Abstract and Applied Analysis

Compactness Conditions in the Theory of Nonlinear Differential and Integral Equations

Guest Editors: Józef Banaś, Mohammad Mursaleen, Beata Rzepka, and Kishin Sadarangani

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Editorial

Compactness Conditions in the Theory of Nonlinear Differential and Integral Equations

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The concept of the compactness appears very frequently in explicit or implicit form in many branches of mathematics. Particulary, it plays a fundamental role in mathematical analysis and topology and creates the basis of several investigations conducted in nonlinear analysis and the theories of functional, differential, and integral equations. In view of the wide applicability of the mentioned branches, the concept of the compactness is one of the most useful in several topics of applied mathematics, engineering, mathematical physics, numerical analysis, and so on.

Reasoning based on concepts and tools associated with the concept of the compactness is very often used in fixed point theory and its applications to the theories of functional, differential, and integral equations of various types. Moreover, that concept is also applied in general operator theory. It is worthwhile mentioning several types of operators defined with help of the compactness, both in strong and in weak topology.

This special issue is devoted to discussing some aspects of the above presented topics. We focus here mainly on the presentation of papers being the outcome of research and study associated with existence results concerning nonlinear differential, integral or functional differential, and functional integral equations. We describe roughly the results contained in the papers covering this special issue.

One of those results, contained in the paper of Q. Li and Y. Li, is concerned with the existence of positive periodic solutions of a second-order functional differential equation with multiple delay. The investigations concerning that equation are located in the space of real functions defined, continuous and periodic (with a fixed period) on the real line. The main tool used in the proofs is the fixed point index theory in cones.

The paper of C. Lizama and J. C. Pozo presents also existence results and is devoted to proving the existence of mild solutions of a semilinear integrodifferential equation in a Banach space. The arguments used in the study are based on theory of semigroups and the technique of measures of noncompactness.

The paper by Z. Liu et al., which we are going to describe briefly, discusses existence results for an integral equation of Volterra type. The authors of that paper used some special measure of noncompactness and the fixed point theorem of Darbo type to obtain a result on the existence of asymptotic stable solutions of the equation in question.

A review article concerning infinite systems of differential equations in the so-called BK spaces presents a brief survey of existence results concerning those systems. The authors, M. Mursaleen and A. Alotaibi, present results obtained with help of the technique of measures of noncompactness.

In this special issue two research articles concerning the existence of solutions of nonlinear fractional differential equations with boundary value problems are included. The paper authored by Y. Zhao et al. discusses a result on the existence of positive solutions of a nonlinear fractional integrodifferential equation with multipoint boundary value problem. The investigations are conducted in an ordered

Banach space. The main tools used in argumentations are related to a fixed point theorem for operators which are strictly condensing with respect to the Kuratowski measure of noncompactness. The research article of H. Ergören and A. Kiliçman deals with existence of solutions of impulsive fractional differential equations with closed boundary conditions. The results of that paper are obtained with help of a Burton-Kirk fixed point theorem for the sum of a contraction and a completely continuous operator.

The technique of measures of noncompactness also creates the main tool utilized in the paper of S. Xie devoted to the existence of mild solutions of nonlinear mixed type integrodifferential functional evolution equations with nonlocal conditions. Apart of the existence, the controllability of the mentioned solutions is also shown.

The issue contains also three papers being only loosely related to compactness conditions. The paper by H. Eltayeb and A. Kiliçman discusses approximate solutions of a nonlinear system of partial differential equations with help of Sumudu decomposition technique and with the use of Adomian decomposition method. The second paper of the mentioned kind, written by M. Masjed-Jamei et al., discusses some new estimates for the error of the Simpson integration rule. That study is conducted in the classical Lebesgue spaces $L^1[a,b]$ and $L^\infty[a,b]$. Finally, the paper of A. Kiliçman and R. Abazari is dedicated to discussing exact solutions of the Schrödinger-Boussinesq system. The authors of that paper construct solutions of the system in question in the form of hyperbolic or trigonometric functions.

This issue includes also a review paper by J. Banaś and K. Sadarangani presenting an overview of several methods involving compactness conditions. Those methods are mainly related to some fixed point theorems (Schauder, Krasnosel'skii-Burton, and Schaefer) and to the technique associated with measures of noncompactness. Examples illustrating the applicability of those methods are also included.

Acknowledgments

The guest editors of this special issue would like to express their gratitude to the authors who have submitted papers for consideration. We believe that the results presented in this issue will be a source of inspiration for researchers working in nonlinear analysis and related areas of mathematics.

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Review Article

Compactness Conditions in the Study of Functional, Differential, and Integral Equations

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We discuss some existence results for various types of functional, differential, and integral equations which can be obtained with the help of argumentations based on compactness conditions. We restrict ourselves to some classical compactness conditions appearing in fixed point theorems due to Schauder, Krasnosel'skii-Burton, and Schaefer. We present also the technique associated with measures of noncompactness and we illustrate its applicability in proving the solvability of some functional integral equations. Apart from this, we discuss the application of the mentioned technique to the theory of ordinary differential equations in Banach spaces.

1. Introduction

The concept of the compactness plays a fundamental role in several branches of mathematics such as topology, mathematical analysis, functional analysis, optimization theory, and nonlinear analysis [1–5]. Numerous mathematical reasoning processes depend on the application of the concept of compactness or relative compactness. Let us indicate only such fundamental and classical theorems as the Weierstrass theorem on attaining supremum by a continuous function on a compact set, the Fredholm theory of linear integral equations, and its generalization involving compact operators as well as a lot of fixed point theorems depending on compactness argumentations [6, 7]. It is also worthwhile mentioning such an important property saying that a continuous mapping transforms a compact set onto compact one.

Let us pay a special attention to the fact that several reasoning processes and constructions applied in nonlinear analysis depend on the use of the concept of the compactness [6]. Since theorems and argumentations of nonlinear analysis are used very frequently in the theories of functional, differential, and integral equations, we focus in this paper on the presentation of some results located in these theories which can be obtained with the help of various compactness conditions.

We restrict ourselves to present and describe some results obtained in the last four decades which are related to some problems considered in the theories of differential, integral, and functional integral equations. Several results using compactness conditions were obtained with the help of the theory of measures of noncompactness. Therefore, we devote one section of the paper to present briefly some basic background of that theory.

Nevertheless, there are also successfully used argumentations not depending of the concept of a measure on noncompactness such as Schauder fixed point principle, Krasnosel'skii-Burton fixed point theorem, and Schaefer fixed point theorem.

Let us notice that our presentation is far to be complete. The reader is advised to follow the most expository monographs in which numerous topics connected with compactness conditions are broadly discussed [6, 8–10].

Finally, let us mention that the presented paper has a review form. It discusses some results described in details in the papers which will be cited in due course.

2. Selected Results of Nonlinear Analysis Involving Compactness Conditions

In order to solve an equation having the form

$$x = Fx, \tag{1}$$

where F is an operator being a self-mapping of a Banach space E, we apply frequently an approach through fixed point theorems. Such an approach is rather natural and, in general, very efficient. Obviously, there exists a huge number of miscellaneous fixed point theorems [6, 7, 11] depending both on order, metric, and topological argumentations.

The most efficient and useful theorems seem to be fixed point theorems involving topological argumentations, especially those based on the concept of compactness. The reader can make an acquaintance with the large theory of fixed point-theorems involving compactness conditions in the above, mentioned monographs, but it seems that the most important and expository fixed point theorem in this fashion is the famous Schauder fixed point principle [12]. Obviously, that theorem was generalized in several directions but till now it is very frequently used in application to the theories of differential, integral, and functional equations.

At the beginning of our considerations we recall two well-known versions of the mentioned Schauder fixed point principle (cf. [13]). To this end assume that $(E, \|\cdot\|)$ is a given Banach space.

Theorem 1. *Let Ω be a nonempty, bounded, closed, and convex subset of E and let $F : \Omega \to \Omega$ be a completely continuous operator (i.e., F is continuous and the image $F\Omega$ is relatively compact). Then F has at least one fixed point in the set Ω (this means that the equation $x = Fx$ has at least one solution in the set Ω).*

Theorem 2. *If Ω is a nonempty, convex, and compact subset of E and $F : \Omega \to \Omega$ is continuous on the set Ω, then the operator F has at least one fixed point in the set Ω.*

Observe that Theorem 1 can be treated as a particular case of Theorem 2 if we apply the well-known Mazur theorem asserting that the closed convex hull of a compact subset of a Banach space E is compact [14]. The basic problem arising in applying the Schauder theorem in the version presented in Theorem 2 depends on finding a convex and compact subset of E which is transformed into itself by operator F corresponding to an investigated operator equation.

In numerous situations, we are able to overcome the above-indicated difficulty and to obtain an interesting result on the existence of solutions of the investigated equations (cf. [7, 15–17]). Below we provide an example justifying our opinion [18].

To do this, let us denote by $\mathbb{R}$ the real line and put $\mathbb{R}_+ = [0, +\infty)$. Further, let $\Delta = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t\}$.

Next, fix a function $p = p(t)$ defined and continuous on $\mathbb{R}_+$ with positive real values. Denote by $C_p = C(\mathbb{R}_+, p(t))$ the space consisting of all real functions defined and continuous on $\mathbb{R}_+$ and such that

$$\sup \{|x(t)|\, p(t) : t \ge 0\} < \infty. \tag{2}$$

It can be shown that C_p forms the Banach space with respect to the norm

$$\|x\| = \sup \{|x(t)|\, p(t) : t \ge 0\}. \tag{3}$$

For our further purposes, we recall the following criterion for relative compactness in the space C_p [10, 15].

Theorem 3. *Let X be a bounded set in the space C_p. If all functions belonging to X are locally equicontinuous on the interval $\mathbb{R}_+$ and if $\lim_{T\to\infty}\{\sup\{|x(t)|p(t) : t \ge T\}\} = 0$ uniformly with respect to X, then X is relatively compact in C_p.*

In what follows, if x is an arbitrarily fixed function from the space C_p and if $T > 0$ is a fixed number, we will denote by $v^T(x, \varepsilon)$ the modulus of continuity of x on the interval $[0, T]$; that is,

$$v^T(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \le \varepsilon\}. \tag{4}$$

Further on, we will investigate the solvability of the nonlinear Volterra integral equation with deviated argument having the form

$$x(t) = d(t) + \int_0^t v(t, s, x(\varphi(s)))\, ds, \tag{5}$$

where $t \in \mathbb{R}_+$. Equation (5) will be investigated under the following formulated assumptions.

(i) $v : \Delta \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $n : \Delta \to \mathbb{R}_+$, $a : \mathbb{R}_+ \to (0, \infty)$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|v(t, s, x)| \le n(t, s) + a(t)\, b(s)\, |x| \tag{6}$$

for all $(t, s) \in \Delta$ and $x \in \mathbb{R}$.

In order to formulate other assumptions, let us put

$$L(t) = \int_0^t a(s)\, b(s)\, ds, \quad t \ge 0. \tag{7}$$

Next, take an arbitrary number $M > 0$ and consider the space C_p, where $p(t) = [a(t)\exp(ML(t) + t)]^{-1}$. Then, we can present other assumptions.

(ii) The function $d : \mathbb{R}_+ \to \mathbb{R}$ is continuous and there exists a nonnegative constant D such that $|d(t)| \le Da(t)\exp(ML(t))$ for $t \ge 0$.

(iii) There exists a constant $N \ge 0$ such that

$$\int_0^t n(t, s)\, ds \le Na(t) \exp(ML(t)) \tag{8}$$

for $t \ge 0$.

(iv) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying the condition

$$L(\varphi(t)) - L(t) \leq K \tag{9}$$

for $t \in \mathbb{R}_+$, where $K \geq 0$ is a constant.

(v) $D+N < 1$ and $a(\varphi(t))/a(t) \leq M(1-D-N)\exp(-MK)$ for all $t \geq 0$.

Now, we can formulate the announced result.

Theorem 4. *Under assumptions* (i)–(v), (5) *has at least one solution x in the space C_p such that $|x(t)| \leq a(t)\exp(ML(t))$ for $t \in \mathbb{R}_+$.*

We give the sketch of the proof (cf. [18]). First, let us define the transformation F on the space C_p by putting

$$(Fx)(t) = d(t) + \int_0^t v(t, s, x(\varphi(s)))\,ds, \quad t \geq 0. \tag{10}$$

In view of our assumptions, the function $(Fx)(t)$ is continuous on $\mathbb{R}_+$.

Further, consider the subset G of the space C_p consisting of all functions x such that $|x(t)| \leq a(t)\exp(ML(t))$ for $t \in \mathbb{R}_+$. Obviously, G is nonempty, bounded, closed, and convex in the space C_p. Taking into account our assumptions, for an arbitrary fixed $x \in G$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned}
|(Fx)(t)| \leq\ & |d(t)| + \int_0^t |v(t, s, x(\varphi(s)))|\,ds \\
\leq\ & Da(t)\exp(ML(t)) \\
& + \int_o^t [n(t,s) + a(t)b(s)|x(\varphi(s))|]\,ds \\
\leq\ & Da(t)\exp(ML(t)) + Na(t)\exp(ML(t)) \\
& + a(t)\int_0^t b(s)a(\varphi(s))\exp(ML(\varphi(s)))\,ds \\
\leq\ & (D+N)a(t)\exp(ML(t)) \\
& + (1-D-N)a(t)\int_0^t Ma(s)b(s) \\
& \times \exp(ML(s))\exp(-MK)\exp(MK)\,ds \\
\leq\ & (D+N)a(t)\exp(ML(t)) \\
& + (1-D-N)a(t)\exp(ML(t)) \\
=\ & a(t)\exp(ML(t)).
\end{aligned} \tag{11}$$

This shows that F transforms the set G into itself.

Next we show that F is continuous on the set G.

To this end, fix $\varepsilon > 0$ and take $x, y \in G$ such that $\|x - y\| \leq \varepsilon$. Next, choose arbitrary $T > 0$. Using the fact that the function $v(t,s,x)$ is uniformly continuous on the set $[0,T]^2 \times [-\alpha(T), \alpha(T)]$, where $\alpha(T) = \max\{a(\varphi(t))\exp(ML(\varphi(t))) : t \in [0,T]\}$, for $t \in [0,T]$, we obtain

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\quad \leq \int_0^t |v(t,s,x(\varphi(s))) - v(t,s,y(\varphi(s)))|\,ds \\
&\quad \leq \beta(\varepsilon),
\end{aligned} \tag{12}$$

where $\beta(\varepsilon)$ is a continuous function with the property $\lim_{\varepsilon\to 0}\beta(\varepsilon) = 0$.

Now, take $t \geq T$. Then we get

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)|\,[a(t)\exp(ML(t)+t)]^{-1} \\
&\quad \leq \{|(Fx)(t)| + |(Fy)(t)|\} \\
&\quad \times [a(t)\exp(ML(t))]^{-1} \cdot e^{-t} \leq 2e^{-t}.
\end{aligned} \tag{13}$$

Hence, for T sufficiently large, we have

$$|(Fx)(t) - (Fy)(t)|\,p(t) \leq \varepsilon \tag{14}$$

for $t \geq T$. Linking (12) and (14), we deduce that F is continuous on the set G.

The next essential step in our proof which enables us to apply the Schauder fixed point theorem (Theorem 1) is to show that the set FG is relatively compact in the space C_p. To this end let us first observe that the inclusion $FG \subset G$ and the description of G imply the following estimate:

$$|(Fx)(t)|\,p(t) \leq e^{-t}. \tag{15}$$

This yields that

$$\lim_{T\to\infty}\{\sup\{|(Fx)(t)|\,p(t) : t \geq T\}\} = 0 \tag{16}$$

uniformly with respect to the set G.

On the other hand, for fixed $\varepsilon > 0$, $T > 0$ and for $t, s \in [0,T]$ such that $|t-s| \leq \varepsilon$, in view of our assumptions, for $x \in G$, we derive the following estimate:

$$\begin{aligned}
&|(Fx)(t) - (Fx)(s)| \\
&\quad \leq |d(t) - d(s)| \\
&\quad + \left|\int_0^t v(t,\tau,x(\varphi(\tau)))\,d\tau - \int_0^s v(t,\tau,x(\varphi(\tau)))\,d\tau\right| \\
&\quad + \left|\int_0^s v(t,\tau,x(\varphi(\tau)))\,d\tau - \int_0^s v(s,\tau,x(\varphi(\tau)))\,d\tau\right| \\
&\quad \leq \nu^T(d,\varepsilon) + \varepsilon\max\{n(t,\tau) + a(\tau)b(\tau) \\
&\qquad \times p(\varphi(\tau)) : 0 \leq \tau \leq t \leq T\} \\
&\quad + T\nu^T(v(\varepsilon, T, \alpha(T))),
\end{aligned} \tag{17}$$

where we denoted

$$\begin{aligned}
&\nu^T(v(\varepsilon,T,\alpha(T))) \\
&\quad = \sup\{|v(t,u,v) - v(s,u,v)| : t,s \in [0,T], \\
&\qquad |t-s| \leq \varepsilon, u \in [0,T], |v| \leq \alpha(T)\}.
\end{aligned} \tag{18}$$

Taking into account the fact that $\lim_{\varepsilon \to 0} \nu^T(d, \varepsilon) = \lim_{\varepsilon \to 0} \nu^T(\upsilon(\varepsilon, T, \alpha(T))) = 0$, we infer that functions belonging to the set FG are equicontinuous on each interval $[0, T]$. Combining this fact with (16), in view of Theorem 3, we conclude that the set FG is relatively compact. Applying Theorem 1, we complete the proof.

Another very useful fixed point theorem using the compactness conditions is the well-known Krasnosel'skii fixed point theorem [19]. That theorem was frequently modified by researchers working in the fixed point theory (cf. [6, 7, 20]), but it seems that the version due to Burton [21] is the most appropriate to be used in applications.

Below we formulate that version.

Theorem 5. *Let S be a nonempty, closed, convex, and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is a contraction, that is, there exists a constant $k \in [0, 1)$ such that $\|Ax - Ay\| \le k\|x - y\|$ for $x, y \in X$,*

(b) *B is completely continuous,*

(c) *$x = Ax + By \Rightarrow x \in S$ for all $y \in S$.*

Then the equation $Ax + Bx = x$ has a solution in S.

In order to show the applicability of Theorem 5, we will consider the following nonlinear functional integral equation [22]:

$$x(t) = f(t, x(\alpha_1(t)), \dots, x(\alpha_n(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))\, ds, \tag{19}$$

where $t \in \mathbb{R}_+$. Here we assume that f and g are given functions.

The above equation will be studied in the space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous, and bounded on the interval $\mathbb{R}_+$ and equipped with the usual supremum norm

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}. \tag{20}$$

Observe that the space $BC(\mathbb{R}_+)$ is a special case of the previously considered space C_p with $p(t) = 1$ for $t \in \mathbb{R}_+$. This fact enables us to adapt the relative compactness criterion contained in Theorem 3 for our further purposes.

In what follows we will impose the following requirements concerning the components involved in (19):

(i) The function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is continuous and there exist constants $k_i \in [0, 1)$ $(i = 1, 2, \dots, n)$ such that

$$|f(t, x_1, x_2, \dots, x_n) - f(t, y_1, y_2, \dots, y_n)| \le \sum_{i=1}^{n} k_i |x_i - y_i| \tag{21}$$

for all $t \in \mathbb{R}_+$ and for all $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(ii) The function $t \to f(t, 0, \dots, 0)$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup\{|f(t, 0, \dots, 0)| : t \in \mathbb{R}_+\}$.

(iii) The functions $\alpha_i, \gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\alpha_i(t) \to \infty$ as $t \to \infty$ $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$.

(iv) The function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

(v) The function $g : \mathbb{R}_+^2 \times \mathbb{R}^m \to \mathbb{R}$ is continuous and there exist continuous functions $q : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x_1, \dots, x_m)| \le q(t, s) + a(t)\, b(s) \sum_{i=1}^{m} |x_i| \tag{22}$$

for all $t, s \in \mathbb{R}_+$ and $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Moreover, we assume that

$$\lim_{t \to \infty} \int_0^{\beta(t)} q(t, s)\, ds = 0, \qquad \lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s)\, ds = 0. \tag{23}$$

Now, let us observe that based on assumption (v), we conclude that the functions $v_1, v_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$v_1(t) = \int_0^{\beta(t)} q(t, s)\, ds, \qquad v_2(t) = a(t) \int_0^{\beta(t)} b(s)\, ds \tag{24}$$

are continuous and bounded on $\mathbb{R}_+$. Obviously this implies that the constants M_1, M_2 defined as:

$$M_i = \sup\{v_i(t) : t \in \mathbb{R}_+\} \quad (i = 1, 2) \tag{25}$$

are finite.

In order to formulate our last assumption, let us denote $k = \sum_{i=1}^{n} k_i$, where the constants k_i $(i = 1, 2, \dots, n)$ appear in assumption (i).

(vi) $k + mM_2 < 1$.

Then we have the following result [22] which was announced above.

Theorem 6. *Under assumptions* (i)–(vi), (19) *has at least one solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of* (19) *are globally attractive.*

Remark 7. In order to recall the concept of the global attractivity mentioned in the above theorem (cf. [22]), suppose that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$ and $Q : \Omega \to BC(\mathbb{R}_+)$ is an operator. Consider the operator equation

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{26}$$

We say that solutions of (26) are *globally attractive* if for arbitrary solutions x, y of this equation, we have that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \tag{27}$$

Let us mention that the above-defined concept was introduced in [23, 24].

Proof of Theorem 6. We provide only the sketch of the proof. Consider the ball B_r in the space $X = BC(\mathbb{R}_+)$ centered at the zero function θ and with radius $r = (F_0+M_1)/[1-(k+mM_2)]$. Next, define two mappings $A : X \to X$ and $B : B_r \to X$ by putting

$$(Ax)(t) = f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))),$$
$$(Bx)(t) = \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), \ldots, x(\gamma_m(s)))\, ds, \tag{28}$$

for $t \in \mathbb{R}_+$. Then (19) can be written in the form

$$x(t) = (Ax)(t) + (Bx)(t), \quad t \in \mathbb{R}_+. \tag{29}$$

Notice that in view of assumptions (i)–(iii), the mapping A is well defined, and for arbitrarily fixed function $x \in X$, the function Ax is continuous and bounded on $\mathbb{R}_+$. Thus A transforms X into itself. Similarly, applying assumptions (iii)–(v), we deduce that the function Bx is continuous and bounded on $\mathbb{R}_+$. This means that B transforms the ball B_r into X.

Now, we check that operators A and B satisfy assumptions imposed in Theorem 5. To this end take $x, y \in X$. Then, in view of assumption (i), for a fixed $t \in \mathbb{R}_+$, we get

$$\begin{aligned}|(Ax)(t) - (Ay)(t)| &\le \sum_{i=1}^{n} k_i |x(\alpha_i(t)) - y(\alpha_i(t))| \\ &\le k\|x - y\|.\end{aligned} \tag{30}$$

This implies that $\|Ax - Ay\| \le k\|x - y\|$, and in view of assumption (vi), we infer that A is a contraction on X.

Next, we prove that B is completely continuous on the ball B_r. In order to show the indicated property of B, we fix $\varepsilon > 0$ and we take $x, y \in B_r$ with $\|x - y\| \le \varepsilon$. Then, taking into account our assumptions, we obtain

$$\begin{aligned}&|(Bx)(t) - (By)(t)| \\ &\quad\le \int_0^{\beta(t)} [|g(t, s, x(\gamma_1(s)), \ldots, x(\gamma_m(s)))| \\ &\qquad + |g(t, s, y(\gamma_1(s)), \ldots, y(\gamma_m(s)))|]\, ds \\ &\quad\le 2(v_1(t) + rmv_2(t)).\end{aligned} \tag{31}$$

Hence, keeping in mind assumption (v), we deduce that there exists $T > 0$ such that $v_1(t) + rmv_2(t) \le \varepsilon/2$ for $t \ge T$. Combining this fact with (31), we get

$$|(Bx)(t) - (By)(t)| \le \varepsilon \tag{32}$$

for $t \ge T$.

Further, for an arbitrary $t \in [0, T]$, in the similar way, we obtain

$$|(Bx)(t) - (By)(t)| \le \int_0^{\beta(t)} \omega_r^T(g, \varepsilon)\, ds \le \beta_T \omega_r^T(g, \varepsilon), \tag{33}$$

where we denoted $\beta_T = \sup\{\beta(t) : t \in [0, T]\}$ and

$$\begin{aligned}\omega_r^T(g, \varepsilon) = \sup\{&|g(t, s, x_1, x_2, \ldots, x_m) \\ &- g(t, s, y_1, y_2, \ldots, y_m)| : t \in [0, T], \\ &s \in [0, \beta_T], x_i, y_i \in [-r, r], \\ &|x_i - y_i| \le \varepsilon\ (i = 1, 2, \ldots, m)\}.\end{aligned} \tag{34}$$

Keeping in mind the uniform continuity of the function g on the set $[0, T] \times [0, \beta_T] \times [-r, r]^m$, we deduce that $\omega_r^T(g, \varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, in view of (32) and (33), we conclude that the operator B is continuous on the ball B_r.

The boundedness of the operator B is a consequence of the inequality

$$|(Bx)(t)| \le v_1(t) + rmv_2(t) \tag{35}$$

for $t \in \mathbb{R}_+$. To verify that the operator B satisfies assumptions of Theorem 3 adapted to the case of the space $BC(\mathbb{R}_+)$, fix arbitrarily $\varepsilon > 0$. In view of assumption (v), we can choose $T > 0$ such that $v_1(t) + rmv_2(t) \le \varepsilon$ for $t \ge T$. Further, take an arbitrary function $x \in B_r$. Then, keeping in mind (35), for $t \ge T$, we infer that

$$|(Bx)(t)| \le \varepsilon. \tag{36}$$

Next, take arbitrary numbers $t, \tau \in [0, T]$ with $|t - \tau| \le \varepsilon$. Then we obtain the following estimate:

$$\begin{aligned}&|(Bx)(t) - (Bx)(\tau)| \\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} |g(t, s, x(\gamma_1(s)), \ldots, x(\gamma_m(s)))|\, ds\right| \\ &\qquad + \int_0^{\beta(\tau)} |g(t, s, x(\gamma_1(s)), \ldots, x(\gamma_m(s))) \\ &\qquad\qquad - g(\tau, s, x(\gamma_1(s)), \ldots, x(\gamma_m(s)))|\, ds \\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} \left[q(t, s) + a(t) b(s) \sum_{i=1}^{m} |x(\gamma_i(s))|\right] ds\right| \\ &\qquad + \int_0^{\beta(\tau)} \omega_1^T(g, \varepsilon; r)\, ds \\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} q(t, s)\, ds\right| \\ &\qquad + a(t) mr \left|\int_{\beta(\tau)}^{\beta(t)} b(s)\, ds\right| + \beta_T \omega_1^T(g, \varepsilon; r),\end{aligned} \tag{37}$$

where we denoted

$$\begin{aligned}\omega_1^T(g, \varepsilon; r) = \sup\{&|g(t, s, x_1, \ldots, x_m) \\ &- g(\tau, s, x_1, \ldots, x_m)| : t, \tau \in [0, T], \\ &|t - s| \le \varepsilon, s \in [0, \beta_T], \\ &|x_1| \le r, \ldots, |x_m| \le r\}.\end{aligned} \tag{38}$$

From estimate (37), we get

$$|(Bx)(t) - (Bx)(\tau)| \leq q_T v^T(\beta, \varepsilon) + rma_T b_T v^T(\beta, \varepsilon) + \beta_T \omega_1^T(g, \varepsilon; r), \tag{39}$$

where $q_T = \max\{q(t,s) : t \in [0,T], s \in [0,\beta_T]\}$, $a_T = \max\{a(t) : t \in [0,T]\}$, $b_T = \max\{b(t) : t \in [0,\beta_T]\}$, and $v^T(\beta,\varepsilon)$ denotes the usual modulus of continuity of the function β on the interval $[0,T]$.

