha},\{z_n\}_{n\in\mathbb{Z}_\alpha})\in A(N,M)$, respectively, (u,v) and (w,z) are bounded positive solutions of the system (1.14) in $A(N,M)$. In order to show that System (1.14) possesses uncountably many bounded positive solutions in $A(N,M)$, we prove only that $(u,v)\ne(w,z)$. By virtue of (2.14) and (2.28)–(2.31), we infer that

$$
\begin{aligned}
&\|(u,v)-(w,z)\|_1 \\
&=\max\{\|u-w\|,\|v-z\|\} \\
&\ge\max\left\{\sup_{n\ge\max\{T_{L_1},T_{L_2}\}}|u_n-w_n|,\ \sup_{n\ge\max\{T_{L_1},T_{L_2}\}}|v_n-z_n|\right\} \\
&\ge\max\Bigg\{|L_1-L_2|-\sum_{l=1}^{\infty}\sum_{m=\max\{T_{L_1},T_{L_2}\}+(2l-1)\tau_1}^{\max\{T_{L_1},T_{L_2}\}+2l\tau_1-1}\big|f_1(m,u_{a_{1m}},\ldots,u_{a_{hm}},v_{b_{1m}},\ldots,v_{b_{km}}) \\
&\qquad\qquad-f_1(m,w_{a_{1m}},\ldots,w_{a_{hm}},z_{b_{1m}},\ldots,z_{b_{km}})\big|,
\end{aligned}
$$

$$
\begin{aligned}
&|L_1 - L_2| - \sum_{l=1}^{\infty} \sum_{m=\max\{T_{L_1},T_{L_2}\}+(2l-1)\tau_2}^{\max\{T_{L_1},T_{L_2}\}+2l\tau_2-1} \Big| f_2(m, u_{c_{1m}}, \ldots, u_{c_{hm}}, v_{d_{1m}}, \ldots, v_{d_{km}}) \\
&\qquad\qquad\qquad\qquad\qquad - f_2\left(m, w_{c_{1m}}, \ldots, w_{c_{hm}}, z_{d_{1m}}, \ldots, z_{d_{km}}\right)\Big| \Bigg\} \\
&\geq \max\Bigg\{ |L_1 - L_2| - \sum_{m=\max\{T_{L_1},T_{L_2}\}+\tau_1}^{\infty} t_{1m} \max\left\{ |u_{a_{im}} - w_{a_{im}}|, \left|v_{b_{jm}} - z_{b_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_K \right\}, \\
&\qquad |L_1 - L_2| - \sum_{m=\max\{T_{L_1},T_{L_2}\}+\tau_2}^{\infty} t_{2m} \max\left\{ |u_{c_{im}} - w_{c_{im}}|, \left|v_{d_{jm}} - z_{d_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_K \right\} \Bigg\} \\
&\geq \max\Bigg\{ |L_1 - L_2| - \sum_{m=\max\{T_{L_1},T_{L_2}\}+\tau_1}^{\infty} t_{1m} \max\{\|u - w\|, \|v - z\|\}, \\
&\qquad |L_1 - L_2| - \sum_{m=\max\{T_{L_1},T_{L_2}\}+\tau_2}^{\infty} t_{2m} \max\{\|u - w\|, \|v - z\|\} \Bigg\} \\
&\geq |L_1 - L_2| - \left(\sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} \max\{t_{im} : i \in \Lambda_2\} \right) \|(u, v) - (w, z)\|_1 \\
&\geq |L_1 - L_2| - \max\{\theta_{L_1}, \theta_{L_2}\} \|(u, v) - (w, z)\|_1,
\end{aligned}
\tag{2.37}
$$

which yields that

$$
\|(u, v) - (w, z)\|_1 \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_{L_1}, \theta_{L_2}\}} > 0, \tag{2.38}
$$

that is, $(u, v) \neq (w, z)$. This completes the proof. □

Theorem 2.3. *Assume that there exist constants $M, N \in \mathbb{R}^+ \setminus \{0\}$, $\overline{p}, \underline{p} \in \mathbb{R}^+$, $n_1 \in \mathbb{N}_{n_0}$, and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.24), *and*

$$
N < \left(1 - \underline{p} - \overline{p}\right) M, \quad \underline{p} + \overline{p} < 1, \quad -\underline{p} \leq p_{in} \leq \overline{p}, \quad \forall n \geq n_1, \ i \in \Lambda_2. \tag{2.39}
$$

Then,

(a) *for each* $L \in (N + \overline{p}M, M(1 - \underline{p}))$*, there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$*, the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by the schemes*

$$x_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)x_n^\mu \\ \quad +\beta_\mu\Bigg\{L - p_{1\mu}x_{n-\tau_1}^\mu \\ \qquad + \displaystyle\sum_{m=n}^{\infty}\Big[f_1\Big(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\Big) - q_{1m}\Big]\Bigg\} \\ \quad +\gamma_\mu\delta_{1n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)x_{T_L}^\mu \\ \quad +\beta_\mu\Bigg\{L - p_{1\mu}x_{T_L-\tau_1}^\mu \\ \qquad + \displaystyle\sum_{m=T_L}^{\infty}\Big[f_1\Big(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\Big) - q_{1m}\Big]\Bigg\} \\ \quad +\gamma_\mu\delta_{1T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L, \end{cases} \tag{2.40}$$

$$y_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)y_n^\mu \\ \quad +\beta_\mu\Bigg\{L - p_{2\mu}y_{n-\tau_2}^\mu \\ \qquad + \displaystyle\sum_{m=n}^{\infty}\Big[f_2\Big(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\Big) - q_{2m}\Big]\Bigg\} \\ \quad +\gamma_\mu\delta_{2n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)y_{T_L}^\mu \\ \quad +\beta_\mu\Bigg\{L - p_{2\mu}y_{T_L-\tau_2}^\mu \\ \qquad + \displaystyle\sum_{m=T_L}^{\infty}\Big[f_2\Big(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\Big) - q_{2m}\Big]\Bigg\} \\ \quad +\gamma_\mu\delta_{2T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L \end{cases} \tag{2.41}$$

converges to a bounded positive solution $(x,y) \in A(N,M)$ *of System* (1.14) *and has the error estimate* (2.7)*, where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$*, and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N,M)$ *are arbitrary sequences with* (2.8) *and* (2.9)*,*

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N,M)$*.*

Proof. (a) Let $L \in (N + \overline{p}M, M(1 - \underline{p}))$. Now we construct a contraction mapping $S_L : A(N,M) \to A(N,M)$ and prove that its fixed point is a bounded positive solution of the system (1.14). Notice that (2.24) and (2.39) mean that there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ satisfying

$$\theta_L = \underline{p} + \overline{p} + \sum_{m=T_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\}, \tag{2.42}$$

$$\sum_{m=T_L}^{\infty} \max\{r_{im} + |q_{im}| : i \in \Lambda_2\} < \min\left\{M\left(1-\underline{p}\right) - L, L - N - \overline{p}M\right\}. \tag{2.43}$$

Define three mappings $S_L : A(N,M) \to l_\alpha^\infty \times l_\alpha^\infty$, $S_{1L}, S_{2L} : A(N,M) \to l_\alpha^\infty$ by (2.14),

$$S_{1L}(x_n, y_n) = \begin{cases} L - p_{1n}x_{n-\tau_1} + \sum_{m=n}^{\infty}\left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m}\right], & n \geq T_L, \\ S_{1L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.44}$$

$$S_{2L}(x_n, y_n) = \begin{cases} L - p_{2n}y_{n-\tau_2} + \sum_{m=n}^{\infty}\left[f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}) - q_{2m}\right], & n \geq T_L, \\ S_{2L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.45}$$

for all $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$.

It follows from (2.1), (2.2), (2.14), and (2.42)–(2.45) that for any $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha})$ and $(u,v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha}, \{v_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$,

$$\begin{aligned}
&\left\|S_L(x,y) - S_L(u,v)\right\|_1 \\
&\quad = \max\left\{\left\|S_{iL}(x,y) - S_{iL}(u,v)\right\| : i \in \Lambda_2\right\} \\
&\quad = \max\left\{\sup_{n\geq T_L}\left|S_{iL}(x_n,y_n) - S_{iL}(u_n,v_n)\right| : i \in \Lambda_2\right\} \\
&\quad \leq \max\left\{\sup_{n\geq T_L}\left\{\left|p_{1n}(x_{n-\tau_1} - u_{n-\tau_1})\right| + \sum_{m=n}^{\infty}\left|f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}})\right.\right.\right. \\
&\qquad\qquad \left.\left. - f_1\ (m, u_{a_{1m}}, \ldots, u_{a_{hm}}, v_{b_{1m}}, \ldots, v_{b_{km}})\right|\right\}, \\
&\qquad \sup_{n\geq T_L}\left\{\left|p_{2n}(y_{n-\tau_2} - v_{n-\tau_2})\right| + \sum_{m=n}^{\infty}\left|f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}})\right.\right. \\
&\qquad\qquad \left.\left.\left. - f_2\ (m, u_{c_{1m}}, \ldots, u_{c_{hm}}, v_{d_{1m}}, \ldots, v_{d_{km}})\right|\right\}\right\} \\
&\quad \leq \max\left\{\left(\underline{p} + \overline{p}\right)\|x-u\| + \sum_{m=T_L}^{\infty} t_{1m}\max\left\{|x_{a_{lm}} - u_{a_{lm}}|, \left|y_{b_{jm}} - v_{b_{jm}}\right| : l \in \Lambda_h, j \in \Lambda_k\right\},\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\quad \left(\underline{p}+\overline{p}\right)\|y-v\| + \sum_{m=T_L}^{\infty} t_{2m} \max\left\{|x_{c_{lm}} - u_{c_{lm}}|, \left|y_{d_{jm}} - v_{d_{jm}}\right| : l \in \Lambda_h, j \in \Lambda_k\right\}\Bigg\} \\
&\quad \le \max\left\{\left(\underline{p}+\overline{p}\right)\|x-u\| + \sum_{m=T_L}^{\infty} t_{1m} \max\{\|x-u\|, \|y-v\|\},\right. \\
&\qquad\quad \left.\left(\underline{p}+\overline{p}\right)\|y-v\| + \sum_{m=T_L}^{\infty} t_{2m} \max\{\|x-u\|, \|y-v\|\}\right\} \\
&\quad \le \left(\underline{p}+\overline{p} + \sum_{m=T_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\}\right)\|(x,y)-(u,v)\|_1 \\
&\quad = \theta_L \|(x,y)-(u,v)\|_1, \\
&S_{1L}(x_n, y_n) \\
&\quad = L - p_{1n} x_{n-\tau_1} + \sum_{m=n}^{\infty} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m}\right] \\
&\quad \le L + \underline{p}M + \sum_{m=T_L}^{\infty} \left(r_{1m} + |q_{1m}|\right) \\
&\quad \le L + \underline{p}M + \min\left\{M\left(1-\underline{p}\right) - L, L - N - \overline{p}M\right\} \\
&\quad \le M, \quad \forall n \ge T_L, \\
&S_{1L}(x_n, y_n) \\
&\quad = L - p_{1n} x_{n-\tau_1} + \sum_{m=n}^{\infty} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m}\right] \\
&\quad \ge L - \overline{p}M - \sum_{m=T_L}^{\infty} \left(r_{1m} + |q_{1m}|\right) \\
&\quad \ge L - \overline{p}M - \min\left\{M\left(1-\underline{p}\right) - L, L - N - \overline{p}M\right\} \\
&\quad \ge N, \quad \forall n \ge T_L, \\
&N \le S_{2L}(x_n, y_n) \le M, \quad \forall n \ge T_L,
\end{aligned}
\tag{2.46}
$$

which give (2.16) and (2.17), which together with $\theta_L \in (0,1)$ guarantee that $S_L : A(N,M) \to A(N,M)$ is a contraction. Thus the Banach fixed point theorem ensures that S_L has a unique fixed point $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$, that is,

$$
\begin{aligned}
x_n &= L - p_{1n} x_{n-\tau_1} + \sum_{m=n}^{\infty} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m}\right], \\
y_n &= L - p_{2n} y_{n-\tau_2} + \sum_{m=n}^{\infty} \left[f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}) - q_{2m}\right], \quad n \ge T_L,
\end{aligned}
\tag{2.47}
$$

which yield that

$$\Delta\left(x_n + p_{1n}x_{n-\tau_1}\right) = -f_1\left(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}}\right) + q_{1n},$$
$$\Delta\left(y_n + p_{2n}y_{n-\tau_2}\right) = f_2\left(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}}\right) + q_{2n}, \quad \forall n \ge T_L, \tag{2.48}$$

that is, (x, y) is a bounded positive solution of System (1.14) in $A(N, M)$.

In light of (2.14), (2.16), (2.40), (2.41), (2.44), and (2.45), we deduce that

$$\begin{aligned}
&\left\|\left(x^{\mu+1}, y^{\mu+1}\right) - (x, y)\right\|_1 \\
&= \max\left\{\sup_{n\ge T_L}\left|x_n^{\mu+1} - x_n\right|, \sup_{n\ge T_L}\left|y_n^{\mu+1} - y_n\right|\right\} \\
&\le \max\Bigg\{\sup_{n\ge T_L}\Bigg[(1-\beta_\mu-\gamma_\mu)\left|x_n^\mu - x_n\right| \\
&\qquad + \beta_\mu\left|L - p_{1n}x_{n-\tau_1} + \sum_{m=n}^{\infty}\left[f_1\left(m, x^\mu_{a_{1m}}, \ldots, x^\mu_{a_{hm}}, y^\mu_{b_{1m}}, \ldots, y^\mu_{b_{km}}\right) - q_{1m}\right] - x_n\right| \\
&\qquad + \gamma_\mu\left|\delta^\mu_{1n} - x_n\right|\Bigg], \\
&\qquad \sup_{n\ge T_L}\Bigg[(1-\beta_\mu-\gamma_\mu)\left|y_n^\mu - y_n\right| \\
&\qquad\quad + \beta_\mu\left|L - p_{2n}y_{n-\tau_2} + \sum_{m=n}^{\infty}\left[f_2\left(m, x^\mu_{c_{1m}}, \ldots, x^\mu_{c_{hm}}, y^\mu_{d_{1m}}, \ldots, y^\mu_{d_{km}}\right) - q_{2m}\right] - y_n\right| \\
&\qquad\quad + \gamma_\mu\left|\delta^\mu_{2n} - y_n\right|\Bigg]\Bigg\} \\
&\le \max\Bigg\{\sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\|x^\mu - x\| + \beta_\mu\left|S_{1L}\left(x_n^\mu, y_n^\mu\right) - S_{1L}(x_n, y_n)\right| + 2M\gamma_\mu\right], \\
&\qquad \sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\|y^\mu - y\| + \beta_\mu\left|S_{2L}\left(x_n^\mu, y_n^\mu\right) - S_{2L}(x_n, y_n)\right| + 2M\gamma_\mu\right]\Bigg\} \\
&\le \left(1 - \beta_\mu(1-\theta_L)\right)\left\|(x^\mu, y^\mu) - (x, y)\right\|_1 + 2M\gamma_\mu, \quad \forall \mu \ge 0,
\end{aligned} \tag{2.49}$$

which implies (2.7). Thus Lemma 1.1 and (2.7)–(2.9) imply that $\lim_{\mu\to\infty}\|(x^\mu, y^\mu) - (x, y)\|_1 = 0$.

(b) Let $L_1, L_2 \in (N, M)$ and $L_1 \neq L_2$. Similar to the proof of (a), we know that for each $i \in \Lambda_2$, there exist a constant $\theta_{L_i} \in (0,1)$, a positive integer $T_{L_i} \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ and mappings S_{L_i}, S_{1L_i}, S_{2L_i} satisfying (2.14) and (2.42)–(2.45), where L, θ_L, and T_L are replaced by L_i, θ_{L_i} and T_{L_i}, respectively, and the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $(u, v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha}, \{v_n\}_{n\in\mathbb{Z}_\alpha})$, $(w, z) = (\{w_n\}_{n\in\mathbb{Z}_\alpha}, \{z_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, which are bounded positive solutions of System (1.14) in $A(N, M)$, respectively. In order to show that System (1.14) possesses uncountably many bounded positive solutions in $A(N, M)$, we prove only that $(u, v) \neq (w, z)$. In terms of (2.14) and (2.42)–(2.45), we get that

$$
\begin{aligned}
&\|(u,v)-(w,z)\|_1 \\
&= \max\{\|u-w\|, \|v-z\|\} \\
&\geq \max\left\{ \sup_{n\geq \max\{T_{L_1},T_{L_2}\}} |u_n - w_n|, \sup_{n\geq \max\{T_{L_1},T_{L_2}\}} |v_n - z_n| \right\} \\
&\geq \max\left\{ \sup_{n\geq \max\{T_{L_1},T_{L_2}\}} \left\{ |L_1 - L_2| - \left|p_{1n}\right| |u_{n-\tau_1} - w_{n-\tau_1}| \right.\right. \\
&\qquad\qquad - \sum_{m=n}^{\infty} \Big| f_1(m, u_{a_{1m}}, \ldots, u_{a_{hm}}, v_{b_{1m}}, \ldots, v_{b_{km}}) \\
&\qquad\qquad\qquad \left. - f_1(m, w_{a_{1m}}, \ldots, w_{a_{hm}}, z_{b_{1m}}, \ldots, z_{b_{km}}) \Big| \right\}, \\
&\qquad \sup_{n\geq \max\{T_{L_1},T_{L_2}\}} \left\{ |L_1 - L_2| - \left|p_{2n}\right| |v_{n-\tau_2} - z_{n-\tau_2}| \right. \\
&\qquad\qquad - \sum_{m=n}^{\infty} \Big| f_2(m, u_{c_{1m}}, \ldots, u_{c_{hm}}, v_{d_{1m}}, \ldots, v_{d_{km}}) \\
&\qquad\qquad\qquad \left.\left. - f_2(m, w_{c_{1m}}, \ldots, w_{c_{hm}}, z_{d_{1m}}, \ldots, z_{d_{km}}) \Big| \right\}\right\} \\
&\geq \max\left\{ |L_1 - L_2| - \left(\underline{p} + \overline{p}\right)\|u - w\| \right. \\
&\qquad - \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{1m} \max\left\{ |u_{a_{im}} - w_{a_{im}}|, \left|v_{b_{jm}} - z_{b_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_K \right\}, \\
&\qquad |L_1 - L_2| - \left(\underline{p} + \overline{p}\right)\|v - z\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{2m} \max\left\{ |u_{c_{im}} - w_{c_{im}}|, \left| v_{d_{jm}} - z_{d_{jm}} \right| : i \in \Lambda_h, j \in \Lambda_K \right\} \Bigg\} \\
&\geq \max\Bigg\{ |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \|u - w\| - \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{1m} \max\{\|u - w\|, \|v - z\|\}, \\
&\qquad |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \|v - z\| - \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{2m} \max\{\|u - w\|, \|v - z\|\} \Bigg\} \\
&\geq |L_1 - L_2| - \left(\underline{p} + \overline{p} + \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} \max\{t_{im} : i \in \Lambda_2\} \right) \|(u,v) - (w,z)\|_1 \\
&\geq |L_1 - L_2| - \max\{\theta_{L_1}, \theta_{L_2}\} \|(u,v) - (w,z)\|_1,
\end{aligned}
\tag{2.50}
$$

which yields that

$$
\|(u,v) - (w,z)\|_1 \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_{L_1}, \theta_{L_2}\}} > 0, \tag{2.51}
$$

that is, $(u,v) \neq (w,z)$. This completes the proof. □

Theorem 2.4. *Assume that there exist constants $M, N, \overline{p}, \underline{p} \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1)*,* (2.2)*,* (2.24)*, and*

$$
0 < N\left(\underline{p} - 1\right) < M(\overline{p} - 1), \quad -\underline{p} \leq p_{in} \leq -\overline{p}, \quad \forall n \geq n_1, \ i \in \Lambda_2. \tag{2.52}
$$

Then,

(a) *for each $L \in (N(\underline{p} - 1), M(\overline{p} - 1))$, there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ such that for any $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, the Mann iterative* *sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by the schemes*

$$x_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)x_n^\mu \\ \quad +\beta_\mu\left\{\dfrac{-L}{p_{1n+\tau_1}} - \dfrac{x_{n+\tau_1}^\mu}{p_{1n+\tau_1}} + \dfrac{1}{p_{1n+\tau_1}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=n+\tau_1}^{\infty}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1n}^\mu, & \mu\ge 0,\ n\ge T_L, \\ (1-\beta_\mu-\gamma_\mu)x_{T_L}^\mu \\ \quad +\beta_\mu\left\{\dfrac{-L}{p_{1T_L+\tau_1}} - \dfrac{x_{T_L+\tau_1}^\mu}{p_{1T_L+\tau_1}} + \dfrac{1}{p_{1T_L+\tau_1}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=T_L+\tau_1}^{\infty}\left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m}\right]\right\} \\ \quad +\gamma_\mu\delta_{1T_L}^\mu, & \mu\ge 0,\ \alpha\le n< T_L, \end{cases} \tag{2.53}$$

$$y_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)y_n^\mu \\ \quad +\beta_\mu\left\{\dfrac{-L}{p_{2n+\tau_2}} - \dfrac{y_{n+\tau_2}^\mu}{p_{2n+\tau_2}} + \dfrac{1}{p_{2n+\tau_2}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=n+\tau_2}^{\infty}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2n}^\mu, & \mu\ge 0,\ n\ge T_L, \\ (1-\beta_\mu-\gamma_\mu)y_{T_L}^\mu \\ \quad +\beta_\mu\left\{\dfrac{-L}{p_{2T_L+\tau_2}} - \dfrac{y_{T_L+\tau_2}^\mu}{p_{2T_L+\tau_2}} + \dfrac{1}{p_{2T_L+\tau_2}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=T_L+\tau_2}^{\infty}\left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m}\right]\right\} \\ \quad +\gamma_\mu\delta_{2T_L}^\mu, & \mu\ge 0,\ \alpha\le n< T_L \end{cases} \tag{2.54}$$

converges to a bounded positive solution $(x,y) \in A(N,M)$ *of System* (1.14) *and has the error estimate* (2.7)*, where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$ *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N,M)$ *are arbitrary sequences with* (2.8) *and* (2.9)*,*

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N,M)$*.*

Proof. (a) Let $L \in (N(\underline{p}-1), M(\overline{p}-1))$. Now we construct a contraction mapping $S_L : A(N,M) \to A(N,M)$ and prove that its fixed point is a bounded positive solution of the system (1.14). Observe that (2.24) and (2.52) guarantee that there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ satisfying

$$\theta_L = \frac{1}{\overline{p}}\left(1 + \sum_{m=\mathrm{T}_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\}\right), \tag{2.55}$$

$$\sum_{m=T_L}^{\infty} \max\{r_{im} + |q_{im}| : i \in \Lambda_2\} < \min\left\{M(\overline{p}-1) - L, \overline{p}\left(\frac{L+N}{\underline{p}} - N\right)\right\}. \tag{2.56}$$

Define three mappings $S_L : A(N,M) \to l_\alpha^\infty \times l_\alpha^\infty$, $S_{1L}, S_{2L} : A(N,M) \to l_\alpha^\infty$ by (2.14),

$$S_{1L}(x_n, y_n) = \begin{cases} \dfrac{-L}{p_{1n+\tau_1}} - \dfrac{x_{n+\tau_1}}{p_{1n+\tau_1}} \\ \quad + \dfrac{1}{p_{1n+\tau_1}} \displaystyle\sum_{m=n+\tau_1}^{\infty} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m}\right], & n \geq T_L, \\ S_{1L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.57}$$

$$S_{2L}(x_n, y_n) = \begin{cases} \dfrac{-L}{p_{2n+\tau_2}} - \dfrac{y_{n+\tau_2}}{p_{2n+\tau_2}} \\ \quad + \dfrac{1}{p_{2n+\tau_2}} \displaystyle\sum_{m=n+\tau_2}^{\infty} \left[f_2(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}) - q_{2m}\right], & n \geq T_L, \\ S_{2L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.58}$$

for all $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$.

It follows from (2.1), (2.2), (2.14), and (2.55)–(2.58) that for any $(x,y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha})$ and $(u,v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha}, \{v_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N,M)$

$$\begin{aligned}
&\left\|S_L(x,y) - S_L(u,v)\right\|_1 \\
&= \max\left\{\left\|S_{iL}(x,y) - S_{iL}(u,v)\right\| : i \in \Lambda_2\right\} \\
&= \max\left\{\sup_{n\ge T_L}\left|S_{iL}(x_n,y_n) - S_{iL}(u_n,v_n)\right| : i \in \Lambda_2\right\} \\
&\le \max\Bigg\{\sup_{n\ge T_L}\Bigg\{\frac{1}{\left|p_{1n+\tau_1}\right|}\Bigg(\left|x_{n+\tau_1} - u_{n+\tau_1}\right| \\
&\qquad\qquad + \sum_{m=n+\tau_1}^{\infty}\Big|f_1\left(m, x_{a_{1m}},\dots,x_{a_{hm}}, y_{b_{1m}},\dots,y_{b_{km}}\right) \\
&\qquad\qquad\qquad - f_1\left(m, u_{a_{1m}},\dots,u_{a_{hm}}, v_{b_{1m}},\dots,v_{b_{km}}\right)\Big|\Bigg)\Bigg\}, \\
&\qquad \sup_{n\ge T_L}\Bigg\{\frac{1}{\left|p_{2n+\tau_2}\right|}\Bigg(\left|y_{n+\tau_2} - v_{n+\tau_2}\right| \\
&\qquad\qquad + \sum_{m=n+\tau_2}^{\infty}\Big|f_2\left(m, x_{c_{1m}},\dots,x_{c_{hm}}, y_{d_{1m}},\dots,y_{d_{km}}\right) \\
&\qquad\qquad\qquad - f_2\left(m, u_{c_{1m}},\dots,u_{c_{hm}}, v_{d_{1m}},\dots,v_{d_{km}}\right)\Big|\Bigg)\Bigg\}\Bigg\} \\
&\le \frac{1}{p}\max\Bigg\{\|x-u\| + \sum_{m=T_L}^{\infty} t_{1m}\max\left\{\left|x_{a_{im}} - u_{a_{im}}\right|, \left|y_{b_{jm}} - v_{b_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_k\right\}, \\
&\qquad \|y-v\| + \sum_{m=T_L}^{\infty} t_{2m}\max\left\{\left|x_{c_{im}} - u_{c_{im}}\right|, \left|y_{d_{jm}} - v_{d_{jm}}\right| : i \in \Lambda_h, j \in \Lambda_k\right\}\Bigg\} \\
&\le \frac{1}{p}\max\Bigg\{\|x-u\| + \sum_{m=T_L}^{\infty} t_{1m}\max\{\|x-u\|, \|y-v\|\}, \\
&\qquad \|y-v\| + \sum_{m=T_L}^{\infty} t_{2m}\max\{\|x-u\|, \|y-v\|\}\Bigg\} \\
&\le \frac{1}{p}\left(1 + \sum_{m=T_L}^{\infty}\max\{t_{im} : i \in \Lambda_2\}\right)\left\|(x,y) - (u,v)\right\|_1 \\
&= \theta_L\left\|(x,y) - (u,v)\right\|_1,
\end{aligned}$$

$$
\begin{aligned}
&S_{1L}(x_n, y_n)\\
&\quad = \frac{-L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right]\\
&\quad \le \frac{L}{\overline{p}} + \frac{M}{\overline{p}} + \frac{1}{\overline{p}} \sum_{m=T_L+\tau_1}^{\infty} \left(r_{1m} + |q_{1m}| \right)\\
&\quad \le \frac{L}{\overline{p}} + \frac{M}{\overline{p}} + \frac{1}{\overline{p}} \min\left\{ M(\overline{p}-1) - L, \overline{p}\left(\frac{L+N}{\underline{p}} - N \right) \right\}\\
&\quad \le M, \quad \forall n \ge T_L,\\
&S_{1L}(x_n, y_n)\\
&\quad = \frac{-L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right]\\
&\quad \ge \frac{L}{\underline{p}} + \frac{N}{\underline{p}} - \frac{1}{\overline{p}} \sum_{m=T_L+\tau_1}^{\infty} \left(r_{1m} + |q_{1m}| \right)\\
&\quad \ge \frac{L}{\underline{p}} + \frac{N}{\underline{p}} - \frac{1}{\overline{p}} \min\left\{ M(\overline{p}-1) - L, \overline{p}\left(\frac{L+N}{\underline{p}} - N \right) \right\}\\
&\quad \ge N, \quad \forall n \ge T_L,\\
&N \le S_{2L}(x_n, y_n) \le M, \quad \forall n \ge T_L,
\end{aligned}
\tag{2.59}
$$

which yield (2.16) and (2.17), which together with $\theta_L \in (0,1)$ ensure that S_L is a contraction in $A(N, M)$. By the Banach fixed point theorem, we deduce that S_L has a unique fixed point $(x, y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, that is,

$$
\begin{aligned}
x_n &= \frac{-L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right], \quad n \ge T_L,\\
y_n &= \frac{-L}{p_{2n+\tau_2}} - \frac{y_{n+\tau_2}}{p_{2n+\tau_2}} + \frac{1}{p_{2n+\tau_2}} \sum_{m=n+\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m} \right], \quad n \ge T_L,
\end{aligned}
\tag{2.60}
$$

which yield that

$$
\begin{aligned}
x_n + p_{1n} x_{n-\tau_1} &= -L + \sum_{m=n}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}\right) - q_{1m} \right], \quad n \ge T_L + \tau_1,\\
y_n + p_{2n} y_{n-\tau_2} &= -L + \sum_{m=n}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}}\right) - q_{2m} \right], \quad n \ge T_L + \tau_2,
\end{aligned}
$$

$$\Delta\left(x_n + p_{1n}x_{n-\tau_1}\right) = -f_1\left(n, x_{a_{1n}}, \dots, x_{a_{hn}}, y_{b_{1n}}, \dots, y_{b_{kn}}\right) + q_{1n}, \quad \forall n \ge T_L + \tau_1,$$
$$\Delta\left(y_n + p_{2n}y_{n-\tau_2}\right) = -f_2\left(n, x_{c_{1n}}, \dots, x_{c_{hn}}, y_{d_{1n}}, \dots, y_{d_{kn}}\right) + q_{2n}, \quad \forall n \ge T_L + \tau_2, \tag{2.61}$$

that is, (x, y) is a bounded positive solution of the system (1.14) in $A(N, M)$.

In light of (2.14), (2.16), (2.53)–(2.55), (2.57), and (2.58), we get that

$$\begin{aligned}
&\left\|\left(x^{\mu+1}, y^{\mu+1}\right) - (x, y)\right\|_1 \\
&= \max\left\{\sup_{n\ge T_L}\left|x_n^{\mu+1} - x_n\right|, \sup_{n\ge T_L}\left|y_n^{\mu+1} - y_n\right|\right\} \\
&\le \max\Bigg\{\sup_{n\ge T_L}\Bigg[(1-\beta_\mu-\gamma_\mu)\left|x_n^\mu - x_n\right| + \beta_\mu \\
&\qquad\qquad \times \left|\frac{-L}{p_{1n+\tau_1}} - \frac{x^\mu_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}}\sum_{m=n}^{\infty}\left[f_1\left(m, x^\mu_{a_{1m}}, \dots, x^\mu_{a_{hm}}, y^\mu_{b_{1m}}, \dots, y^\mu_{b_{km}}\right) - q_{1m}\right] - x_n\right| \\
&\qquad\qquad + \gamma_\mu\left|\delta^\mu_{1n} - x_n\right|\Bigg], \\
&\qquad \sup_{n\ge T_L}\Bigg[(1-\beta_\mu-\gamma_\mu)\left|y_n^\mu - y_n\right| + \beta_\mu \\
&\qquad\qquad \times \left|\frac{-L}{p_{2n+\tau_2}} - \frac{y^\mu_{n+\tau_2}}{p_{2n+\tau_2}} + \frac{1}{p_{2n+\tau_2}}\sum_{m=n}^{\infty}\left[f_2\left(m, x^\mu_{c_{1m}}, \dots, x^\mu_{c_{hm}}, y^\mu_{d_{1m}}, \dots, y^\mu_{d_{km}}\right) - q_{2m}\right] - y_n\right| \\
&\qquad\qquad + \gamma_\mu\left|\delta^\mu_{2n} - y_n\right|\Bigg]\Bigg\} \\
&\le \max\Bigg\{\sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\|x^\mu - x\| + \beta_\mu\left|S_{1L}\left(x_n^\mu, y_n^\mu\right) - S_{1L}(x_n, y_n)\right| + 2M\gamma_\mu\right], \\
&\qquad \sup_{n\ge T_L}\left[(1-\beta_\mu-\gamma_\mu)\|y^\mu - y\| + \beta_\mu\left|S_{2L}\left(x_n^\mu, y_n^\mu\right) - S_{2L}(x_n, y_n)\right| + 2M\gamma_\mu\right]\Bigg\} \\
&\le (1-\beta_\mu(1-\theta_L))\left\|(x^\mu, y^\mu) - (x, y)\right\|_1 + 2M\gamma_\mu, \quad \forall \mu \ge 0,
\end{aligned} \tag{2.62}$$

which implies (2.7). Thus Lemma 1.1 and (2.7)–(2.9) imply that $\lim_{\mu\to\infty}\|(x^\mu, y^\mu) - (x, y)\|_1 = 0$.

(b) Let $L_1, L_2 \in (N, M)$ and $L_1 \ne L_2$. Similar to the proof of (a), we conclude that for each $i \in \Lambda_2$, there exist a constant $\theta_{L_i} \in (0, 1)$, a positive integer $T_{L_i} \ge \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$, and mappings S_{L_i}, S_{1L_i}, S_{2L_i} satisfying (2.14) and (2.55)–(2.58), where L, θ_L, and T_L are replaced by L_i, θ_{L_i}, and T_{L_i}, respectively, and the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $(u, v) = (\{u_n\}_{n\in\mathbb{Z}_\alpha}, \{v_n\}_{n\in\mathbb{Z}_\alpha})$, $(w, z) = (\{w_n\}_{n\in\mathbb{Z}_\alpha}, \{z_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, which are bounded positive solutions of System (1.14) in $A(N,M)$, respectively. In order to show that System (1.14) possesses uncountably many bounded positive solutions in $A(N,M)$, we prove only that $(u,v)\neq(w,z)$. By virtue of (2.14) and (2.55)–(2.58), we get that

$$\begin{aligned}
&\|(u,v)-(w,z)\|_1\\
&=\max\{\|u-w\|,\|v-z\|\}\\
&\geq \max\left\{\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|u_n-w_n|,\ \sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|v_n-z_n|\right\}\\
&\geq \max\left\{\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}\left\{\frac{|L_1-L_2|}{|p_{1n+\tau_1}|}-\frac{|u_{n+\tau_1}-w_{n+\tau_1}|}{|p_{1n+\tau_1}|}\right.\right.\\
&\qquad\qquad -\frac{1}{|p_{1n+\tau_1}|}\sum_{m=n+\tau_1}^{\infty}\left|f_1(m,u_{a_{1m}},\ldots,u_{a_{hm}},v_{b_{1m}},\ldots,v_{b_{km}})\right.\\
&\qquad\qquad\qquad \left.\left. -f_1\left(m,w_{a_{1m}},\ldots,w_{a_{hm}},z_{b_{1m}},\ldots,z_{b_{km}}\right)\right|\right\},\\
&\qquad \sup_{n\geq\max\{T_{L_1},T_{L_2}\}}\left\{\frac{|L_1-L_2|}{|p_{2n+\tau_2}|}-\frac{|u_{n+\tau_2}-w_{n+\tau_2}|}{|p_{2n+\tau_2}|}-\frac{1}{|p_{2n+\tau_2}|}\right.\\
&\qquad\qquad \times\sum_{m=n+\tau_2}^{\infty}\left|f_2(m,u_{c_{1m}},\ldots,u_{c_{hm}},v_{d_{1m}},\ldots,v_{d_{km}})\right.\\
&\qquad\qquad\qquad \left.\left.\left. -f_2\left(m,w_{c_{1m}},\ldots,w_{c_{hm}},z_{d_{1m}},\ldots,z_{d_{km}}\right)\right|\right\}\right\}\\
&\geq \max\left\{\frac{|L_1-L_2|}{\underline{p}}-\frac{\|u-w\|}{\overline{p}}\right.\\
&\qquad -\frac{1}{\overline{p}}\sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty}t_{1m}\max\left\{|u_{a_{im}}-w_{a_{im}}|,\left|v_{b_{jm}}-z_{b_{jm}}\right|: i\in\Lambda_h, j\in\Lambda_K\right\},\\
&\qquad \frac{|L_1-L_2|}{\underline{p}}-\frac{\|v-z\|}{\overline{p}}\\
&\qquad \left.-\frac{1}{\overline{p}}\sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty}t_{2m}\max\left\{|u_{c_{im}}-w_{c_{im}}|,\left|v_{d_{jm}}-z_{d_{jm}}\right|: i\in\Lambda_h, j\in\Lambda_K\right\}\right\}\\
&\geq \max\left\{\frac{|L_1-L_2|}{\underline{p}}-\frac{\|u-w\|}{\overline{p}}-\frac{1}{\overline{p}}\sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty}t_{1m}\max\{\|u-w\|,\|v-z\|\},\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad \frac{|L_1 - L_2|}{\underline{p}} - \frac{\|v - z\|}{\overline{p}} - \frac{1}{\overline{p}} \sum_{m=\max\{T_{L_1}, T_{L_2}\}}^{\infty} t_{2m} \max\{\|u - w\|, \|v - z\|\} \Bigg\} \\
&\geq \frac{|L_1 - L_2|}{\underline{p}} - \frac{1}{\overline{p}} (1 + \max\{\theta_{L_1}, \theta_{L_2}\}) \|(u, v) - (w, z)\|_1,
\end{aligned}
\tag{2.63}
$$

which yields that

$$
\|(u, v) - (w, z)\|_1 \geq \frac{\overline{p}|L_1 - L_2|}{\underline{p}(\overline{p} + \max\{\theta_{L_1}, \theta_{L_2}\})} > 0, \tag{2.64}
$$

that is, $(u, v) \neq (w, z)$. This completes the proof. □

Theorem 2.5. *Assume that there exist constants $M, N, \overline{p}, \underline{p} \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ and four nonnegative sequences $\{r_{1n}\}_{n \in \mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n \in \mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n \in \mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n \in \mathbb{N}_{n_0}}$ satisfying* (2.1)*,* (2.2)*,* (2.24)*, and*

$$
\frac{N\left(\overline{p}^2 - \underline{p}\right)}{\overline{p}} < \frac{M\left(\underline{p}^2 - \overline{p}\right)}{\underline{p}}, \quad 1 < \underline{p} \leq p_{in} \leq \overline{p} < \underline{p}^2, \quad \forall n \geq n_1, \ i \in \Lambda_2. \tag{2.65}
$$

Then,

(a) *for each $L \in (\overline{p}(M/\underline{p} + N), \underline{p}(N/\overline{p} + M))$, there exist $\theta_L \in (0, 1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ such that for any $(x^0, y^0) = (\{x_n^0\}_{n \in \mathbb{Z}_\alpha}, \{y_n^0\}_{n \in \mathbb{Z}_\alpha}) \in A(N, M)$, the Mann iterative sequence with errors $\{(x^\mu, y^\mu)\}_{\mu \in \mathbb{N}_0} = \{(\{x_n^\mu\}_{n \in \mathbb{Z}_\alpha}, \{y_n^\mu\}_{n \in \mathbb{Z}_\alpha})\}_{\mu \in \mathbb{N}_0}$ generated by the schemes*

$$
x_n^{\mu+1} = \begin{cases}
(1 - \beta_\mu - \gamma_\mu) x_n^\mu \\
\quad + \beta_\mu \Bigg\{ \dfrac{L}{p_{1n+\tau_1}} - \dfrac{x_{n+\tau_1}^\mu}{p_{1n+\tau_1}} + \dfrac{1}{p_{1n+\tau_1}} \\
\qquad \times \displaystyle\sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m} \right] \Bigg\} \\
\quad + \gamma_\mu \delta_{1n}^\mu, & \mu \geq 0, \ n \geq T_L, \\
(1 - \beta_\mu - \gamma_\mu) x_{T_L}^\mu \\
\quad + \beta_\mu \Bigg\{ \dfrac{L}{p_{1T_L+\tau_1}} - \dfrac{x_{T_L+\tau_1}^\mu}{p_{1T_L+\tau_1}} + \dfrac{1}{p_{1T_L+\tau_1}} \\
\qquad \times \displaystyle\sum_{m=T_L+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu\right) - q_{1m} \right] \Bigg\} \\
\quad + \gamma_\mu \delta_{1T_L}^\mu, & \mu \geq 0, \ \alpha \leq n < T_L,
\end{cases}
\tag{2.66}
$$

$$y_n^{\mu+1} = \begin{cases} (1-\beta_\mu-\gamma_\mu)y_n^\mu \\ \quad +\beta_\mu \left\{ \dfrac{L}{p_{2n+\tau_2}} - \dfrac{y_{n+\tau_2}^\mu}{p_{2n+\tau_2}} + \dfrac{1}{p_{2n+\tau_2}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=n+\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m} \right] \right\} \\ \quad +\gamma_\mu \delta_{2n}^\mu, & \mu \geq 0,\ n \geq T_L, \\ (1-\beta_\mu-\gamma_\mu)y_{T_L}^\mu \\ \quad +\beta_\mu \left\{ \dfrac{L}{p_{2T_L+\tau_2}} - \dfrac{y_{T_L+\tau_2}^\mu}{p_{2T_L+\tau_2}} + \dfrac{1}{p_{2T_L+\tau_2}} \right. \\ \qquad \left. \times \displaystyle\sum_{m=T_L+\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu\right) - q_{2m} \right] \right\} \\ \quad +\gamma_\mu \delta_{2T_L}^\mu, & \mu \geq 0,\ \alpha \leq n < T_L \end{cases} \tag{2.67}$$

converges to a bounded positive solution $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$.

Proof. (a) Let $L \in (\overline{p}(M/\underline{p} + N), \underline{p}(N/\overline{p} + M))$. Now we construct a contraction mapping $S_L : A(N, M) \to A(N, M)$ and prove that its fixed point is a bounded positive solution of System (1.14). Observe that (2.24) and (2.65) imply that there exist $\theta_L \in (0, 1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ satisfying

$$\theta_L = \frac{1}{\underline{p}} \left(1 + \sum_{m=T_L}^{\infty} \max\{t_{im} : i \in \Lambda_2\} \right), \tag{2.68}$$

$$\sum_{m=T_L}^{\infty} \max\{r_{im} + |q_{im}| : i \in \Lambda_2\} < \min\left\{ \frac{N\underline{p}}{\overline{p}} + M\underline{p} - L, \frac{L\underline{p}}{\overline{p}} - M - N\underline{p} \right\}. \tag{2.69}$$

Define three mappings $S_L : A(N, M) \to l_\alpha^\infty \times l_\alpha^\infty$, $S_{1L}, S_{2L} : A(N, M) \to l_\alpha^\infty$ by (2.14),

$$S_{1L}(x_n, y_n) = \begin{cases} \dfrac{L}{p_{1n+\tau_1}} - \dfrac{x_{n+\tau_1}}{p_{1n+\tau_1}} \\ \quad + \dfrac{1}{p_{1n+\tau_1}} \displaystyle\sum_{m=n+\tau_1}^{\infty} \left[f_1(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}}) - q_{1m} \right], & n \geq T_L, \\ S_{1L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.70}$$

$$S_{2L}(x_n, y_n) = \begin{cases} \dfrac{L}{p_{2n+\tau_2}} - \dfrac{y_{n+\tau_2}}{p_{2n+\tau_2}} \\ \quad + \dfrac{1}{p_{2n+\tau_2}} \displaystyle\sum_{m=n+\tau_2}^{\infty} \left[f_2(m, x_{c_{1m}}, \dots, x_{c_{hm}}, y_{d_{1m}}, \dots, y_{d_{km}}) - q_{2m} \right], & n \geq T_L, \\ S_{2L}(x_{T_L}, y_{T_L}), & \alpha \leq n < T_L, \end{cases} \tag{2.71}$$

for all $(x, y) = (\{x_n\}_{n \in \mathbb{Z}_\alpha}, \{y_n\}_{n \in \mathbb{Z}_\alpha}) \in A(N, M)$.

Using (2.1), (2.2), (2.14), (2.65), and (2.68)–(2.71), we deduce that for any $(x, y) = (\{x_n\}_{n \in \mathbb{Z}_\alpha}, \{y_n\}_{n \in \mathbb{Z}_\alpha})$ and $(u, v) = (\{u_n\}_{n \in \mathbb{Z}_\alpha}, \{v_n\}_{n \in \mathbb{Z}_\alpha}) \in A(N, M)$,

$$\begin{aligned}
&\left\| S_L(x, y) - S_L(u, v) \right\|_1 \\
&= \max\left\{ \left\| S_{iL}(x, y) - S_{iL}(u, v) \right\| : i \in \Lambda_2 \right\} \\
&= \max\left\{ \sup_{n \geq T_L} \left| S_{iL}(x_n, y_n) - S_{iL}(u_n, v_n) \right| : i \in \Lambda_2 \right\} \\
&\leq \max\left\{ \sup_{n \geq T_L} \left\{ \frac{1}{|p_{1n+\tau_1}|} \left(|x_{n+\tau_1} - u_{n+\tau_1}| + \sum_{m=n+\tau_1}^{\infty} \left| f_1(m, x_{a_{1m}}, \dots, x_{a_{hm}}, y_{b_{1m}}, \dots, y_{b_{km}}) \right. \right. \right. \right. \\
&\qquad\qquad \left. \left. \left. - f_1\ (m, u_{a_{1m}}, \dots, u_{a_{hm}}, v_{b_{1m}}, \dots, v_{b_{km}}) \right| \right) \right\}, \\
&\qquad \sup_{n \geq T_L} \left\{ \frac{1}{|p_{2n+\tau_2}|} \left(\left| y_{n+\tau_2} - v_{n+\tau_2} \right| \right. \right. \\
&\qquad\qquad + \sum_{m=n+\tau_2}^{\infty} \left| f_2(m, x_{c_{1m}}, \dots, x_{c_{hm}}, y_{d_{1m}}, \dots, y_{d_{km}}) \right. \\
&\qquad\qquad \left. \left. \left. \left. - f_2\ (m, u_{c_{1m}}, \dots, u_{c_{hm}}, v_{d_{1m}}, \dots, v_{d_{km}}) \right| \right) \right\} \right\} \\
&\leq \frac{1}{\underline{p}} \max\left\{ \|x - u\| + \sum_{m=T_L}^{\infty} t_{1m} \max\left\{ |x_{a_{im}} - u_{a_{im}}|, \left| y_{b_{jm}} - v_{b_{jm}} \right| : i \in \Lambda_h, j \in \Lambda_k \right\}, \right. \\
&\qquad \left. \|y - v\| + \sum_{m=T_L}^{\infty} t_{2m} \max\left\{ |x_{c_{im}} - u_{c_{im}}|, \left| y_{d_{jm}} - v_{d_{jm}} \right| : i \in \Lambda_h, j \in \Lambda_k \right\} \right\} \\
&\leq \frac{1}{\underline{p}} \max\left\{ \|x - u\| + \sum_{m=T_L}^{\infty} t_{1m} \max\{ \|x - u\|, \|y - v\| \}, \|y - v\| + \sum_{m=T_L}^{\infty} t_{2m} \max\{ \|x - u\|, \|y - v\| \} \right\} \\
&\leq \frac{1}{\underline{p}} \left(1 + \sum_{m=T_L}^{\infty} \max\{ t_{1m}, t_{2m} \} \right) \left\| (x, y) - (u, v) \right\|_1 \\
&= \theta_L \left\| (x, y) - (u, v) \right\|_1,
\end{aligned}$$

$$
\begin{aligned}
&S_{1L}(x_n, y_n)\\
&\quad = \frac{L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \dots, x_{a_{hm}}, y_{b_{1m}}, \dots, y_{b_{km}}\right) - q_{1m}\right]\\
&\quad \le \frac{L}{\underline{p}} - \frac{N}{\overline{p}} + \frac{1}{\underline{p}} \sum_{m=T_L+\tau_1}^{\infty} \left(r_{1m} + |q_{1m}|\right)\\
&\quad \le \frac{L}{\underline{p}} - \frac{N}{\overline{p}} + \frac{1}{\underline{p}} \min\left\{ \frac{N\underline{p}}{\overline{p}} + M\underline{p} - L, \frac{L\underline{p}}{\overline{p}} - M - N\underline{p} \right\}\\
&\quad \le M, \quad \forall n \ge T_L,
\end{aligned}
$$

$$
\begin{aligned}
&S_{1L}(x_n, y_n)\\
&\quad = \frac{L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \dots, x_{a_{hm}}, y_{b_{1m}}, \dots, y_{b_{km}}\right) - q_{1m}\right]\\
&\quad \ge \frac{L}{\overline{p}} - \frac{M}{\underline{p}} - \frac{1}{\underline{p}} \sum_{m=T_L+\tau_1}^{\infty} \left(r_{1m} + |q_{1m}|\right)\\
&\quad \ge \frac{L}{\overline{p}} - \frac{M}{\underline{p}} - \frac{1}{\underline{p}} \min\left\{ \frac{N\underline{p}}{\overline{p}} + M\underline{p} - L, \frac{L\underline{p}}{\overline{p}} - M - N\underline{p} \right\}\\
&\quad \ge N, \quad \forall n \ge T_L,\\
&N \le S_{2L}(x_n, y_n) \le M, \quad \forall n \ge T_L,
\end{aligned} \tag{2.72}
$$

which yield (2.16) and (2.17), and which guarantee that S_L is a self-mapping from $A(N, M)$ into itself and is a contraction by $\theta_L \in (0,1)$. Thus the Banach fixed point theorem means that S_L has a unique fixed point $(x, y) = (\{x_n\}_{n\in\mathbb{Z}_\alpha}, \{y_n\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, that is,

$$
\begin{aligned}
x_n &= \frac{L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n+\tau_1}^{\infty} \left[f_1\left(m, x_{a_{1m}}, \dots, x_{a_{hm}}, y_{b_{1m}}, \dots, y_{b_{km}}\right) - q_{1m}\right], \quad n \ge T_L,\\
y_n &= \frac{L}{p_{2n+\tau_2}} - \frac{y_{n+\tau_2}}{p_{2n+\tau_2}} + \frac{1}{p_{2n+\tau_2}} \sum_{m=n+\tau_2}^{\infty} \left[f_2\left(m, x_{c_{1m}}, \dots, x_{c_{hm}}, y_{d_{1m}}, \dots, y_{d_{km}}\right) - q_{2m}\right], \quad n \ge T_L,
\end{aligned} \tag{2.73}
$$

which yield that

$$
\begin{gathered}
x_n + p_{1n} x_{n-\tau_1} = L + \sum_{m=n}^{\infty} \left[f_1 \left(m, x_{a_{1m}}, \ldots, x_{a_{hm}}, y_{b_{1m}}, \ldots, y_{b_{km}} \right) - q_{1m} \right], \quad n \geq T_L + \tau_1, \\
y_n + p_{2n} y_{n-\tau_2} = L + \sum_{m=n}^{\infty} \left[f_2 \left(m, x_{c_{1m}}, \ldots, x_{c_{hm}}, y_{d_{1m}}, \ldots, y_{d_{km}} \right) - q_{2m} \right], \quad n \geq T_L + \tau_2, \\
\Delta \left(x_n + p_{1n} x_{n-\tau_1} \right) = -f_1 \left(n, x_{a_{1n}}, \ldots, x_{a_{hn}}, y_{b_{1n}}, \ldots, y_{b_{kn}} \right) + q_{1n}, \quad \forall n \geq T_L + \tau_1, \\
\Delta \left(y_n + p_{2n} y_{n-\tau_2} \right) = -f_2 \left(n, x_{c_{1n}}, \ldots, x_{c_{hn}}, y_{d_{1n}}, \ldots, y_{d_{kn}} \right) + q_{2n}, \quad \forall n \geq T_L + \tau_2,
\end{gathered} \tag{2.74}
$$

that is, (x, y) is a bounded positive solution of the system (1.14) in $A(N, M)$.

In view of (2.14), (2.16), (2.65)–(2.67), (2.70), and (2.71), we gain that

$$
\begin{aligned}
&\left\| \left(x^{\mu+1}, y^{\mu+1} \right) - (x, y) \right\|_1 \\
&= \max \left\{ \sup_{n \geq T_L} \left| x_n^{\mu+1} - x_n \right|, \sup_{n \geq T_L} \left| y_n^{\mu+1} - y_n \right| \right\} \\
&\leq \max \left\{ \sup_{n \geq T_L} \left[(1 - \beta_\mu - \gamma_\mu) \left| x_n^\mu - x_n \right| + \beta_\mu \right. \right. \\
&\qquad \times \left| \frac{L}{p_{1n+\tau_1}} - \frac{x_{n+\tau_1}^\mu}{p_{1n+\tau_1}} + \frac{1}{p_{1n+\tau_1}} \sum_{m=n}^{\infty} \left[f_1 \left(m, x_{a_{1m}}^\mu, \ldots, x_{a_{hm}}^\mu, y_{b_{1m}}^\mu, \ldots, y_{b_{km}}^\mu \right) - q_{1m} \right] - x_n \right| \\
&\qquad \left. + \gamma_\mu \left| \delta_{1n}^\mu - x_n \right| \right], \\
&\qquad \sup_{n \geq T_L} \left[(1 - \beta_\mu - \gamma_\mu) \left| y_n^\mu - y_n \right| + \beta_\mu \right. \\
&\qquad \times \left| \frac{L}{p_{2n+\tau_2}} - \frac{y_{n+\tau_2}^\mu}{p_{2n+\tau_2}} + \frac{1}{p_{2n+\tau_2}} \sum_{m=n}^{\infty} \left[f_2 \left(m, x_{c_{1m}}^\mu, \ldots, x_{c_{hm}}^\mu, y_{d_{1m}}^\mu, \ldots, y_{d_{km}}^\mu \right) - q_{2m} \right] - y_n \right| \\
&\qquad \left. \left. + \gamma_\mu \left| \delta_{2n}^\mu - y_n \right| \right] \right\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \max\left\{\sup_{n\geq T_L}\left[(1-\beta_\mu-\gamma_\mu)\|x^\mu-x\|+\beta_\mu\left|S_{1L}\left(x_n^\mu,y_n^\mu\right)-S_{1L}(x_n,y_n)\right|+2M\gamma_\mu\right],\right.\\
&\qquad\left.\sup_{n\geq T_L}\left[(1-\beta_\mu-\gamma_\mu)\|y^\mu-y\|+\beta_\mu\left|S_{2L}\left(x_n^\mu,y_n^\mu\right)-S_{2L}(x_n,y_n)\right|+2M\gamma_\mu\right]\right\}\\
&\leq (1-\beta_\mu(1-\theta_L))\left\|(x^\mu,y^\mu)-(x,y)\right\|_1+2M\gamma_\mu,\quad \forall\mu\geq 0,
\end{aligned}\tag{2.75}$$

which implies (2.7). Thus, Lemma 1.1 and (2.7)–(2.9) imply that $\lim_{\mu\to\infty}\|(x^\mu,y^\mu)-(x,y)\|_1=0$.

(b) Let $L_1,L_2\in(N,M)$ and $L_1\neq L_2$. Similar to the proof of (a), we conclude that for each $i\in\Lambda_2$, there exist a constant $\theta_{L_i}\in(0,1)$, a positive integer $T_{L_i}\geq\max\{\tau_1,\tau_2\}+n_1+|\alpha|$, and mappings S_{L_i}, S_{1L_i}, S_{2L_i} satisfying (2.14) and (2.68)–(2.71), where L, θ_L, and T_L are replaced by L_i, θ_{L_i}, and T_{L_i}, respectively, and the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $(u,v)=(\{u_n\}_{n\in\mathbb{Z}_\alpha},\{v_n\}_{n\in\mathbb{Z}_\alpha})$, $(w,z)=(\{x_n\}_{n\in\mathbb{Z}_\alpha},\{y_n\}_{n\in\mathbb{Z}_\alpha})\in A(N,M)$, which are bounded positive solutions of System (1.14) in $A(N,M)$, respectively. In order to show that System (1.14) possesses uncountably many bounded positive solutions in $A(N,M)$, we prove only that $(u,v)\neq(w,z)$. In light of (2.14) and (2.68)–(2.71), we have

$$\begin{aligned}
&\|(u,v)-(w,z)\|_1\\
&=\max\{\|u-w\|,\|v-z\|\}\\
&\geq\max\left\{\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|u_n-w_n|,\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}|v_n-z_n|\right\}\\
&\geq\max\left\{\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}\left\{\frac{|L_1-L_2|}{|p_{1n+\tau_1}|}-\frac{|u_{n+\tau_1}-w_{n+\tau_1}|}{|p_{1n+\tau_1}|}\right.\right.\\
&\qquad-\frac{1}{|p_{1n+\tau_1}|}\sum_{m=n+\tau_1}^{\infty}\left|f_1(m,u_{a_{1m}},\dots,u_{a_{hm}},v_{b_{1m}},\dots,v_{b_{km}})\right.\\
&\qquad\left.\left.-f_1\left(m,w_{a_{1m}},\dots,w_{a_{hm}},z_{b_{1m}},\dots,z_{b_{km}}\right)\right|\right\},\\
&\qquad\sup_{n\geq\max\{T_{L_1},T_{L_2}\}}\left\{\frac{|L_1-L_2|}{|p_{2n+\tau_2}|}-\frac{|u_{n+\tau_2}-w_{n+\tau_2}|}{|p_{2n+\tau_2}|}-\frac{1}{|p_{2n+\tau_2}|}\right.\\
&\qquad\times\sum_{m=n+\tau_2}^{\infty}\left|f_2(m,u_{c_{1m}},\dots,u_{c_{hm}},v_{d_{1m}},\dots,v_{d_{km}})\right.\\
&\qquad\left.\left.\left.-f_2\left(m,w_{c_{1m}},\dots,w_{c_{hm}},z_{d_{1m}},\dots,z_{d_{km}}\right)\right|\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \max\Bigg\{ \frac{|L_1 - L_2|}{\overline{p}} - \frac{\|u-w\|}{\underline{p}} \\
&\qquad - \frac{1}{\underline{p}} \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{1m} \max\Big\{ |u_{a_{lm}} - w_{a_{lm}}|, \Big|v_{b_{jm}} - z_{b_{jm}}\Big| : l \in \Lambda_h, j \in \Lambda_K \Big\}, \\
&\qquad \frac{|L_1 - L_2|}{\overline{p}} - \frac{\|v-z\|}{\underline{p}} \\
&\qquad - \frac{1}{\underline{p}} \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{2m} \max\Big\{ |u_{c_{lm}} - w_{c_{lm}}|, \Big|v_{d_{jm}} - z_{d_{jm}}\Big| : l \in \Lambda_h, j \in \Lambda_K \Big\} \Bigg\} \\
&\geq \max\Bigg\{ \frac{|L_1 - L_2|}{\overline{p}} - \frac{\|u-w\|}{\underline{p}} - \frac{1}{\underline{p}} \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{1m} \max\{\|u-w\|, \|v-z\|\}, \\
&\qquad \frac{|L_1 - L_2|}{\overline{p}} - \frac{\|v-z\|}{\underline{p}} - \frac{1}{\underline{p}} \sum_{m=\max\{T_{L_1},T_{L_2}\}}^{\infty} t_{2m} \max\{\|u-w\|, \|v-z\|\} \Bigg\} \\
&\geq \frac{|L_1 - L_2|}{\overline{p}} - \frac{1}{\underline{p}} (1 + \max\{\theta_{L_1}, \theta_{L_2}\}) \|(u,v) - (w,z)\|_1,
\end{aligned}
\tag{2.76}
$$

which yields that

$$
\|(u,v) - (w,z)\|_1 \geq \frac{\underline{p}|L_1 - L_2|}{\overline{p}\left(\underline{p} + \max\{\theta_{L_1}, \theta_{L_2}\}\right)} > 0, \tag{2.77}
$$

that is, $(u,v) \neq (w,z)$. This completes the proof. □

Remark 2.6. Let (a_1, b_1) and (a_2, b_2) be two arbitrary intervals in $\mathbb{R}$. It is easy to see that

$$
(a_1, b_1) \cap (a_2, b_2) \neq \emptyset \Longleftrightarrow \max\{a_1, a_2\} < \min\{b_1, b_2\}. \tag{2.78}
$$

From Remark 2.6 and Theorems 2.1–2.5, we can obtain the following.

Theorem 2.7. *Assume that there exist constants $M, N \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2),

$$\sum_{n=n_0}^{\infty} n \max\{r_{1n}, t_{1n}, |q_{1n}|\} < +\infty, \tag{2.79}$$

$$\sum_{n=n_0}^{\infty} \max\{r_{2n}, t_{2n}, |q_{2n}|\} < +\infty, \tag{2.80}$$

$$N < M, \qquad p_{1n} = -1, \quad \forall n \geq n_1, \tag{2.81}$$

$$p_{2n} = 1, \quad \forall n \geq n_1. \tag{2.82}$$

Then,

(a) *for each $L \in (N, M)$, there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ such that for any $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, the Mann iterative sequence with errors $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ generated by Schemes* (2.5) *and* (2.27) *converges to a bounded positive solution $(x, y) \in A(N, M)$ of System* (1.14) *and has the error estimate* (2.7), *where $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$ and $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in $A(N, M)$.*

Theorem 2.8. *Assume that there exist constants $M, N \in \mathbb{R}^+ \setminus \{0\}$, $\overline{p}, \underline{p} \in \mathbb{R}^+$, $n_1 \in \mathbb{N}_{n_0}$, and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1), (2.2), (2.79)–(2.81),

$$N < \left(1 - \underline{p} - \overline{p}\right) M, \quad \underline{p} + \overline{p} < 1, \quad -\underline{p} \leq p_{2n} \leq \overline{p}, \quad \forall n \geq n_1. \tag{2.83}$$

Then,

(a) *for each $L \in (N, M) \cap (N + \overline{p}M, M(1 - \underline{p}))$, there exist $\theta_L \in (0,1)$ and $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ such that for any $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, the Mann iterative sequence with errors $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ generated by Schemes* (2.5) *and* (2.41) *converges to a bounded positive solution $(x, y) \in A(N, M)$ of System* (1.14) *and has the error estimate* (2.7), *where $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$ and $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in $A(N, M)$.*

Theorem 2.9. *Assume that there exist constants $M, N, \overline{p}, \underline{p} \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ and four nonnegative sequences $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$ and $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$, satisfying* (2.1), (2.2), (2.79)–(2.81),

$$0 < N\left(\underline{p} - 1\right) < M(\overline{p} - 1), \quad -\underline{p} \leq p_{2n} \leq -\overline{p}, \quad \forall n \geq n_1, \tag{2.84}$$

$$\max\left\{N, N\left(\underline{p} - 1\right)\right\} < \min\{M, M(\overline{p} - 1)\}. \tag{2.85}$$

Then,

(a) *for each* $L \in (N, M) \cap (N(\underline{p}-1), M(\overline{p}-1))$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.5) *and* (2.54) *converges to a bounded positive solution* $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$ *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$.

Theorem 2.10. *Assume that there exist constants* $M, N, M_0, N_0, \overline{p}_0, \underline{p}_0 \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2) *with* $i = 1$, (2.79)–(2.81),

$$\left|f_2(n, u_1, \ldots, u_h, v_1, \ldots, v_k)\right| \leq r_{2n}, \quad \forall (n, u_l, v_j) \in \mathbb{N}_{n_1} \times [N_0, M_0]^2,\ l \in \Lambda_h,\ j \in \Lambda_k, \tag{2.86}$$

$$\begin{gathered}\left|f_2(n, u_1, \ldots, u_h, v_1, \ldots, v_k) - f_2(n, w_1, \ldots, w_h, z_1, \ldots, z_k)\right| \\ \leq t_{2n} \max\left\{|u_l - w_l|, \left|v_j - z_j\right| : l \in \Lambda_h, j \in \Lambda_k\right\}, \\ \forall (n, u_l, w_l, v_j, z_j) \in \mathbb{N}_{n_1} \times [N_0, M_0]^4,\ l \in \Lambda_h,\ j \in \Lambda_k,\end{gathered} \tag{2.87}$$

$$\frac{N_0\left(\overline{p}_0^2 - \underline{p}_0\right)}{\overline{p}_0} < \frac{M_0\left(\underline{p}_0^2 - \overline{p}_0\right)}{\underline{p}_0}, \quad 1 < \underline{p}_0 \leq p_{2n} \leq \overline{p}_0 < \underline{p}_0^2, \quad \forall n \geq n_1, \tag{2.88}$$

$$\max\left\{N, \overline{p}_0\left(\frac{M_0}{\underline{p}_0} + N_0\right)\right\} < \min\left\{M, \underline{p}_0\left(\frac{N_0}{\overline{p}_0} + M_0\right)\right\}. \tag{2.89}$$

Then,

(a) *for each* $L \in (N, M) \cap (\overline{p}_0(M_0/\underline{p}_0 + N_0), \underline{p}_0(N_0/\overline{p}_0 + M_0))$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M, N_0, M_0)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.5) *and* (2.67) *converges to a bounded positive solution* $(x, y) \in A(N, M, N_0, M_0)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M, N_0, M_0)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *the system* (1.14) *has uncountably many bounded positive solutions in* $A(N, M, N_0, M_0)$.

Theorem 2.11. *Assume that there exist constants* $M, N \in \mathbb{R}^+ \setminus \{0\}$, $\overline{p}, \underline{p} \in \mathbb{R}^+$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2), (2.24), (2.83) *and*

$$N < M, \qquad p_{1n} = 1, \quad \forall n \geq n_1. \tag{2.90}$$

Then,

(a) *for each* $L \in (N, M) \cap (N + \overline{p}M, M(1 - \underline{p}))$, *there exist* $\theta_L \in (0, 1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.26) *and* (2.41) *converges to a bounded positive solution* $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0, 1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$.

Theorem 2.12. *Assume that there exist constants* $M, N, \overline{p}, \underline{p} \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2), (2.24), (2.84), (2.85), *and* (2.90). *Then,*

(a) *for each* $L \in (N, M) \cap (N(\underline{p} - 1), M(\overline{p} - 1))$, *there exist* $\theta_L \in (0, 1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.26) *and* (2.54) *converges to a bounded positive solution* $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0, 1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$.

Theorem 2.13. *Assume that there exist* $M, N, M_0, N_0, \overline{p}_0, \underline{p}_0 \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2) *with* $i = 1$, (2.24), *and* (2.86)–(2.90). *Then,*

(a) *for each* $L \in (N, M) \cap (\overline{p}_0(M_0/\underline{p}_0 + N_0), \underline{p}_0(N_0/\overline{p}_0 + M_0))$, *there exist* $\theta_L \in (0, 1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M, N_0, M_0)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.26) *and* (2.67) *converges to a bounded positive solution* $(x, y) \in A(N, M, N_0, M_0)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0, 1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M, N_0, M_0)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M, N_0, M_0)$.

Theorem 2.14. *Assume that there exist constants* $M, N, \overline{p}, \underline{p} \in \mathbb{R}^+ \setminus \{0\}$, $\overline{p}_1, \underline{p}_1 \in \mathbb{R}^+$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}$, $\{r_{2n}\}_{n\in\mathbb{N}_{n_0}}$, $\{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2), (2.24), (2.84),

$$N < \left(1 - \underline{p}_1 - \overline{p}_1\right)M, \quad \underline{p}_1 + \overline{p}_1 < 1, \quad -\underline{p}_1 \leq p_{1n} \leq \overline{p}_1, \quad \forall n \geq n_1, \tag{2.91}$$

$$\max\left\{N + M\overline{p}_1, N\left(\underline{p} - 1\right)\right\} < \min\left\{M\left(1 - \underline{p}_1\right), M(\overline{p} - 1)\right\}. \tag{2.92}$$

Then,

(a) *for each* $L \in (N + M\overline{p}_1, M(1 - \underline{p}_1)) \cap (N(\underline{p} - 1), M(\overline{p} - 1))$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.41) *and* (2.54) *converges to a bounded positive solution* $(x, y) \in A(N, M)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M)$.

Theorem 2.15. *Assume that there exist constants* $M, N, M_0, N_0, \overline{p}_0, \underline{p}_0, \overline{p}_1, \underline{p}_1 \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}, \{r_{2n}\}_{n\in\mathbb{N}_{n_0}}, \{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2) *with* $i = 1$, (2.24), (2.86)–(2.88), (2.91), *and*

$$\max\left\{N + M\overline{p}_1, \overline{p}_0\left(\frac{M_0}{\underline{p}_0} + N_0\right)\right\} < \min\left\{M\left(1 - \underline{p}_1\right), \underline{p}_0\left(\frac{N_0}{\overline{p}_0} + M_0\right)\right\}. \tag{2.93}$$

Then,

(a) *for each* $L \in (N + M\overline{p}_1, M(1 - \underline{p}_1)) \cap (\overline{p}_0(M_0/\underline{p}_0 + N_0), \underline{p}_0(N_0/\overline{p}_0 + M_0))$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M, N_0, M_0)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.41) *and* (2.67) *converges to a bounded positive solution* $(x, y) \in A(N, M, N_0, M_0)$ *of System* (1.14) *and has the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M, N_0, M_0)$ *are arbitrary sequences with* (2.8) *and* (2.9)

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M, N_0, M_0)$.

Theorem 2.16. *Assume that there exist constants* $M, N, M_0, N_0, \overline{p}_0, \underline{p}_0, \overline{p}_1, \underline{p}_1 \in \mathbb{R}^+ \setminus \{0\}$, $n_1 \in \mathbb{N}_{n_0}$ *and four nonnegative sequences* $\{r_{1n}\}_{n\in\mathbb{N}_{n_0}}, \{r_{2n}\}_{n\in\mathbb{N}_{n_0}}, \{t_{1n}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{t_{2n}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (2.1), (2.2) *with* $i = 1$, (2.24), (2.86)–(2.88),

$$0 < N\left(\underline{p}_1 - 1\right) < M(\overline{p}_1 - 1), \quad -\underline{p}_1 \leq p_{1n} \leq -\overline{p}_1, \quad \forall n \geq n_1, \tag{2.94}$$

$$\max\left\{N\left(\underline{p}_1 - 1\right), \overline{p}_0\left(\frac{M_0}{\underline{p}_0} + N_0\right)\right\} < \min\left\{M(\overline{p}_1 - 1), \underline{p}_0\left(\frac{N_0}{\overline{p}_0} + M_0\right)\right\}. \tag{2.95}$$

Then,

(a) *for each* $L \in (N(\underline{p}_1 - 1), M(\overline{p}_1 - 1)) \cap (\overline{p}_0(M_0/\underline{p}_0 + N_0), \underline{p}_0(N_0/\overline{p}_0 + M_0))$, *there exist* $\theta_L \in (0,1)$ *and* $T_L \geq \max\{\tau_1, \tau_2\} + n_1 + |\alpha|$ *such that for any* $(x^0, y^0) = (\{x_n^0\}_{n\in\mathbb{Z}_\alpha}, \{y_n^0\}_{n\in\mathbb{Z}_\alpha}) \in A(N, M, N_0, M_0)$, *the Mann iterative sequence with errors* $\{(x^\mu, y^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{x_n^\mu\}_{n\in\mathbb{Z}_\alpha}, \{y_n^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0}$ *generated by Schemes* (2.54) *and* (2.67) *converges to a bounded positive solution* $(x, y) \in A(N, M, N_0, M_0)$ *of System* (1.14) *and has* *the error estimate* (2.7), *where* $\{\beta_\mu\}_{\mu\in\mathbb{N}_0}, \{\gamma_\mu\}_{\mu\in\mathbb{N}_0} \subset [0,1]$, *and* $\{(\delta_1^\mu, \delta_2^\mu)\}_{\mu\in\mathbb{N}_0} = \{(\{\delta_{1n}^\mu\}_{n\in\mathbb{Z}_\alpha}, \{\delta_{2n}^\mu\}_{n\in\mathbb{Z}_\alpha})\}_{\mu\in\mathbb{N}_0} \subset A(N, M, N_0, M_0)$ *are arbitrary sequences with* (2.8) *and* (2.9),

(b) *System* (1.14) *has uncountably many bounded positive solutions in* $A(N, M, N_0, M_0)$.

Remark 2.17. Theorems 2.1–2.5 extend and improve [14, Theorem 1] and [27, Theorems 2.1–2.7].

3. Examples

In this section we construct fifteen examples to explain the results presented in Section 2.

Example 3.1. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
&\Delta(x_n - x_{n-\tau_1}) + \frac{x_{n-3}^3 y_{n-6} - \sqrt{n}\, x_{n-4} y_{n-5}^2}{n^3 + n x_{n-3}^2 + (n^2+1) x_{n-4}^4 y_{n-6}^2} = \frac{(-1)^{n-1}}{n^4 + \sqrt{n+1}},\\
&\Delta(y_n - y_{n-\tau_2}) + \frac{(n+1) x_{n-2}^2 y_{n-5} + (-1)^n y_{n-2}}{n^4 + (n+2) x_{n-1}^2} = \frac{\sqrt{n+1}}{n^3 + (n+2)\ln n}, \quad n \geq 1,
\end{aligned} \tag{3.1}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$\begin{gathered}
n_0 = n_1 = 1, \quad h = k = 2, \quad M > N > 0, \quad \alpha = \min\{1-\tau_1, 1-\tau_2, -5\}, \quad a_{1n} = n-3,\\
a_{2n} = n-4, \quad b_{1n} = n-6, \quad b_{2n} = n-5, \quad c_{1n} = n-2, \quad c_{2n} = n-1, \quad d_{1n} = n-5,\\
d_{2n} = n-2, \quad p_{1n} = p_{2n} = -1, \quad q_{1n} = \frac{(-1)^{n-1}}{n^4 + \sqrt{n+1}}, \quad q_{2n} = \frac{\sqrt{n+1}}{n^3 + (n+2)\ln n},\\
f_1(n, u_1, u_2, v_1, v_2) = \frac{u_1^3 v_1 - \sqrt{n}\, u_2 v_2^2}{n^3 + n u_1^2 + (n^2+1) u_2^4 v_1^2}, \qquad f_2(n, u_1, u_2, v_1, v_2) = \frac{(n+1) u_1^2 v_2 + (-1)^n v_1}{n^4 + (n+2) u_2^2},\\
r_{1n} = \frac{M^3 (M + \sqrt{n})}{n^3 + nN^2 + (n^2+1) N^6}, \qquad r_{2n} = \frac{M(1 + (n+1)M^2)}{n^4 + (n+2)N^2},\\
t_{1n} = \frac{nM^2 (4n^2 M + 3n^{5/2} + 2M^3 + 5\sqrt{n} M^2 + 16 n M^7 + 14\sqrt{n} M^6)}{(n^3 + nN^2 + (n^2+1)N^6)^2},\\
t_{2n} = \frac{1}{n^7}\left(6n^4 M^2 + n^3 + 30 n M^4 + 9M^2\right), \quad \forall (n, u_1, u_2, v_1, v_2) \in \mathbb{N}_{n_0} \times \mathbb{R}^4.
\end{gathered} \tag{3.2}$$

It is easy to see that (2.1)–(2.4) are satisfied. Thus Theorem 2.1 implies that System (3.1) possesses uncountably many bounded positive solutions in $A(N, M)$. But [14, Theorem 1] and [27, Theorem 2.1] are not valid for System (3.1).

Example 3.2. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
\Delta\left(x_n + x_{n-\tau_1}\right) + \frac{n x_{n-1}^3 - (n-1) y_{n-4}}{n^3 + x_{n-1}^2 y_{n-4}^2} &= \frac{n^2 - 1}{n^4 + 2n + 1}, \\
\Delta\left(y_n + y_{n-\tau_2}\right) + \frac{n^9 x_{n-3} y_{n-2}^2}{n^{12} + (n^6 - 1) x_{n-3}^2} &= \frac{(-1)^n n^2 \ln^3 n}{(n^2 + 1)^2}, \quad n \geq 1,
\end{aligned} \tag{3.3}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$\begin{gathered}
n_0 = n_1 = 1, \qquad h = k = 1, \qquad M > N > 0, \qquad \alpha = \min\{1 - \tau_1, 1 - \tau_2, -3\}, \qquad a_{1n} = n - 1, \\
b_{1n} = n - 4, \qquad c_{1n} = n - 3, \qquad d_{1n} = n - 2, \qquad p_{1n} = p_{2n} = 1, \qquad q_{1n} = \frac{n^2 - 1}{n^4 + 2n + 1}, \\
q_{2n} = \frac{(-1)^n n^2 \ln^3 n}{(n^2 + 1)^2}, \qquad f_1(n, u, v) = \frac{n u^3 - (n-1) v}{n^3 + u^2 v^2}, \qquad f_2(n, u, v) = \frac{n^9 u v^2}{n^{12} + (n^6 - 1) u^2}, \\
r_{1n} = \frac{n M (M+1)}{n^3 + N^4}, \qquad r_{2n} = \frac{n^9 M^3}{n^{12} + (n^6 - 1) N^2}, \qquad t_{1n} = \frac{n\left(n^3 (2M + 1) + 3M^4 + 2M^5\right)}{(n^3 + N^4)^2}, \\
t_{2n} = \frac{3 n^{15} (n^6 + 1) M^2}{\left(n^{12} + (n^6 - 1) N^2\right)^2}, \quad \forall (n, u, v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered} \tag{3.4}$$

It is easy to verify that (2.1), (2.2), (2.24), and (2.25) are fulfilled. Thus Theorem 2.2 implies that System (3.3) possesses uncountably many bounded positive solutions in $A(N, M)$. But in [14, Theorem 1] and [27, Theorem 2.2] are unapplicable for System (3.3).

Example 3.3. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
\Delta\left(x_n + \frac{n^2}{2n^2 + 3} x_{n-\tau_1}\right) + \frac{x_{n-2} - 2 y_{n-6}}{n^2 + \sin\left(x_{n-1} y_{n-6}\right)} &= \frac{\sqrt{n-2}\cos\left(n^3 - 3n^2 + 6\right)}{n^3 + 3n + 1}, \\
\Delta\left(y_n + \frac{(-1)^n}{3n + 1} y_{n-\tau_2}\right) + \frac{n x_{n-1} + (2n - 1) y_{n-3}^2}{n^3 + x_{n-1}^2} &= \frac{(n^2 - 3)\sin\left(n^5 - 2n + 1\right)}{n^4 + \ln^3 n}, \quad n \geq 2,
\end{aligned} \tag{3.5}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$
\begin{gathered}
n_0 = n_1 = 2, \qquad h = k = 1, \qquad M > 6N > 0, \\
\alpha = \min\{2 - \tau_1, 2 - \tau_2, -4\}, \qquad a_{1n} = n - 2, \\
b_{1n} = n - 6, \qquad c_{1n} = n - 1, \qquad d_{1n} = n - 3, \qquad p_{1n} = \frac{n^2}{2n^2 + 3}, \\
p_{2n} = \frac{(-1)^n}{3n + 1}, \qquad \overline{p} = \frac{1}{2}, \qquad \underline{p} = \frac{1}{3}, \\
q_{1n} = \frac{\sqrt{n-2}\cos(n^3 - 3n^2 + 6)}{n^3 + 3n + 1}, \qquad q_{2n} = \frac{(n^2 - 3)\sin(n^5 - 2n + 1)}{n^4 + \ln^3 n}, \\
f_1(n, u, v) = \frac{u - 2v}{n^2 + \sin(uv)}, \\
f_2(n, u, v) = \frac{nu + (2n-1)v^2}{n^3 + u^2}, \qquad r_{1n} = \frac{3M}{n^2 - 1}, \qquad r_{2n} = \frac{nM(1 + 2M)}{n^3 + N^2}, \\
t_{1n} = \frac{6(n^2 + M^2)}{(n^2 - 1)^2}, \quad t_{2n} = \frac{n\left((1 + 4M)n^3 + M^2 + 8M^3\right)}{(n^3 + N^2)^2}, \quad \forall (n, u, v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered} \tag{3.6}
$$

It is easy to see that (2.1), (2.2), (2.24), and (2.39) hold. Thus, Theorem 2.3 ensures that System (3.5) possesses uncountably many bounded positive solutions in $A(N, M)$. But [14, Theorem 1] and [27, Theorems 2.3, 2.5, and 2.6] are not valid for System (3.5).

Example 3.4. Consider the nonlinear difference system with multiple delays:

$$
\begin{gathered}
\Delta\left(x_n + \left(-4 + \sin\left(2n^2\right)\right)x_{n-\tau_1}\right) + \frac{n^5 x_{n-1}^2 y_{n-2}^5}{n^{10} + 5n^8 + 1} = \frac{(-1)^n n^3 \ln(1+n)}{n^9 + n^5 + 1}, \\
\Delta\left(y_n + \left(-4 + \cos\left(5n^3\right)\right)y_{n-\tau_2}\right) + \frac{n^2 x_{n-2} - (n+2)y_{n-3}^2}{n^8 + 1 + n x_{n-2}^2 + y_{n-3}^2} = \frac{n^5 - 6}{(n+1)^7}, \quad n \geq 1,
\end{gathered} \tag{3.7}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$
\begin{gathered}
n_0 = n_1 = h = k = 1, \qquad M > 2N > 0, \qquad \alpha = \min\{1-\tau_1, 1-\tau_2, -2\}, \\
a_{1n} = n-1, \qquad b_{1n} = n-2, \\
c_{1n} = n-2, \qquad d_{1n} = n-3, \qquad p_{1n} = -4 + \sin\left(2n^2\right), \\
p_{2n} = -4 + \cos\left(5n^3\right), \qquad \overline{p} = 3, \qquad \underline{p} = 5, \\
q_{1n} = \frac{(-1)^n n^3 \ln(1+n)}{n^9 + n^5 + 1}, \qquad q_{2n} = \frac{n^5 - 6}{(n+1)^7}, \qquad f_1(n,u,v) = \frac{n^5 u^2 v^5}{n^{10} + 5n^8 + 1}, \\
f_2(n,u,v) = \frac{n^2 u - (n+2)v^2}{n^8 + 1 + nu^2 + v^2}, \qquad r_{1n} = \frac{M^7}{n^5}, \\
r_{2n} = \frac{n^2 M + (n+2)M^2}{n^8 + 1 + nN^2 + N^2}, \qquad t_{1n} = \frac{7M^6}{n^5}, \\
t_{2n} = \frac{n^2\left(2n^8 + 8n^7 M + nM^2 + 8M^3 + 3M^2\right)}{\left(n^8 + 1 + nN^2 + N^2\right)^2}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered}
\tag{3.8}
$$

Clearly (2.1), (2.2), (2.24), and (2.52) hold. Thus, Theorem 2.4 means that System (3.7) possesses uncountably many bounded positive solutions in $A(N, M)$. But [14, Theorem 1] and [27, Theorem 2.4] are inapplicable for System (3.7).

Example 3.5. Consider the nonlinear difference system with multiple delays:

$$
\begin{gathered}
\Delta\left(x_n + \frac{8n^2+1}{n^2+1} x_{n-\tau_1}\right) + \frac{n - (n-1)y_{b_{1n}}^3}{n^3 + x_{a_{1n}}^2} = \frac{\sqrt{n} - (-1)^{n(n+1)/2}\sqrt{3n-2}}{n^2 + \sqrt{2n+1}}, \\
\Delta\left(y_n + \frac{1 + 24\ln^3 n}{2 + 3\ln^3 n} y_{n-\tau_2}\right) + \frac{2nx_{c_{1n}} - 3\sqrt{n+1}\, y_{d_{1n}}^3}{n^4 + x_{c_{1n}}^2 y_{d_{1n}}^2} = \frac{1 + n\ln\left(1+n^2\right)}{n^3 \ln^3\left(1+n^2\right)}, \quad n \geq 1,
\end{gathered}
\tag{3.9}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$
\begin{gathered}
a_{1n} = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even}, \\ n-2 & \text{if } n \text{ is odd}, \end{cases} \qquad b_{1n} = \begin{cases} n-1 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases} \\
c_{1n} = \begin{cases} n-2 & \text{if } n \text{ is even}, \\ n-1 & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} \dfrac{n}{2} - 1 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases}
\end{gathered}
\tag{3.10}
$$

Let

$$
\begin{gathered}
n_0 = 1, \qquad n_1 = 10, \qquad h = k = 1, \qquad M > \frac{399}{328} N > 0, \\
\alpha = \min\{1 - \tau_1, 1 - \tau_2, -1\}, \qquad p_{1n} = \frac{8n^2 + 1}{n^2 + 1}, \\
p_{2n} = \frac{1 + 24 \ln^3 n}{2 + 3 \ln^3 n}, \qquad \overline{p} = 8, \qquad \underline{p} = 7, \\
q_{1n} = \frac{\sqrt{n} - (-1)^{n(n+1)/2} \sqrt{3n - 2}}{n^2 + \sqrt{2n + 1}}, \qquad q_{2n} = \frac{1 + n \ln(1 + n^2)}{n^3 \ln^3 (1 + n^2)}, \\
f_1(n, u, v) = \frac{n - (n - 1) v^3}{n^3 + u^2}, \qquad f_2(n, u, v) = \frac{2nu - 3\sqrt{n + 1} v^3}{n^4 + u^2 v^2}, \\
r_{1n} = \frac{n(1 + M^3)}{n^3 + N^2}, \\
r_{2n} = \frac{nM(2 + 3M^2)}{n^4 + N^4}, \qquad t_{1n} = \frac{nM(2 + 3n^3 M + 5M^3)}{(n^3 + N^2)^2}, \\
t_{2n} = \frac{n(2n^4 + 6M^2 + 9n^4 M^2 + 9M^6)}{(n^4 + N^4)^2}, \quad \forall (n, u, v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered}
\tag{3.11}
$$

Obviously (2.1), (2.2), (2.24), and (2.65) hold. Thus, Theorem 2.5 implies that System (3.9) possesses uncountably many bounded positive solutions in $A(N, M)$. But [14, Theorem 1] and [27, Theorem 2.7] are useless for System (3.9).

Example 3.6. Consider the nonlinear difference system with multiple delays:

$$
\begin{gathered}
\Delta(x_n - x_{n-\tau_1}) + \frac{2n^2 x^3_{a_{1n}} + n - 1}{n^5 + n\left(n x_{a_{1n}} - (3n + 2) y_{b_{1n}}\right)^2} = \frac{(-1)^{(n-1)n(n+1)/3} \sqrt{2n^3 + 4}}{n^4 + \sin^2(2n^4 + 1)}, \\
\Delta\left(y_n + y_{n-\tau_2}\right) + \frac{(-1)^n x^2_{c_{1n}} - n y^3_{d_{1n}}}{(n + 1)^2 (\sqrt{n} + 1)} = \frac{(-1)^{n-1} \ln(n^3 + 2n + 1)}{n^{3/2} + \ln^3(1 + n^2)}, \quad n \geq 1,
\end{gathered}
\tag{3.12}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$
\begin{gathered}
a_{1n} = \begin{cases} n - 1 & \text{if } n \geq 100, \\ n - 2 & \text{if } 1 \leq n < 100, \end{cases} \qquad b_{1n} = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even}, \\ \dfrac{n - 1}{2} & \text{if } n \text{ is odd}, \end{cases} \\
c_{1n} = \begin{cases} \dfrac{n}{2} - 2 & \text{if } n \text{ is even}, \\ \dfrac{n - 1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} n - 3 & \text{if } n \geq 100, \\ n & \text{if } 1 \leq n < 100. \end{cases}
\end{gathered}
\tag{3.13}
$$

Let

$$
\begin{gathered}
n_0 = n_1 = h = k = 1, \qquad M > N > 0, \qquad \alpha = \min\{1-\tau_1, 1-\tau_2, -1\}, \qquad p_{1n} = -1, \\
p_{2n} = 1, \qquad q_{1n} = \frac{(-1)^{(n-1)n(n+1)/3}\sqrt{2n^3+4}}{n^4+\sin^2(2n^4+1)}, \qquad q_{2n} = \frac{(-1)^{n-1}\ln(n^3+2n+1)}{n^{3/2}+\ln^3(1+n^2)}, \\
f_1(n,u,v) = \frac{2n^2u^3+n-1}{n^5+n(nu-(3n+2)v)^2}, \qquad f_2(n,u,v) = \frac{(-1)^n u^2 - nv^3}{(n+1)^2(\sqrt{n}+1)}, \\
r_{1n} = \frac{1+2nM^3}{n^4}, \qquad r_{2n} = \frac{M^2+nM^3}{n^{5/2}}, \\
t_{1n} = \frac{6M(16+n^3M+8(2n+M)M^2+3nM^3)}{n^6}, \\
t_{2n} = \frac{2M+3nM^2}{n^{5/2}}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered}
\tag{3.14}
$$

It is easy to see that (2.1), (2.2), and (2.79)–(2.82) are satisfied. Thus, Theorem 2.7 implies that System (3.12) possesses uncountably many bounded positive solutions in $A(\mathrm{N}, M)$.

Example 3.7. Consider the nonlinear difference system with multiple delays:

$$
\begin{gathered}
\Delta(x_n - x_{n-\tau_1}) + \frac{n x_{a_{1n}}^3 - y_{b_{1n}}}{(n+1)^4 + (\sqrt{n}+1)y_{b_{1n}}^2} = \frac{(-1)^{n(n+1)/2}\left(1-\sqrt{3n+1}\right)}{n^3+\sqrt{3n+2}}, \\
\Delta\left(y_n + \frac{n^2-1}{3n^2+2}y_{n-\tau_2}\right) + \frac{\sqrt{n}(n+1)y_{d_{1n}}^4}{(n+2)^3+nx_{c_{1n}}^2} = \frac{\sqrt{2n+1}(n+3)^2\ln(1+n^4)}{(n+1)^4}, \quad n \geq 1,
\end{gathered}
\tag{3.15}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$
\begin{gathered}
a_{1n} = \begin{cases} n-4 & \text{if } n \text{ is even}, \\ n-1 & \text{if } n \text{ is odd}, \end{cases} \qquad b_{1n} = \begin{cases} n-2 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases} \\
c_{1n} = \begin{cases} n-1 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even}, \\ n-2 & \text{if } n \text{ is odd}. \end{cases}
\end{gathered}
\tag{3.16}
$$

Let

$$n_0 = n_1 = h = k = 1, \qquad M > 6N > 0, \qquad \alpha = \min\{1-\tau_1, 1-\tau_2, -2\},$$
$$p_{1n} = -1, \qquad p_{2n} = \frac{n^2-1}{3n^2+2},$$
$$\overline{p} = \frac{1}{3}, \qquad \underline{p} = \frac{1}{2}, \qquad q_{1n} = \frac{(-1)^{n(n+1)/2}\left(1-\sqrt{3n+1}\right)}{n^3+\sqrt{3n+2}},$$
$$q_{2n} = \frac{\sqrt{2n+1}(n+3)^2\ln\left(1+n^4\right)}{(n+1)^4},$$
$$f_1(n,u,v) = \frac{nu^3 - v}{(n+1)^4 + \left(\sqrt{n}+1\right)v^2}, \qquad f_2(n,u,v) = \frac{\sqrt{n}(n+1)v^4}{(n+2)^3 + nu^2}, \tag{3.17}$$
$$r_{1n} = \frac{\left(nM^2+1\right)M}{(n+1)^4},$$
$$r_{2n} = \frac{\sqrt{n}(n+1)M^4}{(n+2)^3+nN^2}, \qquad t_{1n} = \frac{(n+1)^4\left(3nM^2+1\right) + \left(5nM^2+1\right)\left(\sqrt{n}+1\right)M^2}{\left((n+1)^4 + \left(\sqrt{n}+1\right)N^2\right)^2},$$
$$t_{2n} = \frac{2\sqrt{n}(n+1)M^3\left(2(n+2)^3 + 3nM^2\right)}{\left((n+2)^3+nN^2\right)^2}, \qquad \forall (n,u,v) \in \mathbb{N}_{n_0}\times\mathbb{R}^2.$$

Clearly (2.1), (2.2), (2.79)–(2.81), and (2.83) are satisfied. Thus, Theorem 2.8 guarantees that System (3.15) possesses uncountably many bounded positive solutions in $A(N,M)$.

Example 3.8. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
&\Delta\left(x_n - x_{n-\tau_1}\right) + \frac{n^7 - x_{a_{1n}}^2}{n^{10}+2y_{b_{1n}}^2} = \frac{n^{20} - (-1)^{(n+1)(n+2)/2}\left(3n^5+1\right)^2}{n^{23} + (n+5)^5\ln^8\left(n^2+1\right)},\\
&\Delta\left(y_n - \frac{6n^5}{2n^5+1}y_{n-\tau_2}\right) + \frac{n^3 x_{c_{1n}} y_{d_{1n}}^2}{n^6+(n+1)x_{c_{1n}}^4} = \frac{n^8 - (-1)^{n-1}\left(n^3+1\right)^2}{(n+1)^{11}}, \quad n \ge 2,
\end{aligned} \tag{3.18}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$\begin{aligned}
a_{1n} &= \begin{cases} \dfrac{n}{2}-1 & \text{if } n \text{ is even,} \\ \dfrac{n-3}{2} & \text{if } n \text{ is odd,} \end{cases} \qquad b_{1n} = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n-2 & \text{if } n \text{ is odd,} \end{cases}\\
c_{1n} &= \begin{cases} \dfrac{n}{2}+1 & \text{if } n \text{ is even,} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd,} \end{cases} \qquad d_{1n} = \begin{cases} n-5 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned} \tag{3.19}$$

Let

$$
\begin{gathered}
n_0 = n_1 = 2, \qquad h = k = 1, \qquad M > 6N > 0, \qquad \alpha = \min\{2-\tau_1, 2-\tau_2, -3\}, \qquad p_{1n} = -1, \\
p_{2n} = -\frac{6n^5}{2n^5+1}, \qquad \overline{p} = 2, \qquad \underline{p} = 3, \qquad q_{1n} = \frac{n^{20} - (-1)^{(n+1)(n+2)/2}\left(3n^5+1\right)^2}{n^{23} + (n+5)^5 \ln^8(n^2+1)}, \\
q_{2n} = \frac{n^8 - (-1)^{n-1}\left(n^3+1\right)^2}{(n+1)^{11}}, \qquad f_1(n,u,v) = \frac{n^7 - u^2}{n^{10} + 2v^2}, \qquad f_2(n,u,v) = \frac{n^3 u v^2}{n^6 + (n+1)u^4}, \\
r_{1n} = \frac{n^7 + M^2}{n^{10} + 2N^2}, \qquad r_{2n} = \frac{n^3 M^3}{n^6 + (n+1)N^4}, \qquad t_{1n} = \frac{2M\left(2n^7 + n^{10} + 4M^2\right)}{\left(n^{10} + 2N^2\right)^2}, \\
t_{2n} = \frac{3n^3 M^2\left(n^6 + (n+1)M^4\right)}{\left(n^6 + (n+1)N^4\right)^2}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered}
\tag{3.20}
$$

Obviously (2.1), (2.2), (2.79)–(2.81), (2.84), and (2.85) hold. Thus, Theorem 2.9 implies that System (3.18) possesses uncountably many bounded positive solutions in $A(N, M)$.

Example 3.9. Consider the nonlinear difference system with multiple delays:

$$
\begin{gathered}
\Delta\left(x_n - x_{n-\tau_1}\right) + \frac{n x_{a_{1n}}^2 - y_{b_{1n}}^3}{n^3 \ln^2 n + 1} = \frac{n^2 - (-1)^{n-1}(3n+1)}{n^5 + 8(n+1)^4 + 1}, \\
\Delta\left(y_n + \frac{8n^2}{2n^2+3} y_{n-\tau_2}\right) + \frac{n^2}{n^4 + \left|x_{c_{1n}} y_{d_{1n}}\right|} = \frac{(-1)^{n-1}\sin\left(n^5+1\right)}{n^2+1}, \quad n \ge 1,
\end{gathered}
\tag{3.21}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$
\begin{gathered}
a_{1n} = \begin{cases} n-2 & \text{if } n \text{ is even}, \\ \dfrac{n-3}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad b_{1n} = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases} \\
c_{1n} = \begin{cases} n-1 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} n-4 & \text{if } n \text{ is even}, \\ n-2 & \text{if } n \text{ is odd}. \end{cases}
\end{gathered}
\tag{3.22}
$$

Let

$$
\begin{gathered}
n_0 = 1, \qquad n_1 = 3, \qquad h = k = 1, \qquad M = 3000, \\
N = 2000, \qquad M_0 = 1200, \qquad N_0 = 40, \\
\alpha = \min\{1-\tau_1, 1-\tau_2, -2\}, \qquad p_{1n} = -1, \qquad p_{2n} = \frac{8n^2}{2n^2+3}, \qquad \overline{p}_0 = 4, \qquad \underline{p}_0 = 3, \\
q_{1n} = \frac{n^2 - (-1)^{n-1}(3n+1)}{n^5 + 8(n+1)^4 + 1}, \qquad q_{2n} = \frac{(-1)^{n-1}\sin(n^5+1)}{n^2+1}, \qquad f_1(n,u,v) = \frac{nu^2 - v^3}{n^3\ln^2 n + 1}, \\
f_2(n,u,v) = \frac{n^2}{n^4 + |uv|}, \qquad r_{1n} = \frac{M^2(n+M)}{n^3\ln^2 n + 1}, \qquad r_{2n} = \frac{n^2}{n^4 + N_0^2}, \\
t_{1n} = \frac{M(2n+3M)}{n^3\ln^2 n + 1}, \quad t_{2n} = \frac{2n^2 M_0}{\left(n^4 + N_0^2\right)^2}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered}
\tag{3.23}
$$

It is easy to verify that (2.1), (2.2) with $i = 1$, (2.79)–(2.81), and (2.86)–(2.89) are satisfied. Thus, Theorem 2.10 reveals that System (3.21) possesses uncountably many bounded positive solutions in $A(N, M, N_0, M_0)$.

Example 3.10. Consider the nonlinear difference system with multiple delays:

$$
\begin{aligned}
\Delta(x_n + x_{n-\tau_1}) + \frac{1 - n^5 x_{a_{1n}}^2}{n^7 + (n+3)\ln\left(1 + y_{b_{1n}}^2\right)} &= \frac{n^4 - 4n^3 + n - 1}{n^6 + (n-1)^4}, \\
\Delta\left(y_n - \frac{2n^2}{5n^2+6} y_{n-\tau_2}\right) + \frac{n^2 y_{d_{1n}}}{n^4 + \sin^2\left(x_{c_{1n}}^2 + n y_{d_{1n}}^2\right)} &= \frac{(-1)^n \sqrt{n-1}}{n^3 + 2n^2 + 1}, \quad n \geq 2,
\end{aligned}
\tag{3.24}
$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$
\begin{gathered}
a_{1n} = \begin{cases} n-3 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad b_{1n} = \begin{cases} \dfrac{n}{2} + 1 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}, \end{cases} \\
c_{1n} = \begin{cases} n-1 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} n-2 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases}
\end{gathered}
\tag{3.25}
$$

Let

$$n_0 = n_1 = 2, \qquad h = k = 1, \qquad M > \frac{5N}{3} > 0, \qquad \alpha = \min\{1-\tau_1, 1-\tau_2, -1\}, \qquad p_{1n} = 1,$$
$$p_{2n} = -\frac{2n^2}{5n^2+6}, \qquad \overline{p} = 0, \qquad \underline{p} = \frac{2}{5}, \qquad q_{1n} = \frac{n^4 - 4n^3 + n - 1}{n^6 + (n-1)^4}, \qquad q_{2n} = \frac{(-1)^n\sqrt{n-1}}{n^3+2n^2+1},$$
$$f_1(n,u,v) = \frac{1 - n^5u^2}{n^7 + (n+3)\ln(1+v^2)}, \qquad f_2(n,u,v) = \frac{n^2 v}{n^4 + \sin^2(u^2 + nv^2)},$$
$$r_{1n} = \frac{1 + n^5 M^2}{n^7 + (n+3)\ln(1+N^2)}, \qquad r_{2n} = \frac{M}{n^2},$$
$$t_{1n} = \frac{1}{(n^7 + (n+3)\ln(1+N^2))^2}\left[\frac{2\mathrm{M}(n+3)}{1+N^2} + n^{12} + n^5(n+3)\left(\frac{2M^2}{1+N^2} + \ln\left(1+M^2\right)\right)\right],$$
$$t_{2n} = \frac{n^4 + 4nM^2 + 4M^2 + 1}{n^6}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2. \tag{3.26}$$

Obviously (2.1), (2.2), (2.24), (2.83), and (2.90) hold. Thus, Theorem 2.11 gives that System (3.24) possesses uncountably many bounded positive solutions in $A(N,M)$.

Example 3.11. Consider the nonlinear difference system with multiple delays:

$$\begin{gathered}\Delta(x_n + x_{n-\tau_1}) + \frac{n^2 - x_{a_{1n}}^2 y_{b_{1n}}}{n^4 + (n+1)y_{b_{1n}}^4} = \frac{n^6 - 9n^5 + (-1)^n n^4 - n^3 + n - 1}{n^8 + 3n^7 + n^5 + 4n^4 + n^2 + 1},\\ \Delta\left(y_n - \frac{3n^3}{n^3+1}y_{n-\tau_2}\right) + \frac{n y_{d_{1n}}^2}{n^3 + x_{c_{1n}}^2} = \frac{(-1)^{n(n+1)/2}(2n-5)\ln^2(1+n^2)}{n^3+3n^2+5n+1}, \quad n \ge 1,\end{gathered} \tag{3.27}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed and

$$\begin{gathered}a_{1n} = \begin{cases} n-2 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad b_{1n} = \begin{cases} \dfrac{n}{2} - 1 & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd}, \end{cases}\\ c_{1n} = \begin{cases} n-3 & \text{if } n \text{ is even}, \\ \dfrac{n-3}{2} & \text{if } n \text{ is odd}, \end{cases} \qquad d_{1n} = \begin{cases} n-4 & \text{if } n \text{ is even}, \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases}\end{gathered} \tag{3.28}$$

Let

$$n_0 = n_1 = h = k = 1, \qquad M > 4N > 0, \qquad \alpha = \min\{1-\tau_1, 1-\tau_2, -2\}, \qquad p_{1n} = 1,$$

$$p_{2n} = -\frac{3n^3}{n^3+1}, \qquad \overline{p} = \frac{3}{2}, \qquad \underline{p} = 3, \qquad q_{1n} = \frac{n^6 - 9n^5 + (-1)^n n^4 - n^3 + n - 1}{n^8 + 3n^7 + n^5 + 4n^4 + n^2 + 1},$$

$$q_{2n} = \frac{(-1)^{n(n+1)/2}(2n-5)\ln^2(1+n^2)}{n^3 + 3n^2 + 5n + 1}, \qquad f_1(n,u,v) = \frac{n^2 - u^2 v}{n^4 + (n+1)v^4}, \tag{3.29}$$

$$f_2(n,u,v) = \frac{nv^2}{n^3+u^2}, \qquad r_{1n} = \frac{n^2 + M^3}{n^4 + (n+1)N^4}, \qquad r_{2n} = \frac{nM^2}{n^3 + N^2},$$

$$t_{1n} = \frac{nM^2(8n^2M + 3n^3 + 10M^4)}{\left(n^4 + (n+1)N^4\right)^2}, \quad t_{2n} = \frac{2M(n^3 + 2M^2)}{(n^3+N^2)^2}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.$$

Clearly (2.1), (2.2), (2.24), (2.84), (2.85), and (2.90) hold. Thus, Theorem 2.12 ensures that System (3.27) possesses uncountably many bounded positive solutions in $A(N,M)$.

Example 3.12. Consider the nonlinear difference system with multiple delays:

$$\Delta(x_n + x_{n-\tau_1}) + \frac{n^{25} - (n^{15}+1)y_{n-1}^2}{n^{30} + (n^{18}+1)x_{n-2}^2} = \frac{n^{38} - 19n^{25} + 30n^{18} - 3n^9 + 2n - 1}{n^{40} + 8n^{30} + n^{15} + 6n^2 + 1},$$
$$\Delta\left(y_n + \frac{16n^3+1}{4n^3+131}y_{n-\tau_2}\right) + \frac{x_{n-4}^3 y_{n-3}^2}{n\ln^{10}n + 1} = \frac{n^{34} - 6n^{32} - 7n^{15} + (-1)^n n^7 - 1}{n^{36} + 6n^{30} + 5n^{12} + 6n^6 + n^3 + 1}, \quad n \geq 2, \tag{3.30}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$n_0 = 2, \qquad n_1 = 4, \qquad h = k = 1, \qquad M = 800N_0,$$
$$N = 500N_0, \qquad M_0 = 300N_0 > 0,$$
$$\alpha = \min\{2-\tau_1, 2-\tau_2, -2\}, \qquad a_{1n} = n-2, \qquad b_{1n} = n-1,$$
$$c_{1n} = n-4, \qquad d_{1n} = n-3,$$

$$
\begin{gathered}
p_{1n}=1, \qquad p_{2n}=\frac{16n^3+1}{4n^3+131}, \qquad \overline{p}_0=4, \qquad \underline{p}_0=3, \\
q_{1n}=\frac{n^{38}-19n^{25}+30n^{18}-3n^9+2n-1}{n^{40}+8n^{30}+n^{15}+6n^2+1}, \\
q_{2n}=\frac{n^{34}-6n^{32}-7n^{15}+(-1)^n n^7-1}{n^{36}+6n^{30}+5n^{12}+6n^6+n^3+1}, \qquad f_1(n,u,v)=\frac{n^{25}-(n^{15}+1)v^2}{n^{30}+(n^{18}+1)u^2}, \\
f_2(n,u,v)=\frac{u^3v^2}{n\ln^{10}n+1}, \qquad r_{1n}=\frac{n^{25}+(n^{15}+1)M^2}{n^{30}+(n^{18}+1)N^2}, \qquad r_{2n}=\frac{M_0^5}{n\ln^{10}n+1}, \\
t_{1n}=\frac{4n^{33}M(n^{10}+n^{12}+4M^2)}{(n^{30}+(n^{18}+1)N^2)^2}, \quad t_{2n}=\frac{5M_0^4}{\left(n\ln^{10}n+1\right)^2}, \quad \forall (n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{gathered} \tag{3.31}
$$

It is easy to verify that (2.1), (2.2) with $i=1$, (2.24), and (2.86)–(2.90) hold. Thus, Theorem 2.13 guarantees that System (3.30) possesses uncountably many bounded positive solutions in $A(N,M,N_0,M_0)$.

Example 3.13. Consider the nonlinear difference system with multiple delays:

$$
\begin{aligned}
\Delta\left(x_n+\frac{3(-1)^n n^{10}+2n+1}{9n^{10}+4n^8+1}x_{n-\tau_1}\right)+\frac{x_{n-3}-ny_{n-1}}{n^4+x_{n-3}^2}&=\frac{n^4-\sqrt{2n+5}\ln^5 n}{n^6+6n^2+1}, \\
\Delta\left(y_n-\frac{(4+(-1)^n)n^4+5}{n^4+1}y_{n-\tau_2}\right)+\frac{n^3x_{n-2}y_{n-4}}{n^5+(n+1)y_{n-4}^2}&=\frac{n^7-(-1)^n n^4-1}{n^9+3n^6+5}, \quad n\ge 2,
\end{aligned} \tag{3.32}
$$

where $\tau_1,\tau_2\in\mathbb{N}$ are fixed. Let

$$
\begin{gathered}
n_0=2, \qquad n_1=4, \qquad h=k=1, \qquad M>10N>0, \qquad \alpha=\min\{2-\tau_1,2-\tau_2,-2\}, \\
a_{1n}=n-3, \qquad b_{1n}=n-1, \qquad c_{1n}=n-2, \qquad d_{1n}=n-4, \qquad p_{1n}=\frac{3(-1)^n n^{10}+2n+1}{9n^{10}+4n^8+1}, \\
p_{2n}=-\frac{(4+(-1)^n)n^4+5}{n^4+1}, \qquad \overline{p}_1=\underline{p}_1=\frac{1}{3}, \qquad \overline{p}=3, \qquad \underline{p}=5, \qquad q_{1n}=\frac{n^4-\sqrt{2n+5}\ln^5 n}{n^6+6n^2+1}, \\
q_{2n}=\frac{n^7-(-1)^n n^4-1}{n^9+3n^6+5}, \qquad f_1(n,u,v)=\frac{u-nv}{n^4+u^2}, \qquad f_2(n,u,v)=\frac{n^3uv}{n^5+(n+1)v^2}, \\
r_{1n}=\frac{M(n+1)}{n^4+N^2}, \qquad r_{2n}=\frac{n^3M^2}{n^5+(n+1)N^2}, \qquad t_{1n}=\frac{n^4+M^2+n^5+3nM^2}{(n^4+N^2)^2}, \\
t_{2n}=\frac{2n^3M(n^5+(n+1)M^2)}{(n^5+(n+1)N^2)^2}, \quad \forall (n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{gathered} \tag{3.33}
$$

Obviously (2.1), (2.2), (2.24), (2.84), (2.91), and (2.92) hold. Thus, Theorem 2.14 means that System (3.32) possesses uncountably many bounded positive solutions in $A(N, M)$.

Example 3.14. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
&\Delta\left(x_n + \frac{(-1)^n n + 1}{4n + 100} x_{n-\tau_1}\right) + \frac{x_{n-2} - n y_{n-1}}{n^3 + x_{n-2}^2 y_{n-1}^2} = \frac{n^5 \ln^9 n - (-1)^n}{n^8 + 5n^7 + 1}, \\
&\Delta\left(y_n + \frac{64n^8 + 1}{8n^8 + 3n^4 + 7} y_{n-\tau_2}\right) + \frac{n x_{n-5}^2 - y_{n-3} \sin n}{n^2 \ln^2 n + 1} = \frac{n^6 - (-1)^n n^2 + 1}{n^8 + 7n^7 + 2}, \quad n \geq 2,
\end{aligned} \tag{3.34}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$\begin{gathered}
n_0 = n_1 = 2, \qquad h = k = 1, \qquad M = 40000, \qquad N = 5000, \qquad M_0 = 4480, \qquad N_0 = 80, \\
\alpha = \min\{2 - \tau_1, 2 - \tau_2, -3\}, \qquad a_{1n} = n - 2, \qquad b_{1n} = n - 1, \qquad c_{1n} = n - 5, \qquad d_{1n} = n - 3, \\
p_{1n} = \frac{(-1)^n n + 1}{4n + 100}, \qquad p_{2n} = \frac{64n^8 + 1}{8n^8 + 3n^4 + 7}, \qquad \overline{p}_1 = \underline{p}_1 = \frac{1}{4}, \qquad \overline{p}_0 = 8, \qquad \underline{p}_0 = 7, \\
q_{1n} = \frac{n^5 \ln^9 n - (-1)^n}{n^8 + 5n^7 + 1}, \qquad q_{2n} = \frac{n^6 - (-1)^n n^2 + 1}{n^8 + 7n^7 + 2}, \qquad f_1(n, u, v) = \frac{u - nv}{n^3 + u^2 v^2}, \\
f_2(n, u, v) = \frac{n u^2 - v \sin n}{n^2 \ln^2 n + 1}, \qquad r_{1n} = \frac{M(n+1)}{n^3 + N^4}, \qquad r_{2n} = \frac{M_0(M_0 + 1)}{n^2 \ln^2 n + 1}, \\
t_{1n} = \frac{n^3 + 3M^4 + n^4 + 3nM^2}{\left(n^3 + N^4\right)^2}, \quad t_{2n} = \frac{2M_0 + 1}{\left(n^2 \ln^2 n + 1\right)^2}, \quad \forall (n, u, v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered} \tag{3.35}$$

It is easy to see that (2.1), (2.2) with $i = 1$, (2.24), (2.86)–(2.88), (2.91), and (2.93) hold. Thus, Theorem 2.15 implies that System (3.34) possesses uncountably many bounded positive solutions in $A(N, M, N_0, M_0)$.

Example 3.15. Consider the nonlinear difference system with multiple delays:

$$\begin{aligned}
&\Delta\left(x_n + \frac{-20n + 1}{4n + 9} x_{n-\tau_1}\right) + \frac{n x_{n-1}^3}{n^3 + y_{n-2}^2} = \frac{\sqrt{n^3 + 3n^2\sqrt{n-1} + 1}}{n^5 + 3n^4 + 1}, \\
&\Delta\left(y_n + \frac{6n^2 + 1}{2n^2 + 5n + 3} y_{n-\tau_2}\right) + \frac{x_{n-2} y_{n-4}}{n^2 + \cos(n^2 + 1)} = \frac{n^4 + 5n^2 - 7}{n^6 + 4n^3 + 1}, \quad n \geq 2,
\end{aligned} \tag{3.36}$$

where $\tau_1, \tau_2 \in \mathbb{N}$ are fixed. Let

$$\begin{gathered}
n_0 = 2, \quad n_1 = 10, \quad h = k = 1, \quad M = 2N_0, \quad N = N_0 > 0, \quad M_0 = 30N_0, \\
\alpha = \min\{2-\tau_1, 2-\tau_2, -2\}, \quad a_{1n} = n-1, \quad b_{1n} = n-2, \quad c_{1n} = n-2, \quad d_{1n} = n-4, \\
p_{1n} = \frac{-20n+1}{4n+9}, \quad p_{2n} = \frac{6n^2+1}{2n^2+5n+3}, \quad \overline{p}_1 = 4, \quad \underline{p}_1 = 5, \quad \overline{p}_0 = 3, \quad \underline{p}_0 = 2, \\
q_{1n} = \frac{\sqrt{n^3+3n^2\sqrt{n-1}+1}}{n^5+3n^4+1}, \quad q_{2n} = \frac{n^4+5n^2-7}{n^6+4n^3+1}, \quad f_1(n,u,v) = \frac{nu^3}{n^3+v^2}, \\
f_2(n,u,v) = \frac{uv}{n^2+\cos(n^2+1)}, \quad r_{1n} = \frac{nM^3}{n^3+N^2}, \quad r_{2n} = \frac{M_0^2}{n^2-1}, \\
t_{1n} = \frac{nM^2(3n^3+5M^2)}{(n^3+N^2)^2}, \quad t_{2n} = \frac{2M_0}{(n^2-1)^2}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{gathered} \tag{3.37}$$

Clearly (2.1), (2.2) with $i = 1$, (2.24), (2.86)–(2.88), (2.94), and (2.95) hold. Thus, Theorem 2.16 implies that System (3.36) possesses uncountably many bounded positive solutions in $A(N, M, N_0, M_0)$.

Acknowledgments

The authors thank the editor and reviewers for useful comments and suggestions. This study was supported by research funds from Dong-A University.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 952601, 20 pages
doi:10.1155/2012/952601

Research Article

Asymptotic Behaviour of a Two-Dimensional Differential System with a Finite Number of Nonconstant Delays under the Conditions of Instability

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Received 20 February 2012; Accepted 4 April 2012

Academic Editor: Miroslava Růžičková

The asymptotic behaviour of a real two-dimensional differential system $x'(t) = \mathsf{A}(t)x(t) + \sum_{k=1}^{m} \mathsf{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t)))$ with unbounded nonconstant delays $t-\theta_k(t) \geq 0$ satisfying $\lim_{t\to\infty}\theta_k(t) = \infty$ is studied under the assumption of instability. Here, A, $\mathsf{B}_{k,}$ and h are supposed to be matrix functions and a vector function. The conditions for the instable properties of solutions and the conditions for the existence of bounded solutions are given. The methods are based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties are studied by means of a Lyapunov-Krasovskii functional and the suitable Ważewski topological principle. The results generalize some previous ones, where the asymptotic properties for two-dimensional systems with one constant or nonconstant delay were studied.

1. Introduction

Consider the real two-dimensional system

$$x'(t) = \mathsf{A}(t)x(t) + \sum_{k=1}^{m} \mathsf{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \ldots, x(\theta_m(t))), \tag{1.1}$$

where $\theta_k(t)$ are real functions, $\mathsf{A}(t) = (a_{ij}(t))$, $\mathsf{B}_k(t) = (b_{ijk}(t))(i, j = 1, 2; k = 1, \ldots, m)$ are real square matrices, and $h(t, x, y) = (h_1(t, x, y_1, \ldots, y_m), h_2(t, x, y_1, \ldots, y_m))$ is a real vector function, $x = (x_1, x_2)$, $y_k = (y_{1k}, y_{2k})$. It is supposed that the functions θ_k, a_{ij} are locally absolutely continuous on $[t_0, \infty)$, b_{ijk} are locally Lebesgue integrable on $[t_0, \infty)$, and the function h satisfies Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$.

There are a lot of papers dealing with the stability and asymptotic behaviour of n-dimensional real vector equations with delay. Among others we should mention the recent results [1–13]. Since the plane has special topological properties different from those of n-dimensional space, where $n \geq 3$ or $n = 1$, it is interesting to study the asymptotic behaviour of two-dimensional systems by using tools that are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. For the case of instability, it is useful to add to this combination the version of Ważewski topological principle formulated by Rybakowski in the papers [14, 15]. Using these techniques, we obtain new and easy applicable results on stability, asymptotic stability, instability, or boundedness of solutions of the system (1.1).

The main idea of the investigation, the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, was introduced for ordinary differential equations in the paper by Ráb and Kalas [16] in 1990. The principle was transferred to differential equations with delay by Kalas and Baráková [17] in 2002. The results in the case of instability were obtained for ODEs by Kalas and Osička [18] in 1994 and for delayed differential equations by Kalas [19] in 2005.

We extend such type of results to differential equations with a finite number of nonconstant delays. We introduce the transformation of the considered real system to one equation with complex-valued coefficients. We present sufficient conditions for the instability of a solution and for the existence of a bounded solution. The applicability of the results is demonstrated with several examples.

At the end of this introduction we append a brief overview of notation used in the paper and the transformation of the real system to one equation with complex-valued coefficients.

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}_+$ the set of all positive real numbers,

$\mathbb{R}_+^0$ the set of all nonnegative real numbers,

$\mathbb{R}_-$ the set of all negative real numbers,

$\mathbb{R}_-^0$ the set of all nonpositive real numbers,

$\mathbb{C}$ the set of all complex numbers,

$\mathcal{C}$ the class of all continuous functions $[-r, 0] \to \mathbb{C}$,

$AC_{\text{loc}}(I, M)$ the class of all locally absolutely continuous functions $I \to M$,

$L_{\text{loc}}(I, M)$ the class of all locally Lebesgue integrable functions $I \to M$,

$K(I \times \Omega, M)$ the class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$, Rez the real part of z,

Imz the imaginary part of z, and

$\overline{z}$ the complex conjugate of z.

Introducing complex variables $z = x_1 + ix_2$, $w_1 = y_{11} + iy_{12}$,..., $w_m = y_{m1} + iy_{m2}$, we can rewrite the system (1.1) into an equivalent equation with complex-valued coefficients

$$\begin{aligned} z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{k=1}^{m} [A_k(t)z(\theta_k(t)) + B_k(t)\overline{z}(\theta_k(t))] \\ + g(t, z(t), z(\theta_1(t)), \dots, z(\theta_m(t))), \end{aligned} \tag{1.2}$$

where $\theta_k \in AC_{\text{loc}}(J, \mathbb{R})$ for $k = 1, \dots, m$, $A_k, B_k \in L_{\text{loc}}(J, \mathbb{C})$, $a, b \in AC_{\text{loc}}(J, \mathbb{C})$, $g \in K(J \times \mathbb{C}^{m+1}, \mathbb{C})$, $J = [t_0, \infty)$.

The relations between the functions are as follows:

$$\begin{aligned} a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\ b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \\ A_k(t) &= \frac{1}{2}(b_{11k}(t) + b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) - b_{12k}(t)), \\ B_k(t) &= \frac{1}{2}(b_{11k}(t) - b_{22k}(t)) + \frac{i}{2}(b_{21k}(t) + b_{12k}(t)), \\ g(t, z, w_1, \dots, w_m) &= h_1\left(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w_1 + \overline{w}_1), \dots, \frac{1}{2i}(w_m - \overline{w}_m)\right) \\ &\quad + ih_2\left(t, \frac{1}{2}(z+\overline{z}), \frac{1}{2i}(z-\overline{z}), \frac{1}{2}(w_1+\overline{w}_1), \frac{1}{2i}(w_1-\overline{w}_1), \dots, \frac{1}{2i}(w_m-\overline{w}_m)\right). \end{aligned} \tag{1.3}$$

Conversely, putting

$$\begin{aligned} a_{11}(t) &= \text{Re}[a(t) + b(t)], & a_{12}(t) &= \text{Im}[b(t) - a(t)], \\ a_{21}(t) &= \text{Im}[a(t) + b(t)], & a_{22}(t) &= \text{Re}[a(t) - b(t)], \\ b_{11k}(t) &= \text{Re}[A_k(t) + B_k(t)], & b_{12k}(t) &= \text{Im}[B_k(t) - A_k(t)], \\ b_{21k}(t) &= \text{Im}[A_k(t) + B_k(t)], & b_{22k}(t) &= \text{Re}[A_k(t) - B_k(t)], \\ h_1(t, x, y_1, \dots, y_m) &= \text{Re}\,g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}), \\ h_2(t, x, y_1, \dots, y_m) &= \text{Im}\,g(t, x_1 + ix_2, y_{11} + iy_{12}, \dots, y_{m1} + iy_{m2}), \end{aligned} \tag{1.4}$$

the equation (1.2) can be written in the real form (1.1) as well.

2. Preliminaries

We consider (1.2) in the case when

$$\liminf_{t\to\infty}\left(\left|\operatorname{Im} a(t)\right| - |b(t)|\right) > 0 \tag{2.1}$$

and study the behavior of solutions of (1.2) under this assumption. This situation corresponds to the case when the equilibrium 0 of the autonomous homogeneous system

$$x' = \mathbf{A}x, \tag{2.2}$$

where $\mathbf{A}$ is supposed to be regular constant matrix, is a centre or a focus. See [16] for more details.

Regarding (2.1) and since the delay functions θ_k satisfy $\lim_{t\to\infty}\theta_k(t) = \infty$, there are numbers $T_1 \geq t_0$, $T \geq T_1$, and $\mu > 0$ such that

$$\left|\operatorname{Im} a(t)\right| > |b(t)| + \mu \quad \text{for } t \geq T_1, \qquad t \geq \theta_k(t) \geq T_1 \quad \text{for } t \geq T\,(k = 1, \dots, m). \tag{2.3}$$

Denote

$$\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{\left(\operatorname{Im} a(t)\right)^2 - |b(t)|^2}\operatorname{sgn}\left(\operatorname{Im} a(t)\right), \quad \tilde{c}(t) = -ib(t). \tag{2.4}$$

Notice that the above-defined function $\tilde{\gamma}$ need not be positive.

Since $|\tilde{\gamma}(t)| > |\operatorname{Im} a(t)|$ and $|\tilde{c}(t)| = |b(t)|$, the inequality

$$\left|\tilde{\gamma}(t)\right| > |\tilde{c}(t)| + \mu \tag{2.5}$$

is valid for $t \geq T_1$. It can be easily verified that $\tilde{\gamma}, \tilde{c} \in AC_{\mathrm{loc}}([T_1, \infty), \mathbb{C})$.

For the rest of this section we will denote

$$\tilde{\vartheta}(t) = \frac{\operatorname{Re}\left(\tilde{\gamma}(t)\tilde{\gamma}'(t) - \bar{\tilde{c}}(t)\tilde{c}'(t)\right) - \left|\tilde{\gamma}(t)\tilde{c}'(t) - \tilde{\gamma}'(t)\tilde{c}(t)\right|}{\tilde{\gamma}^2(t) - |\tilde{c}(t)|^2}. \tag{2.6}$$

The instability and boundedness of solutions are studied subject to suitable subsets of the following assumptions.

(i) The numbers $T_1 \geq t_0$, $T \geq T_1$, and $\mu > 0$ are such that (2.3) holds.

(ii) There exist functions $\tilde{\varkappa}, \tilde{\kappa}_k, \varrho : [T, \infty) \to \mathbb{R}$ such that

$$\begin{aligned}&\left|\widetilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\widetilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right|\\&\quad\le\widetilde{\varkappa}(t)\left|\widetilde{\gamma}(t)z+\widetilde{c}(t)\overline{z}\right|+\sum_{k=1}^{m}\widetilde{\kappa}_k(t)\left|\widetilde{\gamma}(\theta_k(t))w_k+\widetilde{c}(\theta_k(t))\overline{w}_k\right|+\varrho(t)\end{aligned}\tag{2.7}$$

for $t\ge T$, $z,w_k\in\mathbb{C}(k=1,\dots,m)$, where ϱ is continuous on $[T,\infty)$.

(ii_n) There exist numbers $R_n\ge 0$ and functions $\widetilde{\varkappa}_n,\widetilde{\kappa}_{nk}:[T,\infty)\to\mathbb{R}$ such that

$$\begin{aligned}&\left|\ \widetilde{\gamma}(t)g(t,z,w_1,\dots,\ w_m)+\widetilde{c}(t)g(t,z,w_1,\dots,w_m)\right|\\&\quad\le\widetilde{\varkappa}_n(t)\left|\widetilde{\gamma}(t)z+\widetilde{c}(t)z\right|+\sum_{k=1}^{m}\widetilde{\kappa}_{nk}(t)\ \left|\widetilde{\gamma}(\theta_k(t))w_k+\widetilde{c}(\theta_k(t))\overline{w}_k\right|\end{aligned}\tag{2.8}$$

for $t\ge\tau_n\ge T,|z|+\sum_{k=1}^{m}|w_k|>R_n$.

(iii) $\widetilde{\beta}\in AC_{\text{loc}}([T,\infty),\mathbb{R}_-^0)$ is a function satisfying

$$\theta_k'(t)\widetilde{\beta}(t)\le-\widetilde{\lambda}_k(t)\quad\text{a.e. on }[T,\infty),\tag{2.9}$$

where $\widetilde{\lambda}_k$ is defined for $t\ge T$ by

$$\widetilde{\lambda}_k(t)=\widetilde{\kappa}_k(t)+(|A_k(t)|+|B_k(t)|)\frac{\left|\widetilde{\gamma}(t)\right|+|\widetilde{c}(t)|}{\left|\widetilde{\gamma}(\theta_k(t))\right|-|\widetilde{c}(\theta_k(t))|}.\tag{2.10}$$

(iii_n) $\widetilde{\beta}_n\in AC_{\text{loc}}[T,\infty),\mathbb{R}_-^0)$ is a function satisfying

$$\theta_k'(t)\widetilde{\beta}_n(t)\le-\widetilde{\lambda}_{nk}(t)\qquad\text{a.e. on }[\tau_n,\infty),\tag{2.11}$$

where $\widetilde{\lambda}_{nk}$ is defined for $t\ge T$ by

$$\widetilde{\lambda}_{nk}(t)=\widetilde{\kappa}_{nk}(t)+(|A_k(t)|+|B_k(t)|)\frac{\left|\widetilde{\gamma}(t)\right|+|\widetilde{c}(t)|}{\left|\widetilde{\gamma}(\theta_k(t))\right|-|\widetilde{c}(\theta_k(t))|}.\tag{2.12}$$

(iv_n) $\widetilde{\Lambda}_n$ is a real locally Lebesgue integrable function satisfying the inequalities $\widetilde{\beta}_n'(t)\ge\widetilde{\Lambda}_n(t)\widetilde{\beta}_n(t)$, $\widetilde{\Theta}_n(t)\ge\widetilde{\Lambda}_n(t)$ for almost all $t\in[\tau_n,\infty)$, where $\widetilde{\Theta}_n$ is defined by

$$\widetilde{\Theta}_n(t)=\text{Re}\,a(t)+\widetilde{\vartheta}(t)-\widetilde{\varkappa}_n(t)+m\widetilde{\beta}_n(t).\tag{2.13}$$

Obviously, if A_k, B_k, $\widetilde{\kappa}_k$, and θ_k' are locally absolutely continuous on $[T,\infty)$ and $\widetilde{\lambda}_k(t)\ge 0$, $\theta_k'(t)>0$, the choice $\widetilde{\beta}(t)=-\max_{k=1,\dots,m}[\widetilde{\lambda}_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii). Similarly, if A_k,

B_k, $\widetilde{\kappa}_{nk}$, and θ'_k are locally absolutely continuous on $[T,\infty)$ and $\widetilde{\lambda}_{nk}(t) \geq 0$, $\theta'_k(t) > 0$, the choice $\widetilde{\beta}_n(t) = -\max_{k=1,\dots,m}[\widetilde{\lambda}_{nk}(t)(\theta'_k(t))^{-1}]$ is admissible in (iii$_{\mathrm{n}}$).

Denote

$$\widetilde{\Theta}(t) = \operatorname{Re} a(t) + \widetilde{\vartheta}(t) - \widetilde{\varkappa}(t). \tag{2.14}$$

From assumption (i) it follows that

$$\begin{aligned}\left|\widetilde{\vartheta}\right| &\leq \frac{\left|\operatorname{Re}\left(\widetilde{\gamma}\widetilde{\gamma}' - \overline{\widetilde{c}}\widetilde{c}'\right)\right| + \left|\widetilde{\gamma}c' - \widetilde{\gamma}'c\right|}{\widetilde{\gamma}^2 - |\widetilde{c}|^2} \leq \frac{\left(\left|\widetilde{\gamma}'\right| + |\widetilde{c}'|\right)\left(\left|\widetilde{\gamma}\right| + |\widetilde{c}|\right)}{\widetilde{\gamma}^2 - |\widetilde{c}|^2} \\ &= \frac{\left|\widetilde{\gamma}'\right| + |\widetilde{c}'|}{\left|\widetilde{\gamma}\right| - |\widetilde{c}|} \leq \frac{1}{\mu}\left(\left|\widetilde{\gamma}'\right| + \left|\widetilde{c}'\right|\right);\end{aligned} \tag{2.15}$$

therefore the function $\widetilde{\vartheta}$ is locally Lebesgue integrable on $[T,\infty)$, assuming that (i) holds true. If the relations $\widetilde{\beta}_n \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_-)$, $\widetilde{\varkappa}_n \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$, and $\widetilde{\beta}'_n(t)/\widetilde{\beta}_n(t) \leq \widetilde{\Theta}_n(t)$ for almost all $t \geq \tau_n$ together with conditions (i) and (ii$_{\mathrm{n}}$) are fulfilled, then we can choose $\widetilde{\Lambda}_n(t) = \widetilde{\Theta}_n(t)$ for $t \in [T,\infty)$ in (iv$_{\mathrm{n}}$).

3. Results

Theorem 3.1. *Let assumptions (i), (ii$_0$), (iii$_0$), and (iv$_0$) be fulfilled for some $\tau_0 \geq T$. Suppose there exist $t_1 \geq \tau_0$ and $\nu \in (-\infty,\infty)$ such that*

$$\inf_{t\geq t_1}\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)\,ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] \geq \nu. \tag{3.1}$$

If $z(t)$ is any solution of (1.2) *satisfying*

$$\min_{\theta(t_1)\leq s\leq t_1}|z(s)| > R_0, \qquad \Delta(t_1) > R_0 e^{-\nu}, \tag{3.2}$$

where

$$\begin{aligned}\theta(t) &= \min_{k=1,\dots,m}\theta_k(t), \\ \Delta(t) &= \left(\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|\right)|z(t)| + \widetilde{\beta}_0(t)\max_{\theta(t)\leq s\leq t}|z(s)|\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}\left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right)ds,\end{aligned} \tag{3.3}$$

then

$$|z(t)| \geq \frac{\Delta(t_1)}{\left|\tilde{\gamma}(t)\right| + |\tilde{c}(t)|} \exp\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)ds\right] \tag{3.4}$$

for all $t \geq t_1$, *for which* $z(t)$ *is defined.*

In the proof we use the following Lemma.

Lemma 3.2. *Let* $a_1, a_2, b_1, b_2 \in \mathbb{C}$, $|a_2| > |b_2|$. *Then,*

$$\operatorname{Re}\frac{a_1 z + b_1 \overline{z}}{a_2 z + b_2 \overline{z}} \geq \frac{\operatorname{Re}\left(a_1 \overline{a}_2 - b_1 \overline{b}_2\right) - |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2} \tag{3.5}$$

for $z \in \mathbb{C}$, $z \neq 0$.

The proof is analogous to that of Lemma 1 in [20, page 101] or to the proof of Lemma in [16, page 131].

Proof of Theorem 3.1. Let $z(t)$ be any solution of (1.2) satisfying (3.2). Consider the Lyapunov functional

$$V(t) = U(t) + \tilde{\beta}_0(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds, \tag{3.6}$$

where

$$U(t) = \left|\tilde{\gamma}(t)z(t) + \tilde{c}(t)\overline{z}(t)\right|. \tag{3.7}$$

For brevity we shall denote $w_k(t) = z(\theta_k(t))$ and we shall write the function of variable t simply without indicating the variable t, for example, $\tilde{\gamma}$ instead of $\tilde{\gamma}(t)$.

In view of (3.6), we have

$$\begin{aligned} V' = U' + \tilde{\beta}_0' \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)ds + m\tilde{\beta}_0 \left|\tilde{\gamma}z + \tilde{c}\overline{z}\right| \\ - \sum_{k=1}^{m} \theta_k' \tilde{\beta}_0 \left|\tilde{\gamma}(\theta_k(t))w_k + \tilde{c}(\theta_k(t))\overline{w}_k\right| \end{aligned} \tag{3.8}$$

for almost all $t \geq t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K} = \{t \geq t_1 : z(t) \text{ exists}, |z(t)| > R_0\}$. Clearly $U(t) \neq 0$ for $t \in \mathcal{K}$. The derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$.

Since $z(t)$ is a solution of (1.2), we obtain

$$
\begin{aligned}
UU' &= \operatorname{Re}\left[\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left(\widetilde{\gamma}'z+\widetilde{\gamma}z'+\widetilde{c}'\overline{z}+\widetilde{c}\overline{z}'\right)\right]\\
&= \operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left[\widetilde{\gamma}'z+\widetilde{c}'\overline{z}+\widetilde{\gamma}\left(az+b\overline{z}+\sum_{k=1}^{m}\left(A_kw_k+B_k\overline{w}_k\right)+g\right)\right.\right.\\
&\qquad\left.\left.+\widetilde{c}\left(\overline{az}+\overline{b}z\right)+\sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k+\overline{B}_kw_k+\overline{g}\right)\right]\right\}\\
&= \operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left[\widetilde{\gamma}'z+\widetilde{c}'\overline{z}+\left(\widetilde{\gamma}a+\widetilde{c}\overline{b}\right)z+\left(\widetilde{\gamma}b+\widetilde{c}\overline{a}\right)\overline{z}\right.\right.\\
&\qquad\left.\left.+\widetilde{\gamma}\left(\sum_{k=1}^{m}\left(A_kw_k+B_k\overline{w}_k\right)+g\right)+\widetilde{c}\left(\sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k+\overline{B}_kw_k\right)+\overline{g}\right)\right]\right\}
\end{aligned}
\tag{3.9}
$$

for almost all $t\in\mathcal{K}$. Taking into account

$$
\left(\widetilde{\gamma}a+\widetilde{c}\overline{b}\right)\widetilde{c}=\left(\widetilde{\gamma}b+\widetilde{c}\overline{a}\right)\widetilde{\gamma},
\tag{3.10}
$$

we get

$$
\begin{aligned}
UU' &\geq \operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left(\widetilde{\gamma}a+\widetilde{c}\overline{b}\right)\left(z+\frac{\widetilde{c}}{\widetilde{\gamma}}\overline{z}\right)\right\}\\
&\quad+\operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left[\widetilde{\gamma}\sum_{k=1}^{m}\left(A_kw_k+B_k\overline{w}_k\right)+\widetilde{c}\sum_{k=1}^{m}\left(\overline{A}_k\overline{w}_k+\overline{B}_kw_k\right)\right]\right\}\\
&\quad+\operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left(\widetilde{\gamma}g+\widetilde{c}\overline{g}\right)\right\}+\operatorname{Re}\left\{\left(\widetilde{\gamma}\overline{z}+\overline{\widetilde{c}}z\right)\left(\widetilde{\gamma}'z+\widetilde{c}'\overline{z}\right)\right\}\\
&\geq U^2\operatorname{Re}\left(a+\frac{\widetilde{c}}{\widetilde{\gamma}}\overline{b}\right)-U\left(\left|\widetilde{\gamma}\right|+\left|\widetilde{c}\right|\right)\left(\sum_{k=1}^{m}\left|A_kw_k+B_k\overline{w}_k\right|\right)\\
&\quad-U\left|\widetilde{\gamma}g+\widetilde{c}\overline{g}\right|+U^2\operatorname{Re}\frac{\widetilde{\gamma}'z+\widetilde{c}'\overline{z}}{\widetilde{\gamma}z+\widetilde{c}\overline{z}}.
\end{aligned}
\tag{3.11}
$$

By the use of Lemma 3.2, we get

$$
\operatorname{Re}\frac{\widetilde{\gamma}'z+\widetilde{c}'\overline{z}}{\widetilde{\gamma}z+\widetilde{c}\overline{z}}\geq\widetilde{\vartheta}.
\tag{3.12}
$$

The last inequality together with (2.12), taken for $n=0$, assumption (ii$_0$), and the relation

$$
\operatorname{Re}\left(a+\frac{\widetilde{c}}{\widetilde{\gamma}}\overline{b}\right)=\operatorname{Re}a
\tag{3.13}
$$

yields

$$
\begin{aligned}
UU' &\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - U\sum_{k=1}^{m}\left(\widetilde{\kappa}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|\right) \\
&\quad - U\left(\left|\widetilde{\gamma}\right| + |\widetilde{c}|\right)\left(\sum_{k=1}^{m}\frac{|A_k||w_k| + |B_k||\overline{w}_k|}{\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|}\left(\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|\right)\right) \\
&\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) \\
&\quad - U\left\{\sum_{k=1}^{m}\left[\widetilde{\kappa}_{0k} + (|A_k| + |B_k|)\frac{\left|\widetilde{\gamma}\right| + |\widetilde{c}|}{\left|\widetilde{\gamma}(\theta_k)\right| - |\widetilde{c}(\theta_k)|}\right]\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|\right\} \\
&\geq U^2\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - U\sum_{k=1}^{m}\widetilde{\lambda}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right|
\end{aligned}
\tag{3.14}
$$

for almost all $t \in \mathcal{K}$.

Consequently,

$$
U' \geq U\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0\right) - \sum_{k=1}^{m}\widetilde{\lambda}_{0k}\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right| \tag{3.15}
$$

for almost all $t \in \mathcal{K}$. Inequality (3.15) together with relation (3.8) gives

$$
\begin{aligned}
V' &\geq U\left(\operatorname{Re} a + \widetilde{\vartheta} - \widetilde{\varkappa}_0 + m\widetilde{\beta}_0\right) - \sum_{k=1}^{m}\left(\widetilde{\lambda}_{0k} + \theta_k'\widetilde{\beta}_0\right)\left|\widetilde{\gamma}(\theta_k)w_k + \widetilde{c}(\theta_k)\overline{w}_k\right| \\
&\quad + \widetilde{\beta}_0'\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}\left|\widetilde{\gamma}(s)z(s) + \widetilde{c}(s)\overline{z}(s)\right|ds.
\end{aligned}
\tag{3.16}
$$

Using (2.11) and (2.13) for $n = 0$, we obtain

$$
V'(t) \geq U(t)\widetilde{\Theta}_0(t) + \widetilde{\beta}_0'(t)\sum_{k=1}^{m}\int_{\theta_k(t)}^{t}U(s)ds. \tag{3.17}
$$

Hence, in view of (iv_0),

$$
V'(t) - \widetilde{\Lambda}_0(t)V(t) \geq 0 \tag{3.18}
$$

for almost all $t \in \mathcal{K}$.

Multiplying (3.18) by $\exp[-\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds]$ and integrating over $[t_1, t]$, we get

$$
V(t)\exp\left[-\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds\right] - V(t_1) \geq 0 \tag{3.19}
$$

on any interval $[t_1, \omega)$, where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. Now, with respect to (3.6), (3.7), and $\widetilde{\beta}_0 \le 0$, we have

$$\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)|z(t)| \ge V(t) \ge V(t_1)\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right] \ge \Delta(t_1)\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right]. \tag{3.20}$$

If (3.2) is fulfilled, there is $R > R_0$ such that $\Delta(t_1) > Re^{-\nu}$. By virtue of (3.1), and (3.2), we can easily see that

$$|z(t)| \ge \frac{\Delta(t_1)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|}\exp\left[\int_{t_1}^{t} \widetilde{\Lambda}_0(s)ds\right] \ge Re^{-\nu}e^{\nu} = R \tag{3.21}$$

for all $t \ge t_1$, for which $z(t)$ is defined. □

To obtain results on the existence of bounded solutions, we shall suppose that (1.2) satisfies the uniqueness property of solutions. Moreover, we suppose that the delays are bounded, that is, that the functions θ_k satisfy the condition

$$t - r \le \theta_k(t) \le t \quad \text{for } t \ge t_0 + r, \tag{3.22}$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T_1 = t_0 + r$, $T \ge T_1$, and $\mu > 0$ such that

$$\left|\mathrm{Im}\, a(t)\right| > |b(t)| + \mu \quad \text{for } t \ge T_1, \quad t \ge \theta_k(t) \ge t - r \quad \text{for } t \ge T (k = 1, \dots, m). \tag{5$'$}$$

In view of this, we replace (2.3) in assumption (i) with (5′). All other assumptions we keep in validity.

In the proof of the following theorem we shall utilize Ważewski topological principle for retarded functional differential equations of Carathéodory type. Details of this theory can be found in the paper of Rybakowski [15].

Theorem 3.3. *Let conditions (i), (ii), and (iii) be fulfilled, and let $\widetilde{\Lambda}$, θ_k' $(k = 1, \dots, m)$ be continuous functions such that the inequality $\widetilde{\Lambda}(t) \le \widetilde{\Theta}(t)$ holds a.e. on $[T, \infty)$, where $\widetilde{\Theta}$ is defined by* (2.14)*. Suppose that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\widetilde{\Lambda}(t) + \widetilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t} \xi(s)ds\right] - \xi(t) > \varrho(t)C^{-1}\exp\left[-\int_{T}^{t} \xi(s)ds\right] \tag{3.23}$$

for $t \in [T, \infty]$ *and some constant* $C > 0$*. Then, there exist* $t_2 > T$ *and a solution* $z_0(t)$ *of* (1.2) *satisfying*

$$|z_0(t)| \le \frac{C}{|\widetilde{\gamma}(t)| - |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s)ds\right] \tag{3.24}$$

for $t \ge t_2$.

Proof. Write (1.2) in the form

$$z' = F(t, z_t), \tag{2'}$$

where $F : J \times \mathcal{C} \to \mathbb{C}$ is defined by

$$\begin{aligned} F(t,\psi) = {} & a(t)\psi(0) + b(t)\overline{\psi}(0) + \sum_{k=1}^{m}\left[A_k(t)\psi(\theta_k(t) - t) + B_k(t)\overline{\psi}(\theta_k(t) - t)\right] \\ & + g(t, \psi(0), \psi(\theta_1(t) - t), \dots, \psi(\theta_m(t) - t)) \end{aligned} \tag{3.25}$$

and z_t is the element of $\mathcal{C}$ defined by the relation $z_t(\widetilde{\theta}) = z(t + \widetilde{\theta})$, $\widetilde{\theta} \in [-r, 0]$. Let $\tau > T$. Put

$$\begin{aligned} \widetilde{U}(t, z, \overline{z}) &= |\widetilde{\gamma}(t)z + \widetilde{c}(t)\overline{z}| - \varphi(t), \\ \varphi(t) &= C \exp\left[\int_T^t \xi(s)ds\right], \\ \Omega^0 &= \left\{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \overline{z}) < 0\right\}, \\ \Omega_{\widetilde{U}} &= \left\{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \overline{z}) = 0\right\}. \end{aligned} \tag{3.26}$$

It can be easily verified that Ω^0 is a polyfacial set generated by the functions $\widehat{U}(t) = \tau - t$, $\widetilde{U}(t, z, \overline{z})$ (see Rybakowski [15, page 134]). It holds that $\Omega_{\widetilde{U}} \subset \partial\Omega^0$. As $(|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|)|z(t)| \ge |\widetilde{\gamma}(t)z + \widetilde{c}(t)\overline{z}|$, we have

$$|z| \ge \frac{\varphi(t)}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} = \frac{C}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s)ds\right] > 0 \tag{3.27}$$

for $(t, z) \in \Omega_{\widetilde{U}}$. It holds that

$$D^+\widehat{U}(t) = \frac{\partial}{\partial t}(\tau - t) = -1 < 0. \tag{3.28}$$

Let $(t^*,\zeta)\in\Omega_{\widetilde{U}}$ and $\phi\in\mathcal{C}$ be such that $\phi(0)=\zeta$ and $(t^*+\widetilde{\theta},\phi(\widetilde{\theta}))\in\Omega^0$ for all $\widetilde{\theta}\in[-r,0)$. If $(t,\varphi)\in(\tau,\infty)\times\mathcal{C}$, then

$$\begin{aligned}
D^+\widetilde{U}(t,\varphi(0),\overline{\varphi}(0)) :&= \limsup_{h\to 0+}\left(\frac{1}{h}\right)\Big[\widetilde{U}\Big(t+h,\varphi(0)+hF(t,\varphi),\overline{\varphi}(0)+h\overline{F}(t,\varphi)\Big)\\
&\qquad -\widetilde{U}(t,\varphi(0),\overline{\varphi}(0))\Big]\\
&= \frac{\partial\widetilde{U}(t,\varphi(0)\ ,\overline{\varphi}(0))}{\partial t}+\frac{\partial\widetilde{U}(t,\varphi(0),\overline{\varphi}(0))}{\partial z}F(t,\varphi)\\
&\quad+\frac{\partial\widetilde{U}(t,\varphi(0),\overline{\varphi}(0))}{\partial\overline{z}}\overline{F}(t,\varphi).
\end{aligned}\tag{3.29}$$

Therefore,

$$\begin{aligned}
D^+\widetilde{U}(t,\varphi(0),\overline{\varphi}(0)) &= \left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|\mathrm{Re}\frac{\widetilde{\gamma}'(t)\varphi(0)+\widetilde{c}'(t)\overline{\varphi}(0)}{\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)}-\varphi'(t)\\
&\quad+\frac{1}{2}\left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|^{-1}\\
&\quad\times\Big\{\Big[\overline{\widetilde{\gamma}}(t)\Big(\widetilde{\gamma}(t)\overline{\varphi}(0)+\overline{\widetilde{c}}(t)\varphi(0)\Big)+\big(\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\big)\overline{\widetilde{c}}(t)\Big]F(t,\varphi)\\
&\qquad+\Big[\widetilde{c}(t)\Big(\widetilde{\gamma}(t)\overline{\varphi}(0)+\overline{\widetilde{c}}(t)\varphi(0)\Big)+\widetilde{\gamma}(t)\big(\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\big)\Big]\overline{F}(t,\varphi)\Big\}
\end{aligned}\tag{3.30}$$

provided that the derivatives $\widetilde{\gamma}'(t)$, $\widetilde{c}'(t)$ exist and that $\varphi(0)\neq 0$. Thus,

$$\begin{aligned}
D^+\widetilde{U}(t,\varphi(0),\overline{\varphi}(0)) &= \left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|\mathrm{Re}\frac{\widetilde{\gamma}'(t)\varphi(0)+\widetilde{c}'(t)\overline{\varphi}(0)}{\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)}-\varphi'(t)\\
&\quad+\left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|^{-1}\mathrm{Re}\Big\{\widetilde{\gamma}(t)\Big(\widetilde{\gamma}(t)\overline{\varphi}(0)+\overline{\widetilde{c}}(t)\varphi(0)\Big)F(t,\varphi)\\
&\qquad+\widetilde{c}(t)\Big(\widetilde{\gamma}(t)\overline{\varphi}(0)+\overline{\widetilde{c}}(t)\varphi(0)\Big)\overline{F}(t,\varphi)\Big\}\\
&= \left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|\mathrm{Re}\frac{\widetilde{\gamma}'(t)\varphi(0)+\widetilde{c}'(t)\overline{\varphi}(0)}{\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)}-\varphi'(t)\\
&\quad+\left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|^{-1}\mathrm{Re}\Big\{\Big(\widetilde{\gamma}(t)\overline{\varphi}(0)+\overline{\widetilde{c}}(t)\varphi(0)\Big)\\
&\qquad\times\Big(\widetilde{\gamma}(t)F(t,\varphi)+\widetilde{c}(t)\overline{F}(t,\varphi)\Big)\Big\}.
\end{aligned}\tag{3.31}$$

Using (3.10), (3.13), and (ii), similarly to the proof of Theorem 3.1, we obtain

$$\begin{aligned}
D^+\widetilde{U}(t,\varphi(0),\overline{\varphi}(0)) \geq{}& \left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|\mathrm{Re}\,a(t)\\
&-\sum_{k=1}^{m}\left|A_k(t)\varphi(\theta_k(t)-t)+B_k(t)\overline{\varphi}(\theta_k(t)-t)\right|\big(\left|\widetilde{\gamma}(t)\right|+|\widetilde{c}(t)|\big)\\
&-\widetilde{\varkappa}(t)\left|\widetilde{\gamma}(t)\varphi(0)+\widetilde{c}(t)\overline{\varphi}(0)\right|
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{m} \widetilde{\kappa}_k(t) \left| \widetilde{\gamma}(\theta_k(t))\varphi(\theta_k(t)-t) + \widetilde{c}(\theta_k(t))\overline{\varphi}(\theta_k(t)-t) \right| \\
&+ \widetilde{\vartheta}(t) \left| \widetilde{\gamma}(t)\varphi(0) + \widetilde{c}(t)\overline{\varphi}(0) \right| - \varrho(t) - \varphi'(t)
\end{aligned}
\tag{3.32}
$$

and consequently, with respect to (iii),

$$
\begin{aligned}
D^+\widetilde{U}(t,\varphi(0),\overline{\varphi}(0)) \geq{}& \left(\mathrm{Re}\,a(t) + \widetilde{\vartheta}(t) - \widetilde{\varkappa}(t)\right) \left| \widetilde{\gamma}(t)\varphi(0) + \widetilde{c}(t)\overline{\varphi}(0) \right| \\
&- \sum_{k=1}^{m} \widetilde{\lambda}_k(t) \left| \widetilde{\gamma}(\theta_k(t))\varphi(\theta_k(t)-t) + \widetilde{c}(\theta_k(t))\overline{\varphi}(\theta_k(t)-t) \right| - \varrho(t) - \varphi'(t) \\
\geq{}& \widetilde{\Theta}(t) \left| \widetilde{\gamma}(t)\varphi(0) + \widetilde{c}(t)\overline{\varphi}(0) \right| \\
&+ \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \left| \widetilde{\gamma}(\theta_k(t))\varphi(\theta_k(t)-t) + \widetilde{c}(\theta_k(t))\overline{\varphi}(\theta_k(t)-t) \right| - \varrho(t) - \varphi'(t) \\
\geq{}& \widetilde{\Lambda}(t) \left| \widetilde{\gamma}(t)\varphi(0) + \widetilde{c}(t)\overline{\varphi}(0) \right| \\
&+ \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \left| \widetilde{\gamma}(\theta_k(t))\varphi(\theta_k(t)-t) + \widetilde{c}(\theta_k(t))\overline{\varphi}(\theta_k(t)-t) \right| - \varrho(t) - \varphi'(t)
\end{aligned}
\tag{3.33}
$$

for almost all $t \in (\tau, \infty)$ and for $\psi \in C$ sufficiently close to ϕ. Replacing t and ψ by t^* and ϕ, respectively, in the last expression, we get

$$
\begin{aligned}
&\widetilde{\Lambda}(t^*) \left| \widetilde{\gamma}(t^*)\phi(0) + \widetilde{c}(t^*)\overline{\phi}(0) \right| + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \left| \widetilde{\gamma}(\theta_k(t^*))\phi(\theta_k(t^*)-t^*) + \widetilde{c}(\theta_k(t^*))\overline{\phi}(\theta_k(t^*)-t^*) \right| \\
&\quad - \varrho(t^*) - \varphi'(t^*) \\
&\geq \widetilde{\Lambda}(t^*) \left| \widetilde{\gamma}(t^*)\zeta + \widetilde{c}(t^*)\overline{\zeta} \right| + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*)\varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\
&\geq \widetilde{\Lambda}(t^*)\varphi(t^*) + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*)\varphi(\theta_k(t^*)) - \varrho(t^*) - \varphi'(t^*) \\
&= \widetilde{\Lambda}(t^*) C \exp\left[\int_T^{t^*} \xi(s)ds\right] + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) C \exp\left[\int_T^{\theta_k(t^*)} \xi(s)ds\right] \\
&\quad - \varrho(t^*) - C\xi(t^*) \exp\left[\int_T^{t^*} \xi(s)ds\right] \\
&= \left\{ \widetilde{\Lambda}(t^*) + \widetilde{\beta}(t^*) \sum_{k=1}^{m} \theta_k'(t^*) \exp\left[-\int_{\theta_k(t^*)}^{t^*} \xi(s)ds\right] - \xi(t^*) \right\} C \exp\left[\int_T^{t^*} \xi(s)ds\right] \\
&\quad - \varrho(t^*) > 0.
\end{aligned}
\tag{3.34}
$$

Therefore, in view of the continuity, $D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) > 0$ holds for ψ sufficiently close to ϕ and almost all t sufficiently close to t^*. Hence, Ω^0 is a regular polyfacial set with respect to (2′).

Choose $Z = \{(t_2,z) \in \Omega^0 \cup \Omega_{\widetilde{U}}\}$, where $t_2 > \tau + r$ is fixed. It can be easily verified that $Z \cap \Omega_{\widetilde{U}}$ is a retract of $\Omega_{\widetilde{U}}$, but $Z \cap \Omega_{\widetilde{U}}$ is not a retract of Z. Let $\eta \in C$ be such that $\eta(0) = 1$ and $0 \le \eta(\theta) < 1$ for $\theta \in [-r,0)$. Define the mapping $p : Z \to C$ for $(t_2,z) \in Z$ by the relation

$$p(t_2,z)(\theta) = \frac{\varphi(t_2+\theta)\eta(\theta)}{\left(\widetilde{\gamma}^2(t_2+\theta) - |\widetilde{c}(t_2+\theta)|^2\right)\varphi(t_2)}\Big[\left(\widetilde{\gamma}(t_2)\widetilde{\gamma}(t_2+\theta) - \overline{\widetilde{c}}(t_2)\widetilde{c}(t_2+\theta)\right)z + \left(\widetilde{\gamma}(t_2+\theta)\widetilde{c}(t_2) - \widetilde{\gamma}(t_2)\widetilde{c}(t_2+\theta)\right)\overline{z}\Big] \tag{3.35}$$

The mapping p is continuous, and it holds that

$$p(t_2,z)(0) = z \quad \text{for } (t_2,z) \in Z, \qquad p(t_2,0)(\theta) = 0 \quad \text{for } \theta \in [-r,0]. \tag{3.36}$$

Since

$$\widetilde{\gamma}(t_2+\theta)p(t_2,z)(\theta) + \widetilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)} = \frac{\varphi(t_2+\theta)\eta(\theta)}{\varphi(t_2)}\left(\widetilde{\gamma}(t_2)z + \widetilde{c}(t_2)\overline{z}\right), \tag{3.37}$$

we have

$$\left|\widetilde{\gamma}(t_2)z + \widetilde{c}(t_2)\overline{z}\right| < \varphi(t_2), \tag{3.38}$$

$$\left|\widetilde{\gamma}(t_2+\theta)p(t_2,z)(\theta) + \widetilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)}\;\right| < \varphi(t_2+\theta) \tag{3.39}$$

for $(t_2,z) \in Z \cap \Omega^0$ and $\theta \in [-r,0]$. Clearly, inequality (3.39) holds also for $(t_2,z) \in Z \cap \Omega_{\widetilde{U}}$ and $\theta \in [-r,0)$.

Using a topological principle for retarded functional differential equations (see Rybakowski [15, Theorem 2.1]), we see that there is a solution $z_0(t)$ of (1.2) such that $(t,z_0(t)) \in \Omega^0$ for all $t \ge t_2$ for which the solution $z_0(t)$ exists. Obviously, $z_0(t)$ exists for all $t \ge t_2$ and

$$\left(\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|\right)|z_0(t)| \le \left|\widetilde{\gamma}(t)z_0(t) + \widetilde{c}(t)\overline{z}_0(t)\right| \le \varphi(t) \quad \text{for } t \ge t_2. \tag{3.40}$$

Hence

$$|z_0(t)| \le \frac{\varphi(t)}{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|} \quad \text{for } t \ge t_2. \tag{3.41}$$

□

Theorem 3.4. *Suppose that hypotheses (i), (ii), (ii$_n$), (iii), (iii$_n$), and (iv$_n$) are fulfilled for $\tau_n \ge T$ and $n \in \mathbb{N}$, where $R_n > 0$, $\inf_{n\in\mathbb{N}} R_n = 0$. Let $\widetilde{\Lambda}$, θ'_k be continuous functions satisfying the inequality*

$\widetilde{\Lambda}(t) \leq \widetilde{\Theta}(t)$ a.e. on $[T, \infty)$, where $\widetilde{\Theta}$ is defined by (2.14)*. Assume that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\widetilde{\Lambda}(t) + \widetilde{\beta}(t) \sum_{k=1}^{m} \theta_k'(t) \exp\left[-\int_{\theta_k(t)}^{t} \xi(s) ds \right] - \xi(t) > \rho(t) C^{-1} \exp\left(-\int_{T}^{t} \xi(s) ds \right) \tag{3.42}$$

for $t \in [T, \infty)$ and some constant $C > 0$. Suppose that

$$\limsup_{t \to \infty} \left[\int_{T}^{t} \left(\widetilde{\Lambda}_n(s) - \xi(s) \right) ds + \ln \frac{|\widetilde{\gamma}(t)| - |\widetilde{c}(t)|}{|\widetilde{\gamma}(t)| + |\widetilde{c}(t)|} \right] = \infty, \tag{3.43}$$

$$\lim_{t \to \infty} \left[\widetilde{\beta}_n(t) \max_{\theta(t) \leq s \leq t} \frac{\exp\left[\int_{T}^{s} \xi(\sigma)\, d\sigma \right]}{|\widetilde{\gamma}(s)| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} \left(|\widetilde{\gamma}(s)| + |\widetilde{c}(s)| \right) ds \right] = 0, \tag{3.44}$$

$$\inf_{\tau_n \leq s \leq t < \infty} \left[\int_{s}^{t} \widetilde{\Lambda}_n(\sigma) d\sigma - \ln\left(|\widetilde{\gamma}(t)| + |\widetilde{c}(t)| \right) \right] \geq \nu \tag{3.45}$$

for $n \in \mathbb{N}$, where $\theta(t) = \min_{k=1,\ldots,m} \theta_k(t)$ and $\nu \in (-\infty, \infty)$. Then, there exists a solution $z_0(t)$ of (1.2) *such that*

$$\lim_{t \to \infty} \min_{\theta(t) \leq s \leq t} |z_0(s)| = 0. \tag{3.46}$$

Proof. By the use of Theorem 3.3 we observe that there is a $t_2 \geq T$ and a solution $z_0(t)$ of (1.2) with property

$$|z_0(t)| \leq \frac{C}{|\widetilde{\gamma}(t)| - |\widetilde{c}(t)|} \exp\left[\int_{T}^{t} \xi(s) ds \right] \tag{3.47}$$

for $t \geq t_2$. Suppose that (3.46) is not satisfied. Then, there is $\varepsilon_0 > 0$ such that

$$\limsup_{t \to \infty} \min_{\theta(t) \leq s \leq t} |z_0(s)| > \varepsilon_0. \tag{3.48}$$

Choose $N \in \mathbb{N}$ such that

$$\max\left\{ R_N, \frac{2}{\mu} R_N e^{-\nu} \right\} < \varepsilon_0. \tag{3.49}$$

It holds that

$$\min_{\theta(\tau) \leq s \leq \tau} |z_0(s)| > \max\left\{ R_N, \frac{2}{\mu} R_N e^{-\nu} \right\} \tag{3.50}$$

for some $\tau > \max\{T, \tau_N, t_2\}$. In view of (3.44), we can suppose that

$$\left|\widetilde{\beta}_N(\tau)\right| C \max_{\theta(\tau)\le s\le\tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\left|\widetilde{\gamma}(s)\right| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds < \frac{1}{2} R_N e^{-\nu}. \tag{3.51}$$

Therefore, taking into account (2.5), (3.47), (3.50), and (3.51) and the nonpositiveness of β_N, we have

$$\begin{aligned}
&\left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| + \widetilde{\beta}_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds \\
&\quad \ge \left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| \\
&\qquad + \widetilde{\beta}_N(\tau) C \max_{\theta(\tau)\le s\le\tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{\left|\widetilde{\gamma}(s)\right| - |\widetilde{c}(s)|} \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds \\
&\quad \ge \mu \frac{2}{\mu} R_N e^{-\nu} - \frac{1}{2} R_N e^{-\nu} > R_N e^{-\nu}.
\end{aligned} \tag{3.52}$$

Moreover, (3.45) implies that

$$\inf_{\tau\le t<\infty} \left[\int_\tau^t \widetilde{\Lambda}_N(s) ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] \ge \nu > -\infty. \tag{3.53}$$

By Theorem 3.1, we obtain an estimation

$$|z_0(t)| \ge \frac{\Psi(\tau)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \exp\left[\int_\tau^t \widetilde{\Lambda}_N(s) ds\right] \tag{3.54}$$

for all $t \ge \tau$, Ψ being defined by

$$\Psi(\tau) = \left(\left|\widetilde{\gamma}(\tau)\right| - |\widetilde{c}(\tau)|\right)|z_0(\tau)| + \widetilde{\beta}_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \sum_{k=1}^{m} \int_{\theta_k(\tau)}^{\tau} \left(\left|\widetilde{\gamma}(s)\right| + |\widetilde{c}(s)|\right) ds. \tag{3.55}$$

Relation (3.47) together with (3.54) yields

$$\frac{\Psi(\tau)}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \exp\left[\int_\tau^t \widetilde{\Lambda}_N(s) ds\right] \le \frac{C}{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|} \exp\left[\int_T^t \xi(s) ds\right], \tag{3.56}$$

that is

$$\int_T^t \left[\widetilde{\Lambda}_N(s) - \xi(s)\right] ds + \ln \frac{\left|\widetilde{\gamma}(t)\right| - |\widetilde{c}(t)|}{\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|} \le \int_T^\tau \widetilde{\Lambda}_N(s) ds - \ln\left[C^{-1}\Psi(\tau)\right] \tag{3.57}$$

for $t \ge \tau$. However, the last inequality contradicts (3.43) and Theorem 3.4 is proved. □

From Theorem 3.1 we easily obtain several corollaries.

Corollary 3.5. *Let the assumptions of Theorem 3.1 be fulfilled with* $R_0 > 0$. *If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] = \varsigma > \nu, \tag{3.58}$$

then for any ε, $0 < \varepsilon < R_0 e^{\varsigma-\nu}$, *there is* $t_2 \geq t_1$ *such that*

$$|z(t)| > \varepsilon \tag{3.59}$$

for all $t \geq t_2$, *for which* $z(t)$ *is defined.*

Proof. Without loss of generality we can assume that $\varepsilon > R_0$. Choose χ, $0 < \chi < 1$ such that $R_0 < \varepsilon < \chi R_0 e^{\varsigma-\nu}$. In view of (3.58), there is $t_2 \geq t_1$ such that

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right) > \varsigma + \ln\chi \tag{3.60}$$

for $t \geq t_2$. Hence,

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right) > \nu + \ln\frac{\varepsilon}{R_0} \tag{3.61}$$

for $t \geq t_2$. Estimation (3.4) together with (3.2) now yields

$$|z(t)| > R_0 e^{-\nu} e^{\nu}\frac{\varepsilon}{R_0} = \varepsilon \tag{3.62}$$

for all $t \geq t_2$, for which $z(t)$ is defined. □

Corollary 3.6. *Let the assumptions of Theorem 3.1 be fulfilled with* $R_0 > 0$. *If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)ds - \ln\left(\left|\widetilde{\gamma}(t)\right| + |\widetilde{c}(t)|\right)\right] = \infty, \tag{3.63}$$

then for any $\varepsilon > 0$ *there exists* $t_2 \geq t_1$ *such that* (3.59) *holds for all* $t \geq t_2$, *for which* $z(t)$ *is defined.*

The efficiency of Theorem 3.1 and Corollary 3.6 is demonstrated in the following example.

Example 3.7. Consider (1.2) where $a(t) \equiv 4 + 3i$, $b(t) \equiv 2$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$ for $k = 1, \ldots, m$, $\theta_k(t) = t + (1/2k)(\cos kt - 1)$, $g(t, z, w_1, \ldots, w_m) = 3z + \sum_{k=1}^{m}(1/2m)e^{-t}w_k$.

Obviously, $t-(1/k) \le \theta_k(t) \le t$ and $1/2 \le \theta_k'(t) \le (3/2)$. Suppose that $t_0 = 1$ and $T \ge 2$. Then, $\tilde{\gamma} \equiv 3+\sqrt{5}$, $\tilde{c} \equiv -2i$. Further,

$$\begin{aligned}\left|\tilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\tilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right| &\le 3\left|\tilde{\gamma}(t)z+\tilde{c}(t)\overline{z}\right| \\ &+\sum_{k=1}^{m}\frac{1}{2m}\,\mathrm{e}^{-t}\left|\tilde{\gamma}(\theta_k(t))w_k+\tilde{c}(\theta_k(t))\overline{w}_k\right|.\end{aligned} \tag{3.64}$$

Taking $\tilde{\varkappa}_0(t) \equiv 3$, $\tilde{\kappa}_{0k}(t) = (1/2m)\mathrm{e}^{-t}$, $\tau_0 = T$, $R_0 = 0$, $\tilde{\vartheta}(t) \equiv 0$, $\tilde{\beta}_0(t) = -(1/m)\mathrm{e}^{-t}$, $\tilde{\Lambda}_0(t) = \tilde{\Theta}_0(t) = 1-\mathrm{e}^{-t}$ (>0) in Theorem 3.1, we have

$$\theta_k'(t)\tilde{\beta}_0(t) \le -\tilde{\lambda}_{0k}(t), \qquad \tilde{\beta}_0'(t) \ge \tilde{\Theta}_0(t)\tilde{\beta}_0(t) \tag{3.65}$$

for $t \in [T,\infty)$ and Theorem 3.1 and Corollary 3.6 are applicable to the considered equation.

As a corollary of Theorem 3.3 we obtain sufficient conditions for the existence of a bounded solution of (1.2) or the existence of a solution $z_0(t)$ of (1.2) satisfying $\lim_{t\to\infty} z_0(t) = 0$.

Corollary 3.8. *Let the assumptions of Theorem 3.3 be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{\left|\tilde{\gamma}(t)\right|-\left|\tilde{c}(t)\right|}\exp\left(\int_T^t \xi(s)ds\right)\right] < \infty, \tag{3.66}$$

then there is a bounded solution $z_0(t)$ of (1.2). If

$$\lim_{t\to\infty}\left[\frac{1}{\left|\tilde{\gamma}(t)\right|-\left|\tilde{c}(t)\right|}\exp\left(\int_T^t \xi(s)ds\right)\right] = 0, \tag{3.67}$$

then there is a solution $z_0(t)$ of (1.2) such that

$$\lim_{t\to\infty} z_0(t) = 0. \tag{3.68}$$

The next example shows how Theorem 3.3 and Corollary 3.8 (namely the first part) can be used.

Example 3.9. Consider (1.2) where $a(t) \equiv 4+3i$, $b(t) \equiv i$, $A_k(t) \equiv 0$, $B_k(t) \equiv 0$, $\theta_k(t) = t - e^{-kt}$ for $k = 1,\dots,m$, $g(t,z,w_1,\dots,w_m) = (1/2)z + \sum_{k=1}^{m}(1/4m)w_k + e^{-t}$.

Obviously $t-1 \le \theta_k(t) \le t$ and $\theta_k'(t) = 1 + k\mathrm{e}^{-kt} \ge 1 > 0$ for $t \ge 0$. Suppose that $t_0 = 1$ and $T \ge 2$. Then,

$$\begin{gathered}\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{\left(\operatorname{Im} a(t)\right)^2 - |b(t)|^2 \operatorname{sgn}\left(\operatorname{Im} a(t)\right)} \equiv 3+2\sqrt{2}, \\ \tilde{c}(t) = -ib(t) \equiv 1.\end{gathered} \tag{3.69}$$

Further,

$$
\begin{aligned}
&\left|\tilde{\gamma}(t)g(t,z,w_1,\dots,w_m)+\tilde{c}(t)\overline{g}(t,z,w_1,\dots,w_m)\right|\\
&\quad\le \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\frac{1}{2}\left|\tilde{\gamma}(t)z+\tilde{c}(t)\overline{z}\right|\\
&\qquad+\frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\sum_{k=1}^{m}\left[\frac{1}{4m}\left|\tilde{\gamma}(\theta_k(t))w_k+\tilde{c}(\theta_k(t))\overline{w}_k\right|\right]+\overset{-t}{\mathrm{e}}\\
&\quad=\frac{\sqrt{2}}{2}\left|\tilde{\gamma}(t)z+\tilde{c}(t)\overline{z}\right|+\sqrt{2}\sum_{k=1}^{m}\left[\frac{1}{4m}\left|\tilde{\gamma}(\theta_k(t))w_k+\tilde{c}(\theta_k(t))\overline{w}_k\right|\right]+\overset{-t}{\mathrm{e}}.
\end{aligned}
\tag{3.70}
$$

If we take $\tilde{\varkappa}(t)\equiv\sqrt{2}/2$, $\tilde{\kappa}_k(t)=\sqrt{2}/4m$, $\tilde{\vartheta}(t)\equiv 0$, $\tilde{\beta}(t)=-\sqrt{2}/4m$, $\tilde{\Lambda}(t)=\tilde{\Theta}(t)=4-(\sqrt{2}/2)$ in Theorem 3.3, we observe that

$$
\theta_k'(t)\tilde{\beta}(t)=-\left(1+k\overset{-kt}{\mathrm{e}}\right)\frac{\sqrt{2}}{4m}\le-\frac{\sqrt{2}}{4m}=-\tilde{\lambda}_k(t)
\tag{3.71}
$$

for $t\in[T,\infty)$. Then, for $\xi\equiv 0$ and $C=1$, we have

$$
\begin{aligned}
&\tilde{\Lambda}(t)+\tilde{\beta}(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t}\xi(s)ds\right]-\xi(t)\\
&\quad=4-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4m}\sum_{k=1}^{m}\left(1+k\overset{-kt}{\mathrm{e}}\right)\ge 4-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4m}\cdot 2m=4-\sqrt{2}>1\\
&\quad>\overset{-t}{\mathrm{e}}=\rho(t)C^{-1}\exp\left(-\int_{T}^{t}\xi(s)ds\right)
\end{aligned}
\tag{3.72}
$$

for $t\in[T,\infty)$ and

$$
\limsup_{t\to\infty}\left[\frac{1}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left(\int_{T}^{t}\xi(s)ds\right)\right]=\frac{1}{2+2\sqrt{2}}<\infty,
\tag{3.73}
$$

and hence the assertions of Theorem 3.3 and the first part of Corollary 3.8 hold true.

4. Summary

We investigated the problem of instability and asymptotic behaviour of real two-dimensional differential system with a finite number of nonconstant delays. We focused on the case corresponding to the situation when the equilibrium 0 of the autonomous system (2.2) is a focus or a centre and it is unstable. We obtained several criteria for instability properties of the solutions as well as conditions for the existence of bounded solutions. We used the methods of complexification, the method of Lyapunov-Krasovskii functional and a Ważewski topological principle for retarded functional differential equations of Carathéodory type. At the end we supplied several corollaries and explanatory examples.

Acknowledgment

Z. Šmarda was supported by Project FEKT-S-11-2(921).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 219040, 27 pages
doi:10.1155/2012/219040

Research Article

Solution of the First Boundary-Value Problem for a System of Autonomous Second-Order Linear Partial Differential Equations of Parabolic Type with a Single Delay

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Received 29 March 2012; Accepted 9 May 2012

Academic Editor: Miroslava Růžičková

The first boundary-value problem for an autonomous second-order system of linear partial differential equations of parabolic type with a single delay is considered. Assuming that a decomposition of the given system into a system of independent scalar second-order linear partial differential equations of parabolic type with a single delay is possible, an analytical solution to the problem is given in the form of formal series and the character of their convergence is discussed. A delayed exponential function is used in order to analytically solve auxiliary initial problems (arising when Fourier method is applied) for ordinary linear differential equations of the first order with a single delay.

1. Introduction

In this paper, we deal with an autonomous second-order system of linear partial differential equations of the parabolic type with a single delay

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= a_{11}\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + a_{12}\frac{\partial^2 v(x,t-\tau)}{\partial x^2} + b_{11}\frac{\partial^2 u(x,t)}{\partial x^2} + b_{12}\frac{\partial^2 v(x,t)}{\partial x^2},\\
\frac{\partial u(x,t)}{\partial t} &= a_{21}\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + a_{22}\frac{\partial^2 v(x,t-\tau)}{\partial x^2} + b_{21}\frac{\partial^2 u(x,t)}{\partial x^2} + b_{22}\frac{\partial^2 v(x,t)}{\partial x^2},
\end{aligned} \tag{1.1}$$

where the matrices of coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{1.2}$$

are constant and $\tau > 0$, $\tau = \text{const}$.

Usually, when systems of differential equations are investigated, the main attention is paid to systems of ordinary differential equations or systems of partial differential equations [1–5]. The analysis of systems of partial differential equations with delay is rather neglected. This investigation is extremely rare.

The first boundary-value problem for (1.1) is solved for A having real eigenvalues λ_1, λ_2, and B having real eigenvalues $\sigma_1 > 0$, $\sigma_2 > 0$. Throughout the paper, we assume that there exists a real constant regular matrix

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{1.3}$$

simultaneously reducing both matrices A and B into diagonal forms

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{1.4}$$

that is,

$$S^{-1}AS = \Lambda, \qquad S^{-1}BS = \Sigma, \tag{1.5}$$

where

$$S^{-1} = \frac{1}{\Delta}\begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix}, \qquad \Delta = s_{11}s_{22} - s_{12}s_{22}. \tag{1.6}$$

For some classes of matrices, suitable transformations are known. Let us mention one of such results [6, Theorem 11′, page 291]. First, we recall that a complex square matrix $\mathcal{A}$ is a normal matrix if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$ where $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}$. If $\mathcal{A}$ is a real matrix, then $\mathcal{A}^* = \mathcal{A}^T$, that is, the real matrix is normal if $\mathcal{A}^T\mathcal{A} = \mathcal{A}\mathcal{A}^T$. A square matrix $\mathcal{U}$ is called unitary if $\mathcal{U}\mathcal{U}^* = E$ where E is the identity matrix.

Theorem 1.1. *If a finite or infinite set of pairwise commuting normal matrices is given, then all these matrices can be carried by one and the same unitary transformation into a diagonal form.*

Let l be a positive constant, and let

$$\begin{aligned} \mu_i &: [-\tau, \infty) \longrightarrow \mathbb{R}, \quad i = 1, 2, \\ \theta_i &: [-\tau, \infty) \longrightarrow \mathbb{R}, \quad i = 1, 2, \\ \varphi &: [0, l] \times [-\tau, 0] \longrightarrow \mathbb{R}, \\ \psi &: [0, l] \times [-\tau, 0] \longrightarrow \mathbb{R} \end{aligned} \tag{1.7}$$

be continuously differentiable functions such that

$$\begin{aligned} \mu_1(t) &= \varphi(0,t), \qquad t \in [-\tau,0], \\ \mu_2(t) &= \varphi(l,t), \qquad t \in [-\tau,0], \\ \theta_1(t) &= \psi(0,t), \qquad t \in [-\tau,0], \\ \theta_2(t) &= \psi(l,t), \qquad t \in [-\tau,0]. \end{aligned} \tag{1.8}$$

Together with system (1.1), we consider the first boundary-value problem, that is, the boundary conditions

$$u(0,t) = \mu_1(t), \qquad t \in [-\tau,\infty), \tag{1.9}$$

$$u(l,t) = \mu_2(t), \qquad t \in [-\tau,\infty), \tag{1.10}$$

$$v(0,t) = \theta_1(t), \qquad t \in [-\tau,\infty), \tag{1.11}$$

$$v(l,t) = \theta_2(t), \qquad t \in [-\tau,\infty) \tag{1.12}$$

and the initial conditions

$$u(x,t) = \varphi(x,t), \quad (x,t) \in [0,l] \times [-\tau,0], \tag{1.13}$$

$$v(x,t) = \psi(x,t), \quad (x,t) \in [0,l] \times [-\tau,0]. \tag{1.14}$$

A solution to the first boundary-value problem (1.1), (1.9)–(1.14) is defined as a pair of functions

$$u,v : [0,l] \times [-\tau,\infty) \longrightarrow \mathbb{R}, \tag{1.15}$$

continuously differentiable with respect to variable t if $(x,t) \in [0,l] \times [0,\infty)$, twice continuously differentiable with respect to x if $(x,t) \in [0,l] \times [0,\infty)$, satisfying the system (1.1) for $(x,t) \in [0,l] \times [0,\infty)$, the boundary conditions (1.9)–(1.12), and the initial conditions (1.13), (1.14). If necessary, we restrict the above definition of the solution to $(x,t) \in [0,l] \times [-\tau,k\tau]$ where k is a positive integer.

The purpose of the paper is to describe a method of constructing a solution of the above boundary-initial problem. Assuming that a decomposition of system (1.1) into a system of independent scalar second-order linear partial differential equations of parabolic type with a single delay is possible, an analytical solution to the problem (1.9)–(1.14) is given in the form of formal series in part 3. Their uniform convergence as well as uniform convergence of the partial derivatives of a formal solution is discussed in part 4. A delayed exponential function (defined in part 2 together with the description of its main properties) is used in order to analytically solve auxiliary initial problems (arising when Fourier method is applied) for ordinary linear differential equations of the first-order with a single delay.

To demonstrate this method, we will use systems of two equations only, although it can simply be extended to systems of n equations.

2. Preliminaries—Representation of Solutions of Linear Differential Equations with a Single Delay

A solution of the systems (1.1) satisfying all boundary and initial conditions (1.9)–(1.14) will be constructed by the classical method of separation of variables (Fourier method). Nevertheless, due to delayed arguments, complications arise in solving analytically auxiliary initial Cauchy problems for first-order linear differential equations with a single delay. We overcome this circumstance by using a special function called a delayed exponential, which is a particular case of the delayed matrix exponential (as defined, e.g., in [7–10]). Here we give a definition of the delayed exponential, its basic properties needed, and a solution of the initial problem for first-order homogeneous and nonhomogeneous linear differential equations with a single delay.

Definition 2.1. Let $b \in \mathbb{R}$. The delayed exponential function $\exp_\tau\{b,t\} : \mathbb{R} \to \mathbb{R}$ is a function continuous on $\mathbb{R} \setminus \{-\tau\}$ defined as

$$\exp_\tau\{b,t\} = \begin{cases} 0 & \text{if } -\infty < t < -\tau, \\ 1 & \text{if } -\tau \le t < 0, \\ 1 + b\dfrac{t}{1!} & \text{if } 0 \le t < \tau, \\ \cdots & \\ 1 + b\dfrac{t}{1!} + b^2\dfrac{(t-\tau)^2}{2!} + \cdots + b^k\dfrac{(t-(k-1)\tau)^k}{k!} & \text{if } (k-1)\tau \le t < k\tau, \\ \cdots & \end{cases} \tag{2.1}$$

where $k = 0, 1, 2, \ldots$.

Lemma 2.2. *For the differentiation of a delayed exponential function, the formula*

$$\frac{d}{dt}\exp_\tau\{b,t\} = b\exp_\tau\{b,t-\tau\} \tag{2.2}$$

holds within every interval $(k-1)\tau \le t < k\tau$, $k = 0, 1, 2, \ldots$.

Proof. Within the intervals $(k-1)\tau \le t < k\tau$, $k = 0, 1, 2, \ldots$, the delayed exponential function is expressed as

$$\exp_\tau\{b,t\} = 1 + b\frac{t}{1!} + b^2\frac{(t-\tau)^2}{2!} + b^3\frac{(t-2\tau)^3}{3!} + \cdots + b^k\frac{(t-(k-1)\tau)^k}{k!}. \tag{2.3}$$

Differentiating this expression, we obtain

$$\begin{aligned}\frac{d}{dt}\exp_\tau\{b,t\} &= b + b^2\frac{t-\tau}{1!} + b^3\frac{(t-2\tau)^2}{2!} + \cdots + b^k\frac{(t-(k-1)\tau)^{k-1}}{(k-1)!}\\ &= b\left[1 + b\frac{t-\tau}{1!} + b^2\frac{(t-2\tau)^2}{2!} + \cdots + b^{k-1}\frac{[t-(k-1)\tau]^{k-1}}{(k-1)!}\right]\\ &= b\exp_\tau\{b,t-\tau\}.\end{aligned} \tag{2.4}$$

□

2.1. First-Order Homogeneous Linear Differential Equations with a Single Delay

Let us consider a linear homogeneous equation with a single delay

$$\dot{x}(t) = bx(t-\tau), \tag{2.5}$$

where $b \in \mathbb{R}$, together with the initial Cauchy condition

$$x(t) = \beta(t), \quad t \in [-\tau, 0]. \tag{2.6}$$

From (2.2), it immediately follows that the delayed exponential $\exp_\tau\{b,t\}$ is a solution of the initial Cauchy problems (2.5), (2.6) with $\beta(t) \equiv 1$, $t \in [-\tau, 0]$.

Theorem 2.3. *Let $\beta : [-\tau, 0] \to \mathbb{R}$ be a continuously differentiable function. Then the unique solution of the initial Cauchy problems* (2.5), (2.6) *can be represented as*

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)ds, \tag{2.7}$$

where $t \in [-\tau, \infty)$.

Proof. The representation (2.7) is a linear functional of the delayed exponential function $\exp_\tau\{b,t\}$ and $\exp_\tau\{b,t-\tau-s\}$. Because by Lemma 2.2 the delayed exponential function is the solution of (2.5), the functional on the right-hand side of (2.7) is a solution of the homogeneous equation (2.5) for arbitrary (differentiable) $\beta(t)$.

We will show that initial condition (2.6) is satisfied as well, that is, we will verify that, for $-\tau \le t \le 0$, the next identity is correct:

$$\beta(t) \equiv \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)ds. \tag{2.8}$$

We rewrite (2.7) as

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{t}\exp_\tau\{b,t-\tau-s\}\beta'(s)ds + \int_{t}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)ds. \tag{2.9}$$

From Definition 2.1, it follows:

$$\begin{gathered}\exp_\tau\{b,t\} \equiv 1 \quad \text{if} \quad -\tau \le t \le 0,\\ \exp_\tau\{b,t-\tau-s\} \equiv 1 \quad \text{if} \quad -\tau \le s \le t,\\ \exp_\tau\{b,t-\tau-s\} \equiv 0 \quad \text{if } t < s < 0.\end{gathered} \tag{2.10}$$

Therefore,

$$x(t) = \beta(-\tau) + \int_{-\tau}^{t} \beta'(s)ds = \beta(-\tau) + \beta(t) - \beta(-\tau) = \beta(t). \tag{2.11}$$

□

Remark 2.4. Computing the integral in formula (2.7) by parts, we obtain for $t \ge \tau$:

$$x(t) = \exp_\tau\{b,t-\tau\}\beta(0) + b\int_{-\tau}^{0} \exp_\tau\{b,t-2\tau-s\}\beta(s)ds. \tag{2.12}$$

We remark that it is possible to prove this formula assuming only continuity of the function β, that is, continuous differentiability of β is, in general, not necessary when we represent x by formula (2.12).

Further we will consider the linear nonhomogeneous differential equation with a single delay

$$\dot{x}(t) = ax(t) + bx(t-\tau), \tag{2.13}$$

where $a, b \in \mathbb{R}$, together with initial Cauchy condition (2.6).

Theorem 2.5. *Let the function β in* (2.6) *be continuously differentiable. Then the unique solution of the initial Cauchy problems* (2.13), (2.6) *can be represented as*

$$x(t) = \exp_\tau\{b_1,t\}e^{a(t+\tau)}\beta(-\tau) + \int_{-\tau}^{0} \exp_\tau\{b_1,t-\tau-s\}e^{a(t-s)}\left[\beta'(s) - a\beta(s)\right]ds, \tag{2.14}$$

where $b_1 = be^{-a\tau}$ and $t \in [-\tau,\infty)$.