Now, let us observe that in view of the standard properties of the functions $\beta = \beta(t)$, $g = g(t,s,x_1,\ldots,x_m)$, we infer that $v^T(\beta,\varepsilon) \to 0$ and $\omega_1^T(g,\varepsilon;r) \to 0$ as $\varepsilon \to 0$. Hence, taking into account the boundedness of the image BB_r and estimates (36) and (39), in view of Theorem 3, we conclude that the set BB_r is relatively compact in the space $BC(\mathbb{R}_+)$; that is, B is completely continuous on the ball B_r.

In what follows fix arbitrary $x \in BC(\mathbb{R}_+)$ and assume that the equality $x = Ax + By$ holds for some $y \in B_r$. Then, utilizing our assumptions, for a fixed $t \in \mathbb{R}_+$, we get

$$\begin{aligned} |x(t)| &\leq |(Ax)(t)| + |(By)(t)| \\ &\leq |f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) - f(t, 0, \ldots, 0)| \\ &\quad + |f(t, 0, \ldots, 0)| \\ &\quad + \int_0^{\beta(t)} q(t,s)\, ds + m\|y\| a(t) \int_0^{\beta(t)} b(s)\, ds \\ &\leq k\|x\| + F_0 + M_1 + mrM_2. \end{aligned} \tag{40}$$

Hence we obtain

$$\|x\| \leq \frac{F_0 + M_1 + mrM_2}{1-k}. \tag{41}$$

On the other hand, we have that $r = (F_0 + M_1 + mrM_2)/(1-k)$. Thus $\|x\| \leq r$ or, equivalently, $x \in B_r$. This shows that assumption (c) of Theorem 5 is satisfied.

Finally, combining all of the above-established facts and applying Theorem 5, we infer that there exists at least one solution $x = x(t)$ of (19).

The proof of the global attractivity of solutions of (19) is a consequence of the estimate

$$\begin{aligned} |x(t) - y(t)| &\leq \sum_{i=1}^{n} k_i \max\{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1,2,\ldots,n\} \\ &\quad + 2\int_0^{\beta(t)} q(t,s)\, ds \\ &\quad + a(t) \int_0^{\beta(t)} \left(b(s) \sum_{i=1}^{m} |x(\gamma_i(s))| \right) ds \\ &\quad + a(t) \int_0^{\beta(t)} \left(b(s) \sum_{i=1}^{m} |y(\gamma_i(s))| \right) ds \\ &\leq k \max\{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1,2,\ldots,n\} \\ &\quad + 2v_1(t) + m(\|x\| + \|y\|) v_2(t), \end{aligned} \tag{42}$$

which is satisfied for arbitrary solutions $x = x(t)$, $y = y(t)$ of (19).

Hence we get

$$\begin{aligned} \limsup_{t\to\infty} |x(t) - y(t)| &\leq k \max_{1\leq i\leq n} \left\{ \limsup_{t\to\infty} |x(\alpha_i(t) - y(\alpha_i(t)))| \right\} \\ &\quad + 2\limsup_{t\to\infty} v_1(t) + m(\|x\| + \|y\|) \limsup_{t\to\infty} v_2(t) \\ &= k \limsup_{t\to\infty} |x(t) - y(t)| \end{aligned} \tag{43}$$

which implies that $\limsup_{t\to\infty}|x(t) - y(t)| = \lim_{t\to\infty}|x(t) - y(t)| = 0$. This means that the solutions of (19) are globally attractive (cf. Remark 7). □

It is worthwhile mentioning that in the literature one can encounter other formulations of the Krasnosel'skii-Burton fixed point theorem (cf. [6, 7, 20, 21]). In some of those formulations and generalizations, there is used the concept of a measure of noncompactness (both in strong and in weak sense) and, simultaneously, the requirement of continuity is replaced by the assumption of weak continuity or weak sequential continuity of operators involved (cf. [6, 20], for instance).

In what follows we pay our attention to another fixed point theorem which uses the compactness argumentation. Namely, that theorem was obtained by Schaefer in [25].

Subsequently that theorem was formulated in other ways and we are going here to present two versions of that theorem (cf. [11, 26]).

Theorem 8. *Let $(E, \|\cdot\|)$ be a normed space and let $T : E \to E$ be a continuous mapping which transforms bounded subsets of E onto relatively compact ones. Then either*

(i) *the equation $x = Tx$ has a solution*

or

(ii) *the set $\bigcup_{0\leq\lambda\leq1}\{x \in E : x = \lambda Tx\}$ is unbounded.*

The below presented version of Schaefer fixed point theorem seems to be more convenient in applications.

Theorem 9. *Let $E, \|\cdot\|$ be a Banach space and let $T : E \to E$ be a continuous compact mapping (i.e., T is continuous and T maps bounded subsets of E onto relatively compact ones). Moreover, one assumes that the set*

$$\bigcup_{0\leq\lambda\leq1} \{x \in E : x = \lambda Tx\} \tag{44}$$

is bounded. Then T has a fixed point.

It is easily seen that Theorem 9 is a particular case of Theorem 8.

Observe additionally, that Schaefer fixed point theorem seems to be less convenient in applications than Schauder fixed point theorem (cf. Theorems 1 and 2). Indeed, Schaefer theorem requires a priori bound on utterly unknown solutions of the operator equation $x = \lambda Tx$ for $\lambda \in [0,1]$. On the other hand, the proof of Schaefer theorem requires the use of the Schauder fixed point principle (cf. [27], for details).

It is worthwhile mentioning that an interesting result on the existence of periodic solutions of an integral equation, based on a generalization of Schaefer fixed point theorem, may be found in [26].

3. Measures of Noncompactness and Their Applications

Let us observe that in order to apply the fundamental fixed point theorem based on compactness conditions, that is, the Schauder fixed point theorem, say, the version of Theorem 2, we are forced to find a convex and compact subset of a Banach space E which is transformed into itself by an operator F. In general, it is a hard task to encounter a set of such a type [16, 28]. On the other hand, if we apply Schauder fixed point theorem formulated as Theorem 1, we have to prove that an operator F is completely continuous. This causes, in general, that we have to impose rather strong assumptions on terms involved in a considered operator equation.

In view of the above-mentioned difficulties starting from seventies of the past century mathematicians working on fixed point theory created the concept of the so-called measure of noncompactness which allowed to overcome the above-indicated troubles. Moreover, it turned out that the use of the technique of measures of noncompactness allows us also to obtain a certain characterization of solutions of the investigated operator equations (functional, differential, integral, etc.). Such a characterization is possible provided we use the concept of a measure of noncompactness defined in an appropriate axiomatic way.

It is worthwhile noticing that up to now there have appeared a lot of various axiomatic definitions of the concept of a measure of noncompactness. Some of those definitions are very general and not convenient in practice. More precisely, in order to apply such a definition, we are often forced to impose some additional conditions on a measure of noncompactness involved (cf. [8, 29]).

By these reasons it seems that the axiomatic definition of the concept of a measure of noncompactness should be not very general and should require satisfying such conditions which enable the convenience of their use in concrete applications.

Below we present axiomatics which seems to satisfy the above-indicated requirements. That axiomatics was introduced by Banaś and Goebel in 1980 [10].

In order to recall that axiomatics, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of a Banach space E and by $\mathfrak{N}_E$ its subfamily consisting of relatively compact sets. Moreover, let $B(x,r)$ stand for the ball with the center at x and with radius r. We write B_r to denote the ball $B(\theta,r)$, where θ is the zero element in E. If X is a subset of E, we write $\overline{X}$, $\operatorname{Conv} X$ to denote the closure and the convex closure of X, respectively. The standard algebraic operations on sets will be denoted by $X+Y$ and λX, for $\lambda \in \mathbb{R}$.

As we announced above, we accept the following definition of the concept of a measure of noncompactness [10].

Definition 10. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a *measure of noncompactness* in the space E if the following conditions are satisfied.

(1^o) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}$.

(2^o) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

(3^o) $\mu(\overline{X}) = \mu(\operatorname{Conv}X) = \mu(X)$.

(4^o) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(X)$ for $\lambda \in [0,1]$.

(5^o) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let us pay attention to the fact that from axiom (5^o) we infer that $\mu(X_\infty) \le \mu(X_n)$ for any $n = 1,2,\ldots$. This implies that $\mu(X_\infty) = 0$. Thus X_∞ belongs to the family $\ker \mu$ described in axiom (1^o). The family $\ker \mu$ is called the *kernel of the measure of noncompactness* μ.

The property of the measure of noncompactness μ mentioned above plays a very important role in applications.

With the help of the concept of a measure of noncompactness, we can formulate the following useful fixed point theorem [10] which is called the fixed point theorem of Darbo type.

Theorem 11. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous operator which is a contraction with respect to a measure of noncompactness μ; that is, there exists a constant k, $k \in [0,1)$, such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of the set Ω. Then the operator F has at least one fixed point in the set Ω.*

In the sequel we show an example of the applicability of the technique of measures of noncompactness expressed by Theorem 11 in proving the existence of solutions of the operator equations.

Namely, we will work on the Banach space $C[a,b]$ consisting of real functions defined and continuous on the interval $[a,b]$ and equipped with the standard maximum norm. For sake of simplicity, we will assume that $[a,b] = [0,1] = I$, so the space on question can be denoted by $C(I)$.

One of the most important and handy measures of noncompactness in the space $C(I)$ can be defined in the way presented below [10].

In order to present this definition, take an arbitrary set $X \in \mathfrak{M}_{C(I)}$. For $x \in X$ and for a given $\varepsilon > 0$, let us put

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in I, |t-s| \le \varepsilon\}. \quad (45)$$

Next, let us define

$$\omega(X,\varepsilon)=\sup\{\omega(x,\varepsilon):x\in X\},$$
$$\omega_0(X)=\lim_{\varepsilon\to 0}\omega(X,\varepsilon). \tag{46}$$

It may be shown that the function $\omega_0(X)$ is the measure of noncompactness in the space $C(I)$ (cf. [10]). This measure has also some additional properties. For example, $\omega_0(\lambda X)=|\lambda|\omega_0(X)$ and $\omega_0(X+Y)\le\omega_0(X)+\omega_0(Y)$ provided $X,Y\in\mathfrak{M}_{C(I)}$ and $\lambda\in\mathbb{R}$ [10].

In what follows we will consider the nonlinear Volterra singular integral equation having the form

$$x(t)=f_1(t,x(t),x(a(t)))+(Gx)(t)\int_0^t f_2(t,s)(Qx)(s)\,ds, \tag{47}$$

where $t\in I=[0,1]$, $a:I\to I$ is a continuous function, and G, Q are operators acting continuously from the space $C(I)$ into itself. Apart from this, we assume that the function f_2 has the form

$$f_2(t,s)=k(t,s)\,g(t,s), \tag{48}$$

where $k:\Delta\to\mathbb{R}$ is continuous and g is monotonic with respect to the first variable and may be discontinuous on the triangle $\Delta=\{(t,s):0\le s\le t\le 1\}$.

Equation (47) will be considered in the space $C(I)$ under the following assumptions (cf. [30]).

(i) $a:I\to I$ is a continuous function.

(ii) The function $f_1:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a nonnegative constant p such that

$$\left|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\right|\le p\max\{|x_1-x_2|,|y_1-y_2|\} \tag{49}$$

for any $t\in I$ and for all $x_1,x_2,y_1,y_2\in\mathbb{R}$.

(iii) The operator G transforms continuously the space $C(I)$ into itself and there exists a nonnegative constant q such that $\omega_0(GX)\le q\omega_0(X)$ for any set $X\in\mathfrak{M}_{C(I)}$, where ω_0 is the measure of noncompactness defined by (46).

(iv) There exists a nondecreasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|Gx\|\le\varphi(\|x\|)$ for any $x\in C(I)$.

(v) The operator Q acts continuously from the space $C(I)$ into itself and there exists a nondecreasing function $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|Qx\|\le\Psi(\|x\|)$ for any $x\in C(I)$.

(vi) $f_2:\Delta\to\mathbb{R}$ has the form (48), where the function $k:\Delta\to\mathbb{R}$ is continuous.

(vii) The function $g(t,s)=g:\Delta\to\mathbb{R}_+$ occurring in the decomposition (48) is monotonic with respect to t (on the interval $[s,1]$), and for any fixed $t\in I$, the function $s\to g(t,s)$ is Lebesgue integrable over the interval $[0,t]$. Moreover, for every $\varepsilon>0$, there exists $\delta>0$ such that for all $t_1,t_2\in I$ with $t_1<t_2$ and $t_2-t_1\le\delta$ the following inequalities are satisfied:

$$\left|\int_0^{t_1}[g(t_2,s)-g(t_1,s)]\,ds\right|\le\varepsilon,$$
$$\int_{t_1}^{t_2}g(t_2,s)\,ds\le\varepsilon. \tag{50}$$

The main result concerning (47), which we are going to present now, will be preceded by a few remarks and lemmas (cf. [30]). In order to present these remarks and lemmas, let us consider the function $h:I\to\mathbb{R}_+$ defined by the formula

$$h(t)=\int_0^t g(t,s)\,ds. \tag{51}$$

In view of assumption (vii), this function is well defined.

Lemma 12. *Under assumption (vii), the function h is continuous on the interval I.*

For proof, we refer to [30].

In order to present the last assumptions needed further on, let us define the constants $\overline{k}$, $\overline{f_1}$, $\overline{h}$ by putting

$$\overline{k}=\sup\{|k(t,s)|:(t,s)\in\Delta\},$$
$$\overline{f_1}=\sup\{|f_1(t,0,0)|:t\in I\}, \tag{52}$$
$$\overline{h}=\sup\{h(t):t\in I\}.$$

The constants $\overline{k}$ and $\overline{f_1}$ are finite in view of assumptions (vi) and (ii), while the fact that $\overline{h}<\infty$ is a consequence of assumption (vii) and Lemma 12.

Now, we formulate the announced assumption.

(viii) There exists a positive solution r_0 of the inequality

$$pr+\overline{f_1}+\overline{kh}\varphi(r)\Psi(r)\le r \tag{53}$$

such that $p+\overline{kh}q\Psi(r_0)<1$.

For further purposes, we definite operators corresponding to (47) and defined on the space $C(I)$ in the following way:

$$(F_1x)(t)=f_1(t,x(t),x(\alpha(t))),$$
$$(F_2x)(t)=\int_0^t f_2(t,s)(Qx)(s)\,ds, \tag{54}$$
$$(Fx)(t)=(F_1x)(t)+(Gx)(t)(F_2x)(t),$$

for $t \in I$. Apart from this, we introduce two functions M and N defined on $\mathbb{R}_+$ by the formulas

$$M(\varepsilon) = \sup\left\{\left|\int_0^{t_1}\left[g(t_2,s)-g(t_1,g)\right]ds\right| : t_1,t_2 \in I, t_1 < t_2, t_2 - t_1 \le \varepsilon\right\}, \tag{55}$$

$$N(\varepsilon) = \sup\left\{\int_{t_1}^{t_2} g(t_2,s)\,ds : t_1,t_2 \in I, t_1 < t_2, t_2 - t_1 \le \varepsilon\right\}. \tag{56}$$

Notice that in view of assumption (vii), we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, we can state the following result.

Lemma 13. *Under assumptions* (i)–(vii), *the operator F transforms continuously the space $C(I)$ into itself.*

Proof. Fix a function $x \in C(I)$. Then $F_1 x \in C(I)$, which is a consequence of the properties of the so-called superposition operator [9]. Further, for arbitrary functions $x, y \in C(I)$, in virtue of assumption (ii), for a fixed $t \in I$, we obtain

$$\begin{aligned}&\left|(F_1x)(t)-(F_1y)(t)\right| \\ &\quad\le p\max\left\{\left|x(t)-y(t)\right|,\left|x(a(t))-y(a(t))\right|\right\}.\end{aligned} \tag{57}$$

This estimate in combination with assumption (i) yields

$$\left\|F_1x - F_1y\right\| \le p\left\|x-y\right\|. \tag{58}$$

Hence we conclude that F_1 acts continuously from the space $C(I)$ into itself.

Next, fix $x \in C(I)$ and $\varepsilon > 0$. Take $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality, we may assume that $t_1 \le t_2$. Then, based on the imposed assumptions, we derive the following estimate:

$$\begin{aligned}&\left|(F_2x)(t_2)-(F_2x)(t_1)\right| \\ &\quad\le \left|\int_0^{t_2} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds - \int_0^{t_1} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds\right| \\ &\qquad+ \left|\int_0^{t_1} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds - \int_0^{t_1} k(t_1,s)\,g(t_2,s)(Qx)(s)\,ds\right| \\ &\qquad+ \left|\int_0^{t_1} k(t_1,s)\,g(t_2,s)(Qx)(s)\,ds - \int_0^{t_1} k(t_1,s)\,g(t_1,s)(Qx)(s)\,ds\right| \\ &\quad\le \int_{t_1}^{t_2}\left|k(t_2,s)\right| g(t_2,s)\left|(Qx)(s)\right| ds \\ &\qquad+ \int_0^{t_1}\left|k(t_2,s)-k(t_1,s)\right| g(t_2,s)\left|(Qx)(s)\right| ds \\ &\qquad+ \int_0^{t_1}\left|k(t_1,s)\right|\left|g(t_2,s)-g(t_1,s)\right|\left|(Qx)(s)\right| ds \\ &\quad\le \bar{k}\Psi(\|x\|)N(\varepsilon) + \omega_1(k,\varepsilon)\Psi(\|x\|)\int_0^{t_2} g(t_2,s)\,ds \\ &\qquad+ \bar{k}\Psi(\|x\|)\int_0^{t_1}\left|g(t_2,s)-g(t_1,s)\right| ds,\end{aligned} \tag{59}$$

where we denoted

$$\begin{aligned}\omega_1(k,\varepsilon) = \sup\{&\left|k(t_2,s)-k(t_1,s)\right| : \\ &(t_1,s),(t_2,s) \in \Delta, \left|t_2-t_1\right| \le \varepsilon\}.\end{aligned} \tag{60}$$

Since the function $t \to g(t,s)$ is assumed to be monotonic, by assumption (vii) we get

$$\int_0^{t_1}\left|g(t_2,s)-g(t_1,s)\right| ds = \left|\int_0^{t_1}\left[g(t_2,s)-g(t_1,s)\right] ds\right|. \tag{61}$$

This fact in conjunction with the above-obtained estimate yields

$$\begin{aligned}\omega(F_2x,\varepsilon) \le{}& \bar{k}\Psi(\|x\|)N(\varepsilon) + \bar{h}\Psi(\|x\|)\omega_1(k,\varepsilon) \\ &+ \bar{k}\Psi(\|x\|)M(\varepsilon),\end{aligned} \tag{62}$$

where the symbol $\omega(y,\varepsilon)$ denotes the modulus of continuity of a function $y \in C(I)$.

Further observe that $\omega_1(k,\varepsilon) \to 0$ as $\varepsilon \to 0$, which is an immediate consequence of the uniform continuity of the function k on the triangle Δ. Combining this fact with the properties of the functions $M(\varepsilon)$ and $N(\varepsilon)$ and taking into account (62), we infer that $F_2x \in C(I)$. Consequently, keeping in mind that $F_1 : C(I) \to C(I)$ and assumption (iii), we conclude that the operator F is a self-mapping of the space $C(I)$.

In order to show that F is continuous on $C(I)$, fix arbitrarily $x_0 \in C(I)$ and $\varepsilon > 0$. Next, take $x \in C(I)$ such that $\|x - x_0\| \le \varepsilon$. Then, for a fixed $t \in I$, we get

$$\begin{aligned}&\left|(Fx)(t)-(Fx_0)(t)\right| \\ &\quad\le \left|(F_1x)(t)-(F_1x_0)(t)\right| \\ &\qquad+ \left|(Gx)(t)(F_2x)(t)-(Gx_0)(t)(F_2x_0)(t)\right| \\ &\quad\le \left\|F_1x-F_1x_0\right\| + \|Gx\|\left|(F_2x)(t)-(F_2x_0)(t)\right| \\ &\qquad+ \left\|F_2x_0\right\|\left\|Gx-Gx_0\right\|.\end{aligned} \tag{63}$$

On the other hand, we have the following estimate:

$$\begin{aligned}\left|(F_2x)(t)-(F_2x_0)(t)\right| &\le \int_0^t |k(t,s)|\, g(t,s) \left|(Qx)(s)-(Qx_0)(s)\right| ds \\ &\le \bar{k}\left(\int_0^t g(t,s)\,ds\right)\|Qx-Qx_0\| \\ &\le \overline{kh}\,\|Qx-Qx_0\|.\end{aligned} \tag{64}$$

Similarly, we obtain

$$\left|(F_2x_0)(t)\right| \le \bar{k}\left(\int_0^t g(t,s)\,ds\right)\|Qx_0\| \le \overline{kh}\,\|Qx_0\|, \tag{65}$$

and consequently

$$\|F_2x_0\| \le \overline{kh}\,\|Qx_0\|. \tag{66}$$

Next, linking (63)–(66) and (58), we obtain the following estimate:

$$\begin{aligned}\|Fx-Fx_0\| \le{}& p\|x-x_0\| + \|Gx\|\,\overline{kh}\,\|Qx-Qx_0\| \\ &+ \overline{kh}\,\|Gx-Gx_0\|\,\|Qx_0\|.\end{aligned} \tag{67}$$

Further, taking into account assumptions (iv) and (v), we derive the following inequality

$$\begin{aligned}\|Fx-Fx_0\| \le{}& p\varepsilon + \varphi\left(\|x_0\|+\varepsilon\right)\overline{kh}\,\|Qx-Qx_0\| \\ &+ \overline{kh}\Psi\left(\|x_0\|\right)\|Gx-Gx_0\|.\end{aligned} \tag{68}$$

Finally, in view of the continuity of the operators G and Q (cf. assumptions (iii) and (v)), we deduce that operator F is continuous on the space $C(I)$. The proof is complete. □

Now, we can formulate the last result concerning (47) (cf. [30]).

Theorem 14. *Under assumptions* (i)–(viii), (47) *has at least one solution in the space* $C(I)$.

Proof. Fix $x \in C(I)$ and $t \in I$. Then, evaluating similarly as in the proof of Lemma 13, we get

$$\begin{aligned}|(Fx)(t)| \le{}& \left|f_1(t,x(t),x(a(t))) - f_1(t,0,0)\right| \\ &+ \left|f_1(t,0,0)\right| \\ &+ |Gx(t)|\left|\int_0^t k(t,s)\,g(t,s)\,(Qx)(s)\,ds\right| \\ \le{}& \overline{f_1} + p\|x\| + \overline{kh}\varphi(\|x\|)\Psi(\|x\|).\end{aligned} \tag{69}$$

Hence, we obtain

$$\|Fx\| \le \overline{f_1} + p\|x\| + \overline{kh}\varphi(\|x\|)\Psi(\|x\|). \tag{70}$$

From the above inequality and assumption (viii), we infer that there exists a number $r_0 > 0$ such that the operator F maps the ball B_{r_0} into itself and $p + \overline{kh}q\Psi(r_0) < 1$. Moreover, by Lemma 13, we have that F is continuous on the ball B_{r_0}.

Further on, take a nonempty subset X of the ball B_{r_0} and a number $\varepsilon > 0$. Then, for an arbitrary $x \in X$ and $t_1, t_2 \in I$ with $|t_2-t_1| \le \varepsilon$, in view of (66) and the imposed assumptions, we obtain

$$\begin{aligned}&|(Fx)(t_2)-(Fx)(t_1)| \\ &\quad\le \left|f_1(t_2,x(t_2),x(a(t_2)))\right. \\ &\qquad\left.-f_1(t_1,x(t_2),x(a(t_2)))\right| \\ &\qquad+\left|f_1(t_1,x(t_2),x(a(t_2)))\right. \\ &\qquad\left.-f_1(t_1,x(t_1),x(a(t_1)))\right| \\ &\qquad+\left|(F_2x)(t_2)\right|\left|(Gx)(t_2)-(Gx)(t_1)\right| \\ &\qquad+\left|(Gx)(t_1)\right|\left|(F_2x)(t_2)-(F_2x)(t_1)\right| \\ &\quad\le \omega_{r_0}(f_1,\varepsilon) + p\max\left\{\left|x(t_2)-x(t_1)\right|,\right. \\ &\qquad\left.\left|x(a(t_2))-x(a(t_1))\right|\right\} \\ &\qquad+\overline{kh}\Psi(r_0)\,\omega(Gx,\varepsilon)+\varphi(r_0)\,\omega(F_2x,\varepsilon),\end{aligned} \tag{71}$$

where we denoted

$$\begin{aligned}\omega_{r_0}(f_1,\varepsilon) ={}& \sup\left\{\left|f_1(t_2,x,y)-f_1(t_1,x,y)\right| : t_1,t_2 \in I,\right. \\ &\left.\left|t_2-t_1\right| \le \varepsilon, x, y \in [-r_0,r_0]\right\}.\end{aligned} \tag{72}$$

Hence, in virtue of (62), we deduce the estimate

$$\begin{aligned}\omega(Fx,\varepsilon) \le{}& \omega_{r_0}(f_1,\varepsilon) + p\max\{\omega(x,\varepsilon),\omega(x,\omega(a,\varepsilon))\} \\ &+\overline{kh}\Psi(r_0)\,\omega(Gx,\varepsilon) \\ &+\varphi(r_0)\Psi(r_0)\left[\bar{k}N(\varepsilon)+\bar{h}\omega_1(k,\varepsilon)+\bar{k}M(\varepsilon)\right].\end{aligned} \tag{73}$$

Finally, taking into account the uniform continuity of the function f_1 on the set $I \times [-r_0,r_0]^2$ and the properties of the functions $k(t,s)$, $a(t)$, $M(\varepsilon)$, and $N(\varepsilon)$ and keeping in mind (46), we obtain

$$\omega_0(FX) \le p\omega_0(X) + \overline{kh}\Psi(r_0)\,\omega_0(GX). \tag{74}$$

Linking this estimate with assumption (iii), we get

$$\omega_0(FX) \le \left(p + \overline{kh}q\Psi(r_0)\right)\omega_0(X). \tag{75}$$

The use of Theorem 11 completes the proof. □

4. Existence Results Concerning the Theory of Differential Equations in Banach Spaces

In this section, we are going to present some classical results concerning the theory of ordinary differential equations in Banach space. We focus on this part of that theory in a which the technique associated with measures of noncompactness is used as the main tool in proving results on the existence of solutions of the initial value problems for ordinary differential equations. Our presentation is based mainly on the papers [31, 32] and the monograph [33].

The theory of ordinary differential equations in Banach spaces was initiated by the famous example of Dieudonné [34], who showed that in an infinite-dimensional Banach space, the classical Peano existence theorem is no longer true. More precisely, Dieudonné showed that if we consider the ordinary differential equation

$$x' = f(t, x) \tag{76}$$

with the initial condition

$$x(0) = x_0, \tag{77}$$

where $f : [0, T] \times B(x_0, r) \to E$ and E is an infinite-dimensional Banach space, then the continuity of f (and even uniform continuity) does not guarantee the existence of solutions of problem (76)-(77).

In light of the example of Dieudonné, it is clear that in order to ensure the existence of solutions of (76)-(77), it is necessary to add some extra conditions. The first results in this direction were obtained by Kisyński [35], Olech [36], and Ważewski [37] in the years 1959-1960. In order to formulate those results, we need to introduce the concept of the so-called Kamke comparison function (cf. [38, 39]).

To this end, assume that T is a fixed number and denote $J = [0, T]$, $J_0 = (0, T]$. Further, assume that Ω is a nonempty open subset of $\mathbb{R}^n$ and x_0 is a fixed element of Ω. Let $f : J \times \Omega \to \mathbb{R}^n$ be a given function.

Definition 15. A function $w : J \times \mathbb{R}_+ \to \mathbb{R}_+$ (or $w : J_0 \times \mathbb{R}_+ \to \mathbb{R}_+$) is called a *Kamke comparison function* provided the inequality

$$\|f(t, x) - f(t, y)\| \le w(t, \|x - y\|) \tag{78}$$

for $x, y \in \Omega$ and $t \in J$ (or $t \in J_0$), together with some additional assumptions concerning the function w, guarantees that problem (76)-(77) has at most one local solution.

In the literature, one can encounter miscellaneous classes of Kamke comparison functions (cf. [33, 39]). We will not describe those classes, but let us only mention that they are mostly associated with the differential equation $u' = w(t, u)$ with initial condition $u(0) = 0$ or the integral inequality $u(t) \le \int_0^t w(s, u(s))ds$ for $t \in J_0$ with initial condition $\lim_{t\to 0} u(t)/t = \lim_{t\to 0} u(t) = 0$. It is also worthwhile recalling that the classical Lipschitz or Nagumo conditions may serve as Kamke comparison functions [39].

The above-mentioned results due to Kisyński et al. [35–37] assert that if $f : J \times B(x_0, r) \to E$ is a continuous function satisfying condition (78) with an appropriate Kamke comparison function, then problem (76)-(77) has exactly one local solution.

Observe that the natural translation of inequality (78) in terms of measures of noncompactness has the form

$$\mu(f(t, X)) \le w(t, \mu(X)), \tag{79}$$

where X denotes an arbitrary nonempty subset of the ball $B(x_0, r)$. The first result with the use of condition (79) for $w(t, u) = Cu$ (C is a constant) was obtained by Ambrosetti [40]. After the result of Ambrosetti, there have appeared a lot of papers containing existence results concerning problem (76)-(77) (cf. [41–44]) with the use of condition (79) and involving various types of Kamke comparison functions. It turned out that generalizations of existence results concerning problem (76)-(77) with the use of more and more general Kamke comparison functions are, in fact, only apparent generalizations [45], since the so-called Bompiani comparison function is sufficient to give the most general result in the mentioned direction.

On the other hand, we can generalize existence results involving a condition like (79) taking general measures of noncompactness [31, 32]. Below we present a result coming from [32] which seems to be the most general with respect to taking the most general measure of noncompactness.

In the beginning, let us assume that μ is a measure of noncompactness defined on a Banach space E. Denote by E_μ the set defined by the equality

$$E_\mu = \{x \in E : \{x\} \in \ker \mu\}. \tag{80}$$

The set E_μ will be called the *kernel set* of a measure μ. Taking into account Definition 10 and some properties of a measure of noncompactness (cf. [10]), it is easily seen that E_μ is a closed and convex subset of the space E.

In the case when we consider the so-called *sublinear measure of noncompactness* [10], that is, a measure of noncompactness μ which additionally satisfies the following two conditions:

(6^o) $\mu(X + Y) \le \mu(X) + \mu(Y)$,

(7^o) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$,

then the kernel set E_μ forms a closed linear subspace of the space E.

Further on, assume that μ is an arbitrary measure of noncompactness in the Banach space E. Let $r > 0$ be a fixed number and let us fix $x_0 \in E_\mu$. Next, assume that $f : J \times B(x_0, r) \to E$ (where $J = [0, T]$) is a given uniformly continuous and bounded function; say, $\|f(t, x)\| \le A$.

Moreover, assume that f satisfies the following comparison condition of Kamke type:

$$\mu(x_0 + f(t, X)) \le w(t, \mu(X)) \tag{81}$$

for any nonempty subset X of the ball $B(x_0, r)$ and for almost all $t \in J$.

Here we will assume that the function $w(t, u) = w : J_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ ($J_0 = (0, T]$) is continuous with respect to u for any t and measurable with respect to t for each u. Apart from this, $w(t, 0) = 0$ and the unique solution of the integral inequality

$$u(t) \le \int_0^t w(s, u(s))\,ds \quad (t \in J_0) \tag{82}$$

such that $\lim_{t\to 0} u(t)/t = \lim_{t\to 0} u(t) = 0$, is $u \equiv 0$.

The following formulated result comes from [32].

Theorem 16. *Under the above assumptions, if additionally* $\sup\{t+a(t) : t \in J\} \le 1$, *where* $a(t) = \sup\{\|f(0, x_0) - f(s, x)\| : s \le t,\ \|x - x_0\| \le As\}$ *and A is a positive constant such that* $AT \le r$, *the initial value problem* (76)-(77) *has at least one local solution* $x = x(t)$ *such that* $x(t) \in E_\mu$ *for* $t \in J$.

The proof of the above theorem is very involved and is therefore omitted (cf. [32, 33]). We restrict ourselves to give a few remarks.

At first, let us notice that in the case when μ is a sublinear measure of noncompactness, condition (81) is reduced to the classical one expressed by (79) provided we assume that $x_0 \in E_\mu$. In such a case, Theorem 16 was proved in [10].

The most frequently used type of a comparison function w is this having the form $w(t, u) = p(t)u$, where $p(t)$ is assumed to be Lebesgue integrable over the interval J. In such a case comparison, condition (81) has the form

$$\mu(x_0 + f(t, X)) \le p(t)\mu(X). \tag{83}$$

An example illustrating Theorem 16 under condition (83) will be given later.

Further, observe that in the case when μ is a sublinear measure of noncompactness such that $x_0 \in E_\mu$, condition (83) can be written in the form

$$\mu(f(t, X)) \le p(t)\mu(X). \tag{84}$$

Condition (84) under additional assumption $\|f(t, x)\| \le P + Q\|x\|$, with some nonnegative constants P and Q, is used frequently in considerations associated with infinite systems of ordinary differential equations [46–48].

Now, we present the above-announced example coming from [33].

Example 17. Consider the infinite system of differential equations having the form

$$x_n' = a_n(t)x_n + f_n(x_n, x_{n+1}, \ldots), \tag{85}$$

where $n = 1, 2, \ldots$ and $t \in J = [0, T]$. System (85) will be considered together with the system of initial conditions

$$x_n(0) = x_0^n \tag{86}$$

for $n = 1, 2, \ldots$. We will assume that there exists the limit $\lim_{n\to\infty} x_0^n = a (a \in \mathbb{R})$.

Problem (85)-(86) will be considered under the following conditions.

(i) $a_n : J \to \mathbb{R}$ $(n = 1, 2, \ldots)$ are continuous functions such that the sequence $(a_n(t))$ converges uniformly on the interval J to the function which vanishes identically on J.

(ii) There exists a sequence of real nonnegative numbers (α_n) such that $\lim_{n\to\infty}\alpha_n = 0$ and $|f_n(x_n, x_{n+1}, \ldots)| \le \alpha_n$ for $n = 1, 2, \ldots$ and for all $x = (x_1, x_2, x_3, \ldots) \in l^\infty$.

(iii) The function $f = (f_1, f_2, f_3, \ldots)$ transforms the space l^∞ into itself and is uniformly continuous.

Let us mention that the symbol l^∞ used above denotes the classical Banach sequence space consisting of all real bounded sequences (x_n) with the supremum norm; that is, $\|(x_n)\| = \sup\{|x_n| : n = 1, 2, \ldots\}$.

Under the above hypotheses, the initial value problem (85)-(86) has at least one solution $x(t) = (x_1(t), x_2(t), \ldots)$ such that $x(t) \in l^\infty$ for $t \in J$ and $\lim_{n\to\infty} x_n(t) = a$ uniformly with respect to $t \in J$, provided $T \le 1 (J = [0, T])$.

As a proof, let us take into account the measure of noncompactness in the space l^∞ defined in the following way:

$$\mu(X) = \limsup_{n\to\infty}\left\{\sup_{x\in X}|x_n - a|\right\} \tag{87}$$

for $X \in \mathfrak{M}_{l^\infty}$ (cf. [10]). The kernel $\ker\mu$ of this measure is the family of all bounded subsets of the space l^∞ consisting of sequences converging to the limit equal to a with the same rate.

Further, take an arbitrary set $X \in \mathfrak{M}_{l^\infty}$. Then we have

$$\begin{aligned}
&\mu(x_0 + f(t, X)) \\
&\quad = \limsup_{n\to\infty}\left\{\sup_{x\in X}\left|x_0^n + a_n(t)x_n + f_n(x_n, x_{n+1}, \ldots) - a\right|\right\} \\
&\quad \le \limsup_{n\to\infty}\left\{\sup_{x\in X}\left[|a_n(t)||x_n| + |f_n(x_n, x_{n+1}, \ldots) + x_0^n - a|\right]\right\} \\
&\quad \le \limsup_{n\to\infty}\left\{\sup_{x\in X} p(t)|x_n - a|\right\} + \limsup_{n\to\infty}\left\{\sup_{x\in X}|a_n(t)||a|\right\} \\
&\qquad + \limsup_{n\to\infty}\left\{\sup_{x\in X}|f_n(x_n, x_{n+1}, \ldots)|\right\} + \limsup_{n\to\infty}\left\{\sup_{x\in X}|x_0^n - a|\right\},
\end{aligned} \tag{88}$$

where we denoted $p(t) = \sup\{|a_n(t)| : n = 1, 2, \ldots\}$ for $t \in J$. Hence we get

$$\mu(x_0 + f(t, X)) \le p(t)\mu(X), \tag{89}$$

which means that the condition (84) is satisfied.

Combining this fact with assumption (iii) and taking into account Theorem 16, we complete the proof.

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Research Article

Existence of Mild Solutions for a Semilinear Integrodifferential Equation with Nonlocal Initial Conditions

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Using Hausdorff measure of noncompactness and a fixed-point argument we prove the existence of mild solutions for the semilinear integrodifferential equation subject to nonlocal initial conditions $u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u(t))$, $t \in [0,1]$, $u(0) = g(u)$, where $A : D(A) \subseteq X \to X$, and for every $t \in [0,1]$ the maps $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0,1]$, and the functions $f : [0,1] \times X \to X$ and $g : C([0,1]; X) \to X$ are X-valued functions which satisfy appropriate conditions.

1. Introduction

The concept of *nonlocal initial condition* has been introduced to extend the study of classical initial value problems. This notion is more precise for describing nature phenomena than the classical notion because additional information is taken into account. For the importance of nonlocal conditions in different fields, the reader is referred to [1–3] and the references cited therein.

The earliest works related with problems submitted to nonlocal initial conditions were made by Byszewski [4–7]. In these works, using methods of semigroup theory and the Banach fixed point theorem the author has proved the existence of mild and strong solutions for the first order Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in [0,1], \\ u(0) &= g(u), \end{aligned} \tag{1.1}$$

where A is an operator defined in a Banach space X which generates a semigroup $\{T(t)\}_{t\geqslant 0}$, and the maps f and g are suitable X-valued functions.

Henceforth, (1.1) has been extensively studied by many authors. We just mention a few of these works. Byszewski and Lakshmikantham [8] have studied the existence and uniqueness of mild solutions whenever f and g satisfy Lipschitz-type conditions. Ntouyas and Tsamatos [9, 10] have studied this problem under conditions of compactness for the semigroup generated by A and the function g. Recently, Zhu et al. [11], have investigated this problem without conditions of compactness on the semigroup generated by A, or the function f.

On the other hand, the study of abstract integrodifferential equations has been an active topic of research in recent years because it has many applications in different areas. In consequence, there exists an extensive literature about integrodifferential equations with nonlocal initial conditions, (cf., e.g., [12–25]). Our work is a contribution to this theory. Indeed, this paper is devoted to study the existence of mild solutions for the following semilinear integrodifferential evolution equation:

$$\begin{aligned} &u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad t \in [0,1], \\ &u(0) = g(u), \end{aligned} \tag{1.2}$$

where $A : D(A) \subseteq X \to X$ and for every $t \in [0,1]$ the mappings $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0,1]$, and the functions $f : [0,1] \times X \to X$ and $g : C([0,1];X) \to X$ are X-valued functions that satisfy appropriate conditions which we will describe later. In order to abbreviate the text of this paper, henceforth we will denote by I the interval $[0,1]$, and $C(I;X)$ is the space of all continuous functions from I to X endowed with the uniform convergence norm.

The classical initial value version of (1.2), that is, $u(0) = u_0$ for some $u_0 \in X$, has been extensively studied by many researchers because it has many important applications in different fields of natural sciences such as thermodynamics, electrodynamics, heat conduction in materials with memory, continuum mechanics and population biology, among others. For more information, see [26–28]. For this reason the study of existence and other properties of the solutions for (1.2) is a very important problem. However, to the best of our knowledge, the existence of mild solutions for the nonlocal initial value problem (1.2) has not been addressed in the existing literature. Most of the authors obtain the existence of solutions and well-posedness for (1.2) by establishing the existence of a resolvent operator $\{R(t)\}_{t\in I}$ and a variation of parameters formula (see, [29, 30]). Using adaptation of the methods described in [11], we are able to prove the existence of mild solutions of (1.2) under conditions of compactness of the function g and continuity of the function $t \mapsto R(t)$ for $t > 0$. Furthermore, in the particular case $B(t) = b(t)A$ for all $t \in [0,1]$, where the operator A is the infinitesimal generator of a C_0-semigroup defined in a Hilbert space H, and the kernel b is a scalar map which satisfies appropriate hypotheses, we are able to give sufficient conditions for the existence of mild solutions only in terms of spectral properties of the operator A and regularity properties of the kernel b. We show that our abstract results can be applied to concrete situations. Indeed, we consider an example with a particular choice of the function b and the operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial \xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial \xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi), \tag{1.3}$$

where the given coefficients a_1, b_1, $\overline{c}$ satisfy the usual uniform ellipticity conditions.

2. Preliminaries

Most of the notations used throughout this paper are standard. So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural integers and real and complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}_0^+ = [0,\infty)$.

In this work X and Y always are complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscript will be dropped when there is no danger of confusion. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X,Y)$. In the case $X = Y$, we will write briefly $\mathcal{L}(X)$. Let A be an operator defined in X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

As we have already mentioned $C(I;X)$ is the vector space of all continuous functions $f : I \to X$. This space is a Banach space endowed with the norm

$$\|f\|_\infty = \sup_{t\in I}\|f(t)\|_X. \tag{2.1}$$

In the same manner, for $n \in \mathbb{N}$ we write $C^n(I;X)$ for denoting the space of all functions from I to X which are n-times differentiable. Further, $C^\infty(I;X)$ represents the space of all infinitely differentiable functions from I to X.

We denote by $L^1(I;X)$ the space of all (equivalent classes of) Bochner-measurable functions $f : I \mapsto X$ such that $\|f(t)\|_X$ is integrable for $t \in I$. It is well known that this space is a Banach space with the norm

$$\|f\|_{L^1(I;X)} = \int_I \|f(s)\|_X ds. \tag{2.2}$$

We next include some preliminaries concerning the theory of resolvent operator $\{R(t)\}_{t\in I}$ for (1.2).

Definition 2.1. Let X be a complex Banach space. A family $\{R(t)\}_{t\in I}$ of bounded linear operators defined in X is called a resolvent operator for (1.2) if the following conditions are fulfilled.

(R_1) For each $x \in X$, $R(0)x = x$ and $R(\cdot)x \in C(I;X)$.

(R_2) The map $R : I \to \mathcal{L}([D(A)])$ is strongly continuous.

(R_3) For each $y \in D(A)$, the function $t \mapsto R(t)y$ is continuously differentiable and

$$\begin{aligned} \frac{d}{dt}R(t)y &= AR(t)y + \int_0^t B(t-s)R(s)y\,ds \\ &= R(t)Ay + \int_0^t R(t-s)B(s)y\,ds, \quad t \in I. \end{aligned} \tag{2.3}$$

In what follows we assume that there exists a resolvent operator $\{R(t)\}_{t\in I}$ for (1.2) satisfying the following property.

(P) The function $t \mapsto R(t)$ is continuous from $(0,1]$ to $\mathcal{L}(X)$ endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

Note that property (P) is also named in different ways in the existing literature on the subject, mainly the theory of C_0-semigroups, namely, norm continuity for $t > 0$, eventually norm continuity, or equicontinuity.

The existence of solutions of the *linear* problem

$$\begin{aligned} &u'(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds + f(t), \quad t \geqslant 0, \\ &u(0) = u_0 \in X \end{aligned} \tag{2.4}$$

has been studied by many authors. Assuming that $f : [0,+\infty) \to X$ is locally integrable, it follows from [29] that the function u given by

$$u(t) = R(t)u_0 + \int_0^t R(t-s)f(s)\,ds, \quad \text{for } t \geqslant 0, \tag{2.5}$$

is a mild solution of the problem (2.4). Motivated by this result, we adopt the following concept of solution.

Definition 2.2. A continuous function $u \in C(I;X)$ is called a mild solution of (1.2) if the equation

$$u(t) = R(t)g(u) + \int_0^t R(t-s)f(s,u(s))\,ds, \quad t \in I, \tag{2.6}$$

is verified.

The main results of this paper are based on the concept of measure of noncompactness. For general information the reader can see [31]. In this paper, we use the notion of Hausdorff measure of noncompactness. For this reason we recall a few properties related with this concept.

Definition 2.3. Let S be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness of S is defined by

$$\eta(S) = \inf\{\varepsilon > 0 : S \text{ has a finite cover by balls of radius } \varepsilon\}. \tag{2.7}$$

Remark 2.4. Let S_1, S_2 be bounded sets of a normed space Y. The Hausdorff measure of noncompactness has the following properties.

(i) If $S_1 \subseteq S_2$, then $\eta(S_1) \leqslant \eta(S_2)$.

(ii) $\eta(S_1) = \eta(\overline{S_1})$, where $\overline{S_1}$ denotes the closure of A.

(iii) $\eta(S_1) = 0$ if and only if S_1 is totally bounded.

(iv) $\eta(\lambda S_1) = |\lambda|\eta(S_1)$ with $\lambda \in \mathbb{R}$.

(v) $\eta(S_1 \cup S_2) = \max\{\eta(S_1), \eta(S_2)\}$.

(vi) $\eta(S_1 + S_2) \leqslant \eta(S_1) + \eta(S_2)$, where $S_1 + S_2 = \{s_1 + s_2 : \ s_1 \in S_1, s_2 \in S_2\}$.

(vii) $\eta(S_1) = \eta(\overline{\text{co}}(S_1))$, where $\overline{\text{co}}(S_1)$ is the closed convex hull of S_1.

We next collect some specific properties of the Hausdorff measure of noncompactness which are needed to establish our results. Henceforth, when we need to compare the measures of noncompactness in X and $C(I;X)$, we will use ζ to denote the Hausdorff measure of noncompactness defined in X and γ to denote the Hausdorff measure of noncompactness on $C(I;X)$. Moreover, we will use η for the Hausdorff measure of noncompactness for general Banach spaces Y.

Lemma 2.5. *Let $W \subseteq C(I;X)$ be a subset of continuous functions. If W is bounded and equicontinuous, then the set $\overline{\text{co}}(W)$ is also bounded and equicontinuous.*

For the rest of the paper we will use the following notation. Let W be a set of functions from I to X and $t \in I$ fixed, and we denote $W(t) = \{w(t) : w \in W\}$. The proof of Lemma 2.6 can be found in [31].

Lemma 2.6. *Let $W \subseteq C(I;X)$ be a bounded set. Then $\zeta(W(t)) \leqslant \gamma(W)$ for all $t \in I$. Furthermore, if W is equicontinuous on I, then $\zeta(W(t))$ is continuous on I, and*

$$\gamma(W) = \sup\{\zeta(W(t)) : t \in I\}. \tag{2.8}$$

A set of functions $W \subseteq L^1(I;X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1(I;\mathbb{R}^+)$ such that $\|w(t)\| \leqslant \kappa(t)$ a.e. for all $w \in W$.

The next property has been studied by several authors; the reader can see [32] for more details.

Lemma 2.7. *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1(I;X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \mapsto \zeta(\{u_n(t)\}_{n\in\mathbb{N}})$ is measurable and*

$$\zeta\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leqslant 2\int_0^t \zeta(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.9}$$

The next result is crucial for our work, the reader can see its proof in [33, Theorem 2].

Lemma 2.8. *Let Y be a Banach space. If $W \subseteq Y$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that*

$$\eta(W) \leqslant 2\eta(\{u_n\}_{n=1}^{\infty}) + \varepsilon. \tag{2.10}$$

The following lemma is essential for the proof of Theorem 3.2, which is the main result of this paper. For more details of its proof, see [34, Theorem 3.1].

Lemma 2.9. *For all $0 \leqslant m \leqslant n$, denote $C_m^n = \binom{n}{m}$. If $0 < \epsilon < 1$ and $h > 0$ and let*

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}, \quad n \in \mathbb{N}, \tag{2.11}$$

then $\lim_{n\to\infty} S_n = 0$.

Clearly, a manner for proving the existence of mild solutions for (1.2) is using fixed-point arguments. The fixed-point theorem which we will apply has been established in [34, Lemma 2.4].

Lemma 2.10. *Let S be a closed and convex subset of a complex Banach space Y, and let $F : S \to S$ be a continuous operator such that $F(S)$ is a bounded set. Define*

$$F^1(S) = F(S), \quad F^n(S) = F\Big(\overline{\text{co}}\Big(F^{n-1}(S)\Big)\Big), \quad n = 2, 3, \ldots. \tag{2.12}$$

If there exist a constant $0 \leqslant r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(F^{n_0}(S)) \leqslant r\eta(S), \tag{2.13}$$

then F has a fixed point in the set S.

3. Main Results

In this section we will present our main results. Henceforth, we assume that the following assertions hold.

$(H1)$ There exists a resolvent operator $\{R(t)\}_{t\in I}$ for (1.2) having the property (P).

$(H2)$ The function $g : C(I; X) \to X$ is a compact map.

$(H3)$ The function $f : I \times X \to X$ satisfies the Carathéodory type conditions; that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.

$(H4)$ There exist a function $m \in L^1(I; \mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \leqslant m(t)\Phi(\|x\|) \tag{3.1}$$

for all $x \in X$ and almost all $t \in I$.

$(H5)$ There exists a function $H \in L^1(I;\mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\zeta(f(t,S)) \leqslant H(t)\zeta(S) \tag{3.2}$$

for almost all $t \in I$.

Remark 3.1. Assuming that the function g satisfies the hypothesis $(H2)$, it is clear that g takes bounded set into bounded sets. For this reason, for each $R \geqslant 0$ we will denote by g_R the number $g_R = \sup\{\|g(u)\| : \|u\|_\infty \leqslant R\}$.

The following theorem is the main result of this paper.

Theorem 3.2. *If the hypotheses* $(H1)$–$(H5)$ *are satisfied and there exists a constant* $R \geqslant 0$ *such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R, \tag{3.3}$$

where $K = \sup\{\|R(t)\| : t \in I\}$*, then the problem* (1.2) *has at least one mild solution.*

Proof. Define $F : C(I;X) \to C(I;X)$ by

$$(Fu)(t) = R(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in I, \tag{3.4}$$

for all $u \in C(I;X)$.

We begin showing that F is a continuous map. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq C(I;X)$ such that $u_n \to u$ as $n \to \infty$ (in the norm of $C(I;X)$). Note that

$$\|F(u_n) - F(u)\| \leqslant K\big\|g(u_n) - g(u)\big\| + K\int_0^1 \big\|f(s,u_n(s)) - f(s,u(s))\big\|ds, \tag{3.5}$$

by hypotheses $(H2)$ and $(H3)$ and by the dominated convergence theorem we have that $\|F(u_n) - F(u)\| \to 0$ when $n \to \infty$.

Let $R \geqslant 0$ and denote $B_R = \{u \in C(I;X) : \|u(t)\| \leqslant R \ \forall t \in I\}$ and note that for any $u \in B_R$ we have

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \big\|R(t)g(u)\big\| + \left\|\int_0^t R(t-s)f(s,u(s))ds\right\| \\ &\leqslant Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R.\end{aligned} \tag{3.6}$$

Therefore $F : B_R \to B_R$ and $F(B_R)$ is a bounded set. Moreover, by continuity of the function $t \mapsto R(t)$ on $(0,1]$, we have that the set $F(B_R)$ is an equicontinuous set of functions.

Define $\mathcal{B} = \overline{\mathrm{co}}(F(B_R))$. It follows from Lemma 2.5 that the set $\mathcal{B}$ is equicontinuous. In addition, the operator $F : \mathcal{B} \to \mathcal{B}$ is continuous and $F(\mathcal{B})$ is a bounded set of functions.