Proof. Transforming x by a substitution

$$x(t) = e^{at}y(t), \tag{2.15}$$

where y is a new unknown function, we obtain

$$ae^{at}y(t) + e^{at}\dot{y}(t) = ae^{at}y(t) + be^{a(t-\tau)}y(t-\tau) \tag{2.16}$$

or

$$\dot{y}(t) = b_1 y(t-\tau). \tag{2.17}$$

Correspondingly, the initial condition for (2.17) is

$$y(t) = e^{-at}\beta(t), \quad t \in [-\tau, 0]. \tag{2.18}$$

As follows, from formula (2.7), the solution of the corresponding initial Cauchy problems (2.17), (2.18) is

$$y(t) = \exp_\tau\{b_1, t\} e^{a\tau}\beta(-\tau) + \int_{-\tau}^{0} \exp_\tau\{b_1, t-\tau-s\} \left[e^{-as}\beta'(s) - ae^{-as}\beta(s)\right] ds. \tag{2.19}$$

Using substitution (2.15), we obtain

$$x(t) = \exp_\tau\{b_1, t\} e^{a(t+\tau)}\beta(-\tau) + \int_{-\tau}^{0} \exp_\tau\{b_1, t-\tau-s\} e^{a(t-s)} \left[\beta'(s) - a\beta(s)\right] ds, \tag{2.20}$$

which is formula (2.14). □

2.2. *First-Order Nonhomogeneous Linear Differential Equations with a Single Delay*

Let a linear non-homogeneous delay equation with a single delay

$$\dot{x}(t) = ax(t) + bx(t-\tau) + f(t) \tag{2.21}$$

be given, where $a, b \in \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$. We consider the Cauchy problem with a zero initial condition

$$x(t) = 0, \quad t \in [-\tau, 0], \tag{2.22}$$

that is, we put $\beta \equiv 0$ in (2.6).

Theorem 2.6. *The unique solution of the problem* (2.21), (2.22) *is given by the formula*

$$x(t) = \int_{0}^{t} \exp_\tau\{b_1, t-\tau-s\} e^{a(t-s)} f(s) ds, \tag{2.23}$$

where $b_1 = be^{-a\tau}$.

Proof. We apply substitution (2.15). Then

$$ae^{at}y(t) + e^{at}\dot{y}(t) = ae^{at}y(t) + be^{a(t-\tau)}y(t-\tau) + f(t), \tag{2.24}$$

or, equivalently,

$$\dot{y}(t) = b_1 y(t-\tau) + e^{-at} f(t). \tag{2.25}$$

We will show that the solution of the non-homogeneous equation (2.25) satisfying a zero initial condition (deduced from (2.15) and (2.22)) is

$$y(t) = \int_0^t \exp_\tau\{b_1, t-\tau-s\} e^{-as} f(s) ds. \tag{2.26}$$

Substituting (2.26) in (2.25), we obtain

$$\begin{aligned} &\exp_\tau\{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} + b_1 \int_0^t \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds \\ &= b_1 \int_0^{t-\tau} \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds + e^{-at} f(t). \end{aligned} \tag{2.27}$$

Since

$$\exp_\tau\{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} = \exp\{b_1, -\tau\} e^{-at} f(t) = e^{-at} f(t), \tag{2.28}$$

we obtain

$$\begin{aligned} &e^{-at} f(t) + b_1 \int_0^{t-\tau} \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds + b_1 \int_{t-\tau}^t \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds \\ &\quad = b_1 \int_0^{t-\tau} \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds + e^{-at} f(t). \end{aligned} \tag{2.29}$$

Hence,

$$\int_{t-\tau}^t \exp_\tau\{b_1, t-2\tau-s\} e^{-as} f(s) ds = 0. \tag{2.30}$$

This equality is true since

$$t-2\tau-s \le t-2\tau-(t-\tau) = -\tau \tag{2.31}$$

and, by formula (2.1) in Definition 2.1,

$$\exp_\tau\{b_1, t-2\tau-s\} \equiv 0 \tag{2.32}$$

if $t-2\tau-s<-\tau$. In accordance with (2.15), we get

$$x(t)=e^{at}y(t)=\int_0^t \exp_\tau\{b_1,t-\tau-s\}e^{a(t-s)}f(s)ds, \tag{2.33}$$

that is, formula (2.23) is proved. □

Combining Theorems 2.5, and 2.6 we get the following Corollary.

Corollary 2.7. *Let the function β in* (2.6) *be continuously differentiable. Then the unique solution of the problems* (2.21), (2.6) *is given as*

$$\begin{aligned} x(t)&=\exp_\tau\{b_1,t\}e^{a(t+\tau)}\beta(-\tau)\\ &\quad+\int_{-\tau}^0 \exp_\tau\{b_1,t-\tau-s\}e^{a(t-s)}\left[\beta'(s)-a\beta(s)\right]ds\\ &\quad+\int_0^t \exp_\tau\{b_1,t-\tau-s\}e^{a(t-s)}f(s)ds, \end{aligned} \tag{2.34}$$

where $b_1=be^{-a\tau}$.

3. Partial Differential Systems with Delay

Now we consider second-order autonomous systems of linear partial homogeneous differential equations of parabolic type with a single delay (1.1) where $0\le x\le l$ and $t\ge-\tau$. The initial conditions (1.13), (1.14) are defined for $(x,t)\in[0,l]\times[-\tau,0]$. Boundary conditions (1.9)–(1.12) are defined for $t\ge-\tau$ and compatibility conditions (1.8) are fulfilled on the interval $-\tau\le t\le 0$.

By the transformation

$$\begin{pmatrix} u(x,t)\\ v(x,t)\end{pmatrix}=S\begin{pmatrix} \xi(x,t)\\ \eta(x,t)\end{pmatrix}, \tag{3.1}$$

the systems (1.1) can be reduced to a form

$$\begin{pmatrix} \dfrac{\partial\xi(x,t)}{\partial t}\\ \dfrac{\partial\eta(x,t)}{\partial t}\end{pmatrix}=\Lambda\begin{pmatrix} \dfrac{\partial^2\xi(x,t-\tau)}{\partial x^2}\\ \dfrac{\partial^2\eta(x,t-\tau)}{\partial x^2}\end{pmatrix}+\Sigma\begin{pmatrix} \dfrac{\partial^2\xi(x,t)}{\partial x^2}\\ \dfrac{\partial^2\eta(x,t)}{\partial x^2}\end{pmatrix}, \tag{3.2}$$

that is, into two independent scalar equations

$$\frac{\partial\xi(x,t)}{\partial t}=\lambda_1\frac{\partial^2\xi(x,t-\tau)}{\partial x^2}+\sigma_1\frac{\partial^2\xi(x,t)}{\partial x^2}, \tag{3.3}$$

$$\frac{\partial\eta(x,t)}{\partial t}=\lambda_2\frac{\partial^2\eta(x,t-\tau)}{\partial x^2}+\sigma_2\frac{\partial^2\eta(x,t)}{\partial x^2}. \tag{3.4}$$

Initial and boundary conditions reduce to

$$\begin{aligned} \xi(0,t) &= \mu_1^*(t), \quad t \in [-\tau,\infty), \\ \xi(l,t) &= \mu_2^*(t), \quad t \in [-\tau,\infty), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \eta(0,t) &= \theta_1^*(t), \quad t \in [-\tau,\infty), \\ \eta(l,t) &= \theta_2^*(t), \quad t \in [-\tau,\infty), \end{aligned} \tag{3.6}$$

and to

$$\xi(x,t) = \varphi^*(x,t), \quad (x,t) \in [0,l] \times [-\tau,0], \tag{3.7}$$

$$\eta(x,t) = \psi^*(x,t), \quad (x,t) \in [0,l] \times [-\tau,0], \tag{3.8}$$

where

$$\begin{aligned} \begin{pmatrix} \mu_1^*(t) \\ \theta_1^*(t) \end{pmatrix} &= S^{-1} \begin{pmatrix} \mu_1(t) \\ \theta_1(t) \end{pmatrix}, \\ \begin{pmatrix} \mu_2^*(t) \\ \theta_2^*(t) \end{pmatrix} &= S^{-1} \begin{pmatrix} \mu_2(t) \\ \theta_2(t) \end{pmatrix}, \\ \begin{pmatrix} \varphi^*(x,t) \\ \psi^*(x,t) \end{pmatrix} &= S^{-1} \begin{pmatrix} \varphi(x,t) \\ \psi(x,t) \end{pmatrix}. \end{aligned} \tag{3.9}$$

3.1. Constructing of a Solution of (3.3)

We will consider (3.3) with the boundary conditions (3.5), (3.6) and the initial condition (3.7). We will construct a solution in the form

$$\xi(x,t) = \xi_0(x,t) + \xi_1(x,t) + \mu_1^*(t) + \frac{x}{l}\left[\mu_2^*(t) - \mu_1^*(t)\right], \tag{3.10}$$

where $(x,t) \in [0,l] \times [-\tau,\infty)$, $\xi_0(x,t)$ is a solution of (3.3) with zero boundary conditions

$$\xi_0(0,t) = 0, \qquad \xi_0(l,t) = 0, \quad t \in [-\tau,\infty) \tag{3.11}$$

and with a nonzero initial condition

$$\xi_0(x,t) = \Phi(x,t) := \varphi^*(x,t) - \mu_1^*(t) - \frac{x}{l}\left[\mu_2^*(t) - \mu_1^*(t)\right], \quad (x,t) \in [0,l] \times [-\tau,0], \tag{3.12}$$

and $\xi_1(x,t)$ is a solution of a non-homogeneous equation

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t-\tau)}{\partial x^2} + \sigma_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + F(x,t), \tag{3.13}$$

where

$$F(x,t) := -\dot{\mu}_1^*(t) - \frac{x}{l}\left[\dot{\mu}_2^*(t) - \dot{\mu}_1^*(t)\right], \tag{3.14}$$

with zero boundary conditions

$$\xi_1(0,t) = 0, \qquad \xi_1(l,t) = 0, \quad t \in [-\tau, \infty) \tag{3.15}$$

and a zero initial condition

$$\xi_1(x,t) = 0, \quad (x,t) \in [0,l] \times [-\tau, 0]. \tag{3.16}$$

3.1.1. *Equation* (3.3)—*Solution of the Problems* (3.11), (3.12)

For finding a solution $\xi = \xi_0(x,t)$ of (3.3), we will use the method of separation of variables. The solution $\xi_0(x,t)$ is seen as the product of two unknown functions $X(x)$ and $T(t)$, that is,

$$\xi_0(x,t) = X(x)T(t). \tag{3.17}$$

Substituting (3.17) into (3.3), we obtain

$$X(x)T'(t) = \lambda_1 X''(x)T(t-\tau) + \sigma_1 X''(x)T(t). \tag{3.18}$$

Separating variables, we have

$$\frac{T'(t)}{\lambda_1 T(t-\tau) + \sigma_1 T(t)} = \frac{X''(x)}{X(x)} = -\kappa^2, \tag{3.19}$$

where κ is a constant. We consider two differential equations

$$T'(t) + \sigma_1\kappa^2 T(t) + \lambda_1\kappa^2 T(t-\tau) = 0, \tag{3.20}$$

$$X''(x) + \kappa^2 X(x) = 0. \tag{3.21}$$

Nonzero solutions of (3.21) that satisfy zero boundary conditions

$$X(0) = 0, \qquad X(l) = 0, \tag{3.22}$$

exist for the choice $\kappa^2 = \kappa_n^2 = (\pi n/l)^2$, $n = 1, 2, \ldots$, and are defined by the formulas

$$X(x) = X_n(x) = A_n \sin \frac{\pi n}{l} x, \quad n = 1, 2, \ldots, \tag{3.23}$$

where A_n are arbitrary constants. Now we consider (3.20) with $\kappa = \kappa_n$:

$$T_n'(t) = -\sigma_1 \left(\frac{\pi n}{l}\right)^2 T_n(t) - \lambda_1 \left(\frac{\pi n}{l}\right)^2 T_n(t - \tau), \quad n = 1, 2, \ldots. \tag{3.24}$$

Each of (3.24) represents a linear first-order delay differential equation with constant coefficients. We will specify initial conditions for each of (3.23), (3.24). To obtain such initial conditions, we expand the corresponding initial condition $\Phi(x, t)$ (see (3.12)) into Fourier series

$$\Phi(x, t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin \frac{\pi n}{l} x, \quad (x, t) \in [0, l] \times [-\tau, 0], \tag{3.25}$$

where

$$\begin{aligned}
\Phi_n(t) &= \frac{2}{l} \int_0^l \Phi(s, t) \sin \frac{\pi n}{l} s ds \\
&= \frac{2}{l} \int_0^l \left[\varphi^*(s, t) - \mu_1^*(t) - \frac{s}{l}\left(\mu_2^*(t) - \mu_1^*(t)\right)\right] \sin \frac{\pi n}{l} s ds \\
&= \frac{2}{l} \int_0^l \varphi^*(s, t) \sin \frac{\pi n}{l} s ds \\
&\quad - \frac{2\mu_1^*(t)}{l} \int_0^l \sin \frac{\pi n}{l} s \, ds - \frac{2\left(\mu_2^*(t) - \mu_1^*(t)\right)}{l^2} \int_0^l s \cdot \sin \frac{\pi n}{l} s ds \\
&= \frac{2}{l} \int_0^l \varphi^*(s, t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n}\left[(-1)^n \mu_2^*(t) - \mu_1^*(t)\right], \quad t \in [-\tau, 0].
\end{aligned} \tag{3.26}$$

We will find an analytical solution of the problem (3.24) with initial function (3.26), that is, we will find an analytical solution of the Cauchy initial problem

$$\begin{gathered}
T_n'(t) = -\sigma_1 \left(\frac{\pi n}{l}\right)^2 T_n(t) - \lambda_1 \left(\frac{\pi n}{l}\right)^2 T_n(t - \tau), \\
T_n(t) = \Phi_n(t), \quad t \in [-\tau, 0],
\end{gathered} \tag{3.27}$$

for every $n = 1, 2, \ldots$. Using the results of Part 2, we will solve the problem (3.27). According to formula (2.14), we get

$$T_n(t) = \exp_\tau\{r_{1n}, t\} e^{-\sigma_1(\pi n/l)^2(t+\tau)} \Phi_n(-\tau) + \int_{-\tau}^{0} \exp_\tau\{r_{1n}, t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} \left[\Phi_n'(s) + \sigma_1\left(\frac{\pi n}{l}\right)^2 \Phi_n(s)\right] ds, \tag{3.28}$$

where

$$r_{1n} = -\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}, \quad n = 1, 2, \ldots. \tag{3.29}$$

Thus, the solution $\xi_0(x,t)$ of the homogeneous equation (3.3) that satisfies zero boundary conditions (3.11) and a nonzero initial condition (3.12) (to satisfy (3.12) we set $A_n = 1$, $n = 1, 2, \ldots$ in (3.23)) is

$$\xi_0(x,t) = \sum_{n=1}^{\infty} \left[\exp_\tau\{r_{1n}, t\} e^{-\sigma_1(\pi n/l)^2(t+\tau)} \Phi_n(-\tau) + \int_{-\tau}^{0} \exp_\tau\{r_{1n}, t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} \left[\Phi_n'(s) + \sigma_1\left(\frac{\pi n}{l}\right)^2 \Phi_n(s)\right] ds\right] \sin\frac{\pi n}{l}x, \tag{3.30}$$

where Φ_n is defined by (3.26), r_{1n} by (3.29), and $(x,t) \in [0,l] \times [-\tau, \infty)$.

3.1.2. Nonhomogeneous Equation (3.13)

Further, we will consider the non-homogeneous equation (3.13) with zero boundary conditions (3.15) and a zero initial condition (3.16). We will try to find the solution in the form of an expansion

$$\xi_1(x,t) = \sum_{n=1}^{\infty} T_n^0(t) \sin\frac{\pi n}{l}x, \tag{3.31}$$

where $(x,t) \in [0,l] \times [-\tau, \infty)$ and $T_n^0 : [-\tau, \infty) \to \mathbb{R}$ are unknown functions. Substituting (3.31) into (3.13) and equating the coefficients of the same functional terms, we will obtain a system of equations:

$$\left(T_n^0\right)'(t) = -\sigma_1\left(\frac{\pi n}{l}\right)^2 T_n^0(t) - \lambda_1\left(\frac{\pi n}{l}\right)^2 T_n^0(t-\tau) + f_n(t), \quad t \in [0,\infty), \; n = 1, 2, \ldots, \tag{3.32}$$

where $f_n : [-\tau, \infty) \to \mathbb{R}$ are Fourier coefficients of the function $F(x,t)$, that is,

$$\begin{aligned} f_n(t) &= \frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l}s ds \\ &= -\frac{2}{l}\int_0^l \left(\dot{\mu}_1^*(t) + \frac{s}{l}\left[\dot{\mu}_2^*(t) - \dot{\mu}_1^*(t)\right]\right)\sin\frac{\pi n}{l}s ds \\ &= -\frac{2}{l}\dot{\mu}_1^*(t)\int_0^l \sin\frac{\pi n}{l}s\, ds - \frac{2}{l^2}\left(\dot{\mu}_2^*(t) - \dot{\mu}_1^*(t)\right)\int_0^l s\cdot\sin\frac{\pi n}{l}s ds \\ &= -\frac{2}{\pi n}\left((-1)^{n+1}\dot{\mu}_2^*(t) + \dot{\mu}_1^*(t)\right). \end{aligned} \tag{3.33}$$

In accordance with (3.15), we assume zero initial conditions

$$T_n^0(t) = 0, \quad t \in [-\tau, 0], \; n = 1, 2, \ldots \tag{3.34}$$

for every equation (3.32). Then, by formula (2.23) in Theorem 2.6, a solution of each of the problems (3.32), (3.34) can be written as

$$T_n^0(t) = \int_0^t \exp_\tau\{r_{1n}, t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} f_n(s)ds, \quad t \in [-\tau, \infty), \; n = 1, 2, \ldots, \tag{3.35}$$

where r_{1n} is defined by formula (3.29).

Hence, the solution of the non-homogeneous equation (3.13) with zero boundary conditions and a zero initial condition is

$$\xi_1(x,t) = \sum_{n=1}^{\infty}\left[\int_0^t \exp_\tau\{r_{1n}, t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} f_n(s)ds\right]\sin\frac{\pi n}{l}x, \tag{3.36}$$

where f_n is given by formula (3.33).

3.2. Formal Solution of the Boundary Value Problem

Now we complete the particular results giving a solution of the boundary value problem of the initial system (1.1) satisfying conditions (1.9)–(1.14) in the form of a formal series. Conditions of their convergence will be discussed in the following Part 4.

Since the solutions $\xi_0(x,t)$ and $\xi_1(x,t)$ of auxiliary problems are formally differentiable once with respect to t and twice with respect to x, and functions $\mu_1^*(t)$, $\mu_2^*(t)$ are once differentiable, we conclude that a formal solution of the first boundary value problem (3.5), (3.6) and (3.7) for (3.3) can be expressed by the formula:

$$
\begin{aligned}
\xi(x,t) = \sum_{n=1}^{\infty}\Big[&\exp_\tau\{r_{1n},t\}e^{-\sigma_1(\pi n/l)^2(t+\tau)}\Phi_n(-\tau) \\
&+\int_{-\tau}^{0}\exp_\tau\{r_{1n},t-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s)}\left[\Phi_n'(s)+\sigma_1\left(\frac{\pi n}{l}\right)^2\Phi_n(s)\right]ds \\
&+\int_0^t \exp_\tau\{r_{1n},t-\tau-s\}\,e^{-\sigma_1(\pi n/l)^2(t-s)}f_n(s)ds\Big]\sin\frac{\pi n}{l}x \\
&+\mu_1^*(t)+\frac{x}{l}\left[\mu_2^*(t)-\mu_1^*(t)\right],
\end{aligned}
\tag{3.37}
$$

where $(x,t)\in[0,l]\times[-\tau,\infty)$, coefficients Φ_n are defined by formulas (3.26), coefficients f_n by formulas (3.33), and the numbers r_{1n} by formula (3.29).

Similarly, a formal solution of the first boundary value problem for (3.4) is given by the formula:

$$
\begin{aligned}
\eta(x,t) = \sum_{n=1}^{\infty}\Big[&\exp_\tau\{r_{2n},t\}e^{-\sigma_2(\pi n/l)^2(t+\tau)}\Psi_n(-\tau) \\
&+\int_{-\tau}^{0}\exp_\tau\{r_{2n},t-\tau-s\}e^{-\sigma_2(\pi n/l)^2(t-s)}\left[\Psi_n'(s)+\sigma_2\left(\frac{\pi n}{l}\right)^2\Psi_n(s)\right]ds \\
&+\int_0^t \exp_\tau\{r_{2n},t-\tau-s\}\,e^{-\sigma_2(\pi n/l)^2(t-s)}g_n(s)ds\Big]\sin\frac{\pi n}{l}x \\
&+\theta_1^*(t)+\frac{x}{l}\left[\theta_2^*(t)-\theta_1^*(t)\right],
\end{aligned}
\tag{3.38}
$$

where $(x,t)\in[0,l]\times[-\tau,\infty)$ and (by analogy with (3.26), (3.33) and (3.29))

$$
\begin{gathered}
\Psi_n(t)=\frac{2}{l}\int_0^l \varphi^*(s,t)\sin\frac{\pi n}{l}s\ ds+\frac{2}{\pi n}\left[(-1)^n\theta_2^*(t)-\theta_1^*(t)\right],\quad t\in[-\tau,0], \\
g_n(t)=-\frac{2}{\pi n}\left((-1)^{n+1}\dot\theta_2^*(t)+\dot\theta_1^*(t)\right),\quad t\in[-\tau,\infty), \\
r_{2n}=-\lambda_2\left(\frac{\pi n}{l}\right)^2 e^{\sigma_2(\pi n/l)^2\tau}.
\end{gathered}
\tag{3.39}
$$

Then, a formal solution of the boundary value problem of the initial system (1.1) satisfying conditions (1.9)–(1.14) is given by the formulas

$$
\begin{aligned}
u(x,t)&=s_{11}\xi(x,t)+s_{12}\eta(x,t),\\
v(x,t)&=s_{21}\xi(x,t)+s_{22}\eta(x,t),
\end{aligned}
\tag{3.40}
$$

where $\xi(x,t)$, $\eta(x,t)$ are defined by (3.37) and (3.38).

4. Convergence of Formal Series

A solution of the first boundary value problem for (3.3), (3.4) is presented in the form of formal series (3.37), (3.38). We will show that, when certain conditions are satisfied, the series (together with its relevant partial derivatives) converges for $(x,t) \in [0,l] \times [-\tau,t^*]$ where $t^* > 0$ is arbitrarily large and, consequently, is a solution of partial delay differential equations (3.3), (3.4).

Theorem 4.1. *Let, for the functions*

$$\Phi_n : [-\tau,0] \longrightarrow \mathbb{R}, \qquad f_n : [-\tau,\infty) \longrightarrow \mathbb{R}, \quad n = 1,2,\dots \tag{4.1}$$

defined by (3.26) *and* (3.33), *for an integer* $k \geq 1$ *and arbitrary* $t^* \in [(k-1)\tau, k\tau]$, *there exist constants* $M \geq 0$, $\alpha > 0$ *such that*

$$e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)} n^{2k} \max_{-\tau\leq t\leq k\tau} \left|f_n(t)\right| \leq \frac{M}{n^{1+\alpha}}, \qquad e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)} n^{2k} \max_{-\tau\leq t\leq 0} |\Phi_n(t)| \leq \frac{M}{n^{3+\alpha}}. \tag{4.2}$$

Then, for $(x,t) \in [0,l] \times [0,k\tau]$, *the formal series on the right-hand side of expression* (3.37) *as well as its first derivative with respect to t and its second derivative with respect to x converge uniformly. Moreover, equality* (3.37) *holds, and the function* $\xi(x,t)$ *is a solution of* (3.3) *for* $(x,t) \in [0,l] \times [-\tau,k\tau]$.

Proof. First we prove that the right-hand side of expression (3.37) uniformly converges. Decompose the function $\xi(x,t)$ as

$$\xi(x,t) = S_1(x,t) + S_2(x,t) + S_3(x,t) + \mu_1^*(t) + \frac{x}{l}\left[\mu_2^*(t) - \mu_1^*(t)\right], \tag{4.3}$$

where

$$\begin{gathered} S_1(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{\pi n}{l}x, \\ A_n(t) := \exp_\tau\{r_{1n},t\} e^{-\sigma_1(\pi n/l)^2(t+\tau)} \Phi_n(-\tau), \\ S_2(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{\pi n}{l}x, \\ B_n(t) := \int_{-\tau}^{0} \exp_\tau\{r_{1n},t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} \left[\Phi_n'(s) + \sigma_1\left(\frac{\pi n}{l}\right)^2 \Phi_n(s)\right] ds, \\ S_3(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{\pi n}{l}x, \\ C_n(t) := \int_{0}^{t} \exp_\tau\{r_{1n},t-\tau-s\} e^{-\sigma_1(\pi n/l)^2(t-s)} f_n(s) ds. \end{gathered} \tag{4.4}$$

In the following parts, we will prove the uniform convergence of each of the series $S_i(x,t)$, $i=1,2,3$ separately.

Throughout the proof, we use the delayed exponential function defined by Definition 2.1, formula (2.1). Note that this function is continuous on $\mathbb{R}\setminus\{-\tau\}$ and at knots $t=k\tau$ where $k=0,1,\ldots$ two lines in formula (2.1) can be applied. We use this property in the proof without any special comment.

Uniform Convergence of the Series $S_1(x,t)$

We consider the coefficients $A_n(t)$, $n=1,2,\ldots$ of the first series $S_1(x,t)$. As follows from Definition 2.1 of the delayed exponential function, the following equality holds:

$$\begin{aligned}
A_n(t^*) &= \exp_\tau\{r_{1n},t^*\}e^{-\sigma_1(\pi n/l)^2(t^*+\tau)}\Phi_n(-\tau)\\
&= e^{-\sigma_1(\pi n/l)^2(t^*+\tau)}\Phi_n(-\tau)\left[1+r_{1n}\frac{t^*}{1!}+r_{1n}^2\frac{(t^*-\tau)^2}{2!}+\cdots+r_{1n}^k\frac{(t^*-(k-1)\tau)^k}{k!}\right]\\
&= e^{-\sigma_1(\pi n/l)^2(t^*+\tau)}\Phi_n(-\tau)\left[1-\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{t^*}{1!}+\lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau}\frac{(t^*-\tau)^2}{2!}\right.\\
&\qquad\left.+\cdots+(-\lambda_1)^k\left(\frac{\pi n}{l}\right)^{2k} e^{k\sigma_1(\pi n/l)^2\tau}\frac{(t^*-(k-1)\tau)^k}{k!}\right].
\end{aligned} \tag{4.5}$$

Therefore,

$$\begin{aligned}
S_1(x,t^*) &= \sum_{n=1}^{\infty}A_n(t^*)\sin\frac{\pi n}{l}x=\sum_{n=1}^{\infty}\exp_\tau\{r_{1n},t^*\}e^{-\sigma_1(\pi n/l)^2(t^*+\tau)}\Phi_n(-\tau)\sin\frac{\pi n}{l}x\\
&= \sum_{n=1}^{\infty}e^{-\sigma_1(\pi n/l)^2(t^*+\tau)}\Phi_n(-\tau)\sin\frac{\pi n}{l}x-\lambda_1\frac{t^*}{1!}\sum_{n=1}^{\infty}e^{-\sigma_1(\pi n/l)^2 t^*}\left(\frac{\pi n}{l}\right)^2\Phi_n(-\tau)\sin\frac{\pi n}{l}x\\
&\quad+\lambda_1^2\frac{(t^*-\tau)^2}{2!}\sum_{n=1}^{\infty}e^{-\sigma_1(\pi n/l)^2(t^*-\tau)}\left(\frac{\pi n}{l}\right)^4\Phi_n(-\tau)\sin\frac{\pi n}{l}x+\cdots\\
&\quad+(-1)^k\lambda_1^k\frac{(t^*-(k-1)\tau)^k}{k!}\sum_{n=1}^{\infty}e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)}\left(\frac{\pi n}{l}\right)^{2k}\Phi_n(-\tau)\sin\frac{\pi n}{l}x.
\end{aligned} \tag{4.6}$$

Due to condition (4.2), we conclude that

$$e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)}n^{2k}|\Phi_n(-\tau)|\le\frac{M}{n^{3+\alpha}}. \tag{4.7}$$

Therefore, the series $S_1(x,t^*)$ converges uniformly with respect to $x\in[0,l]$ and $t^*\in[(k-1)\tau,k\tau]$. If $t^*\in[(k^*-1)\tau,k^*\tau]$ where $k^*\in\{1,2,\ldots,k-1\}$, then the estimations remain valid.

Note that inequalities (4.2) are also valid for $k = k^*$ because

$$\max_{-\tau\le t\le k^*\tau}\left|f_n(t)\right| \le \frac{M}{n^{2k+1+\alpha}} \le \frac{M}{n^{2k^*+1+\alpha}},$$
$$e^{-\sigma_1(\pi n/l)^2(t^*-(k^*-1)\tau)}n^{2k}\max_{-\tau\le t\le 0}|\Phi_n(t)| \le \frac{M}{n^{3+\alpha}}. \tag{4.8}$$

Consequently, it is easy to see that the series $S_1(x,t)$ converges uniformly for $x \in [0,l]$ and $t \in [0,k\tau]$.

Uniform Convergence of the Series $S_2(x,t)$

We consider the coefficients $B_n(t)$, $n = 1,2,\ldots$ of the second series $S_2(x,t)$. In the representation

$$\begin{aligned}B_n(t) &= \sigma_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\{r_{1n},t-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s)}\Phi_n(s)ds\\ &\quad+\int_{-\tau}^{0}\exp_\tau\{r_{1n},t-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s)}\Phi_n'(s)ds,\end{aligned} \tag{4.9}$$

we calculate the second integral by parts and use formula (2.2) in Lemma 2.2:

$$\begin{aligned}B_n(t) &= \sigma_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\{r_{1n},t-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s)}\Phi_n(s)ds\\ &\quad+\ \exp_\tau\{r_{1n},t-\tau\}e^{-\sigma_1(\pi n/l)^2 t}\Phi_n(0)-\exp_\tau\{r_{1n},t\}e^{-\sigma_1(\pi n/l)^2(t+\tau)}\Phi_n(-\tau)\\ &\quad+\int_{-\tau}^{0}\lambda_1\left(\frac{\pi n}{l}\right)^2\exp_\tau\{r_{1n},t-2\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s-\tau)}\Phi_n(s)ds\\ &\quad-\sigma_1\left(\frac{\pi n}{l}\right)^2\int_{-\tau}^{0}\exp_\tau\{r_{1n},t-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s)}\Phi_n(s)ds\\ &= \exp_\tau\{r_{1n},t-\tau\}e^{-\sigma_1(\pi n/l)^2 t}\ \Phi_n(0)-\exp_\tau\{r_{1n},t\}e^{-\sigma_1(\pi n/l)^2(t+\tau)}\Phi_n(-\tau)\\ &\quad+\int_{-\tau}^{0}\lambda_1\left(\frac{\pi n}{l}\right)^2\exp_\tau\{r_{1n},t-2\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s-\tau)}\Phi_n(s)ds\\ &= B_{n1}(t)-B_{n2}(t)+B_{n3}(t),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}B_{n1}(t) &= \exp_\tau\{r_{1n},t-\tau\}e^{-\sigma_1(\pi n/l)^2 t}\ \Phi_n(0),\\ B_{n2}(t) &= \exp_\tau\{r_{1n},t\}e^{-\sigma_1(\pi n/l)^2(t+\tau)}\Phi_n(-\tau),\\ B_{n3}(t) &= \int_{-\tau}^{0}\lambda_1\left(\frac{\pi n}{l}\right)^2\exp_\tau\{r_{1n},t-2\tau-s\}e^{-\sigma_1(\pi n/l)^2(t-s-\tau)}\Phi_n(s)ds.\end{aligned} \tag{4.11}$$

As follows from the Definition 2.1 of the delayed exponential function, for $t^* \in [(k-1)\tau, k\tau]$, the following equality holds:

$$
\begin{aligned}
B_{n1}(t^*) &= \exp_\tau\{r_{1n}, t^* - \tau\} e^{-\sigma_1(\pi n/l)^2 t^*} \Phi_n(0) \\
&= e^{-\sigma_1(\pi n/l)^2 t^*} \Phi_n(0) \times \left[1 - \lambda_1 \left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2 \tau} \frac{t^* - \tau}{1!} + \lambda_1^2 \left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2 \tau} \frac{(t^* - 2\tau)^2}{2!} \right. \\
&\qquad \left. + \cdots + (-1)^{k-1} \lambda_1^{k-1} \left(\frac{\pi n}{l}\right)^{2(k-1)} e^{(k-1)\sigma_1(\pi n/l)^2 \tau} \frac{(t^* - (k-1)\tau)^{k-1}}{(k-1)!} \right].
\end{aligned}
\tag{4.12}
$$

Therefore, for

$$
S_{21}(x, t^*) := \sum_{n=1}^{\infty} B_{n1}(t^*) \sin \frac{\pi n}{l} x, \tag{4.13}
$$

we get

$$
\begin{aligned}
S_{21}(x, t^*) &= \sum_{n=1}^{\infty} B_{n1}(t^*) \sin \frac{\pi n}{l} x = \sum_{n=1}^{\infty} \exp_\tau\{r_{1n}, t^* - \tau\} e^{-\sigma_1(\pi n/l)^2 t^*} \Phi_n(0) \sin \frac{\pi n}{l} x \\
&= \sum_{n=1}^{\infty} e^{-\sigma_1(\pi n/l)^2 t^*} \Phi_n(0) \sin \frac{\pi n}{l} x - \lambda_1 \frac{t^* - \tau}{1!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2 (t^* - \tau)} \Phi_n(0) \sin \frac{\pi n}{l} x \\
&\quad + \lambda_1^2 \frac{(t^* - 2\tau)^2}{2!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^4 e^{-\sigma_1(\pi n/l)^2 (t^* - 2\tau)} \Phi_n(0) \sin \frac{\pi n}{l} x + \cdots \\
&\quad + (-1)^{k-1} \lambda_1^{k-1} \frac{(t^* - (k-1)\tau)^{k-1}}{(k-1)!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^{2(k-1)} e^{-\sigma_1(\pi n/l)^2 (t^* - (k-1)\tau)} \Phi_n(0) \sin \frac{\pi n}{l} x.
\end{aligned}
\tag{4.14}
$$

Due to condition (4.2), we conclude that

$$
e^{-\sigma_1(\pi n/l)^2 (t^* - (k-1)\tau)} n^{2k} |\Phi_n(0)| \leq \frac{M}{n^{3+\alpha}}. \tag{4.15}
$$

The proof of the uniform convergence of the series $S_{21}(x, t)$ for $x \in [0, l]$ and $t \in [0, k\tau]$ can now be performed in a way similar to the proof of the uniform convergence of the series $S_1(x, t)$ for $x \in [0, l]$ and $t \in [0, k\tau]$.

For the coefficients $B_{n2}(t)$, the following holds:

$$\begin{aligned}
B_{n2}(t^*) &= \exp_\tau\{r_{1n}, t^*\} e^{-\sigma_1(\pi n/l)^2(t^*+\tau)} \Phi_n(-\tau) \\
&= e^{-\sigma_1(\pi n/l)^2(t^*+\tau)} \Phi_n(-\tau) \times \Bigg[1 - \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau} \frac{t^*}{1!} + \lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau} \frac{(t^*-\tau)^2}{2!} \\
&\qquad + \cdots + (-1)^k \lambda_1^k \left(\frac{\pi n}{l}\right)^{2k} e^{k\sigma_1(\pi n/l)^2\tau} \frac{(t^*-(k-1)\tau)^k}{k!}\Bigg].
\end{aligned} \tag{4.16}$$

Therefore, for

$$S_{22}(x, t^*) := \sum_{n=1}^{\infty} B_{n2}(t^*) \sin\frac{\pi n}{l}x, \tag{4.17}$$

we get

$$\begin{aligned}
S_{22}(x, t^*) &= \sum_{n=1}^{\infty} B_{n2}(t^*) \sin\frac{\pi n}{l}x = \sum_{n=1}^{\infty} \exp_\tau\{r_{1n}, t^*\} e^{-\sigma_1(\pi n/l)^2(t^*+\tau)} \Phi_n(-\tau) \sin\frac{\pi n}{l}x \\
&= \sum_{n=1}^{\infty} e^{-\sigma_1(\pi n/l)^2(t^*+\tau)} \Phi_n(-\tau) \sin\frac{\pi n}{l}x - \lambda_1 \frac{t^*}{1!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2 t^*} \Phi_n(-\tau) \sin\frac{\pi n}{l}x \\
&\quad + \lambda_1^2 \frac{(t^*-\tau)^2}{2!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^4 e^{-\sigma_1(\pi n/l)^2(t^*-\tau)} \Phi_n(-\tau) \sin\frac{\pi n}{l}x + \cdots \\
&\quad + \lambda_1^k \frac{(t^*-(k-1)\tau)^k}{k!} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^{2k} e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)} \Phi_n(-\tau) \sin\frac{\pi n}{l}x.
\end{aligned} \tag{4.18}$$

Due to (4.7), the proof of the uniform convergence of the series $S_{22}(x, t)$ for $x \in [0, l]$ and $t \in [0, k\tau]$ can now be performed in a way similar to the proof of the uniform convergence of the series $S_1(x, t)$ for $x \in [0, l]$ and $t \in [0, k\tau]$.

Finally, we consider the coefficients $B_{n3}(t)$ at $t = t^* \in [(k-1)\tau, k\tau]$. Substituting $t^* - 2\tau - s = \omega$, we obtain the following:

$$\begin{aligned}
B_{n3}(t^*) &= \int_{-\tau}^{0} \left(\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau} \exp_\tau\{r_{1n}, t^* - 2\tau - s\}\right) e^{-\sigma_1(\pi n/l)^2(t^*-s)} \Phi_n(s)\,ds \\
&= \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau} \int_{t^*-2\tau}^{t^*-\tau} \exp_\tau\{r_{1n}, \omega\} e^{-\sigma_1(\pi n/l)^2(\omega+2\tau)} \Phi_n(t^* - 2\tau - \omega)\,d\omega.
\end{aligned} \tag{4.19}$$

We will split this integral in two:

$$\begin{aligned} B_{n3}(t^*) &= \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\int_{t^*-2\tau}^{(k-2)\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+2\tau)}\Phi_n(t^*-2\tau-\omega)d\omega \\ &\quad + \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\int_{(k-2)\tau}^{t^*-\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+2\tau)}\Phi_n(t^*-2\tau-\omega)d\omega. \end{aligned} \tag{4.20}$$

Therefore, owing to the mean value theorem, there are values ω_1 and ω_2 such that

$$\begin{gathered} t^*-2\tau \le \omega_1 \le (k-2)\tau, \\ (k-2)\tau \le \omega_2 \le t^*-\tau, \end{gathered} \tag{4.21}$$

and (using the Definition 2.1 of the delayed exponential function) we have

$$\begin{aligned} B_{n3}(t^*) &= \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_1+\tau)}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*)\exp_\tau\{r_{1n},\omega_1\} \\ &\quad + \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_2+\tau)}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau)\exp_\tau\{r_{1n},\omega_2\} \\ &= \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_1+\tau)}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*) \\ &\quad \times\Bigg[1-\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_1}{1!}+\lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau}\frac{(\omega_1-\tau)^2}{2!} \\ &\qquad +\cdots+(-1)^{k-2}\lambda_1^{k-2}\left(\frac{\pi n}{l}\right)^{2(k-2)} e^{(k-2)\sigma_1(\pi n/l)^2\tau}\frac{(\omega_1-(k-3)\tau)^{k-2}}{(k-2)!}\Bigg] \\ &\quad + \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_2+\tau)}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau) \\ &\quad \times\Bigg[1-\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_2}{1!}+\lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau}\frac{(\omega_2-\tau)^2}{2!} \\ &\qquad +\cdots+(-1)^{k-1}\lambda_1^{k-1}\left(\frac{\pi n}{l}\right)^{2(k-1)} e^{(k-1)\sigma_1(\pi n/l)^2\tau}\frac{(\omega_2-(k-2)\tau)^{k-1}}{(k-1)!}\Bigg]. \end{aligned} \tag{4.22}$$

Hence, for

$$S_{23}(x,t^*) := \sum_{n=1}^{\infty} B_{n3}(t^*)\sin\frac{\pi n}{l}x, \tag{4.23}$$

we get

$$\begin{aligned}
S_{23}(x,t^*) &= \sum_{n=1}^{\infty} B_{n3}(t^*)\sin\frac{\pi n}{l}x \\
&= \sum_{n=1}^{\infty}\Bigg\{\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_1+\tau)}\Phi_n \\
&\qquad \times (t^*-2\tau-\omega_1)(k\tau-t^*)\Bigg[1-\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_1}{1!}+\lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau} \\
&\qquad\qquad \times\frac{(\omega_1-\tau)^2}{2!}+\cdots+(-1)^{k-2}\lambda_1^{k-2}\left(\frac{\pi n}{l}\right)^{2(k-2)} \\
&\qquad\qquad \times e^{(k-2)\sigma_1(\pi n/l)^2\tau}\frac{(\omega_1-(k-3)\tau)^{k-2}}{(k-2)!}\Bigg] \\
&\qquad + \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{-\sigma_1(\pi n/l)^2(\omega_2+\tau)}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau) \\
&\qquad \times\Bigg[1-\lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_2}{1!}+\lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau}\frac{(\omega_2-\tau)^2}{2!}+\cdots \\
&\qquad\quad +(-1)^{k-1}\lambda_1^{k-1}\left(\frac{\pi n}{l}\right)^{2(k-1)} e^{(k-1)\sigma_1(\pi n/l)^2\tau}\frac{(\omega_2-(k-2)\tau)^{k-1}}{(k-1)!}\Bigg]\Bigg\}\sin\frac{\pi n}{l}x.
\end{aligned} \tag{4.24}$$

After some rearranging, we get

$$\begin{aligned}
S_{23}(x,t^*) = \lambda_1 &\sum_{n=1}^{\infty}\Big[e^{-\sigma_1(\pi n/l)^2(\omega_1+\tau)}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*) \\
&\quad + e^{-\sigma_1(\pi n/l)^2(\omega_2+\tau)}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau)\Big]\left(\frac{\pi n}{l}\right)^2\sin\frac{\pi n}{l}x \\
&- \lambda_1^2\sum_{n=1}^{\infty}\Big[e^{-\sigma_1(\pi n/l)^2\omega_1}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*)\times\frac{\omega_1}{1!} \\
&\quad + e^{-\sigma_1(\pi n/l)^2\omega_2}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau)\frac{\omega_2}{1!}\Big]\left(\frac{\pi n}{l}\right)^4\sin\frac{\pi n}{l}x \\
&+ \lambda_1^3\sum_{n=1}^{\infty}\Bigg[e^{-\sigma_1(\pi n/l)^2(\omega_1-\tau)}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*)\frac{(\omega_1-\tau)^2}{2!} \\
&\quad + e^{-\sigma_1(\pi n/l)^2(\omega_2-\tau)}\Phi_n(t^*-2\tau-\omega_2)(t^*-(k-1)\tau)\frac{(\omega_2-\tau)^2}{2!}\Bigg]\left(\frac{\pi n}{l}\right)^6\sin\frac{\pi n}{l}x \\
&+\cdots
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{k-2}\lambda_1^{k-1}\sum_{n=1}^{\infty}\Bigg[e^{-\sigma_1(\pi n/l)^2(\omega_1-(k-3)\tau)}\Phi_n(t^*-2\tau-\omega_1)(k\tau-t^*)\\
&\qquad\qquad\times\frac{(\omega_1-(k-3)\tau)^{k-2}}{(k-2)!}+e^{-\sigma_1(\pi n/l)^2(\omega_2-(k-3)\tau)}\Phi_n(t^*-2\tau-\omega_2)\\
&\qquad\qquad\times(t^*-(k-1)\tau)\frac{(\omega_2-(k-3)\tau)^{k-2}}{(k-2)!}\Bigg]\left(\frac{\pi n}{l}\right)^{2k-2}\sin\frac{\pi n}{l}x\\
&+(-1)^{k-1}\lambda_1^{k}(t^*-(k-1)\tau)\frac{(\omega_2-(k-2)\tau)^{k-1}}{(k-1)!}\\
&\times\sum_{n=1}^{\infty}\left(\frac{\pi n}{l}\right)^{2k}e^{-\sigma_1(\pi n/l)^2(\omega_2-(k-2)\tau)}\Phi_n(t^*-2\tau-\omega_2)\sin\frac{\pi n}{l}x.
\end{aligned}
\tag{4.25}
$$

Due to condition (4.2), we conclude that

$$
e^{-\sigma_1(\pi n/l)^2(\omega_2-(k-2)\tau)}n^{2k}\max_{-\tau\le t\le 0}|\Phi_n(t)|\le\frac{M}{n^{3+\alpha}}.
\tag{4.26}
$$

The proof of the uniform convergence of the series $S_{23}(x,t)$ for $x\in[0,l]$ and $t\in[0,k\tau]$ can now be performed in a way similar to the proof of uniform convergence of the series $S_1(x,t)$ for $x\in[0,l]$ and $t\in[0,k\tau]$.

Uniform Convergence of the Series $S_3(x,t)$

We will consider the coefficients $C_n(t)$, $n=1,2,\dots$ of the series $S_3(x,t)$ at $t=t^*\in[(k-1)\tau,k\tau]$. Substituting $t^*-\tau-s=\omega$, we obtain

$$
\begin{aligned}
C_n(t^*)&=\int_0^{t^*}\exp_\tau\{r_{1n},t^*-\tau-s\}e^{-\sigma_1(\pi n/l)^2(t^*-s)}f_n(s)ds\\
&=\int_{-\tau}^{t^*-\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+\tau)}f_n(t^*-\tau-\omega)d\omega\\
&=\int_{-\tau}^{0}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+\tau)}f_n(t^*-\tau-\omega)d\omega\\
&\quad+\int_{0}^{\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+\tau)}f_n(t^*-\tau-\omega)d\omega\\
&\quad+\int_{\tau}^{2\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+\tau)}f_n(t^*-\tau-\omega)d\omega\\
&\quad+\cdots+\int_{(k-2)\tau}^{t^*-\tau}\exp_\tau\{r_{1n},\omega\}e^{-\sigma_1(\pi n/l)^2(\omega+\tau)}f_n(t^*-\tau-\omega)d\omega.
\end{aligned}
\tag{4.27}
$$

Owing to the mean value theorem, there are values ω_i, $i = 1, 2, \ldots, k$ such that

$$-\tau \le \omega_1 \le 0,\ \ 0 \le \omega_2 \le \tau, \ldots, (k-2)\tau \le \omega_k \le t^* - \tau$$

$$\begin{aligned}
C_n(t^*) &= \tau e^{-\sigma_1(\pi n/l)^2(\omega_1+\tau)} f_n(t^* - \tau - \omega_1) \\
&\quad + \tau\left[1 - \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_2}{1!}\right] e^{-\sigma_1(\pi n/l)^2(\omega_2+\tau)} f_n(t^* - \tau - \omega_2) \\
&\quad + \tau\left[1 - \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_3}{1!} + \lambda_1^2\left(\frac{\pi n}{l}\right)^4 e^{2\sigma_1(\pi n/l)^2\tau}\frac{(\omega_3-\tau)^2}{2!}\right] e^{-\sigma_1(\pi n/l)^2(\omega_3+\tau)} \\
&\quad \times f_n(t^* - \tau - \omega_3) + \cdots + \tau\left[1 - \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_{k-1}}{1!} + \ldots\right. \\
&\quad\quad + (-1)^{k-2}\lambda_1^{k-2}\left(\frac{\pi n}{l}\right)^{2(k-2)} e^{(k-2)\sigma_1(\pi n/l)^2\tau} \\
&\quad\quad \left.\times \frac{[\omega_{k-1} - (k-3)\tau]^{k-2}}{(k-2)!}\right] e^{-\sigma_1(\pi n/l)^2(\omega_{k-1}+\tau)} \\
&\quad \times f_n(t^* - \tau - \omega_{k-1}) \\
&\quad + [t^* - (k-1)\tau]\left[1 - \lambda_1\left(\frac{\pi n}{l}\right)^2 e^{\sigma_1(\pi n/l)^2\tau}\frac{\omega_k}{1!} + \ldots + (-1)^{k-1}\right. \\
&\quad\quad \left.\times \lambda_1^{k-1}\left(\frac{\pi n}{l}\right)^{2(k-1)} e^{(k-1)\sigma_1(\pi n/l)^2\tau}\frac{[\omega_k - (k-2)\tau]^{k-1}}{(k-1)!}\right] \\
&\quad \times e^{-\sigma_1(\pi n/l)^2(\omega_k+\tau)} f_n(t^* - \tau - \omega_k).
\end{aligned} \tag{4.28}$$

Hence,

$$\begin{aligned}
S_3(x, t^*) &= \sum_{n=1}^{\infty} C_n(t^*) \sin\frac{\pi n}{l}x \\
&= \sum_{n=1}^{\infty}\left[\tau\sum_{i=1}^{k-1} e^{-\sigma_1(\pi n/l)^2(\omega_i+\tau)} f_n(t^* - \tau - \omega_i) + (t^* - (k-1)\tau)\right. \\
&\quad\quad \left.\times e^{-\sigma_1(\pi n/l)^2(\omega_k+\tau)} f_n\left(t^* - \tau - \omega_k\right)\right] \times \sin\frac{\pi n}{l}x \\
&\quad - \lambda_1\sum_{n=1}^{\infty}\left[\tau\sum_{i=2}^{k-1}\frac{\omega_i}{1!} e^{-\sigma_1(\pi n/l)^2\omega_i} f_n(t^* - \tau - \omega_i) + (t^* - (k-1)\tau)\right. \\
&\quad\quad \left.\times \frac{\omega_k}{1!} e^{-\sigma_1(\pi n/l)^2\omega_k} f_n\left(t^* - \tau - \omega_k\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\times\left(\frac{\pi n}{l}\right)^2 \sin\frac{\pi n}{l}x \\
&+\lambda_1^2\sum_{n=1}^{\infty}\left[\tau\sum_{i=3}^{k-1}\frac{(\omega_i-\tau)^2}{2!}e^{-\sigma_1(\pi n/l)^2(\omega_i-\tau)}f_n(t^*-\tau-\omega_i)\right.\\
&\qquad\left.+(t^*-(k-1)\tau)\frac{(\omega_k-\tau)^2}{2!}e^{-\sigma_1(\pi n/l)^2(\omega_k-\tau)}f_n(t^*-\tau-\omega_k)\right]\\
&\times\left(\frac{\pi n}{l}\right)^4 \sin\frac{\pi n}{l}x+\dots+(-1)^{k-2}\lambda_1^{k-2}\\
&\times\sum_{n=1}^{\infty}\left[\tau\frac{(\omega_{k-1}-(k-3)\tau)^{k-2}}{(k-2)!}e^{-\sigma_1(\pi n/l)^2(\omega_{k-1}-(k-3)\tau)}f_n(t^*-\tau-\omega_{k-1})\right.\\
&\qquad\left.+[t^*-(k-1)\tau]\frac{(\omega_k-(k-3)\tau)^{k-2}}{(k-2)!}e^{-\sigma_1(\pi n/l)^2(\omega_k-(k-3)\tau)}f_n(t^*-\tau-\omega_k)\right]\\
&\times\left(\frac{\pi n}{l}\right)^{2(k-2)} \sin\frac{\pi n}{l}x\\
&+(-1)^{k-1}\lambda_1^{k-1}[t^*-(k-1)\tau]\frac{(\omega_k-(k-2)\tau)^{k-1}}{(k-1)!}\\
&\times\sum_{n=1}^{\infty}\left(\frac{\pi n}{l}\right)^{2(k-1)}e^{-\sigma_1(\pi n/l)^2(\omega_k-(k-2)\tau)}f_n(t^*-\tau-\omega_k)\sin\frac{\pi n}{l}x.
\end{aligned} \tag{4.29}$$

□

Due to condition (4.2), we conclude that

$$e^{-\sigma_1(\pi n/l)^2(\omega_k-(k-2)\tau)}n^{2(k-1)}\max_{-\tau\le t\le k\tau}\left|f_n(t)\right|\le\frac{M}{n^{3+\alpha}}. \tag{4.30}$$

$S_3(x,t)x\in[0,l]t\in[0,k\tau]S_1(x,t)x\in[0,l]t\in[0,k\tau]$.

Uniform Convergence of the Formal Series for $\xi(x,t)$

Above, the absolute and uniform convergence of the series $S_1(x,t)$, $S_2(x,t)$, $S_3(x,t)$ was proved. Therefore, the series for $\xi(x,t)$ converges absolutely and uniformly as well.

Uniform Convergence of the Formal Series for $\xi_t'(x,t)$ and $\xi_{xx}''(x,t)$

To prove the uniform convergence of the series for $\xi_t'(x,t)$ and $\xi_{xx}''(x,t)$, we can proceed as in the above proof of the uniform convergence of $\xi(x,t)$. The above scheme can be repeated, and the final inequalities (4.7), (4.15), (4.26), and (4.30) are replaced as follows: inequality (4.7) by

$$e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)}n^{2k+2}|\Phi_n(-\tau)|\le\frac{M}{n^{1+\alpha}}, \tag{4.31}$$

inequality (4.15) by

$$e^{-\sigma_1(\pi n/l)^2(t^*-(k-1)\tau)} n^{2k+2} |\Phi_n(0)| \le \frac{M}{n^{1+\alpha}}, \tag{4.32}$$

inequality (4.26) by

$$e^{-\sigma_1(\pi n/l)^2(\omega_2-(k-2)\tau)} n^{2k+2} \max_{-\tau\le t\le 0} |\Phi_n(t)| \le \frac{M}{n^{1+\alpha}}, \tag{4.33}$$

and inequality (4.30) by

$$e^{-\sigma_1(\pi n/l)^2(\omega_k-(k-2)\tau)} n^{2k} \max_{-\tau\le t\le k\tau} |f_n(t)| \le \frac{M}{n^{1+\alpha}}. \tag{4.34}$$

The proof of the uniform convergence of the series, which represents the solution $\eta(x,t)$ by formula (3.38), and the proof of the uniform convergence of the series for $\eta'_x(x,t)$ and $\eta''_{xx}(x,t)$ are much the same.

Corollary 4.2. *Functions $u(x,t)$, $v(x,t)$ are linear combinations of $\xi(x,t)$ and $\eta(x,t)$. Therefore, representations* (3.40) *are the solutions of the system system* (1.1) *satisfying all boundary and initial conditions* (1.9)–(1.14).

Remark 4.3. Tracing the proof of Theorem 4.1, we see that inequalities (4.2) for functions $\Phi_n : [-\tau,0] \to \mathbb{R}$, $n = 1,2,\ldots$, and $f_n : [-\tau,\infty) \to \mathbb{R}$, $n = 1,2,\ldots$ are too restrictive if $t^* \in [(k^*-1)\tau,(k^*-1)\tau+\varepsilon]$, $k^* \in \{1,2,\ldots,k\}$, where ε is an arbitrarily small positive number. The question whether the series are uniformly convergent for $(x,t) \in [0,l]\times[(k^*-1)\tau,(k^*-1)\tau+\varepsilon]$, $k^* \in \{1,2,\ldots,k\}$ remains open if, for example, $\Phi_n : [-\tau,0] \to \mathbb{R}$, $n = 1,2,\ldots$, and $f_n : [-\tau,\infty) \to \mathbb{R}$, $n = 1,2,\ldots$ satisfy only the inequalities

$$\max_{-\tau\le t\le k\tau} |f_n(t)| \le \frac{M^*}{n}, \qquad \max_{-\tau\le t\le 0} |\Phi_n(t)| \le \frac{M^*}{n} \tag{4.35}$$

for a positive constant M^* because inequalities (4.2) cannot be valid. Nevertheless, in such a case, the series converge at least point-wise for $t \in [(k^*-1)\tau,(k^*-1)\tau+\varepsilon]$, $k^* \in \{1,2,\ldots,k\}$ and uniformly for $x \in [0,l]$. In other words, in such a case, the series converge uniformly for

$$(x,t) \in [0,l] \times \left([0,k\tau] \setminus \bigcup_{k^*=1}^{k} [(k^*-1)\tau,(k^*-1)\tau+\varepsilon]\right), \tag{4.36}$$

(where $\varepsilon > 0$ is fixed but arbitrarily small).

Acknowledgments

The paper was supported by the Grants P201/10/1032 and P201/11/0768 of the Czech Grant Agency (Prague), by the Council of Czech Government Grant MSM 00216 30519, and by the Grant FEKT-S-11-2-921 of Faculty of Electrical Engineering and Communication, Brno, University of Technology.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 819342, 17 pages
doi:10.1155/2012/819342

Research Article

Oscillation Theorems for Second-Order Quasilinear Neutral Functional Differential Equations

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Received 15 March 2012; Revised 14 May 2012; Accepted 14 May 2012

Academic Editor: Agacik Zafer

New oscillation criteria are established for the second-order nonlinear neutral functional differential equations of the form $(r(t)|z'(t)|^{\alpha-1}z'(t))' + f(t, x[\sigma(t)]) = 0$, $t \geq t_0$, where $z(t) = x(t) + p(t)x(\tau(t))$, $p \in C^1([t_0,\infty),[0,\infty))$, and $\alpha \geq 1$. Our results improve and extend some known results in the literature. Some examples are also provided to show the importance of these results.

1. Introduction

This paper is concerned with the oscillation problem of the second-order nonlinear functional differential equation of the following form:

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + f(t, x[\sigma(t)]) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\alpha \geq 1$ is a constant, $z(t) = x(t) + p(t)x[\tau(t)]$.

Throughout this paper, we will assume the following hypotheses:

(A_1) $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t) > 0$ for $t \geq t_0$,

(A_2) $p \in C^1([t_0,\infty),[0,\infty))$,

(A_3) $\tau \in C^2([t_0,\infty),\mathbb{R})$, $\tau'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \infty$,

(A_4) $\sigma \in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty}\sigma(t) = \infty$, $\tau \circ \sigma = \sigma \circ \tau$;

(A_5) $f(t,u) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists a function $q \in C([t_0,\infty),[0,\infty))$ such that

$$f(t,u)\ \operatorname{sgn}\ u \geq q(t)|u|^{\alpha}, \quad \text{for } u \neq 0,\ t \geq t_0. \tag{1.2}$$

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$ which has the property that $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all of its nonconstant solutions are oscillatory.

We note that neutral delay differential equations find numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1]. Therefore, there is constant interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of varietal types of the second-order equations, see, e.g., papers [2–17].

Known oscillation criteria require various restrictions on the coefficients of the studied neutral differential equations.

Agarwal et al. [2], Chern et al. [3], Džurina and Stavroulakis [4], Kusano et al. [5, 6], Mirzov [7], and Sun and Meng [8] observed some similar properties between

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0 \tag{1.3}$$

and the corresponding linear equation

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0. \tag{1.4}$$

Liu and Bai [10], Xu and Meng [11, 12], and Dong [13] established some oscillation criteria for (1.3) with neutral term under the assumption that

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\mathrm{d}t = \infty. \tag{1.5}$$

Han et al. [14] examined the oscillation of second-order linear neutral differential equation

$$\left(r(t)\left[x(t)+p(t)x(\tau(t))\right]'\right)' + q(t)x[\sigma(t)] = 0, \quad t \geq t_0, \tag{1.6}$$

where $\tau'(t) = \tau_0 > 0,\ 0 \leq p(t) \leq p_0 < \infty$, and obtained some oscillation criteria for (1.6) when

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t = \infty. \tag{1.7}$$

Han et al. [15] studied the oscillation of (1.6) under the case $0 \le p(t) \le 1$ and

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \mathrm{d}t < \infty. \tag{1.8}$$

Tripathy [16] considered the nonlinear dynamic equation of the form

$$\left(r(t)\left[\left(x(t)+p(t)x(t-\tau)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + q(t)|x(t-\delta)|^{\gamma} \operatorname{sgn} x(t-\delta) = 0, \tag{1.9}$$

where $0 \le p(t) \le p_0 < \infty$, γ is a the ratios of two positive odd integers, and obtained some oscillation criteria under the following conditions:

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\gamma} \mathrm{d}t = \infty. \tag{1.10}$$

Džurina [17] was concerned with the oscillation behavior of the solutions of the second-order neutral differential equations as follows

$$\left(a(t)\left[\left(x(t)+p(t)x(\tau(t))\right)'\right]^{\gamma}\right)' + q(t)x^{\beta}(\sigma(t)) = 0, \tag{1.11}$$

where $0 \le p(t) \le p_0 < \infty$, γ is a the ratios of two positive odd integers, and obtained some new results under the following conditions

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\gamma} \mathrm{d}t = \infty. \tag{1.12}$$

Our purpose of this paper is to establish some new oscillation criteria for (1.1), and we will also consider the cases (1.5) and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \mathrm{d}t < \infty. \tag{1.13}$$

To the best of my knowledge, there is no result for the oscillation of (1.1) under the conditions both $0 \le p(t) \le p_0 < \infty$ and (1.13).

In this paper, we will use a new inequality to establish some oscillation criteria for (1.1) for the first time. Some examples will be given to show the importance of these results. In Sections 3 and 4, for the sake of convenience, we denote that

$$\begin{gathered} Q(t) := \min\{q(t), q[\tau(t)]\}, \qquad d_+(t) := \max\{0, d(t)\}, \qquad \xi(t) := \frac{\alpha p'[\sigma(t)]\sigma'(t)}{p[\sigma(t)]} - \frac{\tau''(t)}{\tau'(t)}, \\ \zeta(t) := \frac{(\rho'(t))_+}{\rho(t)} + \xi(t), \qquad \varphi(t) := \left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1} + \frac{p^{\alpha}[\sigma(t)](\zeta_+(t))^{\alpha+1}}{\tau'(t)}, \qquad \delta(t) := \int_{\eta(t)}^{\infty} \frac{\mathrm{d}s}{r^{1/\alpha}(s)}. \end{gathered} \tag{1.14}$$

2. Lemma

In this section, we give the following lemma, which we will use in the proofs of our main results.

Lemma 2.1. *Assume that $\alpha \geq 1$, $a, b \in \mathbb{R}$. If $a \geq 0$, $b \geq 0$, then one has*

$$a^\alpha + b^\alpha \geq \frac{1}{2^{\alpha-1}}(a+b)^\alpha. \tag{2.1}$$

Proof. (i) Suppose that $a = 0$ or $b = 0$. Then we have (2.1). (ii) Suppose that $a > 0$, $b > 0$. Define the function g by $g(x) = x^\alpha$, $x \in (0, \infty)$. Then $g''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$ for $x > 0$. Thus, g is a convex function. By the definition of convex function, for $\lambda = 1/2$, $a, b \in (0, \infty)$, we have

$$g\left(\frac{a+b}{2}\right) \leq \frac{g(a)+g(b)}{2}, \tag{2.2}$$

that is,

$$a^\alpha + b^\alpha \geq \frac{1}{2^{\alpha-1}}(a+b)^\alpha. \tag{2.3}$$

This completes the proof. □

3. Oscillation Criteria for the Case (1.5)

In this section, we will establish some oscillation criteria for (1.1) under the case (1.5).

Theorem 3.1. *Suppose that* (1.5) *holds,* $\sigma \in C([t_0, \infty), \mathbb{R})$, $\sigma'(t) > 0$, $\sigma(t) \leq t$, *and* $\sigma(t) \leq \tau(t)$ *for* $t \geq t_0$. *Furthermore, assume that there exists a function* $\rho \in C([t_0, \infty), (0, \infty))$ *such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \rho(s) \left\{ \frac{Q(s)}{2^{\alpha-1}} - \frac{r[\sigma(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\sigma'(s))^\alpha} \right\} ds = \infty. \tag{3.1}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \geq t_1$. By applying (1.1), for all sufficiently large t, we obtain that

$$\begin{aligned} &\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)x^\alpha[\sigma(t)] + q[\tau(t)]p^\alpha[\sigma(t)]x^\alpha(\sigma[\tau(t)]) \\ &\quad + \frac{p^\alpha[\sigma(t)]}{\tau'(t)}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)' \leq 0. \end{aligned} \tag{3.2}$$

Using (2.1) and the definition of z, we conclude that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}[\sigma(t)] + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)' \le 0. \quad (3.3)$$

In view of (1.1), we obtain that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' \le -q(t)x^{\alpha}[\sigma(t)] \le 0, \quad t \ge t_1. \quad (3.4)$$

Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) < 0$ for $t \ge t_2 \ge t_1$. Then, from (3.4), we get

$$r(t)\left|z'(t)\right|^{\alpha-1}z'(t) \le r(t_2)\left|z'(t_2)\right|^{\alpha-1}z'(t_2), \quad t \ge t_2, \quad (3.5)$$

which implies that

$$z(t) \le z(t_2) - r^{1/\alpha}(t_2)\left|z'(t_2)\right|\int_{t_2}^{t} r^{-1/\alpha}(s)\mathrm{d}s. \quad (3.6)$$

Letting $t \to \infty$, by (1.5), we find $z(t) \to -\infty$, which is a contradiction.

(ii) Suppose that $z'(t) > 0$ for $t \ge t_2 \ge t_1$. We define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\sigma(t)])^{\alpha}}, \quad t \ge t_2. \quad (3.7)$$

Then $\omega(t) > 0$. From (3.4), we have

$$z'[\sigma(t)] \ge \left(\frac{r(t)}{r[\sigma(t)]}\right)^{1/\alpha}z'(t). \quad (3.8)$$

Differentiating (3.7), we find that

$$\begin{aligned}\omega'(t) = {}& \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\sigma(t)])^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} \\ & - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{(z[\sigma(t)])^{2\alpha}}.\end{aligned} \quad (3.9)$$

Therefore, by (3.7), (3.8), and (3.9), we see that

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t). \quad (3.10)$$

Similarly, we introduce a Riccati substitution

$$v(t) = \rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\sigma(t)])^{\alpha}}, \quad t \geq t_2. \tag{3.11}$$

Then $v(t) > 0$. From (3.4), we have

$$z'[\sigma(t)] \geq \left(\frac{r[\tau(t)]}{r[\sigma(t)]}\right)^{1/\alpha} z'[\tau(t)]. \tag{3.12}$$

Differentiating (3.11), we find that

$$\begin{aligned} v'(t) = {} & \rho'(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\sigma(t)])^{\alpha}} + \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} \\ & - \alpha\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha} z^{\alpha-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{(z[\sigma(t)])^{2\alpha}}. \end{aligned} \tag{3.13}$$

Therefore, by (3.11), (3.12), and (3.13), we see that

$$v'(t) \leq \frac{\rho'(t)}{\rho(t)}v(t) + \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}v^{(\alpha+1)/\alpha}(t). \tag{3.14}$$

Thus, from (3.10) and (3.14), we have

$$\begin{aligned} \omega'(t) + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}v'(t) \leq {} & \rho(t)\left\{\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}}\right\} + \frac{\rho'(t)}{\rho(t)}\omega(t) \\ & - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}v(t) \\ & - \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}v^{(\alpha+1)/\alpha}(t). \end{aligned} \tag{3.15}$$

It is follows from (3.3) that

$$\begin{aligned} \omega'(t) + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}v'(t) \leq {} & -\frac{1}{2^{\alpha-1}}\rho(t)Q(t) + \frac{\rho'_{+}(t)}{\rho(t)}\omega(t) \\ & - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t) + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'_{+}(t)}{\rho(t)}v(t) \\ & - \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}v^{(\alpha+1)/\alpha}(t). \end{aligned} \tag{3.16}$$

Integrating the above inequality from t_2 to t, we obtain that

$$\begin{aligned}
&\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}v(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}v(t_2)\\
&\quad\le-\int_{t_2}^{t}\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s\\
&\qquad+\int_{t_2}^{t}\left[\frac{\rho'_+(s)}{\rho(s)}\omega(s)-\frac{\alpha\sigma'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\sigma(s)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s\\
&\qquad+\int_{t_2}^{t}\frac{p^{\alpha}[\sigma(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_+(s)}{\rho(s)}+\xi(s)\right]_+v(s)-\frac{\alpha\sigma'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\sigma(s)]}v^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s.
\end{aligned}\tag{3.17}$$

Define

$$A:=\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{\alpha/(\alpha+1)}\omega(t),\qquad B:=\left[\frac{\rho'_+(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.\tag{3.18}$$

Using the following inequality:

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha}-A^{(\alpha+1)/\alpha}\le\frac{1}{\alpha}B^{(\alpha+1)/\alpha},\quad\text{for } A\ge0,\ B\ge0 \text{ are constants},\tag{3.19}$$

we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\sigma(t)]\left(\rho'_+(t)\right)^{\alpha+1}}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}.\tag{3.20}$$

On the other hand, define

$$A:=\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{\alpha/(\alpha+1)}v(t),\qquad B:=\left[\zeta_+(t)\frac{\alpha}{\alpha+1}\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.\tag{3.21}$$

So we have

$$\zeta_+(t)v(t)-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}v^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\sigma(t)](\zeta_+(t))^{\alpha+1}\rho(t)}{(\sigma'(t))^{\alpha}}.\tag{3.22}$$

Thus, from (3.17), we get

$$\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}v(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}v(t_2) \\ \le -\int_{t_2}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}}-\frac{r[\sigma(s)]}{(\alpha+1)^{\alpha+1}(\sigma'(s))^{\alpha}}\left[\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\alpha+1}+\frac{p^{\alpha}[\sigma(s)](\zeta_+(s))^{\alpha+1}}{\tau'(s)}\right]\right\}\mathrm{d}s, \tag{3.23}$$

which contradicts (3.1). This completes the proof. □

When $p(t)\le p_0<\infty,\ \tau'(t)\ge\tau_0>0$, where p_0,τ_0 are constants, we obtain the following result.

Theorem 3.2. *Suppose that* (1.5) *holds,* $p(t)\le p_0<\infty,\ \tau'(t)\ge\tau_0>0,\ \sigma\in C([t_0,\infty),\mathbb{R}),\ \sigma'(t)>0,\ \sigma(t)\le t,\ \sigma(t)\le\tau(t)$ *for* $t\ge t_0$*. Further, assume that there exists a function* $\rho\in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\sigma(s)](\rho'_+(s))^{\alpha+1}}{(\rho(s)\sigma'(s))^{\alpha}}\right]ds=\infty. \tag{3.24}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\ge t_0$ such that $x(t)>0,\ x[\tau(t)]>0$ and $x[\sigma(t)]>0$ for all $t\ge t_1$. Using (1.1), for all sufficiently large t, we obtain that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)'+q(t)x^{\alpha}[\sigma(t)]+p_0^{\alpha}q[\tau(t)]x^{\alpha}(\sigma[\tau(t)]) \\ +\frac{p_0^{\alpha}}{\tau_0}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)'\le 0. \tag{3.25}$$

By applying (2.1) and the definition of z, we conclude that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)'+\frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}[\sigma(t)] \\ +\frac{p_0^{\alpha}}{\tau_0}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)'\le 0. \tag{3.26}$$

The remainder of the proof is similar to that of Theorem 3.1 and hence is omitted. □

Theorem 3.3. *Suppose that* (1.5) *holds,* $\tau(t)\le t,\ \sigma(t)\ge\tau(t)$ *for* $t\ge t_0$*. Furthermore, assume that there exists a function* $\rho\in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}}-\frac{r[\tau(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\right\}ds=\infty. \tag{3.27}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we get (3.3) and (3.4). In view of (3.4), $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. Then, similar to the proof of case (i) of Theorem 3.1, we obtain a contradiction.

(ii) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. We define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\tau(t)])^{\alpha}}, \quad t \geq t_2. \tag{3.28}$$

Then $\omega(t) > 0$. From (3.4), we have

$$z'[\tau(t)] \geq \left(\frac{r(t)}{r[\tau(t)]}\right)^{1/\alpha} z'(t). \tag{3.29}$$

Differentiating (3.28), we find that

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\tau(t)])^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}. \tag{3.30}$$

Therefore, by (3.28), (3.29), and (3.30), we see that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t). \tag{3.31}$$

Similarly, we introduce a Riccati substitution

$$\upsilon(t) = \rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}}, \quad t \geq t_2. \tag{3.32}$$

Then $\upsilon(t) > 0$. Differentiating (3.32), we find that

$$\begin{aligned}\upsilon'(t) = \rho'(t)&\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}} + \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}}\\ &- \alpha\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}.\end{aligned} \tag{3.33}$$

Therefore, by (3.32) and (3.33), we see that

$$\upsilon'(t) = \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.34}$$

Thus, from (3.31) and (3.33), we have

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\le\rho(t)\left\{\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}}+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}}\right\}+\frac{\rho'(t)}{\rho(t)}\omega(t)\\ -\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}\upsilon(t)\\ -\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}\tag{3.35}$$

It follows from (3.3) that

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\le&-\frac{1}{2^{\alpha-1}}\rho(t)Q(t)+\frac{\rho'_{+}(t)}{\rho(t)}\omega(t)\\ &-\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'_{+}(t)}{\rho(t)}\upsilon(t)\\ &-\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}\tag{3.36}$$

Integrating the above inequality from t_2 to t, we obtain that

$$\begin{aligned}&\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}\upsilon(t_2)\\ &\quad\le-\int_{t_2}^{t}\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s\\ &\qquad+\int_{t_2}^{t}\left[\frac{\rho'_{+}(s)}{\rho(s)}\omega(s)-\frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s\\ &\qquad+\int_{t_2}^{t}\frac{p^{\alpha}[\sigma(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_{+}(s)}{\rho(s)}+\xi(s)\right]_{+}\upsilon(s)-\frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\upsilon^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s.\end{aligned}\tag{3.37}$$

Define

$$A:=\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)}\omega(t),\qquad B:=\left[\frac{\rho'_{+}(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.\tag{3.38}$$

Using (3.19), we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)]\left(\rho'_+(t)\right)^{\alpha+1}}{\left(\rho(t)\tau'(t)\right)^{\alpha}}. \tag{3.39}$$

On the other hand, define

$$A := \left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)} \upsilon(t), \qquad B := \left[\zeta_+(t)\frac{\alpha}{\alpha+1}\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}. \tag{3.40}$$

So we have

$$\zeta_+(t)\upsilon(t) - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)](\zeta_+(t))^{\alpha+1}\rho(t)}{(\tau'(t))^{\alpha}}. \tag{3.41}$$

Thus, from (3.37), we get

$$\begin{aligned} &\omega(t) - \omega(t_2) + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon(t) - \frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}\upsilon(t_2) \\ &\quad \le -\int_{t_2}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}} - \frac{r[\tau(s)]}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\left[\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\alpha+1} + \frac{p^{\alpha}[\sigma(s)](\zeta_+(s))^{\alpha+1}}{\tau'(s)}\right]\right\}\mathrm{d}s, \end{aligned} \tag{3.42}$$

which contradicts (3.27). This completes the proof. □

When $p(t) \le p_0 < \infty$, $\tau'(t) \ge \tau_0 > 0$, where p_0, τ_0 are constants, we obtain the following result.

Theorem 3.4. *Suppose that* (1.5) *holds,* $p(t) \le p_0 < \infty$, $\tau'(t) \ge \tau_0 > 0$, $\tau(t) \le t$, $\sigma(t) \ge \tau(t)$ *for* $t \ge t_0$. *Furthermore, assume that there exists a function* $\rho \in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\tau(s)]\left(\rho'_+(s)\right)^{\alpha+1}}{\left(\tau_0\rho(s)\right)^{\alpha}}\right]ds = \infty. \tag{3.43}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \ge t_1$. Using (1.1) and the definition of z, we obtain (3.26) for all sufficiently large t. The remainder of the proof is similar to that of Theorem 3.3 and hence is omitted. □

4. Oscillation Criteria for the Case (1.13)

In this section, we will establish some oscillation criteria for (1.1) under the case (1.13).

In the following, we assume that p_0, τ_0 are constants.

Theorem 4.1. *Suppose that* (1.13) *holds,* $p(t) \leq p_0 < \infty$, $\tau'(t) \geq \tau_0 > 0$, $\sigma(t) \leq t$, $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma'(t) > 0$, $\sigma(t) \leq \tau(t)$ *for* $t \geq t_0$. *Further, assume that there exists a function* $\rho \in C([t_0,\infty),(0,\infty))$ *such that* (3.24) *holds. If there exists a function* $\eta \in C^1([t_0,\infty),\mathbb{R})$, $\eta(t) \geq t$, $\eta'(t) > 0$ *for* $t \geq t_0$ *such that*

$$\limsup_{t'\to\infty}\int_{t_0}^{t'}\left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s)-\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]ds=\infty, \tag{4.1}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.2, we get (3.26). In view of (1.1), we have (3.4). Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. Then, by Theorem 3.2, we obtain a contradiction with (3.24).

(ii) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. We define the function u by

$$u(t) = -\frac{r(t)(-z'(t))^{\alpha}}{z^{\alpha}[\eta(t)]}, \quad t \geq t_2. \tag{4.2}$$

Then $u(t) < 0$. Noting that $r(t)(-z'(t))^{\alpha}$ is increasing, we get

$$r^{1/\alpha}(s)z'(s) \leq r^{1/\alpha}(t)z'(t), \quad s \geq t \geq t_2. \tag{4.3}$$

Dividing the above inequality by $r^{1/\alpha}(s)$, and integrating it from $\eta(t)$ to t', we obtain that

$$z(t') \leq z[\eta(t)] + r^{1/\alpha}(t)z'(t)\int_{\eta(t)}^{t'}\frac{ds}{r^{1/\alpha}(s)}. \tag{4.4}$$

Letting $t' \to \infty$, we have

$$0 \leq z[\eta(t)] + r^{1/\alpha}(t)z'(t)\delta(t), \tag{4.5}$$

that is,

$$-\delta(t)\frac{r^{1/\alpha}(t)z'(t)}{z[\eta(t)]} \leq 1. \tag{4.6}$$

Hence, by (4.2), we get

$$-\delta^{\alpha}(t)u(t) \leq 1. \tag{4.7}$$

Similarly, we define the function v by

$$v(t) = -\frac{r[\tau(t)](-z'[\tau(t)])^{\alpha}}{z^{\alpha}[\eta(t)]}, \quad t \geq t_2. \tag{4.8}$$

Then $v(t) < 0$. Noting that $r(t)(-z'(t))^{\alpha}$ is increasing, we get the following:

$$r(t)\left(-z'(t)\right)^{\alpha} \geq r[\tau(t)]\left(-z'[\tau(t)]\right)^{\alpha}. \tag{4.9}$$

Thus $0 < -v(t) \leq -u(t)$. So by (4.7), we see that

$$-\delta^{\alpha}(t)v(t) \leq 1. \tag{4.10}$$

Differentiating (4.2), we obtain that

$$u'(t) = \frac{\left(-r(t)(-z'(t))^{\alpha}\right)' z^{\alpha}[\eta(t)] + \alpha r(t)(-z'(t))^{\alpha} z^{\alpha-1}[\eta(t)] z'[\eta(t)]\eta'(t)}{z^{2\alpha}[\eta(t)]}, \tag{4.11}$$

by (3.4), and we have $z'[\eta(t)] \leq (r(t)/r[\eta(t)])^{1/\alpha} z'(t)$, so

$$u'(t) \leq \frac{\left(-r(t)(-z'(t))^{\alpha}\right)'}{z^{\alpha}[\eta(t)]} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha}. \tag{4.12}$$

Similarly, we see that

$$v'(t) \leq \frac{\left(-r[\tau(t)](-z'[\tau(t)])^{\alpha}\right)'}{z^{\alpha}[\eta(t)]} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \tag{4.13}$$

Therefore, by (4.12) and (4.13), we get the following:

$$\begin{aligned} u'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq{}& \frac{\left(-r(t)(-z'(t))^{\alpha}\right)'}{z^{\alpha}[\eta(t)]} + \frac{p_0^{\alpha}}{\tau_0} \frac{\left(-r[\tau(t)](-z'[\tau(t)])^{\alpha}\right)'}{z^{\alpha}[\eta(t)]} \\ & - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha} - \frac{\alpha p_0^{\alpha}}{\tau_0} \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \end{aligned} \tag{4.14}$$

Using (3.26) and (4.14), we obtain that

$$u'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq -\frac{Q(t)}{2^{\alpha-1}} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha} - \frac{\alpha p_0^{\alpha}}{\tau_0} \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \tag{4.15}$$

Multiplying (4.15) by $\delta^{\alpha}(t)$, and integrating it from t_2 to t', we have

$$\begin{aligned} &u(t')\delta^{\alpha}(t') - u(t_2)\delta^{\alpha}(t_2) + \alpha\int_{t_2}^{t'} \frac{\delta^{\alpha-1}(s)\eta'(s)u(s)}{r^{1/\alpha}[\eta(s)]}\mathrm{d}s + \alpha\int_{t_2}^{t'} \frac{\eta'(s)\delta^{\alpha}(s)}{r^{1/\alpha}[\eta(s)]}(-u(s))^{(\alpha+1)/\alpha}\mathrm{d}s \\ &\quad + \frac{p_0^{\alpha}}{\tau_0}v(t')\delta^{\alpha}(t') - \frac{p_0^{\alpha}}{\tau_0}v(t_2)\delta^{\alpha}(t_2) + \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_2}^{t'} \frac{\delta^{\alpha-1}(s)\eta'(s)v(s)}{r^{1/\alpha}[\eta(s)]}\mathrm{d}s \\ &\quad + \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_2}^{t'} \frac{\eta'(s)\delta^{\alpha}(s)}{r^{1/\alpha}[\eta(s)]}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s + \int_{t_2}^{t'} \frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s)\mathrm{d}s \le 0. \end{aligned} \tag{4.16}$$

Using (3.19), (4.7), and (4.10), we find that

$$\begin{aligned} \int_{t_2}^{t'} \left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s) - \left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]\mathrm{d}s &\le u(t_2)\delta^{\alpha}(t_2) + \frac{p_0^{\alpha}}{\tau_0}v(t_2)\delta^{\alpha}(t_2) \\ &\quad + 1 + \frac{p_0^{\alpha}}{\tau_0}. \end{aligned} \tag{4.17}$$

Letting $t' \to \infty$, we obtain a contradiction with (4.1). This completes the proof. □

From Theorems 3.4 and 4.1, we have the following result.

Theorem 4.2. *Suppose that* (1.13) *holds,* $p(t) \le p_0 < \infty$, $\tau'(t) \ge \tau_0 > 0$, $\tau(t) \le t$, $\sigma(t) \ge \tau(t)$ *for* $t \ge t_0$. *Further, assume that there exists a function* $\rho \in C([t_0,\infty),(0,\infty))$ *such that* (3.43) *holds. If there exists a function* $\eta \in C^1([t_0,\infty),\mathbb{R})$, $\eta(t) \ge t$, $\eta'(t) > 0$, $\sigma(t) \le \eta(t)$ *for* $t \ge t_0$ *such that* (4.1) *holds, then* (1.1) *is oscillatory.*

5. Examples

In this section, we will give some examples to illustrate the main results.

Example 5.1. Study the second-order neutral differential equation

$$\left[\left|(x(t)+tx(t-\lambda_1))'\right|^{\alpha-1}(x(t)+tx(t-\lambda_1))'\right]' + \beta|x(t-\lambda_2)|^{\alpha-1}x(t-\lambda_2) = 0, \quad t \ge t_0, \tag{5.1}$$

where $\alpha \ge 1$, $0 < \lambda_1 \le \lambda_2 < 1$, $\beta > 0$ are constants.

Let $r(t) = 1$, $p(t) = t$, $\rho(t) = 1$. It is easy to see that all the conditions of Theorem 3.1 hold. Hence, (5.1) is oscillatory.

Example 5.2. Consider the second-order quasilinear neutral differential equation

$$\left[\left|\left(x(t)+p(t)x(\lambda_1 t)\right)'\right|^{\alpha-1}\left(x(t)+p(t)x(\lambda_1 t)\right)'\right]' + \frac{\beta}{t^{\alpha+1}}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t) = 0, \quad t \ge t_0, \tag{5.2}$$

where $\alpha \ge 1$, $\beta > 0$ are constants, $\lambda_1, \lambda_2 \in (0,1), \lambda_2 \le \lambda_1$.

Let $r(t) = 1,\ 0 \le p(t) \le p_0 < \infty,\ q(t) = \beta/t^{\alpha+1},\ \tau(t) = \lambda_1 t,\ \sigma(t) = \lambda_2 t$, and $\rho(t) = t^\alpha$. Then, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}&\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\sigma(s)]\left((\rho'(s)_+)^{\alpha+1}\right)}{\left(\rho(s)\sigma'(s)\right)^\alpha}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_2^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right)\right]\limsup_{t\to\infty}\int_{t_0}^{t}\frac{ds}{s} = \infty,\end{aligned}\tag{5.3}$$

if $\beta > 2^{\alpha-1}\alpha^{\alpha+1}(1+p_0^\alpha/\lambda_1)/[(\alpha+1)^{\alpha+1}\lambda_2^\alpha]$. Hence, by Theorem 3.2, (5.2) is oscillatory if

$$\beta > \frac{2^{\alpha-1}\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_2^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right).\tag{5.4}$$

Example 5.3. Investigate the second-order neutral differential equation

$$\left[x(t) + \frac{(t-1)(t-4\pi)}{t}x(t-4\pi)\right]'' + (t-2\pi)x(t-2\pi) = 0, \quad t \ge t_0.\tag{5.5}$$

Let $r(t) = 1,\ p(t) = (t-1)(t-4\pi)/t,\ q(t) = t-2\pi$, and $\rho(t) = 1$. It is easy to see that all the conditions of Theorem 3.3 hold. Hence, (5.5) is oscillatory, for example, $x(t) = \sin t/t$ is a solution of (5.5).

Example 5.4. Discuss the second-order quasilinear neutral differential equation

$$\left[\left|\left(x(t)+p(t)x(\lambda_1 t)\right)'\right|^{\alpha-1}\left(x(t)+p(t)x(\lambda_1 t)\right)'\right]' + \frac{\beta}{t^{\alpha+1}}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t) = 0, \quad t \ge t_0,\tag{5.6}$$

where $\alpha \ge 1,\ \beta > 0$ are constants, $\lambda_1 \in (0,1), \lambda_2 \in [\lambda_1, \infty)$.

Let $r(t) = 1,\ 0 \le p(t) \le p_0 < \infty,\ q(t) = \beta/t^{\alpha+1},\ \tau(t) = \lambda_1 t,\ \sigma(t) = \lambda_2 t,\ \rho(t) = t^\alpha$. Then, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}&\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\tau(s)]\left(\rho'(s)_+\right)^{\alpha+1}}{\tau_0\left(\rho(s)\right)^\alpha}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_1^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right)\right]\limsup_{t\to\infty}\int_{t_0}^{t}\frac{ds}{s} = \infty\end{aligned}\tag{5.7}$$

if $\beta > 2^{\alpha-1}\alpha^{\alpha+1}(1+p_0^\alpha/\lambda_1)/[(\alpha+1)^{\alpha+1}\lambda_1^\alpha]$. Hence, by Theorem 3.4, (5.6) is oscillatory if

$$\beta > \frac{2^{\alpha-1}\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_1^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right).\tag{5.8}$$

Example 5.5. Examine the second-order quasilinear neutral differential equation

$$\left[t^{2\alpha}\left|(x(t)+p(t)x(\lambda_1 t))'\right|^{\alpha-1}(x(t)+p(t)x(\lambda_1 t))'\right]'+\beta t^{\alpha-1}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t)=0,\quad t\geq t_0,\tag{5.9}$$

where $\alpha \geq 1$, $\beta > 0$ are constants, $\lambda_1, \lambda_2 \in (0,1)$, $\lambda_2 \leq \lambda_1$.

Let $r(t) = t^{2\alpha}$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) = \beta t^{\alpha-1}$, $\tau(t) = \lambda_1 t$, $\sigma(t) = \lambda_2 t$, and $\rho(t) = 1$. Then, $Q(t) = \beta\lambda_1^{\alpha-1}t^{\alpha-1}$. It is easy to see that (3.24) holds. On the other hand, taking $\eta(t) = t$, then $\delta(t) = 1/t$. Therefore, one has

$$\begin{aligned}\limsup_{t'\to\infty}&\int_{t_0}^{t'}\left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s)-\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}}\lambda_1^{\alpha-1}-\frac{1}{2}\left(1+\frac{p_0^{\alpha}}{\lambda_1}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right]\limsup_{t'\to\infty}\int_{t_0}^{t'}\frac{ds}{s}=\infty\end{aligned}\tag{5.10}$$

if $\beta > 2^{\alpha-2}(1+p_0^{\alpha}/\lambda_1)(\alpha/\alpha+1)^{\alpha+1}/\lambda_1^{\alpha-1}$. Thus, by Theorem 4.1, (5.9) oscillates if

$$\beta>\frac{2^{\alpha-2}}{\lambda_1^{\alpha-1}}\left(1+\frac{p_0^{\alpha}}{\lambda_1}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.\tag{5.11}$$

6. Conclusions

Inequality technique plays an important role in studying the oscillatory behavior of differential equations; in this paper, we establish a new inequality (2.1); by using (2.1) and Riccati substitution, we establish some new oscillation criteria for (1.1). Theorem 3.1 can be applied to the case $\tau(t) \geq t$. Specially, taking $\alpha = 1$, our results include and improve the results in [15]; for example, and Theorem 4.1 includes [15, Theorem 3.1], Theorem 4.2 includes [15, Theorem 3.2]. The method can be applied on the second-order Emden-Fowler neutral differential equations

$$\left[r(t)(x(t)+p(t)x(\tau(t)))'\right]'+q(t)|x(\delta(t))|^{\alpha-1}x(\delta(t))=0,\quad t\geq t_0,\tag{6.1}$$

where $\alpha \geq 1$. It would be interesting to find another method to investigate (1.1) when $\tau \circ \sigma \not\equiv \sigma \circ \tau$.

Acknowledgment

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript. This paper is supported by the Natural Science Foundation of China (11071143, 60904024, 61174217), Natural Science Outstanding Youth Foundation of Shandong Province (JQ201119), Shandong Provincial Natural Science Foundation (ZR2010AL002, ZR2009AL003), and by Natural Science Foundation of Educational Department of Shandong Province (J11LA01).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 924107, 11 pages
doi:10.1155/2012/924107

Research Article

The Stability of Nonlinear Differential Systems with Random Parameters

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Received 12 April 2012; Accepted 19 June 2012

Academic Editor: Agacik Zafer

The paper deals with nonlinear differential systems with random parameters in a general form. A new method for construction of the Lyapunov functions is proposed and is used to obtain sufficient conditions for L_2-stability of the trivial solution of the considered systems.

1. Introduction

1.1. The Aim of the Contribution

The method of Lyapunov functions is one of the most effective methods for investigation of self-regulating systems. It is important for determining the fact of stability or instability of given systems among other purposes. A successfully constructed Lyapunov function for given nonlinear self-regulating systems makes it possible to solve all the complex problems important in practical applications such as estimation of changes of a self-regulated variable, estimation of transient processes, estimation of integral criteria of the quality of self-regulation, or estimation of what is called guaranteed domain of stability.

In [1] it is explained why not every positive definite function can serve as a Lyapunov function for a system of differential equations. As experience shows, the most suitable Lyapunov functions have physical meaning. The Lyapunov function method is an effective method for the investigation of stability of linear or nonlinear differential systems that are explicitly independent of time (see, e.g., [1–9]). But there are no universal methods for constructing appropriate Lyapunov functions because, as well-known, in nonlinear differential systems, each case considered requires an individual method for constructing a Lyapunov function.

However, the method of Lyapunov functions is often difficult to apply to the investigation of some kinds of stability of nonstationary differential systems because the concept of Lyapunov stability can make the Lyapunov functions inconvenient to use. This problem was solved by a new definition of what is called L_2-stability of the trivial solution of the nonstationary differential (or difference) systems [10, 11], which is compatible with the method of Lyapunov functions.

In this paper, we deal with much more complicated investigation of the Lyapunov stability of differential systems with random parameters. We define a concept of L_2-stability of the trivial solution of the differential systems with semi-Markov coefficients and give an analogy between the L_2-stability and the stability obtained by Lyapunov functions. A new method of constructing Lyapunov functions is proposed for the study of stability of systems, and Lyapunov functions are derived for systems of differential equations with coefficients depending on a semi-Markov process. Sufficient conditions of stability are given, and it is proved that the condition of L_2-stability implies the existence of Lyapunov functions. In addition to this, the case of the coefficients of the considered systems depending on Markov process is analyzed.

1.2. Systems Considered

In this part, a new concept of semi-Markov function is proposed. It will be used later for the construction of Lyapunov functions.

Consider nonlinear n-dimensional differential system

$$\frac{dX(t)}{dt} = F(t, X(t), \xi(t)), \qquad F(t, 0, \xi) = 0, \tag{1.1}$$

on the probability space $(\Omega \equiv \{\omega\}, \mathfrak{T}, \mathbf{P}, \mathbf{F} \equiv \{\mathbf{F_t} : t \geq 0\})$. A vector-function $X = X(t)$, $t \geq 0$, is called a solution of (1.1) if $X(t)$ is a random vector-function from the set of random vector-functions defined on Ω, there exists mathematical expectation of $\{X^2(t)\}$, and (1.1) is satisfied for $t \geq 0$. The derivative is understood in the meaning of differentiability of a random process [12].

A space of solutions X can be interpreted as a phase space of states of a random environment. Measurable subsets of a random environment form a collection of its states. As a phase space of states serves a complete metric separable space (as a rule the Euclidean space or a finite space equipped with σ-algebra of all subsets of X). Under assumptions of our problem (and in similar problems as well), solutions are defined in the meaning of a strong solution of the Cauchy problem [13].

Together with (1.1), we consider the initial condition

$$X(0) = \varphi(\omega), \qquad \varphi : \Omega \to \mathbb{R}^n. \tag{1.2}$$

In fact, any solution $X(t)$ of (1.1) depends on the random variable ω, that is, $X(t) \equiv X(t, \omega)$.

The random process $\xi(t)$, $t \geq 0$, is a semi-Markov process with the states

$$\theta_1, \theta_2, \dots, \theta_n. \tag{1.3}$$

We assume $\xi(t_0) = 0$ where $t_0 = 0$, and moments of jumps t_j, $j = 0,1,\ldots,n$, $t_0 < t_1 < \cdots < t_n$ of the process ξ are such that $\xi(t_j) = \lim_{t\to t_j+0}\xi(t)$ and $\xi(t) = \theta_s$, $s \in \{1,2,\ldots,n\}$ if $t_j \le t < t_{j+1}$, $j = 0,1,\ldots,n-1$.

The transition from state θ_l to state θ_s is characterized by the intensity $q_{ls}(t)$, $l,s = 1,2,\ldots,n$, and the semi-Markov process is defined by the intensity matrix

$$Q(t) = \left(q_{ls}(t)\right)_{l,s=1}^{n}, \tag{1.4}$$

whose elements satisfy the relationships

$$q_{ls}(t) \ge 0, \qquad \sum_{l=1}^{n}\int_0^{\infty} q_{ls}(t)dt = 1. \tag{1.5}$$

Let mutually different functions $w_s(t,x)$, $s = 1,2,\ldots,n$, be defined for $t > 0$, $x \in \mathbb{R}^n$.

Definition 1.1. The function $w(t,x,\xi(t))$ is called a semi-Markov function if the equalities

$$w(t,x,\xi(t) = \theta_s) = w_s(t - t_j, x), \quad s = 1,2,\ldots,n \tag{1.6}$$

hold for $t_j \le t \le t_{j+1}$.

It means that the semi-Markov function $w(t,x,\xi(t))$ is a functional of a random process $\xi(t)$. The value of $w(t,x,\xi(t))$ is determined by the values $t,x,\xi(t)$ at the time t and also by the value of the jump of the process $\xi(t)$ at time t_j, which precedes time t. In fact, the system (1.1) means n different differential systems in the form

$$\frac{dX(t)}{dt} = F_s(t,X(t)), \quad s = 1,2,\ldots,n, \tag{1.7}$$

where

$$F_s(t,x) \equiv F(t,x,\theta_s). \tag{1.8}$$

We assume that there exists a unique solution of (1.7) for every point (t,x) such that $t \ge 0$, $\|x\| < \infty$ ($\|\cdot\|$ stands for Euclidean norm), continuable on $[0,\infty)$.

1.3. Auxiliaries

In the paper, in addition to what was mentioned above, the following notations and assumptions are introduced:

(1) the functions $F_s(t,x)$, $s = 1,2,\ldots,n$, are Lipschitz functions with the Lipschitz constants ρ_s, that is, the inequalities

$$\left\|F_s(t,x) - F_s(t,y)\right\| \le \rho_s\left\|x - y\right\|, \quad s = 1,2,\ldots,n \tag{1.9}$$

hold.

(2) If $x = 0$, then

$$F_s(t,0) \equiv 0, \quad s = 1,2,\dots,n, \ t \geq 0. \tag{1.10}$$

(3) The inequalities

$$\|N_s(t,x)\| \leq \rho_s e^{-\alpha_s t}, \quad s = 1,\dots,n, \ t \geq 0 \tag{1.11}$$

are valid. Here ρ_s, $s = 1,\dots,n$ are the Lipschitz constants, α_s, $s = 1,\dots,n$ are positive constants, and $N_s(t, X(0))$, $s = 1,\dots,n$ is the solution $X(t)$ of (1.7) in the Cauchy form, that is,

$$X(t) = N_s(t, X(0)), \quad s = 1,2,\dots,n. \tag{1.12}$$

(4) We introduce the Lyapunov functional

$$V = \int_0^\infty E(w(t,x,\xi(t)))dt, \tag{1.13}$$

where $E(\cdot)$ denotes mathematical expectation, and we assume that the integral is convergent.

Definition 1.2. The trivial solution of the differential systems (1.1) is said to be L_2-stable if, for any solution $X(t)$ with bounded initial values of the mathematical expectation

$$E(X(0)X^*(0)), \tag{1.14}$$

the integral

$$J = \int_0^\infty E\left(\|X(t)\|^2\right)dt \tag{1.15}$$

converges.

Remark 1.3. It is easy to see that (1.15) converges if and only if the matrix integral

$$\int_0^\infty E(X(t)X^*(t))dt \tag{1.16}$$

is convergent.

Lemma 1.4. *Let the function $w(t,x,\xi)$ be bounded, that is, there exists a constant β such that the inequalities*

$$0 \leq w(x) \leq w(t,x,\xi) \leq \beta\|x\|^2, \quad \textit{for } \xi = \theta_s, \ s = 1,2,\dots,n, \tag{1.17}$$

or the inequalities

$$0 \le w(x) \le w_s(t,x) \le \beta \|x\|^2, \quad s = 1,\dots,n \tag{1.18}$$

hold where $w(x)$ is a positive definite and differentiable function satisfying the inequality

$$E(w(X(t))) \le E(w(t,X(t),\xi(t))) \le \beta E\left(\|X(t)\|^2\right). \tag{1.19}$$

Let, moreover, the Lyapunov functional (1.13) *exist for the system* (1.7) *with an L_2-stable trivial solution.*

Then the Lyapunov functional (1.13) *can be expressed in the form*

$$V = \int_{E_n} \sum_{s=1}^{n} v_s(x) f_s(0,x)dx, \quad dx \equiv dx_1 \cdots dx_n \tag{1.20}$$

if the particular Lyapunov functions

$$v_s(x) = \int_0^{\infty} E(w(t,X(t),\xi(t)) \mid X(0) = x, \xi(0) = \theta_s)dt, \quad s = 1,\dots,n \tag{1.21}$$

are known.

Proof. The functions $v_s(x)$, $s = 1,\dots,n$, will be defined using auxiliary functions

$$u_s(t,x) = E(w(t,X(t),\xi(t)) \mid X(0) = x, \xi(0) = \theta_s), \quad s = 1,\dots,n. \tag{1.22}$$

The mathematical expectation in (1.22) can be calculated by the transition intensities $q_{ls}(t)$, $l,s = 1,\dots,n$, $t \ge 0$

$$\Psi_s(t) = \int_t^{\infty} q_s(\tau)d\tau, \quad q_s(t) \equiv \sum_{l=1}^{n} q_{ls}(t), \quad s = 1,\dots,n, \tag{1.23}$$

whence the system

$$u_s(t,x) = \Psi_s(t) w_s(t, X_s(t,x)) + \sum_{l=1}^{n} \int_0^t q_{ls}(\tau) u_l(t-\tau, N_s(\tau,x))d\tau, \quad s = 1,\dots,n \tag{1.24}$$

is obtained. Integrating the system of equations (1.24) with respect to t, we get the system of functional equations

$$\begin{aligned} v_s(x) = &\int_0^{\infty} \Psi_s(t) w_s(t, N_s(t,x))dt \\ &+ \sum_{l=1}^{n} \int_0^{\infty} \left[\int_0^t q_{ls}(\tau) v_l(t-\tau, N_s(\tau,x))d\tau \right] dt, \quad s = 1,\dots,n, \end{aligned} \tag{1.25}$$

for

$$v_s(x) = \int_0^{\infty} u_s(t,x)dt, \quad s = 1,\dots,n. \tag{1.26}$$

The system (1.25) thus obtained can be solved by successive approximations

$$\begin{aligned} v_s^0(x) &\equiv 0, \quad s = 1,\dots,n, v_s^{(\alpha+1)}(x) \\ &= \int_0^{\infty} \Psi_s(t) w_s(t, N_s(t,x))dt \\ &\quad + \sum_{l=1}^{n} \int_0^{\infty} q_{ls}(t) v_l^{(\alpha)}(N_s(t,x))dt, \quad s = 1,\dots,n, \alpha = 0,1,2,\dots. \end{aligned} \tag{1.27}$$

□

2. Main Results

2.1. The Case of a Semi-Markovian Random Process $\xi(t)$

Theorem 2.1. *Let the functions* $F_s(t,x)$, $s = 1,2,\dots,n$, *in the system* (1.7) *satisfy conditions* (1.9), (1.11), *let the semi-Markov process* $\xi(t)$ *be determined by the transition intensities* $q_{ls}(t), l, s = 1,\dots,n$, $t \geq 0$ *satisfying* (1.5), *and let the functions* $w(t,x,\xi(t))$ *satisfy* (1.17). *Then the following statements are true.*

(1) *The relationships*

$$v_s^{(\alpha)}(x) \leq C_s^{(\alpha)} \|x\|^2, \quad s = 1,\dots,n, \ \alpha = 0,1,\dots, \tag{2.1}$$

$$\int_0^{\infty} \Psi_s(t) e^{-2\alpha_s t} dt < \infty, \quad \int_0^{\infty} q_s(t) e^{-2\alpha_s t} dt < \infty, \quad s = 1,\dots,n, \tag{2.2}$$

imply that, for the system (1.7), *the particular Lyapunov functions can be established in the form*

$$v_s(x) = \Psi_s(t) w_s(t, N_s(t,x)) + \sum_{l=1}^{n} \int_0^{t} q_{ls}(\tau) v_l(N_s(\tau,x))d\tau, \quad s = 1,\dots,n, \tag{2.3}$$

(2) *If the spectral radius of the matrix* $\Gamma = (\gamma_{ls})_{l,s=1}^{n}$ *is less than one, then the particular Lyapunov functions* $v_s(x)$, $s = 1,\dots,n$, *can be found by the method of successive approximations* (1.27).

(3) *Under assumption* (1.11), *the method of successive approximations* (1.27) *converges and the inequalities*

$$v_s^{(\alpha+1)} \geq v_s^{(\alpha)}(x), \quad v_s^{(\alpha)}(x) \leq v_s(x), \quad s = 1,\dots,n \tag{2.4}$$

hold. Then the sequence of functions $v_s^{(\alpha)}(x)$, $s = 1,\dots,n$, $\alpha = 0,1,2,\dots$, *is monotone increasing and bounded from above by the functions* $v_s(x)$, $s = 1,\dots,n$.

Proof. Applying estimation (2.1) and assumption (2.2) to the successive approximations (1.27), we get

$$\begin{aligned} C_s^{(\alpha+1)} &\leq \beta\rho_s^2\int_0^{\infty}\Psi_s(t)e^{-2\alpha_s t}dt \\ &\quad+\rho_s^2\sum_{l=1}^{n}\int_0^{\infty}q_{ls}(t)e^{-2\alpha_s t}dtC_l^{(\alpha)}, \quad s=1,\dots,n,\ \alpha=0,1,2,\dots. \end{aligned} \tag{2.5}$$

It is sufficient to assume the existence of a bounded solution of the system of inequalities (2.1) whence the existence follows of a positive solution of the system of linear algebraic equations (2.3). Moreover, assumption (2.2) guarantees the convergence of the improper integrals in the system (1.27) and so, for the existence of a positive solution of the system (2.3), it is sufficient that the spectral radius $\rho(\Gamma)$ of the matrix

$$\Gamma=(\gamma_{ls})_{l,s=1}^{n} \tag{2.6}$$

is less than one. For this, it is sufficient that

$$\sum_{l=1}^{n}\gamma_{ls}\equiv\rho_s^2\int_0^{\infty}q_s(t)t^{-2\alpha_s t}dt<1, \quad s=1,\dots,n. \tag{2.7}$$

The convergence of the sequence $v_s^{(\alpha)}(x), s=1,\dots,n, \alpha=0,1,2,\dots$, can be determined by the system

$$\begin{aligned} &v_s^{(\alpha+1)}(x)-v_s^{(\alpha)}(x) \\ &\quad=\sum_{l=1}^{n}\int_0^{\infty}q_{ls}(t)\left[v_l^{(\alpha)}(N_s(t,x))-v_l^{(\alpha-1)}(N_s(t,x))\right]dt, \quad s=1,\dots,n,\ \alpha=1,2,3,\dots. \end{aligned} \tag{2.8}$$

If there exist the inequalities

$$\left|v_s^{(\alpha)}(x)-v_s^{(\alpha-1)}(x)\right|\leq\sum_{l=1}^{n}\int_0^{\infty}q_{ls}(t)\rho_s^2e^{-2\alpha_s t}dt\, d_s^{(\alpha)}\|x\|^2, \quad s=1,\dots,n, \tag{2.9}$$

where

$$d_s^{(\alpha+1)}=\sum_{l=1}^{n}\gamma_{sl}d_l^{(\alpha)}, \quad s=1,\dots,n, \tag{2.10}$$

hold, then estimation (2.4) is true for all $\alpha=2,3,\dots$, $d_s^{(1)}=C_s$, $s=1,\dots,n$.

Under assumptions (2.8), it follows

$$\lim_{\alpha\to+\infty}d_s^{(\alpha)}=0, \quad s=1,\dots,n, \tag{2.11}$$

which implies a uniform convergence of the sequence $v_s^{(\alpha)}(x)$, $s=1,\dots,n$, $\alpha=0,1,2,\dots$. □

Corollary 2.2. *If the trivial solution of the differential systems* (1.1) *is L_2-stable, then there exist particular Lyapunov functions $v_s(x)$, $s = 1, \ldots, n$ that satisfy* (2.3).

Corollary 2.3. *Let the function $w(t, x, \xi(t))$ satisfy the inequality:*

$$\beta_1 \|x\|^2 \leq w(t, x, \xi(t)) \leq \beta \|x\|^2, \quad \beta_1 > 0. \tag{2.12}$$

If there exist the Lyapunov functions $v_s(x)$, $s = 1, \ldots, n$ for the system (1.21), *then the trivial solution of the differential systems* (1.1) *is L_2-stable.*

Corollary 2.4. *Let the semi-Markov process $\xi(t)$ in the system* (1.1) *have jumps at the times t_j, $j = 0, 1, 2, \ldots$, $t_0 = 0$, in the transition from state θ_s to state θ_l, and let the jumps satisfy the equation*

$$X(t_j) = \Phi_{ls}(X(t_j - 0)), \quad \Phi_{ls}(0) = 0, \ j = 1, 2, \ldots, \tag{2.13}$$

where $\Phi_{ls}(x)$ are any continuous Lipschitz vector functions. Then the system (2.3) *has the form*

$$\begin{aligned} v_s(x) = &\int_0^\infty \Psi_s(t) w_s(t, N_s(t, x)) dt \\ &+ \sum_{l=1}^{n} \int_0^\infty q_{ls}(t) v_s(\Phi_{ls}(N_s(t, x))) dt, \quad s = 1, \ldots, n, \end{aligned} \tag{2.14}$$

and its solution can be found by the method of successive approximations.

2.2. The Case of a Markovian Random Process $\xi(t)$

Next result relates to the case of the semi-Markov process $\xi(t)$ being transformed into a Markov process described by the system of ordinary differential equations:

$$\frac{dP}{dt} = AP(t), \qquad A = (a_{ls})_{l,s=1}^{n}, \tag{2.15}$$

under the influence of which the considered system

$$\frac{dX(t)}{dt} = F(X(t), \xi(t)), \quad F(0, \xi(t)) \equiv 0, \tag{2.16}$$

takes the form

$$\frac{dX(t)}{dt} = F_s(X(t)), \quad s = 1, \ldots, n. \tag{2.17}$$

We also assume that, if $t_j \leq t < t_{j+1}$, $\xi(t) = \theta_s$, then

$$w(t, x, \xi(t)) = w_s(x), \quad w_s(0) = 0, \quad s = 1, \ldots, n. \tag{2.18}$$

Then the system of equations (2.14) has the form

$$v_s(x) = \int_0^\infty e^{a_{ss}t} w_s(N_s(t,x))dt + \sum_{\substack{l=1 \\ l\neq s}}^{n} \int_0^\infty a_{ls} e^{a_{ss}t} v_l(N_s(t,x))dt, \quad s = 1,\dots,n. \tag{2.19}$$

Theorem 2.5. *Let the nonlinear differential system* (1.1)*, depending on the Markov process* $\xi(t)$*, be described by* (2.15)*. Then the particular Lyapunov functions* $v_s(x)$ *satisfy the linear differential system:*

$$\frac{Dv_s(x)}{Dx} F_s(x) + w_s(x) + \sum_{l=1}^{n} a_{ls} v_l(x) = 0, \quad s = 1,\dots,n. \tag{2.20}$$

Proof. Let us write the solution of the system (2.17) in the Cauchy form:

$$X(t) = N_s(t-\tau, X(\tau)), \quad s = 1,\dots,n. \tag{2.21}$$

Differentiating (2.21) with respect to τ, we get

$$-\frac{\partial N_s(t-\tau, X(\tau))}{\partial t} + \frac{DN_s(t-\tau, X(\tau))}{DX(\tau)} F_s(X(\tau)) = 0, \tag{2.22}$$

which, for $\tau = 0$, $X(0) = x$, takes the form:

$$\frac{DN_s(t,x)}{Dx} F_s(x) \equiv \frac{\partial N_s(t,x)}{\partial x}, \quad s = 1,\dots,n. \tag{2.23}$$

Then

$$\begin{aligned}
\frac{DN_s(t,x)}{Dx} F_s + a_{ss} v_s(x) = & -\int_0^\infty a_{ss} e^{a_{ss}t} (w_s(N_s(t,x))) + \sum_{\substack{l=1 \\ l\neq s}}^{n} a_{ls} e^{a_{ss}t} v_l(N_s(t,x))dt \\
& + \int_0^\infty e^{a_{ss}t} \frac{\partial}{\partial t} (w_s(N_s(t,x))) + \sum_{\substack{l=1 \\ l\neq s}}^{n} a_{ls} v_l(N_s(t,x))dt \\
= & \int_0^\infty \frac{\partial}{\partial t} \left(e^{a_{ss}t} (w_s(N_s(t,x)))\right) \\
& + \sum_{\substack{l=1 \\ l\neq s}}^{n} a_{ls} v_l(N_s(t,x))dt \\
= & -\sum_{\substack{l=1 \\ l\neq s}}^{n} a_{ls} v_l(x),
\end{aligned} \tag{2.24}$$

which implies (2.20) if

$$\lim_{t\to+\infty} e^{\alpha_{ss}t}\left(w_s(N_s(t,x))\right) + \sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls}v_l(N_s(t,x)) = 0, \quad s = 1,\dots,n \tag{2.25}$$

and the functions $w_s(x)$, $v_s(x)$, $s = 1,\dots,n$ are differentiable. □

Corollary 2.6. *If the solutions of the system* (2.16) *have the same jumps as the solution of the system* (2.14) *and converge to the jumps of the Markov process* $\xi(t)$ *such that* $\Phi_{ls}(x) \equiv E$, $l = 1,\dots,n$, *then the system* (2.19) *takes the form*

$$\begin{aligned} v_s(x) = &\int_0^\infty e^{a_{ss}t} w_s(N_s(t,x))dt \\ &+ \sum_{\substack{l=1\\ l\neq s}}^{n} \int_0^\infty a_{ls} e^{a_{ss}t} v_l(\Phi_{ls}(N_s(t,x)))dt, \quad s = 1,\dots,n, \end{aligned} \tag{2.26}$$

and the system (2.20) *takes the form*

$$\frac{Dv_s(x)}{Dx} = F_s(x) + \sum_{l=1}^{n} a_{ls}v_l(\Phi_{ls}(x)) = -w_s(x), \quad s = 1,\dots,n. \tag{2.27}$$

Example 2.7. Let us investigate the stability of solutions of two-dimensional system

$$\frac{dX(t)}{dt} = (\nu - \lambda - \alpha)X(t) + G(X(t),\xi(t)), \quad \alpha + \lambda > \nu,\ \alpha > 0, \tag{2.28}$$

where $\xi(t)$ is a random Markov process having two states θ_1, θ_2 with probabilities $p_k = P\{\xi(t) = \theta_k\}$, $k = 1, 2$, that satisfy the equations

$$\begin{aligned} \frac{dp_1(t)}{dt} &= -\lambda p_1(t) + \nu p_2(t), \\ \frac{dp_2(t)}{dt} &= \lambda p_1(t) - \nu p_2(t), \end{aligned} \tag{2.29}$$

where $\lambda > 0$. The random matrix function G is known:

$$\begin{aligned} G_1(x) = G(x,\theta_1) &= \begin{pmatrix} -\gamma_1 x_2 & -x_1^3 \\ \gamma_1 x_1 & -x_2^3 \end{pmatrix}, \\ G_2(x) = G(x,\theta_2) &= \begin{pmatrix} \gamma_2 x_2 & -x_1^3 \\ -\gamma_2 x_1 & -x_2^3 \end{pmatrix}. \end{aligned} \tag{2.30}$$

Taking the positive definite functions

$$w_1(x) = w_2(x) = x_1^2 + x_2^2 + \frac{1}{\alpha}\left(x_1^4 + x_2^4\right), \quad x = (x_1, x_2), \tag{2.31}$$

we can verify that the positive definite particular Lyapunov functions

$$v_1(x) = v_2(x) = \frac{1}{2\alpha}\left(x_1^2 + x_2^2\right) \tag{2.32}$$

are the solutions to (2.20). Consequently, since the integral (1.13)

$$v = \int_0^\infty \left\langle x_1^2 + x_2^2 + \frac{1}{\alpha}\left(x_1^4 + x_2^4\right)\right\rangle dt, \tag{2.33}$$

is convergent, the zero solution of the considered system is L_2-stable.

Acknowledgments

This paper was supported by Grant P201/11/0768 of the Czech Grant Agency (Prague), by the Council of Czech Government Grant MSM 00 216 30519, and by Grant no 1/0090/09 of the Grant Agency of Slovak Republic (VEGA).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 103542, 20 pages
doi:10.1155/2012/103542

Research Article

Switched Exponential State Estimation and Robust Stability for Interval Neural Networks with Discrete and Distributed Time Delays

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Received 21 February 2012; Accepted 8 April 2012

Academic Editor: Agacik Zafer

The interval exponential state estimation and robust exponential stability for the switched interval neural networks with discrete and distributed time delays are considered. Firstly, by combining the theories of the switched systems and the interval neural networks, the mathematical model of the switched interval neural networks with discrete and distributed time delays and the interval estimation error system are established. Secondly, by applying the augmented Lyapunov-Krasovskii functional approach and available output measurements, the dynamics of estimation error system is proved to be globally exponentially stable for all admissible time delays. Both the existence conditions and the explicit characterization of desired estimator are derived in terms of linear matrix inequalities (LMIs). Moreover, a delay-dependent criterion is also developed, which guarantees the robust exponential stability of the switched interval neural networks with discrete and distributed time delays. Finally, two numerical examples are provided to illustrate the validity of the theoretical results.

1. Introduction

In the past few decades, the different models of neural networks such as Hopfield neural networks, Cohen-Grossberg neural networks, cellular neural networks, and bidirectional associative memory neural networks have been extensively investigated due to their wide applications in areas like associative memory, pattern classification, reconstruction of moving images, signal processing, solving optimization problems, and so forth, see [1]. In almost all applications about neural networks, a fundamental problem is the stability, which is the prerequisite to ensure that the developed neural network can work [2–12]. In hardware implementation of the neural networks, time delay is inevitably encountered and is usually discrete and distributed due to the finite switching speed of amplifiers. It is known that time delay is often the main cause for instability and poor performance of neural networks. Moreover, due to unavoidable factors, such as modeling error, external perturbation, and parameter fluctuation, the neural networks model certainly involves uncertainties such as perturbations, and component variations, which will change the stability of neural networks. Therefore, it is of great importance to study the robust stability of neural networks with time delays in the presence of uncertainties, see, for example, [11, 13–26] and the references therein. There are mainly two forms of uncertainties, namely, the interval uncertainty and the norm-bounded uncertainty. Recently, some sufficient conditions for the global robust stability of interval neural networks with time delays and parametric uncertainties have been obtained in terms of LMIs [13–16, 18].

A class of hybrid systems have attracted significant attention because it can model several practical control problems that involve the integration of supervisory logic-based control schemes and feedback control algorithms. As a special class of hybrid systems, switched systems are regarded as nonlinear systems, which are composed of a family of continuous-time or discrete-time subsystems and a rule that orchestrates the switching between the subsystems. Recently, switched neural networks, whose individual subsystems are a set of neural networks, have found applications in fields of high-speed signal processing, artificial intelligence, and gene selection in a DNA microarray analysis [27–29]. Therefore, some researchers have studied the stability issues for switched neural networks [19–24]. In [19], based on the Lyapunov-Krasovskii method and LMI approach, some sufficient conditions were derived for global robust exponential stability of a class of switched Hopfield neural networks with time-varying delay under uncertainty. In [20], by combining Cohen-Grossberg neural networks with an arbitrary switching rule, the mathematical model of a class of switched Cohen-Grossberg neural networks with mixed time varying delays were established, and the robust stability for such switched Cohen-Grossberg neural networks were analyzed. In [21], by employing nonlinear measure and LMI techniques, some new sufficient conditions were obtained to ensure global robust asymptotic stability and global robust stability of the unique equilibrium for a class of switched recurrent neural networks with time-varying delay. In [22], a large class of switched recurrent neural networks with time-varying structured uncertainties and time-varying delay were investigated, and some delay-dependent robust periodicity criteria were derived to guarantee the existence, uniqueness, and global asymptotic stability of periodic solution for all admissible parametric uncertainties by taking free weighting matrices and LMIs. In [23], based on multiple Lyapunov functions method and LMI techniques, the authors presented some sufficient conditions in terms of LMIs, which guarantee the robust exponential stability for uncertain switched Cohen-Grossberg neural networks with interval time-varying delay and distributed time-varying delay under the switching rule with the average dwell time property. It should be noted that, almost all results treated the robust stability for switched neural networks with norm-bounded uncertainty in the above literature [19–23], there are few researchers to deal with the global exponential robust stability for switched neural networks with the interval uncertainty in the existing literature, despite its potential and practical importance in many different areas such as system control and error analysis, see [14, 15].

The neuron states in relatively large-scale neural networks are not often completely available in the network outputs. Thus, in many applications, one often needs to estimate the neuron states through available measurements and then utilizes the estimated neuron states to achieve certain design objectives. For example, in [30], a recurrent neural network was applied to model an unknown system, and the neuron states of the designed neural network were then utilized by the control law. Therefore, from the point of view of control, the state estimation problem for neural networks is of significance for many applications. Recently, there are some results for the neuron state estimation problem of neural networks with or without time delays in the existing available [31–35]. In [31, 32], the authors studied state estimation for Markovian jumping recurrent neural networks with time-delays by constructing Lyapunov-Krasovskii functionals and LMIs. In [33], the interconnection matrix of neural networks are assumed to be norm-bounded. Through available output measurements and by using LMI technique, the authors proved that the dynamics of the estimation error was globally exponentially stable for all admissible time-delays for delayed neural networks. In [34], based on the free-weighting matrix approach, a delay-dependent criterion was established to estimate the neuron states through available output measurements such that the dynamics of the estimation error was globally exponentially stable for neural networks with time-varying delays. The results were applicable to the case that the derivative of a time-varying delay takes any value. In [35], by using the Lyapunov-Krasovskii functional approach, the authors presented the existence conditions of the state estimators in terms of the solution to an LMI. In [36], the neuron activation function and perturbed function of the measurement equation were assumed to be sector-bounded, an LMI-based state estimator and a stability criterion for delayed Hopfield neural networks are developed. In [37], based on augmented Lyapunov-Krasovskii functional and passivity theory, the authors proved that the estimation error system was exponentially stable and passive from the control input to the output error. A new delay-dependent state estimator for switched Hopfield neural networks was achieved by solving LMIs obtained. In spite of these advances in studying neural network state estimation, the state estimation problem for switched interval neural networks has not been investigated in the literature, and it is very important in both theories and applications.

Motivated by the preceding discussion, the aim of this paper is to present a new class of neural network models, that is, switched interval neural networks with discrete and distributed time delays, under interval parameter uncertainties by integrating the theory of switched systems with neural networks. Based on Lyapunov stability theory and by using available output measurements, the dynamics of estimation error system will be proved to be globally exponentially stable for all admissible time delays. Both the existence conditions and the explicit characterization of desired estimator are also derived in terms of LMIs. In addition, a delay-dependent criterion will be derived such that the proposed switched interval neural networks is globally robustly exponentially stable. The proposed criterion is represented in terms of LMIs, which can be solved efficiently by using recently developed convex optimization algorithms [38].

The rest of this paper is organized as follows. In Section 2, the model formulation and some preliminaries are given. Section 3 treats switched exponential state estimation problem for interval neural networks with discrete and distributed time delays. In Section 4, the robust exponential stability is discussed and a delay-dependent criterion is developed. Two numerical examples are presented to demonstrate the validity of the proposed results in Section 5. Some conclusions are drawn in Section 6.

Notations. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For any matrix A, $A > 0$ ($A < 0$) means that A is positive definite (negative definite). A^{-1} denotes the inverse of A. A^T denotes the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of A, respectively. Given the vectors $x = (x_1, \ldots, x_n)^T$,

$y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$, $x^T y = \sum_{i=1}^n x_i y_i$. For $r > 0$, $C([-r, 0]; \mathbb{R}^n)$ denotes the family of continuous function φ from $[-r, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-r \le s \le 0} |\varphi(s)|$. $\dot{x}(t)$ denotes the derivative of $x(t)$, $*$ represents the symmetric form of matrix. Matrices, if their dimensions not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Neural Network Model and Preliminaries

The model of interval neural network with discrete and distributed time delays can be described by differential equation system

$$\begin{gathered}
\dot{x}(t) = -Ax(t) + B_1 g_1(x(t-\tau)) + B_2 \int_{t-h}^{t} g_2(x(s))ds + J, \\
y(t) = Cx(t) + f(t, x(t)), \\
A \in A_I, \qquad B_k \in B_I^{(k)}, \qquad k = 1, 2,
\end{gathered} \tag{2.1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the vector of neuron states; $y(t) = (y_1(t), \ldots, y_n(t))^T \in \mathbb{R}^m$ is the output vector; $g_i(x) = (g_{i1}(x_1), \ldots, g_{in}(x_n))^T : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2$, are the vector-valued neuron activation functions; $f(t, x)$ is a mapping from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^m$, which is the neuron-dependent nonlinear disturbances on the network outputs; $C = (c_{ij})_{m \times n}$ is a known constant matrix with appropriate dimension; $J = (J_1, \ldots, J_n)^T$ is a constant external input vector. τ, h denote the discrete and distributed time delays, respectively, and $\tau > 0$, $h > 0$; $A = \mathrm{diag}(a_1, \ldots, a_n)$ is an $n \times n$ constant diagonal matrices, $a_i > 0$, $i = 1, \ldots, n$, are the neural self-inhibitions; $B_k = (b_{ij}^{(k)}) \in \mathbb{R}^{n \times n}$, $k = 1, 2$, are the connection weight matrices; $A_I = [\underline{A}, \overline{A}] = \{A = \mathrm{diag}(a_i) : 0 < \underline{a}_i \le a_i \le \overline{a}_i,\ i = 1, 2, \ldots, n\}$, $B_I^{(k)} = [\underline{B}_k, \overline{B}_k] = \{B_k = (b_{ij}^{(k)}) :\ 0 < \underline{b}_{ij}^{(k)} \le b_{ij}^{(k)} \le \overline{b}_{ij}^{(k)},\ i, j = 1, 2, \ldots, n\}$ with $\underline{A} = \mathrm{diag}(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$, $\overline{A} = \mathrm{diag}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)$, $\underline{B}_k = (\underline{b}_{ij}^{(k)})_{n \times n}$, $\overline{B}_k = (\overline{b}_{ij}^{(k)})_{n \times n}$.

The initial value associated with the system (2.1) is assumed to be $x(s) = \varphi(s)$, $\varphi(s) \in C([-r, 0]; \mathbb{R}^n)$, $r = \max\{\tau, h\}$.

Throughout this paper, the following assumptions are made on $g_i(\cdot)$, $i = 1, 2$, and $f(t, x)$.

($\mathscr{H}_1$) For any two different $s, t \in \mathbb{R}$,

$$0 \le \frac{g_{ij}(s) - g_{ij}(t)}{s - t} \le \sigma_{ij}, \qquad i = 1, 2,\ j = 1, \ldots, n, \tag{2.2}$$

where $\Lambda_i = \mathrm{diag}(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in})$. $\sigma_{ij} > 0$ is a constant, $i = 1, 2,\ j = 1, 2, \ldots, n$.

($\mathscr{H}_2$) For any two different $x, y \in \mathbb{R}^n$,

$$\|f(t, x) - f(t, y)\| \le \|F(x - y)\|, \tag{2.3}$$

where $F \in \mathbb{R}^{n \times n}$ is a constant matrix.

Based on some transformations [15], the system (2.1) can be equivalently written as

$$\dot{x}(t) = -[A_0 + E_A\Sigma_A F_A]x(t) + [B_{10} + E_1\Sigma_1 F_1]g_1(x(t-\tau)) + [B_{20} + E_2\Sigma_2 F_2]\int_{t-h}^{t} g_2(x(s))ds + J,$$
$$y(t) = Cx(t) + f(t, x(t)), \tag{2.4}$$

where $\Sigma_A \in \Sigma,\ \Sigma_k \in \Sigma,\ k = 1,2.$

$$\Sigma = \left\{\operatorname{diag}[\delta_{11}, \ldots, \delta_{1n}, \ldots, \delta_{n1}, \ldots, \delta_{nn}] \in \mathbb{R}^{n^2\times n^2} : \left|\delta_{ij}\right| \le 1,\ i,j = 1,2,\ldots,n\right\},$$
$$A_0 = \frac{\overline{A} + \underline{A}}{2}, \qquad H_A = \left[\alpha_{ij}\right]_{n\times n} = \frac{\overline{A} - \underline{A}}{2} \cdot B_{k0} = \frac{\overline{B}_k + \underline{B}_k}{2}, \qquad H_B^{(k)} = \left[\beta_{ij}\right]_{n\times n} = \frac{\overline{B}_k - \underline{B}_k}{2},$$
$$E_A = \left[\sqrt{\alpha_{11}}e_1, \ldots, \sqrt{\alpha_{1n}}e_1, \ldots, \sqrt{\alpha_{n1}}e_n, \ldots, \sqrt{\alpha_{nn}}e_n\right]_{n\times n^2},$$
$$F_A = \left[\sqrt{\alpha_{11}}e_1, \ldots, \sqrt{\alpha_{1n}}e_n, \ldots, \sqrt{\alpha_{n1}}e_1, \ldots, \sqrt{\alpha_{nn}}e_n\right]_{n^2\times n}^{T},$$
$$E_k = \left[\sqrt{\beta_{11}^{(k)}}e_1, \ldots, \sqrt{\beta_{1n}^{(k)}}e_1, \ldots, \sqrt{\beta_{n1}^{(k)}}e_n, \ldots, \sqrt{\beta_{nn}^{(k)}}e_n\right]_{n\times n^2},$$
$$F_k = \left[\sqrt{\beta_{11}^{(k)}}e_1, \ldots, \sqrt{\beta_{1n}^{(k)}}e_n, \ldots, \sqrt{\beta_{n1}^{(k)}}e_1, \ldots, \sqrt{\beta_{nn}^{(k)}}e_n\right]_{n^2\times n}^{T}, \tag{2.5}$$

where $e_i \in \mathbb{R}^n$ denotes the column vector with ith element to be 1 and others to be 0.

The switched interval neural network with discrete and distributed time delays consists of a set of interval neural network with discrete and distributed time delays and a switching rule. Each of the interval neural networks was regarded as an individual subsystem. The operation mode of the switched neural networks is determined by the switching rule. According to (2.1), the switched interval neural network with discrete and distributed delays can be represented as follows:

$$\dot{x}(t) = -A_{\sigma(t)}x(t) + B_{1_{\sigma(t)}}g_1(x(t-\tau)) + B_{2_{\sigma(t)}}\int_{t-h}^{t} g_2(x(s))ds + J,$$
$$y(t) = C_{\sigma(t)}x(t) + f(t, x(t)), \tag{2.6}$$
$$A_{\sigma(t)} \in A_{l_{\sigma(t)}}, \qquad B_{k_{\sigma(t)}} \in B_{l_{\sigma(t)}}^{(k)}, \quad k = 1,2,$$

where $A_{l_{\sigma(t)}} = [\underline{A}_{\sigma(t)}, \overline{A}_{\sigma(t)}] = \{A_{\sigma(t)} = \operatorname{diag}(a_{i_{\sigma(t)}}) : 0 < \underline{a}_{i_{\sigma(t)}} \le a_{i_{\sigma(t)}} \le \overline{a}_{i_{\sigma(t)}},\ i = 1,2,\ldots,n\}$, $B_{l_{\sigma(t)}}^{(k)} = [\underline{B}_{k_{\sigma(t)}}, \overline{B}_{k_{\sigma(t)}}] = \{B_{k_{\sigma(t)}} = [b_{ij_{\sigma(t)}}^{(k)}] : 0 < \underline{b}_{ij_{\sigma(t)}}^{(k)} \le b_{ij_{\sigma(t)}}^{(k)} \le \overline{b}_{ij_{\sigma(t)}}^{(k)},\ i,j = 1,2,\ldots,n\}$ with $\underline{A}_{\sigma(t)} = \operatorname{diag}(\underline{a}_{1_{\sigma(t)}}, \underline{a}_{2_{\sigma(t)}}, \ldots, \underline{a}_{n_{\sigma(t)}})$, $\overline{A}_{\sigma(t)}$ $\operatorname{diag}(\overline{a}_{1_{\sigma(t)}}, \overline{a}_{2_{\sigma(t)}}, \ldots, \overline{a}_{n_{\sigma(t)}})$, $\underline{B}_{k_{\sigma(t)}} = [\underline{b}_{ij_{\sigma(t)}}^{(k)}]_{n\times n}$, $\overline{B}_{k_{\sigma(t)}} = [\overline{b}_{ij_{\sigma(t)}}^{(k)}]_{n\times n}$

$$A_{0_{\sigma(t)}} = \frac{\overline{A}_{\sigma(t)} + \underline{A}_{\sigma(t)}}{2}, \qquad H_{A_{\sigma(t)}} = \left[\alpha_{ij_{\sigma(t)}}\right]_{n\times n} = \frac{\overline{A}_{\sigma(t)} - \underline{A}_{\sigma(t)}}{2},$$
$$B_{k0_{\sigma(t)}} = \frac{\overline{B}_{k_{\sigma(t)}} + \underline{B}_{k_{\sigma(t)}}}{2}, \qquad H_{B_{\sigma(t)}}^{(k)} = \left[\beta_{ij_{\sigma(t)}}\right]_{n\times n} = \frac{\overline{B}_{k_{\sigma(t)}} - \underline{B}_{k_{\sigma(t)}}}{2},$$

$$
\begin{aligned}
E_{A_{\sigma(t)}} &= \left[\sqrt{\alpha_{11_{\sigma(t)}}}e_1,\dots,\sqrt{\alpha_{1n_{\sigma(t)}}}e_1,\dots,\sqrt{\alpha_{n1_{\sigma(t)}}}e_n,\dots,\sqrt{\alpha_{nn_{\sigma(t)}}}e_n\right]_{n\times n^2},\\
F_{A_{\sigma(t)}} &= \left[\sqrt{\alpha_{11_{\sigma(t)}}}e_1,\dots,\sqrt{\alpha_{1n_{\sigma(t)}}}e_n,\dots,\sqrt{\alpha_{n1_{\sigma(t)}}}e_1,\dots,\sqrt{\alpha_{nn_{\sigma(t)}}}e_n\right]^T_{n^2\times n},\\
E_{k_{\sigma(t)}} &= \left[\sqrt{\beta^{(k)}_{11_{\sigma(t)}}}e_1,\dots,\sqrt{\beta^{(k)}_{1n_{\sigma(t)}}}e_1,\dots,\sqrt{\beta^{(k)}_{n1_{\sigma(t)}}}e_n,\dots,\sqrt{\beta^{(k)}_{nn_{\sigma(t)}}}e_n\right]_{n\times n^2},\\
F_{k_{\sigma(t)}} &= \left[\sqrt{\beta^{(k)}_{11_{\sigma(t)}}}e_1,\dots,\sqrt{\beta^{(k)}_{1n_{\sigma(t)}}}e_n,\dots,\sqrt{\beta^{(k)}_{n1_{\sigma(t)}}}e_1,\dots,\sqrt{\beta^{(k)}_{nn_{\sigma(t)}}}e_n\right]^T_{n^2\times n},
\end{aligned}
\tag{2.7}
$$

$\sigma(t) : [0,+\infty) \to \Gamma = \{1,2,\dots,N\}$ is the switching signal, which is a piecewise constant function of time. For any $i \in \{1,2,\dots,l\}$, $A_i = A_{0_i} + E_{A_i}\Sigma_{A_i}F_{A_i}$, $B_{k_i} = B_{k0_i} + E_{k_i}\Sigma_{k_i}F_{k_i}$, and $\Sigma_{A_i} \in \Sigma$, $\Sigma_{k_i} \in \Sigma$, $k = 1,2$. This means that the matrices $(A_{\sigma(t)}, B_{1_{\sigma(t)}}, B_{2_{\sigma(t)}})$ are allowed to take values, at an arbitrary time, in the finite set $\{(A_1, B_{1_1}, B_{2_1}), (A_2, B_{1_2}, B_{2_2}), \dots, (A_N, B_{1_N}, B_{2_N})\}$. In this paper, it is assumed that the switching rule σ is not known a priori and its instantaneous value is available in real time.

By (2.4), the system (2.6) can be rewritten as

$$
\begin{gathered}
\dot{x}(t) = -A_{0_{\sigma(t)}}x(t) + B_{10_{\sigma(t)}}g_1(x(t-\tau)) + B_{20_{\sigma(t)}}\int_{t-h}^{t} g_2(x(s))ds + E_{\sigma(t)}\Delta_{\sigma(t)}(t) + J,\\
y(t) = C_{\sigma(t)}x(t) + f(t,x(t)),\\
A_{\sigma(t)} \in A_{l_{\sigma(t)}}, \qquad B_{k_{\sigma(t)}} \in B^{(k)}_{l_{\sigma(t)}}, \qquad k = 1,2,
\end{gathered}
\tag{2.8}
$$

where $E_{\sigma(t)} = [E_{A_{\sigma(t)}}, E_{1_{\sigma(t)}}, E_{2_{\sigma(t)}}]$,

$$
\begin{aligned}
\Delta_{\sigma(t)}(t) &= \begin{bmatrix} -\Sigma_{A_{\sigma(t)}}F_{A_{\sigma(t)}}x(t)\\ \Sigma_{1_{\sigma(t)}}F_{1_{\sigma(t)}}g_1(x(t-\tau))\\ \Sigma_{2_{\sigma(t)}}F_{2_{\sigma(t)}}\displaystyle\int_{t-h}^{t}g_2(x(s))ds \end{bmatrix}\\
&= \operatorname{diag}\{\Sigma_{A_{\sigma(t)}}, \Sigma_{1_{\sigma(t)}}, \Sigma_{2_{\sigma(t)}}\}\begin{bmatrix} -F_{A_{\sigma(t)}}x(t)\\ F_{1_{\sigma(t)}}g_1(x(t-\tau))\\ F_{2_{\sigma(t)}}\displaystyle\int_{t-h}^{t}g_2(x(s))ds \end{bmatrix},
\end{aligned}
\tag{2.9}
$$

and $\Delta_{\sigma(t)}(t)$ satisfies the following matrix quadratic inequality:

$$
\Delta^T_{\sigma(t)}(t)\Delta_{\sigma(t)}(t) \le \begin{bmatrix} x(t)\\ g_1(x(t-\tau))\\ \displaystyle\int_{t-h}^{t}g_2(x(s))ds \end{bmatrix}^T \begin{bmatrix} F^T_{A_{\sigma(t)}}\\ F^T_{1_{\sigma(t)}}\\ F^T_{2_{\sigma(t)}} \end{bmatrix} \begin{bmatrix} F^T_{A_{\sigma(t)}}\\ F^T_{1_{\sigma(t)}}\\ F^T_{2_{\sigma(t)}} \end{bmatrix}^T \begin{bmatrix} x(t)\\ g_1(x(t-\tau))\\ \displaystyle\int_{t-h}^{t}g_2(x(s))ds \end{bmatrix}.
\tag{2.10}
$$

Define the indicator function $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$, where

$$\xi_i(t) = \begin{cases} 1, & \text{when the switched system is described by the } i\text{th mode} \\ & A_{0_i}, B_{k0_i},\ k = 1, 2, E_i, \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

where $i = 1, 2, \ldots, N$. Therefore, the system model (2.8) can also be written as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{N} \xi_i(t) \left\{ -A_{0_i} x(t) + B_{10_i} g_1(x(t-\tau)) + B_{20_i} \int_{t-h}^{t} g_2(x(s)) ds + E_i \Delta_i(t) + J \right\}, \\ y(t) &= \sum_{i=1}^{N} \xi_i(t) \{ C_i x(t) + f(t, x(t)) \}, \end{aligned} \tag{2.12}$$

where $\sum_{i=1}^{N} \xi_i(t) = 1$ is satisfied under any switching rules.

In this paper, our main purpose is to develop an efficient algorithm to estimate the neuron states $x(t)$ in (2.12) from the available network outputs $y(t)$ in (2.12). The full-order state estimator is of the form

$$\begin{aligned} \dot{\hat{x}}(t) = \sum_{i=1}^{N} \xi_i(t) \Bigg\{ & -A_{0_i} \hat{x}(t) + B_{10_i} g_1(\hat{x}(t-\tau)) + B_{20_i} \int_{t-h}^{t} g_2(\hat{x}(s)) ds \\ & + E_i \hat{\Delta}_i(t) + K_i [y(t) - C_i \hat{x}(t) - f(t, \hat{x}(t))] + J \Bigg\}, \end{aligned} \tag{2.13}$$

where $\hat{x}(t)$ is the estimation of the neuron state, and the matrix $K_i \in \mathbb{R}^{n \times m}$ is the estimator gain matrix to be designed.

Let the error state be $e(t) = x(t) - \hat{x}(t)$; then it follows from (2.12) and (2.13) that

$$\begin{aligned} \dot{e}(t) = \sum_{i=1}^{N} \xi_i(t) \Bigg\{ & (-A_{0_i} - K_i C_i) e(t) + B_{10_i} [g_1(x(t-\tau)) - g_1(\hat{x}(t-\tau))] \\ & + B_{20_i} \int_{t-h}^{t} [g_2(x(s)) - g_2(\hat{x}(s))] ds + E_i \left[\Delta_i(t) - \hat{\Delta}_i(t) \right] \\ & - K_i [f(t, x(t)) - f(t, \hat{x}(t))] \Bigg\}. \end{aligned} \tag{2.14}$$

For presentation convenience, set $A_{Ki} = -A_{0_i} - K_i C_i$, $\psi_k(t) = g_k(x(t)) - g_k(\hat{x}(t))$, $k = 1, 2$, $\phi(t) = f(t, x(t)) - f(t, \hat{x}(t))$, $\overline{\Delta}_i(t) = \Delta_i(t) - \hat{\Delta}_i(t)$. Then the system (2.14) becomes

$$\dot{e}(t) = \sum_{i=1}^{N} \xi_i(t) \left\{ A_{Ki} e(t) + B_{10_i} \psi_1(t-\tau) + B_{20_i} \int_{t-h}^{t} \psi_2(s) ds + E_i \overline{\Delta}_i(t) - K_i \phi(t) \right\}, \tag{2.15}$$

where

$$\overline{\Delta}_i(t) = \operatorname{diag}\{\Sigma_{A_i}, \Sigma_{1_i}, \Sigma_{2_i}\} \begin{bmatrix} -F_{A_i} e(t) \\ F_{1_i} \psi_1(t-\tau) \\ F_{2_i} \displaystyle\int_{t-h}^{t} \psi_2(s) ds \end{bmatrix}, \tag{2.16}$$

and it satisfies the following quadratic inequality

$$\overline{\Delta}_i^T(t)\overline{\Delta}_i(t) \le \begin{bmatrix} e(t) \\ \psi_1(t-\tau) \\ \displaystyle\int_{t-h}^{t} \psi_2(s) ds \end{bmatrix}^T \begin{bmatrix} F_{A_i}^T \\ F_{1_i}^T \\ F_{2_i}^T \end{bmatrix} \begin{bmatrix} F_{A_i}^T \\ F_{1_i}^T \\ F_{2_i}^T \end{bmatrix}^T \begin{bmatrix} e(t) \\ \psi_1(t-\tau) \\ \displaystyle\int_{t-h}^{t} \psi_2(s) ds \end{bmatrix}. \tag{2.17}$$

The initial value associated with (2.15) is $e(s) = \phi(s)$, $\phi(s) \in C([-r, 0]; \mathbb{R}^n)$, $r = \max\{\tau, h\}$.

To obtain the main results of this paper, the following definitions and lemmas are introduced.

Definition 2.1. For the switched estimation error-state system (2.15), the trivial solution is said to be globally exponentially stable if there exist positive scalars $\alpha > 0$ and $\beta > 0$ such that

$$\|e(t, \phi)\| \le \alpha^{-\beta t} \|\phi\|, \quad t \ge 0, \tag{2.18}$$

where $e(t, \phi)$ is the solution of the system (2.15) with the initial value $e(s) = \phi(s)$, $\phi(s) \in C([-r, 0]; \mathbb{R}^n)$, $r = \max\{\tau, h\}$.

Lemma 2.2 (see [15]). *Let $\Gamma_0(x)$ and $\Gamma_1(x)$ be two arbitrary quadratic forms over $\mathbb{R}^n$, then $\Gamma_0(x) < 0$ for all $x \in \mathbb{R}^n - \{0\}$ satisfying $\Gamma_1(x) \le 0$ if and only if there exists $\varepsilon \ge 0$ such that*

$$\Gamma_0(x) - \varepsilon \Gamma_1(x) < 0, \qquad \forall x \in \mathbb{R}^n - \{0\}. \tag{2.19}$$

Lemma 2.3 (Jensen's Inequality). *For any constant matrix $\Omega \in \mathbb{R}^{n\times n}$, $\Omega = \Omega^T > 0$, scalar $\gamma > 0$, vector function $\omega : [t-\gamma, t] \to \mathbb{R}^n$, $t \ge 0$, such that the integrations concerned are well defined, then*

$$\left(\int_{t-\gamma}^{t} \omega(s) ds\right)^T \Omega \left(\int_{t-\gamma}^{t} \omega(s) ds\right) \le \gamma \left(\int_{t-\gamma}^{t} \omega^T(s) \Omega \omega(s) ds\right). \tag{2.20}$$

Lemma 2.4 (see [38]). *The following LMI*

$$\begin{bmatrix} E(x) & H(x) \\ H^T(x) & F(x) \end{bmatrix} > 0, \tag{2.21}$$

where $E(x) = E^T(x)$, $F(x) = F^T(x)$, *and* $H(x)$ *depend on* x, *is equivalent to each of the following conditions:*

$$\begin{aligned} &(1)\ E(x) > 0, \qquad F(x) - H^T(x)E^{-1}(x)H(x) > 0, \\ &(2)\ F(x) > 0, \qquad E(x) - H(x)F^{-1}(x)H^T(x) > 0. \end{aligned} \tag{2.22}$$

Lemma 2.5. *Given any real matrices* X, Y, *and* $Q > 0$ *with appropriate dimensions, then the following matrix inequality holds:*

$$X^TY + Y^TX \leq X^TQX + Y^TQ^{-1}Y. \tag{2.23}$$

3. Switched Exponential State Estimation for Interval Neural Networks

In this section, we will study the global exponential stability of the system (2.15) under arbitrary switching rule. By constructing a suitable Lyapunov-Krasovskii functional, a delay-dependent criterion for the global exponential stability of the estimation process (2.15) is derived. The following theorem shows that this criterion can be obtained if a quadratic matrix inequality involving several scalar parameters is feasible.

Theorem 3.1. *If there exist scalars* $\beta > 0$ *and* $\epsilon > 0$, *a matrix* $P > 0$ *and two diagonal matrices* $Q_1 > 0$, $Q_2 > 0$ *such that the following quadratic matrix inequalities:*

$$\prod_i = \begin{bmatrix} \prod_{i11} & PB_{10i} + F_{A_i}^T F_{1_i} & PB_{20i} + F_{A_i}^T F_{2_i} & PE_i \\ * & -e^{-\beta\tau}Q_2 + F_{1_i}^T F_{1_i} & F_{1_i}^T F_{2_i} & 0 \\ * & * & -\dfrac{1}{h}e^{-\beta h}Q_2 + F_{2_i}^T F_{2_i} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{3.1}$$

are satisfied, where

$$\begin{aligned} \prod_{i11} = {} & \beta P + (-A_{0_i} - K_iC_i)^T P + P(-A_{0_i} - K_iC_i) + Q_1\Lambda_1^2 + hQ_2\Lambda_2^2 \\ & + F_{A_i}^T F_{A_i} + \epsilon F^T F + \epsilon^{-1} P K_i K_i^T P, \end{aligned} \tag{3.2}$$

then the switched error-state system (2.14) *of the neural network* (2.12) *is globally exponentially stable under any switching rules. Moreover, the estimate of the error-state decay can be given by*

$$\|e(t,\phi)\| \leq \sqrt{\alpha}e^{-(\beta/2)t}\|\phi\|, \tag{3.3}$$

where $\alpha = (\lambda_{\max}(P) + \tau\lambda_{\max}(Q_1\Lambda_1^2) + (h^2/2)\lambda_{\max}(Q_2\Lambda_2^2))/\lambda_{\min}(P)$.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(e(t),t) = e^{\beta t}e^T(t)Pe(t) + \int_{t-\tau}^{t} e^{\beta s}\psi_1^T(s)Q_1\psi_1(s)ds + \int_{t-h}^{t}\int_{\theta}^{t} e^{\beta s}\psi_2^T(s)Q_2\psi_2(s)ds\, d\theta. \tag{3.4}$$

Calculating the time derivative of $V(e(t),t)$ along the solution $e(t,\phi)$ of the system (2.15), it can follow that

$$\begin{aligned}
\dot{V}(e(t),t) &= \beta e^{\beta t}e^T(t)Pe(t) + 2e^{\beta t}e^T(t)P\dot{e}(t) + e^{\beta t}\psi_1^T(t)Q_1\psi_1(t) - e^{\beta(t-\tau)}\psi_1^T(t-\tau)Q_1\psi_1(t-\tau)\\
&\quad - \int_{t-h}^{t} e^{\beta s}\psi_2^T(s)Q_2\psi_2(s)ds + \int_{t-h}^{t} e^{\beta t}\psi_2^T(t)Q_2\psi_2(t)d\theta\\
&= \beta e^{\beta t}e^T(t)Pe(t) + 2e^{\beta t}e^T(t)P\Bigg[\sum_{i=1}^{N}\xi_i(t)\Bigg\{A_{Ki}e(t) + B_{10_i}\psi_1(t-\tau) + B_{20_i}\int_{t-h}^{t}\psi_2(s)ds\\
&\qquad + E_i\overline{\Delta}_i(t) - K_i\phi(t)\Bigg\}\Bigg] + e^{\beta t}\psi_1^T(t)Q_1\psi_1(t)\\
&\quad - e^{\beta(t-\tau)}\psi_1^T(t-\tau)Q_1\psi_1(t-\tau) - \int_{t-h}^{t} e^{\beta s}\psi_2^T(s)Q_2\psi_2(s)ds + \int_{t-h}^{t} e^{\beta t}\psi_2^T(t)Q_2\psi_2(t)d\theta\\
&= \sum_{i=1}^{N}\xi_i(t)\Bigg\{\beta e^{\beta t}e^T(t)Pe(t) + 2e^{\beta t}e^T(t)PA_{Ki}e(t) + 2e^{\beta t}e^T(t)PB_{10_i}\psi_1(t-\tau)\\
&\qquad + 2e^{\beta t}e^T(t)PB_{20_i}\int_{t-h}^{t}\psi_2(s)ds + 2e^{\beta t}e^T(t)PE_i\overline{\Delta}_i(t)\\
&\qquad + 2e^{\beta t}e^T(t)PK_i\phi(t) + e^{\beta t}\psi_1^T(t)Q_1\psi_1(t) - e^{\beta(t-\tau)}\psi_1^T(t-\tau)Q_1\psi_1(t-\tau)\\
&\qquad - \int_{t-h}^{t} e^{\beta s}\psi_2^T(s)Q_2\psi_2(s)ds + he^{\beta t}\psi_2^T(t)Q_2\psi_2(t)\Bigg\}.
\end{aligned} \tag{3.5}$$

By the assumption ($\mathscr{H}_1$) and Lemmas 2.3 and 2.5, we have

$$-\int_{t-h}^{t} e^{\beta s}\psi_2^T(s)Q_2\psi_2(s)ds \le -\frac{1}{h}e^{\beta(t-h)}\left(\int_{t-h}^{t}\psi_2(s)ds\right)Q_2\left(\int_{t-h}^{t}\psi_2(s)ds\right), \tag{3.6}$$

$$\psi_1^T(t)Q_1\psi_1(t) \le e^T(t)Q_1\Lambda_1^2 e(t), \tag{3.7}$$

$$\psi_2^T(t)Q_2\psi_2(t) \le e^T(t)Q_2\Lambda_2^2 e(t), \tag{3.8}$$

$$\begin{aligned}
2e^T(t)PK_i\phi(t) &\le \epsilon\phi^T(t)\phi(t) + \epsilon^{-1}e^T(t)PK_iK_i^TPe(t)\\
&\le e^T(t)\left[\epsilon F^TF + \epsilon^{-1}PK_iK_i^TP\right]e(t).
\end{aligned} \tag{3.9}$$

In the light of (3.7)–(3.10), we obtain that

$$\begin{aligned}\dot{V}(e(t),t) \le \sum_{i=1}^{N}\xi_i(t)e^{\beta t}\Bigg\{&\beta e^T(t)Pe(t) + 2e^T(t)PA_{Ki}e(t) + 2e^T(t)PB_{10_i}\varphi_1(t-\tau)\\&+ 2e^T(t)PB_{20_i}\int_{t-h}^{t}\varphi_2(s)ds + e^T(t)PE_i\overline{\Delta}_i(t) + e^T(t)Q_1\Lambda_1^2e(t)\\&- e^{-\tau}\varphi_1^T(t-\tau)Q_1\varphi_1(t-\tau) + e^T(t)\left[\epsilon F^TF + \epsilon^{-1}PK_iK_i^TP\right]e(t)\\&-\frac{1}{h}e^{-\beta h}\left(\int_{t-h}^{t}\varphi_2(s)ds\right)Q_2\left(\int_{t-h}^{t}\varphi_2(s)ds\right) + he^T(t)Q_2\Lambda_2^2e(t)\Bigg\}.\end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned}\dot{V}(e(t),t) - e^{\beta t}\sum_{i=1}^{N}\xi_i(t)&\left(\overline{\Delta}_i^T(t)\overline{\Delta}_i(t) - \begin{bmatrix} e(t)\\ \varphi_1(t-\tau)\\ \int_{t-h}^{t}\varphi_2(s)ds\end{bmatrix}^T\begin{bmatrix}F_{A_i}^T\\ F_{1_i}^T\\ F_{2_i}^T\end{bmatrix}\begin{bmatrix}F_{A_i}^T\\ F_{1_i}^T\\ F_{2_i}^T\end{bmatrix}^T\begin{bmatrix} e(t)\\ \varphi_1(t-\tau)\\ \int_{t-h}^{t}\varphi_2(s)ds\end{bmatrix}\right)\\&\le e^{\beta t}\sum_{i=1}^{N}\xi_i(t)\digamma^T(t)\Pi_i\digamma(t),\end{aligned} \tag{3.11}$$

where $\digamma(t) = [e^T(t)\varphi_1^T(t-\tau(t))(\int_{t-h}^{t}\varphi_2(s)ds)^T\overline{\Delta}_i^T(t)]^T$. By Lemma 2.2, (2.17) and (3.11), we can obtain that $\dot{V}(e(t),t) < 0$ for all $\digamma(t) \neq 0$. Hence,

$$V(e(t),t) \le V(e(0),0), \quad t > 0. \tag{3.12}$$

From (3.4), it is easy to get

$$V(e(0),0) \le \left[\lambda_{\max}(P) + \tau\lambda_{\max}\left(Q_1\Lambda_1^2\right) + \frac{h^2}{2}\lambda_{\max}\left(Q_2\Lambda_2^2\right)\right]\|\phi\|^2. \tag{3.13}$$

On the other hand, we also have

$$V(e(t),t) \ge \lambda_{\min}(P)e^{\beta t}\|e(t)\|^2. \tag{3.14}$$

By combining (3.12), (3.13), and (3.14), it follows that

$$\|e(t)\| \le \sqrt{\alpha}e^{-(\beta/2)t}\|\phi\|, \tag{3.15}$$

where $\alpha = (\lambda_{\max}(P)+\tau\lambda_{\max}(Q_1\Lambda_1^2)+(h^2/2)\lambda_{\max}(Q_2\Lambda_2^2))/\lambda_{\min}(P)$. The proof is completed. □

The inequalities in (3.1) are nonlinear and coupled, each involving many parameters. Obviously, the inequalities in (3.1) are difficult to solve. A meaningful approach to tackling such a problem is to convert the nonlinearly coupled matrix inequalities into LMIs, while the estimator gain is designed simultaneously. In the following, we will deal with the design problem, that is, giving a practical design procedure for the estimator gain, K_i, such that the set of inequalities (3.1) in Theorem 3.1 are satisfied.

Theorem 3.2. *If there exist two scalars $\beta > 0$, $\epsilon > 0$, a matrix $P > 0$, and two diagonal matrices $Q_1 > 0$, $Q_2 > 0$ such that the following linear matrix inequalities:*

$$\Gamma_i = \begin{bmatrix} \Gamma_{i11} & PB_{10i} + F_{A_i}^T F_{1_i} & PB_{20i} + F_{A_i}^T F_{2_i} & PE_i & R_i \\ * & -e^{-\beta\tau} Q_2 + F_{1_i}^T F_{1_i} & F_{1_i}^T F_{2_i} & 0 & 0 \\ * & * & -\dfrac{1}{h} e^{-\beta h} Q_2 + F_{2_i}^T F_{2_i} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{3.16}$$

are satisfied, where

$$\Gamma_{i11} = \beta P - A_i P - P A_i - R_i C_i - C_i^T \mathbb{R}_i^T + Q_1 \Lambda_1^2 + h Q_2 \Lambda_2^2 + F_{A_i}^T F_{A_i} + \epsilon F^T F, \tag{3.17}$$

and the estimator gain is given by $R_i = PK_i$, then the switched error-state system (2.14) *of the neural network* (2.12) *is globally exponentially stable under any switching rules. Moreover, the estimate of the error-state decay can be given by*

$$\|e(t)\| \le \sqrt{\alpha} e^{-(\beta/2)t} \|\phi\|, \tag{3.18}$$

where $\alpha = (\lambda_{\max}(P) + \tau\lambda_{\max}(Q_1\Lambda_1^2) + (h^2/2)\lambda_{\max}(Q_2\Lambda_2^2))/\lambda_{\min}(P)$.

Proof. Using Lemma 2.4, (3.16) holds if and only if

$$\begin{bmatrix} \Gamma_{i11}^* & PB_{10i} + F_{A_i}^T F_{1_i} & PB_{20i} + F_{A_i}^T F_{2_i} & PE_i \\ * & -e^{-\beta\tau} Q_2 + F_{1_i}^T F_{1_i} & F_{1_i}^T F_{2_i} & 0 \\ * & * & -\dfrac{1}{h} e^{-\beta h} Q_2 + F_{2_i}^T F_{2_i} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{3.19}$$

where $\Gamma_{i11}^* = \beta P - A_i P - P A_i - R_i C_i - C_i^T R_i^T + Q_1\Lambda_1^2 + hQ_2\Lambda_2^2 + F_{A_i}^T F_{A_i} + \epsilon F^T F + \epsilon^{-1} R_i^T R_i$.

Noticing that $K_i = P^{-1} R_i$, it can be easily seen that (3.19) is the same as (3.1). Hence, it follows from Theorem 3.1 that, with the estimator gain given by $K_i = P^{-1} R_i$, the switched error-state system (2.14) of the neural network (2.12) is globally exponentially stable under any switching rules. The proof of Theorem 3.2 is complete. □

4. The Stability of Switched Interval Neural Networks

In the section, we will consider the stability of switched interval neural network (2.1) with discrete and distributed time delays and without the output $y(t)$. It should be noted that the stability of switched interval neural network (2.1) without the output $y(t)$ can be as a by-product, and the main results can be easily derived from the previous section.

Consider the interval neural network (2.1) without the output $y(t)$

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + B_1 g_1(x(t-\tau)) + B_2 \int_{t-h}^{t} g_2(x(s))ds + J,\\ A &\in A_l, \qquad B_k \in B_l^{(k)}, \qquad k = 1,2.\end{aligned} \tag{4.1}$$

The assumptions on the model (4.1) are same as the above in Section 2. With loss of generality, it is assumed that the neural network (4.1) has only one equilibrium point, and denoted by $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$. For the purpose of simplicity, the equilibrium x^* will be shifted to the origin by letting $\mu = x - x^*$ and the system (4.1) can be represented as

$$\begin{gathered}\dot{\mu}(t) = -A\mu(t) + B_1 l_1(\mu(t-\tau)) + B_2 \int_{t-h}^{t} l_2(\mu(s))ds,\\ A \in A_l, \qquad B_k \in B_l^{(k)}, \qquad k = 1,2,\end{gathered} \tag{4.2}$$

where $l_j(\mu(t)) = g_j(\mu(t) + x^*) - g_j(x^*)$, $j = 1,2,\ldots,n$. The initial value associated with (4.1) is changed to be $\mu(s) = \varphi(s) - x^* = \eta(s)$.

The system (4.2) can also be written as an equivalent form

$$\begin{aligned}\dot{\mu}(t) = &-[A_0 + E_A\Sigma_A F_A]\mu(t) + [B_{10} + E_1\Sigma_1 F_1] l_1(\mu(t-\tau))\\ &+ [B_{20} + E_2\Sigma_2 F_2]\int_{t-h}^{t} l_2(\mu(s))ds.\end{aligned} \tag{4.3}$$

Similar to the system (2.8), the switched interval neural network with discrete and distributed time delays and without the output $y(t)$ can be written as

$$\begin{aligned}\dot{\mu}(t) &= -A_{0_{\sigma(t)}}\mu(t) + B_{10_{\sigma(t)}} l_1(\mu(t-\tau)) + B_{20_{\sigma(t)}} \int_{t-h}^{t} l_2(\mu(s))ds + E_{\sigma(t)}\Delta_{\sigma(t)}(t),\\ A_{\sigma(t)} &\in A_{l_{\sigma(t)}}, \qquad B_{k_{\sigma(t)}} \in B_{l_{\sigma(t)}}^{(k)}, \qquad k = 1,2.\end{aligned} \tag{4.4}$$

The following theorem gives a condition, which can ensure that the switched system (4.4) is globally exponentially stable under any switching rules.

Theorem 4.1. *If there exist a scalar $\beta > 0$, a matrix $P > 0$, and two diagonal matrices $Q_1 > 0$, $Q_2 > 0$ such that the following linear matrix inequalities:*

$$\Gamma_i = \begin{bmatrix} \Gamma_{i11} & PB_{10i} + F_{A_i}^T F_{1_i} & PB_{20i} + F_{A_i}^T F_{2_i} & PE_i \\ * & -e^{-\beta\tau} Q_2 + F_{1_i}^T F_{1_i} & F_{1_i}^T F_{2_i} & 0 \\ * & * & -\dfrac{1}{h} e^{-\beta h} Q_2 + F_{2_i}^T F_{2_i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{4.5}$$

are satisfied, where

$$\Gamma_{i11} = \beta P - A_i P - P A_i + Q_1 \Lambda_1^2 + h Q_2 \Lambda_2^2 + F_{A_i}^T F_{A_i}, \tag{4.6}$$

then switched interval neural network system (4.4) *is exponentially stable under any switching rules. Moreover, the estimate of the state decay is given by*

$$\|\mu(t)\| \le \sqrt{\alpha} e^{-(\beta/2)t} \|\eta\|, \tag{4.7}$$

where $\alpha = (\lambda_{\max}(P) + \tau\lambda_{\max}(Q_1\Lambda_1^2) + (h^2/2)\lambda_{\max}(Q_2\Lambda_2^2))/\lambda_{\min}(P)$.

Proof. By using Lyapunov-Krasovskii functional in (3.4), let the matrix $K_i = C_i = F = 0$, and following the similar line of the proof of Theorem 3.1, it is not difficult to get the proof of Theorem 4.1.

When the distributed delays $h = 0$, the system (4.4) changes as the switched interval neural networks with discrete delays

$$\dot{\mu}(t) = \sum_{i=1}^{N} \xi_i(t) \left\{ -A_{0_i} \mu(t) + B_{10_i} l_1(\mu(t-\tau)) + E_i \Delta_i(t) \right\}. \tag{4.8}$$

By Theorem 4.1, it is easy to obtain the following corollary. □

Corollary 4.2. *If there exist scalars $\beta > 0$, $\epsilon > 0$, a matrix $P > 0$, and two diagonal matrices $Q_1 > 0$, $Q_2 > 0$, such that the following linear matrix inequalities:*

$$\Gamma_i = \begin{bmatrix} \Gamma_{i11} & PB_{10i} + F_{A_i}^T F_{1_i} & PB_{20i} + F_{A_i}^T F_{2_i} & PE_i \\ * & -e^{-\beta\tau} Q_2 + F_{1_i}^T F_{1_i} & F_{1_i}^T F_{2_i} & 0 \\ * & * & Q_2 + F_{2_i}^T F_{2_i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{4.9}$$

are satisfied, where

$$\Gamma_{i11} = \beta P - A_i P - P A_i + Q_1 \Lambda_1^2 + F_{A_i}^T F_{A_i}, \tag{4.10}$$

then switched interval neural network system (4.8) *is globally exponentially stable under any switching rules. Moreover, the estimate of the state decay is given by*

$$\|\mu(t)\| \leq \sqrt{\alpha} e^{-(\beta/2)t} \|\eta\|, \tag{4.11}$$

where $\alpha = (\lambda_{\max}(P) + \tau \lambda_{\max}(Q_1 \Lambda_1^2)) / \lambda_{\min}(P)$.

Remark 4.3. In [25], based on homeomorphism mapping theorem and by using Lyapunov functional, some delay-independent stability criteria were obtained to ensure the existence, uniqueness, and global asymptotic stability of the equilibrium point for neural networks with multiple time delays under parameter uncertainties. In [26], authors dealt with the global robust asymptotic stability of a great class of dynamical neural networks with multiple time delays, a new alternative sufficient condition for the existence, uniqueness, and global asymptotic stability of the equilibrium point under parameter uncertainties is proposed by employing a new Lyapunov functional. In this paper, when $N = 1$, the switched system model (4.4) degenerated into the interval neural network model (4.1) with discrete and distributed time delays. It is easy to see that the model studied in [25, 26] is a special case of the model (4.1). Hence, the results obtained in this paper expand and improve the stability results in the existing literature [25, 26].

Remark 4.4. In [39], the authors considered Markovian jumping fuzzy Hopfield neural networks with mixed random time-varying delays. By applying the Lyapunov functional method and LMI technique, delay-dependent robust exponential state estimation and new sufficient conditions guaranteeing the robust exponential stability (in the mean square sense) were proposed. In [40], the authors considered delay-dependent robust asymptotic state estimation for fuzzy Hopfield neural networks with mixed interval time-varying delay. By constructing a Lyapunov-Krasovskii functional containing triple integral term and by employing some analysis techniques, sufficient conditions are derived in terms of LMIs. In the future, based on [39, 40], the model of the switched interval fuzzy Hopfield neural networks with mixed random time-varying delays and the switched interval discrete-time fuzzy complex networks will be expected to be established, the strategy proposed in this paper will be utilized to investigate the state estimation and stability problems.

5. Illustrative Examples

In this section, two illustrative examples will be given to check the validity of the results obtained in Theorems 3.2 and 4.1.

Example 5.1. Consider the second-order switched interval neural network with discrete and distributed delays in (2.6) described by $\sigma(t) : [0, +\infty) \to \Gamma = \{1, 2\}$, $g_i(x) = (1/2)\tan x + (1/2)\sin x$, $i = 1, 2$, $\tau = h = 1$, and $f(t, x) = t \cos x$. Obviously, the assumptions $\mathscr{H}_1$ and

$\mathscr{H}_2$ are satisfied with $\Lambda_1 = \Lambda_2 = F = \mathrm{diag}(1,1)$. The neural network system parameters are defined as

$$\underline{A}_1 = \begin{pmatrix} 3.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \quad \overline{A}_1 = \begin{pmatrix} 4.01 & 0 \\ 0 & 3.01 \end{pmatrix}, \quad \underline{B}_{11} = \begin{pmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{pmatrix}, \quad \overline{B}_{11} = \begin{pmatrix} 1.208 & 0.11 \\ 0.11 & 1.208 \end{pmatrix},$$
$$\underline{B}_{21} = \begin{pmatrix} 0.09 & 0.14 \\ 0.05 & 0.09 \end{pmatrix}, \quad \overline{B}_{21} = \begin{pmatrix} 0.11 & 0.16 \\ 0.07 & 0.11 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0.45 & -0.2 \\ 0.3 & 0.42 \end{pmatrix},$$
$$\underline{A}_2 = \begin{pmatrix} 1.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \quad \overline{A}_2 = \begin{pmatrix} 2.01 & 0 \\ 0 & 3.01 \end{pmatrix}, \quad \underline{B}_{12} = \begin{pmatrix} -0.07 & 0.03 \\ -0.01 & 0.02 \end{pmatrix}, \quad \overline{B}_{12} = \begin{pmatrix} -0.05 & 0.05 \\ -0.04 & 0.04 \end{pmatrix},$$
$$\underline{B}_{22} = \begin{pmatrix} -0.47 & -0.15 \\ 0.11 & -0.54 \end{pmatrix}, \quad \overline{B}_{22} = \begin{pmatrix} -0.45 & -0.13 \\ 0.13 & -0.54 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -0.3 & 0.1 \\ -0.3 & -0.6 \end{pmatrix}. \tag{5.1}$$

In the following, we will design an estimator K_i, $i = 1,2$, for the switched interval neural network in this example. Solving the LMI in (3.16) by using appropriate LMI solver in the Matlab, the scalars $\beta > 0$, $\epsilon > 0$ and feasible positive definite matrices P, Q_i, $i = 1,2$, and the matrices R_i, $i = 1,2$, could be as

$$P = \begin{pmatrix} 3.5939 & -0.3052 \\ -0.3052 & 3.6769 \end{pmatrix}, \qquad R_1 = \begin{pmatrix} 0.0600 & -0.0309 \\ -0.0309 & 0.1288 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} -0.0615 & -0.0665 \\ -0.0665 & -0.1316 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} 0.8083 & 0 \\ 0 & 0.8520 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3.9057 & 0 \\ 0 & 7.1313 \end{pmatrix}, \qquad \beta = 0.5108, \qquad \epsilon = 0.9788. \tag{5.2}$$

Then the the estimator gain K_i, $i = 1,2$, can be designed as

$$K_1 = P^{-1}R_1 = \begin{pmatrix} 0.0161 & -0.0057 \\ -0.0071 & 0.0346 \end{pmatrix}, \qquad K_2 = P^{-1}R_2 = \begin{pmatrix} -0.0188 & -0.0217 \\ -0.0196 & -0.0376 \end{pmatrix}. \tag{5.3}$$

By Theorem 3.2, the switched error-state system of the neural network in this example is globally exponentially stable under any switching rules. Moreover, the estimate of the error-state decay is given by

$$\|e(t,\varphi)\| \le 1.5852e^{-0.2554}\|\varphi\|, \qquad t \ge 0. \tag{5.4}$$

For making numerical simulation for the switched error-state system, set $A_1 = \underline{A}_1$, $B_{11} = \underline{B}_{11}$, $B_{21} = \underline{B}_{21}$, $C = C_1$ and $A_2 = \underline{A}_2$, $B_{12} = \underline{B}_{12}$, $B_{22} = \underline{B}_{22}$, $C = C_2$, and assume that two subsystems are switched every five seconds. Figure 1 displays the trajectories of the error-state $e(t,\varphi)$ with initial value $(\varphi_1(t),\varphi_2(t))^T = ((\tan 2t)^2 - 0.3, (\sin 2t)^2 - 0.6)^T$, $t \in [-1,0]$. It can be seen that these trajectories converge to $e^* = (0,0)^T$. This is in accordance with the conclusion of Theorem 3.2.

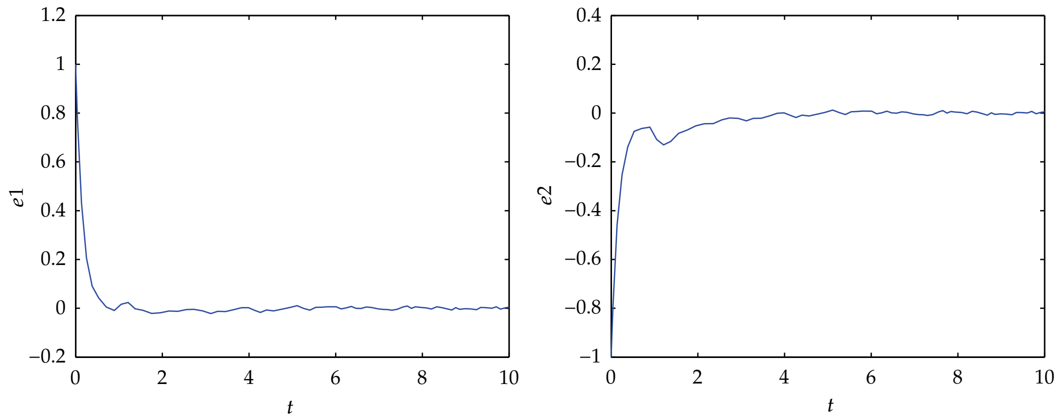

Figure 1: The state trajectories e_1 and e_2 of the network with initial value $(\varphi_1(t), \varphi_2(t))^T = ((\tan 2t)^2 - 0.3, (\sin 2t)^2 - 0.6)^T$, $t \in [-1, 0]$.

Example 5.2. Consider the second-order switched interval neural network with discrete and distributed delays described by

$$\begin{gathered}\dot{\mu}_i(t) = -a_{i_{\sigma(t)}}\mu_i(t) + \sum_{j=1}^{2} b^{(1)}_{ij_{\sigma(t)}} l_j(\mu_j(t-\tau)) + \sum_{j=1}^{2} b^{(2)}_{ij_{\sigma(t)}} \int_{t-h}^{t} l_j(\mu_j(s))ds, \\ a_{i_{\sigma(t)}} \in \left[\underline{a}_{i_{\sigma(t)}}, \overline{a}_{i_{\sigma(t)}}\right], \qquad b^{(k)}_{ij_{\sigma(t)}} \in \left[\underline{b}^{(k)}_{ij_{\sigma(t)}}, \overline{b}^{(k)}_{ij_{\sigma(t)}}\right], \quad k = 1, 2, \\ \mu_i(t) = \eta_i(t), \qquad t \in [-h, 0],\ i, j = 1, 2,\end{gathered} \tag{5.5}$$

where $\sigma(t) : [0, +\infty) \longrightarrow \Gamma = \{1, 2\}$, $l_i(x) = (1/2)\tan x + (1/2)\sin x$, $i = 1, 2$, $\tau = h = 1$. $\mathscr{H}_1$ and $\mathscr{H}_2$ are satisfied with $\Lambda_1 = \Lambda_2 = \operatorname{diag}(1, 1)$. The neural network system parameters are defined as

$$\begin{gathered}\underline{A}_1 = \begin{pmatrix} 3.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \quad \overline{A}_1 = \begin{pmatrix} 4.01 & 0 \\ 0 & 3.01 \end{pmatrix}, \quad \underline{B}_{11} = \begin{pmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{pmatrix}, \quad \overline{B}_{11} = \begin{pmatrix} 1.208 & 0.11 \\ 0.11 & 1.208 \end{pmatrix}, \\ \underline{B}_{21} = \begin{pmatrix} 0.09 & 0.14 \\ 0.05 & 0.09 \end{pmatrix}, \quad \overline{B}_{21} = \begin{pmatrix} 0.11 & 0.16 \\ 0.07 & 0.11 \end{pmatrix}, \\ \underline{A}_2 = \begin{pmatrix} 1.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \quad \overline{A}_2 = \begin{pmatrix} 2.01 & 0 \\ 0 & 3.01 \end{pmatrix}, \quad \underline{B}_{12} = \begin{pmatrix} -0.07 & 0.03 \\ -0.01 & 0.02 \end{pmatrix}, \quad \overline{B}_{12} = \begin{pmatrix} -0.05 & 0.05 \\ -0.04 & 0.04 \end{pmatrix}, \\ \underline{B}_{22} = \begin{pmatrix} -0.47 & -0.15 \\ 0.11 & -0.54 \end{pmatrix}, \quad \overline{B}_{22} = \begin{pmatrix} -0.45 & -0.13 \\ 0.13 & -0.54 \end{pmatrix}.\end{gathered} \tag{5.6}$$

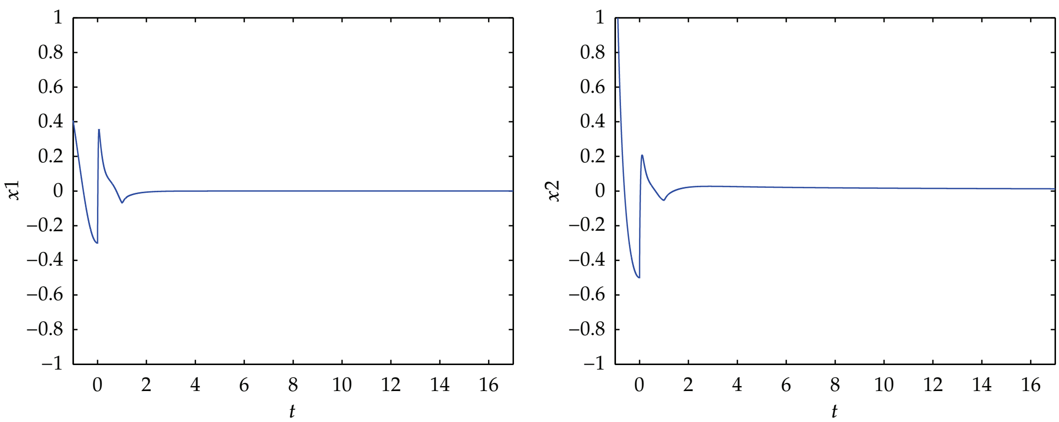

Figure 2: The state trajectories e_1 and e_2 of the network with initial value $(\eta_1(t), \eta_2(t))^T = ((\sin 3t)^2 - 0.3, (\cos 5t)^2 - 0.6)^T$, $t \in [-1, 0]$.

Solving the LMI in (4.5) by using appropriate LMI solver in the Matlab, the scalar β and feasible positive definite matrices P, Q_i, $i = 1, 2$, could be as follows:

$$P = \begin{pmatrix} 3.1369 & -0.0638 \\ -0.0638 & 4.4315 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0.7493 & 0 \\ 0 & 1.1935 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3.2851 & 0 \\ 0 & 9.0429 \end{pmatrix}, \qquad \beta = 0.5108. \tag{5.7}$$

By Theorem 4.1, the switched interval neural network in this example is exponentially stable. Moreover, the state $\mu(t)$ of the system satisfies

$$\|\mu(t)\| \le 1.7997 e^{-0.2554} \|\eta\|, \qquad t \ge 0. \tag{5.8}$$

Let $A_1 = \overline{A}_1$, $B_{11} = \overline{B}_{11}$, $B_{21} = \overline{B}_{21}$, and $A_2 = \overline{A}_2$, $B_{12} = \overline{B}_{12}$, $B_{22} = \overline{B}_{22}$. For numerical simulation, assume that the two subsystems are switched every five seconds. Figure 2 displays the state trajectories of this network with initial value $(\eta_1(t), \eta_2(t))^T = ((\sin 3t)^2 - 0.3, (\cos 5t)^2 - 0.6)^T$, $t \in [-1, 0]$. It can be seen that these trajectories converge to the unique equilibrium $\mu^* = (0, 0)^T$ of the network. This is in accordance with the conclusion of Theorem 4.1.

6. Conclusion

In this paper, a novel class of switched interval neural networks with discrete and distributed delays has been presented by combing the theories of the switched systems and the interval neural networks with discrete and distributed delays. By using feasible output measurements and constructing Lyapunov-Krasovskii functional, the existence conditions and explicit characterization have been obtained for desired estimator and exponential stability criteria for the switched interval neural networks with discrete and distributed delays under arbitrary switching rule in terms of LMIs. Two illustrative examples have been also given to demonstrate the effectiveness and validity of the proposed LMI-based stability criteria.

Acknowledgments

The authors are extremely grateful to the Editor and the Reviewers for their valuable comments and suggestions, which help to enrich the content and improve the presentation of this paper. This paper is supported by the National Science Foundation for Young Scholars of Heilongjiang Province, China (QC2010044), the Natural Science Foundation of Hebei Province of China (A2011203103), and the Hebei Province Education Foundation of China (2009157).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 135903, 18 pages
doi:10.1155/2012/135903

Research Article

Existence of Periodic Solutions for a Class of Difference Systems with p-Laplacian

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Received 15 February 2012; Revised 7 April 2012; Accepted 17 April 2012

Academic Editor: Yuriy Rogovchenko

By applying the least action principle and minimax methods in critical point theory, we prove the existence of periodic solutions for a class of difference systems with p-Laplacian and obtain some existence theorems.

1. Introduction

Consider the following p-Laplacian difference system:

$$\Delta\left(|\Delta u(t-1)|^{p-2}\Delta u(t-1)\right) = \nabla F(t, u(t)), \quad t \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $p \in (1, +\infty)$ such that $1/p + 1/q = 1$, $t \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, and $F(t, x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and T-periodic in t for all $x \in \mathbb{R}^N$.

When $p = 2$, (1.1) reduces to the following second-order discrete Hamiltonian system:

$$\Delta^2 u(t-1) = \nabla F(t, u(t)), \quad t \in \mathbb{Z}. \tag{1.2}$$

Difference equations provide a natural description of many discrete models in real world. Since discrete models exist in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural network, and optimal control, it is of practical importance to investigate the solutions of difference equations. For more details about difference equations, we refer the readers to the books [1–3].

In some recent papers [4–18], the authors studied the existence of periodic solutions and subharmonic solutions of difference equations by applying critical point theory. These papers show that the critical point theory is an effective method to the study of periodic solutions for difference equations. Motivated by the above papers, we consider the existence of periodic solutions for problem (1.1) by using the least action principle and minimax methods in critical point theory.

2. Preliminaries

Now, we first present our main results.

Theorem 2.1. *Suppose that F satisfies the following conditions:*

(F1) *there exists an integer $T > 1$ such that $F(t+T, x) = F(t, x)$ for all $(t, x) \in \mathbb{Z} \times \mathbb{R}^N$;*

(F2) *there exist $f, g \in l^1([1,T], \mathbb{R}^+)$ and $\alpha \in [0, p-1)$ such that*

$$|\nabla F(t,x)| \leq f(t)|x|^{\alpha} + g(t), \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \tag{2.1}$$

where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for every $a, b \in \mathbb{Z}$ with $a \leq b$,

(F3)

$$\liminf_{|x| \to +\infty} |x|^{-q\alpha} \sum_{t=1}^{T} F(t,x) > \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT} \sum_{t=1}^{T} f^q(t), \quad \forall t \in \mathbb{Z}[1,T]. \tag{2.2}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 2.2. *Suppose that F satisfies (F1) and the following conditions:*

$$\sum_{t=1}^{T} f(t) < \frac{T^p}{2^{p-1}(T-1)^{p(1+q)/q}}; \tag{2.3}$$

(F2)′ there exist $f, g \in l^1([1,T], \mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leq f(t)|x|^{p-1} + g(t), \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \tag{2.4}$$

where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for every $a, b \in \mathbb{Z}$ with $a \leq b$;

(F4)

$$\liminf_{|x| \to +\infty} |x|^{-p} \sum_{t=1}^{T} F(t,x) > \frac{2^p T^{q/p} (T-1)^{q(2p-1)/p}}{\left[T^p - 2^{p-1}(T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t)\right]^{q/p}} \sum_{t=1}^{T} f^q(t), \quad \forall t \in \mathbb{Z}[1,T]. \tag{2.5}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 2.3. *Suppose that F satisfies (F1), (F2), and the following condition:*
(F5)

$$\limsup_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(t,x) < -\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT}+\frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}}+\frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T}f^q(t) \quad \forall t\in\mathbb{Z}[1,T]. \tag{2.6}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 2.4. *Suppose that F satisfies (F1),* (2.3)*, (F2)′, and the following condition:*

(F6)

$$\begin{aligned}\limsup_{|x|\to+\infty}&|x|^{-p}\sum_{t=1}^{T}F(t,x)\\ < -&\left[\frac{2^p(pT)^{q/p}(T-1)^{q(2p-1)/p}\left(T^p+2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right)}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^q}\right.\\ &+\frac{2^p(pT)^{1/p}T(T-1)^{2p-1+(2p-1)/p}}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{1+1/p}}\\ &\left.+\frac{2^p(pT)^{(q-1)^2/p}(T-1)^{(q-1)^2(2p-1)/p}}{q\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{(q-1)^2/p}}\right]\sum_{t=1}^{T}f^q(t),\quad \forall t\in\mathbb{Z}[1,T].\end{aligned} \tag{2.7}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Remark 2.5. The lower bounds and the upper bounds of our theorems are more accurate than the existing results in the literature. Moreover, there are functions satisfying our results but not satisfying the existing results in the literature.

Let the Sobolev space E_T be defined by

$$E_T=\left\{u:\mathbb{Z}\longrightarrow\mathbb{R}^N \mid u(t+T)=u(t),\ t\in\mathbb{Z}\right\}. \tag{2.8}$$

For $u\in E_T$, let $\overline{u}=(1/T)\sum_{t=1}^{T}u(t)$, $u=\overline{u}+\tilde{u}$, and $\widetilde{E}_T=\{u\in E_T \mid \overline{u}=0\}$, then $E_T=\mathbb{R}^N\oplus\widetilde{E}_T$. Let

$$\|u\|=\left(|\overline{u}|^p+\sum_{t=1}^{T}|\Delta\tilde{u}(t)|^p\right)^{1/p},\quad u\in E_T. \tag{2.9}$$

As usual, let

$$\|u\|_\infty=\sup\{|u(t)| : t\in\mathbb{Z}[1,T]\},\quad \forall u\in l^\infty\left(\mathbb{Z}[1,T],\mathbb{R}^N\right). \tag{2.10}$$

For any $u \in E_T$, let

$$\varphi(u) = \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}F(t,u(t)) = \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p + \sum_{t=1}^{T}F(t,u(t)). \tag{2.11}$$

To prove our results, we need the following lemma.