Let $\varepsilon > 0$. Since the function g is a compact map, by Lemma 2.8 there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset F(\mathcal{B})$ such that

$$\zeta(F(\mathcal{B})(t)) \leqslant 2\zeta(\{v_n(t)\}_{n=1}^{\infty}) + \varepsilon \leqslant 2\zeta\left(\int_0^t \{R(t-s)f(s,u_n(s))\}_{n=1}^{\infty} ds\right) + \varepsilon. \tag{3.7}$$

By the hypothesis $(H4)$, for each $t \in I$ we have $\|R(t-s)f(s,u_n(s))\| \leqslant K\Phi(R)m(s)$. Therefore, by the condition $(H5)$ we have

$$\begin{aligned}\zeta(F(\mathcal{B})(t)) &\leqslant 4K\int_0^t \zeta(\{f(s,u_n(s))\}_{n=1}^{\infty} ds + \varepsilon \\ &\leqslant 4K\int_0^t H(s)\zeta(\{u_n(s)\}_{n\in\mathbb{N}})\ ds + \varepsilon \\ &\leqslant 4K\gamma(\mathcal{B})\int_0^t H(s)ds + \varepsilon.\end{aligned} \tag{3.8}$$

Since the function $H \in L^1(I;\mathbb{R}^+)$, for $\alpha < 1/4K$ there exists $\varphi \in C(I;\mathbb{R}^+)$ satisfying $\int_0^1 |H(s) - \varphi(s)|ds < \alpha$. Hence,

$$\begin{aligned}\zeta(F(\mathcal{B})(t)) &\leqslant 4K\gamma(\mathcal{B})\left[\int_0^t \left|H(s) - \varphi(s)\right|ds + \int_0^t \varphi(s)ds\right] + \varepsilon \\ &\leqslant 4K\gamma(\mathcal{B})[\alpha + Nt] + \varepsilon,\end{aligned} \tag{3.9}$$

where $N = \|\varphi\|_\infty$. Since $\varepsilon > 0$ is arbitrary, we have

$$\zeta(F(\mathcal{B})(t)) \leqslant (a + bt)\gamma(\mathcal{B}), \quad \text{where } a = 4\alpha K,\ b = 4KN. \tag{3.10}$$

Let $\varepsilon > 0$. Since the function g is a compact map and applying the Lemma 2.8 there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \overline{\text{co}}(F(\mathcal{B}))$ such that

$$\begin{aligned}\zeta\left(F^2(\mathcal{B})(t)\right) &\leqslant 2\zeta\left(\int_0^t \{R(t-s)f(s,w_n(s))\}_{n=1}^{\infty} ds\right) + \varepsilon \\ &\leqslant 4K\int_0^t \zeta\{f(s,w_n(s))\}_{n=1}^{\infty} ds + \varepsilon \\ &\leqslant 4K\int_0^t H(s)\zeta\left(\overline{\text{co}}\left(F^1(\mathcal{B})(s)\right)\right) + \varepsilon = 4K\int_0^t H(s)\zeta\left(F^1(\mathcal{B})(s)\right) + \varepsilon.\end{aligned} \tag{3.11}$$

Using the inequality (3.10) we have that

$$\begin{aligned}\zeta\Big(F^2(\mathcal{B})(t)\Big) &\leqslant 4K\int_0^t \big[|H(s)-\varphi(s)|+|\varphi(s)|\big](a+bs)\gamma(\mathcal{B})ds+\varepsilon \\ &\leqslant 4K(a+bt)\gamma(\mathcal{B})\int_0^t|H(s)-\varphi(s)|ds+4KN\gamma(\mathcal{B})\left(at+\frac{bt^2}{2}\right)+\varepsilon \\ &\leqslant a(a+bt)+b\left(at+\frac{bt^2}{2}\right)+\varepsilon\leqslant\left(a^2+2bt+\frac{(bt)^2}{2}\right)\gamma(\mathcal{B})+\varepsilon.\end{aligned} \tag{3.12}$$

Since $\varepsilon>0$ is arbitrary, we have

$$\zeta\Big(F^2(\mathcal{B})(t)\Big)\leqslant\left(a^2+2bt+\frac{(bt)^2}{2}\right)\gamma(\mathcal{B}). \tag{3.13}$$

By an inductive process, for all $n\in\mathbb{N}$, it holds

$$\zeta(F^n(\mathcal{B})(t))\leqslant\left(a^n+C_1^na^{n-1}bt+C_2^na^{n-2}\frac{(bt)^2}{2!}+\cdots+\frac{(bt)^n}{n!}\right)\gamma(\mathcal{B}), \tag{3.14}$$

where, for $0\leqslant m\leqslant n$, the symbol C_m^n denotes the binomial coefficient $\binom{n}{m}$.

In addition, for all $n\in\mathbb{N}$ the set $F^n(\mathcal{B})$ is an equicontinuous set of functions. Therefore, using the Lemma 2.6 we conclude that

$$\gamma(F^n(\mathcal{B}))\leqslant\left(a^n+C_1^na^{n-1}b+C_2^na^{n-2}\frac{b^2}{2!}+\cdots+\frac{b^n}{n!}\right)\gamma(\mathcal{B}). \tag{3.15}$$

Since $0\leqslant a<1$ and $b>0$, it follows from Lemma 2.7 that there exists $n_0\in\mathbb{N}$ such that

$$\left(a^{n_0}+C_1^{n_0}a^{n_0-1}b+C_2^{n_0}a^{n_0-2}\frac{b^2}{2!}+\cdots+\frac{b^{n_0}}{n_0!}\right)=r<1. \tag{3.16}$$

Consequently, $\gamma(F^{n_0}(\mathcal{B}))\leqslant r\gamma(\mathcal{B})$. It follows from Lemma 2.9 that F has a fixed point in $\mathcal{B}$, and this fixed point is a mild solution of (1.2). □

Our next result is related with a particular case of (1.2). Consider the following Volterra equation of convolution type:

$$\begin{aligned}u'(t)&=Au(t)+\int_0^t b(t-s)Au(s)ds+f(t,u(t)),\quad t\in I,\\ u(0)&=g(u),\end{aligned} \tag{3.17}$$

where A is a closed linear operator defined on a Hilbert space H, the kernel $b \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$, and the function f is an appropriate H-valued map.

Since (3.17) is a convolution type equation, it is natural to employ the Laplace transform for its study.

Let X be a Banach space and $a \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$. We say that the function a is Laplace transformable if there is $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. In addition, we denote by $\widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t}a(t)dt$, for Re $\lambda > \omega$, the Laplace transform of the function a.

We need the following definitions for proving the existence of a resolvent operator for (3.17). These concepts have been introduced by Prüss in [28].

Definition 3.3. Let $a \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$ be Laplace transformable and $k \in \mathbb{N}$. We say that the function a is k-regular if there exists a constant $C > 0$ such that

$$\left|\lambda^n \widehat{a}^{(n)}(\lambda)\right| \leqslant C|\widehat{a}(\lambda)| \tag{3.18}$$

for all Re $\lambda \geqslant \omega$ and $0 < n \leqslant k$.

Convolutions of k-regular functions are again k-regular. Moreover, integration and differentiation are operations which preserve k-regularity as well. See [28, page 70].

Definition 3.4. Let $f \in C^\infty(\mathbb{R}^+;\mathbb{R})$. We say that f is a completely monotone function if and only if $(-1)^n f^{(n)}(\lambda) \geqslant 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$.

Definition 3.5. Let $a \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R})$ such that a is Laplace transformable. We say that a is completely positive function if and only if

$$\frac{1}{\lambda\widehat{a}(\lambda)}, \qquad \frac{-\widehat{a}'(\lambda)}{(\widehat{a}(\lambda))^2} \tag{3.19}$$

are completely monotone functions.

Finally, we recall that a one-parameter family $\{T(t)\}_{t\geqslant 0}$ of bounded and linear operators is said to be exponentially bounded of type (M,ω) if there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant Me^{\omega t}, \quad \forall t \geqslant 0. \tag{3.20}$$

The next proposition guarantees the existence of a resolvent operator for (3.17) satisfying the property (P). With this purpose we will introduce the conditions $(C1)$ and $(C2)$.

$(C1)$ The kernel a defined by $a(t) = 1 + \int_0^t b(s)ds$, for all $t \geqslant 0$, is 2-regular and completely positive.

$(C2)$ The operator A is the generator of a semigroup of type (M,ω) and there exists $\mu_0 > \omega$ such that

$$\lim_{|\mu|\to\infty} \left\| \frac{1}{\widehat{b}(\mu_0 + i\mu) + 1} \left(\frac{\mu_0 + i\mu}{\widehat{b}(\mu_0 + i\mu) + 1} - A \right)^{-1} \right\| = 0. \tag{3.21}$$

Proposition 3.6. *Suppose that A is the generator of a C_0-semigroup of type (M,ω) in a Hilbert space H. If the conditions $(C1)$-$(C2)$are satisfied, then there exists a resolvent operator $\{R(t)\}_{t\in I}$ for* (3.17) *having the property (P).*

Proof. Integrating in time (3.17) we get

$$u(t) = \int_0^t a(t-s)Au(s)ds + \int_0^t f(s,u(s)) + g(u). \tag{3.22}$$

Since the scalar kernel a is completely positive and A generates a C_0-semigroup, it follows from [28, Theorem 4.2] that there exists a family of operators $\{R(t)\}_{t\in I}$ strongly continuous, exponentially bounded which commutes with A, satisfying

$$R(t)x = x + \int_0^t a(t-s)AR(s)xds, \quad \forall x \in D(A). \tag{3.23}$$

On the other hand, using the condition $(C2)$ and since the scalar kernel a is 2-regular, it follows from [35, Theorem 2.2] that the function $t \mapsto R(t)$ is continuous for $t > 0$. Further, since $a \in C^1(\mathbb{R}^+;\mathbb{R})$, it follows from (3.23) that for all $x \in D(A)$ the map $R(\cdot)x$ is differentiable for all $t \geqslant 0$ and satisfies

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t b(t-s)AR(s)x\,ds, \quad t \in I. \tag{3.24}$$

From the quality (3.24), we conclude that $\{R(t)\}_{t\in I}$ is a resolvent operator for (3.17) having the property (P). □

Corollary 3.7. *Suppose that A generates a C_0-semigroup of type (M,ω) in a Hilbert space H. Assume further that the conditions* $(C1)$-$(C2)$ *are fulfilled. If the hypotheses* $(H2)$–$(H5)$ *are satisfied and there exists $R \geqslant 0$ such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R, \quad \text{where } K = \sup\{\|R(t)\| : t \in I\}, \tag{3.25}$$

then (3.17) *has at least one mild solution.*

Proof. It follows from Proposition 3.6 that there exists a resolvent operator $\{R(t)\}_{t\in I}$ for the equation and this resolvent operator has the property (P). Since the hypotheses $(H2)$–$(H5)$ are satisfied, we apply Theorem 3.2 and conclude that (3.17) has at least one mild solution. □

4. Applications

In this section we apply the abstract results which we have obtained in the preceding section to study the existence of solutions for a partial differential equation submitted to nonlocal initial conditions. This type of equations arises in the study of heat conduction in materials with memory (see [26, 27]). Specifically, we will study the following problem:

$$\begin{aligned}
&\frac{\partial w(t,\xi)}{\partial t} = Aw(t,\xi) + \int_0^t \beta e^{-\alpha(t-s)} Aw(s,\xi)ds + p_1(t)p_2(w(t,\xi)), \quad t \in I,\\
&w(t,0) = w(t,2\pi), \quad \text{for } t \in I, \qquad (4.1)\\
&w(0,\xi) = \int_0^1 \int_0^\xi qk(s,\xi)w(s,y)ds\,dy, \quad 0 \leqslant \xi \leqslant 2\pi,
\end{aligned}$$

where $k : I \times [0,2\pi] \to \mathbb{R}^+$ is a continuous function such that $k(t,2\pi) = 0$ for all $t \in I$, the constant $q \in \mathbb{R}^+$ and the constants α, β satisfy the relation $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$. The operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial \xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial \xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi), \qquad (4.2)$$

where the coefficients a_1, b_1, $\overline{c}$ satisfy the usual uniformly ellipticity conditions, and $D(A) = \{v \in L^2([0,2\pi];\mathbb{R}) : v'' L^2([0,2\pi];\mathbb{R})\}$. The functions $p_1 : I \to \mathbb{R}^+$ and $p_2 : \mathbb{R} \to \mathbb{R}$ satisfy appropriate conditions which will be specified later.

Identifying $u(t) = w(t,\cdot)$ we model this problem in the space $X = L^2(\mathbb{T};\mathbb{R})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T};\mathbb{R})$ the space of 2π-periodic and square integrable functions from $\mathbb{R}$ into $\mathbb{R}$. Consequently, (4.1) is rewritten as

$$\begin{aligned}
&u'(t) = Au(t) + \int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I,\\
&u(0) = g(u),
\end{aligned} \qquad (4.3)$$

where the function $g : C(I;X) \to X$ is defined by $g(w)(\xi) = \int_0^1 \int_0^\xi qk(s,\xi)w(s,y)ds\,dy$, and $f(t,u(t)) = p_1(t)p_2(u(t))$, where p_1 is integrable on I, and p_2 is a bounded function satisfying a Lipschitz type condition with Lipschitz constant L.

We will prove that there exists $q > 0$ sufficiently small such that (4.3) has a mild solution on $L^2(\mathbb{T};\mathbb{R})$.

With this purpose, we begin noting that $\|g\| \leqslant q(2\pi)^{1/2}\left(\int_0^{2\pi}\int_0^1 k(s,\xi)^2 ds\,d\xi\right)^{1/2}$. Moreover, it is a well-known fact that the g is a compact map.

Further, the function f satisfies $\|f(t,u(t))\| \leqslant p_1(t)\Phi(\|u(t)\|)$, with $\Phi(\|u(t)\|) \equiv \|p_2\|$ and $\|f(t,u_1(t)) - f(t,u_2(t))\| \leqslant Lp_1(t)\|u_1 - u_2\|$. Thus, the conditions $(H2)$–$(H5)$ are fulfilled.

Define $a(t) = 1 + \int_0^t \beta e^{-\alpha s} ds$, for all $t \in \mathbb{R}_0^+$. Since the kernel b defined by $b(t) = \beta e^{-\alpha t}$ is 2-regular, it follows that a is 2-regular. Furthermore, we claim that a is completely positive. In fact, we have

$$\widehat{a}(\lambda) = \frac{\lambda + \alpha + \beta}{\lambda(\lambda + \alpha)}. \tag{4.4}$$

Define the functions f_1 and f_2 by $f_1(\lambda) = 1/(\lambda \widehat{a}(\lambda))$ and $f_2(\lambda) = -\widehat{a}'(\lambda)/[\widehat{a}(\lambda)]^2$, respectively. In other words

$$f_1(\lambda) = \frac{\lambda + \alpha}{\lambda + \alpha + \beta}, \qquad f_2(\lambda) = \frac{\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + \alpha^2}{(\lambda + \alpha + \beta)^2}. \tag{4.5}$$

A direct calculation shows that

$$f_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(n+1)!}{(\lambda + \alpha + \beta)^{n+1}}, \quad f_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha + \beta)(n+1)!}{(\lambda + \alpha + \beta)^{n+2}} \quad \text{for } n \in \mathbb{N}. \tag{4.6}$$

Since $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$, we have that f_1 and f_2 are completely monotone. Thus, the kernel a is completely positive.

On the other hand, it follows from [36] that A generates an analytic, noncompact semigroup $\{T(t)\}_{t \geqslant 0}$ on $L^2(\mathbb{T}; \mathbb{R})$. In addition, there exists a constant $M > 0$ such that

$$M = \sup\{\|T(t)\| : t \geqslant 0\} < +\infty. \tag{4.7}$$

It follows from the preceding fact and the Hille-Yosida theorem that $z \in \rho(A)$ for all $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. Let $z = \mu_0 + i\mu$. By direct computation we have

$$\operatorname{Re}\left(\frac{\mu_0 + i\mu}{\widehat{b}(\mu_0 + i\mu) + 1}\right) = \frac{\mu_0^3 + \mu_0^2\alpha + \mu_0^2(\alpha + \beta) + \mu_0\alpha(\alpha + \beta) + \mu_0\mu^2 - \mu^2\beta}{(\alpha + \beta)^2 + 2\mu_0(\alpha + \beta) + \mu_0^2 + \mu^2}. \tag{4.8}$$

Hence, $\operatorname{Re}((\mu_0 + i\mu)/(\widetilde{b}(\mu_0 + i\mu) + 1)) > 0$ for all $z = \mu_0 + i\mu$, such that $\mu_0 > 0$. This implies that

$$\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1} \in \mathcal{L}(X), \quad \forall \mu_0 > 0. \tag{4.9}$$

Since the semigroup generated by A is an analytic semigroup we have

$$\left\| \frac{1}{\widehat{b}(\mu_0 + i\mu) + 1}\left(\frac{\mu_0 + i\mu}{\widehat{b}(\mu_0 + i\mu) + 1} - A\right)^{-1} \right\| \leqslant \left\| \frac{M}{\mu_0 + i\mu} \right\|. \tag{4.10}$$

Therefore,

$$\lim_{|\mu|\to\infty}\left\|\frac{1}{\widehat{b}(\mu_0+i\mu)+1}\left(\frac{\mu_0+i\mu}{\widehat{b}(\mu_0+i\mu)+1}-A\right)^{-1}\right\|=0. \tag{4.11}$$

It follows from Proposition 3.6 that (4.3) admits a resolvent operator $\{R(t)\}_{t\in I}$ satisfying property (P).

Let $K=\sup\{\|R(t)\| : t\in I\}$ and $c=(2\pi)^{1/2}\left(\int_0^{2\pi}\int_0^1 k(s,\xi)^2 ds\,d\xi\right)^{1/2}$.

A direct computation shows that for each $R\geqslant 0$ the number g_R is equal to $g_R=qcR$.

Therefore the expression $(Kg_R+K\Phi(R)\int_0^1 m(s)ds)$ is equivalent to $(qcKR+\|p_1\|_1 LK)$.

Since there exists $q>0$ such that $qcK<1$, then there exists $R\geqslant 0$ such that

$$qcKR+\|p_1\|_1 LK\leqslant R. \tag{4.12}$$

From Corollary 3.7 we conclude that there exists a mild solution of (4.1).

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Research Article

On the Existence of Positive Periodic Solutions for Second-Order Functional Differential Equations with Multiple Delays

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The existence results of positive ω-periodic solutions are obtained for the second-order functional differential equation with multiple delays $u''(t) + a(t)u(t) = f(t, u(t), u(t - \tau_1(t)), \ldots, u(t - \tau_n(t)))$, where $a(t) \in C(\mathbb{R})$ is a positive ω-periodic function, $f : \mathbb{R} \times [0, +\infty)^{n+1} \to [0, +\infty)$ is a continuous function which is ω-periodic in t, and $\tau_1(t), \ldots, \tau_n(t) \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic functions. The existence conditions concern the first eigenvalue of the associated linear periodic boundary problem. Our discussion is based on the fixed-point index theory in cones.

1. Introduction

In this paper, we deal with the existence of positive periodic solution of the second-order functional differential equation with multiple delays

$$u''(t) + a(t)u(t) = f(t, u(t), u(t - \tau_1(t)), \ldots, u(t - \tau_n(t))), \quad t \in \mathbb{R}, \tag{1.1}$$

where $a(t) \in C(\mathbb{R})$ is a positive ω-periodic function, $f : \mathbb{R} \times [0, +\infty)^{n+1} \to [0, +\infty)$ is a continuous function which is ω-periodic in t, and $\tau_1(t),\ldots,\tau_n(t) \in C(\mathbb{R}, [0, +\infty))$ are ω-periodic functions $\omega > 0$ is a constant.

In recent years, the existence of periodic solutions for second-order functional differential equations has been researched by many authors see [1–8] and references therein. In some practice models, only positive periodic solutions are significant. In [4–8], the authors obtained the existence of positive periodic solutions for some second-order functional differential equations by using fixed-point theorems of cone mapping. Especially in [5], Wu considered the second-order functional differential equation

$$u''(t) + a(t)u(t) = \lambda f(t, u(t-\tau_1(t)), \ldots, u(t-\tau_n(t))), \quad t \in \mathbb{R}, \tag{1.2}$$

and obtained the existence results of positive periodic solutions by using the Krasnoselskii fixed-point theorem of cone mapping when the coefficient $a(t)$ satisfies the condition that $0 < a(t) < \pi^2/\omega^2$ for every $t \in \mathbb{R}$. And in [8], Li obtained the existence results of positive ω-periodic solutions for the second-order differential equation with constant delays

$$-u''(t) + a(t)u(t) = f(t, u(t-\tau_1), \ldots, u(t-\tau_n)), \quad t \in \mathbb{R}, \tag{1.3}$$

by employing the fixed-point index theory in cones. For the second-order differential equations without delay, the existence of positive periodic solutions has been discussed by more authors, see [9–14].

Motivated by the paper mentioned above, we research the existence of positive periodic solutions of (1.1). We aim to obtain the essential conditions on the existence of positive periodic solution of (1.1) by constructing a special cone and applying the fixed-point index theory in cones.

In this paper, we assume the following conditions:

(H1) $a \in C(\mathbb{R}, (0, +\infty))$ is ω-periodic function and there exists a constant $1 \le p \le +\infty$ such that

$$\|a\|_p \le K(2p^*), \tag{1.4}$$

where $\|a\|_p$ is the p-norm of a in $L^p[0,\omega]$, p^* is the conjugate exponent of p defined by $(1/p) + (1/p^*) = 1$, and the function $K(q)$ is defined by

$$K(q) = \begin{cases} \dfrac{2\pi}{q\omega^{1+2/q}} \left(\dfrac{2}{2+q}\right)^{1-2/q} \left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{if } 1 \le q < +\infty, \\ \dfrac{4}{\omega}, & \text{if } q = +\infty \end{cases} \tag{1.5}$$

in which Γ is the Gamma function.

(H2) $f \in C(\mathbb{R} \times [0, +\infty)^{n+1}, [0, +\infty))$ and $f(t, x_0, x_1, \ldots, x_n)$ is ω-periodic in t.

(H3) $\tau_1(t), \ldots, \tau_n(t) \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions.

In Assumption (H1), if $p = +\infty$, since $K(2) = \pi^2/\omega^2$, then (1.4) implies that a satisfies the condition

$$0 < a(t) \le \frac{\pi^2}{\omega^2}, \quad t \in [0, \omega]. \tag{1.6}$$

This condition includes the case discussed in [5].

The techniques used in this paper are completely different from those in [5]. Our results are more general than those in [5] in two aspects. Firstly, we relax the conditions of the coefficient $a(t)$ appeared in an equation in [5] and expand the range of its values. Secondly, by constructing a special cone and applying the fixed-point index theory in cones, we obtain the essential conditions on the existence of positive periodic solutions of (1.1). The conditions concern the first eigenvalue of the associated linear periodic boundary problem, which improve and optimize the results in [5]. To our knowledge, there are very few works on the existence of positive periodic solutions for the above functional differential equations under the conditions concerning the first eigenvalue of the corresponding linear equation.

Our main results are presented and proved in Section 3. Some preliminaries to discuss (1.1) are presented in Section 2.

2. Preliminaries

In order to discuss (1.1), we consider the existence of ω-periodic solution of the corresponding linear differential equation

$$u'' + a(t)u = h(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $h \in C(\mathbb{R})$ is a ω-periodic function. It is obvious that finding an ω-periodic solution of (2.1) is equivalent to finding a solution of the linear periodic boundary value problem

$$\begin{gathered} u'' + a(t)u = h(t), \quad t \in [0,\omega], \\ u(0) = u(\omega), \quad u'(0) = u'(\omega). \end{gathered} \tag{2.2}$$

In [14], Torres show the following existence resulted.

Lemma 2.1. *Assume that (H1) holds, then for every $h \in C[0,\omega]$, the linear periodic boundary problem* (2.2) *has a unique solution expressed by*

$$u(t) = \int_0^\omega G(t,s)h(s)ds, \quad t \in [0,\omega], \tag{2.3}$$

where $G(t,s) \in C([0,\omega]\times[0,\omega])$ is the Green function of the linear periodic boundary problem (2.2)*, which satisfies the positivity: $G(t,s) > 0$ for every $(t,s) \in [0,\omega] \times [0,\omega]$.*

For the details, see [14, Theorem 2.1 and Corollary 2.3].

For $m \in \mathbb{N}$, we use $C_\omega^m(\mathbb{R})$ to denote the mth-order continuous differentiable ω-periodic functions space. Let $X = C_\omega(\mathbb{R})$ be the Banach space of all continuous ω-periodic functions equipped the norm $\|u\| = \max_{0\le t\le\omega}|u(t)|$.

Let

$$K_0 = \{u \in X \mid u(t) \ge 0,\ t \in \mathbb{R}\} \tag{2.4}$$

be the cone of all nonnegative functions in X. Then X is an ordered Banach space by the cone K_0. K_0 has a nonempty interior

$$\operatorname{int}(K_0) = \{u \in X \mid u(t) > 0,\ t \in \mathbb{R}\}. \tag{2.5}$$

Let

$$\underline{G} = \min_{0 \le t,s \le \omega} G(t,s), \quad \overline{G} = \max_{0 \le t,s \le \omega} G(t,s); \quad \sigma = \underline{G}/\overline{G}. \tag{2.6}$$

Lemma 2.2. *Assume that (H1) holds, then for every $h \in X$,* (2.1) *has a unique ω-periodic solution u. Let $T : h \mapsto u$, then $T : X \to X$ is a completely continuous linear operator, and when $h \in K_0$, Th has the positivity estimate*

$$Th(t) \ge \sigma \|Th\|, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Proof. Let $h \in X$. By Lemma 2.1, the linear periodic boundary problem (2.2) has a unique solution $u \in C^2[0,\omega]$ given by (2.3). We extend u to a ω-periodic function, which is still denoted by u, then $u := Th \in C^2_\omega(\mathbb{R})$ is a unique ω-periodic solution of (2.1). By (2.3),

$$Th(t) = \int_0^\omega G(t,s)h(s)ds, \quad t \in [0,\omega]. \tag{2.8}$$

From this we see that T maps every bounded set in X to a bounded equicontinuous set of X. Hence, by the Ascoli-Arzelà theorem, $T : X \to X$ is completely continuous.