Lemma 2.6 (see [18]). *Let $u \in E_T$. If $\sum_{t=1}^{T} u(t) = 0$, then*

$$\|u\|_\infty \le \frac{(T-1)^{(1+q)/q}}{T}\|\tilde{u}\|, \tag{2.12}$$

$$\|u\|_p^p = \sum_{t=1}^{T}|u(t)|^p \le \frac{(T-1)^{2p-1}}{T^{p-1}}\|\tilde{u}\|^p. \tag{2.13}$$

3. Proofs

For the sake of convenience, we denote

$$M_1 = \left(\sum_{t=1}^{T} f^q(t)\right)^{1/q}, \qquad M_2 = \sum_{t=1}^{T} f(t), \qquad M_3 = \sum_{t=1}^{T} g(t). \tag{3.1}$$

Proof of Theorem 2.1. From (F3), we can choose $a_1 > (T-1)^{(2p-1)/p}/T^{(p-1)/p}$ such that

$$\liminf_{|x|\to+\infty} |x|^{-q\alpha}\sum_{t=1}^{T}F(t,x) > \frac{a_1^q 2^{q\alpha}}{q}M_1^q. \tag{3.2}$$

It follows from (F2), (2.12), and (2.13) that

$$\begin{aligned}
&\left|\sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})]\right| \\
&= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,\overline{u}+s\tilde{u}(t)),\tilde{u}(t))ds\right| \\
&\le \sum_{t=1}^{T}\int_0^1 f(t)|\overline{u}+s\tilde{u}(t)|^\alpha|\tilde{u}(t)|ds + \sum_{t=1}^{T}\int_0^1 g(t)|\tilde{u}(t)|ds \\
&\le 2^\alpha\sum_{t=1}^{T}f(t)\left(|\overline{u}|^\alpha + |\tilde{u}(t)|^\alpha\right)|\tilde{u}(t)| + \sum_{t=1}^{T}g(t)|\tilde{u}(t)| \\
&\le 2^\alpha|\overline{u}|^\alpha\left(\sum_{t=1}^{T}f^q(t)\right)^{1/q}\left(\sum_{t=1}^{T}|\tilde{u}(t)|^p\right)^{1/p} + 2^\alpha\|\tilde{u}\|_\infty^{1+\alpha}\sum_{t=1}^{T}f(t) + \|\tilde{u}\|_\infty\sum_{t=1}^{T}g(t) \\
&= 2^\alpha|\overline{u}|^\alpha \mathrm{M}_1\|\tilde{u}\|_p + 2^\alpha M_2\|\tilde{u}\|_\infty^{1+\alpha} + M_3\|\tilde{u}\|_\infty
\end{aligned}$$

$$\leq \frac{1}{pa_1^p}\|\tilde{u}\|_p^p + \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q + 2^\alpha M_2 \|\tilde{u}\|_\infty^{1+\alpha} + M_3\|\tilde{u}\|_\infty$$
$$\leq \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\|\tilde{u}\|^p + \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} + \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q + \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\|. \tag{3.3}$$

Hence, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T} F(t,u(t)) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})] + \sum_{t=1}^{T} F(t,\overline{u}) \\
&\geq \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} + \sum_{t=1}^{T} F(t,\overline{u}) \\
&\quad - \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\|\tilde{u}\|^p - \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q - \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\| \\
&= \left(\frac{1}{p} - \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\right)\|\tilde{u}\|^p - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} \\
&\quad + |\overline{u}|^{q\alpha}\left(|\overline{u}|^{-q\alpha}\sum_{t=1}^{T} F(t,\overline{u}) - \frac{a_1^q 2^{q\alpha}}{q} M_1^q\right) - \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\|.
\end{aligned} \tag{3.4}$$

The above inequality and (3.2) imply that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. Hence, by the least action principle, problem (1.1) has at least one periodic solution with period T. □

Proof of Theorem 2.2. From (2.3) and (F4), we can choose a constant $a_3 \in \mathbb{R}$ such that

$$\begin{gathered}
a_3 > \frac{T^{1/p}(T-1)^{(2p-1)/p}}{\left[T^p - 2^{p-1} M_2 (T-1)^{p(1+q)/q}\right]^{1/p}} > 0, \\
\liminf_{|x|\to+\infty} |x|^{-p}\sum_{t=1}^{T} F(\mathrm{t},x) > \frac{a_3^q 2^p}{q} M_1^q.
\end{gathered} \tag{3.5}$$

It follows from (F2)′ and Lemma 2.6 that

$$\begin{aligned}
\left|\sum_{\mathrm{t}=1}^{T}[F(t,u(t)) - F(t,\overline{u})]\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,\overline{u} + s\tilde{u}(t)), \tilde{u}(t))ds\right| \\
&\leq \sum_{t=1}^{T}\int_0^1 f(t)|\overline{u} + s\tilde{u}(t)|^{p-1}|\tilde{u}(t)|ds + \sum_{t=1}^{T} g(t)|\tilde{u}(t)|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{t=1}^{T}\int_0^1 2^{p-1}f(t)\left(|\overline{u}|^{p-1}+s^{p-1}|\widetilde{u}(t)|^{p-1}\right)|\widetilde{u}(t)|ds+\sum_{t=1}^{T}g(t)|\widetilde{u}(t)| \\
&= \sum_{t=1}^{T}2^{p-1}f(t)\left(|\overline{u}|^{p-1}+\frac{1}{p}|\widetilde{u}(t)|^{p-1}\right)|\widetilde{u}(t)|+\sum_{t=1}^{T}g(t)|\widetilde{u}(t)| \\
&\le 2^{p-1}|\overline{u}|^{p-1}\left(\sum_{t=1}^{T}f^q(t)\right)^{1/q}\left(\sum_{t=1}^{T}|\widetilde{u}(t)|^p\right)^{1/p}+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&= 2^{p-1}M_1|\overline{u}|^{p-1}\|\widetilde{u}\|_p+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&\le \frac{1}{pa_3^p}\|\widetilde{u}\|_p^p+\frac{a_3^qM_1^q2^p}{q}|\overline{u}|^p+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&\le \left(\frac{(T-1)^{2p-1}}{pa_3^pT^{p-1}}+\frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p+\frac{a_3^qM_1^q2^p}{q}|\overline{u}|^p \\
&\quad+\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|,
\end{aligned}
\tag{3.6}
$$

which implies that

$$
\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p+\sum_{t=1}^{T}[F(t,u(t))-F(t,\overline{u})]+\sum_{t=1}^{T}F(t,\overline{u}) \\
&\ge \left(\frac{1}{p}-\frac{(T-1)^{2p-1}}{pa_3^pT^{p-1}}-\frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p \\
&\quad-\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|+|\overline{u}|^p\left(|\overline{u}|^{-p}\sum_{t=1}^{T}F(t,\overline{u})-\frac{a_3^qM_1^q2^p}{q}\right).
\end{aligned}
\tag{3.7}
$$

The above inequality and (3.5) imply that $\varphi(u)\to+\infty$ as $\|u\|\to\infty$. Hence, by the least action principle, problem (1.1) has at least one periodic solution with period T. □

Proof of Theorem 2.3. First we prove that φ satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of φ; that is, $\varphi'(u_n)\to 0$ as $n\to\infty$ and $\{\varphi(u_n)\}$ is bounded. By (F5), we can choose $a_2>(T-1)^{(2p-1)/p}/T^{(p-1)/p}$ such that

$$
\limsup_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(\mathrm{t},x)<-\left(\frac{2^{q\alpha}a_2^q}{p}+\frac{2^{q\alpha}a_2^{(q-1)^2}}{q}+\frac{2^{q\alpha}a_2(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q.
\tag{3.8}
$$

In a similar way to the proof of Theorem 2.1, we have

$$
\begin{aligned}
\left|\sum_{t=1}^{T}(\nabla F(t,u_n(t)),\widetilde{u}_n(t))\right| &\le \frac{(T-1)^{2p-1}}{pa_2^pT^{p-1}}\|\widetilde{u}_n\|^p+\frac{2^\alpha M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\widetilde{u}_n\|^{1+\alpha} \\
&\quad+\frac{a_2^q2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q+\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\|.
\end{aligned}
\tag{3.9}
$$

Hence, we have

$$\begin{aligned}
\|\tilde{u}_n\|_p &\geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\
&= \sum_{t=1}^{T} |\Delta u_n(t)|^p + \sum_{t=1}^{T} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) \\
&\geq \left(1 - \frac{(T-1)^{2p-1}}{p a_2^p T^{p-1}}\right) \|\tilde{u}_n\|^p - \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\| \\
&\quad - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} - \frac{a_2^q 2^{q\alpha}}{q} |\overline{u}_n|^{q\alpha} M_1^q.
\end{aligned} \tag{3.10}$$

From (2.13), we have

$$\|\tilde{u}_n\|_p = \left(\sum_{t=1}^{T} |\tilde{u}_n(t)|^p\right)^{1/p} \leq \frac{(T-1)^{(2p-1)/p}}{T^{(p-1)/p}} \|\tilde{u}_n\|. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned}
\frac{a_2^q 2^{q\alpha} M_1^q}{q} |\overline{u}_n|^{q\alpha} &\geq \left(1 - \frac{(T-1)^{2p-1}}{p a_2^p T^{p-1}}\right) \|\tilde{u}_n\|^p - \frac{(T-1)^{(2p-1)/p}}{T^{(p-1)/p}} \|\tilde{u}_n\| \\
&\quad - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} - \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\| \\
&\geq \frac{p-1}{p} \|\tilde{u}_n\|^p + C_1 \\
&= \frac{1}{q} \|\tilde{u}_n\|^p + C_1,
\end{aligned} \tag{3.12}$$

where $C_1 = \min_{s \in [0,+\infty)} \{(1/p - (T-1)^{2p-1}/p a_2^p T^{p-1}) s^p - (2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}/T^{1+\alpha}) s^{1+\alpha} - [(T-1)^{(2p-1)/p}/T^{(p-1)/p} + M_3 (T-1)^{(1+q)/q}/T] s\}$. Notice that $a_2 > (T-1)^{(2p-1)/p}/T^{(p-1)/p}$ implies $-\infty < C_1 < 0$. Hence, it follows from (3.12) that

$$\|\tilde{u}_n\|^p \leq 2^{q\alpha} a_2^q M_1^q |\overline{u}_n|^{q\alpha} - q C_1, \tag{3.13}$$

$$\|\tilde{u}_n\| \leq 2^{q\alpha/p} a_2^{q/p} M_1^{q/p} |\overline{u}_n|^{q\alpha/p} + C_2, \tag{3.14}$$

where $C_2 > 0$. By the proof of Theorem 2.1, we have

$$\begin{aligned}
\left|\sum_{t=1}^{T} [F(t, u_n(t)) - F(t, \overline{u}_n)]\right| &\leq 2^\alpha M_1 |\overline{u}_n|^\alpha \|\tilde{u}_n\|_p + 2^\alpha M_2 \|\tilde{u}_n\|_\infty^{1+\alpha} + M_3 \|\tilde{u}_n\|_\infty \\
&\leq \frac{(T-1)^{2p-1}}{p a_2^{q-1} T^{p-1}} \|\tilde{u}_n\|^p + \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} \\
&\quad + \frac{a_2^{(q-1)^2} 2^{q\alpha}}{q} |\overline{u}_n|^{q\alpha} M_1^q + \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\|.
\end{aligned} \tag{3.15}$$

It follows from the boundedness of $\varphi(u_n)$, (3.13)–(3.15) that

$$
\begin{aligned}
C_3 &\leq \varphi(u_n) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta u_n(t)|^p + \sum_{t=1}^{T}\left[F(t,u_n(t)) - F(t,\overline{u}_n)\right] + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\leq \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)\|\widetilde{u}_n\|^p + \frac{2^{\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\widetilde{u}_n\|^{1+\alpha} \\
&\quad + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\| + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\leq \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)\left(2^{q\alpha}a_2^qM_1^q|\overline{u}_n|^{q\alpha} - qC_1\right) + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\quad + \frac{2^{\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right)^{1+\alpha} \\
&\quad \times \frac{M_3(T-1)^{(1+q)/q}}{T}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right) \\
&\leq \left(\frac{2^{q\alpha}a_2^q}{p} + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q} + \frac{a_2 2^{q\alpha}(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q|\overline{u}_n|^{q\alpha} - \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)qC_1 \\
&\quad + \frac{2^{2\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\left(2^{q\alpha(1+\alpha)/p}a_2^{q(1+\alpha)/p}M_1^{q(1+\alpha)/p}|\overline{u}_n|^{q\alpha(1+\alpha)/p} + C_2^{1+\alpha}\right) \\
&\quad + \sum_{t=1}^{T}F(t,\overline{u}_n) + \frac{M_3(T-1)^{(1+q)/q}}{T}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right) \\
&= |\overline{u}_n|^{q\alpha}\left[|\overline{u}_n|^{-q\alpha}\sum_{t=1}^{T}F(t,\overline{u}_n) + \left(\frac{2^{q\alpha}a_2^q}{p} + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q} + \frac{a_2 2^{q\alpha}(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q\right. \\
&\qquad + \frac{2^{2\alpha+q\alpha(1+\alpha)/p}a_2^{q(1+\alpha)/p}M_1^{q(1+\alpha)/p}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}|\overline{u}_n|^{\alpha(1+\alpha-p)(q-1)} \\
&\qquad \left. + \frac{2^{q\alpha/p}a_2^{q/p}M_1^{q/p}M_3(T-1)^{(1+q)/q}}{T}|\overline{u}_n|^{-\alpha}\right] + C_4,
\end{aligned}
\tag{3.16}
$$

where C_3 is a positive constant and C_4 is a constant. The above inequality and (3.8) imply that $\{\overline{u}_n\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.13) and (3.13). Since E_T is finite dimensional, we conclude that φ satisfies (PS) condition.

In order to use the saddle point theorem ([19], Theorem 4.6), we only need to verify the following conditions:

(I1) $\varphi(u) \to -\infty$ as $|u| \to \infty$ in $\mathbb{R}^N$,

(I2) $\varphi(u) \to +\infty$ as $|u| \to \infty$ in $\widetilde{E}_T$, In fact, from (F5), we have

$$\sum_{t=1}^{T} F(t,u) \longrightarrow -\infty \quad \text{as } |u| \longrightarrow \infty \text{ in } \mathbb{R}^N, \tag{3.17}$$

which together with (2.11) implies that

$$\varphi(u) = \sum_{t=1}^{T} F(t,u) \longrightarrow -\infty \quad \text{as } |u| \longrightarrow \infty \text{ in } \mathbb{R}^N. \tag{3.18}$$

Hence, (I1) holds.

Next, for all $u \in \widetilde{E}_T$, by (F2) and (2.12), we have

$$\begin{aligned}
\left|\sum_{t=1}^{T}[F(t,u(t)) - F(t,0)]\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,su(t)),u(t))ds\right| \\
&\le \sum_{t=1}^{T} f(t)|u(t)|^{1+\alpha} + \sum_{t=1}^{T} g(t)|u(t)| \\
&\le M_2\|u\|_\infty^{1+\alpha} + M_3\|u\|_\infty \\
&\le \frac{M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{(1+\alpha)}}\|\widetilde{u}\|^{1+\alpha} + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|,
\end{aligned} \tag{3.19}$$

which implies that

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,0)] + \sum_{t=1}^{T}F(t,0) \\
&\ge \frac{1}{p}\|\widetilde{u}\|^p - \frac{M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{(1+\alpha)}}\|\widetilde{u}\|^{1+\alpha} \\
&\quad - \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\| + \sum_{t=1}^{T}F(t,0),
\end{aligned} \tag{3.20}$$

for all $u \in \widetilde{E}_T$. By Lemma 2.6, $\|u\| \to \infty$ in $\widetilde{E}_T$ if and only if $\|\widetilde{u}\| \to \infty$, so from (3.20), we obtain $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\widetilde{E}_T$; that is, (I2) is verified. Hence, the proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4. First we prove that φ satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of φ; that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (2.3) and (F6), we can choose $a_4 \in \mathbb{R}$ such that

$$a_4 > \frac{p^{1/p}T^{1/p}(T-1)^{(2p-1)/p}}{\left[pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}, \tag{3.21}$$

$$\begin{aligned} &\limsup_{|x|\to+\infty} |x|^{-p}\sum_{t=1}^{T} F(t,x) \\ &\quad < -\left[\frac{2^p a_4^q\left(T^p + 2^{p-1}M_2(T-1)^{p(1+q)/q}\right) + 2^pTa_4(T-1)^{2p-1}}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}} + \frac{2^p a_4^{(q-1)^2}}{q}\right] M_1^q. \end{aligned} \tag{3.22}$$

In a similar way to the proof of Theorem 2.2, we obtain

$$\begin{aligned} \left|\sum_{t=1}^{T}(\nabla F(t,u_n(t)),\tilde{u}_n(t))\right| &\le \left(\frac{(T-1)^{2p-1}}{pa_4^pT^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\tilde{u}_n\|^p \\ &\quad + \frac{a_4^qM_1^q2^p}{q}|\overline{u}_n|^p + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\tilde{u}_n\|. \end{aligned} \tag{3.23}$$

Hence, we have

$$\begin{aligned} \|\tilde{u}_n\|_p &\ge \langle \varphi'(u_n), \tilde{u}_n\rangle \\ &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u_n(t)|^p + \sum_{t=1}^{T}(\nabla F(t,u_n(t)),\tilde{u}_n(t)) \\ &\ge \left(1 - \frac{(T-1)^{2p-1}}{pa_4^pT^{p-1}} - \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\tilde{u}_n\|^p - \frac{a_4^qM_1^q2^p}{q}|\overline{u}_n|^p \\ &\quad - \frac{M_3(T-1)^{(1+q)/q}}{T}\|\tilde{u}_n\|, \end{aligned} \tag{3.24}$$

which together with (3.11) implies that

$$\begin{aligned} \frac{a_4^qM_1^q2^p}{q}|\overline{u}_n|^p &\ge \left(1 - \frac{(T-1)^{2p-1}}{pa_4^pT^{p-1}} - \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\tilde{u}_n\|^p \\ &\quad - \frac{M_3(T-1)^{(1+q)/q}}{T}\|\tilde{u}_n\| - \frac{(T-1)^{(2p-1)/p}}{T^{1/q}}\|\tilde{u}_n\| \\ &\ge \frac{1}{q}\left(1 - \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\tilde{u}_n\|^p + C_5, \end{aligned} \tag{3.25}$$

where $C_5 = \min_{s\in[0,+\infty)}\{(1/p - (T-1)^{2p-1}/pa_4^pT^{p-1} - 2^{p-1}M_2(T-1)^{p(1+q)/q}/p^2T^p)s^p - [M_3(T-1)^{(1+q)/q}/T + (T-1)^{(2p-1)/p}/T^{1/q}]s\}$. It follows from (3.21) that $-\infty < C_5 < 0$, so, we obtain

$$\|\widetilde{u}_n\|^p \le \frac{pT^p a_4^q M_1^q 2^p}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}|\overline{u}_n|^p - \frac{pT^p qC_5}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}, \tag{3.26}$$

$$\|\widetilde{u}_n\| \le \frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}}{\left[pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n| + C_6, \tag{3.27}$$

where C_6 is a positive constant. By the proof of Theorem 2.2, we have

$$\begin{aligned}
&\left|\sum_{t=1}^{T}(\mathrm{F}(t,u_n(t)) - F(t,\overline{u}_n))\right| \\
&\quad\le 2^{p-1}M_1|\overline{u}|^{p-1}\|\widetilde{u}\|_p + \frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p + M_3\|\widetilde{u}\|_\infty \\
&\quad\le \left(\frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}_n\|^p + \frac{a_4^{(q-1)^2}M_1^q2^p}{q}|\overline{u}_n|^p \\
&\qquad+ \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\|.
\end{aligned} \tag{3.28}$$

It follows from the boundedness of $\varphi(u_n)$, (3.26), (3.27), and the above inequality that

$$\begin{aligned}
C_7 &\le \varphi(u_n) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})] + \sum_{t=1}^{T}F(t,\overline{u}) \\
&\le \left[\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right]\|\widetilde{u}_n\|^p + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\quad+ \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\| + \frac{a_4^{(q-1)^2}M_1^q2^p}{q}|\overline{u}_n|^p \\
&\le \left[\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right] \\
&\quad\times\left(\frac{pT^pa_4^qM_1^q2^p}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}|\overline{u}_n|^p - \frac{pT^pqC_5}{pT^p - 2^{p-1}\mathrm{M}_2(T-1)^{p(1+q)/q}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{T}F(t,\overline{u})+\frac{a_4^{(q-1)^2}M_1^q2^p}{q}|\overline{u}_n|^p\\
&+\frac{M_3(T-1)^{(1+q)/q}}{T}\left(\frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}}{\left[pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n|+C_6\right)\\
=|\overline{u}_n|^p&\left\{\left[\frac{2^pa_4^q\left(T^p+2^{p-1}M_2(T-1)^{p(1+q)/q}\right)+2^pTa_4(T-1)^{2p-1}}{pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}}+\frac{2^pa_4^{(q-1)^2}}{q}\right]M_1^q\right.\\
&\left.+|\overline{u}_n|^{-p}\sum_{t=1}^{T}F(t,\overline{u}_n)+\frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}M_3(T-1)^{(1+q)/q}}{T\left[pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n|^{-p+1}\right\}+C_8,
\end{aligned}
\tag{3.29}
$$

where C_7 is a positive constant and C_8 is a constant. The above inequality and (3.22) imply that $\{\overline{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (2.13) and (3.26).

Similar to the proof of Theorem 2.3, we only need to verify (I1) and (I2). It is easy to verify (I1) by (F6). Now, we verify that (I2) holds. For $u\in\widetilde{E}_T$, by (F2)′ and (2.12), we have

$$
\begin{aligned}
\left|\sum_{t=1}^{T}(F(t,u(t))-F(t,0))\right|&=\left|\sum_{t=1}^{T}\int_0^1(\nabla F(t,su(t)),u(t))ds\right|\\
&\le\sum_{t=1}^{T}\int_0^1 f(t)s^{p-1}|u(t)|^pds+\sum_{t=1}^{T}g(t)|u(t)|\\
&\le\frac{M_2}{p}\|u\|_\infty^p+M_3\|u\|_\infty\\
&\le\frac{M_2(T-1)^{p(1+q)/q}}{pT^p}\|\widetilde{u}\|^p+\frac{M_3(T-1)^{(1+q)/q}}{pT}\|\widetilde{u}\|.
\end{aligned}
\tag{3.30}
$$

Thus, we have

$$
\begin{aligned}
\varphi(u)&=\frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p+\sum_{t=1}^{T}(F(t,u(t))-F(t,0))+\sum_{t=1}^{T}F(t,0)\\
&\ge\left(\frac{1}{p}-\frac{M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p-\frac{M_3(T-1)^{(1+q)/q}}{pT}\|\widetilde{u}\|+\sum_{t=1}^{T}F(t,0),
\end{aligned}
\tag{3.31}
$$

for all $u\in\widetilde{E}_T$. By Lemma 2.6, $\|u\|\to\infty$ in $\widetilde{E}_T$ if and only $\|\widetilde{u}\|\to\infty$. So from the above inequality, we have $\varphi(u)\to+\infty$ as $\|u\|\to\infty$, that is (I2) is verified. Hence, the proof of Theorem 2.4 is complete. □

4. Example

In this section, we give four examples to illustrate our results.

Example 4.1. Let $p = 5/2$ and

$$F(t,x) = \sin\left(\frac{2\pi t}{T}\right)|x|^{5/3} + \left(\sin\frac{2\pi t}{T} + 1\right)|x|^{4/3} + (h(t), x), \tag{4.1}$$

where $h \in l^1(\mathbb{Z}[1,T], \mathbb{R}^N)$ and $h(t+T) = h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\leq \frac{5}{3}\left|\sin\frac{2\pi t}{T}\right| |x|^{2/3} + \frac{4}{3}\left|\sin\frac{2\pi t}{T} + 1\right| |x|^{1/3} + |h(t)| \\ &\leq \frac{5}{3}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right)|x|^{2/3} + a(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \end{aligned} \tag{4.2}$$

where $\varepsilon > 0$, and $a(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2) holds with $\alpha = 2/3$ and

$$f(t) = \frac{5}{3}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right), \qquad g(t) = a(\varepsilon) + |h(t)|. \tag{4.3}$$

Moreover, we have

$$\begin{gathered} \liminf_{|x| \to +\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) = T, \\ \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT} \sum_{t=1}^{T} f^q(t) = \frac{3 \times 2^{10/9}(T-1)^{8/3}}{5}\left(\frac{5}{3}\varepsilon\right)^{5/3}. \end{gathered} \tag{4.4}$$

We can choose ε suitable such that

$$\liminf_{|x| \to +\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) = T > \frac{3 \times 2^{10/9}(T-1)^{8/3}}{5}\left(\frac{5}{3}\varepsilon\right)^{5/3} = \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT} \sum_{t=1}^{T} f^q(t), \tag{4.5}$$

which shows that (F3) holds. Then from Theorem 2.1, problem (1.1) has at least one periodic solution with period T.

Example 4.2. Let $p = 2$, then $q = 2$. Let

$$F(t,x) = \frac{1}{6}\left(\frac{1}{2} + \sin\frac{2\pi t}{T}\right)|x|^2 + |x|^{3/2} + (h(t), x), \tag{4.6}$$

where $h \in l^1(\mathbb{Z}[1,T],\mathbb{R}^N)$ and $h(t+T)=h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\leq \frac{1}{3}\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right||x|+\frac{3}{2}|x|^{1/2}+|h(t)| \\ &\leq \frac{1}{3}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right)|x|+b(\varepsilon)+|h(t)|, \quad \forall (t,x)\in\mathbb{Z}[1,T]\times\mathbb{R}^N, \end{aligned} \tag{4.7}$$

where $\varepsilon>0$, and $b(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2)′ holds with

$$f(t)=\frac{1}{3}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right), \qquad g(t)=b(\varepsilon)+|h(t)|. \tag{4.8}$$

Observe that

$$\begin{aligned} |x|^{-p}\sum_{t=1}^{T}F(t,x) &= |x|^{-2}\sum_{t=1}^{T}\left[\frac{1}{6}\left(\frac{1}{2}+\sin\frac{2\pi t}{T}\right)|x|^2+|x|^{3/2}+(h(t),x)\right] \\ &= \frac{T}{12}+T|x|^{-1/2}+\left(\sum_{t=1}^{T}h(t),|x|^{-2}x\right). \end{aligned} \tag{4.9}$$

On the other hand, if we let $T=2$, then we have

$$\begin{gathered} \sum_{t=1}^{T}f(t)=\frac{2}{3}\left(\frac{1}{2}+\varepsilon\right), \qquad \sum_{t=1}^{T}f^2(t)=\frac{1}{9}\sum_{t=1}^{T}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right)^2=\frac{2}{9}\left(\frac{1}{2}+\varepsilon\right)^2, \\ \frac{2^pT^{q/p}(T-1)^{q(2p-1)/p}}{\left[T^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{q/p}}\sum_{t=1}^{T}f^q(t)=\frac{2+8\varepsilon+8\varepsilon^2}{15-6\varepsilon}. \end{gathered} \tag{4.10}$$

We can choose ε sufficiently small such that

$$\begin{aligned} \sum_{t=1}^{T}f(t)=\frac{2}{3}\left(\frac{1}{2}+\varepsilon\right)&<2=\frac{T^p}{2^{p-1}(T-1)^{p-1}}, \\ \liminf_{|x|\to+\infty}|x|^{-p}\sum_{t=1}^{T}F(t,x) &= \frac{1}{6}>\frac{2+8\varepsilon+8\varepsilon^2}{15-6\varepsilon} \\ &= \frac{2^pT^{q/p}(T-1)^{q(2p-1)/p}}{\left[T^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{q/p}}\sum_{t=1}^{T}f^q(t), \end{aligned} \tag{4.11}$$

which shows that (2.3) and (F4) hold. Then from Theorem 2.2, problem (1.1) has at least one periodic solution with period T.

Example 4.3. Let $p=2$, then $q=2$. Let

$$F(t,x)=\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+\left(\sin\frac{2\pi t}{T}-1\right)|x|^{3/2}+(h(t),x), \tag{4.12}$$

where $h \in l^1(\mathbb{Z}[1,T],\mathbb{R}^N)$ and $h(t+T)=h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned}|\nabla F(t,x)| &\le \frac{7}{4}\left|\sin\frac{2\pi t}{T}\right||x|^{3/4}+\frac{3}{2}\left|\sin\frac{2\pi t}{T}-1\right||x|^{1/2}+|h(t)|\\ &\le \frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right|+\varepsilon\right)|x|^{3/4}+c(\varepsilon)+|h(t)|,\quad \forall (t,x)\in\mathbb{Z}[1,T]\times\mathbb{R}^N,\end{aligned}\tag{4.13}$$

where $\varepsilon>0$ and $c(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2) holds with $\alpha = 3/4$ and

$$f(t)=\frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right|+\varepsilon\right),\qquad g(t)=c(\varepsilon)+|h(t)|.\tag{4.14}$$

Observe that

$$\begin{aligned}|x|^{-q\alpha}\sum_{t=1}^{T}F(t,x) &= |x|^{-3/2}\sum_{t=1}^{T}\left[\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+\left(\sin\frac{2\pi t}{T}-1\right)|x|^{3/2}+(h(t),x)\right]\\ &= -T+\left(\sum_{t=1}^{T}h(t),|x|^{-3/2}x\right).\end{aligned}\tag{4.15}$$

On the other hand, we have

$$\begin{aligned}&\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT}+\frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}}+\frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T}f^q(t)\\ &=\left[\frac{\sqrt{2}(T-1)^3}{T}+\frac{\sqrt{2}(T-1)^{3/2}}{T^{1/2}}+\frac{\sqrt{2}(T-1)^{9/2}}{T^{3/2}}\right]\sum_{t=1}^{T}\frac{49}{16}\left(\left|\sin\frac{2\pi t}{T}\right|+\varepsilon\right)^2\\ &=\frac{49\sqrt{2}\varepsilon^2(T-1)^{3/2}\left[T^{1/2}(T-1)^{3/2}+T+(T-1)^3\right]}{16T^{1/2}}.\end{aligned}\tag{4.16}$$

We can choose ε suitable such that

$$\begin{aligned}&\limsup_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(t,x)\\ &=-T\\ &<-\frac{49\sqrt{2}\varepsilon^2(T-1)^{3/2}\left[T^{1/2}(T-1)^{3/2}+T+(T-1)^3\right]}{16T^{1/2}}\\ &=-\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT}+\frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}}+\frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T}f^q(t),\end{aligned}\tag{4.17}$$

which shows that (F5) holds. Then from Theorem 2.3, problem (1.1) has at least one periodic solution with period T.

Example 4.4. Let $p = 2$, then $q = 2$. Let

$$F(t,x) = \frac{1}{3}\left(\sin\frac{2\pi t}{T} - \frac{1}{8}\right)|x|^2 + |x|^{3/2} + (h(t), x), \tag{4.18}$$

where $h \in l^1(\mathbb{Z}[1,T], \mathbb{R}^N)$ and $h(t+T) = h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{2}{3}\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right||x| + \frac{3}{2}|x|^{1/2} + |h(t)| \\ &\leq \frac{2}{3}\left(\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right| + \varepsilon\right)|x| + d(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N,\end{aligned} \tag{4.19}$$

where $\varepsilon > 0$, $d(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2)′ holds with

$$f(t) = \frac{2}{3}\left(\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right| + \varepsilon\right), \quad g(t) = d(\varepsilon) + |h(t)|. \tag{4.20}$$

Observe that

$$\begin{aligned}|x|^{-p}\sum_{t=1}^{T} F(t,x) &= |x|^{-2}\sum_{t=1}^{T}\left[\frac{1}{3}\left(\sin\frac{2\pi t}{T} - \frac{1}{8}\right)|x|^2 + |x|^{3/2} + (h(t), x)\right] \\ &= -\frac{T}{24} + T|x|^{-1/2} + \left(\sum_{t=1}^{T} h(t), |x|^{-2}x\right).\end{aligned} \tag{4.21}$$

On the other hand, if we let $T = 2$, then we have

$$\sum_{t=1}^{T} f(t) = \frac{4}{3}\left(\frac{1}{8} + \varepsilon\right), \qquad \sum_{t=1}^{T} f^2(t) = \frac{4}{9}\sum_{t=1}^{T}\left(\left|\frac{1}{8} + \sin\frac{2\pi t}{T}\right| + \varepsilon\right)^2 = \frac{8}{9}\left(\frac{1}{8} + \varepsilon\right)^2, \tag{4.22}$$

$$
\begin{aligned}
&- \Bigg[\frac{2^p (pT)^{q/p} (T-1)^{q(2p-1)/p} \left(T^p + 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right)}{\left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^q} \\
&\quad + \frac{2^p (pT)^{1/p} T (T-1)^{2p-1+(2p-1)/p}}{\left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^{1+1/p}} \\
&\quad + \frac{2^p (pT)^{(q-1)^2/p} (T-1)^{(q-1)^2(2p-1)/p}}{q \left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^{(q-1)^2/p}} \Bigg] \sum_{t=1}^{T} f^q(t) \\
&= \left[\frac{192 + 128 \times (1/8 + \varepsilon)}{3 \times (8 - (8/3)(1/8 + \varepsilon))^2} + \frac{16}{(8 - (8/3)(1/8 + \varepsilon))^{3/2}} \right. \\
&\quad \left. + \frac{8}{(8 - (8/3)(1/8 + \varepsilon))^{1/2}} \right] \times \frac{8}{9} \left(\frac{1}{8} + \varepsilon \right)^2 .
\end{aligned}
\tag{4.23}
$$

We can choose ε sufficiently small such that

$$
\sum_{t=1}^{T} f(t) = \frac{4}{3} \left(\frac{1}{8} + \varepsilon \right) < 2 = \frac{T^p}{2^{p-1} (T-1)^{p-1}},
\tag{4.24}
$$

$$
\begin{aligned}
\limsup_{|x| \to +\infty} |x|^{-p} \sum_{t=1}^{T} F(t,x) &= -\frac{1}{12} \\
&< \left[\frac{192 + 128 \times (1/8 + \varepsilon)}{3 \times (8 - (8/3)(1/8 + \varepsilon))^2} + \frac{16}{(8 - (8/3)(1/8 + \varepsilon))^{3/2}} \right. \\
&\quad \left. + \frac{8}{(8 - (8/3)(1/8 + \varepsilon))^{1/2}} \right] \times \frac{8}{9} \left(\frac{1}{8} + \varepsilon \right)^2 \\
&= - \Bigg[\frac{2^p (pT)^{q/p} (T-1)^{q(2p-1)/p} \left(T^p + 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right)}{\left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^q} \\
&\quad + \frac{2^p (pT)^{1/p} T (T-1)^{2p-1+(2p-1)/p}}{\left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^{1+1/p}} \\
&\quad + \frac{2^p (pT)^{(q-1)^2/p} (T-1)^{(q-1)^2(2p-1)/p}}{q \left[pT^p - 2^{p-1} (T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t) \right]^{(q-1)^2/p}} \Bigg] \sum_{t=1}^{T} f^q(t),
\end{aligned}
\tag{4.25}
$$

which shows that (F6) holds. Then from Theorem 2.4, problem (1.1) has at least one periodic solution with period T.

Acknowledgment

This work is partially supported by the Scientific Research Foundation of the Guangxi Education Office of China (201203YB093 and 200911MS270).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 241702, 26 pages
doi:10.1155/2012/241702

Research Article

Stochastic Delay Logistic Model under Regime Switching

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Received 2 April 2012; Accepted 14 June 2012

Academic Editor: Elena Braverman

This paper is concerned with a delay logistical model under regime switching diffusion in random environment. By using generalized Itô formula, Gronwall's inequality, and Young's inequality, some sufficient conditions for existence of global positive solutions and stochastically ultimate boundedness are obtained, respectively. Also, the relationships between the stochastic permanence and extinction as well as asymptotic estimations of solutions are investigated by virtue of V-function technique, M-matrix method, and Chebyshev's inequality. Finally, an example is given to illustrate the main results.

1. Introduction

The delay differential equation

$$\frac{dx(t)}{dt} = x(t)[a - bx(t) + cx(t-\tau)] \tag{1.1}$$

has been used to model the population growth of certain species, known as the delay logistic equation. There is an extensive literature concerned with the dynamics of this delay model. We here only mention Gopalsamy [1], Kolmanovskiĭ, and Myshkis [2], Kuang [3] among many others.

In (1.1), the state $x(t)$ denotes the population size of the species. Naturally, we focus on the positive solutions and also require the solutions not to explode at a finite time. To guarantee positive solutions without explosion (i.e., there exists global positive solutions), it is generally assumed that $a > 0$, $b > 0$, and $c < b$ [4] (and the references cited therein).

On the other hand, the population growth is often subject to environmental noise, and the system will change significantly, which may change the dynamical behavior of solutions significantly [5, 6]. It is therefore necessary to reveal how the noise affects on the dynamics of solutions for the delay population model. First of all, let us consider one type of environmental noise, namely, white noise. In fact, recently, many authors have discussed population systems subject to white noise [7–9]. Recall that the parameter a in (1.1) represents the intrinsic growth rate of the population. In practice, we usually estimate it by an average value plus an error term. According to the well-known central limit theorem, the error term follows a normal distribution. In term of mathematics, we can therefore replace the rate a by

$$a + \sigma \dot{w}(t), \tag{1.2}$$

where $\dot{w}(t)$ is a white noise (i.e., $w(t)$ is a Brownian motion) and σ is a positive number representing the intensity of noise. As a result, (1.1) becomes a stochastic differential equation (SDE, in short)

$$dx(t) = x(t)\left[(a - bx(t) + cx(t-\tau))dt + \sigma dw(t)\right]. \tag{1.3}$$

We refer to [4] for more details.

To our knowledge, much attention to environmental noise is paid on white noise ([10–14] and the references cited therein). But another type of environmental noise, namely, color noise or say telegraph noise, has been studied by many authors (see, [15–19]). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which are different in terms of factors such as nutrition or as rain falls [20, 21]. Usually, the switching between different environments is memoryless and the waiting time for the next switch has an exponential distribution. This indicates that we may model the random environments and other random factors in the system by a continuous-time Markov chain $r(t)$, $t \geq 0$ with a finite state space $S = \{1, 2, \ldots, n\}$. Therefore, the stochastic delay logistic (1.3) in random environments can be described by the following stochastic model with regime switching:

$$dx(t) = x(t)\left[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t-\tau))dt + \sigma(r(t))dw(t)\right]. \tag{1.4}$$

The mechanism of ecosystem described by (1.4) can be explained as follows. Assume that initially, the Markov chain $r(0) = \iota \in S$, then the ecosystem (1.4) obeys the SDE

$$dx(t) = x(t)\left[(a(\iota) - b(\iota)x(t) + c(\iota)x(t-\tau))dt + \sigma(\iota)dw(t)\right], \tag{1.5}$$

until the Markov chain $r(t)$ jumps to another state, say, ς. Then the ecosystem satisfies the SDE

$$dx(t) = x(t)\left[(a(\varsigma) - b(\varsigma)x(t) + c(\varsigma)x(t-\tau))dt + \sigma(\varsigma)dw(t)\right], \tag{1.6}$$

for a random amount of time until $r(t)$ jumps to a new state again.

It should be pointed out that the stochastic logistic systems under regime switching have received much attention lately. For instance, the study of stochastic permanence and extinction of a logistic model under regime switching was considered in [18], a new single-species model disturbed by both white noise and colored noise in a polluted environment was developed and analyzed in [22], a general stochastic logistic system under regime switching was proposed and was treated in [23].

Since (1.4) describes a stochastic population dynamics, it is critical to find out whether or not the solutions will remain positive or never become negative, will not explode to infinity in a finite time, will be ultimately bounded, will be stochastically permanent, will become extinct, or have good asymptotic properties.

This paper is organized as follows. In the next section, we will show that there exists a positive global solution with any initial positive value under some conditions. In Sections 3 and 4, we give the sufficient conditions for stochastic permanence or extinction, which show that both have closed relations with the stationary probability distribution of the Markov chain. If (1.4) is stochastically permanent, we estimate the limit of the average in time of the sample path of its solution in Section 5. Finally, an example is given to illustrate our main results.

2. Global Positive Solution

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t),\ t \geq 0$, be a scalar standard Brownian motion defined on this probability space. We also denote by R_+ the interval $(0,\infty)$ and denote by $\overline{R}_+$ the interval $[0,\infty)$. Moreover, let $\tau > 0$ and denote by $C([-\tau,0];R_+)$ the family of continuous functions from $[-\tau,0]$ to R_+.

Let $r(t)$ be a right-continuous Markov chain on the probability space, taking values in a finite state space $S = \{1,2,\dots,n\}$, with the generator $\Gamma = (\gamma_{uv})$ given by

$$P\{r(t+\delta) = v \mid r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \gamma_{uv}\delta + o(\delta), & \text{if } u = v, \end{cases} \tag{2.1}$$

where $\delta > 0$, γ_{uv} is the transition rate from u to v and $\gamma_{uv} \geq 0$ if $u \neq v$, while $\gamma_{uu} = -\sum_{v\neq u}\gamma_{uv}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of jumps in any finite subinterval of $\overline{R}_+$. As a standing hypothesis we assume in this paper that the Markov chain $r(t)$ is irreducible. This is a very reasonable assumption as it means that the system can switch from any regime to any other regime. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \dots, \pi_n) \in R^{1\times n}$ which can be determined by solving the following linear equation:

$$\pi\Gamma = 0, \tag{2.2}$$

subject to

$$\sum_{i=1}^{n} \pi_i = 1, \quad \pi_i > 0,\ \forall\ i \in S. \tag{2.3}$$

We refer to [9, 24] for the fundamental theory of stochastic differential equations.

For convenience and simplicity in the following discussion, define

$$\hat{f} = \min_{i\in S} f(i), \qquad \check{f} = \max_{i\in S} f(i), \qquad \overline{f} = \max_{i\in S} |f(i)|, \tag{2.4}$$

where $\{f(i)\}_{i\in S}$ is a constant vector.

As $x(t)$ in model (1.4) denotes population size at time t, it should be nonnegative. Thus, for further study, we must give some condition under which (1.4) has a unique global positive solution.

Theorem 2.1. *Assume that there are positive numbers $\theta(i)$ $(i = 1, 2, \dots, n)$ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{4\theta(i)}c^2(i) + \check{\theta}\right) \le 0. \tag{2.5}$$

Then, for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, there is a unique solution $x(t)$ to (1.4) on $t \ge -\tau$ and the solution will remain in R_+ with probability 1, namely, $x(t) \in R_+$ for all $t \ge -\tau$ a.s.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to prove $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau\le t\le 0} x(t) \le \max_{-\tau\le t\le 0} x(t) < k_0. \tag{2.6}$$

For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k\right)\right\}, \tag{2.7}$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$, where $\tau_\infty \le \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+$ a.s. for all $t \ge 0$. In other words, we need to show $\tau_\infty = \infty$ a.s. Define a C^2-function $V : R_+ \to R_+$ by

$$V(x) = x - 1 - \log x, \tag{2.8}$$

which is not negative on $x > 0$. Let $k \ge k_0$ and $T > 0$ be arbitrary. For $0 \le t \le \tau_k \wedge T$, it is not difficult to show by the generalized Itô formula that

$$dV(x(t)) = LV(x(t), x(t-\tau), r(t))dt + \sigma(r(t))(x(t) - 1)dw(t), \tag{2.9}$$

where $LV : R_+ \times R_+ \times S \to R$ is defined by

$$LV(x,y,i) = -a(i) + \frac{1}{2}\sigma^2(i) + (a(i)+b(i))x - c(i)y - b(i)x^2 + c(i)xy. \tag{2.10}$$

Using condition (2.5), we compute

$$-b(i)x^2 + c(i)xy \le -b(i)x^2 + \frac{1}{4\theta(i)}c^2(i)x^2 + \theta(i)y^2 \le -\check{\theta}x^2 + \check{\theta}y^2. \tag{2.11}$$

Moreover, there is clearly a constant $K_1 > 0$ such that

$$-a(i) + \frac{1}{2}\sigma^2(i) + (a(i)+b(i))x - c(i)y \le K_1(1+x+y). \tag{2.12}$$

Substituting these into (2.10) yields

$$LV(x,y,i) \le K_1(1+x+y) - \check{\theta}x^2 + \check{\theta}y^2. \tag{2.13}$$

Noticing that $u \le 2(u-1-\log u)+2$ on $u>0$, we obtain that

$$LV(x,y,i) \le K_2(1+V(x)+V(y)) - \check{\theta}x^2 + \check{\theta}y^2, \tag{2.14}$$

where K_2 is a positive constant. Substituting these into (2.9) yields

$$\begin{aligned} dV(x(t)) \le{}& \left[K_2(1+V(x(t))+V(x(t-\tau))) - \check{\theta}x^2(t) + \check{\theta}x^2(t-\tau)\right]dt \\ &+ \sigma(r(t))(x(t)-1)dw(t). \end{aligned} \tag{2.15}$$

Now, for any $t \in [0,T]$, we can integrate both sides of (2.15) from 0 to $\tau_k \wedge t$ and then take the expectations to get

$$EV(x(\tau_k \wedge t)) \le V(x(0)) + E\int_0^{\tau_k\wedge t}\left[K_2(1+V(x(s))+V(x(s-\tau))) - \check{\theta}x^2(s) + \check{\theta}x^2(s-\tau)\right]ds. \tag{2.16}$$

Compute

$$\begin{aligned} E\int_0^{\tau_k\wedge t} V(x(s-\tau))ds &= E\int_{-\tau}^{\tau_k\wedge t-\tau} V(x(s))ds \\ &\le \int_{-\tau}^{0} V(x(s))ds + E\int_0^{\tau_k\wedge t} V(x(s))ds, \end{aligned} \tag{2.17}$$

and, similarly

$$E\int_0^{\tau_k\wedge t} x^2(s-\tau)ds \le \int_{-\tau}^0 x^2(s)ds + E\int_0^{\tau_k\wedge t} x^2(s)ds. \tag{2.18}$$

Substituting these into (2.16) gives

$$\begin{aligned} EV(x(\tau_k\wedge t)) &\le K_3 + 2K_2E\int_0^{\tau_k\wedge t} V(x(s))ds \\ &\le K_3 + 2K_2E\int_0^t V(x(\tau_k\wedge s))ds \\ &= \ K_3 + 2K_2\int_0^t EV(x(\tau_k\wedge s))ds, \end{aligned} \tag{2.19}$$

where $K_3 = V(x(0)) + K_2T + K_2\int_{-\tau}^0 V(x(s))ds + \check{\theta}\int_{-\tau}^0 x^2(s)ds$.

By the Gronwall inequality, we obtain that

$$EV(x(\tau_k\wedge T)) \le K_3e^{2TK_2}. \tag{2.20}$$

Note that for every $\omega \in \{\tau_k \le T\}$, $x(\tau_k,\omega)$ equals either k or $1/k$, thus

$$V(x(\tau_k,\omega)) \ge \left[(k-1-\log k)\wedge\left(\frac{1}{k}-1+\log k\right)\right]. \tag{2.21}$$

It then follows from (2.20) that

$$\begin{aligned} K_3e^{2TK_2} &\ge E\left[1_{\{\tau_k\le T\}}(\omega)V(x(\tau_k\wedge T,\omega))\right] \\ &= E\left[\mathbf{1}_{\{\tau_k\le T\}}(\omega)V(x(\tau_k,\omega))\right] \\ &\ge P\{\tau_k\le T\}\left[(k-1-\log k)\wedge\left(\frac{1}{k}-1+\log k\right)\right], \end{aligned} \tag{2.22}$$

where $\mathbf{1}_{\{\tau_k\le T\}}$ is the indicator function of $\{\tau_k \le T\}$. Letting $k \to \infty$ gives $\lim_{k\to\infty}P\{\tau_k \le T\} = 0$ and hence $P\{\tau_\infty \le T\} = 0$. Since $T > 0$ is arbitrary, we must have $P\{\tau_\infty < \infty\} = 0$, so $P\{\tau_\infty = \infty\} = 1$ as required. □

Corollary 2.2. *Assume that there is a positive number θ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{4\theta}c^2(i) + \theta\right) \le 0. \tag{2.23}$$

Then the conclusions of Theorem 2.1 hold.

The following theorem is easy to verify in applications, which will be used in the sections below.

Theorem 2.3. *Assume that*

$$-\hat{b} + \bar{c} \leq 0. \tag{2.24}$$

Then for any given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$*, there is a unique solution* $x(t)$ *to* (1.4) *on* $t \geq -\tau$ *and the solution will remain in* R_+ *with probability 1, namely,* $x(t) \in R_+$ *for all* $t \geq -\tau$ *a.s.*

Proof. The proof of this theorem is the same as that of the theorem above. Let

$$V(x) = x - 1 - \log x \quad \text{on } x > 0, \tag{2.25}$$

then we have (2.9) and (2.10). By (2.24), we get

$$\begin{aligned} LV(x, y, i) &\leq -a(i) + \frac{1}{2}\sigma^2(i) + (a(i) + b(i))x - c(i)y - b(i)x^2 + c(i)xy \\ &\leq K(1 + x + y) + (-b(i) + \bar{c})x^2 - \frac{1}{2}\bar{c}x^2 + \frac{1}{2}\bar{c}y^2 \\ &\leq K(1 + x + y) - \frac{1}{2}\bar{c}x^2 + \frac{1}{2}\bar{c}y^2, \end{aligned} \tag{2.26}$$

where K is a positive constant. The rest of the proof is similar to that of Theorem 2.1 and omitted. □

Note that condition (2.5) is used to derive (2.13) from (2.10). In fact, there are several different ways to estimate (2.10), which will lead to different alternative conditions for the positive global solution. For example, we know

$$\begin{aligned} c(i)xy &\leq \frac{1}{2\theta(i)}c^2(i)x^2 + \frac{\theta(i)}{2}y^2, \\ -b(i)x^2 + c(i)xy &\leq -b(i)x^2 + \frac{1}{2\theta(i)}c^2(i)x^2 + \frac{\theta(i)}{2}y^2 \\ &= \left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right)x^2 - \frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2. \end{aligned} \tag{2.27}$$

Therefore, if we assume that

$$\max_{i \in S}\left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right) \leq 0, \tag{2.28}$$

then

$$-b(i)x^2 + c(i)xy \leq -\frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2, \tag{2.29}$$

hence

$$LV(x,y,i) \leq K_1(1 + x + y) - \frac{\check{\theta}}{2}x^2 + \frac{\check{\theta}}{2}y^2, \tag{2.30}$$

from which we can show in the same way as in the proof of Theorem 2.1 that the solution of (1.4) is positive and global. In other words, the arguments above can give an alternative result which we describe as a theorem as below.

Theorem 2.4. *Assume that there are positive numbers $\theta(i)$ $(i = 1, 2, \ldots, n)$ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{2\theta(i)}c^2(i) + \frac{\check{\theta}}{2}\right) \leq 0. \tag{2.31}$$

Then for any given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, there is a unique solution $x(t)$ to (1.4) on $t \geq -\tau$ and the solution will remain in R_+ with probability 1, namely, $x(t) \in R_+$ for all $t \geq -\tau$ a.s.

Similarly, we can establish a corollary as follows.

Corollary 2.5. *Assume that there is a positive number θ such that*

$$\max_{i\in S}\left(-b(i) + \frac{1}{2\theta}c^2(i) + \frac{\theta}{2}\right) \leq 0. \tag{2.32}$$

Then the conclusions of Theorem 2.4 hold.

3. Asymptotic Bounded Properties

For convenience and simplicity in the following discussion, we list the following assumptions.

(A1) For each $i \in S, b(i) > 0$, and $-\hat{b} + \bar{c} \leq 0$.

(A1′) For each $i \in S, b(i) > 0$, and $-\hat{b} + \bar{c} < 0$.

(A1″) For each $i \in S, b(i) > 0, c(i) \geq 0$ and $-\hat{b} + \check{c} < 0$.

(A2) For some $u \in S, \gamma_{iu} > 0$ $(\forall i \neq u)$.

(A3) $\sum_{i=1}^{n} \pi_i[a(i) - (1/2)\sigma^2(i)] > 0$.

(A3′) $\sum_{i=1}^{n} \pi_i[a(i) - (1/2)\sigma^2(i)] < 0$.

(A4) For each $i \in S, a(i) - (1/2)\sigma^2(i) > 0$.

(A4′) For each $i \in S, a(i) - (1/2)\sigma^2(i) < 0$.

Definition 3.1. Equation (1.4) is said to be stochastically permanent if for any $\varepsilon \in (0,1)$, there exist positive constants $H = H(\varepsilon)$, $\delta = \delta(\varepsilon)$ such that

$$\liminf_{t\to+\infty} P\{x(t) \le H\} \ge 1-\varepsilon, \qquad \liminf_{t\to+\infty} P\{x(t) \ge \delta\} \ge 1-\varepsilon, \tag{3.1}$$

where $x(t)$ is the solution of (1.4) with any positive initial value.

Definition 3.2. The solutions of (1.4) are called stochastically ultimately bounded, if for any $\varepsilon \in (0,1)$, there exists a positive constant $H = H(\varepsilon)$, such that the solutions of (1.4) with any positive initial value have the property that

$$\limsup_{t\to+\infty} P\{x(t) > H\} < \varepsilon. \tag{3.2}$$

It is obvious that if a stochastic equation is stochastically permanent, its solutions must be stochastically ultimately bounded. So we will begin with the following theorem and make use of it to obtain the stochastically ultimate boundedness of (1.4).

Theorem 3.3. *Let $(A1')$ hold and p is an arbitrary given positive constant. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau,0]; R_+)$, the solution $x(t)$ of (1.4) has the properties that*

$$\limsup_{t\to\infty} E(x^p(t)) \le K_1(p), \tag{3.3}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E\left(x^{p+1}(s)\right) \le K_2(p), \tag{3.4}$$

where both $K_1(p)$ and $K_2(p)$ are positive constants defined in the proof.

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \ge -\tau$ with probability 1. Let

$$-\lambda = \left(1 + \frac{\overline{c}}{p+1}\right)^{-1}\left(-\widehat{b} + \overline{c}\right), \tag{3.5}$$

$$\gamma = \tau^{-1}\log(1+\lambda). \tag{3.6}$$

Define the function $V : R_+ \times R_+ \rightarrow R_+$ by

$$V(x,t) = e^{\gamma t}x^p. \tag{3.7}$$

By the generalized Itô formula, we have

$$dV(x(t),t) = LV(x(t), x(t-\tau), t, r(t))dt + pe^{\gamma t}\sigma(r(t))x^p\, dw(t), \tag{3.8}$$

where $LV : R_+ \times R_+ \times R_+ \times S \to R$ is defined by

$$LV(x,y,t,i) = e^{\gamma t}\left(\gamma + pa(i) + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - pe^{\gamma t}b(i)x^{p+1} + pe^{\gamma t}c(i)x^p y. \tag{3.9}$$

By (3.5) and Young's inequality, we obtain that

$$\begin{aligned}
&LV(x,y,t,i)\\
&\quad\le e^{\gamma t}\left(\gamma + pa(i) + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - pe^{\gamma t}b(i)x^{p+1} + pe^{\gamma t}\bar{c}\left(\frac{p}{p+1}x^{p+1} + \frac{1}{p+1}y^{p+1}\right)\\
&\quad\le e^{\gamma t}\left[\left(\gamma + p\breve{a} + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - \lambda p x^{p+1}\right] + e^{\gamma t}\frac{p\bar{c}}{p+1}\left[-(1+\lambda)x^{p+1} + y^{p+1}\right]\\
&\quad\le H_1 e^{\gamma t} + e^{\gamma t}\frac{p\bar{c}}{p+1}\left(-e^{\gamma\tau}x^{p+1} + y^{p+1}\right),
\end{aligned} \tag{3.10}$$

where $H_1 = \max_{i\in S}\{\sup_{x\in R_+}[(\gamma + p\breve{a} + (1/2)p(p-1)\sigma^2(i))x^p - \lambda p x^{p+1}]\}$. Moreover,

$$\int_0^t e^{\gamma s}x^{p+1}(s-\tau)ds \le e^{\gamma\tau}\int_{-\tau}^0 x^{p+1}(s)ds + e^{\gamma\tau}\int_0^t e^{\gamma s}x^{p+1}(s)ds. \tag{3.11}$$

By (3.10) and (3.11), one has

$$e^{\gamma t}E(x^p(t)) \le x^p(0) + \frac{H_1}{\gamma}(e^{\gamma t} - 1) + \frac{p\bar{c}}{p+1}e^{\gamma\tau}\int_{-\tau}^0 x^{p+1}(s)ds, \tag{3.12}$$

which yields

$$\limsup_{t\to\infty} E(x^p(t)) \le K_1(p), \tag{3.13}$$

where

$$K_1(p) = \max_{i\in S}\left\{\sup_{x\in R_+}\gamma^{-1}\left[\left(\gamma + p\breve{a} + \frac{1}{2}p(p-1)\sigma^2(i)\right)x^p - \lambda p x^{p+1}\right]\right\}. \tag{3.14}$$

By the generalized Itô formula, Young's inequality and (3.5) again, it follows

$$\begin{aligned}
0 &\le E(x^p(t)) \\
&\le x_0^p + E\int_0^t px^p(s)\left[\frac{1}{2}(p-1)\sigma^2(r(s)) + a(r(s))\right] - pb(r(s))x^{p+1}(s) + p\bar{c}x^p(s)x(s-\tau)ds \\
&\le x_0^p + E\int_0^t \left\{\frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p(s)\right. \\
&\qquad \left. +p\left[-\left(1+\frac{\bar{c}}{p+1}\right)\lambda\right]x^{p+1}(s) + \frac{p\bar{c}}{p+1}\left(-x^{p+1}(s) + x^{p+1}(s-\tau)\right)\right\}ds \\
&\le H_2 + E\int_0^t \left\{\frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p + p\left[-\left(1+\frac{\bar{c}}{p+1}\right)\lambda\right]x^{p+1}(s)\right\}ds,
\end{aligned} \tag{3.15}$$

where $H_2 = x^p(0) + (p\bar{c}/(p+1))\int_{-\tau}^0 x^{p+1}(s)ds$. This implies

$$p\lambda E\int_0^t x^{p+1}(s)ds \le H_2 + E\int_0^t \frac{1}{2}p(p-1)\sigma^2(r(s))x^p(s) + pa(r(s))x^p(s) - \frac{\lambda p\bar{c}}{p+1}x^{p+1}(s)ds. \tag{3.16}$$

The inequality above implies

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t x^{p+1}(s) \le \frac{H_3}{p\lambda}, \tag{3.17}$$

where $H_3 = \max_{i\in S}\{\sup_{x\in R_+}[(1/2)p(p-1)\sigma^2(i)x^p + pa(i)x^p - (\lambda p\bar{c}/(p+1))x^{p+1}]\}$ and the desired assertion (3.4) follows by setting $K_2(p) = H_3/p\lambda$. □

Remark 3.4. From (3.3) of Theorem 3.3, there is a $T > 0$ such that

$$E(x^p(t)) \le 2K_1(p) \quad \forall\, t \ge T. \tag{3.18}$$

Since $E(x^p(t))$ is continuous, there is a $\overline{K}_1(p, x_0)$ such that

$$E(x^p(t)) \le \overline{K}_1(p, x_0) \quad \text{for } t \in [0, T]. \tag{3.19}$$

Taking $L(p, x_0) = \max(2K_1(p), \overline{K}_1(p, x_0))$, we havefor the fundamental theory of

$$E(x^p(t)) \le L(p, x_0) \quad \forall\, t \in [0, \infty). \tag{3.20}$$

This means that the pth moment of any positive solution of (1.4) is bounded.

Remark 3.5. Equation (3.4) of Theorem 3.3 shows that the average in time of the pth ($p > 1$) moment of solutions of (1.4) is bounded.

Theorem 3.6. *Solutions of* (1.4) *are stochastically ultimately bounded under* ($A1'$).

Proof. This can be easily verified by Chebyshev's inequality and Theorem 3.3. □

Based on the results above, we will prove the other inequality in the definition of stochastic permanence. For convenience, define

$$\beta(i) = a(i) - \frac{1}{2}\sigma^2(i). \tag{3.21}$$

Under (A3), it has $\sum_{i=1}^{n} \pi_i \beta(i) > 0$. Moreover, let G be a vector or matrix. By $G \gg 0$ we mean all elements of G are positive. We also adopt here the traditional notation by letting

$$Z^{n\times n} = \left\{ A = \left(a_{ij}\right)_{n\times n} : a_{ij} \le 0,\ i \ne j \right\}. \tag{3.22}$$

We will also need some useful results.

Lemma 3.7 (see [24]). *If $A \in Z^{n\times n}$, then the following statements are equivalent.*

(1) *A is a nonsingular M-matrix (see [24] for definition of M-matrix).*

(2) *All of the principal minors of A are positive; that is,*

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \ldots, n. \tag{3.23}$$

(3) *A is semipositive, that is, there exists $x \gg 0$ in R^n such that $Ax \gg 0$.*

Lemma 3.8 (see [18]). *(i) Assumptions* ($A2$) *and* ($A3$) *imply that there exists a constant $\theta > 0$ such that the matrix*

$$A(\theta) = \operatorname{diag}(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta)) - \Gamma \tag{3.24}$$

is a nonsingular M-matrix, where $\xi_i(\theta) = \theta\beta(i) - (1/2)\theta^2\sigma^2(i),\ \forall i \in S$.

(ii) Assumption (A4) implies that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix.

Lemma 3.9. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix and $c(i) \ge 0$ $(i = 1, 2, \ldots, n)$, then the global positive solution $x(t)$ of* (1.4) *has the property that*

$$\limsup_{t\to\infty} E\left(|x(t)|^{-\theta}\right) \le H, \tag{3.25}$$

where H is a fixed positive constant (defined by (3.35) *in the proof).*

Proof. Let $U(t) = x^{-1}(t)$ on $t \geq 0$. Applying the generalized Itô formula, we have

$$dU(t) = U(t)\Big(-a(r(t)) + \sigma^2(r(t)) + b(r(t))x(t) - c(r(t))x(t-\tau)\Big)dt - \sigma U(t)dw(t). \quad (3.26)$$

By Lemma 3.7, for the given θ, there is a vector $\vec{q} = (q_1, q_2, \ldots, q_n)^T \gg 0$ such that

$$\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T = A(\theta)\vec{q} \gg 0, \quad (3.27)$$

namely,

$$q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j > 0 \quad \forall\, 1 \leq i \leq n. \quad (3.28)$$

Define the function $V : R_+ \times S \to R_+$ by $V(U,i) = q_i(1+U)^\theta$. Applying the generalized Itô formula again, we have

$$EV(U(t), r(t)) = V(U(0), r(0)) + E\int_0^t LV(U(s), x(s-\tau), r(s))ds, \quad (3.29)$$

where $LV : R_+ \times R_+ \times S \to R$ is defined by

$$\begin{aligned} LV(U,y,i) = (1+U)^{\theta-2}\Bigg\{ & -U^2\left[q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j\right] \\ & + U\left[q_i\theta\Big(b(i) - a(i) + \sigma^2(i)\Big) + 2\sum_{j=1}^{n}\gamma_{ij}q_j\right] \\ & + \left[q_i\theta b(i) + \sum_{j=1}^{n}\gamma_{ij}q_j - q_i\theta c(i)(1+U)Uy\right]\Bigg\}. \end{aligned} \quad (3.30)$$

Now, choose a constant $\kappa > 0$ sufficiently small such that

$$\vec{\lambda} - \kappa\vec{q} \gg 0, \quad (3.31)$$

that is,

$$q_i\left(\theta\beta(i) - \frac{1}{2}\theta^2\sigma^2(i)\right) - \sum_{j=1}^{n}\gamma_{ij}q_j - \kappa q_i > 0 \quad \forall\, 1 \leq i \leq n. \quad (3.32)$$

Then, by the generalized Itô formula again,

$$E\left[e^{\kappa t}V(U(t),r(t))\right] = V(U(0),r(0)) + E\int_0^t \left[\kappa e^{\kappa t}V(U(s),r(s)) + e^{\kappa t}LV(U(s),x(s-\tau),r(s))\right]ds. \tag{3.33}$$

It is computed that

$$\begin{aligned}
&\kappa e^{\kappa t}V(U,i) + e^{\kappa t}LV(U,y,i) \\
&\quad\le e^{\kappa t}(1+U)^{\theta-2}\Bigg\{-U^2\left[q_i\left(\theta\beta(i)-\frac{1}{2}\theta^2\sigma^2(i)\right)-\sum_{j=1}^n \gamma_{ij}q_j - \kappa q_i\right] \\
&\qquad + U\left[q_i\theta\left(b(i)-a(i)+\sigma^2(i)\right)+2\sum_{j=1}^n\gamma_{ij}q_j + 2\kappa q_i\right] \\
&\qquad + q_i\theta b(i) + \sum_{j=1}^n\gamma_{ij}q_j + \kappa q_i\Bigg\} \\
&\quad\le \hat{q}\kappa H e^{\kappa t},
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
H = \frac{1}{\hat{q}\kappa}\max_{i\in S}\Bigg\{\sup_{U\in R_+}(1+U)^{\theta-2}\Bigg\{&-U^2\left[q_i\left(\theta\beta(i)-\frac{1}{2}\theta^2\sigma^2(i)\right)-\sum_{j=1}^n\gamma_{ij}q_j-\kappa q_i\right] \\
&+U\left[q_i\theta\left(b(i)-a(i)+\sigma^2(i)\right)+2\sum_{j=1}^n\gamma_{ij}q_j+2\kappa q_i\right] \\
&+q_i\theta b(i)+\sum_{j=1}^n\gamma_{ij}q_j+\kappa q_i\Bigg\},1\Bigg\},
\end{aligned} \tag{3.35}$$

which implies

$$\hat{q}E\left[e^{\kappa t}(1+U(t))^{\theta}\right] \le \check{q}\left(1+x^{-1}(0)\right)^{\theta} + \hat{q}He^{\kappa t}. \tag{3.36}$$

Then

$$\limsup_{t\to\infty} E\left(U^{\theta}(t)\right) \le \limsup_{t\to\infty} E\left[(1+U(t))^{\theta}\right] \le H. \tag{3.37}$$

Recalling the definition of $U(t)$, we obtain the required assertion. □

Theorem 3.10. *Under* $(A1'')$, $(A2)$, *and* $(A3)$, (1.4) *is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality, Lemmas 3.8 and 3.9, and Theorem 3.6. Similarly, it is easy to obtain the following result.

Theorem 3.11. *Under* $(A1'')$ *and* $(A4)$, (1.4) *is stochastically permanent.*

Remark 3.12. It is well-known that if $a > 0$, $b > 0$ and $0 \le c < b$, then the solution $x(t)$ of (1.1) is persistent, namely,

$$\liminf_{t\to\infty} x(t) > 0. \tag{3.38}$$

Furthermore, we consider its associated stochastic delay equation (1.4), that is,

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t-\tau))dt + \sigma(r(t))dw(t)], \tag{3.39}$$

where $a(i) > 0$, $b(i) > 0$, $c(i) \ge 0$, for $i \in S$, and $\check{c} < \widehat{b}$. Thus, applying Theorem 3.10 or Theorem 3.11, we can see that (1.4) is stochastically permanent, if the noise intensities are sufficiently small in the sense that

$$\sum_{i=1}^{n}\pi_i\left[a(i) - \frac{1}{2}\sigma^2(i)\right] > 0 \quad \text{or} \quad a(i) - \frac{1}{2}\sigma^2(i) > 0, \quad \text{for each } i \in S. \tag{3.40}$$

Corollary 3.13. *Assume that for some* $i \in S$, $b(i) > 0$, $-b(i) + |c(i)| \le 0$, *and* $a(i) - (1/2)\sigma^2(i) > 0$. *Then the subsystem*

$$dx(t) = x(t)[(a(i) - b(i)x(t) + c(i)x(t-\tau))dt + \sigma(i)dw(t)] \tag{3.41}$$

is stochastically permanent.

4. Extinction

In the previous sections we have shown that under certain conditions, the original (1.1) and the associated SDE (1.4) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to (1.4) will become extinct with probability 1.

Theorem 4.1. *Assume that* $(A1)$ *holds. Then for any given initial data* $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \le \sum_{i=1}^{n}\pi_i\left[a(i) - \frac{1}{2}\sigma^2(i)\right] \quad a.s. \tag{4.1}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \geq -\tau$ with probability 1. We have by the generalized Itô formula and (A1) that

$$d \log x(t) \leq \left(a(r(t)) - \frac{1}{2}\sigma^2(r(t)) - \widehat{b}x(t) + \widehat{b}x(t-\tau) \right) dt + \sigma(r(t)) dw(t), \tag{4.2}$$

where (A1) is used in the last step. Then,

$$\log(V(x(t))) \leq \log(V(x(0))) + \widehat{b} \int_{-\tau}^{0} x(s) ds + \int_{0}^{t} \left(a(r(s)) - \frac{1}{2}\sigma^2(r(s)) \right) ds + M(t), \tag{4.3}$$

where $M(t) = \int_0^t \sigma(r(t)) dw(t)$. The quadratic variation of $M(t)$ is given by

$$\langle M, M \rangle_t = \int_0^t \sigma^2(r(s)) ds \leq \overline{\sigma}^2 t. \tag{4.4}$$

Therefore, applying the strong law of large numbers for martingales [24], we obtain

$$\lim_{t \to \infty} \frac{M(t)}{t} = 0 \quad \text{a.s.} \tag{4.5}$$

It finally follows from (4.3) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t \to \infty} \frac{\log x(t)}{t} \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left[a(r(s)) - \frac{1}{2}\sigma^2(r(s)) \right] ds = \sum_{i=1}^{n} \pi_i \left[a(i) - \frac{1}{2}\sigma^2(i) \right] \quad \text{a.s.}, \tag{4.6}$$

which is the required assertion (4.1). □

Similarly, it is easy to prove the following conclusions.

Theorem 4.2. *Assume that* (A1) *and* (A3′) *hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t \to \infty} \frac{\log x(t)}{t} < 0 \quad a.s. \tag{4.7}$$

That is, the population will become extinct exponentially with probability 1.

Theorem 4.3. *Assume that (Al) and (A4′) hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *has the property that*

$$\limsup_{t \to \infty} \frac{\log x(t)}{t} \leq -\frac{\varphi}{2} \quad a.s., \tag{4.8}$$

where $\varphi = \min_{i \in S}(\sigma^2(i) - 2a(i)) > 0$. *That is, the population will become extinct exponentially with probability 1.*

Remark 4.4. If the noise intensities are sufficiently large in the sense that

$$\sum_{i=1}^{n}\pi_i\left[a(i)-\frac{1}{2}\sigma^2(i)\right]<0 \quad \text{or} \quad a(i)-\frac{1}{2}\sigma^2(i)<0, \quad \text{for each } i\in S, \tag{4.9}$$

then the population $x(t)$ represented by (1.4) will become extinct exponentially with probability 1. However, the original delay equation (1.1) may be persistent without environmental noise.

Remark 4.5. Let A1$''$ and A2 hold, $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]\neq 0$. Then, SDE (1.4) is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]>0$, while it is extinctive if and only if $\sum_{i=1}^{n}\pi_i[a(i)-(1/2)\sigma^2(i)]<0$.

Corollary 4.6. *Assume that for some $i\in S$,*

$$-b(i)+|c(i)|\le 0, \qquad a(i)-\frac{1}{2}\sigma^2(i)<0. \tag{4.10}$$

Then for any given initial data $\{x(t):-\tau\le t\le 0\}\in C([-\tau,0];R_+)$, the solution $x(t)$ of subsystem

$$dx(t)=x(t)[(a(i)-b(i)x(t)+c(i)x(t-\tau))dt+\sigma(i)dw(t)] \tag{4.11}$$

tend to zero a.s.

5. Asymptotic Properties

Lemma 5.1. *Assume that $(A1')$ holds. Then for any given initial data $\{x(t):-\tau\le t\le 0\}\in C([-\tau,0];R_+)$, the solution $x(t)$ of (1.4) has the property*

$$\limsup_{t\to\infty}\frac{\log(x(t))}{\log t}\le 1 \quad a.s. \tag{5.1}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t\ge-\tau$ with probability 1. It is known that

$$\begin{aligned}
dx(t)&\le(\check{a}x(t)+\bar{c}x(t)x(t-\tau))dt+\sigma(r(t))dw(t),\\
E\left(\sup_{t\le u\le t+1}x(u)\right)&\le E(x(t))+\check{a}\int_t^{t+1}E(x(s))ds+\bar{c}\int_t^{t+1}E(x(s)x(s-\tau))ds\\
&\quad+E\left(\sup_{t\le u\le t+1}\int_t^u\sigma(r(s))x(s)dw(s)\right).
\end{aligned} \tag{5.2}$$

From (3.3) of Theorem 3.3, it has

$$\limsup_{t\to\infty} E(x(t)) \le K_1(1),$$
$$\limsup_{t\to\infty} E\left(x^2(t)\right) \le K_1(2). \tag{5.3}$$

By the well-known BDG's inequality [24] and the Hölder's inequality, we obtain

$$\begin{aligned}
E\left(\sup_{t\le u\le t+1}\int_t^u \sigma(r(s))x(s)dB(s)\right) &\le 3E\left[\int_t^{t+1}(\sigma(r(s))x(s))^2 ds\right]^{1/2} \\
&\le E\left(\sup_{t\le u\le t+1} x(u)\cdot 9\check{\sigma}\int_t^{t+1} x(s)ds\right)^{1/2} \\
&\le \frac{1}{2}E\left(\sup_{t\le u\le t+1} x(u)\right) + 9\check{\sigma}^2\int_t^{t+1} E(x(s))ds.
\end{aligned} \tag{5.4}$$

Note that

$$\int_t^{t+1} E[x(s)x(s-\tau)]ds \le \frac{1}{2}\int_t^{t+1} E\left(x^2(s)\right)ds + \frac{1}{2}\int_t^{t+1} E\left(x^2(s-\tau)\right)ds. \tag{5.5}$$

Therefore,

$$\begin{aligned}
E\left(\sup_{t\le u\le t+1} x(u)\right) \le{}& 2E(x(t)) + 2\check{a}\int_t^{t+1} E(x(s))ds + \overline{c}\int_t^{t+1} E\left(x^2(s)\right)ds \\
&+ \overline{c}\int_t^{t+1} E\left(x^2(s-\tau)\right)ds + 18\check{\sigma}^2\int_t^{t+1} E(x(s))ds.
\end{aligned} \tag{5.6}$$

This, together with (5.3), yields

$$\limsup_{t\to\infty} E\left(\sup_{t\le u\le t+1} x(u)\right) \le 2\left(1+\check{a}+18\check{\sigma}^2\right)K_1(1) + 2\overline{c}K_1(2). \tag{5.7}$$

From (5.7), there exists a positive constant M such that

$$E\left(\sup_{k\le t\le k+1} x(t)\right) \le M, \quad k = 1, 2, \ldots. \tag{5.8}$$

Let $\varepsilon > 0$ be arbitrary. Then, by Chebyshev's inequality,

$$P\left(\sup_{k\le t\le k+1} x(u) > k^{1+\varepsilon}\right) \le \frac{M}{k^{1+\varepsilon}}, \quad k = 1,2,\ldots. \tag{5.9}$$

Applying the well-known Borel-Cantelli lemma [24], we obtain that for almost all $\omega \in \Omega$

$$\sup_{k\le t\le k+1} x(u) \le k^{1+\varepsilon}, \tag{5.10}$$

for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (5.10) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \ge k_0$ and $k \le t \le k+1$, then

$$\frac{\log(x(t))}{\log t} \le \frac{(1+\varepsilon)\log k}{\log k} = 1+\varepsilon. \tag{5.11}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\log(x(t))}{\log t} \le 1+\varepsilon \quad \text{a.s.} \tag{5.12}$$

Letting $\varepsilon \to 0$, we obtain the desired assertion (5.1). □

Lemma 5.2. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix and $c(i) \ge 0$ $(i = 1,2,\ldots,n)$, then the global positive solution $x(t)$ of SDE* (1.4) *has the property that*

$$\liminf_{t\to\infty} \frac{\log(x(t))}{\log t} \ge -\frac{1}{\theta} \quad a.s. \tag{5.13}$$

Proof. Applying the generalized Itô formula, for the fixed constant $\theta > 0$, we derive from (3.26) that

$$\begin{aligned} d\left[(1+U(t))^{\theta}\right] \le{}& \theta(1+U(t))^{\theta-2}\left[-U^2(t)\left(\widehat{\beta}-\frac{1}{2}\check{\sigma}^2\right)+U(t)\left(\check{b}+\check{\sigma}^2\right)+\check{b}\right]dt \\ &- \theta\sigma(r(t))(1+U(t))^{\theta-1}U(t)dw(t), \end{aligned} \tag{5.14}$$

where $U(t) = 1/x(t)$ on $t > 0$. By (3.37), there exists a positive constant M such that

$$E\left[(1+U(t))^{\theta}\right] \le M \quad \text{on } t \ge 0. \tag{5.15}$$

Let $\delta > 0$ be sufficiently small for

$$\theta\left\{\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^2\right]\delta+3\check{a}\delta^{1/2}\right\} < \frac{1}{2}. \tag{5.16}$$

Then (5.14) implies that

$$\begin{aligned}
&E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\\
&\quad\le E\left[(1+U((k-1)\delta))^{\theta}\right]\\
&\qquad+E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\left[-U^2(s)\left(\widehat{\beta}-\frac{1}{2}\theta\check{\sigma}^2\right)+U(s)\left(\check{b}+\check{\sigma}^2\right)+\check{b}\right]ds\right|\right\}\\
&\qquad+E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta\sigma(r(s))(1+U(s))^{\theta-1}U(s)dw(s)\right|\right\}.
\end{aligned}\tag{5.17}$$

By directly computing, we have

$$\begin{aligned}
&E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}\theta(1+U(s))^{\theta-2}\left[-U^2(s)\left(\widehat{\beta}-\frac{1}{2}\theta\check{\sigma}^2\right)+U(s)\left(\check{b}+\check{\sigma}^2\right)+\check{b}\right]ds\right|\right\}\\
&\quad\le\theta E\left\{\int_{(k-1)\delta}^{t}\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^2\right](1+U(s))^{\theta}ds\right\}\\
&\quad\le\theta\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^2\right]\delta E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right].
\end{aligned}\tag{5.18}$$

By the BDG's inequality, it follows

$$E\left\{\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{k\delta}\theta\sigma(r(s))(1+U(s))^{\theta-1}U(s)dw(s)\right|\right\}\le 3\theta\check{\sigma}\delta^{1/2}E\left\{\sup_{(k-1)\delta\le t\le k\delta}(1+U(s))^{\theta}\right\}.\tag{5.19}$$

Substituting this and (5.18) into (5.17) gives

$$\begin{aligned}
&E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\\
&\quad\le E\left[(1+U((k-1)\delta))^{\theta}\right]+\theta\left\{\left[\widehat{\beta}+2\check{b}+\frac{1}{2}(\theta+2)\check{\sigma}^2\right]\delta+3\check{\sigma}\delta^{1/2}\right\}E\left\{\sup_{(k-1)\delta\le t\le k\delta}(1+U(s))^{\theta}\right\}.
\end{aligned}\tag{5.20}$$

Making use of (5.15) and (5.16), we obtain

$$E\left[\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\right]\le 2M \quad \text{on } t\ge 0. \tag{5.21}$$

Let $\varepsilon>0$ be arbitrary. Then, we have by Chebyshev's inequality that

$$P\left\{\omega:\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}>(k\delta)^{1+\varepsilon}\right\}\le\frac{2M}{(k\delta)^{1+\varepsilon}},\quad k=1,2,\ldots. \tag{5.22}$$

Applying the Borel-Cantelli lemma, we obtain that for almost all $\omega\in\Omega$,

$$\sup_{(k-1)\delta\le t\le k\delta}(1+U(t))^{\theta}\le(k\delta)^{1+\varepsilon} \tag{5.23}$$

holds for all but finitely many k. Hence, there exists an integer $k_0(\omega)>1/\delta+2$, for almost all $\omega\in\Omega$, for which (5.23) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $(k-1)\delta\le t\le k\delta$,

$$\frac{\log(1+U(t))^{\theta}}{\log t}\le\frac{(1+\varepsilon)\log(k\delta)}{\log((k-1)\delta)}\le 1+\varepsilon. \tag{5.24}$$

Therefore,

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\le 1+\varepsilon \quad \text{a.s.} \tag{5.25}$$

Let $\varepsilon\to 0$, we obtain

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\le 1 \quad \text{a.s.} \tag{5.26}$$

Recalling the definition of $U(t)$, this yields

$$\limsup_{t\to\infty}\frac{\log(1/x^{\theta}(t))}{\log t}\le 1 \quad \text{a.s.,} \tag{5.27}$$

which further implies

$$\liminf_{t\to\infty}\frac{\log(x(t))}{\log t}\ge-\frac{1}{\theta} \quad \text{a.s.} \tag{5.28}$$

This is our required assertion (5.13). □

Theorem 5.3. *Assume that* $(A1'')$, $(A2)$, *and* $(A3)$ *hold. Then for any given initial data* $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{1}{\hat{b}-\bar{c}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s., \tag{5.29}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \ge \frac{1}{\check{b}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s. \tag{5.30}$$

Proof. By Theorem 2.3, the solution $x(t)$ will remain in R_+ for all $t \ge -\tau$ with probability 1. From Lemmas 3.8, 5.1, and 5.2, it follows

$$\lim_{t\to\infty} \frac{\log(x(t))}{t} = 0 \quad \text{a.s.} \tag{5.31}$$

By generalized Itô formula, one has

$$\begin{aligned}\log x(t) = \log x_0 &+ \int_0^t\left(a(r(s))-\frac{1}{2}\sigma^2(r(s))\right)ds - \int_0^t b(r(s))x(s)ds \\ &+ \int_0^t c(r(s))x(s-\tau)ds + \int_0^t \sigma(r(s))dw(s).\end{aligned} \tag{5.32}$$

Dividing by t on both sides, then we have

$$\begin{aligned}\frac{\log x(t)}{t} \le \frac{\log x_0}{t} &+ \frac{1}{t}\int_0^t\left(a(r(s))-\frac{1}{2}\sigma^2(r(s))\right)ds + \left(-\hat{b}+\bar{c}\right)\frac{1}{t}\int_0^t x(s)ds \\ &+ \frac{\bar{c}}{t}\int_{-\tau}^0 x(s)ds + \frac{1}{t}\int_0^t \sigma(r(s))dw(s).\end{aligned} \tag{5.33}$$

Let $t \to \infty$, by the strong law of large numbers for martingales and (5.31), we therefore have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{1}{\hat{b}-\bar{c}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad \text{a.s.,} \tag{5.34}$$

which is the required assertions (5.29). And we also have

$$\frac{\log x(t)}{t} \ge \frac{1}{t}\log x_0 + \frac{1}{t}\int_0^t\left(a(r(s))-\frac{1}{2}\sigma^2(r(s))\right)ds - \frac{\check{b}}{t}\int_0^t x(s)ds + \frac{1}{t}\int_0^t \sigma(r(s))dw(s). \tag{5.35}$$

Let $t \to \infty$, by the strong law of large numbers for martingales and (5.21), we therefore have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{1}{\breve{b}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad \text{a.s.}, \tag{5.36}$$

which is the required assertions (5.30). □

Similarly, by using Lemmas 3.8, 5.1, and 5.2, it is easy to show the following conclusion.

Theorem 5.4. *Assume that* (A1″) *and* (A4) *hold. Then for any given initial data* $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+)$, *the solution* $x(t)$ *of* (1.4) *obeys*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \frac{1}{\hat{b}-\overline{c}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s., \\
&\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{1}{\breve{b}}\sum_{i=1}^{n}\pi_i\left(a(i)-\frac{1}{2}\sigma^2(i)\right) \quad a.s.
\end{aligned} \tag{5.37}$$

Corollary 5.5. *If that for some* $i \in S$,

$$b(i) > 0, \qquad b(i) > |c(i)|, \qquad a(i) - \frac{1}{2}\sigma^2(i) > 0, \tag{5.38}$$

then the solution with positive initial value to subsystem

$$dx(t) = x(t)[(a(i) - b(i)x(t) + c(i)x(t-\tau))dt + \sigma(i)dw(t)] \tag{5.39}$$

has the property that

$$\frac{a(i)-(1/2)\sigma^2(i)}{b(i)} \leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \frac{a(i)-(1/2)\sigma^2(i)}{b(i)-|c(i)|} \quad a.s. \tag{5.40}$$

Remark 5.6. If $\overline{c} = 0$, (1.4) will be written by

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t))dt + \sigma(r(t))dw(t)], \tag{5.41}$$

which is investigated in [18]. It should be pointed out that (1.4) is more difficult to handle than (5.41). Fortunately, it overcomes the difficulties caused by delay term with the help of Young's inequality. Meanwhile, we get the similar results for $\tau \geq 0$.

6. Examples

Example 6.1. Consider a 2-dimensional stochastic differential equation with Markovian switching of the form

$$dx(t) = x(t)[(a(r(t)) - b(r(t))x(t) + c(r(t))x(t-\tau))dt + \sigma(r(t))dw(t)] \quad \text{on } t \le 0, \tag{6.1}$$

where $r(t)$ is a right-continuous Markov chain taking values in $S = \{1,2\}$, and $r(t)$ and $w(t)$ are independent. Here

$$\begin{aligned} &a(1) = 2, \qquad b(1) = 3, \qquad c(1) = 1, \qquad \sigma(1) = 1, \\ &a(2) = 1, \qquad b(2) = 2, \qquad c(2) = \frac{3}{2}, \qquad \sigma(2) = 2. \end{aligned} \tag{6.2}$$

It can be computed that

$$\widehat{b} = 2; \qquad \check{c} = \frac{3}{2}; \qquad a(1) - \frac{1}{2}\sigma^2(1) = \frac{3}{2}; \qquad a(2) - \frac{1}{2}\sigma^2(2) = -1. \tag{6.3}$$

By Theorem 2.3, the solution $x(t)$ of (6.1) will remain in R_+ for all $t \ge -\tau$ with probability 1.

Case 1. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{6.4}$$

By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right). \tag{6.5}$$

Therefore,

$$\sum_{i=1}^{2} \pi_i \left(a(i) - \frac{1}{2}\sigma^2(i) \right) = \frac{2}{3} > 0. \tag{6.6}$$

By Theorems 3.10 and 5.3, (6.1) is stochastically permanent and its solution $x(t)$ with any positive initial value has the following properties:

$$\frac{1}{3} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le \frac{4}{3} \quad \text{a.s.} \tag{6.7}$$

Case 2. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}. \tag{6.8}$$

By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{1}{3}, \frac{2}{3}\right). \tag{6.9}$$

So,

$$\sum_{i=1}^{2} \pi_i \left(a(i) - \frac{1}{2}\sigma^2(i) \right) = -\frac{1}{6} < 0. \tag{6.10}$$

Applying Theorems 4.2, (6.1) is extinctive.

Acknowledgments

The authors are grateful to Editor Professor Elena Braverman and anonymous referees for their helpful comments and suggestions which have improved the quality of this paper. This work is supported by the National Natural Science Foundation of China (no. 10771001), Research Fund for Doctor Station of The Ministry of Education of China (no. 20113401110001 and no. 20103401120002), TIAN YUAN Series of Natural Science Foundation of China (no. 11126177), Key Natural Science Foundation (no. KJ2009A49), and Talent Foundation (no. 05025104) of Anhui Province Education Department, 211 Project of Anhui University (no. KJJQ1101), Anhui Provincial Nature Science Foundation (no. 090416237, no. 1208085QA15), Foundation for Young Talents in College of Anhui Province (no. 2012SQRL021).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 210156, 10 pages
doi:10.1155/2012/210156

Research Article

Finite-Time Attractivity for Diagonally Dominant Systems with Off-Diagonal Delays

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Received 23 April 2012; Accepted 6 June 2012

Academic Editor: Agacik Zafer

We introduce a notion of attractivity for delay equations which are defined on bounded time intervals. Our main result shows that linear delay equations are finite-time attractive, provided that the delay is only in the coupling terms between different components, and the system is diagonally dominant. We apply this result to a nonlinear Lotka-Volterra system and show that the delay is harmless and does not destroy finite-time attractivity.

1. Introduction

Finite-time dynamical systems generated by nonautonomous differential equations which are defined only on a compact interval of time have recently become an active field of research, see, for example, [1–3] and the references therein.

A key ingredient of a qualitative theory is the notion of hyperbolicity of solutions. Finite-time versions of hyperbolicity are introduced and discussed, for example, in [4–8]. Finite-time attractivity is a special case of finite-time hyperbolicity, in case the unstable direction is missing. So far finite-time attractivity has been discussed only for ordinary differential equations, see [9, 10]. For a closely related notion, namely, finite-time stability, we refer to [11, 12] and the references therein for an overview. In this paper we go one step further to extend and investigate finite-time attractivity for delay equations.

For a nonnegative number $r \geq 0$, let $\mathcal{C} := C([-r,0],\mathbb{R}^d)$ denote the space of all continuous functions $\varphi : [-r,0] \to \mathbb{R}^d$. For $\gamma \in \mathbb{R}$, the norm $\|\cdot\|_{\gamma,\infty}$ on $\mathcal{C}$ is defined as follows:

$$\|\varphi\|_{\gamma,\infty} := \max\{e^{\gamma s}\|\varphi(s)\|_{\infty} : s \in [-r,0]\}, \tag{1.1}$$

where $\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, d\}$ for all $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$. Consider a finite-time delay differential equation

$$\dot{x} = f(t, x_t) \quad \text{for } t \in [0, T], \tag{1.2}$$

where $f : [0,T] \times \mathcal{C} \to \mathbb{R}^d$ is assumed to be continuous and Lipschitz in the second argument. For each $\varphi \in \mathcal{C}$, let $x(\cdot, \varphi)$ denote the solution of (2.1) satisfying the initial condition $x(s) = \varphi(s)$ for all $s \in [-r, 0]$. The evolution operator $S : [0,T] \times \mathcal{C} \to \mathcal{C}$ generated by (1.2) is defined as

$$(S(t)\varphi)(s) = x(t+s, \varphi) \quad \forall s \in [-r, 0], \; t \in [0, T]. \tag{1.3}$$

Motivated by recent results on finite-time hyperbolicity (see, e.g., [4, 6, 7]), we introduce in the following an analog notion of finite-time attractivity for delay equations.

Definition 1.1 (finite-time attractivity). The solution $S(\cdot, \varphi)$ is called *finite-time attractive* on $[0, T]$ with respect to the norm $\|\cdot\|_{\gamma,\infty}$ if there exist positive constants α and η such that for all $t, s \in [0, T]$ with $s \le t$ the following estimate holds:

$$\|S(t, \varphi) - S(t, \psi)\|_{\gamma,\infty} \le e^{-\alpha(t-s)} \|S(s, \varphi) - S(s, \psi)\|_{\gamma,\infty} \tag{1.4}$$

for all ψ in the neighborhood $B_\eta(\varphi)$ of φ.

Remark 1.2. In the case that $f : [0, T] \times \mathcal{C} \to \mathbb{R}^d$ is a linear function in the second argument, it is easy to see that the generated semigroup $S : [0, T] \times \mathcal{C} \to \mathcal{C}$ is also linear in the second argument. In particular, for linear systems the following statements are equivalent:

(i) there exists a finite-time attractive solution $S(\cdot, \varphi)$ for a $\varphi \in \mathcal{C}$,

(ii) for all $\varphi \in \mathcal{C}$, the solution $S(\cdot, \varphi)$ is finite-time attractive, and

(iii) there exists $\alpha > 0$ such that for all $0 \le s \le t \le T$ we have

$$\|S(t, \varphi)\|_{\gamma,\infty} \le e^{-\alpha(t-s)} \|S(s, \varphi)\|_{\gamma,\infty} \quad \forall \varphi \in \mathcal{C}. \tag{1.5}$$

In this paper, we prove the finite-time attractivity for linear off-diagonal delay systems in Section 2. Section 3 is devoted to show the finite-time attractivity of the equilibrium for a Lotka-Voltera system.

2. Finite-Time Attractivity for Linear Off-Diagonal Delay Systems

In this section, we consider the following finite-time nonautonomous linear differential equation with *off-diagonal delays* (see, e.g., [13] and the reference therein):

$$\dot{x}_i(t) = a_{ii}(t)x_i(t) + \sum_{j=1, j\neq i}^{d} a_{ij}(t)x_j(t - \tau_{ij}) \quad \text{for } i = 1, \ldots, d, \; t \in [0, T], \tag{2.1}$$

where T is a given positive constant, $a_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j = 1, \dots, d$, are continuous functions and $\tau_{ij} > 0$ for $i, j = 1, \dots, d$ with $i \neq j$. Define

$$r := \max\{\tau_{ij} : i, j = 1, \dots, d,\ i \neq j\}. \tag{2.2}$$

Note that (2.1) is a special case of (1.2). More precisely, the right hand side of (2.1) equals $f(t, x_t)$, where $f = (f_1, \dots, f_d) : [0, T] \times \mathcal{C} \to \mathbb{R}^d$ is defined as follows:

$$f_i(t, \varphi) := a_{ii}(t)\varphi(0) + \sum_{j=1, j\neq i}^{d} a_{ij}(t)\varphi_j(-\tau_{ij}). \tag{2.3}$$

Let $S : [0, T] \times \mathcal{C} \to \mathcal{C}$ denote the evolution operator of (2.1). From (2.3), we see that the function f is linear in the second argument. Therefore, the evolution operator S is also linear in the second argument. Our aim in this section is to provide a sufficient condition for the finite-time attractivity for the zero solution of (2.1) and thus for all solutions of (2.1), see Remark 1.2.

Before presenting the main result, we recall the notion of row diagonal dominance. We refer the reader to [14, Definition 7.10] for a discussion of this notion. System (2.1) is called *row diagonally dominant* if there exists a positive constant δ such that

$$|a_{ii}(t)| \geq \sum_{j=1, j\neq i}^{d} |a_{ij}(t)| + \delta \quad \text{for } t \in [0, T]. \tag{2.4}$$

Theorem 2.1 (finite-time attractivity for delay equations). *Consider system* (2.1) *on a finite-time interval* $[0, T]$. *Suppose that system* (2.1) *is row diagonally dominant with a positive constant* δ *and* $a_{ii}(t) < 0$ *for all* $i = 1, \dots, d$ *and* $t \in [0, T]$. *Define*

$$M := \max\left\{\sum_{j=1, j\neq i}^{d} |a_{ij}(t)| : i = 1, \dots, d,\ t \in [0, T]\right\}, \tag{2.5}$$

and let γ^* *be a positive number satisfying that*

$$\frac{\gamma^*}{2} + \left(e^{\gamma^* r} - 1\right)M = \delta. \tag{2.6}$$

Then for every $\gamma \in [0, \gamma^*]$, *the zero solution of* (2.1) *is finite-time attractive with respect to the norm* $\|\cdot\|_{\gamma,\infty}$ *with exponent* $-\gamma/2$, *that is,*

$$\|S(t, \varphi)\|_{\gamma,\infty} \leq e^{-(\gamma/2)(t-s)} \|S(s, \varphi)\|_{\gamma,\infty} \quad \text{for } 0 \leq s \leq t \leq T. \tag{2.7}$$

Proof. We divide the proof into two steps.

Step 1. We show that for $\varphi \in \mathcal{C}$ the inequality

$$\|x(t, \varphi)\|_{\infty} \leq e^{-(\gamma/2)(t-s)} \|S(s, \varphi)\|_{\gamma,\infty} \quad \forall 0 \leq s \leq t \leq T \tag{2.8}$$

holds. Suppose the opposite, that is, assume that there exists $s \in [0,T]$ such that the set

$$\mathcal{N} := \left\{ t \in [s,T) : \|x(t,\varphi)\|_\infty > e^{-(\gamma/2)(t-s)} \|S(s,\varphi)\|_{\gamma,\infty} \right\} \tag{2.9}$$

is not empty. Define $t_{\inf} = \inf\{t : t \in \mathcal{N}\}$. By continuity of the map $t \mapsto \|x(t,\varphi)\|_\infty$, we get that

$$\|x(t_{\inf},\varphi)\|_\infty = e^{-(\gamma/2)(t_{\inf}-s)} \|S(s,\varphi)\|_{\gamma,\infty}, \tag{2.10}$$

$$\|x(t,\varphi)\|_\infty \le e^{-(\gamma/2)(t-s)} \|S(s,\varphi)\|_{\gamma,\infty} \quad \forall t \in [s,t_{\inf}]. \tag{2.11}$$

Now, we will show that

$$\|x(t_{\inf},\varphi)\|_\infty \ge e^{-\gamma r} \|x(t,\varphi)\|_\infty \quad \forall t \in [t_{\inf}-r,t_{\inf}]. \tag{2.12}$$

Indeed, we consider the following two cases: (i) $t \in [s,t_{\inf}] \cap [t_{\inf}-r,t_{\inf}]$ and (ii) $t \in (-\infty,s] \cap [t_{\inf}-r,t_{\inf}]$.

Case (i). If $t \in [s,t_{\inf}] \cap [t_{\inf}-r,t_{\inf}]$, then, according to (2.10) and (2.11), we obtain that

$$\begin{aligned} \|x(t_{\inf},\varphi)\|_\infty &= e^{-(\gamma/2)(t_{\inf}-s)} \|S(s,\varphi)\|_{\gamma,\infty} \\ &\ge e^{-(\gamma/2)(t_{\inf}-s)} e^{(\gamma/2)(t-s)} \|x(t,\varphi)\|_\infty \\ &\ge e^{-\gamma r} \|x(t,\varphi)\|_\infty, \end{aligned} \tag{2.13}$$

which proves (2.12) in this case.

Case (ii). If $t \in (-\infty,s] \cap [t_{\inf}-r,t_{\inf}]$, then, according to (2.10) and the definition of the norm $\|\cdot\|_{\gamma,\infty}$, we obtain that

$$\begin{aligned} \|x(t_{\inf},\varphi)\|_\infty &= e^{-(\gamma/2)(t_{\inf}-s)} \max_{\omega\in[-r,0]} e^{\gamma\omega} \|S(s,\varphi)(\omega)\|_\infty \\ &= \max_{\omega\in[-r,0]} e^{-(\gamma/2)(t_{\inf}-s-2\omega)} \|x(s+\omega,\varphi)\|_\infty \\ &\ge e^{-(\gamma/2)(t_{\inf}+s-2t)} \|x(t,\varphi)\|_\infty \\ &\ge e^{-\gamma r} \|x(t,\varphi)\|_\infty. \end{aligned} \tag{2.14}$$

Hence, (2.12) is proved. To conclude the proof of this step, we estimate the norm $\|x(t,\varphi)\|_\infty$ for all t in a neighborhood of $t_{\inf}$ in order to show a contradiction to the assumption that the set $\mathcal{N}$ is not empty. To this end, we define the following set:

$$I := \left\{ i \in \{1,\ldots,d\} : |x_i(t_{\inf},\varphi)| = \|x(t_{\inf},\varphi)\|_\infty \right\}. \tag{2.15}$$

The continuity of the functions $t \mapsto x_i(t,\varphi)$ for $i = 1,\dots,d$ implies that there exists a neighborhood $(t_{\inf} - \varepsilon, t_{\inf} + \varepsilon)$ for some $\varepsilon > 0$ such that

$$\|x(t,\varphi)\|_\infty = \max_{i\in I}|x_i(t,\varphi)| \quad \forall t \in (t_{\inf} - \varepsilon, t_{\inf} + \varepsilon). \tag{2.16}$$

By virtue of (2.1), the derivative of the function $t \mapsto x_i^2(t,\varphi)$ is estimated as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}x_i^2(t,\varphi)\Big|_{t=t_{\inf}} &= x_i(t_{\inf},\varphi)\Bigg[a_{ii}(t_{\inf})x_i(t_{\inf},\varphi) \\ &\qquad + \sum_{j=1,j\neq i}^{d} a_{ij}(t_{\inf})x_j(t_{\inf}-\tau_{ij},\varphi)\Bigg] \\ &\le a_{ii}(t_{\inf})x_i(t_{\inf},\varphi)^2 \\ &\quad + |x_i(t_{\inf},\varphi)| \sum_{j=1,j\neq i}^{d} |a_{ij}(t_{\inf})| \|x(t_{\inf}-\tau_{ij},\varphi)\|_\infty,\end{aligned} \tag{2.17}$$

which together with (2.12) and the definition of I implies that for all $i \in I$

$$\frac{1}{2}\frac{d}{dt}x_i(t,\varphi)^2\Big|_{t=t_{\inf}} \le x_i(t_{\inf},\varphi)^2\Bigg[a_{ii}(t_{\inf}) + e^{\gamma r}\sum_{j=1,j\neq i}^{d}|a_{ij}(t_{\inf})|\Bigg]. \tag{2.18}$$

Thus, from the row diagonal dominance (2.4) and bound (2.5), we derive that

$$\frac{1}{2}\frac{d}{dt}x_i(t,\varphi)^2\Big|_{t=t_{\inf}} \le [-\delta + (e^{\gamma r} - 1)M]x_i^2(t_{\inf},\varphi). \tag{2.19}$$

Using (2.6), we obtain that

$$\frac{1}{2}\frac{d}{dt}x_i^2(t,\varphi)\Big|_{t=t_{\inf}} < -\frac{\gamma}{2}x_i^2(t_{\inf},\varphi), \tag{2.20}$$

which yields that there exists a neighborhood $(t_{\inf}, t_{\inf} + \varepsilon_2)$ for an $\varepsilon_2 > 0$ such that for $i \in I$

$$|x_i(t,\varphi)| \le e^{-(\gamma/2)(t-t_{\inf})}|x_i(t_{\inf},\varphi)| \quad \forall t \in (t_{\inf}, t_{\inf} + \varepsilon_2). \tag{2.21}$$

Thus, for all $t \in (t_{\inf}, t_{\inf} + \varepsilon)$ with $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we have

$$\|x(t,\varphi)\|_\infty \le e^{-(\gamma/2)(t-t_{\inf})}\|x(t_{\inf},\varphi)\|_\infty, \tag{2.22}$$

which together with (2.10) implies that

$$\|x(t,\varphi)\|_{\infty} \le e^{-(\gamma/2)(t-s)} \|S(s,\varphi)\|_{\gamma,\infty}. \tag{2.23}$$

Consequently, $\mathcal{N} \cap (t_{\inf}, t_{\inf} + \varepsilon) = \emptyset$, and this is a contradiction to the definition of $\mathcal{N}$ and $t_{\inf}$. Thus, (2.8) is proved.

Step 2. Using (2.8) from Step 1, we show (2.7) by considering two cases: (i) $t \in [s + r, T] \cap [s, T]$ and (ii) $t \in [s, s + r] \cap [s, T]$.

Case (i). If $t \in [s + r, T] \cap [s, T]$, then by virtue of (2.8) we have

$$\begin{aligned} \|S(t,\varphi)\|_{\gamma,\infty} &= \sup_{\omega\in[-r,0]} e^{\gamma\omega} \|x(t+\omega,\varphi)\|_{\infty} \\ &\le \sup_{\omega\in[-r,0]} e^{\gamma\omega} e^{-(\gamma/2)(t+\omega-s)} \|S(s,\varphi)\|_{\gamma,\infty} \\ &\le e^{-(\gamma/2)(t-s)} \|S(s,\varphi)\|_{\gamma,\infty}, \end{aligned} \tag{2.24}$$

which proves (2.7) in this case.

Case (ii). If $t \in [s, s + r] \cap [s, T]$, then we have

$$\begin{aligned} \|S(t,\varphi)\|_{\gamma,\infty} = \max\Bigg\{ &\sup_{\omega\in[-(t-s),0]} e^{\gamma\omega} \|x(t+\omega,\varphi)\|_{\infty}, \\ &\sup_{\omega\in[-r,-(t-s)]} e^{\gamma\omega} \|x(t+\omega,\varphi)\|_{\infty} \Bigg\}. \end{aligned} \tag{2.25}$$

Hence, using (2.8), we obtain that

$$\begin{aligned} \|S(t,\varphi)\|_{\gamma,\infty} \le \max\Bigg\{ &\sup_{\omega\in[-(t-s),0]} e^{(\gamma/2)\omega} e^{-(\gamma/2)(t-s)} \|S(s,\varphi)\|_{\gamma,\infty}, \\ &e^{-\gamma(t-s)} \|S(s,\varphi)\|_{\gamma,\infty} \Bigg\}. \end{aligned} \tag{2.26}$$

Thus, (2.7) is proved, and the proof is complete. □

3. Finite-Time Attractivity of Lotka-Voltera Systems

Consider a Lotka-Voltera system of the following form:

$$\dot{x}_i(t) = x_i(t) \left(r_i + \sum_{j=1}^{d} a_{ij} x_j (t - r_{ij}) \right), \quad i = 1, \ldots, d, \tag{3.1}$$

where $r_{ij} \geq 0$ and $r_{ii} = 0$ for all $i = 1, \ldots, d$. Suppose that $A = (a_{ij})_{i,j=1,\ldots,d}$ is row diagonal dominant with $a_{ii} < 0$ for all $i = 1, \ldots, d$; that is, there exists $\delta > 0$ such that

$$|a_{ii}| \geq \sum_{j=1, j \neq i}^{d} |a_{ij}| + \delta. \tag{3.2}$$

We assume in the following that there exists a positive vector $x^* \in \mathbb{R}_+^d$ such that

$$\mathbf{r} + Ax^* = 0 \quad \text{where } \mathbf{r} = (r_1, \ldots, r_d)^T. \tag{3.3}$$

To shorten the notation, the function $t \mapsto x^*$ for all $t \in [-r, 0]$ is also denoted by x^*. The function x^* is a fixed point of the evolution operator $S(t, \cdot)$ generated by (3.1), that is, $S(t, x^*) = x^*$ for all t. For system (3.1), the result in [13, Theorem 1] showed that the equilibrium x^* is exponentially attractive on the positive real line $\mathbb{R}_+$; that is, there exist positive constants K, α and η such that

$$\|S(t, \varphi) - x^*\|_{0,\infty} \leq Ke^{-\alpha t}\|\varphi - x^*\| \quad \forall \varphi \in B_\eta(x^*). \tag{3.4}$$

However, the constant K is usually greater than 1. Using the result developed in the preceding section, we show in the next theorem that the constant K in (3.4) can be chosen to be equal to 1 on the state space $\mathcal{C}$ with norm $\|\cdot\|_{\gamma,\infty}$ for some $\gamma \geq 0$. As a consequence, the equilibrium solution of system (3.1) is finite-time attractive with respect to these norms.

Theorem 3.1 (finite-time attractive equilibrium of Lotka-Voltera equations). *Consider* (3.1) *on an arbitrary finite-time interval* $[0, T]$ *satisfying* (3.2) *and* (3.3). *Then, there exists a positive weight factor* γ^* *such that for all* $\gamma \in [0, \gamma^*]$ *the positive equilibrium* x^* *is finite-time attractive on* $[0, T]$ *with respect to the norm* $\|\cdot\|_{\gamma,\infty}$.

Proof . The proof is divided into three steps.

Step 1. Construction of the weight factor γ^*. Due to compactness of $[0, T]$ and continuity of solutions of (3.1), there exists $\eta^* > 0$ such that

$$\|S(t, \varphi) - x^*\|_{0,\infty} \leq \frac{\min_{i=1,\ldots,d} x_i^*}{2} \quad \text{for } t \in [0, T],\ \varphi \in B_{\eta^*}(x^*). \tag{3.5}$$

Then, we have

$$|x_i(t, \varphi) - x_i^*| \leq \|S(t, \varphi) - x^*\|_{0,\infty} \leq \frac{\min_{i=1,\ldots,d} x_i^*}{2}, \tag{3.6}$$

which implies that for all $t \in [0, T]$ and $\varphi \in B_{\eta^*}(x^*)$ we have

$$\frac{\min_{i=1,\ldots,d} x_i^*}{2} \leq |x_i(t, \varphi)| \leq \frac{\min_{i=1,\ldots,d} x_i^*}{2} + \max_{i=1,\ldots,d} x_i^*. \tag{3.7}$$

Define

$$\delta^* := \frac{\min_{i=1,\dots,d} x_i^*}{2}\delta, \qquad M := \left(\frac{\min_{i=1,\dots,d} x_i^*}{2} + \max_{i=1,\dots,d} x_i^*\right) \max_{i=1,\dots,d} \sum_{j=1, j\neq i}^{d} |a_{ij}|. \tag{3.8}$$

Now let $\gamma^* > 0$ be the solution of the following equation:

$$\frac{\gamma^*}{2} + \left(e^{\gamma^* r} - 1\right) M = \delta^*. \tag{3.9}$$

Step 2. In this step, we show that $y(t) := x(t, \varphi) - x^*$ is the solution of the delay equation

$$\dot{y}_i(t) = a_{ii}(t) y_i(t) + \sum_{j=1, j\neq i}^{d} a_{ij}(t) y_j\left(t - r_{ij}\right) \quad \text{for } i = 1, \dots, d \tag{3.10}$$

with the initial condition $y(s) = \varphi(s) - x^*$ for $s \in [-r, 0]$, where

$$a_{ij}(t) := a_{ij} x_i\left(t, \varphi\right) \quad \text{for } i, j = 1, \dots, d. \tag{3.11}$$

Indeed, we have

$$\begin{aligned} \dot{y}_i(t) &= x_i\left(t, \varphi\right)\left[r_i + \sum_{j=1}^{d} a_{ij} x_j\left(t - r_{ij}, \varphi\right)\right] \\ &= x_i\left(t, \varphi\right)\left[r_i + \sum_{j=1}^{d} a_{ij} x_j^* + \sum_{j=1}^{d} a_{ij} y_j\left(t - r_{ij}\right)\right], \end{aligned} \tag{3.12}$$

which together with the fact that $\mathbf{r} + Ax^* = 0$ implies that $y(t)$ is a solution of (3.10). Furthermore, we have

$$S^*\left(t, \varphi - x^*\right) = S\left(t, \varphi\right) - x^* \quad \forall t \in [0, T], \tag{3.13}$$

where S^* denotes the evolution operator generated by (3.10).

Step 3. In this step, we show that for all $\gamma \in [0, \gamma^*]$ and $\varphi \in B_{\eta^*}(x^*)$ we have

$$\left\|S(t, \varphi) - x^*\right\|_{\gamma,\infty} \leq e^{-\gamma(t-s)/2} \left\|S(s, \varphi) - x^*\right\|_{\gamma,\infty} \quad \forall 0 \leq s \leq t \leq T. \tag{3.14}$$

Choose and fix $\varphi \in B_{\eta^*}(x^*)$, and we show that (3.10) fulfills all assumptions of Theorem 2.1. Indeed, using (3.7) and the definition of M, we obtain the following upper bound:

$$\begin{aligned}\max_{t\in[0,T],i=1,\dots,d} \sum_{j=1,j\neq i}^{d} \left|a_{ij}(t)\right| &\leq \left(\frac{\min_{i=1,\dots,d} x_i^*}{2} + \max_{i=1,\dots,d} x_i^*\right) \max_{i=1,\dots,d} \sum_{j=1,j\neq i}^{d} \left|a_{ij}\right| \\ &\leq M.\end{aligned} \tag{3.15}$$

Combining (3.2) and (3.7), we also get that

$$\begin{aligned}|a_{ii}(t)| - \sum_{j=1,j\neq i}^{d} \left|a_{ij}(t)\right| &= \left|x_i(t,\varphi)\right| \left(|a_{ii}| - \sum_{j=1,j\neq i}^{d} \left|a_{ij}\right|\right) \\ &\geq \frac{\min_{i=1,\dots,d} x_i^*}{2} \delta = \delta^*.\end{aligned} \tag{3.16}$$

Therefore, system (3.10) fulfills all assumptions of Theorem 2.1. Then, the zero solution of system (3.10) is finite-time attractive with respect to the norm $\|\cdot\|_{\gamma,\infty}$ for all $\gamma \in [0,\gamma^*]$, that is,

$$\left\|S^*(t,\psi)\right\|_{\gamma,\infty} \leq e^{-\gamma(t-s)/2} \left\|S^*(s,\psi)\right\|_{\gamma,\infty} \quad \forall 0 \leq s \leq t \leq T,\ \psi \in C. \tag{3.17}$$

In particular, substituting $\psi = \varphi - x^*$ we get that

$$\left\|S^*(t,\varphi - x^*)\right\|_{\gamma,\infty} \leq e^{-(\gamma(t-s))/2} \left\|S^*(s,\varphi - x^*)\right\|_{\gamma,\infty} \quad \forall 0 \leq s \leq t \leq T, \tag{3.18}$$

which together with (3.13) implies (3.14) and the proof is complete. □

In the rest of the paper, we discuss a planar Lotka-Voltera system, for which we can explicitly compute its equilibrium. Consequently, applying Theorem 3.1 yields a sufficient condition for finite-time attractivity of this equilibrium.

Example 3.2. Consider a planar Lotka-Voltera of the following form:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)(r_1 + a_{11}x_1(t) + a_{12}x_2(t-\tau_{12})), \\ \dot{x}_2(t) &= x_2(t)(r_2 + a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t)),\end{aligned} \tag{3.19}$$

where $r_1, r_2, \tau_{12}, \tau_{21} > 0$ and the coefficients a_{ij} for $i, j = 1, 2$ satisfy the following inequalities for some $\varepsilon > 0$:

$$a_{11}, a_{22} < 0, \quad |a_{11}| \geq |a_{12}| + \varepsilon, \quad |a_{22}| \geq |a_{21}| + \varepsilon. \tag{3.20}$$

Additionally, we assume that the equilibrium point $x^* = (x_1^*, x_2^*)$ is positive, where

$$x_1^* = \frac{r_2 a_{12} - r_1 a_{22}}{a_{11} a_{22} - a_{12} a_{21}}, \quad x_2^* = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11} a_{22} - a_{12} a_{21}}. \tag{3.21}$$

According to Theorem 3.1, for any finite-time interval $[0, T]$, there exists a positive weight factor γ^* such that for all $\gamma \in [0, \gamma^*]$ the positive equilibrium x^* is finite-time attractive on $[0, T]$ with respect to the norm $\| \cdot \|_{\gamma,\infty}$.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 237036, 12 pages
doi:10.1155/2012/237036

Research Article

Bounded Positive Solutions for a Third Order Discrete Equation

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Received 21 February 2012; Accepted 12 April 2012

Academic Editor: Josef Diblík

This paper studies the following third order neutral delay discrete equation $\Delta(a_n\Delta^2(x_n+p_nx_{n-\tau}))+f(n,x_{n-d_{1n}},\ldots,x_{n-d_{ln}})=g_n$, $n\ge n_0$, where $\tau,l\in\mathbb{N}$, $n_0\in\mathbb{N}\cup\{0\}$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, $\{p_n\}_{n\in\mathbb{N}_{n_0}}$, $\{g_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n\neq 0$ for $n\ge n_0$, $\{d_{in}\}_{n\in\mathbb{N}_{n_0}}\subseteq\mathbb{Z}$ with $\lim_{n\to\infty}(n-d_{in})=+\infty$ for $i\in\{1,2,\ldots,l\}$ and $f\in C(\mathbb{N}_{n_0}\times\mathbb{R}^l,\mathbb{R})$. By using a nonlinear alternative theorem of Leray-Schauder type, we get sufficient conditions which ensure the existence of bounded positive solutions for the equation. Three examples are given to illustrate the results obtained in this paper.

1. Introduction and Preliminaries

The oscillatory, nonoscillatory and asymptotic behaviors and existence of solutions for various difference equations have received more and more attentions in recent years. For details, we refer the reader to [1–11] and the references therein.

In 2005, M. Migda and J. Migda [10] studied the asymptotic behavior of solutions for the second order neutral difference equation

$$\Delta^2(x_n+px_{n-k})+f(n,x_n)=0,\quad n\ge 1,\tag{1.1}$$

where $p\in\mathbb{R}$, k is a nonnegative integer and $f:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$. In 2008, Cheng and Chu [7] established sufficientand necessary conditions of oscillation for the second order difference equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n^{\gamma} = 0, \quad n \geq 1, \tag{1.2}$$

where γ is the quotient of two odd positive integers and $p_n, r_n \in (0, +\infty)$ for $n \in \mathbb{N}$. In 2000, Li et al. [9] gave several necessary and/or sufficient conditions of the existence of unbounded positive solution for the nonlinear difference equation

$$\Delta(r_n \Delta x_n) + f(n, x_n) = 0, \quad n \geq n_0, \tag{1.3}$$

where n_0 is a fixed nonnegative integer, $r : \mathbb{N}_{n_0} \to (0, +\infty)$ and $f : \mathbb{N}_{n_0} \times \mathbb{R} \to \mathbb{R}$. In 2003, using the Leray-Schauder's nonlinear alternative theorem, Agarwal et al. [1] presented the existence of nonoscillatory solutions for the discrete equation

$$\Delta\big(a_n \Delta\big(x_n + p x_{n-\tau}\big)\big) + F(n+1, x_{n+1-\sigma}) = 0, \quad n \geq 1, \tag{1.4}$$

where τ, σ are fixed nonnegative integers, $p \in \mathbb{R}, a : \mathbb{N} \to (0, +\infty)$ and $F : \mathbb{N} \times (0, +\infty) \to [0, +\infty)$ is continuous. In 1995, Yan and Liu [11] proved the existence of a bounded nonoscillatory solution for the third order difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-r}) = 0, \quad n \geq n_0 \tag{1.5}$$

by utilizing the Schauder's fixed point theorem. In 2005, Andruch-Sobiło and Migda [2] studied the third order linear difference equations of neutral type

$$\Delta^3\big(x_n - p_n x_{\sigma_n}\big) \pm q_n x_{\tau_n} = 0, \quad n \geq n_0 \tag{1.6}$$

and obtained sufficient conditions under which all solutions of (1.6) are oscillatory.

The aim of this paper is to study the following third order neutral delay discrete equation

$$\Delta\Big(a_n \Delta^2\big(x_n + p_n x_{n-\tau}\big)\Big) + f\big(n, x_{n-d_{1n}}, \ldots, x_{n-d_{ln}}\big) = g_n, \quad n \geq n_0, \tag{1.7}$$

where $\tau, l \in \mathbb{N}, n_0 \in \mathbb{N} \cup \{0\}$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, $\{p_n\}_{n\in\mathbb{N}_{n_0}}$, $\{g_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a_n \neq 0$ for $n \geq n_0$, $\{d_{in}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$ with $\lim_{n\to\infty}(n - d_{in}) = +\infty$ for $i \in \{1, 2, \ldots, l\}$ and $f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^l, \mathbb{R})$. By making use of the Leray-Schauder's nonlinear alternative theorem, we establish the existence results of bounded positive solutions for (1.7), which extend substantially Theorem 2 in [11]. Three nontrivial examples are given to illustrate the superiority and applications of the results presented in this paper.

Let us recall and introduce the below concepts, signs and lemmas. Let $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of all real numbers, integers and positive integers, respectively,

$$\begin{gathered}\mathbb{N}_{n_0} = \{n : n \in \mathbb{N} \text{ with } n \geq n_0\}, \qquad \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\}, \\ \beta = \min\{n_0 - \tau, \inf\{n - d_{in} : 1 \leq i \leq l,\ n \in \mathbb{N}_{n_0}\}\}\end{gathered} \tag{1.8}$$

and l_β^∞ stand for the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty. \tag{1.9}$$

For any constants $M > N > 0$, put

$$\begin{aligned} E(N) &= \left\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l_\beta^\infty : x_n \geq N \text{ for } n \in \mathbb{Z}_\beta\right\}, \\ U(M) &= \{x \in E(N) : \|x\| < M\}. \end{aligned} \tag{1.10}$$

It is easy to verify that $E(N)$ is a nonempty closed convex subset of l_β^∞ and $U(M)$ is a nonempty open subset of $E(N)$.

By a solution of (1.7), we mean a sequence $\{x_n\}_{n \in \mathbb{Z}_\beta}$ with a positive integer $T \geq \tau + |\beta|$ such that (1.7) holds for all $n \geq T$.

For any subset U of a Banach space X, let $\overline{U}$ and ∂U denote the closure and boundary of U in X, respectively.

Lemma 1.1 (see [8]). *A bounded, uniformly Cauchy subset D of l_β^∞ is relatively compact.*

Lemma 1.2 (Leray-Schauder's Nonlinear Alternative Theorem [1]). *Let E be a nonempty closed convex subset of a Banach space X and U be an open subset of E with $p^* \in U$. Also $G : \overline{U} \to E$ is a continuous, condensing mapping with $G(\overline{U})$ bounded. Then either*

(A$_1$) *G has a fixed point in $\overline{U}$; or*

(A$_2$) *there are $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = (1 - \lambda)p^* + \lambda G x$.*

2. Existence of Bounded Positive Solutions

Now we investigate sufficient conditions of the existence of bounded positive solutions for (1.7) by using the Leray-Schauder's Nonlinear Alternative Theorem.

Theorem 2.1. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N, $\overline{p}$ and $\underline{p}$ satisfying*

$$\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\left\{\left|f(j,w_1,\dots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\} < +\infty; \tag{2.1}$$

$$\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|g_j\right| < +\infty; \tag{2.2}$$

$$0 < N < \left(1-\underline{p}-\overline{p}\right)M, \quad \min\left\{\underline{p},\overline{p}\right\} \ge 0, \quad \underline{p}+\overline{p} < 1 \tag{2.3}$$

$$-\underline{p} \le p_n \le \overline{p}, \quad n \ge k_0. \tag{2.4}$$

Then (1.7) *possesses a bounded positive solution in $\overline{U(M)}$.*

Proof. Let $L \in (\overline{p}M + N, M(1-\underline{p}))$. It follows from (2.1)–(2.3) that there exists a positive integer $T > 1 + \tau + k_0 + |\beta|$ sufficiently large satisfying

$$\begin{aligned}&\sum_{s=T}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[\left|g_j\right| + \sup\left\{\left|f(j,w_1,\dots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\}\right]\\&\quad < \min\left\{L-\overline{p}M-N,\ M\left(1-\underline{p}\right)-L\right\}.\end{aligned} \tag{2.5}$$

Choose $p^* = M - \varepsilon_0$ with $\varepsilon_0 \in (0, \min\{L-\overline{p}M-N, M(1-\underline{p}) - L\})$ and

$$\begin{aligned}&\sum_{s=T}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[\left|g_j\right| + \sup\left\{\left|f(j,w_1,\dots,w_l)\right| : w_i \in [N,M],\ 1 \le i \le l\right\}\right]\\&\quad \le \min\left\{L-\overline{p}M-N, M\left(1-\underline{p}\right)-L\right\} - \varepsilon_0.\end{aligned} \tag{2.6}$$

Note that

$$M > M-\varepsilon_0 = p^* > M - \min\left\{L-\overline{p}M-N, M\left(1-\underline{p}\right)-L\right\} \ge N + M(1+\overline{p}) - L > N, \tag{2.7}$$

which implies that $p^* = \{p^*\}_{n\in\mathbb{Z}_\beta} \in U(M)$. Define two mappings $A_L, B_L : \overline{U(M)} \to l_\beta^\infty$

by

$$A_L x_n = \begin{cases} L - p_n x_{n-\tau}, & n \geq T+1 \\ L - p_T x_T, & \beta \leq n \leq T; \end{cases} \tag{2.8}$$

$$B_L x_n = \begin{cases} -\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right], & n \geq T+1 \\ 0, & \beta \leq n \leq T \end{cases} \tag{2.9}$$

for all $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$.

We now show that

$$D_L = A_L + B_L : \overline{U(M)} \longrightarrow E(N). \tag{2.10}$$

For each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, by (2.4)–(2.9), we have

$$\begin{aligned} A_L x_n + B_L x_n &= L - p_n x_{n-\tau} - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right] \\ &\geq L - \overline{p}M - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right] \\ &\geq L - \overline{p}M - \min\left\{L - \overline{p}M - N, M\left(1 - \underline{p}\right) - L\right\} + \varepsilon_0 \\ &\geq N + \varepsilon_0 \\ &> N, \quad n \geq T+1 \end{aligned} \tag{2.11}$$

$$A_L x_n + B_L x_n = L - p_T x_T \geq L - \overline{p}M > N, \quad \beta \leq n \leq T,$$

□

We next assert that

$$B_L : \overline{U(M)} \longrightarrow l_\beta^\infty \text{ is a continuous, compact mapping.} \tag{2.12}$$

Let $\{x^\alpha\}_{\alpha\in\mathbb{N}} \subseteq \overline{U(M)}$ be an arbitrary sequence and $x^0 \in l_\beta^\infty$ with

$$\left\|x^\alpha - x^0\right\| \longrightarrow 0 \quad \text{as } \alpha \longrightarrow \infty. \tag{2.13}$$

Since $\overline{U(M)}$ is closed, it follows that $x^0 \in \overline{U(M)}$. Given $\varepsilon > 0$. Using (2.1), (2.13) and the continuity of f, we infer that there exists $T^{**}, T^* \in \mathbb{N}$ with $T^* > T+1$ satisfying

$$\sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M], 1 \le i \le l\} < \frac{\varepsilon}{16}; \tag{2.14}$$

$$\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\} < \frac{\varepsilon}{16(T^*-T)}; \tag{2.15}$$

$$\sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=T^*}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\} < \frac{\varepsilon}{16}; \tag{2.16}$$

$$\sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{T^*-1}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| < \frac{\varepsilon}{2}, \quad \alpha \ge T^{**}. \tag{2.17}$$

Combining (2.9) and (2.14)–(2.17), we conclude that

$$\begin{aligned}
&\left\|B_L x^{\alpha} - B_L x^0\right\| \\
&= \max\left\{\sup_{\beta\le n\le T}\left|B_L x^{\alpha}_n - B_L x^0_n\right|, \sup_{n\ge T+1}\left|B_L x^{\alpha}_n - B_L x^0_n\right|\right\} \\
&\le \max\left\{0, \sup_{n\ge T+1}\sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right\} \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad + \sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right] \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad + \sum_{s=T}^{T^*-1}\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad + 2\sum_{s=T^*}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M],\ 1 \le i \le l\} \\
&\le \sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=k}^{T^*-1}\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)-f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right| \\
&\quad + \sum_{s=T}^{T^*-1}\sum_{k=s}^{T^*-1}\frac{1}{|a_k|}\sum_{j=T^*}^{\infty}\left[\left|f\left(j,x^{\alpha}_{j-d_{1j}},\ldots,x^{\alpha}_{j-d_{lj}}\right)\right|+\left|f\left(j,x^{0}_{j-d_{1j}},\ldots,x^{0}_{j-d_{lj}}\right)\right|\right] \\
&\quad + 2(T^*-T)\sum_{k=T^*}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\sup\{|f(j,w_1,\ldots,w_l)| : w_i \in [N,M], 1 \le i \le l\} + 2\cdot\frac{\varepsilon}{16} \\
&< \frac{\varepsilon}{2} + 2\cdot\frac{\varepsilon}{16} + 2(T^*-T)\cdot\frac{\varepsilon}{16(T^*-T)} + \frac{\varepsilon}{8} \\
&< \varepsilon, \quad \alpha \ge T^{**},
\end{aligned} \tag{2.18}$$

which means that B_L is continuous in $\overline{U(M)}$. On the other hand, in light of (2.6) and (2.9), we get that for each $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$

$$\begin{aligned}
\|B_L x\| &= \max\left\{ \sup_{\beta\le n\le T} |B_L x_n|, \sup_{n\ge T+1} |B_L x_n| \right\} \\
&\le \sup_{n\ge T+1} \sum_{s=T}^{n-1} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \left| f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right) \right| \right] \\
&\le \sum_{s=T}^{\infty} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \sup\{ |f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \le i \le l \} \right] \\
&\le \min\left\{ L - \overline{p}M - N, M\left(1 - \underline{p}\right) - L \right\} - \varepsilon_0 \\
&\le M,
\end{aligned} \tag{2.19}$$

which yields that $B_L(\overline{U(M)})$ is a bounded subset of l_β^∞. By virtue of (2.1) and (2.2), we deduce that for any $\varepsilon > 0$, there exists $T_0 > T$ satisfying

$$\sum_{s=T_0}^{\infty} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \sup\{ |f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \le i \le l \} \right] < \varepsilon, \tag{2.20}$$

which together with (2.9) gives that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$

$$\begin{aligned}
&|B_L x_n - B_L x_m| \\
&= \left| \sum_{s=T}^{n-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right) \right] - \sum_{s=T}^{m-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right) \right] \right| \\
&\le \sum_{s=n}^{m-1} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \left| f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right) \right| \right] \\
&< \sum_{s=T_0}^{\infty} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \sup\{ |f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \le i \le l \} \right] \\
&< \varepsilon, \quad m > n \ge T_0,
\end{aligned} \tag{2.21}$$

which means that $B_L(\overline{U(M)})$ is uniformly Cauchy. Thus Lemma 1.1 ensures that $B_L(\overline{U(M)})$ is a relatively compact subset of l_β^∞.

Let $x = \{x_n\}_{n\in\mathbb{Z}_\beta}, y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$. In view of (2.3), (2.4) and (2.8), we know that

$$\begin{aligned}
|A_L x_n - A_L y_n| &= |L - p_n x_{n-\tau} - L + p_n y_{n-\tau}| \le |p_n| \|x - y\| \le \left(\underline{p} + \overline{p}\right) \|x - y\|, \quad n \ge T + 1, \\
|A_L x_n - A_L y_n| &= |L - p_T x_{T-\tau} - L + p_T y_{T-\tau}| \le |p_T| \|x - y\| \le \left(\underline{p} + \overline{p}\right) \|x - y\|, \quad \beta \le n \le T,
\end{aligned} \tag{2.22}$$

which implies that

$$\|Ax - Ay\| \le \left(\underline{p} + \overline{p}\right)\|x - y\|, \tag{2.23}$$

which together with (2.10) and (2.12) guarantees that $D_L : \overline{U(M)} \to E(N)$ is a continuous, condensing mapping.

In order to show the existence of a fixed point of D_L, we need to prove that (A_2) in Lemma 1.2 does not hold. Otherwise there exist $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \partial U(M)$ and $\lambda \in (0,1)$ such that $x = (1-\lambda)p^* + \lambda D_L x$. Let

$$\begin{gathered} S_1 = \left\{x \in l_\beta^\infty : N \le x_n \le M,\ \forall n \ge \beta,\ \|x\| = M\right\}, \\ S_2 = \left\{x \in l_\beta^\infty : N \le x_n \le M,\ \forall n \ge \beta,\ \text{and there exists}\ n^* \ge \beta\ \text{satisfying}\ x_{n^*} = N\right\}. \end{gathered} \tag{2.24}$$

It is easy to verify that $\partial U(M) = S_1 \cup S_2$. Now we have to discuss two possible cases as follows:

Case 1. Let $x \in S_1$. It follows from (2.3), (2.4), (2.8) and (2.9) that

$$\begin{aligned} x_n &= (1-\lambda)p^* + \lambda[A_L x_n + B_L x_n] \\ &= (1-\lambda)p^* + \lambda\left[L - p_n x_{n-\tau} - \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right]\right] \\ &\le (1-\lambda)p^* + \lambda\left[L + \underline{p}M + \sum_{s=T}^{n-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \dots, x_{j-d_{lj}}\right)\right|\right]\right] \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda\left[L + \underline{p}M + \min\left\{L - \overline{p}M - N,\ M\left(1-\underline{p}\right) - L\right\} - \varepsilon_0\right] \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda(M-\varepsilon_0) \\ &= M - \varepsilon_0, \quad n \ge T+1, \end{aligned} \tag{2.25}$$

$$\begin{aligned} x_n &= (1-\lambda)p^* + \lambda\left[L - p_T x_T\right] \le (1-\lambda)p^* + \lambda\left(L + \underline{p}M\right) \\ &\le (1-\lambda)(M-\varepsilon_0) + \lambda(M-\varepsilon_0) = M - \varepsilon_0, \quad \beta \le n \le T, \end{aligned} \tag{2.26}$$

which yield that

$$M = \|x\| \le M - \varepsilon_0 < M, \tag{2.27}$$

which is a contradiction;

Case 2. Let $x \in S_2$. If $n^* \geq T+1$, by (2.3), (2.4), (2.8) and (2.9), we deduce that

$$\begin{aligned}
N &= x_{n^*} = (1-\lambda)p^* + \lambda[A_L x_{n^*} + B_L x_{n^*}] \\
&= (1-\lambda)p^* + \lambda\left[L - p_{n^*} x_{n^*-\tau} - \sum_{s=T}^{n^*-1}\sum_{k=s}^{\infty}\frac{1}{a_k}\sum_{j=k}^{\infty}\left[-g_j + f\left(j, x_{j-d_{1j}}, \ldots, x_{j-d_{lj}}\right)\right]\right] \\
&\geq (1-\lambda)(M-\varepsilon_0) + \lambda\left[L - \overline{p}M - \sum_{s=T}^{n^*-1}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \left|f\left(j, x_{j-d_{1j}}, \ldots, x_{j-d_{lj}}\right)\right|\right]\right] \\
&> (1-\lambda)N + \lambda\left[L - \overline{p}M - \min\left\{L - \overline{p}M - N, \quad M\left(1-\underline{p}\right) - L\right\} + \varepsilon_0\right] \\
&\geq (1-\lambda)N + \lambda(N+\varepsilon_0) \\
&= N + \varepsilon_0 \\
&> N,
\end{aligned} \tag{2.28}$$

which is impossible; if $n^* \leq T$, by (2.3), (2.4), (2.8) and (2.9), we arrive at

$$\begin{aligned}
N = x_{n^*} &= (1-\lambda)p^* + \lambda\left[L - p_T x_T\right] \geq (1-\lambda)p^* + \lambda\left(L - \overline{p}M\right) \\
&\geq (1-\lambda)\mathrm{N} + \lambda(N+\varepsilon_0) = N + \varepsilon_0 \\
&> N,
\end{aligned} \tag{2.29}$$

which is absurd.

Consequently Lemma 1.2 ensures that there is $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ such that $D_L x = A_L x + B_L x = x$, which is a bounded positive solution of (1.7). This completes the proof.

Remark 2.2. Under the conditions of Theorem 2.1 we prove also that (1.7) has uncountably many bounded positive solutions in $\overline{U(M)}$.

In fact, as in the proof of Theorem 2.1, for any different $L_1, L_2 \in (\overline{p}M + N, M(1-\underline{p}))$ we conclude that for each $r \in \{1,2\}$, there exist a constant $T_r > 1 + \tau + k_0 + |\beta|$ and two mappings $A_r, B_r : \overline{U(M)} \to l_\beta^\infty$ satisfying (2.6)–(2.9) and

$$\sum_{s=\min\{T_1,T_2\}}^{\infty}\sum_{k=s}^{\infty}\frac{1}{|a_k|}\sum_{j=k}^{\infty}\left[|g_j| + \sup\{|f(j, w_1, \ldots, w_l)| : w_i \in [N, M],\ 1 \leq i \leq l\}\right] \leq \frac{|L_1 - L_2|}{4}, \tag{2.30}$$

where T, L, A_L and B_L are replaced by T_r, r, A_r and B_r, respectively, and $A_r + B_r$ has a fixed point $z^r = \{z_n^r\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, which is a bounded positive solution of (1.7). In order to prove that (1.7) possesses uncountably many bounded positive solutions in $\overline{U(M)}$, we need only to prove that $z^1 \neq z^2$. It follows from (2.8), (2.9) and (2.30) that for $n \geq \min\{T_1, T_2\}$

$$\begin{aligned}
\left|z_n^1 - z_n^2\right| &= \Bigg| L_1 - p_n z_{n-\tau}^1 - \sum_{s=T_1}^{n-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, z_{j-d_{1j}}^1, \dots, z_{j-d_{lj}}^1\right)\right] \\
&\quad - L_2 + p_n z_{n-\tau}^2 + \sum_{s=T_2}^{n-1} \sum_{k=s}^{\infty} \frac{1}{a_k} \sum_{j=k}^{\infty} \left[-g_j + f\left(j, z_{j-d_{1j}}^2, \dots, z_{j-d_{lj}}^2\right)\right] \Bigg| \\
&\geq |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \left\| z^1 - z^2 \right\| \\
&\quad - 2 \sum_{s=\min\{T_1,T_2\}}^{\infty} \sum_{k=s}^{\infty} \frac{1}{|a_k|} \sum_{j=k}^{\infty} \left[|g_j| + \sup\{ |f(j, w_1, \dots, w_l)| : w_i \in [N, M],\ 1 \leq i \leq l \} \right] \\
&\geq |L_1 - L_2| - \left(\underline{p} + \overline{p}\right) \left\| z^1 - z^2 \right\| - \frac{|L_1 - L_2|}{2},
\end{aligned} \tag{2.31}$$

which implies that

$$\left\| z^1 - z^2 \right\| \geq \frac{|L_1 - L_2|}{2\left(1 + \underline{p} + \overline{p}\right)} > 0, \tag{2.32}$$

which yields that $z^1 \neq z^2$.

Remark 2.3. If either $\underline{p} = 0$ or $\overline{p} = 0$, then Theorem 2.1 reduces to the below results, respectively.

Theorem 2.4. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N and $\overline{p}$ satisfying* (2.1), (2.2) *and*

$$0 < N < (1 - \overline{p})M, \quad 0 \leq p_n \leq \overline{p} < 1,\ n \geq k_0. \tag{2.33}$$

Then (1.7) *possesses a bounded positive solution in $\overline{U(M)}$.*

Theorem 2.5. *Assume that there exist constants $k_0 \in \mathbb{N}_{n_0}$ and M, N and $\underline{p}$ satisfying* (2.1), (2.2) *and*

$$0 < N < \left(1 - \underline{p}\right)M, \quad -1 < -\underline{p} \leq p_n \leq 0, \quad n \geq k_0. \tag{2.34}$$

Then (1.7) *possesses a bounded positive solution in $\overline{U(M)}$.*

Remark 2.6. Theorems 2.1–2.3 include Theorem 2 in [11] as special cases. Examples 3.1–3.3 in Section 3 explain that Theorems 2.1–2.3 are genuine generalizations of Theorem 2 in [11].

3. Examples and Applications

Now we construct three nontrivial examples to explain the superiority and applications of Theorems 2.1-2.3, respectively.

Example 3.1. Consider the third order neutral delay discrete equation

$$\Delta\left((-1)^n n^3\Delta^2\left(x_n+\frac{n\sin(n^2)}{3n+1}x_{n-\tau}\right)\right)+\frac{n^3x_{n^2-2}-\sqrt{n}x^3_{n^3-6}}{n^5+5n+nx^2_{2n+5}}=\frac{\cos(n\ln(n^2+1))}{\sqrt{n^3+1}},\quad n\geq 1, \tag{3.1}$$

where $\tau\in\mathbb{N}$ is fixed. Let $l=3,\ n_0=1,\ k_0=2,\ \underline{p}=1/2,\ \overline{p}=1/3,\ M=7,\ N=1$,

$$\begin{gathered}a_n=(-1)^n n^3,\qquad p_n=\frac{n\sin(n^2)}{3n+1},\\ g_n=\frac{\cos(n\ln(n^2+1))}{\sqrt{n^3+1}},\qquad d_{1n}=-n^2+n+2,\\ d_{3n}=-n-5,\qquad f(n,u,v,w)=\frac{n^3u-\sqrt{n}v^3}{n^5+5n+nw^2},\quad \forall(n,u,v,w)\in\mathbb{N}\times\mathbb{R}^3.\end{gathered} \tag{3.2}$$

It is easy to verify that (2.1)–(2.4) hold. It follows from Theorem 2.1 that (3.1) has a bounded positive solution in $\overline{U(M)}$. However Theorem 2 in [11] is useless for (3.1).

Example 3.2. Consider the third order neutral delay discrete equation

$$\Delta\left((n+1)^2\ln^3(n+2)\Delta^2\left(x_n+\frac{3n-4}{4n+2}x_{n-\tau}\right)\right)+\frac{x^2_{n(n+1)/2}x^3_{n(n-1)/2}}{\sqrt{n^3+1}}=\frac{(-1)^n}{n^2},\quad n\geq 1, \tag{3.3}$$

where $\tau\in\mathbb{N}$ is fixed. Let $l=2,\ n_0=1,\ k_0=2,\ \overline{p}=3/4,\ M=40,\ N=8$,

$$\begin{gathered}a_n=(n+1)^2\ln^3(n+2),\qquad p_n=\frac{3n-4}{4n+2},\qquad d_{1n}=\frac{n(1-n)}{2},\qquad d_{2n}=\frac{n(3-n)}{2},\\ g_n=\frac{(-1)^n}{n^2},\qquad f(n,u,v)=\frac{u^2v^3}{\sqrt{n^3+1}},\quad \forall(n,u,v)\in\mathbb{N}\times\mathbb{R}^2.\end{gathered} \tag{3.4}$$

It is clear that (2.1), (2.2) and (2.33) hold. Consequently Theorem 2.4 guarantees that (3.3) has a bounded positive solution in $\overline{U(M)}$. But Theorem 2 in [11] is inapplicable for (3.3).

Example 3.3. Consider the third order neutral delay discrete equation

$$\Delta\left(\sqrt{n^5+1}\Delta^2\left(x_n-\frac{n^3+1}{3n^3+4}x_{n-\tau}\right)\right)+\frac{\sqrt{n}x^6_{n^2-2n}}{n^4+n+1}+\frac{x^3_{2n+3}}{n^2+2}=\frac{\sin(n^2-n)}{n^2+1},\quad n\geq 1, \tag{3.5}$$

where $\tau \in \mathbb{N}$ is fixed. Let $l = 2,\ n_0 = 1,\ k_0 = 3,\ \underline{p} = 1/3,\ M = 30,\ N = 19,$

$$a_n = \sqrt{n^5+1}, \qquad p_n = -\frac{n^3+1}{3n^3+4}, \qquad d_{1n} = n(3-n), \qquad d_{2n} = -n-3,$$
$$g_n = \frac{\sin(n^2-n)}{n^2+1}, \qquad f(n,u,v) = \frac{u^6\sqrt{n}}{n^4+n+1} + \frac{v^3}{n^2+2}, \quad \forall (n,u,v) \in \mathbb{N}\times\mathbb{R}^2. \tag{3.6}$$

Obviously, (2.1), (2.2) and (2.34) hold. Thus Theorem 2.5 ensures that (3.5) has a bounded positive solution in $\overline{U(M)}$. While Theorem 2 in [11] is unfit for (3.5)

Acknowledgment

The authors would like to thank the editor and referees for useful comments and suggestions. This study was supported by research funds from Dong-A University.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 176419, 11 pages
doi:10.1155/2012/176419

Research Article

Adaptive Observer-Based Fault Estimation for Stochastic Markovian Jumping Systems

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Received 15 March 2012; Accepted 4 June 2012

Academic Editor: Miroslava Růžičková

This paper studies the adaptive fault estimation problems for stochastic Markovian jump systems (MJSs) with time delays. With the aid of the selected Lyapunov-Krasovskii functional, the adaptive fault estimation algorithm based on adaptive observer is proposed to enhance the rapidity and accuracy performance of fault estimation. A sufficient condition on the existence of adaptive observer is presented and proved by means of linear matrix inequalities techniques. The presented results are extended to multiple time-delayed MJSs. Simulation results illustrate that the validity of the proposed adaptive faults estimation algorithms.

1. Introduction

Fault detection and isolation (FDI) [1, 2] has been the subject of extensive research since the 1970s and becomes one of the hotspots in control theory presently. With the rising demands of product quality, effectiveness, and safety in modern industries, people expect that they can get the failure information before the fault damages the system. Many techniques have been proposed especially for sensor and actuator failures with application to a wide range of engineering fields. Among these, the most commonly used schemes for fault detection relate to observer-based approaches [2–5]. It should be pointed out that the observer-based approach, which uses a parametric design technique to perform both detection and diagnosis, only works for a small number of sensor faults. In some cases, fault estimation strategies [6–8] are needed to carry on controlling the faulty system. Compared with FDI, fault estimation is a more challenging task because it requires an estimation of the location after the alarm has been set, and the size of the fault should be made. Recently, some results based on adaptive or robust observers [2–8] for fault estimation have been obtained. However, very few results in the literature consider the fault detection and estimation problem for stochastic systems.

In fact, as a special class of stochastic systems [9, 10] that involves both time-evolving and event-driven mechanisms, Markovian jump systems (MJSs) [11] have received considerable attention. This class of systems includes two components which are the mode and the state and the dynamics of jumping modes and continuous states, which are, respectively, modeled by finite-state Markov chains and differential equations. It can be used to model a variety of physical systems, which may experience abrupt changes in structures and parameters due to, for example, sudden environment changes, subsystem switching, system noises, and failures occurred in components or interconnections and executor faults. Some illustrative applications of MJSs can be found for examples in [3, 4, 6, 12–18] and the references therein. In recent years, the FDI problems for MJSs have regained increasing interest, and some results are also available [3, 4, 16–18]. However, very little is known on the problem of fault estimation for time-delay nonlinear MJSs. This problem forms the main purpose of this paper, where an adaptive technique [1, 19–21] is proposed and modified for the estimation of actuator faults.

In this paper, we studied the problem of fault estimation for a class of time-delay MJSs. By comparing with the presented results of time-delay and linear dynamic systems, it can be shown that a derivative term is added on the basis of fault estimation equation in our design, which renders that the conventional adaptive fault estimation algorithm [21] can be treated as a special case of the fast adaptive fault estimation algorithm. The introduction of the derivative term plays a major role in improving the rapidity of fault estimation. The augmented dynamic system is firstly constructed based on the adaptive fault estimation observer, and the observer parameters are designed on the system modes. Sufficient conditions are subsequently established on the existence of the mode-dependent adaptive fault estimation observer. The design criterions are presented in the form of linear matrix inequalities (LMIs) [22], which can be easily checked. The presented results are then extended to multiple time-delayed MJSs case. Finally, a numerical example is included to illustrate the effectiveness of the developed techniques.

Let us introduce some notations. The symbols $\Re^n$ and $\Re^{n\times m}$ stand for an n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively, A^{T} and A^{-1} denote the matrix transpose and matrix inverse, $\mathrm{diag}\{A \;\; B\}$ represents the block-diagonal matrix of A and B, $\sigma_{\max}(C)$ denote the maximal eigenvalue of a positive-define matrix C, $\sum_{i<j}^{N}$ denotes, for example, for $N = 3$, $\sum_{i<j}^{N} a_{ij} \Leftrightarrow a_{12}+a_{13}+a_{23}$, $\|*\|$ denotes the Euclidean norm of vectors, $\mathbf{E}\{*\}$ denotes the mathematics statistical expectation of the stochastic process or vector, $L_2^n(0 \;\; \infty)$ is the space of n dimensional square integrable function vector over $(0 \;\; \infty)$, $P > 0$ (or $P \geq 0$) stands for a positive-definite (or nonnegative-definite) matrix, I is the unit matrix with appropriate dimensions, 0 is the zero matrix with appropriate dimensions, $*$ means that the symmetric terms in a symmetric matrix.

2. Problem Formulation

Given a probability space (Ω, F, ρ), where Ω is the sample space, F is the algebra of events, and ρ is the probability measure defined on F. Let the random form process $\{r_t, t \geq 0\}$ be the continuous-time discrete-state Markov stochastic process taking values in a finite set $\Lambda = \{1, 2, \ldots, N\}$ with transition probabilities given by

$$P_r\{r_{t+\Delta t} = j \mid r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, \end{cases} \tag{2.1}$$

where $\Delta t > 0$ and $\lim_{\Delta t \downarrow 0} o(\Delta t)/\Delta t \to 0$. $\pi_{ij} \geq 0$ are the transition probability rates from mode i at time t to mode $j(i \neq j)$ at time $t + \Delta t$, and $\sum_{j=1, i\neq j}^{N} \pi_{ij} = -\pi_{ii}$.

Consider the following linear MJSs over the probability space (Ω, F, ρ):

$$\begin{gathered} \dot{x}(t) = A(r_t)x(t) + A_d(r_t)x(t-d) + B(r_t)u(t) + B_f(r_t)f(t), \\ y(t) = C(r_t)x(t), \\ x(t) = \eta(t), \quad r(t) = r_0, \quad t \in [-d \ \ 0], \ i = 1, 2, \ldots, S, \end{gathered} \tag{2.2}$$

where $x(t) \in \Re^n$ is the state, $y(t) \in \Re^m$ is the measured output, $u(t) \in \Re^l$ is the controlled input, $f(t) \in \Re^p$ is the unknown actuator fault, and we assume that its derivative is the norm bounded with $\|\dot{f}(t)\| \leq f$, wherein $0 \leq f < \infty$. $d > 0$ is the time delay constant, $\eta(t) \in \Re^n$ is a continuous vector-valued initial function assumed to be continuously differentiable on $[-d \ \ 0]$, and r_0 is the initial mode. $A(r_t)$, $A_d(r_t)$, $B(r_t)$, $B_f(r_t)$, and $C(r_t)$ are known mode-dependent matrices with appropriate dimensions, $B_f(r_t)$ is of full column rank with $\mathrm{rank}[B_f(r_t)] = p$, and r_t represents a continuous-time discrete state Markov stochastic process with values in the finite set Λ.

For presentation convenience, we denote $x(t-d)$, $A(r_t)$, $A_d(r_t)$, $B(r_t)$, $B_f(r_t)$, and $C(r_t)$ as x_d, A_i, A_{di}, B_i, B_{fi}, C_i and, respectively.

Definition 2.1 (see Mao [23]). Let $V(x(t), r_t, t > 0) = V(x(t), i)$ be the positive stochastic functional and define its weak infinitesimal operator as

$$\Im V(x(t), i) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbf{E}\{V(x(t+\Delta t), r_{t+\Delta t}, t+\Delta t) \mid x(t), r_t = i\} - V(x(t), i, t)\right]. \tag{2.3}$$

Refer to observer design [2–4, 7] and consider the following systems:

$$\begin{gathered} \dot{\hat{x}}(t) = A_i\hat{x}(t) + A_{di}\hat{x}_d + B_{fi}\hat{f}(t) + B_iu(t) + L_i[y(t) - \hat{y}(t)] + L_{di}[y_d - \hat{y}_d], \\ \hat{y}(t) = C_i\hat{x}(t), \end{gathered} \tag{2.4}$$

where $\hat{x}(t) \in \Re^n$ is the observer state vector, $\hat{y}(t) \in \Re^m$ is the observer output vector, and $\hat{f}(t) \in \Re^p$ is an estimate of actuator fault of $f(t)$. Denote

$$\begin{gathered} e(t) = x(t) - \hat{x}(t), \\ z(t) = y(t) - \hat{y}(t), \\ e_f(t) = f(t) - \hat{f}(t). \end{gathered} \tag{2.5}$$

Then, we can present the error dynamics (2.4) as

$$\dot{e}(t) = \overline{A}_i e(t) + \overline{A}_{di} e_d + B_{fi} e_f(t),$$
$$z(t) = C_i e(t), \tag{2.6}$$

where $\overline{A}_i = A_i - L_i C_i$, $\overline{A}_{di} = A_{di} - L_{di} C_i$.

In this paper, by comparison with the conventional adaptive fault estimation algorithm [21], we consider the following fast adaptive fault estimation algorithm. In this algorithm, we add a derivative term $\dot{z}(t)$ in estimation equation, that is,

$$\dot{\hat{f}}(t) = \Gamma H_i [\dot{z}(t) + z(t)], \tag{2.7}$$

which can realize $\lim_{t\to\infty} z(t) = 0$, where H_i is a given mode-dependent matrix and $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a prespecified matrix which defines the learning rate for (2.6).

Remark 2.2. In the conventional adaptive fault estimation algorithm $\hat{f}(t) = \Gamma H \int_{t_f}^{t} z(\tau) d\tau$, it is only an integral term in essence. It fails to deal with time-varying faults, that is, $\dot{f}(t) \neq 0$ though it assumes that the constant fault ($\dot{f}(t) = 0$) estimation is unbiased. In this paper, we add a derivative term $\dot{z}(t)$ in estimation equation and improve the conventional adaptive fault estimation algorithm such that the time-varying faults can be considered. For the stochastic modes jumping case, we select the given matrix H_i as a mode-dependent one.

3. Adaptive Fault Estimation Observer Design

Theorem 3.1. *If there exist a set of positive definite symmetric matrices P_i, Q, U and mode-dependent matrices H_i, X_i, and X_{di}, such that the following matrix equations hold for all $i \in \Lambda$:*

$$H_i C_i = B_{fi}^{\mathrm{T}} P_i, \tag{3.1}$$

$$\Pi_i = \begin{bmatrix} \Xi_i & P_i A_{di} - X_{di} C_i & -A_i^{\mathrm{T}} P_i B_{fi} + C_i^{\mathrm{T}} X_i^{\mathrm{T}} B_{fi} \\ * & -Q & -A_{di}^{\mathrm{T}} P_i B_{fi} + C_i^{\mathrm{T}} X_{di}^{\mathrm{T}} B_{fi} \\ * & * & -2B_{fi}^{\mathrm{T}} P_i B_{fi} + U \end{bmatrix} < 0, \tag{3.2}$$

where $\Xi_i = A_i^{\mathrm{T}} P_i + P_i A_i - X_i C_i - C_i^{\mathrm{T}} X_i^{\mathrm{T}} + Q + \sum_{i=1}^{N} \pi_{ij} P_j$. Then the fast adaptive fault estimation algorithm of (2.7) *can be realized. And in the estimated time-interval, it can estimate errors with the uniformly boundedness of the states and faults. Moreover, the observer gains are respectively as*

$$L_i = P_i^{-1} X_i, \qquad L_{di} = P_i^{-1} X_{di}. \tag{3.3}$$

Proof. Let the mode at time t be i, that is, $r_t = i \in \Lambda$. Take the stochastic Lyapunov-Krasovskii function $V(e(t), e_f(t), r_t, t > 0) : \Re^n \times \Re^n \times \Lambda \times \Re_+ \longrightarrow \Re_+$ as

$$V(e(t), e_f(t), i) = V_1(e(t), e_f(t), i) + V_2(e(t), e_f(t), i) + V_3(e(t), e_f(t), i), \tag{3.4}$$

where $V_1(e(t), e_f(t), i) = e^{\mathrm{T}}(t) P_i e(t)$, $V_2(e(t), e_f(t), i) = \int_{t-d}^{t} e^{\mathrm{T}}(t) Q e(t) d\tau$, $V_3(e(t), e_f(t), i) = e_f^{\mathrm{T}}(t) \Gamma^{-1} e_f(t)$, in which $P_i \in \Re^{n\times n}$, $Q \in \Re^{n\times n}$ are the given mode-dependent symmetric positive-definite matrix for each modes $i \in \Lambda$.

According to Definition 2.1 and along the trajectories of the error dynamics MJSs (2.6), we can derive the following:

$$\begin{aligned}
\Im V_1(e(t), e_f(t), i) &= 2e^{\mathrm{T}}(t) P_i \dot{e}(t) + e^{\mathrm{T}}(t) \sum_{i=1}^{N} \pi_{ij} P_j e(t) \\
&= 2e^{\mathrm{T}}(t) P_i \left[\overline{A}_i e(t) + \overline{A}_{di}(r) e_d + B_{fi} e_f(t)\right] + e^{\mathrm{T}}(t) \sum_{i=1}^{N} \pi_{ij} P_j e(t) \\
&= e^{\mathrm{T}}(t) \left[\overline{A}_i^{\mathrm{T}} P_i + P_i \overline{A}_i + \sum_{i=1}^{N} \pi_{ij} P_j\right] e(t) + 2e^{\mathrm{T}}(t) P_i \overline{A}_{di} e_d + 2e^{\mathrm{T}}(t) P_i B_{fi} e_f(t), \\
\Im V_2(e(t), e_f(t), i) &= e^{\mathrm{T}}(t) Q e(t) - e_d^{\mathrm{T}} Q e_d + \sum_{i=1}^{N} \pi_{ij} \int_{t-d}^{t} e^{\mathrm{T}}(t) Q e(t) d\tau \\
&= e^{\mathrm{T}}(t) Q e(t) - e_d^{\mathrm{T}} Q e_d + \left(\sum_{i=1}^{N} \pi_{ij}\right) \left(\int_{t-d}^{t} e^{\mathrm{T}}(t) Q e(t) d\tau\right) \\
&= e^{\mathrm{T}}(t) Q e(t) - e_d^{\mathrm{T}} Q e_d, \\
\Im V_3(e(t), e_f(t), i) &= 2e_f^{\mathrm{T}}(t) \Gamma^{-1} \dot{e}_f(t) = 2e_f^{\mathrm{T}}(t) \Gamma^{-1} \left[\dot{f}(t) - \dot{\hat{f}}(t)\right] \\
&= -2e_f^{\mathrm{T}}(t) H_i [\dot{z}(t) + z(t)] + 2e_f^{\mathrm{T}}(t) \Gamma^{-1} \dot{f}(t) \\
&= -2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i \dot{e}(t) - 2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i e(t) + 2e_f^{\mathrm{T}}(t) \Gamma^{-1} \dot{f}(t) \\
&= -2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i \overline{A}_i e(t) - 2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i \overline{A}_{di} e_d - 2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i B_{fi} e_f(t) \\
&\quad - 2e_f^{\mathrm{T}}(t) B_{fi}^{\mathrm{T}} P_i e(t) + 2e_f^{\mathrm{T}}(t) \Gamma^{-1} \dot{f}(t).
\end{aligned} \tag{3.5}$$

Given a symmetric positive definite matrix U, we can use the following relation:

$$\begin{aligned}
2e_f^{\mathrm{T}}(t) \Gamma^{-1} \dot{f}(t) &\le e_f^{\mathrm{T}}(t) U e_f(t) + \dot{f}^{\mathrm{T}}(t) \Gamma^{-1} U^{-1} \Gamma^{-1} \dot{f}(t) \\
&\le e_f^{\mathrm{T}}(t) U e_f(t) + \dot{f}^2 \sigma_{\max}\left(\Gamma^{-1} U^{-1} \Gamma^{-1}\right).
\end{aligned} \tag{3.6}$$

Then, we can get

$$\begin{aligned}
\Im V\left(e(t),e_f(t),i\right) &= \Im V_1\left(e(t),e_f(t),i\right)+\Im V_2\left(e(t),e_f(t),i\right)+\Im V_3\left(e(t),e_f(t),i\right)\\
&= e^{\mathrm{T}}(t)\left[\overline{A}_i^{\mathrm{T}}P_i+P_i\overline{A}_i+\sum_{i=1}^{N}\pi_{ij}P_j+Q\right]e(t)+2e^{\mathrm{T}}(t)P_i\overline{A}_{di}e_d-e_d^{\mathrm{T}}Qe_d\\
&\quad -2e_f^{\mathrm{T}}(t)B_{fi}^{\mathrm{T}}P_i\overline{A}_ie(t)-2e_f^{\mathrm{T}}(t)B_{fi}^{\mathrm{T}}P_i\overline{A}_{di}e_d-2e_f^{\mathrm{T}}(t)B_{fi}^{\mathrm{T}}P_iB_{fi}e_f(t)\\
&\quad +e_f^{\mathrm{T}}(t)Ue_f(t)+\dot{f}^2\sigma_{\max}\left(\Gamma^{-1}U^{-1}\Gamma^{-1}\right).
\end{aligned}\tag{3.7}$$

By letting $X_i = P_iL_i$ and $X_{di} = P_iL_{di}$, the derivative of $\Im V(e(t),e_f(t),i)$ with respect to time follows that

$$\Im V\left(e(t),e_f(t),i\right)\le\zeta^{\mathrm{T}}(t)\Pi_i\zeta(t)+\eta,\tag{3.8}$$

where $\zeta(t)=\begin{bmatrix} e(t)\\ e_d\\ e_f(t)\end{bmatrix}$, $\eta=\dot{f}^2\sigma_{\max}(\Gamma^{-1}U^{-1}\Gamma^{-1})$.

Thus, it concludes that $\Im V(e(t),e_f(t),i)\le-\lambda\|\zeta(t)\|^2+\eta$, wherein $\lambda=\min_{r\in\Lambda}\sigma_{\min}(-\Pi_i)$. Obviously, we can get $\Im V(e(t),e_f(t),i)<0$ if $\eta<\lambda\|\zeta(t)\|^2$. According to stochastic Lyapunov-Krasovskii stability theory, the trajectory of $\zeta(t)$ will converge to the small set $\Phi=\{\zeta(t)\mid\|\zeta(t)\|^2\le\eta/\lambda\}$, though it is outside set S. Therefore, $\zeta(t)$ is ultimately bounded. This completes the proof. □

Remark 3.2. It is necessary to point out that if the presented faults are constant, that is, $\dot{f}(t)=0$, then the designed adaptive algorithm can achieve asymptotical convergence from (3.8). Then we can get that $\Im V(e(t),e_f(t),i)\le-\lambda\|\zeta(t)\|^2\le 0$, which proves the stability of the origin $e(t)=0$, $e_f(t)=0$ and the uniformly boundedness of $e(t)$ and $e_f(t)$ with $e(t)\in L_2^n(0\ \ \infty)$. Then, $\lim_{t\to\infty}e(t)\to 0$ holds by Barbalat's Lemma.

4. Extension to Multiple Time-Delayed MJSs

Consider the following multiple time-delayed MJSs over the probability space (Ω,F,ρ):

$$\begin{gathered}
\dot{x}(t)=A_ix(t)+\sum_{m=1}^{M}A_{dmi}x(t-d_m)+B_iu(t)+B_{fi}f(t),\\
y(t)=C_ix(t),\\
x(t)=\eta(t),\quad r(t)=r_0,\quad t\in\left[-\max(d_m)\ \ 0\right],
\end{gathered}\tag{4.1}$$

where d_m, $m=1,2,\ldots,M$ are multiple time delays with $0\le d_m<\infty$, and other notations are the same as in Section 2.

Similar to Section 3, the following observer can be constructed:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_i\hat{x}(t) + \sum_{m=1}^{M} A_{dmi}\hat{x}(t-d_m) + B_{fi}\hat{f}(t) + B_iu(t)\\ &\quad + L_i\left[y(t) - \hat{y}(t)\right] + \sum_{m=1}^{M} L_{dmi}\left[y(t-d_m) - \hat{y}(t-d_m)\right],\\ \hat{y}(t) &= C_i\hat{x}(t)\end{aligned} \tag{4.2}$$

where $\hat{x}(t) \in \Re^n$ is the observer state vector, $\hat{y}(t) \in \Re^m$ is the observer output vector, and $\hat{f}(t) \in \Re^p$ is an estimate of actuator fault of $f(t)$.

Then, we can obtain the error dynamics by using the same notations of $e(t)$, $e_f(t)$, and $z(t)$ as follows:

$$\begin{aligned}\dot{e}(t) &= \overline{A}_ie(t) + \sum_{m=1}^{M} \overline{A}_{dmi}e_d(t-d_m) + B_{fi}e_f(t),\\ z(t) &= C_ie(t)\end{aligned} \tag{4.3}$$

where $\overline{A}_i = A_i - L_iC_i$, $\overline{A}_{dmi} = A_{dmi} - L_{dmi}C_i$.

Prior to the design of an adaptive diagnostic law, we can get the following results for multiple time-delayed MJSs (4.1).

Theorem 4.1. *If there exist a set of positive definite symmetric matrices P_i, Q, U and mode-dependent matrices H_i, X_i, and X_{dmi}, $m = 1, 2, \ldots, M$, such that the following matrix equations hold for all $i \in \Lambda$:*

$$H_iC_i = B_{fi}^{\mathrm{T}}P_i, \tag{4.4}$$

$$\Psi_i = \begin{bmatrix} \Xi_i & P_iA_{di} - \sum_{m=1}^{M} X_{dmi}C_i & -A_i^{\mathrm{T}}P_iB_{fi} + C_i^{\mathrm{T}}X_i^{\mathrm{T}}B_{fi} \\ * & -Q & -A_{di}^{\mathrm{T}}P_iB_{fi} + C_i^{\mathrm{T}}\sum_{m=1}^{M} X_{dmi}^{\mathrm{T}}B_{fi} \\ * & * & -2B_{fi}^{\mathrm{T}}P_iB_{fi} + U \end{bmatrix} < 0. \tag{4.5}$$

Then, the fast adaptive fault estimation algorithm of (2.7) *can be realized. And in the estimated time interval, it can estimate errors with the uniformly boundedness of the states and faults. Moreover, the observer gains are respectively as follows:*

$$L_i = P_i^{-1}X_i, \qquad L_{dmi} = P_i^{-1}X_{dmi}, \quad m = 1, 2, \ldots, M. \tag{4.6}$$

When there are difficulties in solving (3.1) or (4.4), we can transform them into the following SDP problems via disciplined convex programming [24]:

$$\begin{aligned}&\min \quad \delta\\ &\text{s.t.} \quad \begin{bmatrix} \delta I & B_{fi}^{\mathrm{T}}P_i - H_iC_i \\ P_iB_{fi} - C_i^{\mathrm{T}}H_i^{\mathrm{T}} & \delta I \end{bmatrix} > 0.\end{aligned} \tag{4.7}$$

In order to make $B_{fi}^{\mathrm{T}}P_i$ approximate to H_iC_i with a satisfactory precision, we can firstly select a sufficiently small scalar $\delta > 0$ to meet (4.7).

Remark 4.2. The solutions of Theorems 3.1 and 4.1 can be obtained by solving an optimization problem with (4.7). By using the Matlab LMI Toolbox, it is straightforward to check the feasibility of LMIs. In order to illustrate the effectiveness of the developed techniques, we will give several numerical examples about fuzzy jump system with time delays in Section 5.

5. Numeral Example

We consider the following time-delayed stochastic MJSs with parameters given by

$$A_1 = \begin{bmatrix} -0.5 & -0.3 \\ -0.1 & -1.0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & -0.4 \\ -0.1 & -1.06 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.05 & -0.1 \\ 0 & 0.05 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.07 & -0.1 \\ 0 & -0.05 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}, \quad B_{f1} = B_{f2} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.2 & 0.4 \end{bmatrix}. \tag{5.1}$$

The transition rate matrix that relates the two operation modes is given as $\Pi = \left[\begin{smallmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{smallmatrix}\right]$. By solving the LMIs in (3.1), (3.2), and (4.7) with $\delta = 8.9252 \times 10^{-4}$, we can get the following solutions:

$$\begin{gathered} L_1 = \begin{bmatrix} 4.5853 \\ 2.2033 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -8.3797 \\ 1.6944 \end{bmatrix}, \\ L_{d1} = \begin{bmatrix} 0.1110 \\ 0.0691 \end{bmatrix}, \quad L_{d2} = \begin{bmatrix} -0.2448 \\ -0.1343 \end{bmatrix}, \\ H_1 = -0.0061, \quad H_2 = 0.0019. \end{gathered} \tag{5.2}$$

To show the effectiveness of the designed methods, the time-delay d is assumed to be 0.2 s, and we consider two kinds of actuator faults $f_1(t)$ and $f_2(t)$ in the simulation over the finite-time interval $t \in [0 \ \ 10]$:

$$\begin{gathered} f_1(t) = \begin{cases} 0, & 0 \le t \le 4, \\ 0.5\sin(5t), & 4 < t \le 10, \end{cases} \\ f_2(t) = \begin{cases} 0.5, & 2k-1 < t \le 2k,\ k \in \{N^+, 1 \le k \le 5\}, \\ 0, & \text{others}. \end{cases} \end{gathered} \tag{5.3}$$

Let $r_0 = 2$ and $\Gamma = 10$, the jumping modes are shown in Figure 1. The estimated faults and estimation errors of $f_1(t)$ and $f_2(t)$ are shown in Figures 2 and 3, respectively. From the simulation results and design algorithm, it can be concluded that the adaptive fault diagnosis observer can enhance the performance of fault estimation for slow and fast time-varying faults.

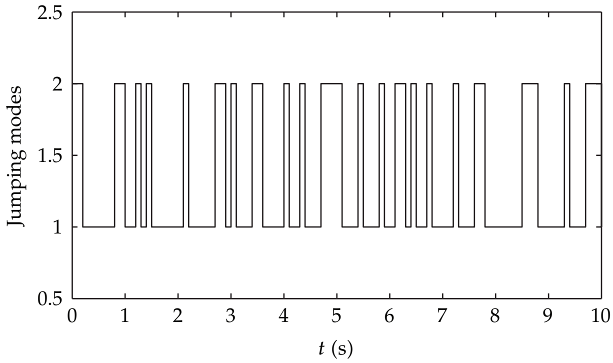

Figure 1: The estimation of changing between modes during the simulation with initial mode 2.

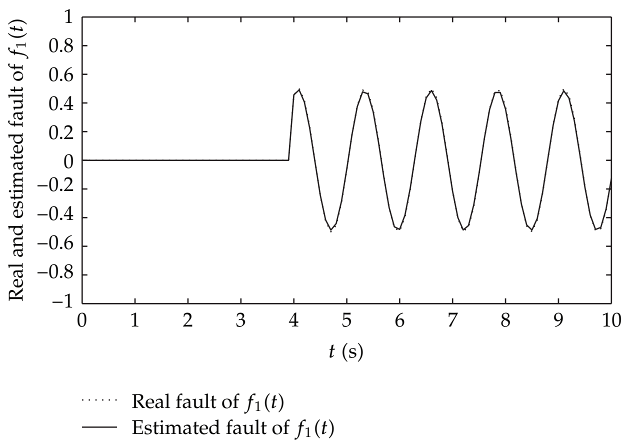

Figure 2: Fault $f_1(t)$ and the estimated $\widehat{f}_1(t)$.

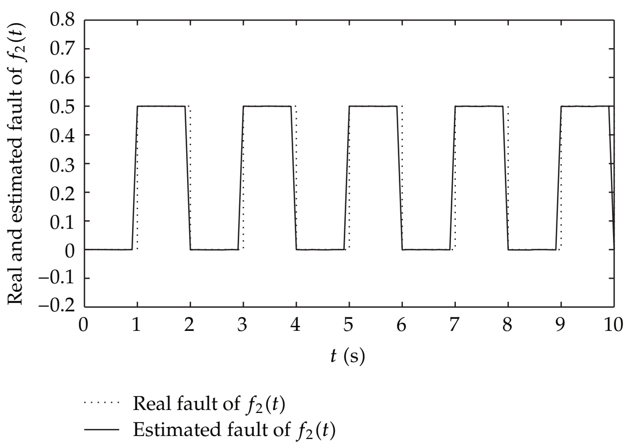

Figure 3: Fault $f_2(t)$ and the estimated $\widehat{f}_2(t)$.

6. Conclusions

In this paper, we have studied the design of adaptive fault estimation observer for time-delayed MJSs. It ensures the rapidity and accuracy performance of fault estimation of the designed observer. By selecting the appropriate Lyapunov-Krasovskii function and applying matrix transformation and variable substitution, the main results are provided in terms of LMIs form and then extended to multiple time-delayed MJSs case. Simulation example demonstrates the effectiveness of the developed techniques.

Acknowledgments

This work was supported in part by the State Key Program of National Natural Science Foundation of China (Grant no. 61134007), the Key Program of Natural Science Foundation of Education Department of Anhui Province (Grant no. KJ2012A014), and Doctor Research Project of Anhui University (Grant no. 32030017).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 623014, 12 pages
doi:10.1155/2012/623014

Research Article

Mean Square Exponential Stability of Stochastic Switched System with Interval Time-Varying Delays

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Received 19 March 2012; Revised 5 May 2012; Accepted 9 May 2012

Academic Editor: Miroslava Růžičková

This paper is concerned with mean square exponential stability of switched stochastic system with interval time-varying delays. The time delay is any continuous function belonging to a given interval, but not necessary to be differentiable. By constructing a suitable augmented Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, a switching rule for the mean square exponential stability of switched stochastic system with interval time-varying delays and new delay-dependent sufficient conditions for the mean square exponential stability of the switched stochastic system are first established in terms of LMIs. Numerical example is given to show the effectiveness of the obtained result.

1. Introduction

Stability analysis of linear systems with time-varying delays $\dot{x}(t) = Ax(t) + Dx(t - h(t))$ is fundamental to many practical problems and has received considarable attention [1–11]. Most of the known results on this problem are derived assuming only that the time-varing delay $h(t)$ is a continuously differentiable function, satisfying some boundedness condition on its derivative: $\dot{h}(t) \le \delta < 1$. In delay-dependent stability criteria, the main concern is to enlarge the feasible region of stability criteria in given time-delay interval. Interval time-varying delay means that a time delay varies in an interval in which the lower bound is not restricted to be zero. By constructing a suitable augmented Lyapunov functionals and utilizing free weight matrices, some less conservative conditions for asymptotic stability are derived in [12–21] for systems with time delay varying in an interval. However, the shortcoming of the method used in these works is that the delay function is assumed to be differential and its derivative is still bounded: $\dot{h}(t) \le \delta$. This paper gives the improved results for the mean square exponential stability of switched stochastic system with interval time-varying delay. The time delay is assumed to be a time-varying continuous function belonging to a given interval, but not necessary to be differentiable. Specifically, our goal is to develop a constructive way to design switching rule to the mean square exponential stability of switched stochastic system with interval time-varying delay. By constructing argumented Lyapunov functional combined with LMI technique, we propose new criteria for the mean square exponential stability of the switched stochastic system. The delay-dependent stability conditions are formulated in terms of LMIs.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Delay-dependent mean square exponential stability conditions of the switched stochastic system and numerical example showing the effectiveness of proposed method are presented in Section 3.

2. Preliminaries

The following notations will be used in this paper. R^+ denotes the set of all real nonnegative numbers; R^n denotes the n-dimensional space with the scalar product $\langle\cdot,\cdot\rangle$ and the vector norm $\|\cdot\|$; $M^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min/\max}(A) = \min/\max\{\operatorname{Re}\lambda;\ \lambda\in\lambda(A)\}$; $x_t := \{x(t+s) : s\in[-h,0]\}$, $\|x_t\| = \sup_{s\in[-h,0]}\|x(t+s)\|$; $C([0,t],R^n)$ denotes the set of all R^n-valued continuous functions on $[0,t]$; matrix A is called semipositive definite $(A\ge 0)$ if $\langle Ax,x\rangle\ge 0$, for all $x\in R^n$; A is positive definite $(A>0)$ if $\langle Ax,x\rangle>0$ for all $x\neq 0$; $A>B$ means $A-B>0$. $*$ denotes the symmetric term in a matrix.

Consider a switched stochastic system with interval time-varying delay of the form

$$\begin{gathered}\dot{x}(t) = A_\gamma x(t) + D_\gamma x(t-h(t)) + \sigma_\gamma(x(t),x(k-h(t)),t)\omega(t),\quad t\in R^+,\\ x(t) = \phi(t),\quad t\in[-h_2,0],\end{gathered}\tag{2.1}$$

where $x(t)\in R^n$ is the state; $\gamma(\cdot) : R^n \to \mathcal{N} := \{1,2,\ldots,N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(t)) = i$ implies that the system realization is chosen as the ith system, $i=1,2,\ldots,N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(t)$ hits predefined boundaries. A_i, $D_i\in M^{n\times n}$, $i=1,2,\ldots,N$ are given constant matrices, and $\phi(t)\in C([-h_2,0],R^n)$ is the initial function with the norm $\|\phi\| = \sup_{s\in[-h_2,0]}\|\phi(s)\|$.

$\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega,\mathcal{F},\mathcal{P})$ with

$$E\{\omega(t)\} = 0,\quad E\left\{\omega^2(t)\right\} = 1,\quad E\{\omega(i)\omega(j)\} = 0\quad (i\neq j),\tag{2.2}$$

and σ_i: $R^n\times R^n\times R \to R^n$, $i=1,2,\ldots,N$ is the continuous function and is assumed to satisfy that

$$\begin{aligned}\sigma_i^T(x(t),x(t-h(t)),t)\sigma_i(x(t),x(t-h(t)),\ t) \le \rho_{i1}x^T(t)x(t) + \rho_{i2}x^T(t-h(t))x(t-h(t)),\\ x(t),\ x(t-h(t))\in R^n,\end{aligned}\tag{2.3}$$

where $\rho_{i1} > 0$ and $\rho_{i2} > 0$, $i = 1, 2, \ldots, N$ are known constant scalars. For simplicity, we denote $\sigma_i(x(t),\ x(t-h(t)),\ t)$ by σ_i, respectively.

The time-varying delay function $h(t)$ satisfies

$$0 \le h_1 \le h(t) \le h_2, \quad t \in R^+. \tag{2.4}$$

The stability problem for switched stochastic system (2.1) is to construct a switching rule that makes the system exponentially stable.

Definition 2.1. Given $\alpha > 0$, the switched stochastic system (2.1) is α-exponentially stable in the mean square if there exists a switching rule $\gamma(\cdot)$ such that every solution $x(t, \phi)$ of the system satisfies the following condition:

$$\exists N > 0 : E\{\|x(t, \phi)\|\} \le E\{N e^{-\alpha t}\|\phi\|\}, \quad \forall t \in R^+. \tag{2.5}$$

We end this section with the following technical well-known propositions, which will be used in the proof of the main results.

Proposition 2.2 (Cauchy inequality). *For any symmetric positive definite marix $N \in M^{n\times n}$ and $a, b \in R^n$ one has*

$$\pm a^T b \le a^T N a + b^T N^{-1} b. \tag{2.6}$$

Proposition 2.3 (see [22]). *For any symmetric positive definite matrix $M \in M^{n\times n}$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$\left(\int_0^\gamma \omega(s)ds\right)^T M \left(\int_0^\gamma \omega(s)ds\right) \le \gamma\left(\int_0^\gamma \omega^T(s) M \omega(s)ds\right). \tag{2.7}$$

Proposition 2.4 (see [23]). *Let E, H, and F be any constant matrices of appropriate dimensions and $F^T F \le I$. For any $\epsilon > 0$, one has*

$$EFH + H^T F^T E^T \le \epsilon E E^T + \epsilon^{-1} H^T H. \tag{2.8}$$

Proposition 2.5 (Schur complement lemma [24]). *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^T$, $Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.9}$$

3. Main Results

Let us set

$$
\begin{aligned}
\mathcal{M}_i &= \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{pmatrix}, \\
\lambda_1 &= \lambda_{\min}(P), \\
\lambda_2 &= \lambda_{\max}(P) + 2h_2^2\lambda_{\max}(R), \\
M_{11} &= A_i^T P + PA_i - S_1A_i - A_i^T S_1^T + 2\alpha P \\
&\quad - e^{-2\alpha h_1}R - e^{-2\alpha h_2}R + 2\rho_{i1}, \\
M_{12} &= e^{-2\alpha h_1}R - S_2A_i, \\
M_{13} &= e^{-2\alpha h_2}R - S_3A_i, \\
M_{14} &= PD_i - S_1D_i - S_4A_i, \\
M_{15} &= S_1 - S_5A_i, \\
M_{22} &= -e^{-2\alpha h_1}R, \\
M_{24} &= S_2D_i, \\
M_{33} &= -e^{-2\alpha h_2}R, \\
M_{34} &= -S_3D_i, \\
M_{44} &= -S_4D_i + 2\rho_{i2}, \\
M_{45} &= S_4 - S_5D_i, \\
M_{55} &= S_5 + S_5^T + h_1^2R + h_2^2R.
\end{aligned}
\tag{3.1}
$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *Given $\alpha > 0$, the zero solution of the switched stochastic system* (2.1) *is α-exponentially stable in the mean square if there exist symmetric positive definite matrices P, R, and matrices S_i, $i = 1, 2, \dots, 5$ satisfying the following conditions:*

(i) $\mathcal{M}_i < 0$, $i = 1, 2, \dots, N$.