Let $h \in K_0$. For every $t \in [0,\omega]$, from (2.8) it follows that

$$Th(t) = \int_0^\omega G(t,s)h(s)ds \le \overline{G} \int_0^\omega h(s)ds, \tag{2.9}$$

and therefore,

$$\|Th\| \le \overline{G} \int_0^\omega h(s)ds. \tag{2.10}$$

Using (2.8) and this inequality, we have that

$$\begin{aligned} Th(t) &= \int_0^\omega G(t,s)h(s)ds \ge \underline{G} \int_0^\omega h(s)ds \\ &= \left(\underline{G}/\overline{G}\right) \cdot \overline{G} \int_0^\omega h(s)ds \\ &\ge \sigma \|Th\|. \end{aligned} \tag{2.11}$$

Hence, by the periodicity of u, (2.7) holds for every $t \in \mathbb{R}$. □

From (2.7) we easily see that $T(K_0) \subset \text{int}(K_0)$; namely, $T : X \to X$ is a strongly positive linear operator. By the well-known Krein-Rutman theorem, the spectral radius $r(T) > 0$ is a simple eigenvalue of T, and T has a corresponding positive eigenfunction $\phi \in K_0$; that is,

$$T\phi = r(T)\phi. \tag{2.12}$$

Since ϕ can be replaced by $c\phi$, where $c > 0$ is a constant, we can choose $\phi \in K_0$ such that

$$\int_0^\omega \phi(t)dt = 1. \tag{2.13}$$

Set $\lambda_1 = 1/r(T)$, then $\phi = T(\lambda_1\phi)$. By Lemma 2.2 and the definition of T, $\phi \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$\phi''(t) + a(t)\phi(t) = \lambda_1\phi(t), \quad t \in \mathbb{R}. \tag{2.14}$$

Thus, λ_1 is the minimum positive real eigenvalue of the linear equation (2.1) under the ω-periodic condition. Summarizing these facts, we have the following lemma.

Lemma 2.3. *Assume that (H1) holds, then there exist $\phi \in K_0 \cap C^2_\omega(\mathbb{R})$ such that* (2.13) *and* (2.14) *hold.*

Let $f : \mathbb{R} \times [0, \infty)^{n+1} \to [0, \infty)$ satisfy the assumption (H2). For every $u \in X$, set

$$F(u)(t) := f(t, u(t), u(t - \tau_1), \dots, u(t - \tau_n)), \quad t \in \mathbb{R}. \tag{2.15}$$

Then $F : K_0 \to K_0$ is continuous. Define a mapping $A : K_0 \to X$ by

$$A = T \circ F. \tag{2.16}$$

By the definition of operator T, the ω-periodic solution of (1.1) is equivalent to the fixed point of A. Choose a subcone of K_0 by

$$K = \{u \in K_0 \mid u(t) \geq \sigma\|u\|_C,\ t \in \mathbb{R}\}. \tag{2.17}$$

By the strong positivity (2.7) of T and the definition of A, we easily obtain the following.

Lemma 2.4. *Assume that (H1) holds, then $A(K_0) \subset K$ and $A : K \to K$ is completely continuous.*

Hence, the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. We will find the nonzero fixed point of A by using the fixed-point index theory in cones.

We recall some concepts and conclusions on the fixed-point index in [15, 16]. Let X be a Banach space and $K \subset X$ a closed convex cone in X. Assume that Ω is a bounded open subset of X with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed-point index $i(A, K \cap \Omega, K)$ has definition. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas in [16] are needed in our argument.

Lemma 2.5. *Let Ω be a bounded open subset of X with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\mu Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.6. *Let Ω be a bounded open subset of X and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

3. Main Results

We consider the existence of positive ω-periodic solutions of (1.1). Assume that $f : \mathbb{R} \times [0, \infty)^{n+1} \to [0, \infty)$ satisfy (H2). To be convenient, we introduce the notations

$$\begin{aligned} f_0 &= \liminf_{\overline{x} \to 0^+} \min_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\overline{x}}, \\ f^0 &= \limsup_{\overline{x} \to 0^+} \max_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\underline{x}}, \\ f_\infty &= \liminf_{\underline{x} \to +\infty} \min_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\overline{x}}, \\ f^\infty &= \limsup_{\underline{x} \to +\infty} \max_{t \in [0,\omega]} \frac{f(t, x_0, x_1, \ldots, x_n)}{\underline{x}}, \end{aligned} \tag{3.1}$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Our main results are as follows.

Theorem 3.1. *Suppose that (H1)–(H3) hold. If f satisfies the condition*

(H4) $f^0 < \lambda_1 < f_\infty$,

then (1.1) *has at least one positive ω-periodic solution.*

Theorem 3.2. *Suppose that (H1)–(H3) hold. If f satisfies the condition*

(H5) $f^\infty < \lambda_1 < f_0$,

then (1.1) *has at least one positive ω-periodic solution.*

In Theorem 3.1, the condition (H4) allows $f(t, x_0, x_1, \ldots, x_n)$ to be superlinear growth on $x_0, x_1, \ldots, x_n$. For example,

$$f(t, x_0, x_1, \ldots, x_n) = b_0(t)x_0^2 + b_1(t)x_1^2 + \cdots + b_n(t)x_n^2 \tag{3.2}$$

satisfies (H4) with $f^0 = 0$ and $f_\infty = +\infty$, where $b_0, b_1, \ldots, b_n \in C_\omega(\mathbb{R})$ are positive ω-periodic functions.

In Theorem 3.2, the condition (H5) allows $f(t, x_0, x_1, \ldots, x_n)$ to be sublinear growth on $x_0, x_1, \ldots, x_n$. For example,

$$f(t, x_0, x_1, \ldots, x_n) = c_0(t)\sqrt{|x_0|} + c_1(t)\sqrt{|x_1|} + \cdots + c_n(t)\sqrt{|x_n|} \tag{3.3}$$

satisfies (H5) with $f_0 = +\infty$ and $f^\infty = 0$, where $c_0, c_1, \ldots, c_n \in C_\omega(\mathbb{R})$ are positive ω-periodic solution.

Applying Theorems 3.1 and 3.2 to (1.2), we have the following.

Corollary 3.3. *Suppose that (H1)–(H3) hold. If the parameter λ satisfies one of the following conditions*

(1) $\lambda_1/f_\infty < \lambda < \lambda_1/f^0$,

(2) $\lambda_1/f_0 < \lambda < \lambda_1/f^\infty$,

then (1.2) *has at least one positive ω-periodic solution.*

This result improves and extends [5, Theorem 1.3].

Proof of Theorem 3.1. Let $K \subset X$ be the cone defined by (2.17) and $A : K \to K$ the operator defined by (2.16). Then the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in X \mid \|u\| < r\}, \qquad \Omega_2 = \{u \in X \mid \|u\| < R\}. \tag{3.4}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Since $f^0 < \lambda_1$, by the definition of f^0, there exist $\varepsilon \in (0, \lambda_1)$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \ldots, x_n) \le (\lambda_1 - \varepsilon)\underline{x}, \quad t \in [0, \omega],\ \overline{x} \in (0, \delta), \tag{3.5}$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Choosing $r \in (0, \delta)$, we prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$; namely, $\mu Au \ne u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \mu \le 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \le 1$ such that $\mu_0 A u_0 = u_0$ and since $u_0 = T(\mu_0 F(u_0))$, by definition of T and Lemma 2.2, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + a(t)u_0(t) = \mu_0 F(u_0)(t), \quad t \in \mathbb{R}, \tag{3.6}$$

where $F(u)$ is defined by (2.15). Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 < \sigma\ \|u_0\| \le u_0(\tau) \le \|u_0\| = r < \delta, \quad \forall \tau \in \mathbb{R}. \tag{3.7}$$

This implies that

$$0 < \max\{u_0(t), u_0(t - \tau_1(t)), \ldots, u_0(t - \tau_n(t))\} < \delta, \quad t \in \mathbb{R}. \tag{3.8}$$

From this and (3.5), it follows that

$$\begin{aligned}F(u_0)(t) &= f(t, u_0(t), u_0(t-\tau_1(t)), \dots, u_0(t-\tau_n(t)))\\&\le (\lambda_1 - \varepsilon)\cdot \min\{u_0(t), u_0(t-\tau_1(t)), \dots, u_0(t-\tau_n(t))\}\\&\le (\lambda_1-\varepsilon)u_0(t), \quad t \in \mathbb{R}.\end{aligned} \tag{3.9}$$

By this inequality and (3.6), we have

$$u_0''(t) + a(t)u_0(t) = \mu_0 F(u_0)(t) \le (\lambda_1 - \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.10}$$

Let $\phi \in K \cap C^2_\omega(\mathbb{R})$ be the function given in Lemma 2.4. Multiplying the inequality (3.10) by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^\omega \left[u_0''(t) + a(t)u_0(t)\right]\phi(t)dt \le (\lambda_1-\varepsilon)\int_0^\omega u_0(t)\phi(t)dt. \tag{3.11}$$

For the left side of the above inequality using integration by parts, then using the periodicity of u_0 and ϕ and (2.14), we have

$$\begin{aligned}\int_0^\omega \left[u_0''(t) + a(t)u_0(t)\right]\phi(t)dt &= \int_0^\omega u_0(t)\left[\phi''(t) + a(t)\phi(t)\right]dt\\&= \lambda_1\int_0^\omega u_0(x)\phi(t)dt.\end{aligned} \tag{3.12}$$

Consequently, we obtain that

$$\lambda_1\int_0^\omega u_0(x)\phi(t)dt \ \le (\lambda_1-\varepsilon)\int_0^\omega u_0(x)\phi(t)dt. \tag{3.13}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definition of K and (2.13),

$$\int_0^\omega u_0(x)\phi(t)dt \ge \sigma\|u_0\|\int_0^\omega \phi(t)dt = \sigma\|u_0\| > 0. \tag{3.14}$$

From this and (3.13), we conclude that $\lambda_1 \le \lambda_1 - \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$. By Lemma 2.5, we have

$$i(A, K\cap\Omega_1, K) = 1. \tag{3.15}$$

On the other hand, since $f_\infty > \lambda_1$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$f(t, x_0, x_1, \dots, x_n) \ge (\lambda_1 + \varepsilon_1)\ \overline{x}, \quad t\in[0,\omega],\ \underline{x} > H, \tag{3.16}$$

where $\overline{x} = \max\{x_0, x_1, \dots, x_n\}$ and $\underline{x} = \min\{x_0, x_1, \dots, x_n\}$. Choose $R > \max\{H/\sigma, \delta\}$ and $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$; namely, $u - Au \neq \mu\phi$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$, since $u_1 - \mu_1 e = Au_1 = T(F(u_1))$, by the definition of T and Lemma 2.2, $u_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$u_1''(t) + a(t)\big(u_1(t) - \mu_1\big) = F(u_1)(t), \quad t \in \mathbb{R}. \tag{3.17}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definitions of K and Ω_2, we have

$$u_1(\tau) \geq \sigma\|u_1\| = \sigma R > H, \quad \forall\tau \in \mathbb{R}. \tag{3.18}$$

This means that

$$\min\{u_1(t), u_1(t - \tau_1(t)), \dots, u_1(t - \tau_n(t))\} > H, \quad t \in \mathbb{R}. \tag{3.19}$$

Combining this with (3.16), we have that

$$\begin{aligned} F(u_1)(t) &= f(t, u_1(t), u_1(t - \tau_1(t)), \dots, u_1(t - \tau_n(t))) \\ &\geq (\lambda_1 + \varepsilon_1) \cdot \max\{u_1(t), u_1(t - \tau_1(t)), \dots, u_1(t - \tau_n(t))\} \\ &\geq (\lambda_1 + \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \end{aligned} \tag{3.20}$$

From this inequality and (3.17), it follows that

$$u_1''(t) + a(t)u_1(t) = \mu_1 a(t) + F(u_1)(t) \geq (\lambda_1 + \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \tag{3.21}$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^\omega \Big[u_1''(t) + a(t)u_1(t)\Big]\phi(t)dt \geq (\lambda_1 + \varepsilon_1)\int_0^\omega u_1(t)\phi(t)dt. \tag{3.22}$$

For the left side of the above inequality using integration by parts and (2.14), we have

$$\begin{aligned} \int_0^\omega [u_1''(t) + a(t)u_1(t)]\phi(t)dt &= \int_0^\omega u_1(t)\Big[\phi''(t) + a(t)\phi(t)\Big]dt \\ &= \lambda_1 \int_0^\omega u_1(x)\phi(t)dt. \end{aligned} \tag{3.23}$$

From this and (3.22), it follows that

$$\lambda_1 \int_0^\omega u_1(x)\phi(t)dt \geq (\lambda_1 + \varepsilon_1)\int_0^\omega u_1(x)\phi(t)dt. \tag{3.24}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K and (2.13), we have

$$\int_0^{\omega} u_1(x)\phi(t)dt \geq \sigma\|u_1\| \int_0^{\omega} \phi(t)dt = \sigma\|u_1\| > 0. \tag{3.25}$$

Hence, from (3.24) it follows that $\lambda_1 \geq \lambda_1 + \varepsilon_1$, which is a contradiction. Therefore, A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$. By Lemma 2.6, we have

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.26}$$

Now by the additivity of the fixed-point index (3.15), and (3.26), we have

$$i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1. \tag{3.27}$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

Proof of Theorem 3.2. Let $\Omega_1, \Omega_2 \subset X$ be defined by (3.4). We prove that the operator A defined by (2.16) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R is large enough.

By $f_0 > \lambda_1$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \ldots, x_n) \geq (\lambda_1 + \varepsilon)\overline{x}, \quad t \in [0, \omega], \ \overline{x} \in (0, \delta), \tag{3.28}$$

where $\overline{x} = \max\{x_0, x_1, \ldots, x_n\}$. Let $r \in (0, \delta)$ and $e(t) \equiv 1$. We prove that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$; namely, $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_1$ and $\mu \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0 e$ and since $u_0 - \mu_0 e = Au_0 = T(F(u_0))$, by the definition of T and Lemma 2.2, $u_0 \in C_{\omega}^2(\mathbb{R})$ satisfies the differential equation

$$u_0''(t) + a(t)\left(u_0(t) - \mu_0\right) = F(u_0)(t), \quad t \in \mathbb{R}. \tag{3.29}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.7), and hence (3.8) holds. From (3.8) and (3.28), it follows that

$$\begin{aligned} F(u_0)(t) &= f(t, u_0(t), u_0(0 - \tau_1(t)), \ldots, u_0(t - \tau_n(t))) \\ &\geq (\lambda_1 + \varepsilon) \cdot \max\{u_0(t), u_0(t - \tau_1(t)), \ldots, u_0(t - \tau_n(t))\} \\ &\geq (\lambda_1 + \varepsilon)u_0(t), \quad t \in \mathbb{R}. \end{aligned} \tag{3.30}$$

By this and (3.29), we obtain that

$$u_0''(t) + a(t)u_0(t) = \mu_0 a(t) + F(u_0)(t) \geq (\lambda_1 + \varepsilon)u_0(t), \quad t \in \mathbb{R}. \tag{3.31}$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^{\omega} \left[u_0''(t) + a(t)u_0(t)\right]\phi(t)dt \geq (\lambda_1 + \varepsilon) \int_0^{\omega} u_0(t)\phi(t)dt. \tag{3.32}$$

For the left side of this inequality using integration by parts and (2.14), we have

$$\begin{aligned}\int_0^\omega \left[u_0''(t) + a(t)u_0(t)\right]\phi(t)dt &= \int_0^\omega u_0(t)\left[\phi''(t) + a(t)\phi(t)\right]dt \\ &= \lambda_1 \int_0^\omega u_0(x)\phi(t)dt.\end{aligned} \tag{3.33}$$

From this and (3.32), it follows that

$$\lambda_1 \int_0^\omega u_1(x)\phi(t)dt \geq (\lambda_1 + \varepsilon_1) \int_0^\omega u_1(x)\phi(t)dt. \tag{3.34}$$

Since $u_0 \in K \cap \partial\Omega_1$, from the definition of K and (2.13) it follows that (3.14) holds. By (3.14) and (3.34), we see that $\lambda_1 \geq \lambda_1 + \varepsilon$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$. By Lemma 2.6, we have

$$i(A, K \cap \Omega_1, K) = 0. \tag{3.35}$$

Since $f^\infty < \lambda_1$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, \lambda_1)$ and $H > 0$ such that

$$f(t, x_0, x_1, \ldots, x_n) \leq (\lambda_1 - \varepsilon_1)\underline{x}, \quad t \in [0, \omega], \ \underline{x} > H, \tag{3.36}$$

where $\underline{x} = \min\{x_0, x_1, \ldots, x_n\}$. Choosing $R > \max\{H/\sigma, \delta\}$, we show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$; namely, $\mu Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \mu \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \leq 1$ such that $\mu_1 Au_1 = u_1$, since $u_1 = T(\mu_1 F(u_1))$, by the definition of T and Lemma 2.2, $u_1 \in C_\omega^2(\Omega)$ satisfies the differential equation

$$u_1''(t) + a(t)u_1(t) = \mu_1 F(u_1)(t), \quad t \in \mathbb{R}. \tag{3.37}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definitions of K and Ω_2, u_1 satisfies (3.18), and hence (3.19) holds. By (3.19) and (3.36), we have

$$\begin{aligned}F(u_1)(t) &= f(t, u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))) \\ &\leq (\lambda_1 - \varepsilon_1) \cdot \min\{u_1(t), u_1(t - \tau_1(t)), \ldots, u_1(t - \tau_n(t))\} \\ &\leq (\lambda_1 - \varepsilon_1)u_1(t), \quad t \in \mathbb{R}.\end{aligned} \tag{3.38}$$

From this inequality and (3.37), it follows that

$$u_1''(t) + a(t)u_1(t) = \mu_1 F(u_1)(t) \leq (\lambda_1 - \varepsilon_1)u_1(t), \quad t \in \mathbb{R}. \tag{3.39}$$

Multiplying this inequality by $\phi(t)$ and integrating on $[0, \omega]$, we have

$$\int_0^\omega \left[u_1''(t) + a(t)u_1(t)\right]\phi(t)dt \leq (\lambda_1 - \varepsilon_1) \int_0^\omega u_1(t)\phi(t)dt. \tag{3.40}$$

For the left side of this inequality using integration by parts and (2.14), we have

$$\begin{aligned}\int_0^{\omega}\left[u_1''(t)+a(t)u_1(t)\right]\phi(t)dt &= \int_0^{\omega}u_1(t)\left[\phi''(t)+a(t)\phi(t)\right]dt\\ &= \lambda_1\int_0^{\omega}u_1(x)\phi(t)dt.\end{aligned}\tag{3.41}$$

Consequently, we obtain that

$$\lambda_1\int_0^{\omega}u_1(x)\phi(t)dt \le (\lambda_1-\varepsilon_1)\int_0^{\omega}u_1(x)\phi(t)dt.\tag{3.42}$$

Since $u_1 \in K\cap\partial\Omega_2$, by the definition of K and (2.13) we see that (3.25) holds. From (3.25) and (3.42), we see that $\lambda_1 \le \lambda_1-\varepsilon_1$, which is a contradiction. Hence, A satisfies the condition of Lemma 2.5 in $K\cap\partial\Omega_2$. By Lemma 2.5 we have

$$i(A, K\cap\Omega_2, K) = 1.\tag{3.43}$$

Now, from (3.35) and (3.43) it follows that

$$i\left(A, K\cap\left(\Omega_2\setminus\overline{\Omega}_1\right), K\right) = i(A, K\cap\Omega_2, K) - i(A, K\cap\Omega_1, K) = 1.\tag{3.44}$$

Hence, A has a fixed point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

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Research Article

Solvability of Nonlinear Integral Equations of Volterra Type

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This paper deals with the existence of continuous bounded solutions for a rather general nonlinear integral equation of Volterra type and discusses also the existence and asymptotic stability of continuous bounded solutions for another nonlinear integral equation of Volterra type. The main tools used in the proofs are some techniques in analysis and the Darbo fixed point theorem via measures of noncompactness. The results obtained in this paper extend and improve essentially some known results in the recent literature. Two nontrivial examples that explain the generalizations and applications of our results are also included.

1. Introduction

It is well known that the theory of nonlinear integral equations and inclusions has become important in some mathematical models of real processes and phenomena studied in mathematical physics, elasticity, engineering, biology, queuing theory economics, and so on (see, [1–3]). In the last decade, the existence, asymptotical stability, and global asymptotical stability of solutions for various Volterra integral equations have received much attention, see, for instance, [1, 4–22] and the references therein.

In this paper, we are interested in the following nonlinear integral equations of Volterra type:

$$x(t) = f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t, s, x(c(s)))ds\right), \quad \forall t \in \mathbb{R}_+, \tag{1.1}$$

$$x(t) = h\left(t, x(t), \int_0^{\alpha(t)} u(t, s, x(c(s)))ds\right), \quad \forall t \in \mathbb{R}_+, \tag{1.2}$$

where the functions f, h, u, a, b, c, α and the operator H appearing in (1.1) are given while $x = x(t)$ is an unknown function.

To the best of our knowledge, the papers dealing with (1.1) and (1.2) are few. But some special cases of (1.1) and (1.2) have been investigated by a lot of authors. For example, Arias et al. [4] studied the existence, uniqueness, and attractive behaviour of solutions for the nonlinear Volterra integral equation with nonconvolution kernels

$$x(t) = \int_0^t k(t,s)g(x(s))ds, \quad \forall t \in \mathbb{R}_+. \tag{1.3}$$

Using the monotone iterative technique, Constantin [13] got a sufficient condition which ensures the existence of positive solutions of the nonlinear integral equation

$$x(t) = L(t) + \int_0^t \left[M(t,s)x(s) + K(t,s)g(x(s))\right]ds, \quad \forall t \in \mathbb{R}_+. \tag{1.4}$$

Roberts [21] examined the below nonlinear Volterra integral equation

$$x(t) = \int_0^t k(t-s)G((x(s),s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.5}$$

which arose from certain models of a diffusive medium that can experience explosive behavior; utilizing the Darbo fixed point theorem and the measure of noncompactness in [7], Banaś and Dhage [6], Banaś et al. [8], Banaś and Rzepka [9, 10], Hu and Yan [16] and Liu and Kang [19] investigated the existence and/or asymptotic stability and/or global asymptotic stability of solutions for the below class of integral equations of Volterra type:

$$x(t) = (Tx)(t)\int_0^t u(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.6}$$

$$x(t) = f(t,x(t)) + \int_0^t u(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.7}$$

$$x(t) = f(t,x(t))\int_0^t u(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.8}$$

$$x(t) = g(t,x(t)) + x(t)\int_0^t u(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.9}$$

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^t u(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+, \tag{1.10}$$

$$x(t) = f(t,x(\alpha(t))) + \int_0^{\beta(t)} g\left(t,s,x(\gamma(s))\right)ds, \quad \forall t \in \mathbb{R}_+, \tag{1.11}$$

respectively. By means of the Schauder fixed point theorem and the measure of noncompactness in [7], Banaś and Rzepka [11] studied the existence of solutions for the below nonlinear quadratic Volterra integral equation:

$$x(t) = p(t) + f(t,x(t))\int_0^t v(t,s,x(s))ds, \quad \forall t \in \mathbb{R}_+. \tag{1.12}$$

Banaś and Chlebowicz [5] got the solvability of the following functional integral equation

$$x(t) = f_1\left(t, \int_0^t k(t,s) f_2(s, x(s)) ds\right), \quad \forall t \in \mathbb{R}_+ \tag{1.13}$$

in the space of Lebesgue integrable functions on $\mathbb{R}_+$. El-Sayed [15] studied a differential equation of neutral type with deviated argument, which is equivalent to the functional-integral equation

$$x(t) = f\left(t, \int_0^{H(t)} x(s) ds, x(h(t))\right), \quad \forall t \in \mathbb{R}_+ \tag{1.14}$$

by the technique linking measures of noncompactness with the classical Schauder fixed point principle. Using an improvement of the Krasnosel'skii type fixed point theorem, Taoudi [22] discussed the existence of integrable solutions of a generalized functional-integral equation

$$x(t) = g(t, x(t)) + f_1\left(t, \int_0^t k(t,s) f_2(s, x(s)) ds\right), \quad \forall t \in \mathbb{R}_+. \tag{1.15}$$

Dhage [14] used the classical hybrid fixed point theorem to establish the uniform local asymptotic stability of solutions for the nonlinear quadratic functional integral equation of mixed type

$$x(t) = f(t, x(\alpha(t)))\left(q(t) + \int_0^{\beta(t)} u(t, s, x(\gamma(s))) ds\right), \quad \forall t \in \mathbb{R}_+. \tag{1.16}$$

The purpose of this paper is to prove the existence of continuous bounded solutions for (1.1) and to discuss the existence and asymptotic stability of continuous bounded solutions for (1.2). The main tool used in our considerations is the technique of measures of noncompactness [7] and the famous fixed point theorem of Darbo [23]. The results presented in this paper extend proper the corresponding results in [6, 9, 10, 15, 16, 19]. Two nontrivial examples which show the importance and the applicability of our results are also included.

This paper is organized as follows. In the second section, we recall some definitions and preliminary results and prove a few lemmas, which will be used in our investigations. In the third section, we state and prove our main results involving the existence and asymptotic stability of solutions for (1.1) and (1.2). In the final section, we construct two nontrivial examples for explaining our results, from which one can see that the results obtained in this paper extend proper several ones obtained earlier in a lot of papers.

2. Preliminaries

In this section, we give a collection of auxiliary facts which will be needed further on. Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. Assume that $(E, \|\cdot\|)$ is an infinite dimensional Banach space with zero element θ and B_r stands for the closed ball centered at θ and with radius r. Let $B(E)$ denote the family of all nonempty bounded subsets of E.

Definition 2.1. Let D be a nonempty bounded closed convex subset of the space E. A operator $f : D \to E$ is said to be a *Darbo operator* if it is continuous and satisfies that $\mu(fA) \leq k\mu(A)$ for each nonempty subset A of D, where $k \in [0,1)$ is a constant and μ is a measure of noncompactness on $B(E)$.

The Darbo fixed point theorem is as follows.

Lemma 2.2 (see [23]). *Let D be a nonempty bounded closed convex subset of the space E and let $f : D \to D$ be a Darbo operator. Then f has at least one fixed point in D.*

Let $BC(\mathbb{R}_+)$ denote the Banach space of all bounded and continuous functions $x : \mathbb{R}_+ \to \mathbb{R}$ equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}. \tag{2.1}$$

For any nonempty bounded subset X of $BC(\mathbb{R}_+)$, $x \in X, t \in \mathbb{R}_+, T > 0$ and $\varepsilon \geq 0$, define

$$\begin{aligned}
&\omega^T(x,\varepsilon) = \sup\{|x(p) - x(q)| : p, q \in [0,T] \text{ with } |p-q| \leq \varepsilon\},\\
&\omega^T(X,\varepsilon) = \sup\left\{\omega^T(x,\varepsilon) : x \in X\right\}, \qquad \omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon),\\
&\omega_0(X) = \lim_{T\to+\infty}\omega_0^T(X), \qquad X(t) = \{x(t) : x \in X\},\\
&\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\},\\
&\mu(X) = \omega_0(X) + \limsup_{t\to+\infty} \operatorname{diam} X(t).
\end{aligned} \tag{2.2}$$

It can be shown that the mapping μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ [4].

Definition 2.3. Solutions of an integral equation are said to be *asymptotically stable* if there exists a ball B_r in the space $BC(\mathbb{R}_+)$ such that for any $\varepsilon > 0$, there exists $T > 0$ with

$$|x(t) - y(t)| \leq \varepsilon \tag{2.3}$$

for all solutions $x(t), y(t) \in B_r$ of the integral equation and any $t \geq T$.

It is clear that the concept of asymptotic stability of solutions is equivalent to the concept of uniform local attractivity [9].

Lemma 2.4. *Let φ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be functions with*

$$\lim_{t\to+\infty}\psi(t) = +\infty, \qquad \limsup_{t\to+\infty}\varphi(t) < +\infty. \tag{2.4}$$

Then

$$\limsup_{t\to+\infty}\varphi(\psi(t)) = \limsup_{t\to+\infty}\varphi(t). \tag{2.5}$$

Proof. Let $\limsup_{t\to+\infty}\varphi(t) = A$. It follows that for each $\varepsilon > 0$, there exists $T > 0$ such that

$$\varphi(t) < A + \varepsilon, \quad \forall t \geq T. \tag{2.6}$$

Equation (2.4) means that there exists $C > 0$ satisfying

$$\psi(t) \geq T, \quad \forall t \geq C. \tag{2.7}$$

Using (2.6) and (2.7), we infer that

$$\varphi(\psi(t)) < A + \varepsilon, \quad \forall t \geq C, \tag{2.8}$$

that is, (2.5) holds. This completes the proof. □

Lemma 2.5. *Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a differential function. If for each $T > 0$, there exists a positive number $\overline{a}_T$ satisfying*

$$0 \leq a'(t) \leq \overline{a}_T, \quad \forall t \in [0, T], \tag{2.9}$$

then

$$\omega^T(x \circ a, \varepsilon) \leq \omega^{a(T)}(x, \overline{a}_T\varepsilon), \quad \forall (x, \varepsilon) \in BC(R_+) \times (0, +\infty). \tag{2.10}$$

Proof. Let $T > 0$. It is clear that (2.9) yields that the function a is nondecreasing in $[0, T]$ and for any $t, s \in [0, T]$, there exists $\xi \in (0, T)$ satisfying

$$|a(t) - a(s)| = a'(\xi)|t - s| \leq \overline{a}_T|t - s| \tag{2.11}$$

by the mean value theorem. Notice that (2.9) means that $a(t) \in [a(0), a(T)] \subseteq [0, a(T)]$ for each $t \in [0, T]$, which together with (2.11) gives that

$$\begin{aligned}
\omega^T(x \circ a, \varepsilon) &= \sup\{|x(a(t)) - x(a(s))| : t, s \in [0, T], |t - s| \leq \varepsilon\} \\
&\leq \sup\{\left|x(p) - x(q)\right| : p, q \in [a(0), a(T)], \left|p - q\right| \leq \overline{a}_T\varepsilon\} \\
&\leq \omega^{a(T)}(x, \overline{a}_T\varepsilon), \quad \forall (x, \varepsilon) \in BC(R_+) \times (0, +\infty),
\end{aligned} \tag{2.12}$$

which yields that (2.10) holds. This completes the proof. □

Lemma 2.6. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\lim_{t\to+\infty}\varphi(t) = +\infty$ and X be a nonempty bounded subset of $BC(\mathbb{R}_+)$. Then*

$$\omega_0(X) = \lim_{T\to+\infty}\omega_0^{\varphi(T)}(X). \tag{2.13}$$

Proof. Since X is a nonempty bounded subset of $BC(\mathbb{R}_+)$, it follows that $\omega_0(X) = \lim_{T\to+\infty}\omega_0^T(X)$. That is, for given $\varepsilon > 0$, there exists $M > 0$ satisfying

$$\left|\omega_0^T(X) - \omega_0(X)\right| < \varepsilon, \quad \forall T > M. \tag{2.14}$$

It follows from $\lim_{t\to+\infty}\varphi(t) = +\infty$ that there exists $L > 0$ satisfying

$$\varphi(T) > M, \quad \forall T > L. \tag{2.15}$$

By means of (2.14) and (2.15), we get that

$$\left|\omega_0^{\varphi(T)}(X) - \omega_0(X)\right| < \varepsilon, \quad \forall T > L, \tag{2.16}$$

which yields (2.13). This completes the proof. □

3. Main Results

Now we formulate the assumptions under which (1.1) will be investigated.

(H1) $f : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ is continuous with $f(t,0,0,0) \in BC(\mathbb{R}_+)$ and $\overline{f} = \sup\{|f(t,0,0,0)| : t \in \mathbb{R}_+\}$;

(H2) $a, b, c, \alpha : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy that a and b have nonnegative and bounded derivative in the interval $[0,T]$ for each $T > 0$, c and α are continuous and α is nondecreasing and

$$\lim_{t\to+\infty} a(t) = \lim_{t\to+\infty} b(\mathrm{t}) = +\infty; \tag{3.1}$$

(H3) $u : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H4) there exist five positive constants r, M, M_0, M_1, and M_2 and four continuous functions $m_1, m_2, m_3, g : \mathbb{R}_+ \to \mathbb{R}_+$ such that g is nondecreasing and

$$\begin{gathered}\left|f(t,v,w,z) - f(t,p,q,y)\right| \le m_1(t)\left|v-p\right| + m_2(t)\left|w-q\right| + m_3(t)\left|z-y\right|, \\ \forall t \in \mathbb{R}_+,\ v,p \in [-r,r],\ w,q \in \left[-g(r), g(r)\right],\ z,y \in [-M,M],\end{gathered} \tag{3.2}$$

$$\lim_{t\to+\infty} \sup\left\{m_3(t)\int_0^{\alpha(t)} |u(t,s,v(c(s))) - u(t,s,w(c(s)))|ds : v,w \in B_r\right\} = 0, \tag{3.3}$$

$$\sup\left\{m_3(t)\int_0^{\alpha(t)} |u(t,s,v(c(s)))|ds : t \in \mathbb{R}_+, v \in B_r\right\} \le M_0, \tag{3.4}$$

$$\sup\left\{\int_0^{\alpha(t)} |u(t,s,v(c(s)))|ds : t \in \mathbb{R}_+, v \in B_r\right\} \le M, \tag{3.5}$$

$$\sup\{m_i(t) : t \in \mathbb{R}_+\} \le M_i, \quad i \in \{1,2\}, \tag{3.6}$$

$$M_1 r + M_2 g(r) + M_0 + \overline{f} \le r, \qquad M_1 + QM_2 < 1; \tag{3.7}$$

(H5) $H : BC(\mathbb{R}_+) \to BC(R_+)$ satisfies that $H : B_r \to BC(\mathbb{R}_+)$ is a Darbo operator with respect to the measure of noncompactness of μ with a constant Q and

$$|(Hx)(b(t))| \le g(|x(b(t))|), \quad \forall (x,t) \in B_r \times \mathbb{R}_+. \tag{3.8}$$

Theorem 3.1. *Under Assumptions* (H1)–(H5), (1.1) *has at least one solution* $x = x(t) \in B_r$.

Proof. Let $x \in B_r$ and define

$$(Fx)(t) = f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t,s,x(c(s)))ds\right), \quad \forall t \in \mathbb{R}_+. \tag{3.9}$$

It follows from (3.9) and Assumptions (H1)–(H5) that Fx is continuous on $\mathbb{R}_+$ and that

$$\begin{aligned}
|(Fx)(t)| &\le \left| f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t,s,x(c(s)))ds\right) - f(t,0,0,0)\right| + |f(t,0,0,0)| \\
&\le m_1(t)|x(a(t))| + m_2(t)|(Hx)(b(t))| + m_3(t)\left|\int_0^{\alpha(t)} u(t,s,x(c(s)))ds\right| + \overline{f} \\
&\le M_1 r + M_2 g(r) + M_0 + \overline{f} \\
&\le r, \quad \forall t \in \mathbb{R}_+,
\end{aligned} \tag{3.10}$$

which means that Fx is bounded on $\mathbb{R}_+$ and $F(B_r) \subseteq B_r$.

We now prove that

$$\mu(FX) \le (M_1 + M_2 Q)\mu(X), \quad \forall X \subseteq B_r. \tag{3.11}$$

Let X be a nonempty subset of B_r. Using (3.2), (3.6), and (3.9), we conclude that

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\quad = \left| f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t,s,x(c(s)))ds\right)\right. \\
&\qquad \left. - f\left(t, y(a(t)), (Hy)(b(t)), \int_0^{\alpha(t)} u(t,s,y(c(s)))ds\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq m_1(t)\big|x(a(t))-y(a(t))\big| + m_2(t)\big|(Hx)(b(t))-(Hy)(b(t))\big| \\
&\quad + m_3(t)\int_0^{\alpha(t)}\big|u(t,s,x(c(s)))-u(t,s,y(c(s)))\big|ds \\
&\leq M_1\big|x(a(t))-y(a(t))\big| + M_2\big|(Hx)(b(t))-(Hy)(b(t))\big| \\
&\quad + \sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,w(c(s)))-u(t,s,z(c(s)))|ds : w,z\in B_r\right\}, \quad \forall x,y\in X,\ t\in\mathbb{R}_+,
\end{aligned} \tag{3.12}$$

which yields that

$$\begin{aligned}
\operatorname{diam}(FX)(t) &\leq M_1 \operatorname{diam} X(a(t)) + M_2 \operatorname{diam}(HX)(b(t)) \\
&\quad + \sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,w(c(s)))-u(t,s,z(c(s)))|ds : w,z\in B_r\right\}, \\
&\qquad \forall t\in\mathbb{R}_+,
\end{aligned} \tag{3.13}$$

which together with (3.3), Assumption (H2) and Lemma 2.4 ensures that

$$\begin{aligned}
&\limsup_{t\to+\infty} \operatorname{diam}(FX)(t) \\
&\quad \leq M_1 \limsup_{t\to+\infty} \operatorname{diam} X(a(t)) + M_2 \limsup_{t\to+\infty} \operatorname{diam}(HX)(b(t)) \\
&\qquad + \limsup_{t\to+\infty} \sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,w(c(s)))-u(t,s,z(c(s)))|ds : w,z\in B_r\right\} \\
&\quad = M_1 \limsup_{t\to+\infty} \operatorname{diam} X(t) + M_2 \limsup_{t\to+\infty} \operatorname{diam}(HX)(t),
\end{aligned} \tag{3.14}$$

that is,

$$\limsup_{t\to+\infty} \operatorname{diam}(FX)(t) \leq M_1 \limsup_{t\to+\infty} \operatorname{diam} X(t) + M_2 \limsup_{t\to+\infty} \operatorname{diam}(HX)(t). \tag{3.15}$$

For each $T>0$ and $\varepsilon>0$, put

$$\begin{aligned}
&M_{3T} = \sup\{m_3(p) : p\in[0,T]\}, \\
&u_r^T = \sup\{|u(p,q,v)| : p\in[0,T],\ q\in[0,\alpha(T)],\ v\in[-r,r]\}, \\
&\omega_r^T(u,\varepsilon) = \sup\{|u(p,\tau,v)-u(q,\tau,v)| : p,q\in[0,T],\ |p-q|\leq\varepsilon,\ \tau\in[0,\alpha(T)],\ v\in[-r,r]\}, \\
&\omega^T(u,\varepsilon,r) = \sup\{|u(p,q,v)-u(p,q,w)| : p\in[0,T],\ q\in[0,\alpha(T)],\ v,w\in[-r,r],\ |v-w|\leq\varepsilon\}, \\
&\omega_r^T(f,\varepsilon,g(r)) = \sup\{|f(p,v,w,z)-f(q,v,w,z)| : p,\ q\in[0,T],\ |p-q|\leq\varepsilon, \\
&\qquad\qquad v\in[-r,r],\ w\in[-g(r),g(r)],\ z\in[-M,M]\}.
\end{aligned} \tag{3.16}$$

Let $T > 0$, $\varepsilon > 0$, $x \in X$ and $t, s \in [0, T]$ with $|t - s| \le \varepsilon$. It follows from (H2) that there exist $\overline{a}_T$ and $\overline{b}_T$ satisfying

$$0 \le a'(t) \le \overline{a}_T, \quad 0 \le b'(t) \le \overline{b}_T, \quad \forall t \in [0, T]. \tag{3.17}$$

In light of (3.2), (3.6), (3.9), (3.16), (3.17), and Lemma 2.5, we get that

$$\begin{aligned}
&|(Fx)(t) - (Fx)(s)| \\
&\quad \le \left| f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t, \tau, x(c(\tau)))d\tau \right) \right. \\
&\qquad \left. - f\left(t, x(a(s)), (Hx)(b(s)), \int_0^{\alpha(s)} u(s, \tau, x(c(\tau)))d\tau \right) \right| \\
&\qquad + \left| f\left(t, x(a(s)), (Hx)(b(s)), \int_0^{\alpha(s)} u(s, \tau, x(c(\tau)))d\tau \right) \right. \\
&\qquad \left. - f\left(s, x(a(s)), (Hx)(b(s)), \int_0^{\alpha(s)} u(s, \tau, x(c(\tau)))d\tau \right) \right| \\
&\quad \le m_1(t)|x(a(t)) - x(a(s))| + m_2(t)|(Hx)(b(t)) - (Hx)(b(s))| \\
&\qquad + m_3(t)\left[\left| \int_{\alpha(s)}^{\alpha(t)} |u(t, \tau, x(c(\tau)))|d\tau \right| + \int_0^{\alpha(s)} |u(t, \tau, x(c(\tau))) - u(s, \tau, x(c(\tau)))|d\tau \right] \\
&\qquad + \sup\{ |f(p, v, w, z) - f(q, v, w, z)| : p, q \in [0, T],\ |p - q| \le \varepsilon, \\
&\qquad\qquad v \in [-r, r],\ w \in [-g(r), g(r)],\ z \in [-M, M] \} \\
&\quad \le M_1 \omega^T(x \circ a, \varepsilon) + M_2 \omega^T((Hx) \circ b, \varepsilon) \\
&\qquad + M_{3T}|\alpha(t) - \alpha(s)| \sup\{ |u(p, \tau, v)| : p \in [0, T],\ \tau \in [0, \alpha(T)],\ v \in [-r, r] \} \\
&\qquad + M_{3T}\alpha(T) \sup\{ |u(p, \tau, v) - u(q, \tau, v)| : p, q \in [0, T],\ |p - q| \le \varepsilon,\ \tau \in [0, \alpha(T)],\ v \in [-r, r] \} \\
&\qquad + \omega_r^T(f, \varepsilon, g(r)) \\
&\quad \le M_1 \omega^{a(T)}(x, \overline{a}_T \varepsilon) + M_2 \omega^{b(T)}\left(Hx, \overline{b}_T \varepsilon \right) + M_{3T} \omega^T(\alpha, \varepsilon) u_r^T \\
&\qquad + M_{3T}\alpha(T)\omega_r^T(u, \varepsilon) + \omega_r^T(f, \varepsilon, g(r)),
\end{aligned} \tag{3.18}$$

which implies that

$$\begin{aligned}
\omega^T(Fx, \varepsilon) &\le M_1 \omega^{a(T)}(x, \overline{a}_T \varepsilon) + M_2 \omega^{b(T)}\left(Hx, \overline{b}_T \varepsilon \right) + M_{3T} \omega^T(\alpha, \varepsilon) u_r^T \\
&\quad + M_{3T}\alpha(T)\omega_r^T(u, \varepsilon) + \omega_r^T(f, \varepsilon, g(r)), \quad \forall T > 0,\ \varepsilon > 0,\ x \in X.
\end{aligned} \tag{3.19}$$

Notice that Assumptions (H1)–(H3) imply that the functions $\alpha = \alpha(t)$, $f = f(t,p,q,v)$ and $u = u(t,y,z)$ are uniformly continuous on the sets $[0,T]$,$[0,T] \times [-r,r] \times [-g(r),g(r)] \times [-M,M]$ and $[0,T] \times [0,\alpha(T)] \times [-r,r]$, respectively. It follows that

$$\lim_{\varepsilon\to 0}\omega^T(\alpha,\varepsilon) = \lim_{\varepsilon\to 0}\omega_r^T(u,\varepsilon) = \lim_{\varepsilon\to 0}\omega_r^T(f,\varepsilon,g(r)) = 0. \tag{3.20}$$

In terms of (3.19) and (3.20), we have

$$\omega_0^T(FX) \le M_1\omega_0^{a(T)}(X) + M_2\omega_0^{b(T)}(HX), \tag{3.21}$$

letting $T \to +\infty$ in the above inequality, by Assumption (H2) and Lemma 2.6, we infer that

$$\omega_0(FX) \le M_1\omega_0(X) + M_2\omega_0(HX). \tag{3.22}$$

By means of (3.15), (3.22), and Assumption (H5), we conclude immediately that

$$\begin{aligned}
\mu(FX) &= \omega_0(FX) + \limsup_{t\to+\infty}\operatorname{diam}(FX)(t)\\
&\le M_1\omega_0(X) + M_2\omega_0(HX) + M_1\limsup_{t\to+\infty}\operatorname{diam}X(t) + M_2\limsup_{t\to+\infty}\operatorname{diam}(HX)(t)\\
&= M_1\mu(X) + M_2\Big(\omega_0(HX) + \limsup_{t\to+\infty}\operatorname{diam}(HX)(t)\Big)\\
&\le (M_1 + M_2Q)\mu(X),
\end{aligned} \tag{3.23}$$

that is, (3.11) holds.

Next we prove that F is continuous on the ball B_r. Let $x \in B_r$ and $\{x_n\}_{n\ge1} \subset B_r$ with $\lim_{n\to\infty} x_n = x$. It follows from (3.3) that for given $\varepsilon > 0$, there exists a positive constant T such that

$$\sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s))) - u(t,s,w(c(s)))|ds : v,w \in B_r\right\} < \frac{\varepsilon}{3}, \quad \forall t > T. \tag{3.24}$$

Since $u = u(t,s,v)$ is uniformly continuous in $[0,T] \times [0,\alpha(T)] \times [-r,r]$, it follows from (3.16) that there exists $\delta_0 > 0$ satisfying

$$\omega^T(u,\delta,r) < \frac{\varepsilon}{1 + 3M_{3T}\alpha(T)}, \quad \forall\delta \in (0,\delta_0). \tag{3.25}$$

By Assumption (H5) and $\lim_{n\to\infty} x_n = x$, we know that there exists a positive integer N such that

$$(1 + M_1)\|x_n - x\| + M_2\|Hx_n - Hx\| < \frac{1}{2}\min\{\varepsilon,\delta_0\}, \quad \forall n > N. \tag{3.26}$$

In view of (3.2), (3.6), (3.9), (3.24)–(3.26), and Assumption (H2), we gain that for any $n > N$ and $t \in \mathbb{R}_+$

$$
\begin{aligned}
&|(Fx_n)(t) - (Fx)(t)| \\
&= \Bigg| f\left(t, x_n(a(t)), (Hx_n)(b(t)), \int_0^{\alpha(t)} u(t,s,x_n(c(s)))ds\right) \\
&\qquad - f\left(t, x(a(t)), (Hx)(b(t)), \int_0^{\alpha(t)} u(t,s,x(c(s)))ds\right)\Bigg| \\
&\le m_1(t)|x_n(a(t)) - x(a(t))| + m_2(t)|(Hx_n)(b(t)) - (Hx)(b(t))| \\
&\quad + m_3(t)\int_0^{\alpha(t)} |u(t,s,x_n(c(s))) - u(t,s,x(c(s)))|ds \\
&\le M_1\|x_n - x\| + M_2\|Hx_n - Hx\| \\
&\quad + \max\Bigg\{\sup_{\tau > T}\sup\left\{m_3(\tau)\int_0^{\alpha(\tau)} \big|u(\tau,s,z(c(s))) - u(\tau,s,y(c(s)))\big|ds : z, y \in B_r\right\}, \\
&\qquad\qquad \sup_{\tau\in[0,T]}\sup\Bigg\{m_3(\tau)\int_0^{\alpha(\tau)} \big|u(\tau,s,z(c(s))) - u(\tau,s,y(c(s)))\big|ds : z, \\
&\qquad\qquad\qquad y \in B_r, \|z - y\| \le \frac{\delta_0}{2}\Bigg\}\Bigg\} \\
&< \frac{\varepsilon}{2} + \max\left\{\frac{\varepsilon}{3}, \sup_{\tau\in[0,T]}\left\{m_3(\tau)\int_0^{\alpha(\tau)} \omega^T\left(u, \frac{\delta_0}{2}, r\right)ds\right\}\right\} \\
&< \frac{\varepsilon}{2} + \max\left\{\frac{\varepsilon}{3}, \frac{M_{3T}\alpha(T)\varepsilon}{1 + 3M_{3T}\alpha(T)}\right\} \\
&= \frac{5\varepsilon}{6},
\end{aligned}
\tag{3.27}
$$

which yields that

$$
\|Fx_n - Fx\| < \varepsilon, \quad \forall n > N, \tag{3.28}
$$

that is, F is continuous at each point $x \in B_r$.

Thus Lemma 2.2 ensures that F has at least one fixed point $x = x(t) \in B_r$. Hence (1.1) has at least one solution $x = x(t) \in B_r$. This completes the proof. □

Now we discuss (1.2) under below hypotheses:

(H6) $h : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with $h(t,0,0) \in BC(\mathbb{R}_+)$ and $\overline{h} = \sup\{|h(t,0,0)| : t \in \mathbb{R}_+\}$;

(H7) $c, \alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and α is nondecreasing;

(H8) there exist two continuous functions $m_1, m_3 : \mathbb{R}_+ \to \mathbb{R}_+$ and four positive constants r, M, M_0 and M_1 satisfying (3.3)–(3.5),

$$\begin{gathered}\left|h(t,v,w) - h(t,p,q)\right| \le m_1(t)\left|v-p\right| + m_3(t)\left|w-q\right|, \\ \forall t \in \mathbb{R}_+,\ v,\ p \in [-r,r],\ w,q \in [-M,M],\end{gathered} \tag{3.29}$$

$$\sup\{m_1(t) : t \in \mathbb{R}_+\} \le M_1, \tag{3.30}$$

$$M_0 + \overline{h} \le r(1 - M_1). \tag{3.31}$$

Theorem 3.2. *Under Assumptions* (H3) *and* (H6)–(H8), (1.2) *has at least one solution* $x = x(t) \in B_r$. *Moreover, solutions of* (1.2) *are asymptotically stable.*

Proof. As in the proof of Theorem 3.1, we conclude that (1.2) possesses at least one solution in B_r.

Now we claim that solutions of (1.2) are asymptotically stable. Note that r, M_0, and M_1 are positive numbers and $\overline{h} \ge 0$, it follows from (3.31) that $M_1 < 1$. In terms of (3.3), we infer that for given $\varepsilon > 0$, there exists $T > 0$ such that

$$\begin{aligned}&\sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s))) - u(t,s,w(c(s)))|ds : v,w \in B_r\right\} \\ &\quad< \varepsilon(1 - M_1), \quad \forall t \ge T.\end{aligned} \tag{3.32}$$

Let $z = z(t), y = y(t)$ be two arbitrarily solutions of (1.2) in B_r. According to (3.29)–(3.32), we deduce that

$$\begin{aligned}&\left|z(t) - y(t)\right| \\ &\quad= \left|h\left(t, z(t), \int_0^{\alpha(t)} u(t,\tau,z(c(\tau)))d\tau\right) - h\left(t, y(t), \int_0^{\alpha(t)} u(t,\tau,y(c(\tau)))d\tau\right)\right| \\ &\quad\le m_1(t)\left|z(t) - y(t)\right| + m_3(t)\int_0^{\alpha(t)}\left|u(t,\tau,z(c(\tau))) - u(t,\tau,y(c(\tau)))\right|d\tau \\ &\quad\le M_1\left|z(t) - y(t)\right| + \sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,\tau,v(c(\tau))) - u(t,\tau,w(c(\tau)))|d\tau : v,w \in B_r\right\} \\ &\quad< M_1\left|z(t) - y(t)\right| + \varepsilon(1 - M_1), \quad \forall t \ge T,\end{aligned} \tag{3.33}$$

which means that

$$\left|z(t) - y(t)\right| < \varepsilon \tag{3.34}$$

whenever z, y are solutions of (1.2) in B_r and $t \ge T$. Hence solutions of (1.2) are asymptotically stable. This completes the proof. □

Remark 3.3. Theorems 3.1 and 3.2 generalize Theorem 3.1 in [6], Theorem 2 in [9], Theorem 3 in [10], Theorem 1 in [15], Theorem 2 in [16], and Theorem 3.1 in [19]. Examples 4.1 and 4.2 in the fourth section show that Theorems 3.1 and 3.2 substantially extend the corresponding results in [6, 9, 10, 15, 16, 19].

4. Examples

In this section, we construct two nontrivial examples to support our results.