The switching rule is chosen as $\gamma(x(t)) = i$. Moreover, the solution $x(t, \phi)$ of the switched stochastic system satisfies

$$
E\{\|x(t,\phi)\|\} \le E\left\{\sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|\right\}, \quad \forall t \in R^{+}. \tag{3.2}
$$

Proof. We consider the following Lyapunov-Krasovskii functional for the system (2.1):

$$E\{V(t,x_t)\} = \sum_{i=1}^{3} E\{V_i\}, \tag{3.3}$$

where

$$\begin{aligned} V_1 &= x^T(t)Px(t), \\ V_2 &= h_1 \int_{-h_1}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) R \dot{x}(\tau) d\tau\, ds, \\ V_3 &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) R \dot{x}(\tau) d\tau\, ds. \end{aligned} \tag{3.4}$$

It easy to check that

$$E\left\{\lambda_1 \|x(t)\|^2\right\} \le E\{V(t,x_t)\} \le E\left\{\lambda_2 \|x_t\|^2\right\}, \quad \forall t \ge 0, \tag{3.5}$$

Taking the derivative of Lyapunov-Krasovskii functional along the solution of system (2.1) and taking the mathematical expectation, we obtained

$$\begin{aligned} E\{\dot{V}_1\} &= E\left\{2x^T(t)P\dot{x}(t)\right\} \\ &= E\left\{x^T(t)\left[A_i^T P + A_i P\right]x(t) + 2x^T(t)PD_i x(t-h(t)) + 2x^T(t)P\sigma_i\omega(t)\right\}, \\ E\{\dot{V}_2\} &= E\left\{h_1^2 \dot{x}^T(t)R\dot{x}(t) - h_1 e^{-2\alpha h_1} \int_{t-h_1}^{t} \dot{x}^T(s)R\dot{x}(s)ds - 2\alpha V_2\right\}, \\ E\{\dot{V}_3\} &= E\left\{h_2^2 \dot{x}^T(t)R\dot{x}(t) - h_2 e^{-2\alpha h_2} \int_{t-h_2}^{t} \dot{x}^T(s)R\dot{x}(s)ds - 2\alpha V_3\right\}. \end{aligned} \tag{3.6}$$

Applying Proposition 2.3 and the Leibniz-Newton formula, we have

$$\begin{aligned} E\left\{-h_i \int_{t-h_i}^{t} \dot{x}^T(s)R\dot{x}(s)ds\right\} &\le E\left\{-\left[\int_{t-h_i}^{t} \dot{x}(s)ds\right]^T R \left[\int_{t-h_i}^{t} \dot{x}(s)ds\right]\right\} \\ &\le E\left\{-[x(t)-x(t-h_i)]^T R[x(t)-x(t-h_i)]\right\} \\ &= E\left\{-x^T(t)Rx(t) + 2x^T(t)Rx(t-h_i) - x^T(t-h_i)Rx(t-h_i)\right\}, \end{aligned} \tag{3.7}$$

Therefore, we have

$$\begin{aligned}
E\{\dot{V}(\cdot)+2\alpha V(\cdot)\} \le\ & E\Big\{x^T(t)\Big[A_i^T P + A_i P + 2\alpha P\Big]x(t)\Big\} \\
& + E\Big\{2x^T(t)PD_i x(t-h(t)) + 2x^T(t)P\sigma_i\omega(t)\Big\} \\
& + E\Big\{\dot{x}^T(t)\Big[\Big(h_1^2+h_2^2\Big)R\Big]\dot{x}(t)\Big\} \\
& - E\Big\{e^{-2\alpha h_1}[x(t)-x(t-h_1)]^T R[x(t)-x(t-h_1)]\Big\} \\
& - E\Big\{e^{-2\alpha h_2}[x(t)-x(t-h_2)]^T R[x(t)-x(t-h_2)]\Big\}.
\end{aligned} \tag{3.8}$$

By using the following identity relation

$$\dot{x}(t) - A_i x(t) - D_i x(t-h(t)) = 0, \tag{3.9}$$

we have

$$\begin{aligned}
& 2x^T(t)S_1\dot{x}(t) - 2x^T(t)S_1A_ix(t) - 2x^T(t)S_1D_ix(t-h(t)) - 2x^T(t)S_1\sigma_i\omega(t) = 0, \\
& 2x^T(t-h_1)S_2\dot{x}(t) - 2x^T(t-h_1)S_2A_ix(t) - 2x^T(t-h_1)S_2D_ix(t-h(t)) \\
& \quad - 2x^T(t-h_1)S_2\sigma_i\omega(t) = 0, \\
& 2x^T(t-h_2)S_3\dot{x}(t) - 2x^T(t-h_2)S_3A_ix(t) - 2x^T(t-h_2)S_3D_ix(t-h(t)) \\
& \quad - 2x^T(t-h_2)S_3\sigma_i\omega(t) = 0, \\
& 2x^T(t-h(t))S_4\dot{x}(t) - 2x^T(t-h(t))S_4A_ix(t) - 2x^T(t-h(t))S_4D_ix(t-h(t)) \\
& \quad - 2x^T(t-h(t))S_4\sigma_i\omega(t) = 0, \\
& 2\dot{x}^T(t)S_5\dot{x}(t) - 2\dot{x}^T(t)S_5A_ix(t) - 2\dot{x}^T(t)S_5D_ix(t-h(t)) - 2\dot{x}^T(t)S_5\sigma_i\omega(t) = 0, \\
& 2\omega^T(t)\sigma_i^T\dot{x}(t) - 2\omega^T(t)\sigma_i^TA_ix(t) - 2\omega^T(t)\sigma_i^TD_ix(t-h(t)) - 2\omega^T(t)\sigma_i^T\sigma_i\omega(t) = 0.
\end{aligned} \tag{3.10}$$

Adding all the zero items of (3.10) into (3.8), we obtain

$$\begin{aligned}
E\{\dot{V}(\cdot)+2\alpha V(\cdot)\} \le\ & E\Big\{x^T(t)\Big[A_i^T P + PA_i + 2\alpha P - e^{-2\alpha h_1}R\Big]x(t)\Big\} \\
& - E\Big\{x^T(t)\Big[e^{-2\alpha h_2}R - S_1A_i - A_i^TS_1^T\Big]x(t)\Big\} \\
& + E\Big\{2x^T(t)\Big[e^{-2\alpha h_1}R - S_2A_i\Big]x(t-h_1)\Big\} \\
& + E\Big\{2x^T(t)\Big[e^{-2\alpha h_2}R - S_3A_i\Big]x(t-h_2)\Big\} \\
& + E\Big\{2x^T(t)[PD_i - S_1D_i - S_4A_i]x(t-h(t))\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+E\Big\{2x^T(t)[S_1-S_5A_i]\dot{x}(t)\Big\}\\
&+E\Big\{2x^T(t)\Big[P\sigma_i-S_1\sigma_i-A_i^T\sigma_i\Big]\omega(t)\Big\}\\
&+E\Big\{x^T(t-h_1)\Big[-e^{-2\alpha h_1}R\Big]x(t-h_1)\Big\}\\
&+E\Big\{2x^T(t-h_1)[-S_2D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h_1)S_2\dot{x}(t)+2x^T(t-h_1)[-S_2\sigma_i]\omega(t)\Big\}\\
&+E\Big\{x^T(t-h_2)\Big[-e^{-2\alpha h_2}R\Big]x(t-h_2)\Big\}\\
&+E\Big\{x^T(t-h_2)[-S_3D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h_2)S_3\dot{x}(t)+2x^T(t-h_2)[-S_3\sigma_i]\omega(t)\Big\}\\
&+E\Big\{x^T(t-h(t))[-S_4D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h(t))[S_4-S_5D_i]\dot{x}(t)\Big\}\\
&+E\Big\{2x^T(t-h(t))\Big[-S_4\sigma_i-D_i^T\sigma_i\Big]\omega(t)\Big\}\\
&+E\Big\{\dot{x}^T(t)\Big[S_5+S_5^T+h_1^2R+h_2^2R\Big]\dot{x}(t)\Big\}\\
&+E\Big\{2\dot{x}^T(t)\Big[\sigma_i^T-S_5\sigma_i\Big]\omega(t)\Big\}\\
&+E\Big\{2\omega^T(t)[-\sigma_i\sigma_i]\omega(t)\Big\}.
\end{aligned}
\tag{3.11}
$$

By assumption (2.2), we have

$$
\begin{aligned}
E\{\dot{V}(\cdot)+2\alpha V(\cdot)\}\le\;& E\Big\{x^T(t)\Big[A_i^TP+PA_i+2\alpha P-e^{-2\alpha h_1}R\Big]x(t)\Big\}\\
&-E\Big\{x^T(t)\Big[e^{-2\alpha h_2}R-S_1A_i-A_i^TS_1^T\Big]x(t)\Big\}\\
&+E\Big\{2x^T(t)\Big[e^{-2\alpha h_1}R-S_2A_i\Big]x(t-h_1)\Big\}\\
&+E\Big\{2x^T(t)\Big[e^{-2\alpha h_2}R-S_3A_i\Big]x(t-h_2)\Big\}\\
&+E\Big\{2x^T(t)[PD_i-S_1D_i-S_4A_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t)[S_1-S_5A_i]\dot{x}(t)\Big\}\\
&+E\Big\{x^T(t-h_1)\Big[-e^{-2\alpha h_1}R\Big]x(t-h_1)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+E\Big\{2x^T(t-h_1)[-S_2D_i]x(t-h(t))\Big\}\\
&+E\Big\{x^T(t-h_2)\Big[-e^{-2\alpha h_2}R\Big]x(t-h_2)\Big\}\\
&+E\Big\{x^T(t-h_2)[-S_3D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h_2)S_3\dot{x}(t)\Big\}\\
&+E\Big\{x^T(t-h(t))[-S_4D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h(t))[S_4-S_5D_i]\dot{x}(t)\Big\}\\
&+E\Big\{\dot{x}^T(t)\Big[S_5+S_5^T+h_1^2R+h_2^2R\Big]\dot{x}(t)\Big\}\\
&+E\Big\{2\Big[-\sigma_i^T\sigma_i\Big]\Big\}.
\end{aligned}
\tag{3.12}
$$

Applying assumption (2.3), the following estimations hold:

$$
\begin{aligned}
E\{\dot{V}(\cdot)+2\alpha V(\cdot)\} \le{}& E\Big\{x^T(t)\Big[A_i^TP+PA_i+2\alpha P-e^{-2\alpha h_1}R\Big]x(t)\Big\}\\
&-E\Big\{x^T(t)\Big[e^{-2\alpha h_2}R-S_1A_i-A_i^TS_1^T+2\rho_{i1}I\Big]x(t)\Big\}\\
&+E\Big\{2x^T(t)\Big[e^{-2\alpha h_1}R-S_2A_i\Big]x(t-h_1)\Big\}\\
&+E\Big\{2x^T(t)\Big[e^{-2\alpha h_2}R-S_3A_i\Big]x(t-h_2)\Big\}\\
&+E\Big\{2x^T(t)[PD_i-S_1D_i-S_4A_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t)[S_1-S_5A_i]\dot{x}(t)\Big\}\\
&+E\Big\{x^T(t-h_1)\Big[-e^{-2\alpha h_1}R\Big]x(t-h_1)\Big\}\\
&+E\Big\{2x^T(t-h_1)[-S_2D_i]x(t-h(t))\Big\}\\
&+E\Big\{x^T(t-h_2)\Big[-e^{-2\alpha h_2}R\Big]x(t-h_2)\Big\}\\
&+E\Big\{x^T(t-h_2)[-S_3D_i]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h_2)S_3\dot{x}(t)\Big\}\\
&+E\Big\{x^T(t-h(t))[-S_4D_i+2\rho_{i2}I]x(t-h(t))\Big\}\\
&+E\Big\{2x^T(t-h(t))[S_4-S_5D_i]\dot{x}(t)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ E\Big\{\dot{x}^T(t)\Big[S_5 + S_5^T + h_1^2 R + h_2^2 R\Big]\dot{x}(t)\Big\} \\
&= E\Big\{\zeta^T(t)\mathcal{M}_i\zeta(t)\Big\},
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
&\zeta(t) = [x(t), x(t-h_1), x(t-h_2), x(t-h(t)), \dot{x}(t)], \\
&\mathcal{M}_i = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{bmatrix}, \\
&M_{11} = A_i^T P + PA_i - S_1 A_i - A_i^T S_1^T \\
&\qquad + 2\alpha P - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R + 2\rho_{i1}, \\
&M_{12} = e^{-2\alpha h_1} R - S_2 A_i, \\
&M_{13} = e^{-2\alpha h_2} R - S_3 A_i, \\
&M_{14} = PD_i - S_1 D_i - S_4 A_i, \\
&M_{15} = S_1 - S_5 A_i, \\
&M_{22} = -e^{-2\alpha h_1} R, \\
&M_{24} = -S_2 D_i, \\
&M_{33} = -e^{-2\alpha h_2} R, \\
&M_{34} = -S_3 D_i, \\
&M_{44} = -S_4 D_i + 2\rho_{i2} I, \\
&M_{45} = S_4 - S_5 D_i, \\
&M_{55} = S_5 + S_5^T + h_1^2 R + h_2^2 R.
\end{aligned}
\tag{3.14}
$$

Therefore, we finally obtain from (3.13) and the condition (i) that

$$
E\big\{\dot{V}(\cdot) + 2\alpha V(\cdot)\big\} < 0, \quad \forall i = 1, 2, \ldots, N,\ t \in R^+,
\tag{3.15}
$$

and hence

$$
E\big\{\dot{V}(t, x_t)\big\} \le -E\{2\alpha V(t, x_t)\}, \quad \forall t \in R^+.
\tag{3.16}
$$

Integrating both sides of (3.16) from 0 to t, we obtain

$$E\{V(t,x_t)\} \le E\left\{V(\phi)e^{-2\alpha t}\right\}, \quad \forall t \in R^+. \tag{3.17}$$

Furthermore, taking condition (3.5) into account, we have

$$E\left\{\lambda_1\|x(t,\phi)\|^2\right\} \le E\{V(x_t)\} \le E\left\{V(\phi)e^{-2\alpha t}\right\} \le E\left\{\lambda_2 e^{-2\alpha t}\|\phi\|^2\right\}, \tag{3.18}$$

then

$$E\{\|x(t,\phi)\|\} \le E\left\{\sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|\right\}, \quad t \in R^+, \tag{3.19}$$

By Definition 2.1, the system (2.1) is exponentially stable in the mean square. The proof is complete. □

To illustrate the obtained result, let us give the following numerical examples.

Example 3.2. Consider the following the switched stochastic systems with interval time-varying delay (2.1), where the delay function $h(t)$ is given by

$$\begin{gathered} h(t) = 0.1 + 0.8311\sin^2 3t, \\ A_1 = \begin{pmatrix} -1 & 0.01 \\ 0.02 & -2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1.1 & 0.02 \\ 0.01 & -2 \end{pmatrix}, \\ D_1 = \begin{pmatrix} -0.1 & 0.01 \\ 0.02 & -0.3 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} -0.1 & 0.02 \\ 0.01 & -0.2 \end{pmatrix}. \end{gathered} \tag{3.20}$$

It is worth noting that the delay function $h(t)$ is nondifferentiable. Therefore, the methods used is in [2–15] are not applicable to this system. By LMI toolbox of MATLAB, by using LMI Toolbox in MATLAB, the LMI (i) is feasible with $h_1 = 0.1,\ h_2 = 0.9311,\ \alpha = 0.1,\ \rho_{11} = 0.01,\ \rho_{12} = 0.01,\ \rho_{21} = 0.01,\ \rho_{22} = 0.01$, and

$$\begin{gathered} P = \begin{pmatrix} 2.0788 & -0.0135 \\ -0.0135 & 1.5086 \end{pmatrix}, \qquad R = \begin{pmatrix} 1.0801 & -0.0042 \\ -0.0042 & 0.8450 \end{pmatrix}, \\ S_1 = \begin{pmatrix} -0.6210 & -0.0335 \\ 0.0499 & -0.3576 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} -0.3602 & 0.0170 \\ 0.0298 & -0.3550 \end{pmatrix}, \\ S_3 = \begin{pmatrix} -0.3602 & 0.0170 \\ 0.0298 & -0.3550 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 0.6968 & -0.0401 \\ -0.0525 & 0.7040 \end{pmatrix}, \qquad S_5 = \begin{pmatrix} -1.4043 & 0.0265 \\ -0.0028 & -0.9774 \end{pmatrix} \end{gathered} \tag{3.21}$$

By Theorem 3.1 the switched stochastic systems (2.1) are 0.1-exponentially stable in the mean square and the switching rule is chosen as $\gamma(x(t)) = i$. Moreover, the solution $x(t,\phi)$ of the system satisfies

$$E\{\|x(t,\phi)\|\} \leq E\left\{1.8731e^{-0.1t}\|\phi\|\right\}, \quad \forall t \in R^{+}. \tag{3.22}$$

4. Conclusions

In this paper, we have proposed new delay-dependent conditions for the mean square exponential stability of switched stochastic system with non-differentiable interval time-varying delay. By constructing a set of improved Lyapunov-Krasovskii functionals and Newton-Leibniz formula, the conditions for the exponential stability of the systems have been established in terms of LMIs.

Acknowledgments

This work was supported by the Thai Research Fund Grant, the Higher Education Commission and Faculty of Science, Maejo University, Thailand.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 589202, 14 pages
doi:10.1155/2012/589202

Research Article

The Permanence and Extinction of the Single Species with Contraception Control and Feedback Controls

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Received 8 February 2012; Accepted 3 April 2012

Academic Editor: Elena Braverman

Population control has become a major problem in many wildlife species. Sterility control through contraception has been proposed as a method for reducing population size. In this paper, the single species with sterility control and feedback controls is considered. Sufficient conditions are obtained for the permanence and extinction of the system. The results show that the feedback controls do not influence the permanence of the species.

1. Introduction

Control of wildlife pest populations has usually relied on methods like chemical pesticides, biological pesticides, remote sensing and measure, computers, atomic energy, and so forth. Some brilliant achievements have been obtained. However, the warfare will never be over. Although a large variety of pesticide were used to control wildlife pest populations, the wildlife pests impairing crops are increasing especially because of resistance to the pesticide. So the pesticides are invalid. Moreover, wildlife pests will continue. On the other hand, the chemical pesticide kills not only wildlife pests but also their natural enemies. Therefore, wildlife pests are rampant. Now, sterile control to suppress wildlife pests is one of the most important measures in wildlife pest control. Sterile control [1–5] is especially for the purpose of suppressing the abundance of the pest in a new target region to a level at which it no longer causes economic damage. This can be achieved by releasing sterile insects into the environment in very large numbers in order to mate with the native insects that are present in the environment. A native female that mates with a sterile male will produce eggs, but the eggs will not hatch (the same effect will occur for the reciprocal cross). If there is a sufficiently high number of sterile insects than most of the crosses are sterile, and as time goes on, the number of native insects decreases and the ratio of sterile to normal insects increases, thus driving the native population to extinction. Sterile male techniques were first used successfully in 1958 in Florida to control Screwworm fly (Cochliomya omnivorax). A number of mathematical models have been done to assist the effectiveness of the SIT (see, e.g., [2–4]). Recently, Liu and Li [6] considered the following contraception control model:

$$\begin{aligned}
x_1'(t) &= x_1(t)\big(b - k(x_1(t) + x_2(t)) - \mu\big),\\
x_2'(t) &= \mu x_1(t) - [d + k(x_1(t) + x_2(t))]x_2(t),
\end{aligned} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ represent, respectively, the density of the fertile species and the sterile species at time t. The authors proved the equilibrium point of system (1.1) is stable under appropriate conditions. In view of the effects of a periodically changing environment, we consider the following nonautonomous contraception model:

$$\begin{aligned}
x_1'(t) &= x_1(t)\big(b(t) - k(t)(x_1(t) + x_2(t)) - \mu(t)\big),\\
x_2'(t) &= \mu(t)x_1(t) - [d(t) + k(t)(x_1(t) + x_2(t))]x_2(t).
\end{aligned} \tag{1.2}$$

However, we note that ecosystem in the real world are continuously disturbed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecosystem is the question of whether or not an ecosystem can withstand those unpredictable forces which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. So it is necessary to study models with control variables which are so-called disturbance functions, and to find some suitable conditions to prevent a particular species from dying out. In 1993, Gopalsamy and Weng [7] introduced a feedback control variable into the delay logistic model and discussed the asymptotic behavior of solution in logistic models with feedback controls, in which the control variables satisfy certain differential equation.

In recent years, the population dynamical systems with feedback controls have been studied in many articles, for example, see [7–12] and references cited therein. However, to the best of the authors knowledge, to this day, still less scholars consider the nonautonomous single species with contraception control and feedback controls.

Motivated by the above works, we focus our attention on the permanence of species for the following single species nonautonomous systems with delays and feedback control:

$$\begin{aligned}
x_1'(t) &= x_1(t)\big(b(t) - k(t)(x_1(t) + x_2(t)) - \mu(t) - c_1(t)u(t - \sigma_1(t))\big),\\
x_2'(t) &= \mu(t)x_1(t) - [d(t) + k(t)(x_1(t) + x_2(t)) + c_2(t)u(t - \sigma_2(t))]x_2(t),\\
u'(t) &= -e(t)u(t) + f_1(t)x_1(t - \tau_1(t)) + f_2(t)x_2(t - \tau_2(t)),
\end{aligned} \tag{1.3}$$

where $x_1(t)$, $x_2(t)$ represent, respectively, the density of the fertile population and the sterile population at time t. $u(t)$ is the control variable at time t. $b(t)$, $k(t)$ represent, respectively, the intrinsic growth rate and density-dependent rate of the species at time t, respectively. $\mu(t)$ is the migration rates from the fertile population to the sterile population. The function $b(t)$ is bounded continuous defined on $R_+ = [0, \infty)$; functions $\mu(t)$, $k(t)$, $d(t)$, $e(t)$, $c_i(t)$,

$f_i(t)$, $\sigma_i(t)$, and $\tau_i(t)$ $(i = 1,2)$ are continuous, bounded, and nonnegative defined on R_+. Let $\tau = \sup\{\tau_1(t), \tau_2(t), \sigma_1(t), \sigma_2(t) : t \geq 0\}$.

We will consider (1.3) together with initial conditions:

$$x_i(\theta) = \phi_i(\theta), \qquad u(\theta) = \psi(\theta), \tag{1.4}$$

where $\phi_i(\theta), \psi(\theta) \in BC^+[-\tau, 0]$ and

$$BC^+[-\tau, 0] = \{\phi \in C([-\tau, 0], [0, +\infty]) : \phi(0) > 0, \phi\ (\theta) \text{ is bounded}\}. \tag{1.5}$$

By the fundamental theory of functional differential equations [12], it is not difficult to see that the solution $(x_1(t), x_2(t), u(t))$ of (1.3) is unique and positive if initial functions satisfy initial condition (1.4). So, in this paper, the solution of (1.3) satisfying initial conditions (1.4) is said to be positive.

The main purpose of this paper is to establish a new general criterion for the permanence and the extinction of (1.3), which is described by integral form and independent feedback controls. The paper is organized as follows. In the next section, we will give some assumptions and useful lemmas. In Section 3, some new sufficient conditions which guarantee the permanence of all positive solutions for (1.3) are obtained. In Section 4, we obtained some new sufficient conditions which guarantee the extinction of all positive solutions for (1.3).

2. Preliminaries

Throughout this paper, we will introduce the following assumptions:

(H_1) there exist constants $\omega_i > 0$ $(i = 1,2)$ such that

$$\liminf_{t\to\infty} \int_t^{t+\omega_1} [b(s) - \mu(s)]ds > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\omega_2} \mu(s)ds > 0; \tag{2.1}$$

(H_2) there exist constants $\lambda_i > 0$ $(i = 1,2)$, such that

$$\liminf_{t\to\infty} \int_t^{t+\lambda_1} k(s)ds > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\lambda_2} d(s)ds > 0; \tag{2.2}$$

(H_3) there exists a constant $\gamma > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+\gamma} e(s)ds > 0. \tag{2.3}$$

In addition, for a function $g(t)$ defined on set $I \subset R$, we denote

$$g^L = \inf_{t\in I} g(t), \qquad g^M = \sup_{t\in I} g(t). \tag{2.4}$$

Now, we state several lemmas which will be useful in the proving of main results in this paper.

First, we consider the following nonautonomous logistic equation:

$$x'(t) = x(t)(r(t) - a(t)x(t)), \tag{2.5}$$

where functions $a(t)$, $r(t)$ are bounded and continuous on R_+. Furthermore, $a(t) \geq 0$ for all $t \geq 0$. We have the following result which is given in [13] by Teng and Li.

Lemma 2.1. *Suppose that there exist constants λ, σ such that*

$$\liminf_{t\to\infty} \int_t^{t+\lambda} r(s)ds > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\sigma} a(s)ds > 0. \tag{2.6}$$

Then,

(a) *there exist positive constants m and M such that*

$$m \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M. \tag{2.7}$$

for any positive solution $x(t)$ of (2.5);

(b) $\lim_{t\to\infty}(x^{(1)}(t) - x^{(2)}(t)) = 0$ *for any two positive solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ of* (2.5).

Further, consider the following nonautonomous linear equation:

$$u'(t) = r(t) - e(t)u(t), \tag{2.8}$$

where functions $r(t)$ and $e(t)$ are bounded continuous defined on R_+, and $r(t) \geq 0$ for all $t \geq 0$. One has the following result.

Lemma 2.2. *Suppose that (H_3) holds. Then,*

(a) *there exists a positive constant U such that $\limsup_{t\to\infty} u(t) \leq U$ for any positive solution $u(t)$ of* (2.8);

(b) $\lim_{t\to\infty}(u^{(1)}(t) - u^{(2)}(t)) = 0$ *for any two positive solutions $u^{(1)}(t)$ and $u^{(2)}(t)$ of* (2.8).

The proof of Lemma 2.2 is very simple by making a transformation with $u(t) = 1/x(t)$. This produces the calculations: $u'(t) = -(1/x^2(t))x'(t)$ and $x'(t) = x(t)(e(t) - r(t)x(t))$. Then, according to the Lemma 2.1 one can obtain Lemma 2.2.

Lemma 2.3. *Suppose that (H_3) holds. Then, for any constants $\epsilon > 0$ and $M > 0$, there exist constants $\delta = \delta(\epsilon) > 0$ and $T = T(\epsilon, M) > 0$ such that for any $t_0 \in R_+$ and $u_0 \in R$ with $|u_0| \leq M$, when $|r(t)| < \delta$ for all $t \geq t_0$, one has*

$$|u(t, t_0, u_0)| < \epsilon \quad \forall t \geq t_0 + T, \tag{2.9}$$

where $u(t, t_0, u_0)$ is the solution of (2.8) *with initial condition $u(t_0) = u_0$.*

The proof of Lemma 2.3 can be found as Lemma 2.4 in [11] by Wang et al.

3. Main Results

In this section, we study the permanence of species $x_1(t)$, $x_2(t)$ of (1.3). First, we have the theorem on the ultimate boundedness of all positive solutions of (1.3).

Theorem 3.1. *Suppose that assumptions* (H_1)–(H_3) *hold. Then, any positive solution of* (1.3) *is ultimate bounded, in the sense that there exists a positive constant* $M > 0$ *such that*

$$\limsup_{t\to\infty} x_1(t) < M, \qquad \limsup_{t\to\infty} x_2(t) < M, \qquad \limsup_{t\to\infty} u(t) < M, \tag{3.1}$$

for any positive solution $(x_1(t), x_2(t), u(t))$ *of* (1.3).

Proof. Let $(x_1(t), x_2(t), u(t))$ be any positive solution of (1.3). We first prove that the component $x_1(t)$ of (1.3) is ultimately bounded. From the first equation of (1.3), we have

$$\frac{dx_1(t)}{dt} \le x_1(t)\big(b(t) - \mu(t) - k(t)x_1(t)\big). \tag{3.2}$$

We consider the following auxiliary equation:

$$\frac{dy(t)}{dt} = y(t)\big(b(t) - \mu(t) - k(t)y(t)\big), \tag{3.3}$$

then by (H_1) and applying Lemma 2.1, there exists a constant M_1 such that

$$\limsup_{t\to\infty} y(t) < M_1 \tag{3.4}$$

for any positive solution $y(t)$ of (3.3). Let $y^*(t)$ be the solution of (3.3) satisfying initial condition $y^*(t_0) = x_1(t_0)$. Further, from comparison theorem, it follows that

$$x_1(t) < y^*(t) \quad \forall t > t_0. \tag{3.5}$$

Thus, we finally obtain that

$$\limsup_{t\to\infty} x_1(t) < M_1. \tag{3.6}$$

From inequality (3.6), we obtain that there exists a positive constant T_1 such that

$$x_1(t) < M_1 \quad \forall t \ge T_1. \tag{3.7}$$

Hence, from the second equation of (1.3), one has

$$\frac{dx_2(t)}{dt} \leq \mu(t)M_1 - d(t)x_2(t) \tag{3.8}$$

for all $t \geq T_1$. Further, consider the following auxiliary equation:

$$\frac{dv(t)}{dt} = \mu(t)M_1 - d(t)v(t), \tag{3.9}$$

from assumptions (H_1) and (H_2) and according to Lemma 2.2, there exists constant $M_2 > 0$ such that

$$\limsup_{t\to\infty} v(t) < M_2 \tag{3.10}$$

for the solution $v(t)$ of (3.9) with initial condition $v(T_1) = x_2(T_1)$. By the comparison theorem, we have

$$x_2(t) \leq v(t) \quad \forall t \geq T_1. \tag{3.11}$$

From this, we further obtain

$$\limsup_{t\to\infty} x_2(t) < M_2. \tag{3.12}$$

Then, we obtain that there exists constant $T_2 > T_1$ such that

$$x_2(t) < M_2 \quad \forall t \geq T_2. \tag{3.13}$$

From the third equation of (1.3), we have

$$\frac{du(t)}{dt} \leq -e(t)u(t) + M_1 f_1(t) + M_2 f_2(t) \tag{3.14}$$

for all $t \geq T_2 + \tau$. Consider the following auxiliary equation:

$$\frac{dv(t)}{dt} = -e(t)v(t) + M_1 f_1(t) + M_2 f_2(t). \tag{3.15}$$

By assumption (H_3) and conclusions of Lemma 2.2, we can get that there exists a constant $M_3 > 0$ such that

$$\limsup_{t\to\infty} v(t) < M_3 \tag{3.16}$$

for the solution $v(t)$ of (3.15) with initial condition $v(T_2+\tau) = u(T_2+\tau)$. By the comparison theorem, we have

$$u(t) \leq v(t) \quad \forall t \geq T_2 + \tau. \tag{3.17}$$

Hence, we further obtain

$$\limsup_{t\to\infty} u(t) < M_3. \tag{3.18}$$

Choose the constant $M = \max\{M_1, M_2, M_3\}$, then we finally obtain

$$\limsup_{t\to\infty} x_1(t) < M, \qquad \limsup_{t\to\infty} x_2(t) < M, \qquad \limsup_{t\to\infty} u(t) < M. \tag{3.19}$$

This completes the proof. □

Theorem 3.2. *Suppose that assumptions* (H_1)*–*(H_3) *hold. Then, there exists a constant* $\eta > 0$*, which is independent of any solution of* (1.3)*, such that*

$$\liminf_{t\to\infty} x_1(t) > \eta, \qquad \liminf_{t\to\infty} x_2(t) > \eta, \tag{3.20}$$

for any positive solution $(x_1(t), x_2(t), u(t))$ *of* (1.3).

Proof. Let $(x_1(t), x_2(t), u(t))$ be a solution of (1.3) satisfying initial condition (1.4). In view of Theorem 3.1, there exists a T_0 such that for all $t > T_0$ we have $x_i(t) < M,\ u(t) < M\ (i = 1, 2)$. According to (H_1), we can choose constants $\varepsilon_0 > 0$ and $T_1 > 0$ such that, for all $t \geq T_1$, we have

$$\int_t^{t+\omega_1} (b(s) - \mu(s) - 2k(s)\varepsilon_0 - c_1(s)\varepsilon_0)\,ds > \varepsilon_0. \tag{3.21}$$

Next, we consider the following equation:

$$\frac{du(t)}{dt} = -e(t)u(t) + f_1(t)\alpha_0 + f_2(t)\alpha_0, \tag{3.22}$$

where α_0 was given in the later. By (H_3), we have (3.22) satisfying all the conditions of Lemma 2.3. So, we can obtain that, for given constants $\varepsilon_0 > 0$ and $M > 0$ (M was given in Theorem 3.1), there exist constants $\delta_0 = \delta_0(\varepsilon_0) > 0$ and $T^* = T^*(\varepsilon_0, M) > 0$ such that for any $t_0 \in R_+$ and $0 \leq u_0 \leq M$, when $\alpha_0 f_1(t) + \alpha_0 f_2(t) < \delta_0$ for all $t \geq t_0$, we have

$$u(t, t_0, u_0) < \varepsilon_0 \quad \forall t \geq t_0 + T^*, \tag{3.23}$$

where $u(t, t_0, u_0)$ is the solution of (3.22) with initial condition $u(t_0) = u_0$.

Further, consider the following equation:

$$\frac{dz(t)}{dt} = \mu(t)\alpha_0 - d(t)z(t), \tag{3.24}$$

where α_0 was given in (3.22). By (H_2), we have (3.24) satisfying all the conditions of Lemma 2.3, so by Lemma 2.3, for given constants $\alpha_0 > 0$ and $M > 0$, there exist constants $\delta_1 = \delta_1(\alpha_0) > 0$ and $T^{**} = T^{**}(\alpha_0, M) > 0$ such that for any $t_0 \in R_+$ and $0 \le z_0 \le M$, when $\mu(t)\alpha_0 < \delta_1$ for all $t \ge t_0$, we have

$$z(t, t_0, z_0) < \alpha_0 \quad \forall t \ge t_0 + T^{**}, \tag{3.25}$$

where $z(t, t_0, z_0)$ is the solution of (3.24) with initial condition $z(t_0) = z_0$.

Let $\alpha_0 \le \min\{\varepsilon_0, (\delta_1/(\mu^M + 1)), \ (\delta_0/(f_1^M + f_2^M + 1))\}$ such that for all $t \ge T_1$, we have

$$\int_t^{t+\omega_1} (b(s) - \mu(s) - 2k(s)\alpha_0 - f_1(s)\varepsilon_0)ds > \varepsilon_0. \tag{3.26}$$

We first prove that

$$\limsup_{t\to\infty} x_1(t) \ge \alpha_0. \tag{3.27}$$

In fact, if (3.27) is not true, then there exist a positive solution $(x_1(t), x_2(t), u(t))$ of (1.3) and a constant $T_2 \ge T_1$ such that

$$x_1(t) < \alpha_0, \tag{3.28}$$

for all $t \ge T_2$.

From the second equation of (1.3), we have

$$\frac{dx_2(t)}{dt} \le \mu(t)\alpha_0 - d(t)x_2(t) \tag{3.29}$$

for all $t \ge T_2$. Let $\widehat{z}(t)$ be the solution of (3.24) with initial condition $\widehat{z}(T_2) = x_2(T_2)$, by the comparison theorem, we have $x_2(t) \le \widehat{z}(t)$ for all $t \ge T_2$. In (3.25), we choose $t_0 = T_2$ and $\widehat{z}_0 = x_2(T_2)$, since $\mu(t)\alpha_0 < \delta_1$, we obtain $\widehat{z}(t, t_0, \widehat{z}_0) < \alpha_0$ for all $t \ge T_2 + T^{**}$. Hence, we further obtain

$$x_2(t, T_2, x_2(T_2)) < \alpha_0 \quad \forall t \ge T_2 + T^{**}. \tag{3.30}$$

By applying (3.28) and (3.30) to the third equation of (1.3), it follows that

$$\frac{du(t)}{dt} \le -e(t)u(t) + f_1(t)\alpha_0 + f_2(t)\alpha_0, \tag{3.31}$$

for all $t \geq T_2 + T^{**} + \tau$. Let $\widehat{u}(t)$ be the solution of (3.22) with initial condition $\widehat{u}(T_2 + T^{**} + \tau) = u(T_2 + T^{**} + \tau)$, by the comparison theorem, we have

$$u(t) \leq \widehat{u}(t) \quad \forall t \geq T_2 + T^{**} + \tau. \tag{3.32}$$

In (3.23), we choose $t_0 = T_2 + T^{**} + \tau$ and $\widehat{u}_0 = u(T_2 + T^{**} + \tau)$, since $\alpha_0 f_1(t) + \alpha_0 f_2(t) < \delta_0$ for all $t \geq T_2 + T^{**} + \tau$, we obtain

$$\widehat{u}(t, T_2 + T^{**} + \tau, u(T_2 + T^{**} + \tau)) < \varepsilon_0 \quad \forall t \geq T_2 + T^{**} + \tau + T^*. \tag{3.33}$$

Hence, from (3.32), we further obtain

$$u(t, T_2 + T^{**} + \tau, u(T_2 + T^{**} + \tau)) < \varepsilon_0 \quad \forall t \geq T_2 + T^{**} + \tau + T^*. \tag{3.34}$$

Hence, by (3.30) and (3.34) it follows that

$$\frac{dx_1(t)}{dt} \geq x_1(t)\big(b(t) - \mu(t) - 2k(t)\alpha_0 - c_1(t)\varepsilon_0\big), \tag{3.35}$$

for any $t > T_2 + T^{**} + T^* + 2\tau$. Integrating (3.35) from $T_3 = T_2 + T^{**} + T^* + 2\tau$ to t we have

$$x_1(t) \geq x_1(T_3) \exp \int_{T_3}^{t} \big(b(s) - \mu(s) - 2k(s)\alpha_0 - c_1(s)\varepsilon_0\big) ds. \tag{3.36}$$

Thus, from (3.26), we have $x_1(t) \to \infty$ as $t \to \infty$, which leads to a contradiction. So, (3.27) holds.

Now, we prove the conclusion of Theorem 3.2. In fact, if it is not true, then there exists an initial functions sequence $\{Z^{(m)}\} = \{(\phi_1^{(m)}(\theta), \phi_2^{(m)}(\theta), \psi^{(m)}(\theta))\}, \theta \in [-\tau, 0]$ such that

$$\liminf_{t\to\infty} x_1\left(t, Z^{(m)}\right) < \frac{\alpha_0}{(m+1)^2} \quad \forall\, m = 1, 2, \ldots, \tag{3.37}$$

for the solution $(x_1(t, Z^{(m)}), x_2(t, Z^{(m)}), u(t, Z^{(m)}))$ of (1.3). From (3.27) and (3.37), for every m, there are two time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$, satisfying $0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots$ and $\lim_{q\to\infty} s_q^{(m)} = \infty$, such that

$$x_1\left(s_q^{(m)}, Z^{(m)}\right) = \frac{\alpha_0}{m+1}, \qquad x_1\left(t_q^{(m)}, Z^{(m)}\right) = \frac{\alpha_0}{(m+1)^2}, \tag{3.38}$$

$$\frac{\alpha_0}{(m+1)^2} \leq x_1\left(t, Z^{(m)}\right) \leq \frac{\alpha_0}{m+1} \quad \forall t \in \left(s_q^{(m)}, t_q^{(m)}\right). \tag{3.39}$$

From Theorem 3.1, we can choose a positive constant $T^{(m)}$ such that $x_1(t, Z^{(m)}) < M$ and $x_2(t, Z^{(m)}) < M$ and $u(t, Z^{(m)}) < M$, for all $t > T^{(m)}$. Further, there is an integer $K_1^{(m)} > 0$ such that $s_q^{(m)} > T^{(m)} + \tau$ for all $q \geq K_1^{(m)}$. Let $q \geq K_1^{(m)}$, for any $t \in (s_q^{(m)}, t_q^{(m)})$, we have

$$\begin{aligned}\frac{dx_1(t, Z^{(m)})}{dt} &\geq x_1\left(t, Z^{(m)}\right)\left[b(t) - \mu(t) - k(t)M - k(t)M - c_1(t)M\right]\\ &\geq -\gamma_1 x_1\left(t, Z^{(m)}\right),\end{aligned} \tag{3.40}$$

where $\gamma_1 = \sup_{t\in R_+}\{|b(t) - \mu(t) - 2k(t)M - c_1(t)M|\}$. Integrating the above inequality from $s_q^{(m)}$ to $t_q^{(m)}$, we further have

$$\begin{aligned}\frac{\alpha_0}{(m+1)^2} &= x_1\left(t_q^{(m)}, Z^{(m)}\right) \geq x_1\left(s_q^{(m)}, Z^{(m)}\right)\exp\left[-\gamma_1\left(t_q^{(m)} - s_q^{(m)}\right)\right]\\ &= \frac{\alpha_0}{m+1}\exp\left[-\gamma_1\left(t_q^{(m)} - s_q^{(m)}\right)\right].\end{aligned} \tag{3.41}$$

Consequently,

$$t_q^{(m)} - s_q^{(m)} \geq \frac{\ln(m+1)}{\gamma_1} \quad \forall q \geq K_1^{(m)},\ m = 1, 2, \ldots, \tag{3.42}$$

we can choose a large enough N_0 such that

$$t_q^{(m)} - s_q^{(m)} \geq T^{**} + T^{*} + 2\tau + \omega_1 \quad \forall m \geq N_0,\ q \geq K_1^{(m)}. \tag{3.43}$$

For any $m \geq N_0, q \geq K_1^{(m)}$, and $t \in [s_q^{(m)}, t_q^{(m)}]$, from (3.39), we can obtain

$$\begin{aligned}\frac{dx_2(t, Z^{(m)})}{dt} &\leq \mu(t)\frac{\alpha_0}{m+1} - d(t)x_2\left(t, Z^{(m)}\right)\\ &\leq \mu^M\alpha_0 - d(t)x_2\left(t, Z^{(m)}\right).\end{aligned} \tag{3.44}$$

Let $\widehat{z}(t)$ be solution of (3.24) with initial condition $\widehat{z}(s_q^{(m)}) = x_2(s_q^{(m)})$, by the comparison theorem, we have $x_2(t) \leq \widehat{z}(t)$ for all $t \geq s_q^{(m)}$. In (3.25), we choose $t_0 = s_q^{(m)}$ and $\widehat{z}_0 = z(s_q^{(m)})$, since $\mu^M\alpha_0 < \delta_1$ and $x_2(s_q^{(m)} < M$, we obtain $\widehat{z}(t, s_q^{(m)}, z(s_q^{(m)})) < \alpha_0$ for all $t \geq s_q^{(m)} + T^{**}$. By Lemma 2.3, we can obtain that $T^{**} = T^{**}(\alpha_0, M)$ is independent of m. Hence, we further obtain

$$x_2\left(t, s_q^{(m)}, x_2\left(s_q^{(m)}\right)\right) < \alpha_0, \tag{3.45}$$

for all $t > s_q^{(m)} + T^{**}$. From the third equation of (1.3), we obtain

$$\begin{aligned}\frac{du(t,Z^{(m)})}{dt} &= -e(t)u\left(t,Z^{(m)}\right) + f_1(t)x_1\left(t-\tau_1(t),Z^{(m)}\right) + f_2(t)x_2\left(t-\tau_2(t),Z^{(m)}\right) \\ &\le -e(t)u\left(t,Z^{(m)}\right) + f_1(t)\alpha_0 + f_2(t)\alpha_0,\end{aligned} \tag{3.46}$$

for all $t \ge s_q^{(m)} + T^{**} + \tau$. Assume that $\tilde{u}(t)$ is the solution of (3.22) with initial condition $\tilde{u}(s_q^{(m)} + T^{**} + \tau) = u(s_q^{(m)} + T^{**} + \tau)$, then we have

$$u\left(t,Z^{(m)}\right) \le \tilde{u}(t) \quad \forall t \in \left[s_q^{(m)} + T^{**} + \tau, t_q^{(m)}\right],\ m \ge N_0,\ q \ge K_1^{(m)}. \tag{3.47}$$

In (3.23), we choose $t_0 = s_q^{(m)} + T^{**} + \tau$ and $u_0 = u(s_q^{(m)} + T^{**} + \tau)$. Obviously, $\alpha_0(f_1(t)+f_2(t)) < \delta_0$ for all $t \ge s_q^{(m)} + T^{**} + \tau$. So, we have

$$\tilde{u}(t) = \tilde{u}\left(t, s_q^{(m)} + T^{**} + \tau, u\left(s_q^{(m)} + T^{**} + \tau\right)\right) < \varepsilon_0 \tag{3.48}$$

for all $t \in [s_q^{(m)} + T^{**} + \tau + T^*, t_q^{(m)}]$. Using the comparison theorem, it follows that

$$u\left(t,Z^{(m)}\right) < \varepsilon_0 \tag{3.49}$$

for all $t \in [s_q^{(m)} + T^{**} + \tau + T^*, t_q^{(m)}]$, $q \ge K_1^{(m)}$, and $m \ge N_0$.

So, for any $m \ge N_0$, $q \ge K_1^{(m)}$, and $t \in [s_q^{(m)} + T^{**} + T^* + 2\tau, t_q^{(m)}]$, from (3.26), (3.45), and (3.49), it follows

$$\begin{aligned}\frac{dx_1(t,Z^{(m)})}{dt} &= x_1\left(t,Z^{(m)}\right)\left(b(t)-\mu(t)-k(t)\left(x_1\left(t,Z^{(m)}\right)+x_2\left(t,Z^{(m)}\right)\right)-c_1(t)u\left(t-\sigma_1(t),Z^{(m)}\right)\right) \\ &\ge x_1\left(t,Z^{(m)}\right)\left(b(t)-\mu(t)-2k(t)\alpha_0-c_1(t)\varepsilon_0\right).\end{aligned} \tag{3.50}$$

Integrating the above inequality from $t_q^{(m)} - \omega_1$ to $t_q^{(m)}$, then from (3.26), we obtain

$$\begin{aligned}\frac{\alpha_0}{(m+1)^2} &= x_1\left(t_q^{(m)}, Z^{(m)}\right) \\ &\ge x_1\left(t_q^{(m)} - \omega_1, Z^{(m)}\right)\exp\int_{t_q^{(m)}-\omega_1}^{t_q^{(m)}}\left(b(t)-\mu(t)-2k(t)\alpha_0-c_1(t)\varepsilon_0\right)dt \\ &\ge \frac{\alpha_0}{(m+1)^2}\exp(\alpha_0) > \frac{\alpha_0}{(m+1)^2},\end{aligned} \tag{3.51}$$

which leads to a contradiction. Therefore, this contradiction shows that there exists constant $\eta_1 > 0$ such that

$$\liminf_{t\to\infty} x_1(t) > \eta_1, \tag{3.52}$$

for any positive solution $(x_1(t), x_2(t), u(t))$ of (1.3). Therefore, there exists constant $\eta_2 > 0$ such that

$$\liminf_{t\to\infty} x_2(t) > \eta_2, \tag{3.53}$$

for any positive solution $(x_1(t), x_2(t), u(t))$ of (1.3). This completes the proof. □

Remark 3.3. In Theorem 3.2, we note that (H_1)–(H_3) are decided by (1.3), which is independent of the feedback controls. So, the feedback controls have no influence on the permanence of (1.3).

4. Extinction

In this section, we discuss the extinction of the component $x_1(t), x_2(t)$ of (1.3).

Theorem 4.1. *Suppose that there exist constants σ_1, σ_2, such that*

$$\limsup_{n\to\infty} \int_t^{t+\sigma_1} \big(b(s) - \mu(s)\big) ds < 0; \qquad \liminf_{t\to\infty} \int_t^{t+\sigma_2} k(s) ds > 0 \tag{4.1}$$

hold. Then,

$$\lim_{t\to\infty} x_1(t) = 0, \qquad \lim_{t\to\infty} x_2(t) = 0, \tag{4.2}$$

for any positive solution $(x_1(t), x_2(t), u(t))$ of (1.3).

Proof. By the condition, for every given positive constant ε, there exist constants ε_1 and T_1 such that

$$\int_t^{t+\sigma_1} \big(b(s) - \mu(s) - k(s)\varepsilon\big) ds < -\varepsilon_1, \tag{4.3}$$

for all $t > T_1$. First, we show that there exists a $T_2 > T_1$, such that $x_1(T_2) < \varepsilon$. Otherwise, we have

$$x_1(t) \geq \varepsilon, \qquad \forall t > T_1. \tag{4.4}$$

Hence, for all $t \geq T_1$, one has

$$x_1'(t) < x_1(t)\big(b(t) - k(t)\varepsilon - \mu(t)\big). \tag{4.5}$$

Thus, as $t \to +\infty$, we have

$$\begin{aligned} \varepsilon \le x_1(t) &< x_1(T_1)\exp\left\{\int_{T_1}^{t}(b(s)-\mu(s)-k(s)\varepsilon)ds\right\} \\ &< x_1(T_1)\exp\{-\varepsilon_1(t-T_1)\} \longrightarrow 0. \end{aligned} \tag{4.6}$$

So, $\varepsilon < 0$, which leads to a contradiction. Therefore, there exists a $T_2 > T_1$, such that $x_1(T_2) < \varepsilon$.

Second, we show that

$$x_1(t) < \varepsilon\exp\{\mu_1\sigma_1\} \quad \forall t > T_2, \tag{4.7}$$

where $\mu_1 = \max_{t\in[0,+\infty]}\{b(t)+\mu(t)+k(t)\varepsilon\}$. Otherwise, there exists a $T_3 > T_2$, such that $x_1(T_3) > \varepsilon\exp\{\mu_1\sigma_1\}$. By the continuity of $x_1(t)$, there must exist $T_4 \in (T_2, T_3)$ such that $x_1(T_4) = \varepsilon$ and $x_1(t) > \varepsilon$ for $t \in (T_4, T_3)$. Let P_1 be the nonnegative integer such that $T_3 \in (T_4 + P_1\sigma_1, T_4 + (P_1 + 1)\sigma_1)$. Further, we obtain that

$$\begin{aligned} \varepsilon\exp\{\mu_1\sigma_1\} < x_1(T_3) &< x_1(T_4)\exp\left\{\int_{T_4}^{T_3}(b(s)-\mu(s)-k(s)\varepsilon)ds\right\} \\ &= x_1(T_4)\exp\left\{\int_{T_4}^{T_4+P_1\sigma_1}(b(s)-\mu(s)-k(s)\varepsilon)ds \right. \\ &\qquad\qquad \left. + \int_{T_4+P_1\sigma_1}^{T_3}(b(s)-\mu(s)-k(s)\varepsilon)ds\right\} \\ &< \varepsilon\exp\{\mu_1\sigma_1\}, \end{aligned} \tag{4.8}$$

which leads to contradiction. This shows that (4.7) holds. By the arbitrariness of ε, it immediately follows that $x_1(t) \to 0$ as $t \to +\infty$. Further, we can obtain that $x_2(t) \to 0$ as $t \to +\infty$. This completes the proof of Theorem 4.1. □

Acknowledgment

This work is supported by the National Natural Science Foundation of China (11071283), Research Project supported by Shanxi Scholarship Council of China (2011-093) and the Sciences Foundation of Shanxi (2009011005-3).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 351709, 23 pages
doi:10.1155/2012/351709

Research Article

Interval Oscillation Criteria of Second Order Mixed Nonlinear Impulsive Differential Equations with Delay

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Received 19 December 2011; Revised 10 April 2012; Accepted 11 April 2012

Academic Editor: Agacik Zafer

We study the following second order mixed nonlinear impulsive differential equations with delay $(r(t)\Phi_\alpha(x'(t)))'+p_0(t)\Phi_\alpha(x(t))+\sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t)$, $t \geq t_0$, $t \neq \tau_k$, $x(\tau_k^+) = a_k x(\tau_k)$, $x'(\tau_k^+) = b_k x'(\tau_k)$, $k = 1,2,\ldots$, where $\Phi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence, and $\tau_{k+1} - \tau_k > \sigma$. Some sufficient conditions for the interval oscillation criteria of the equations are obtained. The results obtained generalize and improve earlier ones. Two examples are considered to illustrate the main results.

1. Introduction

We consider the following second order impulsive differential equations with delay

$$
\begin{aligned}
&\left(r(t)\Phi_\alpha\left(x'(t)\right)\right)' + p_0(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(t-\sigma)) = e(t), \quad t \geq t_0,\ t \neq \tau_k, \\
&x\left(\tau_k^+\right) = a_k x(\tau_k), \qquad x'\left(\tau_k^+\right) = b_k x'(\tau_k), \quad k = 1,2,\ldots,
\end{aligned}
\tag{1.1}
$$

where $\Phi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence, and $\tau_{k+1} - \tau_k > \sigma$, for all $k \in \mathbb{N}$.

Let $J \subset \mathbb{R}$ be an interval, and we define

$$\begin{aligned}\mathrm{PLC}(J,\mathbb{R}) := \{y : J \longrightarrow \mathbb{R} \mid y \text{ is continuous everywhere except each } \tau_k \text{ at which } y(\tau_k^+)\\ \text{and } y(\tau_k^-) \text{ exist and } y(\tau_k^-) = y(\tau_k),\ k \in \mathbb{N}\}.\end{aligned} \tag{1.2}$$

For given t_0 and $\phi \in \mathrm{PLC}([t_0 - \sigma, t_0], \mathbb{R})$, we say $x \in \mathrm{PLC}([t_0 - \sigma, \infty), \mathbb{R})$ is a solution of (1.1) with initial value ϕ if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x(t) = \phi(t)$ for $t \in [t_0 - \sigma, t_0]$.

A solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Impulsive differential equation is an adequate mathematical apparatus for the simulation of processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, and so forth. Because it has more richer theory than its corresponding without impulsive differential equation, much research has been done on the qualitative behavior of certain impulsive differential equations (see [1, 2]).

In the last decades, there is constant interest in obtaining new sufficient conditions for oscillation or nonoscillation of the solutions of various impulsive differential equations, see, for example, [1–9] and the references cited therein.

In recent years, interval oscillation of impulsive differential equations was also arousing the interest of many researchers. In 2007, Özbekler and Zafer [10] investigated the following equations:

$$\begin{aligned}&(m(t)\varphi_\alpha(y'))' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,\\ &\Delta(m(t)\varphi_\alpha(y')) + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i,\ (i \in \mathbb{N}),\end{aligned} \tag{1.3}$$

$$\begin{aligned}&(m(t)y')' + s(t)y' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,\\ &\Delta(m(t)y') + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i,\ (\beta \geq 1),\end{aligned} \tag{1.4}$$

where $\varphi_*(u) = |u|^{*-1}u$, $\beta \geq \alpha$, $\{q_i\}$ and $\{f_i\}$ are sequences of real numbers. In 2009, they further gave a research [11] for equations of the form

$$\begin{aligned}&(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)\varphi_\beta(x) = e(t), \quad t \neq \theta_i,\\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\varphi_\beta(x) = e_i, \quad t = \theta_i,\end{aligned} \tag{1.5}$$

and obtained some interval oscillation results which improved and extended the earlier ones for the equations without impulses.

For the mixed type Emden-Fowler equations

$$\begin{aligned}&(r(t)x'(t))' + p(t)x(t) + \sum_{i=1}^{n} p_i(t)|x(t)|^{\alpha_i - 1}x(t) = e(t), \quad t \neq \tau_k,\\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \quad k \in \mathbb{N},\end{aligned} \tag{1.6}$$

Liu and Xu [12] established some interval oscillation results. Recently, Özbekler and Zafer [13] investigated the more general cases

$$
\begin{aligned}
&\left(r(t)\Phi_\alpha\left(x'(t)\right)\right)' + q(t)\Phi_\alpha(x(t)) + \sum_{k=1}^{n} q_k(t)\Phi_{\beta_k}(x(t)) = e(t), \quad t \neq \theta_i, \\
&x\left(\theta_i^+\right) = a_i x(\theta_i), \qquad x'\left(\theta_i^+\right) = b_i x'(\theta_i),
\end{aligned}
\tag{1.7}
$$

where $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$.

However, for the impulsive equations, almost all of interval oscillation results in the existing literature were established only for the case of "without delay." In other words, for the case of "with delay" the study on the interval oscillation is very scarce. To the best of our knowledge, Huang and Feng [14] gave the first research in this direction recently. They considered second order delay differential equations with impulses

$$
\begin{aligned}
&x''(t) + p(t)f(x(t-\tau)) = e(t), \quad t \geq t_0,\ t \neq t_k,\ k = 1,2,\ldots, \\
&x\left(t_k^+\right) = a_k x(t_k), \qquad x'\left(t_k^+\right) = b_k x'(t_k), \quad k = 1,2,\ldots
\end{aligned}
\tag{1.8}
$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses [15–17].

Motivated mainly by [13, 14], in this paper, we study the interval oscillation of the delay impulsive (1.1). By using some inequalities, Riccati transformation and $\mathscr{H}$ functions (introduced first by Philos [18]), we establish some interval oscillation criteria which generalize and improve some known results. Moreover, examples are considered to illustrate the main results.

2. Main Results

Throughout the paper, we always assume that the following conditions hold:

(A_1) the exponents satisfy that $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$;

(A_2) $r(t) \in C([t_0,\infty),(0,\infty))$ is nondecreasing, $e(t)$, $p_i(t) \in \mathrm{PLC}([t_0,\infty),\mathbb{R})$, $i = 0,1,\ldots,n$;

(A_3) $\{a_k\}$ and $\{b_k\}$ are real constant sequences such that $b_k \geq a_k > 0$, $k \in \mathbb{N}$.

It is clear that all solutions of (1.1) are oscillatory if there exists a subsequence $\{k_i\}$ of $\{k\}$ such that $a_{k_i} \leq 0$ for all $i \in \mathbb{N}$. So, we assume $a_k > 0$ for all $k \in \mathbb{N}$ in condition (A_3).

In this section, intervals $[c_1,d_1]$ and $[c_2,d_2]$ are considered to establish oscillation criteria. For convenience, we introduce the following notations (see [12]). Let

$$
\begin{aligned}
&k(s) = \max\{i : t_0 < \tau_i < s\}, \qquad r_j = \max\{r(t) : t \in [c_j,d_j]\}, \\
&\Omega(c_j,d_j) = \left\{w_j \in C^1[c_j,d_j] : w_j(t) \not\equiv 0,\ w_j(c_j) = w_j(d_j) = 0\right\}, \quad j = 1,2.
\end{aligned}
\tag{2.1}
$$

For two constants $c, d \notin \{\tau_k\}$ with $c < d$ and $k(c) < k(d)$ and a function $\varphi \in C([c,d],\mathbb{R})$, we define an operator $Q : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$Q_c^d[\varphi] = \varphi(\tau_{k(c)+1}) \frac{b_{k(c)+1}^{\alpha} - a_{k(c)+1}^{\alpha}}{a_{k(c)+1}^{\alpha}(\tau_{k(c)+1} - c)^{\alpha}} + \sum_{i=k(c)+2}^{k(d)} \varphi(\tau_i) \frac{b_i^{\alpha} - a_i^{\alpha}}{a_i^{\alpha}(\tau_i - \tau_{i-1})^{\alpha}}, \tag{2.2}$$

where $\sum_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\sigma)$, we need to consider the following cases for $k(c_j) < k(d_j)$

(S1) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma > d_j$,

(S2) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma < d_j$,

(S3) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma > d_j$,

(S4) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma < d_j$,

and the cases for $k(c_j) = k(d_j)$

($\overline{\text{S}}$1) $\tau_{k(c_j)} + \sigma < c_j$,

($\overline{\text{S}}$2) $c_j < \tau_{k(c_j)} + \sigma < d_j$,

($\overline{\text{S}}$3) $\tau_{k(c_j)} + \sigma > d_j$.

Combining (S$*$) with ($\overline{\text{S}}*$), we can get 12 cases. In order to save space, throughout the paper, we study (1.1) under the case of combination of (S1) with ($\overline{\text{S}}$1) only. The discussions for other cases are similar and omitted.

The following preparatory lemmas will be useful to prove our theorems. The first is derived from [19] and second is from [20].

Lemma 2.1. *For any given n-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying (A_1), then there exists an n-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \qquad \sum_{i=1}^{n} \eta_i = \lambda, \quad 0 < \eta_i < 1, \tag{2.3}$$

where $\lambda \in (0,1]$.

Lemma 2.2. *Suppose X and Y are nonnegative, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda}, \quad \lambda > 1, \tag{2.4}$$

where equality holds if and if $X = Y$.

Let $\alpha > 0$, $B \ge 0$, $A > 0$, and $y \ge 0$. Put

$$\lambda = 1 + \frac{1}{\alpha}, \qquad X = A^{\alpha/(\alpha+1)} y, \qquad Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} B^{\alpha} A^{-\alpha^2/(\alpha+1)}. \tag{2.5}$$

It follows from Lemma 2.2 that

$$By - Ay^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}. \tag{2.6}$$

Theorem 2.3. *Assume that for any $T \ge t_0$, there exist $c_j, d_j \notin \{\tau_k\}$, $j = 1,2$, such that $T < c_1 < d_1 \le c_2 < d_2$ and for $j = 1,2$*

$$\begin{gathered} p_i(t) \ge 0, \quad t \in [c_j - \sigma, d_j] \setminus \{\tau_k\}, \quad i = 0,1,2,\dots,n; \\ (-1)^j e(t) \ge 0, \quad t \in [c_j - \sigma, d_j] \setminus \{\tau_k\}. \end{gathered} \tag{2.7}$$

If there exist $w_j(t) \in \Omega(c_j, d_j)$ and $\rho(t) \in C^1([c_j, d_j], (0,\infty))$ such that, for $k(c_j) < k(d_j)$, $j = 1,2$,

$$\begin{aligned} &\int_{c_j}^{\tau_{k(c_j)+1}} W_j(t) \frac{\left(t - \tau_{k(c_j)} - \sigma\right)^\alpha}{\left(t - \tau_{k(c_j)}\right)^\alpha} dt \\ &\quad + \sum_{i=k(c_j)+1}^{k(d_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} W_j(t) \frac{(t-\tau_i)^\alpha}{b_i^\alpha (t+\sigma-\tau_i)^\alpha} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_j(t) \frac{(t-\tau_i-\sigma)^\alpha}{(t-\tau_i)^\alpha} dt \right] \\ &\quad + \int_{\tau_{k(d_j)}}^{d_j} W_j(t) \frac{\left(t - \tau_{k(d_j)}\right)^\alpha}{b_{k(d_j)}^\alpha \left(t + \sigma - \tau_{k(d_j)}\right)^\alpha} dt \\ &\quad + \int_{c_j}^{d_j} \rho(t) \left(p_0(t) |w_j(t)|^{\alpha+1} - r(t) \left(|w_j'(t)| + \frac{|\rho'(t)||w_j(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} \right) dt \ge \rho_j r_j Q_{c_j}^{d_j}\left[|w_j|^{\alpha+1} \right], \end{aligned} \tag{2.8}$$

where ρ_j is maximum value of $\rho(t)$ on $[c_j, d_j]$ and, for $k(c_j) = k(d_j)$, $j = 1,2$,

$$\int_{c_i}^{d_i} \left(W_j(t) \frac{(t-c_i)^\alpha}{(t-c_i+\sigma)^\alpha} + \rho(t) \left(p_0(t) |w_j(t)|^{\alpha+1} - r(t) \left(|w_j'(t)| + \frac{|\rho'(t)||w_j(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} \right) \right) dt \ge 0, \tag{2.9}$$

where $W_j(t) = \eta_0^{-\eta_0} |e(t)|^{\eta_0} \prod_{i=1}^n \eta_i^{-\eta_i} (p_i(t))^{\eta_i} |w_j(t)|^{\alpha+1}$ with $\eta_0 = 1 - \sum_{i=1}^n \eta_i$ and $\eta_1, \eta_2, \dots, \eta_n$ are positive constants satisfying conditions of Lemma 2.1, then (1.1) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \ge t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. Define

$$u(t) = \rho(t) \frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}, \quad t \in [c_1, d_1]. \tag{2.10}$$

It follows, for $t \neq \tau_k$, that

$$u'(t) = -\rho(t)p_0(t) - \rho(t)\left[\sum_{i=1}^{n} p_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma)) + \frac{|e(t)|}{\Phi_\alpha(x(t-\sigma))}\right]\frac{\Phi_\alpha(x(t-\sigma))}{\Phi_\alpha(x(t))} + \frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}. \tag{2.11}$$

Now, let

$$v_0 = \eta_0^{-1}\frac{|e(t)|}{\Phi_\alpha(x(t-\sigma))}, \qquad v_i = \eta_i^{-1}p_i(t)\Phi_{\beta_i-\alpha}(x(t-\sigma)), \quad i = 1,2,\ldots,n, \tag{2.12}$$

where $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying conditions of Lemma 2.1 and $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$. Employing in (2.11) the arithmetic-geometric mean inequality (see [20])

$$\sum_{i=0}^{n}\eta_i v_i \geq \prod_{i=0}^{n} v_i^{\eta_i} \tag{2.13}$$

and in view of (2.3), we have that

$$u'(t) \leq -\rho(t)p_0(t) - \rho(t)\varphi(t)\frac{\Phi_\alpha(x(t-\sigma))}{\Phi_\alpha(x(t))} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha} + \frac{\rho'(t)}{\rho(t)}u(t), \tag{2.14}$$

where

$$\varphi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}. \tag{2.15}$$

First, we consider the case $k(c_1) < k(d_1)$.

In this case, we assume impulsive moments in $[c_1, d_1]$ are $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)}$. Choosing $w_1(t) \in \Omega(c_1, d_1)$, multiplying both sides of (2.14) by $|w_1(t)|^{\alpha+1}$ and then integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
&\left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right)u'(t)|w_1(t)|^{\alpha+1}dt \\
&\quad\leq \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right)\left(\frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}\right)|w_1(t)|^{\alpha+1}dt \\
&\qquad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad - \int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt,
\end{aligned} \tag{2.16}$$

where $W_1(t) = \rho(t)\psi(t)|w_1(t)|^{\alpha+1}$. Using the integration by parts formula in the left-hand side of above inequality and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} |w_1(\tau_i)|^{\alpha+1}\left[u(\tau_i) - u(\tau_i^+)\right] \\
&\quad\le \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right) \\
&\qquad\times \left[(\alpha+1)\Phi_\alpha(w_1(t))w_1'(t)u(t) + \frac{\rho'(t)}{\rho(t)}u(t)|w_1(t)|^{\alpha+1}\right. \\
&\qquad\qquad \left. -\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1}\right]dt \\
&\qquad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad - \int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt, \\
&\quad\le \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} + \int_{\tau_{k(d_1)}}^{d_1}\right) \\
&\qquad\times \left[(\alpha+1)|w_1(t)|^\alpha|u(t)|\left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)\right. \\
&\qquad\qquad \left. -\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1}\right]dt \\
&\qquad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}}\right] + \int_{\tau_{k(d_1)}}^{d_1}\right)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}W_1(t)dt \\
&\qquad - \int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.
\end{aligned} \tag{2.17}$$

Letting $y = |w_1(t)|^\alpha|u(t)|$, $B = (\alpha+1)(|w_1'(t)| + |\rho'(t)||w_1(t)|/(\alpha+1)|\rho(t)|)$, $A = \alpha/(\rho(t)r(t))^{1/\alpha}$ and using (2.6), we have for the integrand function in above inequality that

$$\begin{aligned}
&(\alpha+1)|w_1(t)|^\alpha\left|w_1'(t)\right||u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{(1+\alpha)/\alpha}|w_1(t)|^{\alpha+1} \\
&\quad\le \rho(t)r(t)\left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}.
\end{aligned} \tag{2.18}$$

In view of the impulse condition in (1.1) and the definition of u we have, for $t = \tau_k$, $k = 1, 2, \ldots$, that

$$u\left(\tau_k^+\right) = \frac{b_k^\alpha}{a_k^\alpha} u(\tau_k). \tag{2.19}$$

From (2.19), we have

$$\sum_{i=k(c_1)+1}^{k(d_1)} |w_1(\tau_i)|^{\alpha+1} \left[u(\tau_i) - u\left(\tau_i^+\right)\right] = \sum_{i=k(c_1)+1}^{k(d_1)} \left(1 - \frac{b_i^\alpha}{a_i^\alpha}\right) |w_1(\tau_i)|^{\alpha+1} u(\tau_i). \tag{2.20}$$

Therefore, we get

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \left(1 - \frac{b_i^\alpha}{a_i^\alpha}\right) |w_1(\tau_i)|^{\alpha+1} u(\tau_i) \\
&\quad \leq \int_{c_1}^{d_1} \rho(t) r(t) \left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right| |w_1(t)|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} dt \\
&\qquad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} + \int_{\tau_i+\sigma}^{\tau_{i+1}} \right] + \int_{\tau_{k(d_1)}}^{d_1} \right) \frac{x^\alpha(t-\sigma)}{x^\alpha(t)} W_1(t) dt \\
&\qquad - \int_{c_1}^{d_1} \rho(t) p_0(t) |w_1(t)|^{\alpha+1} dt.
\end{aligned} \tag{2.21}$$

On the other hand, for $t \in [c_1, d_1] \setminus \{\tau_i\}$,

$$\left(r(t)\Phi_\alpha(x'(t))\right)' = e(t) - p_0(t)\Phi_\alpha(x(t)) - \sum_{i=1}^{n} p_i(t) \Phi_{\beta_i}(x(t-\sigma)) \leq 0. \tag{2.22}$$

Hence $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on $[c_1, d_1] \setminus \{\tau_k\}$.

Because there are different integration intervals in (2.21), we will estimate $x(t-\sigma)/x(t)$ in each interval of t as follows.

Case 1. $t \in (\tau_i, \tau_{i+1}] \subset [c_1, d_1]$, for $i = k(c_1) + 1, \ldots, k(d_1) - 1$.

Subcase 1. If $\tau_i + \sigma < t \leq \tau_{i+1}$, then $(t-\sigma, t) \subset (\tau_i, \tau_{i+1}]$. Thus there is no impulsive moment in $(t-\sigma, t)$. For any $s \in (t-\sigma, t)$, we have

$$x(s) - x\left(\tau_i^+\right) = x'(\xi_1)(s - \tau_i), \qquad \xi_1 \in (\tau_i, s). \tag{2.23}$$

Since $x(\tau_i^+) > 0$, $r(s)$ is nondecreasing, function $\Phi_\alpha(\cdot)$ is an increasing function and $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on (τ_i, τ_{i+1}), we have

$$\begin{aligned}\Phi_\alpha(x(s)) &\geq \frac{r(\xi_1)}{r(s)}\Phi_\alpha(x(s)) > \frac{r(\xi_1)}{r(s)}\Phi_\alpha\left(x'(\xi_1)(s-\tau_i)\right) = \frac{r(\xi_1)\Phi_\alpha(x'(\xi_1))}{r(s)}(s-\tau_i)^\alpha \\ &\geq \frac{r(s)\Phi_\alpha(x'(s))}{r(s)}(s-\tau_i)^\alpha = \Phi_\alpha\left(x'(s)(s-\tau_i)\right), \quad \xi_1 \in (\tau_i, s).\end{aligned} \tag{2.24}$$

Therefore,

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i}. \tag{2.25}$$

Integrating both sides of the above inequality from $t-\sigma$ to t, we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i-\sigma}{t-\tau_i}, \quad t \in (\tau_i+\sigma, \tau_{i+1}]. \tag{2.26}$$

Subcase 2. If $\tau_i < t < \tau_i + \sigma$, then $\tau_i - \sigma < t - \sigma < \tau_i < t < \tau_i + \sigma$. There is an impulsive moment τ_i in $(t-\sigma, t)$. For any $t \in (\tau_i, \tau_i + \sigma)$, we have

$$x(t) - x\left(\tau_i^+\right) = x'(\xi_2)(t-\tau_i), \quad \xi_2 \in (\tau_i, t). \tag{2.27}$$

Using the impulsive condition of (1.1) and the monotone properties of $r(t)$, $\Phi_\alpha(\cdot)$ and $r(t)\Phi_\alpha(x'(t))$, we get

$$\begin{aligned}\Phi_\alpha(x(t) - a_i x(\tau_i)) &= \frac{r(\xi_2)\Phi_\alpha(x'(\xi_2))}{r(\xi_2)}(t-\tau_i)^\alpha \leq \frac{r(\tau_i)\Phi_\alpha\left(x'\left(\tau_i^+\right)\right)}{r(\xi_2)}(t-\tau_i)^\alpha \\ &= \frac{r(\tau_i)\Phi_\alpha(b_i x'(\tau_i)(t-\tau_i))}{r(\xi_2)}.\end{aligned} \tag{2.28}$$

Since $x(\tau_i) > 0$, we have

$$\Phi_\alpha\left(\frac{x(t)}{x(\tau_i)} - a_i\right) \leq \frac{r(\tau_i)}{r(\xi_2)}\Phi_\alpha\left(b_i \frac{x'(\tau_i)}{x(\tau_i)}(t-\tau_i)\right). \tag{2.29}$$

In addition,

$$x(\tau_i) > x(\tau_i) - x(\tau_i - \sigma) = x'(\xi_3)\sigma, \quad \xi_3 \in (\tau_i - \sigma, \tau_i). \tag{2.30}$$

Using the same analysis as (2.24) and (2.25), we have

$$\frac{x'(\tau_i)}{x(\tau_i)} < \frac{1}{\sigma}. \tag{2.31}$$

From (2.29) and (2.31) and note that the monotone properties of $\Phi_\alpha(\cdot)$ and $r(t)$, we get

$$\Phi_\alpha\left(\frac{x(t)}{x(\tau_i)} - a_i\right) < \frac{r(\tau_i)}{r(\xi_2)}\Phi_\alpha\left(\frac{b_i}{\sigma}(t-\tau_i)\right) \le \Phi_\alpha\left(\frac{b_i}{\sigma}(t-\tau_i)\right). \tag{2.32}$$

Then,

$$\frac{x(t)}{x(\tau_i)} < a_i + \frac{b_i}{\sigma}(t-\tau_i). \tag{2.33}$$

In view of (A_3), we have

$$\frac{x(\tau_i)}{x(t)} > \frac{\sigma}{\sigma a_i + b_i(t-\tau_i)} \ge \frac{\sigma}{b_i(t+\sigma-\tau_i)} > 0. \tag{2.34}$$

On the other hand, similar to the above analysis, we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i+\sigma}, \quad s \in (\tau_i - \sigma, \tau_i). \tag{2.35}$$

Integrating (2.35) from $t-\sigma$ to τ_i, where $t \in (\tau_i, \tau_i+\sigma)$, we have

$$\frac{x(t-\sigma)}{x(\tau_i)} > \frac{t-\tau_i}{\sigma} \ge 0. \tag{2.36}$$

From (2.34) and (2.36), we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i}{b_i(t+\sigma-\tau_i)}, \quad t \in (\tau_i, \tau_i+\sigma). \tag{2.37}$$

Case 2 ($t \in [c_1, \tau_{k(c_1)+1}]$). Since $\tau_{k(c_1)}+\sigma < c_1$, then $t-\sigma \in [c_1-\sigma, \tau_{k(c_1)+1}-\sigma] \subset (\tau_{k(c_1)}, \tau_{k(c_1)+1}-\sigma]$. So, there is no impulsive moment in $(t-\sigma, t)$. Similar to (2.26) of Subcase 1, we have

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_{k(c_1)}-\sigma}{t-\tau_{k(c_1)}}, \quad t \in \left[c_1, \tau_{k(c_1)+1}\right]. \tag{2.38}$$

Case 3 ($t \in (\tau_{k(d_1)}, d_1]$). Since $\tau_{k(d_1)}+\sigma > d_1$, then $t-\sigma \in (\tau_{k(d_1)}-\sigma, d_1-\sigma] \subset (\tau_{k(d_1)}-\sigma, \tau_{k(d_1)})$. Hence, there is an impulsive moment $\tau_{k(d_1)}$ in $(t-\sigma, t)$. Making a similar analysis of Subcase 2, we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_{k(d_1)}}{b_{k(d_1)}\left(t+\sigma-\tau_{k(d_1)}\right)} \ge 0, \quad t \in \left(\tau_{k(d_1)}, d_1\right]. \tag{2.39}$$

From (2.21), (2.26), (2.37), (2.38), and (2.39) we get

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)}\left(1-\frac{b_i^\alpha}{a_i^\alpha}\right)|w_1(\tau_i)|^{\alpha+1}u(\tau_i)\\
&\quad<\int_{c_1}^{d_1}\rho(t)r(t)\left(\left|w_1'(t)\right|+\frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt-\int_{c_1}^{\tau_{k(c_1)+1}}W_1(t)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^\alpha}{\left(t-\tau_{k(c_1)}\right)^\alpha}dt\\
&\qquad-\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}W_1(t)\frac{(t-\tau_i)^\alpha}{b_i^\alpha(t+\sigma-\tau_i)^\alpha}dt-\int_{\tau_i+\sigma}^{\tau_{i+1}}W_1(t)\frac{(t-\tau_i-\sigma)^\alpha}{(t-\tau_i)^\alpha}dt\right]\\
&\qquad-\int_{\tau_{k(d_1)}}^{d_1}W_1(t)\frac{\left(t-\tau_{k(d_1)}\right)^\alpha}{b_{k(d_1)}^\alpha\left(t+\sigma-\tau_{k(d_1)}\right)^\alpha}dt-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.
\end{aligned} \tag{2.40}$$

On the other hand, for $t\in(\tau_{i-1},\tau_i]\subset[c_1,d_1]$, $i=k(c_1)+2,\ldots,k(d_1)$, we have

$$x(t)-x(\tau_{i-1})=x'(\xi)(t-\tau_{i-1}),\quad \xi\in(\tau_{i-1},t). \tag{2.41}$$

In view of $x(\tau_{i-1})>0$ and the monotone properties of $\Phi_\alpha(\cdot)$, $r(t)\Phi_\alpha(x'(t))$ and $r(t)$, we obtain

$$\Phi_\alpha(x(t))>\Phi_\alpha\left(x'(\xi)\right)\Phi_\alpha(t-\tau_{i-1})\geq\frac{r(t)}{r(\xi)}\Phi_\alpha\left(x'(t)\right)\Phi_\alpha(t-\tau_{i-1}). \tag{2.42}$$

This is

$$\rho(t)\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}<\frac{\rho_1 r(\xi)}{(t-\tau_{i-1})^\alpha}. \tag{2.43}$$

Letting $t\to\tau_i^-$, we have

$$u(\tau_i)=\rho(t)\frac{r(\tau_i)\Phi_\alpha(x'(\tau_i))}{\Phi_\alpha(x(\tau_i))}<\frac{\rho_1 r_1}{(\tau_i-\tau_{i-1})^\alpha},\qquad i=k(c_1)+2,\ldots,k(d_1). \tag{2.44}$$

Using similar analysis on $(c_1,\tau_{k(c_1)+1}]$, we get

$$u\left(\tau_{k(c_1)+1}\right)<\frac{\rho_1 r_1}{\left(\tau_{k(c_1)+1}-c_1\right)^\alpha}. \tag{2.45}$$

Then from (2.44), (2.45), and (A_3), we have

$$\begin{aligned}
\sum_{i=k(c_1)+1}^{k(d_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)|w_1(\tau_i)|^{\alpha+1}u(\tau_i)&<\rho_1 r_1\left[\left|w_1\left(\tau_{k(c_1)+1}\right)\right|^{\alpha+1}\theta(c_1)+\sum_{i=k(c_1)+2}^{k(d_1)}|w_1(\tau_i)|^{\alpha+1}\zeta(\tau_i)\right]\\
&=\rho_1 r_1 Q_{c_1}^{d_1}\left[|w_1|^{\alpha+1}\right],
\end{aligned} \tag{2.46}$$

where $\theta(c_1)=(b_{k(c_1)+1}^\alpha-a_{k(c_1)+1}^\alpha)/a_{k(c_1)+1}^\alpha(\tau_{k(c_1)+1}-c_1)^\alpha$ and $\zeta(\tau_i)=(b_i^\alpha-a_i^\alpha)/a_i^\alpha(\tau_i-\tau_{i-1})^\alpha$.

From (2.40) and (2.46), we obtain

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} W_1(t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}} W_1(t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(d_1)}}^{d_1} W_1(t)\frac{\left(t-\tau_{k(d_1)}\right)^{\alpha}}{b_{k(d_1)}^{\alpha}\left(t+\sigma-\tau_{k(d_1)}\right)^{\alpha}}dt \\
&\quad+\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt-\int_{c_1}^{d_1}\rho(t)r(t)\left(\left|w_1'(t)\right|+\frac{\left|\rho'(t)\right||w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt \\
&< \rho_1 r_1 Q_{c_1}^{d_1}\left[|w_1|^{\alpha+1}\right].
\end{aligned} \tag{2.47}$$

This contradicts (2.8).

Next we consider the case $k(c_1)=k(d_1)$. From the condition $(\overline{\mathrm{S}}1)$ we know that there is no impulsive moment in $[c_1,d_1]$. Multiplying both sides of (2.47) by $|w_1(t)|^{\alpha+1}$ and integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
\int_{c_1}^{d_1} u'(t)|w_1(t)|^{\alpha+1}dt \le &-\int_{c_1}^{d_1}\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}|w_1(t)|^{\alpha+1}dt \\
&-\int_{c_1}^{d_1}\frac{x^{\alpha}(t-\sigma)}{x^{\alpha}(t)}W_1(t)dt-\int_{c_1}^{d_1}\rho(t)p_0(t)|w_1(t)|^{\alpha+1}dt.
\end{aligned} \tag{2.48}$$

Similar to the proof of (2.21), we have

$$\int_{c_1}^{d_1}\left[\frac{x^{\alpha}(t-\sigma)}{x^{\alpha}(t)}W_1(t)+\rho(t)p_0(t)\left|w_j(t)\right|^{\alpha+1}-\rho(t)r(t)\left(\left|w_j'(t)\right|+\frac{\left|\rho'(t)\right|\left|w_j(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right]dt\le 0. \tag{2.49}$$

Using same way as Subcase 1, we get

$$\frac{x(t-\sigma)}{x(t)}>\frac{t-c_1}{t-c_1+\sigma},\qquad t\in[c_1,d_1]. \tag{2.50}$$

From (2.49) and (2.50) we obtain

$$\int_{c_1}^{d_1}\left[W_1(t)\frac{(t-c_1)^{\alpha}}{(t-c_1+\sigma)^{\alpha}}+\rho(t)p_0(t)\left|w_j(t)\right|^{\alpha+1}-\rho(t)r(t)\left(\left|w_j'(t)\right|+\frac{\left|\rho'(t)\right|\left|w_j(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right]dt<0. \tag{2.51}$$

This contradicts condition (2.9).

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore, we complete the proof.

□

Remark 2.4. In article [14], the authors obtained the following inequalities:

$$\frac{x(t)}{x(t_j)} > a_j \frac{t_j + \tau - t}{\tau} \geq 0, \quad t \in (t_j, t_j + \tau), \tag{2.52}$$

See [14, equation (2.9)],

$$\frac{x(t) - \tau}{x(t_j)} > \frac{t - t_j}{\tau} \geq 0, \quad t \in (t_j, t_j + \tau). \tag{2.53}$$

See [14, equation (2.10)].

Dividing [14, equation (2.10)] by [14, equation (2.9)], they obtained

$$\frac{x(t) - \tau}{x(t)} > \frac{t - t_j}{a_j(t_j + \tau - t)} \geq 0, \quad t \in (t_j, t_j + \tau). \tag{2.54}$$

See [14, equation (2.11)]

This is an error. Moreover, similar errors appeared many times in the later arguments, for example, in inequalities (2.15), (2.19), and (2.20) in [14]. Moreover, the above substitution can lead to some divergent integrals, for example, the integrals in (2.22), (2.24) in [14]. Therefore, the conditions of their Theorems 2.1–2.5 must be defective. In the proof of our Theorem 2.3, this error is remedied.

Remark 2.5. When $\sigma = 0$, that is, the delay disappears, (1.1) reduces to (1.7) studied by Özbekler and Zafer [13]. In this case, our result with $\rho(t) = 1$ is Theorem 2.1 of [13].

Remark 2.6. When $\sigma = 0$, that is, the delay disappears in (1.1) and $\alpha = 1$, our result reduces to Theorem 2.1 of [12].

Remark 2.7. When $a_k = b_k = 1$ for all $k = 1, 2, \ldots$ and $\sigma = 0$, that is, both impulses and delay disappear in (1.1), our result with $\alpha = 1$ and $\rho(t) = 1$ reduces to Theorem 1 of [21].

In the following we will establish a Kong-type interval oscillation criteria for (1.1) by the ideas of Philos [18] and Kong [22].

Let $D = \{(t, s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$, then a pair function H_1, H_2 is said to belong to a function set $\mathscr{H}$, defined by $(H_1, H_2) \in \mathscr{H}$, if there exist $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R})$ satisfying the following conditions:

(C_1) $H_1(t, t) = H_2(t, t) = 0$, $H_1(t, s) > 0, H_2(t, s) > 0$ for $t > s$;

(C_2) $(\partial/\partial t)H_1(t, s) = h_1(t, s)H_1(t, s)$, $(\partial/\partial s)H_2(t, s) = h_2(t, s)H_2(t, s)$.

We assume that there exist $c_j, d_j, \delta_j \notin \{\tau_k, k = 1, 2, \ldots.\}(j = 1, 2)$ such that $T < c_1 < \delta_1 < d_1 \leq c_2 < \delta_2 < d_2$ for any $T \geq t_0$. Noticing whether or not there are impulsive moments of $x(t)$ in $[c_j,\delta_j]$ and $[\delta_j,d_j]$, we should consider the following four cases, namely, (S5) $k(c_j) < k(\delta_j) < k(d_j)$; (S6) $k(c_j) = k(\delta_j) < k(d_j)$; (S7) $k(c_j) < k(\delta_j) = k(d_j)$ and (S8) $k(c_j) = k(\delta_j) = k(d_j)$. Moreover, in the discussion of the impulse moments of $x(t-\sigma)$, it is necessary to consider the following two cases: $(\bar{\text{S}}5)$ $\tau_{k(\delta_j)} + \sigma > \delta_j$ and $(\bar{\text{S}}6)$ $\tau_{k(\delta_j)} + \sigma \le \delta_j$. In the following theorem, we only consider the case of combination of (S5) with $(\bar{\text{S}}5)$. For the other cases, similar conclusions can be given and the proofs will be omitted here.

For convenience in the expression below, we define, for $j = 1,2$,

$$
\begin{aligned}
\Pi_{1,j} =: &\frac{1}{H_1(\delta_j,c_j)} \\
&\times\left\{\int_{c_j}^{\tau_{k(c_j)+1}} \widetilde{H}_1(t,c_j)\frac{\left(t-\tau_{k(c_j)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_j)}\right)^{\alpha}}dt\right.\\
&\quad+\sum_{i=k(c_j)+1}^{k(\delta_j)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_1(t,c_j)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_1(t,c_j)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right]\\
&\quad+\int_{\tau_{k(\delta_j)}}^{\delta_j}\widetilde{H}_1(t,c_j)\frac{\left(t-\tau_{k(\delta_j)}\right)^{\alpha}}{b_{k(\delta_j)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_j)}\right)^{\alpha}}dt+\int_{c_j}^{\delta_j}\rho(t)p_0(t)H_1(t,c_j)dt\\
&\quad\left.-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_j}^{\delta_j}\rho(t)r(t)H_1(t,c_j)\left|h_1(t,c_j)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt\right\},\\
\Pi_{2,j} =: &\frac{1}{H_2(d_j,\delta_j)}\left\{\int_{\delta_j}^{\tau_{k(\delta_j)}+\sigma}\widetilde{H}_2(d_j,t)\frac{\left(t-\tau_{k(\delta_j)}\right)^{\alpha}}{b_{k(\delta_j)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_j)}\right)^{\alpha}}dt\right.\\
&\quad+\int_{\tau_{k(\delta_j)}+\sigma}^{\tau_{k(\delta_j)+1}}\widetilde{H}_2(d_j,t)\frac{\left(t-\tau_{k(\delta_j)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(\delta_j)}\right)^{\alpha}}dt\\
&\quad+\sum_{i=k(\delta_j)+1}^{k(d_j)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_2(d_j,t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_2(d_j,t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right]\\
&\quad+\int_{\tau_{k(d_j)}}^{d_j}\widetilde{H}_2(d_j,t)\frac{\left(t-\tau_{k(d_j)}\right)^{\alpha}}{b_{k(d_j)}^{\alpha}\left(t+\sigma-\tau_{k(d_j)}\right)^{\alpha}}dt\\
&\quad-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_j}^{d_j}\rho(t)r(t)H_2(d_j,t)\left|h_2(d_j,t)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt\\
&\quad\left.+\int_{\delta_j}^{d_j}\rho(t)p_0(t)H_2(d_j,t)dt\right\},
\end{aligned}
\tag{2.55}
$$

where $\widetilde{H}_1(t,c_j)= H_1(t,c_j)\varphi(t)$, $\widetilde{H}_2(d_j,t) = H_2(d_j,t)\varphi(t)$ and $\varphi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}$ with $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$ and $\eta_1,\eta_2,\ldots,\eta_n$ are positive constants satisfying conditions of Lemma 2.1.

Theorem 2.8. *Assume* (2.7) *holds. If there exists a pair of* $(H_1, H_2) \in \mathscr{H}$ *such that*

$$\Pi_{1,j} + \Pi_{2,j} > \frac{\rho_j r_j}{H_1(\delta_j, c_j)} Q_{c_j}^{\delta_j}[H_1(\cdot, c_j)] + \frac{\rho_j r_j}{H_2(d_j, \delta_j)} Q_{\delta_j}^{d_j}[H_2(d_j, \cdot)], \quad j = 1,2, \tag{2.56}$$

then (1.1) *is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. Similar to the proof of Theorem 2.3, we can get (2.14) and (2.19). Multiplying both sides of (2.14) by $H_1(t, c_1)$ and integrating it from c_1 to δ_1, we have

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt \leq{}& \int_{c_1}^{\delta_1} H_1(t,c_1)\left(\frac{\rho'(t)}{\rho(t)}u(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(1+\alpha)/\alpha}\right)dt \\ &- \int_{c_1}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt - \int_{c_1}^{\delta_1} H_1(t,c_1)\rho(t)p_0(t)dt,\end{aligned} \tag{2.57}$$

where $\widetilde{H}_1(t,c_1) = H_1(t,c_1)\rho(t)\varphi(t)$, $\varphi(t) = \eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}$ with $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$ and $\eta_1,\eta_2,\ldots,\eta_n$ are positive constants satisfying conditions of Lemma 2.1.

Noticing impulsive moments $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(\delta_1)}$ are in $[c_1, \delta_1]$ and using the integration by parts formula on the left-hand side of above inequality, we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt ={}& \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)du(t) \\ ={}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(1 - \frac{b_i^\alpha}{a_i^\alpha}\right)H_1(\tau_i,c_1)u(\tau_i) + H_1(\delta_1,c_1)u(\delta_1) \\ &- \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)h_1(t,c_1)u(t)dt.\end{aligned} \tag{2.58}$$

Substituting (2.58) into (2.57), we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt \le{}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1)\\&+\left(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\cdots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\right)H_1(t,c_1)\\&\times\left[\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right||u(t)|-\frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}|u(t)|^{(\alpha+1)/\alpha}\right]dt\\&-\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt.\end{aligned}\tag{2.59}$$

Letting $A=\alpha/(\rho(t)r(t))^{1/\alpha}$, $B=|h_1(t,c_1)+\rho'(t)/\rho(t)|$, $y=|u(t)|$ and using (2.6) to the right-hand side of above inequality, we have

$$\begin{aligned}\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt \le{}& \sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1)\\&+\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_1}^{\delta_1}\rho(t)r(t)H_1(t,c_1)\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt\\&-\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt.\end{aligned}\tag{2.60}$$

Similar to the proof of Theorem 2.3, we need to divide the integration interval $[c_1,\delta_1]$ into several subintervals for estimating the function $x(t-\sigma)/x(t)$. Using the methods of (2.26), (2.37), (2.38), and (2.39) we estimate the left-hand side of above inequality as follows:

$$\begin{aligned}&\int_{c_1}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{x^\alpha(t-\sigma)}{x^\alpha(t)}dt\\&\quad>\int_{c_1}^{\tau_{k(c_1)+1}}\widetilde{H}_1(t,c_1)\frac{(t-\tau_{k(c_1)}-\sigma)^\alpha}{(t-\tau_{k(c_1)})^\alpha}dt\\&\qquad+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^\alpha}{b_i^\alpha(t+\sigma-\tau_i)^\alpha}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^\alpha}{(t-\tau_i)^\alpha}dt\right]\\&\qquad+\int_{\tau_{k(\delta_1)}}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{(t-\tau_{k(\delta_1)})^\alpha}{b_{k(\delta_1)}^\alpha(t+\sigma-\tau_{k(\delta_1)})^\alpha}dt.\end{aligned}\tag{2.61}$$

From (2.60) and (2.61), we have

$$
\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(c_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(\delta_1)}}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}{b_{k(\delta_1)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\alpha}}dt \\
&\quad-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{c_1}^{\delta_1}\rho(t)r(t)H_1(t,c_1)\left|h_1(t,c_1)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt+\int_{c_1}^{\delta_1}\rho(t)p_0(t)H_1(t,c_1)dt \\
&\quad<\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1).
\end{aligned}
\tag{2.62}
$$

On the other hand, multiplying both sides of (2.14) by $H_2(d_1,t)$ and using similar analysis to the above, we can obtain

$$
\begin{aligned}
&\int_{\delta_1}^{\tau_{k(\delta_1)}+\sigma} \widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}{b_{k(\delta_1)}^{\alpha}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\alpha}}+\int_{\tau_{k(\delta_1)}+\sigma}^{\tau_{k(\delta_1)+1}} \widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}-\sigma\right)^{\alpha}}{\left(t-\tau_{k(\delta_1)}\right)^{\alpha}}dt \\
&\quad+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_2(d_1,t)\frac{(t-\tau_i)^{\alpha}}{b_i^{\alpha}(t+\sigma-\tau_i)^{\alpha}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_2(d_1,t)\frac{(t-\tau_i-\sigma)^{\alpha}}{(t-\tau_i)^{\alpha}}dt\right] \\
&\quad+\int_{\tau_{k(d_1)}}^{d_1} \widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(d_1)}\right)^{\alpha}}{b_{k(d_1)}^{\alpha}\left(t+\sigma-\tau_{k(d_1)}\right)^{\alpha}}dt+\int_{\delta_1}^{d_1}\rho(t)p_0(t)H_2(d_1,t)dt \\
&\quad-\frac{1}{(\alpha+1)^{\alpha+1}}\int_{\delta_1}^{d_1}\rho(t)r(t)H_2(d_1,t)\left|h_2(d_1,t)+\frac{\rho'(t)}{\rho(t)}\right|^{\alpha+1}dt \\
&\quad<\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_2(d_1,\tau_i)u(\tau_i)+H_2(d_1,\delta_1)u(\delta_1).
\end{aligned}
\tag{2.63}
$$

Dividing (2.62) and (2.63) by $H_1(\delta_1,c_1)$, and $H_2(d_1,\delta_1)$ respectively, and adding them, we get

$$
\begin{aligned}
\Pi_{1,1}+\Pi_{2,1}<{}&\frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_1(\tau_i,c_1)u(\tau_i) \\
&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}\left(\frac{b_i^{\alpha}}{a_i^{\alpha}}-1\right)H_2(d_1,\tau_i)u(\tau_i).
\end{aligned}
\tag{2.64}
$$

Using the same methods as (2.46), we have

$$\begin{aligned} &\sum_{i=k(c_1)+1}^{k(\delta_1)} \left(\frac{b_i^\alpha}{a_i^\alpha} - 1\right) H_1(\tau_i, c_1) u(\tau_i) \leq \rho_1 r_1 Q_{c_1}^{\delta_1}[H_1(\cdot, c_1)], \\ &\sum_{i=k(\delta_1)+1}^{k(d_1)} \left(\frac{r_i b_i^\alpha}{a_i^\alpha} - 1\right) H_2(d_1, \tau_i) u(\tau_i) \leq \rho_1 r_1 Q_{\delta_1}^{d_1}[H_2(d_1, \cdot)]. \end{aligned} \tag{2.65}$$

From (2.64), (2.65), we can obtain a contradiction to the condition (2.56).

When $x(t) < 0$, we choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore, we complete the proof. □

Remark 2.9. When $\sigma = 0$, that is, the delay disappears and $\alpha = 1$ in (1.1), our result Theorem 2.8 reduces to Theorem 2.2 of [12].

3. Examples

In this section, we give two examples to illustrate the effectiveness and nonemptiness of our results.

Example 3.1. Consider the following delay differential equation with impulse:

$$\begin{aligned} &x''(t) + \mu_1 \left| x\left(t - \frac{\pi}{12}\right)\right|^{3/2} x\left(t - \frac{\pi}{12}\right) + \mu_2 \left| x\left(t - \frac{\pi}{12}\right)\right|^{-1/2} x\left(t - \frac{\pi}{12}\right) = -\sin(2t), \quad t \neq \tau_k, \\ &x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \quad t = \tau_k, \end{aligned} \tag{3.1}$$

where $\tau_k : \tau_{n,1} = 2n\pi + 5\pi/18,\ \tau_{n,2} = 2n\pi + 11\pi/18, n \in \mathbb{N}$ and μ_1, μ_2 are positive constants.

For any $T > 0$, we can choose large n_0 such that $T < c_1 = 2n\pi + \pi/6,\ d_1 = 2n\pi + \pi/3,\ c_2 = 2n\pi + \pi/2,\ d_2 = 2n + 2\pi/3,\ n = n_0, n_0 + 1, \ldots$. There are impulsive moments $\tau_{n,1}$ in $[c_1, d_1]$ and $\tau_{n,2}$ in $[c_2, d_2]$. From $\tau_{n,2} - \tau_{n,1} = \pi/3 > \pi/12$ and $\tau_{n+1,1} - \tau_{n,2} = 5\pi/3 > \pi/12$ for all $n > n_0$, we know that condition $\tau_{k+1} - \tau_k > \sigma$ is satisfied. Moreover, we also see the conditions (S1) and (2.7) are satisfied.

We can choose $\eta_0 = \eta_1 = \eta_2 = 1/3$ such that Lemma 2.1 holds. Let $w_1(t) = w_2(t) = \sin(6t)$ and $\rho(t) = 1$. It is easy to verify that $W_1(t) = 3(\mu_1\mu_2)^{1/3} |\sin(2t)|^{1/3} \sin^2(6t)$. By a simple calculation, the left side of (2.8) is the following:

$$\begin{aligned} &\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t) \frac{\left(t - \tau_{k(c_1)} - \sigma\right)^\alpha}{\left(t - \tau_{k(c_1)}\right)^\alpha} dt \\ &\quad + \sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} W_1(t) \frac{(t-\tau_i)^\alpha}{b_i^\alpha (t + \sigma - \tau_i)^\alpha} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_1(t) \frac{(t - \tau_i - \sigma)^\alpha}{(t-\tau_i)^\alpha} dt\right] \\ &\quad + \int_{\tau_{k(d_1)}}^{d_1} W_1(t) \frac{\left(t - \tau_{k(d_1)}\right)^\alpha}{b_{k(d_1)}^\alpha \left(t + \sigma - \tau_{k(d_1)}\right)^\alpha} dt + \int_{c_1}^{d_1} \rho(t) p_0(t) |w_1(t)|^{\alpha+1} dt \end{aligned}$$

$$
\begin{aligned}
&-\int_{c_1}^{d_1} \rho(t) r(t)\left(\left|w_1'(t)\right| + \frac{\left|\rho'(t)\right| |w_1(t)|}{(\alpha+1)\rho(t)}\right)^{\alpha+1} dt \\
&= \int_{2n\pi+\pi/6}^{2n\pi+5\pi/18} W_1(t) \frac{t-2(n-1)\pi - 11\pi/18 - \pi/12}{t-2(n-1)\pi - 11\pi/18} dt \\
&\quad + \int_{2n\pi+5\pi/18}^{2n\pi+\pi/3} W_1(t) \frac{t-2n\pi - 5\pi/18}{b_{n,1}(t-2n\pi-5\pi/18+\pi/12)} dt - \int_{2n\pi+\pi/6}^{2n\pi+\pi/3} 36\cos^2 6t \, dt \\
&= \int_{\pi/6}^{5\pi/18} W_1(t) \frac{t+47\pi/36}{t+25\pi/18} dt + \int_{5\pi/18}^{\pi/3} W_1(t) \frac{t-5\pi/18}{b_{n,1}(t-7\pi/36)} dt - 3\pi \\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.199 + \frac{0.715}{b_{n,1}}\right) - 3\pi.
\end{aligned}
\tag{3.2}
$$

On the other hand, we have

$$
Q_{c_1}^{d_1}\left[w_1^2\right] = \frac{27}{4\pi} \frac{b_{n,1} - a_{n,1}}{a_{n,1}}. \tag{3.3}
$$

Thus if

$$
3\sqrt[3]{\mu_1\mu_2}\left(0.199 + \frac{0.715}{b_{n,1}}\right) \geq 3\pi + \frac{27}{4\pi} \frac{b_{n,1} - a_{n,1}}{a_{n,1}}, \tag{3.4}
$$

the condition (2.8) is satisfied in $[c_1, d_1]$. Similarly, we can show that for $t \in [c_2, d_2]$ the condition (2.8) is satisfied if

$$
3\sqrt[3]{\mu_1\mu_2}\left(0.057 + \frac{0.003}{b_{n,2}}\right) \geq 3\pi + \frac{27}{4\pi} \frac{b_{n,2} - a_{n,2}}{a_{n,2}}. \tag{3.5}
$$

Hence, by Theorem 2.3, (3.1) is oscillatory, if (3.4) and (3.5) hold. Particularly, let $a_k = b_k$, for all $k \in \mathbb{N}$, condition (3.4) and (3.5) become

$$
\begin{aligned}
&\sqrt[3]{\mu_1\mu_2}\left(0.199 + \frac{0.715}{b_{n,1}}\right) \geq \pi, \\
&\sqrt[3]{\mu_1\mu_2}\left(0.057 + \frac{0.003}{b_{n,2}}\right) \geq \pi.
\end{aligned}
\tag{3.6}
$$

Example 3.2. Consider the following equation:

$$
\begin{aligned}
&x''(t) + \mu_1 p_1(t)\left|x\left(t-\frac{2}{3}\right)\right|^{3/2} x\left(t-\frac{2}{3}\right) + \mu_2 p_2(t)\left|x\left(t-\frac{2}{3}\right)\right|^{-1/2} x\left(t-\frac{2}{3}\right) = e(t), \quad t \neq \tau_k, \\
&x\left(\tau_k^+\right) = a_k x(\tau_k), \qquad x'\left(\tau_k^+\right) = b_k x'(\tau_k), \quad k = 1, 2, \ldots,
\end{aligned}
\tag{3.7}
$$

where μ_1, μ_2 are positive constants; $\tau_k : \tau_{n,1} = 9n + 3/2$, $\tau_{n,2} = 9n + 5/2$, $\tau_{n,3} = 9n + 15/2$, $\tau_{n,4} = 9n + 17/2$ $(n = 0, 1, 2, \ldots)$ and $\tau_{k+1} - \tau_k > \sigma = 2/3$. In addition, let

$$p_1(t) = p_2(t) = \begin{cases} (t-9n)^3, & t \in [9n, 9n+3], \\ 3^3, & t \in [9n+3, 9n+6], \\ (9n+9-t)^3, & t \in [9n+6, 9n+9], \end{cases}$$
$$e(t) = (t-9n-3)^3, \quad t \in [9n, 9n+9]. \tag{3.8}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 9n$, and let $[c_1, d_1] = [9n+1, 9n+3]$, $[c_2, d_2] = [9n+7, 9n+9]$, $\delta_1 = 9n+2$ and $\delta_2 = 9n+8$. It is easy to see that condition (2.7) in Theorem 2.8 is satisfied. Letting $H_1(t,s) = H_2(t,s) = (t-s)^3$, we get $h_1(t,s) = -h_2(t,s) = 3/(t-s)$. By simple calculation, we have

$$\begin{aligned}
\Pi_{1,1} &= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_{9n+1}^{9n+3/2}(t-9n-1)^3(9n+3-t)(t-9n)^2\frac{t-9n-1/6}{t-9n+1/2}dt\right. \\
&\quad \left.+\int_{9n+3/2}^{9n+2}(t-9n-1)^3(9n+3-t)(t-9n)^2\frac{t-9-3/2}{b_{n,1}(t-9n-5/6)}dt\right\}-\frac{9}{8} \\
&= 3\sqrt[3]{\mu_1\mu_2}\left(\int_1^{3/2}\frac{u^2(u-1)^3(3-u)(u-1/6)}{u+1/2}du+\int_{3/2}^{2}\frac{u^2(u-1)^3(3-u)(u-3/2)}{b_{n,1}(u-5/6)}du\right)-\frac{9}{8} \\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(0.32+\frac{0.290}{b_{n,1}}\right)-\frac{9}{8}, \\
\Pi_{2,1} &= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_{9n+2}^{9n+13/6}(9n+3-t)^4(t-9n)^2\frac{t-9n-3/2}{b_{n,1}(t-9n+5/6)}dt\right. \\
&\quad +\int_{9n+13/6}^{9n+5/2}(9n+3-t)^4(t-9n)^2\frac{t-9n-13/6}{t-9n-3/2}dt \\
&\quad \left.+\int_{9n+5/2}^{9n+3}(9n+3-t)^4(t-9n)^2\frac{t-9n-5/2}{b_{n,2}(t-9n-11/6)}dt\right\}-\frac{9}{8} \\
&= 3\sqrt[3]{\mu_1\mu_2}\left\{\int_2^{13/6}\frac{u^2(3-u)^4(u-3/2)}{b_{n,1}(u+5/6)}dt+\int_{13/6}^{5/2}\frac{u^2(3-u)^4(u-13/6)}{u-3/2}dt\right. \\
&\quad \left.+\int_{5/2}^{3}\frac{u^2(3-u)^4(u-5/2)}{b_{n,2}(u-11/6)}dt\right\}-\frac{9}{8} \\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(\frac{0.102}{b_{n,1}}+0.056+\frac{0.005}{b_{n,2}}\right)-\frac{9}{8}.
\end{aligned} \tag{3.9}$$

Then the left-hand side of the inequality (2.56) is

$$\Pi_{1,1} + \Pi_{2,1} \approx 3\sqrt[3]{\mu_1\mu_2}\left(0.376 + \frac{0.392}{b_{n,1}} + \frac{0.005}{b_{n,2}}\right) - \frac{9}{4}. \tag{3.10}$$

Because $r_1 = r_2 = 1$, $\tau_{k(c_1)+1} = \tau_{k(\delta_1)} = \tau_{n,1} = 9n + 3/2 \in (c_1, \delta_1)$ and $\tau_{k(\delta_1)+1} = \tau_{k(d_1)} = \tau_{n,2} = 9n + 5/2 \in (\delta_1, d_1)$, it is easy to get that the right-hand side of the inequality (2.56) for $j = 1$ is

$$\frac{r_1}{H_1(\delta_1, c_1)} Q_{c_1}^{\delta_1}[H_1(\cdot, c_1)] + \frac{r_1}{H_2(d_1, \delta_1)} Q_{\delta_1}^{d_1}[H_2(d_1, \cdot)] = \frac{b_{n,1} - a_{n,1}}{4a_{n,1}} + \frac{b_{n,2} - a_{n,2}}{4a_{n,2}}. \tag{3.11}$$

Thus (2.56) is satisfied with $j = 1$ if

$$3\sqrt[3]{\mu_1\mu_2}\left(0.376 + \frac{0.392}{b_{n,1}} + \frac{0.005}{b_{n,2}}\right) > \frac{9}{4} + \frac{b_{n,1} - a_{n,1}}{4a_{n,1}} + \frac{b_{n,2} - a_{n,2}}{4a_{n,2}}. \tag{3.12}$$

When $j = 2$, with the same argument as above we get that the left-hand side of inequality (2.56) is

$$\begin{aligned}
&\Pi_{1,2} + \Pi_{2,2} \\
&\quad = 3\sqrt[3]{\mu_1\mu_2}\left\{\int_7^{15/2} \frac{(u-9)^2(u-3)(u-7)^3(u-19/6)}{u-5/2}\,du \right.\\
&\qquad\qquad + \int_{15/2}^{8} \frac{(u-9)^2(u-3)(u-7)^3(u-15/2)}{b_{n,3}(u-41/6)}\,du \\
&\qquad\qquad + \int_{8}^{49/6} \frac{(u-3)(9-u)^5(u-15/2)}{b_{n,3}(u-41/6)}\,du + \int_{49/6}^{17/2} \frac{(u-3)(9-u)^5(u-19/6)}{u-5/2}\,du \\
&\qquad\qquad \left. + \int_{17/2}^{9} \frac{(u-3)(9-u)^5(u-17/2)}{b_{n,4}(u-47/6)}\,du \right\} - \frac{9}{4} \\
&\quad \approx 3\sqrt[3]{\mu_1\mu_2}\left(0.400 + \frac{0.724}{b_{n,3}} + \frac{0.001}{b_{n,4}}\right) - \frac{9}{4},
\end{aligned} \tag{3.13}$$

and the right-hand side of the inequality (2.56) is

$$\frac{r_2}{H_2(\delta_2, c_2)} Q_{c_2}^{\delta_2}[H_1(\cdot, c_2)] + \frac{r_2}{H_2(d_2, \delta_2)} Q_{\delta_2}^{d_2}[H_2(d_2, \cdot)] = \frac{b_{n,3} - a_{n,3}}{4a_{n,3}} + \frac{b_{n,4} - a_{n,4}}{4a_{n,4}}. \tag{3.14}$$

Therefore, (2.51) is satisfied with $j = 2$ if

$$3\sqrt[3]{\mu_1\mu_2}\left(0.400 + \frac{0.724}{b_{n,3}} + \frac{0.001}{b_{n,4}}\right) > \frac{9}{4} + \frac{b_{n,3} - a_{n,3}}{4a_{n,3}} + \frac{b_{n,4} - a_{n,4}}{4a_{n,4}}. \tag{3.15}$$

Hence, by Theorem 2.8, (3.7) is oscillatory if

$$
\begin{aligned}
&3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)>\frac{9}{4}+\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}+\frac{b_{n,2}-a_{n,2}}{4a_{n,2}},\\
&3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)>\frac{9}{4}+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}.
\end{aligned}
\tag{3.16}
$$

Particularly, when $a_k = b_k$, for all $k \in \mathbb{N}$, condition (3.16) becomes

$$
\begin{aligned}
&3\sqrt[3]{\mu_1\mu_2}\left(0.376+\frac{0.392}{b_{n,1}}+\frac{0.005}{b_{n,2}}\right)>\frac{9}{4},\\
&3\sqrt[3]{\mu_1\mu_2}\left(0.400+\frac{0.724}{b_{n,3}}+\frac{0.001}{b_{n,4}}\right)>\frac{9}{4}.
\end{aligned}
\tag{3.17}
$$

Acknowledgments

The authors thank the anonymous reviewers for their detailed and insightful comments and suggestions for improvement of the paper. They were supported by the NNSF of China (11161018), the NSF of Guangdong Province (10452408801004217).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 492025, 36 pages
doi:10.1155/2012/492025

Research Article

Dirichlet Problems with an Indefinite and Unbounded Potential and Concave-Convex Nonlinearities

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Received 5 March 2012; Accepted 18 April 2012

Academic Editor: Yuriy Rogovchenko

We consider a parametric semilinear Dirichlet problem with an unbounded and indefinite potential. In the reaction we have the competing effects of a sublinear (concave) term and of a superlinear (convex) term. Using variational methods coupled with suitable truncation techniques, we prove two multiplicity theorems for small values of the parameter. Both theorems produce five nontrivial smooth solutions, and in the second theorem we provide precise sign information for all the solutions.

1. Introduction

Let $\Omega \in \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric nonlinear Dirichlet problem:

$$\begin{gathered} -\Delta u(z) + \beta(z)u(z) = \lambda g(z,u(z)) + f(z,u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad \lambda > 0. \end{gathered} \tag{$(P)_\lambda$}$$

Here $\beta \in L^s(\Omega)$ with $s > N/2$ ($N \geqslant 2$) is a potential function which may change sign (indefinite potential). Also $\lambda > 0$ is a parameter, and $g, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e., for all $\zeta \in \mathbb{R}$, functions $z \mapsto g(z,\zeta)$ and $z \mapsto f(z,\zeta)$ are measurable and for almost all $z \in \Omega$, functions $\zeta \mapsto g(z,\zeta)$ and $\zeta \mapsto f(z,\zeta)$ are continuous). We assume that for almost all $z \in \Omega$, the function $\zeta \mapsto g(z,\zeta)$ is strictly sublinear near $\pm\infty$, while the function $\zeta \mapsto f(z,\zeta)$ is superlinear near $\pm\infty$. So, problem $((P)_\lambda)$ exhibits competing nonlinearities of the concave-convex type. This situation was first studied with $\beta \equiv 0$ and the right hand side nonlinearity being

$$\lambda|\zeta|^{q-2}\zeta + |\zeta|^{r-2}\zeta, \tag{1.1}$$

with

$$1 < q < 2 < r < 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 2, \end{cases} \tag{1.2}$$

by Ambrosetti et al. [1]. In [1], the authors focus on positive solutions and proved certain bifurcation-type phenomena as $\lambda > 0$ varies. Further results in this direction can be found in the works of I'lyasov [2], Li et al. [3], Lubyshev [4], and Rădulescu and Repovš [5]. In all the aforementioned works $\beta \equiv 0$. We should also mention the recent work of Motreanu et al. [6], where the authors consider equations driven by the p-Laplacian, with concave term of the form $|\zeta|^{q-2}\zeta$ (where $1 < q < p$) and a perturbation exhibiting an asymmetric behaviour at $+\infty$ and at $-\infty$ ($(p-1)$-superlinear near $+\infty$ and $(p-1)$-sublinear near $-\infty$). They prove multiplicity results producing four nontrivial solutions with sign information.

In this work, we prove two multiplicity results for problem $((P)_\lambda)$ when the parameter $\lambda > 0$ is small. In both results we produce five nontrivial smooth solutions, and in the second we provide precise sign information for all the solutions. For the superlinear ("convex") nonlinearity $f(z,\cdot)$, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition. Instead, we use a more general condition, which incorporates in our framework also superlinear perturbations with "slow" growth near $\pm\infty$, which do not satisfy the Ambrosetti-Rabinowitz condition. We should point out that none of the works mentioned earlier provide sign information for all the solutions (in particular, none of them produced a nodal (sign changing) solution) and all use the Ambrosetti-Rabinowitz condition to express the superlinearity of the "convex" contribution in the reaction.

Our approach is variational based on the critical point theory which is combined with suitable truncation techniques. In the next section we recall the main mathematical tools we will use in the analysis of problem $((P)_\lambda)$. We also introduce the hypotheses on the terms g and f.

2. Mathematical Background and Hypotheses

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition*, if the following is true.

Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*, \tag{2.1}$$

admits a strongly convergent subsequence.

This compactness-type condition is in general weaker than the usual Palais-Smale condition. However, the Cerami condition suffices to prove a deformation theorem and from it we derive the minimax theory for certain critical values of $\varphi \in C^1(X)$ (see Gasiński and Papageorgiou [7] and Motreanu and Rădulescu [8]). In particular, we can have the following theorem, known in the literature as the *mountain pass theorem*.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the Cerami condition, $x_0, x_1 \in X$, $\varrho > 0\|x_1 - x_0\| > \varrho$:*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \varrho\} = \eta_\varrho, \\ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{2.2}$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}. \tag{2.3}$$

Then $c \geqslant \eta_\varrho$ and c are a critical value of φ.

In the analysis of problem $((P)_\lambda)$, in addition to the Sobolev space $H_0^1(\Omega)$, we will also use the Banach space

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : u|_{\partial\Omega} = 0\right\}. \tag{2.4}$$

This is an ordered Banach space with positive cone:

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \geqslant 0\ \forall z \in \overline{\Omega}\right\}. \tag{2.5}$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0\ \forall z \in \Omega, \frac{\partial u}{\partial n}(z) < 0\ \forall z \in \partial\Omega\right\}, \tag{2.6}$$

where $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

In the proof of the second multiplicity theorem and in order to produce a nodal (sign changing) solution, we will also use critical groups. So, let us recall their definition. Let $\varphi \in C^1(X)$ and let $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leqslant c\}, \\ K_\varphi = \{x \in X : \varphi'(x) = 0\}. \tag{2.7}$$

Also, if (Y_1, Y_2) is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$ and $k \geqslant 0$ is an integer, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_2, Y_1) with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in X$ of φ with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \quad \forall k \geqslant 0, \tag{2.8}$$

where U is a neighbourhood of x_0, such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighbourhood U.

Using the spectral theorem for compact self-adjoint operators, we can show that the differential operator

$$H_0^1(\Omega) \ni u \longmapsto -\Delta u + \beta u \tag{2.9}$$

has a sequence of distinct eigenvalues $\{\widehat{\lambda}_k\}_{k\geqslant 1}$, such that

$$\widehat{\lambda}_k \longrightarrow +\infty \quad \text{as } k \longrightarrow +\infty. \tag{2.10}$$

The first eigenvalue is simple and admits the following variational characterization:

$$\widehat{\lambda}_1 = \inf\left\{ \frac{\sigma(u)}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0 \right\}, \tag{2.11}$$

where

$$\sigma(u) = \|\nabla u\|_2^2 + \int_\Omega \beta(z) u(z)^2 dz \quad \forall u \in H_0^1(\Omega). \tag{2.12}$$

Moreover, the corresponding eigenfunction $\widehat{u} \in C_0^1(\overline{\Omega})$ does not change sign, and in fact we can take $\widehat{u}(z) > 0$ for all $z \in \Omega$ (see Gasiński and Papageorgiou [9]). Using (2.11) and this property of the principal eigenfunction, we can have the following lemma (see Gasiński and Papageorgiou [9, Lemma 2.1]).

Lemma 2.2. *If $\eta \in L^s(\Omega)$, $\eta(z) \leqslant \widehat{\lambda}_1$ for almost all $z \in \Omega$, $\eta \neq \widehat{\lambda}_1$, then there exists $\widehat{c} > 0$, such that*

$$\sigma(u) - \int_\Omega \eta u^2 \, dz \geqslant \widehat{c}\|u\|^2 \quad \forall u \in H_0^1(\Omega). \tag{2.13}$$

Also, from Gasiński and Papageorgiou [9] we know that there exist $\widehat{c}_0, \widehat{c}_1 > 0$, such that

$$\|u\|^2 \leqslant \widehat{c}_0 \left(\sigma(u) + \widehat{c}_1 \|u\|_2^2 \right) \quad \forall u \in H_0^1(\Omega), \tag{2.14}$$

with $\|u\| = \|\nabla u\|_2$, the norm of the Sobolev space $H_0^1(\Omega)$.

Next we state the hypotheses on the two components g and f of the reaction in problem $((P)_\lambda)$.

Let

$$G(z,\zeta) = \int_0^\zeta g(z,s) ds, \qquad F(z,\zeta) = \int_0^\zeta f(z,s) \, ds. \tag{2.15}$$

$\underline{H_g}$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $g(z,0) = 0$ for almost all $z \in \Omega$.

(i) For every $\varrho > 0$, there exists a function $a_\varrho \in L^\infty(\Omega)_+$, such that

$$\left|g(z,\zeta)\right| \leqslant a_\varrho(z) \quad \text{for almost all } z \in \Omega, \text{all } |\zeta| \leqslant \varrho. \tag{2.16}$$

(ii) We have

$$\lim_{\zeta \to \pm\infty} \frac{g(z,\zeta)}{\zeta} = 0 \quad \text{uniformly for almost all } z \in \Omega. \tag{2.17}$$

(iii) There exist constants $c_1, c_2 > 0$, $1 < q < \mu < 2$ and $\delta_0 > 0$, such that

$$\begin{gathered} c_1|\zeta|^q \leqslant g(z,\zeta)\zeta \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \\ \mu G(z,\zeta) - g(z,\zeta)\zeta \geqslant c_2|\zeta|^q \text{ for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta_0. \end{gathered} \tag{2.18}$$

(iv) For every $\varrho > 0$, we can find $\gamma_\varrho > 0$, such that for almost all $z \in \Omega$

$$\text{the map} \left[-\varrho, \varrho\right] \ni \zeta \longmapsto g(z,\zeta) + \gamma_\varrho |\zeta|^{q-2}\zeta \text{ is nondecreasing.} \tag{2.19}$$

Remark 2.3. Hypothesis H_g(ii) implies that for almost all $z \in \Omega$, the function $g(z,\cdot)$ is strictly sublinear near $\pm\infty$. Hence $g(z,\cdot)$ is the "concave" component in the reaction of $((P)_\lambda)$ (the terminology "concave" and "convex" nonlinearities is due to Ambrosetti [1]). Note that hypothesis H_g(iii) implies that for almost all $z \in \Omega$, the function $g(z,\cdot)$ has a similar growth near 0 that is, we have a concave term near zero. Hypothesis H_g(iv) is weaker than assuming the monotonicity of $g(z,\cdot)$ for almost all $z \in \Omega$.

$\underline{H_f}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$.

(i) There exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in (2, 2^*)$, such that

$$\left|f(z,\zeta)\right| \leqslant a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{2.20}$$

(ii) We have

$$\lim_{\zeta \to \pm\infty} \frac{F(z,\zeta)}{\zeta^2} = +\infty \quad \text{uniformly for almost all } z \in \Omega. \tag{2.21}$$

(iii) There exist functions $\eta_0, \widehat{\eta}_0 \in L^\infty(\Omega)_+$, such that $\eta_0(z) \leqslant \widehat{\lambda}_1$ for almost all $z \in \Omega$, $\eta_0 \neq \widehat{\lambda}_1$ and

$$\widehat{\eta}_0(z) \leqslant \liminf_{\zeta \to 0} \frac{f(z,\zeta)}{\zeta} \leqslant \limsup_{\zeta \to 0} \frac{f(z,\zeta)}{\zeta} \leqslant \eta_0(z) \tag{2.22}$$

uniformly for almost all $z \in \Omega$.

(iv) For every $\varrho > 0$, we can find $\widehat{\gamma}_\varrho > 0$, such that for almost all $z \in \Omega$, we have

$$\text{the map } [-\varrho, \varrho] \ni \zeta \longmapsto f(z,\zeta) + \widehat{\gamma}_\varrho \zeta \text{ is nondecreasing.} \tag{2.23}$$

Remark 2.4. Hypothesis H_f(ii) implies that for almost all $z \in \Omega$, the function $F(z,\cdot)$ is superquadratic near $\pm\infty$. Evidently, this is satisfied if the function $f(z,\cdot)$ is superlinear near $\pm\infty$, that is, when

$$\lim_{\zeta \to \pm\infty} \frac{f(z,\zeta)}{\zeta} = +\infty \quad \text{uniformly for almost all } z \in \Omega. \tag{2.24}$$

So, $f(z,\zeta)$ is the "convex" component of the reaction which "competes" with the "concave" component $g(z,\zeta)$.

Note that in H_f, we did not include the Ambrosetti-Rabinowitz condition to characterize the superlinearity of $f(z,\cdot)$. We recall that the Ambrosetti-Rabinowitz condition says that there exist $\tau > 2$ and $M > 0$, such that

$$0 < \tau F(z,\zeta) \leqslant f(z,\zeta)\zeta \text{ uniformly for almost all } z \in \Omega, \text{ all } |\zeta| \geqslant M, \tag{2.25}$$

$$\operatorname*{ess\,inf}_{\Omega} F(\cdot, M) > 0. \tag{2.26}$$

Integrating (2.25) and using (2.26), we obtain the weaker condition

$$c_3|\zeta|^\tau \leqslant F(z,\zeta) \quad \text{uniformly for almost all } z \in \Omega, \text{ all } |\zeta| \geqslant M. \tag{2.27}$$

Therefore, for almost all $z \in \Omega$, the function

$$\zeta \longmapsto \lambda G(z,\zeta) + F(z,\zeta) \tag{2.28}$$

is superquadratic with at least τ-growth near $\pm\infty$. Hence the Ambrosetti-Rabinowitz condition excludes superlinear perturbations with "slower" growth near $\pm\infty$. For this reason, here we employ a weaker condition. So, let

$$\xi_\lambda(z,\zeta) = \left(\lambda g(z,\zeta) + f(z,\zeta)\right)\zeta - 2(\lambda G(z,\zeta) + F(z,\zeta)). \tag{2.29}$$

We employ the following hypothesis.

$\underline{H_0}$: For every $\lambda > 0$, there exists a function $\vartheta_\lambda^* \in L^1(\Omega)_+$, such that

$$\begin{aligned} \xi_\lambda(z,\zeta) &\leqslant \xi_\lambda(z,y) + \vartheta_\lambda^*(z) \quad \text{for almost all } z \in \Omega, \text{ all } 0 \leqslant \zeta \leqslant y, \\ \xi_\lambda(z,\zeta) &\leqslant \xi_\lambda(z,y) + \vartheta_\lambda^*(z) \quad \text{for almost all } z \in \Omega, \text{ all } y \leqslant \zeta \leqslant 0. \end{aligned} \tag{2.30}$$

Remark 2.5. Hypothesis H_0 is a generalized version of a condition first introduced by Li and Yang [10], where the reader can find other possible extensions of the Ambrosetti-Rabinowitz condition and comparisons between them. Hypothesis H_0 is a quasimonotonicity condition on $\xi_\lambda(z,\cdot)$, and it is satisfied if there exists $M > 0$, such that for almost all $z \in \Omega$, the function

$$\zeta \longmapsto \frac{\lambda g(z,\zeta) + f(z,\zeta)}{\zeta} \tag{2.31}$$

is increasing on $[M, +\infty]$ and is decreasing on $[-\infty, -M]$ (see Li and Yang [10]).

Example 2.6. The following pairs of functions satisfy hypotheses H_g, H_f, and H_0 (for the sake of simplicity we drop the z-dependence):

$$\begin{gathered} g_1(\zeta) = |\zeta|^{q-2}\zeta, \qquad f_1(\zeta) = |\zeta|^{p-2}\zeta + \eta_0\zeta, \\ g_2(\zeta) = |\zeta|^{q-2}\zeta, \qquad f_2(\zeta) = |\zeta|^{p-2}\zeta\ln(1+|\zeta|) + \eta_0\zeta, \\ g_3(\zeta) = |\zeta|^{q-2}\zeta - |\zeta|^{\tau-2}\zeta, \qquad f_3(\zeta) = \begin{cases} \eta_0\left(\zeta - |\zeta|^{r-2}\zeta\right) & \text{if } |\zeta| \leqslant 1, \\ |\zeta|^{p-2}\zeta\ln|\zeta| & \text{if } |\zeta| > 1, \end{cases} \end{gathered} \tag{2.32}$$

with $1 < q < \tau < 2 < p < 2^*$, $2 < r$, $\eta_0 < \widehat{\lambda}_1$.

Note that f_2 and f_3 do not satisfy the Ambrosetti-Rabinowitz condition (see (2.25)-(2.26)).

For every $u \in H_0^1(\Omega)$, we set

$$\|u\| = \|\nabla u\|_2 \tag{2.33}$$

(by virtue of the Poincaré inequality). We mention that the notation $\|\cdot\|$ will be also used to denote the $\mathbb{R}^N$-norm. It will always be clear from the context which norm is used. For $\zeta \in \mathbb{R}$, let $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We have $u^\pm \in H_0^1(\Omega)$, $|u| = u^+ + u^-$, and $u = u^+ - u^-$. For a given measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (e.g., a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.34}$$

Finally, let $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ be the operator, defined by

$$\langle A(u),\ y\rangle = \int_\Omega (\nabla u, \nabla y)_{\mathbb{R}^N} dz \quad \forall u, y \in H_0^1(\Omega). \tag{2.35}$$

For the properties of the operator A we refer to Gasiński and Papageorgiou [11, Proposition 3.1, page 852].

3. Solutions of Constant Sign

In this section, for $\lambda > 0$ small, we generate four nontrivial smooth solutions of constant sign (two positive and two negative). To this end, we introduce the following modifications of the nonlinearities $g(z,\cdot)$ and $f(z,\cdot)$:

$$g_{\pm}(z,\zeta) = g\left(z,\pm\zeta^{\pm}\right), \qquad \widehat{f}_{\pm}(z,\zeta) = f\left(z,\pm\zeta^{\pm}\right) + \widehat{c}_1\left(\pm\zeta^{\pm}\right), \tag{3.1}$$

with $\widehat{c}_1 > 0$ as in (2.14). These modifications are Carathéodory functions. We set

$$G_{\pm}(z,\zeta) = \int_0^{\zeta} g_{\pm}(z,s)ds, \qquad \widehat{F}_{\pm}(z,\zeta) = \int_0^{\zeta} \widehat{f}_{\pm}(z,s)ds \tag{3.2}$$

and consider the C^1-functionals $\widehat{\varphi}_{\lambda}^{\pm} : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\widehat{\varphi}_{\lambda}^{\pm}(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \lambda\int_{\Omega} G_{\pm}(z,u(z)) - \int_{\Omega} \widehat{F}_{\pm}(z,u(z))dz \quad \forall u \in H_0^1(\Omega). \tag{3.3}$$

Proposition 3.1. *If hypotheses H_g, H_f, and H_0 hold and $\lambda > 0$, then the functionals $\widehat{\varphi}_{\lambda}^{\pm}$ satisfy the Cerami condition.*

Proof. We do the proof for $\widehat{\varphi}_{\lambda}^{+}$, the proof for $\widehat{\varphi}_{\lambda}^{-}$ being similar.

So, let $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ be a sequence, such that

$$\left|\widehat{\varphi}_{\lambda}^{+}(u_n)\right| \leqslant M_1 \quad \forall n \geqslant 1, \tag{3.4}$$

for some $M_1 > 0$ and

$$(1 + \|u_n\|)\left(\widehat{\varphi}_{\lambda}^{+}\right)'(u_n) \longrightarrow 0 \quad \text{in } H^{-1}(\Omega). \tag{3.5}$$

From (3.5), we have

$$\begin{aligned} &\left|\langle A(u_n), h\rangle + \int_{\Omega}(\beta + \widehat{c}_1)u_n h\,dz - \lambda\int_{\Omega} g_{+}(z,u_n)h\,dz - \int_{\Omega}\widehat{f}_{+}(z,u_n)h\,dz\right| \\ &\quad \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \forall h \in H_0^1(\Omega), \end{aligned} \tag{3.6}$$

with $\varepsilon_n \searrow 0$. In (3.6) we choose $h = -u_n^- \in H_0^1(\Omega)$. Then

$$\left|\sigma\left(u_n^-\right) + \widehat{c}_1\left\|u_n^-\right\|_2^2\right| \leqslant \varepsilon_n \quad \forall n \geqslant 1, \tag{3.7}$$

so

$$\frac{1}{\widehat{c}_0}\left\|u_n^-\right\|^2 \leqslant \varepsilon_n \quad \forall n \geqslant 1 \tag{3.8}$$

(see (2.14)), and hence

$$u_n^- \longrightarrow 0 \quad \text{in } H_0^1(\Omega). \tag{3.9}$$

From (3.4) and (3.9), we have

$$\sigma(u_n^+) - \lambda \int_\Omega 2G_+(z,u_n^+)dz - \int_\Omega 2F(z,u_n^+)dz \leqslant M_2 \quad \forall n \geqslant 1, \tag{3.10}$$

for some $M_2 > 0$. Also, if in (3.6) we choose $h = u_n^+ \in H_0^1(\Omega)$, then

$$-\sigma(u_n^+) + \lambda \int_\Omega g(z,u_n^+)u_n^+ dz + \int_\Omega f(z,u_n^+)u_n^+\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.11}$$

Adding (3.10) and (3.11), we obtain

$$\int_\Omega \xi_\lambda(z,u_n^+)\ dz \ \leq M_3 \quad \forall n \geqslant 1, \tag{3.12}$$

for some $M_3 > 0$ (see (2.29) for the definition of ξ_λ).

Claim 1. The sequence $\{u_n^+\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ is bounded.

Arguing by contradiction, suppose that the claim is not true. Then by passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \longrightarrow +\infty. \tag{3.13}$$

Let

$$y_n = \frac{u_n^+}{\|u_n^+\|} \quad \forall n \geqslant 1. \tag{3.14}$$

Then

$$\|y_n\| = 1 \quad \forall n \geqslant 1. \tag{3.15}$$

And so, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } H_0^1(\Omega), \tag{3.16}$$

$$y_n \longrightarrow y \quad \text{in } L^r(\Omega). \tag{3.17}$$

If $y \neq 0$, then

$$u_n^+(z) \longrightarrow +\infty \quad \text{for almost all } z \in \Omega_+ = \{y > 0\} \tag{3.18}$$

(recall that $y \geqslant 0$). So, by virtue of hypotheses H_g(ii) and H_f(ii), for almost all $z \in \Omega_+$, we have

$$\lim_{n\to+\infty} \frac{G(z,u_n^+(z))}{\|u_n^+\|^2} = \lim_{n\to+\infty} \frac{G(z,u_n^+(z))}{u_n^+(z)^2} y_n(z)^2 = 0,$$
$$\lim_{n\to+\infty} \frac{F(z,u_n^+(z))}{\|u_n^+\|^2} = \lim_{n\to+\infty} \frac{F(z,u_n^+(z))}{u_n^+(z)^2} y_n(z)^2 = +\infty. \tag{3.19}$$

Then Fatou's lemma implies that

$$\lim_{n\to+\infty} \left(\lambda \int_\Omega \frac{G(z,u_n^+)}{\|u_n^+\|^2} dz + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2} dz \right) = +\infty. \tag{3.20}$$

But from (3.4) and (3.9), we have

$$\lambda \int_\Omega G(z,u_n^+)\, dz + \int_\Omega F(z,u_n^+)\, dz \leqslant M_4 + |\sigma(u_n^+)| \quad \forall n \geqslant 1, \tag{3.21}$$

for some $M_4 > 0$, so

$$\lambda \int_\Omega \frac{G(z,u_n^+)}{\|u_n^+\|^2} dz + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^2} dz \leqslant M_5 \quad \forall n \geqslant 1, \tag{3.22}$$

for some $M_5 > 0$ (since the sequence $\{\sigma(y_n)\}_{n\geqslant 1}$ is bounded in $\mathbb{R}$). Comparing (3.20) and (3.22), we reach a contradiction.

So, we have $y = 0$. We fix $\mu > 0$ and set

$$v_n = (2\mu)^{1/2} y_n \quad \forall n \geqslant 1. \tag{3.23}$$

Evidently

$$v_n \longrightarrow 0 \text{ in } L^r(\Omega) \tag{3.24}$$

(see (3.17)). Hence by Krasnoselskii's theorem (see Gasiński and Papageorgiou [12, Proposition 1.4.14, page 87] and hypotheses H_g(i) and H_f(i)), we have

$$\int_\Omega G(z,v_n(z))dz \longrightarrow 0, \qquad \int_\Omega F(z,v_n(z))dz \longrightarrow 0. \tag{3.25}$$

Since $\|u_n^+\| \to +\infty$, we can find $n_0 \geqslant 1$, such that

$$0 < (2\mu)^{1/2} \frac{1}{\|u_n^+\|} < 1 \quad \forall n \geqslant n_0. \tag{3.26}$$

Let $t_n \in [0,1]$ be such that

$$\widehat{\varphi}_\lambda^+(t_n u_n^+) = \max_{0\leqslant t\leqslant 1} \widehat{\varphi}_\lambda^+(t u_n^+). \tag{3.27}$$

By virtue of (3.26), we have

$$\begin{aligned}\widehat{\varphi}_\lambda^+(t_n u_n^+) &\geqslant \widehat{\varphi}_\lambda^+(v_n) \\ &= 2\mu\sigma(y_n) - \lambda\int_\Omega G(z,v_n)dz - \int_\Omega F(z,v_n)dz \\ &= 2\mu + 2\mu\int_\Omega \beta y_n^2\,dz - \lambda\int_\Omega G(z,v_n)dz - \int_\Omega F(z,v_n)dz.\end{aligned} \tag{3.28}$$

Note that

$$\int_\Omega \beta y_n^2\,dz \longrightarrow 0. \tag{3.29}$$

This fact together with (3.25) and (3.28) implies that

$$\widehat{\varphi}_\lambda^+(t_n u_n^+) \geqslant \mu \quad \forall n \geqslant n_1, \tag{3.30}$$

for some $n_1 \geqslant n_0$. Since $\mu > 0$ is arbitrary, we infer that

$$\widehat{\varphi}_\lambda^+(t_n u_n^+) \longrightarrow +\infty. \tag{3.31}$$

Note that

$$\widehat{\varphi}_\lambda^+(0) = 0, \qquad \widehat{\varphi}_\lambda^+(u_n^+) \leqslant M_6 \quad \forall n \geqslant 1, \tag{3.32}$$

for some $M_6 > 0$ (see (3.4) and (3.9)).

Hence (3.31) implies that there exists $n_2 \geqslant n_1$, such that

$$t_n \in (0,1) \quad \forall n \geqslant n_2. \tag{3.33}$$

And so from the choice of t_n, we have

$$\frac{d}{dt}\widehat{\varphi}_\lambda^+(t u_n^+)\big|_{t=t_n} = 0 \quad \forall n \geqslant n_2, \tag{3.34}$$

so

$$\left\langle (\widehat{\varphi}_\lambda^+)'(t_n u_n), u_n \right\rangle = 0 \quad \forall n \geqslant n_2, \tag{3.35}$$

thus

$$\left\langle (\widehat{\varphi}_\lambda^+)'(t_n u_n), t_n u_n \right\rangle = 0 \quad \forall n \geqslant n_2, \tag{3.36}$$

and hence

$$\sigma(t_n u_n^+) = \lambda \int_\Omega g(z, t_n u_n^+) t_n u_n^+ \, dz + \int_\Omega f(z, t_n u_n^+) t_n u_n^+ \, dz. \tag{3.37}$$

Hypothesis H_0 implies that

$$\int_\Omega \xi_\lambda(z, t_n u_n^+) dz \leqslant \int_\Omega \xi_\lambda(z, u_n^+) dz + \left\| \vartheta_\lambda^* \right\|_1 \quad \forall n \geqslant 1, \tag{3.38}$$

so

$$2\widehat{\varphi}_\lambda^+(t_n u_n^+) \leqslant \int_\Omega \xi_\lambda(z, u_n^+) dz + \left\| \vartheta_\lambda^* \right\|_1 \quad \forall n \geqslant 1 \tag{3.39}$$

(see (3.37)), and thus

$$2\widehat{\varphi}_\lambda^+(t_n u_n^+) \leqslant M_3 + \left\| \vartheta_\lambda^* \right\|_1 \quad \forall n \geqslant 1 \tag{3.40}$$

(see (3.12)).

Comparing (3.31) and (3.40), we reach a contradiction. This proves the claim.

By virtue of the claim and (3.9), we have that the sequence $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ is bounded. So, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } H_0^1(\Omega), \tag{3.41}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{3.42}$$

In (3.6) we choose $h = u_n - u$, pass to the limit as $n \to +\infty$, and use (3.42). Then

$$\lim_{n \to +\infty} \langle A(u_n), u_n - u \rangle = 0, \tag{3.43}$$

so

$$\|u_n\| = \|\nabla u_n\|_2 \longrightarrow \|\nabla u\|_2 = \|u\| \tag{3.44}$$

(see Gasiński and Papageorgiou [11, Proposition 3.1, page 852]), and thus

$$u_n \longrightarrow u \quad \text{in } H_0^1(\Omega) \tag{3.45}$$

(by the Kadec-Klee property of Hilbert spaces). This proves that $\widehat{\varphi}_{\lambda}^{+}$ satisfies the Cerami condition.

Similarly we show that $\widehat{\varphi}_{\lambda}^{-}$ also satisfies the Cerami condition. □

Our aim is to apply Theorem 2.1 (the mountain pass theorem) to two functionals $\widehat{\varphi}_{\lambda}^{+}$ and $\widehat{\varphi}_{\lambda}^{-}$. We have checked that both functionals satisfy the Cerami condition. So, it remains to show that they satisfy the mountain pass geometry as it is described in Theorem 2.1.

The next proposition is a crucial step in satisfying the mountain pass geometry for the two functionals $\widehat{\varphi}_{\lambda}^{+}$ and $\widehat{\varphi}_{\lambda}^{-}$.

Proposition 3.2. *If hypotheses H_g and H_f hold, then there exist $\lambda_{\pm}^{*} > 0$, such that for every $\lambda \in (0, \lambda_{\pm}^{*})$, we can find $\varrho_{\lambda}^{\pm} > 0$, such that*

$$\inf\left\{\widehat{\varphi}_{\lambda}^{\pm}(u) : \|u\| = \varrho_{\lambda}^{\pm}\right\} = \widehat{\eta}_{\lambda}^{\pm} > 0. \tag{3.46}$$

Proof. Hypotheses H_g(i) and (ii) imply that for a given $\varepsilon > 0$, we can find $c_4 = c_4(\varepsilon) > 0$, such that

$$g(z,\zeta) \leqslant \varepsilon\zeta + c_4\zeta^{q-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0, \tag{3.47}$$

so

$$G(z,\zeta) \leqslant \frac{\varepsilon}{2}\zeta^2 + \frac{c_4}{q}\zeta^q \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{3.48}$$

Similarly hypotheses H_f(i) and (iii) imply that for a given $\varepsilon > 0$, we can find $c_5 = c_5(\varepsilon) > 0$, such that

$$f(z,\zeta) \leqslant \left(\eta_0(z) + \varepsilon\right)\zeta + c_5\zeta^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0, \tag{3.49}$$

so

$$F(z,\zeta) \leqslant \frac{1}{2}\left(\eta_0(z) + \varepsilon\right)\zeta^2 + \frac{c_5}{r}\zeta^r \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{3.50}$$

Then, for $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} \widehat{\varphi}_{\lambda}^{+}(u) &= \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \lambda\int_{\Omega} G_{+}(z,u)dz - \int_{\Omega}\widehat{F}_{+}(z,u)dz \\ &\geqslant \frac{1}{2}\sigma(u^+) - \frac{1}{2}\int_{\Omega}\eta_0(u^+)^2dz - \frac{\varepsilon}{2}\|u^+\|_2^2 - \frac{c_5}{r}\|u^+\|_r^r \\ &\quad - \frac{\lambda\varepsilon}{2}\|u^+\|_2^2 - \frac{\lambda c_4}{q}\|u^+\|_q^q + \frac{1}{2}\sigma(u^-) + \frac{\widehat{c}_1}{2}\left\|u^-\right\|_2^2 \\ &\geqslant \frac{1}{2}\left(\widehat{c} - \frac{\varepsilon(\lambda+1)}{\widehat{\lambda}_1}\right)\|u^+\|^2 + \frac{1}{2\widehat{c}_0}\left\|u^-\right\|^2 - c_6\left(\|u\|^r + \lambda\|u\|^q\right), \end{aligned} \tag{3.51}$$

for some $c_6 > 0$ (see (3.48), (3.50), (2.11), (2.14), and Lemma 2.2).

Choosing $\varepsilon \in (0, \widehat{\lambda}_1 \widehat{c}/(\lambda+1))$, we have

$$\begin{aligned}\widehat{\varphi}_\lambda^+(u) &\geqslant c_7\|u\|^2 - c_6\left(\|u\|^r + \lambda\|u\|^q\right) \\ &= \left(c_7 - c_6\left(\|u\|^{r-2} + \lambda\|u\|^{q-2}\right)\right)\|u\|^2,\end{aligned} \tag{3.52}$$

for some $c_7 > 0$. Let

$$\mu_\lambda(t) = t^{r-2} + \lambda t^{q-2} \quad \forall t > 0. \tag{3.53}$$

Since $q < 2 < r$, we have

$$\lim_{t\to 0^+} \mu_\lambda(t) = \lim_{r\to +\infty} \mu_\lambda(t) = +\infty. \tag{3.54}$$

Also μ_λ is continuous in $(0, +\infty)$. Therefore, we can find $t_0 \in (0, +\infty)$, such that

$$\mu_\lambda(t_0) = \inf_{t>0} \mu_\lambda(t), \tag{3.55}$$

so

$$\mu_\lambda'(t_0) = 0, \tag{3.56}$$

and thus

$$t_0 = t_0(\lambda) = \left(\frac{\lambda(2-q)}{r-2}\right)^{1/(r-q)}. \tag{3.57}$$

Evidently

$$\mu_\lambda(t_0) \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+. \tag{3.58}$$

Hence we can find $\lambda_+^* > 0$, such that

$$\mu_\lambda(t_0) < \frac{c_7}{c_6} \quad \forall \lambda \in (0, \lambda_+^*), \tag{3.59}$$

so

$$\inf\left\{\widehat{\varphi}_\lambda^+(u) : \|u\| = \varrho_\lambda^+ = t_0(\lambda)\right\} = \widehat{\eta}_\lambda^+ > 0 \tag{3.60}$$

(see (3.52)).

Similarly, for $\widehat{\varphi}_\lambda^-$, we can find $\lambda_-^* > 0$, such that for all $\lambda \in (0, \lambda_-^*)$ there exists $\varrho_\lambda^- > 0$, such that

$$\inf\{\widehat{\varphi}_\lambda^-(u) : \|u\| = \varrho_\lambda^-\} = \widehat{\eta}_\lambda^- > 0. \tag{3.61}$$

□

With the next proposition we complete the mountain pass geometry for problem $((P)_\lambda)$.

Proposition 3.3. *If hypotheses H_g and H_f hold, $\lambda > 0$ and $\widetilde{u} \in \operatorname{int} C_+$ with $\|\widetilde{u}\|_2 = 1$, then*

$$\widehat{\varphi}_\lambda^\pm(t\widetilde{u}) \longrightarrow -\infty \quad \text{as } t \longrightarrow \pm\infty. \tag{3.62}$$

Proof. By virtue of hypotheses H_g(i) and (ii), for a given $\varepsilon > 0$, we can find $c_8 = c_8(\varepsilon) > 0$, such that

$$G(z,\zeta) \geqslant -\frac{\varepsilon}{2}\zeta^2 - c_8, \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.63}$$

Similarly, hypotheses H_f(i) and (ii) imply that for any given $\xi > 0$, we can find $c_9 = c_9(\xi) > 0$, such that

$$F(z,\zeta) \geqslant \frac{\xi}{2}\zeta^2 - c_9, \quad \text{for almost all } z \in \Omega, \text{all } \zeta \in \mathbb{R}. \tag{3.64}$$

Then, we have

$$\begin{aligned} \widehat{\varphi}_\lambda^+(t\widetilde{u}) &= \frac{t^2}{2}\sigma(\widetilde{u}) + \left(\frac{\lambda\varepsilon - \xi}{2}\right)t^2 + c_{10} \\ &\leqslant \frac{t^2}{2}\left(\|\widetilde{u}\|^2 + \|\beta\|_s\|\widetilde{u}\|_{2s'} + \lambda\varepsilon - \xi\right) + c_{10} \end{aligned} \tag{3.65}$$

for some $c_{10} > 0$ (see (3.63), (3.64) and recall that $\widetilde{u} \in \operatorname{int} C_+$, $\|\widetilde{u}\|_2 = 1$ and $1/s + 1/s' = 1$).

Since $\xi > 0$ is arbitrary, choosing

$$\xi > \lambda\varepsilon + \|\widetilde{u}\|^2 + \|\beta\|_s\|\widetilde{u}\|_{2s'}, \tag{3.66}$$

from (3.65), we infer that

$$\widehat{\varphi}_\lambda^+(t\widetilde{u}) \longrightarrow -\infty \quad \text{as } t \longrightarrow \pm\infty. \tag{3.67}$$

□

Now we are ready to produce the first two nontrivial smooth solutions of constant sign. In what follows, we set

$$\lambda^* = \min\{\lambda_-^*, \lambda_+^*\}. \tag{3.68}$$

Proposition 3.4. *If hypotheses H_g, H_f, and H_0 hold and $\lambda \in (0, \lambda^*]$, then problem $((P)_\lambda)$ has at least two nontrivial smooth solutions of constant sign:*

$$u_0, v_0 \in C_0^1\left(\overline{\Omega}\right), \quad \text{with } v_0(z) < 0 < u_0(z) \quad \forall z \in \Omega. \tag{3.69}$$

Proof. Propositions 3.1, 3.2, and 3.3 permit the application of the mountain pass theorem (see Theorem 2.1) for the functional $\widehat{\varphi}_\lambda^+$, and so we obtain $u_0 \in H_0^1(\Omega)$, such that

$$\widehat{\varphi}_\lambda^+(0) = 0 < \widehat{\eta}_\lambda^+ \leqslant \widehat{\varphi}_\lambda^+(u_0), \tag{3.70}$$

$$\left(\widehat{\varphi}_\lambda^+\right)'(u_0) = 0. \tag{3.71}$$

From (3.70), we see that $u_0 \neq 0$. From (3.71), we have

$$A(u_0) + \beta u_0 + \widehat{c}_1 u_0 = \lambda N_{g_+}(u_0) + N_{\widehat{f}_+}(u_0). \tag{3.72}$$

On (3.72) we act with $-u_0^- \in H_0^1(\Omega)$ and obtain

$$\sigma\left(u_0^-\right) + \widehat{c}_1 \left\|u_0^-\right\|_2^2 = 0, \tag{3.73}$$

so

$$\frac{1}{\widehat{c}_0}\left\|u_0^-\right\|^2 \leqslant 0 \tag{3.74}$$

(see (2.14)); hence $u_0 \geqslant 0$, $u_0 \neq 0$.

Therefore (3.72) becomes

$$A(u_0) + \beta u_0 = \lambda N_g(u_0) + N_f(u_0), \tag{3.75}$$

so

$$\begin{gathered} -\Delta u_0(z) + \beta(z) u_0(z) = \lambda g(z, u_0(z)) + f(z, u_0(z)) \quad \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{gathered} \tag{3.76}$$

From the regularity theory for Dirichlet problems (see Struwe [13, pp. 217–219]), we have that $u_0 \in C_0^1(\overline{\Omega})$. Moreover, invoking the weak Harnack inequality of Pucci and Serrin [14, page 154], we have that $u_0(z) > 0$ for all $z \in \Omega$.

Similarly working with $\widehat{\varphi}_\lambda^-$, this time we obtain a nontrivial smooth negative solution $v_0 \in C_0^1(\overline{\Omega})$ with $v_0(z) < 0$ for all $z \in \Omega$. □

We can improve the conclusion of this proposition by strengthening the condition on the potential β:

$\underline{H_\beta}$: $\beta \in L^s(\Omega)$ with $s > N/2$ and $\beta^+ \in L^\infty(\Omega)_+$.

Remark 3.5. So, the potential function is bounded from above but in general can be unbounded from below.

Proposition 3.6. *If hypotheses H_g, H_f, H_0, and H_β hold and $\lambda \in (0, \lambda^*)$, then problem $((P)_\lambda)$ has at least two nontrivial smooth solutions of constant sign:*

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+. \tag{3.77}$$

Proof. From Proposition 3.4, we already have two solutions:

$$u_0, v_0 \ \in \ C_0^1\left(\overline{\Omega}\right), \quad \text{with } v_0(z) < 0 < u_0(z) \quad \forall z \in \Omega. \tag{3.78}$$

We have

$$\begin{aligned} -\Delta u_0(z) + \beta(z) u_0(z) &= \lambda g(z, u_0(z)) + f(z, u_0(z)) \\ &\geqslant f(z, u_0(z)) \text{ for almost all } z \in \Omega \end{aligned} \tag{3.79}$$

(see hypothesis H_g(iii)). Let $\varrho = \|u_0\|_\infty$ and let $\widehat{\gamma}_\varrho > 0$ be as postulated by hypothesis H_f(iv). Then from (3.79), we have

$$-\Delta u_0(z) + \left(\beta(z) + \widehat{\gamma}_\varrho\right) u_0(z) \geqslant f(z, u_0(z)) + \widehat{\gamma}_\varrho u_0(z), \quad \text{for almost all } z \in \Omega, \tag{3.80}$$

so

$$\Delta u_0(z) \leqslant \left(\left\|\beta^+\right\|_\infty + \widehat{\gamma}_\varrho\right) u_0(z) \quad \text{for almost all } z \in \Omega. \tag{3.81}$$

and thus $u_0 \in \operatorname{int} C_+$ (see Vázquez [15] and Pucci and Serrin [14, page 120]).

Similarly for the negative solution v_0. □

To continue and produce additional nontrivial smooth solutions of constant sign, we need to keep hypotheses H_β.

Proposition 3.7. *If hypotheses H_g, H_f, H_0 and H_β hold and $\lambda \in (0, \lambda^*)$, then problem $((P)_\lambda)$ has at least four nontrivial smooth solutions of constant sign:*

$$\begin{aligned} u_0, \widehat{u} &\in \operatorname{int} C_+, \qquad \widehat{u} - u_0 \in \operatorname{int} C_+, \\ v_0, \widehat{v} &\in -\operatorname{int} C_+, \qquad v_0 - \widehat{v} \in \operatorname{int} C_+. \end{aligned} \tag{3.82}$$

Proof. From Proposition 3.6, we already have two solutions:

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+. \tag{3.83}$$

We introduce the following truncation perturbation of the reaction of the problem $((P)_\lambda)$:

$$h_\lambda^+(z,\zeta) = \begin{cases} \lambda g(z,u_0(z)) + f(z,u_0(z)) + \widehat{c}_1 u_0(z) & \text{if } \zeta \leqslant u_0(z), \\ \lambda g(z,\zeta) + f(z,\zeta) + \widehat{c}_1 \zeta & \text{if } u_0(z) < \zeta. \end{cases} \tag{3.84}$$

This is a Carathéodory function. We set

$$H_\lambda^+(z,\zeta) = \int_0^\zeta h_\lambda^+(z,s)ds \tag{3.85}$$

and consider the C^1-functional $\varphi_\lambda^+ : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\varphi_\lambda^+(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega H_\lambda^+(z,u(z))dz \quad \forall u \in H_0^1(\Omega). \tag{3.86}$$

Claim 2. We have $K_{\varphi_\lambda^+} \subseteq [u_0)$, where

$$[u_0) = \left\{ u \in H_0^1(\Omega) : u_0(z) \leqslant u(z) \text{ for almost all } z \in \Omega \right\}. \tag{3.87}$$

Let $\widetilde{u} \in K_{\varphi_\lambda^+}$. Then

$$A(\widetilde{u}) + (\beta + \widehat{c}_1)\widetilde{u} = N_{h_\lambda^+}(\widetilde{u}). \tag{3.88}$$

On (3.88) we act with $(u_0 - \widetilde{u})^+ \in H_0^1(\Omega)$. Then

$$\begin{aligned} &\left\langle A(\widetilde{u}), (u_0 - \widetilde{u})^+ \right\rangle + \int_\Omega (\beta + \widehat{c}_1)\widetilde{u}(u_0 - \widetilde{u})^+ dz \\ &\quad = \int_\Omega h_\lambda^+(z,\widetilde{u})(u_0 - \widetilde{u})^+ dz \\ &\quad = \int_\Omega \left(\lambda g(z,u_0) + f(z,u_0) + \widehat{c}_1 u_0\right)(u_0 - \widetilde{u})^+ dz \\ &\quad = \langle A(u_0), (u_0 - \widehat{u})^+ \rangle + \int_\Omega (\beta + \widehat{c}_1)u_0(u_0 - \widetilde{u})^+ dz \end{aligned} \tag{3.89}$$

(see (3.84)), so

$$\left\langle A(u_0 - \widetilde{u}), (u_0 - \widetilde{u})^+ \right\rangle + \int_\Omega \beta(u_0 - \widetilde{u})(u_0 - \widetilde{u})^+ dz + \widehat{c}_1 \left\| (u_0 - \widetilde{u})^+ \right\|_2^2 = 0, \tag{3.90}$$

thus

$$\sigma\left((u_0 - \widetilde{u})^+\right) + \widehat{c}_1 \left\| (u_0 - \widetilde{u})^+ \right\|_2^2 = 0, \tag{3.91}$$

hence

$$\frac{1}{\widehat{c}_0}\left\|(u_0-\widetilde{u})^+\right\|^2 \leqslant 0 \tag{3.92}$$

(see (2.14)), and so finally $u_0 \leqslant \widetilde{u}$. This proves Claim 2.

Claim 3. We may assume that u_0 is a local minimizer of φ_λ^+.

Let $\mu \in (\lambda, \lambda^*)$, and consider problem $(P)_\mu$. As we did in the proof of Proposition 3.4, via the mountain pass theorem, we obtain a nontrivial smooth positive solution u_μ, and by virtue of the strong maximum principle, we have $u_\mu \in \operatorname{int} C_+$ (see the proof of Proposition 3.7). Then

$$\begin{aligned} -\Delta u_\mu(z)+\beta(z)u_\mu(z) &= \mu g(z,u_\mu(z))+f(z,u_\mu(z)) \\ &\geqslant \lambda g(z,u_\mu(z))+f(z,u_\mu(z)) \quad \text{for almost all } z\in\Omega \end{aligned} \tag{3.93}$$

(see H_g(iii) and recall that $\lambda < \mu$).

We consider the following truncation perturbation of the reaction of problem $((P)_\lambda)$:

$$\gamma_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \lambda g(z,\zeta)+f(z,\zeta)+\widehat{c}_1\zeta & \text{if } 0 \leqslant \zeta \leqslant u_\mu(z), \\ \lambda g(z,u_\mu(z))+f(z,u_\mu(z))+\widehat{c}_1 u_\mu(z) & \text{if } u_\mu(z) < \zeta. \end{cases} \tag{3.94}$$

This is a Carathéodory function. We set

$$\Gamma_\lambda^+(z,\zeta) = \int_0^\zeta \gamma_\lambda^+(z,s)\,ds \tag{3.95}$$

and consider the C^1-functional $\widehat{\varphi}_\lambda^+ : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\widehat{\varphi}_\lambda^+(u) = \frac{1}{2}\sigma(u)+\frac{\widehat{c}_1}{2}\|u\|_2^2-\int_\Omega \Gamma_\lambda^+(z,u(z))\,dz \quad \forall u\in H_0^1(\Omega). \tag{3.96}$$

From (3.94), it is clear that $\widehat{\varphi}_\lambda^+$ is coercive. Also, using the Sobolev embedding theorem, we check that $\widehat{\varphi}_\lambda^+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in H_0^1(\Omega)$, such that

$$\widehat{\varphi}_\lambda^+(u_\lambda) = \inf_{u\in H_0^1(\Omega)} \widehat{\varphi}_\lambda^+(u). \tag{3.97}$$

By virtue of hypothesis H_f(iii), we can find $c_{11} > 0$ and $\widehat{\delta}_0 > 0$, such that

$$F(z,\zeta) \geqslant -\frac{c_{11}}{2}\zeta^2 \quad \text{for almost all } z\in\Omega,\ \text{all } |\zeta| \leqslant \widehat{\delta}_0. \tag{3.98}$$

Let $\tilde{u} \in \operatorname{int} C_+$ and let $t \in (0,1)$ be small, such that

$$t\tilde{u} \leqslant u_\mu, \qquad t\tilde{u}(z) \leqslant \widehat{\delta}_0 \quad \forall z \in \overline{\Omega} \tag{3.99}$$

(recall that $u_\mu \in \operatorname{int} C_+$). Then, we have

$$\begin{aligned} \widehat{\varphi}_\lambda^+(t\tilde{u}) &= \frac{t^2}{2}\sigma(\tilde{u}) + \frac{t^2\widehat{c}_1}{2}\left\|\tilde{u}^-\right\|_2^2 - \lambda\int_\Omega G(z,t\tilde{u})dz - \int_\Omega F(z,t\tilde{u})dz \\ &\leqslant \frac{t^2}{2}\left(\sigma(\tilde{u}) + (\widehat{c}_1 + c_{11})\|\tilde{u}\|_2^2\right) - \frac{\lambda t^q c_1}{q}\|\tilde{u}\|_q^q \end{aligned} \tag{3.100}$$

(see (3.94), hypothesis H_g(iii), and (3.98)). Since $q < 2$, by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_\lambda^+(t\tilde{u}) < 0, \tag{3.101}$$

so

$$\widehat{\varphi}_\lambda^+(u_\lambda) < 0 = \widehat{\varphi}_\lambda^+(0) \tag{3.102}$$

(see (3.97)); hence $u_\lambda \neq 0$.

From (3.97), we have

$$\left(\widehat{\varphi}_\lambda^+\right)'(u_\lambda) = 0, \tag{3.103}$$

so

$$A(u_\lambda) + (\beta + \widehat{c}_1)u_\lambda = N_{\gamma_\lambda^+}(u_\lambda). \tag{3.104}$$

Acting on (3.104) with $-u_\lambda^- \in H_0^1(\Omega)$, we obtain that $u_\lambda \geqslant 0$, $u_\lambda \neq 0$. Also, acting on (3.104) with $(u_\lambda - u_\mu)^+ \in H_0^1(\Omega)$, we have

$$\begin{aligned} &\left\langle A(u_\lambda), (u_\lambda - u_\mu)^+\right\rangle + \int_\Omega (\beta + \widehat{c}_1)u_\lambda(u_\lambda - u_\mu)^+ dz \\ &\quad = \int_\Omega \gamma_\lambda^+(z,u_\lambda)(u_\lambda - u_\mu)^+ dz \\ &\quad = \int_\Omega (\lambda g(z,u_\mu) + f(z,u_\mu) + \widehat{c}_1 u_\mu)(u_\lambda - u_\mu)^+ dz \\ &\quad \leqslant \left\langle A(u_\mu), (u_\lambda - u_\mu)^+\right\rangle + \int_\Omega (\beta + \widehat{c}_1)u_\mu(u_\lambda - u_\mu)^+ dz \end{aligned} \tag{3.105}$$

(see (3.94) and (3.93)), so

$$\left\langle A(u_\lambda - u_\mu),\ (u_\lambda - u_\mu)^+\right\rangle + \int_\Omega (\beta + \widehat{c}_1)(u_\lambda - u_\mu)(u_\lambda - u_\mu)^+ dz \leqslant 0, \tag{3.106}$$

thus

$$\sigma\Big(\big(u_\lambda - u_\mu\big)^+\Big) + \widehat{c}_1 \Big\| \big(u_\lambda - u_\mu\big)^+ \Big\|_2^2 \leqslant 0, \tag{3.107}$$

hence

$$\frac{1}{\widehat{c}_0} \Big\| \big(u_\lambda - u_\mu\big)^+ \Big\|^2 \leqslant 0 \tag{3.108}$$

(see (2.14)) and so finally

$$u_\lambda \leqslant u_\mu. \tag{3.109}$$

So, we have proved that

$$u_\lambda \in \big[0, u_\mu\big] = \Big\{ u \in H_0^1(\Omega) : 0 \leqslant u(z) \leqslant u_\mu(z) \text{ for almost all } z \in \Omega \Big\}. \tag{3.110}$$

Hence (3.104) becomes

$$A(u_\lambda) + \beta u_\lambda = \lambda N_g(u_\lambda) + N_f(u_\lambda) \tag{3.111}$$

(see (3.94)), so

$$\begin{gathered} -\Delta u_\lambda(z) + \beta(z) u_\lambda(z) = \lambda g(z, u_\lambda(z)) + f(z, u_\lambda(z)) \text{ in } \Omega, \\ u_\lambda|_{\partial\Omega} = 0, \end{gathered} \tag{3.112}$$

thus $u_\lambda \in \operatorname{int} C_+$ (as in the proof of Proposition 3.6) and it is a solution of $((P)_\lambda)$.

If $u_\lambda \neq u_0$, then this is the desired second nontrivial positive smooth solution of $((P)_\lambda)$.

So, we may assume that $u_\lambda = u_0$ and that there is no other solution of $((P)_\lambda)$ in the order interval

$$[u_0, u_\lambda] = \Big\{ u \in H_0^1(\Omega) : u_0(z) \leqslant u(z) \leqslant u_\mu(z) \text{ for almost all } z \in \Omega \Big\}. \tag{3.113}$$

We introduce the following truncation of $\gamma_\lambda^+(z, \cdot)$:

$$\widetilde{\gamma}_\lambda^+(z, \zeta) = \begin{cases} \gamma_\lambda^+(z, u_0(z)) & \text{if } \zeta < u_0(z), \\ \gamma_\lambda^+(z, \zeta) & \text{if } u_0(z) \leqslant \zeta. \end{cases} \tag{3.114}$$

This is a Carathéodory function. We set

$$\widetilde{\Gamma}_\lambda^+(z, \zeta) = \int_0^\zeta \widetilde{\gamma}_\lambda^+(z, s) ds \tag{3.115}$$

and consider the C^1-functional $\widetilde{\varphi}_\lambda^+ : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\widetilde{\varphi}_\lambda^+(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega \widetilde{\Gamma}_\lambda^+(z, u(z))dz \quad \forall u \in H_0^1(\Omega). \tag{3.116}$$

From (3.94) and (3.114), it follows that $\widetilde{\varphi}_\lambda^+$ is coercive. Also, it is sequentially weakly lower semicontinuous. Hence, we can find $\widetilde{u}_\lambda \in H_0^1(\Omega)$, such that

$$\widetilde{\varphi}_\lambda^+(\widetilde{u}_\lambda) = \inf_{u \in H_0^1(\Omega)} \widetilde{\varphi}_\lambda^+(u), \tag{3.117}$$

so

$$\left(\widetilde{\varphi}_\lambda^+\right)'(\widetilde{u}_\lambda) = 0, \tag{3.118}$$

and thus

$$A(\widetilde{u}_\lambda) + \left(\beta + \widehat{c}_1\right)\widetilde{u}_\lambda = N_{\widetilde{\gamma}_\lambda^+}(\widetilde{u}_\lambda). \tag{3.119}$$

As before, acting on (3.119) with $(u_0 - \widetilde{u}_\lambda)^+ \in H_0^1(\Omega)$ and with $(\widetilde{u}_\lambda - u_\mu)^+ \in H_0^1(\Omega)$, we show that $\widetilde{u}_\lambda \in [u_0, u_\mu]$. Then, from (3.94) and (3.114), it follows that

$$A(\widetilde{u}_\lambda) + \beta\widetilde{u}_\lambda = \lambda N_g(\widetilde{u}_\lambda) + N_f(\widetilde{u}_\lambda), \tag{3.120}$$

so $\widetilde{u}_\lambda \in \operatorname{int} C_+$ is a solution of $((P)_\lambda)$ in $[u_0, u_\mu]$, hence $\widetilde{u}_\lambda = u_0$.

Let $\varrho = \|u_0\|_\infty$ and let $\gamma_\varrho > 0$ and $\widehat{\gamma}_\varrho > 0$ be as postulated by hypotheses H_g(iv) and H_f(iv), respectively. We have

$$\begin{aligned}
&-\Delta u_0(z) + \left(\beta(z) + \widehat{\gamma}_\varrho\right)u_0(z) + \lambda\gamma_\varrho u_0(z)^{q-1} \\
&\quad = \lambda g(z, u_0(z)) + \lambda\gamma_\varrho u_0(z)^{q-1} + f(z, u_0(z)) + \widehat{\gamma}_\varrho u_0(z) \\
&\quad \leqslant \lambda g\left(z, u_\mu(z)\right) + \lambda\gamma_\varrho u_\mu(z)^{q-1} + f\left(z, u_\mu(z)\right) + \widehat{\gamma}_\varrho u_\mu(z) \\
&\quad \leqslant \mu g\left(z, u_\mu(z)\right) + \mu\gamma_\varrho u_\mu(z)^{q-1} + f\left(z, u_\mu(z)\right) + \widehat{\gamma}_\varrho u_\mu(z) \\
&\quad = -\Delta u_\mu(z) + \left(\beta(z) + \widehat{\gamma}_\varrho\right)u_\mu(z) + \mu\gamma_\varrho u_\mu(z)^{q-1} \text{ for almost all } z \in \Omega
\end{aligned} \tag{3.121}$$

(see H_g(iv) and (iii), H_f(iv) and recall that $u_0 \leqslant u_\mu$ and $\lambda < \mu$), so there exists $c_{12} > 0$, such that

$$\Delta\left(u_\mu - u_0\right)(z) \leqslant \left(\left\|\beta^+\right\|_\infty + \widehat{\gamma}_\varrho\right)\left(u_\mu - u_0\right)(z) + \mu\gamma_\varrho c_{12}\left(u_\mu - u_0\right)(z), \tag{3.122}$$

for almost all $z \in \Omega$ (recall that the function $\zeta \mapsto \zeta^{q-1}$ is locally Lipschitz); thus

$$u_\mu - u_0 \;\in\; \operatorname{int} C_+ \tag{3.123}$$

(see Struwe [13] and Pucci and Serrin [14, page 120]). So, we have that

$$u_\lambda = u_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [0, u_\mu]. \tag{3.124}$$

Note that

$$\widetilde{\varphi}_\lambda^+|_{[0,u_\mu]} = \varphi_\lambda^+|_{[0,u_\mu]} \tag{3.125}$$

(see (3.94) and (3.114)), so

$$u_0 \text{ is alocal } C_0^1\left(\overline{\Omega}\right)\text{-minimizer of } \varphi_\lambda^+ \tag{3.126}$$

(see (3.124)), and thus

$$u_0 \text{ is alocal } H_0^1(\Omega)\text{-minimizer of } \varphi_\lambda^+ \tag{3.127}$$

(Brézis and Nirenberg [16]). This proves Claim 3.

By virtue of Claim 3, as in Gasiński and Papageorgiou [17, proof of Theorem 3.4], we can find $\varrho_\lambda \in (0,1)$ small, such that

$$\varphi_\lambda^+(u_0) < \inf\{\varphi_\lambda^+(u) : \|u - u_0\| = \varrho_\lambda\} = \eta_\lambda^+. \tag{3.128}$$

As in Proposition 3.4, for $\widetilde{u} \in \operatorname{int} C_+$ with $\|\widetilde{u}\|_2 = 1$, we have

$$\varphi_\lambda^+(t\widetilde{u}) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \tag{3.129}$$

Note that $\widehat{\varphi}_\lambda^+ = \varphi_\lambda^+ - \xi_\lambda^+$ with $\xi_\lambda^+ \in \mathbb{R}$. Hence by virtue of Proposition 3.1, φ_λ^+ satisfies the Cerami condition. This fact together with (3.128) and (3.129) permits the use of the mountain pass theorem (see Theorem 2.1). So, we can find $\widehat{u} \in H_0^1(\Omega)$, such that

$$\varphi_\lambda^+(u_0) < \eta_\lambda^+ \leqslant \varphi_\lambda^+(\widehat{u}), \tag{3.130}$$

$$\left(\varphi_\lambda^+\right)'(\widehat{u}) = 0. \tag{3.131}$$

From (3.130) we have $\widehat{u} \neq u_0$. From (3.131) and Claim 2, we have that

$$u_0 \leqslant \widehat{u}. \tag{3.132}$$

Hence

$$A(\widehat{u}) + \beta\widehat{u} = \lambda N_g(\widehat{u}) + N_f(\widehat{u}) \tag{3.133}$$

(see (3.84)), and so $\widehat{u} \in \operatorname{int} C_+$ solves problem $((P)_\lambda)$.

Moreover, as before, using the strong maximum principle (see Vázquez [15] and Pucci and Serrin [14, page 120]), we have

$$\widehat{u} - u_0 \in \operatorname{int} C_+. \tag{3.134}$$

In a similar way, using $v_0 \in -\operatorname{int} C_+$, we define

$$h_\lambda^-(z,\zeta) = \begin{cases} \lambda g(z,\zeta) + f(z,\zeta) + \widehat{c}_1\zeta & \text{if } \zeta < v_0(z), \\ \lambda g(z,v_0(z)) + f(z,v_0(z)) + \widehat{c}_1 v_0(z) & \text{if } v_0(z) \leqslant \zeta. \end{cases} \tag{3.135}$$

This is a Carathéodory function. We set

$$H_\lambda^-(z,\zeta) = \int_0^\zeta h_\lambda^-(z,s)ds \tag{3.136}$$

and consider the C^1-functional $\varphi_\lambda^- : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\varphi_\lambda^-(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega H_\lambda^-(z,u(z))dz \quad \forall u \in H_0^1(\Omega). \tag{3.137}$$

Reasoning as above, using this time φ_λ^-, we obtain a second negative smooth solution $\widehat{v} \in -\operatorname{int} C_+$ of problem $((P)_\lambda)$, such that

$$v_0 - \widehat{v} \in \operatorname{int} C_+. \tag{3.138}$$

□

4. Five Solutions

In this section, we prove two multiplicity theorems, establishing five nontrivial smooth solutions when $\lambda \in (0,\lambda^*)$. In the second multiplicity theorem, we provide sign information for all the solutions (i.e., we show that the fifth solution is actually nodal).

Theorem 4.1. *If hypotheses H_g, H_f, H_0, and H_β hold and $\lambda \in (0,\lambda^*)$, then problem $((P)_\lambda)$ has at least five nontrivial smooth solutions:*

$$\begin{gathered} u_0, \widehat{u} \in \operatorname{int} C_+, \qquad \widehat{u} - u_0 \in \operatorname{int} C_+, \qquad v_0, \widehat{v} \in -\operatorname{int} C_+, \qquad v_0 - \widehat{v} \in \operatorname{int} C_+ \\ y_0 \in C_0^1\left(\overline{\Omega}\right), \qquad u_0 - y_0 \in \operatorname{int} C_+, \qquad y_0 - v_0 \in \operatorname{int} C_+. \end{gathered} \tag{4.1}$$

Proof. From Proposition 3.7, we already have four nontrivial smooth solutions of constant sign:

$$u_0, \widehat{u} \in \operatorname{int} C_+, \qquad \widehat{u} - u_0 \in \operatorname{int} C_+, v_0, \qquad \widehat{v} \in -\operatorname{int} C_+, \qquad v_0 - \widehat{v} \in \operatorname{int} C_+. \tag{4.2}$$

We consider the following truncation perturbation of the reaction of problem $((P)_\lambda)$:

$$h_\lambda^*(z,\zeta)=\begin{cases}\lambda g(z,v_0(z))+f(z,v_0(z))+\widehat{c}_1v_0(z) & \text{if } \zeta<v_0(z),\\ \lambda g(z,\zeta)+f(z,\zeta) & \text{if } v_0(z)\leqslant\zeta\leqslant u_0(z),\\ \lambda g(z,u_0(z))+f(z,u_0(z))+\widehat{c}_1u_0(z) & \text{if } u_0(z)<\zeta.\end{cases}\tag{4.3}$$

This is a Carathéodory function. We set

$$H_\lambda^*(z,\zeta)=\int_0^\zeta h_\lambda^*(z,s)ds\tag{4.4}$$

and consider the C^1-functional $\varphi_\lambda^*:H_0^1(\Omega)\to\mathbb{R}$, defined by

$$\varphi_\lambda^*(u)=\frac{1}{2}\sigma(u)+\frac{\widehat{c}_1}{2}\|u\|_2^2-\int_\Omega H_\lambda^*(z,u(z))dz\quad\forall u\in H_0^1(\Omega).\tag{4.5}$$

From (4.3), it follows that φ_λ^* is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $y_0\in H_0^1(\Omega)$, such that

$$\varphi_\lambda^*(y_0)=\inf_{u\in H_0^1(\Omega)}\varphi_\lambda^*(u).\tag{4.6}$$

As in the proof of Proposition 3.7, using (3.98), we show that

$$\varphi_\lambda^*(y_0)<0=\varphi_\lambda^*(0),\tag{4.7}$$

hence $y_0\neq 0$. From (4.6), we have

$$(\varphi_\lambda^*)'(y_0)=0,\tag{4.8}$$

so

$$A(y_0)+(\beta+\widehat{c}_1)y_0=N_{h_\lambda^*}(y_0).\tag{4.9}$$

On (4.9) we act with $(y_0-u_0)^+\in H_0^1(\Omega)$. Then

$$\begin{aligned}&\left\langle A(y_0),(y_0-u_0)^+\right\rangle+\int_\Omega(\beta+\widehat{c}_1)y_0(y_0-u_0)^+dz\\ &\quad=\int_\Omega(\lambda g(z,u_0)+f(z,u_0)+\widehat{c}_1u_0)(y_0-u_0)^+dz\\ &\quad=\left\langle A(u_0),(y_0-u_0)^+\right\rangle+\int_\Omega(\beta+\widehat{c})u_0(y_0-u_0)^+dz\end{aligned}\tag{4.10}$$

(see (4.3)), so

$$\left\langle A(y_0-u_0),(y_0-u_0)^+\right\rangle+\int_\Omega(\beta+\widehat{c}_1)(y_0-u_0)(y_0-u_0)^+dz=0, \tag{4.11}$$

thus

$$\sigma\left((y_0-u_0)^+\right)+\widehat{c}_1\left\|(y_0-u_0)^+\right\|_2^2=0, \tag{4.12}$$

and hence

$$\frac{1}{\widehat{c}_0}\left\|(y_0-u_0)^+\right\|^2\leqslant 0 \tag{4.13}$$

(see (2.14)); hence $y_0\leqslant u_0$.

Similarly, acting on (4.9) with $(v_0-y_0)^+\in H_0^1(\Omega)$, we show that $v_0\leqslant y_0$. Therefore,

$$y_0\in[v_0,u_0]=\left\{u\in H_0^1(\Omega):v_0(z)\leqslant u(z)\leqslant u_0(z)\ \text{for almost all}\ z\in\Omega\right\}, \tag{4.14}$$

and so (4.9) becomes

$$A(y_0)+\beta y_0=\lambda N_g(y_0)+N_f(y_0) \tag{4.15}$$

(see (4.3)); thus

$$y_0\in C_0^1\left(\overline{\Omega}\right) \tag{4.16}$$

(regularity theory; see Struwe [13]), and it solves problem $((P)_\lambda)$.

Moreover, as in the proof of Proposition 3.7, using hypotheses H_g(iv) and H_f(iv), we also show that

$$u_0-y_0\in\operatorname{int}C_+,\qquad y_0-v_0\in\operatorname{int}C_+. \tag{4.17}$$

□

Next we will improve the conclusion of Theorem 4.1 and show that the fifth solution y_0 is nodal (sign changing). To do this we need to strengthen a little bit the hypotheses on $f(z,\cdot)$.

The new hypotheses of f are the following:

$\underline{H_f'}$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, such that $f(z,0)=0$ for almost all $z\in\Omega$,

(i), (iii), (iv) are the same as the corresponding hypotheses H_f(i), (iii), (iv),

(ii) we have

$$\lim_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{\zeta}=+\infty \quad \text{uniformly for almost all } z\in\Omega. \tag{4.18}$$

Remark 4.2. Hypothesis H_f' is a slight restricted version of hypothesis H_f. Note that if

$$\widehat{\xi}(z,\zeta)=f(z,\zeta)\zeta-2F(z,\zeta) \tag{4.19}$$

and there exists a function $\widehat{\vartheta}^*\in L^1(\Omega)_+$, such that

$$\widehat{\xi}(z,\zeta)\leqslant\widehat{\xi}(z,y)+\widehat{\vartheta}^*(z) \quad \text{for almost all } z\in\Omega, \text{ all } 0\leqslant\zeta\leqslant y \text{ or } y\leqslant\zeta\leqslant 0, \tag{4.20}$$

then (4.20) and H_f (ii) imply H_f' (ii) (see Li and Yang [10]). Also, note that hypotheses H_f(i), (ii), and (iii) imply that there exists $c_*>0$, such that

$$f(z,\zeta)\zeta+c_*\zeta^2 \geqslant 0 \quad \text{for almost all } z\in\Omega, \text{ all } \zeta\in\mathbb{R}. \tag{4.21}$$

We consider the following auxiliary Dirichlet problem:

$$\begin{gathered}-\Delta u(z)+\beta(z)u(z)=\lambda c_1|u(z)|^{q-2}u(z)-c_*u(z) \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0.\end{gathered} \tag{$(Q)_\lambda$}$$

Here $c_1>0$ and $q\in(1,2)$ are as in hypothesis H_g(iii) and $c_*>0$ is as in (4.21).

Proposition 4.3. *For every $\lambda>0$, problem $((Q)_\lambda)$ has a unique nontrivial positive solution $\underline{u}_\lambda\in$ int C_+, and by oddness, we have that $-\underline{u}_\lambda=\overline{v}_\lambda\in -$int C_+ is the unique negative solution of $((Q)_\lambda)$.*

Proof. Let $k_\lambda^+:\Omega\times\mathbb{R}\to\mathbb{R}$ be the Carathéodory function, defined by

$$k_\lambda^+(z,\zeta)=\begin{cases}0 & \text{if } \zeta\leqslant 0,\\ \lambda c_1\zeta^{q-1}-(\widehat{c}_1-c_*)\zeta & \text{if } \zeta>0.\end{cases} \tag{4.22}$$

Clearly, we can always assume that $c_*\geqslant\widehat{c}_1$ (see (4.21)). Let

$$K_\lambda^+(z,\zeta)=\int_0^\zeta k_\lambda^+(z,s)ds, \tag{4.23}$$

and consider the C^1-functional $\theta_\lambda^+:H_0^1(\Omega)\to\mathbb{R}$, defined by

$$\theta_\lambda^+(u)=\frac{1}{2}\sigma(u)+\frac{\widehat{c}_1}{2}\|u\|_2^2-\int_\Omega K_\lambda^+(z,u(z))dz \quad \forall u\in H_0^1(\Omega). \tag{4.24}$$

Using (2.14) and (4.22), we have

$$\theta_\lambda^+(u) \geqslant \frac{1}{2\widehat{c}_0}\|u\|^2 + \frac{c_* - \widehat{c}_1}{2}\|u^+\|_2^2 - \frac{\lambda c_1}{q}\|u^+\|_q^q. \tag{4.25}$$

Since $q < 2$, from (4.25), we infer that θ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous (recall that $c_* \geqslant \widehat{c}_1$). So, by the Weierstrass theorem, we can find $\underline{u}_\lambda \in H_0^1(\Omega)$, such that

$$\theta_\lambda^+(\underline{u}_\lambda) = \inf_{u\in H_0^1(\Omega)} \theta_\lambda^+(u). \tag{4.26}$$

As before (see the proof of Proposition 3.7), since $q < 2$, we have

$$\theta_\lambda^+(\underline{u}_\lambda) < 0 = \theta_\lambda^+(0), \tag{4.27}$$

hence $\underline{u}_\lambda \neq 0$. From (4.26), we have

$$\left(\theta_\lambda^+\right)'(\underline{u}_\lambda) = 0, \tag{4.28}$$

so

$$A(\underline{u}_\lambda) + (\beta + \widehat{c}_1)\underline{u}_\lambda = N_{k_\lambda^+}(\underline{u}_\lambda). \tag{4.29}$$

Acting on (4.29) with $-\underline{u}_\lambda^- \in H_0^1(\Omega)$, we show that $\underline{u}_\lambda \geqslant 0$, $\underline{u}_\lambda \neq 0$ (see (2.14)). Then (4.29) becomes

$$A(\underline{u}_\lambda) + (\beta + c_*)\underline{u}_\lambda = \lambda c_1 \underline{u}_\lambda^{q-1}, \tag{4.30}$$

so

$$-\Delta \underline{u}_\lambda(z) + (\beta(z) + c_*)\underline{u}_\lambda(z) \geqslant 0 \quad \text{for almost all } z \in \Omega, \tag{4.31}$$

thus

$$\Delta \underline{u}_\lambda(z) \leqslant \left(\left\|\beta^+\right\|_\infty + c_*\right)\underline{u}_\lambda(z) \quad \text{for almost all } z \in \Omega \tag{4.32}$$

(see hypothesis H_β), and hence

$$\underline{u}_\lambda \in \operatorname{int} C_+ \tag{4.33}$$

(see Vázquez [15] and Pucci and Serrin [14, page 120]).

We claim that this solution is the unique nontrivial positive solution of $((Q)_\lambda)$. To this end, let $u, v \in \operatorname{int} C_+$ be two positive solutions of $((Q)_\lambda)$. We have

$$\begin{aligned}
&\int_\Omega \frac{\lambda c_1 u^{q-1} - c_* u}{u}\left(u^2 - v^2\right) dz \\
&\quad = \int_\Omega \left(\lambda c_1 u^{q-1} - c_* u\right)\left(u - \frac{v^2}{u}\right) dz \\
&\quad = \int_\Omega (-\Delta u)\left(u - \frac{v^2}{u}\right) dz + \int_\Omega \beta\left(u^2 - v^2\right) dz \\
&\quad = \int_\Omega \left(\nabla u, \nabla u - \nabla\left(\frac{v^2}{u}\right)\right)_{\mathbb{R}^N} dz + \int_\Omega \beta\left(u^2 - v^2\right) dz \\
&\quad = \|\nabla u\|_2^2 - \int_\Omega \left(\nabla u, \frac{2v}{u}\nabla v - \frac{v^2}{u^2}\nabla u\right)_{\mathbb{R}^N} dz + \int_\Omega \beta\left(u^2 - v^2\right) dz \\
&\quad = \|\nabla u\|_2^2 - \int_\Omega \frac{2v}{u}(\nabla u, \nabla v)_{\mathbb{R}^N} dz \\
&\qquad + \int_\Omega \frac{v^2}{u^2}\|\nabla u\|^2\, dz + \int_\Omega \beta\left(u^2 - v^2\right) dz.
\end{aligned} \tag{4.34}$$

Interchanging the roles of u and v in the above argument, we also have

$$\begin{aligned}
&\int_\Omega \frac{\lambda c_1 v^{q-1} - c_* v}{v}\left(v^2 - u^2\right) dz \\
&\quad = \|\nabla v\|_2^2 - \int_\Omega \frac{2u}{v}(\nabla v, \nabla u)_{\mathbb{R}^N} dz + \int_\Omega \frac{u^2}{v^2}\|\nabla v\|^2 dz + \int_\Omega \beta\left(v^2 - u^2\right) dz.
\end{aligned} \tag{4.35}$$

Adding (4.34) and (4.35), we obtain

$$\int_\Omega \left(\frac{\lambda c_1 u^{q-1} - c_* u}{u} - \frac{\lambda c_1 v^{q-1} - c_* v}{v}\right)\left(u^2 - v^2\right)\, dz = \left\|\nabla u - \frac{u}{v}\nabla v\right\|_2^2 + \left\|\nabla v - \frac{v}{u}\nabla u\right\|_2^2 \geqslant 0. \tag{4.36}$$

Since $q < 2$, the function $\zeta \mapsto (\lambda c_1 \zeta^{q-1} - c_* \zeta)/\zeta$ is strictly decreasing on $(0, +\infty)$. Hence, from (4.36), we infer that $u = v$. This proves the uniqueness of the nontrivial positive solution $\underline{u}_\lambda \in \operatorname{int} C_+$ of problem $((Q)_\lambda)$.

The oddness of problem $((Q)_\lambda)$ implies that $\overline{v}_\lambda = -\underline{u}_\lambda \in -\operatorname{int} C_+$ is the unique nontrivial negative solution of $((Q)_\lambda)$. □

Using Proposition 4.3, we can show that problem $((P)_\lambda)$ (for $\lambda \in (0, \lambda^*)$) has extremal constant sign solutions; that is, it has a smallest nontrivial positive solution and a biggest nontrivial negative solution.

Proposition 4.4. *If hypotheses H_g, H_f', H_0, and H_β hold and $\lambda \in (0, \lambda^*)$, then problem $((P)_\lambda)$ has a smallest nontrivial positive solution $u_+^\lambda \in \operatorname{int} C_+$ and a biggest nontrivial negative solution $v_-^\lambda \in \operatorname{int} -C_+$.*

Proof. Let $\overline{u}$ be a nontrivial positive solution of $((P)_\lambda)$. From the proof of Proposition 3.6, we know that $\overline{u} \in \operatorname{int} C_+$. We have

$$-\Delta\overline{u}(z) + \beta(z)\overline{u}(z) = \lambda g(z, \overline{u}(z)) + f(z, \overline{u}(z)) \geqslant \lambda c_1 \overline{u}(z)^{q-1} - c_* \overline{u}(z), \tag{4.37}$$

for almost all $z \in \Omega$ (see hypothesis H_g(iii) and (4.21)).

We consider the following Carathéodory function:

$$\widehat{k}_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta \leqslant 0, \\ \lambda c_1 \zeta^{q-1} + (\widehat{c}_1 - c_*)\zeta & \text{if } 0 \leqslant \zeta \leqslant \overline{u}, \\ \lambda c_1 \overline{u}(z)^{q-1} + (\widehat{c}_1 - c_*)\overline{u}(z) & \text{if } \overline{u} < \zeta. \end{cases} \tag{4.38}$$

Let

$$\widehat{K}_\lambda^+(z,\zeta) = \int_0^\zeta \widehat{k}_\lambda^+(z,s)\,ds, \tag{4.39}$$

and consider the C^1-functional $\xi_\lambda^+ : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\xi_\lambda^+(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega \widehat{K}_\lambda^+(z, u(z))\,dz \quad \forall u \in H_0^1(\Omega). \tag{4.40}$$

From (4.38) and (2.14), it is clear that ξ_λ^+ is coercive. Also ξ_λ^+ is sequentially weakly lower semicontinuous. Thus we can find $w_0 \in H_0^1(\Omega)$, such that

$$\xi_\lambda^+(w_0) = \inf_{u \in H_0^1(\Omega)} \xi_\lambda^+(u). \tag{4.41}$$

As before, the presence of the "concave" term $\lambda c_1 \zeta^{q-1}$ implies that

$$\xi_\lambda^+(w_0) < 0 = \xi_\lambda^+(0), \tag{4.42}$$

that is, $w_0 \neq 0$. From (4.41), we have

$$\left(\xi_\lambda^+\right)'(w_0) = 0, \tag{4.43}$$

so

$$A(w_0) + \left(\beta + \widehat{c}_1\right)w_0 = N_{\widehat{k}_\lambda^+}(w_0). \tag{4.44}$$

On (4.44) we act with $-w_0^- \in H_0^1(\Omega)$ and obtain

$$\sigma\left(w_0^-\right) + \widehat{c}_1 \left\|w_0^-\right\|_2^2 = 0 \tag{4.45}$$

(see (4.38)), so

$$\frac{1}{\widehat{c}_0}\left\|w_0^-\right\|^2 \leqslant 0 \tag{4.46}$$

(see (2.14)), and hence $w_0 \geqslant 0$, $w_0 \neq 0$.