Example 4.1. Consider the following nonlinear integral equation of Volterra type:

$$\begin{aligned} x(t) = {} & \frac{1}{10+\ln^3(1+t^3)} + \frac{tx^2(1+3t^2)}{200(1+t)} + \frac{1}{10\sqrt{3}+20t+3|x(t^3)|} \\ & + \frac{1}{9+\sqrt{1+t}}\left(\int_0^{t^2} \frac{s^2x(4s)\sin\left(\sqrt{1+t^3s^5}-(t-s)^4x^3(4s)\right)}{1+t^{12}+\left|stx^3(4s)-3s^3+7(t-2s)^2\right| x^2(4s)}ds\right)^3, \quad \forall t\in\mathbb{R}_+. \end{aligned} \tag{4.1}$$

Put

$$\begin{aligned} f(t,v,w,z) &= \frac{1}{10+\ln^3(1+t^3)} + \frac{tv^2}{200(1+t)} + \frac{1}{10\sqrt{3}+20t+|w|} + \frac{z^3}{9+\sqrt{1+t}}, \\ u(t,s,p) &= \frac{s^2p\sin\left(\sqrt{1+t^3s^5}-(t-s)^4p^3\right)}{1+t^{12}+\left|stp^3-3s^3+7(t-2s)^2\right|p^2}, \\ a(t) &= 1+3t^2, \quad b(t)=t^3, \quad c(t)=4t, \quad \alpha(t)=t^2, \quad \forall t,s\in\mathbb{R}_+,\ v,w,z,p\in\mathbb{R}. \end{aligned} \tag{4.2}$$

Let

$$\begin{aligned} & r\in\left[\frac{9-\sqrt{51}}{2},\frac{9+\sqrt{51}}{2}\right], \quad M=3, \quad M_0=\frac{9r}{10}, \quad M_1=\frac{r}{100}, \quad M_2=\frac{1}{300}, \quad \overline{f}=\frac{1}{10}, \\ & Q=3, \quad m_1(t)=\frac{rt}{100(1+t)}, \quad m_2(t)=\frac{1}{\left(10\sqrt{3}+20t\right)^2}, \quad m_3(t)=\frac{3M^2}{9+\sqrt{1+t}}, \\ & (Hy)(t)=3y(t), \quad g(t)=3t, \quad \forall t\in\mathbb{R}_+, y\in BC(\mathbb{R}_+). \end{aligned} \tag{4.3}$$

It is easy to verify that (3.6) and Assumptions (H1)–(H3) and (H5) are satisfied. Notice that

$$\begin{aligned} & M_1r+M_2g(r)+M_0+\overline{f}\le r \iff r\in\left[\frac{9-\sqrt{51}}{2},\frac{9+\sqrt{51}}{2}\right], \\ & \left|f(t,v,w,z)-f(t,p,q,y)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t}{200(1+t)}\left|v^2-p^2\right| + \left|\frac{1}{10\sqrt{3}+20t+|w|} - \frac{1}{10\sqrt{3}+20t+|q|}\right| \\
&\quad + \frac{1}{9+\sqrt{1+t}}\left|z^3-y^3\right| \leq m_1(t)|v-p| + m_2(t)|w-q| + m_3(t)|z-y|, \\
&\qquad \forall t\in\mathbb{R}_+,\ v,p\in[-r,r],\ w,q\in[-g(r),g(r)],\ z,y\in[-M,M], \\
&\sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s)))-u(t,s,w(c(s)))|ds : v,w\in B_r\right\} \\
&= \sup\left\{\frac{3M^2}{9+\sqrt{1+t}}\int_0^{t^2}\left|\frac{s^2v(4s)\sin\left(\sqrt{1+t^3s^5}-(t-s)^4v^3(4s)\right)}{1+t^{12}+\left|stv^3(4s)-3s^3+7(t-2s)^2\right|v^2(4s)}\right.\right. \\
&\qquad\qquad \left.\left. -\frac{s^2w(4s)\sin\left(\sqrt{1+t^3s^5}-(t-s)^4w^3(4s)\right)}{1+t^{12}+\left|stw^3(4s)-3s^3+7(t-2s)^2\right|w^2(4s)}\right|ds : v,w\in B_r\right\} \\
&\leq \frac{3M^2}{9+\sqrt{1+t}}\int_0^{t^2}\frac{2s^2}{(1+t^{12})^2}\left[r\left(1+t^{12}\right)+r^3\left(r^3t^3+3r^6+7\left(t+2t^2\right)^2\right)\right]ds \\
&\leq \frac{2M^2t^6}{\left(9+\sqrt{1+t}\right)(1+t^{12})^2}\left\{r\left(1+t^{12}\right)+r^3\left[r^3t^3+3r^6+7\left(t+2t^2\right)^2\right]\right\} \longrightarrow 0 \quad \text{as } t\longrightarrow+\infty, \\
&\sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s)))|ds : t\in\mathbb{R}_+, v\in B_r\right\} \\
&= \sup\left\{\frac{3M^2}{9+\sqrt{1+t}}\int_0^{t^2}\left|\frac{s^2v(4s)\sin\left(\sqrt{1+t^3s^5}-(t-s)^4v^3(4s)\right)}{1+t^{12}+\left|stv^3(4s)-3s^3+7(t-2s)^2\right|v^2(4s)}\right|ds : t\in\mathbb{R}_+, v\in B_r\right\} \\
&\leq \sup\left\{\frac{3M^2}{10}\int_0^t\frac{s^2ds}{1+t^{12}} : t\in\mathbb{R}^+\right\} \\
&= \frac{rM^2}{20} < M_0, \\
&\sup\left\{\int_0^{\alpha(t)}|u(t,s,v(c(s)))|ds : t\in\mathbb{R}_+, v\in B_r\right\} \\
&= \sup\left\{\frac{3M^2}{9+\sqrt{1+t}}\int_0^{t^2}\left|\frac{s^2v(4s)\sin\left(\sqrt{1+t^3s^5}-(t-s)^4v^3(4s)\right)}{1+t^{12}+\left|stv^3(4s)-3s^3+7(t-2s)^2\right|v^2(4s)}\right|ds : t\in\mathbb{R}_+, v\in B_r\right\} \\
&\leq \sup\left\{\frac{3M^2}{10}\int_0^t\frac{s^2ds}{1+t^{12}} : t\in\mathbb{R}^+\right\} \\
&= \frac{r}{6} \leq M,
\end{aligned}
\tag{4.4}
$$

that is, (3.2)–(3.5) and (3.7) hold. Hence all Assumptions of Theorem 3.1 are fulfilled. Consequently, Theorem 3.1 ensures that (4.1) has at least one solution $x = x(t) \in B_r$. However Theorem 3.1 in [6], Theorem 2 in [9],Theorem 3 in [10], Theorem 1 in [15], Theorem 2 in [16], and Theorem 3.1 in [19] are unapplicable for (4.1).

Example 4.2. Consider the following nonlinear integral equation of Volterra type:

$$\begin{aligned} x(t) = {} & \frac{3+\sin^4\left(\sqrt{1+t^2}\right)}{16} + \frac{x^2(t)}{8+t^2} \\ & + \frac{1}{1+t}\left(\int_0^{\sqrt{t}} \frac{sx^3(1+s^2)}{1+t^2+s\cos^2(1+t^3s^7x^2(1+s^2))}ds\right)^2, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{4.5}$$

Put

$$\begin{aligned} & h(t,v,w) = \frac{3+\sin^4\left(\sqrt{1+t^2}\right)}{16} + \frac{v^2}{8+t^2} + \frac{w^2}{1+t}, \\ & u(t,s,p) = \frac{sp^3}{1+t^2+s\cos^2(1+t^3s^7p^2)}, \\ & \alpha(t) = \sqrt{t}, \quad c(t) = 1+t^2, \quad m_1(t) = \frac{1}{8+t^2}, \quad m_3(t) = \frac{4}{1+t}, \quad \forall t,s \in \mathbb{R}_+, v,w,p \in \mathbb{R}, \\ & r = \frac{1}{2}, \qquad M = 2, \qquad M_0 = M_1 = \frac{1}{8}, \qquad \overline{h} = \frac{1}{4}. \end{aligned} \tag{4.6}$$

It is easy to verify that (3.30), (3.31) and Assumptions (H6) and (H7) are satisfied. Notice that

$$\begin{aligned} & \left|h(t,v,w) - h(t,p,q)\right| \\ & \quad \le \frac{1}{8+t^2}\left|v^2-p^2\right| + \frac{1}{1+t}\left|w^2-q^2\right| \\ & \quad \le m_1(t)\left|v-p\right| + m_3(t)\left|z-y\right|, \quad \forall t \in \mathbb{R}_+, \ v, \ p \in [-r,r], \ w, \ q \in [-M,M], \\ & \sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s))) - u(t,s,w(c(s)))|ds : x, \ y \in B_r\right\} \\ & \quad = \sup\left\{\frac{4}{1+t}\int_0^{\sqrt{t}}\left|\frac{sv^3(1+s^2)}{1+t^2+s\cos^2(1+t^3s^7v^2(1+s^2))}\right.\right. \\ & \qquad\qquad \left.\left. - \frac{sw^3(1+s^2)}{1+t^2+s\cos^2(1+t^3s^7w^2(1+s^2))}ds\right|ds : v, \ w \in B_r\right\} \\ & \quad \le \frac{4r^3t\left(1+\sqrt{t}+t^2\right)}{(1+t)(1+t^2)^2} \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty, \end{aligned}$$

$$\sup\left\{m_3(t)\int_0^{\alpha(t)}|u(t,s,v(c(s)))|ds : t\in\mathbb{R}_+,\ v\in B_r\right\}$$
$$= \sup\left\{\frac{4}{1+t}\int_0^{\sqrt{t}}\left|\frac{sv^3(1+s^2)}{1+t^2+s\cos^2(1+t^3s^7v^2(1+s^2))}ds\right| : t\in\mathbb{R}_+,\ v\in B_r\right\}$$
$$\leq \sup\left\{\frac{4r^3}{1+t}\int_0^{\sqrt{t}}\frac{s}{1+t^2}ds : t\in\mathbb{R}^+\right\}$$
$$\leq \sup\left\{\frac{t}{4(1+t)(1+t^2)} : t\in\mathbb{R}^+\right\}$$
$$\leq \frac{1}{8} = M_1,$$
$$\sup\left\{\int_0^{\alpha(t)}|u(t,s,v(c(s)))|ds : t\in\mathbb{R}_+,\ v\in B_r\right\}$$
$$= \sup\left\{\int_0^{\sqrt{t}}\left|\frac{sv^3(1+s^2)}{1+t^2+s\cos^2(1+t^3s^7v^2(1+s^2))}ds\right| : t\in\mathbb{R}_+,\ v\in B_r\right\}$$
$$\leq \sup\left\{r^3\int_0^{\sqrt{t}}\frac{s}{1+t^2}ds : t\in\mathbb{R}^+\right\}$$
$$= \frac{1}{32} < M, \tag{4.7}$$

which yield (3.3)–(3.5). That is, all Assumptions of Theorem 3.2 are fulfilled. Therefore, Theorem 3.2 guarantees that (4.5) has at least one solution $x = x(t) \in B_r$. Moreover, solutions of (4.5) are asymptotically stable. But Theorem 2 in [9], Theorem 3 in [10], Theorem 1 in [15], Theorem 2 in [16], and Theorem 3.1 in [19] are invalid for (4.5).

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Research Article

Existence of Solutions for Fractional Integro-Differential Equation with Multipoint Boundary Value Problem in Banach Spaces

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By means of the fixed-point theorem in the cone of strict-set-contraction operators, we consider the existence of a nonlinear multi-point boundary value problem of fractional integro-differential equation in a Banach space. In addition, an example to illustrate the main results is given.

1. Introduction

The purpose of this paper is to establish the existence results of positive solution to nonlinear fractional boundary value problem

$$\begin{gathered} D_{0^+}^{q}u(t)+f\left(t,u,u',\dots,u^{(n-2)},Tu,Su\right)=\theta,\quad 0<t<1,\ n-1<q\le n,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=\theta,\qquad u^{(n-2)}(1)=\sum_{i=1}^{m-2}a_iu^{(n-2)}(\eta_i) \end{gathered}\tag{1.1}$$

in a Banach space E, where θ is the zero element of E, and $n\ge 2,\ 0<\eta_1<\cdots<\eta_{m-2}<1,\ a_i>0\ (i=1,2,\dots,m-2)$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and

$$Tu(t)=\int_0^t K(t,s)u(s)ds,\qquad Su(t)=\int_0^1 H(t,s)u(s)ds,\tag{1.2}$$

where $K \in C[B, R_+]$, $B = \{(t,s) \in I \times I : t \geq s\}$, $H \in C[I \times I, R_+]$, $I = [0,1]$, and R_+ denotes the set of all nonnegative numbers.

Fractional differential equations have gained importance due to their numerous applications in many fields of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, and probability. For details see [1–3] and the references therein. In recent years, there are some papers dealing with the existence of the solutions of initial value problems or linear boundary value problems for fractional differential equations by means of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, lower and upper solutions method, and so forth), see for example, [4–23].

In [8], by means of the fixed-point theorem for the mixed monotone operator, the authors considers unique existence of positive to singular boundary value problems for fractional differential equation

$$\begin{gathered} D_{0^+}^q u(t) + a(t) f\left(t, u, u', \ldots, u^{(n-2)}\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{gathered} \tag{1.3}$$

where $D_{0^+}^q$ is Riemann-Liouville fractional derivative of order $n-1 < q \leq n$, $n \geq 2$.

In [11], El-Shahed and Nieto study the existence of nontrivial solutions for a multi-point boundary value problem for fractional differential equations

$$\begin{gathered} D_{0^+}^q u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \ n-1 < q \leq n, \ n \in N, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{gathered} \tag{1.4}$$

where $n \geq 2$, $\eta_i \in (0,1)$, $a_i > 0$ $(i = 1, 2, \ldots, m-2)$, and $D_{0^+}^q$ is Riemann-Liouville fractional derivative. Under certain growth conditions on the nonlinearity, several sufficient conditions for the existence of nontrivial solution are obtained by using Leray-Schauder nonlinear alternative. And then, Goodrich [24] was concerned with a partial extension of the problem (1.3)

$$\begin{gathered} D_{0^+}^q u(t) = f(t, u(t)), \quad 0 < t < 1, \ n-n < q \leq n-2, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \qquad D_{0^+}^p u(1) = 0, \quad 1 \leq p \leq n-2, \end{gathered} \tag{1.5}$$

and the authors derived the Green function for the problem (1.5) and showed that it satisfies certain properties.

By the contraction mapping principle and the Krasnoselskii's fixed-point theorem, Zhou and Chu [13] discussed the existence and uniqueness results for the following fractional differential equation with multi-point boundary conditions:

$$\begin{gathered} {}^C D_{0^+}^q u(t) + f(t, u, Ku, Su) = 0, \quad 0 < t < 1, \ 1 < q < 2, \\ a_1 u(0) - b_1 u'(0) = d_1 u(\eta_1), \qquad a_2 u(1) - b_2 u'(1) = d_2 u(\eta_2), \end{gathered} \tag{1.6}$$

where ${}^{C}D_{0^+}^{q}$ is the Caputo's fractional derivative, a_1, a_2, b_1, b_2, d_1, and d_2 are real numbers, $0 < \eta_1$, and $\eta_2 < 1$.

In [20], Staněk has discussed the existence of positive solutions for the singular fractional boundary value problem

$$\begin{gathered} D^{q}u(t) + f\left(t, u, u', D^{p}u\right) = 0, \quad 2 < q < 3,\ 0 < p < 1, \\ u(0) = 0, \qquad u'(0) = u'(1) = 0. \end{gathered} \tag{1.7}$$

However, to the best of the author's knowledge, a few papers can be found in the literature dealing with the existence of solutions to boundary value problems of fractional differential equations in Banach spaces. In [25], Salem investigated the existence of Pseudo solutions for the following nonlinear m-point boundary value problem of fractional type

$$\begin{gathered} D_{0^+}^{q}u(t) + a(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \zeta_i u(\eta_i) \end{gathered} \tag{1.8}$$

in a reflexive Banach space E, where $D_{0^+}^{q}$ is the Pseudo fractional differential operator of order $n - 1 < q \leq n$, $n \geq 2$.

In [26], by the monotone iterative technique and mönch fixed-point theorem, Lv et al. investigated the existence of solution to the following Cauchy problems for differential equation with fractional order in a real Banach space E

$${}^{C}D^{q}u(t) = f(t, u(t)), \qquad u(0) = u_0, \tag{1.9}$$

where ${}^{C}D^{q}u(t)$ is the Caputo's derivative order, $0 < q < 1$.

By means of Darbo's fixed-point theorem, Su [27] has established the existence result of solutions to the following boundary value problem of fractional differential equation on unbounded domain $[0, +\infty)$

$$\begin{gathered} D_{0^+}^{q}u(t) = f(t, u(t)), \quad t \in [0, +\infty),\ 1 < q \leq 2, \\ u(0) = \theta, \qquad D_{0^+}^{q-1}u(\infty) = u_\infty \end{gathered} \tag{1.10}$$

in a Banach space E. $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative.

Motivated by the above mentioned papers [8, 13, 24, 25, 27, 28] but taking a quite different method from that in [26–29]. By using fixed-point theorem for strict-set-contraction operators and introducing a new cone Ω, we obtain the existence of at least two positive solutions for the BVP (1.1) under certain conditions on the nonlinear term in Banach spaces. Our results are different from those of [8, 13, 24, 25, 28, 30]. Note that the nonlinear term f depends on u and its derivatives $u', u'', \ldots, u^{(n-2)}$.

2. Preliminaries and Lemmas

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E; that is, $u \leq v$ if and only if $v - u \in P$; and P is said to be normal if there exists a positive constant N such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$, where the smallest N is called the normal constant of P. For details on cone theory, see [31].

The basic space used in this paper is $C[I, E]$. For any $u \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|_C)$ is a Banach space with norm $\|u\|_C = \sup_{t\in I}|u(t)|$, and $P = \{u \in C[I, E] : u(t) \geq \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$.

Definition 2.1 (see [31]). Let V be a bounded set in a real Banach space E, and $\alpha(V) = \inf\{\delta > 0 : V = \cup_{i=1}^{m} V_i,$ all the diameters of $V_i \leq \delta\}$. Clearly, $0 \leq \alpha(V) < \infty$. $\alpha(V)$ is called the Kuratovski measure of noncompactness.

We use α, α_C to denote the Kuratowski noncompactness measure of bounded sets in the spaces E, $C(I, E)$, respectively.

Definition 2.2 (see [31]). Let E_1, E_2 be real Banach spaces, $S \subset E_1$. $T : S \to E_2$ is a continuous and bounded operator. If there exists a constant k, such that $\alpha(T(S)) \leq k\alpha(S)$, then T is called a k-set contraction operator. When $k < 1$, T is called a strict-set-contraction operator.

Lemma 2.3 (see [31]). *If $D \subset C[I, E]$ is bounded and equicontinuous, then $\alpha(D(t))$ is continuous on I and*

$$\alpha_C(D) = \max_{t\in I} \alpha(D(t)), \quad \alpha\left(\left\{\int_I u(t)dt : u \in D\right\}\right) \leq \int_I \alpha(D(t))dt, \tag{2.1}$$

where $D(t) = \{u(t) : u \in D, t \in I\}$.

Definition 2.4 (see [2, 3]). The left-sided Riemann-Liouville fractional integral of order $q > 0$ of a function $y : R_+^0 \to R$ is given by

$$I_{0+}^{q} y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)ds. \tag{2.2}$$

Definition 2.5 (see [2, 3]). The fractional derivative of order $q > 0$ of a function $y : R_+^0 \to R$ is given by

$$D_{0+}^{q} y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s)ds, \tag{2.3}$$

where $n = [q] + 1, [q]$ denotes the integer part of number q, provided that the right side is pointwise defined on R_+^0.

Lemma 2.6 (see [2, 3]). *Let $q > 0$. Then the fractional differential equation*

$$D_{0+}^{q} y(t) = 0 \tag{2.4}$$

has a unique solution $y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n}$, $c_i \in R$, $i = 1, 2, \ldots, n$; here $n - 1 < q \leq n$.

Lemma 2.7 (see [2, 3]). *Let $q > 0$. Then the following equality holds for $y \in L(0,1)$, $D_{0+}^{q}y \in L(0,1)$,*

$$I_{0+}^{q}D_{0+}^{q}y(t) = y(t) + c_1t^{q-1} + c_2t^{q-2} + \cdots + c_Nt^{q-N} \tag{2.5}$$

for some $c_i \in R$, $i = 1,2,\ldots,N$; here N is the smallest integer greater than or equal to q.

Lemma 2.8 (see [31]). *Let K be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a strict-set-contraction operator such that either:*

(i) $\|Tx\| \leq \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \geq \|x\|$, $x \in K \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \geq \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \leq \|x\|$, $x \in K \cap \partial\Omega_2$,

then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

For convenience, we list some following assumptions.

(H1) There exist $a \in C[I, R_+]$ and $h \in C[R_+^{n+1}, R_+]$ such that

$$\left\|f(t, u_1, \ldots, u_{n+1})\right\| \leq a(t)h(\|u_1\|, \ldots, \|u_{n+1}\|), \quad \forall t \in I,\ u_k \in P,\ k = 1, \ldots, n+1. \tag{3.1}$$

(H2) $f : I \times P_r^{n+1} \to P$, for any $r > 0$, f is uniformly continuous on $I \times P_r^{n+1}$ and there exist nonnegative constants L_k, $k = 1, \ldots, n+1$, with

$$\frac{2}{\Gamma(q-n+2)\eta^*}\left(\sum_{k=1}^{n-2}\frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^*}{(n-3)!}L_n + \frac{b^*}{(n-3)!}L_{n+1}\right) < 1 \tag{3.2}$$

such that

$$\alpha\left(f(t, D_1, D_2, \ldots, D_{n+1})\right) \leq \sum_{k=1}^{n+1} L_k\alpha(D_k), \quad \forall t \in I,\ \text{bounded sets } D_k \in P_r, \tag{3.3}$$

where $a^* = \max\{K(t,s) : (t,s) \in B\}$, $b^* = \max\{H(t,s) : (t,s) \in I \times I\}$, $P_r = \{u \in P : \|u\| \leq r\}$, $\eta^* = 1 - \sum_{i=1}^{m-2} a_i\eta_i^{q-n+1}$.

Lemma 3.1. *Given $y \in C[I, E]$ and $1 - \sum_{i=1}^{m-2} a_i\eta_i^{q-n+1} \neq 0$ hold. Then the unique solution of*

$$\begin{gathered} D_{0^+}^{q-n+2}x(t) + y(t) = 0, \quad 0 < t < 1,\ n-1 < q \leq n,\ n \geq 2, \\ x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), \end{gathered} \tag{3.4}$$

is

$$x(t) = \int_0^1 G(t,s)y(s)ds, \tag{3.5}$$

where

$$G(t,s) = g(t,s) + \frac{\sum_{i=1}^{m-2} a_i g(\eta_i, s)}{\eta^*} t^{q-n+1}, \tag{3.6}$$

$$g(t,s) = \begin{cases} \dfrac{(t(1-s))^{q-n+1} - (t-s)^{q-n+1}}{\Gamma(q-n+2)}, & s \leq t, \\ \dfrac{(t(1-s))^{q-n+1}}{\Gamma(q-n+2)}, & t \leq s. \end{cases} \tag{3.7}$$

Proof. Deduced from Lemma 2.7, we have

$$x(t) = -I_{0+}^{q-n+2} y(t) + c_1 t^{q-n+1} + c_2 t^{q-n} \tag{3.8}$$

for some $c_1, c_2 \in R$. Consequently, the general solution of (3.4) is

$$x(t) = -\int_0^t \frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds + c_1 t^{q-n+1} + c_2 t^{q-n}. \tag{3.9}$$

By boundary value conditions $x(0) = 0,\ x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i)$, there is $c_2 = 0$, and

$$c_1 = \frac{1}{1 - \sum_{i=1}^{m-2} a_i \eta_i^{q-n+1}} \int_0^1 \frac{(1-s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds - \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i \eta_i^{q-n+1}} \int_0^{\eta_i} \frac{(\eta_i - s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds. \tag{3.10}$$

Therefore, the solution of problem (3.4) is

$$\begin{aligned} x(t) = & -\int_0^t \frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds + \frac{t^{q-n+1}}{1 - \sum_{i=1}^{m-2} a_i {\eta_i}^{q-n+1}} \int_0^1 \frac{(1-s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds \\ & - \frac{\sum_{i=1}^{m-2} a_i t^{q-n+1}}{1 - \sum_{i=1}^{m-2} a_i \eta_i^{q-n+1}} \int_0^{\eta_i} \frac{(\eta_i - s)^{q-n+1}}{\Gamma(q-n+2)} y(s)ds \\ = & \int_0^1 G(t,s)y(s)ds. \end{aligned} \tag{3.11}$$

The proof is complete. □

Moreover, there is one paper [8] in which the following statement has been shown.

Lemma 3.2. *The function $g(t,s)$ defined in (3.7) satisfying the following properties:*

(1) *$g(t,s) \geq 0$ is continuous on $[0,1]\times[0,1]$, and $g(t,s) \leq t^{q-n+1}/\Gamma(q-n+2)$, $g(t,s) \leq g(s,s)$ for all $0 \leq t,s \leq 1$;*

(2) *there exists a positive function $\rho_0 \in C(0,1)$ such that $\min_{\gamma\leq t\leq\delta} g(t,s) \geq \rho_0(s)g(t,s), s \in (0,1)$, where $0<\gamma<\delta<1$ and*

$$\rho_0(s) = \begin{cases} \dfrac{(\delta(1-s))^{q-n+1} - (\delta-s)^{q-n+1}}{(s(1-s))^{q-n+1}}, & s \in (0,\xi], \\ \left(\dfrac{\gamma}{s}\right)^{q-n+1}, & s \in [\xi,1), \end{cases} \tag{3.12}$$

where $\gamma<\xi<\delta$ is the solution of

$$(\delta(1-\xi))^{q-n+1} - (\delta-\xi)^{q-n+1} = \left(\gamma(1-\xi)\right)^{q-n+1}. \tag{3.13}$$

For our purpose, one assumes that

(H3) *$\eta^* = 1 - \sum_{i=1}^{m-2} a_i\eta_i^{q-n+1} > 0$ and $0<\gamma\leq\min\{2\delta-1,\delta/2\}$, $2/3\leq\delta<1$, where γ, δ are the constants in (2) of Lemma 3.2.*

Remark 3.3. We note that if (H3) holds, then the function $G(t,s)$ defined in (3.4) is satisfying the following properties:

(i) $G(t,s) \geq 0$ is continuous on $[0,1]\times[0,1]$, and $G(t,s) \leq \Delta t^{q-n+1}$, for all $0\leq t$, $s\leq 1$, where $\Delta^{-1} = \eta^*\Gamma(q-n+2)$;

(ii) $G(t,s) \leq G(s)$ for all $0 \leq t$, $s \leq 1$, where

$$G(s) = g(s,s) + \frac{\sum_{i=1}^{m-2} a_i g(\eta_i,s)}{\eta^*}. \tag{3.14}$$

Indeed, it is obvious from (1) of Lemma 3.2 and (3.6) that

$$\begin{aligned} G(t,s) &\leq g(s,s) + \frac{1}{\eta^*}\sum_{i=1}^{m-2} a_i g(\eta_i,s) = G(s) \\ &\leq \frac{t^{q-n+1}}{\Gamma(q-n+2)} + \frac{\sum_{i=1}^{m-2} a_i\eta_i^{q-n+1}}{\eta^*\Gamma(q-n+2)} t^{q-n+1} \leq \Delta t^{q-n+1}. \end{aligned} \tag{3.15}$$

Lemma 3.4. *Let $u(t) = I_{0^+}^{n-2}x(t)$, $x \in C[I,E]$. Then the problem (1.1) can be transformed into the following modified problem:*

$$\begin{gathered} D_{0^+}^{q-n+2}x(t) + f\left(s, I_{0^+}^{n-2}x(s), \ldots, I_{0^+}^{1}x(s), x(s), T\left(I_{0^+}^{n-2}x(s)\right), S\left(I_{0^+}^{n-2}x(s)\right)\right) = \theta, \\ x(0) = \theta, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), \end{gathered} \tag{3.16}$$

where $0 < t < 1$, $n-1 < q \leq n$, $n \geq 2$. *Moreover, if* $x \in C[I, E]$ *is a solutions of problem* (3.16), *then the function* $u(t) = I_{0^+}^{n-2} x(t)$ *is a solution of* (1.1).