Also, on (4.44) we act with $(w_0 - \overline{u})^+ \in H_0^1(\Omega)$. Then

$$\begin{aligned} &\langle A(w_0), (w_0 - \overline{u})^+ \rangle + \int_\Omega (\beta + \widehat{c}_1) w_0 (w_0 - \overline{u})^+ dz \\ &\quad = \int_\Omega \widehat{k}_\lambda^+(z, w_0)(w_0 - \overline{u})^+ dz \\ &\quad = \int_\Omega \left(\lambda c_1 \overline{u}^{q-1} - c_* \overline{u}\right)(w_0 - \overline{u})^+ dz \\ &\quad \leqslant \langle A(\overline{u}), (w_0 - \overline{u})^+ \rangle + \int_\Omega \beta\left((w_0 - \overline{u})^+\right) dz + \widehat{c}_1 \left\|(w_0 - \overline{u})^+\right\|_2^2 \end{aligned} \tag{4.47}$$

(see (4.38) and (4.37)), so

$$\sigma\left((w_0 - \overline{u})^+\right) + \widehat{c}_1 \left\|(w_0 - \overline{u})^+\right\|_2^2 \leqslant 0, \tag{4.48}$$

thus

$$\frac{1}{\widehat{c}_0}\left\|(w_0 - \overline{u})^+\right\|^2 \leqslant 0 \tag{4.49}$$

(see (2.14)), and hence

$$w_0 \leqslant \overline{u}. \tag{4.50}$$

So, we have proved that

$$w_0 \in [0, \overline{u}] = \left\{u \in H_0^1(\Omega) : 0 \leqslant u(z) \leqslant \overline{u}(z) \text{ for almost all } z \in \Omega\right\}. \tag{4.51}$$

This means that (4.44) becomes

$$A(w_0) + \beta w_0 = \lambda c_1 w_0^{q-1} - c_* w_0, \tag{4.52}$$

so $w_0 \in \operatorname{int} C_+$ (regularity theory of Struwe [13] and strong maximum principle due to Vázquez [15] and Pucci Serrin [14, page 120]) and it solves problem $((Q)_\lambda)$. Thus

$$w_0 = \underline{u}_\lambda \tag{4.53}$$

(see Proposition 4.3), and

$$\underline{u}_\lambda \leqslant \underline{u}. \tag{4.54}$$

This shows that every nontrivial positive solution u of $((P)_\lambda)$ satisfies

$$\underline{u}_\lambda \leqslant u. \tag{4.55}$$

Similarly, we show that every nontrivial negative solution v of problem $((P)_\lambda)$ satisfies

$$v \leqslant \overline{v}_\lambda = -\underline{u}_\lambda. \tag{4.56}$$

Let $S_+(\lambda)$ (resp., $S_-(\lambda)$) be the set of nontrivial positive (resp., negative) solutions of problem $((P)_\lambda)$. Let $C \subseteq S_+(\lambda)$ be a chain (i.e., a totally ordered subset of $S_+(\lambda)$). From Dunford and Schwartz [18, page 336], we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq C$, such that

$$\inf C = \inf_{n\geqslant 1} u_n. \tag{4.57}$$

Lemma 1.5 of Heikkilä and Lakshmikantham [19, page 15] implies that we can have the sequence $\{u_n\}_{n\geqslant 1} \subseteq C$ to be decreasing. Then we have

$$A(u_n) + \beta u_n = \lambda N_g(u_n) + N_f(u_n), \quad \underline{u}_\lambda \leqslant u_n \leqslant u_1 \ \forall n \geqslant 1, \tag{4.58}$$

so

$$\text{the sequnece } \{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega) \text{ is bounded.} \tag{4.59}$$

Hence by passing to a suitable subsequence if necessary, we may assume that

$$u_n \longrightarrow u_* \text{ weakly in } H_0^1(\Omega), \tag{4.60}$$

$$u_n \longrightarrow u_* \text{ in } L^{2s'}(\Omega) \text{ and in } L^r(\Omega). \tag{4.61}$$

So, passing to the limit as $n \to +\infty$ in (4.58) and using (4.61), we obtain

$$A(u_*) + \beta u_* = \lambda N_g(u_*) + N_f(u_*), \quad \underline{u}_\lambda \leqslant u_*, \tag{4.62}$$

so

$$\inf C = u_* \in S_+(\lambda). \tag{4.63}$$

Since C was an arbitrary chain, invoking the Kuratowski-Zorn lemma, we infer that $S_+(\lambda)$ has a minimal element $u_+^\lambda \in \operatorname{int} C_+$. From Gasiński and Papageorgiou [17, Lemma 4.2, page 5763], we know that $S_+(\lambda)$ is downward directed (i.e., if $u_1, u_2 \in S_+(\lambda)$; then we can find $u \in S_+(\lambda)$, such that $u \leqslant u_1$ and $u \leqslant u_2$). So, it follows that u_+^λ is the smallest nontrivial positive solution of problem $((P)_\lambda)$.

Similarly, we introduce the biggest nontrivial negative solution $v_-^\lambda \in -\operatorname{int} C_+$ of problem $((P)_\lambda)$. Note that $S_-(\lambda)$ is upward directed (i.e., if $v_1, v_2 \in S_-(\lambda)$, then we can find $v \in S_-(\lambda)$, such that $v_1 \leqslant v$ and $v_2 \leqslant v$; see Gasiński and Papageorgiou [17, Lemma 4.3, page 5764]). □

Now that we have these extremal constant sign solutions, we can produce a nodal solution of problem $((P)_\lambda)$ (with $\lambda \in (0, \lambda^*)$).

Theorem 4.5. *If hypotheses H_g, H_f', H_0, and H_β hold and $\lambda \in (0, \lambda^*)$, then problem $((P)_\lambda)$ has at least five nontrivial smooth solutions:*

$$\begin{gathered} u_0, \widehat{u} \in \operatorname{int} C_+, \qquad \widehat{u} - u_0 \in \operatorname{int} C_+, \qquad v_0, \widehat{v} \in -\operatorname{int} C_+, \qquad v_0 - \widehat{v} \in \operatorname{int} C_+ \\ y_0 \in C_0^1\left(\overline{\Omega}\right) \textit{ nodal with } u_0 - y_0 \in \operatorname{int} C_+, \quad y_0 - v_0 \in \operatorname{int} C_+. \end{gathered} \tag{4.64}$$

Moreover, problem $((P)_\lambda)$ has a smallest nontrivial positive solution and a biggest negative solution.

Proof. The existence of extremal nontrivial constant sing solutions is guaranteed by Proposition 4.4. Let $u_+^\lambda \in \operatorname{int} C_+$ and $v_-^\lambda \in -\operatorname{int} C_+$ be these two extremal solutions.

We introduce the following truncation perturbation of the reaction of problem $((P)_\lambda)$:

$$\gamma_\lambda(z, \zeta) = \begin{cases} \lambda g\left(z, v_-^\lambda(z)\right) + f\left(z, v_-^\lambda(z)\right) + \widehat{c}_1 v_-^\lambda(z) & \text{if } \zeta < v_-^\lambda(z), \\ \lambda g(z, \zeta) + f(z, \zeta) + \widehat{c}_1 \zeta & \text{if } v_-^\lambda(z) \leqslant \zeta \leqslant u_+^\lambda(z), \\ \lambda g(z, u_+^\lambda(z)) + f(z, u_+^\lambda(z)) + \widehat{c}_1 u_+^\lambda(z) & \text{if } u_+^\lambda(z) < \zeta. \end{cases} \tag{4.65}$$

This is a Carathéodory function. We set

$$\Gamma_\lambda(z, \zeta) = \int_0^\zeta \gamma_\lambda(z, s)\, ds \tag{4.66}$$

and consider the C^1-functional $\chi_\lambda : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\chi_\lambda(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega \Gamma_\lambda(z, u(z))\, dz \quad \forall u \in H_0^1(\Omega). \tag{4.67}$$

Also, we introduce

$$\gamma_\lambda^\pm(z,\zeta) = \gamma_\lambda(z, \pm\zeta^\pm), \qquad \Gamma_\lambda^\pm(z,\zeta) = \int_0^\zeta \gamma_\lambda^\pm(z,s)ds \tag{4.68}$$

and consider the C^1-functionals $\chi_\lambda^\pm : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\chi_\lambda^\pm(u) = \frac{1}{2}\sigma(u) + \frac{\widehat{c}_1}{2}\|u\|_2^2 - \int_\Omega \Gamma_\lambda^\pm(z,u(z))dz \quad \forall u \in H_0^1(\Omega). \tag{4.69}$$

As in the proof of Theorem 4.1, we show that

$$K_{\chi_\lambda} \subseteq \left[v_-^\lambda, u_+^\lambda\right], \qquad K_{\chi_\lambda^+} \subseteq \left[0, u_+^\lambda\right], \qquad K_{\chi_\lambda^-} \subseteq \left[v_-^\lambda, 0\right] \tag{4.70}$$

(see (4.65)). The extremality of the solutions v_-^λ and u_+^λ implies that

$$K_{\chi_\lambda} \subseteq \left[v_-^\lambda, u_+^\lambda\right], \qquad K_{\chi_\lambda^+} = \left\{0, u_+^\lambda\right\}, \qquad K_{\chi_\lambda^-} = \left\{v_-^\lambda, 0\right\}. \tag{4.71}$$

Claim 4. Solutions u_+^λ and v_-^λ are both local minimizers of χ_λ.

Evidently χ_λ^+ is coercive (see (4.65)) and sequentially weakly lower semicontinuous. So, we can find $u_+ \in H_0^1(\Omega)$, such that

$$\chi_\lambda^+(u_+) = \inf_{u \in H_0^1(\Omega)} \chi_\lambda^+(u). \tag{4.72}$$

As before (see the proof of Proposition 3.7), the presence of the "concave" term implies that

$$\chi_\lambda^+(u_+) < 0 = \chi_\lambda^+(0), \tag{4.73}$$

hence $u_+ \neq 0$, and so $u_+ = u_+^\lambda$ (see (4.71)). Since

$$\chi_\lambda\big|_{C_+} = \chi_\lambda^+\big|_{C_+}, \tag{4.74}$$

it follows that $u_+ = u_+^\lambda \in \operatorname{int} C_+$ is local $C_0^1(\overline{\Omega})$-minimizers of χ_λ; hence by Brézis and Nirenberg [16], it is also local $H_0^1(\Omega)$-minimizers of χ_λ.

Similarly this is for v_-^λ using this time the functional χ_λ^-. This proves the claim.

Without any loss of generality, we may assume that

$$\chi_\lambda\left(v_-^\lambda\right) \leqslant \chi_\lambda\left(u_+^\lambda\right). \tag{4.75}$$

The analysis is similar, if the opposite inequality holds. Because of the claim, we can find $\varrho \in (0,1)$ small, such that

$$\chi_\lambda\left(v_-^\lambda\right) \leqslant \chi_\lambda\left(u_+^\lambda\right) < \inf\left\{\chi_\lambda(u) : \left\|u - u_+^\lambda\right\| = \varrho\right\} = \eta_\lambda \tag{4.76}$$

(see Gasiński and Papageorgiou [17, proof of Theorem 3.4]).

Since χ_λ is coercive (see (4.65)), it satisfies the Cerami condition. This fact and (4.76) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $y_0 \in K_{\chi_\lambda} \subseteq [v_-^\lambda, u_+^\lambda]$ (see (4.71)), such that

$$\eta_\lambda \leqslant \chi_\lambda(y_0), \tag{4.77}$$

so

$$y_0 \notin \left\{v_-^\lambda, u_+^\lambda\right\} \tag{4.78}$$

(see (4.76)).

Since y_0 is a critical point of χ_λ of mountain pass type, we have

$$C_1(\chi_\lambda, y_0) \neq 0 \tag{4.79}$$

(see e.g., Chang [20]). On the other hand, hypothesis H_f(iii) implies that we can find $\widehat{\xi}_1 > 0$, such that

$$f(z,\zeta)\zeta - \mu F(z,\zeta) \leqslant \widehat{\xi}_1 \zeta^2 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta_1, \tag{4.80}$$

for some $\delta_1 \leqslant \delta_0$. This combined with hypothesis H_g(iii) implies that

$$\begin{aligned} \mu\lambda G(z,\zeta) + \mu F(z,\zeta) &\geqslant \lambda g(z,\zeta)\zeta + f(z,\zeta)\zeta \\ &> 0 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta_2, \end{aligned} \tag{4.81}$$

for some $\delta_2 \leqslant \delta_1$ and

$$\operatorname*{ess\,sup}_{\Omega} \lambda G(\cdot,\delta_2) + F(\cdot,\delta_2) > 0. \tag{4.82}$$

Hence invoking Proposition 2.1 of Jiu and Su [21], we infer that

$$C_k(\chi_\lambda, 0) = 0 \quad \forall k \geqslant 0. \tag{4.83}$$

Combining (4.79) and (4.83), we have that $y_0 \neq 0$. Since $y_0 \in [v_-^\lambda, u_+^\lambda]$, the extremality of v_-^λ and u_+^λ implies that y_0 must be a nodal solution of problem $((P)_\lambda)$, and the regularity theory (see Struwe [13]) implies that $y_0 \in C_0^1(\overline{\Omega})$.

□

Acknowledgments

This research has been partially supported by the Ministry of Science and Higher Education of Poland under Grants no. N201 542438 and N201 604640.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 191254, 21 pages
doi:10.1155/2012/191254

Research Article

Existence of Bounded Positive Solutions for Partial Difference Equations with Delays

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Received 3 February 2012; Accepted 14 March 2012

Academic Editor: Agacik Zafer

This paper deals with solvability of the third-order nonlinear partial difference equation with delays $\Delta_n(a_{m,n}\Delta_m^2(x_{m,n}+b_{m,n}x_{m-\tau_0,n-\sigma_0}))+f(m,n,x_{m-\tau_{1,m},n-\sigma_{1,n}},\ldots,x_{m-\tau_{k,m},n-\sigma_{k,n}})=c_{m,n},\ m\geq m_0,\ n\geq n_0$. With the help of the Banach fixed-point theorem, the existence results of uncountably many bounded positive solutions for the partial difference equation are given; some Mann iterative schemes with errors are suggested, and the error estimates between the iterative schemes and the bounded positive solutions are discussed. Three nontrivial examples illustrating the results presented in this paper are also provided.

1. Introduction and Preliminaries

In the past twenty years many authors studied the oscillation, nonoscillation, asymptotic behavior, and solvability for various neutral delay difference and partial difference equations; see, for example, [1–14] and the references cited therein.

By using the Banach fixed-point theorem, Cheng [2] investigated the existence of a nonoscillatory solution for the second-order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n+px_{n-m})+p_nx_{n-k}-q_nx_{n-l}=0,\quad n\geq n_0 \tag{1.1}$$

under the condition $p \in \mathbb{R}\setminus\{-1\}$. Applying a nonlinear alternative of Leray-Schauder type for condensing operators, Agarwal et al. [1] discussed the existence of a bounded nonoscillatory solution for the discrete equation:

$$\Delta\left(a_n \Delta\left(x_n + p x_{n-\tau}\right)\right) + F\left(n+1, x_{n+1-\sigma}\right) = 0, \quad n \geq 0. \tag{1.2}$$

Liu et al. [6] introduced the second-order nonlinear neutral delay difference equation

$$\Delta(a_n \Delta(x_n + b x_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq 0 \tag{1.3}$$

with respect to all $b \in \mathbb{R}$ and gave the existence of uncountably many bounded nonoscillatory solutions for (1.3) by utilizing the Banach fixed-point theorem. Kong et al. [3] investigated a class of BVPs for the third-order functional difference equation

$$\Delta^3 x_n + a_n f\left(n, x_{w(n)}\right) = 0, \quad n \geq 0 \tag{1.4}$$

and established the existence of positive solutions for (1.4) under certain conditions. Using the Schauder fixed-point theorem, Yan and Liu [12] studied the existence of a bounded nonoscillatory solution for third order nonlinear delay difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-r}) = 0, \quad n \geq n_0 \tag{1.5}$$

and provided also a necessary and sufficient condition for the existence of a bounded nonoscillatory solution of (1.5).

Karpuz and Öcalan [4] discussed the first-order linear partial difference equation:

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + p_{m,n} x_{m-k,n-l} = 0, \quad (m,n) \in \mathbb{Z}_{0,0}, \tag{1.6}$$

where $\{p_{m,n}\}_{(m,n)\in\mathbb{Z}_{0,0}}$ is a nonnegative sequence and $k, l \in \mathbb{N}_1$ and obtained sufficient conditions under which every solution of (1.6) is oscillatory. Yang and Zhang [14] considered oscillations of the partial difference equation with several nonlinear terms of the form

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + \sum_{i=1}^{h} p_i(m,n) \left|x_{m-k_i,n-l_i}\right|^{\alpha_i} \operatorname{sgn} x_{m-k_i,n-l_i} = 0 \tag{1.7}$$

and established some new oscillatory criteria by making use of frequency measures. Wong and Agarwal [10] considered the partial difference equations

$$x_{m+1,n} + \beta_{m,n} x_{m,n+1} - \delta_{m,n} x_{m,n} + p(m,n,x_{m-k,n-l}) = Q(m,n,x_{m-k,n-l}), \quad m \geq m_0,\ n \geq n_0, \tag{1.8}$$

$$x_{m+1,n} + \beta_{m,n} x_{m,n+1} - \delta_{m,n} x_{m,n} + \sum_{i=1}^{\tau} p_i(m,n,x_{m-k,n-l}) = \sum_{i=1}^{\tau} Q(m,n,x_{m-k,n-l}), \quad m \geq m_0,\ n \geq n_0 \tag{1.9}$$

and offered sufficient conditions for the oscillation of all solutions for (1.8) and (1.9), respectively. Wong [9] established the existence of eventually positive and monotone decreasing solutions for the partial difference inequalities

$$\Delta_m \Delta_n x_{m,n} + \sum_{i=1}^{r} p_i\left(m, n, x_{g_i(m),h_i(n)}\right) \geq (\leq) \sum_{i=1}^{r} Q_i\left(m, n, x_{g_i(m),h_i(n)}\right), \quad m \geq m_0,\ n \geq n_0, \tag{1.10}$$

where $g_i(m)$ and $h_i(m)$ are some deviating arguments for $1 \leq i \leq \tau$.

However, to the best of our knowledge, there is no literature referred to the following third order nonlinear partial difference equation with delays:

$$\begin{aligned} &\Delta_n\Big(a_{m,n}\Delta_m^2(x_{m,n} + b_{m,n}x_{m-\tau_0,n-\sigma_0})\Big) + f\left(m, n, x_{m-\tau_{1,m},n-\sigma_{1,n}}, \ldots, x_{m-\tau_{k,m},n-\sigma_{k,n}}\right) \\ &\quad = c_{m,n}, \quad m \geq m_0,\ n \geq n_0, \end{aligned} \tag{1.11}$$

where $m_0, n_0 \in \mathbb{N}_0$, $k, \tau_0, \sigma_0 \in \mathbb{N}$, $\{a_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$, $\{b_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$, $\{c_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ are real sequences with $a_{m,n} \neq 0$, $b_{m,n} \neq \pm 1$ for $(m,n) \in \mathbb{N}_{m_0,n_0}$, $f : \mathbb{N}_{m_0,n_0} \times \mathbb{R}^k \to \mathbb{R}$ and $\{\tau_{l,m}, \sigma_{l,n} : (m,n) \in \mathbb{N}_{m_0,n_0},\ l \in \{1, 2, \ldots, k\}\} \subseteq \mathbb{Z}$ with

$$\lim_{m\to\infty}(m - \tau_{l,m}) = \lim_{n\to\infty}(n - \sigma_{l,n}) = +\infty, \quad l \in \{1, 2, \ldots, k\}. \tag{1.12}$$

The aim of this paper is to establish three sufficient conditions of the existence of uncountably many bounded positive solutions for (1.11) by using the Banach fixed-point theorem, to suggest some Mann iterative methods with errors for these bounded positive solutions and to compute the error estimates between the bounded positive solutions and the sequences generated by the Mann iterative methods with errors. In order to explain the results presented in this paper, three nontrivial examples are constructed.

Throughout this paper, the forward partial difference operators Δ_m and Δ_n are defined by $\Delta_m x_{m,n} = x_{m+1,n} - x_{m,n}$ and $\Delta_n x_{m,n} = x_{m,n+1} - x_{m,n}$, respectively the second and third-order partial difference operators are defined by $\Delta_m^2 x_{m,n} = \Delta_m(\Delta_m x_{m,n})$ and $\Delta_n\Delta_m^2 x_{m,n} = \Delta_n(\Delta_m^2 x_{m,n})$, respectively. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all positive integers and integers, respectively,

$$\begin{gathered} \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ \mathbb{N}_s = \{n : n \in \mathbb{N}_0 \text{ with } n \geq s\}, \quad s \in \mathbb{N}_0, \\ \mathbb{N}_{s,t} = \{(m,n) : m, n \in \mathbb{N}_0 \text{ with } m \geq s, n \geq t\}, \quad s, t \in \mathbb{N}_0, \\ \mathbb{Z}_{s,t} = \{(m,n) :\ m, n \in \mathbb{Z} \text{ with } m \geq s, n \geq t\}, \quad s, t \in \mathbb{Z}, \\ \alpha = \min\{m - \tau_0, m - \tau_{l,m} : 1 \leq l \leq k, m \in \mathbb{N}_{m_0}\}, \\ \beta = \min\{n - \sigma_0, n - \sigma_{l,n} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\}. \end{gathered} \tag{1.13}$$

$l^{\infty}_{\alpha,\beta}$ represents the Banach space of all bounded sequences on $\mathbb{Z}_{\alpha,\beta}$ with the norm

$$\|x\| = \sup_{m,n\in\mathbb{Z}_{\alpha,\beta}} |x_{m,n}| \quad \text{for } x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^{\infty}_{\alpha,\beta},$$
$$A(N,M) = \left\{x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in l^{\infty}_{\alpha,\beta} : N \le x_{m,n} \le M, (m,n) \in \mathbb{Z}_{\alpha,\beta}\right\} \quad \text{for } M > N > 0. \tag{1.14}$$

It is not difficult to see that $A(N,M)$ is a bounded closed and convex subset of the Banach space $l^{\infty}_{\alpha,\beta}$. By a solution of (1.11), we mean a sequence $\{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ with positive integers $m_1 \ge m_0 + \tau_0 + |\alpha|$ and $n_1 \ge n_0 + \sigma_0 + |\beta|$ such that (1.11) is satisfied for all $m \ge m_1$ and $n \ge n_1$.

Lemma 1.1 (see [15]). *Let $\{\alpha(n)\}_{n\in\mathbb{N}_0}$, $\{\beta(n)\}_{n\in\mathbb{N}_0}$, $\{\gamma(n)\}_{n\in\mathbb{N}_0}$, and $\{t(n)\}_{n\in\mathbb{N}_0}$ be nonnegative sequences satisfying the inequality*

$$\alpha(n+1) \le (1-t(n))\alpha(n) + t(n)\beta(n) + \gamma(n), \quad n \in \mathbb{N}_0, \tag{1.15}$$

where $\{t(n)\}_{n\in\mathbb{N}_0} \subset [0,1]$ with $\sum_{n=0}^{\infty} t(n) = +\infty$, $\lim_{n\to\infty}\beta(n) = 0$ and $\sum_{n=0}^{\infty}\gamma(n) < +\infty$. Then $\lim_{n\to\infty}\alpha(n) = 0$.

2. Existence of Uncountably Many Bounded Positive Solutions and Mann Iterative Schemes with Errors

Utilizing the Banach fixed-point theorem, we now investigate the existence of uncountably many bounded positive solutions for (1.11), suggest the Mann type iterative schemes with errors and discuss the error estimates between the bounded positive solutions and the sequences generated by the Mann iterative schemes.

Theorem 2.1. *Assume that there exists positive constants M and N, nonnegative constants b_1 and b_2, and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying*

$$b_1 + b_2 < 1, \quad N < [1-(b_1+b_2)]M, \tag{2.1}$$

$$-b_2 \le b_{m,n} \le b_1, \quad \textit{eventually}, \tag{2.2}$$

$$\begin{gathered}\left|f(m,n,u_1,u_2,\ldots,u_k) - f(m,n,\overline{u}_1,\overline{u}_2,\ldots,\overline{u}_k)\right| \le P_{m,n}\max\{|u_l-\overline{u}_l| : 1 \le l \le k\},\\ (m,n,u_l,\overline{u}_l) \in \mathbb{N}_{m_0,n_0} \times [N,M]^2, \quad 1 \le l \le k,\end{gathered} \tag{2.3}$$

$$\left|f(m,n,u_1,u_2,\ldots,u_k)\right| \le Q_{m,n}, \quad (m,n,u_l) \in \mathbb{N}_{m_0,n_0} \times [N,M],\ 1 \le l \le k; \tag{2.4}$$

$$\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_0}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\} < +\infty. \tag{2.5}$$

Then

(a) *for any* $L \in (N + b_1 M, (1 - b_2)M)$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for any* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by the scheme:*

$$x_{m,n}(s+1) = \begin{cases} \left[1-\alpha(s)-\beta(s)\right]x_{m,n}(s)+\alpha(s) \\ \quad \times\left\{ L - b_{m,n}x_{m-\tau_0,n-\sigma_0}(s) + \displaystyle\sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}} \right. \\ \qquad \left. \times\displaystyle\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right)-c_{i,t}\right] \right\} \\ \quad +\beta(s)\gamma_{m,n}(s), \quad (m,n)\in\mathbb{Z}_{m_1,n_1},\ s\in\mathbb{N}_0, \\ \left[1-\alpha(s)-\beta(s)\right]_{x_{m_1,n_1}}(s)+\alpha(s) \\ \quad \times\left\{ L - b_{m_1,n_1}x_{m_1-\tau_0,n_1-\sigma_0}(s) + \displaystyle\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n_1}}\sum_{t=n_1}^{\infty} \right. \\ \qquad \left. \times\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right)-c_{i,t}\right] \right\} \\ \quad +\beta(s)\gamma_{m_1,n_1}(s), \quad (m,n)\in\mathbb{Z}_{\alpha,\beta}\setminus\mathbb{Z}_{m_1,n_1},\ s\in\mathbb{N}_0, \end{cases} \tag{2.6}$$

converges to a bounded positive solution $x \in A(M,N)$ *of* (1.11) *and has the following error estimate:*

$$\|x(s+1)-x\| \leq [1-(1-\theta)\alpha(s)]\|x(s)-x\| + 2M\beta(s), \quad s\in\mathbb{N}_0, \tag{2.7}$$

where $\{\gamma(s)\}_{s\in\mathbb{N}_0}$ *is an arbitrary sequence in* $A(M,N)$, $\{\alpha(s)\}_{s\in\mathbb{N}_0}$ *and* $\{\beta(s)\}_{s\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *such that*

$$\sum_{s=0}^{\infty}\alpha(s) = +\infty, \tag{2.8}$$

$$\begin{gathered} \sum_{s=0}^{\infty}\beta(s) < +\infty \text{ or there exists a sequence } \{\xi(s)\}_{s\in\mathbb{N}_0} \subseteq [0,+\infty) \text{ satisfying} \\ \beta(s) = \xi(s)\alpha(s), \quad s\in\mathbb{N}_0, \qquad \lim_{s\to\infty}\xi(s) = 0; \end{gathered} \tag{2.9}$$

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. First of all we show that (a) holds. Set $L \in (N + b_1 M, (1 - b_2)M)$. It follows from (2.1), (2.2), and (2.5) that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ such that

$$\theta = b_1 + b_2 + \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\}, \tag{2.10}$$

$$-b_2 \leq b_{m,n} \leq b_1, \quad (m,n) \in \mathbb{N}_{m_1,n_1}, \tag{2.11}$$

$$\sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \leq \min\{(1 - b_2)M - L, L - b_1 M - N\}. \tag{2.12}$$

Define a mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_L x_{m,n} = \begin{cases} L - b_{m,n} x_{m-\tau_0,n-\sigma_0} + \displaystyle\sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \\ \quad \times \displaystyle\sum_{t=n}^{\infty} \left[f\left(i, t, x_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_1,n_1}, \\ T_L x_{m_1,n_1}, \quad (m,n) \in \mathbb{Z}_{\alpha,\beta} \setminus \mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.13}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$. By employing (2.1)–(2.4) and (2.10)–(2.13), we infer that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}, y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned} \left|T_L x_{m,n} - T_L y_{m,n}\right| &= \Bigg| b_{m,n}\left(x_{m-\tau_0,n-\sigma_0} - y_{m-\tau_0,n-\sigma_0}\right) \\ &\quad - \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i, t, x_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right. \\ &\quad \left. - f\left(i, t, y_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, y_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right] \Bigg| \\ &\leq |b_{m,n}| \left| x_{m-\tau_0,n-\sigma_0} - y_{m-\tau_0,n-\sigma_0} \right| \\ &\quad + \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left| f\left(i, t, x_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right. \\ &\quad \left. - f\left(i, t, y_{i-\tau_{1,i},t-\sigma_{1,t}}, \ldots, y_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| \\ &\leq (b_1 + b_2)\|x - y\| + \sum_{j=m}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n}|} \\ &\quad \times \sum_{t=n}^{\infty} P_{i,t} \max\left\{ \left| x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}} \right| : 1 \leq l \leq k \right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \left(b_1 + b_2 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\|x-y\| = \theta\|x-y\|,\\
T_L x_{m,n} &= L - b_{m,n}x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)-c_{i,t}\right]\\
&\leq L + b_2M + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left[\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right|+|c_{i,t}|\right]\\
&\leq L + b_2M + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left(Q_{i,t}+|c_{i,t}|\right)\right\}\\
&\leq L + b_2M + \min\{(1-b_2)M-L, L-b_1M-N\} \leq M,\\
T_L x_{m,n} &= L - b_{m,n}x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)-c_{i,t}\right]\\
&\geq L - b_1M - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left[\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right|+|c_{i,t}|\right]\\
&\geq L - b_1M - \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left(Q_{i,t}+|c_{i,t}|\right)\right\}\\
&\geq L - b_1M - \min\{(1-b_2)M-L, L-b_1M-N\} \geq N,
\end{aligned}
\tag{2.14}
$$

which lead to

$$
T_L(A(N,M)) \subseteq A(N,M), \qquad \|T_Lx - T_Ly\| \leq \theta\|x-y\|, \quad x,y \in A(N,M). \tag{2.15}
$$

Consequently, (2.15) means that T_L is a contraction mapping in $A(N,M)$ and it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.13) gives that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
x_{m,n} = L - b_{m,n}x_{m-\tau_0,n-\sigma_0} + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)-c_{i,t}\right], \tag{2.16}
$$

which yields that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
\Delta_m\left(x_{m,n} + b_{m,n}x_{m-\tau_0,n-\sigma_0}\right) &= -\sum_{i=m}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)-c_{i,t}\right],\\
\Delta_m^2\left(x_{m,n} + b_{m,n}x_{m-\tau_0,n-\sigma_0}\right) &= \frac{1}{a_{m,n}}\sum_{t=n}^{\infty}\left[f\left(m,t,x_{m-\tau_{1,m},t-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right)-c_{m,t}\right],\\
\Delta_n\left(a_{m,n}\Delta_m^2\left(x_{m,n} + b_{m,n}x_{m-\tau_0,n-\sigma_0}\right)\right) &= -f\left(m,n,x_{m-\tau_{1,m},n-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right)+c_{m,n},
\end{aligned}
\tag{2.17}
$$

that is, $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$.

Using (2.6), (2.13), and (2.15), we infer that for any $s \in \mathbb{N}_0$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
|x_{m,n}(s+1) - x_{m,n}| = \Bigg| &[1-\alpha(s)-\beta(s)]x_{m,n}(s) + \alpha(s) \\
&\times \Bigg\{ L - b_{m,n}x_{m-\tau_0,n-\sigma_0}(s) \\
&\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t}\right]\Bigg\} \\
&+\beta(s)\gamma_{m,n}(s) - x_{m,n}\Bigg| \\
\le{}& [1-\alpha(s)-\beta(s)]|x_{m,n}(s) - x_{m,n}| + \alpha(s)|T_L x_{m,n}(s) - T_L x_{m,n}| \\
&+\beta(s)|\gamma_{m,n}(s) - x_{m,n}| \\
\le{}& [1-\alpha(s)-\beta(s)]\|x(s)-x\| + \alpha(s)\theta\|x(s)-x\| + 2M\beta(s) \\
\le{}& [1-(1-\theta)\alpha(s)]\|x(s)-x\| + 2M\beta(s),
\end{aligned} \tag{2.18}$$

which yields that

$$\|x(s+1) - x\| \le [1-(1-\theta)\alpha(s)]\|x(s)-x\| + 2M\beta(s), \quad s \in \mathbb{N}_0. \tag{2.19}$$

That is, (2.7) holds. Consequently, Lemma 1.1 and (2.7)–(2.9) imply that $\lim_{s\to\infty}x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (N+b_1M, (1-b_2)M)$ and let $L_1 \neq L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist $\theta_i,\ m_{i+1},\ n_{i+1}$ and T_{L_i} satisfying (2.10)–(2.13), where θ, m_1, n_1, L and T_L are replaced by $\theta_i,\ m_{i+1},\ n_{i+1},\ L_i,$ and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11), that is,

$$\begin{aligned}
x^1_{m,n} &= L_1 - b_{m,n}x^1_{m-\tau_0,n-\sigma_0} \\
&\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t}\right], \quad (m,n) \in \mathbb{Z}_{m_2,n_2}, \\
x^2_{m,n} &= L_2 - b_{m,n}x^2_{m-\tau_0,n-\sigma_0} \\
&\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\left[f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t}\right], \quad (m,n) \in \mathbb{Z}_{m_3,n_3}.
\end{aligned} \tag{2.20}$$

In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.10), (2.11), (2.20) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \Bigg| L_1 - L_2 - b_{m,n}\left(x^1_{m-\tau_0,n-\sigma_0} - x^2_{m-\tau_0,n-\sigma_0}\right) \\
&\quad + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n}}\sum_{t=n}^{\infty}\Big[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \\
&\quad - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\Big]\Bigg| \\
&\geq |L_1 - L_2| - |b_{m,n}|\left|x^1_{m-\tau_0,n-\sigma_0} - x^2_{m-\tau_0,n-\sigma_0}\right| \\
&\quad - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\Big[\Big| f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \\
&\quad - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\Big|\Big] \\
&\geq |L_1 - L_2| - (b_1 + b_2)\left\|x^1 - x^2\right\| \\
&\quad - \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\max\left\{\left|x^1_{i-\tau_{l,i},t-\sigma_{l,t}} - x^2_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1 \leq l \leq k\right\} \\
&\geq |L_1 - L_2| - \left(b_1 + b_2 + \sum_{j=m}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right)\left\|x^1 - x^2\right\| \\
&\geq |L_1 - L_2| - \left(b_1 + b_2 + \sum_{j=\max\{m_2,m_3\}}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{\max\{n_2,n_3\}}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\left\|x^1 - x^2\right\| \\
&\geq |L_1 - L_2| - \max\{\theta_1,\theta_2\}\left\|x^1 - x^2\right\|,
\end{aligned} \tag{2.21}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1,\theta_2\}} > 0, \tag{2.22}$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.2. *Assume that there exist positive constants M and N, negative constants b_1 and b_2 and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3)–(2.5) *and*

$$b_1 < -1, \qquad N(1+b_2) > M(1+b_1); \tag{2.23}$$

$$b_2 \leq b_{m,n} \leq b_1, \quad \text{eventually}. \tag{2.24}$$

Then

(a) *for any* $L \in (M(1+b_1), N(1+b_2))$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for each* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by the scheme:*

$$x_{m,n}(s+1) = \begin{cases} [1-\alpha(s)-\beta(s)]x_{m,n}(s) + \alpha(s) \\ \quad \times \left\{ \dfrac{L}{b_{m+\tau_0,n+\sigma_0}} - \dfrac{x_{m+\tau_0,n+\sigma_0}(s)}{b_{m+\tau_0,n+\sigma_0}} + \dfrac{1}{b_{m+\tau_0,n+\sigma_0}} \displaystyle\sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \right. \\ \quad\quad \left. \times \displaystyle\sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\ \quad +\beta(s)\gamma_{m,n}(s), \quad (m,n)\in\mathbb{Z}_{m_1,n_1},\ s\in\mathbb{N}_0, \\ [1-\alpha(s)-\beta(s)]x_{m_1,n_1}(s) + \alpha(s) \\ \quad \times \left\{ \dfrac{L}{b_{m_1+\tau_0,n_1+\sigma_0}} - \dfrac{x_{m_1+\tau_0,n_1+\sigma_0}(s)}{b_{m_1+\tau_0,n_1+\sigma_0}} + \dfrac{1}{b_{m_1+\tau_0,n_1+\sigma_0}} \displaystyle\sum_{j=m_1+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n_1+\sigma_0}} \right. \\ \quad\quad \left. \times \displaystyle\sum_{t=n_1+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\ \quad +\beta(s)\gamma_{m_1,n_1}(s), \quad (m,n)\in\mathbb{Z}_{\alpha,\beta}\setminus\mathbb{Z}_{m_1,n_1},\ s\in\mathbb{N}_0 \end{cases} \tag{2.25}$$

converges to a bounded positive solution $x \in A(N,M)$ *of* (1.11) *and has the error estimate* (2.7), *where* $\{\gamma(s)\}_{s\in\mathbb{N}_0}$ *is an arbitrary sequence in* $A(N,M)$, $\{\alpha(s)\}_{s\in\mathbb{N}_0}$ *and* $\{\beta(s)\}_{s\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *satisfying* (2.8) *and* (2.9);

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. First of all we show (a). Taking $L \in (M(1+b_1), N(1+b_2))$, from (2.5), (2.23), and (2.24) we infer that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ such that

$$\theta = -\frac{1}{b_1}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right), \tag{2.26}$$

$$b_2 \leq b_{m,n} \leq b_1, \quad (m,n)\in\mathbb{N}_{m_1,n_1}, \tag{2.27}$$

$$\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left(Q_{i,t}+|c_{i,t}|\right)\right\} \leq \min\left\{L - M(1+b_1), b_1N\left(1+\frac{1}{b_2}\right) - \frac{b_1L}{b_2}\right\}. \tag{2.28}$$

Define a mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ by

$$T_Lx_{m,n} = \begin{cases} \dfrac{L}{b_{m+\tau_0,n+\sigma_0}} - \dfrac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \dfrac{1}{b_{m+\tau_0,n+\sigma_0}} \displaystyle\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}} \\ \quad \times \displaystyle\sum_{t=n+\sigma_0}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t}\right], \quad (m,n)\in\mathbb{Z}_{m_1,n_1}, \\ T_Lx_{m_1,n_1}, \quad (m,n)\in\mathbb{Z}_{\alpha,\beta}\setminus\mathbb{Z}_{m_1,n_1} \end{cases} \tag{2.29}$$

for each $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$. It follows from (2.3), (2.4), (2.23), (2.24), and (2.26)–(2.29) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$, $y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n)\in\mathbb{Z}_{m_1,n_1}$:

$$\begin{aligned}
\left|T_L x_{m,n} - T_L y_{m,n}\right| = \Bigg| & \frac{x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\
& \qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right]\Bigg| \\
\le & -\frac{\left|x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\
& \qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| \\
\le & -\frac{\|x-y\|}{b_1} - \frac{1}{b_1}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
& \times \sum_{t=n+\sigma_0}^{\infty} P_{i,t}\max\left\{\left|x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1\le l\le k\right\} \\
\le & -\frac{1}{b_1}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\|x-y\| = \theta\|x-y\|,
\end{aligned}$$

$$\begin{aligned}
T_L x_{m,n} = & \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t}\right] \\
\le & \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left[\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| + |c_{i,t}|\right] \\
\le & \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|}\sum_{t=n+\sigma_0}^{\infty}\left(Q_{i,t} + |c_{i,t}|\right) \\
\le & \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1}\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\left(Q_{i,t} + |c_{i,t}|\right)\right\} \\
\le & \frac{L}{b_1} - \frac{M}{b_1} - \frac{1}{b_1}\min\left\{L - M(1+b_1), b_1 N\left(1 + \frac{1}{b_2}\right) - \frac{b_1 L}{b_2}\right\} \le M,
\end{aligned}$$

$$\begin{aligned}
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} \left(Q_{i,t} + |c_{i,t}|\right) \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n\in\mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} \left(Q_{i,t} + |c_{i,t}|\right) \right\} \\
&\geq \frac{L}{b_2} - \frac{N}{b_2} + \frac{1}{b_1} \min\left\{ L - M(1+b_1), b_1 N\left(1 + \frac{1}{b_2}\right) - \frac{b_1 L}{b_2} \right\} \geq N,
\end{aligned} \tag{2.30}$$

which imply that (2.15) holds. Consequently, (2.15) ensures that T_L is a contraction mapping in $A(N,M)$ and it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which together with (2.29) gives that

$$\begin{aligned}
x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_1,n_1},
\end{aligned} \tag{2.31}$$

which yields that for $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
\Delta_m\left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}\right) &= -\sum_{i=m}^{\infty} \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \\
\Delta_m^2\left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}\right) &= \frac{1}{a_{m,n}} \sum_{t=n}^{\infty} \left[f\left(m,t,x_{m-\tau_{1,m},t-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) - c_{m,t} \right], \\
\Delta_n\left(a_{m,n} \Delta_m^2\left(x_{m,n} + b_{m,n} x_{m-\tau_0,n-\sigma_0}\right)\right) &= -f\left(m,n,x_{m-\tau_{1,m},n-\sigma_{1,t}},\ldots,x_{m-\tau_{k,m},t-\sigma_{k,t}}\right) + c_{m,n},
\end{aligned} \tag{2.32}$$

which implies that $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$.

It follows from (2.15), (2.25) and (2.29) that for any $s \in \mathbb{N}_0$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$
\begin{aligned}
|x_{m,n}(s+1) - x_{m,n}| = \Bigg| & [1-\alpha(s)-\beta(s)]x_{m,n}(s) + \alpha(s) \\
& \times \left\{ \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}(s)}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \right. \\
& \left. \times \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}}(s),\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}(s)\right) - c_{i,t} \right] \right\} \\
& + \beta(s)\gamma_{m,n}(s) - x_{m,n} \Bigg| \\
\le{} & [1-\alpha(s)-\beta(s)]|x_{m,n}(s) - x_{m,n}| + \alpha(s)|T_L x_{m,n}(s) - T_L x_{m,n}| \\
& + \beta(s)|\gamma_{m,n}(s) - x_{m,n}| \\
\le{} & [1-\alpha(s)-\beta(s)]\|x(s) - x\| + \alpha(s)\theta\|x(s) - x\| + 2M\beta(s) \\
\le{} & [1-(1-\theta)\alpha(s)]\|x(s) - x\| + 2M\beta(s),
\end{aligned}
\tag{2.33}
$$

which yields (2.7). Thus Lemma 1.1 and (2.7)–(2.9) ensure that $\lim_{s\to\infty} x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (M(1+b_1), N(1+b_2))$ let and $L_1 \ne L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist θ_i, m_{i+1}, n_{i+1} and T_{L_i} satisfying (2.26)–(2.29), where θ, m_1, n_1, L and T_L are replaced by θ_i, m_{i+1}, n_{i+1}, L_i and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11), that is:

$$
\begin{aligned}
x^1_{m,n} = {} & \frac{L_1}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_2,n_2},
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
x^2_{m,n} = {} & \frac{L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \quad (m,n) \in \mathbb{Z}_{m_3,n_3}.
\end{aligned}
\tag{2.35}
$$

In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.26), (2.27), (2.34), and (2.35) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \Bigg| \frac{L_1 - L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right] \Bigg| \\
&\geq -\frac{|L_1 - L_2|}{b_{m+\tau_0,n+\sigma_0}} + \frac{\left|x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left| f\left(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - f\left(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{\left\|x^1 - x^2\right\|}{b_1} + \frac{1}{b_1} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} P_{i,t} \max\left\{ \left| x^1_{i-\tau_{1,i},t-\sigma_{1,t}} - x^2_{i-\tau_{1,i},t-\sigma_{1,t}} \right| : 1 \leq l \leq k \right\} \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{1}{b_1}\left(1 + \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \sum_{t=n+\sigma_0}^{\infty} P_{i,t}\right) \left\|x^1 - x^2\right\| \\
&\geq -\frac{|L_1 - L_2|}{b_2} + \frac{1}{b_1}\left(1 + \sum_{j=\max\{m_2,m_3\}}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{\max\{n_2,n_3\}}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} P_{i,t} \right\}\right) \left\|x^1 - x^2\right\| \\
&\geq -\frac{|L_1 - L_2|}{b_2} - \max\{\theta_1,\theta_2\} \left\|x^1 - x^2\right\|,
\end{aligned} \tag{2.36}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq -\frac{|L_1 - L_2|}{b_2(1 + \max\{\theta_1,\theta_2\})} > 0, \tag{2.37}$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.3. *Assume that there exist positive constants M and N, nonnegative constants b_1 and b_2, and nonnegative sequences $\{P_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{N}_{m_0,n_0}}$ satisfying* (2.3)–(2.5), (2.24) *and*

$$1 < b_2, \qquad b_1 < b_2^2, \qquad M b_1\left(b_2^2 - b_1\right) > N b_2\left(b_1^2 - b_2\right). \tag{2.38}$$

Then

(a) *for any* $L \in (b_1 N + b_1 M/b_2\ ,\ b_2 M + b_2 N/b_1\ )$, *there exist* $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ *and* $n_1 \geq n_0 + \sigma_0 + |\beta|$ *such that for each* $x(0) = \{x_{m,n}(0)\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, *the Mann iterative sequence with errors* $\{x(s)\}_{s\in\mathbb{N}_0} = \{x_{m,n}(s)\}_{(m,n,s)\in\mathbb{Z}_{\alpha,\beta}\times\mathbb{N}_0}$ *generated by* (2.25) *converges to a bounded positive solution* $x \in A(N,M)$ *of* (1.11) *and has the error estimate* (2.7), *where* $\{\gamma(s)\}_{s\geq 0}$ *is an arbitrary sequence in* $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ *and* $\{\beta(s)\}_{s\geq 0}$ *are any sequences in* $[0,1]$ *satisfying* (2.8) *and* (2.9);

(b) (1.11) *possesses uncountably many bounded positive solutions in* $A(M,N)$.

Proof. Set $L \in (b_1 N + b_1 M/b_2), b_2 M + b_2 N/b_1)$. It follows from (2.5), (2.24), and (2.38) that there exist $\theta \in (0,1)$, $m_1 \geq m_0 + \tau_0 + |\alpha|$ and $n_1 \geq n_0 + \sigma_0 + |\beta|$ satisfying (2.27):

$$\theta = \frac{1}{b_2}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right), \tag{2.39}$$

$$\sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}(Q_{i,t} + |c_{i,t}|)\right\} \leq \min\left\{b_2 M - L + \frac{b_2 N}{b_1}, \frac{b_2 L}{b_1} - M - b_2 N\right\}. \tag{2.40}$$

Let the mapping $T_L : A(N,M) \to l^{\infty}_{\alpha,\beta}$ be defined by (2.29). It follows from (2.3), (2.4), (2.24), (2.27), (2.29), and (2.38)–(2.40) that for $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$, $y = \{y_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$ and $(m,n) \in \mathbb{Z}_{m_1,n_1}$

$$\begin{aligned}
\left|T_L x_{m,n} - T_L y_{m,n}\right| = \Bigg| & \frac{x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\
& \qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right]\Bigg| \\
\leq & \frac{\left|x_{m+\tau_0,n+\sigma_0} - y_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
& \times \sum_{t=n+\sigma_0}^{\infty}\left|f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right. \\
& \qquad \left.-f\left(i,t,y_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,y_{i-\tau_{k,i},t-\sigma_{k,t}}\right)\right| \\
\leq & \frac{\|x-y\|}{b_2} + \frac{1}{b_2}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
& \times \sum_{t=n+\sigma_0}^{\infty}P_{i,t}\max\left\{\left|x_{i-\tau_{l,i},t-\sigma_{l,t}} - y_{i-\tau_{l,i},t-\sigma_{l,t}}\right| : 1 \leq l \leq k\right\} \\
\leq & \frac{1}{b_2}\left(1 + \sum_{j=m_1}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{n_1}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\|x-y\| = \theta\|x-y\|,
\end{aligned}$$

$$
\begin{aligned}
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_{0,m}}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{n_1}} \left\{ \frac{1}{|a_{i,n}|} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \\
&\le \frac{L}{b_2} - \frac{N}{b_1} + \frac{1}{b_2} \min\left\{ b_2 M - L + \frac{b_2 N}{b_1}, \frac{b_2 L}{b_1} - M - b_2 N \right\} \le M, \\
T_L x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_{0,m}}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right] \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{|a_{i,n+\sigma_0}|} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[\left| f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) \right| + |c_{i,t}| \right] \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \sum_{j=m_1}^{\infty} \sum_{i=j}^{\infty} \sup_{n \in \mathbb{N}_{n_1}} \left\{ \frac{1}{a_{i,n}} \sum_{t=n}^{\infty} (Q_{i,t} + |c_{i,t}|) \right\} \\
&\ge \frac{L}{b_1} - \frac{M}{b_2} - \frac{1}{b_2} \min\left\{ b_2 M - L + \frac{b_2 N}{b_1}, \frac{b_2 L}{b_1} - M - b_2 N \right\} \ge N,
\end{aligned}
\tag{2.41}
$$

which imply that (2.15) holds. Consequently (2.15) ensures that T_L is a contraction mapping, and hence it has a unique fixed-point $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which gives that

$$
\begin{aligned}
x_{m,n} &= \frac{L}{b_{m+\tau_0,n+\sigma_0}} - \frac{x_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}} \sum_{j=m+\tau_0}^{\infty} \sum_{i=j}^{\infty} \frac{1}{a_{i,n+\sigma_0}} \\
&\quad \times \sum_{t=n+\sigma_0}^{\infty} \left[f\left(i,t,x_{i-\tau_{1,i},t-\sigma_{1,t}},\ldots,x_{i-\tau_{k,i},t-\sigma_{k,t}}\right) - c_{i,t} \right], \ (m,n) \in \mathbb{Z}_{m_1,n_1}.
\end{aligned}
\tag{2.42}
$$

As in the proof of Theorem 2.2, it is easy to verify that $x = \{x_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}}$ is a bounded positive solution of (1.11) in $A(N,M)$; (2.7) holds and $\lim_{s\to\infty} x(s) = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (b_1N + b_1M/b_2\ , b_2M + b_2N/b_1)$ and $L_1 \neq L_2$. As in the proof of (a), we infer that for each $i \in \{1,2\}$, there exist $\theta_i, m_{i+1}, n_{i+1}$ and T_{L_i} satisfying (2.27), (2.29), (2.39), and (2.40), where $\theta,\ m_1,\ n_1,\ L$ and T_L are replaced by $\theta_i,\ m_{i+1},\ n_{i+1},\ L_i$, and T_{L_i}, respectively, and the mapping T_{L_i} has a fixed-point $x^i = \{x^i_{m,n}\}_{(m,n)\in\mathbb{Z}_{\alpha,\beta}} \in A(N,M)$, which is a bounded positive solution of (1.11) and satisfies (2.34) and (2.35). In order to show that the set of bounded positive solutions of (1.11) is uncountable, it is sufficient to prove that $x^1 \neq x^2$. It follows from (2.3), (2.27), (2.34), (2.35), and (2.39) that for $(m,n) \in \mathbb{Z}_{\max\{m_2,m_3\},\max\{n_2,n_3\}}$

$$\begin{aligned}
\left|x^1_{m,n} - x^2_{m,n}\right| &= \Bigg|\frac{L_1 - L_2}{b_{m+\tau_0,n+\sigma_0}} - \frac{x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}}{b_{m+\tau_0,n+\sigma_0}} + \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{a_{i,n+\sigma_0}} \\
&\qquad\times\sum_{t=n+\sigma_0}^{\infty}\Big[f\Big(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\Big) \\
&\qquad\qquad -f\Big(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\Big)\Big]\Bigg| \\
&\geq \frac{|L_1 - L_2|}{b_{m+\tau_0,n+\sigma_0}} - \frac{\left|x^1_{m+\tau_0,n+\sigma_0} - x^2_{m+\tau_0,n+\sigma_0}\right|}{b_{m+\tau_0,n+\sigma_0}} - \frac{1}{b_{m+\tau_0,n+\sigma_0}}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
&\qquad\times\sum_{t=n+\sigma_0}^{\infty}\left|f\Big(i,t,x^1_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^1_{i-\tau_{k,i},t-\sigma_{k,t}}\Big) - f\Big(i,t,x^2_{i-\tau_{1,i},t-\sigma_{1,t}},\dots,x^2_{i-\tau_{k,i},t-\sigma_{k,t}}\Big)\right| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{\left\|x^1 - x^2\right\|}{b_2} - \frac{1}{b_2}\sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|} \\
&\qquad\times\sum_{t=n+\sigma_0}^{\infty}P_{i,t}\max\left\{\left|x^1_{i-\tau_{1,i},t-\sigma_{1,t}} - x^2_{i-\tau_{1,i},t-\sigma_{1,t}}\right| : 1 \leq l \leq k\right\} \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{1}{b_2}\left(1 + \sum_{j=m+\tau_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{|a_{i,n+\sigma_0}|}\sum_{t=n+\sigma_0}^{\infty}P_{i,t}\right)\left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \frac{1}{b_2}\left(1 + \sum_{j=\max\{m_2,m_3\}}^{\infty}\sum_{i=j}^{\infty}\sup_{n\in\mathbb{N}_{\max\{n_2,n_3\}}}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}P_{i,t}\right\}\right)\left\|x^1 - x^2\right\| \\
&\geq \frac{|L_1 - L_2|}{b_1} - \max\{\theta_1,\theta_2\}\left\|x^1 - x^2\right\|,
\end{aligned}\tag{2.43}$$

which implies that

$$\left\|x^1 - x^2\right\| \geq \frac{|L_1 - L_2|}{b_1(1 + \max\{\theta_1,\theta_2\})} > 0,\tag{2.44}$$

that is, $x^1 \neq x^2$. This completes the proof. □

3. Examples

Now we illustrate the results presented in Section 2 with the following three examples. Note that none of the known results can be applied to the examples.

Example 3.1. Consider the third-order nonlinear partial difference equation with delays:

$$\begin{aligned}&\Delta_n\left((-1)^{mn}m^4n^3\Delta_m^2\left(x_{m,n}+\frac{(-1)^{m+n}}{3}x_{m-\tau_0,n-\sigma_0}\right)\right)+\frac{\sqrt{n}}{m^2(n^2+1)}x^3_{m^2,n(n+1)/2}\\&-\frac{\cos\left(m^3n^5-\ln m\right)}{(m+1)^2n^2}x^2_{m^3-2m,n^2-n}=\frac{(-1)^n\sin\left(m^2-2n\right)}{\sqrt{m^5n^4+1}},\quad m\geq 1,\ n\geq 1,\end{aligned}\tag{3.1}$$

where $\tau_0,\sigma_0\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $k=2$, $b_1=b_2=1/3$, $\alpha=\min\{1-\tau_0,-1\}$, $\beta=1-\sigma_0$, M and let N be two positive constants with $M>3N$ and

$$\begin{gathered}a_{m,n}=(-1)^{mn}m^4n^3,\qquad b_{m,n}=\frac{(-1)^{m+n}}{3},\qquad c_{m,n}=\frac{(-1)^{m+n}\sin\left(m^2-2n\right)}{\sqrt{m^5n^4+1}},\\ f(m,n,u,v)=\frac{\sqrt{n}}{m^2(n^2+1)}u^3-\frac{\cos\left(m^3n^5-\ln m\right)}{(m+1)^2n^2}v^2,\\ \tau_{1,m}=m(1-m),\qquad \tau_{2,m}=m\left(3-m^2\right),\qquad \sigma_{1,n}=\frac{n(1-n)}{2},\qquad \sigma_{2,n}=n(2-n),\\ P_{m,n}=\frac{3M^2\sqrt{n}}{m^2(n^2+1)}+\frac{2M}{(m+1)^2n^2},\qquad Q_{m,n}=\frac{M^3\sqrt{n}}{m^2(n^2+1)}+\frac{M^2}{(m+1)^2n^2},\\ (m,n,u,v)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}^2.\end{gathered}\tag{3.2}$$

It is easy to verify that (2.1)–(2.4) hold. Note that

$$\begin{aligned}&\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\}\\&\quad=\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^4n^3}\sum_{t=n}^{\infty}\max\left\{\frac{3M^2\sqrt{t}}{i^2(t^2+1)}+\frac{2M}{(i+1)^2t^2},\frac{M^3\sqrt{t}}{i^2(t^2+1)}+\frac{M^2}{(i+1)^2t^2},\frac{\left|\sin\left(i^2-2t\right)\right|}{\sqrt{i^5t^4+1}}\right\}\right\}\\&\quad<\left(1+2M+4M^2+M^3\right)\left(\sum_{t=n_0}^{\infty}\frac{1}{\sqrt{t^3}}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^4}<+\infty.\end{aligned}\tag{3.3}$$

Hence the conditions of Theorem 2.1 are fulfilled. It follows from Theorem 2.1 that (3.1) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N+(1/3)M,(2/3)M)$, there exist $\theta\in(0,1)$ and $m_1\geq m_0+\tau_0+|\alpha|$, $n_1\geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.6) converges to a bounded positive solution $x\in A(N,M)$ of (3.1) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

Example 3.2. Consider the third-order nonlinear partial difference equation with delays:

$$\begin{aligned}
&\Delta_n\left((-1)^n m^3 \ln^2(m+n)\Delta_m^2\left(x_{m,n}-\frac{4n+(-1)^m n}{n+1}x_{m-\tau_0,n-\sigma_0}\right)\right)\\
&\quad+\frac{x^2_{m-2,n-3}x^3_{3m^2-2,n^5-1}}{m^3n^2}=\frac{\cos\left(nm^3-\sqrt{m}\right)}{\sqrt{n^3+1}},\quad m\geq 1,\ n\geq 1,
\end{aligned}\tag{3.4}$$

where $\tau_0,\sigma_0\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $k=2$, $b_1=-2$, $b_2=-5$, $\alpha=\min\{1-\tau_0,-1\}$, $\beta=\min\{1-\sigma_0,-2\}$, M and let N be two positive constants with $M>4N$ and

$$\begin{gathered}
a_{m,n}=(-1)^n m^3\ln^2(m+n),\qquad b_{m,n}=-\frac{4n+(-1)^m n}{n+1},\qquad c_{m,n}=\frac{\cos\left(nm^3-\sqrt{m}\right)}{\sqrt{n^3+1}},\\
f(m,n,u,v)=\frac{u^2v^3}{m^3n^2},\qquad \tau_{1,m}=2,\qquad \tau_{2,m}=-3m^2+m+2,\qquad \sigma_{1,n}=3,\qquad \sigma_{2,n}=-n^5+n+1,\\
P_{m,n}=\frac{5M^4}{m^3n^2},\qquad Q_{m,n}=\frac{M^5}{m^3n^2},\qquad (m,n,u,v)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}^2.
\end{gathered}\tag{3.5}$$

It is clear that (2.3), (2.4), (2.23), and (2.24) hold. Observe that

$$\begin{aligned}
&\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\}\\
&\quad=\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^3\ln^2(i+n)}\sum_{t=n}^{\infty}\max\left\{\frac{5M^4}{i^3t^2},\frac{M^5}{i^3t^2},\frac{\left|\cos\left(ti^3-\sqrt{i}\right)\right|}{\sqrt{t^3+1}}\right\}\right\}\\
&\quad<\frac{1+5M^4+M^5}{\ln^2 2}\left(\sum_{t=n_0}^{\infty}\frac{1}{\sqrt{t^3}}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^3}<+\infty.
\end{aligned}\tag{3.6}$$

That is, the conditions of Theorem 2.2 are fulfilled. Thus Theorem 2.2 ensures that (3.4) has uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(-M,-4N)$, there exist $\theta\in(0,1)$ and $m_1\geq m_0+\tau_0+|\alpha|$, $n_1\geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.25) converges to a bounded positive solution $x\in A(N,M)$ of (3.4) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

Example 3.3. Consider the third-order nonlinear partial difference equation with delays:

$$\Delta_n\left((-1)^n\left(m^5 n\right)\Delta_m^2\left(x_{m,n}+\frac{2mn+3}{mn+1}x_{m-\tau_0,n-\sigma_0}\right)\right)+\frac{\left((-1)^m-1/n\right)x_{m-4,n-3}^4}{m^3n^2+x_{m^2-2m,n^3-n}^2}$$
$$=\frac{(-1)^{n+m}\left(m^2-3n^3\right)}{m^5\left(n^7+1\right)},\quad m\geq 1,\ n\geq 1, \tag{3.7}$$

where $\tau_0,\sigma_0\in\mathbb{N}$ are fixed. Let $m_0=n_0=1$, $k=2$, $b_1=3$, $b_2=2$, $\alpha=\min\{1-\tau_0,-3\}$, $\beta=\min\{1-\sigma_0,-2\}$, M and N be two positive constants with $M>(14/3)N$ and

$$a_{m,n}=(-1)^n\left(m^5n\right),\qquad b_{m,n}=\frac{2mn+3}{mn+1},\qquad c_{m,n}=\frac{(-1)^{n+m}\left(m^2-3n^3\right)}{m^5\left(n^7+1\right)},$$
$$f(m,n,u,v)=\frac{\left((-1)^m-1/n\right)u^4}{m^3n^2+v^2},\qquad \tau_{1,m}=4,\qquad \tau_{2,m}=m(3-m),\qquad \sigma_{1,n}=3,\qquad \sigma_{2,n}=-n^3+2n,$$
$$P_{m,n}=\frac{2M^3\left(2m^3n^2+2M^2+1\right)}{\left(m^3n^2+N^2\right)^2},\qquad Q_{m,n}=\frac{2M^4}{m^3n^2+N^2},\qquad (m,n,u,v)\in\mathbb{N}_{m_0,n_0}\times\mathbb{R}^2. \tag{3.8}$$

Clearly (2.3), (2.4), (2.24), and (2.38) hold. Notice that

$$\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{|a_{i,n}|}\sum_{t=n}^{\infty}\max\{P_{i,t},Q_{i,t},|c_{i,t}|\}\right\}$$
$$=\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\sup_{n\geq n_0}\left\{\frac{1}{i^5n}\sum_{t=n}^{\infty}\max\left\{\frac{2M^3\left(2i^3t^2+2M^2+1\right)}{\left(i^3t^2+N^2\right)^2},\frac{2M^4}{i^3t^2+N^2},\frac{\left|i^2-3t^3\right|}{i^5\left(t^7+1\right)}\right\}\right\} \tag{3.9}$$
$$<\max\left\{4,2M^4,4M^3\left(1+M^2\right)\right\}\left(\sum_{t=n_0}^{\infty}\frac{1}{t^2}\right)\sum_{j=m_0}^{\infty}\sum_{i=j}^{\infty}\frac{1}{i^5}<+\infty.$$

Hence the conditions of Theorem 2.3 are fulfilled. Consequently Theorem 2.3 implies that (3.7) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(3N+(3/2)M,2M+(2/3)N)$, there exist $\theta\in(0,1)$ and $m_1\geq m_0+\tau_0+|\alpha|$, $n_1\geq n_0+\sigma_0+|\beta|$ such that the Mann iterative sequence with errors $\{x(s)\}_{s\geq 0}$ generated by (2.25) converges to a bounded positive solution $x\in A(N,M)$ of (3.7) and has the error estimate (2.7), where $\{\gamma(s)\}_{s\geq 0}$ is an arbitrary sequence in $A(N,M)$, $\{\alpha(s)\}_{s\geq 0}$ and $\{\beta(s)\}_{s\geq 0}$ are any sequences in $[0,1]$ satisfying (2.8) and (2.9).

Acknowledgment

The third author was supported by research funds from Dong-A University.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 906893, 8 pages
doi:10.1155/2012/906893

Research Article

Uniqueness Theorems on Entire Functions and Their Difference Operators or Shifts

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Received 23 December 2011; Accepted 24 March 2012

Academic Editor: István Györi

We study the uniqueness problems on entire functions and their difference operators or shifts. Our main result is a difference analogue of a result of Jank-Mues-Volkmann, which is concerned with the uniqueness of the entire function sharing one finite value with its derivatives. Two relative results are proved, and examples are provided for our results.

1. Introduction and Main Results

Throughout this paper, we assume the reader is familiar with the standard notations and fundamental results of Nevanlinna theory of meromorphic functions (see, e.g., [1–3]). In what follows, a meromorphic function always means meromorphic in the whole complex plane, and c always means a nonzero complex constant. For a meromorphic function $f(z)$, we define its shift by $f(z+c)$ and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^n f(z) = \Delta_c^{n-1}\left(\Delta_c f(z)\right), \quad n \in \mathbb{N},\ n \geq 2. \tag{1.1}$$

For a meromorphic function $f(z)$, we use $S(f)$ to denote the family of all meromorphic functions $a(z)$ that satisfy $T(r,a) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Functions in the set $S(f)$ are called small functions with respect to $f(z)$.

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ IM, provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros (ignoring multiplicities), and we say that $f(z)$ and $g(z)$ share $a(z)$ CM, provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities.

Uniqueness theory of meromorphic functions is an important part of the Nevanlinna theory. In the past 40 years, a very active subject is the investigation on the uniqueness of the entire function sharing values with its derivatives, which was initiated by Rubel and Yang [4]. We first recall the following result by Jank et al. [5].

Theorem A (see [5]). *Let f be a nonconstant meromorphic function, and let $a \not\equiv 0$ be a finite constant. If f, f', and f'' share the value a CM, then $f \equiv f'$.*

Recently, value distribution in difference analogues of meromorphic functions has become a subject of some interest (see, e.g., [6–11]). In particular, a few authors started to consider the uniqueness of meromorphic functions sharing small functions with their shifts or difference operators (see, e.g., [12, 13]).

In this paper, we consider difference analogues of Theorem A.

Theorem 1.1. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f$, and $\Delta_c^2 f$ share $a(z)$ CM, then $\Delta_c^2 f = \Delta_c f$.*

Example 1.2. Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c f$, and $\Delta_c^2 f$ share a CM and can easily see that $\Delta_c^2 f = \Delta_c f$. This example satisfies Theorem 1.1.

Remark 1.3. In Example 1.2, we have $\Delta_c^2 f = \Delta_c f = f$. However, it remains open whether the claim $\Delta_c^2 f = \Delta_c f$ in Theorem 1.1 can be replaced by $\Delta_c f = f$ in general. In fact, the next example resulted from our efforts to find an entire function $f(z)$ satisfying Theorem 1.1, while $\Delta_c f \neq f$.

Example 1.4. Let $f(z) = e^{z \ln 2} - 2$, $a = -1$, $b = 1$, and $c = 1$. Then we observe that $f(z) - a = e^{z \ln 2} - 1$, $\Delta_c f - b = e^{z \ln 2} - 1$, and $\Delta_c^2 f - b = e^{z \ln 2} - 1$ share 0 CM. Here, we also get $\Delta_c^2 f = \Delta_c f$.

From this example, it is natural to ask what happens if $f(z) - a(z)$, $\Delta_c f - b(z)$, and $\Delta_c^2 f - b(z)$ share 0 CM, where $a(z)$ and $b(z)$ are two (not necessarily distinct) small periodic entire functions. Considering this question, we prove the following Theorem 1.5, whose proof is omitted as it is similar to the proof of Theorem 1.1.

Theorem 1.5. *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)$, $b(z)(\not\equiv 0) \in S(f)$ be periodic entire functions with period c. If $f(z) - a(z)$, $\Delta_c f - b(z)$, and $\Delta_c^2 f - b(z)$ share 0 CM, then $\Delta_c^2 f = \Delta_c f$.*

Now it would be interesting to know what happens if the difference operators of $f(z)$ are replaced by shifts of $f(z)$ in Theorem 1.5. We prove the following result concerning this question.

Theorem 1.6. *Let $f(z)$ be a nonconstant entire function of finite order, let $a(z), b(z) \in S(f)$ be two distinct periodic entire functions with period c, and let n and m be positive integers satisfying $n > m$. If $f(z) - a(z)$, $f(z + mc) - b(z)$, and $f(z + nc) - b(z)$ share 0 CM, then $f(z + mc) = f(z + nc)$ for all $z \in \mathbb{C}$.*

Example 1.7. Let $f(z) = \sin z + 1$, $a = 0$, $b = 2$, and $c = \pi$. Then we notice that $f(z) - a = \sin z + 1$, $f(z + c) - b = -\sin z - 1$, and $f(z + 3c) - b = -\sin z - 1$ share 0 CM and can easily see that $f(z + c) = f(z + 3c)$ for all $z \in \mathbb{C}$. This example satisfies Theorem 1.6.

Example 1.8. Let $f(z) = e^{z^2} + a(z)$, where $a(z) \in S(f)$ is a periodic entire function with period 1. Then $f(z) - a(z) = e^{z^2}$, $f(z+1) - a(z) = e^{(z+1)^2}$, and $f(z+3) - a(z) = e^{(z+3)^2}$ share 0 CM, while $f(z+c) - f(z+3c) \not\equiv 0$. This example shows that the condition that $a(z)$ and $b(z)$ are distinct in Theorem 1.6 cannot be deleted.

2. Proof of Theorem 1.1

Lemma 2.1 (see [8, Theorem 2.1]). *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a nonzero complex constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r). \tag{2.1}$$

Lemma 2.2 (see [10, Lemma 2.3]). *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f-a}\right) = S(r, f), \tag{2.2}$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Proof of Theorem 1.1. Suppose, on the contrary, the assertion that $\Delta_c^2 f \neq \Delta_c f$. Note that $f(z)$ is a nonconstant entire function of finite order. By Lemma 2.1, $\Delta_c f$ and $\Delta_c^2 f$ are entire functions of finite order.

Since $f(z)$, $\Delta_c f$, and $\Delta_c^2 f$ share $a(z)$ CM, then we have

$$\frac{\Delta_c^2 f - a(z)}{f(z) - a(z)} = e^{\alpha(z)}, \qquad \frac{\Delta_c f - a(z)}{f(z) - a(z)} = e^{\beta(z)}, \tag{2.3}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials.

Set

$$\varphi(z) = \frac{\Delta_c^2 f - \Delta_c f}{f(z) - a(z)}. \tag{2.4}$$

From (2.3), we get $\varphi(z) = e^{\alpha(z)} - e^{\beta(z)}$. Then by supposition and (2.4), we see that $\varphi(z) \not\equiv 0$. By Lemma 2.2, we deduce that

$$\begin{aligned} T(r, \varphi) &= m(r, \varphi) \\ &\leq m\left(r, \frac{\Delta_c^2 f}{f(z) - a(z)}\right) + m\left(r, \frac{\Delta_c f}{f(z) - a(z)}\right) + \log 2 = S(r, f). \end{aligned} \tag{2.5}$$

Note that $e^{\alpha}/\varphi - e^{\beta}/\varphi = 1$. By using the second main theorem and (2.5), we have

$$\begin{aligned} T\left(r, \frac{e^{\alpha}}{\varphi}\right) &\le \overline{N}\left(r, \frac{e^{\alpha}}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{e^{\alpha}/\varphi - 1}\right) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\ &= \overline{N}\left(r, \frac{e^{\alpha}}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\alpha}}\right) + \overline{N}\left(r, \frac{\varphi}{e^{\beta}}\right) + S\left(r, \frac{e^{\alpha}}{\varphi}\right) \\ &= S(r, f) + S\left(r, \frac{e^{\alpha}}{\varphi}\right). \end{aligned} \tag{2.6}$$

Thus, by (2.5) and (2.6), we have $T(r, e^{\alpha}) = S(r, f)$. Similarly, $T(r, e^{\beta}) = S(r, f)$.

By Lemma 2.2 and the first equation in (2.3), we deduce that $a(z)/(f(z) - a(z)) = \Delta_c^2 f/(f(z) - a(z)) - e^{\alpha(z)}$ and

$$\begin{aligned} m\left(r, \frac{1}{f(z) - a(z)}\right) &= m\left(r, \frac{1}{a(z)}\left(\frac{\Delta_c^2 f}{f(z) - a(z)} - e^{\alpha(z)}\right)\right) \\ &\le m\left(r, \frac{\Delta_c^2 f}{f(z) - a(z)}\right) + m\left(r, e^{\alpha(z)}\right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{2.7}$$

From (2.7), we see that

$$\begin{aligned} N\left(r, \frac{1}{f(z) - a(z)}\right) &= T(r, f(z)) - m\left(r, \frac{1}{f(z) - a(z)}\right) + S(r, f) \\ &= T(r, f(z)) + S(r, f). \end{aligned} \tag{2.8}$$

Now we rewrite the second equation in (2.3) as $\Delta_c f = e^{\beta(z)}(f(z) - a(z)) + a(z)$ and deduce that

$$\begin{aligned} \Delta_c^2 f &= \Delta_c\left(e^{\beta(z)}(f(z) - a(z)) + a(z)\right) \\ &= e^{\beta(z+c)}(f(z+c) - a(z+c)) + a(z+c) - e^{\beta(z)}(f(z) - a(z)) - a(z) \\ &= e^{\beta(z+c)}(f(z+c) - a(z)) - e^{\beta(z)}(f(z) - a(z)). \end{aligned} \tag{2.9}$$

This together with the first equation in (2.3) gives

$$\begin{aligned} f(z+c) = &\left(e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)}\right) f(z) \\ &- a(z)\left(e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)} - 1 - e^{-\beta(z+c)}\right), \end{aligned} \tag{2.10}$$

that is,

$$\begin{aligned}\Delta_c f &= \left(e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)} - 1\right) f(z) \\ &\quad - a(z)\left(e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)} - 1 - e^{-\beta(z+c)}\right).\end{aligned} \tag{2.11}$$

Thus, (2.11) can be rewritten as

$$\Delta_c f = \gamma(z) f(z) + \delta(z), \tag{2.12}$$

where

$$\begin{aligned}\gamma(z) &= e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)} - 1, \\ \delta(z) &= -a(z)\left(e^{\alpha(z)-\beta(z+c)} + e^{\beta(z)-\beta(z+c)} - 1 - e^{-\beta(z+c)}\right) \\ &= -a(z)\gamma(z) + a(z)e^{-\beta(z+c)},\end{aligned} \tag{2.13}$$

which satisfy $T(r,\gamma) = S(r,f)$ and $T(r,\delta) = S(r,f)$.

Now we rewrite $\Delta_c f = \gamma(z) f(z) + \delta(z)$ as

$$\Delta_c f - a(z) - \gamma(z)\left(f(z) - a(z)\right) = \gamma(z)a(z) + \delta(z) - a(z). \tag{2.14}$$

Suppose that $\gamma(z)a(z)+\delta(z)-a(z) \not\equiv 0$. Let z_0 be a zero of $f(z)-a(z)$ with multiplicity k. Since $f(z)$, $\Delta_c f$ share $a(z)$ CM, then z_0 is a zero of $\Delta_c f - a(z)$ with multiplicity k. Thus, z_0 is a zero of $\Delta_c f - a(z) - \gamma(z)(f(z) - a(z))$ with multiplicity at least k. Then, by (2.8) and (2.14), we see that

$$\begin{aligned}N\left(r, \frac{1}{\gamma(z)a(z) + \delta(z) - a(z)}\right) &= N\left(r, \frac{1}{\Delta_c f - a(z) - \gamma(z)\left(f(z) - a(z)\right)}\right) \\ &\geq N\left(r, \frac{1}{f(z) - a(z)}\right) \\ &= T(r, f(z)) + S(r,f).\end{aligned} \tag{2.15}$$

On the other hand, we have

$$N\left(r, \frac{1}{\gamma(z)a(z) + \delta(z) - a(z)}\right) \leq T\left(r, \frac{1}{\gamma(z)a(z) + \delta(z) - a(z)}\right) = S(r,f). \tag{2.16}$$

Then, by (2.15) and (2.16), we get $T(r,f) \leq S(r,f)$, which is a contradiction.

Thus, $\gamma(z)a(z)+\delta(z)-a(z) \equiv 0$. Noting that $\delta(z) = -a(z)\gamma(z)+a(z)e^{-\beta(z+c)}$, we deduce that $e^{-\beta(z+c)} \equiv 1$. So, $e^{\beta(z)} \equiv e^{\beta(z+c)} \equiv 1$, since $\beta(z)$ is a polynomial.

By the second equation in (2.3), we obtain $\Delta_c f = f$, which leads to $\Delta_c^2 f = \Delta_c f$. This is a contradiction. The proof is thus completed. □

3. Proof of Theorem 1.6

Lemma 3.1 (see [9, Corollary 2.2]). *Let $f(z)$ be a nonconstant meromorphic function of finite order, $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^{\delta}}\right), \tag{3.1}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Proof of Theorem 1.6. Suppose, on the contrary, the assertion that $f(z+mc)-f(z+nc)\not\equiv 0$. Since $f(z)-a(z)$, $f(z+mc)-b(z)$, and $f(z+nc)-b(z)$ share 0 CM, then we have

$$\frac{f(z+nc)-b(z)}{f(z)-a(z)} = e^{\alpha(z)}, \qquad \frac{f(z+mc)-b(z)}{f(z)-a(z)} = e^{\beta(z)}, \tag{3.2}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials.

By (3.2), we obtain

$$\frac{f(z+nc)-f(z+mc)}{f(z)-a(z)} = e^{\alpha(z)} - e^{\beta(z)}. \tag{3.3}$$

Set $\psi(z) = \mathrm{e}^{\alpha(z)} - e^{\beta(z)}$. Then by supposition, we see that $\psi(z)\not\equiv 0$. By Lemma 3.1, we deduce that

$$\begin{aligned} T(r,\psi) &= m\left(r, \frac{f(z+nc)-f(z+mc)}{f(z)-a(z)}\right) \\ &\le m\left(r, \frac{f(z+nc)-a(z+nc)}{f(z)-a(z)}\right) + m\left(r, \frac{f(z+mc)-a(z+mc)}{f(z)-a(z)}\right) + \log 2 \\ &= S(r,f). \end{aligned} \tag{3.4}$$

Note that $e^{\alpha}/\psi - e^{\beta}/\psi = 1$. Thus, using a similar method as in the proof of Theorem 1.1, we get $T(r,e^{\alpha}) = S(r,f)$ and $T(r,e^{\beta}) = S(r,f)$.

By Lemma 3.1 and the first equation in (3.2), we deduce that

$$\begin{aligned} m\left(r, \frac{1}{f(z)-a(z)}\right) &= m\left(r, \frac{1}{b(z)-a(z)}\left(\frac{f(z+nc)-a(z)}{f(z)-a(z)} - e^{\alpha(z)}\right)\right) \\ &\le m\left(r, \frac{f(z+nc)-a(z+nc)}{f(z)-a(z)}\right) + m(r,e^{\alpha}) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{3.5}$$

From (3.5), we see that

$$N\left(r, \frac{1}{f(z) - a(z)}\right) = T(r, f(z)) + S(r, f). \tag{3.6}$$

Now we rewrite the second equation in (3.2) as $f(z + mc) = e^{\beta(z)}(f(z) - a(z)) + b(z)$ and deduce that

$$\begin{aligned} f(z + nc) = {} & e^{\beta(z+(n-m)c)}(f(z + (n - m)c) - a(z + (n - m)c)) \\ & + b(z + (n - m)c). \end{aligned} \tag{3.7}$$

This together with the first equation in (3.2) gives

$$\begin{aligned} f(z + (n - m)c) = {} & e^{\alpha(z)-\beta(z+(n-m)c)} f(z) \\ & - a(z)e^{\alpha(z)-\beta(z+(n-m)c)} + a(z + (n - m)c), \end{aligned} \tag{3.8}$$

that is,

$$\begin{aligned} f(z + nc) = {} & e^{\alpha(z+mc)-\beta(z+nc)} f(z + mc) \\ & - a(z + mc)e^{\alpha(z+mc)-\beta(z+nc)} + a(z + nc) \\ = {} & e^{\alpha(z+mc)-\beta(z+nc)} f(z + mc) - a(z)e^{\alpha(z+mc)-\beta(z+nc)} + a(z). \end{aligned} \tag{3.9}$$

Now we rewrite (3.9) as

$$\begin{aligned} & f(z + nc) - b(z) - e^{\alpha(z+mc)-\beta(z+nc)}(f(z + mc) - b(z)) \\ & \quad = (b(z) - a(z))\left(e^{\alpha(z+mc)-\beta(z+nc)} - 1\right). \end{aligned} \tag{3.10}$$

Suppose that $e^{\alpha(z+mc)-\beta(z+nc)} \equiv 1$; then, by (3.9), we get $f(z + nc) = f(z + mc)$, which is a contradiction.

Now we have $e^{\alpha(z+mc)-\beta(z+nc)} - 1 \not\equiv 0$. Then using a similar method as in the proof of Theorem 1.1, we can also get a contradiction and obtain that $f(z + mc) = f(z + nc)$ for all $z \in \mathbb{C}$. Thus, Theorem 1.6 is proved. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11171119). The authors would like to thank the referee for valuable suggestions.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 185692, 12 pages
doi:10.1155/2012/185692

Research Article

Positive Periodic Solutions for First-Order Neutral Functional Differential Equations with Periodic Delays

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Received 15 February 2012; Accepted 20 March 2012

Academic Editor: István Győri

In this paper, two classes of first-order neutral functional differential equations with periodic delays are considered. Some results on the existence of positive periodic solutions for the equations are obtained by using the Krasnoselskii fixed point theorem. Four examples are included to illustrate our results.

1. Introduction and Preliminaries

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions, maximal and minimal periodic solutions and positive periodic solutions for several classes of functional differential equations with periodic delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models, and other models, see, for example, [1–5] and the references therein.

In 2004, Wan et al. [5] studied the first-order functional differential equation with periodic delays

$$x'(t) = -a(t)x(t) + f(t, x(t-\tau(t))), \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $a \in C(\mathbb{R}, \mathbb{R}^+ \setminus \{0\}), \tau \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic, and $f \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ is ω-periodic with respect to the first variable. By using a fixed point theorem in cones, they proved the existence of a periodic solution and a positive periodic solution of (1.1), respectively, under certain conditions. In 2005, Kang and Zhang [2] used the partial ordering and topological degree theory to establish the existence of a nontrivial periodic solution of (1.1). In 2010, Kang et al. [1] gave the existence of maximal and minimal periodic solutions of (1.1) by utilizing the method of lower and upper solutions. By means of the continuation theorem of coincidence degree principle, Serra [4] discussed the existence of periodic solutions for the following neutral functional differential equation

$$[x(t) + cx(t - \tau)]' = f(t, x(t)), \quad \forall t \in \mathbb{R}, \tag{1.2}$$

where $|c| \leq 1$ and $\tau > 0$ are constants. In 2008, Luo et al. [3] employed the Krasnoselskii fixed point theorem to prove the existence of positive periodic solutions for two kinds of neutral functional differential equations with periodic delays

$$\begin{gathered}
[x(t) - cx(t - \tau(t))]' = -a(t)x(t) + f(t, x(t - \tau(t))), \quad \forall t \in \mathbb{R}, \\
\left[x(t) - c\int_{-\infty}^{0} Q(r)x(t + r)dr\right]' = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)f(t, x(t + r))dr, \quad \forall t \in \mathbb{R},
\end{gathered} \tag{1.3}$$

where $\omega \in \mathbb{R}^+ \setminus \{0\}$ and $|c| < 1$ are constants, $\tau \in C(\mathbb{R}, \mathbb{R})$, $a, b \in C(\mathbb{R}, \mathbb{R}^+ \setminus \{0\})$, $f \in C(\mathbb{R}^2, \mathbb{R})$, τ, a, and b are ω-periodic and f is ω-periodic with respect to the first variable, $Q \in C(\mathbb{R}_-, \mathbb{R}^+)$, and $\int_{-\infty}^{0} Q(r)dr = 1$.

Motivated by the papers [1–5] and the references therein, we consider two new kinds of first-order neutral functional differential equations with periodic delays:

$$[g(t)(x(t) + c(t)x(t - \tau(t)))]' = -a(t)x(t) + f(t, x(t - \tau(t))), \quad \forall t \in \mathbb{R}, \tag{1.4}$$

$$\begin{aligned}
&\left[g(t)\left(x(t) + c(t)\int_{-\infty}^{0} Q(r)x(t + h(r))dr\right)\right]' \\
&\quad = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)f(t, x(t + h(r)))dr, \quad \forall t \in \mathbb{R},
\end{aligned} \tag{1.5}$$

where $\omega \in \mathbb{R}^+ \setminus \{0\}$ is a constant, $\tau, a, b, c \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}^2, \mathbb{R})$, $h \in C(\mathbb{R}_-, \mathbb{R})$, $g \in C^1(\mathbb{R}, \mathbb{R}^+ \setminus \{0\})$, τ, a, b, c, and g are ω-periodic functions and f is ω-periodic with respect to the first variable, $Q \in C(\mathbb{R}_-, \mathbb{R}^+)$, and $\int_{-\infty}^{0} Q(r)dr = 1$. It is evident that (1.4) and (1.5) include, respectively, (1.1)–(1.3) as special cases. To the best of our knowledge, the existence of periodic solutions for (1.4) and (1.5) have not been investigated till now. The aim of this paper is, by applying the Krasnoselskii fixed point theorem and some new techniques, to establish a set of sufficient conditions which guarantee the existence of positive periodic solutions of (1.4) and (1.5). Four examples are given to show the efficiency and applications of our results.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{N}$ denotes the set of all positive integers, $P = \min_{t\in[0,\omega]} g(t)$,

$$G(t,s) = \frac{\exp\left(\int_t^s \left(\left(g'(r) + a(r)\right)/g(r)\right)dr\right)}{g(s)\left[\exp\left(\int_0^\omega \left(\left(g'(r) + a(r)\right)/g(r)\right)dr\right) - 1\right]}, \quad \forall (t,s) \in \mathbb{R}^2,$$
$$X = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t) = x(t+\omega),\ \forall t \in \mathbb{R}\}. \tag{1.6}$$

It is well known that X is a Banach space with the norm

$$\|x\| = \sup_{t\in[0,\omega]} |x(t)|, \quad \text{for each } x \in X. \tag{1.7}$$

Let

$$A(N, M) = \{x \in X : N \le x(t) \le M,\ \forall t \in [0,\omega]\}, \quad \text{for any } M > N \ge 0. \tag{1.8}$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of the Banach space X.

Lemma 1.1 (the Krasnoselskii fixed point theorem). *Let Y be a nonempty bounded closed convex subset of a Banach space Z and f, g mappings from Y into Z such that $fx + gy \in Y$ for every pair $x, y \in Y$. If f is a contraction mapping and g is completely continuous, then the equation $fx + gx = x$ has at least one solution in Y.*

2. Main Results

Now we use the Krasnoselskii fixed point theorem to show the existence of positive solutions for (1.4) and (1.5).

Theorem 2.1. *Assume that there exist constants N, M, G_1, G_2, c_1, and c_2 satisfying*

$$0 < N < M, \quad c_1 \ge 0, \quad c_2 \ge 0, \quad c_1 + c_2 < 1, \quad -c_1 \le c(t) \le c_2, \quad \forall t \in [0,\omega], \tag{2.1}$$

$$0 < G_1 \le g'(t) + a(t) \le G_2, \quad \forall t \in [0,\omega], \tag{2.2}$$

$$(N + c_2 M)G_2 \le f(t,s) + a(t)c(t)s \le (1 - c_1)MG_1, \quad \forall (t,s) \in [0,\omega] \times [N, M]. \tag{2.3}$$

Then (1.5) *has at least one positive ω-periodic solution in $A(N, M)$.*

Proof. It is obvious that (1.4) has a solution $x(t)$ if and only if the integral equation

$$x(t) = \int_t^{t+\omega} G(t,s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds - c(t)x(t-\tau(t)), \quad \forall t \in \mathbb{R}, \tag{2.4}$$

has a solution $x(t)$. Define two mappings T and $S : A(N,M) \to X$ by

$$\begin{gathered}(Tx)(t) = \int_t^{t+\omega} G(t,s)\left[f(s,x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds, \quad \forall t \in \mathbb{R}, \\ (Sx)(t) = -c(t)x(t-\tau(t)), \quad \forall t \in \mathbb{R},\end{gathered} \tag{2.5}$$

for each $x \in A(N,M)$. It follows from (2.5) that for any $x \in A(N,M)$ and $t \in \mathbb{R}$

$$\begin{aligned}(Tx)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega,s)\left[f(s,x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds \\ &= \int_t^{t+\omega} G(t+\omega,u+\omega)\left[f(u+\omega,x(u+\omega-\tau(u+\omega)))\right. \\ &\qquad\qquad \left. + a(u+\omega)c(u+\omega)x(u+\omega-\tau(u+\omega))\right]du \\ &= \int_t^{t+\omega} G(t,u)\left[f(u,x(u-\tau(u))) + a(u)c(u)x(u-\tau(u))\right]du = (Tx)(t), \\ (Sx)(t+\omega) &= -c(t+\omega)x(t+\omega-\tau(t+\omega)) = -c(t)x(t-\tau(t)) = (Sx)(t),\end{aligned} \tag{2.6}$$

which mean that

$$T(A(N,M)) \subseteq X, \qquad S(A(N,M)) \subseteq X. \tag{2.7}$$

Using (2.1)–(2.3) and (2.5), we infer that for all $x, y \in A(N,M)$ and $t \in \mathbb{R}$

$$\begin{aligned}(Tx)(t) + (Sy)(t) &= \int_t^{t+\omega} G(t,s)\left[f(s,x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds - c(t)y(t-\tau(t)) \\ &\le (1-c_1)MG_1 \int_t^{t+\omega} G(t,s)ds + c_1 M \\ &\le (1-c_1)M \int_t^{t+\omega} G(t,s)\left[g'(s) + a(s)\right]ds + c_1 M \\ &= \frac{(1-c_1)M}{\left[\exp\left(\int_0^{\omega}((g'(r)+a(r))/g(r))dr\right) - 1\right]} \\ &\quad \times \int_t^{t+\omega} \exp\left(\int_t^s \left(\frac{g'(r)+a(r)}{g(r)}\right)dr\right)\left(\frac{g'(s)+a(s)}{g(s)}\right)ds + c_1 M \\ &= \frac{(1-c_1)M}{\left[\exp\left(\int_0^{\omega}((g'(r)+a(r))/g(r))dr\right) - 1\right]} \\ &\quad \times \left[\exp\left(\int_t^{t+\omega}\left(\frac{g'(r)+a(r)}{g(r)}\right)dr\right) - 1\right] + c_1 M \\ &= (1-c_1)M + c_1 M = M,\end{aligned} \tag{2.8}$$

$$(Tx)(t) + (Sy)(t) \geq (N + c_2 M) G_2 \int_t^{t+\omega} G(t,s)\,ds - c_2 M \geq N + c_2 M - c_2 M = N, \tag{2.9}$$

$$\left|(Sx)(t) - (Sy)(t)\right| = |c(t)| \left|x(t-\tau(t)) - y(t-\tau(t))\right| \leq (c_1 + c_2)\|x - y\|, \tag{2.10}$$

which imply that

$$Tx + Sy \in A(N,M), \quad \|Sx - Sy\| \leq (c_1 + c_2)\|x - y\|, \quad \forall x, y \in A(N,M). \tag{2.11}$$

Now we show that T is a completely continuous mapping in $A(N,M)$. First, we claim that T is continuous in $A(N,M)$. Let $\{y_k\}_{k\in N} \subset A(N,M)$ and $y \in A(N,M)$ with $\lim_{k\to\infty} y_k = y$. Note that $f \in C(\mathbb{R}^2, \mathbb{R})$. It follows from the uniform continuity of f in $[0,\omega] \times [N,M]$ that for given $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\left|f(t_1, s_1) - f(t_2, s_2)\right| < \frac{G_1 \varepsilon}{2}, \quad \begin{array}{c} \forall (t_1, t_2, s_1, s_2) \in [0,\omega]^2 \times [N,M]^2, \\ \text{with } \max\{|t_1 - t_2|, |s_1 - s_2|\} < \delta. \end{array} \tag{2.12}$$

Since $\lim_{k\to\infty} y_k = y$, it follows that there exists $N_1 \in \mathbb{N}$ satisfying

$$\|y_k - y\| < \frac{G_1 \min\{\varepsilon, \delta\}}{2(1 + G_2)(1 + \|a\|)}, \quad \forall k \geq N_1. \tag{2.13}$$

In view of (2.1), (2.2), (2.5), (2.12), and (2.13), we get that

$$\begin{aligned}
\|Ty_k - Ty\| &= \sup_{t\in[0,\,\omega]} \left| \int_t^{t+\omega} G(t,s)\left[f(s, y_k(s-\tau(s))) + a(s)c(s)y_k(s-\tau(s))\right] ds \right. \\
&\qquad \left. - \int_t^{t+\omega} G(t,s)\left[f(s, y(s-\tau(s))) + a(s)c(s)y(s-\tau(s))\right] ds \right| \\
&\leq \sup_{t\in[0,\,\omega]} \int_t^{t+\omega} G(t,s)\left[\left|f(s, y_k(s-\tau(s))) - f(s, y(s-\tau(s)))\right|\right. \\
&\qquad \left. + |a(s)c(s)| \left|y_k(s-\tau(s)) - y(s-\tau(s))\right|\right] ds \\
&< G_1 \left[\frac{\varepsilon}{2} + \frac{\|a\|(c_1 + c_2)\min\{\varepsilon, \delta\}}{2(1 + G_2)(1 + \|a\|)}\right] \sup_{t\in[0,\omega]} \int_t^{t+\omega} G(t,s)\,ds < \varepsilon, \quad \forall k \geq N_1,
\end{aligned} \tag{2.14}$$

which yields that $\lim_{k\to\infty} Ty_k = Ty$, that is, T is continuous in $A(N,M)$.

Second, we claim that $T(A(N,M))$ is relatively compact. It is sufficient to show that $T(A(N,M))$ is uniformly bounded and equicontinuous in $[0,\omega]$. Notice that (2.1)–(2.3) and (2.5) ensure that

$$\begin{aligned}
\|Tx\| &= \sup_{t\in[0,\,\omega]}\left|\int_t^{t+\omega} G(t,s)\left[f(s,x(s-\tau(s)))+a(s)c(s)x(s-\tau(s))\right]ds\right| \\
&\le (1-c_1)MG_1\sup_{t\in[0,\omega]}\int_t^{t+\omega} G(t,s)ds \le (1-c_1)M, \quad \forall x\in A(N,M), \\
\left|(Tx)'(t)\right| &= \left|-\frac{g'(t)+a(t)}{g(t)}(Tx)(t)+G(t,t+\omega)\right. \\
&\quad\times\left[f(t+\omega,x(t+\omega-\tau(t+\omega)))+a(t+\omega)c(t+\omega)x(t+\omega-\tau(t+\omega))\right] \\
&\quad\left.-G(t,t)\left[f(t,x(t-\tau(t)))+a(t)c(t)x(t-\tau(t))\right]\right| \\
&\le \frac{g'(t)+a(t)}{g(t)}|(Tx)(t)|+|G(t,t+\omega)-G(t,t)|\left|f(t,x(t-\tau(t)))+a(t)c(t)x(t-\tau(t))\right| \\
&\le \frac{G_2}{P}(1-c_1)M+\frac{\exp(\int_0^\omega((g'(r)+a(r))/g(r))dr)-1}{g(t)\left[\exp(\int_0^\omega((g'(r)+a(r))/g(r))dr)-1\right]}(1-c_1)MG_1 \\
&\le \frac{(1-c_1)M(G_1+G_2)}{P}, \quad \forall (x,t)\in A(N,M)\times[0,\omega],
\end{aligned} \tag{2.15}$$

which give that $T(A(N,M))$ is uniformly bounded and equicontinuou sin $[0,\omega]$, which together with (2.7), (2.11), and Lemma 1.1 yields that there is $x_0\in A(N,M)$ with $Tx_0+Sx_0=x_0$. It follows from (2.4) and (2.5) that x_0 is a positive ω-periodic solution of (1.4). This completes the proof. □

Theorem 2.2. *Assume that there exist constants N, M, G_1, G_2, c_1, c_2, and $t_0\in[0,\omega]$ satisfying* (2.2), (2.3)*:*

$$0\le N<M,\quad c_1\ge 0,\quad c_2\ge 0,\quad c_1+c_2<1,\quad -c_1\le c(t)\le c_2,\quad \forall t\in[0,\omega], \tag{2.16}$$

and either

$$f(t_0,s)+a(t_0)c(t_0)s>(N+c_2M)G_2,\quad \forall s\in[N,M], \tag{2.17}$$

or

$$g'(t_0)+a(t_0)<G_2. \tag{2.18}$$

Then (1.4) *has at least one positive ω-periodic solution $x\in A(N,M)$ with $N<x(t)\le M$ for each $t\in[0,\omega]$.*

Proof. As in the proof of Theorem 2.1, we conclude similarly that (1.4) has an ω-periodic solution $x \in A(N, M)$. Now we assert that $x(t) > N$ for all $t \in [0, \omega]$. Otherwise, there exists $t^* \in [0, \omega]$ satisfying $x(t^*) = N$. In view of (2.4), (2.5), and (2.16), we have

$$\begin{aligned} N &= \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds - c(t^*)x(t^* - \tau(t^*)) \\ &\geq \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds - c_2 M, \end{aligned} \tag{2.19}$$

which implies that

$$\begin{aligned} 0 &\geq \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s))\right]ds - (N + c_2 M) \\ &= \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s)) - (N + c_2 M)\left(g'(s) + a(s)\right)\right]ds. \end{aligned} \tag{2.20}$$

Assume that (2.17) holds. By means of (2.2), (2.3), (2.17), and the continuity of G, f, a, c, g, g', τ, and x, we get that

$$\begin{aligned} &\int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s)) - (N + c_2 M)\left(g'(s) + a(s)\right)\right]ds \\ &\quad\geq \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s)) - (N + c_2 M)G_2\right]ds > 0, \end{aligned} \tag{2.21}$$

which contradicts (2.20).

Assume that (2.18) holds. In light of (2.2), (2.3), (2.18), and the continuity of G, f, a, c, g, g',τ, and x, we infer that

$$\begin{aligned} &\int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s)) - (N + c_2 M)\left(g'(s) + a(s)\right)\right]ds \\ &\quad> \int_{t^*}^{t^*+\omega} G(t^*, s)\left[f(s, x(s-\tau(s))) + a(s)c(s)x(s-\tau(s)) - (N + c_2 M)G_2\right]ds \geq 0, \end{aligned} \tag{2.22}$$

which contradicts (2.20). This completes the proof. □

Theorem 2.3. *Assume that there exist constants N, M, G_1, G_2, c_1, and c_2 satisfying* (2.1), (2.2), *and*

$$(N + c_2 M)G_2 \leq b(t)f(t, s) + a(t)c(t)s \leq (1 - c_1)MG_1, \quad \forall (t, s) \in [0, \omega] \times [N, M]. \tag{2.23}$$

Then (1.5) *has at least one positive ω-periodic solution in $A(N, M)$.*

Proof. It is obvious that (1.5) has a solution $x(t)$ if and only if the integral equation

$$\begin{aligned} x(t) = \int_t^{t+\omega} G(t,s)\left[b(s)\int_{-\infty}^{0} Q(r)f(s,x(s+h(r)))dr + a(s)c(s)\int_{-\infty}^{0} Q(r)x(s+h(r))dr\right]ds \\ - c(t)\int_{-\infty}^{0} Q(r)x(t+h(r))dr, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{2.24}$$

has a solution $x(t)$. Define two mappings T and $S : A(N,M) \to X$ by

$$\begin{aligned} (Tx)(t) = \int_t^{t+\omega} G(t,s)\left[b(s)\int_{-\infty}^{0} Q(r)f(s,x(s+h(r)))dr + a(s)c(s)\int_{-\infty}^{0} Q(r)x(s+h(r))dr\right]ds, \\ (Sx)(t) = -c(t)\int_{-\infty}^{0} Q(r)x(t+h(r))dr, \end{aligned} \tag{2.25}$$

for each $(x,t) \in A(N,M) \times \mathbb{R}$. The rest of the proof is similar to that of Theorem 2.1, and is omitted. This completes the proof. □

Theorem 2.4. *Assume that there exist constants N, M, G_1, G_2, c_1, c_2, and $t_0 \in [0,\omega]$ satisfying* (2.2), (2.16), (2.23), *and either* (2.18) *or*

$$b(t_0)f(t_0,s) + a(t_0)c(t_0)s > (N + c_2M)G_2, \quad \forall s \in [N,M]. \tag{2.26}$$

Then (1.5) *has at least one positive ω-periodic solution $x \in A(N,M)$ with $N < x(t) \le M$ for each $t \in [0,\omega]$.*

The proof of Theorem 2.4 is similar to that of Theorems 2.2 and 2.3 and is omitted.

Remark 2.5. Even if $g(t) \equiv 1$, $c(t) \equiv c$ and $h(r) = r$ for all $r \in \mathbb{R}_-$, the conditions of Theorems 2.2 and 2.4 in this paper are different from these conditions of Theorems 2.1–2.4 in [3], respectively.

3. Examples

Now we construct four examples which illustrate the results obtained in Section 2. Note that none of the known results can be applied to the examples.

Example 3.1. Consider the first-order neutral functional differential equation with periodic delays

$$\begin{aligned} &\left[\left(1+\frac{\cos t}{100}\right)\left(x(t)+\frac{1+2\sin t}{100}x(t-3\sin t-2\cos t)\right)\right]' \\ &\quad = -\left(1+\frac{\sin t}{50}\right)x(t) + 20 + \cos^2 t + \sin^2\left(x^5(t-3\sin t-2\cos t)\cos t\right), \quad \forall t \in \mathbb{R}. \end{aligned} \tag{3.1}$$

Let $\omega = 2\pi$, $M = 100$, $N = 1$, $c_1 = 1/100$, $c_2 = 3/100$, $G_1 = 99/100$, $G_2 = 101/100$, and

$$\begin{gathered} g(t) = 1 + \frac{\cos t}{100}, \quad c(t) = \frac{1 + 2\sin t}{100}, \quad a(t) = 1 + \frac{\sin t}{50}, \quad \tau(t) = 3\sin t + 2\cos t, \\ f(t,s) = 20 + \cos^2 t + \sin^2\left(s^5 \cos t\right), \quad \forall (t,s) \in \mathbb{R}^2. \end{gathered} \tag{3.2}$$

It is easy to see that (2.1) and (2.2) hold. Notice that

$$\begin{aligned} (N + c_2 M)G_2 = 4.04 &< 20 + \left(1 + \frac{1}{50}\right)\frac{1-2}{100} \cdot 100 \le f(t,s) + a(t)c(t)s \\ &\le 22 + \left(1 + \frac{1}{50}\right)\frac{1+2}{100} \cdot 100 < 98.01 = (1 - c_1)MG_1, \quad \forall (t,s) \in [0,\omega] \times [N,M], \end{aligned} \tag{3.3}$$

that is, (2.3) is satisfied. Thus Theorem 2.1 yields that (3.1) has a positive ω-periodic solution in $A(N,M)$.

Example 3.2. Consider the first-order neutral functional differential equation with periodic delays

$$\begin{aligned} &\left[\frac{3 + 2\cos t + \sin t}{100}\left(x(t) + \frac{2 + 2\sin t + \cos t}{20} x\left(t - \sin^3 t\right)\right)\right]' \\ &\quad = -\left(\frac{100 + 2\sin t + 3\cos t}{100}\right)x(t) + 60 + \frac{x\left(t - \sin^3 t\right)\sin\sqrt{\left|t + x^8\left(t - \sin^3 t\right)\right| + 1}}{50 + 10\cos\left(t - x^5\left(t - \sin^3 t\right)\right)}, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{3.4}$$

Let $\omega = 2\pi$, $M = 100$, $N = 0$, $c_1 = 1/20$, $c_2 = 1/4$, $G_1 = 24/25$, $G_2 = 26/25$, $t_0 = \pi/2$, and

$$\begin{gathered} g(t) = \frac{3 + 2\cos t + \sin t}{100}, \quad c(t) = \frac{2 + 2\sin t + \cos t}{20}, \quad \tau(t) = \sin^3 t, \\ a(t) = \frac{100 + 2\sin t + 3\cos t}{100}, \quad f(t,s) = 60 + \frac{s \sin\sqrt{|t + s^8| + 1}}{50 + 10\cos(t - s^5)}, \quad \forall (t,s) \in \mathbb{R}^2. \end{gathered} \tag{3.5}$$

It is clear that (2.2), (2.16), and (2.18) hold. It follows that

$$\begin{aligned} (N + c_2 M)G_2 = 26 &< 60 + \frac{100(-1)}{50 - 10} + \frac{105}{100} \cdot \frac{-1}{20} \cdot 100 \le f(t,s) + a(t)c(t)s \\ &\le 60 + \frac{100}{50 - 10} + 26.25 < 91.2 = (1 - c_1)MG_1, \quad \forall (t,s) \in [0,\omega] \times [N,M], \end{aligned} \tag{3.6}$$

that is, (2.3) holds. Obviously (2.17) follows from the above inequalities. Hence, Theorem 2.2 ensures that (3.4) has a positive ω-periodic solution $x \in A(N, M)$ with $N < x(t) \le M$ for all $t \in [0, \omega]$.

Example 3.3. Consider the first-order neutral functional differential equation with periodic delays

$$\left[\frac{5+\cos t+3\sin t}{50+\sin t}\left(x(t)+\frac{2+3\sin t}{6(50-\sin t)^2}\int_{-\infty}^{0}\exp(r)x(t-r\cos r)dr\right)\right]'$$
$$= -\frac{151+50\sin t}{(50+\sin t)^2}x(t)+\frac{1}{3+2\sin t}$$
$$\times\int_{-\infty}^{0}\exp(r)\left[4.53+3\sin t+\frac{3+2\sin t}{10000+\cos t}\left[x(t-r\cos r)\cos^2 t+\cos\left(t+x^{100}(t-r\cos r)\right)\right]\right]dr,$$
$$\forall t \in \mathbb{R}. \tag{3.7}$$

Let $\omega = 2\pi$, $M = 1440.6$, $N = 1$, $c_1 = 1/14406$, $c_2 = 5/14406$, $G_1 = 5/2601$, $G_2 = 295/2401$, and

$$g(t) = \frac{5+\cos t+3\sin t}{50+\sin t},\quad c(t) = \frac{2+3\sin t}{6(50-\sin t)^2},\quad a(t) = \frac{151+50\sin t}{(50+\sin t)^2},\quad b(t) = \frac{1}{3+2\sin t},$$
$$f(t,s) = 4.53+3\sin t+\frac{(3+2\sin t)\left[s\ \cos^2 t+\cos\left(t+s^{100}\right)\right]}{10000+\cos t},\quad \forall (t,s) \in \mathbb{R}^2,$$
$$Q(r) = \exp(r),\quad h(r) = -r\cos r,\quad \forall r \in \mathbb{R}_-. \tag{3.8}$$

Clearly, (2.1) and (2.2) hold. Note that

$$(N+c_2M)G_2 = \frac{885}{4802} < \frac{4.53+3}{3+2}+\frac{-1}{10000-1}+\frac{302-553}{37470006}\cdot 1440.6 \le b(t)f(t,s)+a(t)c(t)s$$
$$\le \frac{4.53-3}{3-2}+\frac{1441.6}{10000-1}+\frac{1005}{37470006}\cdot 1440.6 < \frac{14404}{5282}$$
$$= (1-c_1)MG_1,\quad \forall (t,s) \in [0,\omega]\times[N,M], \tag{3.9}$$

that is, (2.23) is fulfilled. Thus Theorem 2.3 yields that (3.7) has a positive ω-periodic solution in $A(N, M)$.

Example 3.4. Consider the first-order neutral functional differential equation with periodic delays

$$\left[\ln(100+2\sin t)\left(x(t)+\frac{2+3\cos t}{100-2\sin t}\int_{-\infty}^{0}\exp(r)x(t+r)dr\right)\right]'$$
$$=-\left(\frac{3}{100+2\sin t}\right)x(t)$$
$$+\left(1+\frac{\cos t+\sin t}{100}\right)\int_{-\infty}^{0}\exp(r)\left[4+\frac{2x(t+r)\sin t+\cos^2 t}{5000+5\sin^2[t-\ln(1+x^2(t+r))]}\right]dr,\quad \forall t\in\mathbb{R}. \tag{3.10}$$

Let $\omega=2\pi$, $M=1000$, $N=1$, $c_1=1/98$, $c_2=5/98$, $G_1=1/102$, $G_2=5/98$, $t_0=\pi/2$, and

$$g(t)=\ln(100+2\sin t),\quad c(t)=\frac{2+3\cos t}{100-2\sin t},\quad a(t)=\frac{3}{100+2\sin t},\quad b(t)=1+\frac{\cos t+\sin t}{100},$$
$$f(t,s)=4+\frac{2s\sin t+\cos^2 t}{5000+5\sin^2[t-\ln(1+s^2)]},\quad \forall(t,s)\in\mathbb{R}^2,\quad Q(r)=\exp(r),\quad h(r)=r,\quad \forall r\in\mathbb{R}_-. \tag{3.11}$$

Obviously, (2.2), (2.16), and (2.18) hold. A simple calculation yields that

$$\begin{aligned}(N+c_2M)G_2&=\frac{12745}{4802}<\frac{336103}{104125}=\left(1+\frac{-1-1}{100}\right)\left(4+\frac{-2000+0}{5000}\right)+\frac{6-9}{10000-4}\cdot 1000\\
&\le b(t)f(t,s)+a(t)c(t)s\le\left(1+\frac{1+1}{100}\right)\left(4+\frac{2000+1}{5000}\right)+\frac{6+9}{10000-4}\cdot 1000\\
&=\frac{623563}{104125}<\frac{24250}{2499}=(1-c_1)MG_1,\quad \forall(t,s)\in[0,\omega]\times[N,M];\end{aligned} \tag{3.12}$$

that is, (2.23) holds. Clearly (2.26) follows from the above inequalities. Thus Theorem 2.4 ensures that (3.10) has a positive ω-periodic solution $x\in A(N,M)$ with $N<x(t)\le M$ for all $t\in[0,\omega]$.

Acknowledgment

This study was supported by research funds from the Dong-A University.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 609862, 10 pages
doi:10.1155/2012/609862

Research Article

A Linear Homogeneous Partial Differential Equation with Entire Solutions Represented by Laguerre Polynomials

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Received 11 November 2011; Revised 19 March 2012; Accepted 19 March 2012

Academic Editor: Agacik Zafer

We study a homogeneous partial differential equation and get its entire solutions represented in convergent series of Laguerre polynomials. Moreover, the formulae of the order and type of the solutions are established.

1. Introduction and Main Results

The existence and behavior of global meromorphic solutions of homogeneous linear partial differential equations of the second order

$$a_0\frac{\partial^2 u}{\partial t^2} + 2a_1\frac{\partial^2 u}{\partial t\partial z} + a_2\frac{\partial^2 u}{\partial z^2} + a_3\frac{\partial u}{\partial t} + a_4\frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}$$

where $a_k = a_k(t,z)$ are polynomials for $(t,z) \in \mathbb{C}^2$, have been studied by Hu and Yang [1]. Specially, in [1, 2], they have studied the following cases of (1.1)

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + (2t+2)\frac{\partial u}{\partial t} - 2z\frac{\partial u}{\partial z} = 0, \tag{1.2}$$

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} - z\frac{\partial u}{\partial z} + t^2 u = 0 \tag{1.3}$$

and showed that the solutions of (1.2) and (1.3) are closely related to Bessel functions and Bessel polynomials, respectively. Hu and Li [3] studied meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables:

$$\left(1-t^2\right)\frac{\partial^2 u}{\partial t^2}+z^2\frac{\partial^2 u}{\partial z^2}-\{\alpha-\beta+(\alpha+\beta+2)t\}\frac{\partial u}{\partial t}+(\alpha+\beta+2)z\frac{\partial u}{\partial z}=0, \tag{1.4}$$

where $\alpha, \beta \in \mathbb{C}$. Equation (1.4) has a lot of entire solutions on $\mathbb{C}^2$ represented by Jacobian polynomials. Global solutions of some first-order partial differential equations (or system) were studied by Berenstein and Li [4], Hu and Yang [5], Hu and Li [6], Li [7], Li and Saleeby [8], and so on.

In this paper, we concentrate on the following partial differential equation (PDE)

$$t\frac{\partial^2 u}{\partial t^2}+(\alpha+1-t)\frac{\partial u}{\partial t}+z\frac{\partial u}{\partial z}=0 \tag{1.5}$$

for a real $\alpha > 0$. We will characterize the entire solutions of (1.5), which are related to Laguerre polynomials. Further, the formulae of the order and type of the solutions are obtained.

It is well known that the Laguerre polynomials are defined by

$$L_n(\alpha,t)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-t)^k}{k!}, \tag{1.6}$$

which are solutions of the following ordinary differential equations (ODE):

$$t\frac{d^2\omega}{dt^2}+(\alpha+1-t)\frac{d\omega}{dt}+n\omega=0. \tag{1.7}$$

Moreover, Hu [9] pointed out that the generating function of $L_n(\alpha,t)$

$$F(\alpha,t,z)=(1-z)^{-\alpha-1}e^{-tz/(1-z)}=\sum_{n=0}^{\infty}L_n(\alpha,t)z^n \tag{1.8}$$

is a solution of the PDE (1.5). Based on the methods from Hu and Yang [2], we get the following results.

Theorem 1.1. *The partial differential equation* (1.5) *has an entire solution* $u=f(t,z)$ *on* $\mathbb{C}^2$, *if and only if* $u=f(t,z)$ *has a series expansion*

$$f(t,z)=\sum_{n=0}^{\infty}c_nL_n(\alpha,t)z^n \tag{1.9}$$

such that

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{1.10}$$

If $f(t,z)$ is an entire function on $\mathbb{C}^2$, set

$$M(r,f) = \max_{|t|\le r, |z|\le r} \left| f(t,z) \right|, \tag{1.11}$$

we define its order by

$$\operatorname{ord}(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}, \tag{1.12}$$

where

$$\log^+ x = \begin{cases} \log x, & \text{if } x \ge 1, \\ 0, & \text{if } x < 1. \end{cases} \tag{1.13}$$

Theorem 1.2. *If $f(t,z)$ is defined by* (1.9) *and* (1.10), *then*

$$\rho \le \operatorname{ord}(f) \le \max(1,\rho), \tag{1.14}$$

where

$$\rho = \limsup_{n\to\infty} \frac{\log n}{\log\left(1/\sqrt[n]{|c_n|}\right)}. \tag{1.15}$$

Valiron [10] showed that each entire solution of a homogeneous linear ODE with polynomial coefficients was of finite order. By studying (1.2) and (1.3), Hu and Yang showed that Valiron's theorem was not true for general partial differential equations. Here by using Theorems 1.1 and 1.2, we can construct entire solution of (1.5) with arbitrary order ρ $(\rho \ge 1)$.

If $0 < \lambda = \operatorname{ord}(f) < \infty$, we define the type of f by

$$\operatorname{typ}(f) = \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^\lambda}. \tag{1.16}$$

Theorem 1.3. *If $f(t,z)$ is defined by* (1.9) *and* (1.10), *and $1 < \lambda = \operatorname{ord}(f) < \infty$, then the type $\sigma = \operatorname{typ}(f)$ satisfies*

$$e\sigma\lambda = \limsup_{n\to\infty} n \sqrt[n]{|c_n|^\lambda}. \tag{1.17}$$

Lindelöf-Pringsheim theorem [11] gave the expression of order and type for one complex variable entire function, and for two variable entire function the formulae of order and type were obtained by Bose and Sharma in [12]. Hu and Yang [2] established an analogue of Lindelöf-Pringsheim theorem for the entire solution of PDE (1.2). But from Theorems 1.2 and 1.3, we find that the analogue theorem for the entire solution of (1.5) is different from the results due to Hu and Yang.

2. An Estimate of Laguerre Polynomials

Before we prove our theorems, we give an upper bound of $L_n(\alpha,t)$, which will play an important role in this paper. The following asymptotic properties of $L_n(\alpha,t)$ can be found in [13]:

(a)

$$L_n(\alpha,t) = \frac{1}{2\sqrt{\pi}} e^{t/2}(-t)^{-\alpha/2-1/4} n^{\alpha/2-1/4} e^{2\sqrt{-nt}}\left(1+O\left(n^{-1/2}\right)\right) \quad (n \longrightarrow \infty) \tag{2.1}$$

holds for t in the complex plane cut along the positive real semiaxis; thus, for $|t| \le r$, we obtain that

$$|L_n(\alpha,t)| \le n^{\alpha/2-1/4} e^{r/2} r^{-\alpha/2-1/4} e^{2\sqrt{nr}} \tag{2.2}$$

holds when n is large enough.

(b)

$$\frac{L_n(\alpha,t)}{n^{\alpha/2}} = e^{t/2} t^{-\alpha/2} J_\alpha\left(2\sqrt{nt}\right) + O\left(n^{-3/4}\right) \quad (n \longrightarrow \infty) \tag{2.3}$$

holds uniformly on compact subsets of $(0,+\infty)$, where J_α is the Bessel function and

$$J_\alpha\left(2\sqrt{nt}\right) = \frac{2}{\sqrt{\pi}\Gamma(\alpha+1/2)}\left(\sqrt{nt}\right)^\alpha \int_0^{\pi/2} \cos\left(\sqrt{nt}\cos x\right)\sin^{2\alpha} x \, dx, \tag{2.4}$$

combining with (2.3), for $|t| \le r$ we can deduce that

$$|L_n(\alpha,t)| \le \frac{\pi n^\alpha}{\sqrt{\pi}\Gamma(\alpha+1/2)} e^{t/2} \le \frac{\pi n^\alpha}{\sqrt{\pi}\Gamma(\alpha+1/2)} e^{r/2} \le n^{\alpha/2-1/4} e^{r/2} r^{-\alpha/2-1/4} e^{2\sqrt{nr}} \tag{2.5}$$

holds when n is large enough. Then (2.2) and (2.5) imply

$$M(r, L_n) \le n^{\alpha/2-1/4} e^{r/2} r^{-\alpha/2-1/4} e^{2\sqrt{nr}}, \tag{2.6}$$

where

$$M(r, L_n) = \max_{|t|\le r} |L_n(\alpha,t)|. \tag{2.7}$$

3. Proof of Theorem 1.1

Assuming that $u = f(t,z)$ is an entire solution on $\mathbb{C}^2$ satisfying (1.5), we have Taylor expansion

$$f(t,z) = \sum_{n=0}^{\infty} \frac{w_n(t)}{n!} z^n, \tag{3.1}$$

where

$$w_n(t) = \frac{\partial^n f}{\partial z^n}(t,0). \tag{3.2}$$

Hence $w_n(t)$ is an entire solution of (1.7).

By the method of Frobenius (see [14]), we can get a second independent solution $X_n(\alpha,t)$ of (1.7) which is

$$X_n(\alpha,t) = qL_n(\alpha,t)\log t + \sum_{i=0}^{\infty} p_i t^i, \tag{3.3}$$

where $q(\neq 0)$, p_i are constants.

So there exist c_n and b_n satisfying

$$w_n(t) = n! c_n L_n(\alpha,t) + b_n X_n(\alpha,t). \tag{3.4}$$

Because of the singularity of $X_n(\alpha,t)$ at $t = 0$, we obtain $b_n = 0$. That shows

$$f(t,z) = \sum_{n=0}^{\infty} c_n L_n(\alpha,t) z^n. \tag{3.5}$$

Now we need to estimate the terms of c_n. Since

$$f(0,z) = \sum_{n=0}^{\infty} c_n L_n(\alpha,0) z^n \tag{3.6}$$

is an entire function, we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n| L_n(\alpha,0)} = 0. \tag{3.7}$$

Since

$$L_n(\alpha,0) = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}, \tag{3.8}$$

we easily get

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{3.9}$$

Conversely, the relations (1.7), (1.9), and (1.10) imply that

$$t\frac{\partial^2 f}{\partial t^2} + (\alpha + 1 - t)\frac{\partial f}{\partial t} + z\frac{\partial f}{\partial z} = \sum_{n=0}^{\infty} c_n \left\{ t\frac{d^2 L_n(\alpha,t)}{dt^2} + (\alpha+1-t)\frac{dL_n(\alpha,t)}{dt} + nL_n(\alpha,t) \right\} z^n = 0 \tag{3.10}$$

holds for all $(t, z) \in \mathbb{C}^2$. Since (2.6) implies

$$\limsup_{n\to\infty} \sqrt[n]{M(r, L_n)} \le 1, \tag{3.11}$$

we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n L_n(\alpha,t)|} \le \limsup_{n\to\infty} \sqrt[n]{|c_n| M(r, L_n)} \le \limsup_{n\to\infty} \sqrt[n]{|c_n|}. \tag{3.12}$$

Combining (1.10), (3.10) with (3.12), we can get that $u = f(t, z)$ is obviously an entire solution of (1.5) on $\mathbb{C}^2$.

4. Proof of Theorem 1.2

Firstly, we prove $\rho \le \operatorname{ord}(f)$. If $\rho = 0$, the result is trivial. Now we assume $0 < \rho \le \infty$ and prove $\operatorname{ord}(f) \ge k_1$ for any $0 < k_1 < \rho$. The relation (1.15) implies that there exists a sequence $n_j \to \infty$ such that

$$n_j \log n_j \ge k_1 \log \frac{1}{\left|c_{n_j}\right|}. \tag{4.1}$$

By using Cauchy's inequality of holomorphic functions, we have

$$\left| \frac{\partial^n f}{\partial z^n}(0,0) \right| \le n! r^{-n} M(r, f), \tag{4.2}$$

together with the formula of the coefficients of the Taylor expansion

$$\frac{\partial^n f}{\partial z^n}(0,0) = c_n L_n(\alpha, 0) n!, \tag{4.3}$$

we obtain $M(r, f) \ge |c_n L_n(\alpha, 0)| r^n$. Since $|L_n(\alpha, 0)| \ge 1$, we have

$$|c_n| r^n \le M(r, f), \tag{4.4}$$

then

$$\log M(r,f) \geq \log|c_n| + n\log r \geq n_j\left(\log r - \frac{1}{k_1}\log n_j\right). \tag{4.5}$$

Putting $r_j = (en_j)^{1/k_1}$, we have

$$\log M(r_j, f) \geq \frac{r_j^{k_1}}{ek_1}, \tag{4.6}$$

which means $\operatorname{ord}(f) \geq k_1$. Then we can get $\operatorname{ord}(f) \geq \rho$.

Next, we will prove $\operatorname{ord}(f) \leq \max(1,\rho)$. Set $\rho' = \max(1,\rho)$. The result is easy for $\rho' = \infty$; then we assume $\rho' < \infty$. For any $\varepsilon > 0$, (1.15) implies that there exists $n_0 > 0$, when $n > n_0$, we have

$$|c_n| < n^{-n(1+2\varepsilon)/\rho'(1+3\varepsilon)} = n^{-n(1+2\varepsilon)/k_2}, \tag{4.7}$$

where $k_2 = \rho'(1+3\varepsilon) > 1$. For any $\alpha \geq 0$, there exists $n_1 > n_0$ such that when $n > n_1$,

$$n^{(\alpha/2-1/4)} < n^{n(\varepsilon/k_2)}, \tag{4.8}$$

combining with (2.6) and (4.7), we get

$$\begin{aligned} M(r,f) &\leq \sum_{n=0}^{\infty}|c_n|M(r,L_n)r^n \\ &\leq Cr^{2n_1} + C\sum_{n>n_1}^{\infty}|c_n|n^{\alpha/2-1/4}e^{r/2}e^{2\sqrt{nr}}r^n \\ &\leq Cr^{2n_1} + Ce^{r/2}\sum_{n>n_1}^{\infty}n^{-(1+\varepsilon)n/k_2}e^{2\sqrt{nr}}r^n, \end{aligned} \tag{4.9}$$

where C is a constant but not necessary to be the same every time.

Set $m_1(r) = ((1/\varepsilon)(2\sqrt{r}/\log r))^2$, which means that $e^{2\sqrt{nr}} < r^{\varepsilon n}$ for $n > m_1(r)$. Further set $m_2(r) = (2r)^{k_2}$, which yields that when $n \geq m_2(r)$,

$$\left(n^{-1/k_2}r\right)^{(1+\varepsilon)n} \leq \left(\frac{1}{2}\right)^{(1+\varepsilon)n} < \left(\frac{1}{2}\right)^n. \tag{4.10}$$

Obviously, we can choose $r_0 > 0$ such that $m_2(r) > m_1(r)$ for $r > r_0$. Then

$$\sum_{n>m_2(r)} n^{-(1+\varepsilon)n/k_2}e^{2\sqrt{nr}}r^n \leq \sum_{n>m_2(r)}^{\infty} n^{-(1+\varepsilon)n/k_2}r^{(1+\varepsilon)n} \leq \sum_{n>m_2(r)}^{\infty}\left(\frac{1}{2}\right)^n \leq 1. \tag{4.11}$$

We also have

$$\begin{aligned}\sum_{n_1<n\le m_2(r)} n^{-(1+\varepsilon)n/k_2}e^{2\sqrt{nr}}r^n &\le \sum_{n_1<n\le m_2(r)} n^{-(1+\varepsilon)n/k_2}e^{2\sqrt{m_2(r)r}}r^{m_2(r)}\\ &\le \sum_{n_1<n\le m_2(r)} n^{-(1+\varepsilon)n/k_2}r^{m_2(r)}r^{\varepsilon m_2(r)}\\ &\le r^{(1+\varepsilon)m_2(r)}\sum_{n_1<n\le m_2(r)} n^{-(1+\varepsilon)n/k_2}\\ &\le Cr^{(1+\varepsilon)m_2(r)}\\ &= Cr^{(1+\varepsilon)(2r)^{k_2}}.\end{aligned} \tag{4.12}$$

Therefore, when $|t| = r \ge r_0$, we have

$$M(r,f) \le Cr^{2n_1} + Ce^{r/2} + Ce^{r/2}r^{(1+\varepsilon)(2r)^{k_2}}, \tag{4.13}$$

which means $\operatorname{ord}(f) \le k_2$. Hence $\operatorname{ord}(f) \le \rho' = \max(1,\rho)$ follows by letting $\varepsilon \to 0$.

5. Proof of Theorem 1.3

Set

$$\kappa = \limsup_{n\to\infty} n\sqrt[n]{|c_n|^{\lambda}}. \tag{5.1}$$

At first, we prove $e\lambda\sigma \ge \kappa$. The result is trivial for $\kappa = 0$, we assume $0 < \kappa \le \infty$ and take ε with $0 < \varepsilon < \kappa$, set

$$k_3 = \begin{cases} \kappa - \varepsilon, & \text{if } \kappa < \infty,\\ \dfrac{1}{\varepsilon}, & \text{if } \kappa = \infty.\end{cases} \tag{5.2}$$

Equation (5.1) implies that there exists a sequence $n_j \to \infty$ satisfying

$$\left|c_{n_j}\right| > \left(\frac{k_3}{n_j}\right)^{n_j/\lambda}, \tag{5.3}$$

combining with (4.4), we can deduce that

$$M(r,f) \ge \left|c_{n_j}\right| r^{n_j} \ge \left(\frac{k_3}{n_j}\right)^{n_j/\lambda} r^{n_j} = \left(\frac{k_3}{n_j}r^{\lambda}\right)^{n_j/\lambda}. \tag{5.4}$$

Taking $r_j^{\lambda} = en_j/k_3$, we get $M(r_j,f) > e^{k_3 r_j^{\lambda}/e\lambda}$, which yields $\sigma \ge k_3/e\lambda$, so $e\lambda\sigma \ge \kappa$.

Next, we prove $e\lambda\sigma \le \kappa$. We may assume $\kappa < \infty$. Equation (5.1) implies that for any $\varepsilon > 0$, there exists $n_0 > 0$, such that when $n > n_0$,

$$|c_n| < \left(\frac{\kappa + (\varepsilon/2)}{n}\right)^{n/\lambda}. \tag{5.5}$$

For any $\alpha \ge 0$, we choose $n_1 (> n_0)$ such that when $n > n_1$,

$$n^{\alpha/2-1/4} < \left(\frac{\kappa+\varepsilon}{\kappa + (\varepsilon/2)}\right)^{n/\lambda}, \tag{5.6}$$

combining with (2.6), we have

$$\begin{aligned} M(r,f) &\le \sum_{n=0}^{\infty} |c_n| M(r, L_n) r^n \\ &\le Cr^{2n_1} + C\sum_{n>n_1}^{\infty} |c_n| n^{\alpha/2-1/4} e^{r/2} e^{2\sqrt{nr}} r^n \\ &\le Cr^{2n_1} + Ce^{r/2} \sum_{n>n_1}^{\infty} \left(\frac{\kappa+\varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n. \end{aligned} \tag{5.7}$$

Set $m_3(r) = 16r\lambda^2$, when $n > m_3(r)$, we deduce $e^{2\sqrt{nr}} < e^{n/2\lambda}$. Set $m_4(r) = 2(\kappa+\varepsilon)r^\lambda$, it is obvious that $(\kappa+\varepsilon)r^\lambda/n < 1/2$ for $n > m_4(r)$. Since $\lambda > 1$, there exists r_1 such that when $r > r_1$,

$$m_4(r) = 2(\kappa+\varepsilon)r^\lambda > 16r\lambda^2 = m_3(r). \tag{5.8}$$

Then

$$\sum_{n>m_4(r)}^{\infty} \left(\frac{\kappa+\varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n = \sum_{n>m_4(r)}^{\infty} \left(\frac{(\kappa+\varepsilon)r^\lambda}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} \le \sum_{n>m_4(r)}^{\infty} \left(\frac{e}{4}\right)^{n/2\lambda} \le C. \tag{5.9}$$

We note that for $a > 0$, $b > 0$, $\max_{x>0}(a/x)^{x/b} = e^{a/eb}$, then we have

$$\left(\frac{\kappa+\varepsilon}{n}\right)^{n/\lambda} r^n = \left(\frac{(\kappa+\varepsilon)r^\lambda}{n}\right)^{n/\lambda} \le e^{(\kappa+\varepsilon)r^\lambda/e\lambda}. \tag{5.10}$$

This shows

$$\begin{aligned} \sum_{n_1<n\le m_4(r)} \left(\frac{\kappa+\varepsilon}{n}\right)^{n/\lambda} e^{2\sqrt{nr}} r^n &\le m_4(r) e^{2\sqrt{m_4(r) r}} e^{(\kappa+\varepsilon)r^\lambda/e\lambda} \\ &\le 2(\kappa+\varepsilon) r^\lambda e^{2\sqrt{2(\kappa+\varepsilon)r^{\lambda+1}}} e^{(\kappa+\varepsilon)r^\lambda/e\lambda}. \end{aligned} \tag{5.11}$$

Therefore when $|t| = r \geq r_1$,

$$M(r, f) \leq Ce^{r/2}(\kappa + \varepsilon)r^{\lambda}e^{2\sqrt{2(\kappa+\varepsilon)r^{\lambda+1}}}e^{(\kappa+\varepsilon)r^{\lambda}/e\lambda} + Cr^{2n_1} + Ce^{r/2}. \tag{5.12}$$

Together with $\lambda > 1$ and the definition of type, we can get $\sigma \leq (\kappa+\varepsilon)/e\lambda$, which yields $e\lambda\sigma \leq \kappa$ by letting $\varepsilon \to 0$.

Acknowledgments

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original paper. The first author was partially supported by Natural Science Foundation of China (11001057), and the third author was partially supported by Natural Science Foundation of Shandong Province.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 180894, 17 pages
doi:10.1155/2012/180894

Research Article

Almost Periodic (Type) Solutions to Parabolic Cauchy Inverse Problems

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Received 23 December 2011; Revised 1 March 2012; Accepted 12 March 2012

Academic Editor: István Györi

We first show the existence and uniqueness of (pseudo) almost periodic solutions of some types of parabolic equations. Then, we apply the results to a type of Cauchy parabolic inverse problems and show the existence, uniqueness, and stability.

1. Introduction

Zhang in [1, 2] defined pseudo almost periodic functions. As almost periodic functions, pseudo almost periodic functions are applied to many mathematical areas, particular to the theory of ordinary differential equations. (e.g., see [3–26] and references therein). However, the study of the related topic on partial differential equations has only a few important developments. On the other hand, almost periodic functions to various problems have been investigated (e.g., see [27–32] and references therein), but little has been done about the inverse problems except for our work in [33–36]. In [36], we study pseudo almost periodic solutions to parabolic boundary value inverse problems. In this paper, we devote such solutions to cauchy problems.

To this end, we need first to define the spaces in a more general setting. Let $J \in \{\mathbf{R}, \mathbf{R}^n\}$. Let $C(J)$ (resp., $C(J \times \Omega)$, where $\Omega \subset \mathbf{R}^m$) denote the C^*-algebra of bounded continuous complex-valued functions on J (resp. $J \times \Omega$) with the supremum norm. For $f \in C(J)$ (resp., $C(J\times\Omega)$) and $s \in J$, the translation of f by s is the function $R_s f(t) = f(t+s)$ (resp., $R_s f(t, Z) = f(t+s, Z)$, $(t, Z) \in J \times \Omega$).

Definition 1.1. (1) A function $f \in C(J)$ is called almost periodic if for every $\epsilon > 0$ the set

$$T(f, \epsilon) = \{\tau \in J : \|R_\tau f - f\| < \epsilon\} \tag{1.1}$$

is relatively dense in J. Denote by $\mathcal{AP}(J)$ the set of all such functions. The number (vector) τ is called ϵ-translation number (vector) of f.

(2) A function $f \in C(J \times \Omega)$ is said to be almost periodic in $t \in J$ and uniform on compact subsets of Ω if $f(\cdot, Z) \in \mathcal{AP}(J)$ for each $Z \in \Omega$ and is uniformly continuous on $J \times K$ for any compact subset $K \subset \Omega$. Denote by $\mathcal{AP}(J \times \Omega)$ the set of all such functions. For convenience, such functions are also called uniformly almost periodic.

(3) A function $f \in C(J)(C(J \times \Omega))$ is called pseudo almost periodic if

$$f = g + \varphi, \tag{1.2}$$

where $g \in \mathcal{AP}(J)(\mathcal{AP}(J \times \Omega))$ and $\varphi \in \mathcal{PAP}_0(J)(\mathcal{PAP}_0(J \times \Omega))$,

$$\begin{gathered}
\mathcal{PAP}_0(J) = \left\{ \varphi \in C(J) : \lim_{r \to \infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} |\varphi(x)| dx = 0 \right\}, \\
\mathcal{PAP}_0(J \times \Omega) = \left\{ \varphi \in C(J \times \Omega) : \lim_{r \to \infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} |\varphi(x, Z)| dx = 0 \right\},
\end{gathered} \tag{1.3}$$

uniformly with respect to $Z \in K$, where K is any compact subset of Ω. Denote by $\mathcal{PAP}(J)(\mathcal{PAP}(J \times \Omega))$ the set of all such functions.

Set

$$\begin{gathered}
\mathcal{APT}(J) \in \{\mathcal{AP}(J),\ \mathcal{PAP}(J)\}, \\
\mathcal{APT}(J \times \Omega) \in \{\mathcal{AP}(J \times \Omega),\ \mathcal{PAP}(J \times \Omega)\}.
\end{gathered} \tag{1.4}$$

Members of $\mathcal{APT}(J)(\mathcal{APT}(J \times \Omega))$ are called almost periodic type.

We will use the notations throughout the paper: $\mathbf{R}_T^m = \mathbf{R}^m \times (0, T)$, $\|F\|_T = \sup\{|F(x,t)| : x \in \mathbf{R}^n,\ 0 \le t \le T\}$. $F \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}_T^m)$ means that $F(x^{(1)}, x^{(2)}, t)$ is almost periodic type in $x^{(1)} \in \mathbf{R}^n$ and uniformly for $(x^{(2)}, t) \in \mathbf{R}_T^m$; $F \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}^m)$ means that $F(x^{(1)}, x^{(2)})$ is almost periodic type in $x^{(1)} \in \mathbf{R}^n$ and uniformly for $x^{(2)} \in \mathbf{R}^m$.

Let

$$Z(x, t; \xi, s) = \frac{1}{\left(2\sqrt{\pi(t-s)}\right)^{n+m}} \exp\left\{ -\frac{\sum (x_i - \xi_i)^2}{4(t-s)} \right\}, \quad (x,\ \xi \in \mathbf{R}^{n+m}), \tag{1.5}$$

be the fundamental solution of the heat equation [37].

In the next section, we will show the existence and uniqueness of some type of parabolic equations. Sections 3 is devoted to a type of Cauchy Problem respectively.

2. Solutions of Parabolic Equations

Lemma 2.1. *Let $T > 0$. If $\varphi \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}^m)$ and*

$$u(x,t;s) = \int_{\mathbf{R}^{n+m}} \varphi(\xi) Z(x,t;\xi,s) d\xi, \tag{2.1}$$

then for each fixed $s \in [0,T)$ $u \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$.

Proof. First consider the case that $\varphi \in \mathcal{AP}(\mathbf{R}^n \times \mathbf{R}^m)$. Let $\tau \in \mathbf{R}^n$ be an ϵ-translation vector of φ:

$$\begin{aligned}
&u\left(x^{(1)} + \tau, x^{(2)}, t; s\right) - u\left(x^{(1)}, x^{(2)}, t; s\right) \\
&\quad = \int_{\mathbf{R}^{n+m}} \varphi\left(\xi^{(1)}, \xi^{(2)}\right)\left[Z\left(x^{(1)} + \tau, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s\right) - Z\left(x^{(1)}, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s\right)\right] d\xi^{(1)} d\xi^{(2)} \\
&\quad = \int_{\mathbf{R}^{n+m}} \left[\varphi\left(x^{(1)} + \tau + \xi^{(1)}, x^{(2)} + \xi^{(2)}\right) - \varphi\left(x^{(1)} + \xi^{(1)}, x^{(2)} + \xi^{(2)}\right)\right] Z(0,t;\xi s) d\xi,
\end{aligned} \tag{2.2}$$

where $0 \in \mathbf{R}^{n+m}$ is the zero vector. Note that $\int_{\mathbf{R}^{n+m}} Z(0,t;\xi,s)d\xi = 1$, we get

$$\|R_\tau u - u\| \le \|R_\tau \varphi - \varphi\| \int_{\mathbf{R}^{n+m}} Z(0,t;\xi,s) d\xi < \epsilon, \tag{2.3}$$

where $t \in [s,T]$ and $x^{(2)} \in B$ with B a bounded subset of $\mathbf{R}^m$. This shows that $u \in \mathcal{AP}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$.

To show that $u \in \mathcal{PAP}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$ if $\varphi \in \mathcal{PAP}(\mathbf{R}^n \times \mathbf{R}^m)$, we only need to show that $u \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$ if $\varphi \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m)$. That is,

$$\lim_{r \to \infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} \left| u\left(x^{(1)}, x^{(2)}, t; s\right) \right| ds^{(1)} = 0, \tag{2.4}$$

uniformly with respect to $(x^{(2)}, t) \in \Omega$, here Ω is any compact subset of $\mathbf{R}^m \times [s,T]$.

Since $\varphi \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m)$, for $\epsilon > 0$ there exist positive numbers A and r_0 such that, when $r \ge r_0$ for all $\xi^{(1)} \in [-A,A]^n$ and $\xi^{(2)} \in \Omega \cap \mathbf{R}^m$, one has

$$\begin{gathered}
\frac{1}{(2r)^n} \int_{[-r,r]^n} \left| \varphi\left(x^{(1)} + \xi^{(1)}, \xi^{(2)}\right) \right| dx^{(1)} < \frac{\epsilon}{3}, \\
\int_{-\infty}^{-A} + \int_A^\infty \left| \varphi\left(x^{(1)} + \xi^{(1)}, \xi^{(2)}\right) \right| Z\left(0, x^{(2)}, t; \xi, s\right) d\xi < \frac{2\epsilon}{3}.
\end{gathered} \tag{2.5}$$

Therefore,

$$\begin{aligned}
&\frac{1}{(2r)^n}\int_{[-r,r]^n}\left|u\left(x^{(1)},x^{(2)},t;s\right)\right|dx^{(1)}\\
&\le\frac{1}{(2r)^n}\int_{[-r,r]^n}\int_{\mathbf{R}^{n+m}}\left|\varphi\left(x^{(1)}+\xi^{(1)},\xi^{(2)}\right)\right|Z\left(0,x^{(2)},t;\xi,s\right)d\xi dx^{(1)}\\
&=\int_{-\infty}^{-A}+\int_{-A}^{A}+\int_{A}^{\infty}Z\left(0,x^{(2)},t;\xi,s\right)d\xi\frac{1}{(2r)^n}\int_{[-r,r]^n}\left|\varphi\left(x^{(1)}+\xi^{(1)},\xi^{(2)}\right)\right|dx^{(1)}<\epsilon
\end{aligned}\tag{2.6}$$

uniformly with respect to $(x^{(2)},t)\in\Omega$, where by $\int_a^b F(\xi)d\xi$ we mean that

$$\int_a^b F(\xi)\xi=\int_{[a,b]^{n+m}}F(\xi)d\xi=\int_a^b\cdots\int_a^b F(\xi)d\xi_1 d\xi_2\cdots d\xi_{n+m}.\tag{2.7}$$

This shows that $u\in \mathcal{PAP}_0(\mathbf{R}^n\times\mathbf{R}^m)$. The proof is complete. □

Corollary 2.2. *Let $\varphi,\partial\varphi/\partial x_i\in\mathcal{APT}(\mathbf{R}^n\times\mathbf{R}^m)$, and let u be as in Lemma 2.1. Then, $\partial u/\partial x_i\in\mathcal{APT}(\mathbf{R}^n\times\mathbf{R}^m\times[s,T])$.*

Proof. Note that

$$\begin{aligned}
\frac{\partial u(x,t;s)}{\partial x_i}&=\int_{\mathbf{R}^{n+m}}\varphi(\xi)\frac{\partial Z(x,t;\xi,s)}{\partial x_i}d\xi\\
&=-\int_{\mathbf{R}^{n+m}}\varphi(\xi)\frac{\partial Z(x,t;\xi,s)}{\partial \xi_i}d\xi\\
&=\int_{\mathbf{R}^{n+m}}\frac{\partial\varphi(\xi)}{\partial\xi_i}Z(x,t;\xi,s)d\xi.
\end{aligned}\tag{2.8}$$

□

By Lemma 2.1 we get the conclusion.

Lemma 2.3. *If $f(x,t)\in\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}^m_T})$ and*

$$u(x,t)=\int_0^t ds\int_{\mathbf{R}^{n+m}}f(\xi,s)Z(x,t;\xi,s)d\xi,\tag{2.9}$$

then u and $\partial u(x,t)/\partial x_i(i=1,2,\ldots,n+m)$ are all in $\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}^m_T})$.

The proof is similar to that of Lemma 2.1, so we omit it.

Theorem 2.4. *Consider the heat problem*

$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n+m}\left[\frac{\partial^2 u}{\partial x_i^2}+b_i(x,t)\frac{\partial u}{\partial x_i}\right]-c(x,t)u=f(x,t),\quad (x,t)\in \mathbf{R}_T^{n+m},$$
$$u(x,0)=\varphi(x),\quad x\in\mathbf{R}^{n+m}. \tag{2.10}$$

If $f(x,t), b_i(x,t), \partial b_i/\partial x_j$ $(i,j=1,2,\dots,n+m)$, $c(x,t)$ *are in* $\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}_T^m})$ *and* $\varphi, \partial\varphi/\partial x_i$ $(i=1,2,\dots,n+m)$ *are in* $\mathcal{APT}(\mathbf{R}^n\times\mathbf{R}^m)$*, then* (2.10) *has a unique solution* $u\in\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}_T^m})$.

Proof. Problem (2.10) has the standard solution (see [37]):

$$u(x,t)=\int_{\mathbf{R}^{n+m}}\varphi(\xi)\Gamma(x,t;\xi,0)d\xi+\int_0^t ds\int_{\mathbf{R}^{n+m}}f(\xi,s)\Gamma(x,t;\xi,s)d\xi=I_1+I_2, \tag{2.11}$$

where

$$\Gamma(x,t;\xi,s)=Z(x,t;\xi,s)+\int_s^t\int_{\mathbf{R}^{n+m}}Z(x,t;y,\eta)\cdot\Phi(y,\eta;\xi,s)dyd\eta,$$
$$\Phi(y,\eta;\xi,s)=\sum_{l=1}^{\infty}(LZ)_l(y,\eta;\xi,s),\quad (LZ)_1=LZ, \tag{2.12}$$
$$(LZ)_{l+1}(y,\eta;\xi,s)=\int_s^t\int_{\mathbf{R}^{n+m}}\left[LZ(y,\eta;v,\sigma)\right](LZ)_l(v,\sigma;\xi,s)dvd\sigma,$$

and L is the parabolic operator

$$L=\sum_{i=1}^{n+m}\left[\frac{\partial^2}{\partial x_i^2}+b_i(x,t)\frac{\partial}{\partial x_i}\right]+c(x,t)-\frac{\partial}{\partial t}. \tag{2.13}$$

Now, we show that $u\in\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}_T^m})$:

$$\begin{aligned}I_1&=\int_{\mathbf{R}^{n+m}}\varphi(\xi)Z(x,t;\xi,0)d\xi\\&\quad+\int_{\mathbf{R}^{n+m}}\varphi(\xi)d\xi\int_0^t d\eta\int_{\mathbf{R}^{n+m}}Z(x,t;y,\eta)\sum_{l=1}^{\infty}(LZ)_l(y,\eta;\xi,0)dy\\&=u_1(x,t)+\sum_{l=1}^{\infty}\int_0^t d\eta\int_{\mathbf{R}^{n+m}}Z(x,t;y,\eta)dy\int_{\mathbf{R}^{n+m}}\varphi(\xi)(LZ)_l(y,\eta;\xi,0)d\xi,\end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s) Z(x,t;\xi,s) d\xi \\
&\quad + \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s) d\xi \int_s^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta) \sum_{l=1}^{\infty} (LZ)_l Z(y,\eta;\xi,s) dy \\
&= u_2(x,t) + \sum_{l=1}^{\infty} \int_0^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta) dy \int_0^{\eta} ds \int_{\mathbf{R}^{n+m}} f(\xi,s)(LZ)_l(y,\eta;\xi,s) d\xi.
\end{aligned}
\tag{2.14}
$$

By Lemmas 2.1 and 2.3, $u_i \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$ $(i = 1, 2,)$.

Obviously,

$$
LZ(x,t;\xi,s) = \sum_{i=1}^{n+m} b_i(x,t) \frac{\partial Z(x,t;\xi,s)}{\partial x_i} + c(x,t) Z(x,t;\xi,s). \tag{2.15}
$$

By Lemmas 2.1 and 2.3 we only need to show that the functions

$$
\begin{aligned}
w_l(x,t) &= \int_{\mathbf{R}^{n+m}} \varphi(\xi)(LZ)_l(x,t;\xi,0) d\xi, \\
v_l(x,t) &= \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s)(LZ)_l(x,t;\xi,s) d\xi,
\end{aligned}
\tag{2.16}
$$

are in $\mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. We do this by induction. By Lemmas 2.1 and 2.3 and Corollary 2.2, it is true for the case $l = 1$. Suppose that $w_l(x,t),\ v_l(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}_T^m)$. Then,

$$
\begin{aligned}
w_{l+1}(x,t) &= \int_{\mathbf{R}^{n+m}} \varphi(\xi) d\xi \int_0^t d\eta \int_{\mathbf{R}^{n+m}} LZ(x,t;y,\eta)(LZ)_l(y,\eta;\xi,0) dy \\
&= \int_0^t d\eta \int_{\mathbf{R}^{n+m}} LZ(x,t;y,\eta) dy \int_{\mathbf{R}^{n+m}} \varphi(\xi)(LZ)_l(y,\eta;\xi,0) d\xi \\
&= \int_0^t d\eta \int_{\mathbf{R}^{n+m}} w_l(y,\eta) LZ(x,t;y,\eta) dy.
\end{aligned}
\tag{2.17}
$$

By the induction assumption and Lemma 2.3, we have $w_{l+1}(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. Similarly one shows that $v_{l+1}(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. The proof is complete. □

3. Cauchy Problem

Starting this section we will apply the results of the last section to inverse problems of partial differential equations. We will investigate two types of initial value problems in this and the next sections, respectively. We will keep the notation in Section 2 and, at the same time, introduce the following new notation:

$$x = (x_1, x_2, \ldots, x_{n-1}), \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}),$$
$$X = (x, x_n), \qquad \zeta = (\xi, \xi_n). \tag{3.1}$$

The following estimates are easily obtained:

$$\left\| \int_0^t ds \int_{\mathbf{R}^n} Z(X, t; \zeta, s) d\zeta \right\|_T \leq m(T), \tag{3.2}$$

where $m(T)$ are positive and increasing for $T \geq 0$ and $m(T) \to 0$ as $T \to 0$.

To show the main results of this and the next sections, the following lemmas are needed. The first lemma is the Gronwall-Bellman lemma; the convenient reference should be an ODE text, for instance, it is proved on page 15 of [38].

Lemma 3.1. *Let φ, ϕ, and χ be real, continuous functions on $[0, T]$ with $\chi \geq 0$. If*

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\varphi(s) ds \quad (t \in [0, T]), \tag{3.3}$$

then

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\phi(s) \exp\left\{ \int_s^t \chi(\rho) d\rho \right\} ds \quad (t \in [0, T]). \tag{3.4}$$

Lemma 3.2. *Let φ be a continuous function on $[0, T]$. If ϕ, χ_1, and χ_2 are nondecreasing and non-negative on $[0, T]$ and*

$$\varphi(t) \leq \phi(t) + \chi_1(t) \int_0^t \varphi(s) ds + \chi_2(t) \int_0^t \frac{\varphi(s)}{\sqrt{t-s}} ds \quad (t \in [0, T]), \tag{$*$}$$

then

$$\varphi(t) \leq \phi(t) \left[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t) \right] e^{t\chi(t)}, \tag{3.5}$$

where

$$\chi(t) = t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t). \tag{3.6}$$

Proof. Replacing $\varphi(s)$ in the two integrals of $(*)$ by the expression on the right-hand side in (26), changing the integral order of the resulting inequality, and making use of the monotonicity of ϕ, χ_1, and χ_2, one gets

$$\varphi(t) \leq \phi(t)\left[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t)\right] + \left[t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t)\right]\int_0^t \varphi(s)Ds. \tag{3.7}$$

Using Lemma 3.1 leads to the conclusion. □

Lemma 3.3. *Let $F(X,t) \in C(\overline{\mathbf{R}_T^n})$ and $\varphi \in C(\mathbf{R}^n)$. If $u(X,t)$ is a solution of the problem*

$$\begin{gathered} u_t - \Delta u + qu = F(X,t), \quad (X,t) \in \mathbf{R}_T^n, \\ u(X,0) = \varphi(X), \quad X \in \mathbf{R}^n, \end{gathered} \tag{3.8}$$

then

$$\|u\|_T \leq K(T)\left(T\|F\|_T + \|\varphi\|\right), \tag{3.9}$$

where $K(T) = 1 + T\|q\|_T e^{T\|q\|_T}$.

One sees that $K(T)$ depends on $\|q\|_T$ only and is bounded near zero.

Proof. The solution u can been written as

$$\begin{aligned} u(X,t) = & \int_{\mathbf{R}^n} \varphi(\zeta)Z(X,t;\zeta,0)d\zeta + \int_0^t ds \int_{\mathbf{R}^n} F(\zeta,s)Z(X,t;\zeta,s)d\zeta \\ & - \int_0^t ds \int_{\mathbf{R}^n} q(\xi,s)u(\zeta,s)Z(X,t;\zeta,s)d\zeta, \end{aligned} \tag{3.10}$$

so,

$$\|u\|_t \leq \|\varphi\| + \int_0^t \|F\|_s ds + \int_0^t \|q\|_s \|u\|_s ds. \tag{3.11}$$

By Lemma 3.1 one gets the desired result. The proof is complete. □

Problem 1. Find functions $u(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APC}(\overline{\mathbf{R}_T^{n-1}})$ such that

$$u_t - \Delta u + qu = F(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.12}$$

$$u(X,0) = \varphi(X), \quad X \in \mathbf{R}^n, \tag{3.13}$$

$$u(x,0,t) = h(x,t), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}, \tag{3.14}$$

where $\varphi(x,x_n)$, $\varphi_{x_n x_n}(x,x_n) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \mathbf{R})$, $h(x,t) \geq \text{const} > 0$, $h(x,t)$, $(\Delta h - h_t) \in \mathcal{APC}(\overline{\mathbf{R}_{T_0}^{n-1}})$, and $F(x,x_n,t)$, $F_{x_n x_n}(x,x_n,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_{T_0}})$. $T_0 > T > 0$ are constants.

By (3.13) and (3.14), one sees that $h(x,0) = \varphi(x,0)$.

We have the following additional problem.

Problem 2. Find functions $W(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APC}(\overline{\mathbf{R}_T^{n-1}})$ such that

$$W_t - \Delta W + qW = F_{x_n x_n}(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.15}$$

$$W(X,0) = \varphi_{x_n x_n}(X), \quad X \in \mathbf{R}^n, \tag{3.16}$$

$$W(x,0,t) = h_t - \Delta h + qh - F(x,0,t), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}. \tag{3.17}$$

The Cauchy problems with unknown coefficient belong to inverse problems [39]. "In the last two decades, the field of inverse problems has certainly been one of the fastest growing areas in applied mathematics. This growth has largely been driven by the needs of applications both in other sciences and in industry." [40]. For the two problems above, we have the following.

Lemma 3.4. *Problems 1 and 2 are equivalent to each other.*

Proof. Let $V(X,t) = u_{x_n}(X,t)$. Then, $V(X,t)$ satisfies

$$V_t - \Delta V + qV = F_{x_n}(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.18}$$

$$V(X,0) = \varphi_{x_n}(X), \quad X \in \mathbf{R}^n, \tag{3.19}$$

$$V_{x_n}(x,0,t) = h_t - \Delta h + qh - F(x,0,t), \qquad \varphi(x,0) = h(x,0), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}. \tag{3.20}$$

So, if Problem 1 has a solution (u,q), then Problems (3.18)–(3.20) have the solution (V,q) with $V(X,t) = u_{x_n}(X,t)$. Obviously $V(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ if $u(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

On the other hand, if $V(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APC}(\overline{\mathbf{R}_T^{n-1}})$ satisfy (3.18)–(3.20), then we will show that Problem 1 has a unique solution (u,q) and $u(X,t) \in \mathcal{APC}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

The uniqueness comes from the uniqueness of Cauchy Problem (1)-(2). For the existence, note the fact that if (u,q) is a solution of (3.12)–(3.14), then $V = u_{x_n}$. Thus, we define

$$u(X,t) = \int_0^{x_n} V(x,y,t)\,dy + \Phi(x,t). \tag{3.21}$$

It follows from (3.14) that $\Phi = h$. Now, u satisfies (3.13) because

$$u(X,0) = \int_0^{x_n} V(x,y,0)\,dy + h(x,0) = \varphi(X) - \varphi(x,0) + h(x,0) = \varphi(X). \tag{3.22}$$

It follows from (3.18), (3.20), and (3.21) that

$$\begin{aligned} u_t - \Delta u + qu &= \int_0^{x_n} (V_t - \Delta V + qV)dy + \int_0^{x_n} \frac{\partial^2 V(x,y,t)}{\partial y^2} dy \\ &\quad - \frac{\partial^2}{\partial x_n^2} \int_0^{x_n} V(x,y,t)dy + h_t - \nabla h + qh \\ &= F(x,x_n,t) - F(x,0,t) + V_{x_n}(x,x_n,t) - V_{x_n}(x,0,t) - V_{x_n}(x,x_n,t) \\ &\quad + h_t - \Delta h + qh = F(X,t). \end{aligned} \tag{3.23}$$

Thus, u satisfies (3.12) and (u,q) is a unique solution of Problem 1.

It follows from $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T},\ h(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, and (3.21) that $u(X,t)$ in (3.21) is in $\mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

Since we have shown that Problem 1 is equivalent to (3.18)–(3.20), to show the lemma we only need to show that Problem 2, equivalent to (3.18)–(3.20) too.

If (V,q) is a solution of (3.18)–(3.20), let $W(X,t) = V_{x_n}(X,t)$. Then one can directly calculate that (W,q) is a solution of (3.15)–(3.17) and $W(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

On the other hand, if (W,q) is a solution of (3.15)–(3.17), let

$$V(X,t) = \int_0^{x_n} W(x,y,t)dy + \Phi(x,t), \tag{3.24}$$

where Φ is the solution of the Cauchy problem

$$\begin{aligned} \Phi_t - \Delta\Phi + q\Phi &= W_{x_n}(x,0,t) + F_{x_n}(x,0,t), \quad (x,t) \in \mathbf{R}_T^{n-1} \\ \Phi(x,0) &= \varphi_{x_n}(x,0), \quad x \in \mathbf{R}^{n-1}. \end{aligned} \tag{3.25}$$

Since $W(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$, $W_{x_n}(x,0,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. By Theorem 2.4, $\Phi(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ and so $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

Obviously, $V_{x_n}(x,0,t) = W(x,0,t) = h_t - \Delta h + qh - F(x,0,t)$, and this shows that V satisfies (3.20). V satisfies (3.19) because

$$\begin{aligned} V(X,0) &= \int_0^{x_n} W(x,y,0)dy + \Phi(x,0) = \int_0^{x_n} \varphi_{x_n x_n}(x,y)dy + \Phi(x,0) \\ &= \varphi_{x_n}(x,x_n) - \varphi_{x_n}(x,0) + \Phi(x,0) = \varphi_{x_n}(X). \end{aligned} \tag{3.26}$$

Finally,

$$
\begin{aligned}
V_t - \Delta V + qV &= \int_0^{x_n} (W_t - \Delta W + qW)\,dy + \int_0^{x_n} \frac{\partial^2 W(x,y,t)}{\partial y^2}\,dy \\
&\quad - \frac{\partial^2}{\partial x_n^2}\int_0^{x_n} W(x,\eta,t)\,d\eta + \Phi_t - \Delta\Phi + q\Phi \\
&= F_{x_n}(x,x_n,t) - F_{x_n}(x,0,t) + W_{x_n}(x,x_n,t) - W_{x_n}(x,0,t) - W_{x_n}(x,x_n,t) \\
&\quad + \Phi_t - \Delta\Phi + q\Phi = F_{x_n}(X,t).
\end{aligned} \tag{3.27}
$$

This shows that V satisfies (3.18). The proof is complete. □

By (3.15)-(3.16) we have the integral equation about $W(X,t)$:

$$
\begin{aligned}
W(X,t) &= \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(X,t;\zeta,0)\,d\zeta + \int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(X,t;\zeta,s)\,d\zeta \\
&\quad - \int_0^t ds \int_{\mathbf{R}^n} q(\xi,s) W(\zeta,s) Z(X,t;\zeta,s)\,d\zeta,
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
q &= Lq \\
&= [\Delta h - h_t + F(x,0,t)]h^{-1} + h^{-1}\int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0)\,d\zeta \\
&\quad + h^{-1}\int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \\
&\quad - h^{-1}\int_0^t ds \int_{\mathbf{R}^n} q(\xi,s) W(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \quad (x,t) \in \mathbf{R}_T^{n-1},
\end{aligned} \tag{3.29}
$$

where W is determined by (3.28).

It is readily to show that (3.15)–(3.17) are equivalent to (3.28)-(3.29).

Note that, for a given $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, Theorem 2.4 shows the Cauchy problem (3.15) and (3.16) (or equivalently (3.28)) has a unique solution $W \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T})$. Thus, (3.29) does define an operator L. To show that Problem 2 (and so Problem 1) has a unique solution, we only need to show that (3.29) has a solution $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. That is, L has a fixed point in $\mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. To this end, let

$$
\begin{aligned}
&\Big\| [\Delta h - h_t + F(x,0,t)]h^{-1} + h^{-1}\int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0)\,d\zeta \\
&\quad + h^{-1}\int_0^T ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \Big\|_{T_0} = \frac{M}{2}.
\end{aligned} \tag{3.30}
$$

Set $B(M,T) = \{q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}}) : \|q\|_T \leq M\}$. Now, we show that for small T the operator L in (3.29) is a contraction from $B(M,T)$ into itself.

If $q \in B(M,T)$, then, according to Theorem 2.4, the function W determined by (3.15)-(3.16) and therefore by (3.28) belongs to $\mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$. Note that $\varphi_{x_n x_n} \in \mathcal{APT}(\mathbf{R}^{n-1} \times \mathbf{R})$, $(\Delta h - h_t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, and $F_{x_n x_n} \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$. It follows from Lemma 2.3, Theorem 2.4, and (3.29) that $Lq \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ and

$$\begin{aligned} \|Lq\|_T \leq & \left\| [\Delta h - t_t + F(x,0,t)]h^{-1} + h^{-1} \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0) d\zeta \right. \\ & \left. + h^{-1} \int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s) d\zeta \right\|_T \\ & + \left\| h^{-1} \right\|_T \|q\|_T \|W\|_T m(T) \\ \leq & \frac{1}{2} M + \left\| h^{-1} \right\|_T M K_0(T) \left(\left\| \varphi_{x_n x_n} \right\| + T \|F_{x_n x_n}\|_T \right) m(T), \end{aligned} \tag{3.31}$$

where K_0 comes from Lemma 3.3. Noting that $m(T) \to 0$ ad $T \to 0$, we choose $T_1 \leq T_0$ such that when $T < T_1$ one has

$$\left\| h^{-1} \right\|_T K_0(T) \left(\left\| \varphi_{x_n x_n} \right\| + T \|F_{x_n x_n}\|_T \right) m(T) \leq \frac{1}{2}. \tag{3.32}$$

So, $Lq \in B(M,T)$. For $q_1, q_2 \in B(M,T)$, by (3.29)

$$\begin{aligned} \|Lq_1 - Lq_2\| & \leq \left\| h^{-1} \right\|_T \|W_1 q_1 - W_2 q_2\|_T m(T) \\ & \leq m(T) \left\| h^{-1} \right\|_T \left[\|W_1\|_T \|q_1 - q_2\|_T + \|q_2\|_T \|W_1 - W_2\|_T \right]. \end{aligned} \tag{3.33}$$

The function $V = W_1 - W_2$ is a solution of the Cauchy problem

$$\begin{gathered} V_t - \Delta V + q_1 V = W_2(q_2 - q_1), \quad (X,t) \in \mathbf{R}_T^n, \\ V(X,0) = 0, \quad X \in \mathbf{R}^n. \end{gathered} \tag{3.34}$$

Thus, by Lemma 3.3

$$\|V\|_T \leq K_1(T) \|W_2\|_T \|q_2 - q_1\|. \tag{3.35}$$

Applying Lemma 3.3 to (W_1, q_1) and (W_2, q_2), respectively, one gets

$$\|W_1\| \le K_1(T)\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right),$$
$$\|W_2\| \le K_2(T)\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right). \tag{3.36}$$

If we choose $T_2 \le T_1$ so that when $T \le T_2$

$$2m(T)\left\|h^{-1}\right\|_T\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right)K_1(T)[1 + MK_2(T)] \le 1, \tag{3.37}$$

then

$$\begin{aligned}\|Lq_1 - Lq_2\|_T &\le m(T)\|h^{-1}\|_T[\|W_1\| + MK_1(T)\|W_2\|]\|q_2 - q_1\|_T \\ &\le m(T)\|h^{-1}\|_T K_1(T)(1 + MK_2(T))(\|\varphi_{x_n x_n}\| + T\|F_{x_n x_n}\|_T)\|q_2 - q_1\|_T \\ &\le \frac{1}{2}\|q_2 - q_1\|_T.\end{aligned}$$

One sees that for such T, the operator L is a contraction from $B(M,T)$ into itself and, therefore, has a unique fixed point in $B(M,T)$. Thus, we have shown.

Theorem 3.5. *If functions F, φ, and h satisfy the conditions of Problem 1, M and T are determined by* (3.30) *and* (3.32), (3.37) *respectively, then in $\mathbf{R}^n_T$, Problem 1 has a unique solution (u, q) with $u \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q \in \mathcal{APT}(\overline{\mathbf{R}^{n-1}_T})$.*

Furthermore, we have the following.

Theorem 3.6. *Let F, φ, and h be as in Problem 1. Then, there exists an almost periodic type solution for Problem 1 in $\mathbf{R}^n_{T_0}$.*

Proof. We show that the conclusion of Theorem 3.5 can be extended to $\mathbf{R}^n_{T_0}$. Let $T = \sup\{s :$ Problem 1 has solution in $\mathbf{R}^n_s\}$. By Theorem 3.5, $T > 0$. Suppose that $T < T_0$. Consider the problem

$$u_t - \Delta u + qu = F(X,t), \quad X \in \mathbf{R}^n,\ T \le t \le T_0,$$
$$u(X,T) = f(X), \quad X \in \mathbf{R}^n, \tag{3.38}$$
$$u(x,0,t) = h(x,t), \quad (x,t) \in \mathbf{R}^{n-1} \times [T, T_0].$$

□

For q we can write the integral equation similar to (3.29), but this time its domain is $x \in \mathbf{R}^{n-1},\ t \in [T, T_0]$. As the proof above, define the ball $B_1(M,S)$ in $\mathcal{APT}(\mathbf{R}^{n-1} \times [T, T_0])$; then there exists a $t_0 > 0$ such that the operator L_{t_0} is a contraction from $B_1(M,S)$ into itself. So, (3.29) has a solution for the domain $x \in \mathbf{R}^{n-1},\ t \in [T, T + t_0]$. This contradicts the definition of T. We must have $T = T_0$.

For the stability, we have the following.

Theorem 3.7. *Let functions h_i, φ_i, and $F_i (i = 1, 2)$ be as in Problem 1. If $W_i(X, t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q_i \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ $(i = 1, 2)$ are solutions to (3.15)–(3.17), then*

$$\begin{aligned} \|q_2 - q_1\|_t \leq c_1 \|h_2 - h_1\|_t + c_2 \left\| \left(\frac{\partial}{\partial t} - \Delta \right) (h_2 - h_1) \right\|_t \\ + c_3 \left\| \frac{\partial^2}{\partial x_n^2} (\varphi_2 - \varphi_1) \right\| + c_4 \|F_2 - F_1\|_t + c_5 \left\| \frac{\partial^2}{\partial x_n^2} (F_2 - F_1) \right\|_t, \end{aligned} \tag{3.39}$$

where c_j $(1 \leq j \leq 5)$ depends on T, $\|h_1\|_t$, $\|(\partial^2/\partial x_n^2)\varphi_1\|$, $\|F_1\|_t$, $\|(\partial^2/\partial x_n^2)F_1\|_t$, and $\|q_i\|_t$ $(i = 1, 2)$ only.

Proof. By (3.29),

$$\begin{aligned} h_1(q_2 - q_1) = {} & -q_2(h_2 - h_1) - \left(\frac{\partial}{\partial t} - \Delta \right)(h_2 - h_1) + F_2(x, 0, t) - F_1(x, 0, t) \\ & + \int_{\mathbf{R}^n} \frac{\partial^2}{\partial \xi_n^2} (\varphi_2(\zeta) - \varphi_1(\zeta)) Z(x, 0, t; \zeta, 0) d\zeta \\ & + \int_0^t ds \int_{\mathbf{R}^n} \frac{\partial^2}{\partial \xi_n^2} (F_2 - F_1) Z(x, 0, t; \zeta, s) d\zeta \\ & - \int_0^t ds \int_{\mathbf{R}^n} [(W_2 - W_1)q_2 + W_1(q_2 - q_1)] Z(x, 0, t; \zeta, s) d\zeta. \end{aligned} \tag{3.40}$$

By Lemma 3.3,

$$\|W_1\|_t \leq K_1(t) \left[\left\| \frac{\partial^2 \varphi_1}{\partial x_n^2} \right\| + t \left\| \frac{\partial^2 F_1}{\partial x_n^2} \right\|_t \right]. \tag{3.41}$$

Since the function $V = W_2 - W_1$ is the solution of the Cauchy problem

$$\begin{aligned} V_t - \Delta V + q_2 V &= \frac{\partial^2}{\partial x_n^2} (F_2 - F_1) - W_1(q_2 - q_1), \quad (X, t) \in \mathbf{R}_T^n \\ V(X, 0) &= \frac{\partial^2}{\partial x_n^2} [\varphi_2(X) - \varphi_1(X)], \quad X \in \mathbf{R}^n, \end{aligned} \tag{3.42}$$

one has

$$\begin{aligned} \|V\|_t \leq K_2(T) \Bigg[& \left\| \frac{\partial^2}{\partial x_n^2} (\varphi_2 - \varphi_1) \right\| + t \left\| \frac{\partial^2}{\partial x_n^2} (F_2 - F_1) \right\|_t \\ & + K_1(t) \left(\left\| \frac{\partial^2}{\partial x_n^2} \varphi_1 \right\| + t \left\| \frac{\partial^2}{\partial x_n^2} F_1 \right\| \right) \|q_2 - q_1\|_t \Bigg]. \end{aligned} \tag{3.43}$$

Therefore,

$$\begin{aligned}
\|q_2-q_1\|_t \le \left\|h_1^{-1}\right\|_t \Bigg\{ &\|q_2\|_t\|h_2-h_1\|_t + \left\|\left(\frac{\partial}{\partial t}-\Delta\right)(h_2-h_1)\right\|_t + \|F_2(x,0,t)-F_1(x,0,t)\|_t \\
&+ \left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2-\varphi_1)\right\| + T\left\|\frac{\partial^2}{\partial x_n^2}(F_2-F_1)\right\|_t \\
&+ \int_0^t K_2(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2-\varphi_1)\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}(F_2-F_1)\right\|_s\right]\|q_2\|_s ds \\
&+ \int_0^t K_1(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_s\right]\|q_2-q_1\|_s ds \\
&+ \int_0^t K_1(s)K_2(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_s\right]\|q_2-q_1\|_s\|q_2\|_s ds \Bigg\}.
\end{aligned} \tag{3.44}$$

Using Lemma 3.1, we get the estimates desired if we let

$$\begin{aligned}
c_1 &= \left\|h_1^{-1}\right\|_t\|q_2\|_t\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\
c_2 &= \left\|h_1^{-1}\right\|_t\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\
c_3 &= \left\|h_1^{-1}\right\|_t\left[1+\int_0^t\|q_2\|_s K_2(s)ds\right]\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\
c_4 &= c_2 \qquad c_5 = c_3,
\end{aligned} \tag{3.45}$$

where

$$\chi(t) = \left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + t\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_t\right]K_1(t)\left(1+K_2(t)\|q_2\|_t\right). \tag{3.46}$$

The proof is complete. □

Corollary 3.8. *If Problem 1 has a solution in $\mathbf{R}^n_{T_0}$, then it has a unique one.*

Acknowledgment

The research is supported by the NSF of China (no. 11071048).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 546302, 33 pages
doi:10.1155/2012/546302

Research Article

Bounded Oscillation of a Forced Nonlinear Neutral Differential Equation

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Received 20 December 2011; Accepted 5 March 2012

Academic Editor: Miroslava Růžičková

This paper is concerned with the nth-order forced nonlinear neutral differential equation $[x(t)-p(t)x(\tau(t))]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\ldots,x(\sigma_{ik_i}(t)))=g(t),\ t\ge t_0$. Some necessary and sufficient conditions for the oscillation of bounded solutions and several sufficient conditions for the existence of uncountably many bounded positive and negative solutions of the above equation are established. The results obtained in this paper improve and extend essentially some known results in the literature. Five interesting examples that point out the importance of our results are also included.

1. Introduction

Consider the following nth-order forced nonlinear neutral differential equation:

$$\left[x(t)-p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\ldots,x(\sigma_{ik_i}(t)))=g(t),\quad t\ge t_0,\tag{1.1}$$

where $t_0\in\mathbb{R}$ and $n,m,k_i\in\mathbb{N}$ are constants for $1\le i\le m$. In what follows, we assume that

$(A1)$ $p,g,\tau,\sigma_{ij}\in C([t_0,+\infty),\mathbb{R})$ and $q_i\in C([t_0,+\infty),\mathbb{R}^+)$ satisfy that

$$\lim_{t\to+\infty}\tau(t)=\lim_{t\to+\infty}\sigma_{ij}(t)=+\infty,\quad 1\le j\le k_i,\ 1\le i\le m,\tag{1.2}$$

and there exists $1 \le i_0 \le m$ such that q_{i_0} is positive eventually:

$(A2)$ τ is strictly increasing and $\tau(t) < t$ in $[t_0, +\infty)$;

$(A3)$ $f_i \in C(\mathbb{R}^{k_i}, \mathbb{R})$ satisfies that

$$\begin{aligned} &f_i(u_1, u_2, \ldots, u_{k_i}) > 0, \quad \forall (u_1, u_2, \ldots, u_{k_i}) \in (\mathbb{R}^+ \setminus \{0\})^{k_i}, \\ &f_i(u_1, u_2, \ldots, u_{k_i}) < 0, \quad \forall (u_1, u_2, \ldots, u_{k_i}) \in (\mathbb{R}_- \setminus \{0\})^{k_i} \end{aligned} \tag{1.3}$$

for $1 \le i \le m$.

During the last decades, the oscillation criteria and the existence results of nonoscillatory solutions for various linear and nonlinear differential equations have been studied extensively, for example, see [1–28] and the references cited therein. In particular, Zhang and Yan [25] obtained some sufficient conditions for the oscillation of the first-order linear neutral delay differential equation with positive and negative coefficients:

$$\left[x(t) - p(t)x(t-\tau)\right]' + q(t)x(t-\sigma) - r(t)x(t-\delta) = 0, \quad t \ge t_0, \tag{1.4}$$

where $p, q, r \in C([t_0, +\infty), \mathbb{R}^+)$, $\tau > 0$, and $\sigma \ge \delta \ge 0$. Das and Misra [7] studied the nonhomogeneous neutral delay differential equation:

$$[x(t) - cx(t-\tau)]' + q(t)f(x(t-\sigma)) = g(t), \quad t \ge t_0, \tag{1.5}$$

where $q, g \in C([T, +\infty), \mathbb{R}^+ \setminus \{0\})$, $\sigma > 0$, $\tau > 0$, $c \in [0, 1)$, $f : \mathbb{R} \to \mathbb{R}$, $tf(t) > 0$ for $t \neq 0$, f is nondecreasing, Lipschitzian, and satisfies $\int_0^k (1/f(t))dt < +\infty$ for every $k > 0$, and they obtained a necessary and sufficient condition for the solutions of (1.5) to be oscillatory or tend to zero asymptotically. Parhi and Rath [18] extended Das and Misra's result to the following forced first-order neutral differential equation with variable coefficients:

$$\left[x(t) - p(t)x(t-\tau)\right]' + q(t)f(x(t-\sigma)) = g(t), \quad t \ge 0, \tag{1.6}$$

where $p \in C(\mathbb{R}^+, \mathbb{R})$, and they got necessary and sufficient conditions which ensures every solution of (1.6) is oscillatory or tends to zero or to $\pm\infty$ as $t \to +\infty$. By using Banach's fixed point theorem, Zhang et al. [24] proved the existence of a nonoscillatory solution for the first-order linear neutral delay differential equation:

$$\left[x(t) + p(t)x(t-\tau)\right]' + \sum_{i=1}^{n} f_i(t)x(t-\sigma_i) = 0, \quad t \ge t_0, \tag{1.7}$$

where $p \in C([t_0, +\infty), \mathbb{R})$, $\tau > 0$, $\sigma_i \in \mathbb{R}^+$, and $f_i \in C([t_0, +\infty), \mathbb{R})$ for $1 \le i \le m$. Çakmak and Tiryaki [6] showed several sufficient conditions for the oscillation of the forced second-order nonlinear differential equations with delayed argument in the form:

$$x''(t) + p(t)f(x(\alpha(t))) = g(t), \quad t \ge t_0 \ge 0, \tag{1.8}$$

where $p, \alpha, g \in C([t_0, +\infty), \mathbb{R})$, $\alpha(t) \le t$, $\lim_{t\to+\infty}\alpha(t) = +\infty$, and $f \in C(\mathbb{R}, \mathbb{R})$. Travis [20] investigated the oscillatory behavior of the second-order differential equation with functional argument:

$$x''(t) + p(t)f(x(t), x(\alpha(t))) = 0, \quad t \ge t_0, \tag{1.9}$$

where $p, \alpha \in C([t_0, +\infty), \mathbb{R})$ and $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies that $f(s,t)$ has the same sign of s and t when they have the same sign. Lin [12] got some sufficient conditions for oscillation and nonoscillation of the second order nonlinear neutral differential equation:

$$\left[x(t) - p(t)x(t-\tau)\right]'' + q(t)f(x(t-\sigma)) = 0, \quad t \ge 0, \tag{1.10}$$

where $p, q \in C(\mathbb{R}^+, \mathbb{R})$, $\overline{p} \in [0,1)$ with $0 \le p(t) \le \overline{p}$ eventually, $f \in C(\mathbb{R}, \mathbb{R})$, f is nondecreasing and $tf(t) > 0$ for $t \neq 0$. Kulenović and Hadžiomerspahić [9] deduced the existence of a nonoscillatory solution for the neutral delay differential equation of second order with positive and negative coefficients:

$$[x(t) + cx(t-\tau)]'' + q_1(t)x(t-\sigma_1) - q_2(t)x(t-\sigma_2) = 0, \quad t \ge t_0, \tag{1.11}$$

where $c \neq \pm 1$, $\tau > 0$, $\sigma_i \in \mathbb{R}^+$, $q_i \in C([t_0, +\infty), \mathbb{R}^+)$, and $\int_{t_0}^{+\infty} q_i(t)dt < +\infty$ for $i \in \{1,2\}$. Utilizing the fixed point theorems due to Banach, Schauder and Krasnoselskii, and Zhou and Zhang [27], and Zhou et al. [28] established some sufficient conditions for the existence of a nonoscillatory solution of the following higher-order neutral functional differential equations:

$$\begin{gathered}
[x(t) + cx(t-\tau)]^{(n)} + (-1)^{n+1}[P(t)x(t-\sigma) - Q(t)x(t-\delta)] = 0, \quad t \ge t_0, \\
\left[x(t) + p(t)x(t-\tau)\right]^{(n)} + \sum_{i=1}^{m} q_i(t)f_i(x(t-\sigma_i)) = g(t), \quad t \ge t_0,
\end{gathered} \tag{1.12}$$

where $c \in \mathbb{R} \setminus \{\pm 1\}$, $\tau, \sigma, \delta, \sigma_i \in \mathbb{R}^+$, $P, Q \in C([t_0, +\infty), \mathbb{R}^+)$, and $p, g, f_i \in C([t_0, +\infty), \mathbb{R})$ for $1 \le i \le m$. Li et al. [11] investigated the existence of an unbounded positive solution, bounded oscillation, and nonoscillation criteria for the following even-order neutral delay differential equation with unstable type:

$$\left[x(t) - p(t)x(t-\tau)\right]^{(n)} - q(t)|x(t-\sigma)|^{\alpha-1}x(t-\sigma) = 0, \quad t \ge t_0, \tag{1.13}$$

where $\tau > 0$, $\sigma > 0$, $\alpha \ge 1$, and $p, q \in C([t_0, +\infty), \mathbb{R}^+)$. Zhang and Yan [22] obtained some sufficient conditions for oscillation of all solutions of the even-order neutral differential equation with variable coefficients and delays:

$$\left[x(t) + p(t)x(\tau(t))\right]^{(n)} + q(t)x(\sigma(t)) = 0, \quad t \ge t_0, \tag{1.14}$$

where n is even, $p, q, \tau, \sigma \in C([t_0, +\infty), \mathbb{R}^+)$, $p(t) < 1$, $\tau(t) \le t$ and $\sigma(t) \le t$ for $t \in [t_0, +\infty)$, and $\lim_{t\to+\infty}\tau(t) = \lim_{t\to+\infty}\sigma(t) = +\infty$. Yilmaz and Zafer [21] discussed sufficient conditions for the existence of positive solutions and the oscillation of bounded solutions of the nth-order neutral type differential equations:

$$\begin{gathered}[x(t)+cx(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0,\\ \left[x(t)+p(t)x(\tau(t))\right]^{(n)}+q(t)f(x(\sigma(t)))=g(t), \quad t\geq t_0,\end{gathered} \tag{1.15}$$

where $c\in\mathbb{R}\setminus\{\pm1\}$, $\tau,\sigma\in C([t_0,+\infty),\mathbb{R}^+)$, $p,q,g\in C([t_0,+\infty),\mathbb{R})$, and $f\in C(\mathbb{R},\mathbb{R})$. Bolat and Akin [4, 5] got sufficient criteria for oscillatory behaviour of solutions for the higher-order neutral type nonlinear forced differential equations with oscillating coefficients:

$$\begin{gathered}\left[x(t)+p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=0, \quad t\geq t_0,\\ \left[x(t)+p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=g(t), \quad t\geq t_0,\end{gathered} \tag{1.16}$$

where $n\in\mathbb{N}\setminus\{1\}$, $m\in\mathbb{N}$, $p,f_i,g,\tau,\sigma_i\in C([t_0,+\infty),\mathbb{R})$, f_i is nondecreasing and $uf_i(u)>0$ for $u\neq0$, $\sigma_i\in C^1([t_0,+\infty),\mathbb{R})$, $\sigma_i'(t)>0$, $\sigma_i(t)\leq t$ for $t\in[t_0,+\infty)$, $\lim_{t\to+\infty}\tau(t)=\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $1\leq i\leq m$, and p and g are oscillating functions. Zhou and Yu [26] attempted to extend the result of Bolat and Akin [4] and established a necessary and sufficient condition for the oscillation of bounded solutions of the higher-order nonlinear neutral forced differential equation of the form:

$$\left[x(t)-p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=g(t), \quad t\geq t_0, \tag{1.17}$$

where $n\in\mathbb{N}\setminus\{1\}$, $m\in\mathbb{N}$, and

(C_1) $p,q_i,\tau,g\in C([t_0,+\infty),\mathbb{R})$ for $i=1,2,\ldots,m$ and $\lim_{t\to+\infty}\tau(t)=+\infty$;

(C_2) p and g are oscillating functions;

(C_3) $\sigma_i\in C([t_0,+\infty),\mathbb{R})$, $\sigma_i'(t)>0$, $\sigma_i(t)\leq t$ and $\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $i=1,2,\ldots,m$;

(C_4) $f_i\in C(\mathbb{R},\mathbb{R})$ is nondecreasing function, $uf_i(u)>0$ for $u\neq0$ and $i=1,2,\ldots,m$.

That is, they claimed the following result.

Theorem 1.1 (see [26, Theorem 2.1]). *Assume that*

(C_5) *there is an oscillating function* $r\in C([t_0,+\infty),\mathbb{R})$ *such that* $r^{(n)}(t)=g(t)$ *and* $\lim_{t\to+\infty}r(t)=0$;

(C_6) p *is an oscillating function and* $|p(t)|\leq p_0<1/2$;

(C_7) $q_i(t)\geq0$, $i=1,2,\ldots,m$.

Then, every bounded solution of (1.17) *either oscillates or tends to zero if and only if*

$$\int_{t_0}^{+\infty}s^{n-1}q_i(s)ds=+\infty, \quad i=1,2,\ldots,m. \tag{1.18}$$

We, unfortunately, point out that the necessary part in Theorem 1.1 is false, see Remark 4.2 and Example 4.7 below. It is clear that (1.1) includes (1.4)–(1.17) as special cases. To the best of our knowledge, there is no literature referred to the oscillation and existence of uncountably many bounded nonoscillatory solutions of (1.1). The aim of this paper is to establish the bounded oscillation and the existence of uncountably many bounded positive and negative solutions for (1.1) without the monotonicity of the nonlinear term f_i. Our results extend and improve substantially some known results in [4, 5, 9, 10, 20, 24, 26–28] and correct Theorem 2.1 in [26].

The paper is organized as follows. In Section 2, a few notation and lemmas are introduced and proved, respectively. In Section 3, by employing Krasnoselskii's fixed point theorem and some techniques, the existence of uncountably many bounded positive and negative solutions for (1.1) are given, and some necessary and sufficient conditions for all bounded solutions of (1.1) to be oscillatory or tend to zero as $t \to +\infty$ are provided. In Section 4, a number of examples which clarify advantages of our results are constructed.

2. Preliminaries

It is assumed throughout this paper that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$ and

$$\beta = \min\{t_0, \inf\{\tau(t), \sigma_{ij}(t) : t \in [t_0, +\infty), 1 \le j \le i_k, 1 \le i \le m\}\}. \tag{2.1}$$

By a solution of (1.1), we mean a function $x \in C([\beta, +\infty), \mathbb{R})$ for some $T \ge t_0 + \beta$, such that $x(t) - p(t)x(\tau(t))$ is n times continuously differentiable in $[T, +\infty)$ and such that (1.1) is satisfied for $t \ge T$. As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is nonoscillatory, that is, if it is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Let $BC([\beta, +\infty), \mathbb{R})$ stand for the Banach space of all bounded continuous functions in $[\beta, +\infty)$ with the norm $\|x\| = \sup_{t \ge \beta}|x(t)|$ for each $x \in BC([\beta, +\infty), \mathbb{R})$ and

$$A(N, M) = \{x \in BC([\beta, +\infty), \mathbb{R}) : N \le x(t) \le M, \ t \ge \beta\} \quad \text{for } M, N \in \mathbb{R} \text{ with } M > N. \tag{2.2}$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of the Banach space $BC([\beta, +\infty), \mathbb{R})$.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $x \in C^n([t_0, +\infty), \mathbb{R})$ be bounded. If $x^{(n)}(t) \le 0$ eventually, then*

(a) $\lim_{t \to +\infty} x(t)$ *exists and* $\lim_{t \to +\infty} x^{(i)}(t) = 0$ *for* $1 \le i \le n-1$; *furthermore, there exists* $\theta = 0$ *for n odd and* $\theta = 1$ *for n even such that*

(b) $(-1)^{\theta+i} x^{(i)}(t) \ge 0$ *eventually for* $1 \le i \le n$;

(c) $(-1)^{\theta+i} x^{(i)}$ *is nonincreasing eventually for* $0 \le i \le n-1$.

Proof. Now, we consider two possible cases below.

Case 1. Assume that $n = 1$. Let $\theta = 0$. Note that $x'(t) \le 0$ eventually. It follows that there exists a constant $t_1 > t_0$ satisfying $x'(t) \le 0$, for all $t \ge t_1$, which yields that x is nonincreasing in $[t_1, +\infty)$. Since x is bounded in $[t_0, +\infty)$, it follows that $\lim_{t \to +\infty} x(t)$ exists.

Case 2. Assume that $n \geq 2$. Notice that $\theta+n$ is odd. It follows that $(-1)^{\theta+n}x^{(n)}(t) \geq 0$ eventually, which implies that there exists a constant $t_1 > t_0$ satisfying

$$(-1)^{\theta+n}x^{(n)}(t) \geq 0, \quad \forall t \geq t_1, \tag{2.3}$$

which means that

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \text{ is nonincreasing in } [t_1, +\infty). \tag{2.4}$$

Suppose that there exists a constant $t_2 \geq t_1$ satisfying $(-1)^{\theta+n-1}x^{(n-1)}(t_2) < 0$, which together with (2.4) gives that

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \leq (-1)^{\theta+n-1}x^{(n-1)}(t_2) < 0, \quad \forall t \geq t_2, \tag{2.5}$$

which guarantees that $(-1)^{\theta+n-2}x^{(n-2)}(t)$ is increasing in $[t_2, +\infty)$ and

$$\begin{aligned} &(-1)^{\theta+n-1}x^{(n-2)}(t) - (-1)^{\theta+n-1}x^{(n-2)}(t_2) \\ &\quad = \int_{t_2}^{t} (-1)^{\theta+n-1}x^{(n-1)}(s)ds \leq (-1)^{\theta+n-1}x^{(n-1)}(t_2)(t-t_2) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \end{aligned} \tag{2.6}$$

that is,

$$\lim_{t\to+\infty} x^{(n-2)}(t) = -\infty, \tag{2.7}$$

which means that

$$\lim_{t\to+\infty} x^{(n-3)}(t) = \lim_{t\to+\infty} x^{(n-4)}(t) = \cdots = \lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} x(t) = -\infty, \tag{2.8}$$

which contradicts the boundedness of x. Consequently, we have

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \geq 0, \quad \forall t \geq t_1. \tag{2.9}$$

Combining (2.4) and (2.9), we conclude easily that there exists a constant $L \geq 0$ with

$$\lim_{t\to+\infty} (-1)^{\theta+n-1}x^{(n-1)}(t) = L. \tag{2.10}$$

Next, we claim that $L = 0$. Otherwise, there exists a constant $b > t_1$ satisfying

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \geq \frac{L}{2} > 0, \quad \forall t \geq b, \tag{2.11}$$

which yields that

$$\begin{aligned}&(-1)^{\theta+n-1}x^{(n-2)}(t)-(-1)^{\theta+n-1}x^{(n-2)}(b)\\&\quad=\int_b^t(-1)^{\theta+n-1}x^{(n-1)}(s)ds\geq\frac{L(t-b)}{2}\longrightarrow+\infty\quad\text{as } t\longrightarrow+\infty,\end{aligned}\tag{2.12}$$

which gives that

$$\lim_{t\to+\infty}x^{(n-2)}(t)=+\infty,\tag{2.13}$$

which means that

$$\lim_{t\to+\infty}x^{(n-3)}(t)=\lim_{t\to+\infty}x^{(n-4)}(t)=\cdots=\lim_{t\to+\infty}x'(t)=\lim_{t\to+\infty}x(t)=+\infty,\tag{2.14}$$

which contradicts the boundedness of x in $[t_0,+\infty)$. Hence, $L=0$, that is,

$$\lim_{t\to+\infty}x^{(n-1)}(t)=0.\tag{2.15}$$

Repeating the proof of (2.3)–(2.15), we deduce similarly that

$$\begin{gathered}(-1)^{\theta+j}x^{(j)}\text{ is nonincreasing and nonnegative in }[t_1,+\infty),\\\lim_{t\to+\infty}x^{(j)}(t)=0,\quad 1\leq j\leq n-1,\end{gathered}\tag{2.16}$$

which together with the boundedness of x implies that $(-1)^{\theta}x$ is nonincreasing in $[t_1,+\infty)$ and $\lim_{t\to+\infty}x(t)$ exists.

Thus, (2.3) and (2.16) yield (a)–(c). This completes the proof. □

Lemma 2.2. *Let $x,p,\tau,r,y\in C([t_0,+\infty),\mathbb{R})$ satisfy $(A2)$ and*

$$y(t)=x(t)-p(t)x(\tau(t))-r(t),\quad\forall t\geq t_0;\tag{2.17}$$

$$x\text{ is bounded and }\lim_{t\to+\infty}\tau(t)=+\infty;\tag{2.18}$$

$$\lim_{t\to+\infty}y(t)=\lim_{t\to+\infty}r(t)=0,\qquad|p(t)|\geq p_0>1\text{ eventually},\tag{2.19}$$

where p_0 is a fixed constant. Then, $\lim_{t\to+\infty}x(t)=0$.

Proof. Since τ is a strictly increasing continuous function, $\tau(t)<t$ in $[t_0,+\infty)$ and $\lim_{t\to+\infty}\tau(t)=+\infty$, it follows that the inverse function τ^{-1} of τ is also strictly increasing continuous, $\tau^{-1}(t)>t$ in $[\tau(t_0),+\infty)$ and $\lim_{j\to\infty}\tau^{-j}(t)=+\infty$, where $\tau^{-j}=\tau^{-(j-1)}(\tau^{-1})$ for all $j\in\mathbb{N}$. Equation (2.18) implies that there exists a constant $B>0$ with

$$|x(t)|\leq B,\quad\forall t\geq t_0.\tag{2.20}$$

Using (2.18) and (2.19), we deduce that, for any $\varepsilon > 0$, there exist sufficiently large numbers $T > 1 + |t_0|$ and $K \in \mathbb{N}$ satisfying

$$\frac{B}{p_0^K} < \frac{\varepsilon}{4}, \qquad \max\{|y(t)|, |r(t)|\} < \frac{\varepsilon(p_0 - 1)}{4}, \qquad |p(t)| \geq p_0, \quad \forall t \geq T. \tag{2.21}$$

In view of (2.17), (2.20), and (2.21), we infer that for all $t \geq T$

$$\begin{aligned}
|x(t)| &= \frac{|x(\tau^{-1}(t)) - y(\tau^{-1}(t)) - r(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} \\
&\leq \frac{|x(\tau^{-1}(t))| + |y(\tau^{-1}(t))| + |r(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} \\
&< \frac{1}{p_0}\left|x\left(\tau^{-1}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0} \\
&\leq \frac{1}{p_0}\left[\frac{1}{p_0}\left|x\left(\tau^{-2}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\right] + \frac{\varepsilon(p_0 - 1)}{2p_0} \\
&= \frac{1}{p_0^2}\left|x\left(\tau^{-2}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\left(1 + \frac{1}{p_0}\right) \\
&\leq \cdots \\
&\leq \frac{1}{p_0^K}\left|x\left(\tau^{-K}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\left(1 + \frac{1}{p_0} + \cdots + \frac{1}{p_0^{K-1}}\right) \\
&\leq \frac{B}{p_0^K} + \frac{\varepsilon(p_0 - 1)}{2p_0} \cdot \frac{1}{1 - 1/p_0} \\
&< \varepsilon,
\end{aligned} \tag{2.22}$$

which gives that $\lim_{t \to +\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3. *Let x, p, τ, r, and y be in $C([t_0, +\infty), \mathbb{R})$ satisfying $(A2)$, (2.17), (2.18), and*

$$\lim_{t \to +\infty} |y(t)| = d > 0, \qquad \lim_{t \to +\infty} r(t) = 0; \tag{2.23}$$

$$p_1 \geq |p(t)| \geq p_0 > 1 \text{ eventually}, \qquad p_0^2 > p_0 + p_1, \tag{2.24}$$

where d, p_0, and p_1 are constants. Then, there exists $L > 0$ such that $|x(t)| \geq L$ eventually.

Proof. Obviously, (2.20) holds. It follows from (2.18), (2.23), and (2.24) that for $\varepsilon = d[p_0(p_0 - 1) - p_1]/(p_0(p_0 - 1) + p_1) > 0$, there exist $K \in \mathbb{N}$ and $T > 1 + |t_0|$ satisfying

$$\frac{B}{p_0^K} < \frac{\varepsilon}{4p_1}, \quad d - \frac{\varepsilon}{4} < |y(t)| < d + \frac{\varepsilon}{4}, \quad |r(t)| < \frac{\varepsilon}{4p_0}, \quad p_1 \geq |p(t)| \geq p_0, \quad \forall t \geq T. \tag{2.25}$$

Put $L = d[p_0(p_0-1)-p_1]/2p_1p_0(p_0-1)$. In light of (2.17), we conclude that for each $t \geq T$

$$\begin{aligned}
x(t) &= \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&= \frac{1}{p(\tau^{-1}(t))}\left[\frac{x(\tau^{-2}(t))}{p(\tau^{-2}(t))} - \frac{y(\tau^{-2}(t))}{p(\tau^{-2}(t))} - \frac{r(\tau^{-2}(t))}{p(\tau^{-2}(t))}\right] - \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&= \frac{x(\tau^{-2}(t))}{\prod_{i=1}^{2} p(\tau^{-i}(t))} - \sum_{j=1}^{2}\frac{y(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} - \sum_{j=1}^{2}\frac{r(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} \\
&= \cdots \\
&= \frac{x(\tau^{-K}(t))}{\prod_{i=1}^{K} p(\tau^{-i}(t))} - \sum_{j=1}^{K}\frac{y(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} - \sum_{j=1}^{K}\frac{r(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))},
\end{aligned} \tag{2.26}$$

which together with (2.20) and (2.25) yields that for any $t \geq T$

$$\begin{aligned}
|x(t)| &\geq \frac{|y(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} - \frac{|x(\tau^{-K}(t))|}{\prod_{i=1}^{K} |p(\tau^{-i}(t))|} - \sum_{j=2}^{K}\frac{|y(\tau^{-j}(t))|}{\prod_{i=1}^{j} |p(\tau^{-i}(t))|} - \sum_{j=1}^{K}\frac{|r(\tau^{-j}(t))|}{\prod_{i=1}^{j} |p(\tau^{-i}(t))|} \\
&\geq \frac{d-\varepsilon/4}{p_1} - \frac{B}{p_0^K} - \left(d+\frac{\varepsilon}{4}\right)\sum_{j=2}^{K}\frac{1}{p_0^j} - \frac{\varepsilon}{4p_0}\sum_{j=1}^{K}\frac{1}{p_0^j} \\
&\geq \frac{d-\varepsilon/4}{p_1} - \frac{\varepsilon}{4p_1} - \left(d+\frac{\varepsilon}{4}\right)\cdot\frac{1/p_0^2}{1-1/p_0} - \frac{\varepsilon}{4p_0}\cdot\frac{1/p_0}{1-1/p_0} \\
&= \frac{d-\varepsilon/2}{p_1} - \frac{d+\varepsilon/2}{p_0(p_0-1)} = \frac{d[p_0(p_0-1)-p_1]-(\varepsilon/2)[p_0(p_0-1)+p_1]}{p_1p_0(p_0-1)} \\
&= L.
\end{aligned} \tag{2.27}$$

This completes the proof. □

Similar to the proof of Lemma 3.2 in [26], we have the following two lemmas.

Lemma 2.4. *Let x, p, τ, r, and y be in $C([t_0,+\infty),\mathbb{R})$ satisfying $(A2)$, (2.17), (2.18), and*

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} r(t) = 0; \tag{2.28}$$

$$|p(t)| \leq p_0 < \frac{1}{2} \text{ eventually}, \tag{2.29}$$

where p_0 is a constant. Then, $\lim_{t\to+\infty} x(t) = 0$.

Lemma 2.5. *Let x, p, τ, r, and y be in $C([t_0,+\infty),\mathbb{R})$ satisfying $(A2)$,* (2.17), (2.18), (2.23), *and* (2.29). *Then, there exists $L > 0$ such that $|x(t)| \geq L$ eventually.*

Lemma 2.6 (Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Y be a nonempty bounded closed convex subset of X, and let f, g be mappings of Y into X such that $fx + gy \in Y$ for every pair $x, y \in Y$. If f is a contraction mapping and g is completely continuous, then the mapping $f + g$ has a fixed point in Y.*

3. Main Results

First, we use the Krasnoselskii's fixed point theorem to show the existence and multiplicity of bounded positive and negative solutions of (1.1).

Theorem 3.1. *Let $(A1)$, $(A2)$, and $(A3)$ hold. Assume that there exist $p_0, p_1 \in \mathbb{R}^+\backslash\{0\}$, $r_0, r_1 \in \mathbb{R}^+$, and $r \in C^n([t_0,+\infty),\mathbb{R})$ satisfying*

$$p_1 \geq p(t) \geq p_0 > 1 \text{ eventually}, \qquad p_0^2 > p_0 + p_1; \tag{3.1}$$

$$r^{(n)}(t) = g(t), \qquad -r_0 \leq r(t) \leq r_1 \text{ eventually}; \tag{3.2}$$

$$\int_{t_0}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds < +\infty. \tag{3.3}$$

Then, the following hold:

(a) *for arbitrarily positive constants M and N with*

$$(p_0 - 1)M > (p_1 - 1)N + \frac{p_1 r_1}{p_0} + r_0, \tag{3.4}$$

equation (1.1) *has uncountably many bounded positive solutions $x \in A(N, M)$ with*

$$N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M; \tag{3.5}$$

(b) *for arbitrarily positive constants M and N with*

$$(p_0 - 1)N > (p_1 - 1)M + \frac{p_1 r_0}{p_0} + r_1, \tag{3.6}$$

equation (1.1) *has uncountably many bounded negative solutions $x \in A(-N, -M)$ with*

$$-N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq -M. \tag{3.7}$$

Proof. It follows from (3.1) and (3.2) that there exists an enough large constant T_0 with $\tau^{-1}(T_0) > 1 + |t_0| + |\beta|$ satisfying

$$p_0 \le p(t) \le p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \le r(t) \le r_1, \quad \forall t \ge T_0. \tag{3.8}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.4). Let $D \in ((p_1 - 1)N + (p_1 r_1/p_0)\ ,\ (p_0 - 1)M - r_0)$. First of all, we prove that there exist two mappings $F_D, G_D : A(N,M) \to BC([\beta,+\infty),\mathbb{R})$ and a constant $T_D > \tau^{-1}(T_0)$ such that $F_D + G_D$ has a fixed point $x \in A(N,M)$, which is also a bounded positive solution of (1.1) with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$. Put

$$B = \max\left\{\left|f_i(u_1, u_2, \ldots, u_{k_i})\right| : u_j \in [N,M], 1 \le j \le k_i, 1 \le i \le m\right\}. \tag{3.9}$$

In light of (3.3), (3.9), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B}{p_0(n-1)!}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\left\{M - \frac{D+M+r_0}{p_0}, \frac{D+N}{p_1} - \frac{r_1}{p_0} - N\right\}. \tag{3.10}$$

Define two mappings $F_D, G_D : A(N,M) \to C([\beta,+\infty),\mathbb{R})$ by

$$(F_D x)(t) = \begin{cases} \dfrac{D}{p(\tau^{-1}(t))} + \dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \dfrac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \ge T_D \\ (F_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.11}$$

$$(G_D x)(t) = \begin{cases} \dfrac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\ \quad \times \displaystyle\int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \\ \quad \times \displaystyle\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \ldots, x(\sigma_{ik_i}(s)))ds, & t \ge T_D, \\ (G_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.12}$$

for each $x \in A(N,M)$. In view of (3.1), (3.8), and (3.10)–(3.12), we conclude that for any $x,u \in A(N,M)$ and $t \ge T_D$

$$
\begin{aligned}
&|(F_Dx)(t)-(F_Du)(t)| = \left|\frac{x(\tau^{-1}(t))-u(\tau^{-1}(t))}{p(\tau^{-1}(t))}\right| \le \frac{1}{p_0}\|x-u\|,\\
&(F_Dx)(t)+(G_Du)(t)\\
&\quad= \frac{D}{p(\tau^{-1}(t))}+\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}+\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!}\\
&\qquad\times\int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(u(\sigma_{i1}(s)),u(\sigma_{i2}(s)),\dots,u(\sigma_{ik_i}(s)))ds\\
&\quad\le \frac{D}{p_0}+\frac{M}{p_0}+\frac{r_0}{p_0}+\frac{B}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&\quad< \frac{D+M+r_0}{p_0}+\min\left\{M-\frac{D+M+r_0}{p_0},\frac{D+N}{p_1}-\frac{r_1}{p_0}-N\right\}\\
&\quad\le M,\\
&(F_Dx)(t)+(G_Du)(t)\\
&\quad= \frac{D}{p(\tau^{-1}(t))}+\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}+\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!}\\
&\qquad\times\int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(u(\sigma_{i1}(s)),u(\sigma_{i2}(s)),\dots,u(\sigma_{ik_i}(s)))ds\\
&\quad\ge \frac{D}{p_1}+\frac{N}{p_1}-\frac{r_1}{p_0}-\frac{B}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&\quad> \frac{D+N}{p_1}-\frac{r_1}{p_0}-\min\left\{M-\frac{D+M+r_0}{p_0},\frac{D+N}{p_1}-\frac{r_1}{p_0}-N\right\}\\
&\quad\ge N,\\
&|(G_Du)(t)|\\
&\quad= \left|\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!}\right.\\
&\qquad\left.\times\int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(u(\sigma_{i1}(s)),u(\sigma_{i2}(s)),\dots,u(\sigma_{ik_i}(s)))ds\right|\\
&\quad\le \frac{B}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&\quad< \min\left\{M-\frac{D+M+r_0}{p_0},\frac{D+N}{p_1}-\frac{r_1}{p_0}-N\right\}\\
&\quad< M,
\end{aligned}
\tag{3.13}
$$

which ensures that

$$\|F_D x - F_D u\| = \sup_{t \geq T_D} |(F_D x)(t) - (F_D u)(t)| \leq \frac{1}{p_0}\|x - u\|, \quad \forall x, u \in A(N, M), \tag{3.14}$$

$$F_D x + G_D u \in A(N, M), \quad \forall x, u \in A(N, M), \tag{3.15}$$

$$\|G_D u\| \leq M, \quad \forall u \in A(N, M). \tag{3.16}$$

It follows from (3.11), (3.12), (3.15), and (3.16) that F_D and G_D map $A(N, M)$ into $BC([\beta, +\infty), \mathbb{R})$, respectively.

Now, we show that G_D is continuous in $A(N, M)$. Let $\{x_l\}_{l \in \mathbb{N}} \subset A(N, M)$ and $x \in A(N, M)$ with $\lim_{l \to \infty} x_l = x$, given $\varepsilon > 0$. It follows from the uniform continuity of f_i in $[N, M]^{k_i}$ for $1 \leq i \leq m$ and $\lim_{l \to \infty} x_l = x$ that there exist $\delta > 0$ and $K \in \mathbb{N}$ satisfying

$$\begin{aligned} &\left| f_i(u_{i1}, u_{i2}, \ldots, u_{ik_i}) - f_i(v_{i1}, v_{i2}, \ldots, v_{ik_i}) \right| \\ &\quad < \frac{\varepsilon}{1 + (1/p_0(n-1)!) \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^m q_i(s) ds}, \quad \forall u_{ij}, v_{ij} \in [N, M], \\ &\qquad\qquad\qquad\qquad \left| u_{ij} - v_{ij} \right| < \delta, \ 1 \leq j \leq k_i, \ 1 \leq i \leq m, \\ &\|x_l - x\| < \delta, \quad \forall l \geq K. \end{aligned} \tag{3.17}$$

In view of (3.8), (3.12), (3.17), we arrive at

$$\begin{aligned} &\|G_D x_l - G_D x\| \\ &\quad = \sup_{t \geq T_D} \Bigg| \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\ &\qquad\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^m q_i(s) \left[f_i(x_l(\sigma_{i1}(s)), x_l(\sigma_{i2}(s)), \ldots, x_l(\sigma_{ik_i}(s))) \right. \\ &\qquad\qquad\qquad \left. - f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \ldots, x(\sigma_{ik_i}(s))) \right] ds \Bigg| \\ &\quad \leq \sup_{t \geq T_D} \frac{1}{p_0(n-1)!} \\ &\qquad\quad \times \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^m q_i(s) \left| f_i(x_l(\sigma_{i1}(s)), x_l(\sigma_{i2}(s)), \ldots, x_l(\sigma_{ik_i}(s))) \right. \\ &\qquad\qquad\qquad \left. - f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \ldots, x(\sigma_{ik_i}(s))) \right| ds \\ &\quad \leq \frac{1}{p_0(n-1)!} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^m q_i(s) ds \cdot \frac{\varepsilon}{1 + 1/p_0(n-1)! \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^m q_i(s) ds} \\ &\quad < \varepsilon, \quad \forall l \geq K, \end{aligned} \tag{3.18}$$

which means that G_D is continuous in $A(N, M)$.

Next, we show that $G_D(A(N,M))$ is equicontinuous in $[\beta,+\infty)$. Let $\varepsilon>0$. Taking into account (3.3) and $(A2)$, we know that there exists $T^*>T_D$ satisfying

$$\frac{1}{p_0(n-1)!}\int_{\tau^{-1}(T^*)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds<\frac{\varepsilon}{4}. \tag{3.19}$$

Put

$$B_1=\max\left\{s^{n-1}\sum_{i=1}^{m} q_i(s):\tau^{-1}(T_D)\le s\le \tau^{-1}(T^*)\right\}. \tag{3.20}$$

It follows from the uniform continuity of $p\tau^{-1}$ and τ^{-1} in $[T_D,T^*]$ that there exists $\delta>0$ satisfying

$$\begin{aligned}
&\left|p\left(\tau^{-1}(t_1)\right)-p\left(\tau^{-1}(t_2)\right)\right|<\frac{\varepsilon p_0^2(n-1)!}{4\left[1+B\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\right]},\\
&\qquad\qquad\qquad \forall t_1,t_2\in[T_D,T^*] \text{ with } |t_1-t_2|<\delta;\\
&\left|\tau^{-1}(t_1)-\tau^{-1}(t_2)\right|<\frac{\varepsilon p_0(n-1)!}{4B\left[1+B_1+(n-1)\int_{\tau^{-1}(T_D)}^{+\infty} u^{n-1}\sum_{i=1}^{m} q_i(s)ds\right]},\\
&\qquad\qquad\qquad \forall t_1,t_2\in[T_D,T^*] \text{ with } |t_1-t_2|<\delta.
\end{aligned} \tag{3.21}$$

Let $x\in A(N,M)$ and $t_1,t_2\in[\beta,+\infty)$ with $|t_1-t_2|<\delta$. We consider three possible cases.

Case 1. Let $t_1,t_2\in[T^*,+\infty)$. In view of (3.8), (3.9), (3.12), and (3.19), we conclude that

$$\begin{aligned}
&|(G_Dx)(t_1)-(G_Dx)(t_2)|\\
&\quad=\frac{1}{(n-1)!}\left|\frac{1}{p(\tau^{-1}(t_1))}\right.\\
&\qquad\times\int_{\tau^{-1}(t_1)}^{+\infty}\left(s-\tau^{-1}(t_1)\right)^{n-1}\sum_{i=1}^{m} q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds\\
&\qquad-\frac{1}{p(\tau^{-1}(t_2))}\\
&\qquad\left.\times\int_{\tau^{-1}(t_2)}^{+\infty}\left(s-\tau^{-1}(t_2)\right)^{n-1}\sum_{i=1}^{m} q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{B}{p_0(n-1)!}\left[\int_{\tau^{-1}(t_1)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds + \int_{\tau^{-1}(t_2)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\right] \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{3.22}$$

Case 2. Let $t_1, t_2 \in [T_D, T^*]$. In terms of (3.8), (3.9), (3.12), (3.21), we arrive at

$$\begin{aligned}
&|(G_D x)(t_1) - (G_D x)(t_2)| \\
&\quad = \frac{1}{(n-1)!}\Bigg| \frac{1}{p(\tau^{-1}(t_1))} \\
&\qquad\qquad \times \int_{\tau^{-1}(t_1)}^{+\infty} \left(s-\tau^{-1}(t_1)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \\
&\qquad\qquad - \frac{1}{p(\tau^{-1}(t_2))} \\
&\qquad\qquad \times \int_{\tau^{-1}(t_2)}^{+\infty} \left(s-\tau^{-1}(t_2)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \Bigg| \\
&\quad \leq \frac{1}{(n-1)!}\Bigg\{ \left| \frac{1}{p(\tau^{-1}(t_1))} - \frac{1}{p(\tau^{-1}(t_2))} \right| \\
&\qquad\qquad \times \int_{\tau^{-1}(t_1)}^{+\infty} \left(s-\tau^{-1}(t_1)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \\
&\qquad\qquad + \frac{1}{p(\tau^{-1}(t_2))} \\
&\qquad\qquad \times \Bigg[\left| \int_{\tau^{-1}(t_1)}^{\tau^{-1}(t_2)} \left(s-\tau^{-1}(t_1)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \right| \\
&\qquad\qquad\quad + \int_{\tau^{-1}(t_2)}^{+\infty} \left| \left(s-\tau^{-1}(t_1)\right)^{n-1} - \left(s-\tau^{-1}(t_2)\right)^{n-1} \right| \\
&\qquad\qquad\quad \times \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \Bigg] \Bigg\} \\
&\quad \leq \frac{B}{(n-1)!}\Bigg\{ \frac{|p(\tau^{-1}(t_1)) - p(\tau^{-1}(t_2))|}{p(\tau^{-1}(t_1))p(\tau^{-1}(t_2))} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds + \frac{1}{p_0} \\
&\qquad\qquad \times \Bigg[\left| \int_{\tau^{-1}(t_1)}^{\tau^{-1}(t_2)} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \right| \\
&\qquad\qquad\quad + \int_{\tau^{-1}(t_2)}^{+\infty} (n-1)s^{\max\{n-2,0\}} \left|\tau^{-1}(t_1) - \tau^{-1}(t_2)\right| \sum_{i=1}^{m} q_i(s)ds \Bigg] \Bigg\}
\end{aligned}$$

$$\begin{aligned}&\leq \frac{B}{p_0^2(n-1)!}\left|p\left(\tau^{-1}(t_1)\right)-p\left(\tau^{-1}(t_2)\right)\right|\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\&\quad+\frac{B}{p_0(n-1)!}\left[B_1+(n-1)\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m}q_i(s)ds\right]\left|\tau^{-1}(t_1)-\tau^{-1}(t_2)\right|\\&<\frac{\varepsilon}{2}.\end{aligned} \tag{3.23}$$

Case 3. Let $t_1, t_2 \in [\beta, T_D]$. By (3.12), we have

$$|(G_Dx)(t_1)-(G_Dx)(t_2)| = |(G_Dx)(T_D)-(G_Dx)(T_D)| = 0 < \varepsilon. \tag{3.24}$$

Thus, $G_D(A(N,M))$ is equicontinuous in $[\beta,+\infty)$. Consequently, $G_D(A(N,M))$ is relatively compact by (3.16) and the continuity of G_D. By means of (3.14), (3.15), and Lemma 2.6, we infer that $F_D + G_D$ possesses a fixed point $x \in A(N,M)$, that is,

$$\begin{aligned}x(t) &= \frac{D}{p(\tau^{-1}(t))}+\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}+\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!}\\&\quad\times\int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\ldots,x(\sigma_{ik_i}(s)))ds,\quad \forall t\geq T_D,\end{aligned} \tag{3.25}$$

which gives that

$$\begin{aligned}x(t)-p(t)x(\tau(t)) &= -D+r(t)+\frac{(-1)^{n-1}}{(n-1)!}\\&\quad\times\int_{t}^{+\infty}(s-t)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\ldots,x(\sigma_{ik_i}(s)))ds,\\&\qquad\forall t\geq \tau^{-1}(T_D),\\ \left[x(t)-p(t)x(\tau(t))\right]^{(n)} &= g(t)-\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\ldots,x(\sigma_{ik_i}(t))),\quad \forall t\geq\tau^{-1}(T_D),\end{aligned} \tag{3.26}$$

which mean that $x \in A(N,M)$ is a bounded positive solution of (1.1) with

$$N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M. \tag{3.27}$$

Let D_1 and D_2 be two arbitrarily different numbers in $((p_1-1)N+(p_1r_1/p_0),\ (p_0-1)M-r_0)$. Similarly, we conclude that for each $l \in \{1,2\}$ there exist two mappings F_{D_j}, G_{D_j} :

$A(N,M) \to BC([\beta,+\infty),\mathbb{R})$ and a sufficiently large number $T_{D_l} > \tau^{-1}(T_0)$ satisfying (3.8)–(3.12), where D, T_D, F_D, and G_D are replaced by D_l, T_{D_l}, F_{D_l}, and G_{D_l}, respectively, and $F_{D_l} + G_{D_l}$ has a fixed point $x_l \in A(N,M)$, which is also a bounded positive solution with $N \le \liminf_{t\to+\infty} x_l(t) \le \limsup_{t\to+\infty} x_l(t) \le M$, that is,

$$x_l(t) = \frac{D_l}{p(\tau^{-1}(t))} + \frac{x_l(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x_l(\sigma_{i1}(s)), x_l(\sigma_{i2}(s)), \dots, x_l(\sigma_{ik_i}(s)))ds, \quad \forall t \ge T_{D_l}. \tag{3.28}$$

It follows from (3.3) that there exists $T_3 > \max\{T_{D_1}, T_{D_2}\}$ satisfying

$$\frac{B}{p_0(n-1)!}\int_{\tau^{-1}(T_3)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \frac{|D_1-D_2|}{4p_1}. \tag{3.29}$$

Combining (3.8), (3.28), and (3.29), we conclude easily that

$$\begin{aligned}
&|x_1(t)-x_2(t)| \\
&= \Bigg| \frac{D_1-D_2}{p(\tau^{-1}(t))} + \frac{x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \\
&\qquad \times \sum_{i=1}^{m} q_i(s)\left[f_i(x_1(\sigma_{i1}(s)), x_1(\sigma_{i2}(s)), \dots, x_1(\sigma_{ik_i}(s)))\right. \\
&\qquad\qquad \left. - f_i(x_2(\sigma_{i1}(s)), x_2(\sigma_{i2}(s)), \dots, x_2(\sigma_{ik_i}(s)))\right]ds \Bigg| \\
&\ge \frac{|D_1-D_2|}{p(\tau^{-1}(t))} - \frac{\left|x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))\right|}{p(\tau^{-1}(t))} - \frac{2B}{p(\tau^{-1}(t))(n-1)!}\int_{\tau^{-1}(t)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&\ge \frac{|D_1-D_2|}{p_1} - \frac{\|x_1-x_2\|}{p_0} - \frac{2B}{p_0(n-1)!}\int_{\tau^{-1}(T_3)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&> \frac{|D_1-D_2|}{p_1} - \frac{\|x_1-x_2\|}{p_0} - \frac{|D_1-D_2|}{2p_1} \\
&= \frac{|D_1-D_2|}{2p_1} - \frac{\|x_1-x_2\|}{p_0}, \quad \forall t \ge T_3,
\end{aligned} \tag{3.30}$$

which guarantees that

$$\|x_1 - x_2\| \geq \frac{p_0|D_1 - D_2|}{2p_1(1+p_0)} > 0, \tag{3.31}$$

that is, $x_1 \neq x_2$. Hence, (1.1) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M$.

(b) Assume that M and N are arbitrary positive constants satisfying (3.6) and put

$$B_2 = \max\left\{\left|f_i(u_1, u_2, \ldots, u_{k_i})\right| : u_j \in [-N, -M], 1 \leq j \leq k_i, 1 \leq i \leq m\right\}. \tag{3.32}$$

Let $D \in ((1-p_0)N + r_1, (1-p_1)M - (p_1 r_0/p_0))$. It follows from (3.3), (3.8), (3.32), and $(A2)$ that there exists $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B_2}{p_0(n-1)!}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\left\{-M + \frac{M-D}{p_1} - \frac{r_0}{p_0}, N + \frac{D-N-r_1}{p_0}\right\}. \tag{3.33}$$

Let the mappings $F_D, G_D : A(-N, -M) \to C([\beta, +\infty), \mathbb{R})$ be defined by (3.11) and (3.12), respectively.

Using (3.1), (3.8), (3.11), (3.12), and (3.33), we deduce that for any $x, u \in A(-N, -M)$ and $t \geq T_D$

$$\begin{aligned}
&(F_D x)(t) + (G_D u)(t)\\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!}\\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty}\left(s - \tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s)))ds\\
&\quad \leq \frac{D}{p_1} - \frac{M}{p_1} + \frac{r_0}{p_0} + \frac{B_2}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\\
&\quad < \frac{D-M}{p_1} + \frac{r_0}{p_0} + \min\left\{-M + \frac{M-D}{p_1} - \frac{r_0}{p_0},\ N + \frac{D-N-r_1}{p_0}\right\}\\
&\quad \leq -M,
\end{aligned}$$

$$
\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s)))ds \\
&\quad \geq \frac{D}{p_0} - \frac{N}{p_0} - \frac{r_1}{p_0} - \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds \\
&\quad > \frac{D-N-r_1}{p_0} - \min\left\{ -M + \frac{M-D}{p_1} - \frac{r_0}{p_0},\ N + \frac{D-N-r_1}{p_0} \right\} \\
&\quad \geq -N,
\end{aligned}
\tag{3.34}
$$

which give that

$$
F_D x + G_D u \in A(-N,-M), \quad \forall x, u \in A(-N,-M). \tag{3.35}
$$

The rest of the proof is similar to the proof of (a) and is omitted. This completes the proof. □

Theorem 3.2. *Let* $(A1)$, $(A2)$, *and* $(A3)$, *hold. Assume that there exist* $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$, $r_0, r_1 \in \mathbb{R}^+$, *and* $r \in C^n([t_0,+\infty),\mathbb{R})$ *satisfying* (3.2), (3.3), *and*

$$
p_1 \geq -p(t) \geq p_0 > 1 \text{ eventually.} \tag{3.36}
$$

Then, the following hold:

(a) *for arbitrarily positive constants* M *and* N *with*

$$
N < M, \qquad \left(p_0^2 - p_1\right)M > \left(p_1 - \frac{p_0}{p_1}\right)p_0 N + p_0 r_1 + p_1 r_0, \tag{3.37}
$$

equation (1.1) *has uncountably many bounded positive solutions* $x \in A(N,M)$ *with*

$$
N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M; \tag{3.38}
$$

(b) *for arbitrarily positive constants* M *and* N *with*

$$
M < N, \qquad \left(p_0^2 - p_1\right)N > \left(p_1 - \frac{p_0}{p_1}\right)p_0 M + p_1 r_1 + p_0 r_0, \tag{3.39}
$$

equation (1.1) *has uncountably many bounded negative solutions* $x \in A(-N,-M)$ *with*

$$-N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le -M. \tag{3.40}$$

Proof. It follows from (3.2) and (3.36) that there exists a constant T_0 with $\tau(T_0) > 1 + |t_0| + |\beta|$ satisfying

$$p_0 \le -p(t) \le p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \le r(t) \le r_1, \quad \forall t \ge T_0. \tag{3.41}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.37). Let $D \in (p_1((M+r_0)/p_0 + N),\ p_0(N/p_1 + M) - r_1)$ and B be defined by (3.9). In light of (3.3), (3.9), and $(A2)$, there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B}{p_0(n-1)!}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\left\{M - \frac{D+r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M+r_0}{p_0} - N\right\}. \tag{3.42}$$

Define two mappings $F_D, G_D : A(N,M) \to C([\beta,+\infty),\mathbb{R})$ by (3.12) and

$$(F_Dx)(t) = \begin{cases} -\dfrac{D}{p(\tau^{-1}(t))} + \dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \dfrac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \ge T_D \\ (F_Dx)(T_D), & \beta \le t < T_D \end{cases} \tag{3.43}$$

for each $x \in A(N,M)$. In view of (3.12), (3.36), and (3.41)–(3.43), we conclude that for any $x, u \in A(N,M)$ and $t \ge T_D$

$$\begin{aligned}
&(F_Dx)(t) + (G_Du)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s)))ds \\
&\quad \le \frac{D}{p_0} - \frac{N}{p_1} + \frac{r_1}{p_0} + \frac{B}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&\quad < \frac{D}{p_0} - \frac{N}{p_1} + \frac{r_1}{p_0} + \min\left\{M - \frac{D+r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M+r_0}{p_0} - N\right\} \\
&\quad \le M,
\end{aligned}$$

$$\begin{aligned}
&(F_Dx)(t) + (G_Du)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s))) ds \\
&\quad \geq \frac{D}{p_1} - \frac{M}{p_0} - \frac{r_0}{p_0} - \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad > \frac{D}{p_1} - \frac{M+r_0}{p_0} - \min\left\{ M - \frac{D+r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M+r_0}{p_0} - N \right\} \\
&\quad \geq N,
\end{aligned} \tag{3.44}$$

which imply (3.15). The rest of the proof is similar to that of Theorem 3.1 and is omitted.

(b) Assume that M and N are arbitrary positive constants satisfying (3.39). Let $D \in (-p_0(N+(M/p_1))M+r_0,\ -Mp_1-(p_1/p_0)(N+r_1))$ and B_2 be defined by (3.32). Note that (3.3), (3.32), and $(A2)$ yield that there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds < \min\left\{ -M - \frac{D}{p_1} - \frac{N+r_1}{p_0}, N + \frac{D-r_0}{p_0} + \frac{M}{p_1} \right\}. \tag{3.45}$$

Let the mappings $F_D, G_D : A(-N,-M) \to C([\beta,+\infty),\mathbb{R})$ be defined by (3.12) and (3.43), respectively.

Using (3.12), (3.36), (3.41), and (3.45), we infer that for any $x, u \in A(N,M)$ and $t \geq T_D$

$$\begin{aligned}
&(F_Dx)(t) + (G_Du)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s))) ds \\
&\quad \leq \frac{D}{p_1} + \frac{N}{p_0} + \frac{r_1}{p_0} + \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad < \frac{D}{p_1} + \frac{N}{p_0} + \frac{r_1}{p_0} + \min\left\{ -M - \frac{D}{p_1} - \frac{N+r_1}{p_0}, N + \frac{D-r_0}{p_0} + \frac{M}{p_1} \right\} \\
&\quad \leq -M,
\end{aligned}$$

$$
\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s)\ f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\quad \geq \frac{D}{p_0} + \frac{M}{p_1} - \frac{r_0}{p_0} - \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad > \frac{D}{p_0} + \frac{M}{p_1} - \frac{r_0}{p_0} - \min\left\{ -M - \frac{D}{p_1} - \frac{N + r_1}{p_0}, N + \frac{D - r_0}{p_0} + \frac{M}{p_1} \right\} \\
&\quad \geq -N,
\end{aligned}
\tag{3.46}
$$

which give (3.15). The rest of the proof is similar to the proof of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.3. *Let* $(A1)$ *and* $(A3)$ *hold. Assume that there exist* $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$, $r_0, r_1 \in \mathbb{R}^+$, *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (3.2), (3.3), *and*

$$-p_0 \leq p(t) \leq p_1 \text{ eventually}, \qquad p_0 + p_1 < 1. \tag{3.47}$$

Then, the following hold:

(a) for arbitrarily positive constants M *and* N *with*

$$r_0 + r_1 + N < (1 - p_0 - p_1)M, \tag{3.48}$$

equation (1.1) *has uncountably many bounded positive solutions* $x \in A(N, M)$ *with*

$$N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq M; \tag{3.49}$$

for arbitrarily positive constants M *and* N *with*

$$r_0 + r_1 + M < (1 - p_0 - p_1)N, \tag{3.50}$$

equation (1.1) *has uncountably many bounded negative solutions* $x \in A(-N, -M)$ *with*

$$-N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq -M. \tag{3.51}$$

Proof. It follows from (3.2) and (3.47) that there exists a constant $T_0 > 1 + |t_0| + |\beta|$ satisfying

$$-p_0 \leq p(t) \leq p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \leq r(t) \leq r_1, \quad \forall t \geq T_0. \tag{3.52}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.48). Let $D \in (p_0M + r_0 + N,\ (1 - p_1)M_1 - r_1)$ and B be defined by (3.9). In light of (3.3), (3.9), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \max\{T_0, \tau(T_0)\}$ satisfying

$$\frac{B}{p_0(n-1)!}\int_{T_D}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\{M - D - p_1M - r_1, D - p_0M - r_0 - N\}. \tag{3.53}$$

Define two mappings $F_D, G_D : A(N, M) \to C([\beta, +\infty), \mathbb{R})$ by

$$(F_Dx)(t) = \begin{cases} D + p(t)x(\tau(t)) + r(t), & t \geq T_D, \\ (F_Dx)(T_D), & \beta \leq t < T_D, \end{cases} \tag{3.54}$$

$$(G_Dx)(t)\begin{cases} \dfrac{(-1)^{n-1}}{(n-1)!} \\ \quad\times\displaystyle\int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \ldots, x(\sigma_{ik_i}(s)))ds, & t \geq T_D \\ (G_Dx)(T_D), & \beta \leq t < T_D, \end{cases} \tag{3.55}$$

for each $x \in A(N, M)$. In view of (3.47) and (3.52)–(3.55), we conclude that for any $x, u \in A(N, M)$ and $t \geq T_D$

$$\begin{aligned}
|(F_Dx)(t) - (F_Du)(t)| &\leq |p(t)(x(\tau(t)) - u(\tau(t)))| \leq (p_0 + p_1)\|x - u\|, \\
(F_Dx)(t) + (G_Du)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad \times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s)))ds \\
&\leq D + p_1M + r_1 + \frac{B}{(n-1)\,!}\int_t^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&< D + p_1M + r_1 + \min\{M - D - p_1M - r_1, D - p_0M - r_0 - N\} \leq M, \\
(F_Dx)(t) + (G_Du)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad \times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s)))ds
\end{aligned}$$

$$\begin{aligned}
&\geq D-p_0M-r_0-\frac{B}{(n-1)!}\int_t^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&> D-p_0M-r_0-\min\{M-D-p_1M-r_1,D-p_0M-r_0-N\}\\
&\geq N,
\end{aligned}\tag{3.56}$$

which yield (3.15). The rest of the proof is similar to that of Theorem 3.1 and is omitted.

(b) Assume that M and N are arbitrary positive constants satisfying (3.50). Let $D \in (r_0-(1-p_1)N-M-N,\ p_0-r_1)$ and B_2 be defined by (3.32). In light of (3.3), (3.32), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \max\{T_0, \tau(T_0)\}$ satisfying

$$\frac{B_2}{p_0(n-1)!}\int_{T_D}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds<\min\{-M-D-p_0N-r_1,D+N(1-p_1)-r_0\}.\tag{3.57}$$

Define two mappings $F_D, G_D : A(-N,-M) \rightarrow C([\beta,+\infty),\mathbb{R})$ by (3.54) and (3.55). In view of (3.47), (3.52), (3.54), (3.55), and (3.57), we conclude that (3.56) holds and

$$\begin{aligned}
(F_Dx)(t)+(G_Du)(t) &= D+p(t)x(\tau(t))+r(t)+\frac{(-1)^{n-1}}{(n-1)!}\\
&\quad\times\int_t^{+\infty}(s-t)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(u(\sigma_{i1}(s)),u(\sigma_{i2}(s)),\ldots,u(\sigma_{ik_i}(s)))ds\\
&\leq D+p_0N+r_1+\frac{B}{(n-1)\,!}\int_t^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&< D+p_0N+r_1+\min\{-M-D-p_0N-r_1,D+N(1-p_1)-r_0\}\\
&\leq -M,\quad \forall x,u\in A(N,M),\ t\geq T_D,\\
(F_Dx)(t)+(G_Du)(t) &= D+p(t)x(\tau(t))+r(t)+\frac{(-1)^{n-1}}{(n-1)!}\\
&\quad\times\int_t^{+\infty}(s-t)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(u(\sigma_{i1}(s)),u(\sigma_{i2}(s)),\ldots,u(\sigma_{ik_i}(s)))ds\\
&\geq D-p_1N-r_0-\frac{B}{(n-1)!}\int_t^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\\
&> D-p_1N-r_0-\min\{-M-D-p_0N-r_1,D+N(1-p_1)-r_0\}\\
&\geq -N,\quad \forall x,u\in A(N,M),\ t\geq T_D.
\end{aligned}\tag{3.58}$$

Thus, (3.15) follows from (3.58). The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Second, we provide necessary and sufficient conditions for the oscillation of bounded solutions of (1.1).

Theorem 3.4. *Let* $(A1)$, $(A2)$, *and* $(A3)$ *hold. Assume that there exist* $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$ *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (2.24) *and*

$$\lim_{t \to +\infty} r(t) = 0, \qquad r^{(n)}(t) = g(t) \text{ eventually.} \tag{3.59}$$

Then, each bounded solution of (1.1) *either oscillates or tends to 0 as* $t \to +\infty$ *if and only if*

$$\int_{t_0}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds = +\infty. \tag{3.60}$$

Proof.

Sufficiency. Suppose, without loss of generality, that (1.1) possesses a bounded eventually positive solution x with $\limsup_{t \to +\infty} x(t) > 0$, which together with $(A1)$, $(A3)$, (2.17), (2.24), and (3.60), yields that there exist constants $M > 0$ and $T > 1 + |t_0| + |\beta|$ satisfying

$$0 < x(t) \leq M, \quad \forall t \geq T; \tag{3.61}$$

$$y^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) f_i(x(\sigma_{i1}(t)), x(\sigma_{i2}(t)), \ldots, x(\sigma_{ik_i}(t))) < 0, \quad \forall t \geq T. \tag{3.62}$$

Obviously (2.17), (2.24), (3.59), and the boundedness of x imply that y is bounded. It follows from (2.17), (3.62), Lemmas 2.1 and 2.2 that there exists a constant L satisfying

$$\lim_{t \to +\infty} y(t) = L \neq 0, \quad \lim_{t \to +\infty} y^{(i)}(t) = 0, \quad 1 \leq i \leq n-1. \tag{3.63}$$

Thus, $(A1)$, (3.61), (3.63), and Lemma 2.3 imply that there exist constants N and $T_1 \geq T_0 \geq T$ satisfying

$$\begin{gathered} \inf\{\sigma_{ij}(t) : t \geq T_1, 1 \leq j \leq k_i, 1 \leq i \leq m\} \geq T_0, \\ [7pt] 0 < N \leq x(t), \quad \left|y(t) - L\right| < 1, \quad \forall t \geq T_1. \end{gathered} \tag{3.64}$$

Put

$$B_3 = \min\{f_i(u_1, u_2, \ldots, u_{k_i}) : u_j \in [N, M],\ 1 \leq j \leq k_i,\ 1 \leq i \leq m\}. \tag{3.65}$$

Clearly, $(A3)$ guarantees that $B_3 > 0$. Integrating (3.62) from t to $+\infty$, by (3.63) and (3.64), we have

$$y^{(n-1)}(t) = (-1)^2 \int_t^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1)))du_1, \quad \forall t \geq T_1, \tag{3.66}$$

repeating this procedure, we obtain that

$$\begin{aligned}
y^{(n-2)}(t) &= (-1)^3 \int_t^{+\infty} du_2 \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1)))du_1, \quad \forall t \geq T_1, \\
&\dots \\
y'(t) &= (-1)^n \int_t^{+\infty} du_{n-1} \int_{u_{n-1}}^{+\infty} du_{n-2} \cdots \\
&\quad \times \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1)))du_1, \quad \forall t \geq T_1, \\
L - y(t) &= \lim_{u \to +\infty} y(u) - y(t) \\
&= (-1)^n \int_t^{+\infty} du_n \int_{u_n}^{+\infty} du_{n-1} \cdots \\
&\quad \times \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1)))du_1 \\
&= \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds, \quad \forall t \geq T_1,
\end{aligned} \tag{3.67}$$

which together with (3.64) and $(A3)$ means that

$$\begin{aligned}
1 > \left|L - y(t)\right| &= \left| \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds \right| \\
&\geq \frac{B_3}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s)ds, \quad \forall t \geq T_1,
\end{aligned} \tag{3.68}$$

which gives that

$$\int_{T_1}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds < +\infty, \tag{3.69}$$

which contradicts (3.60).

Necessity. Suppose that (3.60) does not hold. Observe that $\lim_{t\to+\infty} r(t) = 0$ implies that there exist two positive constants r_0 and r_1 satisfying

$$-r_0 \le r(t) \le r_1 \text{ eventually.} \tag{3.70}$$

It follows from Theorem 3.1 or Theorem 3.2 that, for any positive constants M and N satisfying (3.4) or (3.37), (1.1) possesses uncountably many bounded positive solutions $x \in A(N, M)$ with $M \ge \limsup_{t\to+\infty} x(t) \ge \liminf_{t\to+\infty} x(t) \ge N$. This is a contradiction. This completes the proof. □

As in the proof of Theorem 3.4, by means of Lemmas 2.1, 2.4, and 2.5, we have

Theorem 3.5. *Let* $(A1)$ *and* $(A3)$ *hold. Assume that there exist* $p_0 \in \mathbb{R}^+ \setminus \{0\}$ *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (2.29) *and* (3.59). *Then, each bounded solution of* (1.1) *either oscillates or tends to 0 as* $t \to +\infty$ *if and only if* (3.60) *holds.*

4. Remarks and Examples

Now, we compare the results in Section 3 with some known results in the literature. In order to illustrate the advantage and applications of our results, five nontrivial examples are constructed.

Remark 4.1. Theorems 3.1–3.3 extend and improve the Theorem in [9], Theorem 8.4.2 in [10], Theorem 1 in [21], Theorems 1–3 in [24], Theorem 2.2 in [26], and Theorems 1–4 in [27, 28].

Remark 4.2. The sufficient part of Theorem 3.5 is a generalization of Theorem 3.1 in [4, 5]. Theorem 3.5 corrects and perfects Theorem 2.1 in [26].

The examples below show that our results extend indeed the corresponding results in [4, 5, 9, 10, 21, 24, 26–28]. Notice that none of the known results can be applied to these examples.

Example 4.3. Consider the nth-order forced nonlinear neutral differential equation:

$$\begin{aligned}
&\left[x(t) - \frac{3+4t^n}{1+t^n}x\left(\sqrt{t}\right)\right]^{(n)} + \frac{\left(1+\sqrt{3+2t}\right)\left[x^5(3t+\sin t) + x^3(t-1/t)\right]}{(1+t^{n+3})\left[1+\left|x^8(3t^2) - 2x^{21}\left(t-\sqrt{t-1}\right)\right|\right]} \\
&\quad + \frac{tx(3t-\ln t)x^4(t^2-t)x^6(t-2) + 5tx\left(t(1+1/t)^t\right)}{(1+3t^{3n+1})\left[1+|x^3(4t-\cos^3 t) - 4x^4(t-1)|\right]} = \frac{1}{2}\sin\left(t+\frac{n\pi}{2}\right), \quad t \ge 2,
\end{aligned} \tag{4.1}$$

where $t_0 = 2$, $m = 2$, and $n \in \mathbb{N}$. Put $k_1 = 4$, $k_2 = 6$, $\beta = 0$, $r_0 = r_1 = 1/2$, $p_0 = 3$, $p_1 = 4$,

$$\begin{gathered}
p(t) = \frac{3+4t^n}{1+t^n}, \qquad q_1(t) = \frac{1+\sqrt{3+2t}}{1+t^{n+3}}, \qquad q_2(t) = \frac{t}{1+3t^{3n+1}}, \\
g(t) = \frac{1}{2}\sin\left(t+\frac{n\pi}{2}\right), \qquad r(t) = \frac{1}{2}\sin t, \qquad \tau(t) = \sqrt{t}, \qquad \sigma_{11}(t) = 3t+\sin t,
\end{gathered}$$

$$\sigma_{12}(t)=t-\frac{1}{t},\qquad \sigma_{13}(t)=3t^2,\qquad \sigma_{14}(t)=t-\sqrt{t-1},\qquad \sigma_{21}(t)=3t-\ln t,$$
$$\sigma_{22}(t)=t^2-t,\qquad \sigma_{23}(t)=t-2,\qquad \sigma_{24}(t)=t\left(1+\frac{1}{t}\right)^t,\qquad \sigma_{25}(t)=4t-\cos^3 t,$$
$$\sigma_{26}(t)=t-1,\qquad f_1(u,v,w,z)=\frac{u^5+v^3}{1+|w^8-2z^{21}|},$$
$$f_2(u,v,w,z,y,s)=\frac{uv^4w^6+5z}{1+|y^3-4s^4|},\quad \forall(t,u,v,w,z,y,s)\in[t_0,+\infty)\times\mathbb{R}^6. \tag{4.2}$$

Clearly $(A1)$, $(A2)$, $(A3)$, and (3.1)–(3.3) hold.

Let M and N be arbitrarily positive constants satisfying $M>(3/2)N+7/12$. It is easy to verify that (3.4) holds. It follows from Theorem 3.1 that (4.1) has uncountably many bounded positive solutions $x\in A(N,M)$ with $N\le\liminf_{t\to+\infty}x(t)\le\limsup_{t\to+\infty}x(t)\le M$.

Let M and N be arbitrarily positive constants satisfying $N>3M/2+7/12$. It is easy to verify that (3.6) holds. It follows from Theorem 3.1 that (4.1) has uncountably many bounded negative solutions $x\in A(-N,-M)$ with $-N\le\liminf_{t\to+\infty}x(t)\le\limsup_{t\to+\infty}x(t)\le -M$.

Example 4.4. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t)+\frac{8+10t^5}{2+t^5}x\left(\sqrt{t^2+1}-1\right)\right]^{(n)}+\frac{(t^2+3t^3)x^7(3t^2)x(t-1)x^3(t\ln t)}{\left(2+\sin^3(t^2)+t^{n+5}\right)\left[1+x^2(t-1)x^4(t\ln t)\right]}$$
$$+\frac{(3+t^2)\left[x^5(t^2+1)+7x^3(t^4-2)+x^9\left(t+\sqrt{t}\right)x^8(t-4)\right]}{\left(\sqrt{t+1}+t^{n+3}\right)\left[1+\left(x^5\left(t+\sqrt{t}\right)-4x^4(t-4)-3\right)^6\right]}=\frac{(-1)^n n!}{t^{n+1}}+\frac{n!}{(1-t)^{n+1}},\quad t\ge 3, \tag{4.3}$$

where $t_0=3$, $m=2$, and $n\in\mathbb{N}$. Put $k_1=3$, $k_2=4$, $\beta=-1$, $r_0=1/2$, $r_1=0$, $p_0=4$, $p_1=10$,

$$p(t)=-\frac{8+10t^5}{2+t^5},\qquad q_1(t)=\frac{t^2+3t^3}{2+\sin^3(t^2)+t^{n+5}},\qquad q_2(t)=\frac{3+t^2}{\sqrt{t+1}+t^{n+3}},$$
$$g(t)=\frac{(-1)^n n!}{t^{n+1}}+\frac{n!}{(1-t)^{n+1}},\qquad r(t)=\frac{1}{t(1-t)},\qquad \tau(t)=\sqrt{t^2+1}-1,$$
$$\sigma_{11}(t)=3t^2,\qquad \sigma_{12}(t)=t-1,\qquad \sigma_{13}(t)=t\ln t,\qquad \sigma_{21}(t)=t^2+1, \tag{4.4}$$
$$\sigma_{22}(t)=t^4-2,\qquad \sigma_{23}(t)=t+\sqrt{t},\qquad \sigma_{24}(t)=t-4,\qquad f_1(u,v,w)=\frac{u^7vw^3}{1+v^2w^4},$$
$$f_2(u,v,w,z)=\frac{u^5+7v^3+w^9z^8}{1+\left(w^5-4z^4-3\right)^6},\quad \forall(t,u,v,w,z)\in[t_0,+\infty)\times\mathbb{R}^4.$$

Clearly $(A1)$, $(A2)$, $(A3)$, (3.2), (3.3), and (3.36) hold.

Let M and N be arbitrarily positive constants satisfying $M > (32/5)N + 5/6$. It is easy to verify that (3.37) holds. It follows from Theorem 3.2 that (4.3) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$.

Let M and N be arbitrarily positive constants satisfying $N > (32/5)M + 1/3$. It is easy to verify that (3.39) holds. It follows from Theorem 3.2 that (4.3) has uncountably many bounded negative solutions $x \in A(-N, -M)$ with $-N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le -M$.

Example 4.5. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t) - \frac{2\sin\left(t^2-\sqrt{t}\right)}{5+\sin\left(t^2-\sqrt{t}\right)} x\left((t-5)^2\right)\right]^{(n)} + \frac{\left(3\sqrt{t-1}+t^5\right)x^3(t-4)}{\left(\sqrt{t^2+1}+t^{n+6}\right)\left[2+\cos^5 x\left(\sqrt{t+1}-3\right)\right]}$$
$$+ \frac{\left(1-\sqrt{t+1}\ln^2 t+t^4\right)x^9\left(2t+\sin(t^2+1)\right)}{(1-2t^3+3t^4+t^{n+5})\ln\left[2+x^2\left(t^2\sqrt{1+2t}\right)\right]} = (-1)^n\cos\left(t+\frac{n\pi}{2}\right), \quad t \ge 1, \tag{4.5}$$

where $t_0 = 1$, $m = 2$, and $n \in \mathbb{N}$. Put $k_1 = k_2 = 2$, $\beta = -4$, $r_0 = r_1 = 1$, $p_0 = 1/2$, $p_1 = 1/3$,

$$\begin{gathered} p(t) = \frac{2\sin\left(t^2-\sqrt{t}\right)}{5+\sin\left(t^2-\sqrt{t}\right)}, \qquad q_1(t) = \frac{3\sqrt{t-1}+t^5}{\sqrt{t^2+1}+t^{n+6}}, \qquad q_2(t) = \frac{1-\sqrt{t+1}\ln^2 t+t^4}{1-2t^3+3t^4+t^{n+5}}, \\ g(t) = (-1)^n\cos\left(t+\frac{n\pi}{2}\right), \qquad r(t) = (-1)^n\cos t, \qquad \tau(t) = (t-4)^2 \qquad \sigma_{11}(t) = t-5, \\ \sigma_{12}(t) = \sqrt{t+1}-3, \qquad \sigma_{21}(t) = 2t+\sin\left(t^2+1\right), \qquad \sigma_{22}(t) = t^2\sqrt{1+2t}, \\ f_1(u,v) = \frac{u^3}{2+\cos^5 v}, \qquad f_2(u,v) = \frac{u^9}{\ln(2+v^2)}, \quad \forall (t,u,v) \in [t_0,+\infty)\times\mathbb{R}^2. \end{gathered} \tag{4.6}$$

Clearly $(A1)$, $(A3)$, (3.2), (3.3), and (3.47) hold.

Let M and N be arbitrarily positive constants satisfying $M > 6N + 12$. It is easy to verify that (3.48) holds. It follows from Theorem 3.3 that (4.5) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$.

Let M and N be arbitrarily positive constants satisfying $N > 6M + 12$. It is easy to verify that (3.50) holds. It follows from Theorem 3.3 that (4.5) has uncountably many bounded negative solutions $x \in A(-N, -M)$ with $-N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le -M$.

Example 4.6. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t) - \frac{(-1)^n\left(5+9\ln^2 t\right)}{1+\ln^2 t} x\left(\sqrt{t}-1\right)\right]^{(n)} + \left(t^8+9t^5+3\right)\left[2x^3\left(t\ln^2 t\right)+5x^7(t-16)\right]$$

$$+t^2\left(2+\sin\left(t^3-5t\right)\right)x^9\left(t-\frac{\sin t}{t}\right)\left[x^4(t-\cos t)+4x^6\left(\frac{1+t+t^2+t^3}{1+t+t^2}\right)\right]^2$$
$$+\frac{\left[(t+1)^2-\sqrt{t}\right]x^5\left(t\arctan(t^3+1)/\left(1+\sqrt{t+1}\right)\right)\ln\left(1+x^6(t+1)/(1+x^2(t-2))\right)}{\left[2t^{n+3}+\sqrt{t}\sin(3t^5-1)\right]\left[1+x^2(t^2-t)x^4(t^2+t)\right]}$$
$$=\frac{(-1)^n n!\left(\ln t-\sum_{i=1}^{n}(1/i)\right)}{t^{n+1}},\quad t\geq 4, \tag{4.7}$$

where $t_0=4$, $m=3$, and $n\in\mathbb{N}$. Put $k_1=2$, $k_2=3$, $k_3=5$, $\beta=-12$, $p_0=5$, $p_1=9$,

$$\begin{gathered}
p(t)=\frac{(-1)^n\left(5+9\ln^2 t\right)}{1+\ln^2 t},\qquad q_1(t)=t^8+9t^5+3,\qquad q_2(t)=t^2\left(2+\sin\left(t^3-5t\right)\right),\\
q_3=\frac{(t+1)^2-\sqrt{t}}{2t^{n+3}+\sqrt{t}\sin(3t^5-1)},\qquad g(t)=\frac{(-1)^n n!\left(\ln t-\sum_{i=1}^{n}1/i\right)}{t^{n+1}},\qquad r(t)=\frac{\ln t}{t},\\
\tau(t)=\sqrt{t}-1,\qquad \sigma_{11}(t)=t\ln^2 t,\qquad \sigma_{12}(t)=t-16,\qquad \sigma_{21}(t)=t-\frac{\sin t}{t},\\
\sigma_{22}(t)=t-\cos t,\qquad \sigma_{23}(t)=\frac{1+t+t^2+t^3}{1+t+t^2},\qquad \sigma_{31}(t)=\frac{t\arctan(t^3+1)}{1+\sqrt{t+1}},\\
\sigma_{32}(t)=t+1,\qquad \sigma_{33}(t)=t-2,\qquad \sigma_{34}(t)=t^2-t,\qquad \sigma_{35}(t)=t^2+t,\\
f_1(u,v)=2u^3+5v^7,\qquad f_2(u,v,w)=u^9\left(v^4+4w^6\right)^2,\\
f_3(u,v,w,y,z)=\frac{u^5\ln\left(1+v^6/(1+w^2)\right)}{1+y^2z^4},\quad \forall(t,u,v,w,y,z)\in[t_0,+\infty)\times\mathbb{R}^5.
\end{gathered} \tag{4.8}$$

Clearly $(A1)$, $(A2)$, $(A3)$, (2.24), (3.59), and (3.60) hold. It follows from Theorem 3.4 that each bounded solution of (4.7) either oscillates or tends to 0 as $t\to+\infty$.

Example 4.7. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t)-\frac{(-1)^n\cos^3(3t-1)}{4+\cos^3(3t-1)}x(t-\sin t)\right]^{(n)}+\frac{\left(t^3+2t^2-\sqrt{t}+1\right)x^5\left(\sqrt{t-2}-1\right)}{1+x^2\left(\sqrt{t-2}-1\right)}$$
$$+\frac{\sqrt{t^2-1}\left[x^3(t-1/t)+5x^7(t-1/t)\right]\ln\left(2+x^6(t-1/t)\right)}{t^{2n+1}+2t^n\ln^3(1+t^2)+1}=\frac{2^n\sin\left(\sqrt{2}t+n\pi/4\right)}{e^{\sqrt{2}t}},\quad t\geq 6, \tag{4.9}$$

where $t_0 = 6,\ m = 2$, and $n \in \mathbb{N}$. Put $k_1 = k_2 = 1,\ \beta = 1,\ p_0 = 1/3$,

$$p(t) = \frac{(-1)^n \cos^3(3t-1)}{4+\cos^3(3t-1)}, \qquad q_1(t) = t^3 + 2t^2 - \sqrt{t} + 1,$$
$$q_2(t) = \frac{\sqrt{t^2-1}}{t^{2n+1} + 2t^n \ln^3(1+t^2) + 1}, \qquad g(t) = \frac{2^n \sin\left(\sqrt{2}t + n\pi/4\right)}{e^{\sqrt{2}t}}, \qquad r(t) = \frac{\sin\left(\sqrt{2}t\right)}{e^{\sqrt{2}t}},$$
$$\tau(t) = t - \sin t, \qquad \sigma_1(t) = \sqrt{t-2} - 1 \qquad \sigma_2(t) = t - \frac{1}{t}, \qquad f_1(u) = \frac{u^5}{1+u^2},$$
$$f_2(u) = u^3 + 5u^7 \ln\left(2+u^6\right), \quad \forall (t,u) \in [t_0, +\infty) \times \mathbb{R}. \tag{4.10}$$

Clearly $(A1)$, $(A3)$, (2.29), (3.59), and (3.60) hold. It follows from Theorem 3.5 that each bounded solution of (4.9) either oscillates or tends to 0 as $t \to +\infty$.

Next, we prove that the necessary part of Theorem 2.1 in [26] does not hold by means of (4.9). It is easy to verify that the conditions of Theorem 2.1 in [26] are fulfilled. Suppose that the necessary part of Theorem 2.1 in [26] is true. Because each bounded solution of (4.9) either oscillates or tends to 0 as $t \to +\infty$, it follows that the necessary part of Theorem 2.1 in [26] gives that

$$\int_{t_0}^{+\infty} s^{n-1} q_i(s)\,ds = +\infty, \quad i \in \{1,2\}, \tag{4.11}$$

which yields that

$$+\infty = \int_{t_0}^{+\infty} s^{n-1} q_2(s)\,ds = \int_{t_0}^{+\infty} \frac{s^{n-1}\sqrt{s^2-1}}{s^{2n+1} + 2s^n \ln^3(1+s^2) + 1}\,ds \le \int_{t_0}^{+\infty} \frac{1}{s^{n+1}}\,ds < +\infty, \tag{4.12}$$

which is a contradiction.

Acknowledgments

The authors would like to thank the referees for useful comments and suggestions. This study was supported by research funds from Dong-A University.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 764351, 12 pages
doi:10.1155/2012/764351

Research Article

Asymptotic Upper and Lower Estimates of a Class of Positive Solutions of a Discrete Linear Equation with a Single Delay

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Received 20 February 2012; Accepted 1 March 2012

Academic Editor: Elena Braverman

We study a frequently investigated class of linear difference equations $\Delta v(n) = -p(n)v(n-k)$ with a positive coefficient $p(n)$ and a single delay k. Recently, it was proved that if the function $p(n)$ is bounded above by a certain function, then there exists a positive vanishing solution of the considered equation, and the upper bound was found. Here we improve this result by finding even the lower bound for the positive solution, supposing the function $p(n)$ is bounded above and below by certain functions.

1. Introduction

Throughout this paper, we use the following notation: for an integer q, we define

$$\mathbb{Z}_q^\infty := \{q, q+1, \ldots\}. \tag{1.1}$$

We investigate the asymptotic behavior as $n \to \infty$ of the solutions of the discrete delayed equation of the $(k+1)$-th order

$$\Delta v(n) = -p(n)v(n-k), \tag{1.2}$$

where n is the independent variable assuming values from the set $\mathbb{Z}_a^\infty$ with a fixed $a \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The number $k \in \mathbb{N}$, $k \geq 1$ is the fixed delay, $\Delta v(n) = v(n+1) - v(n)$, and $p : \mathbb{Z}_a^\infty \to \mathbb{R}^+ = (0, \infty)$.

Along with (1.2), we consider $k+1$ initial conditions

$$v(a+s-k) = v^{a+s-k} \in \mathbb{R}, \quad s = 0, 1, \ldots, k. \tag{1.3}$$

Initial problem (1.2), (1.3) obviously has a unique solution, defined for every $n \in \mathbb{Z}_{a-k}^\infty$. Moreover, the solution of (1.2) continuously depends on initial conditions (1.3).

Equation (1.2) is investigated very frequently. It was analyzed, for example, in papers [1–3] (where the comparison method [4, 5] was used) and [6]. Similar problems for differential and dynamic equations are studied, for example, in [7–10].

In a recent work of the authors [6], it is proved that if the function $p(n)$ is bounded above by a certain function, then there exists a positive vanishing (i.e., tending to 0 as $n \to \infty$) solution of the considered equation. Moreover, its upper bound was found. Our aim is to improve this result and to show that if the coefficient $p(n)$ is between two functions $p_\ell(n) - \psi(n)$ and $p_\ell(n) + \omega(n)$ (see (2.3), (2.6), and (2.7) below) then (1.2) has a positive vanishing solution which is bounded from below by the function $\alpha_\ell(n)$ (see (2.5)) and from above by the function $\nu_\ell(n)$ (see (2.4)). Due to the linearity of equation considered it becomes clear that a similar result holds for a one-parametric family of positive vanishing solutions of (1.2).

To prove this, we will use Theorem 1.1 which is one of the main results of [6]. This theorem is valid for any delayed difference equation of the form:

$$\Delta v(n) = f(n, v(n), v(n-1), \ldots, v(n-k)). \tag{1.4}$$

Theorem 1.1. *Let $b(n)$, $c(n)$, $b(n) < c(n)$ be real functions defined on $\mathbb{Z}_{a-k}^\infty$. Further, let $f : \mathbb{Z}_a^\infty \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function and let the inequalities:*

$$b(n) + f(n, b(n), v_2, \ldots, v_{k+1}) < b(n+1), \tag{1.5}$$

$$c(n) + f(n, c(n), v_2, \ldots, v_{k+1}) > c(n+1), \tag{1.6}$$

hold for every $n \in \mathbb{Z}_a^\infty$ and every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1. \tag{1.7}$$

Then there exists a solution $v = v^*(n)$ of (1.4) satisfying the inequalities

$$b(n) < v^*(n) < c(n), \tag{1.8}$$

for every $n \in \mathbb{Z}_{a-k}^\infty$.

For related comparison theorems for solutions of difference equations as well as related methods and their applications, see, for example, [1, 11–21] and the related references therein. Investigation of positive solutions (and connected problems of oscillating solutions) attracted recently large attention. Except the references given above, one refers as well to

[11, 22–33], and to the references therein. Existence of positive solutions of some classes of difference equations has been also studied in papers [12–16]. The existence of unbounded solutions by some comparison methods can be found, for example, in [17, 18].

2. Auxiliary Functions and Lemmas

Define the expression $\ln_q n$, $q \in \mathbb{N} \setminus \{0\}$, as

$$\ln_q n := \ln(\ln_{q-1} n), \tag{2.1}$$

where $\ln_0 n := n$. We will write only $\ln n$ instead of $\ln_1 n$. Further, for a fixed integer $\ell \geq 0$ define auxiliary functions:

$$\mu_\ell(n) := \frac{1}{8n^2} + \frac{1}{8(n\ln n)^2} + \cdots + \frac{1}{8(n\ln n \cdots \ln_\ell n)^2}, \tag{2.2}$$

$$p_\ell(n) := \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right), \tag{2.3}$$

$$\nu_\ell(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n\ln n \ln_2 n \cdots \ln_\ell n}, \tag{2.4}$$

$$\alpha_\ell(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n\ln n \ln_2 n \cdots \ln_\ell n \ln_{\ell+1}^{-\sigma} n}, \tag{2.5}$$

where $\sigma \in \mathbb{R}$, $\sigma > 0$, is a constant. Notice that if a is sufficiently large, all these functions are well defined for $n \in \mathbb{Z}_a^\infty$.

Finally, let functions $\psi, \omega : \mathbb{Z}_a^\infty \to \mathbb{R}^+$ satisfy for $n \in \mathbb{Z}_a^\infty$ the inequalities:

$$\psi(n) \leq \left(\frac{k}{k+1}\right)^k \cdot \frac{\delta}{(n\ln n \cdots \ln_\ell n)^2 \ln_{\ell+1}^\beta n}, \tag{2.6}$$

$$\omega(n) \leq \varepsilon\left(\frac{k}{k+1}\right)^k \cdot \frac{k(2k-1)}{16n^3}, \tag{2.7}$$

for fixed $\delta > 0$, $\beta > 2$ and $\varepsilon \in (0,1)$.

In [3], it was proved that if $p(n)$ in (1.2) is a positive function bounded above by $p_\ell(n)$ for some $\ell \geq 0$, then there exists a positive solution of (1.2) bounded above by the function $\nu_\ell(n)$ for n sufficiently large. Since $\lim_{n\to\infty}\nu_\ell(n) = 0$, such solution will vanish as $n \to \infty$. This result was further improved in [6], where it was shown that (1.2) has a positive solution bounded above by $\nu_\ell(n)$ even if the coefficient $p(n)$ satisfies a less restrictive inequality, namely, $p(n) < p_\ell(n) + \omega(n)$. Here we will prove that function α_ℓ provides the lower estimate of the solution, supposing $p_\ell(n) - \psi(n) \leq p(n) \leq p_\ell(n) + \omega(n)$. The proof of this statement will be based on the following four lemmas. The symbols "o" and "O" stand for the Landau order symbols and are used as $n \to \infty$.

Lemma 2.1. *For fixed $r \in \mathbb{R} \setminus \{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representation:*

$$\begin{aligned}\ln_q(n-r) = \ln_q n &- \frac{r}{n \ln n \cdots \ln_{q-1} n} - \frac{r^2}{2n^2 \ln n \cdots \ln_{q-1} n} \\ &- \frac{r^2}{2(n\ln n)^2 \ln_2 n \cdots \ln_{q-1} n} - \cdots - \frac{r^2}{2(n \ln n \cdots \ln_{q-1} n)^2} + o\left(\frac{1}{n^3}\right),\end{aligned} \tag{2.8}$$

holds as $n \to \infty$.

Proof. Relation (2.8) can be proved by induction with respect to q, for details, see [6]. □

Lemma 2.2. *For fixed $r, s \in \mathbb{R} \setminus \{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representations:*

$$\begin{aligned}\left(\frac{\ln_q(n-r)}{\ln_q n}\right)^s = 1 &- \frac{rs}{n \ln n \cdots \ln_q n} - \frac{r^2 s}{2n^2 \ln n \cdots \ln_q n} - \frac{r^2 s}{2(n \ln n)^2 \ln_2 n \cdots \ln_q n} - \cdots \\ &- \frac{r^2 s}{2(n \ln n \cdots \ln_{q-1} n)^2 \ln_q n} + \frac{r^2 s(s-1)}{2(n \ln n \cdots \ln_q n)^2} + o\left(\frac{1}{n^3}\right),\end{aligned} \tag{2.9}$$

$$\sqrt{\frac{n-r}{n}} = 1 - \frac{r}{2n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right), \tag{2.10}$$

hold as $n \to \infty$.

Proof. Both these relations are simple consequences of the asymptotic formula:

$$(1-x)^s = 1 - sx + \frac{s(s-1)}{2}x^2 - \frac{s(s-1)(s-2)}{6}x^3 + o\left(x^3\right) \quad \text{as } x \longrightarrow 0. \tag{2.11}$$

and of Lemma 2.1 (for formula (2.9)).

In the case of relation (2.10), we put $x = r/n$ and $s = 1/2$.
To prove relation (2.9), first notice that dividing (2.8) by $\ln_q n$, we get

$$\begin{aligned}\frac{\ln_q(n-r)}{\ln_q n} = 1 &- \frac{r}{n \ln n \cdots \ln_{q-1} n \ln_q n} - \frac{r^2}{2n^2 \ln n \cdots \ln_{q-1} n \ln_q n} \\ &- \frac{r^2}{2(n \ln n)^2 \ln_2 n \cdots \ln_{q-1} n \ln_q n} - \cdots - \frac{r^2}{2(n \ln n \cdots \ln_{q-1} n)^2 \ln_q n} + o\left(\frac{1}{n^3}\right).\end{aligned} \tag{2.12}$$

Thus, putting

$$\begin{aligned} x &= 1 - \frac{\ln_q(n-r)}{\ln_q n} \\ &= \frac{r}{n \ln n \cdots \ln_q n} + \frac{r^2}{2n^2 \ln n \cdots \ln_q n} + \cdots + \frac{r^2}{2(n \ln n \cdots \ln_{q-1} n)^2 \ln_q n} + o\left(\frac{1}{n^3}\right) \end{aligned} \tag{2.13}$$

and using (2.11), we get (2.9). □

The following lemma is proved in [6].

Lemma 2.3. *For fixed $r \in \mathbb{R} \setminus \{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representation:*

$$\begin{aligned} &\sqrt{\frac{(n-r)}{n} \cdot \frac{\ln(n-r)}{\ln n} \cdots \frac{\ln_q(n-r)}{\ln_q n}} \\ &\quad = 1 - r\left(\frac{1}{2n} + \frac{1}{2n \ln n} + \cdots + \frac{1}{2n \ln n \cdots \ln_q n}\right) - r^2 \mu_q(n) - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right), \end{aligned} \tag{2.14}$$

holds as $n \to \infty$.

Lemma 2.4. *For fixed $r \in \mathbb{R} \setminus \{0\}$, $\sigma \in \mathbb{R}^+$ and $q \in \mathbb{N}$, the asymptotic representation:*

$$\begin{aligned} &\sqrt{\frac{(n-r)}{n} \cdot \frac{\ln(n-r)}{\ln n} \cdots \frac{\ln_q(n-r)}{\ln_q n} \cdot \frac{\ln_{q+1}^{-\sigma}(n-r)}{\ln_{q+1}^{-\sigma} n}} \\ &\quad = 1 - r\left(\frac{1}{2n} + \frac{1}{2n \ln n} + \cdots + \frac{1}{2n \ln n \cdots \ln_q n} - \frac{\sigma}{2n \ln n \cdots \ln_q n \ln_{q+1} n}\right) \\ &\qquad - r^2 \mu_q(n) + \frac{\sigma(\sigma+2)}{8} \cdot \frac{r^2}{(n \ln n \cdots \ln_{q+1} n)^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right), \end{aligned} \tag{2.15}$$

holds as $n \to \infty$.

Proof. Using Lemma 2.2 with $s = -\sigma/2$ and $q+1$ instead of q, we get for $n \to \infty$

$$\begin{aligned} \sqrt{\frac{\ln_{q+1}^{-\sigma}(n-r)}{\ln_{q+1}^{-\sigma} n}} &= 1 + \frac{r\sigma}{2n \ln n \cdots \ln_{q+1} n} + \frac{r^2\sigma}{4n^2 \ln n \cdots \ln_{q+1} n} + \frac{r^2\sigma}{4(n \ln n)^2 \ln_2 n \cdots \ln_{q+1} n} + \cdots \\ &\quad + \frac{r^2\sigma}{4(n \ln n \cdots \ln_q n)^2 \ln_{q+1} n} + \frac{r^2\sigma(\sigma+2)}{8(n \ln n \cdots \ln_{q+1} n)^2} + o\left(\frac{1}{n^3}\right). \end{aligned} \tag{2.16}$$

Multiplying the asymptotic representations (2.14) and (2.16), we get (2.15). □

3. Main Result

Now we are ready to prove that there exists a positive solution of (1.2) which is bounded below and above. Remind the functions p_ℓ, ν_ℓ, α_ℓ, ψ, and ω were defined by (2.3)–(2.6) and (2.7), respectively.

Theorem 3.1. *Suppose that there exist numbers $a, \ell \in \mathbb{N}$, and $\sigma > 0$, such that the function p in* (1.2) *satisfies the inequalities*

$$0 < p_\ell(n) - \psi(n) \le p(n) \le p_\ell(n) + \omega(n), \tag{3.1}$$

for every $n \in \mathbb{Z}_a^\infty$. Then there exists a solution $v = v^*(n),\ n \in \mathbb{Z}_{a-k}^\infty$ of (1.2) *such that for n sufficiently large the inequalities:*

$$\alpha_\ell(n) < v^*(n) < \nu_\ell(n), \tag{3.2}$$

hold.

Proof. Show that all the assumptions of Theorem 1.1 are fulfilled. For (1.2),

$$f(n, v_1, \ldots, v_{k+1}) = -p(n) v_{k+1}. \tag{3.3}$$

This is a continuous function. Put

$$b(n) := \alpha_\ell(n), \qquad c(n) := \nu_\ell(n). \tag{3.4}$$

We have to prove that for every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1, \tag{3.5}$$

the inequalities (1.5) and (1.6) hold for n sufficiently large. Start with (1.5). That gives that for

$$b(n-k) < v_{k+1} < c(n-k), \tag{3.6}$$

it has to be

$$\alpha_\ell(n) - p(n) \cdot v_{k+1} < \alpha_\ell(n+1), \tag{3.7}$$

which is equivalent to the inequality

$$-p(n) v_{k+1} < \alpha_\ell(n+1) - \alpha_\ell(n). \tag{3.8}$$

Denote the left-hand side of (3.8) as $L_{(3.8)}$. As $v_{k+1} > b(n-k) = \alpha_\ell(n-k)$ and as by (2.3), (2.6), and (3.1)

$$p(n) \geq \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right) - \left(\frac{k}{k+1}\right)^k \cdot \frac{\delta}{(n\ln n \cdots \ln_\ell n)^2 \ln_{\ell+1}^\beta n}, \tag{3.9}$$

we have

$$\begin{aligned} L_{(3.8)} &< -\left(\frac{k}{k+1}\right)^k \left(\frac{1}{k+1} + k\mu_\ell(n) - \frac{\delta}{(n\ln n \cdots \ln_\ell n)^2 \ln_{\ell+1}^\beta n}\right) \\ &\quad \times \left(\frac{k}{k+1}\right)^{n-k} \sqrt{(n-k)\ln(n-k)\cdots \ln_\ell(n-k)\ln_{\ell+1}^{-\sigma}(n-k)} \\ &= -\left(\frac{k}{k+1}\right)^n \left(\frac{1}{k+1} + k\mu_\ell(n) - \frac{\delta}{(n\ln n \cdots \ln_\ell n)^2 \ln_{\ell+1}^\beta n}\right) \\ &\quad \times \sqrt{(n-k)\ln(n-k)\cdots \ln_\ell(n-k)\ln_{\ell+1}^{-\sigma}(n-k)}. \end{aligned} \tag{3.10}$$

Further, we can easily see that

$$\begin{aligned} \alpha_\ell(n+1) - \alpha_\ell(n) &= \left(\frac{k}{k+1}\right)^n \sqrt{n\ln n \cdots \ln_\ell n \ln_{\ell+1}^{-\sigma} n} \\ &\quad \times \left(\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_\ell(n+1)}{\ln_\ell n}\frac{\ln_{\ell+1}^{-\sigma}(n+1)}{\ln_{\ell+1}^{-\sigma} n}} - 1\right). \end{aligned} \tag{3.11}$$

Thus, to prove (3.8), it suffices to show that for n sufficiently large, the following inequality holds:

$$\begin{aligned} &-\left(\frac{1}{k+1} + k\mu_\ell(n) - \frac{\delta}{(n\ln n \cdots \ln_\ell n)^2 \ln_{\ell+1}^\beta n}\right)\sqrt{\frac{(n-k)}{n}\frac{\ln(n-k)}{\ln n}\cdots\frac{\ln_\ell(n-k)}{\ln_\ell n}\frac{\ln_{\ell+1}^{-\sigma}(n-k)}{\ln_{\ell+1}^{-\sigma} n}} \\ &\quad < \frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_\ell(n+1)}{\ln_\ell n}\frac{\ln_{\ell+1}^{-\sigma}(n+1)}{\ln_{\ell+1}^{-\sigma} n}} - 1. \end{aligned} \tag{3.12}$$

Denote the left-hand side of inequality (3.12) as $L_{(3.12)}$ and the right-hand side as $R_{(3.12)}$. In the following computation we will use the fact that $\beta > 2$ and

$$\mu_\ell(n) = \frac{1}{8n^2} + O\left(\frac{1}{n^2\ln^2 n}\right), \tag{3.13}$$

and we will omit all the terms which are of order $o(1/n^3)$. Applying Lemma 2.4 with $r = k$ and $q = \ell$, we can write

$$
\begin{aligned}
L_{(3.12)} = &-\left(\frac{1}{k+1} + k\mu_\ell(n) - \frac{\delta}{(n\ln n\cdots\ln_\ell n)^2 \ln_{\ell+1}^{\beta} n}\right) \\
&\times\left(1 - k\left(\frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \frac{\sigma}{2n\ln n\cdots\ln_\ell n\ln_{\ell+1} n}\right)\right. \\
&\qquad\left. -k^2\mu_\ell(n) + \frac{\sigma(\sigma+2)}{8}\cdot\frac{k^2}{(n\ln n\cdots\ln_{\ell+1} n)^2} - \frac{k^3}{16n^3} + o\left(\frac{1}{n^3}\right)\right) \\
= &-\frac{1}{k+1} + \frac{k}{k+1}\left(\frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \frac{\sigma}{2n\ln n\cdots\ln_\ell n\ln_{\ell+1} n}\right) \\
&+ \frac{k^2}{k+1}\mu_\ell(n) - \frac{\sigma(\sigma+2)}{8(k+1)}\cdot\frac{k^2}{(n\ln n\cdots\ln_{\ell+1} n)^2} + \frac{k^3}{16n^3(k+1)} \\
&- k\mu_\ell(n) + \frac{k^2}{16n^3} + \frac{\delta}{(n\ln n\cdots\ln_\ell n)^2\ln_{\ell+1}^{\beta} n} + o\left(\frac{1}{n^3}\right) \\
= &-\frac{1}{k+1} + \frac{k}{k+1}\left(\frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \frac{\sigma}{2n\ln n\cdots\ln_\ell n\ln_{\ell+1} n}\right) \\
&- \frac{k}{k+1}\mu_\ell(n) - \frac{\sigma(\sigma+2)}{8(k+1)}\cdot\frac{k^2}{(n\ln n\cdots\ln_{\ell+1} n)^2} + \frac{2k^3+k^2}{16n^3(k+1)} \\
&+ \frac{\delta}{(n\ln n\cdots\ln_\ell n)^2\ln_{\ell+1}^{\beta} n} + o\left(\frac{1}{n^3}\right).
\end{aligned}
\tag{3.14}
$$

Using Lemma 2.4 with $r = -1$ and $q = \ell$, we get for $R_{(3.12)}$

$$
\begin{aligned}
R_{(3.12)} = &\frac{k}{k+1}\left(1 + \frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \frac{\sigma}{2n\ln n\cdots\ln_\ell n\ln_{\ell+1} n}\right. \\
&\qquad\left. -\mu_\ell(n) + \frac{\sigma(\sigma+2)}{8}\cdot\frac{1}{(n\ln n\cdots\ln_{\ell+1} n)^2} + \frac{1}{16n^3} + o\left(\frac{1}{n^3}\right)\right) - 1 \\
= &\frac{-1}{k+1} + \frac{k}{k+1}\left(\frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \frac{\sigma}{2n\ln n\cdots\ln_\ell n\ln_{\ell+1} n}\right) \\
&- \frac{k}{k+1}\cdot\mu_\ell(n) + \frac{\sigma(\sigma+2)k}{8(k+1)}\cdot\frac{1}{(n\ln n\cdots\ln_{\ell+1} n)^2} + \frac{k}{16n^3(k+1)} + o\left(\frac{1}{n^3}\right).
\end{aligned}
\tag{3.15}
$$

It is easy to see that the inequality (3.12) reduces to

$$
\begin{aligned}
&-\frac{\sigma(\sigma+2)k^2}{8(k+1)}\cdot\frac{1}{(n\ln n\cdots\ln_{\ell+1}n)^2}+\frac{2k^3+k^2}{16n^3(k+1)}+\frac{\delta}{(n\ln n\cdots\ln_{\ell}n)^2\ln_{\ell+1}^{\beta}n}+o\left(\frac{1}{n^3}\right)\\
&\quad<\frac{\sigma(\sigma+2)k}{8(k+1)}\cdot\frac{1}{(n\ln n\cdots\ln_{\ell+1}n)^2}+\frac{k}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right).
\end{aligned}
\tag{3.16}
$$

This inequality is equivalent to

$$
-\frac{\sigma(\sigma+2)k}{8}\cdot\frac{1}{(n\ln n\cdots\ln_{\ell+1}n)^2}+\frac{k(2k-1)}{16n^3}+\frac{\delta}{(n\ln n\cdots\ln_{\ell}n)^2\ln_{\ell+1}^{\beta}n}+o\left(\frac{1}{n^3}\right)<0.
\tag{3.17}
$$

The last inequality holds for n sufficiently large because $k\geq 1$, $\sigma>0$, $\beta>2$, and as $n\to\infty$,

$$
\frac{\delta}{(n\ln n\cdots\ln_{\ell}n)^2\ln_{\ell+1}^{\beta}n},\qquad\frac{1}{n^3}
\tag{3.18}
$$

tend to zero faster than

$$
\frac{1}{(n\ln n\cdots\ln_{\ell+1}n)^2}
\tag{3.19}
$$

does.

Thus, we have proved that inequality (1.5) holds.

Next, according to (1.6), we have to prove that

$$
\nu_{\ell}(n)-p(n)v_{k+1}>\nu_{\ell}(n+1),
\tag{3.20}
$$

which is equivalent to the inequality:

$$
-p(n)v_{k+1}>\nu_{\ell}(n+1)-\nu_{\ell}(n).
\tag{3.21}
$$

Denote the left-hand side of (3.21) as $L_{(3.21)}$. As $v_{k+1}<c(n-k)=\nu_{\ell}(n-k)$ and as by (2.3), (3.1), and (2.7)

$$
p(n)\leq\left(\frac{k}{k+1}\right)^k\cdot\left(\frac{1}{k+1}+k\mu_{\ell}(n)\right)+\varepsilon\left(\frac{k}{k+1}\right)^k\cdot\frac{k(2k-1)}{16n^3},
\tag{3.22}
$$

we have

$$
\begin{aligned}
L_{(3.21)} &> -\left(\frac{k}{k+1}\right)^{k}\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k(2k-1)}{16n^{3}}\right) \\
&\quad\times\left(\frac{k}{k+1}\right)^{n-k}\sqrt{(n-k)\ln(n-k)\cdots\ln_{\ell}(n-k)} \\
&= -\left(\frac{k}{k+1}\right)^{n}\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k(2k-1)}{16n^{3}}\right)\cdot\sqrt{(n-k)\ln(n-k)\cdots\ln_{\ell}(n-k)}.
\end{aligned}
\tag{3.23}
$$

Further, we can easily see that

$$
\nu_{\ell}(n+1)-\nu_{\ell}(n)=\left(\frac{k}{k+1}\right)^{n}\sqrt{n\ln n\cdots\ln_{\ell}n}\left(\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\cdot\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_{\ell}(n+1)}{\ln_{\ell}n}}-1\right).
\tag{3.24}
$$

Thus, to prove (3.21), it suffices to show that for n sufficiently large, the following inequality holds:

$$
\begin{aligned}
&-\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k(2k-1)}{16n^{3}}\right)\sqrt{\frac{(n-k)}{n}\cdot\frac{\ln(n-k)}{\ln n}\cdots\frac{\ln_{\ell}(n-k)}{\ln_{\ell}n}} \\
&\quad > \frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\cdot\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_{\ell}(n+1)}{\ln_{\ell}n}}-1.
\end{aligned}
\tag{3.25}
$$

Denote the left-hand side of inequality (3.25) as $L_{(3.25)}$ and the right-hand side as $R_{(3.25)}$. Using Lemma 2.3 with $r=k$ and $q=\ell$, we can write

$$
\begin{aligned}
L_{(3.25)} &= -\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k(2k-1)}{16n^{3}}\right) \\
&\quad\times\left(1-k\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right)-k^{2}\mu_{\ell}(n)-\frac{k^{3}}{16n^{3}}+o\left(\frac{1}{n^{3}}\right)\right) \\
&= -\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right) \\
&\quad+\frac{k^{2}}{k+1}\mu_{\ell}(n)+\frac{k^{3}}{16n^{3}(k+1)}-k\mu_{\ell}(n)+\frac{k^{2}}{16n^{3}}-\varepsilon\cdot\frac{k(2k-1)}{16n^{3}}+o\left(\frac{1}{n^{3}}\right) \\
&= -\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right) \\
&\quad-\frac{k}{k+1}\mu_{\ell}(n)+\frac{k^{3}}{16n^{3}(k+1)}+\frac{k(k-\varepsilon(2k-1))}{16n^{3}}+o\left(\frac{1}{n^{3}}\right).
\end{aligned}
\tag{3.26}
$$

Using Lemma 2.3 with $r = -1$ and $q = \ell$, we get for $R_{(3.25)}$

$$
\begin{aligned}
R_{(3.25)} &= \frac{k}{k+1}\left(1 + \frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n} - \mu_\ell(n) + \frac{1}{16n^3} + o\left(\frac{1}{n^3}\right)\right) - 1 \\
&= \frac{-1}{k+1} + \frac{k}{k+1}\left(\frac{1}{2n} + \frac{1}{2n\ln n} + \cdots + \frac{1}{2n\ln n\cdots\ln_\ell n}\right) \\
&\quad - \frac{k}{k+1}\cdot\mu_\ell(n) + \frac{k}{16n^3(k+1)} + o\left(\frac{1}{n^3}\right).
\end{aligned}
\tag{3.27}
$$

It is easy to see that the inequality (3.25) reduces to

$$
\frac{k^3}{16n^3(k+1)} + \frac{k(k-\varepsilon(2k-1))}{16n^3} + o\left(\frac{1}{n^3}\right) > \frac{k}{16n^3(k+1)} + o\left(\frac{1}{n^3}\right). \tag{3.28}
$$

This inequality is equivalent to

$$
\frac{k(2k-1)(1-\varepsilon)}{16n^3} + o\left(\frac{1}{n^3}\right) > 0. \tag{3.29}
$$

The last inequality holds for n sufficiently large because $k \geq 1$ and $1 - \varepsilon \in (0,1)$. We have proved that all the assumptions of Theorem 1.1 are fulfilled and hence there exists a solution of (1.2) satisfying conditions (1.8), that is, in our case, conditions (3.2). □

Acknowledgments

The research was supported by the Grant P201/10/1032 of the Czech Grant Agency (Prague), by the Council of Czech Government Grant MSM 00216 30519 and by the Grant FEKT-S-11-2-921 of Faculty of Electrical Engineering and Communication, Brno University of Technology.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 805846, 21 pages
doi:10.1155/2012/805846

Research Article

Existence and Global Exponential Stability of Periodic Solution to Cohen-Grossberg BAM Neural Networks with Time-Varying Delays

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Received 16 December 2011; Revised 4 February 2012; Accepted 6 February 2012

Academic Editor: Agacik Zafer

We investigate first the existence of periodic solution in general Cohen-Grossberg BAM neural networks with multiple time-varying delays by means of using degree theory. Then using the existence result of periodic solution and constructing a Lyapunov functional, we discuss global exponential stability of periodic solution for the above neural networks. Our result on global exponential stability of periodic solution is different from the existing results. In our result, the hypothesis for monotonicity ineqiality conditions in the works of Xia (2010) Chen and Cao (2007) on the behaved functions is removed and the assumption for boundedness in the works of Zhang et al. (2011) and Li et al. (2009) is also removed. We just require that the behaved functions satisfy sign conditions and activation functions are globally Lipschitz continuous.

1. Introduction

In 1983, Cohen and Grossberg [1] constructed a kind of simplified neural networks that are now called Cohen-Grossberg neural networks (CGNNs); they have received increasing interesting due to their promising potential applications in many fields such as pattern recognition, parallel computing, associative memory, and combinatorial optimization. Such applications heavily depend on the dynamical behaviors. Thus, the qualitative analysis of the dynamical behaviors is a necessary step for the practical design and application of neural networks (or neural system [2–4]). The stability of Cohen-Grossberg neural network with or without delays has been widely studied by many researchers, and various interesting results have been reported [5–14].

On the other hand, since the pioneering work of Kosko [15, 16], a series of neural networks related to bidirectional associative memory models have been proposed. These models generalized the single-layer autoassociative Hebbian correlator to a class of two-layer pattern-matched heteroassociative circuits. Bidirectional associative memory neural networks have also been used in many fields such as pattern recognition and automatic control and image and signal processing. During the last years, many authors have discussed the existence and global stability of BAM neural networks [17–20]. In recent years, a few authors [17, 21–26] discussed global stability of Cohen-Grossberg BAM neural networks.

As is well known, the studies on neural dynamical system not only involve a discussion of stability properties but also involve other dynamic behavior, such as periodic oscillatory behavior, chaos, and bifurcation. In many applications, periodic oscillatory behavior is of great interest; it has been found in applications in learning theory. Hence, it is of prime importance to study periodic oscillatory solutions of neural networks.

This motivates us to consider periodic solutions of Cohen-Grossberg BAM neural networks. Recently, a few authors discussed the existence and stability of periodic solution to Cohen-Grossberg BAM neural networks with delays [27–31].

In [27], the authors proposed a class of bidirectional Cohen-Grossberg neural networks with distributed delays as follows:

$$\begin{aligned}
&\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\\
&= -a_i(x_i(t))\left[b_i(t,x_i(t)) - \sum_{j=1}^{m} p_{ij}(t)\int_0^{\infty} K_{ji}(u)\times f_j\big(t,\lambda_j y_j(t-u)\big)\mathrm{d}u - I_i(t)\right], \quad i=1,2,\ldots,n,\\
&\frac{\mathrm{d}y_j(t)}{\mathrm{d}t}\\
&= -c_j\big(y_j(t)\big)\left[d_j\big(t,y_j(t)\big) - \sum_{i=1}^{n} q_{ji}(t)\int_0^{\infty} L_{ij}(u)\times g_i\big(t,\mu_i x_i(t-u)\big)\mathrm{d}u - J_j(t)\right], \quad j=1,2,\ldots,m.
\end{aligned} \tag{1.1}$$

By using the Lyapunov functional method and some analytical techniques, some sufficient conditions were obtained for global exponential stability of periodic solutions to these networks.

In [28], the authors discussed the following Cohen-Grossberg-type BAM neural networks with time-varying delays:

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} p_{ij}(t) f_j\big(\lambda_j y_j\big(t-\tau_{ij}(t)\big)\big) - I_i(t)\right], \quad i=1,2,\ldots,n,\\
\frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -c_j\big(y_j(t)\big)\left[d_j\big(y_j(t)\big) - \sum_{i=1}^{n} q_{ji}(t) g_i\big(\mu_i x_i\big(t-\sigma_{ji}(t)\big)\big) - J_j(t)\right], \quad j=1,2,\ldots,m,
\end{aligned} \tag{1.2}$$

where $n, m \geq 2$ are the number of neurons in the networks with initial value conditions:

$$x_i(\theta) = \phi_i(\theta), \quad \theta \in [-r_1, 0], \quad y_j(\theta) = \phi_j(\theta), \quad \theta \in [-r_2, 0], \tag{1.3}$$

where $r_1 = \max_{1\leq i\leq n,\ 1\leq j\leq m,\ 0\leq t\leq \omega}\{\sigma_{ji}(t)\}$, $r_2 = \max_{1\leq i\leq n, 1\leq j\leq m, 0\leq t\leq\omega}\{\tau_{ij}(t)\}$, $a_i(x_i(t))$, $b_i(x_i(t))$, $c_j(y_j(t))$, $d_j(y_j(t))$ are continuous functions, $f_j(\lambda_j y_j(t-\tau_{ji}(t)))$, $g_i(\mu_i x_i(t-\delta_{ij}(t)))$ are continuous functions, λ_j, μ_i are parameters, $I_i(t)$ and $J_j(t)$ are continuous functions, x_i and y_j denote the state variables of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V at time t, respectively, $a_i(x_i(t)) > 0$, $c_j(y_j(t)) > 0$ represent amplification functions of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, $b_i(x_i(t)), d_j(y_j(t))$ are appropriately behaved functions of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, f_j, g_i are the activation functions of the jth neurons from the neural field F_V and the ith neurons from the neural field F_U, respectively, I_i, J_j are the exogenous inputs of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively, p_{ij} and q_{ji} are the connection weights, which denote the strengths of connectivity between the neuron j from the neural field F_V and the neuron i from the neural field F_U, and $\tau_{ij}(t)$, $\sigma_{ij}(t)$ correspond to the transmission time delays.

By using the analysis method and inequality technique, some sufficient conditions were obtained to ensure the existence, uniqueness, global attractivity, and exponential stability of the periodic solution to this neural networks.

In [29, 30], the authors discussed, respectively, two Cohen-Grossberg BAM neural networks on time scales. When time scale T becomes R, the existence and global exponential stability of periodic solution are obtained in [29, 30] under the assumptions that activation functions satisfy global Lipschitz conditions and boundedness conditions and behaved functions satisfy some inequality conditions.

In [31], the authors discussed the following Cohen-Grossberg BAM neural networks of neutral type with delays:

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} a_{ij}(t) f_j\big(y_j(t-\tau_{ij}(t))\big)\right. \\ &\qquad \left. - \sum_{j=1}^{m} b_{ij}(t) f_j\big(y_j(t-\sigma_{ij}(t))\big) - I_i(t)\right], \quad i = 1,2,\ldots,n, \\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -c_j\big(y_j(t)\big)\left[d_j\big(y_j(t)\big) - \sum_{i=1}^{n} c_{ji}(t) g_i\big(x_i(t-p_{ji}(t))\big)\right. \\ &\qquad \left. - \sum_{i=1}^{n} d_{ji}(t) g_i\big(x_i(t-q_{ji}(t))\big) - J_j(t)\right], \quad j = 1,2,\ldots,m. \end{aligned} \tag{1.4}$$

Under the assumptions that activation functions satisfy global Lipschitz conditions and behaved functions satisfy some inequality conditions, global exponential stability of periodic solution is obtained for system (1.4).

In this paper, our purpose is to obtain a new sufficient condition for the existence and global exponential stability of periodic solution of system (1.2). The paper is organized as follows. In Section 2, we discuss the existence of periodic solution of system (1.2) by using coincidence degree theory and inequality technique. In Section 3, we study the global exponential stability of periodic solution of system (1.2) by using the existence result of periodic solution and constituting Lyapunov functional. Our result on global exponential stability of periodic solution is different from the existing results. In our result, the hypotheses for monotonicity inequalities in [27, 28] on behaved functions are replaced with sign conditions and the assumption for boundedness in [29, 30] on activation functions is removed.

2. Existence of Periodic Solution

In this section, we first establish the existence of at least a periodic solution by applying the coincidence degree theory. To establish the existence of at least a periodic solution by applying the coincidence degree theory, we recall some basic tools in the frame work of Mawhin's coincidence degree [32] that will be used to investigate the existence of periodic solutions.

Let X, Z be Banach spaces, L: $\mathrm{Dom}\,L \subset X \to Z$ a linear mapping, and $N : X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}\,L = \mathrm{codim}\ \mathrm{Im}\ L < \infty$ and $\mathrm{Im}L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to \mathrm{Ker}\,L$ and $Q : Z \to Z/\,\mathrm{Im}\,L$ such that $\mathrm{Im}\,P = \mathrm{Ker}\,L$ and $\mathrm{Im}\,L = \mathrm{Ker}\,Q = \mathrm{Im}(I-Q)$. It follows that $L/_{\mathrm{Dom}\,L \cap \mathrm{Ker}\,P} : (I-P)X \to \mathrm{Im}\,L$ is invertible. We denote the inverse of the map $L/_{\mathrm{Dom}\,L\cap \mathrm{Ker}\,P}$ by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $(QN)(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\mathrm{Im}\ Q$ is isomorphic to $\mathrm{Ker}\ L$, there exists an isomorphism $J : \mathrm{Im}\,Q \to \mathrm{Ker}\,L$.

In the proof of our existence theorem, we will use the continuation theorem of Gaines and Mawhin [32].

Lemma 2.1 (continuation theorem). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(a) $Lx \neq \lambda N(x)$, *for all* $\lambda \in (0,1)$, $x \in \partial\Omega$,

(b) $QN(x) \neq 0$, *for all* $x \in \mathrm{Ker}\,L \cap \partial\Omega$,

(c) $\deg(JQNx, \Omega \cap \mathrm{Ker}\,L, 0) \neq 0$.

Then, $Lx = Nx$ has at least one solution in $\mathrm{Dom}\,L \cap \overline{\Omega}$.

For the sake of convenience, we introduce some notations.
$|\cdot|$ denotes the norm in R, $\overline{f} = \max_{0\le t\le\omega}|f(t)|$, $\underline{f} = \min_{0\le t\le\omega}|f(t)|$, where $f(t)$ is a continuously periodic function with common period ω. Our main result on the existence of at least a periodic solution for system (1.2) is stated in the following theorem.

Theorem 2.2. *One assume that the following conditions holds:*

(i) $p_{ij}(t)$, $q_{ji}(t)$, $I_i(t)$, $J_j(t)$ *are continuously periodic functions on* $t \in [0,+\infty)$ *with common period* $\omega > 0$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$;

(ii) $a_i(\cdot)$ *and* $c_j(\cdot)$ *are continuously bounded, that is, there exist positive constants* l_i, l_i^*, k_j, k_j^* $(i = 1,\ldots,n,\ j = 1,\ldots,m)$ *such that*

$$\begin{aligned} l_i &\le a_i \le l_i^*, \\ k_j &\le c_j \le k_j^*; \end{aligned} \tag{2.1}$$

(iii) $b_i(x_i(t))$ *and* $d_j(y_j(t))$ *are continuous and there exist positive constants* M_i, N_j $(i = 1,\ldots,n, j = 1,\ldots,m)$ *such that for all* $x, y \neq x \in R$,

$$\begin{gathered}\operatorname{sign}(x-y)\left[b_i(x)-b_i(y)\right] \geq M_i|x-y|,\\ \operatorname{sign}(x-y)\left[d_j(x)-d_j(y)\right] \geq N_j|x-y|;\end{gathered} \tag{2.2}$$

(iv) *there exist positive constants* A_j, B_i $(i = 1,\ldots,n,\ j = 1,2,\ldots,m)$ *such that for all* $x, y \in R$,

$$\begin{gathered}\left|f_j(x)-f_j(y)\right| \leq A_j|x-y|,\\ \left|g_i(x)-g_i(y)\right| \leq B_i|x-y|;\end{gathered} \tag{2.3}$$

(v) *there exist two positive constants* $r_i > 1,\ i = 1,2$ *with* $\tau'_{ij} < \min\{1, 1-r_1^{-1}\} < 1$ *and* $\sigma'_{ji} < \min\{1, 1-r_2^{-1}\} < 1$ *such that for* $i = 1,\ldots,n;\ j = 1,\ldots,m$,

$$\begin{gathered}l_iM_i > \sum_{j=1}^{m} l_i^* \overline{p_{ij}} A_j \lambda_j \sqrt{r_1},\\ k_jN_j > \sum_{i=1}^{n} k_j^* \overline{q_{ji}} B_i \mu_i \sqrt{r_2}.\end{gathered} \tag{2.4}$$

Then, system (1.2) *has at least one* ω*-periodic solution.*

Proof. In order to apply Lemma 2.1 to system (1.2), let

$$\begin{gathered}X = \left\{u = \left(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\right)^T \in C\left(R, R^{m+n}\right) : u(t+\omega) = u(t)\right\},\\ Z = \left\{z \in C\left(R, R^{m+n}\right) : z(t+\omega) = z(t)\right\}.\end{gathered} \tag{2.5}$$

Define

$$\|u\| = \max_{t\in[0,\omega]} \sum_{i=1}^{n} |x_i(t)| + \max_{t\in[0,\omega]} \sum_{j=1}^{m} \left|y_j(t)\right|, \quad u \in X \text{ or } Z. \tag{2.6}$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces.

Let for $u \in X$

$$\begin{gathered}Nu = \begin{pmatrix} H_i(t)\\ K_j(t)\end{pmatrix} = \begin{pmatrix} -a_i(x_i(t))\left[b_i(x_i(t)) - \displaystyle\sum_{j=1}^{m} p_{ij}(t) f_j\left(\lambda_j y_j\left(t-\tau_{ij}(t)\right)\right) - I_i(t)\right]\\ -c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right) - \displaystyle\sum_{i=1}^{n} q_{ji}(t) g_i\left(\mu_i x_i\left(t-\sigma_{ji}(t)\right)\right) - J_j(t)\right]\end{pmatrix},\\ Lu = u' = \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \quad Pu = \frac{1}{\omega}\int_0^{\omega} u(t)\mathrm{d}t, \quad u \in X, \quad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\mathrm{d}t, \quad z \in Z.\end{gathered} \tag{2.7}$$

Then, it follows that $\operatorname{Ker} L = R^{(m+n)}$, $\operatorname{Im} L = \{z \in Z : \int_0^{\omega} z(t)dt = 0\}$ is closed in Z, $\dim \operatorname{Ker} L = m + n = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{2.8}$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_{p}\colon \operatorname{Im} L \rightarrow \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{s} z(t)\,\mathrm{d}t\,\mathrm{d}s. \tag{2.9}$$

Then,

$$QNu = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} H_1(s)\mathrm{d}s \\ \frac{1}{\omega}\int_0^{\omega} H_2(s)\mathrm{d}s \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} H_n(s)\mathrm{d}s \\ \frac{1}{\omega}\int_0^{\omega} K_1(s)\mathrm{d}s \\ \frac{1}{\omega}\int_0^{\omega} K_2(s)\mathrm{d}s \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} K_m(s)\mathrm{d}s \end{pmatrix},$$

$$K_p(I-Q)Nu = \begin{pmatrix} \int_0^t H_1(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{t} H_1(s)\mathrm{d}s\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} H_1(s)\mathrm{d}s \\ \int_0^t H_2(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{t} H_2(s)\mathrm{d}s\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} H_2(s)\mathrm{d}s \\ \vdots \\ \int_0^t H_n(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{t} H_n(s)\mathrm{d}s\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} H_n(s)\mathrm{d}s \\ \int_0^t K_1(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{t} K_1(s)\mathrm{d}s\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} K_1(s)\mathrm{d}s \\ \vdots \\ \int_0^t K_m(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^{t} K_m(s)\mathrm{d}s\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} K_m(s)\mathrm{d}s \end{pmatrix}. \tag{2.10}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_p(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Condition (iii) in Theorem 2.2 implies that for all $x \in R$

$$\begin{aligned} \operatorname{sign} x b_i(x) &\geq M_i|x| + \operatorname{sign} x b_i(0), \\ \operatorname{sign} x d_j(x) &\geq N_j|x| + \operatorname{sign} x d_j(0). \end{aligned} \tag{2.11}$$

Condition (iv) in Theorem 2.2 implies that for all $x \in R$

$$\begin{aligned} \left|f_j(x)\right| &\leq A_j|x| + \left|f_j(0)\right|, \\ \left|g_i(x)\right| &\leq B_i|x| + \left|g_j(0)\right|. \end{aligned} \tag{2.12}$$

Corresponding to the operator equation $Lx = \lambda Nx,\ \lambda \in (0,1)$, we have for $i = 1,2,\ldots,n,\ j = 1,\ldots,m$

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= \lambda H_i(t), \\ \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= \lambda K_j(t). \end{aligned} \tag{2.13}$$

Assume that $u \in X$ is a solution of system (2.13) for some $\lambda \in (0,1)$. Multiplying the first equation of system (2.13) by $x_i(t)$ and integrating over $[0,\omega]$, we have

$$\begin{aligned} &\int_0^\omega x_i(t) \operatorname{sign} x_i(t) \operatorname{sign} x_i(t) \\ &\quad \times \left\{ a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^m p_{ij}(t)\ f_j\left(\lambda_j y_j\left(t - \tau_{ij}(t)\right)\right) - I_i(t) \right] \right\} \mathrm{d}t = 0. \end{aligned} \tag{2.14}$$

Multiplying the second equation of system (2.13) by $y_j(t)$ and integrating over $[0,\omega]$, we have

$$\begin{aligned} &\int_0^\omega y_j(t) \operatorname{sign} y_j(t) \operatorname{sign} y_j(t) \\ &\quad \times \left\{ c_j\left(y_j(t)\right) \left[d_j\left(y_j(t)\right) - \sum_{i=1}^n q_{ji}(t) g_i\left(\mu_i x_i\left(t - \sigma_{ji}(t)\right)\right) - J_j(t) \right] \right\} \mathrm{d}t = 0. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \leq l_i^* \int_0^{\omega} |x_i(t)| \left\{ -a_i \operatorname{sign} x_i(t) b_i(0) + \sum_{j=1}^{m} \overline{p_{ij}} \left(A_j \lambda_j \left| y_j\left(t-\tau_{ij}(t)\right)\right| + \left|f_j(0)\right| \right) + \overline{I_i} \right\} \mathrm{d}t, \tag{2.16}$$

$$k_i N_j \int_0^{\omega} \left|y_j(t)\right|^2 \mathrm{d}t \leq k_j^* \int_0^{\omega} \left|y_j(t)\right| \left\{ -c_j \operatorname{sign} y_j(t) d_j(0) + \sum_{i=1}^{n} \overline{q_{ji}} \left(B_i \mu_i \left| x_i\left(t-\sigma_{ji}(t)\right)\right| + \left|g_i(0)\right| \right) + \overline{J_j} \right\} \mathrm{d}t. \tag{2.17}$$

Hence,

$$\begin{aligned} l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t &\leq l_i^* \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \right)^{1/2} \\ &\times \left\{ \sum_{j=1}^{m} \overline{p_{ij}} \left[A_j \lambda_j \left(\int_0^{\omega} \left| y_j\left(t-\tau_{ij}(t)\right)\right|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left|f_j(0)\right| \right] + l_i^* |b_i(0)| + \sqrt{\omega}\, \overline{I_i} \right\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} k_j N_j \int_0^{\omega} \left|y_j(t)\right|^2 \mathrm{d}t &\leq k_j^* \left(\int_0^{\omega} \left|y_j(t)\right|^2 \mathrm{d}t \right)^{1/2} \\ &\times \left\{ \sum_{i=1}^{n} \overline{q_{ji}} \left[B_i \mu_i \left(\int_0^{\omega} \left| x_i\left(t-\sigma_{ji}(t)\right)\right|^2 \mathrm{d}t \right)^{1/2} + \sqrt{\omega} \left|g_i(0)\right| \right] + k_j^* \left|d_j(0)\right| + \sqrt{\omega}\, \overline{J_j} \right\} \end{aligned} \tag{2.19}$$

Denoting $s = t - \tau_{ij}(t) = g(t),\ \sigma = t - \sigma_{ji}(t) = h(t)$, then

$$\left(\int_0^{\omega} \left| y_j\left(t-\tau_{ij}(t)\right)\right|^2 \mathrm{d}t \right)^{1/2} = \left(\int_0^{\omega} \frac{\left|y_j(s)\right|^2}{1-\tau_{ij}'\left(g^{-1}(s)\right)} \mathrm{d}s \right)^{1/2}, \tag{2.20}$$

$$\left(\int_0^{\omega} \left| x_i\left(t-\sigma_{ji}(t)\right)\right|^2 \mathrm{d}t \right)^{1/2} = \left(\int_0^{\omega} \frac{|x_i(\sigma)|^2}{1-\sigma_{ji}'\left(h^{-1}(\sigma)\right)} \mathrm{d}\sigma \right)^{1/2}. \tag{2.21}$$

Substituting (2.20) into (2.18) and substituting (2.21) into (2.19) give for $i = 1,\ldots,n,\ j = 1,\ldots,m$

$$\begin{aligned}
&l_i M_i \int_0^{\omega} |x_i(t)|^2 \mathrm{d}t \\
&\quad \le l_i^* \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t\right)^{1/2} \\
&\times \left\{ \sum_{j=1}^{m} \overline{p_{ij}} A_j \lambda_j \sqrt{r_1} \left(\int_0^{\omega} |y_j(t)|^2 \mathrm{d}t\right)^{1/2} + \sqrt{\omega}\left(\sum_{j=1}^{m} \overline{p_{ij}} |f_j(0)| + |b_i(0)| + \overline{I_i} \right) \right\},
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&k_j N_j \int_0^{\omega} |y_j(t)|^2 \mathrm{d}t \\
&\quad \le k_j^* \left(\int_0^{\omega} |y_j(t)|^2\right)^{1/2} \\
&\times \quad \left\{ \sum_{i=1}^{n} \overline{q_{ji}} B_i \mu_i \sqrt{r_2} \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t\right)^{1/2} + \sqrt{\omega}\left(\sum_{i=1}^{n} \overline{q_{ji}} |g_i(0)| + |d_j(0)| + \overline{J_j} \right) \right\}.
\end{aligned} \tag{2.23}$$

Denoting for the sake of convenience

$$\begin{aligned}
\max_{1\le i\le n}\left\{ \left(\int_0^{\omega} |x_i(t)|^2 \mathrm{d}t\right)^{1/2} \right\} &= \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2}, \\
\max_{1\le j\le m}\left\{ \left(\int_0^{\omega} |y_j(t)|^2 \mathrm{d}t\right)^{1/2} \right\} &= \left(\int_0^{\omega} |y_{j_0}(t)|^2 \mathrm{d}t\right)^{1/2},
\end{aligned} \tag{2.24}$$

where, $i_0 \in \{1,2,\ldots,n\},\ j_0 \in \{1,2,\ldots,m\}$, and from (2.22) and (2.23), we obtain

$$\begin{aligned}
l_{i_0} M_{i_0} \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} \le{}& l_{i_0}^* \sum_{j=1}^{m} \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1} \left(\int_0^{\omega} |y_{j_0}(t)|^2 \mathrm{d}t\right)^{1/2} \\
&+ l_{i_0}^* \sqrt{\omega} \left(\sum_{j=1}^{m} \overline{p_{i_0 j}} |f_j(0)| + |b_{i_0}(0)| + \overline{I_{i_0}} \right),
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
k_{j_0} N_{j_0} \left(\int_0^{\omega} |y_{j_0}(t)|^2 \mathrm{d}t\right)^{1/2} \le{}& k_{j_0}^* \sum_{i=1}^{n} \overline{q_{j_0 i}} B_i \mu_i \sqrt{r_2} \left(\int_0^{\omega} |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} \\
&+ k_{j_0}^* \sqrt{\omega} \left(\sum_{i=1}^{n} \overline{q_{j_0 i}} |g_i(0)| + |d_{j_0}(0)| + \overline{J_{j_0}} \right).
\end{aligned} \tag{2.26}$$

Now we consider two possible cases for (2.26) and (2.25):

$$\begin{aligned}
&\text{(i)}\ \left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \le \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2}, \\
&\text{(ii)}\ \left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} > \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{2.27}$$

When $(\int_0^\omega |y_{j_0}(t)|^2 \mathrm{d}t)^{1/2} \le (\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t)^{1/2}$, from (2.25), we have

$$\left(l_{i_0} M_{i_0} - l_{i_0}^* \sum_{j=1}^m \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1}\right)\left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} \le l_{i_0}^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{i_0 j}} \left|f_j(0)\right| + |b_{i_0}(0)| + \overline{I_{i_0}}\right). \tag{2.28}$$

Thus,

$$\begin{aligned}
\left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} &\le \frac{l_{i_0}^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{i_0 j}} \left|f_j(0)\right| + |b_{i_0}(0)| + \overline{I_{i_0}}\right)}{l_{i_0} M_{i_0} - l_{i_0}^* \sum_{j=1}^m \overline{p_{i_0 j}} A_j \lambda_j \sqrt{r_1}} \\
&\le \max_{1 \le i \le n} \left\{ \frac{l_i^* \sqrt{\omega}\left(\sum_{j=1}^m \overline{p_{ij}} \left|f_j(0)\right| + |b_i(0)| + \overline{I_i}\right)}{l_i M_i - l_i^* \sum_{j=1}^m \overline{p_{ij}} A_j \lambda_j \sqrt{r_1}} \right\} \\
&\overset{\text{def}}{=} d_1.
\end{aligned} \tag{2.29}$$

Therefore,

$$\begin{aligned}
\left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} &\le \left(\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t\right)^{1/2} \\
&\le d_1.
\end{aligned} \tag{2.30}$$

(ii) When $(\int_0^\omega |y_{j_0}(t)|^2 \mathrm{d}t)^{1/2} > (\int_0^\omega |x_{i_0}(t)|^2 \mathrm{d}t)^{1/2}$, from (2.26), we have

$$\left(k_{j_0} N_{j_0} - k_{j_0}^* \sum_{i=1}^n \overline{q_{j_0 i}} B_i \mu_i \sqrt{r_2}\right)\left(\int_0^\omega \left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \le k_{j_0}^* \sqrt{\omega}\left(\sum_{i=1}^n \overline{q_{j_0 i}} \left|g_i(0)\right| + \left|d_{j_0}(0)\right| + \overline{J_{j_0}}\right). \tag{2.31}$$

Thus,

$$\left(\int_0^{\omega}\left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \leq \frac{k_{j_0}^*\sqrt{\omega}\left(\sum_{i=1}^{n}\overline{q_{j_0 i}}\left|g_i(0)\right|+\left|d_{j_0}(0)\right|+\overline{J_{j_0}}\right)}{k_{j_0}N_{j_0}-k_{j_0}^*\sum_{i=1}^{n}\overline{q_{j_0 i}}B_i\mu_i\sqrt{r_2}}$$
$$\leq \max_{1\leq j\leq m}\left\{\frac{k_j^*\sqrt{\omega}\left(\sum_{i=1}^{n}\overline{q_{ji}}\left|g_i(0)\right|+\left|d_j(0)\right|+\overline{J_j}\right)}{k_jN_j-k_j^*\sum_{i=1}^{n}\overline{q_{ji}}B_i\mu_i\sqrt{r_2}}\right\}$$
$$\stackrel{\text{def}}{=} d_2. \tag{2.32}$$

Therefore,

$$\left(\int_0^{\omega}\left|x_{i_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \leq \left(\int_0^{\omega}\left|y_{j_0}(t)\right|^2 \mathrm{d}t\right)^{1/2} \leq d_2. \tag{2.33}$$

Hence, from (2.30) and (2.33), we have for $i=1,2,\dots,n,\ j=1,2,\dots,m,\ t\in[0,\omega]$

$$\left(\int_0^{\omega}\left|x_i(t)\right|^2 \mathrm{d}t\right)^{1/2} < \max\{d_1,d_2\} \stackrel{\text{def}}{=} d, \tag{2.34}$$

$$\left(\int_0^{\omega}\left|y_j(t)\right|^2 \mathrm{d}t\right)^{1/2} < \max\{d_1,d_2\} = d. \tag{2.35}$$

Multiplying the first equation of system (2.13) by $x_i'(t)$ and integrating over $[0,\omega]$, from (2.20) and (2.35) and the fact that

$$\int_0^{\omega} a_i(x_i(t))b_i(x_i(t))x_i'(t)\,\mathrm{d}t = 0, \tag{2.36}$$

it follows that

$$\left(\int_0^{\omega}\left|x_i'(t)\right|^2 \mathrm{d}t\right)^{1/2} \leq l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\left(\int_0^{\omega}\left|y_j(t-\tau_{ij}(t))\right|\mathrm{d}t\right)^{1/2}+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right)$$
$$\leq l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\sqrt{r_1}\left(\int_0^{\omega}\left|y_j(t)\right|^2\mathrm{d}t\right)^{1/2}+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right)$$
$$<\max_{1\leq i\leq n}\left\{l_i^*\sum_{j=1}^{m}\overline{p_{ij}}A_j\lambda_j\sqrt{r_1}d+l_i^*\sqrt{\omega}\left(\sum_{j=1}^{m}\overline{p_{ij}}\left|f_j(0)\right|+\overline{I_i}\right)\right\}\stackrel{\text{def}}{=} c_1. \tag{2.37}$$

Similarly, multiplying the second equation of system (2.13) by $y_j(t)$ and integrating over $[0,\omega]$, from (2.21) and (2.34) and the fact that

$$\int_0^{\omega} c_j(y_j(t))d_j(y_j(t))y_j'(t)\mathrm{d}t = 0, \tag{2.38}$$

it follows that there exists a positive constant c_2 such that

$$\left(\int_0^{\omega} \left|y_j'(t)\right|^2 \mathrm{d}t\right)^{1/2} < c_2. \tag{2.39}$$

From (2.34) and (2.35), it follows that there exist points t_i and $\overline{t_j}$ such that

$$|x_i(t_i)| < \frac{d}{\sqrt{\omega}}, \tag{2.40}$$

$$\left|y_j\left(\overline{t_j}\right)\right| < \frac{d}{\sqrt{\omega}}. \tag{2.41}$$

Since for all $t \in [0,\omega]$,

$$\begin{aligned} |x_i(t)| &\le |x_i(t_i)| + \int_0^{\omega} \left|x_i'(t)\right|\mathrm{d}t \\ &\le |x_i(t_i)| + \sqrt{\omega}\left(\int_0^{\omega} \left|x_i'(t)\right|^2\right)^{1/2}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} \left|y_j(t)\right| &\le \left|y_j(t_i)\right| + \int_0^{\omega} \left|y_j'(t)\right|\mathrm{d}t \\ &\le \left|y_j(t)\right| + \sqrt{\omega}\left(\int_0^{\omega} \left|y_j'(t)\right|^2\right)^{1/2}, \end{aligned} \tag{2.43}$$

then from (2.40)–(2.43), we have for $t \in [0,\omega]$, $i = 1,\dots,n, j = 1,\dots,m$

$$\begin{aligned} |x_i(t)| &\le \frac{d}{\sqrt{\omega}} + \sqrt{\omega}c_1, \\ \left|y_j(t)\right| &\le \frac{d}{\sqrt{\omega}} + \sqrt{\omega}c_2. \end{aligned} \tag{2.44}$$

Obviously, $d/\sqrt{\omega}$, $\sqrt{\omega}c_1$, and $\sqrt{\omega}c_2$ are all independent of λ. Now let

$$\begin{aligned} \Omega = \Bigg\{ & u = (x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_m)^T \in X : \\ & \|u\| < n\left(\frac{d}{\sqrt{\omega}} + r_1 + \sqrt{\omega}c_1\right) + m\left(\frac{d}{\sqrt{\omega}} + r_2 + \sqrt{\omega}c_2\right)\Bigg\}, \end{aligned} \tag{2.45}$$

where $r_1,\ r_2$ are two chosen positive constants such that the bound of Ω is larger. Then, Ω is bounded open subset of X. Hence, Ω satisfies requirement (a) in Lemma 2.1. We prove that (b) in Lemma 2.1 holds. If it is not true, then when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$ we have

$$\begin{aligned}
&QNu \\
&\quad = \left(\frac{1}{\omega}\int_0^\omega H_1(t)\mathrm{d}t, \frac{1}{\omega}\int_0^\omega H_2(t)\mathrm{d}t, \ldots, \frac{1}{\omega}\int_0^\omega H_n(t)\mathrm{d}t; \frac{1}{\omega}\int_0^\omega K_1(t)\mathrm{d}t, \ldots, \frac{1}{\omega}\int_0^\omega K_m(t)\mathrm{d}t\right)^T \\
&\quad = (0, \ldots, 0)^T.
\end{aligned} \tag{2.46}$$

Therefore, there exist points ξ_i $(i = 1, 2, \ldots, n)$ and η_j $(j = 1, 2, \ldots, m)$ such that

$$\begin{gathered}
H_i(\xi_i) = 0, \\
K_j(\eta_j) = 0.
\end{gathered} \tag{2.47}$$

From this and following the arguments of (2.40) and (2.41), we have for forall $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, t \in [0, \omega]$

$$\begin{gathered}
|x_i(t)| < \frac{d}{\sqrt{\omega}}, \\
|y_j(t)| < \frac{d}{\sqrt{\omega}}.
\end{gathered} \tag{2.48}$$

Hence,

$$\|u\| < n\frac{d}{\sqrt{\omega}} + m\frac{d}{\sqrt{\omega}}. \tag{2.49}$$

Thus, $u \in \Omega \cap R^{(m+n)}$. This contradicts the fact that $u \in \partial\Omega \cap R^{(m+n)}$. Hence, this proves that (b) in Lemma 2.1 holds. Finally, we show that (c) in Lemma 2.1 holds. We only need to prove that $\deg\{-JQNu, \Omega \cap \operatorname{Ker} L, (0,0)^T\} \neq (0, 0, \ldots, 0)^T$. Now, we show that

$$\begin{aligned}
&\deg\left\{-JQNu, \Omega \cap \operatorname{Ker} L, (0, 0, \ldots, 0)^T\right\} \\
&\quad = \deg\left\{(l_1 M_1 x_1, l_2 M_2 x_2, \ldots, l_n M_n x_n; k_1 N_1 y_1, \ldots, k_m N_m y_m)^T,\ \Omega \cap \operatorname{Ker} L, (0, \ldots, 0)^T\right\}.
\end{aligned} \tag{2.50}$$

To this end, we define a mapping $\phi : \operatorname{Dom} L \times [0, 1] \to X$ by

$$\begin{aligned}
&\phi(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m, \mu) \\
&\quad = -\frac{\mu}{\omega}\left(\int_0^\omega H_1(t)\mathrm{d}t, \int_0^\omega H_2(t)\mathrm{d}t, \ldots, \int_0^\omega H_n(t)\mathrm{d}t, \int_0^\omega K_1(t)\mathrm{d}t, \ldots, \int_0^\omega K_m(t)\mathrm{d}t\right) \\
&\qquad + (1 - \mu)(l_1 M_1 x_1, l_2 M_2 x_2, \ldots, l_n M_n x_n;\ k_1 N_1 y_1, \ldots, k_m N_m y_m),
\end{aligned} \tag{2.51}$$

where $\mu \in [0,1]$ is a parameter. We show that when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$, $\phi(x_1, x_2, \ldots, x_n;\ y_1, \ldots, y_m, \mu) \neq (0, 0, \ldots, 0)^T$. If it is not true, then when $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{(m+n)}$, $\phi(x_1, x_2, \ldots, x_n; y_1, \ldots, y_m, \mu) = (0, 0, \ldots, 0)^T$. Thus, constant vector u with $u \in \partial\Omega$ satisfies for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$,

$$\begin{gathered}
\frac{\mu}{\omega}\int_0^{\omega}\left\{a_i(x_i)\left[b_i(x_i) - a_i(x_i)\sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t + (1-\mu) l_i M_i x_i = 0, \\
\frac{\mu}{\omega}\int_0^{\omega}\left\{c_j(y_j)\left[d_j(y_j) - c_j(y_j)\sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i u_i) - J_j(t)\right]\right\}\mathrm{d}t + (1-\mu) k_j N_j y_j = 0.
\end{gathered} \tag{2.52}$$

That is,

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} x_i\left\{a_i(x_i)(b_i(x_i) - b_i(0)) + a_i(x_i) b_i(0) - a_i(x_i)\left[\sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t \\
&\qquad + (1-\mu) l_i M_i |x_i| = 0,
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} y_j\left\{c_j(y_j)(d_j(y_j) - d_j(0)) + c_j(y_j) d_j(0) - c_j(y_j)\left[\sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i u_i) - J_j(t)\right]\right\}\mathrm{d}t \\
&\qquad + (1-\mu) k_j N_j |y_j| = 0.
\end{aligned} \tag{2.54}$$

Denote $|y_{j_0}| = \max_{1\le j\le m}\{|y_j|\}, |x_{i_0}| = \max_{1\le i\le n}\{|x_i|\}$. □

Claim 1. We claim that $|x_{i_0}| < (d/\sqrt{\omega}) + \sqrt{\omega} c_1 + r_1$, otherwise, $|x_{i_0}| \ge (d/\sqrt{\omega}) + \sqrt{\omega} c_1 + r_1$. We consider two possible cases: (i) $|y_{j_0}| \le |x_{i_0}|$ and (ii) $|y_{j_0}| > |x_{i_0}|$.

(i) When $|y_{j_0}| \le |x_{i_0}|$, we have

$$\begin{aligned}
&\frac{\mu}{\omega}\int_0^{\omega}\operatorname{sign} x_{i_0}\left\{a_i(x_{i_0})(b_i(x_{i_0}) - b_i(0)) + a_i(x_{i_0})\left[b_i(0) - \sum_{j=1}^{m} p_{ij}(t) f_j(\lambda_j y_j) - I_i(t)\right]\right\}\mathrm{d}t \\
&\quad + (1-\mu) l_i M_i |x_{i_0}| \\
&\qquad \ge \mu l_i M_i |x_{i_0}| - l_i^{*}\left[|b_i(0)| + \sum_{j=1}^{m}\overline{p_{ij}}\left(\lambda_j A_j |y_j| + |f_j(0)|\right) + \overline{I_i}\right] + (1-\mu) l_i M_i |x_{i_0}|
\end{aligned}$$

$$
\begin{aligned}
&\geq l_i M_i |x_{i_0}| - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&\geq \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) |x_{i_0}| - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&\geq \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) \left(\frac{d_1}{\sqrt{\omega}} + \sqrt{\omega} c_1 + r_1 \right) - l_i^* \left[|b_i(0)| + \sum_{j=1}^{m} \overline{p_{ij}} \left(\lambda_j A_j \left| y_{j_0} \right| + \left| f_j(0) \right| \right) + \overline{I_i} \right] \\
&> \left(l_i M_i - l_i^* \sum_{j=1}^{m} A_j \lambda_j \overline{p_{ij}} \right) r_1 \\
&> 0,
\end{aligned}
\tag{2.55}
$$

which contradicts (2.53).

(ii) When $|y_{j_0}| > |x_{i_0}|$, we have

$$
\begin{aligned}
&\frac{\mu}{\omega} \int_0^{\omega} \operatorname{sign} y_{j_0} \left\{ c_j(y_{j_0}) \left(d_j(y_{j_0}) - d_j(0) \right) + c_j(y_{j_0}) \left[d_j(0) - \sum_{i=1}^{n} q_{ji}(t) g_i(\mu_i x_i) - J_j(t) \right] \right\} \mathrm{d}t \\
&\quad + (1-\mu) k_j N_j \left| y_{j_0} \right| \\
&\quad \geq \mu k_j N_j \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_i| + \left| g_i(0) \right| \right) + \overline{J_j} \right] + (1-\mu) k_j N_j \left| y_{j_0} \right| \\
&\quad \geq k_j N_j \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad \geq \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) \left| y_{j_0} \right| - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad \geq \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) \left(\frac{d_2}{\sqrt{\omega}} + \sqrt{\omega} c_1 + r_1 \right) - k_j^* \left[\left| d_j(0) \right| + \sum_{i=1}^{n} \overline{q_{ji}} \left(\mu_i B_i |x_{i_0}| + \left| g_i(0) \right| \right) + \overline{J_j} \right] \\
&\quad > \left(k_j N_j - k_j^* \sum_{i=1}^{n} B_i \mu_i \overline{q_{ji}} \right) r_1 \\
&\quad > 0,
\end{aligned}
\tag{2.56}
$$

which contradicts (2.54). From the discussion of (i) and (ii), Claim 1 holds.

Claim 2. We claim that $|y_{j_0}| < (d/\sqrt{\omega}) + \sqrt{\omega} c_2 + r_2$, otherwise, $|y_{j_0}| \geq (d/\sqrt{\omega}) + \sqrt{\omega} c_2 + r_2$. We consider two possible cases: (i) $|x_{i_0}| \leq |y_{j_0}|$ and (ii) $|x_{i_0}| > |y_{j_0}|$.

The proofs of (i) and (ii) are similar to those of (ii) and (1) in Claim 1, respectively, therefore Claim 2 holds.

Thus, $|x_i| < (d_1/\sqrt{\omega}) + c_1\sqrt{\omega} + r_1$ and $|y_j| < (d_2/\sqrt{\omega}) + \sqrt{\omega}c_2 + r_2$. Thus, $u \in \Omega \cap R^{(m+n)}$. This contradicts the fact that $u \in \partial\Omega \cap R^{(m+n)}$. According to the topological degree theory and by taking $J = I$ since $\operatorname{Ker} L = \operatorname{Im} Q$, we obtain

$$\begin{aligned}
&\deg\left\{-JQNu, \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\phi(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m, 1), \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\phi(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m, 0), \Omega \cap \operatorname{Ker} L, (0,0)^T\right\} \\
&\quad = \deg\left\{\left(l_1M_1x_1, l_2M_2x_2, \dots, l_nM_nx_n; k_1N_1y_1, \dots, k_mN_my_m\right)^T, \Omega \cap \operatorname{Ker} L, (0,\dots,0)^T\right\} \\
&\quad \neq 0.
\end{aligned} \tag{2.57}$$

So far, we have proved that Ω satisfies all the assumptions in Lemma 2.1. Therefore, system (1.2) has at least one ω-periodic solution.

3. Global Exponential Stability of Periodic Solution

In this section, by constructing a Lyapunov functional, we derive new sufficient conditions for global exponential stability of a periodic solution of system (1.2).

Theorem 3.1. *In addition to all conditions in Theorem 2.2, one assumes further that the following conditions hold:*

(H_1) *there exists two positive constants $r_i \geq 1$ $(i = 1, 2)$ with $M_i > \sum_{j=1}^{m} \overline{q_{ji}}\mu_i B_i r_2$ and $N_j > \sum_{i=1}^{n} \overline{p_{ij}}\lambda_j A_j r_1$ such that $\tau'_{ij} < \min\{1, 1 - r_1^{-1}\} < 1$ and $\sigma'_{ji} < \min\{1, 1 - r_2^{-1}\} < 1$;*

(H_2) *there exist constants τ_{ij} and σ_{ji}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, such that*

$$0 < \tau_{ij}(t) < \tau_{ij}, \quad 0 < \sigma_{ji}(t) < \sigma_{ji}. \tag{3.1}$$

Then, the ω periodic solution of system (1.2) *is globally exponentially stable.*

Proof. By Theorem 2.2, system (1.2) has at least one ω periodic solution, say, $u^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t); y_1^*(t), \dots, y_m^*(t))^T$. Suppose that $u(t) = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ is an arbitrary ω periodic solution of system (1.2). From (H_1), we can choose a suitable θ such that

$$\begin{aligned}
M_i &> \frac{\theta}{l_i} + \sum_{j=1}^{m} \overline{q_{ji}}\mu_i B_i r_2 \exp(\theta\tau_{ij}), \\
N_j &> \frac{\theta}{k_j} + \sum_{i=1}^{n} \overline{p_{ij}}\lambda_j A_j r_i \exp(\theta\sigma_{ji}).
\end{aligned} \tag{3.2}$$

We define a Lyapunov functional as follows for $t > 0, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$:

$$
\begin{aligned}
V(t) = \exp(\theta t)&\left\{ \sum_{i=1}^{n} \left| \int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s \right| + \sum_{j=1}^{m} \left| \int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \right| \mathrm{d}s \right\} \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j \int_{t-\tau_{ij}(t)}^{t} \exp\left[\theta\left(\sigma + \tau_{ij}\left(g^{-1}(\sigma) \right) \right) \right] \frac{\left| y_j(\sigma) - y_j^*(\sigma) \right|}{1 - \tau_{ij}'(g^{-1}(\sigma))} \mathrm{d}\sigma \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{q_{ji}} \mu_i B_i \int_{t-\sigma_{ji}(t)}^{t} \exp\left[\theta\left(\sigma + \sigma_{ji}\left(h^{-1}(\sigma) \right) \right) \right] \frac{\left| x_i(\sigma) - x_i^*(\sigma) \right|}{1 - \sigma_{ji}'(h^{-1}(\sigma))} \mathrm{d}\sigma,
\end{aligned} \tag{3.3}
$$

where $g(t) = t - \tau_{ij}(t),\ h(t) = t - \sigma_{ji}(t),\ i = 1, 2, \ldots, n,\ j = 1, \ldots, m$. Calculating the upper right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (1.2), we obtain

$$
\begin{aligned}
D^+V(t) \le \exp(\theta t) \sum_{i=1}^{n} &\left\{ \theta \left| \int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s \right| - M_i \left| x_i(t) - x_i^*(t) \right| \right. \\
&\qquad \left. + \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j \left| y_j(t - \tau_{ij}(t)) - y_j^*(t - \tau_{ij}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{j=1}^{m} &\left\{ \theta \left| \int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s \right| - N_j \left| y_j(t) - y_j^*(t) \right| \right. \\
&\qquad \left. + \sum_{i=1}^{n} \overline{q_{ji}} \mu_i B_i \left| x_i(t - \sigma_{ji}(t)) - x_i^*(t - \tau_{ji}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{p_{ij}} \lambda_j A_j &\left\{ \frac{\left| y_j(t) - y_j^*(t) \right| \exp\left[\theta \tau_{ij}(s^{-1}(t)) \right]}{1 - \tau_{ij}'(s^{-1}(t))} \right. \\
&\qquad \left. - \left| y_j(t - \tau_{ij}(t)) - y_j^*(t - \tau_{ij}(t)) \right| \right\} \\
+ \exp(\theta t) \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{q_{ji}} \mu_i B_i &\left\{ \frac{\left| x_i(t) - x_i^*(t) \right| \exp\left[\theta \sigma_{ji}(h^{-1}(t)) \right]}{1 - \sigma_{ji}'(h^{-1}(t))} \right. \\
&\qquad \left. - \left| x_i(t - \sigma_{ji}(t)) - x_i^*(t - \sigma_{ji}(t)) \right| \right\}.
\end{aligned} \tag{3.4}
$$

Since there exist points ξ_i, η_j such that

$$\left|\int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s\right| = \frac{1}{a_i(\xi_i)} \left|x_i(t) - x_i^*(t)\right|,$$
$$\left|\int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s\right| = \frac{1}{c_j(\eta_j)} \left|y_j(t) - y_j^*(t)\right|, \tag{3.5}$$

from (3.4), we have

$$\begin{aligned} D^+V(t) \leq &-\exp(\theta t)\sum_{j=1}^{m}\left\{N_j - \frac{\theta}{k_j} - \sum_{i=1}^{n}\overline{p_{ij}}\lambda_j A_j r_1 \exp(\theta\sigma_{ji})\right\}\left|y_j(t) - y_j^*(t)\right| \\ &- \exp(\theta t)\sum_{i=1}^{n}\left\{M_i - \frac{\theta}{l_i} - \sum_{j=1}^{m}\overline{q_{ji}}\mu_i B_i r_2 \exp(\theta\tau_{ij})\right\}\left|x_i(t) - x_i^*(t)\right|. \end{aligned} \tag{3.6}$$

In view of (3.2), it follows that $V(t) < V(0)$. Therefore,

$$\exp(\theta t)\left\{\sum_{i=1}^{n}\left|\int_{x_i^*(t)}^{x_i(t)} \frac{1}{a_i(s)} \mathrm{d}s\right| + \sum_{j=1}^{m}\left|\int_{y_j^*(t)}^{y_j(t)} \frac{1}{c_j(s)} \mathrm{d}s\right|\right\} < V(t) < V(0). \tag{3.7}$$

Equation (3.3) implies that

$$\begin{aligned} V(0) < &\sum_{i=1}^{n}\left\{\frac{1}{l_i} + \sum_{j=1}^{m}\overline{w_{ji}}\mu_i B_i r_2 \int_{-\sigma_{ji}(0)}^{0} \exp\left[\theta\left(\sigma + \sigma_{ji}\left(h^{-1}(\sigma)\right)\right)\right]\mathrm{d}\sigma\right\} \sup_{0\leq\leq\omega}\left|x_i(s) - x_i^*(s)\right| \\ &+ \sum_{j=1}^{m}\left\{\frac{1}{k_j} + \sum_{i=1}^{n}\overline{h_{ij}}\lambda_j A_j r_1 \int_{-\tau_{ij}}^{0} \exp\left[\theta\left(\sigma + \tau_{ij}\left(g^{-1}(\sigma)\right)\right)\right]\mathrm{d}\sigma\right\} \sup_{0\leq s\leq\omega}\left|y_j(s) - y_j^*(s)\right|. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) gives

$$\begin{aligned} &\sum_{i=1}^{n}\left|x_i(t) - x_i^*(t)\right| + \sum_{j=1}^{m}\left|y_j(t) - y_j^*(t)\right| \\ &\quad < \frac{M}{N}\exp(-\theta t)\left\{\sum_{i=1}^{n}\sup_{0\leq s\leq\omega}\left|x_i(s) - x_i^*(s)\right| + \sum_{j=1}^{m}\sup_{0\leq s\leq\omega}\left|y_j(s) - y_j^*(s)\right|\right\}, \end{aligned} \tag{3.9}$$

where

$$M = \max_{1\le i\le n, 1\le j\le m}\left\{\frac{1}{l_i} + \sum_{i=1}^{n}\overline{h_{ij}}\lambda_j A_j r_1 \int_{-\tau_{ij}(0)}^{0} \exp\left[\theta + \tau_{ij}\left(g^{-1}(\sigma)\right)\right]\mathrm{d}\sigma,\right.$$
$$\left.\frac{1}{k_j} + \sum_{j=1}^{m}\overline{w_{ji}}\mu_i B_i r_2 \int_{-\sigma_{ji}(0)}^{0} \exp\left[\theta + \sigma_{ji}\left(h^{-1}(\sigma)\right)\right]\mathrm{d}\sigma\right\}, \quad (3.10)$$
$$N = \min\left\{\frac{1}{l_i^*}, \frac{1}{k_j^*}\right\}.$$

The proof of Theorem 3.1 is complete. □

4. An Example

Example 4.1. Consider the following Cohen-Grossberg BAM neural networks with time-varying delays:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(2+\sin x_1)\left\{200x_1(t) + 100\sin x_1(t) - (2+\sin t)\left|y_1\left[t-\left(1+\frac{\sin t}{2}\right)\right]\right| - \sin t\right\},$$
$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -(3+\cos y_1)\left\{200y_1(t) + 100\sin y_1(t) - (2+\cos t)\left|x_1\left[t-\left(1+\frac{\sin t}{3}\right)\right]\right| - \cos t\right\}. \quad (4.1)$$

In Theorem 3.1,

$$A_1 = 1,\quad B_1 = 1,\quad l_1 = 1,\quad l_1^* = 3,\quad k_1 = 2,\quad k_1^* = 4,\quad M_1 = 100,$$
$$N_1 = 100,\quad \overline{p_{11}} = 3,\quad \overline{q_{11}} = 3,\quad \lambda_1 = \mu_1 = 1, \quad (4.2)$$
$$\tau_{11}' = \frac{\cos t}{2},\quad \sigma_{11}' = \frac{\cos t}{3}.$$

Since

$$1 - \frac{\cos t}{2} \ge 1 - \frac{|\cos t|}{2} \ge \frac{1}{2},\qquad 1 - \frac{\cos t}{3} \ge 1 - \frac{|\cos t|}{3} \ge \frac{2}{3}, \quad (4.3)$$

then $r_1 = 2$, $r_2 = 3/2$.

Since

$$M_1 = 100 > \overline{q_{11}}\mu_1 B_1 r_2 = \frac{9}{2},\qquad N_1 = 100 > \overline{p_{11}}\lambda_1 A_1 r_1 = 6,$$
$$l_1 M_1 = 100 > l_1^*\overline{p_{11}} A_1\sqrt{r_1} = 9\sqrt{2},\qquad k_1 N_1 = 200 > \overline{q_{11}} B_1 \mu_1 \sqrt{r_2} = 12\sqrt{\frac{3}{2}}, \quad (4.4)$$

then conditions (H_1), (H_2), and (v) are satisfied. It is easy to prove that the rest of the conditions in Theorem 3.1 are satisfied. By Theorem (3.2) , system (4.1) has a unique ω periodic solution that is globally exponentially stable.

5. Conclusion

We investigate first the existence of the periodic solution in general Cohen-Grossberg BAM neural networks with multiple time-varying delays by means of using degree theory. Then, using the existence result of periodic solution and constructing a Lyapunov functional, we discuss global exponential stability of periodic solution for the above neural networks. In our result, the hypotheses for monotonicity in [27, 28] on the behaved functions are replaced with sign conditions and the assumption for boundedness on activation functions is removed. We just require that the behaved functions satisfy sign conditions and activation functions are globally Lipschitz continuous.

Acknowledgment

This paper is supported by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry of China (no. 20091341, 2011508) and the Fund of National Natural Science of China (no. 61065008).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 358362, 14 pages
doi:10.1155/2012/358362

Research Article

Globally Exponential Stability of Periodic Solutions to Impulsive Neural Networks with Time-Varying Delays

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Received 12 January 2012; Accepted 26 February 2012

Academic Editor: Josef Diblík

By using Schaeffer's theorem and Lyapunov functional, sufficient conditions of the existence and globally exponential stability of positive periodic solution to an impulsive neural network with time-varying delays are established. Applications, examples, and numerical analysis are given to illustrate the effectiveness of the main results.

1. Introduction

It is well known that in implementation of neural networks, time delays are inevitably encountered because of the finite switching speed of amplifiers. Specially in electronic neural networks, delays are usually time-varying and often become sources of instability. So it is important to investigate the dynamics of neural networks with delays [1–7]. Recently, the study of the existence of periodic solutions of neural networks has received much attention. The common approaches are based on using Mawhin continuation theorem [1, 2, 8–10], Banach's fixed point theorem [11–13], fixed point theorem in a cone [14], Schaeffer's theorem [15, 16], and so on. On the other hand, studies on neural dynamical systems not only involve the existence of periodic solutions, but also involve other dynamical behaviors such as stability of periodic solutions, bifurcations, and chaos. In recent years, the stability of solutions of neural networks has attracted attention of many researchers and many nice results have been obtained [1–3, 5–13, 16–25]. For example, M. Tan and Y. Tan [1] considered the following neural network with variable coefficients and time-varying delays:

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) + J_i(t), \qquad i = 1,2,\ldots,n. \tag{1.1}$$

By using the Mawhin continuation theorem, they discussed the existence and globally exponential stability of periodic solutions.

However, in real world, many physical systems often undergo abrupt changes at certain moments due to instantaneous perturbations, which lead to impulsive effects. In fact, impulsive differential equation represents a more natural framework for mathematical modelling of many real world phenomena such as population dynamic and neural networks. The theory of impulsive differential equations is now being recognized to be richer than the corresponding theory of differential equations without impulse, and various kinds of impulsive differential equations have been extensively studied, see [8–16, 18, 19, 21–25] and references therein. Then, considering impulsive effects, it is necessary and interesting for us to study further the dynamics of system (1.1). Furthermore, as pointed by Gopalsamy and Sariyasa [4], it would be of great interest to study neural networks in periodic environment. On the other hand, to the best of our knowledge, few authors considered the existence of periodic solutions by using Schaeffer's theorem. Hence, in this paper, by using Schaeffer's theorem and Lyapunov functional, we aim to discuss the existence and exponential stability of periodic solutions to a class of impulsive neural networks with periodic coefficients and time-varying delays. The model is as follows:

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) + J_i(t), \quad t \neq t_k, \\ x_i(t^+) &= \left(1 + q_i^k\right) x_i(t), \quad t = t_k, \end{aligned} \tag{1.2}$$

with initial conditions

$$x_i(s) = \phi_i(s), \quad \phi_i(s) \in C([-\tau, 0], R^n), \ i = 1,2,\ldots,n, \tag{1.3}$$

where $x_i(t)$ corresponds to the state of the ith unit, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $f_j(x_j(t))$ denotes the output of the jth unit, $a_{ij}(t)$ and $b_{ij}(t)$ denote the strength of the jth unit on the ith unit, respectively, $J_i(t)$ is the external bias on the ith unit, $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the jth unit, t_k denotes the impulsive moment, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$, $C([-\tau, 0], R^n)$ denotes the Banach space of continuous mapping from $[-\tau, 0]$ to R^n equipped with the norm $\|\phi\| = \max_{1\le i\le n} \sup_{t\in[0,\omega]} |\phi_i(t)|$ for all $\phi = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T \in C([-\tau, 0], R^n)$, where $\tau = \max_{1\le i\le n} \sup_{t\in[0,\omega]} \tau_{ij}(t)$.

Throughout this paper, we always assume the following.

(H_1) $c_i(t) > 0, a_{ij}(t), b_{ij}(t), J_i(t), \tau_{ij}(t)$ are all continuous ω-periodic functions for $i, j = 1, 2, \ldots, n$.

(H_2) $f_j : R \to R$ is continuous and there exists positive constant k_j such that $f_j(u) - f_j(v) \leq k_j|u - v|$ for any $u, v \in R$ and $j = 1, 2, \ldots, n$.

(H_3) There exists positive integer p such that $t_{k+p} = t_k + \omega,\ q_i^k = q_i^{k+p}$. Then

$$[0, \omega] \cap \{t_k, k = 1, 2, \ldots\} = \{t_1, t_2, \ldots, t_p\}. \tag{1.4}$$

For convenience, we use the following notations:

$$\overline{f} = \frac{1}{\omega}\int_0^{\omega} |f(t)| dt, \qquad f^M = \max_{t\in[0,\omega]} |f(t)|, \qquad f^m = \min_{t\in[0,\omega]} |f(t)|, \tag{1.5}$$

where $f(t)$ is continuous and ω-periodic function.

The rest of this paper is organized as follows. In Section 2, by using Schaeffer's theorem, sufficient conditions of the existence of ω-periodic solution to system (1.2) with initial conditions (1.3) are established. In Section 3, by using Lyapunov functional, we derive the conditions under which the periodic solution is globally exponentially stable. In Section 4, applications, illustrative examples, and simulations are given to show the effectiveness of the main results. Finally, some conclusions are drawn in Section 5.

2. Existence of Periodic Solution

First we make some preparations. As usual in the theory of impulsive differential equation, by a solution of model (1.2), it means the following.

(i) $x(t) = (x_1(t), x_2(t), \ldots x_n(t))^T \in R^n,\ x_i(t)$ is piecewise continuous such that $x_i(t_k^-) = x_i(t_k),\ x_i(t_k^+)$ exists, and $x_i(t)$ is differentiable on (t_{k-1}, t_k) for $i = 1, 2, \ldots, n,\ k = 1, 2, \ldots$.

(ii) $x_i(t)$ satisfies (1.2) for $i = 1, 2, \ldots, n$.

Definition 2.1. The set A is said to be quasi-equicontinuous in $[0, \omega]$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in A$, $t \in Z$, t', $t'' \in (t_{k-1}, t_k) \cap [0, \omega]$ and $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \epsilon$.

Lemma 2.2 (see [26, Compactness criterion]). *The set $A \subset X$ is relatively compact if and only if*

(i) *A is bounded, that is, $\|x\| \leq M$ for each $x \in A$ and some $M > 0$,*

(ii) *A is quasi-equicontinuous in $[0, \omega]$.*

The following lemma is fundamental to our discussion. The method is similar to that of [13, 16], so the proof is omitted here.

Lemma 2.3. *$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an ω-periodic solution of system (1.2) which is equivalent to $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an ω-periodic solution of the following equation:*

$$\begin{aligned} x_i(t) = {} & \int_0^{\omega} G_i(t,s)\left(\sum_{j=1}^{n} a_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) + J_i(s)\right) ds \\ & + \sum_{k=1}^{p} G_i(t,t_k) q_i^k x_i(t_k), \end{aligned} \tag{2.1}$$

where $G(t,s) = (G_1(t,s), G_2(t,s), \dots, G_n(t,s))^T$, and

$$G_i(t,s) = \begin{cases} \dfrac{e^{\int_0^{\omega} c_i(u)du - \int_s^t c_i(u)du}}{e^{\int_0^{\omega} c_i(u)du} - 1}, & 0 \le s \le t \le \omega, \\ \dfrac{e^{\int_t^{s} c_i(u)du}}{e^{\int_0^{\omega} c_i(u)du} - 1}, & 0 \le t \le s \le \omega. \end{cases} \tag{2.2}$$

It is easy to show that $G_i(t+\omega, s+\omega) = G_i(t,s)$, $G_i(t, t+\omega) - G_i(t,t) = 1$ and

$$\frac{1}{\sigma_i - 1} \le G_i(t,s) \le \frac{\sigma_i}{\sigma_i - 1}, \tag{2.3}$$

where $\sigma_i = e^{\int_0^{\omega} c_i(u)du}$ and $i = 1, 2, \dots, n$.

Lemma 2.4 (see [27, Schaeffer's theorem]). *Let X be a normed space and $\phi : X \to X$ be a compact operator. Define*

$$H(\phi) = \{x \mid x \in X, x = \lambda\phi x, 0 < \lambda < 1\}. \tag{2.4}$$

Then either

(i) *set $H(\phi)$ is unbounded, or*

(ii) *operator ϕ has a fixed point in X.*

In order to use Lemma 2.4, let

$$\begin{aligned} & \mathrm{PC}([0,\omega], R^n) \\ & \quad = \left\{ x : [0,\omega] \longrightarrow R^n \mid \lim_{s\to t} x(s) = x(t), t \ne t_k, \lim_{t\to t_k^-} x(t) = x(t_k), \lim_{t\to t_k^+} x(t) \text{ exists},\ k = 1, 2, \dots, p \right\}, \end{aligned} \tag{2.5}$$

with the norm $\|x\| = \max_{1\le i\le n} \sup_{t\in[0,\omega]} |x_i(t)|$, then $\mathrm{PC}([0,\omega], R^n)$ is a Banach space.

Define a mapping $\phi : \mathrm{PC}([0,\omega], R^n) \to \mathrm{PC}([0,\omega], R^n)$ *by* $(\phi x)(t) = x(t)$, *where* $(\phi x)(t) = ((\phi x)_1(t), (\phi x)_2(t), \ldots, (\phi x)_n(t))^T$ *and*

$$(\phi x)_i(t) = \int_t^{t+\omega} G_i(t,s)\left(\sum_{j=1}^{n} a_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) + J_i(s)\right) ds + \sum_{k=1}^{p} G_i(t,t_k) q_i^k x_i(t_k). \tag{2.6}$$

By Lemma 2.3, it is easy to see that the existence of ω*-periodic solution of* (1.2) *is equivalent to the existence of fixed point of the mapping* ϕ *in* $\mathrm{PC}([0,\omega], R^n)$.

Theorem 2.5. *Suppose that* (H_1)–(H_3) *hold. Further,*

(H_4) $\max_{1\le i\le n}(\sigma_i/\sigma_i - 1)(\sum_{j=1}^{n}(\overline{a_{ij}} + \overline{b_{ij}})k_j + \sum_{k=1}^{p}|q_i^k|) := \theta < 1$.

Then system (1.2) *admits an* ω*-periodic solution.*

Proof. By Lemma 2.3, it suffices to prove that the mapping ϕ admits a fixed point in $\mathrm{PC}([0,\omega], R^n)$.

For any constant $H > 0$, let $\Omega = \{x \mid x \in \mathrm{PC}([0,\omega], R^n), \|x\| < H\}$. For $x \in \Omega$, from (2.3) and (H_2), we have

$$\begin{aligned}
\|\phi x\| &= \max_{1\le i\le n}\sup_{t\in[0,\omega]}\left|\int_t^{t+\omega} G_i(t,s)\left(\sum_{j=1}^{n} a_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s-\tau_{ij}(s))) + J_i(s)\right) ds \right.\\
&\qquad\left. + \sum_{k=1}^{p} G_i(t,t_k) q_i^k x_i(t_k)\right|\\
&\le \max_{1\le i\le n}\sup_{t\in[0,\omega]}\frac{\sigma_i}{\sigma_i - 1}\int_0^{\omega}\left(\sum_{j=1}^{n}\left|a_{ij}(t) f_j(x_j(t))\right| + \sum_{j=1}^{n}\left|b_{ij}(t) f_j(x_j(t-\tau_{ij}(t)))\right| + |J_i(t)|\right) dt\\
&\quad + \frac{\sigma_i}{\sigma_i - 1}\sum_{k=1}^{p}\left|q_i^k x_i(t_k)\right|\\
&\le \max_{1\le i\le n}\sup_{t\in[0,\omega]}\frac{\sigma_i}{\sigma_i - 1}\int_0^{\omega}\left(\sum_{j=1}^{n}\left|a_{ij}(t)\right|\left(k_j\left|x_j(t)\right| + \left|f_j(0)\right|\right)\right.\\
&\qquad\left. + \sum_{j=1}^{n}\left|b_{ij}(t)\right|\left(k_j\left|x_j(t-\tau_{ij}(t))\right| + \left|f_j(0)\right|\right)\right) dt\\
&\quad + \int_0^{\omega}|J_i(t)|dt + \frac{\sigma_i}{\sigma_i - 1}\sum_{k=1}^{p}\left|q_i^k x_i(t_k)\right|
\end{aligned}$$

$$\le \max_{1\le i\le n}\frac{\sigma_i}{\sigma_i-1}\left(\sum_{j=1}^{n}\left(\overline{a_{ij}}+\overline{b_{ij}}\right)k_j+\sum_{k=1}^{p}\left|q_i^k\right|\right)\|x\|+\frac{\sigma_i}{\sigma_i-1}\left(\overline{J_i}+\sum_{j=1}^{n}\left|f_j(0)\right|\left(\overline{a_{ij}}+\overline{b_{ij}}\right)\right)$$
$$\le \max_{1\le i\le n}\frac{\sigma_i}{\sigma_i-1}\left(\sum_{j=1}^{n}\left(\overline{a_{ij}}+\overline{b_{ij}}\right)k_j+\sum_{k=1}^{p}\left|q_i^k\right|\right)H+\frac{\sigma_i}{\sigma_i-1}\left(\overline{J_i}+\sum_{j=1}^{n}\left|f_j(0)\right|\left(\overline{a_{ij}}+\overline{b_{ij}}\right)\right):=R. \tag{2.7}$$

It implies that $\phi(\Omega)$ is uniformly bounded.

For any $t\in[0,\omega], x\in\Omega$, we have

$$(\phi x)_i'(t)=\frac{d}{dt}\left(\int_t^{t+\omega}G_i(t,s)\left(\sum_{j=1}^{n}a_{ij}(s)f_j(x_j(s))+\sum_{j=1}^{n}b_{ij}(t)f_j(x_j(s-\tau_{ij}(s)))+J_i(s)\right)ds\right.$$
$$\left.+\sum_{k=1}^{p}G_i(t,t_k)q_ix_i(t_k)\right)$$
$$=-c_i(t)(\phi x)_i(t)+\left(\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))+J_i(t)\right). \tag{2.8}$$

If $t=t_k$, it is obvious that $(\phi x)_i'(t)=\lim_{t\to t_k^-}(\phi x)_i'(t)$. Hence, from (2.7) and (2.8), we have

$$\left|(\phi x)_i'(t)\right|\le c_i^MR+\sum_{j=1}^{n}\left(\left(a_{ij}^M+b_{ij}^M\right)k_j\right)H+\left(J_i^M+\sum_{j=1}^{n}\left|f_j(0)\right|\left(a_{ij}^M+b_{ij}^M\right)\right):=\beta_i. \tag{2.9}$$

Therefore, $\phi(\Omega)\subset \mathrm{PC}([0,\omega],R^n)$ is a family of uniformly bounded and equicontinuous subset. By Lemma 2.2, the mapping ϕ is compact.

Let $x\in \mathrm{PC}([0,\omega],R^n)$, and considering the following operator equation:

$$x=\lambda(\phi x),\qquad \lambda\in(0,1). \tag{2.10}$$

If x is a solution of (2.10), then

$$\|x\|\le\|\phi x\|\le\theta\|x\|+\max_{1\le i\le n}\frac{\sigma_i}{\sigma_i-1}\left(\overline{J_i}+\sum_{j=1}^{n}\left|f_j(0)\right|\left(\overline{a_{ij}}+\overline{b_{ij}}\right)\right). \tag{2.11}$$

According to (H_4), we deduce that

$$\|x\|\le\frac{\max_{1\le i\le n}(\sigma_i/(\sigma_i-1))\left(\overline{J_i}+\sum_{j=1}^{n}\left|f_j(0)\right|\left(\overline{a_{ij}}+\overline{b_{ij}}\right)\right)}{1-\theta}:=M. \tag{2.12}$$

It implies that $\|x\|$ is bounded, which is independent of $\lambda \in (0,1)$. By Lemma 2.4, we obtain that the mapping ϕ admits a fixed point in $\mathrm{PC}([0,\omega], R^n)$. Hence system (1.2) admits an ω-periodic solution such that $\|x\| \leq M$. This completes the proof. □

3. Globally Exponentially Stable

In this section, the sufficient conditions ensuring that (1.2) admits a unique ω-periodic solution and all solutions of (1.2) exponentially converge to the unique ω-periodic solution are to be established.

Definition 3.1. Let $x^*(t)$ be an ω-periodic solution of system (1.2) with initial value ϕ^*. If there exist constants $\alpha > 0, P \geq 1$, for every solution $x(t)$ of (1.2) with initial ϕ, such that

$$\left\|x_i(t) - x_i^*(t)\right\| \leq P\left\|\phi - \phi^*\right\|e^{-\alpha t} \quad \text{for any } t > 0,\ i = 1,2,\ldots,n, \tag{3.1}$$

then $x^*(t)$ is said to be globally exponentially stable.

Theorem 3.2. *Suppose that* (H_1)–(H_4) *hold. Further,*

(H_5) $-c_i^m + \sum_{j=1}^n (a_{ij}^M + b_{ij}^M)k_j < 0,\ i = 1,2,\ldots,n,$

(H_6) $\ln \max_{1\leq i\leq n}|1 + q_i^k|/t_k - t_{k-1} \leq \zeta < \alpha,$

where $\max_{1\leq i\leq n}|1 + q_i^k| \geq 1,\ \zeta > 0$ *is a constant,* α *is a constant determined in* (3.5).

Then system (1.2) *admits a unique* ω*-periodic solution, which is globally exponentially stable.*

Proof. By Theorem 2.5, system (1.2) admits an ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ with initial value ϕ^*. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ be an arbitrary solution of (1.2) with initial value ϕ. Define $z_i(t) = x_i^*(t) - x_i(t)$ and $g_j(z_j(t)) = f_j(z_j(t) + x_j(t)) - f_j(x_j(t))$, then we have

$$\begin{gathered} z_i'(t) = -c_i(t)z_i(t) + \sum_{j=1}^n a_{ij}(t)g_j\left(z_j(t)\right) + \sum_{j=1}^n b_{ij}(t)g_j\left(z_j\left(t - \tau_{ij}(t)\right)\right), \quad t \neq t_k, \\ z_i(t^+) = \left(1 + q_i^k\right)z_i(t), \quad t = t_k. \end{gathered} \tag{3.2}$$

By (H_5), we have $-c_i^m + \sum_{j=1}^n (a_{ij}^M + b_{ij}^M)k_j < 0$ for $i = 1,2,\ldots,n$. Let

$$h_i(\lambda) = \lambda - c_i^m + e^{\lambda\tau}\sum_{j=1}^n \left(a_{ij}^M + b_{ij}^M\right)k_j. \tag{3.3}$$

It is clear that $h_i(\lambda)$ is continuous on R and $h_i(0) < 0,\ i = 1,2,\ldots,n$. In addition,

$$\frac{d(h_i(\lambda))}{d\lambda} = 1 + \tau e^{\lambda\tau}\sum_{j=1}^n \left(a_{ij}^M + b_{ij}^M\right)k_j > 0, \tag{3.4}$$

and $h_i(+\infty) = +\infty$, then $h_i(\lambda)$ is strictly monotone increasing. Therefore, there exists a unique $\lambda_i > 0$ such that $\lambda_i - c_i^m + e^{\lambda_i \tau} \sum_{j=1}^{n} (a_{ij}^M + b_{ij}^M) k_j = 0$ for $i = 1, 2, \ldots, n$. Let

$$\alpha = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \tag{3.5}$$

then

$$h_i(\alpha) = \alpha - c_i^m + e^{\alpha\tau} \sum_{j=1}^{n} \left(a_{ij}^M + b_{ij}^M \right) k_j \leq 0, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Obviously, for $t \in [-\tau, 0]$ and the above α, we have

$$|z_i(t)| \leq \|\phi - \phi^*\| \leq \|\phi - \phi^*\| e^{-\alpha t}, \quad i = 1, 2, \ldots, n, \tag{3.7}$$

where $\|\phi - \phi^*\| = \max_{1 \leq i \leq n} \sup_{-\tau \leq s \leq 0} |\phi_i(s) - \phi_i^*(s)|$.

Define $V(t) = (V_1(t), V_2(t), \ldots, V_n(t))^T$ by

$$V_i(t) = e^{\alpha t} |z_i(t)|, \quad i = 1, 2, \ldots, n. \tag{3.8}$$

In view of (3.2) and (3.8), for $t \neq t_k$, we have

$$\begin{aligned} \frac{d^+ V_i(t)}{dt} &= \alpha e^{\alpha t} |z_i(t)| + e^{\alpha t} \operatorname{sgn} z_i(t) \left\{ -c_i(t) z_i(t) + \sum_{j=1}^{n} a_{ij}(t) g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}(t) g_j(z_j(t - \tau_{ij}(t))) \right\} \\ &\leq (\alpha - c_i(t)) e^{\alpha t} |z_i(t)| + e^{\alpha t} \sum_{j=1}^{n} |a_{ij}(t)| k_j |z_j(t)| + e^{\alpha t} \sum_{j=1}^{n} |b_{ij}(t)| k_j |z_j(t - \tau_{ij}(t))| \\ &\leq \left(\alpha - c_i^m \right) V_i(t) + \sum_{j=1}^{n} a_{ij}^M k_j V_j(t) + e^{\alpha\tau} \sum_{j=1}^{n} b_{ij}^M k_j V_j(t - \tau_{ij}(t)). \end{aligned} \tag{3.9}$$

We claim that

$$V_i(t) = e^{\alpha t} |z_i(t)| \leq \|\phi - \phi^*\| \quad \text{for } t \in (0, t_1), \; i = 1, 2, \ldots, n. \tag{3.10}$$

If not, then there exist $i_0 \in \{1, 2, \ldots, n\}$ and $0 < \bar{t} < t_1$ such that

$$V_{i_0}\left(\bar{t}\right) = \|\phi - \phi^*\|, \qquad \frac{d^+ V_{i_0}\left(\bar{t}\right)}{dt} > 0, \qquad V_i(t) \leq \|\phi - \phi^*\|, \tag{3.11}$$

for $t \in (-\tau, \bar{t}],\ i = 1, 2, \ldots, n$. Then, it follows from (3.9) and (3.11) that

$$0 < \frac{d^+ V_{i_0}\left(\bar{t}\right)}{dt} \le (\alpha - c_i^m) V_i\left(\bar{t}\right) + \sum_{j=1}^{n} a_{ij}^M k_j V_j\left(\bar{t}\right) + e^{\alpha\tau} \sum_{j=1}^{n} b_{ij}^M k_j V_j\left(\bar{t} - \tau_{ij}\left(\bar{t}\right)\right) \\ \le \left(\alpha - c_{i_0}^m + e^{\alpha\tau} \sum_{j=1}^{n} \left(a_{ij}^M + b_{ij}^M\right) k_j\right) \|\phi - \phi^*\|. \tag{3.12}$$

Equation (3.12) leads to

$$\alpha - c_{i_0}^m + e^{\alpha\tau} \sum_{j=1}^{n} \left(a_{i_0 j}^M + b_{i_0 j}^M\right) k_j > 0, \tag{3.13}$$

which contradicts (3.6). Thus (3.10) holds, that is,

$$z_i(t) \le \|\phi - \phi^*\| e^{-\alpha t}, \quad \text{for any } t \in [0, t_1),\ i = 1, 2, \ldots, n. \tag{3.14}$$

If $t = t_1$, we have

$$\left|z_i\left(t_1^+\right)\right| = \left|\left(1 + q_i^1\right) z_i(t_1)\right| = \left|1 + q_i^1\right| \lim_{t \to t_1^-} |z_i(t)| \le \left|1 + q_i^1\right| \|\phi - \phi^*\| e^{-\alpha t_1}, \tag{3.15}$$

for $i = 1, 2, \ldots, n$. Similar to the steps of (3.10)–(3.14), we can derive that

$$|z_i(t)| \le \left|1 + q_i^1\right| \|\phi - \phi^*\| e^{-\alpha t}, \quad \text{for } t \in [t_1, t_2),\ i = 1, 2, \ldots, n. \tag{3.16}$$

If $t = t_2$, then

$$\left|z_i\left(t_2^+\right)\right| = \left|\left(1 + q_i^2\right) z_i(t_2)\right| \le \left|\left(1 + q_i^1\right)\left(1 + q_i^2\right)\right| \|\phi - \phi^*\| e^{-\alpha t_2}. \tag{3.17}$$

By repeating the same procedure, then

$$|z_i(t)| \le \left|\left(1 + q_i^1\right)\left(1 + q_i^2\right) \cdots \left(1 + q_i^p\right)\right| \cdot \|\phi - \phi^*\| e^{-\alpha t}, \quad t \in (t_p, t_{p+1}),\ i = 1, 2, \ldots, n. \tag{3.18}$$

It follows from (H_6) that $|1 + q_i^k| \le e^{\zeta(t_k - t_{k-1})}$, which leads to

$$\left|\left(1 + q_i^1\right)\left(1 + q_i^2\right) \cdots \left(1 + q_i^p\right)\right| \le e^{\zeta(t_1 - t_0)} e^{\zeta(t_2 - t_1)} \cdots e^{\zeta(t_p - t_{p-1})} \le e^{\zeta t} e^{\zeta(\omega - t_p)}, \tag{3.19}$$

for any $t \in [t_k, t_{k+1}),\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots$. So the combination (3.18) and (3.19) gives

$$|z_i(t)| \le e^{\zeta(\omega - t_p)} \|\phi - \phi^*\| e^{-(\alpha - \zeta)t}, \quad t \in [t_k, t_{k+1}),\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots. \tag{3.20}$$

In addition, it is clear that

$$|z_i(t)| \le e^{\zeta(\omega-t_p)}\|\phi-\phi^*\|e^{-(\alpha-\zeta)t}, \quad t\in[0,t_1),\ i=1,2,\ldots,n. \tag{3.21}$$

Therefore, from (3.20) and (3.21), for any $t>0$, we have

$$\left|x_i(t)-x_i^*(t)\right| = |z_i(t)| \le e^{\zeta(\omega-t_p)}\|\phi-\phi^*\|e^{-(\alpha-\zeta)t}, \quad i=1,2,\ldots,n. \tag{3.22}$$

It implies that the ω-periodic solution $x^*(t)$ of (1.2) is globally exponentially stable. Hence, (1.2) admits a unique ω-periodic solution, which is globally exponentially stable. This completes the proof. □

Remark 3.3. Theorem 3.2 implies that the impulse q_i^k affects the existence and exponential stability of the periodic solution of system (1.2). It shows the dynamics of impulsive differential system (1.2) is richer than the corresponding system (1.1) without impulse.

4. Applications and Examples

In (1.2), if $a_{ij}(t)\equiv 0$, then (1.2) reads:

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t)+\sum_{j=1}^n b_{ij}(t)f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)+J_i(t), \quad t\ne t_k, \\ x_i(t^+) &= \left(1+q_i^k\right)x_i(t), \quad t=t_k. \end{aligned} \tag{4.1}$$

For system (4.1), we have the following result.

Proposition 4.1. *Suppose that* (H_1)–(H_3) *hold. Further,*

(H_7) $\max_{1\le i\le n}(\sigma_i/(\sigma_i-1))(\sum_{j=1}^n \overline{b_{ij}}k_j+\sum_{k=1}^p|q_i^k|) := \theta<1$,

(H_8) $-c_i^m+\sum_{j=1}^n b_{ij}^M k_j<0$,

(H_9) $\ln\max_{1\le i\le n}|1+q_i^k|/(t_k-t_{k-1})\le\zeta<\alpha$,

where $\max_{1\le i\le n}|1+q_i^k|\ge 1, \zeta>0$ *is a constant,* α *is determined in Theorem 3.2.*

Then system (4.1) *admits a unique* ω*-periodic solution, which is globally exponentially stable.*

If the impulses are absent in system (1.2), that is, $q_i^k\equiv 0$, then (1.2) leads to (1.1). Similarly we have the following.

Proposition 4.2. *Suppose that* (H_1)–(H_3) *hold. Further,*

(H_{10}) $\max_{1\le i\le n}(\sigma_i/(\sigma_i-1))\sum_{j=1}^n(\overline{a_{ij}}+\overline{b_{ij}})k_j := \theta<1$,

(H_{11}) $-c_i^m+\sum_{j=1}^n(a_{ij}^M+b_{ij}^M)k_j<0$,

then system (1.1) *admits a unique* ω*-periodic solution, which is globally exponentially stable.*

Remark 4.3. Proposition 4.2 implies that the sufficient conditions of the existence and globally exponential stability of periodic solution to (1.1) are independent of the time-varying delays, while the corresponding results obtained by authors [5] are dependent on delays. Without effect from time-varying delays, our results are better for people to keep the stability of system (1.1). Although the authors [1] also established similar conditions which are independent of delays, their employed tool and analysis techniques are very different so that their main results are different from ours. Particularly, (1.1) is the special case of (1.2) without impulse. Hence, in this sense, results of this paper complement or improve some previously known results [1, 5].

Finally, two examples and numerical analysis are given to show the usefulness of the main results.

Example 4.4. Let

$$\begin{aligned}
x_1'(t) &= -c_1(t)x_1(t) + b_{11}(t)f(x_1(t-\tau_{11}(t))) + b_{12}(t)f(x_2(t-\tau_{12}(t))) + J_1(t),\\
x_2'(t) &= -c_2(t)x_2(t) + b_{21}(t)f(x_1(\mathrm{t}-\tau_{21}(t))) + b_{22}(t)f(x_2(t-\tau_{22}(t))) + J_2(t),\\
\Delta x_i(t) &= \left(1+q_i^k\right)x_i(t), \quad i=1,2,
\end{aligned} \tag{4.2}$$

where $f(x) = x$, $c_1(t) = 1+\sin \pi t/4$, $c_2(t) = 2+\cos \pi t/4$, $a_{11}(t) = a_{12}(t) = a_{21}(t) = a_{22}(t) = 0$, $b_{11}(t) = 1/8+\sin \pi t/16$, $b_{12}(t) = 1/8+\cos \pi t/16$, $b_{21}(t) = 1/16-\cos \pi t/32$, $b_{22}(t) = 1/16-\sin \pi t/24$, $\tau_{11}(t) = \tau_{12}(t) = \cos \pi t$, $\tau_{21}(t) = \tau_{22}(t) = 1/2-\sin \pi t/3$, $J_1(t) = 5+2\cos \pi t$, $J_2(t) = 7-\sin \pi t$, $q_i^k = 1/8$, $t_k = k-1/2$. Then $k_1 = k_2 = 1, \omega = 2, \tau = 1, \{t_k, k = 1,2,\ldots\} \cap [0,2] = \{t_1, t_2\}$.

By easy computation, $\sigma_1 = e^2, \sigma_2 = e^4$, and $\theta \approx 0.8674 < 1$, which implies (H_4) holds. On the other hand, it is easy to verify that (H_5) holds. By verification, $\alpha > 1/4 > \ln(1+(1/8))$, namely, (H_6) holds too. From Theorems 2.5 and 3.2, we obtain that (4.2) has a unique 2-periodic solution, which is globally exponentially stable, see Figure 1.

Example 4.5. Let

$$\begin{aligned}
x_1'(t) = &-\left(\frac{1}{2}+\frac{\sin 2\pi t}{4}\right)x_1(t) + \left(\frac{1}{4}+\frac{\cos 2\pi t}{6}\right)f(x_1(t-(2+\sin 2\pi t)))\\
&+\left(\frac{1}{4}-\frac{\sin 2\pi t}{8}\right)f(x_2(t-(3-\sin 2\pi t)) + \cos 2\pi t),\\
x_2'(t) = &-\left(\frac{1}{2}+\frac{\cos 2\pi t}{4}\right)x_2(t) + \left(\frac{1}{3}+\frac{\cos 2\pi t}{4}\right)f(x_1(t-(5-\sin 2\pi t)))\\
&+\left(\frac{1}{6}-\frac{\cos 2\pi t}{8}\right)f(x_2(t-(1+\cos 2\pi t)) + \sin 2\pi t),
\end{aligned} \tag{4.3}$$

where $f(x) = (1/4)x$ for $x \in R$, $c_1(t) = 1/2+\sin 2\pi t/4$,$c_2(t) = 1/2+\cos 2\pi t/4$,$a_{11}(t) = a_{12}(t) = a_{21}(t) = a_{22}(t) = 0$, $b_{11}(t) = 1/4+\cos 2\pi t/6$, $b_{12}(t) = 1/4-\sin 2\pi t/8$, $b_{21}(t) = 1/3+\cos 2\pi t/4$, $b_{22}(t) = 1/6-\cos 2\pi t/8$. Then $k_1 = k_2 = (1/4), \omega = 1$.

By computation, $\theta \approx 0.318 < 1$, which implies that (H_{10}) holds. It is easy to verify that (H_{11}) holds too. From Proposition 4.2, system (4.3) has a unique 1-periodic solution, which is globally exponentially stable, see Figure 2. However, by calculation, conditions of the results

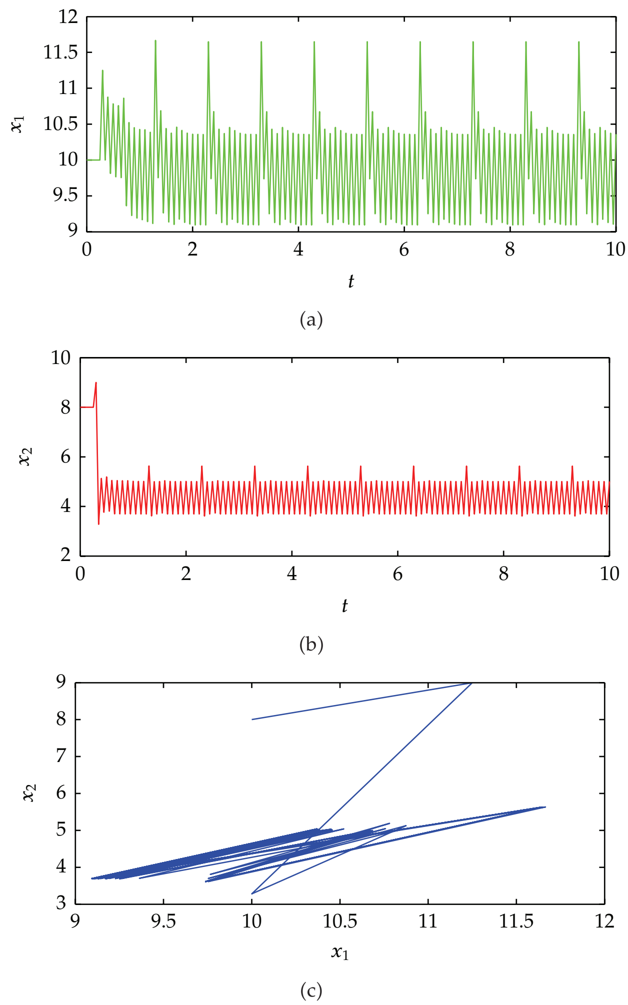

Figure 1: Dynamics of (4.2)—(a) time series of x_1, (b) time series of x_2, and (c) portrait of (x_1, x_2).

of [1] fail, then one cannot obtain the existence of periodic solution of system (4.3) by results of reference [1], which further shows that the results complement or improve previously known results.

5. Conclusions

In this paper, the existence and globally exponential stability of the periodic solution of system (1.2) are studied. Model (1.2) is very general, including such models as continuous bidirectional associative memory networks, cellular neural networks, and Hopfield-type neural networks (see, e.g., [6, 7, 28]). The main methods employed here are Schaeffer' theorem, differential inequality techniques, and Lyapunov functional, which are very different from [1]. The sufficient conditions obtained here are new and complement or improve

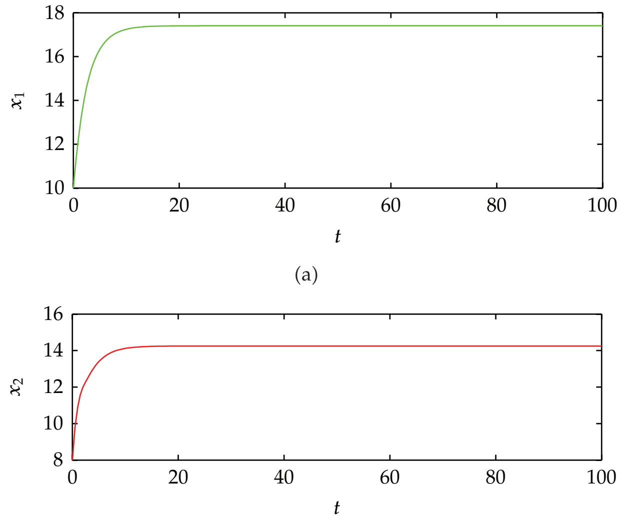

Figure 2: Dynamics of (4.3)—(a) time series of x_1, (b) time series of x_2.

the previously known results [1, 5–7]. Finally, applications, two illustrative examples and simulations, are given to show the effectiveness of the main results.

Acknowledgments

The authors would like to thank the reviewers for their valuable comments and constructive suggestions, which are very useful for improving the quality of this paper. This paper is supported by National Natural Science Foundation of China (11161015) and Doctoral Foundation of Guilin University of Technology (2010).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 493271, 17 pages
doi:10.1155/2012/493271

Research Article

Monotone-Iterative Method for the Initial Value Problem with Initial Time Difference for Differential Equations with "Maxima"

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Received 13 February 2012; Accepted 6 March 2012

Academic Editor: Josef Diblík

The object of investigation of the paper is a special type of functional differential equations containing the maximum value of the unknown function over a past time interval. An improved algorithm of the monotone-iterative technique is suggested to nonlinear differential equations with "maxima." The case when upper and lower solutions of the given problem are known at different initial time is studied. Additionally, all initial value problems for successive approximations have both initial time and initial functions different. It allows us to construct sequences of successive approximations as well as sequences of initial functions, which are convergent to the solution and to the initial function of the given initial value problem, respectively. The suggested algorithm is realized as a computer program, and it is applied to several examples, illustrating the advantages of the suggested scheme.

1. Introduction

Equations with "maxima" find wide applications in the theory of automatic regulation. As a simple example of mathematical simulations by means of such equations, we shall consider the system of regulation of the voltage of a generator of constant current ([1]). The object of regulation is a generator of constant current with parallel stimulation, and the quantity regulated is the voltage on the clamps of the generator feeding an electric circuit with different loads. An equation with maxima is used if the regulator is constructed such that the maximal deviation of the quantity is regulated on the segment $[t-r, t]$. The equation describing the work of the regulator has the form

$$Tu'(t) + u(t) + q \max_{s\in[t-r,t]} u(s) = f(t), \tag{1.1}$$

where T and q are constant characterizing the object, $u(t)$ is the voltage regulated, and $f(t)$ is the perturbing effect.

Note that the above given model as well as all types of differential equations with "maxima" could be considered as delay functional differential equations. Differential equations with delay are studied by many authors (see, e.g., [2–14]). At the same time, the presence of maximum function in the equations causes impossibilities of direct application of almost all known results for delay functional differential equations (see the monograph [15]). Also, in the most cases, differential equations with "maxima," including the some linear scalar equations, are not possible to be solved in an explicit form. That requires the application of approximate methods. In the recent years, several effective approximate methods, based on the upper and lower solutions of the given problem, was proved for various problems of differential equations in [16–21].

In the current paper, an approximate method for solving the initial value problem for nonlinear differential equations with "maxima" is considered. This method is based on the method of lower and upper solutions. Meanwhile, in studying the initial value problems the authors usually keep the initial time unchanged. But, in the real repeated experiments it is difficult to keep this time fixed because of all kinds of disturbed factors. It requires the changing of initial time to be taken into consideration. Note that several qualitative investigations of the solutions of ordinary differential equations with initial time difference are studied in [22–30]. Also some approximate methods for various types of differential equations with initial time difference are proved in [23, 26, 31–33].

In this paper, an improved algorithm of monotone-iterative techniques is suggested to nonlinear differential equations with "maxima." The case when upper and lower solutions of the given problem are known at different initial time is studied. The behavior of the lower/upper solutions of the initial value problems with different initial times is studied. Also, some other improvements in the suggested algorithm are given. The main one is connected with the initial functions. In the known results for functional differential equations, the initial functions in the corresponding linear problems for successive approximations are the same (see, e.g., [20, 34–42]). In our case, it causes troubles with obtaining the solutions, which are successive approximations. To avoid these difficulties, we consider linear problems with different initial functions at any step. It allows us to chose appropriate initial data. Additionally, in connection with the presence of the maximum function in the equation and computer application of the suggested algorithm, an appropriate computer program is realized and it is applied to some examples illustrating the advantages and the practical application of the considered algorithm.

2. Preliminary Notes and Definitions

Consider the following initial value problem for the nonlinear differential equation with "maxima" (IVP):

$$x' = f\left(t, x(t), \max_{s\in[t-r,t]} x(s)\right) \quad \text{for } t \in [t_0, \infty), \tag{2.1}$$

$$x(t) = \varphi(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{2.2}$$

where $x \in \mathbb{R}$, $t_0 \geq 0$, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi(t) : [-r, 0] \to \mathbb{R}$, and $r > 0$ is a fixed constant.

In the paper, we will study the differential equation with "maxima" (2.1) with an initial condition at two different initial points. For this purpose, we consider the differential equation (2.1) with the initial condition

$$x(t) = \varphi(t - \tau_0) \quad \text{for } t \in [\tau_0 - r, \tau_0], \tag{2.3}$$

where $\tau_0 \geq 0$.

Assume there exist a solution $x(t; t_0, \varphi)$ of the IVP (2.1), (2.2) defined on $[t_0 - r, t_0 + T]$, and there exists a solution $y(t; \tau_0, \varphi)$ of the IVP (2.1), (2.3) defined on $[\tau_0 - r, \tau_0 + T]$, where $T > 0$ is a given constant. In the paper, we will compare the behavior of both solutions.

Definition 2.1. The function $\alpha \in C([t_0 - r, t_0 + T), \mathbb{R}) \cup C^1([t_0, t_0 + T), \mathbb{R})$ is called a lower (upper) solution of the IVP (2.1), (2.2) for $t \geq t_0 - h$ if the following inequalities are satisfied:

$$\begin{gathered} \alpha'(t) \leq (\geq) f\left(t, \alpha(t), \max_{s \in [t-r,t]} \alpha(s)\right) \quad \text{for } t \in [t_0,\ t_0 + T], \\ \alpha(t) \leq (\geq) \varphi(t - t_0) \quad \text{for } t \in [t_0 - r, t_0]. \end{gathered} \tag{2.4}$$

Note that the function $\alpha(t)$ is a lower (upper) solution of the IVP (2.1), (2.3) for $t \in [\tau_0 - r, \tau_0 + T]$ if the point t_0 in the inequalities (2.4) is replaced by τ_0.

3. Comparison Results

Often in the real world applications, the lower and upper solutions of one and the same differential equation are obtained at different initial time intervals.

The following result is a comparison result for lower and upper solutions with initial conditions given on different initial time intervals.

Theorem 3.1. *Let the following conditions be satisfied.*

(1) *Let $t_0, \tau_0 \geq 0$ be fixed such that $\eta = \tau_0 - t_0 > 0$.*

(2) *The function $\alpha \in C([t_0 - r, t_0 + T], \mathbb{R}) \cup C^1([t_0, t_0 + T], \mathbb{R})$ is a lower solution of the IVP* (2.1), (2.2).

(3) *The function $\beta \in C([\tau_0 - r, \tau_0 + T], \mathbb{R}) \cup C^1([\tau_0, \tau_0 + T], \mathbb{R})$ is an upper solution of the IVP* (2.1), (2.3).

(4) *The function $f(t, u, v) : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing in its both first and third argument t for any $u \in \mathbb{R}$, and there exist positive constants L_1, L_2 such that for $t \in \mathbb{R}_+$ and $u_1 \geq u_2$ and $v_1 \geq v_2$, the inequality*

$$f(t, u_1, v_1) - f(t, u_2, v_2) \leq L_1(u_1 - u_2) + L_2(v_1 - v_2) \tag{3.1}$$

holds.

(5) *The function $\varphi(t) \in C([-r, 0], \mathbb{R})$.*

Then $\alpha(t) \leq \beta(t + \eta)$ for $t \in [t_0, t_0 + T]$.

Proof. Choose a positive number M such that $M > L_1 + L_2$ and a positive number ϵ. Define a function $w(t) = \beta(t+\eta) + \epsilon e^{Mt}$ for $t \in [t_0, t_0+T]$ and $w(t) = \beta(t+\eta) + \epsilon e^{Mt_0}$ for $t \in [t_0-r, t_0)$. Then for $t \in [t_0, t_0+T]$, the following inequalities

$$\begin{gathered}
w(t) > \beta(t+\eta), \\
\max_{s\in[t-r,t]} w(s) = \max_{s\in[t-r,t]} \left(\beta(s+\eta) + \epsilon e^{Ms}\right) \geq \max_{s\in[t-r,t]} (\beta(s+\eta)) = \max_{s\in[t+\eta-r,t+\eta]} \beta(s), \\
\max_{s\in[t-r,t]} w(s) \leq \max_{s\in[t-r,t]} (\beta(s+\eta)) + \epsilon e^{Mt} = \max_{s\in[t+\eta-r,t+\eta]} \beta(s) + \epsilon e^{Mt}
\end{gathered} \tag{3.2}$$

hold.

Therefore, we obtain

$$\begin{aligned}
w'(t) &\geq f\left(t+\eta, \beta(t+\eta), \max_{s\in[t+\eta-r,t+\eta]} \beta(s)\right) + \epsilon M e^{Mt} \\
&\geq -L_1(w(t) - \beta(t+\eta)) - L_2\left(\max_{s\in[t-r,t]} w(s) - \max_{s\in[t+\eta-r,t+\eta]} \beta(s)\right) \\
&\quad + f\left(t+\eta, w(t), \max_{s\in[t-r,t]} w(s)\right) + \epsilon M e^{Mt} \\
&\geq f\left(t+\eta, w(t), \max_{s\in[t-r,t]} w(s)\right) + \epsilon e^{Mt}(M - L_1 - L_2) \\
&> f\left(t+\eta, w(t), \max_{s\in[t-r,t]} w(s)\right) \geq f\left(t, w(t), \max_{s\in[t-r,t]} w(s)\right).
\end{aligned} \tag{3.3}$$

Note that for $t \in [t_0-r, t_0]$ we have $\alpha(t) \leq \varphi(t-t_0) = \varphi(t+\eta-\tau_0) \leq \beta(t+\eta) < w(t)$. We will prove that

$$\alpha(t) \leq w(t) \quad \text{for } t \in [t_0-r, t_0+T]. \tag{3.4}$$

Assume the contrary, that is, there exists a point $t_1 > t_0$ such that $\alpha(t) < w(t)$ for $t \in [t_0-r, t_1)$, $\alpha(t_1) = w(t_1)$ and $\alpha(t) \geq w(t)$ for $t \in (t_1, t_2)$, where t_2 is sufficiently close to t_1. Therefore $\alpha'(t_1) \geq w'(t_1)$. Then $w'(t_1) \leq \alpha'(t_1) \leq f(t_1, \alpha(t_1), \max_{s\in[t_1-r,t_1]}\alpha(s)) \leq f(t_1, w(t_1), \max_{s\in[t_1-r,t_1]} w(s)) < w'(t_1)$. The obtained contradiction proves the claim.

Therefore, $\alpha(t) \leq w(t) < \beta(t+\eta)$ for $t \in [t_0-r, t_0+T]$. □

Corollary 3.2. *Let the conditions of Theorem 3.1 be fulfilled. Then $\alpha(t-\eta) \leq \beta(t)$ for $t \in [\tau_0-r, \tau_0+T]$.*

The comparison results are true if the inequality $\tau_0 < t_0$ holds.

Theorem 3.3 (comparison result). *Let the following conditions be satisfied.*

(1) *The conditions 2, 3, 5 of Theorem 3.1 are satisfied.*

(2) *Let $t_0, \tau_0 \geq 0$ be fixed such that $\eta = \tau_0 - t_0 < 0$.*

(3) *The function* $f(t,u,v) : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is nonincreasing in its first argument and is nondecreasing in its third argument for any* $u \in \mathbb{R}$, *and there exist positive constants* $L_1,\ L_2$ *such that for* $t \in \mathbb{R}_+$ *and* $u_1 \geq u_2$ *and* $v_1 \geq v_2$ *the inequality*

$$f(t,u_1,v_1) - f(t,u_2,v_2) \leq L_1(u_1 - u_2) + L_2(v_1 - v_2) \tag{3.5}$$

holds.

Then $\alpha(t) \leq \beta(t+\eta)$ *for* $t \geq t_0$.

The proof of Theorem 3.3 is similar to the proof of Theorem 3.1 and we omit it.

In what follows, we shall need that the functions $\alpha, \beta \in C([t_0 - r, T + t_0], \mathbb{R})$ are such that $\alpha(t) \leq \beta(t+\eta)$.

Consider the sets:

$$\begin{aligned} S(\alpha,\beta,t_0,\eta) &= \{u \in C([t_0 - r, T + t_0], \mathbb{R}) : \alpha(t) \leq u(t) \leq \beta(t+\eta) \text{ for } t \in [t_0 - r, T + t_0]\}, \\ \widetilde{S}(\alpha,\beta,\tau_0,\eta) &= \{u \in C([\tau_0 - r, T + \tau_0], \mathbb{R}) : \alpha(t-\eta) \leq u(t) \leq \beta(t) \text{ for } t \in [\tau_0 - r, T + \tau_0]\}, \\ \Omega(\alpha,\beta,t_0) &= \left\{ (t,x,y) \in [t_0, T + t_0] \times \mathbb{R}^2 : \alpha(t) \leq x \leq \beta(t),\ \max_{s\in[t-r,t]} \alpha(s) \leq y \leq \max_{s\in[t-r,t]} \beta(s) \right\}. \end{aligned} \tag{3.6}$$

In our further investigations, we will need the following comparison result on differential inequalities with "maxima."

Lemma 3.4 (see [43, Lemma 2.1]). *Let the function* $m \in C([t_0 - r, T + t_0], \mathbb{R}) \cup C^1([t_0, T + t_0], \mathbb{R})$ *satisfies the inequalities*

$$\begin{gathered} m' \leq -L_1 m(t) - L_2 \min_{s\in[t-r,t]} m(s) \quad \textit{for } t \in [t_0, T + t_0], \\ m(t) \leq 0 \quad \textit{for } t \in [t_0 - r, t_0], \end{gathered} \tag{3.7}$$

where the positive constants L_1, L_2 *are such that* $(L_1 + L_2)T \leq 1$.

Then the inequality $m(t) \leq 0$ *holds on* $[t_0 - r, T + t_0]$.

In our further investigations, we will use the following result, which is a partial case of Theorem 3.1 [44].

Lemma 3.5 (existence and uniqueness result). *Let the following conditions be fulfilled.*

(1) *The functions* $Q \in C([t_0, t_0 + T], \mathbb{R}),\ q \in C([t_0, t_0 + T], \mathbb{R})$.

(2) *The functions* $\varphi \in C([-r, 0], \mathbb{R})$.

Then the initial value problem for the linear scalar equation

$$\begin{gathered} u' = L_1 u(t) + L_2 \max_{s\in[t-r,t]} u(s) + Q(t) \quad \textit{for } t \in [t_0, T + t_0], \\ u(t) = \varphi(t - t_0) \quad \textit{for } t \in [t_0 - r, t_0] \end{gathered} \tag{3.8}$$

has a unique solution $u(t - t_0)$ *on the interval* $[t_0 - r, T + t_0]$.

4. Main Results

Denote $J_1 = [t_0 - r, T + t_0]$ and $J_2 = [\tau_0 - r, T + \tau_0]$, where $t_0, \tau_0 \geq 0$ are fixed numbers.

Case 1. Let $t_0 \leq \tau_0$.

Theorem 4.1. *Let the following conditions be fulfilled.*

(1) *The points $t_0, \tau_0 \geq 0$ are such that $\eta = \tau_0 - t_0 \geq 0$.*

(2) *The function $\alpha_0(t) \in C([t_0 - r, T + t_0], \mathbb{R}) \cup C^1([t_0, T + t_0], \mathbb{R})$ is a lower solution of the IVP* (2.1), (2.2) *in J_1.*

(3) *The function $\beta_0(t) \in C([\tau_0 - r, T + \tau_0], \mathbb{R}) \cup C^1([\tau_0, T + \tau_0], \mathbb{R})$ is an upper solution of the IVP* (2.1), (2.3) *in J_2.*

(4) *The function $\varphi \in C([-r, 0], \mathbb{R})$.*

(5) *The function $f \in C(\Omega(\alpha_0, \beta_0, t_0), \mathbb{R})$ is nondecreasing in its first argument and satisfies the one side Lipschitz condition*

$$f(t, u_1, v_1) - f(t, u_2, v_2) \geq L_1(u_2 - u_1) + L_2(v_2 - v_1), \tag{4.1}$$

for $u_1 \leq u_2$ and $v_1 \leq v_2$, where $L_1, L_2 > 0$ are such that

$$(L_1 + L_2)T \leq 1. \tag{4.2}$$

Then there exist two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ such that

(a) *the functions $\alpha_n(t) = 1, 2, \ldots$ are lower solutions of the IVP* (2.1), (2.2) *on $[t_0 - r, T + t_0]$;*

(b) *the functions $\beta_n(t) = 1, 2, \ldots$ are upper solutions of the IVP* (2.1), (2.3) *on $[\tau_0 - r, T + \tau_0]$;*

(c) *the sequence $\{\alpha_n(t)\}_0^\infty$ is increasing;*

(d) *the sequence $\{\beta_n(t)\}_0^\infty$ is decreasing;*

(e) *the inequalities*

$$\begin{aligned} &\alpha_0(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t + \eta) \leq \cdots \leq \beta_0(t + \eta), \quad t \in [t_0 - r, T + t_0], \\ &\alpha_0(t - \eta) \leq \cdots \leq \alpha_n(t - \eta) \leq \beta_n(t) \leq \cdots \leq \beta_0(t), \quad t \in [\tau_0 - r, T + \tau_0] \end{aligned} \tag{4.3}$$

hold;

(f) *both sequences uniformly converge and $x(t) = \lim_{n\to\infty}\alpha_n(t)$ is a solution of the IVP* (2.1), (2.2) *in $S(\alpha_0, \beta_0, t_0, \eta)$, and $y(t) = \lim_{n\to\infty}\beta_n(t)$ is a solution of the IVP* (2.1), (2.3) *in $\widetilde{S}(\alpha_0, \beta_0, \tau_0, \eta)$.*

Proof. According to Theorem 3.1, the inequality $\alpha_0(t) \leq \beta_0(t + \eta)$ holds on $[t_0 - r, T + t_0]$.

Let $L_0 = \min_{s\in[t_0-r,t_0]}(\varphi(s - t_0) - \alpha_0(s)) \geq 0$. Choose a number $k_0 \in [0, 1)$ such that

$$k_0 \leq L_0. \tag{4.4}$$

Therefore, $\varphi(t-t_0)-\alpha_0(t) \geq L_0 > k_0 L_0$ or $\alpha_0(t) \leq \varphi(t-t_0)-k_0L_0$ on $[t_0-r, t_0]$.

We consider the linear differential equation with "maxima"

$$\begin{aligned} x'(t) = f\left(t, \alpha_0(t), \max_{s\in[t-r,t]} \alpha_0(s)\right) - L_1(x(t)-\alpha_0(t)) \\ - L_2\left(\max_{s\in[t-r,t]} x(s) - \max_{s\in[t-r,t]} \alpha_0(s)\right) \quad \text{for } t \in [t_0, T+t_0], \end{aligned} \tag{4.5}$$

with the initial condition

$$x(t) = \varphi(t-t_0) - k_0 L_0 \quad \text{for } t \in [t_0 - r, t_0]. \tag{4.6}$$

According to Lemma 3.5, the linear initial value problem (4.5), (4.6) has a unique solution $\alpha_1(t)$ in J_1.

We will prove that $\alpha_0(t) \leq \alpha_1(t)$ on J_1.

From the choice of L_0, k_0 and equality (4.6), it follows that $\alpha_0(t) \leq \varphi(t-t_0)-k_0L_0 = \alpha_1(t)$ for $t \in [t_0-r, t_0]$.

Now, let $t \in [t_0, T+t_0]$. Consider the function $u(t) = \alpha_0(t) - \alpha_1(t)$ defined on J_1. It is clear that $u(t) \leq 0$ on $[t_0-r, t_0]$. From the definition of the function $\alpha_0(t)$ and (4.5), we have

$$u'(t) \leq -L_1 u(t) - L_2\left(\max_{s\in[t-r,t]} \alpha_0(s) - \max_{s\in[t-r,t]} \alpha_1(s)\right) \quad \text{for } t \in [t_0, T+t_0]. \tag{4.7}$$

Now from inequality (4.7) and

$$\begin{aligned} \max_{s\in[t-h,t]} \alpha_0(s) - \max_{s\in[t-r,t]} \alpha_1(s) &= \max_{s\in[t-r,t]} \alpha_0(s) - \alpha_1(\xi) \\ &\geq \alpha_0(\xi) - \alpha_1(\xi) \\ &\geq \min_{s\in[t-r,t]} (\alpha_0(s) - \alpha_1(s)) \\ &= \min_{s\in[t-r,t]} u(s), \end{aligned} \tag{4.8}$$

we obtain

$$u'(t) \leq -L_1 u(t) - L_2 \min_{s\in[t-r,t]} u(s), \quad t \in [t_0, T+t_0]. \tag{4.9}$$

According to Lemma 3.4, the inequality $u(t) \leq 0$ holds on J_1, that is, $\alpha_0(t) \leq \alpha_1(t)$.

We will prove that the function $\alpha_1(t)$ is a lower solution of the IVP (2.1), (2.2) on J_1.

From equality (4.6), it follows the validity of the inequality $\alpha_1(t) \leq \varphi(t-t_0)$ on $[t_0-r, t_0]$. Now, let $t \in [t_0, T+t_0]$. Then, since $\alpha_0(t) \leq \alpha_1(t)$, according to the one side Lipschitz condition 2 of Theorem 4.1, we have

$$\begin{aligned} \alpha_1'(t) &= Q(t) - L_1(\alpha_1(t)-\alpha_0(t)) - L_2\left(\max_{s\in[t-r,t]} \alpha_1(s) - \max_{s\in[t-r,t]} \alpha_0(s)\right) \\ &= f\left(t, \alpha_1(t), \max_{s\in[t-r,t]} \alpha_1(s)\right) + f\left(t, \alpha_0(t), \max_{s\in[t-r,t]} \alpha_0(s)\right) \end{aligned}$$

$$
\begin{aligned}
&- f\Big(t, \alpha_1(t), \max_{s\in[t-r,t]} \alpha_1(s)\Big) - L_1(\alpha_1(t) - \alpha_0(t)) \\
&- L_2\Big(\max_{s\in[t-r,t]} \alpha_1(s) - \max_{s\in[t-r,t]} \alpha_0(s)\Big) \\
\le{}& f\Big(t, \alpha_1(t), \max_{s\in[t-r,t]} \alpha_1(s)\Big) \quad \text{for } t \in [t_0, T + t_0],
\end{aligned}
\tag{4.10}
$$

where $Q(t) = f(t, \alpha_0(t), \max_{s\in[t-r,t]} \alpha_0(s))$.

Next, according to Lemma 3.4 for the function $u(t) = \alpha_1(t) - \beta_0(t - \eta)$, the inequality $\alpha_1(t) \le \beta_0(t + \eta)$ holds on J_1, that is, the inclusion $\alpha_1 \in S(\alpha_0, \beta_0, t_0, \eta)$ is valid.

Let $C_0 = \min_{s\in[\tau_0 - r, \tau_0]}(\beta_0(s) - \varphi(s - \tau_0)) \ge 0$. Choose a number $p_0 \in [0, 1)$ such that

$$p_0 \le C_0. \tag{4.11}$$

Therefore, $\beta_0(t) - \varphi(t - \tau_0) \ge C_0 > p_0 C_0$ or $\beta_0(t) \ge \varphi(t - \tau_0) + p_0 C_0$ on $[\tau_0 - r, \tau_0]$.

We consider the linear differential equation with "maxima"

$$
\begin{aligned}
x'(t) = f\Big(t, \beta_0(t), \max_{s\in[t-r,t]} \beta_0(s)\Big) - L_1(x(t) - \beta_0(t)) \\
- L_2\Big(\max_{s\in[t-r,t]} x(s) - \max_{s\in[t-r,t]} \beta_0(s)\Big) \quad \text{for } t \in [\tau_0, T + \tau_0]
\end{aligned}
\tag{4.12}
$$

with the initial condition

$$x(t) = \varphi(t + \tau_0) + p_0 C_0 \quad \text{for } t \in [\tau_0 - r, \tau_0]. \tag{4.13}$$

There exists a unique solution $\beta_1(t)$ of the IVP (4.12), (4.13), which is defined on J_2.

The function $\beta_1(t)$ is an upper solution of the IVP (2.1), (2.3) on J_2, and the inclusion $\beta_1 \in \widetilde{S}(\alpha_0, \beta_0, \tau_0, \eta)$ is valid. The proofs are similar to the ones about the function α_1. We omit the proofs.

Functions $\alpha_1(t), \beta_1(t)$ are a lower and an upper solution of the IVP (2.1), (2.2) and (2.1), (2.3) correspondingly. According to Lemma 3.4 the inequality $\alpha_1(t) \le \beta_1(t + \eta)$ on J_1 holds.

Similarly, recursively, we can construct two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$. In fact, if the functions $\alpha_n(t)$ and $\beta_n(t)$ are known, and $L_n = \min_{s\in[-r,0]}(\varphi(s) - \alpha_n(s))$, $C_n = \min_{s\in[-r,0]}(\beta_n(s) - \varphi(s))$, and the numbers $k_n, p_n \in [0, 1)$ are such that

$$k_n \le L_n, \qquad p_n \le C_n, \tag{4.14}$$

then the function $\alpha_{n+1}(t)$ is the unique solution of the initial value problem for the linear differential equation with "maxima"

$$\begin{aligned} x' = {} & f\left(t, \alpha_n(t), \max_{s\in[t-r,t]} \alpha_n(s)\right) - L_1(x - \alpha_n(t)) \\ & - L_2\left(\max_{s\in[t-r,t]} x(s) - \max_{s\in[t-r,t]} \alpha_n(s)\right) \quad \text{for } t \in [t_0, T + t_0], \end{aligned} \tag{4.15}$$

$$x(t) = \varphi(t) - k_n L_n \quad \text{for } t \in [t_0 - r, t_0], \tag{4.16}$$

and the function $\beta_{n+1}(t)$ is the unique solution of the initial value problem

$$\begin{aligned} x'(t) = {} & f\left(t, \beta_n(t), \max_{s\in[t-r,t]} \beta_n(s)\right) - L_1(x - \beta_n(t)) \\ & - L_2\left(\max_{s\in[t-h,t]} x(s) - \max_{s\in[t-r,t]} \beta_n(s)\right) \quad \text{for } t \in [\tau_0, T + \tau_0], \end{aligned} \tag{4.17}$$

$$x(t) = \varphi(t) + p_n C_n \quad \text{for } t \in [\tau_0 - r, \tau_0]. \tag{4.18}$$

Now following exactly as for the case $n = 0$, it can be proved that the function $\alpha_{n+1}(t)$ is a lower solution of the IVP (2.1), (2.2) on J_1, the function $\beta_{n+1}(t)$ is an upper solution of the IVP (2.1), (2.3) on J_2, the inclusions $\alpha_{n+1} \in S(\alpha_n, \beta_n, t_0, \eta)$, $\beta_{n+1} \in \widetilde{S}(\alpha_n, \beta_n, \tau_0, \eta)$ are valid, and the inequalities (4.3) hold.

Therefore, the sequence $\{\alpha_n(t)\}_0^\infty$ is uniformly convergent on J_1 and $\{\beta_n(t)\}_0^\infty$ is uniformly convergent on J_2.

Denote

$$\begin{aligned} \lim_{n\to\infty} \alpha_n(t) = u(t), \quad t \in [t_0 - r, T + t_0], \\ \lim_{n\to\infty} \beta_n(t) = v(t), \quad t \in [\tau_0 - r, T + \tau_0]. \end{aligned} \tag{4.19}$$

From the uniform convergence and the definition of the functions $\alpha_n(t)$ and $\beta_n(t)$, it follows that the following inequalities hold

$$\alpha_0(t) \le u(t) \le v(t + \eta) \le \beta_0(t + \eta), \quad t \in [t_0 - r, T + t_0]. \tag{4.20}$$

□

Case 2. Let $t_0 > \tau_0$.

In this case, we could approximate again the solution of the given initial value problem, starting from lower and upper solutions given at two different initial points. Since the proofs are similar, we will set up only the results.

Theorem 4.2. *Let the following conditions be fulfilled.*

(1) *The points $t_0, \tau_0 \ge 0$ are such that $\eta = t_0 - \tau_0 \ge 0$.*

(2) *The conditions 2, 3, 4 of Theorem 4.1 are satisfied.*

(3) *The function* $f \in C(\Omega(\alpha_0, \beta_0, \tau_0), \mathbb{R})$ *is nonincreasing in its first argument and satisfies the one side Lipschitz condition*

$$f(t, u_1, v_1) - f(t, u_2, v_2) \geq L_1(u_2 - u_1) + L_2(v_2 - v_1) \tag{4.21}$$

for $u_1 \leq u_2$ *and* $v_1 \leq v_2$, *where* $L_1, L_2 > 0$ *are such that*

$$(L_1 + L_2)T \leq 1. \tag{4.22}$$

Then there exist two sequences of functions $\{\alpha_n(t)\}_0^\infty$ *and* $\{\beta_n(t)\}_0^\infty$ *such that*

(a) *the functions* $\alpha_n(t) = 1, 2, \ldots$ *are lower solutions of the IVP* (2.1), (2.2) *on* $[t_0 - r, T + t_0]$;

(b) *the functions* $\beta_n(t) = 1, 2, \ldots$ *are upper solutions of the IVP* (2.1), (2.3) *on* $[\tau_0 - r, T + \tau_0]$;

(c) *the sequence* $\{\alpha_n(t)\}_0^\infty$ *is increasing;*

(d) *the sequence* $\{\beta_n(t)\}_0^\infty$ *is decreasing;*

(e) *the inequalities*

$$\begin{aligned} &\alpha_0(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t + \eta) \leq \cdots \leq \beta_0(t + \eta), \quad t \in [t_0 - r, T + t_0], \\ &\alpha_0(t - \eta) \leq \cdots \leq \alpha_n(t - \eta) \leq \beta_n(t) \leq \cdots \leq \beta_0(t), \quad t \in [\tau_0 - r, T + \tau_0] \end{aligned} \tag{4.23}$$

hold;

(f) *both sequences uniformly converge and* $x(t) = \lim_{n\to\infty}\alpha_n(t)$ *is a solution of the IVP* (2.1), (2.2) *in* $S(\alpha_0, \beta_0, t_0, \eta)$ *and* $y(t) = \lim_{n\to\infty}\beta_n(t)$ *is a solution of the IVP* (2.1), (2.3) *in* $\widetilde{S}(\alpha_0, \beta_0, \tau_0, \eta)$.

5. Computer Realization

Since the set of differential equations, which could be solved in an explicit form, is very narrow, we will realize the above suggested algorithm numerically. At present, we cannot solve numerically differential equations with "maxima" by the existing ready-made systems, such as *Mathematica, Mathlab*, and so forth, because of the maximum of the unknown function over a past time interval. It requires proposed and computer realized new algorithms for solving such kind of equations. In [43], an algorithm for solving a class of differential equations with "maxima," based on the trapezoid's method, is presented. The main problem in the suggested algorithm is obtaining the maximum of the unknown function over a past time interval. It is based on the idea that the local maximum depends on values that are initially inserted and later they are removed within the range. A special structure is applied to help quicker obtaining the local maximum ([43]). The suggested scheme could be based on any numerical method for solving differential equations. In this paper, we will use Euler method in the application of the algorithm ([43]) for solving differential equations with "maxima." We combine the algorithm of Euler method and obtaining of maximum of the function, and we get the following modification:

$$\begin{aligned} t_{k+1} &= t_k + h, \\ y_{n+1}(t_{k+1}) &= y_{n+1}(t_k) + hf\left(t_k, y_{n+1}(t_k), y_n(t_k), \max y_{n,k}, \max y_{n+1,k}\right), \end{aligned} \tag{5.1}$$

where the values $\max y_{n,k} = \max_{s\in[t_k-r,t_k]} y_n(s)$ and $\max y_{n+1,k} = \max_{s\in[t_k-r,t_k]} y_{n+1}(s)$ are obtained by the algorithm suggested in [43], and the initial values of $y_{n+1}(t_0)$ are found by initial conditions (4.16) and (4.18).

6. Applications

Now we will illustrate the employment of the suggested above scheme to a particular nonlinear scalar differential equations with "maxima."

Initially we will consider the case of one and the same initial points of both initial value problems. By this way, we will emphasize our considerations to the advantages of the involved in the initial conditions constants.

Example 6.1. Consider the following scalar nonlinear differential equation with "maxima":

$$x' = \frac{1}{1-x(t)} - 2\max_{s\in[t-0.1,t]} x(s) - 1, \quad \text{for } t \in [0, 0.35], \tag{6.1}$$

with the initial condition

$$x(t) = 0, \quad t \in [-0.1, 0], \tag{6.2}$$

where $x \in \mathbb{R}$. It is easy to check that the initial value problem (6.1), (6.2) has a zero solution.

In this case $f(t,x,y) \equiv 1/(1-x) - 2y - 1$, $t_0 = \tau_0 = 0$, $T = 0.35$, $r = 0.1$, $J_1 = J_2 = [-0.1, 0.35]$. Choose $\alpha_0(t) = -1/4$ and $\beta_0(t) = 1/4$. Then $\alpha_0(t)$ is a lower solution and $\beta_0(t)$ is an upper solution of the IVP (6.1), (6.2). In this case, $\Omega(\alpha_0,\beta_0,t_0) = \{(t,u,v) \in [0,0.35]\times\mathbb{R}^2 : -1/4 \le u, v \le 1/4\}$. Let $(t,u_1,v_1), (t,u_2,v_2) \in \Omega(\alpha_0,\beta_0,t_0)$ and $u_1 \le u_2$ and $v_1 \le v_2$. Then

$$\begin{aligned}
f(t,u_1,v_1) - f(t,u_2,v_2) &= \frac{1}{1-u_1} - \frac{1}{1-u_2} + 2(v_2 - v_1) \\
&= \frac{1}{(1-u_1)(1-u_2)}(u_2 - u_1) + 2(v_2 - v_1) \\
&\ge \frac{16}{25}(u_2 - u_1) + 2(v_2 - v_1).
\end{aligned} \tag{6.3}$$

That is, $L_1 = 0.64$, $L_2 = 2$, and the inequality $(L_1 + L_2)T < 1$ holds.

The successive approximations $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ will be the unique solutions of the linear problems

$$\begin{gathered}
x' = \frac{1}{1-\alpha_n(t)} + 0.64\alpha_n(t) - 1 - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35], \\
x(t) = -k_n L_n \quad \text{for } t \in [-0.1, 0], \\
x'(t) = \frac{1}{1-\beta_n(t)} - 1 + 0.64\beta_n(t) - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35], \\
x(t) = p_n C_n \quad \text{for } t \in [-0.1, 0],
\end{gathered} \tag{6.4}$$

where $L_n = \min_{s\in[-0.1,0]}(-\alpha_n(s)) \geq 0$, $C_n = \min_{s\in[-0.1,0]}(\beta_n(s)) \geq 0$, and the numbers $k_n, p_n \in [0,1)$ are chosen such that $k_n \leq L_n$, $p_n \leq C_n$, $n = 1,2,3,\ldots$.

Now we will construct an increasing sequence of lower solutions and a decreasing sequence of upper solutions, which will be convergent to the zero solution.

The first lower approximation $\alpha_1(t)$ is a solution of the IVP

$$x' = -\frac{9}{25} - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35],$$
$$x(t) = -\frac{k_0}{4}, \quad t \in [-0.1, 0]. \tag{6.5}$$

Choose $k_0 = 6/11$. Then the IVP (6.5) has an exact solution $\alpha_1(t) = -3/22$.

The second lower approximation $\alpha_2(t)$ is a solution of the IVP

$$x' = -\frac{57}{275} - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35],$$
$$x(t) = -k_1\frac{3}{22}, \quad t \in [-0.1, 0]. \tag{6.6}$$

Choose $k_1 = 19/33$. Then the IVP (6.6) has an exact solution $\alpha_2(t) = -57/725$. It is clear that $\alpha_0(t) \leq \alpha_1(t) \leq \alpha_2(t) < 0 = x(t)$.

The first upper approximation $\beta_1(t)$ is a solution of the IVP

$$x'(t) = \frac{37}{75} - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35],$$
$$x(t) = \frac{p_0}{4}, \quad t \in [-0.1, 0]. \tag{6.7}$$

Choose $p_0 = 74/99$. Then the IVP (6.7) has an exact solution $\beta_1(t) = 37/198 \approx 0.1869$.

The second upper approximation $\beta_2(t)$ is a solution of the IVP

$$x' = -\frac{114481}{398475} - 0.64x - 2\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t \in [0, 0.35],$$
$$x(t) = p_1\frac{37}{198}, \quad t \in [-0.1, 0]. \tag{6.8}$$

Choose $p_1 = 114481/196581$. Then the IVP (6.8) has an exact solution $\beta_2(t) = 114481/1051974 \approx 0.108825$.

It is obviouse that $\beta_0(t) > \beta_1(t) > \beta_2(t) > 0 = x(t) > \alpha_2(t) > \alpha_1(t) > \alpha_0(t)$ on the interval $[-0.1, 0.35]$.

Example 6.1 illustrates that the presence of the constants p_n and k_n allow us easily to construct each successive approximation in an explicit form.

Now we will illustrate the suggested method in the case when lower and upper solutions are defined for different initial points and on different intervals.

Example 6.2. Consider the following IVP for the scalar nonlinear differential equation with "maxima":

$$\begin{gathered} x' = \frac{t}{1+x(t)} - 0.5 \max_{s\in[t-0.1,t]} x(s) - 2 \quad \text{for } t \in [1.5, 1.75], \\ x(t) = 0, \quad t \in [1.4, 1.5], \end{gathered} \tag{6.9}$$

$$\begin{gathered} x' = \frac{t}{1+x(t)} - 0.5 \max_{s\in[t-0.1,t]} x(s) - 2 \quad \text{for } t \in [2.35, 2.6], \\ x(t) = 0, \quad t \in [2.25, 2.35], \end{gathered} \tag{6.10}$$

where $x \in \mathbb{R}$.

In this case, $f(t,x,y) \equiv t/(1+x) - 0.5y - 2$, $t_0 = 1.5$, $\tau_0 = 2.35$, $\eta = \tau_0 - t_0 = 0.85$, $T = 0.25$, $r = 0.1$, $J_1 = [1.4, 1.75]$, $J_2 = [2.25, 2.6]$.

Choose $\alpha_0(t) = -1/4$. Then for $t \in [1.5, 1.75]$ the inequality

$$f\left(t, \alpha_0(t), \max_{s\in[t-0.1,t]} \alpha_0(s)\right) = \frac{t}{1-1/4} + \frac{1}{8} - 2 = \frac{4t}{3} - \frac{15}{8} \geq 1.5\frac{4}{3} - \frac{15}{8} = 0.125 > 0 \tag{6.11}$$

holds. Therefore, $\alpha_0(t)$ is a lower solution of the IVP (6.9) in $[1.4, 1.75]$.

Let $\beta_0(t) = 1/4$. Then for $t \in [2.35, 2.6]$ the inequality

$$f\left(t, \beta_0(t), \max_{s\in[t-0.1,t]} \beta_0(s)\right) = \frac{t}{1+1/4} - \frac{1}{8} - 2 = \frac{4t}{5} - \frac{17}{8} \leq 2.6\frac{4}{5} - \frac{17}{8} = -0.045 < 0 \tag{6.12}$$

holds. Therefore, $\beta_0(t)$ is an upper solution of the IVP (6.10) on $[2.25, 2.6]$.

In this case, $\tau_0 > t_0$ and $\Omega(\alpha_0, \beta_0, \tau_0) = \{(t,u,v) \in [1.5, 1.75] \times \mathbb{R}^2 : \ -1/4 \leq u, v \leq 1/4\}$.

Let $t \in [1.5, 1.76]$, $-1/4 \leq u_1 \leq u_2 \leq 1/4$ and $-1/4 \leq v_1 \leq v_2 \leq 1/4$. Then from the inequality $t/((1+u_1)(1+u_2)) \leq (16/9)t \leq 3.1111 < 3.2$, we obtain

$$\begin{aligned} f(t,u_1,v_1) - f(t,u_2,v_2) &= \frac{t}{1+u_1} - \frac{t}{1+u_2} + 0.5(v_2 - v_1) \\ &= \frac{t}{(1+u_1)(1+u_2)}(u_2 - u_1) + 0.5(v_2 - v_1) \leq 3.2(u_2 - u_1) + (v_2 - v_1). \end{aligned} \tag{6.13}$$

That is, $L_1 = 3.2$, $L_2 = 0.5$, and the inequality $(L_1 + L_2)T < 1$ holds.

According to Theorem 4.1, there exist two convergent monotone sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ such that (4.3) holds and their limits are solutions of the IVP (6.9) and (6.10).

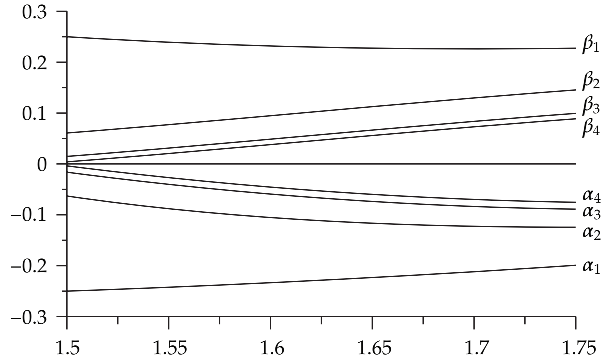

Figure 1: (graphs of $\alpha_k(t)$ and $\beta_k(t+.85)$, $k=1,2,3,4$).

The successive approximations $\alpha_{n+1}(t)$ will be the unique solutions of the linear problems

$$\begin{aligned}x' &= \frac{t}{1+\alpha_n(t)} - 0.5\alpha_n(t) - 2 - 3.2(x-\alpha_n(t)) - 0.5\left(\max_{s\in[t-0.1,t]} x(s) - \max_{s\in[t-0.1,t]} \alpha_n(s)\right)\\ &= \frac{t}{1+\alpha_n(t)} + 2.7\alpha_n(t) + 0.5\max_{s\in[t-0.1,t]}\alpha_n(s) - 2 - 3.2x - 0.5\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t\in[1.5,1.75],\\ x(t) &= -k_nL_n \quad \text{for } t\in[1.4,1.5],\end{aligned} \tag{6.14}$$

and the successive approximations $\beta_{n+1}(t)$ will be the unique solutions of the linear problems

$$\begin{aligned}x'(t) &= \frac{t}{1+\beta_n(t)} + 2.7\beta_n(t) + 0.5\max_{s\in[t-0.1,t]}\beta_n(s) - 2 - 3.2x - 0.5\max_{s\in[t-0.1,t]} x(s) \quad \text{for } t\in[2.35,2.6],\\ x(t) &= p_nC_n \quad \text{for } t\in[2.25,2.35],\end{aligned} \tag{6.15}$$

where $\alpha_0(t) = -1/4$, $\beta_0(t) = 1/4$, $L_n = C_n = 0.25$, $k_n = 0.25^n$, $p_n = 0.25^n$, and $n = 1,2,3,\ldots$.

Note the above IVPs are linear, but we are not able to obtain their solutions in explicit form because of the presence of maximum function.

We will use the computer realization of the considered method, explained in Section 4, to solve the initial value problems (6.14) and (6.15) for $n = 1,2,3,4,5$. In this case at any step, the function $y_n(t_k)$ in (5.1) is replaced by the functions $\alpha_n(t_k)$ or $\beta_n(t_k)$, respectively, the function $y_{n+1}(t_k)$ is the next approximation $\alpha_{n+1}(t_k)$ or $\beta_{n+1}(t_k)$, respectively, and the function f is the right part of (6.14) or (6.15), respectively, and $h = 0.00001$. Some of the numerical values of the successive approximations $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$, which are lower/upper solutions of the given problem, are shown in the Table 1. Also, the obtained successive approximations are graped on the Figure 1. Both, Table 1 and Figure 1, illustrate us the monotonicity of the sequences $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ and the validity of the inequalities

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \beta_n(t+.85) \le \cdots \le \beta_0(t+.85), \quad t\in[1.5,1.75]. \tag{6.16}$$

Table 1: Values of the successive lower/upper approximations $\alpha_k(t)$ and $\beta_k(t+.85)$.

t	1.50	1.55	1.60	1.65	1.70	1.75
$\beta_1(t+.85)$	0.2500000	0.2396285	0.2326388	0.2286608	0.2273177	0.2281468
$\beta_2(t+.85)$	0.0625000	0.0795487	0.0954878	0.1138875	0.1307175	0.1464314
$\beta_3(t+.85)$	0.0156250	0.0322634	0.0490684	0.0671041	0.0850914	0.1029515
$\beta_4(t+.85)$	0.0039063	0.0211106	0.0383552	0.0558882	0.0734248	0.0909491
$\beta_5(t+.85)$	0.0009766	0.0183982	0.0358232	0.0532996	0.0707523	0.0881732
$\alpha_5(t)$	−0.0009766	−0.0236676	−0.0416446	−0.0552493	−0.0648509	−0.0708277
$\alpha_4(t)$	−0.0039063	−0.0269991	−0.0452915	−0.0591247	−0.0688717	−0.0749110
$\alpha_3(t)$	−0.0156250	−0.0399773	−0.0590208	−0.0731539	−0.0828101	−0.0884511
$\alpha_2(t)$	−0.0625000	−0.0871172	−0.1048556	−0.1164338	−0.1225992	−0.1243204
$\alpha_1(t)$	−0.2500000	−0.2427252	−0.2336358	−0.2230384	−0.2111877	−0.1982954

Acknowledgments

This paper was partially supported by Fund "Scientific Research" NI11FMI004/30.05.2011, Plovdiv University and BG051PO001/3.3-05-001 Science and Business, financed by the Operative Program "Development of Human Resources," European Social Fund.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 678536, 12 pages
doi:10.1155/2012/678536

Research Article

Exponential Stability of Impulsive Stochastic Functional Differential Systems

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Received 2 January 2012; Revised 25 February 2012; Accepted 26 February 2012

Academic Editor: Josef Diblík

This paper is concerned with stabilization of impulsive stochastic delay differential systems. Based on the Razumikhin techniques and Lyapunov functions, several criteria on pth moment and almost sure exponential stability are established. Our results show that stochastic functional differential systems may be exponentially stabilized by impulses.

1. Introduction

In the past decades, many authors have obtained various results of deterministic functional differential systems (see [1–6] and the references therein). But it is well known that there are many stochastic factors in the realistic environment, and it is necessary to consider stochastic models. In fact, stochastic functional differential systems (SFDSs) have received more attention in recent years. The properties of SFDSs including stability have been studied in [7–10], which can be widely used in science and engineering (see [11] and the references therein). Furthermore, besides stochastic effects, impulsive effects likewise exist in many evolution processes in which system states change abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics, and telecommunications, and so forth. The impulsive control theory comes to play an important role in science and industry [12]. So the stability investigation of impulsive stochastic differential systems (ISDSs) and impulsive stochastic functional differential systems (ISFDSs) is interesting to many authors [13–20].

Recently, the Razumikhin-type asymptotical stability theorems for ISFESs were established [21, 22]. However, little work has been done on generally exponential stability of ISFESs [23, 24]. In this paper, stability criteria for impulsive stochastic function differential systems are investigated by Razumikhin technique and Lyapunov functions. It is shown that an unstable stochastic delay system can be successfully stabilized by impulses and the results can be easily applied.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))^T$ means a d-dimensional Brownian motion defined on this probability space. R denotes the set of real numbers, R_+ is the set of nonnegative real numbers, and R^n denotes the n-dimensional real space equipped with Euclidean norm $|\cdot|$. If A is a vector or matrix, its transpose is denoted by A^T and its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R_+)$ the family of continuous functions from $[-\tau, 0]$ to R_+. Let N denote the set of positive integers, that is, $N = \{1, 2, \ldots\}$.

For $-\infty < a < b < +\infty$, a function from $[a, b]$ to R^n is called piecewise continuous, if the function has at most a finite number of jump discontinuities on $(a, b]$, which is continuous from the right for all points in $[a, b)$. Given $\tau > 0$, $PC([-\tau, 0]; R^n)$ denotes the family of piecewise continuous functions from $[-\tau, 0]$ to R^n. A norm on $PC([-\tau, 0]; R^n)$ is defined as $\|\phi\| = \sup_{-\tau\leq s\leq 0}|\phi(s)|$ for $\phi \in PC([-\tau, 0]; R^n)$.

For $p > 0$ and $t \geq 0$, let $PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote the family of all $\mathcal{F}_t$-measurable $PC([-\tau, 0]; R^n)$-value random variables ϕ such that $\sup_{-\tau\leq\theta\leq 0} E|\phi(\theta)|^p < \infty$ and $PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote the family of $PC([-\tau, 0]; R^n)$-value random variables that are bounded and $\mathcal{F}_t$-measurable.

In this paper, we consider the following ISFDS:

$$
\begin{gathered}
dx(t) = f(x_t, t)dt + g(x_t, t)dw(t), \quad t \neq t_k,\ t \geq t_0, \\
\Delta x(t_k) = I_k\left(x_{t_k^-}, t_k\right), \quad k \in N, \\
x_{t_0} = \xi,
\end{gathered}
\tag{2.1}
$$

where the initial value $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, $x(t) = (x_1(t), \ldots, x_n(t))^T$, x_t is regarded as a $PC([-\tau, 0]; R^n)$-value process and $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Similarly, x_{t^-} is defined by $x_{t^-}(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0)$ and $x_{t^-}(0) = \lim_{s\to t^-} x(s)$. Both $f : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^n$ and $g : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^{n\times d}$ are Borel measurable, and $I_k : PC^b_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R_+ \to R^n$ represents the impulsive perturbation of x at time t_k. The fixed moments of impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ (as $k \to \infty$), $\Delta x(t_k) = x(t_k) - x(t_k^-)$. Moreover, f, g, and I_k are assumed to satisfy necessary assumptions so that, for any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, system (2.1) has a unique global solution, denoted by $x(t; t_0, \xi)$ (e.g., see [25] for existence and uniqueness results for general impulsive hybrid stochastic delay systems including (2.1)). For the purpose of stability in this note, we also assume the $f(0, t) \equiv 0$, $g(t, 0) \equiv 0$ and $I_k(0, t) \equiv 0$ for all $t \geq t_0$, $k \in N$, then system (2.1) admits a trivial solution.

Definition 2.1. The trivial solution of system (2.1) is said to be pth ($p > 0$) moment exponentially stable if there is a pair of positive constants λ, C such that

$$
E|x(t; t_0, \xi)|^p \leq C\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \geq t_0,
\tag{2.2}
$$

for all $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$. When $p = 2$, it is usually said to be exponentially stable in mean square. It follows from (2.2) that

$$\limsup_{t \to \infty} \frac{1}{t} \log E|x(t; t_0, \xi)|^p \leq -\lambda. \tag{2.3}$$

The left-hand side of (2.3) is called the pth moment Lyapunov exponent of the solution.

Definition 2.2. The trivial solution of system (2.1) is said to be almost exponentially stable if there is a pair of positive constants λ, C such that for $t \geq t_0$

$$|x(t; t_0, \xi)|^p \leq C\|\xi\| e^{-\lambda(t-t_0)}, \quad \text{a.s.}, \tag{2.4}$$

for all $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$. It follows from (2.4) that

$$\limsup_{t \to \infty} \frac{1}{t} \log|x(t; t_0, \xi)| \leq -\lambda. \tag{2.5}$$

The left-hand side of (2.5) is called the Lyapunov exponent of the solution.

Definition 2.3. Let $C^{2,1}(R^n \times [t_0, \infty); R_+)$ denote the family of all nonnegative functions $V(x, t)$ on $R^n \times [t_0 - \tau, \infty)$ that are continuously twice differential in x and once in t. If $V \in C^{2,1}(R^n \times [t_0, \infty); R_+)$, define the operator $\mathcal{L}V : PC([-\tau, 0]; R^n) \times [t_0, \infty) \to R$ for system (2.1) by

$$\mathcal{L}V(x_t, t) = V_t(x, t) + V_x(x, t) f(x_t, t) + \frac{1}{2}\text{trace}\left[g^T(x_t, t) V_{xx}(x, t) g(x_t, t)\right], \tag{2.6}$$

where $V_t(x, t) = \partial V(x, t)/\partial t$, $V_x(x, t) = (\partial V(x, t)/\partial x_1, \ldots, \partial V(x, t)/\partial x_n)$, $V_{xx}(x, t) = (\partial^2 V(x, t)/\partial x_i \partial x_j)_{n \times n}$.

3. Main Results

In this section, we will establish some criteria on the pth moment exponential stability and almost exponential stability for system (2.1) by using the Razumikhin technique and Lyapunov functions. We begin with the following lemma, which concerns with the continuity of $EV(x(t), t)$.

Lemma 3.1. *Let $V(x, t) \in C^{2,1}(R^n \times [t_0, \infty); R_+)$, and let $x(t)$ be a solution of system* (2.1)*. If there exists $c > 0$ such that $V(x, t) \leq c|x|^p$, then $EV(x(t), t)$ is continuous on $[t_{k-1}, t_k)$, $k \in N$.*

Proof. By the Itô formula,

$$V(x(t), t) = V(x(t_{k-1}), t_{k-1}) + \int_{t_{k-1}}^{t} \mathcal{L}V(x_s, s) ds + \int_{t_{k-1}}^{t} V_x(x(s), s) g(x_s, s) dw(s) \tag{3.1}$$

for all $t \in [t_{k-1}, t_k)$, where $k \in N$. Since $x_{t_{k-1}} \in PC^b_{\mathcal{F}_{t_{k-1}}}([-\tau, 0]; R^n)$, we can find an integer l_0 such that $\|x_{t_{k-1}}\| < l_0$ a.s. For any integer $l > l_0$, define the stopping time

$$\rho_l = \inf\{t \in [t_{k-1}, t_k) : |x(t)| \geq l\}, \tag{3.2}$$

where $\inf \emptyset = \infty$ as usual. Since $x(t)$ is continuous on $[t_{k-1}, t_k)$, $|x(t)|$ is also continuous on $[t_{k-1}, t_k)$. Clearly, $\rho_l \to \infty$ a.s. as $l \to \infty$. Moreover, it has $EV(x(t_{k-1}), t_k) \leq cl_0$, following from $x_{t_{k-1}} \in PC^b_{\mathcal{F}_{t_{k-1}}}([-\tau, 0]; R^n)$. It then follows from the definition of ρ_l above that

$$EV\left(x(t'_l), t'_l\right) = EV(x(t_{k-1}), t_{k-1}) + E\left[\int_{t_{k-1}}^{t'_l} \mathcal{L}V(x_s, s)ds\right], \tag{3.3}$$

where $t'_l = t \wedge \rho_l$. So, letting $l \to \infty$, by the dominated convergence theorem and Fubini's theorem, we have

$$\begin{aligned} EV(x(t), t) &= EV(x(t_{k-1}), t_{k-1}) + E\left[\int_{t_{k-1}}^{t} \mathcal{L}V(x_s, s)ds\right] \\ &= EV(x(t_{k-1}), t_{k-1}) + \int_{t_{k-1}}^{t} E[\mathcal{L}V(x_s, s)]ds, \end{aligned} \tag{3.4}$$

for $t \in [t_{k-1}, t_k)$. This implies that $EV(x(t), t)$ is continuous on $[t_{k-1}, t_k)$, $k \in N$. □

Theorem 3.2. *Let $V \in C^{2,1}(R^n \times [t_0 - \tau, \infty); R_+)$ and $u : [t_0, \infty) \to R_+$ be a piecewise continuous function. Suppose there exist some positive constants p, c_1, c_2, and λ such that*

(i) *for all $(x, t) \in R^n \times [t_0 - \tau, \infty)$,*

$$c_1|x|^p \leq V(x, t) \leq c_2|x|^p, \tag{3.5}$$

(ii) *for all $k \in N$, and $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,*

$$EV\left(\phi(0^-) + I(t_k, \phi), t_k\right) \leq d_k EV\left(\phi(0^-), t_k^-\right), \tag{3.6}$$

where $0 < d_k < \exp\{-\lambda(t_{k+1} - t_k) - \int_{t_k}^{t_{k+1}} u(s)ds\}$,

(iii) *for all $t \geq t_0$, $t \neq t_k$, $k \in N$ and $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,*

$$E\left[\mathcal{L}V(\phi, t)\right] \leq u(t) EV(\phi(0), t) \tag{3.7}$$

whenever

$$EV(\phi, t + \theta) < qEV(\phi(0), t), \quad \theta \in [-\tau, 0], \tag{3.8}$$

where $q > \max_{k \in N}\{d_k^{-1} e^{\lambda\tau}\} \vee \exp\{\int_{t_0}^{t_1} u(s)ds\}$.

Then the trivial solution of system (2.1) is pth moment exponentially stable and its pth moment Lyapunov exponent is not greater than $-\lambda$.

Proof. Given any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau,0];R^n)$, the global solution $x(t;t_0,\xi) = x(t)$ of (2.1) is written as $x(t)$ in this proof. Without loss of generality, assume that the initial date ξ is nontrivial so that $x(t)$ is not a trivial solution. Choose M such that

$$c_2 e^{\lambda(t_1-t_0)+\int_{t_0}^{t_1} u(s)ds} < M < c_2 q e^{\lambda(t_1-t_0)}. \tag{3.9}$$

Then it follows from condition (i) and (3.9) that

$$EV(x(t),t) \le c_2\|\xi\|^p < M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0-\tau, t_0]. \tag{3.10}$$

In the following, we will show that

$$EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in N. \tag{3.11}$$

In order to do so, we first prove that

$$EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{3.12}$$

If (3.12) is not true, then there exist some $t \in [t_0,t_1)$ such that $EV(x(t),t) > M\|\xi\|^p e^{-\lambda(t_1-t_0)}$. Set $t^* = \inf\{t \in [t_0,t_1) : EV(x(t),t) > M\|\xi\|^p e^{-\lambda(t_1-t_0)}\}$. Then $t^* \in (t_0,t_1)$ and also, by the continuity of $EV(x(t),t))$ (see Lemma 3.1),

$$EV(x(t),t) < EV(x(t^*),t^*) = M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0-\tau, t_*). \tag{3.13}$$

In view of (3.10), define $t_* = \sup\{t \in [t_0-\tau,t^*) : EV(x(t),t) \le c_2\|\xi\|^p\}$. Then $t_* \in [t_0,t^*)$ and, by the continuity of $EV(x(t),t)$,

$$EV(x(t),t) > EV(x(t_*),t_*) = c_2\|\xi\|^p, \quad t \in (t_*, t^*]. \tag{3.14}$$

Now in view of (3.9), (3.13), and (3.14), one has, for $t \in [t_*,t^*]$ and $\theta \in [-\tau,0]$,

$$EV(x(t+\theta),t+\theta) \le M\|\xi\|^p e^{-\lambda(t_1-t_0)} < qEV(x(t_*),t_*) \le qEV(x(t),t). \tag{3.15}$$

By the Razumikhin-type condition (iii),

$$E[\mathcal{L}V(x_t,t)] \le u(t)EV(x(t),t), \quad \forall t \in [t_*, t^*]. \tag{3.16}$$

Applying Itô formula and by (3.16), one obtains that

$$EV(x(t^*),t^*) \le EV(x(t_*),t_*) + \int_{t_*}^{t^*} u(s)EV(x(s),s)ds. \tag{3.17}$$

Finally, by (3.9), (3.13), (3.14), and the Gronwall inequality,

$$
\begin{aligned}
EV(x(t^*),t^*) &\le EV(x(t_*),t_*)e^{\int_{t_*}^{t^*}u(s)ds} \le c_2\|\xi\|^p e^{\int_{t_0}^{t_1}u(s)ds} \\
&< M\|\xi\|^p e^{-\lambda(t_1-t_0)} = EV(x(t^*),t^*),
\end{aligned} \tag{3.18}
$$

which is a contradiction. So inequality (3.12) holds and (3.11) is true for $k=1$.

Now assume that

$$
EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_k-t_0)}, \quad \forall t \in [t_{k-1},t_k),\ k \in N, \tag{3.19}
$$

for all $k \le m$, where $k,m \in N$. We proceed to show that

$$
EV(x(t),t) \le M\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}, \quad \forall t \in [t_m,t_{m+1}). \tag{3.20}
$$

Suppose (3.20) is not true, set $\bar{t} = \inf\{t \in [t_m,t_{m+1}) : EV(x(t),t) > M\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}\}$. By condition (ii) and (3.20), we know

$$
EV(x(t_m),t_m) \le d_m EV\left(x\left(t_m^-\right),t_m^-\right) \le d_m M\|\xi\|^p e^{-\lambda(t_m-t_0)} < M\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}. \tag{3.21}
$$

From this, together with $EV(x(t),t)$ being continuous on $t \in [t_m,t_{m+1})$, we know that $\bar{t} \in (t_m,t_{m+1})$ and

$$
EV(x(t),t) < EV\left(x\left(\bar{t}\right),\bar{t}\right) = M\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}, \quad \forall t \in \left[t_m,\bar{t}\right). \tag{3.22}
$$

Define $\underline{t} = \sup\{t \in [t_0,\bar{t}] : EV(x(t),t) \le d_m M\|\xi\|^p e^{-\lambda(t_m-t_0)}\}$, then $\underline{t} \in [t_m,\bar{t})$ and

$$
EV(x(t),t) > EV\left(x\left(\underline{t}\right),\underline{t}\right) = d_m M\|\xi\|^p e^{-\lambda(t_m-t_0)}, \quad \forall t \in \left(\underline{t},\bar{t}\right]. \tag{3.23}
$$

For $t \in [\underline{t},\bar{t}]$ and $\theta \in [-\tau,0]$, when $t+\theta \ge t_m$, then (3.22) and (3.23) imply that

$$
\begin{aligned}
EV(x(t+\theta),t+\theta) &\le M\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)} < M\|\xi\|^p e^{-\lambda(t+\theta-t_0)} \\
&\le Me^{\lambda\tau}\|\xi\|^p e^{-\lambda(t-t_0)} \le Me^{\lambda\tau}\|\xi\|^p e^{-\lambda(t_m-t_0)} \\
&\le qEV\left(x\left(\underline{t}\right),\underline{t}\right).
\end{aligned} \tag{3.24}
$$

If $t+\theta < t_m$ for some $\theta \in [-\tau,0)$, we assume that, without loss of generality, $t+\theta \in [t_l,t_{l+1})$ for some $l \in N,\ l \le m-1$, then from (3.19) and (3.23),

$$
\begin{aligned}
EV(x(t+\theta),t+\theta) &\le M\|\xi\|^p e^{-\lambda(t_{l+1}-t_0)} < M\|\xi\|^p e^{-\lambda(t+\theta-t_0)} \\
&\le Me^{\lambda\tau}\|\xi\|^p e^{-\lambda(t-t_0)} \le Me^{\lambda\tau}\|\xi\|^p e^{-\lambda(t_m-t_0)} \\
&\le qEV\left(x\left(\underline{t}\right),\underline{t}\right).
\end{aligned} \tag{3.25}
$$

Therefore,

$$EV(x(t+\theta), t+\theta) < qEV(x(t), t), \quad t \in \left[\underline{t}, \bar{t}\right], \ \theta \in [-\tau, 0]. \tag{3.26}$$

Then, it follows from condition (iii) that

$$E[\mathcal{L}V(x_t, t)] \le u(t)EV(x(t), t), \quad \forall t \in \left[\underline{t}, \bar{t}\right]. \tag{3.27}$$

Combining Itô formula with (3.27), we can check that

$$EV\left(x\left(\bar{t}, \bar{t}\right)\right) \le EV\left(x\left(\underline{t}\right), \underline{t}\right) + \int_{\bar{t}}^{\underline{t}} u(s)EV(x(s), s)ds. \tag{3.28}$$

Finally, by (3.22), (3.23), and the Gronwall inequality,

$$\begin{aligned} EV\left(x\left(\bar{t}\right), \bar{t}\right) &\le EV\left(x\left(\underline{t}\right), \underline{t}\right)e^{\int_{\underline{t}}^{\bar{t}} u(s)ds} \le EV\left(x\left(\underline{t}\right), \underline{t}\right)e^{\int_{t_m}^{t_{m+1}} u(s)ds} \\ &= d_m M\|\xi\|^p e^{-\lambda(t_m - t_0)} e^{\int_{t_m}^{t_{m+1}} u(s)ds} < EV\left(x\left(\bar{t}\right), \bar{t}\right), \end{aligned} \tag{3.29}$$

which is a contradiction. So inequality (3.20) holds. By mathematical induction, we obtain that (3.11) holds for all $k \in N$. Furthermore, from condition (i), we have

$$E|x(t)|^p \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t_k - t_0)} \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ k \in N, \tag{3.30}$$

which implies

$$E\|x\|^p \le \frac{c_1}{c_2} M\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{3.31}$$

that is, system (2.1) is pth moment exponentially stable. The proof is complete. □

Remark 3.3. If $u(t) \equiv c > 0$, then Theorem 3.1 of [23] follows from Theorem 3.2 immediately.

Theorem 3.4. *Let $V \in C^{2,1}(R^n \times [t_0 - \tau, \infty); R_+)$, and let $u : [t_0, \infty) \to R_+$ be a piecewise continuous function. Suppose there exist some positive constants p, c_1, c_2, and λ such that*

(i) *for all $(x, t) \in R^n \times [t_0 - \tau, \infty)$,*

$$c_1|x|^p \le V(x, t) \le c_2|x|^p, \tag{3.32}$$

(ii) *for all $k \in N$ and $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,*

$$EV\left(\phi(0^-) + I(t_k, \phi)\right) \le \rho d_k EV\left(\phi(0^-), t_k^-\right), \tag{3.33}$$

where $0 < \rho < \max\{e^{-\lambda(t_{m+1} - t_m)}\}$ and $d_k > 0$ with $\widehat{d} = \sup_{n\in N} \Pi_{k=1}^n d_k < \infty$,

(iii) *for all* $t \geq t_0$, $t \neq t_k$, $k \in N$, *and* $\phi \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$,

$$E[\mathcal{L}V(\phi, t)] \leq u(t)EV(\phi(0), t) \tag{3.34}$$

whenever

$$EV(\phi, t+\theta) < qEV(\phi(0), t), \quad \theta \in [-\tau, 0], \tag{3.35}$$

where $q > (\rho^{-1}e^{\lambda\tau}) \vee (\rho^{-1}e^{\lambda\tau}\widehat{d}^{-1})$.

Then the trivial solution of system (2.1) is pth moment exponentially stable and its pth moment Lyapunov exponent is not greater than $-\lambda$.

Proof. Given any initial data $\xi \in PC^b_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$, the global solution $x(t; t_0; \xi) = x(t)$ of (2.1) is written as $x(t)$ in this proof. Without loss of generality, assume that the initial date ξ is nontrivial so that $x(t)$ is not a trivial solution. Choose M such that

$$c_2 e^{\lambda(t_1-t_0)+\int_{t_0}^{t_1} u(s)ds} < M < c_2 q e^{\lambda(t_1-t_0)}. \tag{3.36}$$

Then it follows from condition (i) and (3.36) that

$$EV(x(t), t) \leq c_2\|\xi\|^p < M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0-\tau, t_0]. \tag{3.37}$$

In the following, we will show that

$$EV(x(t), t) \leq M_k\|\xi\|^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k), \tag{3.38}$$

where $k \in N$ and M_k is defined as $M_1 = M$ and $M_k = M\Pi_{1\leq l\leq k-1} d_l$. Similarly, as the proof in Theorem 3.2, one can prove that

$$EV(x(t), t) \leq M\|\xi\|^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{3.39}$$

Now assume that

$$EV(x(t), t) \leq M_k\|\xi\|^p e^{-\lambda(t_k-t_0)}, \quad \forall t \in [t_{k-1}, t_k), \ k \in N, \tag{3.40}$$

for all $k \leq m$, where $k, m \in N$. We proceed to show that

$$EV(x(t), t) \leq M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}, \quad \forall t \in [t_m, t_{m+1}). \tag{3.41}$$

Suppose (3.41) is not true, set $\bar{t} = \inf\{t \in [t_m, t_{m+1}) : EV(x(t), t) > M_k\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}\}$. By condition (ii),

$$EV(x(t_m), t_m) \le \rho d_m EV\left(x(t_m^-), t_m^-\right) \le \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)} \\ < M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}. \tag{3.42}$$

From this, together with $EV(x(t), t)$ being continuous on $t \in [t_m, t_{m+1})$, we know that $\bar{t} \in (t_m, t_{m+1})$ and

$$EV(x(t), t) < EV\left(x\left(\bar{t}\right), \bar{t}\right) = M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)}, \quad \forall t \in \left[t_m, \bar{t}\right). \tag{3.43}$$

Define $\underline{t} = \sup\{t \in [t_0, \bar{t}] : EV(x(t), t) \le \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)}\}$, then $\underline{t} \in [t_m, \bar{t})$ and

$$EV(x(t), t) > EV\left(x\left(\underline{t}\right), \underline{t}\right) = \rho M_{m+1}\|\xi\|^p e^{-\lambda(t_m - t_0)}, \quad \forall t \in \left(\underline{t}, \bar{t}\right]. \tag{3.44}$$

For $t \in [\underline{t}, \bar{t}]$ and $\theta \in [-\tau, 0]$, when $t + \theta \ge t_m$, then (3.44) implies that

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\le M_{m+1}\|\xi\|^p e^{-\lambda(t_{m+1}-t_0)} \\ &= \rho^{-1} e^{-\lambda(t_{m+1}-t_m)} EV\left(x\left(\underline{t}\right), \underline{t}\right) \\ &< qEV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.45}$$

If $t + \theta < t_m$ for some $\theta \in [-\tau, 0)$, we assume that, without loss of generality, $t + \theta \in [t_l, t_{l+1})$ for some $l \in N,\ l \le m-1$, then from (3.41) and (3.44), we obtain

$$\begin{aligned} EV(x(t+\theta), t+\theta) &\le M_{l+1}\|\xi\|^p e^{-\lambda(t_{l+1}-t_0)} < M_{l+1}\|\xi\|^p e^{-\lambda(t+\theta-t_0)} \\ &\le M_{l+1} e^{\lambda\tau}\|\xi\|^p e^{-\lambda(t-t_0)} \le \rho^{-1} e^{\lambda\tau}\frac{M_{l+1}}{M_{m+1}} EV\left(x\left(\underline{t}\right), \underline{t}\right) \\ &= \rho^{-1} e^{\lambda\tau}\hat{d}^{-1} EV\left(x\left(\underline{t}\right), \underline{t}\right) \le qEV\left(x\left(\underline{t}\right), \underline{t}\right). \end{aligned} \tag{3.46}$$

Therefore,

$$EV(x(t+\theta), t+\theta) < qEV(x(t), t), \quad t \in \left[\underline{t}, \bar{t}\right],\ \theta \in [-\tau, 0]. \tag{3.47}$$

The rest of the proof is similar to that of Theorem 3.2 and omitted here. □

Remark 3.5. Let $\overline{u}$ and δ be positive constants. Assume that the conditions of Theorem 3.4 hold, function $u : [t_0, \infty) \to R_+$ satisfies $\int_t^{t+\delta} u(s)ds \le \overline{u}\delta$ and $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} = \delta < -(\ln\rho/(\lambda + \overline{u}))$. Then Theorem 3.1 of [24] follows immediately.

Remark 3.6. It is not strictly required by condition (ii) of Theorem 3.4 that each impulse contributes to stabilize the system, as long as the overall contribution of the impulses are stabilizing. Without these d_k (i.e., $d_k \equiv 1$), it is required that each impulse is a stabilizing factor ($\rho < 1$), which is more restrictive.

Remark 3.7. It is clear that Theorems 3.2 and 3.4 allow the continuous dynamics of system (2.1) to be unstable, since the function $u(t)$, which characterizes the changing rate of $V(x(t),t)$ at t, is assumed to be nonnegative. Theorems 3.2 and 3.4 show that an unstable stochastic delay system can be successfully stabilized by impulses.

The following theorems show that the trivial solutions of system (2.1) are also almost surely exponentially stable, under some additional conditions.

Assumption 3.8. Suppose the impulsive instances t_k satisfy

$$\sup_{k\in N}\{t_k - t_{k-1}\} < \infty, \qquad \inf_{k\in N}\{t_k - t_{k-1}\} > 0. \tag{3.48}$$

Assumption 3.9. Assume that there is a constant $L > 0$ such that, for all $(\phi, t) \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times [t_0, \infty)$,

$$E\left[\left|f(\phi,t)\right|^p + \left|g(\phi,t)\right|^p\right] \leq L \sup_{-\tau\leq\theta\leq 0} E\left|\phi(\theta)\right|. \tag{3.49}$$

Lemma 3.10 (see [23]). *Let $p \geq 1$, and let Assumptions 3.8 and 3.9 hold. Then* (3.31) *implies that, for all $t \geq t_0$,*

$$|x(t;\xi,t_0)| \leq Ce^{-(\lambda/p)(t-t_0)}\|\xi\|^p \quad a.s., \tag{3.50}$$

where C is a positive constant. In other words, under Assumptions 3.8 and 3.9, the pth moment exponential stability implies the almost exponential stability for system (2.1).

By using Theorems 3.2 and 3.4 and Lemma 3.10, it is easy to show the following conclusions.

Theorem 3.11. *Suppose that $p \geq 1$, Assumptions 3.8 and 3.9 and the same conditions as in Theorem 3.2 hold. Then the trivial solution of system* (2.1) *is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\lambda/p$.*

Theorem 3.12. *Suppose that $p \geq 1$, Assumptions 3.8 and 3.9 and the same conditions as in Theorem 3.4 hold. Then the trivial solution of system* (2.1) *is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\lambda/p$.*

4. An Example

Example 4.1. Consider a scalar ISDDs of the form

$$\begin{gathered} dx(t) = x(t)dt + \frac{1}{4}\sqrt{x^2(t) + x^2(t-2)}\,dw(t), \qquad t \neq t_k,\ t \geq t_0, \\ \Delta x(t_k) = -0.4x(t_k^-), \qquad k \in N. \end{gathered} \tag{4.1}$$

It is easy to check that the corresponding system without impulses is not mean square exponentially stable. In fact, if $V(x,t) = x^2$, then it follows from the Itô formula that

$E[\mathcal{L}V(x(t), x(t-2), t)] \geq 2E|x(t)|^2 = 2EV(x(t), t)$. This leads to $E|x(t)|^2 = EV(x(t), t) \geq EV(x(0), 0)e^{2t} = E|x(0)|^2 e^{2t}$ for all $t \geq 0$. But, in the following, we will show that system (4.1) is mean square exponentially stable and almost exponentially stable.

If $V(x(t), t) = x^2$, then condition (i) of Theorem 3.2 holds with $c_1 = c_2 = 1, p = 2$, and condition (ii) holds with $d_k = 0.36$. By calculating, we have $E[\mathcal{L}V(x(t), x(t-2), t)] \leq (33/16)EV(x(t), t) + (1/16)EV(x(t-2), t)$. By taking $q = 5$, $\lambda = 0.5$, and $t_k - t_{k-1} = 0.3$, it is easy to verify that condition (iii) of Theorem 3.2 is satisfied, which means system (4.1) is mean square exponentially stable. Applying Theorem 3.11, we can derive that system (4.1) is almost exponentially stable.

Acknowledgments

The authors are grateful to Editor Professor Josef Diblík and anonymous referees for their helpful comments and suggestions which have improved the quality of this paper. This work is supported by Natural Science Foundation of China (no. 10771001), Research Fund for Doctor Station of Ministry of Education of China (no. 20113401110001, no. 20103401120002), TIAN YUAN Series of Natural Science Foundation of China (no. 11126177), Key Natural Science Foundation (no. KJ2009A49), Talent Foundation (no. 05025104) of Anhui Province Education Department, 211 Project of Anhui University (no. KJJQ1101), Anhui Provincial Nature Science Foundation (no. 090416237, no. 1208085QA15), and Foundation for Young Talents in College of Anhui Province (no. 2012SQRL021).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 954352, 13 pages
doi:10.1155/2012/954352

Research Article

Smooth Solutions of a Class of Iterative Functional Differential Equations

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Received 14 December 2011; Accepted 5 March 2012

Academic Editor: Miroslava Růžičková

By Faà di Bruno's formula, using the fixed-point theorems of Schauder and Banach, we study the existence and uniqueness of smooth solutions of an iterative functional differential equation $x'(t) = 1/(c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t))$.

1. Introduction

There has been a lot of monographs and research articles to discuss the kinds of solutions of functional differential equations since the publication of Jack Hale's paper [1]. Several papers discussed the iterative functional differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), \ldots, x^{[m]}(t)\right), \tag{1.1}$$

where $x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[k]}(t) = x(x^{[k-1]}(t)),\ k = 2, \ldots, m$. More specifically, Eder [2] considered the functional differential equation

$$x'(t) = x^{[2]}(t) \tag{1.2}$$

and proved that every solution either vanishes identically or is strictly monotonic. Furthermore, Fečkan [3] and Wang [4] studied the equation

$$x'(t) = f\left(x^{[2]}(t)\right) \tag{1.3}$$

with different conditions. Staněk [5] considered the equation

$$x'(t) = x(t) + x^{[2]}(t) \tag{1.4}$$

and obtained every solution either vanishes identically or is strictly monotonic. Si and his coauthors [6, 7] studied the following equations:

$$\begin{aligned} x'(t) &= x^{[m]}(t), \\ x'(t) &= \frac{1}{x^{[m]}(t)}, \end{aligned} \tag{1.5}$$

$$x'(t) = \frac{1}{c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t)} \tag{1.6}$$

and established sufficient conditions for the existence of analytic solutions. Especially in [8, 9], the smooth solutions of the following equations:

$$\begin{aligned} x'(t) &= \sum_{j=1}^{m} a_j x^{[j]}(t) + F(t), \\ x'(t) &= \sum_{j=1}^{m} a_j(t) x^{[j]}(t) + F(t), \end{aligned} \tag{1.7}$$

have been studied by the fixed-point theorems of Schauder and Banach.

A smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. Let $x \in C^n$ if $x', \dots, x^{(n)}$ are continuous, $C^n(I, I)$ is the set in which $x \in C^n$ and maps a closed interval I into I. As in [9], we, using the same symbols, denote the norm

$$\|x\|_n = \sum_{k=0}^{n} \left\| x^{(k)} \right\|, \quad \|x\| = \max_{t \in I}\{|x(t)|\}, \tag{1.8}$$

then $C^n(I, R)$ with $\|\cdot\|_n$ is a Banach space, and $C^n(I, I)$ is a subset of $C^n(I, R)$. For given $M_i > 0$ $(i = 1, 2, \dots, n+1)$, let

$$\begin{aligned} \Omega(M_1, \dots, M_{n+1}; I) = \Big\{ x \in C^n(I, I) : \left| x^{(i)}(t) \right| \le M_i, i = 1, 2, \dots, n; \\ \left| x^{(n)}(t_1) - x^{(n)}(t_2) \right| \le M_{n+1} |t_1 - t_2|,\ t, t_1, t_2 \in I \Big\}. \end{aligned} \tag{1.9}$$

For convenience, we will make use of the notation

$$x_{ij}(t) = x^{(i)}\left(x^{[j]}(t)\right), \qquad x_{*jk}(t) = \left(x^{[j]}(t)\right)^{(k)}, \tag{1.10}$$

where i, j, and k are nonnegative integers. Let I be a closed interval in R. By induction, we may prove that

$$x_{*jk}(t) = P_{jk}\left(x_{10}(t), \ldots, x_{1,j-1}(t); \ldots; x_{k0}(t), \ldots, x_{k,j-1}(t)\right), \tag{1.11}$$

$$\beta_{jk} = P_{jk}\left(\overbrace{x'(\xi), \ldots, x'(\xi)}^{j \text{ terms}}; \ldots; \overbrace{x^{(k)}(\xi), \ldots, x^{(k)}(\xi)}^{j \text{ terms}}\right), \tag{1.12}$$

$$H_{jk} = P_{jk}\left(\overbrace{1, \ldots, 1}^{j \text{ terms}}; \overbrace{M_2, \ldots, M_2}^{j \text{ terms}}; \ldots; \overbrace{M_k, \ldots, M_k}^{j \text{ terms}}\right), \tag{1.13}$$

where P_{jk} is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can be found in [8].

In order to seek a solution $x(t)$ of (1.6), in $C^n(I, I)$ such that ξ is a fixed point of the function $x(t)$, that is, $x(\xi) = \xi$, it is natural to seek an interval I of the form $[\xi - \delta, \xi + \delta]$ with $\delta > 0$.

Let us define

$$\begin{aligned} &X(\xi; \xi_0, \ldots, \xi_n; 1, M_2, \ldots, M_{n+1}; I) \\ &\quad = \left\{x \in \Omega(1, M_2, \ldots, M_{n+1}; I) : x(\xi) = \xi_0 = \xi, x^{(i)}(\xi) = \xi_i,\ i = 1, 2, \ldots, n\right\}. \end{aligned} \tag{1.14}$$

2. Smooth Solutions of (1.6)

In this section, we will prove the existence theorem of smooth solutions for (1.6). First of all, we have the inequalities in the following for all $x(t), y(t) \in X$:

$$\left|x^{[j]}(t_1) - x^{[j]}(t_2)\right| \le |t_1 - t_2|, \quad t_1, t_2 \in I,\ j = 0, 1, \ldots, m, \tag{2.1}$$

$$\left\|x^{[j]} - x^{[j]}\right\| \le j\|x - y\|, \quad j = 1, \ldots, m, \tag{2.2}$$

$$\|x - y\| \le \delta^n \left\|x^{(n)} - y^{(n)}\right\|, \tag{2.3}$$

and the proof can be found in [9].

Theorem 2.1. *Let $I = [\xi - \delta, \xi + \delta]$, where ξ and δ satisfy*

$$\xi \ge \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \qquad 0 < \delta \le \xi - \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \tag{2.4}$$

where $|c_0| > \sum_{i=0}^m |c_i|$, then (1.6) has a solution in

$$X(\xi; \xi_0, \ldots, \xi_n; 1, M_2, \ldots, M_{n+1}; I), \tag{2.5}$$

provided the following conditions hold:

(i)

$$\xi_1 = \xi^{-1}\left(\sum_{i=0}^{m} c_i\right)^{-1}, \tag{2.6}$$

$$\begin{aligned}\xi_k = \sum \frac{(-1)^s\ (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!}\left(\xi\sum_{i=0}^{m} c_i\right)^{-s-1}\left(\frac{\sum_{i=0}^{m} c_i\beta_{i1}}{1!}\right)^{s_1}\\ \times\left(\frac{\sum_{i=0}^{m} c_i\beta_{i2}}{2!}\right)^{s_2}\cdots\left(\frac{\sum_{i=0}^{m} c_i\beta_{ik-1}}{(k-1)!}\right)^{s_{k-1}},\end{aligned} \tag{2.7}$$

where $k = 2,\ldots,n$, and the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$,

(ii)

$$\begin{aligned}\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!}(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\frac{\sum_{i=0}^{m}|c_i|H_{i1}}{1!}\right)^{s_1}\\ \times\left(\frac{\sum_{i=0}^{m}|c_i|H_{i2}}{2!}\right)^{s_2}\cdots\left(\frac{\sum_{i=0}^{m}|c_i|H_{ik-1}}{(k-1)!}\right)^{s_{k-1}} \le M_k, \quad k = 2,\ldots,n,\end{aligned} \tag{2.8}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$,

(iii)

$$\begin{aligned}&\sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}}\\ &\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1+1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\\ &\quad\times\cdots\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}} + s_1(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1}\\ &\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\left(\sum_{i=0}^{m}|c_i|H_{i3}\right)^{s_3}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}}\\ &\quad+\cdots+s_{n-1}(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-2}\right)^{s_{n-2}}\\ &\quad\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in}\right)\Bigg]\\ &\le M_{n+1},\end{aligned} \tag{2.9}$$

where $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} = n-1$ and $s = s_1 + s_2 + \cdots + s_{n-1}$, *Proof.* Define an operator T from X into $C^n(I,I)$ by

$$(Tx)(t) = \xi + \int_{\xi}^{t} \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds. \tag{2.10}$$

We will prove that for any $x \in X$, $Tx \in X$,

$$|(Tx)(t) - \xi| = \left| \int_{\xi}^{t} \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(s)} ds \right| \le \left((\xi - \delta) \left(|c_0| - \sum_{i=1}^{m} |c_i| \right) \right)^{-1} |t - \xi| \le \delta, \tag{2.11}$$

where the second inequality is from (2.4) and $x(I) \subseteq I$. Thus, $(Tx)(I) \subseteq I$.

It is easy to see that

$$(Tx)'(t) = \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)}, \tag{2.12}$$

and by Faà di Bruno's formula, for $k = 2, \ldots, n$, we have

$$\begin{aligned}
(Tx)^{(k)}(t) &= \left(\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)} \right)^{(k-1)} = \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t) \right)^{s+1}} \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t) \right)'}{1!} \right)^{s_1} \\
&\quad \times \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t) \right)''}{2!} \right)^{s_2} \cdots \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t) \right)^{(k-1)}}{(k-1)!} \right)^{s_{k-1}} \\
&= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t) \right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i x_{*i1}(t)}{1!} \right)^{s_1} \\
&\quad \times \left(\frac{\sum_{i=0}^{m} c_i x_{*i2}(t)}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i x_{*ik-1}(t)}{(k-1)!} \right)^{s_{k-1}},
\end{aligned} \tag{2.13}$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k - 1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Furthermore, note $(Tx)(\xi) = \xi$, by (2.6) and (2.7),

$$
\begin{aligned}
(Tx)'(\xi) &= \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(\xi)} = \frac{1}{\xi \sum_{i=0}^{m} c_i} = \xi_1, \\
(Tx)^{(k)}(\xi) &= \sum \frac{(-1)^s (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left(\xi \sum_{i=0}^{m} c_i\right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i x_{*i1}(\xi)}{1!}\right)^{s_1} \\
&\quad \times \left(\frac{\sum_{i=0}^{m} c_i x_{*i2}(\xi)}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i x_{*ik-1}(\xi)}{(k-1)!}\right)^{s_{k-1}} \\
&= \sum \frac{(-1)^s (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left(\xi \sum_{i=0}^{m} c_i\right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i \beta_{i1}}{1!}\right)^{s_1} \\
&\quad \times \left(\frac{\sum_{i=0}^{m} c_i \beta_{i2}}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i \beta_{ik-1}}{(k-1)!}\right)^{s_{k-1}} \\
&= \xi_k, \quad k = 2, \dots, n,
\end{aligned}
\tag{2.14}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. Thus, $(Tx)^{(k)}(\xi) = \xi_k$ for $k = 0, 1, \dots, n$,

$$
\left|(Tx)'(t)\right| = \left|\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)}\right| \le \left((\xi - \delta)\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)\right)^{-1} \le 1 = M_1,
\tag{2.15}
$$

By (2.8), we have

$$
\begin{aligned}
\left|(Tx)^{(k)}(t)\right| &\le \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left|\sum_{i=0}^{m} c_i x^{[i]}(t)\right|^{s+1}} \left(\frac{\sum_{i=0}^{m}|c_i||x_{*i1}(t)|}{1!}\right)^{s_1} \\
&\quad \times \left(\frac{\sum_{i=0}^{m}|c_i||x_{*i2}(t)|}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|}{(k-1)!}\right)^{s_{k-1}} \\
&\le \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1} \left(\frac{\sum_{i=0}^{m}|c_i| H_{i1}}{1!}\right)^{s_1} \\
&\quad \times \left(\frac{\sum_{i=0}^{m}|c_i| H_{i2}}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m}|c_i| H_{ik-1}}{(k-1)!}\right)^{s_{k-1}} \\
&\le M_k, \quad k = 2, \dots, n,
\end{aligned}
\tag{2.16}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Finally,

$$
\begin{aligned}
&\left|(Tx)^{(n)}(t_1)-(Tx)^{(n)}(t_2)\right| \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_1)\right)^{-s-1}\left(\sum_{i=0}^{m}c_i x_{*i1}(t_1)\right)^{s_1}\left(\sum_{i=0}^{m}c_i x_{*i2}(t_1)\right)^{s_2}\cdots\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_1)\right)^{s_{n-1}}\right. \\
&\qquad\quad\left.-\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\left(\sum_{i=0}^{m}c_i x_{*i1}(t_2)\right)^{s_1}\left(\sum_{i=0}^{m}c_i x_{*i2}(t_2)\right)^{s_2}\cdots\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_2)\right)^{s_{n-1}}\right| \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left[\left|\frac{1}{\left(\sum_{i=0}^{m}c_i x^{[i]}(t_1)\right)^{s+1}}-\frac{1}{\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{s+1}}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t_1)|\right)^{s_1}\right. \\
&\qquad\quad\times\cdots\left(\sum_{i=0}^{m}|c_i||x_{*in-1}(t_1)|\right)^{s_{n-1}}+\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\right| \\
&\qquad\times\left|\left(\sum_{i=0}^{m}c_i x_{*i1}(t_1)\right)^{s_1}-\left(\sum_{i=0}^{m}c_i x_{*i1}(t_2)\right)^{s_1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i2}(t_1)|\right)^{s_2}\cdots \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i||x_{*in-1}(t_1)|\right)^{s_{n-1}}+\cdots+\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t_2)|\right)^{s_1} \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i||x_{*i2}(t_2)|\right)^{s_2}\cdots\left(\sum_{i=0}^{m}|c_i||x_{*in-2}(t_2)|\right)^{s_{n-2}} \\
&\qquad\times\left.\left|\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_1)\right)^{s_{n-1}}-\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_2)\right)^{s_{n-1}}\right|\right] \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1+1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\right. \\
&\qquad\quad\times\cdots\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}}+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1} \\
&\qquad\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\left(\sum_{i=0}^{m}|c_i|H_{i3}\right)^{s_3}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}} \\
&\qquad\quad+\cdots+s_{n-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-2}\right)^{s_{n-2}} \\
&\qquad\quad\left.\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in}\right)\right]|t_1-t_2|,
\end{aligned}
\tag{2.17}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. By (2.9), we see that

$$\left|(Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2)\right| \le M_{n+1}|t_1 - t_2|. \tag{2.18}$$

Now, we can say that T is an operator from X into itself.

Next, we will show that T is continuous. Let $x, y \in X$, then

$$\begin{aligned}
\|Tx - Ty\|_n &= \|Tx - Ty\| + \left\|(Tx)' - (Ty)'\right\| + \sum_{k=2}^{n}\left\|(Tx)^{(k)} - (Ty)^{(k)}\right\| \\
&= \max_{t\in I}\left|\int_{\xi}^{t}\left(\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(s)} - \frac{1}{\sum_{i=0}^{m} c_i y^{[i]}(s)}\right)ds\right| \\
&\quad + \max_{t\in I}\left|\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)} - \frac{1}{\sum_{i=0}^{m} c_i y^{[i]}(t)}\right| \\
&\quad + \sum_{k=2}^{n}\max_{t\in I}\left|\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}}\right. \\
&\quad \times \left[\frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{s+1}}\left(\sum_{i=0}^{m} c_i x_{*i1}(t)\right)^{s_1}\left(\sum_{i=0}^{m} c_i x_{*i2}(t)\right)^{s_2}\cdots\right. \\
&\quad \times \left(\sum_{i=0}^{m} c_i x_{*ik-1}(t)\right)^{s_{k-1}} - \frac{1}{\left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{s+1}}\left(\sum_{i=0}^{m} c_i y_{*i1}(t)\right)^{s_1} \\
&\quad \times \left.\left.\left(\sum_{i=0}^{m} c_i y_{*i2}(t)\right)^{s_2}\cdots\left(\sum_{i=0}^{m} c_i y_{*ik-1}(t)\right)^{s_{k-1}}\right]\right| \\
&\le \delta(\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]} - y^{[i]}\right\|\right) \\
&\quad + (\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]} - y^{[i]}\right\|\right) \\
&\quad + \sum_{k=2}^{n}\max_{t\in I}\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \\
&\quad \times \left[\left|\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{-s-1} - \left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t)|\right)^{s_1}\cdots\right. \\
&\qquad \times \left(\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|\right)^{s_{k-1}} + \left|\left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{-s-1}\right| \\
&\qquad \times \left|\left(\sum_{i=0}^{m} c_i x_{*i1}(t)\right)^{s_1} - \left(\sum_{i=0}^{m} c_i y_{*i1}(t)\right)^{s_1}\right|
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{i=0}^{m}|c_i||x_{*i2}(t)|\right)^{s_2}\cdots\left(\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|\right)^{s_{k-1}}\\
&+\cdots+\left|\left(\sum_{i=0}^{m}c_iy^{[i]}(t)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i|\left|y_{*i1}(t)\right|\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|\left|y_{*ik-2}(t)\right|\right)^{s_{k-2}}\\
&\times\left|\left(\sum_{i=0}^{m}c_ix_{*ik-1}(t)\right)^{s_{k-1}}-\left(\sum_{i=0}^{m}c_iy_{*ik-1}(t)\right)^{s_{k-1}}\right|\Bigg]\\
\leq{}&(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\|\\
&+\sum_{k=2}^{n}\sum\frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}}\\
&\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\\
&\quad+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1}\\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\\
&\quad+\cdots+s_{k-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{ik-2}\right)^{s_{k-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}-1}\left(\sum_{i=0}^{m}c_iH_{ik}\right)\\
&\quad\times\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\Bigg]\\
&+\delta^{n+1}(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\left\|x^{(n)}-y^{(n)}\right\|,
\end{aligned}
\tag{2.19}
$$

where $s_1+2s_2+\cdots+(k-1)s_{k-1}=k-1$ and $s=s_1+s_2+\cdots+s_{k-1}$.

Moreover, we can find some constants P_k such that

$$\begin{aligned}
&\sum_{k=2}^{n}\sum\frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}}\\
&\quad\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\times\\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\\
&\qquad+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1}\\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\\
&\qquad+\cdots+s_{k-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{ik-2}\right)^{s_{k-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}-1}\left(\sum_{i=0}^{m}c_iH_{ik}\right)\\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\Bigg]\\
&\leq\sum_{k=1}^{n-1}P_k\left(\xi,\delta,c_i,H_{ij}\right)\left\|x^{(k)}-y^{(k)}\right\|,
\end{aligned}\tag{2.20}$$

where

$$P_k\left(\xi,\delta,c_i,H_{ij}\right)=P(\xi,\delta;c_1,\ldots,c_m;H_{11},\ldots,H_{1k+1};\ldots;H_{m1},\ldots,H_{mk+1};)\tag{2.21}$$

are the positive constants depend on ξ,δ,c_i, and $H_{ij},\ i=1,\ldots,m;\ j=1,\ldots,k+1$. Then

$$\begin{aligned}
\|Tx-Ty\|_n&\leq(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\|\\
&\quad+\sum_{k=1}^{n-1}P_k\left(\xi,\delta,c_i,H_{ij}\right)\left\|x^{(k)}-y^{(k)}\right\|\\
&\quad+\delta^{n+1}(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\left\|x^{(n)}-y^{(n)}\right\|\\
&\leq\Gamma\|x-y\|_n.
\end{aligned}\tag{2.22}$$

Here,

$$\Gamma = \max\left\{ (\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0| + \sum_{i=1}^{m} i|c_i|\right);\ \max_{1\le k\le n-1}\{P_k(\xi,\delta,c_i,H_{ij})\};\right.$$
$$\left.\delta^{n+1}(\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0| + \sum_{i=1}^{m} i|c_i|\right)\right\}, \tag{2.23}$$

$k = 1,\dots,n-1$. So we can say that T is continuous.

It is easy to see that X is closed and convex. We now show that X is a relatively compact subset of $C^n(I,I)$. For any $x = x(t) \in X$,

$$\|x\|_n \le \|x\| + \sum_{k=1}^{n}\left\|x^{(k)}\right\| \le |\xi| + \delta + 1 + \sum_{k=2}^{n} M_k. \tag{2.24}$$

Next, for any t_1, t_2 in I, we have

$$|x(t_1) - x(t_2)| \le |t_1 - t_2|. \tag{2.25}$$

Hence, X is bounded in $C^n(I,I)$ and equicontinuous on I, and by the Arzela-Ascoli theorem, we know X is relatively compact in $C^n(I,I)$, since $C^n(I,I)$ is the subset of $C^n(I,R)$, and we can say that X is relatively compact in $C^n(I,R)$.

From Schauder's fixed-point theorem, we conclude that

$$x(t) = \xi + \int_{\xi}^{t} \frac{1}{c_0x^{[0]}(s) + c_1x^{[1]}(s) + \cdots + c_mx^{[m]}(s)}ds, \tag{2.26}$$

for some $x = x(t)$ in X. By differentiating both sides of the above equality, we see that x is the desired solution of (1.6). This completes the proof. □

Theorem 2.2. *Let* $I = [\xi-\delta, \xi+\delta]$, *where* ξ *and* δ *satisfy* (2.4), *then* (1.6) *has a unique solution in*

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \tag{2.27}$$

provided the conditions (2.6)–(2.9) *hold and* $\Gamma < 1$ *in* (2.23).

Proof. Since $\Gamma < 1$, we see that T defined by (2.10) is contraction mapping on the close subset X of $C^n(I,I)$. Thus, the fixed point x in the proof of Theorem 2.1 must be unique. This completes the proof. □

Remark 2.3. By Theorem 2.1 or Theorem 2.2, the existence and uniqueness of smooth solutions of an iterative functional differential equation of the form (1.6) can be obtained. If $n \to +\infty$, we can also find that the solution is C^∞-smooth.

Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the following equation:

$$x'(t) = \frac{1}{t + (1/2)x(t) + (1/4)x(x(t))}, \tag{2.28}$$

where $c_0 = 1$, $c_1 = (1/2)$, $c_2 = (1/4)$, and $\xi \geq 4$. Moreover, we take $0 < \delta \leq \xi - 4$. Then, (2.4) is satisfied, and ξ, δ define the interval $I = [\xi - \delta, \xi + \delta]$. Now, take $\xi_0 = \xi$,

$$\begin{gathered}\xi_1 = \frac{4}{7}\xi^{-1}, \\ \xi_2 = -\frac{16}{49}\xi^{-2}\left(1 + \frac{1}{2}\xi_1 + \frac{1}{4}\xi_1^2\right), \\ \xi_3 = \frac{4}{343}\xi^{-3}\left(4 + 2\xi_1 + \xi_1^2\right)^2 - \frac{4}{49}\xi^{-2}\xi_2\left(2 + \xi_1 + \xi_1^2\right),\end{gathered} \tag{2.29}$$

then (2.6) and (2.7) are satisfied.

Finally, if we take

$$M_1 = 1, \quad M_2 = 28(\xi - \delta)^{-2}, \quad M_3 = 392(\xi - \delta)^{-3} + 16(\xi - \delta)^{-2}M_2 \tag{2.30}$$

as positive, and

$$M_4 = 8232(\xi - \delta)^{-4} + 576(\xi - \delta)^{-3}M_2 + 8(\xi - \delta)^{-2}\left(6M_2^2 + 5M_3\right), \tag{2.31}$$

then (2.8) and (2.9) are satisfied.

Thus, we have shown that when $\xi_0, \ldots, \xi_3$ and $M_1, \ldots, M_4$ are defined as above, then there will be a solution for (2.28) in $X(\xi; \xi_0, \ldots, \xi_3; 1, \ldots, M_4; I)$.

Acknowledgment

This work was partially supported by the Natural Science Foundation of Chongqing Normal University (Grant no. 12XLB003).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 951251, 12 pages
doi:10.1155/2012/951251

Research Article

Necessary and Sufficient Condition for the Existence of Solutions to a Discrete Second-Order Boundary Value Problem

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Received 20 December 2011; Accepted 22 February 2012

Academic Editor: Yuriy Rogovchenko

This paper is concerned with the existence of solutions for the discrete second-order boundary value problem $\Delta^2 u(t-1) + \lambda_1 u(t) + g(\Delta u(t)) = f(t)$, $t \in \{1, 2, \dots, T\}$, $u(0) = u(T+1) = 0$, where $T > 1$ is an integer, $f : \{1, \dots, T\} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and λ_1 is the first eigenvalue of the eigenvalue problem $\Delta^2 u(t-1) + \lambda u(t) = 0$, $t \in \mathbb{T}$, $u(0) = u(T+1) = 0$.

1. Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ and $p : [0, \pi] \to \mathbb{R}$ be continuous. The nonlinear two-point boundary value problem of ordinary differential equation

$$\begin{gathered} u''(t) + u(t) + g(u') = p(t), \quad t \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

is very important in applications. Let us mention the problems arising in viscosity, nonlinear oscillations, electric circuits, and so forth. The term $g(u')$ may be regarded as a nonlinear damping term in resonance problems and its appears, for example, in Rayleigh's equation (which is closely connected with a theory of oscillation of violin string), in oscillations of a simple pendulum under the action of viscous damping, in dry (Coulomb) friction (which occurs when the surfaces of two solids are contact and relative motion without lubrication), and in some cases of van der Pol oscillator, see [1–4] and the references therein.

Since the pioneer work of Landesman and Lazer [5], the problems of the type

$$u''(t) + u(t) + g(u) = f(t), \quad t \in (0,\pi),$$
$$u(0) = u(\pi) = 0, \tag{1.2}$$

(where g is independent of u') have been extensively studied in the past forty years, see Iannacci and Nkashama [6] and the references therein.

It has been remarked (see [7, 8]) that conditions of the Landesmen-Lazer type are not appropriated to yield the existence of solutions to (1.1). Thus, it is usually much more difficult to deal with (1.1) than to deal with (1.2), see Kannan et al. [7], Cañada and Drábek [8], Habets and Sanchez [9], Drábek et al. [10], and Del Toro and Roca [11].

In [8], Cañada and Drábek used the well-known Lyapunov-Schmidt method and the Schauder fixed point theorem to find a necessary and sufficient condition for the existence of solutions of (1.1). To wit, they proved

Theorem A (See [8, Theorem 3.1]). *Let $p : [0,\pi] \to \mathbb{R}$ be continuous and let*

$$p(t) = s\sqrt{\frac{2}{\pi}}\sin t + \widetilde{p}(t), \quad s \in \mathbb{R}, \qquad \int_0^\pi \widetilde{p}(t)\sin t\, dt = 0. \tag{1.3}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and bounded with $g(-\infty) = g(+\infty)$ and $g(\xi) < g(+\infty)$ for $\xi \in \mathbb{R}$, where

$$g(-\infty) := \lim_{s\to-\infty} g(s), \qquad g(+\infty) := \lim_{s\to+\infty} g(s). \tag{1.4}$$

Then for any $\widetilde{p} \in C[0,\pi]$ with $\int_0^\pi \widetilde{p}(t)\sin t\, dt = 0$, there exists a real number $g_{\widetilde{p}} < 2\sqrt{2/\pi}\, g(+\infty)$ such that (1.1) has at least one solution $u \in C^2[0,\pi]$ if and only if

$$s \in \left[g_{\widetilde{p}},\ 2\sqrt{\frac{2}{\pi}}g(+\infty)\right). \tag{1.5}$$

It is the purpose of this paper to establish the similar results for the discrete analogue of (1.1) of the form

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g(\Delta u(t)) = f(t), \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0, \tag{1.6}$$

where $T > 1$ is an integer, $\mathbb{T} := \{1,\dots, T\}$, $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, $f : \mathbb{T} \to \mathbb{R}$, λ_1 is the first eigenvalue of the linear eigenvalue problem

$$\Delta^2 u(t-1) + \lambda u(t) = 0, \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0. \tag{1.7}$$

Finally, it is worth remarking that the existence of solutions for nonlinear problem

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g(u(t)) = f(t), \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \tag{1.8}$$

which is a discrete analogue of (1.2), has been studied by Rodriguez [12] and Ma [13]. For other recent results on the existence of solutions of discrete problems, see [14–21] and the reference therein.

The rest of this paper is arranged as follows. In Section 2, we give some preliminaries and develop the methods of lower and upper solutions for the more generalized problems, that is, the case of the nonlinearity $g = g(t, u, \Delta u)$; in Section 3, we state our main result and provide the proof.

2. Preliminaries

Recall that $\mathbb{T} = \{1, 2, \ldots, T\}$. Let $\widehat{\mathbb{T}} = \{0, 1, \ldots, T+1\}$. Let $X := \{u \mid u : \widehat{\mathbb{T}} \to \mathbb{R}\}$, $Y := \{u \mid u : \mathbb{T} \to \mathbb{R}\}$ be equipped with the norm

$$\|u\|_X = \max_{k \in \widehat{\mathbb{T}}} |u(k)|, \qquad \|u\|_Y = \max_{k \in \mathbb{T}} |u(k)|, \tag{2.1}$$

respectively. It is easy to see that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

Assume that $g_0 : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, bounded by a constant $M > 0$:

$$|g_0(t, \eta, \xi)| \le M \tag{2.2}$$

for $t \in \mathbb{T}$ and $(\eta, \xi) \in \mathbb{R}^2$. Consider the following problem:

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g_0(t, u(t), \Delta u(t)) = f(t), \quad t \in \mathbb{T}, \tag{2.3}$$

$$u(0) = u(T+1) = 0. \tag{2.4}$$

Definition 2.1. If $x \in X$ satisfies

$$\Delta^2 x(t-1) + \lambda_1 x(t) \ge f(t) - g_0(t, x(t), \Delta x(t)), \quad t \in \mathbb{T}, \\ x(0) \le 0, \quad x(T+1) \le 0, \tag{2.5}$$

then one says $x(t)$ is a lower solution of (2.3), (2.4). If $y \in X$ satisfies

$$\Delta^2 y(t-1) + \lambda_1 y(t) \le f(t) - g_0(t, y(t), \Delta y(t)), \quad t \in \mathbb{T}, \\ y(0) \ge 0, \quad y(T+1) \ge 0, \tag{2.6}$$

then one says $y(t)$ is an upper solution of (2.3), (2.4).

Theorem 2.2. *Suppose that $x(t)$, $y(t)$ are the lower and upper solutions of* (2.3), (2.4), *respectively, and $x(t) \leq y(t)$, $t \in \mathbb{T}$. Then BVP* (2.3) *and* (2.4) *have at least one solution $u(t)$ satisfies*

$$x(t) \leq u(t) \leq y(t). \tag{2.7}$$

Proof. Define the function $p : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ by

$$p(t, u(t)) = \begin{cases} x(t), & u(t) < x(t), \\ u(t), & x(t) \leq u(t) \leq y(t), \\ y(t), & u(t) > y(t). \end{cases} \tag{2.8}$$

Set $f^*(t, u, v) = f(t) - g_0(t, u, v) - \lambda_1 u$. Consider the auxiliary problems:

$$\begin{gathered} \Delta^2 u(t-1) = f^*(t, p(t,u), \Delta p(t,u)), \qquad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0. \end{gathered} \tag{2.9}$$

From (2.8) and the boundness of g_0, we know $f^*(t, p(t,u), \Delta p(t,u))$ is bounded. So, by the Schauder fixed point theorem, (2.9) has a solution $u \in X$.

Now, we only prove $u(t) \leq y(t)$, the other case $u(t) \geq x(t)$ is similar.

Set $z(t) = u(t) - y(t)$. Suppose that $z(t) > 0$, for $t \in \{t_0+1, t_0+2, \ldots, t_0+p\}$, and $z(t_0) \leq 0$, $z(t_0+p+1) \leq 0$, where $t_0 \in \{0, 1, \ldots, T\}$, $p \in \{1, 2, \ldots, T\}$.

On the other hand, by the definition of upper solution, for $t \in \{t_0+1, t_0+2, \ldots, t_0+p\}$,

$$\begin{aligned} \Delta^2 y(t-1) &\leq f^*(t, y(t), \Delta y(t)) \\ &= f(t) - \lambda_1 y(t) - g_0(t, y(t), \Delta y(t)) \\ &= f(t) - \lambda_1 p(t,u) - g_0(t, p(t, u(t)), \Delta p(t, u(t))) \\ &= \Delta^2 u(t-1). \end{aligned} \tag{2.10}$$

Then

$$\begin{gathered} \Delta^2 z(t-1) \geq 0, \quad t \in \{t_0+1, t_0+2, \ldots, t_0+p\}, \\ z(t_0) \leq 0, \quad z(t_0+p+1) \leq 0. \end{gathered} \tag{2.11}$$

Now, by the convexity of z on $\{t_0+1, t_0+2, \ldots, t_0+p\}$, we get $z(t) \leq 0$, $t \in \{t_0+1, t_0+2, \ldots, t_0+p\}$, that is, $u(t) \leq y(t), t \in \{t_0+1, t_0+2, \ldots, t_0+p\}$. This contradicts $u(t) > y(t)$, $t \in \{t_0+1, t_0+2, \ldots, t_0+p\}$. Thus, $u(t) \leq y(t)$, $t \in \widehat{\mathbb{T}}$. □

Lemma 2.3. *See* $\sum_{t=1}^{T} \sin^2(\pi t/(T+1)) = (T+1)/2$.

Proof. Let $\omega = \cos(2\pi/(T+1)) + i\sin(2\pi/(T+1))$. Then $\omega^{T+1} = 1$ and $(1-\omega)(1+\omega+\omega^2+\cdots+\omega^T) = 0$. Since $1-\omega \neq 0$, we have $\sum_{t=1}^{T} \cos(2\pi t/(T+1)) = -1$. This together with the fact that $\sum_{t=1}^{T} \sin^2(\pi t/(T+1)) = \sum_{t=1}^{T} (1-\cos(2\pi t/(T+1)))/2$ implies the assertion holds. □

Now, let $\varphi_1(t) := \sqrt{2/(T+1)}\sin(\pi t/(T+1))$, $t \in \mathbb{T}$, denote the positive eigenfunction corresponding to the first eigenvalue $\lambda_1 = 4\sin^2(\pi/2(T+1))$ of (1.7). Then by Lemma 2.3, $\sum_{t=1}^{T}\varphi_1^2(t) = 1$.

Since $\varphi_1(t)$ is located on $\sqrt{2/(T+1)}\sin t,\ t \in [0,\pi]$, by the direct computation, we can obtain the following result.

Lemma 2.4. *If T is an odd number, then*

$$\Delta\varphi_1(t) > 0, \quad \text{for } t \in \left\{0, \ldots \left[\frac{T}{2}\right]\right\}, \qquad \Delta\varphi_1(t) < 0, \quad \text{for } t \in \left\{\left[\frac{T}{2}\right]+1, \ldots, T\right\}, \tag{2.12}$$

if T is an even number, then $\Delta\varphi_1(T/2) = 0$,

$$\Delta\varphi_1(t) > 0, \quad \text{for } t \in \left\{0, \ldots \frac{T}{2}-1\right\}, \qquad \Delta\varphi_1(t) < 0, \quad \text{for } t \in \left\{\frac{T}{2}+1, \ldots, T\right\}. \tag{2.13}$$

Define the operator $L : D(L) \subset X \to Y$ by

$$Lu(t) = \Delta^2 u(t-1) + \lambda_1 u(t), \tag{2.14}$$

where $D(L) = \{u \in X \mid u(0) = u(T+1) = 0\}$.

Define $N : X \to Y$ by

$$(Nu)(t) = f(t) - g_0(t, u(t), \Delta u(t)). \tag{2.15}$$

Then (2.3), (2.4) is equivalent to the operator equation $Lu = Nu$.

In Theorem 2.2, we established the methods of lower and upper solutions under well order. Now, we can also develop the methods of lower and upper solutions for (2.3), (2.4) when $x(t) \le y(t)$ is not necessary, its proofs are based on the following lemma, that is, the connectivity properties of the solution sets of parameterized families of compact vector fields, they are a direct consequence of Mawhin [22, Lemma 2.3].

Lemma 2.5 (see [22, Lemma 2.3]). *Let E be a Banach space and $C \subset E$ a nonempty, bounded, closed convex subset. Suppose that $T : [a,b] \times C \to C$ is completely continuous. Then the set*

$$S = \{(\lambda, x) \mid T(\lambda, x) = x, \quad \lambda \in [a,b]\} \tag{2.16}$$

contains to be a closed connected subset Σ which connects $\{a\} \times C$ to $\{b\} \times C$.

Theorem 2.6. *Assume that $x(t)$, $y(t)$ are the lower solution and the upper solution of* (2.3), (2.4), *respectively. Then* (2.3) *and* (2.4) *have at least one solution.*

Proof. Define the projections $P : X \to X$, $Q : Y \to \mathbb{R}$ by

$$(Pu)(t) = \left[\sum_{t=1}^{T} u(t)\varphi_1(t)\right]\varphi_1(t), \qquad (Qy)(t) = \sum_{t=1}^{T} y(t)\varphi_1(t). \tag{2.17}$$

Then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, and $X = (\operatorname{Ker} P \oplus \operatorname{Ker} L), Y = (\operatorname{Im} L \oplus \operatorname{Im} Q)$. Now, the operator equation $Lu = Nu$ is equivalent to the alternative system

$$\begin{gathered} u - Pu = K(I-Q)Nu, \\ QNu = 0, \end{gathered} \tag{2.18}$$

where K is the inverse of mapping $L : (D(L) \cap \operatorname{Ker} P) \to \operatorname{Im} L$.

Writing $u \in D(L)$ in the form $u(t) = c\psi_1(t) + w(t)$, $c \in \mathbb{R}$, $\sum_{t=1}^{T} \psi_1(t)w(t) = 0$, (2.3) and (2.4) are equivalent to the system

$$w = K(I-Q)N\big(c\psi_1(\cdot) + w\big), \tag{2.19}$$

$$QN\big(c\psi_1(\cdot) + w\big) = 0. \tag{2.20}$$

Since X is finite dimensional, it is easy to see that $K(I-Q)N$ is completely continuous, by the Schauder fixed point theorem and the fact $N(c\psi_1(t) + w(t))$ is bounded, we get that for any fixed $c \in \mathbb{R}$, $W(c) := \{w \in X \cap \operatorname{Ker} P \mid (c,w) \text{ satisfies } (2.19)\} \neq \emptyset$ and $W(c)$ is bounded. Then there exist positive constants $\alpha > \tau$ such that $-\tau\psi_1 \le w \le \tau\psi_1$ for all $w \in W(c)$, $(\alpha-\tau)\psi_1 \ge x(t)$ and $-(\alpha-\tau)\psi_1 \le y(t)$. Let

$$\gamma(c,w) = \sum_{t=1}^{T} N\big(c\psi_1(t) + w(t)\big)\psi_1(t) \tag{2.21}$$

for all $(c,w) \in \mathbb{R} \times W(c)$. Observe that Lemma 2.5 is applicable. Hence there exists a connected subset of $\{(c,w) \in R \times (X \cap \operatorname{Ker} P) \mid (c,w) \text{ satisfies } (2.19)\}$, $\Sigma_\alpha(\psi_1)$, which connected $\{-\alpha\} \times W(-\alpha)$ and $\{\alpha\} \times W(\alpha)$. Since $\gamma : \Sigma_\alpha(\psi_1) \to \mathbb{R}$ is continuous, $I := \gamma(\Sigma_\alpha(\psi_1))$ is an interval. If $0 \in I$, then (2.3) and (2.4) have a solution. If $I \subset (0,\infty)$, then every $c\psi_1 + w$ with $(c,w) \in \Sigma_\alpha(\psi_1)$ is an upper solution. Indeed, it is obvious that

$$\begin{gathered} L\big(c\psi_1(t) + w(t)\big) - N\big(c\psi_1(t) + w(t)\big) = -\gamma(c,w) \le 0, \qquad t \in \mathbb{T}, \\ \big(c\psi_1 + w\big)(0) = \big(c\psi_1 + w\big)(T+1) = 0. \end{gathered} \tag{2.22}$$

By construction, $\alpha\psi_1 + w$ with $(\alpha,w) \in \Sigma_\alpha(\psi_1)$ satisfies $x(t) \le \alpha\psi_1 + w$. Hence, from Theorem 2.2, (2.3) and (2.4) have a solution. A similar argument applies if $I \subset (-\infty,0)$. □

Theorem 2.7. *Suppose that f satisfies*

$$f(t) = s\psi_1(t) + \tilde{f}(t), \qquad s \in \mathbb{R}, \tag{2.23}$$

where $\tilde{f}$ satisfies

$$\sum_{t=1}^{T} \tilde{f}(t)\psi_1(t) = 0. \tag{2.24}$$

Then, there exists a nonempty, connected, and bounded set $J_{\tilde{f}} \subset \mathbb{R}$ *such that* (2.3) *and* (2.4) *have at least one solution* $u \in X$ *if and only if* $s \in J_{\tilde{f}}$.

Proof. As the proof of Theorem 2.6, (2.3) and (2.4) are equivalent to the system (2.19), (2.20). Since N is bounded, applying the Schauder fixed point theorem we obtain that for any fixed $c \in \mathbb{R}$, there exits at least one $w_c \in X$ such that (2.19) holds.

Now, (2.20) becomes

$$\sum_{t=1}^{T} g_0\big(t, c\varphi_1(t) + w_c(t), c\Delta\varphi_1(t) + \Delta w_c(t)\big)\varphi_1(t) = s. \tag{2.25}$$

Hence, for a given $\tilde{f}$, $\sum_{t=1}^{T} \tilde{f}(t)\varphi_1(t) = 0$, (2.3), (2.4) with $f(t) = c\varphi_1(t) + \tilde{f}(t)$ has at least one solution if and only if s belongs to the range of the (multivalued, in general) function $\Gamma_{\tilde{f}} : \mathbb{R} \to \Gamma_{\tilde{f}}(\mathbb{R})$,

$$\Gamma_{\tilde{f}}(c) = \sum_{t=1}^{T} g_0\big(t, c\varphi_1(t) + w_c(t), c\Delta\varphi_1(t) + \Delta w_c(t)\big)\varphi_1(t), \tag{2.26}$$

where $w_c \in \{w \in D(L) : w \text{ is a solution of (2.19) for fixed } c\}$. But $J_{\tilde{f}} \equiv \Gamma_{\tilde{f}}(\mathbb{R})$ is a connected set. In fact, let s_1 and s_2 belong to $J_{\tilde{f}}$ and $s_1 \le s_2$. Then (2.3), (2.4) with $f_1 = s_1\varphi_1 + \tilde{f}$ and $f_2 = s_2\varphi_1 + \tilde{f}$ has solutions u_1 and u_2, respectively. If we consider (2.3), (2.4) with $f = s\varphi_1 + \tilde{f}$, where $s \in [s_1, s_2]$, then u_1 is an upper solution and u_2 is a lower solution to this problem. By Theorem 2.6, there exists at least one solution, that is, s belongs to $J_{\tilde{f}}$. Moreover, since g is bounded, the range of $\Gamma_{\tilde{f}}$ is bounded. □

3. Main Results

In this section, we deal with (1.6). First, let us make the following assumptions:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function and satisfies $g(+\infty) = g(-\infty)$ and $g(\xi) < g(+\infty)$ for any $\xi \in \mathbb{R}$,

(H2) $f : \mathbb{T} \to \mathbb{R}$ satisfies

$$f(t) = s\varphi_1(t) + \tilde{f}(t), \quad s \in \mathbb{R}, \qquad \sum_{t=1}^{T} \tilde{f}(t)\varphi_1(t) = 0. \tag{3.1}$$

Theorem 3.1. *Suppose that* (H1), (H2) *hold. Then there exists a real number* $g_{\tilde{f}}$, $g_{\tilde{f}} < \sqrt{2/(T+1)}\,g(+\infty)\sum_{t=1}^{T}\sin(\pi t/(T+1))$, *such that* (1.6) *has at least one solution* $u \in X$ *if and only if*

$$s \in \left[g_{\tilde{f}}, \sqrt{\frac{2}{T+1}}\,g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}\right). \tag{3.2}$$

Proof. Note that $\psi_1(t) = \sqrt{2/(T+1)}\sin(\pi t/(T+1))$. Due to the consideration in the the proof of Theorem 2.7. It is sufficient to show that for a given $\tilde{f}$ with $\sum_{t=1}^{T}\tilde{f}(t)\psi_1(t) = 0$, we have

$$\Gamma_{\tilde{f}}(\mathbb{R}) = \left[g_{\tilde{f}}, \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}\right). \tag{3.3}$$

The (possibly multivalued) function $\Gamma_{\tilde{f}}$ has the following form:

$$\Gamma_{\tilde{f}}(c) = \sqrt{\frac{2}{T+1}}\sum_{t=1}^{T}g\big(c\Delta\psi_1(t) + \Delta w_c(t)\big)\sin\frac{\pi t}{T+1}, \tag{3.4}$$

where $c \in \mathbb{R}$ and w_c verify (2.19). From the boundedness of g and (2.19), there exists a constant $D > 0$ (independent of c) such that $\|w_c\|_X \le D$ for any $c \in \mathbb{R}$, furthermore,

$$\max_{t\in\{0,1,\dots,T\}}|\Delta w_c(t)| \le 2D. \tag{3.5}$$

Now, we divide the proof into two cases.

Case 1. T is an odd number. By Lemma 2.4, we obtain that

$$\begin{aligned}
&\sum_{t=1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad= \sum_{t=1}^{[T/2]}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+ \sum_{t=[T/2]+1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad\longrightarrow \sum_{t=1}^{[T/2]}g(\pm\infty)\sin\frac{\pi t}{T+1} + \sum_{s=[T/2]+1}^{T}g(\mp\infty)\sin\frac{\pi t}{T+1},
\end{aligned} \tag{3.6}$$

as $c \longrightarrow \pm\infty$. Due to $g(+\infty) = g(-\infty)$, we get

$$\Gamma_{\tilde{f}}(c) \longrightarrow \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}. \tag{3.7}$$

The assumption $g(\xi) < g(+\infty),\ \xi \in \mathbb{R}$, and (3.5) yields

$$\Gamma_{\tilde{f}}(c) < \sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1} \tag{3.8}$$

for any $c \in \mathbb{R}$.

Case 2. T is an even number. By Lemma 2.4, we know that

$$\Delta\varphi_1\left(\frac{T}{2}\right)=\sqrt{\frac{2}{T+1}}\Delta\sin\frac{\pi T}{2(T+1)}=2\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2T/2+1)}{2(T+1)}=0. \tag{3.9}$$

Hence,

$$\begin{aligned}
&\sum_{t=1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)}+\Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\quad=\sum_{t=1}^{(T/2)-1}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)}+\Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+\sum_{t=(T/2)+1}^{T}g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)}+\Delta w_c(t)\right)\sin\frac{\pi t}{T+1}\\
&\qquad+g\left(\Delta w_c\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}.
\end{aligned} \tag{3.10}$$

By (3.5) and the assumption $g(\xi)<g(+\infty)$, $\xi\in\mathbb{R}$, we know that for any $c\in\mathbb{R}$, $g(\Delta w_c(T/2))<g(+\infty)$. Thus, for any $c\in\mathbb{R}$,

$$\Gamma_{\tilde{f}}(c)<\sqrt{\frac{2}{T+1}}g(+\infty)\sum_{t=1}^{T}\sin\frac{\pi t}{T+1}. \tag{3.11}$$

It is sufficient to prove that this infimum is achieved. Let us denote

$$g_{\tilde{f}}=\inf_{c\in\mathbb{R}}\Gamma_{\tilde{f}}(c). \tag{3.12}$$

Suppose that $\{s_n\}\subset\Gamma_{\tilde{f}}(\mathbb{R})$ satisfies $s_n\to g_{\tilde{f}}$ and $\{c_n\}$ is the corresponding minimizing sequence, that is, $u_n(t)=c_n\sqrt{2/(T+1)}\sin(\pi t/(T+1))+w_{c_n}(t)$, are the solution of (1.1), with the right-hand sides $f_n(t)=s_n\sqrt{2/(T+1)}\sin(\pi t/(T+1))+\tilde{f}(t)$.

We claim that $\{c_n\}$ is bounded. In fact, if $c_n\to\infty$ as $n\to\infty$, then we can get two contradictions in the following two cases.

Case 1. If T is an odd number, then by (2.20),

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T}g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)}+\Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1}=s_n, \tag{3.13}$$

letting $n \to \infty$ in (3.13), we get

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T} g(+\infty)\sin\frac{\pi t}{T+1} = g_{\tilde{f}}. \tag{3.14}$$

From (H1), we arrive for any $\xi \in \mathbb{R}$, $g(\xi) < g(+\infty)$, which together with (3.14) implies that

$$\sqrt{\frac{2}{T+1}}\sum_{t=1}^{T} g\left(2c\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_c(t)\right)\sin\frac{\pi t}{T+1} < g_{\tilde{f}}. \tag{3.15}$$

This contradicts (3.12).

Case 2. If T is an even number, then by (2.20) and $\Delta\varphi_1(T/2) = 0$, we get

$$\begin{aligned} s_n = {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\, g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}. \end{aligned} \tag{3.16}$$

This implies that

$$\begin{aligned} g_{\tilde{f}} \geq {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g(+\infty)\sin\frac{\pi t}{T+1} + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g(+\infty)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\inf_{w_{c_n}\in X} g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)}. \end{aligned} \tag{3.17}$$

On the other hand, by (3.12) and (H1), we get that for any fixed $n \in \mathbb{N}$,

$$\begin{aligned} g_{\tilde{f}} \leq {} & \sqrt{\frac{2}{T+1}}\sum_{t=1}^{(T/2)-1} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\sum_{t=(T/2)+1}^{T} g\left(2c_n\sqrt{\frac{2}{T+1}}\sin\frac{\pi}{2(T+1)}\cos\frac{\pi(2t+1)}{2(T+1)} + \Delta w_{c_n}(t)\right)\sin\frac{\pi t}{T+1} \\ & + \sqrt{\frac{2}{T+1}}\inf_{w_{c_n}\in X} g\left(\Delta w_{c_n}\left(\frac{T}{2}\right)\right)\sin\frac{\pi T}{2(T+1)} \end{aligned}$$

$$< \sqrt{\frac{2}{T+1}} \sum_{t=1}^{(T/2)-1} g(+\infty) \sin \frac{\pi t}{T+1} + \sqrt{\frac{2}{T+1}} \sum_{t=(T/2)+1}^{T} g(+\infty) \sin \frac{\pi t}{T+1}$$
$$+ \sqrt{\frac{2}{T+1}} \inf_{w_{c_n} \in X} g\left(\Delta w_{c_n} \left(\frac{T}{2} \right) \right) \sin \frac{\pi T}{2(T+1)}. \tag{3.18}$$

Now, we obtain a contradiction. Thus, c_n is bounded.

Since X is finite dimensional and w_{c_n} is bounded, we obtain that $c_n \to c$, $w_{c_n} \to w_c$ (at least for a subsequence), and $u(t) = c\sqrt{2/(T+1)} \sin(\pi t/(T+1)) + w_c(t)$ is a solution of (1.1) with $f(t) = g_{\tilde{f}} \sqrt{2/(T+1)} \sin(\pi t/(T+1)) + \tilde{f}(t)$. Hence, the infimum is achieved in c. □

Acknowledgments

This paper is supported by NSFC (11061030, 11101335, 11126296) and the Fundamental Research Funds for the Gansu Universities.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 760918, 8 pages
doi:10.1155/2012/760918

Research Article

Periodic Solutions for Duffing Type p-Laplacian Equation with Multiple Constant Delays

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Received 19 November 2011; Accepted 17 January 2012

Academic Editor: Elena Braverman

Using inequality techniques and coincidence degree theory, new results are provided concerning the existence and uniqueness of periodic solutions for the Duffing type p-Laplacian equation with multiple constant delays of the form $(\varphi_p(x'(t)))' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) = e(t)$. Moreover, an example is provided to illustrate the effectiveness of the results in this paper.

1. Introduction

Referring to the work of Esmailzadeh and Nakhaie-Jazar [1], Duffing type equation is the simplest case of a vibrating system with nonlinear restoring force generator element. This is equivalent to a mechanical vibrating system with either a hard or soft spring. Thus, this equation and its modifications have been extensively and intensively studied. In particular, the existence of periodic solutions for Duffing type equations with and without delays have been discussed by various researchers (see, e.g., [2–8] and the references given therein). However, to the best of our knowledge, the existence and uniqueness of periodic solutions of Duffing type p-Laplacian equation whose delays more than two have not been sufficiently researched. Motivated by this, we shall consider the Duffing type p-Laplacian equations with multiple constant delays of the form

$$\left(\varphi_p\left(x'(t)\right)\right)' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) = e(t), \tag{1.1}$$

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, C and τ_k are constants, $e : R \to R$ and $g_0, g_k : R \times R \to R$ are continuous functions, τ_k and e are T-periodic, g_0 and g_k are T-periodic in the first argument, $T > 0$ and $k = 1, 2, \dots, n$. The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1). The results of this paper are new and complement previously known results. Moreover, we give an example to illustrate the results.

2. Preliminary Results

Throughout this paper, we will denote

$$C_T^1 := \left\{x \in C^1(R) : x \text{ is } T\text{-periodic}\right\},$$
$$|x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k} \quad (k > 0), \qquad |x|_\infty = \max_{t\in[0,T]}|x(t)|. \tag{2.1}$$

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = \tilde{f}(t, x, x'), \qquad x(0) = x(T), \qquad x'(0) = x'(T), \tag{2.2}$$

where $\tilde{f} \in C(R^3, R)$ is T-periodic in the first variable, we have the following lemma.

Lemma 2.1 (see [9]). *Let Ω be an open bounded set in C_T^1, if the following conditions hold.*

(i) *For each $\lambda \in (0, 1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda\tilde{f}(t, x, x'), \qquad x(0) = x(T), \qquad x'(0) = x'(T), \tag{2.3}$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(t, a, 0)dt = 0 \tag{2.4}$$

has no solution on $\partial\Omega \bigcap R$.

(iii) *The Brouwer degree of F*

$$\deg\left(F, \Omega \bigcap R, 0\right) \neq 0. \tag{2.5}$$

Then, the periodic boundary value problem (2.2) *has at least one T-periodic solution on $\overline{\Omega}$.*

We can easily obtain the homotopic equation of (1.1) as follows:

$$(\varphi_p(x'(t)))' + \lambda Cx'(t) + \lambda\left[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k))\right] = \lambda e(t), \quad \lambda \in (0, 1). \tag{2.6}$$

Lemma 2.2. *Assume that the following conditions are satisfied.*
(A_1) There exists a constant $d > 0$ such that

(1) $\sum_{k=0}^{n} g_k(t, x_k) - e(t) < 0$ for $x_k > d$, $t \in R$, $k = 0, 1, 2, \ldots, n$,

(2) $\sum_{k=0}^{n} g_k(t, x_k) - e(t) > 0$ for $x_k < -d$, $t \in R$, $k = 0, 1, 2, \ldots, n$.

Moveover, if $x(t)$ is a T-periodic solution of (2.6)*, then*

$$|x|_\infty \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.7}$$

Proof. Let $x(t)$ be a T-periodic solution of (2.6). Then, integrating (2.6) over $[0, T]$, we have

$$\int_0^T \left[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k)) - e(t) \right] dt = 0. \tag{2.8}$$

Using the integral mean-value theorem, it follows that there exists $t_1 \in [0, T]$ such that

$$g_0(t_1, x(t_1)) + \sum_{k=1}^{n} g_k(t_1, x(t_1 - \tau_k)) - e(t_1) = 0. \tag{2.9}$$

We now prove that there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \le d. \tag{2.10}$$

Indeed, suppose otherwise. Then,

$$|x(t)| > d \quad \forall t \in R. \tag{2.11}$$

Let $\tau_0 = 0$. From (A_1), (2.9), and (2.11), we see that there exist $0 \le i,\ j \le n$ such that

$$x(t_1 - \tau_i) = \max_{0 \le k \le n} x(t_1 - \tau_k) \ge \min_{0 \le k \le n} x(t_1 - \tau_k) = x\left(t_1 - \tau_j\right), \tag{2.12}$$

which, together with (2.11), implies

$$-d > x(t_1 - \tau_i) = \max_{0 \le k \le n} x(t_1 - \tau_k) \quad \text{or} \quad x\left(t_1 - \tau_j\right) = \min_{0 \le k \le n} x(t_1 - \tau_k) > d. \tag{2.13}$$

Without loss of generality, we may assume that $x(t_1 - \tau_j) > d$ (the situation is analogous for $-d > x(t_1 - \tau_i)$). Then, we have

$$x(t_1 - \tau_i) \ge x(t_1 - \tau_k) \ge x\left(t_1 - \tau_j\right) > d, \quad k = 0, 1, 2, \ldots, n. \tag{2.14}$$

According to (2.14) and (A_1), we obtain

$$0 > \sum_{k=0}^{n} g_k(t_1, x(t_1 - \tau_k)) - e(t_1), \tag{2.15}$$

this contradicts the fact (2.9); thus, (2.10) is true.

Let $t_2 = mT + t_0$ where $t_0 \in [0, T]$ and m is an integer. Then, by the same approach used in the proof of inequality (3.3) of [7], we have

$$|x|_\infty = \max_{t\in[t_0,t_0+T]} |x(t)| \le \max_{t\in[t_0,t_0+T]} \left\{ d + \frac{1}{2}\left(\int_{t_0}^{t} |x'(s)| ds + \int_{t-T}^{t_0} |x'(s)| ds \right) \right\} \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.16}$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let (A_1) holds. Assume that the following condition is satisfied.*

(A_2) There exist nonnegative constants $\overline{b_0}, b_0, b_1, b_2, \dots, b_n$ such that

$$\begin{gathered} \overline{b_0}|x_1 - x_2|^2 \le -(g_0(t, x_1) - g_0(t, x_2))(x_1 - x_2), \\ \overline{b_0} > b_1 + b_2 + \cdots + b_n, \ \left|g_k(t, x_1) - g_k(t, x_2)\right| \le b_k|x_1 - x_2|, \end{gathered} \tag{2.17}$$

for all $t, x_1, x_2 \in R$, $k = 0, 1, 2, \dots, n$.

Then, (1.1) *has at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we obtain

$$\begin{aligned} &(\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))' + C(x_1'(t) - x_2'(t)) + [g_0(t, x_1(t)) - g_0(t, x_2(t))] \\ &\quad + \sum_{k=1}^{n} [g_k(t, x_1(t - \tau_k)) - g_k(t, x_2(t - \tau_k))] = 0. \end{aligned} \tag{2.18}$$

Multiplying $Z(t)$ and (2.18) and then integrating it from 0 to T, from (A_2) and Schwarz inequality, we get

$$\begin{aligned} \overline{b_0}|Z|_2^2 &= \overline{b_0} \int_0^T |Z(t)|^2 dt \\ &\le -\int_0^T (x_1(t) - x_2(t))[g_0(t, x_1(t)) - g_0(t, x_2(t))] dt \\ &= -\int_0^T (\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))(x_1'(t) - x_2'(t)) dt \\ &\quad + \sum_{k=1}^{n} \int_0^T [g_k(t, x_1(t - \tau_k)) - g_k(t, x_2(t - \tau_k))] Z(t) dt \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} b_k \int_0^T |x_1(t-\tau_k) - x_2(t-\tau_k)||Z(t)|dt \\
&\leq \sum_{k=1}^{n} b_k \left(\int_0^T |x_1(t-\tau_k) - x_2(t-\tau_k)|^2 dt \right)^{1/2} |Z|_2 \\
&= \sum_{k=1}^{n} b_k |Z|_2^2.
\end{aligned}
\tag{2.19}
$$

Since $\overline{b_0} > b_1 + b_2 + \cdots + b_n$, we have

$$
Z(t) \equiv 0 \quad \forall t \in R. \tag{2.20}
$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in R$. Therefore, (1.1) has at most one T-periodic solution. The proof of Lemma 2.3 is now complete. □

3. Main Results

Theorem 3.1. *Let (A_1) and (A_2) hold. Then,* (1.1) *has a unique T-periodic solution in C_T^1.*

Proof. By Lemma 2.3, it is easy to see that (1.1) has at most one T-periodic solution in C_T^1. Thus, in order to prove Theorem 3.1, it suffices to show that (1.1) has at least one T-periodic solution in C_T^1. To do this, we are going to apply Lemma 2.1. Firstly, we claim that the set of all possible T-periodic solutions of (2.6) in C_T^1 is bounded.

Let $x(t) \in C_T^1$ be a T-periodic solution of (2.6). Multiplying $x(t)$ and (2.6) and then integrating it from 0 to T, we have

$$
-\int_0^T \varphi_p(x'(t))x'(t)dt + \lambda \int_0^T x(t)\left[g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) - e(t) \right] dt = 0. \tag{3.1}
$$

Since $x(0) = x(T)$, then there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$. And since $\varphi_p(0) = 0$, we have

$$
|\varphi_p(x'(t))| = \left| \int_{t_0}^{t} (\varphi_p(x'(s)))' ds \right| \leq \lambda \int_{t_0}^{t_0+T} \left| g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) - e(t) \right| dt, \tag{3.2}
$$

where $t \in [t_0, t_0 + T]$.

In view of (3.1), (A_2), and Schwarz inequality, we get

$$
\begin{aligned}
\overline{b_0}|x|_2^2 &= b_0 \int_0^T |x(t)|^2 dt \\
&\leq -\int_0^T (x(t) - 0)(g_0(t, x(t)) - g_0(t, 0))dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\lambda}\int_0^T \varphi_p(x'(t))x'(t)dt + \sum_{k=1}^n \int_0^T \left[g_k(t, x(t-\tau_k)) - g_k(t,0)\right]x(t)dt \\
&\quad + \sum_{k=0}^n \int_0^T g_k(t,0)x(t)dt - \int_0^T x(t)e(t)dt \\
&\le \sum_{k=1}^n b_k \int_0^T |x(t-\tau_k)||x(t)|dt + \sum_{k=0}^n \int_0^T \left|g_k(t,0)\right||x(t)|dt + \sqrt{T}|e|_\infty |x|_2 \\
&\le \sum_{k=1}^n b_k \left(\int_0^T |x(t-\tau_k)|^2 dt\right)^{1/2} |x|_2 + \sqrt{T}\sum_{k=0}^n \left|g_k(t,0)\right|_\infty |x|_2 + \sqrt{T}|e|_\infty |x|_2 \\
&= \sum_{k=1}^n b_k |x|_2^2 + \sqrt{T}\sum_{k=0}^n \left|g_k(t,0)\right|_\infty |x|_2 + \sqrt{T}|e|_\infty |x|_2.
\end{aligned}
\tag{3.3}
$$

It follows that

$$
|x|_2 \le \frac{\sqrt{T}\,\sum_{k=0}^n \left|g_k(t,0)\right|_\infty + \sqrt{T}|e|_\infty}{\overline{b_0} - \sum_{k=1}^n b_k} := \theta. \tag{3.4}
$$

Again from (A_2) and Schwarz inequality, (3.2) and (3.4) yield

$$
\begin{aligned}
\left|x'\right|_\infty^{p-1} &= \max_{t\in[t_0,\ t_0+T]}\left\{\left|\varphi_p(x'(t))\right|\right\} = \max_{t\in[t_0,t_0+T]}\left\{\left|\int_{t_0}^t (\varphi_p(x'(s)))' ds\right|\right\} \\
&\le \int_{t_0}^{t_0+T}\left|g_0(t,x(t)) + \sum_{k=1}^n g_k(t,x(t-\tau_k)) - e(t)\right| dt \\
&= \int_0^T \left|g_0(t,x(t)) + \sum_{k=1}^n g_k(t,x(t-\tau_k)) - e(t)\right| dt \\
&\le \int_0^T \left|g_0(t,x(t)) - g_0(t,0)\right|dt + \sum_{k=1}^n \int_0^T \left|g_k(t,x(t-\tau_k)) - g_k(t,0)\right|dt \\
&\quad + \sum_{k=0}^n \int_0^T \left|g_k(t,0)\right|dt + T|e|_\infty \\
&\le b_0 \int_0^T |x(t)|dt + \sum_{k=1}^n \int_0^T b_k |x(t-\tau_k)|dt + \sum_{k=0}^n T\left|g_k(t,0)\right|_\infty + T|e|_\infty \\
&\le b_0\sqrt{T}|x|_2 + \sum_{k=1}^n b_k\sqrt{T}|x|_2 + \sum_{k=0}^n T\left|g_k(t,0)\right|_\infty + T|e|_\infty
\end{aligned}
$$

$$\begin{aligned}&\leq b_0\sqrt{T}\theta+\sum_{k=1}^{n}b_k\sqrt{T}\theta+\sum_{k=0}^{n}T\left|g_k(t,0)\right|_\infty+T|e|_\infty\\&:=\overline{\eta},\end{aligned}\tag{3.5}$$

which, together with (2.7), implies that there exists a positive constant $M>1+(\overline{\eta})^{1/(p-1)}$ such that, for all $t\in R$,

$$\left|x'\right|_\infty<M,\qquad |x|_\infty\leq d+\frac{1}{2}\sqrt{T}\left|x'\right|_2\leq d+\frac{1}{2}T\left|x'\right|_\infty<M.\tag{3.6}$$

Set

$$\Omega=\left\{x\in C_T^1:|x|_\infty\leq M,\left|x'\right|_\infty\leq M\right\},\tag{3.7}$$

then we know that (2.6) has no T-periodic solution on $\partial\Omega$ as $\lambda\in(0,1)$ and when $x(t)\in\partial\Omega\bigcap R,\ x(t)=M$ or $x(t)=-M$, from (A_2), we can see that

$$\begin{aligned}\frac{1}{T}\int_0^T\left\{-g_0(t,M)-\sum_{k=1}^{n}g_k(t,M)+e(t)\right\}dt>0,\\\frac{1}{T}\int_0^T\left\{-g_0(t,-M)-\sum_{k=1}^{n}g_k(t,-M)+e(t)\right\}dt<0,\end{aligned}\tag{3.8}$$

so condition (ii) of Lemma 2.1 is also satisfied. Set

$$H(x,\mu)=\mu x-(1-\mu)\frac{1}{T}\int_0^T\left[g_0(t,x)+\sum_{k=1}^{n}g_k(t,x)-e(t)\right]dt,\tag{3.9}$$

and when $x\in\partial\Omega\bigcap R,\ \mu\in[0,1]$, we have

$$xH(x,\mu)=\mu x^2-(1-\mu)x\frac{1}{T}\int_0^T\left[g_0(t,x)+\sum_{k=1}^{n}g_k(t,x)-e(t)\right]dt>0.\tag{3.10}$$

Thus, $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned}\deg\left\{F,\Omega\bigcap R,0\right\}&=\deg\left\{-\frac{1}{T}\int_0^T\left[g_0(t,x)+\sum_{k=1}^{n}g_k(t,x)-e(t)\right]dt,\Omega\bigcap R,0\right\}\\&=\deg\left\{x,\Omega\bigcap R,0\right\}\neq0,\end{aligned}\tag{3.11}$$

so condition (iii) of Lemma 2.1 is satisfied. In view of the previous Lemma 2.1, (1.1) has at least one solution with period T. This completes the proof. □

4. Example and Remark

Example 4.1. Let $p = 4$, $g_0(t,x) = -10e^{20+\sin t}x$, $g_1(t,x) = -2e^{2+\sin t}\sin x$, and $g_2(t,x) = -3e^{3+\cos t}\cos x$ for all $t,x \in R$. Then, the following Liénard type p-Laplacian equation with two constant delays

$$(\varphi_p x'(t))' + 55x'(t) + g_0(t,x(t)) + g_1(t,x(t-1)) + g_2(t,x(t-2)) = \cos t \tag{4.1}$$

has a unique 2π-periodic solution.

Proof. From (4.1), it is straight forward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, (4.1) has at least one 2π-periodic solution. □

Remark 4.2. Obviously, the results in [2–5] obtained on Duffing type p-Laplacian equation with single delay and without multiple delays cannot be applicable to (4.1). This implies that the results of this paper are essentially new.

Acknowledgments

The authors would like to express their sincere appreciation to the editor and anonymous referee for their valuable comments which have led to an improvement in the presentation of the paper. This work was supported by the construct program of the key discipline in Hunan province (Mechanical Design and Theory), the Scientific Research Fund of Hunan Provincial Natural Science Foundation of PR China (Grant no. 11JJ6006), the Natural Scientific Research Fund of Hunan Provincial Education Department of PR China (Grants no. 11C0916, 11C0915, 11C1186), the Natural Scientific Research Fund of Zhejiang Provincial of P.R. China (Grant no. Y6110436), and the Natural Scientific Research Fund of Zhejiang Provincial Education Department of P.R. China (Grant no. Z201122436).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 851691, 25 pages
doi:10.1155/2012/851691

Research Article

Study of Solutions to Some Functional Differential Equations with Piecewise Constant Arguments

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Received 12 October 2011; Revised 6 January 2012; Accepted 9 January 2012

Academic Editor: Josef Diblík

We provide optimal conditions for the existence and uniqueness of solutions to a nonlocal boundary value problem for a class of linear homogeneous second-order functional differential equations with piecewise constant arguments. The nonlocal boundary conditions include terms of the state function and the derivative of the state function. A similar nonhomogeneous problem is also discussed.

1. Introduction

In the study of second-order functional differential equations, there is a wide range of works dealing with periodic boundary value problems and piecewise continuous functional dependence. We mention, for instance, [1], where an equation independent of the first derivative is analyzed, and many other works as [2–14], where existence and stability results are provided.

Most of works on this field deal with nonconstructive existence results. However, in [7, 11], explicit solutions are found for second-order functional differential equations with piecewise constant arguments, through the calculus of the Green's function. Other works in relation with first and higher-order differential equations with delay are [15–18].

In [19], a class of linear second-order differential equations with piecewise constant arguments is considered under the nonlocal conditions $x(0) = \varphi$, $x(T) = x(\mu) + \psi$, where

$\varphi, \psi \in \mathbb{R}$, that is, an initial position is assumed and the boundary conditions are independent of the derivative of the function.

In this paper, we study the following nonlocal boundary value problem for homogeneous linear second-order functional differential equations:

$$\begin{gathered}
x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0, \quad t \in J = [0, T], \\
x(T) = x(\mu) + \varphi, \\
x'(T) = x'(\mu) + \psi,
\end{gathered} \tag{1.1}$$

for $a, b, c, d, \varphi, \psi \in \mathbb{R}$, $T > 0$ and $\mu \in (0, T)$, where the functional dependence is given by the greatest integer part $[t]$, and the nonlocal boundary conditions involve both the state function and its derivative, which is the main difference from the study in [19]. A discussion for nonhomogeneous equations is also included. We consider the existence and uniqueness of solution to this problem, providing optimal conditions and calculating the exact expression of solutions.

To better illustrate the significant differences between the results included in this paper and those in [19], we remark that [19] is devoted to the study of the same class of linear second-order differential equations with piecewise constant arguments but considering a nonlocal boundary value problem where the value of the unknown function x is fixed at the initial instant $t = 0$ (i.e., an initial condition is imposed) and the boundary condition also involves the value of the sought solution at the right endpoint of the interval T and an intermediate point μ. None of the conditions imposed in [19] involve the value of the rate of change of the solution at any points. Thus, the nonlocal conditions in [19] can be reduced, by using the expression of the solution, to a boundary value problem affecting only the state of the solution, being independent of the rate of change of the state of the system. On the other hand, the nonlocal problem considered in this paper does not fix a certain initial position and, moreover, the boundary condition introduces the dependence, not only on the state, but also on the variation of the state of the system. Indeed, the value of the unknown function and its derivative is determined by the value of the corresponding magnitudes at an intermediate point of the interval of interest. Moreover, since the results in these two papers are also extensible to the special case where the intermediate point is identical to the initial instant ($\mu = 0$), we compare the consequences of both works to explain better the implications of the study of each problem. If we consider $\mu = 0$, the results in [19] are applicable to obtain the solution to a class of linear second-order differential equations with piecewise constant arguments subject to the boundary conditions $x(0) = \varphi$, $x(T) = x(0) + \psi = \varphi + \psi$ and, therefore, we solve a problem with fixed values of the function at the end-points of the interval. However, again for $\mu = 0$, the results presented in this paper allow to characterize the existence of solution (and provide its explicit expression) imposing boundary conditions of the type $x(T) = x(0) + \varphi$, $x'(T) = x'(0) + \psi$, which include, as a particular case, periodic boundary conditions (on x and x').

The main results are stated after a few preliminary results are recalled, and, finally, examples are included to show the applicability of these results.

2. Preliminaries

Consider the equation

$$x''(t) + ax'(t) + bx(\mathrm{t}) + cx'([t]) + dx([t]) = 0, \qquad t \in J = [0, T], \tag{2.1}$$

where a, b, c, d and $T > 0$. The following concepts and results come from [7].

Definition 2.1. [7] Let the spaces

$$\begin{aligned} \Lambda \ &:= \{y : J \longrightarrow \mathbb{R} : y \text{ is continuous in } J \setminus \{1, 2, \ldots, [T]\}, \\ &\qquad \text{and there exist } y(n^-) \in \mathbb{R},\ y(n^+) = y(n),\ \forall n \in \{1, 2, \ldots, [T]\}\}, \\ E &:= \{x : J \longrightarrow \mathbb{R} : x, x' \text{ are continuous and } x'' \in \Lambda\}. \end{aligned} \tag{2.2}$$

A solution to (2.1) is a function $x \in E$ which satisfies (2.1), taking $x''(n) = x''(n^+)$, for all $n \in \{0, 1, 2, \ldots, [T]\}$.

For the constants $a, b, c, d \in \mathbb{R}$, we define $h_1(s)$ as

$$\begin{gathered} 1 - \frac{d}{a}s + \frac{d}{a^2}\left(1 - e^{-as}\right), \quad \text{if } b = 0,\ a \neq 0, \\ 1 - \frac{d}{2}s^2, \quad \text{if } b = 0,\ a = 0, \\ \left(1 + \frac{d}{b}\right)\left(1 + \frac{a}{2}s\right)e^{-(a/2)s} - \frac{d}{b}, \quad \text{if } b \neq 0,\ a^2 = 4b, \\ \left(1 + \frac{d}{b}\right)\frac{\beta e^{\alpha s} - \alpha e^{\beta s}}{\beta - \alpha} - \frac{d}{b}, \quad \text{if b} \neq 0,\ a^2 > 4b, \\ \left(1 + \frac{d}{b}\right)e^{-(a/2)s}\left\{\cos\sqrt{b - \frac{a^2}{4}}s + \frac{a}{2\sqrt{b - a^2/4}}\sin\sqrt{b - \frac{a^2}{4}}s\right\} - \frac{d}{b}, \quad \text{if } b \neq 0,\ a^2 < 4b. \end{gathered} \tag{2.3}$$

Consider also $h_2(s)$ given by

$$\begin{gathered} \frac{1}{a}\left(1 - e^{-as} - cs + \frac{c}{a}\left(1 - e^{-as}\right)\right), \quad \text{if } b = 0,\ a \neq 0, \\ s - \frac{c}{2}s^2, \quad \text{if } b = 0,\ a = 0, \\ e^{-(a/2)s}\left\{\frac{c}{b}\left(1 + \frac{a}{2}s\right) + s\right\} - \frac{c}{b}, \quad \text{if } b \neq 0,\ a^2 = 4b, \end{gathered}$$

$$\frac{(\beta c/b-1)e^{\alpha s}+(1-\alpha c/b)e^{\beta s}}{\beta-\alpha}-\frac{c}{b},\quad \text{if } b\neq 0,\ a^2>4b,$$

$$e^{-(a/2)s}\left\{\frac{c}{b}\cos\sqrt{b-\frac{a^2}{4}}s+\frac{1+ac/(2b)}{\sqrt{b-a^2/4}}\sin\sqrt{b-\frac{a^2}{4}}s\right\}-\frac{c}{b},\quad \text{if } b\neq 0,\ a^2<4b. \tag{2.4}$$

In the definitions of functions h_1 and h_2, we denote, for $b\neq 0,\ a^2>4b$,

$$\alpha=-\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b},\qquad \beta=-\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}. \tag{2.5}$$

It is easy to check [7] that $h_1(0)=1,\ h_2(0)=0,\ h_1'(0)=0,\ h_2'(0)=1$.

Theorem 2.2 (see [7, Theorem 2.1]). *The initial value problem*

$$\begin{gathered} v''(t)+av'(t)+bv(t)+cv'([t])+dv([t])=0,\qquad t\in[0,+\infty),\\ v(0)=v_0,\\ v'(0)=v_0', \end{gathered} \tag{2.6}$$

for $v_0,v_0'\in\mathbb{R}$, has the solution

$$v(t)=\begin{pmatrix}h_1(t-n) & h_2(t-n)\end{pmatrix}\begin{pmatrix}C_1 & C_2\\ C_1' & C_2'\end{pmatrix}^n\begin{pmatrix}v_0\\ v_0'\end{pmatrix},\qquad t\in[n,n+1), \tag{2.7}$$

where $n\in\mathbb{Z}^+$,

$$\begin{aligned} C_1&=h_1(1), & C_2&=h_2(1),\\ C_1'&=h_1'(1), & C_2'&=h_2'(1). \end{aligned} \tag{2.8}$$

To simplify calculus, for $z\in[0,1]$, one denotes

$$\begin{gathered} H(z)=\begin{pmatrix}h_1(z) & h_2(z)\\ h_1'(z) & h_2'(z)\end{pmatrix},\\ C=H(1)=\begin{pmatrix}C_1 & C_2\\ C_1' & C_2'\end{pmatrix}. \end{gathered} \tag{2.9}$$

3. Main Results

First, we consider the nonlocal boundary value problem

$$\begin{gathered} x''(t)+ax'(t)+bx(t)+cx'([t])+dx([t])=0, \quad t\in J=[0,T], \\ x(T)=x(\mu)+\varphi, \\ x'(T)=x'(\mu)+\psi, \end{gathered} \tag{3.1}$$

for $a,b,c,d,\varphi,\psi\in\mathbb{R}$, $T>0$ and $\mu\in(0,T)$.

If $\mu=0=\varphi$, the condition $x(T)=x(\mu)+\varphi$ is reduced to a periodic boundary condition of Dirichlet type. On the other hand, if $\mu=0$, we obtain $x'(T)=x'(0)+\psi$, and, moreover, if $\psi=0$, we deduce the periodicity of the derivative of the function.

Theorem 3.1. *If $T>0$ and $\mu\in(0,T)$, then problem* (3.1) *is solvable for each $\binom{\varphi}{\psi}$ in the image of the mapping*

$$\mathcal{F}:\begin{pmatrix}u\\v\end{pmatrix}\longrightarrow\left[H(T-[T])C^{[T]-[\mu]}-H(\mu-[\mu])\right]C^{[\mu]}\begin{pmatrix}u\\v\end{pmatrix}. \tag{3.2}$$

Hence, there exists a unique solution to (3.1) *for every $\binom{\varphi}{\psi}\in\mathbb{R}^2$ if and only if the matrix*

$$\left[H(T-[T])C^{[T]-[\mu]}-H(\mu-[\mu])\right]C^{[\mu]} \tag{3.3}$$

is nonsingular. In this case, the solution is given by

$$v(t)=\begin{pmatrix}h_1(t-n) & h_2(t-n)\end{pmatrix}C^n\begin{pmatrix}v_0\\v_0'\end{pmatrix}, \quad t\in[n,n+1),\ n\in\mathbb{Z}^+, \tag{3.4}$$

where $\binom{v_0}{v_0'}$ is the inverse image of $\binom{\varphi}{\psi}$ by $\mathcal{F}$.

If the matrix (3.3) *is singular, then there exists an infinite number of solutions to problem* (3.1) *for every $\binom{\varphi}{\psi}$ in the image of $\mathcal{F}$, taking as initial position and initial slope in* (3.4) *any preimage of $\binom{\varphi}{\psi}$ by $\mathcal{F}$, and there exist no solutions for the remaining pairs $\binom{\varphi}{\psi}\in\mathbb{R}^2$.*

Proof. We consider the initial value problem

$$\begin{gathered} v''(t)+av'(t)+bv(t)+cv'([t])+dv([t])=0, \quad t\in[0,+\infty), \\ v(0)=v_0, \\ v'(0)=v_0', \end{gathered} \tag{3.5}$$

for $v_0, v_0' \in \mathbb{R}$, whose solution is, by Theorem 2.2 [7, Theorem 2.1],

$$v(t) = \left(h_1(t-n) \;\; h_2(t-n)\right) C^n \begin{pmatrix} v_0 \\ v_0' \end{pmatrix}, \quad t \in [n, n+1), \tag{3.6}$$

where $n \in \mathbb{Z}^+$. We analyze under which conditions the boundary conditions are fulfilled, taking into account that

$$v'(t) = \left(h_1'(t-n) \;\; h_2'(t-n)\right) C^n \begin{pmatrix} v_0 \\ v_0' \end{pmatrix}, \quad t \in [n, n+1), \tag{3.7}$$

where $n \in \mathbb{Z}^+$.

First, we consider the case $T \notin \mathbb{Z}$, and $\mu \notin \mathbb{Z}$. To obtain the solution to (3.1), we calculate v_0 and v_0' from the expressions of (3.6) and (3.7), in order to satisfy $v(T) = v(\mu) + \varphi$ and $v'(T) = v'(\mu) + \psi$. Hence the boundary conditions are written as

$$\begin{gathered} \left[\left(h_1(T-[T]) \;\; h_2(T-[T])\right) C^{[T]-[\mu]} - \left(h_1(\mu-[\mu]) \;\; h_2(\mu-[\mu])\right)\right] C^{[\mu]} \begin{pmatrix} v_0 \\ v_0' \end{pmatrix} = \varphi, \\ \left[\left(h_1'(T-[T]) \;\; h_2'(T-[T])\right) C^{[T]-[\mu]} - \left(h_1'(\mu-[\mu]) \;\; h_2'(\mu-[\mu])\right)\right] C^{[\mu]} \begin{pmatrix} v_0 \\ v_0' \end{pmatrix} = \psi, \end{gathered} \tag{3.8}$$

that is,

$$\left[H(T-[T]) C^{[T]-[\mu]} - H(\mu-[\mu])\right] C^{[\mu]} \begin{pmatrix} v_0 \\ v_0' \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{3.9}$$

The properties of functions h_1 and h_2 produce that the boundary conditions, in the cases where $T \in \mathbb{Z}$, or $\mu \in \mathbb{Z}$ (or both), can be derived from the expression obtained in the case studied $T \notin \mathbb{Z}$, and $\mu \notin \mathbb{Z}$.

Therefore, this system has a solution only for $\binom{\varphi}{\psi}$ in the image of the mapping

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \left[H(T-[T]) C^{[T]-[\mu]} - H(\mu-[\mu])\right] C^{[\mu]} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.10}$$

Therefore, the solution is unique if the matrix (3.3) is nonsingular, in which case, the initial condition and initial slope are given as

$$\begin{aligned} \begin{pmatrix} v_0 \\ v_0' \end{pmatrix} &= \left(\left[H(T-[T]) C^{[T]-[\mu]} - H(\mu-[\mu])\right] C^{[\mu]}\right)^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \\ &= \left(C^{[\mu]}\right)^{-1} \left[H(T-[T]) C^{[T]-[\mu]} - H(\mu-[\mu])\right]^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \end{aligned} \tag{3.11}$$

On the other hand, if the matrix (3.3) is singular, for $\binom{\varphi}{\psi}$ in the image of $\mathcal{F}$, the infinitely many solutions are calculated from (3.6) taking as initial conditions any preimage of $\binom{\varphi}{\psi}$ by $\mathcal{F}$, which proves the result. □

Remark 3.2. The existence and uniqueness condition (3.3) is reduced, if $[\mu] > 0$, to the nonsingularity of the matrices C and

$$\left[H(T-[T])C^{[T]-[\mu]} - H(\mu - [\mu])\right]. \tag{3.12}$$

On the other hand, if $[\mu] = 0$, it is just reduced to the nonsingularity of the matrix $[H(T - [T])C^{[T]} - H(\mu)]$.

Remark 3.3. In Theorem 3.1, if we consider $\mu = 0$, then the boundary conditions are $x(T) = x(0)+\varphi, x'(T) = x'(0)+\psi$, and the condition of existence and uniqueness of solution is reduced to the nonsingularity of

$$\left[H(T-[T])C^{[T]} - I\right], \tag{3.13}$$

which coincides with condition (15) in [7, Theorem 2.2]. Moreover, if $\varphi = \psi = 0$, the abovementioned nonsingularity condition provides that the unique solution to the homogeneous equation subject to periodic boundary value conditions is the trivial solution (see [7, Theorem 2.2]).

Remark 3.4. In Theorem 3.1, the order 2 matrix (3.3) can be written in a simplified manner in the following particular cases.

(i) If $T \notin \mathbb{Z}$ and $T - [T] = \mu - [\mu]$,

$$H(T-[T])\left[C^{[T]-[\mu]} - I\right]C^{[\mu]}. \tag{3.14}$$

(ii) If $T \notin \mathbb{Z}$ and $\mu \in \mathbb{Z}$,

$$\left[H(T-[T])C^{[T]-\mu} - I\right]C^{\mu}. \tag{3.15}$$

(iii) If $T \in \mathbb{Z}$ and $\mu \notin \mathbb{Z}$,

$$\left[C^{T-[\mu]} - H(\mu - [\mu])\right]C^{[\mu]}. \tag{3.16}$$

(iv) If $T,\ \mu \in \mathbb{Z}$,

$$\left[C^{T-\mu} - I\right]C^{\mu}. \tag{3.17}$$

Remark 3.5. Summarizing, the study of the solvability of (3.1) is reduced to the discussion of system (3.9). In the case of existence of an infinite number of solutions, we must analyze the rank of the matrix in (3.3) to determine whether any pair is admissible as initial position and slope or the space of initial conditions is one-dimensional. If the rank of (3.3) is zero, problem (3.1) is solvable only for $\binom{\varphi}{\psi} = \binom{0}{0}$, that is, problem

$$\begin{gathered} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0, \quad t \in J = [0,T], \\ x(T) = x(\mu), \\ x'(T) = x'(\mu), \end{gathered} \tag{3.18}$$

for $a, b, c, d \in \mathbb{R}$, $T > 0$ and $\mu \in (0,T)$, has an infinite number of solutions, given by (3.4), for any initial point $\binom{v_0}{v_0'} \in \mathbb{R}^2$.

On the other hand, and denoting by G the matrix in (3.3), if $\operatorname{rank}(G) = 1$ and $\binom{\varphi}{\psi}$ depends linearly on each column of G, then we have a one-dimensional space of solutions, whose starting conditions $V_0 = \binom{v_0}{v_0'}$ are determined from the row of the system $G\binom{v_0}{v_0'} = \binom{\varphi}{\psi}$ corresponding to a nonzero minor of G.

Next, with the purpose of extending Theorem 3.1 to the nonhomogeneous case, we consider the following nonlocal boundary value problem for a nonhomogeneous equation:

$$\begin{gathered} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t), \quad t \in J = [0,T], \\ x(T) = x(\mu) + \varphi, \\ x'(T) = x'(\mu) + \psi, \end{gathered} \tag{3.19}$$

for $a, b, c, d, \varphi, \psi \in \mathbb{R}$, $T > 0$, $\mu \in (0,T)$, and $\sigma \in \Lambda$. Using the expression of the solution for the corresponding initial value problem provided by [19, Theorem 5.2], we prove the following existence (and uniqueness) result.

Theorem 3.6. *Consider $a, b, c, d \in \mathbb{R}$, $T > 0$, $\mu \in (0,T)$, $\sigma \in \Lambda$ and φ, $\psi \in \mathbb{R}$. Then problem* (3.19) *has a unique solution if and only if the matrix* (3.3) *is nonsingular. Under this assumption, the unique solution to* (3.19) *is given by*

$$\begin{aligned} x(t) = {} & \big(h_1(t-n) \quad h_2(t-n)\big) C^n V_0 \\ & + \sum_{k=0}^{n-1} \int_k^{k+1} \big(h_1(t-n) \quad h_2(t-n)\big) C^{n-1-k} \begin{pmatrix} g(k+1-s) \\ g'(k+1-s) \end{pmatrix} \sigma(s)\,ds \\ & + \int_n^t g(t-s)\sigma(s)\,ds, \quad t \in [n, n+1), \ n \in \mathbb{Z}^+, \end{aligned} \tag{3.20}$$

with g defined as

$$g(z) = \begin{cases} \dfrac{1}{a}\left(1 - e^{-az}\right), & \text{if } b = 0,\ a \neq 0, \\ z, & \text{if } b = 0,\ a = 0, \\ z e^{-(a/2)z}, & \text{if } b \neq 0,\ a^2 = 4b, \\ \dfrac{e^{\beta z} - e^{\alpha z}}{\beta - \alpha}, & \text{if } b \neq 0,\ a^2 > 4b, \\ \dfrac{1}{\sqrt{b - a^2/4}} e^{-(a/2)z} \sin\sqrt{b - \dfrac{a^2}{4}}\, z, & \text{if } b \neq 0,\ a^2 < 4b, \end{cases} \tag{3.21}$$

and taking, as the initial condition V_0,

$$V_0 = \left[H(T - [T])C^{[T]} - H(\mu - [\mu])C^{[\mu]}\right]^{-1} \left(\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \mathcal{M}_{T,\mu,\sigma} \right), \tag{3.22}$$

where

$$\begin{aligned} \mathcal{M}_{T,\mu,\sigma} = {} & \sum_{k=0}^{[\mu]-1} \int_k^{k+1} \left[H(T - [T])C^{[T]-[\mu]} - H(\mu - [\mu])\right] C^{[\mu]-1-k} \begin{pmatrix} g(k+1-s) \\ g'(k+1-s) \end{pmatrix} \sigma(s)\,ds \\ & + \sum_{k=[\mu]}^{[T]-1} \int_k^{k+1} H(T - [T])C^{[T]-1-k} \begin{pmatrix} g(k+1-s) \\ g'(k+1-s) \end{pmatrix} \sigma(s)\,ds \\ & + \int_{[T]}^{T} \begin{pmatrix} g(T-s) \\ g'(T-s) \end{pmatrix} \sigma(s)\,ds - \int_{[\mu]}^{\mu} \begin{pmatrix} g(\mu-s) \\ g'(\mu-s) \end{pmatrix} \sigma(s)\,ds. \end{aligned} \tag{3.23}$$

On the other hand, if the matrix (3.3) *is singular, the number of solutions to* (3.19) *is determined by the discussion of the linear system*

$$\left[H(T - [T])C^{[T]} - H(\mu - [\mu])C^{[\mu]}\right] V_0 = \left(\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \mathcal{M}_{T,\mu,\sigma} \right), \tag{3.24}$$

and, for each solution V_0 *to this system (in case it exists), the expression* (3.20) *provides a solution to* (3.19).

Proof. The result follows from the expression of the solution (3.20) for the corresponding initial value problem, whose derivative is given by

$$x'(t) = \left(h_1'(t-n)\ \ h_2'(t-n)\right)C^n V_0 + \sum_{k=0}^{n-1}\int_k^{k+1}\left(h_1'(t-n)\ \ h_2'(t-n)\right)C^{n-1-k}\begin{pmatrix} g(k+1-s) \\ g'(k+1-s)\end{pmatrix}\sigma(s)ds + \int_n^t g'(t-s)\sigma(s)ds, \quad t\in[n,n+1),\ n\in\mathbb{Z}^+, \tag{3.25}$$

and the fact that the restrictions represented by the boundary conditions produce, respectively,

$$\begin{aligned}(1\ \ 0)\left[H(T-[T])C^{[T]} - H(\mu-[\mu])C^{[\mu]}\right]V_0 &= \varphi - (1\ \ 0)\mathcal{M}_{T,\mu,\sigma},\\ (0\ \ 1)\left[H(T-[T])C^{[T]} - H(\mu-[\mu])C^{[\mu]}\right]V_0 &= \psi - (0\ \ 1)\mathcal{M}_{T,\mu,\sigma}.\end{aligned} \tag{3.26}$$

□

4. Examples

We present some examples where different situations are analyzed in order to decide if the existence and uniqueness condition (nonsingularity of (3.3)) holds and, in case of nonuniqueness, the dimension of the space of solutions. The exact expression of the solution (or solutions) is given explicitly.

We recall that, for $s\in[0,1]$, $H(s)$ is given, depending on the values of the coefficients a,b,c,d as follows.

(i) If $b=0,\ a\neq 0$,

$$H(s) = \begin{pmatrix} 1-\dfrac{d}{a}s+\dfrac{d}{a^2}(1-e^{-as}) & \dfrac{1}{a}\left(1-e^{-as}-cs+\dfrac{c}{a}(1-e^{-as})\right) \\ \dfrac{d}{a}(-1+e^{-as}) & \dfrac{1}{a}((a+c)e^{-as}-c)\end{pmatrix}, \tag{4.1}$$

thus

$$C = H(1) = \begin{pmatrix} C_1 & C_2 \\ C_1' & C_2'\end{pmatrix} = \begin{pmatrix} 1-\dfrac{d}{a}+\dfrac{d}{a^2}(1-e^{-a}) & \dfrac{1}{a}\left(1-e^{-a}-c+\dfrac{c}{a}(1-e^{-a})\right) \\ \dfrac{d}{a}(-1+e^{-a}) & \dfrac{1}{a}((a+c)e^{-a}-c)\end{pmatrix}. \tag{4.2}$$

Note that the determinant of the matrix C is equal to

$$\det(C) = \frac{e^{-a}\left((e^{a} - a - 1)d + (a - a\ e^{a})c + a^{2}\right)}{a^{2}}. \tag{4.3}$$

(ii) If $b = 0$, $a = 0$,

$$H(s) = \begin{pmatrix} 1 - \frac{d}{2}s^{2} & s - \frac{c}{2}s^{2} \\ -ds & 1 - cs \end{pmatrix}, \tag{4.4}$$

thus

$$C = H(1) = \begin{pmatrix} C_1 & C_2 \\ C_1' & C_2' \end{pmatrix} = \begin{pmatrix} 1 - \frac{d}{2} & 1 - \frac{c}{2} \\ -d & 1 - c \end{pmatrix}. \tag{4.5}$$

The determinant of the matrix C is equal to

$$\det(C) = \frac{d}{2} - c + 1. \tag{4.6}$$

(iii) If $b \neq 0$, $a^{2} = 4b$,

$$H(s) = \begin{pmatrix} \left(1 + \frac{d}{b}\right)\left(1 + \frac{a}{2}s\right)e^{-(a/2)s} - \frac{d}{b} & e^{-(a/2)s}\left\{\frac{c}{b}\left(1 + \frac{a}{2}s\right) + s\right\} - \frac{c}{b} \\ \left(1 + \frac{d}{b}\right)\left(-\frac{a^{2}}{4}s\right)e^{-(a/2)s} & e^{-(a/2)s}\left\{\frac{-a^{2}c}{4b}s - \frac{a}{2}s + 1\right\} \end{pmatrix}, \tag{4.7}$$

thus

$$C = H(1) = \begin{pmatrix} \left(1 + \frac{d}{b}\right)\left(1 + \frac{a}{2}\right)e^{-a/2} - \frac{d}{b} & e^{-a/2}\left\{\frac{c}{b}\left(1 + \frac{a}{2}\right) + 1\right\} - \frac{c}{b} \\ \left(1 + \frac{d}{b}\right)\left(-\frac{a^{2}}{4}\right)e^{-a/2} & e^{-a/2}\left\{\frac{-a^{2}c}{4b} - \frac{a}{2} + 1\right\} \end{pmatrix}. \tag{4.8}$$

The determinant of the matrix C is equal to

$$\det(C) = e^{-a}\ \frac{\left((2a - 4)e^{a/2} + 4\right)d - a^{2}\ e^{a/2}c + 4b}{4b}. \tag{4.9}$$

(iv) If $b \neq 0$, $a^2 > 4b$, and denoting

$$\alpha = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \qquad \beta = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b},$$

$$H(s) = \begin{pmatrix} \left(1 + \frac{d}{b}\right)\frac{\beta e^{\alpha s} - \alpha e^{\beta s}}{\beta - \alpha} - \frac{d}{b} & \frac{(\beta c/b - 1)e^{\alpha s} + (1 - \alpha c/b)e^{\beta s}}{\beta - \alpha} - \frac{c}{b} \\ (b + d)\frac{e^{\alpha s} - e^{\beta s}}{\beta - \alpha} & \frac{(c - \alpha)e^{\alpha s} + (\beta - c)e^{\beta s}}{\beta - \alpha} \end{pmatrix}, \tag{4.10}$$

thus

$$C = H(1) = \begin{pmatrix} \left(1 + \frac{d}{b}\right)\frac{\beta e^{\alpha} - \alpha e^{\beta}}{\beta - \alpha} - \frac{d}{b} & \frac{(\beta c/b - 1)e^{\alpha} + (1 - \alpha c/b)e^{\beta}}{\beta - \alpha} - \frac{c}{b} \\ \left(1 + \frac{d}{b}\right)\frac{b(e^{\alpha} - e^{\beta})}{\beta - \alpha} & \frac{(c - \alpha)e^{\alpha} + (\beta - c)e^{\beta}}{\beta - \alpha} \end{pmatrix}. \tag{4.11}$$

In this case, the determinant of the matrix C is equal to

$$\begin{aligned} \det(C) = &\left(1 + \frac{d}{b}\right)\left(\frac{(\beta e^{\alpha} - \alpha e^{\beta})((c - \alpha)e^{\alpha} + (\beta - c)e^{\beta})}{(\beta - \alpha)^2} - \frac{(e^{\alpha} - e^{\beta})(\beta c - b)e^{\alpha} + (b - \alpha c)e^{\beta}}{(\beta - \alpha)^2}\right) \\ &- \frac{d}{b}\frac{(c - \alpha)e^{\alpha} + (\beta - c)e^{\beta}}{\beta - \alpha} + c\left(1 + \frac{d}{b}\right)\frac{e^{\alpha} - e^{\beta}}{\beta - \alpha} \\ = &\left(1 + \frac{d}{b}\right)\frac{\alpha^2 + \beta^2 - 2b}{(\beta - \alpha)^2}e^{\alpha + \beta} + \frac{1}{\beta - \alpha}\left(\frac{d}{b}\alpha e^{\alpha} - \frac{d}{b}\beta e^{\beta} + ce^{\alpha} - ce^{\beta}\right) \\ = &\left(1 + \frac{d}{b}\right)e^{-a} + \frac{1}{\beta - \alpha}\left(\left(\frac{d}{b}\alpha + c\right)e^{\alpha} - \left(\frac{d}{b}\beta + c\right)e^{\beta}\right). \end{aligned} \tag{4.12}$$

(v) If $b \neq 0$, $a^2 < 4b$, and denoting $R = \sqrt{b - a^2/4}$,

$$H(s) = \begin{pmatrix} \left(1 + \frac{d}{b}\right)e^{-(a/2)s}\left\{\cos Rs + \frac{a}{2R}\sin Rs\right\} - \frac{d}{b} & e^{-(a/2)s}\left\{\frac{c}{b}\cos Rs + \frac{1 + ac/(2b)}{R}\sin Rs\right\} - \frac{c}{b} \\ -\frac{1}{R}(b + d)\ e^{-(a/2)s}\sin Rs & e^{-(a/2)s}\left\{\cos Rs - \frac{a + 2c}{2R}\sin Rs\right\} \end{pmatrix}, \tag{4.13}$$

thus

$$C = H(1) = \begin{pmatrix} \left(1+\dfrac{d}{b}\right) e^{-a/2}\left\{\cos R + \dfrac{a}{2R}\sin R\right\} - \dfrac{d}{b} & e^{-a/2}\left\{\dfrac{c}{b}\cos R + \dfrac{1+ac/(2b)}{R}\sin R\right\} - \dfrac{c}{b} \\ -\dfrac{1}{R}(b+d)\ e^{-a/2}\sin R & e^{-a/2}\left\{\cos R - \dfrac{a+2c}{2R}\sin R\right\} \end{pmatrix}. \tag{4.14}$$

The determinant of C is given by

$$\begin{aligned} \det(C) = {} & \left(1+\frac{d}{b}\right)e^{-a}\left\{\cos^2 R - \frac{c}{R}\sin R\cos R - \frac{a(a+2c)}{4R^2}\sin^2 R\right\} \\ & - \frac{d}{b}e^{-a/2}\cos R + \frac{d}{b}\frac{a+2c}{2R}\sin R + \frac{1}{R}\left(1+\frac{d}{b}\right)ce^{-a}\sin R\cos R \\ & + \frac{b+d}{R^2}e^{-a}\left(1+\frac{ac}{2b}\right)\sin^2 R - \frac{c}{R}\left(1+\frac{d}{b}\right)e^{-a/2}\sin R \\ = {} & \left(1+\frac{d}{b}\right)e^{-a}\cos^2 R + \left(1+\frac{d}{b}\right)e^{-a}\left(-\frac{a(a+2c)}{4R^2} + \frac{b}{R^2}\left(1+\frac{ac}{2b}\right)\right)\sin^2 R \\ & - \frac{d}{b}e^{-a/2}\cos R + \frac{1}{R}\left(\frac{d(a+2c)}{2b} - c\left(1+\frac{d}{b}\right)e^{-a/2}\right)\sin R \\ = {} & \left(1+\frac{d}{b}\right)e^{-a} - \frac{d}{b}e^{-a/2}\cos R + \frac{1}{R}\left(\frac{d(a+2c)}{2b} - c\left(1+\frac{d}{b}\right)e^{-a/2}\right)\sin R. \end{aligned} \tag{4.15}$$

Example 4.1. Consider the problem

$$\begin{gathered} x''(t) + x'(t) + dx([t]) = 0, \qquad t \in J = \left[0, \frac{3}{2}\right], \\ x\left(\frac{3}{2}\right) = x\left(\frac{1}{2}\right) + \varphi, \\ x'\left(\frac{3}{2}\right) = x'\left(\frac{1}{2}\right) + \psi, \end{gathered} \tag{4.16}$$

where $d, \varphi, \psi \in \mathbb{R}$. In this case, $b = c = 0$, $a = 1 \neq 0$, $T = 3/2$ and $\mu = 1/2$. Therefore, we get $H(s) = \begin{pmatrix} d(1-e^{-s})-ds+1 & 1-e^{-s} \\ de^{-s}-d & e^{-s} \end{pmatrix}$ so that matrix $G = [H(T-[T])C^{[T]-[\mu]} - H(\mu-[\mu])]C^{[\mu]} = H(1/2)[C-I]$ is reduced to

$$G = \begin{pmatrix} -\dfrac{e^{-3/2}\left(\sqrt{e}(d^2+2ed) - 2d^2 + (2-2e)d\right)}{2} & \dfrac{e^{-3/2}\left(\sqrt{e}(e-1)d + (2-2e)d + 2e - 2\right)}{2} \\ e^{-3/2}\left(\sqrt{e}d^2 - d^2 + (1-e)d\right) & -e^{-3/2}\left(\sqrt{e}(e-1)d + (1-e)d + e - 1\right) \end{pmatrix}, \tag{4.17}$$

whose determinant is calculated as

$$\begin{aligned}\det(G) &= -\frac{e^{-2}\left(\sqrt{e}\left((3e-3)d^2+(2-2e)d\right)+\left(2e-2e^2\right)d^2\right)}{2}\\ &= -\frac{\left(e^{-3/2}(3e-3)+2e^{-1}-2\right)d^2+e^{-3/2}(2-2e)d}{2}.\end{aligned} \tag{4.18}$$

Hence, G is nonsingular if and only if $d \neq 0$ and $d \neq 2/(3-2e^{1/2})$. In this case, problem (4.16) has a unique solution for every $\binom{\varphi}{\psi} \in \mathbb{R}^2$, given by (see (3.4))

$$\begin{aligned}x(t) &= \left(d\left(1-e^{-t}\right)-dt+1\right)v_0+\left(1-e^{-t}\right)v_0', \quad t \in [0,1),\\ x(t) &= \left(1-e^{1-t}\right)\left(\left(e^{-1}d-d\right)v_0+e^{-1}v_0'\right)\\ &\quad+\left(d\left(1-e^{1-t}\right)-d(t-1)+1\right)\left(\left(\left(1-e^{-1}\right)d-d+1\right)v_0+\left(1-e^{-1}\right)v_0'\right)\\ &= -e^{-t-1}\left(\left(\left(\left(ed-d^2\right)t+2d^2-ed-e\right)e^t-ed^2+ed\right)v_0\right.\\ &\quad\left.+\left(\left((e-1)dt+(2-2e)d-e\right)e^t+\left(e^2-e\right)d+e\right)v_0'\right), \quad t \in \left[1,\frac{3}{2}\right],\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}\begin{pmatrix} v_0 \\ v_0' \end{pmatrix} &= G^{-1}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\\ &= \begin{pmatrix} -\dfrac{2(\sqrt{e}-1)d+2}{(2\sqrt{e}-3)d^2+2d} & -\dfrac{(\sqrt{e}-2)d+2}{(2\sqrt{e}-3)d^2+2d} \\ -\dfrac{2(\sqrt{e}-1)d-2e+2}{(\sqrt{e}(2e-2)+(3-3e))d+2e-2} & -\dfrac{(\sqrt{e}-2)d+2\sqrt{e}e-2e+2}{(\sqrt{e}(2e-2)+(3-3e))d+2e-2} \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix},\end{aligned} \tag{4.20}$$

that is,

$$\begin{aligned}v_0 &= \frac{-(2(\sqrt{e}-1)d+2)\varphi-((\sqrt{e}-2)d+2)\psi}{(2\sqrt{e}-3)d^2+2d}\\ v_0' &= \frac{-2((\sqrt{e}-1)d-2e+2)\varphi-((\sqrt{e}-2)d+2\sqrt{e}e-2e+2)}{(\sqrt{e}(2e-2)+(3-3e))d+2e-2}.\end{aligned} \tag{4.21}$$

For the particular case where $d = 2$, $\varphi = 1$, and $\psi = -1$, then

$$\begin{aligned}\begin{pmatrix} v_0 \\ v_0' \end{pmatrix} &= G^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -\dfrac{2\sqrt{e}-1}{4\sqrt{e}-4} & -\dfrac{1}{4} \\ -\dfrac{-e+2\sqrt{e}-1}{\sqrt{e}(2e-2)-2e+2} & -\dfrac{e+1}{2e-2} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{4}\dfrac{\sqrt{e}}{\sqrt{e}-1} \\ \dfrac{1}{2}\dfrac{\sqrt{e}}{\sqrt{e}-1} \end{pmatrix},\end{aligned} \tag{4.22}$$

and hence the unique solution to

$$\begin{gathered} x''(t) + x'(t) + 2x([t]) = 0, \quad t \in J = \left[0, \frac{3}{2}\right], \\ x\left(\frac{3}{2}\right) = x\left(\frac{1}{2}\right) + 1, \\ x'\left(\frac{3}{2}\right) = x'\left(\frac{1}{2}\right) - 1, \end{gathered} \tag{4.23}$$

is given by

$$\begin{aligned} x(t) &= \frac{\sqrt{e}(2t-1) - 2et + e}{-4e + 8\sqrt{e} - 4}, \quad t \in [0,1), \\ x(t) &= -\frac{\sqrt{e}\left(\left((2e^2+6e)t - 7e^2 - 21e\right)e^t + 4e^3 + 12e^2\right)}{(4e^2+24e+4)e^{t+1/2} + (-16e^2-16e)e^t} \\ &\quad - \frac{\left((-6e^2-2e)t + 21e^2 + 7e\right)e^t - 12e^3 - 4e^2}{(4e^2+24e+4)e^{t+1/2} + (-16e^2-16e)e^t}, \quad t \in \left[1, \frac{3}{2}\right]. \end{aligned} \tag{4.24}$$

See Figure 1.

If $d = 0$, then $H(s) = \left(\begin{smallmatrix} 1 & 1-e^{-s} \\ 0 & e^{-s} \end{smallmatrix}\right)$ and the matrix

$$G = \left[H(T - [T])C^{[T]-[\mu]} - H(\mu - [\mu])\right]C^{[\mu]} = H\left(\frac{1}{2}\right)[C - I] \tag{4.25}$$

is reduced to $G = \left(\begin{smallmatrix} 0 & e^{-3/2}\,(e-1) \\ 0 & -e^{-3/2}\,(e-1) \end{smallmatrix}\right)$, which has rank one. Since G is singular, there exists an infinite number of solutions to problem (4.16) for every $\left(\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right)$ in the image of the mapping given by the matrix G, that is, for every $\left(\begin{smallmatrix} \varphi \\ -\varphi \end{smallmatrix}\right)$, where $\varphi \in \mathbb{R}$. On each of these cases, the solutions

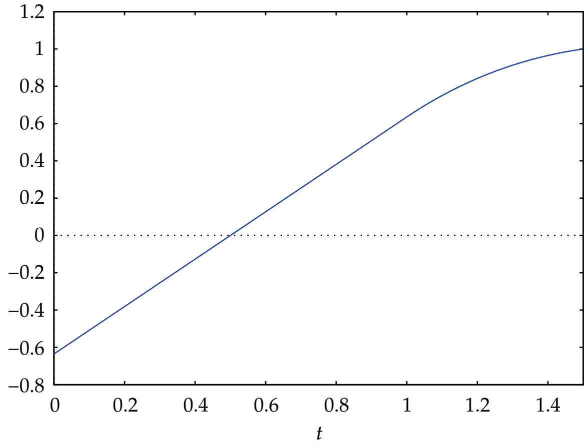

Figure 1: Solution to problem (4.23).

are given by (3.4) taking as initial position and initial slope any preimage of $\left(\begin{smallmatrix}\varphi \\ -\varphi\end{smallmatrix}\right)$, that is, the initial slope can be taken as

$$v_0' = \frac{e^{3/2}\varphi}{e-1}, \tag{4.26}$$

and the initial position can be chosen as any real number v_0.

Hence, the solutions to the problem

$$\begin{gathered} x''(t) + x'(t) = 0, \qquad t \in J = \left[0, \frac{3}{2}\right], \\ x\left(\frac{3}{2}\right) = x\left(\frac{1}{2}\right) + \varphi, \\ x'\left(\frac{3}{2}\right) = x'\left(\frac{1}{2}\right) - \varphi, \end{gathered} \tag{4.27}$$

where $\varphi \in \mathbb{R}$, are given by

$$x(t) = v_0 + \left(1 - e^{-t}\right)v_0', \qquad t \in [0,1), \tag{4.28}$$

and the same expression

$$\begin{aligned} x(t) &= \left(1 \quad \left(1 - e^{-(t-1)}\right)\right)\begin{pmatrix} 1 & 1-e^{-1} \\ 0 & e^{-1}\end{pmatrix}\begin{pmatrix} v_0 \\ v_0'\end{pmatrix} \\ &= v_0 + \left(1 - e^{-t}\right)v_0', \qquad t \in \left[1, \frac{3}{2}\right], \end{aligned} \tag{4.29}$$

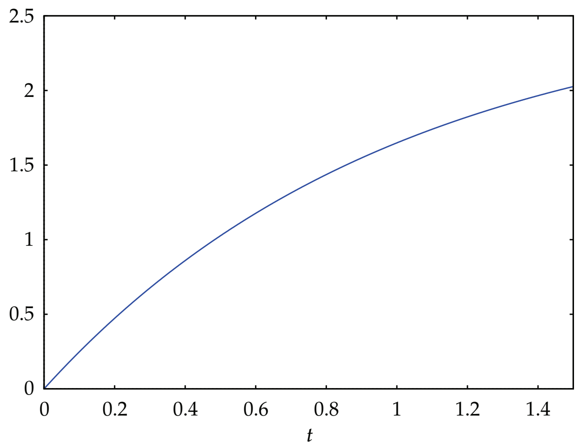

Figure 2: Solution which generates by vertical shifts the remaining solutions to problem (4.27) with $\varphi = 1$.

where $v_0 \in \mathbb{R}$ and $v_0' = e^{3/2}\varphi/(e-1)$. Hence, for each φ fixed, all the vertical shifts of the function $x(t) = (1-e^{-t})\,(e^{3/2}\varphi/(e-1))$, $t \in [0,3/2]$ are solutions to (4.27). In Figure 2, we show the graph of a solution which generates, by vertical shifting, all the solutions to problem (4.27) for $\varphi = 1$.

If $\psi \neq -\varphi$, there is no solution for problem (4.16) with $d = 0$.

Finally, if $d = 2/(3-2e^{1/2})$, then

$$
\begin{aligned}
H(s) &= \begin{pmatrix} \dfrac{5-2\sqrt{e}-2s-2e^{-s}}{3-2\sqrt{e}} & 1-e^{-s} \\ \dfrac{2(e^{-s}-1)}{3-2\sqrt{e}} & e^{-s} \end{pmatrix}, \\
G &= H\left(\frac{1}{2}\right)[C-I] \\
&= \begin{pmatrix} \dfrac{4-2\sqrt{e}-2e^{-1/2}}{3-2\sqrt{e}} & 1-e^{-1/2} \\ \dfrac{2(e^{-1/2}-1)}{3-2\sqrt{e}} & e^{-1/2} \end{pmatrix} \begin{pmatrix} \dfrac{-2e^{-1}}{3-2\sqrt{e}} & 1-e^{-1} \\ \dfrac{2(e^{-1}-1)}{3-2\sqrt{e}} & e^{-1}-1 \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{-2e^{-3/2}+2e^{-1}+6e^{-1/2}-10+4\sqrt{e}}{(3-2\sqrt{e})^2} & \dfrac{-e^{-3/2}+e^{-1}+e^{-1/2}-1}{3-2\sqrt{e}} \\ \dfrac{2e^{-3/2}-6e^{-1/2}+4}{(3-2\sqrt{e})^2} & \dfrac{e^{-3/2}-e^{-1/2}}{3-2\sqrt{e}} \end{pmatrix}
\end{aligned}
\tag{4.30}
$$

has rank one. Hence problem (4.16), for $d = 2/(3-2e^{1/2})$, has an infinite number of solutions for $\binom{\varphi}{\psi}$ in the image of the mapping given by the matrix G, that is, for the values of φ, and ψ satisfying the system

$$\begin{gathered}\left(-2e^{-3/2}+2e^{-1}+6e^{-1/2}-10+4\sqrt{e}\right)v_0+\left(-3e^{-3/2}+5e^{-1}+e^{-1/2}-5+2e^{1/2}\right)v_0'=\varphi\left(3-2\sqrt{e}\right)^2,\\ \left(2e^{-3/2}-6e^{-1/2}+4\right)v_0+\left(3e^{-3/2}-3e^{-1/2}-2e^{-1}+2\right)v_0'=\psi\left(3-2\sqrt{e}\right)^2,\end{gathered} \tag{4.31}$$

for some values of $v_0,\ v_0'\in\mathbb{R}$. For the existence of these values of $v_0,\ v_0'$, it is necessary and sufficient that the rank of the matrix of the system (equal to one) coincides with the rank of the matrix

$$\begin{pmatrix}-2e^{-3/2}+2e^{-1}+6e^{-1/2}-10+4\sqrt{e} & -3e^{-3/2}+5e^{-1}+e^{-1/2}-5+2e^{1/2} & \varphi\left(3-2\sqrt{e}\right)^2\\ 2e^{-3/2}-6e^{-1/2}+4 & 3e^{-3/2}-3e^{-1/2}-2e^{-1}+2 & \psi\left(3-2\sqrt{e}\right)^2\end{pmatrix}, \tag{4.32}$$

that is, for instance,

$$\det\begin{pmatrix}-3e^{-3/2}+5e^{-1}+e^{-1/2}-5+2e^{1/2} & \varphi\\ 3e^{-3/2}-3e^{-1/2}-2e^{-1}+2 & \psi\end{pmatrix}=0, \tag{4.33}$$

equivalent to

$$\psi=\frac{3e^{-3/2}-3e^{-1/2}-2e^{-1}+2}{-3e^{-3/2}+5e^{-1}+e^{-1/2}-5+2e^{1/2}}\varphi. \tag{4.34}$$

Under this assumption, there exists an infinite number of solutions to problem

$$\begin{gathered}x''(t)+x'(t)+\frac{2}{3-2e^{1/2}}x([t])=0,\qquad t\in J=\left[0,\frac{3}{2}\right],\\ x\left(\frac{3}{2}\right)=x\left(\frac{1}{2}\right)+\varphi,\\ x'\left(\frac{3}{2}\right)=x'\left(\frac{1}{2}\right)+\psi,\end{gathered} \tag{4.35}$$

which are given by

$$
\begin{aligned}
x(t) &= \frac{5-2\sqrt{e}-2t-2e^{-t}}{3-2\sqrt{e}}v_0 + \left(1-e^{-t}\right)v_0', \quad t\in[0,1),\\
x(t) &= \begin{pmatrix} \dfrac{5-2\sqrt{e}-2(t-1)-2e^{-(t-1)}}{3-2\sqrt{e}} & 1-e^{-(t-1)} \end{pmatrix} \begin{pmatrix} \dfrac{3-2\sqrt{e}-2e^{-1}}{3-2\sqrt{e}} & 1-e^{-1} \\ \dfrac{2\left(e^{-1}-1\right)}{3-2\sqrt{e}} & e^{-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_0' \end{pmatrix}\\
&= \frac{-2e^{-t}+4e^{1/2-t}+4\sqrt{e}t+4e^{-1}t-6t+4e-16\sqrt{e}-8e^{-1}+15}{\left(3-2\sqrt{e}\right)^2}v_0\\
&\quad + \frac{-e^{-t}-2e^{1-t}+2e^{1/2-t}+2e^{-1}t-2t-2\sqrt{e}-4e^{-1}+7}{3-2\sqrt{e}}v_0', \quad t\in\left[1,\frac{3}{2}\right],
\end{aligned}
\tag{4.36}
$$

where v_0 and v_0' satisfy (one of) the equations in (4.31).

On the other hand, if (4.34) fails, there is no solution to (4.35).

Example 4.2. Next, consider the problem

$$
\begin{gathered}
x''(t)+cx'([t])+dx([t])=0, \qquad t\in J=\left[0,\frac{3}{2}\right],\\
x\left(\frac{3}{2}\right)=x\left(\frac{1}{2}\right)+\varphi,\\
x'\left(\frac{3}{2}\right)=x'\left(\frac{1}{2}\right)+\psi,
\end{gathered}
\tag{4.37}
$$

where $c,d,\varphi,\psi\in\mathbb{R}$. In this case, $a=b=0$, $T=3/2$, and $\mu=1/2$. Hence, the matrix

$$
G=\left[H(T-[T])C^{[T]-[\mu]}-H(\mu-[\mu])\right]C^{[\mu]}=H\left(\frac{1}{2}\right)[C-I]
\tag{4.38}
$$

is reduced to

$$
G=\begin{pmatrix} \dfrac{d^2+(2c-16)d}{16} & \dfrac{(c-2)d+2c^2-16c+16}{16} \\ \dfrac{d^2+(2c-4)d}{4} & \dfrac{(c-2)d+2c^2-4c}{4} \end{pmatrix},
\tag{4.39}
$$

whose determinant is $\det(G) = d((d+8-4c)/8)$. If $d \neq 0$ and $d \neq 4c-8$, then G is invertible and problem (4.37) has a unique solution for every $\binom{\varphi}{\psi} \in \mathbb{R}^2$. Since

$$G^{-1} = \frac{1}{d-4c+8}\begin{pmatrix} \dfrac{(2c-4)d+4c^2-8c}{d} & -\dfrac{(c-2)d+2c^2-16c+16}{2d} \\ -(2d+4c-8) & \dfrac{d+2c-16}{2} \end{pmatrix}, \tag{4.40}$$

the unique solution to (4.37), for $\binom{\varphi}{\psi}$ fixed, is given by

$$\begin{aligned} v(t) &= \left(1-\frac{d}{2}t^2\right)v_0 + \left(t-\frac{c}{2}t^2\right)v_0', \quad t \in [0,1), \\ v(t) &= \left(1-\frac{d}{2}(t-1)^2 \quad (t-1)-\frac{c}{2}(t-1)^2\right)\begin{pmatrix} 1-\dfrac{d}{2} & 1-\dfrac{c}{2} \\ -d & 1-c \end{pmatrix}\begin{pmatrix} v_0 \\ v_0' \end{pmatrix} \\ &= \left(\left(1-\frac{d}{2}(t-1)^2\right)\left(1-\frac{d}{2}\right) + \left((t-1)-\frac{c}{2}(t-1)^2\right)(-d)\right)v_0 \\ &\quad + \left(\left(1-\frac{d}{2}(t-1)^2\right)\left(1-\frac{c}{2}\right) + \left((t-1)-\frac{c}{2}(t-1)^2\right)(1-c)\right)v_0', \quad t \in \left[1,\frac{3}{2}\right], \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} v_0 &= \frac{(2c-4)d+4c^2-8c}{d(d+8-4c)}\varphi - \frac{(c-2)d+2c^2-16c+16}{2d(d+8-4c)}\psi, \\ v_0' &= -\frac{2d+4c-8}{d-4c+8}v_0 + \frac{d+2c-16}{2(d-4c+8)}v_0'. \end{aligned} \tag{4.42}$$

For instance, for $c = 2$ and $d = 1$, then $H(s) = \begin{pmatrix} 1-s^2/2 & s-s^2 \\ -s & 1-2s \end{pmatrix}$ and $G = \begin{pmatrix} -11/16 & -1/2 \\ 1/4 & 0 \end{pmatrix}$, which is invertible with inverse $G^{-1} = \begin{pmatrix} 0 & 4 \\ -2 & -11/2 \end{pmatrix}$. Then the unique solution to (4.37), for $c = 2$ and $d = 1$ and $\binom{\varphi}{\psi}$ fixed, is given by

$$\begin{aligned} v(t) &= 2\left(t^2-t\right)\varphi + \frac{1}{2}\left(7t^2-11t+8\right)\psi, \quad t \in [0,1), \\ v(t) &= -\frac{1}{2}\left(4t^2-12t+8\right)\varphi + \frac{1}{2}\left(-5t^2+13t-4\right)\psi, \quad t \in \left[1,\frac{3}{2}\right]. \end{aligned} \tag{4.43}$$

See Figure 3 for the solution to (4.37) with $c = 2$, $d = 1$, $\varphi = 1$, and $\psi = -1$:

$$v(t) = \begin{cases} \dfrac{1}{2}\left(-3\,t^2+7t-8\right), & t \in [0,1), \\ \dfrac{1}{2}\left(t^2-t-4\right), & t \in \left[1,\dfrac{3}{2}\right]. \end{cases} \tag{4.44}$$

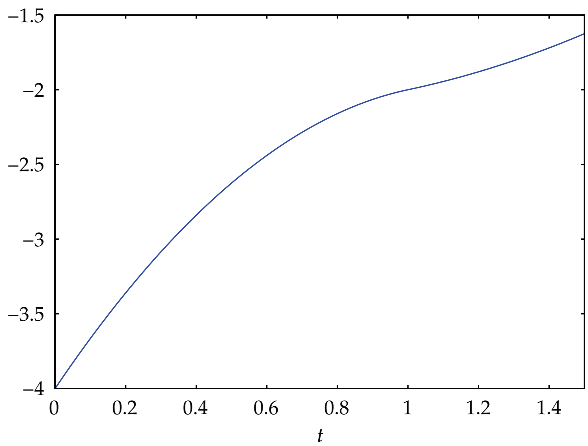

Figure 3: Solution to (4.37) for $c = 2$, $d = 1$, $\varphi = 1$, and $\psi = -1$.

On the other hand, if $d = 0$, then

$$H(s) = \begin{pmatrix} 1 & s - \dfrac{cs^2}{2} \\ 0 & 1 - cs \end{pmatrix},$$
$$G = H\left(\frac{1}{2}\right)[C - I] = \begin{pmatrix} 0 & \dfrac{c^2 - 8c + 8}{8} \\ 0 & \dfrac{c^2 - 2c}{2} \end{pmatrix} \tag{4.45}$$

is singular with rank 1, since $G_{12} = 0$ if and only if $c = 4 \pm 2\sqrt{2}$ and $G_{22} = 0$ if and only if $c = 0$ or $c = 2$.

Hence, there exists an infinite number of solutions to problem (4.37) with $d = 0$, for every $\binom{\varphi}{\psi}$ in the image of the mapping given by the matrix G, that is, for every $\binom{\varphi}{\psi} \in \mathbb{R}^2$ such that $4(c^2 - 2c)\varphi = (c^2 - 8c + 8)\psi$. Under this assumption, the solutions to (4.37) with $d = 0$ are given by

$$v(t) = v_0 + \left(t - \frac{c}{2}t^2\right)v_0', \quad t \in [0, 1),$$
$$v(t) = \left(1 \quad (t-1) - \frac{c}{2}(t-1)^2\right)\begin{pmatrix} 1 & 1 - \dfrac{c}{2} \\ 0 & 1 - c \end{pmatrix}\begin{pmatrix} v_0 \\ v_0' \end{pmatrix}$$
$$= v_0 + \left(1 - \frac{c}{2} + (1 - c)\left(t - 1 - \frac{c}{2}(t-1)^2\right)\right)v_0', \quad t \in \left[1, \frac{3}{2}\right], \tag{4.46}$$

where the initial position v_0 can be chosen as any real number and the initial slope must satisfy

$$v_0' = \frac{8\varphi}{c^2 - 8c + 8}, \quad \text{if } c \neq 4 \pm 2\sqrt{2},$$
$$v_0' = \frac{2\psi}{c^2 - 2c}, \quad \text{if } c \neq 0,\ c \neq 2. \tag{4.47}$$

For instance, if $d = 0$ and $c = 1$, then $H(s) = \begin{pmatrix} 1 & s-s^2/2 \\ 0 & 1-s \end{pmatrix}$, so that $G = \begin{pmatrix} 0 & 1/8 \\ 0 & -1/2 \end{pmatrix}$. Therefore, there exists an infinite number of solutions to problem (4.37) with $d = 0$ and $c = 1$, for every $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ such that $\psi = -4\varphi$. In this case, the solutions to (4.37) with $d = 0$ and $c = 1$ are given by

$$v(t) = v_0 + \begin{cases} 8\left(t - \dfrac{1}{2}t^2\right)\varphi, & t \in [0,1), \\ 4\varphi, & t \in \left[1, \dfrac{3}{2}\right], \end{cases} \tag{4.48}$$

where v_0 is any real number. Note that $v_0' = 8\varphi$. All the solutions are obtained as vertical shifts of the function

$$f(t) = \begin{cases} 8\left(t - \dfrac{1}{2}t^2\right)\varphi, & t \in [0,1), \\ 4\varphi, & t \in \left[1, \dfrac{3}{2}\right]. \end{cases} \tag{4.49}$$

See Figure 4 for the graph of function f for $\varphi = 1$, which generates by vertical shifting the one-dimensional space of solutions to problem (4.37) with $d = 0$, $c = 1$, $\varphi = 1$, and $\psi = -4$.

Finally, if $d = 4c - 8$, then

$$H(s) = \begin{pmatrix} 1 - (2c-4)s^2 & s - \dfrac{c}{2}s^2 \\ (-4c+8)s & 1 - cs \end{pmatrix},$$
$$G = \begin{pmatrix} \dfrac{3c^2 - 18c + 24}{2} & \dfrac{3c^2 - 16c + 16}{8} \\ 6c^2 - 24c + 24 & \dfrac{3c^2 - 10c + 8}{2} \end{pmatrix}, \tag{4.50}$$

which has rank 1. Note that $G_{11} \neq 0$ for $c \neq 2$ and $c \neq 4$; $G_{12} \neq 0$ for $c \neq 4/3$ and $c \neq 4$; $G_{21} \neq 0$ for $c \neq 2$; and $G_{22} \neq 0$ for $c \neq 2$ and $c \neq 4/3$.

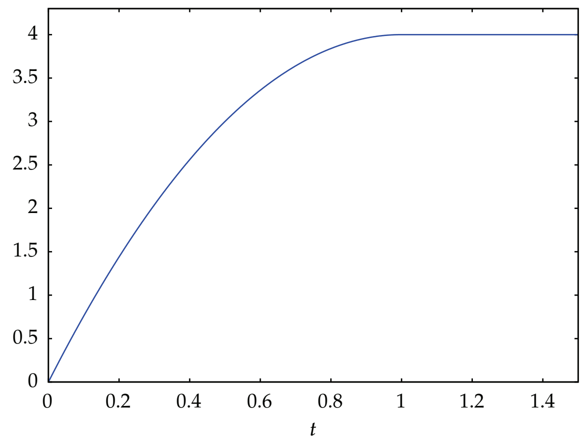

Figure 4: Solution which generates by vertical shifting the remaining solutions to problem (4.37) with $d = 0$, $c = 1$, $\varphi = 1$, and $\psi = -4$.

Therefore, problem (4.37) for $d = 4c - 8$ has solution (an infinite number of solutions) if and only if

$$\frac{3c^2 - 18c + 24}{2}\psi = \left(6c^2 - 24c + 24\right)\varphi, \tag{4.51}$$

$$\frac{3c^2 - 16c + 16}{8}\psi = \frac{3c^2 - 10c + 8}{2}\varphi, \tag{4.52}$$

in which case the expressions of the solutions are given by (4.41), replacing the value of d, where v_0 and v_0' satisfy

$$\frac{3c^2 - 18c + 24}{2}v_0 + \frac{3c^2 - 16c + 16}{8}v_0' = \varphi, \tag{4.53}$$

$$\left(6c^2 - 24c + 24\right)v_0 + \frac{3c^2 - 10c + 8}{2}v_0' = \psi. \tag{4.54}$$

Indeed,

(i) if $c = 2$, then $G = \left(\begin{smallmatrix} 0 & -1/2 \\ 0 & 0 \end{smallmatrix}\right)$, and the problem has an infinite number of solutions for $\psi = 0$ (see (4.52) since (4.51) is trivially satisfied), where $v_0 \in \mathbb{R}$ and $v_0' = -2\varphi$ (from (4.53) since (4.54) is trivially satisfied),

(ii) if $c = 4/3$, then $G = \left(\begin{smallmatrix} 8/3 & 0 \\ 8/3 & 0 \end{smallmatrix}\right)$, and the problem has an infinite number of solutions for $\varphi = \psi$ (see (4.51) since (4.52) is trivially satisfied), where $v_0 = (3/8)\varphi = (3/8)\psi$ (from (4.53) or (4.54)) and $v_0' \in \mathbb{R}$,

(iii) if $c \neq 2$ and $c \neq 4/3$, then the elements in the second row of G are nonzero and, hence, the problem has an infinite number of solutions for $\varphi = ((3c^2-18c+24)/(2(6c^2-24c+24)))\psi$ (from (4.51) or the same expression $\varphi = ((3c^2 - 16c + 16)/(4(3c^2 - 10c + 8)))\psi$ from (4.52)). The initial position and slope are taken satisfying (4.54).

If we take $c = 4$ (see the last case distinguished), then $G = \begin{pmatrix} 0 & 0 \\ 24 & 16 \end{pmatrix}$, and conditions (4.51) and (4.52) are reduced to $\varphi = 0$ and (4.54) is written as $24v_0 + 16v_0' = \psi$.

5. Conclusions

In this work, we have analyzed the existence and uniqueness of solutions to the nonlocal boundary value problem for linear second-order functional differential equations with piecewise constant arguments:

$$\begin{gathered} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t), \qquad t \in J = [0, T], \\ x(T) = x(\mu) + \varphi, \\ x'(T) = x'(\mu) + \psi, \end{gathered} \tag{5.1}$$

where $a, b, c, d, \varphi, \psi \in \mathbb{R}$, $T > 0$, $\mu \in (0, T)$ and $\sigma \in \Lambda$, by reducing the study to the discussion of a linear system of order 2. The nonlocal boundary conditions considered involve terms of the state function and the derivative of the state function, and the problem of interest also includes, as a particular case, periodic boundary value problems for second-order linear differential equations with functional dependence given by the greatest integer part.

Besides providing conditions on the existence and uniqueness of solutions, this approach also allows to obtain the expression of solutions explicitly, process which is illustrated with some examples.

The differences between the results in this paper and those in [19] have been included in the introduction.

Acknowledgments

The authors thank the editor and the anonymous referees for their helpful suggestions towards the improvement of the paper. This research is partially supported by Ministerio de Ciencia e Innovación and FEDER, Project MTM2010-15314.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 401923, 14 pages
doi:10.1155/2012/401923

Research Article

Positive Solutions for Neumann Boundary Value Problems of Second-Order Impulsive Differential Equations in Banach Spaces

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Received 8 September 2011; Revised 28 December 2011; Accepted 6 January 2012

Academic Editor: Yuriy Rogovchenko

The existence of positive solutions for Neumann boundary value problem of second-order impulsive differential equations $-u''(t) + Mu(t) = f(t, u(t), t \in J, t \neq t_k, -\Delta u'|_{t=t_k} = I_k(u(t_k)),$ $k = 1, 2, \ldots, m$, $u'(0) = u'(1) = \theta$, in an ordered Banach space E was discussed by employing the fixed point index theory of condensing mapping, where $M > 0$ is a constant, $J = [0, 1]$, $f \in C(J \times K, K)$, $I_k \in C(K, K)$, $k = 1, 2, \ldots, m$, and K is the cone of positive elements in E. Moreover, an application is given to illustrate the main result.

1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, chemical, biological, engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades; see [1]. Correspondingly, boundary value problems of second-order impulsive differential equations have been considered by many authors, and some basic results have been obtained; see [2–7]. But many of them obtained extremal solutions by monotone iterative technique coupled with the method of upper and lower solutions; see [2–4]. The research on positive solutions is seldom and most in real space $\mathbb{R}$; see [5–7].

In this paper, we consider the existence of positive solutions to the second-order impulsive differential equation Neumann boundary value problem in an ordered Banach space E:

$$\begin{gathered} -u''(t) + Mu(t) = f(t, u(t)), \quad t \in J, \ t \neq t_k, \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u'(0) = u'(1) = \theta, \end{gathered} \tag{1.1}$$

where $M > 0$ is a constant, $f \in C(J \times E, E)$, $J = [0,1]$; $0 < t_1 < t_2 < \cdots < t_m < 1$; $I_k \in C(E, E)$ is an impulsive function, $k = 1,2,\ldots,m$. $\Delta u'|_{t=t_k}$ denotes the jump of $u'(t)$ at $t = t_k$, that is, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right and left limits of $u'(t)$ at $t = t_k$, respectively.

In the special case where $E = \mathbb{R}^+ = [0,+\infty)$, $I_k = 0$, $k = 1,2,\ldots,m$, NBVP (1.1) has been proved to have positive solutions; see [8, 9]. Motivated by the aforementioned facts, our aim is to study the positive solutions for NBVP (1.1) in a Banach space by fixed point index theory of condensing mapping. Moreover, an application is given to illustrate the main result. As far as we know, no work has been done for the existence of positive solutions for NBVP (1.1) in Banach spaces.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $K = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N. Let $J = [0,1]$; $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$; $J_k = [t_{k-1}, t_k]$, $k = 1,2,\ldots,m+1$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$. Let $PC^1(J,E) = \{u \in C(J,E) \mid u'(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u'(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. Evidently, $PC^1(J,E)$ is a Banach space with the norm $\|u\|_{PC^1} = \max\{\|u(t)\|_C, \|u'\|_{PC}\}$, where $\|u\|_C = \sup_{t\in J}\|u(t)\|$, $\|u'\|_{PC} = \sup_{t\in J}\|u'(t)\|$; see [2]. An abstract function $u \in PC^1(J,E) \cap C^2(J',E)$ is called a solution of NBVP (1.1) if $u(t)$ satisfies all the equalities of (1.1).

Let $C(J,E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C = \sup_{t\in J}\|u(t)\|$. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [10, 11]. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{u(t) \mid u \in B\} \subset E$. If B is bounded in $C(J,E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

Now, we first give the following lemmas in order to prove our main results.

Lemma 2.1 (see [12]). *Let $B \subset C(J,E)$ be equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)). \tag{2.1}$$

Lemma 2.2 (see [13]). *Let $B = \{u_n\} \subset C(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\left(\left\{\int_J u_n(t)dt \mid n \in \mathbb{N}\right\}\right) \leq 2\int_J \alpha(B(t))dt. \tag{2.2}$$

Lemma 2.3 (see [14]). *Let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

To prove our main results, for any $h \in C(J,E)$, we consider the Neumann boundary value problem (NBVP) of linear impulsive differential equation in E:

$$
\begin{gathered}
-u''(t) + Mu(t) = h(t), \quad t \in J', \\
-\Delta u'|_{t=t_k} = y_k, \quad k = 1,2,\ldots,m, \\
u'(0) = u'(1) = \theta,
\end{gathered}
\tag{2.3}
$$

where $M > 0,\ y_k \in E,\ k = 1,2,\ldots,m$.

Lemma 2.4. *For any $h \in C(J,E)$, $M > 0$, and $y_k \in E$, $k = 1,2,\ldots,m$, the linear NBVP* (2.3) *has a unique solution $u \in PC^1(J,E) \cap C^2(J',E)$ given by*

$$
u(t) = \int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{m} G(t,t_k)y_k,
\tag{2.4}
$$

where

$$
G(t,s) = \begin{cases} \dfrac{\cosh\sqrt{M}(1-t)\cosh\sqrt{M}s}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le t < s \le 1. \end{cases}
\tag{2.5}
$$

Proof. Suppose that $u(t)$ is a solution of (2.3); then

$$
\begin{gathered}
u''(t) - Mu(t) = -h(t), \\
\left[e^{-2\sqrt{M}t}\left(e^{\sqrt{M}t}u(t)\right)'\right]' = -Me^{-\sqrt{M}t}u(t) + e^{-\sqrt{M}t}u''(t) = -e^{-\sqrt{M}t}h(t).
\end{gathered}
\tag{2.6}
$$

Let $y(t) = e^{-2\sqrt{M}t}(e^{\sqrt{M}t}u(t))'$; then

$$
y'(t) = -e^{-\sqrt{M}t}h(t), \qquad \Delta y|_{t=t_k} = -e^{-\sqrt{M}t_k}y_k.
\tag{2.7}
$$

Integrating (2.7) from 0 to t_1, we have

$$
y(t_1) - y(0) = -\int_0^{t_1} e^{-\sqrt{M}s}h(s)ds.
\tag{2.8}
$$

Again, integrating (2.7) from t_1 to t, where $t \in (t_1,t_2]$, then

$$
y(t) = y(t_1^+) - \int_{t_1}^{t} e^{-\sqrt{M}s}h(s)ds = y(0) - \int_0^t e^{-\sqrt{M}s}h(s)ds - e^{-\sqrt{M}t_1}y_1.
\tag{2.9}
$$

Repeating the aforementioned procession, for $t \in J$, we have

$$y(t) = y(0) - \int_0^t e^{-\sqrt{M}s} h(s)ds - \sum_{0<t_k<t} e^{-\sqrt{M}t_k} y_k. \tag{2.10}$$

Hence,

$$\left(e^{\sqrt{M}t} u(t)\right)' = e^{2\sqrt{M}t}\left(y(0) - \int_0^t e^{-\sqrt{M}s} h(s)ds - \sum_{0<t_k<t} e^{-\sqrt{M}t_k} y_k\right). \tag{2.11}$$

For $t \in J$, integrating (2.11) from 0 to t, we have

$$\begin{aligned}
u(t) &= e^{-\sqrt{M}t}\left(u(0) + \int_0^t e^{2\sqrt{M}s} y(0)ds - \int_0^t e^{2\sqrt{M}s} \int_0^s e^{-\sqrt{M}\tau} h(\tau)d\tau\, ds \right.\\
&\qquad \left. - \int_0^t e^{2\sqrt{M}s} \sum_{0<t_k<s} e^{-\sqrt{M}t_k} y_k ds\right)\\
&= e^{-\sqrt{M}t}\left\{u(0) + \frac{1}{2\sqrt{M}}\left[y(0)\left(e^{2\sqrt{M}t} - 1\right) - e^{2\sqrt{M}t}\int_0^t e^{-\sqrt{M}s} h(s)ds \right.\right.\\
&\qquad \left.\left. + \int_0^t e^{\sqrt{M}s} h(s)ds - \sum_{0<t_k<t}\left(e^{2\sqrt{M}t} - e^{2\sqrt{M}t_k}\right) e^{-\sqrt{M}t_k} y_k\right]\right\}.
\end{aligned} \tag{2.12}$$

Notice that $y(0) = \sqrt{M}u(0) + u'(0)$; thus, for $t \in J$, we have

$$\begin{aligned}
u(t) &= \frac{1}{2\sqrt{M}}\left[e^{-\sqrt{M}t} 2\sqrt{M}u(0) + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t}\right.\\
&\qquad - \left(\sqrt{M}u(0) + u'(0)\right)e^{-\sqrt{M}t} + e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s} h(s)ds\\
&\qquad \left. - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s} h(s)ds - \sum_{0<t_k<t}\left(e^{2\sqrt{M}t} - e^{2\sqrt{M}t_k}\right) e^{-\sqrt{M}(t+t_k)} y_k\right]\\
&= \frac{1}{2\sqrt{M}}\left[\left(\sqrt{M}u(0) - u'(0)\right)e^{-\sqrt{M}t} + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t}\right.
\end{aligned}$$

$$
\begin{aligned}
&+ e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}h(s)ds - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds \\
&- \sum_{0<t_k<t}\left(e^{\sqrt{M}(t-t_k)}\, y_k - e^{\sqrt{M}(t_k-t)}y_k\right)\Bigg],
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
u'(t) = \frac{1}{2}\Bigg[&-\left(\sqrt{M}u(0) - u'(0)\right)e^{-\sqrt{M}t} + \left(\sqrt{M}u(0) + u'(0)\right)e^{\sqrt{M}t} \\
&- e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}h(s)ds - e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}h(s)ds \\
&- \sum_{0<t_k<t}\left(e^{\sqrt{M}(t-t_k)}y_k + e^{-\sqrt{M}(t-t_k)}y_k\right)\Bigg].
\end{aligned}
\tag{2.14}
$$

In view of that $u'(0) = u'(1) = \theta$, we have

$$
\begin{aligned}
u(0) &= \int_0^1 \frac{e^{\sqrt{M}(1-s)} + e^{-\sqrt{M}(1-s)}}{\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}h(s)ds + \sum_{k=1}^{m}\frac{e^{\sqrt{M}(1-t_k)} + e^{-\sqrt{M}(1-t_k)}}{\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)} \\
&= \int_0^1 \frac{\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}h(s)ds + \sum_{k=1}^{m}\frac{\cosh\sqrt{M}(1-t_k)}{\sqrt{M}\sinh\sqrt{M}}y_k.
\end{aligned}
\tag{2.15}
$$

Substituting (2.15) into (2.13), for $t \in J$, we obtain

$$
\begin{aligned}
u(t) = &\int_0^t \frac{\left(e^{\sqrt{M}(1-t)} + e^{-\sqrt{M}(1-t)}\right)\left(e^{\sqrt{M}s} + e^{-\sqrt{M}s}\right)}{2\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}h(s)ds \\
&+ \sum_{0<t_k<t}\frac{\left(e^{\sqrt{M}(1-t)} + e^{-\sqrt{M}(1-t)}\right)\left(e^{\sqrt{M}t_k} + e^{-\sqrt{M}t_k}\right)}{2\sqrt{M}\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}y_k \\
&+ \int_t^1 \frac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-s)}{\sqrt{M}\sinh\sqrt{M}}h(s)ds \\
&+ \sum_{t\le t_k<1}\frac{\cosh\sqrt{M}t\cosh\sqrt{M}(1-t_k)}{\sqrt{M}\sinh\sqrt{M}}y_k \\
= &\int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{m}G(t,t_k)y_k.
\end{aligned}
\tag{2.16}
$$

Inversely, we can verify directly that the function $u \in PC^1(J,E)\cap C^2(J',E)$ defined by (2.4) is a solution of the linear NBVP (2.3). Therefore, the conclusion of Lemma 2.4 holds. $\square$

By (2.5), it is easy to verify that $G(t,s)$ has the following property:

$$\frac{1}{\sqrt{M}\sinh\sqrt{M}} \leq G(t,s) \leq \frac{\cosh^2\sqrt{M}}{\sqrt{M}\sinh\sqrt{M}}. \tag{2.17}$$

Evidently, $C(J,E)$ is also an ordered Banach space with the partial order $\leq$ reduced by the positive cone $C(J,K) = \{u \in C(J,E) \mid u(t) \geq \theta,\ t \in J\}$. $C(J,K)$ is also normal with the same normal constant N.

Define an operator $A : C(J,K) \to C(J,K)$ as follows:

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds + \sum_{k=1}^{m} G(t,t_k) I_k(u(t_k)). \tag{2.18}$$

Clearly, $A : C(J,K) \to C(J,K)$ is continuous, and the positive solution of NBVP (1.1) is the nontrivial fixed point of operator A. However, the integral operator A is noncompactness in general Banach space. In order to employ the topological degree theory and the fixed point theory of condensing mapping, there demands that the nonlinear f and impulsive function I_k satisfy some noncompactness measure condition. Thus, we suppose the following.

$(P0)$ For any $R > 0$, $f(J \times K_R)$ and $I_k(K_R)$ are bounded and

$$\alpha\big(f(t,D)\big) \leq L\alpha(D), \quad \alpha(I_k(D)) \leq M_k\alpha(D), \quad k = 1,2,\ldots,m, \tag{2.19}$$

where $K_R = K \cap B(\theta,R)$, $D \subset K$ is arbitrarily countable set, $L > 0$ and $M_k \geq 0$ are counsants and satisfy $(4L/M) + (2\cosh^2\sqrt{M}\ \sum_{k=1}^{m} M_k/\sqrt{M}\ \sinh\sqrt{M}) < 1$.

Lemma 2.5. *Suppose that condition* $(P0)$ *is satisfied; then* $A : C(J,K) \to C(J,K)$ *is condensing.*

Proof. Since $A(B)$ is bounded and equicontinuous for any bounded and nonrelative compact set $B \subset C(J,K)$, by Lemma 2.3, there exists a countable set $B_1 = \{u_n\} \subset B$, such that

$$\alpha(A(B)) \leq 2\alpha(A(B_1)). \tag{2.20}$$

By assumption $(P0)$ and Lemma 2.1,

$$\begin{aligned}
\alpha(A(B_1)(t)) &= \alpha\left(\left\{\int_0^1 G(t,s) f(s,u_n(s))ds + \sum_{k=1}^{m} G(t,t_k) I_k(u_n(t_k))\right\}\right) \\
&\leq \alpha\left(\left\{\int_0^1 G(t,s) f(s,u_n(s))ds\right\}\right) + \alpha\left(\left\{\sum_{k=1}^{m} G(t,t_k) I_k(u_n(t_k))\right\}\right) \\
&\leq 2\int_0^1 G(t,s)\alpha\big(f(s,B_1(s))\big)ds + \sum_{k=1}^{m} G(t,t_k)\alpha(I_k(B_1(t_k)))
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_0^1 G(t,s)L\alpha(B_1(s))ds + \sum_{k=1}^{m} G(t,t_k)M_k\alpha(B_1(t_k)) \\
&\leq 2L\int_0^1 G(t,s)ds\alpha(B_1) + \sum_{k=1}^{m} M_k G(t,t_k)\alpha(B_1) \\
&\leq \frac{2L}{M}\alpha(B_1) + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\alpha(B_1) \\
&= \left(\frac{2L}{M} + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B_1).
\end{aligned}
\tag{2.21}
$$

Since $A(B_1)$ is equicontinuous, by Lemma 2.1, we have

$$
\alpha(A(B_1)) = \max_{t\in J}\alpha(A(B_1)(t)) \leq \left(\frac{2L}{M} + \frac{\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B_1). \tag{2.22}
$$

Combining (2.20) and $(P0)$, we have

$$
\alpha(A(B)) \leq 2\alpha(A(B_1)) \leq \left(\frac{4L}{M} + \frac{2\cosh^2\sqrt{M}\sum_{k=1}^{m} M_k}{\sqrt{M}\sinh\sqrt{M}}\right)\alpha(B). \tag{2.23}
$$

Hence, $A : C(J,K) \to C(J,K)$ is condensing. □

Let P be a cone in $C(J,K)$ defined by

$$
P = \{u \in C(J,K) \mid u(t) \geq \sigma u(\tau),\ \forall t,\tau \in J\}, \tag{2.24}
$$

where $\sigma = 1/\cosh^2\sqrt{M}$.

Lemma 2.6. *For any $f(J,K) \subset K$, $A(C(J,K)) \subset P$.*

Proof. For any $u \in C(J,K)$, $t,\tau \in J$, by (2.18) and the second inequality of (2.17), we have

$$
\begin{aligned}
A(u(\tau)) &= \int_0^1 G(\tau,s)f(s,u(s))ds + \sum_{k=1}^{m} G(\tau,t_k)I_k(u(t_k)) \\
&\leq \frac{\cosh^2\sqrt{M}}{\sqrt{M}\sinh\sqrt{M}}\left(\int_0^1 f(s,u(s))ds + \sum_{k=1}^{m} I_k(u(t_k))\right).
\end{aligned}
\tag{2.25}
$$

By this, (2.18), and the first inequality of (2.17), we have

$$\begin{aligned} A(u(t)) &= \int_0^1 G(t,s)f(s,u(s))ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)) \\ &\geq \frac{1}{\sqrt{M}\sinh\sqrt{M}}\left(\int_0^1 f(s,u(s))ds + \sum_{k=1}^{m} I_k(u(t_k))\right) \\ &\geq \sigma A(u(\tau)). \end{aligned} \tag{2.26}$$

Hence, $A(C(J,K)) \subset P$. □

Thus, for any $f(J,K) \subset K$, $A : P \to P$ is condensing mapping; the positive solution of NBVP (1.1) is equivalent to the nontrivial fixed point of A in P. For $0 < r < R < \infty$, let $P_r = \{u \in P \mid \|u\|_C < r\}$, and $\partial P_r = \{u \in P \mid \|u\|_C = r\}$, which is the relative boundary bound of P_r in P. Denote that $P_{r,R} = P_R \setminus \overline{P_r}$; then the fixed point of A in $P_{r,R}$ is the positive solution of NBVP (1.1). We will use the fixed point theory of condensing mapping to find the fixed point of A in $P_{r,R}$.

Let X be a Banach space and let $P \subset X$ be a cone in X. Assume that Ω is a bounded open subset of X and let $\partial\Omega$ be its bound. Let $Q : P \cap \overline{\Omega} \to P$ be a condensing mapping. If $Qu \neq u$ for every $u \in P \cap \partial\Omega$, then the fixed point index $i(Q, P\cap\Omega, P)$ is defined. If $i(Q, P\cap\Omega, P) \neq 0$, then Q has a fixed point in $P \cap \Omega$. As the fixed point index theory of completely continuous mapping, see [10, 11], we have the following lemmas that are needed in our argument for condensing mapping.

Lemma 2.7. *Let $Q : P \to P$ be condensing mapping; if*

$$u \neq \lambda Au, \quad \forall u \in \partial P_r,\ 0 < \lambda \leq 1, \tag{2.27}$$

then $i(Q, P_r, P) = 1$.

Lemma 2.8. *Let $Q : P \to P$ be condensing mapping; if there exists $v_0 \in P$, $v_0 \neq \theta$, such that*

$$u - Au \neq \tau v_0, \quad \forall u \in \partial P_r,\ \tau \geq 0, \tag{2.28}$$

then $i(Q, P_r, P) = 0$.

3. Main Results

($P1$) (i) There exist $\delta > 0$, $a, a_k > 0$, such that for all $x \in P_\delta$ and $t \in J$, $f(t,x) \leq ax$, $I_k(x) \leq a_k x$, and $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$.
(ii) There exist $b, b_k > 0$, $h_0 \in C(J,K)$, and $y_k \in K$, such that for all $x \in P$ and $t \in J$, $f(t,x) \geq bx - h_0(t)$, $I_k(x) \geq b_k x - y_k$, and $b + \sigma^2 \sum_{k=1}^{m} b_k > M$.

($P2$) (i) There exist $\delta > 0$, $b, b_k > 0$, such that for all $x \in P_\delta$ and $t \in J$, $f(t,x) \geq bx$, $I_k(x) \geq b_k x$, and $b + \sigma^2 \sum_{k=1}^{m} b_k > M$.
(ii) There exist $a, a_k > 0$, $h_0 \in C(J,K)$, and $y_k \in K$, such that for all $x \in P$ and $t \in J$, $f(t,x) \leq ax + h_0(t)$, $I_k(x) \leq a_k x + y_k$, and $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$.

Theorem 3.1. *Let E be an ordered Banach space, whose positive* cone *K is normal, $f \in C(J \times K, K)$, and $I_k \in C(K, K)$, $k = 1, 2, \ldots, m$. Suppose that conditions $(P0)$ and $(P1)$ or $(P2)$ are satisfied; then the NBVP* (1.1) *has at least one positive solution.*

Proof. We show, respectively, that the operator A defined by (2.18) has a nontrivial fixed point in two cases that $(P1)$ is satisfied and $(P2)$ is satisfied.

Case 1. Assume that $(P1)$ is satisfied; let $0 < r < \delta$, where δ is the constant in condition $(P1)$, to prove that A satisfies

$$u \neq \lambda A u, \quad \forall u \in \partial P_r, \ 0 < \lambda \leq 1. \tag{3.1}$$

If (3.1) is not true, then there exist $u_0 \in \partial P_r$ and $0 < \lambda_0 \leq 1$, such that $u_0 = \lambda_0 A u_0$; by the definition of A, $u_0(t)$ satisfies

$$\begin{gathered} -u_0''(t) + M u_0(t) = \lambda_0 f(t, u_0(t)), \quad t \in J, \ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = \lambda_0 I_k(u_0(t_k)), \quad k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{gathered} \tag{3.2}$$

Integrating (3.2) from 0 to 1, using (i) of assumption $(P1)$, we have

$$(M - a) \int_0^1 u_0(t) dt \leq \sum_{k=1}^{m} a_k u_0(t_k). \tag{3.3}$$

Since $u_0 \in P$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$, $u_0(t_k) \leq (1/\sigma) u_0(s)$, and thus

$$\sigma(M - a) u_0(s) \leq \frac{\sum_{k=1}^{m} a_k}{\sigma} u_0(s), \tag{3.4}$$

that is; $(M - (a + (\sum_{k=1}^{m} a_k / \sigma^2)))\ u_0(s) \leq \theta$. So we obtain that $u_0(s) \leq \theta$ in J, which contracts with $u_0 \in \partial P_r$. Hence (3.1) is satisfied; by Lemma 2.7, we have

$$i(A, P_r, P) = 1. \tag{3.5}$$

Let $e \in C(J, K)$, $\|e\| = 1$, $v_0(t) \equiv e$, and obviously $v_0 \in P$. We show that if R is large enough, then

$$u - Au \neq \tau v_0, \quad \forall u \in \partial P_R, \ \tau \geq 0. \tag{3.6}$$

In fact, if there exist $u_0 \in \partial P_R$, $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 v_0$, then $Au_0 = u_0 - \tau_0 v_0$; by the definition of A, $u_0(t)$ satisfies

$$\begin{gathered} -u_0''(t) + Mu_0(t) - M\tau_0 v_0 = f(t, u_0(t)), \quad t \in J,\ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = I_k(u_0(t_k)), \quad k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{gathered} \tag{3.7}$$

By (ii) of assumption $(P1)$, we have

$$-u_0''(t) + Mu_0(t) = f(t, u_0(t)) + M\tau_0 v_0 \geq bu_0(t) - h_0(t), \quad t \in J'. \tag{3.8}$$

Integrating on J and using (ii) of assumption $(P1)$, we have

$$(b - M)\int_0^1 u_0(t)dt + \sum_{k=1}^m b_k u_0(t_k) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.9}$$

If $b > M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$; $u_0(t_k) \geq \sigma u_0(s)$; thus

$$\left(\sigma(b - M) + \sigma\sum_{k=1}^m b_k\right)u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.10}$$

By $\sigma(b - M) + \sigma\sum_{k=1}^m b_k > 0$ and the normality of cone K, we have

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^m \|y_k\|\right)}{\sigma(b - M) + \sigma\sum_{k=1}^m b_k} \triangleq R_1. \tag{3.11}$$

If $b \leq M$, then for any $t, s \in J$, by the definition of P, we have $u_0(t) \leq (1/\sigma)u_0(s)$, $u_0(t_k) \geq \sigma u_0(s)$; thus

$$\left(\frac{(b - M)}{\sigma} + \sigma\sum_{k=1}^m b_k\right)u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^m y_k. \tag{3.12}$$

By $b + \sigma^2\sum_{k=1}^m b_k > M$, and the normality of K, we have

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^m \|y_k\|\right)}{((b - M)/\sigma) + \sigma\sum_{k=1}^m b_k} \triangleq R_2. \tag{3.13}$$

Let $R > \max\{R_1, R_2, r\}$; then (3.6) is satisfied; by Lemma 2.8, we have

$$i(A, P_R, P) = 0. \tag{3.14}$$

Combining (3.5), (3.14), and the additivity of fixed point index, we have

$$i(A, P_{r,R}, P) = i(A, P_R, P) - i(A, P_r, P) = -1 \neq 0. \tag{3.15}$$

Therefore A has a fixed point in $P_{r,R}$, which is the positive solution of NBVP (1.1).

Case 2. Assume that $(P2)$ is satisfied; let $0 < r < \delta$, where δ is the constant in condition $(P2)$, to proof that A satisfies

$$u - Au \neq \tau v_0, \quad \forall u \in \partial P_r,\ \tau \geq 0, \tag{3.16}$$

where $v_0(t) = e \in P,\ e \neq \theta$. In fact, if there exists $u_0 \in \partial P_r$ and $\tau_0 \geq 0$, such that $u_0 - Au_0 = \tau_0 v_0$, then u_0 satisfies (3.7) and (i) of condition $(P2)$, and we have

$$-u_0''(t) + Mu_0(t) = f(t, u_0(t)) + M\tau_0 v_0 \geq bu_0(t), \quad t \in J'. \tag{3.17}$$

Integrating on J and using (i) of assumption $(P2)$, we have

$$(b - M)\int_0^1 u_0(t)dt + \sum_{k=1}^{m} b_k u_0(t_k) \leq \theta. \tag{3.18}$$

If $b > M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s),\ u_0(t_k) \geq \sigma u_0(s)$, for all $t, s \in J$, and thus

$$\left(\sigma(b - M) + \sigma\sum_{k=1}^{m} b_k\right)u_0(s) \leq \theta. \tag{3.19}$$

By $\sigma(b - M) + \sigma\sum_{k=1}^{m} b_k > 0$, we obtain that $u_0(s) \leq \theta$, which contracts with $u_0 \in \partial P_r$. Hence (3.16) is satisfied.

If $b \leq M$, for any $t, s \in J$, by the definition of P, we have $u_0(t) \leq (1/\sigma)u_0(s)$, $u_0(t_k) \geq \sigma u_0(s)$, for all $t, s \in J$, and thus

$$\left(\frac{1}{\sigma}(b - M) + \sigma\sum_{k=1}^{m} b_k\right)u_0(s) \leq \theta. \tag{3.20}$$

By $b + \sigma^2\sum_{k=1}^{m} b_k > M$, we obtain that $u_0(s) \leq \theta$, which contracts with $u_0 \in \partial P_r$. Hence (3.16) is satisfied.

Hence, by Lemma 2.8, we have

$$i(A, P_r, P) = 0. \tag{3.21}$$

Next, we show that if R is large enough, then

$$u \neq \lambda Au, \quad \forall u \in \partial P_R,\ 0 < \lambda \leq 1. \tag{3.22}$$

In fact, if there exists $u_0 \in \partial P_R$ and $0 < \lambda_0 \leq 1$ such that $u_0 = \lambda_0 A u_0$, then u_0 satisfies (3.2). Integrating (3.2) on J, and using (ii) of $(P2)$, we have

$$(M-a)\int_0^1 u_0(t)dt - \sum_{k=1}^{m} a_k u_0(t_k) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k. \tag{3.23}$$

For any $t, s \in J$, by the definition of P, we have $u_0(t) \geq \sigma u_0(s)$, $u_0(t_k) \leq (1/\sigma)u_0(s)$, for all $t, s \in J$, and thus

$$\left(\sigma(M-a) - \frac{1}{\sigma}\sum_{k=1}^{m} a_k\right) u_0(s) \leq \int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k. \tag{3.24}$$

By $a + (\sum_{k=1}^{m} a_k/\sigma^2) < M$, we have

$$u_0(s) \leq \frac{\int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k}. \tag{3.25}$$

By the normality of K, we have

$$\|u_0(s)\| \leq \frac{N\left\|\int_0^1 h_0(t)dt + \sum_{k=1}^{m} y_k\right\|}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k}. \tag{3.26}$$

Thus

$$\|u_0\|_C \leq \frac{N\left(\|h_0\|_C + \sum_{k=1}^{m}\|y_k\|\right)}{\sigma(M-a) - (1/\sigma)\sum_{k=1}^{m} a_k} \triangleq R_3. \tag{3.27}$$

Let $R > \max\{R_3, r\}$; then (3.22) is satisfied; by Lemma 2.7, we have

$$i(A, P_R, P) = 1. \tag{3.28}$$

Combining (3.21), (3.28), and the additivity of fixed index, we have

$$i(A, P_{r,R}, P) = i(A, P_R, P) - i(A, P_r, P) = 1. \tag{3.29}$$

Therefore A has a fixed point in $P_{r,R}$, which is the positive solution of NBVP (1.1). □

Remark 3.2. The conditions $(P1)$ and $(P2)$ are a natural extension of suplinear condition and sublinear condition in Banach space E. Hence if $I_k = \theta$, then Theorem 3.1 improves and extends the main results in [8, 9].

4. Example

We provide an example to illustrate our main result.

Example 4.1. Consider the following problem:

$$\begin{gathered}
-\frac{\partial^2}{\partial t^2}w(t,x)+w(t,x)=\int_0^1 e^{(t-s)}w^2(s,x)ds+\frac{1}{100}w(t,x), \quad t\in J',\ x\in I,\\
-\Delta\frac{\partial}{\partial t}w(t,x)|_{t=1/2}=\frac{1}{100}w\left(\frac{1}{2},x\right), \quad x\in I,\\
\frac{\partial}{\partial t}w(0,x)=\frac{\partial}{\partial t}w(1,x)=0, \quad x\in I,
\end{gathered} \tag{4.1}$$

where $J=[0,1]$, $J'=J\setminus\{1/2\}$, $I=[0,T]$, and $T>0$ is a constant.

Conclusion

Problem (4.1) has at least one positive solution.

Proof. Let $E=C(I)$, and $K=\{w\in C(I)\mid w(x)\geq 0,\ x\in I\}$; then E is a Banach space with norm $\|w\|=\max_{t\in I}|w(x)|$, and K is a positive cone of E. Let $u(t)=w(t,\cdot)$; then the problem (4.1) can be transformed into the form of NBVP (1.1), where $M=1$, $f(t,u)=\int_0^1 e^{(t-s)}u^2(s)ds+(1/100)u(t)$, and $I_1(u(1/2))=(1/100)u(1/2)$. Evidently $C(J,E)$ is a Banach space with norm $\|u\|_C=\max_{t\in J}\|u(t)\|$, and $C(J,K)$ is positive cone of $C(J,E)$. Let $P=\{u\in C(J,K)\mid u(t)\geq\sigma u(s),\ t,s\in J\}$, where $\sigma=1/\cosh^2 1$; then P is a cone in $C(J,K)$, and for any $u\in P$, $t\in J$, we have $\sigma\|u\|_C\leq u(t)\leq\|u\|_C$.

Next, we will verify that the conditions $(P0)$ and $(P1)$ in Theorem 3.1 are satisfied.

It is easy to verify that for any $R>0$, $f(J\times K_R)$ and $I_1(K_R)$ are bounded. Let $g(t,u)=\int_0^1 e^{(t-s)}u^2(s)ds$; then g is completely continuous. So for any countable bounded set $D\subset K$, we have

$$\alpha\big(f(t,D)\big)\leq\alpha\big(g(t,D)\big)+\frac{1}{100}\alpha(D)=\frac{1}{100}\alpha(D), \quad \alpha(I_1(D))=\frac{1}{100}\alpha(D), \tag{4.2}$$

and for $L=1/100$, and $M_1=1/100$, by simple calculations, $(4L/M)+(2\cosh^2\sqrt{M}\,M_1/\sqrt{M}\ \sinh\sqrt{M})<1$. So condition $(P0)$ is satisfied.

Let $\delta=1/100$; then for $u\in P_\delta$, we have

$$\begin{gathered}
f(t,u)\leq\frac{\delta\int_0^1 e^{(t-s)}ds}{\sigma}u(t)+\frac{1}{100}u(t)\leq\left(\frac{\delta(e-1)}{\sigma}+\frac{1}{100}\right)u(t),\\
I_1\left(u\left(\frac{1}{2}\right)\right)=\frac{1}{100}u\left(\frac{1}{2}\right)\leq\frac{1}{100\sigma}u(t).
\end{gathered} \tag{4.3}$$

Let $a=(\delta(e-1)/\sigma)+(1/100)$, $a_1=(1/100)\sigma$; by simple calculations, we have $a+(a_1/\sigma^2)<1$. So (i) of condition $(P1)$ is satisfied.

Let $R = 100$, for $u \in P$, $\|u\|_C \geq R$; we have $f(t,u) \geq (1/2\,\sigma^2 R + (1/100))u(t)$. For $u \in P$, $0 \leq \|u\|_C \leq R$, we have $f(t,u) \leq R^2 e^t + 1$. Hence, let $h_0(t) = 10000e^t + 1$, $y_1 = 0$; then for any $u \in P$, we have

$$\begin{gathered} f(t,u) \geq \left(\frac{1}{2}\sigma^2 R + \frac{1}{100}\right)u(t) - h_0(t) \\ I_1\left(u\left(\frac{1}{2}\right)\right) \geq \frac{1}{100}\sigma u(t). \end{gathered} \tag{4.4}$$

Let $b = (1/2)\sigma^2 R + (1/100)$, and $b_1 = (1/100)\sigma$; by simple calculations, we have $b + \sigma^2 b_1 > 1$, so (ii) of condition $(P1)$ is satisfied.

By Theorem 3.1, Problem (4.1) has at least one positive solution. □

Acknowledgment

This work was supported by NNSF of China (no. 10871160, 11061031).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 918082, 17 pages
doi:10.1155/2012/918082

Research Article

Existence and Multiplicity of Positive Solutions of a Nonlinear Discrete Fourth-Order Boundary Value Problem

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Received 16 September 2011; Revised 14 December 2011; Accepted 25 December 2011

Academic Editor: Yuriy Rogovchenko

we show the existence and multiplicity of positive solutions of the nonlinear discrete fourth-order boundary value problem $\Delta^4 u(t-2) = \lambda h(t) f(u(t))$, $t \in \mathbb{T}_2$, $u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0$, where $\lambda > 0$, $h : \mathbb{T}_2 \to (0, \infty)$ is continuous, and $f : \mathbb{R} \to [0, \infty)$ is continuous, $T > 4$, $\mathbb{T}_2 = \{2, 3, \ldots, T\}$. The main tool is the Dancer's global bifurcation theorem.

1. Introduction

It's well known that the fourth order boundary value problem

$$\begin{gathered} u''''(t) = f(t, u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{1.1}$$

can describe the stationary states of the deflection of an elastic beam with both ends hinged, (it also models a rotating shaft). The existence and multiplicity of positive solutions of the boundary value problem (1.1) have been considered extensively in the literature, see [1–10]. The existence and multiplicity of positive solutions of the parameterized boundary value problem

$$\begin{gathered} u''''(t) = \lambda h(t) f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{1.2}$$

have also been studied by several authors, see Bai and Wang [11], Cid et al. [12], and the references therein.

However, relatively little is known about the corresponding discrete fourth-order problems. Let

$$T > 4, \quad \mathbb{T}_0 = \{0, 1, \ldots, T+2\}, \quad \mathbb{T}_1 = \{1, 2, \ldots, T+1\}, \quad \mathbb{T}_2 = \{2, 3, \ldots, T\}. \tag{1.3}$$

Zhang et al. [13], and He and Yu [14] used the fixed point index theory in cones to study the following discrete analogue

$$\Delta^4 u(t-2) = \lambda h(t) f(u(t)), \quad t \in \mathbb{T}_2, \tag{1.4}$$

$$u(0) = u(T+2) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \tag{1.5}$$

where $\Delta^4 u(t-2)$ denote the fourth forward difference operator and $\Delta u(t) = u(t+1) - u(t)$. It has been pointed out in [13, 14] that (1.4), (1.5) are equivalent to the equation of the form:

$$u(t) = \lambda \sum_{s=1}^{T+1} G(t,s) \sum_{j=2}^{T} G_1(s,j) h(j) f(u(j)) =: A_0 u(t), \quad t \in \mathbb{T}_0, \tag{1.6}$$

where

$$\begin{gathered} G(t,s) = \frac{1}{T+2} \begin{cases} s(T+2-t), & 1 \le s \le t \le T+2, \\ t(T+2-s), & 0 \le t \le s \le T+1, \end{cases} \\ G_1(s,j) = \frac{1}{T} \begin{cases} (T+1-s)(j-1), & 2 \le j \le s \le T+1, \\ (T+1-j)(s-1), & 1 \le s \le j \le T. \end{cases} \end{gathered} \tag{1.7}$$

Notice that two distinct Green's functions used in (1.6) make the construction of cones and the verification of strong positivity of A_0 become more complex and difficult. Therefore, Ma and Xu [15] considered (1.4) with the boundary condition

$$u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \tag{1.8}$$

and introduced the definition of *generalized positive solutions*:

Definition 1.1. A function $y : \mathbb{T}_0 \to \mathbb{R}^+$ is called a *generalized positive solution* of (1.4), (1.8), if y satisfies (1.4), (1.8), and $y(t) \ge 0$ on $\mathbb{T}_1$ and $y(t) > 0$ on $\mathbb{T}_2$.

Remark 1.2. Notice that the fact $y : \mathbb{T}_0 \to \mathbb{R}^+$ is a generalized positive solution of (1.4), (1.8) does not means that $y(t) \ge 0$ on $\mathbb{T}_0$. In fact, y satisfies

(1) $y(t) \ge 0$ for $t \in \mathbb{T}_2$;

(2) $y(1) = y(T+1) = 0$;

(3) $y(0) = -y(2), \quad y(T+2) = -y(T)$.

Ma and Xu [15] also applied the fixed point theorem in cones to obtain some results on the existence of generalized positive solutions.

It is the purpose of this paper to show some new results on the existence and multiplicity of generalized positive solutions of (1.4), (1.8) by Dancer's global bifurcation theorem. To wit, we get the following.

Theorem 1.3. *Let* $h : \mathbb{T}_2 \to (0,\infty)$, $f \in C(\mathbb{R},[0,\infty))$, *and*

$$\lim_{s\to 0^+}\frac{f(s)}{s} = f_0 \in (0,\infty), \quad \lim_{s\to\infty}\frac{f(s)}{s} = f_\infty = +\infty. \tag{1.9}$$

Assume that there exists $B \in [0,+\infty]$ *such that* f *is nondecreasing on* $[0,B)$. *Then*

(i) (1.4), (1.8) *have at least one generalized positive solution if* $0 < \lambda < \lambda_1/f_0$;

(ii) (1.4), (1.8) *have at least two generalized positive solutions if*

$$\frac{\lambda_1}{f_0} < \lambda < \sup_{s\in(0,B)}\frac{s}{\gamma^* f(s)}, \tag{1.10}$$

where $\gamma^* = \max_{t\in\mathbb{T}_1}\sum_{s=2}^{T} K(t,s)h(s)$, $K(t,s)$ *is defined as* (2.3) *and* λ_1 *is the first eigenvalue of*

$$\begin{gathered}\Delta^4 u(t-2) = \lambda h(t)u(t), \quad t \in \mathbb{T}_2,\\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0.\end{gathered} \tag{1.11}$$

The "dual" of Theorem 1.3 is as follows.

Theorem 1.4. *Let* $h : \mathbb{T}_2 \to (0,\infty)$, $f \in C(\mathbb{R},[0,\infty))$, *and*

$$\lim_{s\to 0^+}\frac{f(s)}{s} = f_0 \in (0,\infty), \quad \lim_{s\to\infty}\frac{f(s)}{s} = f_\infty = 0. \tag{1.12}$$

Assume that there exists $B \in [0,+\infty]$ *such that* f *is nondecreasing on* $[0,B)$. *Then*

(i) (1.4), (1.8) *have at least a generalized positive solution provided*

$$\lambda > \inf_{s\in(0,c_1 B)}\frac{s}{c_1\gamma_* f(s)}, \tag{1.13}$$

where $\gamma_* = \min_{t\in\mathbb{T}_2}\sum_{s=2}^{T} K(t,s)h(s)$;

(ii) (1.4), (1.8) *have at least two generalized positive solutions provided*

$$\inf_{s\in(0,c_1B)} \frac{s}{c_1\gamma_* f(s)} < \lambda < \frac{\lambda_1}{f_0}. \tag{1.14}$$

The rest of the paper is organized as follows: in Section 2, we present the form of the Green's function of (1.4), (1.8) and its properties, and we enunciate the Dancer's global bifurcation theorem ([16, Corollary 15.2]). In Section 3, we use the Dancer's bifurcation theorem to prove Theorems 1.3 and 1.4 and in Section 4, we finish the paper presenting a couple of illustrative examples.

Remark 1.5. For other results on the existence and multiplicity of positive solutions and nodal solutions for fourth-order boundary value problems based on bifurcation techniques, see [17–21].

2. Preliminaries and Dancer's Global Bifurcation Theorem

Lemma 2.1. *Let $h : \mathbb{T}_2 \to \mathbb{R}$. Then the linear boundary value problem*

$$\begin{aligned} &\Delta^4 u(t-2) = h(t), \quad t \in \mathbb{T}_2, \\ &u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \end{aligned} \tag{2.1}$$

has a solution

$$u(t) = \sum_{s=2}^{T} K(t,s)h(s), \quad t \in \mathbb{T}_1, \tag{2.2}$$

where

$$K(t,s) = \begin{cases} \dfrac{(s-1)(T+1-t)\left(2T(t-1)-(t-1)^2-(s-2)s\right)}{6T}, & 2 \le s \le t \le T+1, \\ \dfrac{(t-1)(T+1-t)\left(2T(s-1)\ -(s-1)^2-(t-2)t\right)}{6T}, & 1 \le t \le s \le T. \end{cases} \tag{2.3}$$

Proof. By a simple summing computation and $u(1) = \Delta^2 u(0) = 0$, we can obtain

$$u(t) = \Delta u(0)(t-1) + \frac{t(t-1)(t-2)}{6}\Delta^3 u(0) + \sum_{s=2}^{t-1} \frac{(t-s)(t-s-1)(t-s+1)}{6} h(s). \tag{2.4}$$

This together with $u(T+1) = \Delta^2 u(T) = 0$, it follows that

$$\begin{aligned} u(t) &= \sum_{s=2}^{T} \frac{(T+1-s)(t-1)\left[2T(s-1)-(s-1)^2-t(t-2)\right]}{6T} h(s) \\ &\quad + \sum_{s=2}^{t-1} \frac{(t-s)(t-s-1)(t-s+1)}{6T} h(s) \\ &= \sum_{s=t}^{T} \frac{(T+1-s)(t-1)\left[2T(s-1)-(s-1)^2-t(t-2)\right]}{6T} h(s) \\ &\quad + \sum_{s=2}^{t-1} \frac{(T+1-t)(s-1)\left[2T(t-1)-(t-1)^2-s(s-2)\right]}{6T} h(s). \end{aligned} \tag{2.5}$$

Therefore, (2.2) holds. □

Remark 2.2. It has been pointed out in [15] that (2.1) is equivalent to the summation equation of the form

$$u(t) = \sum_{s=2}^{T} G_1(t,s) \sum_{j=2}^{T} G_1(s,j) h(j), \quad t \in \mathbb{T}_1. \tag{2.6}$$

It is easy to verify that (2.2) and (2.6) are equivalent.

By a similar method in [9], it follows that $K(t,s)$ satisfies

$$\begin{aligned} &K(t,s) \le \Phi(s) \quad \text{for } s \in \mathbb{T}_1, \quad t \in \mathbb{T}_1, \\ &K(t,s) \ge c(t)\Phi(s) \quad \text{for } s \in \mathbb{T}_1, \quad t \in \mathbb{T}_1, \end{aligned} \tag{2.7}$$

where

$$\Phi(s) = \begin{cases} \dfrac{\sqrt{3}}{27T}(s-1)\left(T^2-(s-2)s\right)^{3/2}, & 1 \le s \le \dfrac{T}{2}+1, \\ \dfrac{\sqrt{3}}{27T}(T+1-s)(2T(s-1)-(s-2)s)^{3/2}, & \dfrac{T}{2}+1 < s \le T+1, \end{cases} \tag{2.8}$$

$$c(t) = \begin{cases} \dfrac{3\sqrt{3}\left[T^2-t(t-2)\right](t-1)}{2(T^2+1)^{3/2}}, & 1 \le t \le \dfrac{T}{2}+1, \\ \dfrac{3\sqrt{3}(T+1-t)[2T(t-1)-t(t-2)]}{2(T^2+1)^{3/2}}, & \dfrac{T}{2}+1 < t \le T+1. \end{cases} \tag{2.9}$$

Moreover, we have that

$$K(t,s) \geq c_1\Phi(s), \quad \text{for } s \in \mathbb{T}_1, \quad t \in \mathbb{T}_2. \tag{2.10}$$

here $c_1 = 3\sqrt{3}T^2/2(T^2+1)^{3/2}$.

Let X be a real Banach space with a cone K such that $X = K - K$. Let us consider the equation:

$$x = \mu(Lx + Nx), \quad \mu \in \mathbb{R}, \quad x \in X \tag{2.11}$$

under the assumptions:

(A1) The operators $L, N : X \to X$ are compact. Furthermore, L is linear, $\|Nx\|_X/\|x\|_X \to 0$ as $\|x\|_X \to 0$, and $(L+N)(K) \subseteq K$.

(A2) The spectral radius $r(L)$ of L is positive. Denote $\mu_0 = r(L)^{-1}$.

(A3) L is strongly positive.

Dancer's global bifurcation theorem is the following.

Theorem 2.3 (see [16, Corollary 15.2]). *Let*

$$S_+ := \{(\mu, x) \in \mathbb{R} \times X \mid (\mu, x) \ \text{ is a solution of } (2.11) \text{ with } x > 0 \text{ and } \mu > 0\}. \tag{2.12}$$

If (A1) and (A2) are satisfied, then $(\mu_0, 0)$ is a bifurcation point of (2.11) *and $\overline{S}_+$ has an unbounded solution component $\mathcal{C}_+$ which passes through $(\mu_0, 0)$. Additionally, if (A3) is satisfied, then $(\mu, x) \in \mathcal{C}_+$ and $\mu \neq \mu_0$ always implies $x > 0$ and $\mu > 0$.*

3. Proof of the Main Results

Before proving Theorem 1.3, we state some preliminary results and notations. Let

$$\rho := 4\sin^2\frac{\pi}{2T}, \qquad e(t) := \sin\frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_1, \tag{3.1}$$

$$X := \left\{u \ \mid u : \mathbb{T}_0 \longrightarrow \mathbb{R}, \quad u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\right\}. \tag{3.2}$$

Then X is a Banach space under the normal:

$$\|u\|_X := \inf\left\{\frac{\gamma}{\rho} \mid -\gamma e(t) \leq -\Delta^2 u(t-1) \leq \gamma e(t), \quad t \in \mathbb{T}_1\right\}. \tag{3.3}$$

See [22] for the detail.

Let

$$K := \left\{u \in X \mid \Delta^2 u(t-1) \leq 0, \ u(t) \geq 0, \ t \in \mathbb{T}_1\right\}. \tag{3.4}$$

Then K is normal and has a nonempty interior and $X = K - K$.

Let $Y = \{u \mid u : \mathbb{T}_2 \to \mathbb{R}\}$. Then Y is a Banach space under the norm:

$$\|u\|_\infty = \max_{t\in\mathbb{T}_2}|u(t)|. \tag{3.5}$$

Define $\mathcal{L} : X \to Y$ by setting

$$\mathcal{L}u := \Delta^4 u(t-2), \quad u \in X. \tag{3.6}$$

It is easy to check that $\mathcal{L}^{-1} : Y \to X$ is compact.

Lemma 3.1. *Let $h \in Y$ with $h \geq 0$ and $h(t_0) > 0$ for some $t_0 \in \mathbb{T}_2$, and*

$$\mathcal{L}u - h = 0. \tag{3.7}$$

Then $u \in \operatorname{int} K$.

Proof. It is enough to show that there exist two constants $r_1,\ r_2 \in (0,\infty)$ such that

$$r_1 e(t) \leq -\Delta^2 u(t-1) \leq r_2 e(t), \quad t \in \mathbb{T}_1. \tag{3.8}$$

In fact, we have from (3.7) that

$$-\Delta^2 u(t-1) = \sum_{s=2}^{T} G_1(t,s)h(s), \quad t \in \mathbb{T}_1. \tag{3.9}$$

This together with the relation $((t-1)(T+1-t)/T)G_1(s,s) \leq G_1(t,s) \leq (t-1)(T+1-t)/T$ implies that

$$\left[\sum_{s=2}^{T} G_1(s,s)h(s)\right]\frac{(t-1)(T+1-t)}{T} \leq \sum_{s=2}^{T} G_1(t,s)h(s) \leq \|h\|_\infty \frac{(t-1)(T+1-t)}{T}. \tag{3.10}$$

Combining (3.9) with (3.10) and the fact that

$$c_1 \sin\frac{\pi(t-1)}{T} \leq \frac{(t-1)(T+1-t)}{T} \leq c_2 \sin\frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_1 \tag{3.11}$$

for some constants $c_1, c_2 \in (0,\infty)$, we conclude that (3.8) is true. □

Let $\zeta \in C(\mathbb{R},\mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u), \tag{3.12}$$

clearly

$$\lim_{|u|\to 0}\frac{\zeta(u)}{u}=0. \tag{3.13}$$

Let us consider

$$\mathcal{L}u=\lambda\left(h(\cdot)f_0u+h(\cdot)\zeta(u)\right) \tag{3.14}$$

as a bifurcation problem from the trivial solution $u\equiv 0$.

By (1.4), (1.8), it follows that if $u(t)\in X$ is one solution of (1.4), (1.8), then $u(t)$ satisfies $u(0)=-u(2),\quad u(T+2)=-u(T)$. So, $(u(0),0,u(2),\dots,u(T),0,u(T+2))$ is a solution of (1.4), (1.8), if and only if, $(0,u(2),\dots,u(T),0)$ solves the operator equation

$$u(t)=\lambda\sum_{s=2}^{T}G(t,s)h(s)f(u(s)),\quad t\in\mathbb{T}_1. \tag{3.15}$$

Now, let $J:Y\to X$ be the linear operator:

$$J(u(2),u(3),\dots,u(T))=(-u(2),0,u(2),u(3),\dots,u(T),0,-u(T)),\quad u\in Y. \tag{3.16}$$

Let $L,N:X\to X$ be the operators:

$$Lu:=(J\circ\mathcal{L})^{-1}\left(h(\cdot)f_0u\right), \tag{3.17}$$

$$Nu:=(J\circ\mathcal{L})^{-1}(h(\cdot)\zeta(u)), \tag{3.18}$$

respectively. Then Lemma 3.1 yields that $L:X\to X$ is strongly positive. Moreover, [16, Theorem 7.c] implies $r(L)>0$.

Now, it follows from Theorem 2.3 that there exists a continuum

$$\mathcal{C}_+\subseteq\overline{\left\{(\mu,x)\in\mathbb{R}\times X\mid(\mu,x)\text{ is a solution of (1.4), (1.8) with }x>0\text{ and }\mu>0\right\}}, \tag{3.19}$$

which joins $(r(L)^{-1},0)$ with infinity in $(0,\infty)\times K$ and

$$(\mu,x)\in\mathcal{C}_+,\quad\mu\neq r(L)^{-1}\implies x>0,\quad\mu>0. \tag{3.20}$$

It is easy to check that

$$r(L)^{-1}=\frac{\lambda_1}{f_0}. \tag{3.21}$$

Lemma 3.2. *Let $h_1, h_2 \in Y$ with $h_1 \geq h_2 > 0$. Then the eigenvalue problems*

$$\mathcal{L}u(t) = \lambda h_i(t)u(t), \quad t \in \mathbb{T}_2, \quad i = 1,\ 2 \tag{3.22}$$

have the principal eigenvalue $\overline{\lambda}_i$, $i = 1,\ 2$ such that $\overline{\lambda}_1 \leq \overline{\lambda}_2$. Moreover, the corresponding eigenfunctions φ_i are positive in $\mathbb{T}_2$.

Proof. Let $L_i : X \to X$ be the operator

$$L_i u := \lambda (J \circ \mathcal{L})^{-1}(h_i(\cdot)u), \quad i = 1, 2. \tag{3.23}$$

Then Lemma 3.1 yields that $L_i : X \to X$ is strongly positive. By Krein-Rutman theorem [16, Theorem 7.c] the spectral radius $r(L_i) > 0$ and there exist $\varphi_i \in X$ with $\varphi_i > 0$ on $\mathbb{T}_2$ such that

$$L_i \varphi_i(t) = r(L_i)\varphi_i(t), \quad i = 1,\ 2. \tag{3.24}$$

That is, the eigenvalue problems (3.22) have the principal eigenvalues $\overline{\lambda}_i = 1/r(L_i)$, and $\varphi_i(t)$ is the corresponding eigenfunctions of $\overline{\lambda}_i$, $i = 1,\ 2$.

Next, we prove $\overline{\lambda}_1 \leq \overline{\lambda}_2$. Since $\sum_{t=2}^{T} \Delta^4 \varphi_1(t-2)\ \varphi_2(t) = \sum_{t=2}^{T} \varphi_1(t)\Delta^4\varphi_2(t-2)$, it follows that

$$\begin{aligned}
\sum_{t=2}^{T} h_1(t)\varphi_1(t)\varphi_2(t) &\geq \sum_{t=2}^{T} \frac{\overline{\lambda}_2}{\overline{\lambda}_2} h_2(t)\varphi_2(t)\varphi_1(t) = \sum_{t=2}^{T} \frac{1}{\overline{\lambda}_2}\Delta^4\varphi_2(t-2)\varphi_1(t) \\
&= \sum_{t=2}^{T} \frac{1}{\overline{\lambda}_2}\varphi_2(t)\Delta^4\varphi_1(t-2) = \sum_{t=2}^{T} \frac{\varphi_2(t)}{\overline{\lambda}_2}\overline{\lambda}_1 h_1(t)\varphi_1(t) \\
&= \frac{\overline{\lambda}_1}{\overline{\lambda}_2}\sum_{t=2}^{T} h_1(t)\varphi_1(t)\varphi_2(t).
\end{aligned} \tag{3.25}$$

Therefore, $\overline{\lambda}_1 \leq \overline{\lambda}_2$. □

Suppose that $\mathbb{T}_a = \{a+1, a+2, \ldots, b-1\}$ is a strict subset of $\mathbb{T}_2$ and h_a denote the restriction of h on $\mathbb{T}_a$. Consider the linear eigenvalue problems:

$$\begin{gathered}
\Delta^4 u(t-2) = \lambda h_a(t) f_0 u(t), \quad t \in \mathbb{T}_a, \\
u(a) = u(b) = \Delta^2 u(a-1) = \Delta^2 u(b-1) = 0.
\end{gathered} \tag{3.26}$$

Then we get the following result.

Lemma 3.3. *Let $\widetilde{\lambda}_1$ is the principal eigenvalue of* (3.17), *then* (3.26) *has only one principal eigenvalue λ_a such that $0 < \widetilde{\lambda}_1 < \lambda_a$.*

Proof. It is not difficult to prove that (3.26) has only one principal eigenvalue $\lambda_a > 0$ by Lemma 3.1, and the corresponding eigenfunction $\psi_a > 0$ on $\mathbb{T}_a$. So we only to verify that $0 < \widetilde{\lambda}_1 < \lambda_a$.

Let ψ_1 be the corresponding eigenfunction of $\widetilde{\lambda}_1$, we have that

$$\begin{aligned}\sum_{t=a+1}^{b-1} \Delta^4\psi_a(t-2)\psi_1(t) &= \sum_{t=a+1}^{b-1} \Delta^4\psi_1(t-2)\psi_a(t) - \psi_1(b)\Delta^2\psi_a(b-2) - \psi_a(b-1)\Delta^2\psi_1(b-1) \\ &\quad - \Delta^2\psi_a(a)\psi_1(a) - \psi_a(a+1)\Delta^2\psi_1(a-1) \\ &> \sum_{t=a+1}^{b-1} \Delta^4\psi_1(t-2)\psi_a(t).\end{aligned} \tag{3.27}$$

So

$$\begin{aligned}\sum_{t=a+1}^{b-1} h(t)\psi_a(t)\psi_1(t) &= \sum_{t=a+1}^{b-1} \frac{1}{\lambda_a}\Delta^4\psi_a(t-2)\psi_1(t) \\ &> \sum_{t=a+1}^{b-1} \frac{1}{\lambda_a}\Delta^4\psi_1(t-2)\psi_a(t) \\ &= \sum_{t=a+1}^{b-1} \frac{\psi_a(t)}{\lambda_a}\widetilde{\lambda}_1 h(t)\psi_1(t) \\ &= \frac{\widetilde{\lambda}_1}{\lambda_a}\sum_{t=a+1}^{b-1} h(t)\ \psi_a(t)\psi_1(t).\end{aligned} \tag{3.28}$$

Thus $0 < \widetilde{\lambda}_1 < \lambda_a$. □

Proof of Theorem 1.3. We divide the proof into three steps.

Let $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ be such that

$$|\mu_n| + \|y_n\|_X \to \infty, \quad n \to \infty. \tag{3.29}$$

Then

$$\begin{gathered}\Delta^4 y_n(t-2) = \mu_n h(t) f(y_n(t)), \quad t \in \mathbb{T}_2, \\ y_n(1) = y_n(T+1) = \Delta^2 y_n(0) = \Delta^2 y_n(T) = 0.\end{gathered} \tag{3.30}$$

Step 1. We show that there exists a constant M such that $\mu_n \in (0, M]$ for all n.

Suppose on the contrary that

$$\lim_{n\to\infty} \mu_n = \infty. \tag{3.31}$$

Let $v_n = y_n/\|y_n\|_X$. Then it follows from (3.30) that

$$\Delta^4 v_n(t-2) = \mu_n h(t)\frac{f(y_n(t))}{y_n(t)}v_n(t), \quad t \in \mathbb{T}_2,$$
$$v_n(1) = v_n(T+1) = \Delta^2 v_n(0) = \Delta^2 v_n(T) = 0. \tag{3.32}$$

Since

$$\inf\left\{\frac{f(s)}{s} \mid s > 0\right\} := M_0 > 0, \tag{3.33}$$

there exists a constant $M_0 > 0$, such that

$$\frac{f(y_n(t))}{y_n(t)} > M_0 > 0. \tag{3.34}$$

Let λ^* be the principal eigenvalue of the linear eigenvalue problems:

$$\Delta^4 v(t-2) = \lambda h(t)\ M_0 v(t), \quad t \in \mathbb{T}_2,$$
$$v(1) = v(T+1) = \Delta^2 v(0) = \Delta^2 v(T) = 0. \tag{3.35}$$

Combining (3.31) and (3.34) with the relation (3.32), using Lemma 3.2, we get

$$0 < \mu_n \le \lambda^*. \tag{3.36}$$

This contradicts (3.31). So $\mu_n \in (0, M]$ for all n.

Step 2. We show that $\mathcal{C}_+$ joins $(\lambda_1/f_0, 0)$ with $(0, \infty)$.

Assume that there exist $\delta > 0$ and $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ such that

$$0 < \delta \le \mu_n \le M; \quad \|y_n\|_X \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{3.37}$$

First, we show that

$$\|y_n\|_X \longrightarrow \infty \implies \|y_n\|_\infty \longrightarrow \infty. \tag{3.38}$$

Suppose on the contrary that

$$\|y_n\|_\infty \le M_1 \tag{3.39}$$

for some $M_1 > 0$ (independent on n). Then it follows from (3.30) and $0 < \delta \le |\mu_n| \le M$ that

$$\left\|\Delta^4 y_n\right\|_\infty \le M\|h\|_\infty \sup\{f(s) \mid 0 < s \le M_1\}, \tag{3.40}$$

and subsequently, $\{\|y_n\|_X\}$ is bounded. This is a contradiction. So, (3.38) holds.

Next, we show that

$$\|y_n\|_\infty \longrightarrow \infty \;\Rightarrow\; \min\{y_n(t) \mid t \in \mathbb{T}_2\} \longrightarrow \infty. \tag{3.41}$$

In fact,

$$y_n(t) = \mu_n \sum_{s=2}^{T} K(t,s)h(s)f(y_n(s)), \quad t \in \mathbb{T}_1. \tag{3.42}$$

This together with (2.7) imply that (3.41) is valid.

Finally, we have from the facts that $\min\{y_n(t) \mid t \in \mathbb{T}_2\} \to \infty$ and $0 < \delta \le |\mu_n| \le M$ that

$$\mu_n \frac{f(y_n(t))}{y_n(t)} \longrightarrow \infty, \qquad n \longrightarrow \infty \text{ for any } t \in \mathbb{T}_a. \tag{3.43}$$

Consider the following linear eigenvalue problems:

$$\begin{gathered} \Delta^4 v(t-2) = \lambda h_a(t)\, v(t), \qquad t \in \mathbb{T}_a, \\ v(a) = v(b) = \Delta^2 v(a-1) = \Delta^2 v(b-1) = 0. \end{gathered} \tag{3.44}$$

By Lemma 3.3 and (3.32), (3.44) has a positive principal eigenvalue λ_a, and

$$\mu_n \frac{f(y_n(t))}{y_n(t)} \le \lambda_a, \tag{3.45}$$

which contradicts (3.43). Thus $\lim_{n\to\infty}\mu_n = 0$.

Step 3. Fixed λ such that

$$0 < \lambda < \sup_{s\in(0,B)} \frac{s}{\gamma^* f(s)}. \tag{3.46}$$

Then there exists $b \in (0, B]$ such that

$$0 < \lambda < \frac{b}{\gamma^* f(b)}. \tag{3.47}$$

We show that there is no $(\mu, u) \in \mathcal{C}_+$ such that

$$\|u\|_\infty = b, \quad 0 < \mu < \frac{b}{\gamma^* f(b)}. \tag{3.48}$$

In fact, if there exists $(\eta, y) \in \mathcal{C}_+$ satisfying (3.48), then

$$\begin{aligned} y(t) &= \eta \sum_{s=2}^{T} K(t,s)h(s)f(y(s)) \\ &\leq \eta\gamma^* f(b) \\ &= \eta\gamma^* \frac{f(b)}{b} \cdot b \end{aligned} \tag{3.49}$$

for $t \in \mathbb{T}_1$, and subsequently, $\eta \geq b/\gamma^* \ f(b)$. Therefore, no $(\mu, u) \in \mathcal{C}_+$ satisfies (3.48).

Now, combining the conclusions in Steps 2 and 3, using the fact that no $(\mu, u) \in \mathcal{C}_+$ satisfies (3.48), it concludes that for every $\lambda \in (\lambda_1/f_0, b/\gamma^* f(b))$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. For arbitrary $\lambda \in (0, \sup_{s\in(0,B)}(s/\gamma^* f(s)))$, we may find $b = b(\lambda)$ satisfying (3.47). So, for every $\lambda \in (\lambda_1/f_0, \sup_{s\in(0,B)}(s/\gamma^* f(s)))$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. □

Proof of Theorem 1.4. We divide the proof into three steps.

Step 1. We show that there exists a positive constant $\beta > 0$ such that

$$\inf \{\mu \mid (\mu, u) \in \mathcal{C}_+\} =: \beta > 0. \tag{3.50}$$

Suppose on the contrary that there exists $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ such that

$$\mu_n \to 0^+, \quad \text{as} \quad n \to \infty. \tag{3.51}$$

Then we have from (3.32), (3.51), $f_0 \in (0, \infty)$ and $f_\infty = 0$ that

$$\|v_n\|_X \to 0, \quad \text{as} \quad n \to \infty. \tag{3.52}$$

However, this contradicts with the fact that $\|v_n\|_X = 1$ for all $n \in \mathbb{N}$. Therefore, (3.50) holds.

Step 2. We show that for any closed interval $I \subset [\beta, \infty)$, there exists $M_I > 0$ such that

$$\sup\{\|u\| \mid (\mu, u) \in \mathcal{C}_+\} \leq M_I. \tag{3.53}$$

Suppose on the contrary that there exists $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ with

$$\{\mu_n\} \subset I, \quad \|y_n\|_X \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty. \tag{3.54}$$

Then by (3.38),

$$\|y_n\|_\infty \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{3.55}$$

and subsequently

$$\min_{t\in\mathbb{T}_2} y_n(t) \geq c_1\|y_n\|_\infty \longrightarrow \infty. \tag{3.56}$$

This together with (3.32) and $f_0 \in (0,\infty)$ and $f_\infty = 0$ that

$$\|(v_n|_{\mathbb{T}_2})\|_\infty \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.57}$$

However, this contradicts with the fact that

$$\min_{t\in\mathbb{T}_2} v_n(t) \geq c_1, \quad n \in \mathbb{N}. \tag{3.58}$$

Therefore, (3.53) holds.

Step 3. Fixed λ such that

$$\lambda > \inf_{s\in(0,c_1B)} \frac{s}{c_1\gamma_* f(s)}. \tag{3.59}$$

Then there exists $l \in (0, c_1B)$ such that

$$\lambda > \frac{l}{\gamma_* c_1 f(l)}. \tag{3.60}$$

We show that there is no $(\eta, y) \in C_+$ such that

$$\|y\|_\infty = \frac{l}{c_1} \quad \eta > \frac{l}{\gamma_* c_1 f(l)}. \tag{3.61}$$

Suppose on the contrary that there exists $(\eta, y) \in C_+$ satisfying (3.61). Then for $t \in \mathbb{T}_2$,

$$\begin{aligned} y(t) &= \eta \sum_{s=2}^{T} K(t,s)h(s)f(y(s)) \\ &\geq \eta \sum_{s=2}^{T} K(t,s)h(s)f(c_1\|y\|_\infty) \\ &= \eta \sum_{s=2}^{T} K(t,s)h(s)f(l) \\ &\geq \eta\gamma_* f(l) = \eta\gamma_* \frac{f(l)}{l}\cdot l, \end{aligned} \tag{3.62}$$

and subsequently, $\eta \leq l/c_1\gamma_* f(l)$. Therefore, there is no $(\eta, y) \in C_+$ such that (3.61) holds.

Now, combining the conclusions in Steps 2 and 3, using the fact that no $(\mu, u) \in \mathcal{C}_+$ satisfying (3.61), it concludes that for every $\lambda \in (l/c_1\gamma_* f(l), \lambda_1/f_0)$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. For arbitrary $\lambda \in (\inf_{s\in(0,c_1B)}(s/c_1\gamma_* f(s)),\ \infty)$, we may find $l = l(\lambda)$ satisfying (3.60). So, for every $\lambda \in (\inf_{s\in(0,c_1B)}(s/\gamma_* f(s)),\ \lambda_1/f_0)$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. □

4. Some Examples

In this section, we will apply our results to two examples.

For convenience, set $T = 12$, then $\mathbb{T}_1 = \{1, 2, \ldots, 13\}$, $\mathbb{T}_2 = \{2, 3, \ldots, 12\}$.

Example 4.1. Let us consider the boundary value problem

$$\begin{gathered} \Delta^4 u(t-2) = \lambda f(u(t)), \quad t \in \mathbb{T}_2, \\ u(1) = u(13) = \Delta^2 u(0) = \Delta^2 u(12) = 0, \end{gathered} \tag{4.1}$$

where

$$f(u) = \begin{cases} \arctan u, & u \in (0, 1000], \\ (u-1000)^2 + \arctan 1000, & u \in (1000, \infty). \end{cases} \tag{4.2}$$

Clearly, $f(u)$ is nondecreasing, $f_0 = 1$, $f_\infty = \infty$. Take $B = 1000$. By a simple computation, it follows that $\inf_{s\in(0,1000)}(f(s)/s) = \arctan 1000/1000 \approx 0.00157$, $\lambda_1 = 16\sin^4(\pi/24) \approx 0.0048$ and $\gamma^* = \max_{t\in\mathbb{T}_1}\sum_{s=2}^{12} K(t,s) = 1629/6$, then

$$0.0048 \approx 16\sin^4\frac{\pi}{24} = \frac{\lambda_1}{f_0} < \sup_{s\in(0,1000)}\frac{s}{f(s)\gamma^*} = \frac{6}{1629\inf_{s\in(0,1000)}(f(s)/s)} \approx 2.34631. \tag{4.3}$$

So, Theorem 1.3(i) implies that (4.1) has at least one generalized positive solution for

$$0 < \lambda < \frac{\lambda_1}{f_0} \approx 0.0048; \tag{4.4}$$

Theorem 1.3(ii) implies that (4.1) has at least two generalized positive solutions for

$$\frac{\lambda_1}{f_0} < \lambda < \frac{1}{\gamma^*\left(16\sin^4(\pi/24) - 1\right)} \approx 2.34631. \tag{4.5}$$

Example 4.2. Let us consider the boundary value problem:

$$\begin{gathered} \Delta^4 u(t-2) = \lambda \widetilde{f}(u(t)), \quad t \in \mathbb{T}_2, \\ u(1) = u(13) = \Delta^2 u(0) = \Delta^2 u(12) = 0, \end{gathered} \tag{4.6}$$

where

$$\tilde{f}(u) = \begin{cases} \dfrac{e^u - 1}{u}, & u \in (0, 30], \\ \sqrt{u - 30} + \dfrac{e^{30} - 1}{30}, & u \in (30, \infty). \end{cases} \tag{4.7}$$

Obviously, $\tilde{f}(u)$ is nondecreasing in $[0, \infty)$, so $\tilde{f}_0 = \lim_{u \to 0}(\tilde{f}(u)/u) = 1$, $\tilde{f}_\infty = \lim_{u \to 0}(\tilde{f}(u)/u) = 0$. By a simple computation, it follows that $\lambda_1 = 16 \sin^4(\pi/24)$ and $\gamma_* = \min_{t \in \mathbb{T}_2} \sum_{s=2}^{12} K(t,s) = 143/2$, $c_1 = 216\sqrt{3}/145\sqrt{145} \approx 0.214$. Take $B = 50$. Since $\sup_{s \in (0, 10.715\,)}(\tilde{f}(s)/s) = (e^{10.715} - 1)/10.715 \approx 4202.0726$, it follows that

$$0.0000155 \approx \inf_{s \in (0, c_1 B)} \frac{s}{c_1 \gamma_* \tilde{f}(s)} = \frac{1}{\sup_{s \in (0, 10.715)}\left(\tilde{f}(s)/s\right) c_1 \gamma_*} < \frac{\lambda_1}{\tilde{f}_0} = 16 \sin^4 \frac{\pi}{24} \approx 0.0048. \tag{4.8}$$

Therefore, (i) of Theorem 1.4 implies that (4.6) has at least one generalized positive solution for

$$\lambda > \inf_{s \in (0, c_1 B)} \frac{s}{c_1 \gamma_* \tilde{f}(s)} \approx 0.0000155; \tag{4.9}$$

(ii) of Theorem 1.4 implies that (4.6) has at least two generalized positive solutions for

$$0.0000155 < \lambda < 16 \sin^4 \frac{\pi}{24}. \tag{4.10}$$

Acknowledgments

This paper was written when the first author visited Tabuk University, Tabuk, during May 16-June 13, 2011 and he is very thankful to the administration of Tabuk University for providing him the hospitalities during the stay. The second author gratefully acknowledges the partial financial support from the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah. The authors thank the referees for their valuable comments.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 387193, 7 pages
doi:10.1155/2012/387193

Research Article

Fixed Point Theorems and Uniqueness of the Periodic Solution for the Hematopoiesis Models

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Received 20 September 2011; Revised 26 December 2011; Accepted 26 December 2011

Academic Editor: Elena Braverman

We present some results to the existence and uniqueness of the periodic solutions for the hematopoiesis models which are described by the functional differential equations with multiple delays. Our methods are based on the equivalent norm techniques and a new fixed point theorem in the continuous function space.

1. Introduction

In this paper, we aim to establish the existence and uniqueness result for the periodic solutions to the following functional differential equations with multiple delays:

$$x'(t) = -a(t)x(t) + f(t, x(t-\tau_1(t)), \dots, x(t-\tau_m(t))), \tag{1.1}$$

where $a, \tau_i \in C(R, R^+)$, $f \in C(R^{m+1}, R)$ are T-periodic functions on variable t for $T > 0$ and m is a positive integer.

Recently, many authors investigate the dynamics for the various hematopoiesis models, which includ the attractivity and uniqueness of the periodic solutions. For examples, Mackey and Glass in [1] have built the following delay differential equation:

$$x'(t) = -ax(t) + \frac{\beta\theta^n}{\theta^n + x^n(t-\tau)}, \tag{1.2}$$

where $a, n, \beta, \theta, \tau$ are positive constants, $x(t)$ denotes the density of mature cells in blood circulation, and τ is the time between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream; Liu et al. [2],Yang [3], Saker [4], Zaghrout et al. [5], and references therein, also investigate the attractivity and uniqueness of the periodic solutions for some hematopoiesis models.

This paper is organized as follows. In Section 2, we present two new fixed point theorems in continuous function spaces and establish the existence and uniqueness results for the periodic solutions of (1.1). An illustrative example to the hematopoiesis models is exhibited in the Section 3.

2. Fixed Point Theorems and Existence Results

2.1. Fixed Point Theorems

In this subsection, we will present two new fixed point theorems in continuous function spaces. More details about the fixed point theorems in continuous function spaces can be found in the literature [6–8] and references therein.

Let E be a Banach space equipped with the norm $\|\cdot\|_E$. $BC(R,E)$ which denotes the Banach space consisting of all bounded continuous mappings from R into E with norm $\|u\|_C = \max\{\|u(t)\|_E : t \in R\}$ for $u \in BC(R,E)$.

Theorem 2.1. *Let F be a nonempty closed subset of $BC(R,E)$ and $A : F \to F$ an operator. Suppose the following:*

(H_1) there exist $\beta \in [0,1)$ and $G : R \times R \to R$ such that for any $u, v \in F$,

$$\|Au(t) - Av(t)\|_E \le \beta\|u(t) - v(t)\|_E + \int_{t-T}^{t} G(t,s)\|u(s) - v(s)\|_E ds \quad \text{for } t \in R, \tag{2.1}$$

(H_2) there exist an $\alpha \in [0, 1-\beta)$ and a positive bounded function $y \in C(R,R)$ such that

$$\int_{t-T}^{t} G(t,s)y(s)ds \le \alpha y(t) \quad \forall t \in R. \tag{2.2}$$

Then A has a unique fixed point in F.

Proof. For any given $x_0 \in F$, let $x_n = Ax_{n-1}$, $(n = 1, 2, \ldots)$. By (H_1), we have

$$\|Ax_{n+1}(t) - Ax_n(t)\|_E \le \beta\|x_{n+1}(t) - x_n(t)\|_E + \int_{t-T}^{t} G(t,s)\|x_{n+1}(s) - x_n(s)\|_E ds. \tag{2.3}$$

Set $a_n(t) = \|x_{n+1}(t) - x_n(t)\|_E$, then we get

$$a_{n+1}(t) \le \beta a_n(t) + \int_{t-T}^{t} G(t,s)a_n(s)ds. \tag{2.4}$$

In order to prove that the sequence $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$, we introduce an equivalent norm and show that $\{x_n\}$ is a Cauchy sequence with respect to the new one. Basing on the condition (H_2), we see that there are two positive constants M and m such that $m \leq y(t) \leq M$ for all $t \in R$. Define the new norm $\|\cdot\|_1$ by

$$\|u\|_1 = \sup\left\{\frac{1}{y(t)}\|u(t)\|_E : t \in R\right\}, \quad u \in BC(R,E). \tag{2.5}$$

Then,

$$\frac{1}{M}\|u\|_C \leq \|u\|_1 \leq \frac{1}{m}\|u\|_C. \tag{2.6}$$

Thus, the two norms $\|\cdot\|_1$ and $\|\cdot\|_C$ are equivalent.

Set $a_n = \|x_{n+1} - x_n\|_1$, then we have $a_n(t) \leq y(t)a_n$ for $t \in R$. By (2.13), we have

$$\begin{aligned}\frac{1}{y(t)}a_{n+1}(t) &\leq \beta a_n + \frac{1}{y(t)}\int_{t-T}^{t} G(t,s)a_n(s)ds \\ &\leq \beta a_n + \frac{a_n}{y(t)}\int_{t-T}^{t} G(t,s)y(s)ds \leq (\beta+\alpha)a_n.\end{aligned} \tag{2.7}$$

Thus,

$$a_{n+1} \leq (\beta+\alpha)a_n \leq (\beta+\alpha)^2 a_{n-1} \leq \cdots \leq (\beta+\alpha)^{n+1}a_0. \tag{2.8}$$

This means $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_1$. Therefore, also, $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$. Thus, we see that $\{x_n\}$ has a limit point in F, say u. It is known that u is the fixed point of A in F.

Suppose both u and v $(u \neq v)$ are the fixed points of A, then $Au = u$, $Av = v$. Following the similar arguments, we prove that

$$\|u-v\|_1 = \|Au - Av\|_1 \leq (\beta+\alpha)\|u-v\|_1. \tag{2.9}$$

It is impossible. Thus the fixed point of A is unique. This completes the proof of Theorem 2.1. □

Let $PC(R,E)$ be a Banach space consisting of all T-periodic functions in $BC(R,E)$ with the norm $\|u\|_P = \max\{\|u(t)\|_E : t \in [0,T]\}$ for $u \in PC(R,E)$. Then, following the similar arguments in Theorem 2.1, we deduce Theorem 2.2 which is a useful result for achieving the existence of periodic solutions of functional differential equations.

Theorem 2.2. *Let $A : PC(R,E) \to PC(R,E)$ be an operator. Suppose the following:*

$(\widetilde{H}_1)$ *there exist $\beta \in [0,1)$ and $G : R \times R \to R$ such that for any $u, v \in PC(R,E)$,*

$$\|Au(t) - Av(t)\|_E \le \beta \|u(t) - v(t)\|_E + \int_{t-T}^{t} G(t,s) \sum_{i=1}^{n} \left\| u(\eta_i(s)) - v(\eta_i(s)) \right\|_E ds, \tag{2.10}$$

where $\eta_i \in C([0,T], R^+)$ with $\eta_i(s) \le s$ and n is a positive integer;

$(\widetilde{H}_2)$ *there exist two constants α, K and a positive function $y \in C(R,R)$ such that $nK\alpha \in [0, 1-\beta)$, $y(\eta_i(s)) \le Ky(s)$, and*

$$\int_{t-T}^{t} G(t,s) y(s) ds \le \alpha y(t) \quad \forall t \in [0,T]. \tag{2.11}$$

Then A has a unique fixed point in $PC(R,E)$.

Proof. For any given $x_0 \in F$, let $x_k = Ax_{k-1}$, $(k = 1, 2, \ldots)$. By $(\widetilde{H}_1)$, we have

$$\|Ax_{k+1}(t) - Ax_k(t)\|_E \le \beta \|x_{k+1}(t) - x_k(t)\|_E + \int_{t-T}^{t} G(t,s) \sum_{i=1}^{n} \left\| x_{k+1}(\eta_i(s)) - x_k(\eta_i(s)) \right\|_E ds. \tag{2.12}$$

Set $a_k(t) = \|x_{k+1}(t) - x_k(t)\|_E$, then we get

$$a_{k+1}(t) \le \beta a_k(t) + \int_{t-T}^{t} G(t,s) \sum_{i=1}^{n} a_k(\eta_i(s)) ds. \tag{2.13}$$

Basing on the condition $(\widetilde{H}_2)$, we see that there are two positive constants M and m such that $m \le y(t) \le M$ for all $t \in [0,T]$. Define the new norm $\|\cdot\|_2$ by

$$\|u\|_2 = \sup\left\{ \frac{1}{y(t)} \|u(t)\|_E : t \in [0,T] \right\}, \quad u \in PC(R,E). \tag{2.14}$$

Then,

$$\frac{1}{M} \|u\|_p \le \|u\|_2 \le \frac{1}{m} \|u\|_p. \tag{2.15}$$

Thus, the two norms $\|\cdot\|_2$ and $\|\cdot\|_p$ are equivalent.

Set $a_k = \|x_{k+1} - x_k\|_2$, then we have $a_k(t) \leq y(t)a_k$ for $t \in [0,T]$. By (2.13), we have

$$\begin{aligned}\frac{1}{y(t)}a_{k+1}(t) &\leq \beta a_k + \frac{1}{y(t)}\int_{t-T}^{t} G(t,s)\sum_{i=1}^{n} a_k(\eta_i(s))ds \\ &\leq \beta a_k + \frac{a_k}{y(t)}\int_{t-T}^{t} G(t,s)\sum_{i=1}^{n} y(\eta_i(s))ds \leq (\beta + nK\alpha)a_k.\end{aligned} \tag{2.16}$$

Thus,

$$a_{k+1} \leq (\beta + nK\alpha)a_k \leq (\beta + nK\alpha)^2 a_{k-1} \leq \cdots \leq (\beta + nK\alpha)^{k+1} a_0. \tag{2.17}$$

This means $\{x_k\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_2$. Therefore, $\{x_k\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_p$. Therefore, we see that $\{x_k\}$ has a limit point in $PC(R,E)$, say u. It is easy to prove that u is the fixed point of A in $PC(R,E)$. The uniqueness of the fixed point is obvious. This completes the proof of Theorem 2.2. □

2.2. Existence and Uniqueness of the Periodic Solution

In order to show the existence of periodic solutions of (1.1), we assume that the function f is fulfilling the following conditions:

(H_f) there exist $L_i > 0$ $(i = 1,2,\ldots,m)$ such that for any $x_i, y_i \in R$,

$$|f(t,x_1,\ldots,x_m) - f(t,y_1,\ldots,y_m)| \leq \sum_{i=1}^{m} L_i|x_i - y_i|, \tag{2.18}$$

$(H_{f\tau})$ for all $t \in [0,T]$, $t \geq \tau_i(t) \geq 0$ $(i = 1,2,\ldots,m)$.

Theorem 2.3. *Suppose* (H_f) *and* $(H_{f\tau})$ *hold. Then the equation* (1.1) *has a unique T-periodic solution in* $C[0,T]$.

Proof. By direction computations, we see that $\varphi(t)$ is the T-periodic solution if and only if $\varphi(t)$ is solution of the following integral equation:

$$x(t) = \frac{e^{\lambda T}}{e^{\lambda T} - 1}\int_{t-T}^{t} e^{-\lambda(t-s)}\left[(\lambda - a(s))x(s) + f(s, x(s-\tau_1(s)),\ldots,x(s-\tau_m(s)))\right]ds, \tag{2.19}$$

where $\lambda = \max\{|a(t)| : t \in [0,T]\}$.

Thus, we would transform the existence of periodic solution of (1.1) into a fixed point problem. Considering the map $A : PC(R,R) \to PC(R,R)$ defined by, for $t \in [0,T]$,

$$(Ax)(t) = \frac{e^{\lambda T}}{e^{\lambda T} - 1}\int_{t-T}^{t} e^{-\lambda(t-s)}\left[(\lambda - a(s))x(s) + f(s, x(s-\tau_1(s)),\ldots,x(s-\tau_m(s)))\right]ds. \tag{2.20}$$

Then, u is a T-periodic solution of (1.1) if and only if u is a fixed point of the operator A in $PC(R,R)$.

At this stage, we should check that A fulfill all conditions of Theorem 2.2. In fact, for $x, y \in PC(R, R)$, by assumption (H_f), we have

$$\begin{aligned} &\left|(Ax)(t) - (Ay)(t)\right| \\ &\quad \le \frac{e^{\lambda T}}{e^{\lambda T}-1}\int_{t-T}^{t} e^{-\lambda(t-s)}\left[2\lambda\left|x(s)-y(s)\right| + \sum_{i=1}^{m} L_i\left|x(s-\tau_i(s)) - y(s-\tau_i(s))\right|\right]ds \\ &\quad \le \frac{Le^{\lambda T}}{e^{\lambda T}-1}\int_{t-T}^{t} e^{-\lambda(t-s)}\left[\sum_{i=1}^{m+1}\left|x(\eta_i(s)) - y(\eta_i(s))\right|\right]ds, \end{aligned} \tag{2.21}$$

where $\eta_i(s) = s - \tau_i(s)$ $(i = 1, 2, \ldots, m)$, $\eta_{m+1}(s) = s$ and $L = \max\{2\lambda, L_1, \ldots, L_m\}$.

Thus, the condition $(\widetilde{H}_1)$ in Theorem 2.2 holds for $\beta = 0, n = m + 1$, and $G(t, s) = (Le^{\lambda T}/(e^{\lambda T} - 1))e^{-\lambda(t-s)}$.

On the other hand, we choose a constant $c > 0$ such that $0 < (m + 1)(Le^{\lambda T}/(e^{\lambda T} - 1))(1/(c + \lambda)) < 1$. Take $\alpha = (Le^{\lambda T}/(e^{\lambda T} - 1))(1/(c + \lambda))$ and $y(t) = e^{ct}$ for $t \in [0, T]$, then $y(\eta_i(t)) \le y(t)$, and we have

$$\int_{t-T}^{t} G(t,s)y(s)ds = \int_{t-T}^{t} \frac{Le^{\lambda T}}{e^{\lambda T}-1} e^{-\lambda(t-s)} e^{cs} ds \le \alpha y(t). \tag{2.22}$$

This implies the condition $(\widetilde{H}_2)$ in Theorem 2.2 holds for $K = 1$.

Following Theorem 2.2, we conclude that the operator A has a unique fixed point, say φ, in $PC(R, R)$. Thus, (1.1) has a unique T-periodic solution in $PC(R, R)$. This completes the proof of Theorem 2.3. □

3. Application to the Hematopoiesis Model

In this section, we consider the periodic solution of following hematopoiesis model with delays:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \tag{3.1}$$

where $a, b_i, \tau_i \in C(R, R^+)$ are T-periodic functions, τ_i satisfies conditions $(H_{f\tau})$, and $n \ge 1$ is a real number $(i = 1, \ldots, m)$.

Theorem 3.1. *The delayed hematopoiesis model* (3.1) *has a unique positive T-periodic solution.*

Proof. Let $C_T^+ = \{y : y \in C_T \text{ and } y(t) \ge 0 \text{ for } t \ge 0\}$, define the operator $F : C_T^+ \to C_T^+$ by

$$(Ax)(t) = \frac{e^{\lambda T}}{e^{\lambda T}-1}\int_{t-T}^{t} e^{-\lambda(t-s)}\left[(\lambda - a(s))x(s) + \sum_{i=1}^{m}\frac{b_i(s)}{1 + x^n(s - \tau_i(s))}\right]ds. \tag{3.2}$$

It is easy to show that A is welldefined. Furthermore, since the function

$$g(t, x_1, \ldots, x_m) = \sum_{i=1}^{m} \frac{b_i(t)}{1 + x_i^n} \quad \text{for } x_i \in R^+ \tag{3.3}$$

with the bounded partial derivative

$$\frac{\partial g(t, x_1, \ldots, x_m)}{\partial x_i} \le \max\left\{ \frac{b_i(t) n x_i^{n-1}}{1 + x_i^n} : t \in [0, T], 1 \le i \le m \right\}, \tag{3.4}$$

then it is easy to prove that the condition (H_f) holds. Following the similar arguments of Theorem 2.3, we claim that the operator A has a unique fixed point in C_T^+, which is the unique positive T-periodic solution for equation (3.1). This completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 exhibits that the periodic coefficients hematopoiesis model admits a unique positive periodic solution without additional restriction. Also, Theorem 3.1 improves Theorem 2.1 in [2] and Corollary 1 in [3].

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 307968, 14 pages
doi:10.1155/2012/307968

Research Article

Existence of Positive Solutions of Neutral Differential Equations

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Received 10 November 2011; Accepted 14 December 2011

Academic Editor: Josef Diblík

The paper contains some suffcient conditions for the existence of positive solutions which are bounded below and above by positive functions for the nonlinear neutral differential equations of higher order. These equations can also support the existence of positive solutions approaching zero at infinity.

1. Introduction

This paper is concerned with the existence of a positive solution of the neutral differential equations of the form:

$$\frac{d^n}{dt^n}[x(t) - a(t)x(t-\tau)] = (-1)^{n+1}p(t)f(x(t-\sigma)), \quad t \geq t_0, \tag{1.1}$$

where $n > 0$ is an integer, $\tau > 0$, $\sigma \geq 0$, $a \in C([t_0, \infty), (0, \infty))$, $p \in C(R, (0, \infty))$, $f \in C(R, R)$, f is a nondecreasing function and $xf(x) > 0$, $x \neq 0$.

By a solution of (1.1) we mean a function $x \in C([t_1 - \tau, \infty), R)$ for some $t_1 \geq t_0$, such that $x(t) - a(t)x(t-\tau)$ is n-times continuously differentiable on $[t_1, \infty)$ and such that (1.1) is satisfied for $t \geq t_1$.

The problem of the existence of solutions of neutral differential equations has been studied and discussed by several authors in the recent years. For related results we refer the reader to [1–17] and the references cited therein. However, there is no conception which guarantees the existence of positive solutions which are bounded below and above by positive functions. Maybe it is due to the technical difficulties arising in the analysis of the problem. In this paper we presented some conception. The method also supports the existence of positive solutions which approaching zero at infinity. Some examples illustrating the results.

The existence and asymptotic behavior of solutions of the nonlinear neutral differential equations and systems have been also solved in [1–7, 12, 15].

As much as we know for (1.1) in the literature, there is no result for the existence of solutions which are bounded by positive functions. Only the existence of solutions which are bounded by constants is treated and discussed, for example, in [10, 15, 17]. It seems that conditions of theorems are rather complicate, but cannot be simpler due to Corollaries 2.4, 2.8, and 3.3.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 (see [7, 10, 12] Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous then the equation:*

$$S_1x + S_2x = x \tag{1.2}$$

has a solution in Ω.

2. The Existence of Positive Solution

In this section, we will consider the existence of a positive solution for (1.1) which is bounded by two positive functions. We will use the notation $m = \max\{\tau, \sigma\}$.

Theorem 2.1. *Suppose that there exist bounded functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$, and $t_1 \geq t_0 + m$ such that*

$$u(t) \leq v(t), \quad t \geq t_0, \tag{2.1}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1, \tag{2.2}$$

$$\begin{aligned} &\frac{1}{u(t-\tau)}\left(u(t) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}p(s)f(v(s-\sigma))ds\right) \\ &\quad \leq a(t) \leq \frac{1}{v(t-\tau)}\left(v(t) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}p(s)f(u(s-\sigma))ds\right) \\ &\quad \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.3}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. Let $C([t_0, \infty), R)$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t\geq t_0}|x(t)|$. Then $C([t_0, \infty), R)$ is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), R) : u(t) \leq x(t) \leq v(t),\ t \geq t_0\}. \tag{2.4}$$

We now define two maps S_1 and $S_2 : \Omega \to C([t_0, \infty), R)$ as follows:

$$
\begin{aligned}
(S_1x)(t) &= \begin{cases} a(t)x(t-\tau), & t \geq t_1, \\ (S_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases} \\
(S_2x)(t) &= \begin{cases} -\dfrac{1}{(n-1)!}\displaystyle\int_t^{\infty}(s-t)^{n-1}p(s)f(x(s-\sigma))ds, & t \geq t_1, \\ (S_2x)(t_1) + v(t) - v(t_1), & t_0 \leq t \leq t_1. \end{cases}
\end{aligned}
\tag{2.5}
$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$. For every $x, y \in \Omega$ and $t \geq t_1$ we obtain

$$(S_1x)(t) + (S_2y)(t) \leq a(t)v(t-\tau) - \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}p(s)f(u(s-\sigma))ds \leq v(t). \tag{2.6}$$

For $t \in [t_0, t_1]$ we have

$$
\begin{aligned}
(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\
&\leq v(t_1) + v(t) - v(t_1) = v(t).
\end{aligned}
\tag{2.7}
$$

Furthermore for $t \geq t_1$ we get

$$(S_1x)(t) + (S_2y)(t) \geq a(t)u(t-\tau) - \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}p(s)f(v(s-\sigma))ds \geq u(t). \tag{2.8}$$

Finally let $t \in [t_0, t_1]$ and with regard to (2.2) we get

$$v(t) - v(t_1) + u(t_1) \geq u(t), \quad t_0 \leq t \leq t_1. \tag{2.9}$$

Then for $t \in [t_0, t_1]$ and any $x, y \in \Omega$ we get

$$
\begin{aligned}
(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\
&\geq u(t_1) + v(t) - v(t_1) \geq u(t).
\end{aligned}
\tag{2.10}
$$

Thus, we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq t_1$ we have

$$\left|(S_1x)(t) - (S_1y)(t)\right| = |a(t)|\left|x(t-\tau) - y(t-\tau)\right| \leq c\left\|x - y\right\|. \tag{2.11}$$

This implies that

$$\left\|S_1x - S_1y\right\| \leq c\left\|x - y\right\|. \tag{2.12}$$

Also for $t \in [t_0, t_1]$ the inequality above is valid. We conclude that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$\begin{aligned}
&|(S_2x_k)(t) - (S_2x)(t)| \\
&\quad \leq \frac{1}{(n-1)!}\left|\int_t^{\infty}(s-t)^{n-1}p(s)\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]ds\right| \\
&\quad \leq \frac{1}{(n-1)!}\int_{t_1}^{\infty}(s-t_1)^{n-1}p(s)\left|f(x_k(s-\sigma)) - f(x(s-\sigma))\right|ds.
\end{aligned} \tag{2.13}$$

According to (2.8) we get

$$\int_{t_1}^{\infty}(s-t_1)^{n-1}p(s)f(v(s-\sigma))ds < \infty. \tag{2.14}$$

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k\to\infty}\|(S_2x_k)(t) - (S_2x)(t)\| = 0. \tag{2.15}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show, according to Levitan result [8], that for any given $\varepsilon > 0$ the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. With regard to the condition (2.14), for $x \in \Omega$ and any $\varepsilon > 0$ we take $t^* \geq t_1$ large enough so that

$$\frac{1}{(n-1)!}\int_{t^*}^{\infty}(s-t_1)^{n-1}p(s)f(x(s-\sigma))ds < \frac{\varepsilon}{2}. \tag{2.16}$$

Then for $x \in \Omega,\ T_2 > T_1 \geq t^*$ we have

$$\begin{aligned}
&|(S_2x)(T_2) - (S_2x)(T_1)| \\
&\quad \leq \frac{1}{(n-1)!}\int_{T_2}^{\infty}(s-t_1)^{n-1}p(s)f(x(s-\sigma))ds + \frac{1}{(n-1)!}\int_{T_1}^{\infty}(s-t_1)^{n-1}p(s)f(x(s-\sigma))ds \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{2.17}$$

For $x \in \Omega$, $t_1 \leq T_1 < T_2 \leq t^*$ and $n \geq 2$ we get

$$\begin{aligned}
|(S_2x)(T_2) - (S_2x)(T_1)| &= \frac{1}{(n-1)!}\left|\int_{T_1}^{\infty}(s-T_1)^{n-1}p(s)f(x(s-\sigma))ds\right. \\
&\qquad \left. - \int_{T_2}^{\infty}(s-T_2)^{n-1}p(s)f(x(s-\sigma))ds\right| \\
&= \frac{1}{(n-1)!}\left|\int_{T_1}^{T_2}(s-T_1)^{n-1}p(s)f(x(s-\sigma))ds\right. \\
&\qquad + \int_{T_2}^{\infty}(s-T_1)^{n-1}p(s)f(x(s-\sigma))ds \\
&\qquad \left. - \int_{T_2}^{\infty}(s-T_2)^{n-1}p(s)f(x(s-\sigma))ds\right| \\
&\leq \frac{1}{(n-1)!}\int_{T_1}^{T_2}s^{n-1}p(s)f(x(s-\sigma))ds \\
&\quad + \frac{1}{(n-1)!}\int_{T_2}^{\infty}\left[(s-T_1)^{n-1} - (s-T_2)^{n-1}\right]p(s)f(x(s-\sigma))ds \\
&\leq \max_{t_1 \leq s \leq t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}(T_2 - T_1) \\
&\quad + \frac{1}{(n-1)!}\int_{T_2}^{\infty}[(s-T_1) - (s-T_2)] \\
&\quad \times \left[(s-T_1)^{n-2} + (s-T_1)^{n-3}(s-T_2) + \cdots + (s-T_1)(s-T_2)^{n-3}\right. \\
&\qquad \left. + (s-T_2)^{n-2}\right]p(s)f(x(s-\sigma))ds \\
&\leq \max_{t_1 \leq s \leq t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}(T_2 - T_1) \\
&\quad + \frac{1}{(n-2)!}\int_{T_2}^{\infty}(T_2 - T_1)(s-T_1)^{n-2}p(s)f(x(s-\sigma))ds.
\end{aligned} \tag{2.18}$$

With regard to the condition (2.14) we have that

$$\frac{1}{(n-2)!}\int_{T_2}^{\infty}(s-T_1)^{n-2}p(s)f(x(s-\sigma))ds < B, \quad B > 0. \tag{2.19}$$

Then we obtain

$$|(S_2x)(T_2)-(S_2x)(T_1)| < \left(\max_{t_1\le s\le t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}+B\right)(T_2-T_1). \tag{2.20}$$

Thus there exists a $\delta_1=\varepsilon/(M+B)$, where

$$M=\max_{t_1\le s\le t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}, \tag{2.21}$$

such that

$$|(S_2x)(T_2)-(S_2x)(T_1)|<\varepsilon \quad \text{if } 0<T_2-T_1<\delta_1. \tag{2.22}$$

For $n=1$ we proceed by the similar way as above. Finally for any $x\in\Omega$, $t_0\le T_1<T_2\le t_1$ there exists a $\delta_2>0$ such that

$$\begin{aligned}|(S_2x)(T_2)-(S_2x)(T_1)| &= |v(T_1)-v(T_2)| = \left|\int_{T_1}^{T_2}v'(s)ds\right| \\ &\le \max_{t_0\le s\le t_1}\left\{\left|v'(s)\right|\right\}(T_2-T_1)<\varepsilon \quad \text{if } 0<T_2-T_1<\delta_2.\end{aligned} \tag{2.23}$$

Then $\{S_2x : x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$ and hence $S_2\Omega$ is relatively compact subset of $C([t_0,\infty),R)$. By Lemma 1.1 there is an $x_0\in\Omega$ such that $S_1x_0+S_2x_0=x_0$. We conclude that $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.2. *Suppose that all conditions of Theorem 2.1 are satisfied and*

$$\lim_{t\to\infty}v(t)=0. \tag{2.24}$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 2.3. *Suppose that there exist bounded functions $u,v\in C^1([t_0,\infty),(0,\infty))$, constant $c>0$ and $t_1\ge t_0+m$ such that* (2.1), (2.3) *hold and*

$$v'(t)-u'(t)\le 0, \quad t_0\le t\le t_1. \tag{2.25}$$

Then (1.1) *has a positive solution which is bounded by the functions u,v.*

Proof. We only need to prove that condition (2.25) implies (2.2). Let $t\in[t_0,t_1]$ and set

$$H(t)=v(t)-v(t_1)-u(t)+u(t_1). \tag{2.26}$$

Then with regard to (2.25), it follows that $H'(t) = v'(t) - u'(t) \le 0$, $t_0 \le t \le t_1$. Since $H(t_1) = 0$ and $H'(t) \le 0$ for $t \in [t_0, t_1]$, this implies that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \ge 0, \quad t_0 \le t \le t_1. \tag{2.27}$$

Thus all conditions of Theorem 2.1 are satisfied. □

Corollary 2.4. *Suppose that there exists a bounded function $v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \ge t_0 + m$ such that*

$$a(t) = \frac{1}{v(t-\tau)}\left(v(t) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}p(s)f(v(s-\sigma))ds\right) \le c < 1, \quad t \ge t_1. \tag{2.28}$$

Then (1.1) *has a solution $x(t) = v(t)$, $t \ge t_1$.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.1. □

Theorem 2.5. *Suppose that p is bounded and there exist bounded functions $u, v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \ge t_0 + m$ such that* (2.1), (2.2) *hold and*

$$\begin{aligned}&\frac{1}{u(t-\tau)}\left(u(t) - \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(u(s-\sigma))ds\right)\\ &\quad\le a(t) \le \frac{1}{v(t-\tau)}\left(v(t) - \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\right)\\ &\quad\le c < 1, \quad t \ge t_1,\end{aligned} \tag{2.29}$$

if n is odd,

$$\begin{aligned}&\frac{1}{u(t-\tau)}\left(u(t) + \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\right)\\ &\quad\le a(t) \le \frac{1}{v(t-\tau)}\left(v(t) + \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(u(s-\sigma))ds\right)\\ &\quad\le c < 1, \quad t \ge t_1,\end{aligned} \tag{2.30}$$

if n is even, and

$$\int_{t_1}^{t}(t-s)^{n-2}p(s)f(v(s-\sigma))ds \le K, \quad t \ge t_1,\ K > 0,\ n \ge 2. \tag{2.31}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. Let $C([t_0,\infty),R)$ be the set as in the proof of Theorem 2.1. We define a closed, bounded, and convex subset Ω of $C([t_0,\infty),R)$ as in the proof of Theorem 2.1. We define two maps S_1 and $S_2:\Omega\to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t)=\begin{cases}a(t)x(t-\tau), & t\ge t_1,\\ (S_1x)(t_1), & t_0\le t\le t_1,\end{cases}$$
$$(S_2x)(t)=\begin{cases}\dfrac{(-1)^{n+1}}{(n-1)!}\displaystyle\int_{t_1}^{t}(t-s)^{n-1}p(s)f(x(s-\sigma))ds, & t\ge t_1,\\ (S_2x)(t_1)+v(t)-v(t_1), & t_0\le t\le t_1.\end{cases} \tag{2.32}$$

We shall show that for any $x,y\in\Omega$ we have $S_1x+S_2y\in\Omega$. For n odd, every $x,y\in\Omega$ and $t\ge t_1$ we obtain

$$(S_1x)(t)+\big(S_2y\big)(t)\le a(t)v(t-\tau)+\frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\le v(t). \tag{2.33}$$

For $t\in[t_0,t_1]$, we have

$$\begin{aligned}(S_1x)(t)+\big(S_2y\big)(t)&=(S_1x)(t_1)+\big(S_2y\big)(t_1)+v(t)-v(t_1)\\&\le v(t_1)+v(t)-v(t_1)=v(t).\end{aligned} \tag{2.34}$$

Furthermore for $t\ge t_1$, we get

$$(S_1x)(t)+\big(S_2y\big)(t)\ge a(t)u(t-\tau)+\frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(u(s-\sigma))ds\ge u(t). \tag{2.35}$$

Let $t\in[t_0,t_1]$ and according to (2.2) we have

$$v(t)-v(t_1)+u(t_1)\ge u(t). \tag{2.36}$$

Then for $t\in[t_0,t_1]$ and any $x,y\in\Omega$ we get

$$\begin{aligned}(S_1x)(t)+\big(S_2y\big)(t)&=(S_1x)(t_1)+\big(S_2y\big)(t_1)+v(t)-v(t_1)\\&\ge u(t_1)+v(t)-v(t_1)\ge u(t).\end{aligned} \tag{2.37}$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

For n even by the similar way as above we can prove that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

As in the proof of Theorem 2.1, we can show that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First, we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$|(S_2x_k)(t) - (S_2x)(t)| \leq \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)\big|f(x_k(s-\sigma)) - f(x(s-\sigma))\big|ds. \tag{2.38}$$

According to (2.33) there exists a positive constant M such that

$$\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds \leq M \quad \text{for } t \geq t_1. \tag{2.39}$$

The inequality above also holds for n even.

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k\to\infty}\|(S_2x_k)(t) - (S_2x)(t)\| = 0. \tag{2.40}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For $n \geq 2$ and with regard to (2.31) we have

$$\begin{aligned}\left|\frac{d}{dt}(S_2x)(t)\right| &= \frac{1}{(n-2)!}\int_{t_1}^{t}(t-s)^{n-2}p(s)f(x(s-\sigma))ds \\ &\leq \frac{1}{(n-2)!}\int_{t_1}^{t}(t-s)^{n-2}p(s)f(v(s-\sigma)) \leq M_1,\end{aligned} \tag{2.41}$$

and for $n = 1$ we obtain

$$\left|\frac{d}{dt}(S_2x)(t)\right| = p(t)f(v(t-\sigma)) \leq M_2, \tag{2.42}$$

for $t \geq t_1$, $M_2 > 0$ and $|(d/dt)(S_2x)(t)| = |v'(t)| \leq M_3$ for $t_0 \leq t \leq t_1$, $M_3 > 0$, which shows the equicontinuity of the family $S_2\Omega$, (cf. [7, page 265]). Hence $S_2\Omega$ is relatively compact and therefore S_2 is completely continuous. By Lemma 1.1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Thus $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.6. *Suppose that all conditions of Theorem 2.5 are satisfied and*

$$\lim_{t\to\infty} v(t) = 0. \tag{2.43}$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 2.7. *Suppose that p is bounded and there exist bounded functions $u, v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.29), (2.30), (2.31) *hold and*

$$v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1. \tag{2.44}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. The proof is similar to that of Corollary 2.3 and we omit it. □

Corollary 2.8. *Suppose that p is bounded and there exists a bounded function $v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.31) *holds and*

$$\begin{aligned} a(t) &= \frac{1}{v(t-\tau)}\left(v(t) + \frac{(-1)^n}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\right) \\ &\leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.45}$$

Then (1.1) *has a solution $x(t) = v(t)$, $t \geq t_1$.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.5. □

3. Applications and Examples

In this section, we give some applications of the theorems above.

Theorem 3.1. *Suppose that $0 < k_1 \leq k_2$ and there exist $\gamma \geq 0$, $c > 0$, $t_1 \geq t_0 + m$ such that*

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} p(t)dt\right) \geq 1, \tag{3.1}$$

$$\begin{aligned} &\exp\left(-k_2\int_{t-\tau}^{t}p(s)ds\right) + \frac{1}{(n-1)!}\exp\left(k_2\int_{t_0-\gamma}^{t-\tau}p(s)ds\right) \\ &\quad\times\int_t^{\infty}(s-t)^{n-1}p(s)f\left(\exp\left(-k_1\int_{t_0-\gamma}^{s-\sigma}p(\xi)d\xi\right)\right)ds \leq a(t) \\ &\leq \exp\left(-k_1\int_{t-\tau}^{t}p(s)ds\right) + \frac{1}{(n-1)!}\exp\left(k_1\int_{t_0-\gamma}^{t-\tau}p(s)ds\right) \\ &\quad\times\int_t^{\infty}(s-t)^{n-1}p(s)f\left(\exp\left(-k_2\int_{t_0-\gamma}^{s-\sigma}p(\xi)d\xi\right)\right)ds \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{3.2}$$

Then (1.1) *has a positive solution which is bounded by two exponential functions.*

Proof. We set

$$u(t) = \exp\left(-k_2 \int_{t_0-\gamma}^{t} p(s)ds\right), \quad v(t) = \exp\left(-k_1 \int_{t_0-\gamma}^{t} p(s)ds\right), \quad t \geq t_0. \tag{3.3}$$

We will show that the conditions of Corollary 2.3 are satisfied. With regard to (3.1) for $t \in [t_0, t_1]$ we get

$$\begin{aligned}
v'(t) - u'(t) &= -k_1 p(t)v(t) + k_2 p(t)u(t) \\
&= p(t)v(t)\left[-k_1 + k_2 u(t) \exp\left(k_1 \int_{t_0-\gamma}^{t} p(s)ds\right)\right] \\
&= p(t)v(t)\left[-k_1 + k_2 \exp\left((k_1 - k_2) \int_{t_0-\gamma}^{t} p(s)ds\right)\right] \\
&\leq p(t)v(t)\left[-k_1 + k_2 \exp\left((k_1 - k_2) \int_{t_0-\gamma}^{t_0} p(s)ds\right)\right] \leq 0.
\end{aligned} \tag{3.4}$$

Other conditions of Corollary 2.3 are also satisfied. The proof is complete. □

Corollary 3.2. *Suppose that all conditions of Theorem 3.1 are satisfied and*

$$\int_{t_0}^{\infty} p(t)dt = \infty. \tag{3.5}$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 3.3. *Suppose that* $k > 0,\ c > 0,\ t_1 \geq t_0 + m$ *and*

$$\begin{aligned}
a(t) = {} & \exp\left(-k \int_{t-\tau}^{t} p(s)ds\right) + \frac{1}{(n-1)!} \exp\left(k \int_{t_0}^{t-\tau} p(s)ds\right) \\
& \times \int_{t}^{\infty} (s-t)^{n-1} p(s) f\left(\exp\left(-k \int_{t_0}^{s-\sigma} p(\xi)d\xi\right)\right) ds \leq c < 1, \quad t \geq t_1.
\end{aligned} \tag{3.6}$$

Then (1.1) *has a solution:*

$$x(t) = \exp\left(-k \int_{t_0}^{t} p(s)ds\right), \quad t \geq t_1. \tag{3.7}$$

Proof. We put $k_1 = k_2 = k,\ \gamma = 0$ and apply Theorem 3.1. □

Example 3.4. Consider the nonlinear neutral differential equation:

$$[x(t) - a(t)x(t-2)]' = px^3(t-1), \quad t \geq t_0, \tag{3.8}$$

where $p \in (0, \infty)$. We will show that the conditions of Theorem 3.1 are satisfied. The condition (3.1) has a form:

$$\frac{k_1}{k_2} \exp\left((k_2 - k_1)p\gamma\right) \geq 1, \tag{3.9}$$

$0 < k_1 \leq k_2$, $\gamma \geq 0$. For function $a(t)$, we obtain

$$\begin{aligned}
&\exp(-2pk_2) + \frac{1}{3k_1} \exp\left(p\left[k_2(\gamma - t_0 - 2) - 3k_1(\gamma - t_0 - 1) + (k_2 - 3k_1)t\right]\right) \\
&\quad \leq a(t) \leq \exp(-2pk_1) \\
&\qquad + \frac{1}{3k_2} \exp\left(p\left[k_1(\gamma - t_0 - 2) - 3k_2(\gamma - t_0 - 1) + (k_1 - 3k_2)t\right]\right), \quad t \geq t_0.
\end{aligned} \tag{3.10}$$

For $p = 1,\ k_1 = 1,\ k_2 = 2,\ \gamma = 1,\ t_0 = 1$, the condition (3.9) is satisfied and

$$e^{-4} + \frac{1}{3e} e^{-t} \leq a(t) \leq e^{-2} + \frac{e^4}{6} e^{-5t}, \quad t \geq t_1 \geq 3. \tag{3.11}$$

If the function $a(t)$ satisfies (3.11), then (3.8) has a solution which is bounded by the functions $u(t) = \exp(-2t),\ v(t) = \exp(-t),\ t \geq 3$.

Example 3.5. Consider the nonlinear differential equation:

$$[x(t) - a(t)x(t-\pi)]' = p(t) f(x(t-\pi)), \quad t \geq 0, \tag{3.12}$$

where $f(x) = \sqrt{x},\ x > 0,\ p(t) = 0.8 \exp(\pi - t + 0.05 \cos t),\ t \geq 0$, and

$$e^{0.1\cos t}\left(e^{0.1\cos t} + 0.8\left(e^{\pi - t} - 1\right)\right) \leq a(t) \leq e^{0.1\cos t}\left(e^{0.1\cos t} + \frac{0.8}{\sqrt{b}}\left(e^{\pi - t} - 1\right)\right) < 1, \tag{3.13}$$

for $t \geq \pi,\ b \in [1, 2]$. Set

$$u(t) = e^{0.1\cos t}, \quad v(t) = be^{0.1\cos t}, \quad t \geq 0. \tag{3.14}$$

Then we have

$$\begin{aligned}
v'(t) - u'(t) &= -0.1be^{0.1\cos t} \sin t + 0.1e^{0.1\cos t} \sin t \\
&= -0.1(b-1)e^{0.1\cos t} \sin t \leq 0 \quad \text{for } t \in [0, \pi].
\end{aligned} \tag{3.15}$$

By Corollary 2.7, (3.12) has a solution which is bounded by the functions $e^{0.1\cos t}$ and $be^{0.1\cos t}$, $t \geq \pi$. If

$$a(t) = e^{0.1\cos t}\left(e^{0.1\cos t} + 0.8\left(e^{\pi - t} - 1\right)\right) \quad \text{for } t \geq \pi, \tag{3.16}$$

then (3.12) has the positive periodic solution $x(t) = u(t) = e^{0.1\cos t}$, $t \geq \pi$.

Acknowledgment

The research was supported by the Grants 1/0090/09 and 1/1260/12 of the Scientific Grant Agency of the Ministry of Education of the Slovak Republic.

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 489353, 12 pages
doi:10.1155/2012/489353

Research Article

Monotonic Positive Solutions of Nonlocal Boundary Value Problems for a Second-Order Functional Differential Equation

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Received 13 October 2011; Accepted 5 December 2011

Academic Editor: István Györi

We study the existence of at least one monotonic positive solution for the nonlocal boundary value problem of the second-order functional differential equation $x''(t) = f(t, x(\phi(t)))$, $t \in (0,1)$, with the nonlocal condition $\sum_{k=1}^{m} a_k x(\tau_k) = x_0$, $x'(0) + \sum_{j=1}^{n} b_j x'(\eta_j) = x_1$, where $\tau_k \in (a,d) \subset (0,1)$, $\eta_j \in (c,e) \subset (0,1)$, and $x_0, x_1 > 0$. As an application the integral and the nonlocal conditions $\int_a^d x(t)dt = x_0$, $x'(0) + x(e) - x(c) = x_1$ will be considered.

1. Introduction

The nonlocal boundary value problems of ordinary differential equations arise in a variety of different areas of applied mathematics and physics.

The study of nonlocal boundary value problems was initiated by Il'in and Moiseev [1, 2]. Since then, the non-local boundary value problems have been studied by several authors. The reader is referred to [3–22] and references therein.

In most of all these papers, the authors assume that the function $f : [0,1] \times R^+ \to R^+$ is continuous. They all assume that

$$\begin{aligned} \lim_{x \to \infty} \frac{f(x)}{x} &= 0 \quad \text{or } \infty, \\ \lim_{x \to 0} \frac{f(x)}{x} &= 0 \quad \text{or } \infty. \end{aligned} \tag{1.1}$$

These assumptions are restrictive, and there are many functions that do not satisfy these assumptions.

Here we assume that the function $f : [0,1] \times R^+ \to R^+$ is measurable in $t \in [0,1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0,1]$ is and there exists an integrable function $a \in L_1[0,1]$ and a constant $b > 0$ such that

$$\left| f(t,x) \right| \le |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D. \tag{1.2}$$

Our aim here is to study the existence of at least one monotonic positive solution for the nonlocal problem of the second-order functional differential equation

$$x''(t) = f\left(t, x\left(\phi(t)\right)\right), \quad t \in (0,1), \tag{1.3}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^{n} b_j x'\left(\eta_j\right) = x_1, \tag{1.4}$$

where $\tau_k \in (a,d) \subset (0,1), \eta_j \in (c,e) \subset (0,1),$ and $x_0, x_1 > 0$.

As an application, the problem with the integral and nonlocal conditions

$$\int_a^d x(t)dt = x_0, \qquad x'(0) + x(e) - x(c) = x_1, \tag{1.5}$$

is studied.

It must be noticed that the nonlocal conditions

$$\begin{gathered} x(\tau) = x_0, \quad \tau \in (a,d)\ , \qquad x'(0) + x'(\eta) = x_1, \quad \eta \in (c,e), \\ \sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,d), \qquad x'(0) + \sum_{j=1}^{n} b_j x'\left(\eta_j\right) = 0, \quad \eta_j \in (c,e), \\ \int_a^d x(t)dt = 0, \qquad x'(0) + x(e) = x(c) \end{gathered} \tag{1.6}$$

are special cases of our the nonlocal and integral conditions.

2. Integral Equation Representation

Consider the functional differential equation (1.3) with the nonlocal condition (1.4) with the following assumptions.

(i) $f : [0,1] \times R^+ \to R^+$ is measurable in $t \in [0,1]$ for all $x \in R^+$ and continuous in $x \in R^+$ for almost all $t \in [0,1]$ and there exists an integrable function $a \in L_1[0,1]$, and a constant $b > 0$ such that

$$\left|f(t,x)\right| \leq |a(t)| + b|x|, \quad \forall (t,x) \in [0,1] \times D. \tag{2.1}$$

(ii) $\phi : (0,1) \to (0,1)$ is continuous.

(iii) $b < 1/(3-B)$, $B = (\sum_{j=1}^{n} b_j + 1)^{-1}$.

(iv)

$$\sum_{k=1}^{m} a_k > 0, \quad \forall k = 1,2,\ldots,m, \qquad \sum_{j=1}^{n} b_j > 0, \quad \forall j = 1,2,\ldots,n. \tag{2.2}$$

Now, we have the following Lemma.

Lemma 2.1. *The solution of the nonlocal problem* (1.3)-(1.4) *can be expressed by the integral equation*

$$\begin{aligned} x(t) = {} & A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ & + B\left(t - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\ & + \int_0^t (t-s) f(s, x(\phi(s)))ds, \end{aligned} \tag{2.3}$$

where $A = (\sum_{k=1}^{m} a_k)^{-1}$, $B = (\sum_{j=1}^{n} b_j + 1)^{-1}$.

Proof. Integrating (1.3), we get

$$x'(t) = x'(0) + \int_0^t f(s, x(\phi(s)))ds. \tag{2.4}$$

Integrating (2.4), we obtain

$$x(t) = x(0) + x'(0)t + \int_0^t (t-s) f(s, x(\phi(s)))ds. \tag{2.5}$$

Let $t = \tau_k$, in (2.5), we get

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{n} a_k x(0) + \sum_{k=1}^{n} a_k \tau_k x'(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds, \tag{2.6}$$

and we deduce that

$$x(0) = A\left\{x_0 - \sum_{k=1}^{m} a_k \tau_k x'(0) - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\}, \quad A = \left(\sum_{k=1}^{m} a_k\right)^{-1}. \tag{2.7}$$

Substitute from (2.7) into (2.5), we obtain

$$\begin{aligned} x(t) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} + x'(0)\left(t - A\sum_{k=1}^{m} a_k \tau_k\right) \\ &+ \int_0^t (t-s) f(s, x(\phi(s)))ds. \end{aligned} \tag{2.8}$$

Let $t = \eta_j$, in (2.4), we obtain

$$\begin{aligned} \sum_{j=1}^{n} b_j x'(\eta_j) &= \sum_{j=1}^{n} b_j x'(0) + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds, \\ x_1 - x'(0) &= x'(0)\sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds, \end{aligned} \tag{2.9}$$

and we deduce that

$$x'(0) = B\left(x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right), \quad B = \left(\sum_{j=1}^{n} b_j + 1\right)^{-1}. \tag{2.10}$$

Substitute from (2.10) into (2.8), we obtain

$$\begin{aligned} x(t) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ &+ B\left(t - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\}, \\ &+ \int_0^t (t-s) f(s, x(\phi(s)))ds, \end{aligned} \tag{2.11}$$

which proves that the solution of the nonlocal problem (1.3)-(1.4) can be expressed by the integral equation (2.3). □

3. Existence of Solution

We study here the existence of at least one monotonic nondecreasing solution $x \in C[0,1]$ for the integral equation (2.3).

Theorem 3.1. *Assume that (i)–(iv) are satisfied. Then the nonlocal problem* (1.3)-(1.4) *has at least one solution* $x \in C[0,1]$.

Proof. Define the subset $Q_r \subset C(0,1)$ by $Q_r = \{x \in C : |x(t)| \le r, r = (Ax_0 + Bx_1 + (3-B)\|a\|)/(1-(3-B)b), r > 0\}$. Clear the set Q_r which is nonempty, closed, and convex.

Let H be an operator defined by

$$\begin{aligned}(Hx)(t) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\}\\ &+ B\left(t - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\}\\ &+ \int_0^t (t-s) f(s, x(\phi(s)))ds.\end{aligned} \tag{3.1}$$

Let $x \in Q_r$, then

$$\begin{aligned}|(Hx)(t)| \le A&\left\{x_0 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\left|f(s, x(\phi(s)))\right|ds\right\}\\ &+ B\left(t - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 + \sum_{j=1}^{n} b_j \int_0^{\eta_j} \left|f(s, x(\phi(s)))\right|ds\right\}\\ &+ \int_0^t (t-s)\left|f(s, x(\phi(s)))\right|ds\\ \le A&\left\{x_0 + \sum_{k=1}^{m} a_k \int_0^{1} \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\right\}\\ &+ B\left\{x_1 + \sum_{j=1}^{n} b_j \int_0^{1} \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\right\}\\ &+ \int_0^1 \left[|a(s)| + b\left|x(\phi(s))\right|\right]ds\\ \le A&x_0 + \|a\| + b\sup_{t\in I}\left|x(\phi(t))\right| + Bx_1 + B\sum_{j=1}^{n} b_j\|a\|\\ &+ bB\sum_{j=1}^{n} b_j \sup_{t\in I}\left|x(\phi(t))\right| + \|a\| + b\sup_{t\in I}\left|x(\phi(t))\right|\end{aligned}$$

$$\begin{aligned}&\leq Ax_0 + Bx_1 + 2\|a\| + 2b\|x\| + (1-B)\|a\| + b(1-B)\|x\| \\ &\leq Ax_0 + Bx_1 + (3-B)\|a\| + (3-B)br \leq r,\end{aligned} \tag{3.2}$$

then $H : Q_r \to Q_r$ and $\{Hx(t)\}$ is uniformly bounded in Q_r.

Also for $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}(Hx)(t_2) - (Hx)(t_1) = {}& B\left(t_2 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(t)))ds\right\} \\ &+ \int_0^{t_2}(t_2 - s)f(s, x(\phi(t)))ds \\ &- B\left(t_1 - A\sum_{k=1}^{m} a_k\tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(t)))ds\right\} \\ &- \int_0^{t_1}(t_1 - s)f(s, x(\phi(t)))ds \\ = {}& B(t_2 - t_1)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(t)))ds\right\} \\ &+ \int_0^{t_1}(t_2 - t_1)f(s, x(\phi(t)))ds \\ &+ \int_{t_1}^{t_2}(t_2 - s)f(s, x(\phi(t)))ds.\end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}|(Hx)(t_2) - (Hx)(t_1)| \leq {}& B|t_2 - t_1|\left\{x_1 + \sum_{j=1}^{n} b_j \int_0^{\eta_j} \big[|a(s)| + b\big|x(\phi(s))\big|\big]ds\right\} \\ &+ |t_2 - t_1|\int_0^{t_1}\big[|a(s)| + b\big|x(\phi(s))\big|\big]ds \\ &+ \int_{t_1}^{t_2}(t_2 - s)\big[|a(s)| + b\big|x(\phi(s))\big|\big]ds\end{aligned}$$

$$\begin{aligned}&\le B|t_2-t_1|x_1+\sum_{j=1}^{n}b_j[\|a\|+br]\\&\quad+|t_2-t_1|[\|a\|+br]+\int_{t_1}^{t_2}\|a\|ds+br[t_2-t_1].\end{aligned}\tag{3.4}$$

The above inequality shows that

$$|(Hx)(t_2)-(Hx)(t_1)|\longrightarrow 0\quad \text{as } t_2\longrightarrow t_1.\tag{3.5}$$

Therefore $\{Hx(t)\}$ is equicontinuous. By the Arzelà-Ascoli theorem, $\{Hx(t)\}$ is relatively compact.

Since all conditions of the Schauder theorem hold, then H has a fixed point in Q_r which proves the existence of at least one solution $x\in C[0,1]$ of the integral equation (2.3), where

$$\begin{aligned}\lim_{t\to 0^+}x(t)&=A\left\{x_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}(\tau_k-s)f(s,x(\phi(s)))ds\right\}\\&\quad-BA\sum_{k=1}^{m}a_k\tau_k\left\{x_1-\sum_{j=1}^{n}b_j\int_0^{\eta_j}f(s,x(\phi(s)))ds\right\}=x(0),\\ \lim_{t\to 1^-}x(t)&=A\left\{x_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}(\tau_k-s)f(s,x(\phi(s)))ds\right\}\\&\quad+B\left(1-A\sum_{k=1}^{m}a_k\tau_k\right)\left\{x_1-\sum_{j=1}^{n}b_j\int_0^{\eta_j}f(s,x(\phi(s)))ds\right\}\\&\quad+\int_0^1(1-s)f(s,x(\phi(s)))ds=x(1).\end{aligned}\tag{3.6}$$

To complete the proof, we prove that the integral equation (2.3) satisfies nonlocal problem (1.3)-(1.4). Differentiating (2.3), we get

$$x'(t)=B\left\{x_1-\sum_{j=1}^{n}b_j\int_0^{\eta_j}f(s,x(\phi(s)))ds\right\}+\int_0^t f(s,x(\phi(s)))ds,\tag{3.7}$$

$$x''(t)=f(t,x(\phi(t))).\tag{3.8}$$

Let $t = \tau_k$ in (2.3), we obtain

$$x(\tau_k) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f\left(s, x\left(\phi(s)\right)\right) ds\right\} + \int_0^{\tau_k} (\tau_k - s) f\left(s, x\left(\phi(s)\right)\right) ds, \tag{3.9}$$

which proves

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0. \tag{3.10}$$

Also let $t = \eta_j$ in (3.7), we obtain

$$x'\left(\eta_j\right) = B\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f\left(s, x\left(\phi(s)\right)\right) ds\right\} + \int_0^{\eta_j} f\left(s, x\left(\phi(s)\right)\right) ds, \tag{3.11}$$

then

$$\sum_{j=1}^{n} b_j x'\left(\eta_j\right) = B \sum_{j=1}^{n} b_j \left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f\left(s, x\left(\phi(s)\right)\right) ds\right\} + \sum_{j=1}^{n} b_j \int_0^{\eta_j} f\left(s, x\left(\phi(s)\right)\right) ds. \tag{3.12}$$

Let $t = 0$ in (3.7), we obtain

$$x'(0) = B\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f\left(s, x\left(\phi(s)\right)\right) ds\right\}. \tag{3.13}$$

Adding (3.12) and (3.13), we obtain

$$x'(0) + \sum_{j=1}^{n} b_j x'\left(\eta_j\right) = x_1. \tag{3.14}$$

This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.3) and (1.4). This completes the proof. □

Corollary 3.2. *The solution of the problem* (1.3)-(1.4) *is monotonic nondecreasing.*

Proof. Let $t_1 < t_2$, we deduce from (2.3) that

$$\begin{aligned} x(t_1) = A&\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ &+ B\left(t_1 - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\ &+ \int_0^{t_1} (t_1 - s) f(s, x(\phi(s)))ds \\ &< A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} \\ &+ B\left(t_2 - A\sum_{k=1}^{m} a_k \tau_k\right)\left\{x_1 - \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s, x(\phi(s)))ds\right\} \\ &+ \int_0^{t_2} (t_2 - s) f(s, x(\phi(s)))ds = x(t_2), \end{aligned} \tag{3.15}$$

which proves that the solution x of the problem (1.3)-(1.4) is monotonic nondecreasing. □

3.1. Positive Solution

Let $b_j = 0, j = 1, 2, \ldots n$ and $x_1 = 0$, then the nonlocal problem condition (1.4) will be

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \qquad x'(0) = 0. \tag{3.16}$$

Theorem 3.3. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the solution of the nonlocal problem* (1.3)–(3.16) *is positive* $t \in [d, 1]$.

Proof. Let $b_j = 0, j = 1, 2, \ldots n$ and $x_1 = 0$ in the integral equation (2.3) and the nonlocal condition (1.4), then the solution of the nonlocal problem (1.3)–(3.16) will be given by the integral equation

$$x(t) = A\left\{x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s)))ds\right\} + \int_0^{t} (t - s)\ f(s, x(\phi(s)))ds, \tag{3.17}$$

where $A = \left(\sum_{k=1}^{m} a_k\right)^{-1}$.

Let $t \in [d,1]$, then

$$\begin{gathered}
\int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \le \int_0^t (t-s) f(s, x(\phi(s))) ds, \quad \tau_k \le t, \\
\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds \le \sum_{k=1}^m a_k \int_0^t (t-s) f(s, x(\phi(s))) ds.
\end{gathered} \tag{3.18}$$

Multiplying by $A = (\sum_{k=1}^m a_k)^{-1}$, we obtain

$$\begin{aligned}
A\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s) f(s, x(\phi(s))) ds &\le A\sum_{k=1}^m a_k \int_0^t (t-s) f(s, x(\phi(s))) ds \\
&= \int_0^t (t-s) f(s, x(\phi(s))) ds,
\end{aligned} \tag{3.19}$$

and the solution x of the nonlocal problem (1.3) and (3.16), given by the integral equation (3.17), is positive for $t \in [d,1]$. This complete the proof. □

Example 3.4. Consider the nonlocal problem of the second-order functional differential equation (1.3) with two-point boundary condition

$$x'(0) = 0, \qquad x(\eta) = x_0, \quad \eta \in (a,d) \subset (0,1). \tag{3.20}$$

Applying our results here, we deduce that the two-point boundary value problem (1.3)–(3.20) has at least one monotonic nondecreasing solution $x \in C[0,1]$ represented by the integral equation

$$x(t) = x_0 - \int_0^{\eta} (\eta - s) f(s, x(\phi(s))) ds + \int_0^t (t-s) f(s, x(\phi(s))) ds. \tag{3.21}$$

This the solution is positive with $t > \eta$.

4. Nonlocal Integral Condition

Let $x \in C[0,1]$ be the solution of the nonlocal problem (1.3) and (1.4).

Let $a_k = t_k - t_{k-1}, \tau_k \in (t_{k-1}, t_k) \subset (a,d) \subset (0,1)$ and let $b_j = \xi_j - \xi_{j-1}, \eta_j \in (\xi_{j-1}, \xi_j) \subset (c,e) \subset (0,1)$, then

$$\sum_{k=1}^m (t_k - t_{k-1}) x(\tau_k) = x_0, \qquad x'(0) + \sum_{j=1}^n (\xi_j - \xi_{j-1}) x'(\eta_j) = x_1. \tag{4.1}$$

From the continuity of the solution x of the nonlocal problem (1.3) and (1.4), we obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}(t_k - t_{k-1})x(\tau_k) = \int_a^d x(s)ds,$$
$$x'(0) + \lim_{n\to\infty}\sum_{j=1}^{n}(\xi_j - \xi_{j-1})x'(\eta_j) = x'(0) + \int_c^e x'(s)ds, \tag{4.2}$$

and the nonlocal condition (1.4) transformed to the integral condition

$$\int_a^d x(s)ds = x_0, \qquad x'(0) + x(e) - x(c) = x_1, \tag{4.3}$$

and the solution of the integral equation (2.3) will be

$$\begin{aligned} x(t) &= (d-a)^{-1}\left\{x_0 - \int_a^d\int_0^t (t-s)f(s,x(\phi(s)))ds\,dt\right\} \\ &\quad + ((b-c)+1)^{-1}(t-1)\left\{x_1 - \int_c^e\int_0^t f(s,x(\phi(s)))ds\,dt\right\} \\ &\quad + \int_0^t f(s,x(\phi(s)))ds. \end{aligned} \tag{4.4}$$

Now, we have the following theorem.

Theorem 4.1. *Let the assumptions (i)–(iv) of Theorem 3.1 be satisfied. Then the nonlocal problem*

$$x''(t) = f(t,x(\phi(t))), \quad t \in (0,1),$$
$$\int_a^d x(s)ds = x_0, \qquad x'(0) + x(e) - x(c) = x_1 \tag{4.5}$$

has at least one monotonic nondecreasing solution $x \in C[0,1]$ *represented by* (4.4).

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 437912, 13 pages
doi:10.1155/2012/437912

Research Article

Existence of One-Signed Solutions of Discrete Second-Order Periodic Boundary Value Problems

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Received 2 September 2011; Accepted 24 October 2011

Academic Editor: Agacik Zafer

We prove the existence of one-signed periodic solutions of second-order nonlinear difference equation on a finite discrete segment with periodic boundary conditions by combining some properties of Green's function with the fixed-point theorem in cones.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Z}$ be the integers set, $T, a, b \in \mathbb{Z}$ with $T > 2$, $a > b$, and $[a,b]_{\mathbb{Z}} = \{a, a+1, \dots, b\}$.

In recent years, the existence and multiplicity of positive solutions of periodic boundary value problems for difference equations have been studied extensively, see [1–5] and the references therein. In 2003, Atici and Cabada [2] studied the existence of solutions of second-order difference equation boundary value problem

$$\begin{gathered}\Delta^2 y(n-1) + a(n)y(n) + f\left(n, y(n)\right) = 0, \quad n \in [1,T]_{\mathbb{Z}}, \\ y(0) = y(T), \qquad \Delta y(0) = \Delta y(T),\end{gathered} \tag{1.1}$$

where a, f satisfy

(H1) $a : [1,T]_{\mathbb{Z}} \to (-\infty, 0]$ and $a(\cdot) \not\equiv 0$;

(H2) $f : [1,T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $y \in \mathbb{R}$.

The authors obtained the existence results of solutions of (1.1) under conditions (H1), (H2), and the used tool is upper and lower solutions techniques.

Naturally, whether there exists the Green function $G(t,s)$ of the homogeneous linear boundary value problem corresponding to (1.1) if $a(n) \geq 0$? Moreover, if the answer is positive, whether $G(t,s)$ keeps its sign? To the knowledge of the authors, there are very few works on the case $a(n) \geq 0$.

Recently, in 2003, Torres [6] investigated the existence of one-signed periodic solutions for second-order differential equation boundary value problem

$$\begin{aligned} x''(t) &= f(t, x(t)), \quad t \in [1, T], \\ x(0) = x(T), &\qquad x'(0) = x'(T), \end{aligned} \tag{1.2}$$

by applying the fixed-point theorem in cones, and constructed Green's function of

$$\begin{aligned} x''(t) + a(t)x(t) &= 0, \quad t \in [1, T], \\ x(0) = x(T), &\qquad x'(0) = x'(T), \end{aligned} \tag{1.3}$$

where $a \in L^p(0, T)$ satisfies either

(H3) $a \le 0$, $a(\cdot) \not\equiv 0$ on $[0, T]$;

(H4) $a \ge 0$, $a(\cdot) \not\equiv 0$ on $[0, T]$ and $\|a\|_p \le K(2p^*)$ for some $1 \le p \le +\infty$.

Motivated by Torres [6], in Section 2, the paper gives the new expression of Green's function of the linear boundary value problem

$$\Delta^2 y(t-1) + a(t)y(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}, \tag{1.4}$$

$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \tag{1.5}$$

where $a \in \Lambda^+ \cup \Lambda^-$ and

$$\begin{gathered} \Lambda^- = \{a \mid a : [1, T]_{\mathbb{Z}} \longrightarrow (-\infty, 0],\ a(\cdot) \not\equiv 0\}, \\ \Lambda^+ = \left\{a \mid a : [1, T]_{\mathbb{Z}} \longrightarrow [0, \infty),\ a(\cdot) \not\equiv 0,\ \max_{t \in [1,T]_{\mathbb{Z}}} |a(t)| < 4\sin^2\frac{\pi}{2T}\right\}, \end{gathered} \tag{1.6}$$

and obtains the sign properties of Green's function of (1.4), (1.5).

In Section 3, we obtain the existence of one-signed periodic solutions of the discrete second-order nonlinear periodic boundary value problem

$$\begin{aligned} \Delta^2 y(t-1) &= f(t, y(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ y(0) = y(T), &\qquad \Delta y(0) = \Delta y(T), \end{aligned} \tag{1.7}$$

where $f : [1, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous. For related results on the associated differential equations, see Torres [6].

2. Preliminaries

Let

$$E = \{y \mid y : [0, T+1]_{\mathbb{Z}} \longrightarrow \mathbb{R},\ y(0) = y(T),\ y(1) = y(T+1)\} \tag{2.1}$$

be a Banach space endowed with the norm $\|y\| = \max_{t \in [0,T+1]_{\mathbb{Z}}} |y(t)|$.

We say that the linear boundary value problem (1.4), (1.5) is nonresonant when its unique solution is the trivial one. If (1.4), (1.5) is nonresonant, and let $h : [1,T]_{\mathbb{Z}} \to \mathbb{R}$, by the virtue of the Fredholm's alternative theorem, we can get that the discrete second-order periodic boundary value problem

$$\Delta^2 y(t-1) + a(t)y(t) = h(t), \quad t \in [1,T]_{\mathbb{Z}} \tag{2.2}$$

$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T) \tag{2.3}$$

has a unique solution y,

$$y(t) = \sum_{s=1}^{T} G(t,s)h(s), \quad t \in [0,T+1]_{\mathbb{Z}}, \tag{2.4}$$

where $G(t,s)$ is Green's function related to (1.4), (1.5).

Definition 2.1 (see [7]). We say that a solution y of (1.4) has a generalized zero at t_0 provided that $y(t_0) = 0$ if $t_0 = 0$ and if $t_0 > 0$ either $y(t_0) = 0$ or $y(t_0-1)y(t_0) < 0$.

Theorem 2.2. *Assume that the distance between two consecutive generalized zeros of a nontrivial solution of* (1.4) *is greater than* T. *Then Green's function* $G(t,s)$ *has constant sign.*

Proof. Obviously, G is well defined on $[0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$. We only need to prove that G has no generalized zero in any point. Suppose on the contrary that there exists $(t_0,s_0) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$ such that (t_0,s_0) is a generalized zero of $G(t,s)$. It is well known that for a given $s_0 \in [1,T]_{\mathbb{Z}}$, $G(t,s_0)$ as a function of t is a solution of (1.4) in the intervals $[0,s_0-1]_{\mathbb{Z}}$ and $[s_0+1,T+1]_{\mathbb{Z}}$ such that

$$G(0,s_0) = G(T,s_0), \qquad G(1,s_0) = G(T+1,s_0). \tag{2.5}$$

Case 1 ($G(t_0,s_0) = 0$, $(t_0,s_0) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$). If $t_0 \in [s_0+1,T+1]_{\mathbb{Z}}$, we can construct

$$y(t) = \begin{cases} G(t,s_0), & t \in [s_0,T+1]_{\mathbb{Z}}, \\ G(t-T,s_0), & t \in [T+1,s_0+T]_{\mathbb{Z}}. \end{cases} \tag{2.6}$$

Consequently, y is a solution of (1.4) in the whole interval $[s_0,s_0+T]_{\mathbb{Z}}$. Since $y(t_0) = 0$, we have $\Delta^2 y(t_0-1) = -a(t)y(t_0) = 0$, that is, $y(t_0-1)y(t_0+1) < 0$. Moreover, $y(s_0) = y(s_0+T)$, so there at least exists another generalized zero $t_1 \in [s_0+1,s_0+T]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, which is a contradiction.

Analogously, if $t_0 \in [0,s_0-1]_{\mathbb{Z}}$, we get a contradiction by the same reasoning with

$$y(t) = \begin{cases} G(t+T,s_0), & t \in [s_0-T,0]_{\mathbb{Z}}, \\ G(t,s_0), & t \in [0,s_0]_{\mathbb{Z}}. \end{cases} \tag{2.7}$$

If $t_0 = s_0$, we can apply y as defined (2.7). Since $y(t_0) = y(s_0) = 0$ and $y(s_0-T) = y(s_0)$, which contradicts with the hypothesis.

Case 2 $(G(t_0-1,s_0)G(t_0,s_0)<0,\ (t_0,s_0)\in[1,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}})$. If $t_0\in[s_0+1,T+1]_{\mathbb{Z}}$, we can construct y defined as (2.6). It is not difficult to verify that y is a solution of (1.4) in the whole interval $[s_0,s_0+T]_{\mathbb{Z}}$. Also, we have that $y(t_0-1)y(t_0)<0$, that is, t_0 is a generalized zero of y. Moreover, $y(s_0)=y(s_0+T)$, so there at least exists another generalized zero $t_1\in[s_0+1,s_0+T]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, which is a contradiction.

Similarly, if $t_0\in[1,s_0-1]_{\mathbb{Z}}$, we can get a contradiction by the same reasoning as y defined (2.7).

If $t_0=s_0$, we can construct y defined by (2.7). Since $y(t_0-1)y(t_0)=y(s_0-1)y(s_0)<0$ and $y(s_0-T)=y(s_0)$, it is clear that there exists another generalized zero $t_1\in[s_0-T,s_0]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, this contradicts with the hypothesis. □

To apply the above result, we are going to study the two following cases.

Corollary 2.3. *If $a\in\Lambda^-$, then $G(t,s)<0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.*

Proof. If $a\in\Lambda^-$, by [7, Corollary 6.7], it is easy to verify that (1.4) is disconjugate on $[0,T+1]_{\mathbb{Z}}$, and any nontrivial solution of (1.4) has at most one generalized zero on $[0,T+1]_{\mathbb{Z}}$. Hence, by Theorem 2.2, Green's function $G(t,s)$ has constant sign. We claim that the sign is negative. In fact, $y(t)=\sum_{s=1}^{T}G(t,s)$ is the unique T-periodic solution of the equation

$$\Delta^2 y(t-1)+a(t)y(t)=1, \tag{2.8}$$

and summing both sides of (2.8) from $t=1$ to $t=T$, we can get

$$\sum_{t=1}^{T}a(t)y(t)=T>0. \tag{2.9}$$

Since $a(t)<0,\ y(t)<0$ for some $t\in[1,T]_{\mathbb{Z}}$, and as a consequence $G(t,s)<0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$. □

Remark 2.4. If $a(\cdot)\equiv a_0$ (a_0 is a negative constant), then by computing we can obtain

$$G(t,s)=\begin{cases}-\dfrac{\lambda_1^{t-s}+\lambda_1^{T-t+s}}{(\lambda_1-\lambda_1^{-1})(\lambda_1^T-1)}, & 1\le s\le t\le T+1,\\[2ex] -\dfrac{\lambda_1^{s-t}+\lambda_1^{T-s+t}}{(\lambda_1-\lambda_1^{-1})(\lambda_1^T-1)}, & 0\le t\le s\le T,\end{cases} \tag{2.10}$$

where $\lambda_1=(2-a_0+\sqrt{a_0^2-4a_0})/2>1$. Obviously, $G(t,s)<0,\ (t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.

If $a\ge 0$, then the solutions of (1.4) are oscillating, that is, there are infinite zeros, and to get the required distance between generalized zeros, a should satisfy Λ^+.

Corollary 2.5. *If $a\in\Lambda^+$, then $G(t,s)>0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.*

Proof. We claim that the distance between two consecutive generalized zeros of a nontrivial solution y of (1.4) is strictly greater than T. In fact, it is not hard to verify that

$\Delta^2 y(t-1) + \|a\| y(t) = 0$ is disconjugate on $[0, T+1]_{\mathbb{Z}}$ under assumption $\|a\| < 4\sin^2(\pi/2T)$. Since $a(t) \le \|a\|$, $t \in [1, T]_{\mathbb{Z}}$, by Sturm comparison theorem [7, Theorem 6.19], (1.4) is disconjugate on $[0, T+1]_{\mathbb{Z}}$, that is, any nontrivial solution of (1.4) has at most one generalized zero on $[0, T+1]$.

Hence, by Theorem 2.2, $G(t, s)$ has constant sign on $[0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$, and the positive sign of G is determined as the proof process of Corollary 2.3. □

Remark 2.6. If $a(\cdot) \equiv \overline{a}$ ($\overline{a}$ is apositive constant), and $0 < \overline{a} < 4\sin^2(\pi/2T)$, then by computing we can obtain

$$G(t, s) = \begin{cases} \dfrac{\sin[\theta(t-s)] + \sin[\theta(T-t+s)]}{2\sin\theta(1-\cos(\theta T))}, & 1 \le s \le t \le T+1, \\ \dfrac{\sin[\theta(s-t)] + \sin[\theta(T-s+t)]}{2\sin\theta(1-\cos(\theta T))}, & 0 \le t \le s \le T, \end{cases} \tag{2.11}$$

where $\theta = \arccos((2-\overline{a})/2)$ and $0 < \theta < \pi/T$. Clearly, $G(t, s) > 0$, $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$.

If $a(\cdot) \equiv \overline{a}$ and $\overline{a} = 4\sin^2(\pi/2T)$, then $\theta = \pi/T$, and by computing we get

$$G(t, s) = \begin{cases} \dfrac{1}{2\sin(\pi/T)} \sin\left[\dfrac{\pi}{T}(t-s)\right], & 1 \le s \le t \le T+1, \\ \dfrac{1}{2\sin(\pi/T)} \sin\left[\dfrac{\pi}{T}(s-t)\right], & 0 \le t \le s \le T. \end{cases} \tag{2.12}$$

Obviously, Green's function $G(t, s) = 0$ for $t = s$ and $G(t, s) > 0$ for $t \ne s$.

If $a(\cdot) \equiv \overline{a}$ and $\overline{a} = 4\sin^2(\pi/T)$, then $\theta = 2\pi/T$, and it is not difficult to verify that

$$\varphi(t) = \sin\left(\frac{2\pi}{T}t\right), \qquad \varphi(t) = \cos\left(\frac{2\pi}{T}t\right), \quad t \in [0, T+1]_{\mathbb{Z}} \tag{2.13}$$

are nontrivial solutions of (1.4), (1.5). That is, the problem (1.4), (1.5) has no Green's function.

If $a(\cdot) \equiv \overline{a}$ and $4\sin^2(\pi/2T) < \overline{a} < 4\sin^2(\pi/T)$, then Green's function may change its sign. For example, let $T = 6$, $\overline{a} = 4\sin^2(\pi/8) = 2 - \sqrt{2}$, it is easy to verify that $2 - \sqrt{3} = 4\sin^2(\pi/12) < \overline{a} < 4\sin^2(\pi/6) = 1$ and $\theta = \pi/4$, thus

$$G(t, s) = \begin{cases} \sin\left[\dfrac{\pi}{4}(t-s-1)\right], & 1 \le s \le t \le T+1, \\ \sin\left[\dfrac{\pi}{4}(s-t-1)\right], & 0 \le t \le s \le T. \end{cases} \tag{2.14}$$

Clearly, $G(t, s) = -\sin(\pi/4) < 0$ for $t = s$, $G(t, s) = 0$ for $|t-s| = 1$, and $G(t, s) = \sin(\pi/4) > 0$ for $|t-s| = 2$.

Consequently, $a \in \Lambda^+$ is the optimal condition of $G(t, s) > 0$, $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$.

Next, we provide a way to get the expression of $G(t, s)$. Let u be the unique solution of the initial value problem

$$\Delta^2 u(t-1) + a(t)u(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}, \quad u(0) = 0, \quad \Delta u(0) = 1, \tag{2.15}$$

and v be the unique solution of the initial value problem

$$\Delta^2 v(t-1) + a(t)v(t) = 0, \quad t \in [1,T]_{\mathbb{Z}}, \quad v(T) = 0, \quad \Delta v(T) = -1. \tag{2.16}$$

Lemma 2.7. *Let $a \in \Lambda^- \cup \Lambda^+$. Then Green's function $G(t,s)$ of* (1.4), (1.5) *is explicitly given by*

$$G(t,s) = \frac{[u(s)+v(s)][u(t)+v(t)]}{v(0)[2+v(1)-u(T+1)]} - \frac{1}{v(0)}\begin{cases} u(t)v(s), & 0 \le t \le s \le T, \\ u(s)v(t), & 1 \le s \le t \le T+1. \end{cases} \tag{2.17}$$

Proof. Suppose that Green's function of (1.4), (1.5) is of the form

$$G(t,s) = \left[\alpha(s)u(t) + \beta(s)v(t)\right] - \frac{1}{v(0)}\begin{cases} u(t)v(s), & 0 \le t \le s \le T, \\ u(s)v(t), & 1 \le s \le t \le T+1, \end{cases} \tag{2.18}$$

where $\alpha(s),\ \beta(s)$ can be determined by imposing the boundary conditions.

From the basis theory of Green's function, we know that

$$G(0,s) = G(T,s), \qquad G(1,s) = G(T+1,s), \quad \forall s \in [1,T]_{\mathbb{Z}},\ v(0) = u(T). \tag{2.19}$$

Hence, $\beta(s)v(0) = G(0,s) = G(T,s) = \alpha(s)u(T),\ s \in [1,T]_{\mathbb{Z}}$, combining with $v(0) = u(T)$, we can get

$$\alpha(s) = \beta(s), \quad s \in [1,T]_{\mathbb{Z}}. \tag{2.20}$$

Moreover, since $G(1,s) = G(T+1,s)$, it follows that

$$\alpha(s) = \frac{u(s)+v(s)}{v(0)[2+v(1)-u(T+1)]}. \tag{2.21}$$

□

Note that $\alpha(\cdot)$ has the same sign with $a(\cdot)$. In fact, by the comparison theorem [7, Theorem 6.6], it is easy to prove that $u, v \ge 0$ on $[0,T]_{\mathbb{Z}}$. If $a(t) \ge 0$, then

$$\Delta^2 u(t-1) = -a(t)u(t) \le 0, \qquad \Delta u(t) \le \Delta u(t-1), \quad t \in [1,T]_{\mathbb{Z}}. \tag{2.22}$$

Thus $\Delta u(T) < \Delta u(0) = 1$. Similarly, we can get that $\Delta v(0) > \Delta v(T) = -1$. Since $v(0) = u(T)$, we have

$$2 + v(1) - u(T+1) = 2 + \Delta v(0) - \Delta u(T) > 0. \tag{2.23}$$

That is $\alpha(t) = (u(t)+v(t))/(v(0)[2+v(1)-u(T+1)]) > 0$.

If $a(\cdot) \le 0$, by the similar method, we can prove $\alpha(\cdot) < 0$.

Lemma 2.8. *Let $a \in \Lambda^- \cup \Lambda^+$. Then the periodic boundary value problem* (2.2), (2.3) *has the unique solution*

$$y(t) = \sum_{s=1}^{T} G(t,s)h(s), \quad t \in [0, T+1]_{\mathbb{Z}}, \tag{2.24}$$

where $G(t,s)$ *is defined by* (2.17).

Proof. We check that y satisfies (2.2). In fact,

$$\begin{aligned}
y(t) &= \sum_{s=1}^{T} (u(t)+v(t))\alpha(s)h(s) - \frac{1}{v(0)}\sum_{s=1}^{t-1} u(t)v(s)h(s) - \frac{1}{v(0)}\sum_{s=t}^{T} u(s)v(t)h(s) \\
&= (u(t)+v(t))\sum_{s=1}^{T}\alpha(s)h(s) - \frac{u(t)}{v(0)}\sum_{s=1}^{t-1} v(s)h(s) - \frac{v(t)}{v(0)}\sum_{s=t}^{T} u(s)h(s), \\
y(t+1) &= (u(t+1)+v(t+1))\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{u(t+1)}{v(0)}\sum_{s=1}^{t} v(s)h(s) - \frac{v(t+1)}{v(0)}\sum_{s=t+1}^{T} u(s)h(s), \\
y(t-1) &= (u(t-1)+v(t-1))\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{u(t-1)}{v(0)}\sum_{s=1}^{t-2} v(s)h(s) - \frac{v(t-1)}{v(0)}\sum_{s=t-1}^{T} u(s)h(s), \\
\Delta^2 y(t-1) + a(t)y(t) &= y(t+1) - (2-a(t))y(t) + y(t-1) \\
&= \left[\Delta^2 u(t-1) + a(t)u(t) + \Delta^2 v(t-1) + a(t)v(t)\right]\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{\Delta^2 u(t-1) + a(t)u(t)}{v(0)}\sum_{s=1}^{t-2} v(s)h(s) \\
&\quad - \frac{\Delta^2 v(t-1) + a(t)v(t)}{v(0)}\sum_{s=t+1}^{T} u(s)h(s) \\
&\quad - \frac{u(t-1)h(t-1)}{v(0)}\left[\Delta^2 v(t-1) + a(t)v(t)\right] \\
&\quad - \frac{u(t)h(t)}{v(0)}\left[\Delta^2 v(t-1) + a(t)v(t)\right] \\
&\quad + \frac{u(t)v(t-1)h(t) - u(t-1)v(t)h(t)}{v(0)}
\end{aligned}$$

$$= \frac{h(t)}{v(0)} \begin{vmatrix} u(t) & v(t) \\ u(t-1) & v(t-1) \end{vmatrix} = \frac{h(t)}{v(0)} \begin{vmatrix} u(1) & v(1) \\ u(0) & v(0) \end{vmatrix} = h(t). \tag{2.25}$$

On the other hand, it is easy to verify that $y(0) = y(T)$, $y(1) = y(T+1)$. □

Denote that

$$m = \min_{t,s\in[1,T]_{\mathbb{Z}}} G(t,s), \qquad M = \max_{t,s\in[1,T]_{\mathbb{Z}}} G(t,s). \tag{2.26}$$

As a direct application, we can compute the maximum and the minimum of the Green's function when $a(\cdot) \equiv a_0$, it follows that

$$0 > m \geq -\frac{2\lambda_1^{T/2}}{(\lambda_1 - \lambda_1^{-1})(\lambda_1^T - 1)}, \qquad M = -\frac{\lambda_1^T + 1}{(\lambda_1 - \lambda_1^{-1})(\lambda_1^T - 1)}, \tag{2.27}$$

where λ_1 is defined in Remark 2.4. Similarly, when $0 < a(\cdot) \equiv \overline{a} < 4\sin^2(\pi/2T)$, we can get

$$m = \frac{1}{2\sin\theta}\cot\left(\frac{\theta T}{2}\right) > 0, \qquad M \leq \frac{1}{2\sin\theta\sin(\theta T/2)}, \tag{2.28}$$

where θ is defined in Remark 2.6.

3. Main Results

In this section, we consider the existence of one-signed solutions of (1.7). The following well-known fixed-point theorem in cones is crucial to our arguments.

Theorem 3.1 (see [8]). *Let E be a Banach space and $K \subset E$ be a cone. Suppose Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Assume that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Assume that there exist $a \in \Lambda^+$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \geq 0, \quad \forall y \in \left[\frac{m}{M}r, \frac{M}{m}R\right], \ t \in [1,T]_{\mathbb{Z}}. \tag{3.1}$$

If one of the following conditions holds:

(i)

$$f(t,y)+a(t)y \geq \frac{M}{Tm^2}y, \quad \forall y \in \left[\frac{m}{M}r, r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y)+a(t)y \leq \frac{1}{TM}y, \quad \forall y \in \left[R, \frac{M}{m}R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.2}$$

(ii)

$$f(t,y)+a(t)y \leq \frac{1}{TM}y, \quad \forall y \in \left[\frac{m}{M}r, r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y)+a(t)y \geq \frac{M}{Tm^2}y, \quad \forall y \in \left[R, \frac{M}{m}R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.3}$$

then problem (1.7) *has a positive solution.*

Proof. From Corollary 2.5, we get that $M > m > 0$. It is easy to see that the equation $\Delta^2 y(t-1) = f(t, y(t))$ is equivalent to

$$\Delta^2 y(t-1) + a(t)y(t) = f(t, y(t)) + a(t)y(t). \tag{3.4}$$

Define the open sets

$$\Omega_1 = \{y \in E : \|y\| < r\}, \qquad \Omega_2 = \left\{y \in E : \|y\| < \frac{M}{m}R\right\}, \tag{3.5}$$

and the cone P in E,

$$P = \left\{y \in E : \min_{t \in [0,T+1]_{\mathbb{Z}}} y(t) > \frac{m}{M}\|y\|\right\}. \tag{3.6}$$

Clearly, if $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then $(m/M)r \leq y(t) \leq (M/m)R$, for all $t \in [0, T+1]_{\mathbb{Z}}$.

From Lemma 2.8, we define the operator $A : E \to E$ by

$$(Ay)(t) = \sum_{s=1}^{T} G(t,s)[f(s, y(s)) + a(s)y(s)], \quad t \in [0, T+1]_{\mathbb{Z}}. \tag{3.7}$$

From (3.1), if $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$\begin{aligned} Ay(t) &\geq \frac{m}{M} M \sum_{s=1}^{T} [f(s, y(s)) + a(s)y(s)] \\ &> \frac{m}{M} \max_{t \in [0,T+1]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)[f(s, y(s)) + a(s)y(s)] = \frac{m}{M}\|Ay\|. \end{aligned} \tag{3.8}$$

Thus $A(P\cap(\overline{\Omega}_2\setminus\Omega_1))\subset P$. Moreover, E is a finite space, it is easy to prove that $A: P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ is a completely continuous operator. Clearly, y is the solution of problem (1.7) if and only if y is the fixed point of the operator A.

We only prove (i). (ii) can be obtained by the similar method. If $y\in\partial\Omega_1\cap P$, then $\|y\|=r$ and $(m/M)r\le y(t)\le r$ for all $t\in[0,T+1]_{\mathbb{Z}}$. Therefore, from (i),

$$Ay(t)\ge m\sum_{s=1}^{T}\left[f(s,y(s))+a(s)y(s)\right]\ge\frac{M}{Tm}\sum_{s=1}^{T}y(s)\ge r=\|y\|. \tag{3.9}$$

If $y\in\partial\Omega_2\cap P$, then $\|y\|=(M/m)R$ and $R\le y(t)\le(M/m)R$ for all $t\in[0,T+1]_{\mathbb{Z}}$. As a consequence,

$$Ay(t)\le M\sum_{s=1}^{T}\left[f(s,y(s))+a(s)y(s)\right]\le\frac{1}{T}\sum_{s=1}^{T}y(s)\le\|y\|. \tag{3.10}$$

From Theorem 3.1, A has a fixed point $y\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$ and satisfies

$$\frac{m}{M}r\le y(t)\le\frac{M}{m}R. \tag{3.11}$$

Therefore, y is a positive solution of (1.7). □

Similar to the proof of Theorem 3.2, we can prove the following.

Corollary 3.3. *Assume that there exist $a\in\Lambda^+$ and $0<r<R$ such that*

$$f(t,y)+a(t)y\le 0,\quad \forall y\in\left[-\frac{M}{m}R,-\frac{m}{M}r\right],\ t\in[1,T]_{\mathbb{Z}}. \tag{3.12}$$

If one of the following conditions holds

(i)

$$\begin{aligned}
&f(t,y)+a(t)y\le\frac{M}{Tm^2}y,\quad \forall y\in\left[-r,-\frac{m}{M}r\right],\ t\in[1,T]_{\mathbb{Z}},\\
&f(t,y)+a(t)y\ge\frac{1}{TM}y,\quad \forall y\in\left[-\frac{M}{m}R,-R\right],\ t\in[1,T]_{\mathbb{Z}},
\end{aligned} \tag{3.13}$$

(ii)

$$\begin{aligned}
&f(t,y)+a(t)y\ge\frac{1}{TM}y,\quad \forall y\in\left[-r,-\frac{m}{M}r\right],\ t\in[1,T]_{\mathbb{Z}},\\
&f(t,y)+a(t)y\le\frac{M}{Tm^2}y,\quad \forall y\in\left[-\frac{M}{m}R,-R\right],\ t\in[1,T]_{\mathbb{Z}},
\end{aligned} \tag{3.14}$$

then (1.7) *has a negative solution.*

Applying the sign properties of $G(t,s)$ when $a \in \Lambda^-$ and the similar argument to prove Theorem 3.2 with obvious changes, we can prove the following.

Theorem 3.4. *Assume that there exist $a \in \Lambda^-$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \le 0, \quad \forall y \in \left[\frac{M}{m}r, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}. \tag{3.15}$$

If one of the following conditions holds

(i)

$$\begin{aligned} f(t,y) + a(t)y &\le \frac{m}{TM^2}y, \quad \forall y \in \left[\frac{M}{m}r, r\right],\ t \in [1,T]_{\mathbb{Z}}, \\ f(t,y) + a(t)y &\ge \frac{1}{Tm}y, \quad \forall y \in \left[R, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}, \end{aligned} \tag{3.16}$$

(ii)

$$\begin{aligned} f(t,y) + a(t)y &\ge \frac{1}{Tm}y, \quad \forall y \in \left[\frac{M}{m}r, r\right],\ t \in [1,T]_{\mathbb{Z}}, \\ f(t,y) + a(t)y &\le \frac{m}{TM^2}y, \quad \forall y \in \left[R, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}, \end{aligned} \tag{3.17}$$

then (1.7) *has a positive solution.*

Proof. Since $m < M < 0$, define the open sets

$$\Omega_1 = \{y \in E : \|y\| < r\}, \qquad \Omega_2 = \left\{y \in E : \|y\| < \frac{m}{M}R\right\}, \tag{3.18}$$

and define the cone P in E,

$$P = \left\{y \in E : \min_{t \in [0,T+1]_{\mathbb{Z}}} y(t) > \frac{M}{m}\|y\|\right\}. \tag{3.19}$$

If $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$\frac{M}{m}r \le y(t) \le \frac{m}{M}R, \quad \forall t \in [0,T+1]_{\mathbb{Z}}. \tag{3.20}$$

Define the operator A as (3.7), and the proof is analogous to that of Theorem 3.2 and is omitted. □

Corollary 3.5. *Assume that there exist $a \in \Lambda^-$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \ge 0, \quad \forall y \in \left[-\frac{m}{M}R, -\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}}. \tag{3.21}$$

If one of the following conditions holds

(i)

$$f(t,y)+a(t)y \geq \frac{m}{TM^2}y, \quad \forall y \in \left[-r,-\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y)+a(t)y \leq \frac{1}{Tm}y, \quad \forall y \in \left[-\frac{m}{M}R,-R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.22}$$

(ii)

$$f(t,y)+a(t)y \leq \frac{1}{Tm}y, \quad \forall y \in \left[-r,-\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y)+a(t)y \geq \frac{m}{TM^2}y, \quad \forall y \in \left[-\frac{m}{M}R,-R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.23}$$

then (1.7) *has a negative solution.*

Example 3.6. Let us consider the periodic boundary value problem

$$\Delta^2 y(n-1) = f(n,y(n)), \quad n \in [1,T]_{\mathbb{Z}},$$
$$y(0)=y(T), \qquad \Delta y(0)=\Delta y(T), \tag{3.24}$$

where

$$f(n,y)=\begin{cases} 2n^2+3y, & y \in \left[\dfrac{\sqrt{2}}{2}r,r\right],\ n \in [1,T]_{\mathbb{Z}}, \\ (2n^2+3y)\dfrac{R-y}{R-r}-0.25y\dfrac{y-r}{R-r}, & y \in [r,R],\ n \in [1,T]_{\mathbb{Z}}, \\ -0.25y, & y \in \left[R,\sqrt{2}R\right],\ n \in [1,T]_{\mathbb{Z}}. \end{cases} \tag{3.25}$$

Consider the auxiliary problem

$$\Delta^2 y(n-1)+a(n)y(n) = f(n,y(n))+a(n)y(n), \quad n \in [1,T]_{\mathbb{Z}},$$
$$y(0)=y(T), \qquad \Delta y(0)=\Delta y(T), \tag{3.26}$$

take $r=2\sqrt{2}$, $R=8\sqrt{2}$, $T=3, a(n) \equiv \overline{a} = 2-\sqrt{3} < 4\sin^2(\pi/2T), \theta=\pi/6, \cos\theta = (2-\overline{a})/2=\sqrt{3}/2, \sin\theta=1/2$, $m=(1/2\sin\theta)\cot(\theta T/2)=1$, $M=1/(2\sin\theta\sin(\theta T/2))=\sqrt{2}$. By computing, $f(n,y)+a(n)y \geq 0$, $y \in [(\sqrt{2}/2)r,\sqrt{2}R]$, $n \in [1,T]_{\mathbb{Z}}; f(n,y)+a(n)y \geq (\sqrt{2}/3)y$, $y \in [(\sqrt{2}/2)r,r], n \in [1,T]_{\mathbb{Z}}$; $f(n,y)+a(n)y \leq (1/3\sqrt{2})y, y \in [R,\sqrt{2}R]$, $n \in [1,T]_{\mathbb{Z}}$. Consequently, from Theorem 3.2, the problem (3.24) has a positive solution.

Acknowledgment

This work was supported by the NSFC (no. 11061030) and the Fundamental Research Funds for the Gansu Universities.

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