The proof follows by routine calculations.

To obtain a positive solution, we construct a cone Ω by

$$\Omega = \left\{ x(t) \in P : x(t) \geq \frac{\lambda}{3} x(s),\ t \in I^*,\ s \in I \right\}, \tag{3.17}$$

where $P = \{x \in C[I, E], x(t) \geq \theta, t \in I\}$, $\lambda = \min\{\min_{\gamma \leq t \leq \delta} \rho(t), \gamma^{q-n+1}\}$, $I^* = [\gamma, \delta]$.

Let

$$(Ax)(t) = \int_0^1 G(t,s) f\left(s, I_{0^+}^{n-2} x(s), \ldots, x(s), T\left(I_{0^+}^{n-2} x(s)\right), S\left(I_{0^+}^{n-2} x(s)\right)\right) ds, \quad 0 \leq t \leq 1. \tag{3.18}$$

Lemma 3.5. *Assume that (H1)–(H3) hold. Then* $A : \Omega \to \Omega$ *is a strict-set-contraction operator.*

Proof. Let $x \in \Omega$. Then, it follows from Remark 3.3. that

$$\begin{aligned}
(Ax)(t) &\leq \int_0^1 G(s) f\left(s, I_{0^+}^{n-2} x_j(s), \ldots, x_j(s), T\left(I_{0^+}^{n-2} x_j(s)\right), S\left(I_{0^+}^{n-2} x_j(s)\right)\right) ds \\
&= \left(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\right) G(s) f\left(s, I_{0^+}^{n-2} x_j(s), \ldots, x_j(s), T\left(I_{0^+}^{n-2} x_j(s)\right), S\left(I_{0^+}^{n-2} x_j(s)\right)\right) ds \\
&\leq 3 \int_\gamma^\delta G(s) f\left(s, I_{0^+}^{n-2} x_j(s), \ldots, x_j(s), T\left(I_{0^+}^{n-2} x_j(s)\right), S\left(I_{0^+}^{n-2} x_j(s)\right)\right) ds,
\end{aligned} \tag{3.19}$$

here, by (H3), we know that $\gamma < \delta - \gamma$ and $\delta - \gamma > 1 - \delta$.

From (3.6) and (3.18), we obtain

$$\begin{aligned}
\min_{t \in [\gamma, \delta]} (Ax)(t) &= \min_{t \in [\gamma, \delta]} \int_0^1 G(t,s) f\left(s, I_{0^+}^{n-2} x_j(s), \ldots, x_j(s), T\left(I_{0^+}^{n-2} x_j(s)\right), S\left(I_{0^+}^{n-2} x_j(s)\right)\right) ds \\
&\geq \int_\gamma^\delta \left(g(t,s) + \frac{t^{q-n+1}}{\eta^*} \sum_{i=1}^{m-2} a_i g(\eta_i, s) \right) \\
&\quad \times f\left(s, I_{0^+}^{n-2} x_j(s), \ldots, x_j(s), T\left(I_{0^+}^{n-2} x_j(s)\right), S\left(I_{0^+}^{n-2} x_j(s)\right)\right) ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{\gamma}^{\delta}\left(\rho_0(s)g(s,s)+\frac{\gamma^{q-n+1}}{\eta^*}\sum_{i=1}^{m-2}a_i g(\eta_i,s)\right)\\
&\quad\times f\left(s,I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds\\
&\geq \lambda\int_{\gamma}^{\delta}G(s)f\left(s,I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds\\
&\geq \frac{\lambda}{3}(Ax)(t'),\quad t'\in I,
\end{aligned}\tag{3.20}$$

which implies that $(Ax)(t)\in\Omega$; that is, $A(\Omega)\subset\Omega$.

Next, we prove that A is continuous on Ω. Let $\{x_j\}$, $\{x\}\subset\Omega$, and $\|x_j-x\|_\Omega\to 0$ $(j\to\infty)$. Hence $\{x_j\}$ is a bounded subset of Ω. Thus, there exists $r>0$ such that $r=\sup_j\|x_j\|_\Omega<\infty$ and $\|x\|_\Omega\leq r$. It is clear that

$$\begin{aligned}
\left\|(Ax_j)(t)-(Ax)(t)\right\| &= \int_0^1 G(t,s)\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\qquad\left.-f\left(s,I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|ds\\
&\leq \Delta t^{q-n+1}\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\qquad\left.-f\left(s,I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|ds.
\end{aligned}\tag{3.21}$$

According to the properties of f, for all $\varepsilon>0$, there exists $J>0$ such that

$$\begin{aligned}
&\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\quad\left.-f\left(s,I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|<\frac{\varepsilon}{\Delta},
\end{aligned}\tag{3.22}$$

for $j\geq J$, for all $t\in I$.

Therefore, for all $\varepsilon>0$, for any $t\in I$ and $j\geq J$, we get

$$\left\|(Ax_j)(t)-(Ax)(t)\right\|<t^{q-n+1}\varepsilon\leq\varepsilon.\tag{3.23}$$

This implies that A is continuous on Ω.

By the properties of continuous of $G(t,s)$, it is easy to see that A is equicontinuous on I.

Finally, we are going to show that A is a strict-set-contraction operator. Let $D \subset \Omega$ be bounded. Then by condition (H1), Lemma 3.1 implies that $\alpha_C(AD) = \max_{t\in I} \alpha((AD)(t))$. It follows from (3.18) that

$$\begin{aligned}
\alpha((AD)(t)) &\le \alpha\Big(\overline{co}\Big\{G(t,s)f\Big(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\Big(I_{0^+}^{n-2}x(s)\Big),S\Big(I_{0^+}^{n-2}x(s)\Big)\Big) \\
&\qquad : s\in[0,t],\ t\in I,\ x\in D\Big\}\Big) \\
&\le \Delta\cdot\alpha\Big(\Big\{f\Big(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\Big(I_{0^+}^{n-2}x(s)\Big),S\Big(I_{0^+}^{n-2}x(s)\Big)\Big) \\
&\qquad : s\in[0,t],\ t\in I,\ x\in D\Big\}\Big) \\
&\le \Delta\cdot\alpha\Big(f\Big(I\times\Big(I_{0^+}^{n-2}D\Big)(I)\times\cdots\times D(I)\times T\Big(I_{0^+}^{n-2}D\Big)(I)\times S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Big) \\
&\le \Delta\cdot\Bigg\{\sum_{k=1}^{n-2}L_k\alpha\Big(\Big(I_{0^+}^{n-1-k}D\Big)(I)\Big)+L_{n-1}\alpha(D(I))+a^*L_n\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)(I)\Big) \\
&\qquad +b^*L_{n+1}\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Bigg\},
\end{aligned} \tag{3.24}$$

which implies

$$\begin{aligned}
\alpha_C(AD) \le \Delta\cdot\Bigg\{&\sum_{k=1}^{n-2}L_k\alpha\Big(\Big(I_{0^+}^{n-1-k}D\Big)(I)\Big)+L_{n-1}\alpha(D(I)) \\
&+a^*L_n\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)(I)\Big)+b^*L_{n+1}\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Bigg\}.
\end{aligned} \tag{3.25}$$

Obviously,

$$\begin{aligned}
\alpha\Big(I_{0^+}^{n-1-k}D\Big)(I) &= \alpha\left(\left\{\int_0^s\frac{(s-\tau)^{n-2-k}}{(n-2-k)!}x(\tau)d\tau : \tau\in[0,s],\ s\in I,\ k=1,\dots,n-2\right\}\right) \\
&\le \frac{1}{(n-2-k)!}\alpha(D(I)),
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)\Big)(I) &= \alpha\left(\left\{\int_0^t K(t,s)\left(\int_0^s\frac{(s-\tau)^{n-2}}{(n-3)!}u(\tau)d\tau\right)ds : u\in D,\ t\in I\right\}\right) \\
&\le \frac{a^*}{(n-3)!}\alpha(\{u(t):t\in I,\ u\in D\})\le\frac{a^*}{(n-3)!}\alpha(D(I)),
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)\Big)(I) &= \alpha\left(\left\{\int_0^1 H(t,s)\left(\int_0^s\frac{(s-\tau)^{n-2}}{(n-3)!}u(\tau)d\tau\right)ds : u\in D, t\in I\right\}\right) \\
&\le \frac{b^*}{(n-3)!}\alpha(\{u(t):t\in I,\ u\in D\})\ \le\frac{b^*}{(n-3)!}\alpha(D(I)).
\end{aligned} \tag{3.28}$$

Using a similar method as in the proof of Theorem 2.1.1 in [31], we have

$$\alpha(D(I)) \leq 2\alpha_C(D). \tag{3.29}$$

Therefore, it follows from (3.26)–(3.29) that

$$\alpha_C(AD) \leq 2\Delta \cdot \left(\sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^* L_n}{(n-3)!} + \frac{b^* L_{n+1}}{(n-3)!}\right) \alpha_C(D). \tag{3.30}$$

Noticing that (3.3), we obtain that T is a strict-set-contraction operator. The proof is complete. □

Theorem 3.6. *Let cone P be normal and conditions $(H1)$~$(H3)$ hold. In addition, assume that the following conditions are satisfied.*

(H4) *There exist $u^* \in P \setminus \{\theta\}$, $c_1 \in C[I^*, R_+]$ and $h_1 \in C[P^{n+1}, R_+]$ such that*

$$\begin{gathered} f(t, u_1, \ldots, u_{n+1}) \geq c_1(t) h_1(u_1, \ldots, u_{n-1}) u^*, \quad \forall t \in I^*, \ u_k \in P, \\ \frac{h_1(u_1, \ldots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_k\|} \longrightarrow \infty, \quad as \ \sum_{k=1}^{n-1} \|u_k\| \longrightarrow \infty, \ u_k \in P. \end{gathered} \tag{3.31}$$

(H5) *There exist $u_* \in P \setminus \{\theta\}$, $c_2 \in C[I^*, R_+]$, and $h_2 \in C[P^{n-1}, R_+]$ such that*

$$\begin{gathered} f(t, u_1, \ldots, u_{n+1}) \geq c_2(t) h_2(u_1, \ldots, u_{n-1}) u_*, \quad \forall t \in I^*, \ u_k \in P, \\ \frac{h_2(u_1, \ldots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_i\|} \longrightarrow \infty, \quad as \ \sum_{k=1}^{n-1} \|u_k\| \longrightarrow 0, \ u_k \in P. \end{gathered} \tag{3.32}$$

(H6) *There exists a $\beta > 0$ such that*

$$N M_\beta \int_0^1 G(s) a(s) ds < \beta, \tag{3.33}$$

where $M_\beta = \max_{u_k \in P_\beta} \{h(\|u_1\|, \ldots, \|u_{n+1}\|)\}$. Then problem (1.1) *has at least two positive solutions.*

Proof. Consider condition (H4), there exists an $r_1 > 0$, such that

$$h_1(u_1, \ldots, u_{n-1}) \geq \frac{3N^2 \sum_{k=1}^{n-1} \|u_k\|}{\lambda^2 \int_\gamma^\delta G(s) c_1(s) ds \cdot \|u^*\|}, \quad \forall u_k \in P, \ \sum_{k=1}^{n-1} \|u_k\| \geq r_1. \tag{3.34}$$

Therefore,

$$f(t,u_1,\cdot,u_{n+1}) \geq \frac{3N^2\sum_{k=1}^{n-1}\|u_k\|}{\lambda^2\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\cdot c_1(t)u^*, \quad \forall u_k\in P,\ \sum_{k=1}^{n-1}\|u_k\|\geq r_1. \tag{3.35}$$

Take

$$r_0 > \max\left\{3N\lambda^{-1}r_1,\ \beta\right\}. \tag{3.36}$$

Then for $t\in[\gamma,\delta]$, $\|x\|_\Omega = r_0$, we have, by (3.18),

$$\|x(t)\| \geq \frac{\lambda}{3N}\|x\|_\Omega \geq \frac{\lambda}{3N}r_0 > r_1. \tag{3.37}$$

Hence,

$$\begin{aligned}
(Ax)(t) &\geq \int_\gamma^\delta G(t,s)f\left(s,I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)ds\\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\int_\gamma^\delta G(s)\left(\sum_{k=1}^{n-2}\left\|I_{0^+}^{n-1-k}x(s)\right\|+\|x(s)\|\right)c_1(s)ds\cdot u^*\\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\int_\gamma^\delta G(s)c_1(s)\|x(s)\|ds\cdot u^*\\
&\geq \frac{N}{\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\|x\|_\Omega\left(\int_\gamma^\delta G(s)c_1(s)ds\right)\cdot u^*\\
&= \frac{1}{\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\left(\int_\gamma^\delta G(s)c_1(s)ds\|u^*\|\right)\cdot\frac{N\|x\|_\Omega}{\|u^*\|}u^*\\
&\geq \frac{N\|x\|_\Omega}{\|u^*\|}\cdot u^*,
\end{aligned} \tag{3.38}$$

and consequently,

$$\|Ax\|_\Omega \geq \|x\|_\Omega, \quad \forall x\in\Omega, \|x\|_\Omega = r_0. \tag{3.39}$$

Similarly, by condition (H5), there exists $r_2 > 0$, such that

$$h_2(u_1,\dots,u_{n-1}) \geq \frac{3N^2\sum_{k=1}^{n-1}\|u_k\|}{\lambda\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}, \quad \forall u_k\in P,\ 0<\sum_{k=1}^{n-1}\|u_k\|\leq r_2, \tag{3.40}$$

where ξ is given in (2) of Lemma 3.2. Therefore,

$$f(t,u_1,\dots,u_{n+1}) \geq \frac{3N^2 \sum_{k=1}^{n-1}\|u_k\|}{\lambda \int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot \|u_*\|} \cdot c_2(t)u_*, \quad \forall u_k \in P,\ 0 < \sum_{k=1}^{n-1}\|u_k\| \leq r_2. \tag{3.41}$$

Choose

$$0 < r < \min\left\{ \left(\sum_{k=0}^{n-2} \frac{1}{k!} \right)^{-1} r_2,\ \beta \right\}. \tag{3.42}$$

Then for $t \in [\gamma,\delta]$, $x \in \Omega$, $\|x\|_\Omega = r$, we have

$$\begin{aligned}
(Ax)(\xi) &= \int_0^1 G(\xi,s) f\left(s, I_{0^+}^{n-2}x(s),\dots,x(s), T\left(I_{0^+}^{n-2}x\right)(s), S\left(I_{0^+}^{n-2}x\right)(s)\right)ds \\
&\geq \int_\gamma^\delta G(\xi,s) f\left(s, I_{0^+}^{n-2}x(s),\dots,x(s), T\left(I_{0^+}^{n-2}x\right)(s), S\left(I_{0^+}^{n-2}x\right)(s)\right)ds \\
&\geq \frac{3N^2}{\lambda \int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot \|u_*\|} \int_\gamma^\delta G(\xi,s) \left(\sum_{k=1}^{n-2} \left\| I_{0^+}^{n-1-k}x(s) \right\| + \|x(s)\| \right) c_2(s)ds \cdot u_* \\
&\geq \frac{3N^2}{\lambda \int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot \|u_*\|} \int_\gamma^\delta G(\xi,s)\|x(s)\|c_2(s)ds \cdot u_* \\
&\geq \frac{N}{\int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot \|u_*\|} \|x(s)\|_\Omega \int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot u_* \\
&= \frac{1}{\int_\gamma^\delta G(\xi,s)c_2(s)ds \cdot \|u_*\|} \|x(s)\|_\Omega \left(\int_\gamma^\delta G(\xi,s)c_2(s)ds\|u_*\| \right) \cdot \frac{N\|x(s)\|_\Omega}{\|u_*\|} u_* \\
&\geq \frac{N\|x(s)\|_\Omega}{\|u_*\|} \cdot u_*,
\end{aligned} \tag{3.43}$$

which implies

$$\|(Ax)(\xi)\|_\Omega \geq \|x(s)\|_\Omega, \quad \forall x \in \Omega, \|x\|_\Omega = r, \tag{3.44}$$

that is,

$$\|Ax\|_\Omega \geq \|x(s)\|_\Omega, \quad \forall x \in \Omega, \|x\|_\Omega = r. \tag{3.45}$$

On the other hand, according to (ii) of Remark 3.3 and (3.18), we get

$$(Ax)(t) \leq \int_0^1 G(s) f\left(s, I_{0^+}^{n-2} x(s), \ldots, x(s), T\left(I_{0^+}^{n-2} x\right)(s), S\left(I_{0^+}^{n-2} x\right)(s)\right) ds. \tag{3.46}$$

By condition (H1), for $t \in I$, $x \in \Omega$, $\|x\|_\Omega = \beta$, we have

$$\left\| f(t, u_1, \ldots, u_{n+1}) \right\| \leq a(t) h(\|u_1\|, \ldots, \|u_{n+1}\|) \leq M_\beta a(t). \tag{3.47}$$

Therefore,

$$\|(Ax)(t)\|_\Omega \leq N M_\beta \cdot \int_0^1 G(s) a(s) ds < \beta = \|x\|_\Omega. \tag{3.48}$$

Applying Lemma 2.7 to (3.39), (3.45), and (3.48) yields that T has a fixed-point $x^* \in \overline{\Omega}_{r,\beta}$, $r \leq \|x^*\| \leq \beta$, and a fixed-point $x^{**} \in \overline{\Omega}_{\beta,r_0}$, $\beta \leq \|x^{**}\| \leq r_0$. Noticing (3.48), we get $\|x^*\| \neq \beta$ and $\|x^{**}\| \neq \beta$. This and Lemma 3.4 complete the proof. □

Theorem 3.7. *Let cone P be normal and conditions $(H1)$~$(H4)$ hold. In addition, assume that the following condition is satisfied:*

(H7)

$$\frac{h(\|u_1\|, \ldots, \|u_{n+1}\|)}{\sum_{k=1}^{n+1} \|u_k\|} \longrightarrow 0, \quad \textit{as } u_k \in P, \ \sum_{k=1}^{n+1} \|u_k\| \longrightarrow 0^+. \tag{3.49}$$

Then problem (1.1) *has at least one positive solution.*

Proof. By (H4), we can choose $r_0 > 3N\lambda^{-1} r_1$. As in the proof of Theorem 3.6, it is easy to see that (3.39) holds. On the other hand, considering (3.49), there exists $r_3 > 0$ such that

$$h(\|u_1\|, \ldots, \|u_{n+1}\|) \leq \varepsilon_0 \sum_{k=1}^{n+1} \|u_k\|, \quad \text{for } t \in I, \ u_k \in P, \ 0 < \sum_{k=1}^{n+1} \|u_k\| \leq r_3, \tag{3.50}$$

where $\varepsilon_0 > 0$ satisfies

$$\varepsilon_0 = \left(N \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} + \frac{a^* + b^*}{(n-3)!} \right\} \int_0^1 G(s) a(s) ds \right)^{-1}. \tag{3.51}$$

Choose $0 < r^* < \min\{(\sum_{k=1}^{n-1}(1/k!)+(a^*+b^*)/(n-3)!)^{-1}r_3, r_0\}$. Then for $t \in I$, $x \in \Omega$, $\|x\|_\Omega = r^*$, it follows from (3.46) that

$$\begin{aligned}
\|(Ax)(t)\| &\le N\int_0^1 G(s)\left\|f\left(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)\right\|ds \\
&\le N\int_0^1 G(s)a(s)h\left(\left\|I_{0^+}^{n-2}x(s)\right\|,\dots,\|x(s)\|,\left\|T\left(I_{0^+}^{n-2}x\right)(s)\right\|,\left\|S\left(I_{0^+}^{n-2}x\right)(s)\right\|\right)ds \\
&\le N\varepsilon_0\int_0^1 G(s)a(s)\left(\sum_{k=1}^{n-1}\left\|I_{0^+}^{n-1-k}x(s)\right\|+\left\|T\left(I_{0^+}^{n-2}x\right)(s)\right\|+\left\|S\left(I_{0^+}^{n-2}x\right)(s)\right\|\right)ds \\
&\le N\varepsilon_0\left(\sum_{k=1}^{n-1}\frac{1}{k!}+\frac{a^*+b^*}{(n-3)!}\right)r^*\int_0^1 G(s)a(s)ds = r^*
\end{aligned} \tag{3.52}$$

and consequently,

$$\|(Ax)(t)\|_\Omega \le \|x\|_\Omega, \quad \forall x \in \Omega,\ \|x\|_\Omega \le r^*. \tag{3.53}$$

Since $0 < r^* < r_0$, applying Lemma 2.7 to (3.39) and (3.53) yield that T has a fixed-point $x^* \in \overline{\Omega}_{r^*,r_0}$, $r^* \le \|x^*\| \le r_0$. This and Lemma 3.4 complete the proof. □

4. An Example

Consider the following system of scalar differential equations of fractional order

$$\begin{aligned}
-D^{5/2}u_k(t) = {}& \frac{(1+t)^3}{960k^3}\left\{\left[u_{2k}(t)+u'_{3k}(t)+\sum_{j=1}^{\infty}u_{2j}(t)+\sum_{j=1}^{\infty}u'_j(t)\right]^3\right. \\
&\left.+\left(3u_k(t)+3u'_{k+1}(t)+\sum_{j=1}^{\infty}u_j(t)+\sum_{j=1}^{\infty}u'_{2j}(t)\right)^{1/2}\right\} \\
&+\frac{1+t^3}{36k^5}\left(\int_0^t e^{-(1+t)s}u_k(s)ds\right)^{2/3} \\
&+\frac{1+t^2}{24k^4}\left(\int_0^1 e^{-s}\sin^2(t-s)\pi u_{2k}(s)ds\right), \quad t \in I, \\
u_k(0) = u'_k(0) = 0,& \quad u'_k(1) = \frac{1}{8}u'_k\left(\frac{1}{4}\right)+\frac{1}{2}u'_k\left(\frac{4}{9}\right), \quad k = 1,2,3,\dots.
\end{aligned} \tag{4.1}$$

Conclusion. The problem (4.1) has at least two positive solutions.

Proof. Let $E = l^1 = \{u = (u_1, u_2, \ldots, u_k, \ldots) : \sum_{k=1}^{\infty} |u_k| < \infty\}$ with the norm $\|u\| = \sum_{k=1}^{\infty} |u_k|$, and $P = \{(u_1, \ldots, u_k, \ldots) : u_k \geq 0, k = 1, 2, 3, \ldots\}$. Then P is a normal cone in E with normal constant $N = 1$, and system (4.1) can be regarded as a boundary value problem of the form (1.1). In this situation, $q = 5/2$, $n = 3$, $a_1 = 1/8$, $a_2 = 1/2$, $\eta_1 = 1/4$, $\eta_2 = 4/9$, $\eta^* = 29/48$, $K(t,s) = e^{-(1+t)s}$, $H(t,s) = e^{-s}\sin^2(t-s)\pi$, $u = (u_1, \ldots, u_k, \ldots)$, $f = (f_1, f_2, \ldots, f_k, \ldots)$, in which

$$f_k(t,u,v,x,y) = \frac{(1+t)^3}{960k^3}\left\{\left(u_{2k} + v_{3k} + \sum_{j=1}^{\infty} u_{2j} + \sum_{j=1}^{\infty} v_j\right)^3 + \left(3u_k + 3v_{k+1} + \sum_{j=1}^{\infty} u_j + \sum_{j=1}^{\infty} v_{2j}\right)^{1/2}\right\} + \frac{1+t^3}{36k^5} x_k^{1/5} + \frac{1+t^2}{24k^4} y_{2k}. \tag{4.2}$$

Observing the inequality $\sum_{k=1}^{\infty}(1/k^3) < 3/2$, we get, by (4.2),

$$\begin{aligned} \|f(t,u,v,x,y)\| &= \sum_{k=1}^{\infty} |f_k(t,u,v,x,y)| \\ &\leq \frac{(1+t)^3}{2}\left(\frac{1}{40}(\|u\| + \|v\|)^3 + \frac{1}{160}(\|u\| + \|v\|)^{1/2} + \frac{1}{12}\|x\|^{1/5} + \frac{1}{8}\|y\|\right). \end{aligned} \tag{4.3}$$

Hence (H1) is satisfied for $a(t) = (1+t)^3/2$ and

$$h(u,v,x,y) = \frac{1}{40}(u+v)^3 + \frac{1}{160}(u+v)^{1/2} + \frac{1}{12}x^{1/5} + \frac{1}{8}y. \tag{4.4}$$

Now, we check condition (H2). Obviously, $f : I \times P_r^4 \to P$, for any $r > 0$, and f is uniformly continuous on $I \times P_r^4$. Let $f = f^{(1)} + f^{(2)}$, where $f^{(1)} = (f_1^{(1)}, \ldots, f_k^{(1)}, \ldots)$ and $f^{(2)} = (f_1^{(2)}, \ldots, f_k^{(2)}, \ldots)$, in which

$$\begin{aligned} f_k^{(1)}(t,u,v,x,y) &= \frac{(1+t)^3}{960k^3}\left\{\left(u_{2k} + v_{3k} + \sum_{j=1}^{\infty} u_{2j} + \sum_{j=1}^{\infty} v_j\right)^3 + \left(3u_k + 3v_{k+1} + \sum_{j=1}^{\infty} u_j + \sum_{j=1}^{\infty} v_{2j}\right)^{1/2}\right\} \\ &\quad + \frac{1+t^3}{36k^5} x_k^{1/5}, \quad (k = 1, 2, 3, \ldots), \\ f_k^{(2)}(t,u,v,x,y) &= \frac{1+t^2}{24k^4} y_{2k}, \quad (k = 1, 2, 3, \ldots). \end{aligned} \tag{4.5}$$

For any $t \in I$ and bounded subsets $D_i \subset E$, $i = 1,2,3,4$, from (4.5) and by the diagonal method, we have

$$\alpha\left(f^{(1)}(t, D_1, D_2, D_3, D_4)\right) = 0, \quad \forall t \in I, \text{ bounded sets } D_i \subset E, \ i = 1,2,3,4,$$
$$\alpha\left(f^{(2)}(I, D_1, D_2, D_3, D_4)\right) \le \frac{1}{12}\alpha(D_4), \quad \forall t \in I, D_i \subset E, \ i = 1,2,3,4. \tag{4.6}$$

It follows from (4.6) that

$$\alpha(f(I, D_1, D_2, D_3, D_4)) \le \frac{1}{12}\alpha(D_4), \quad \forall t \in I, D_i \subset E, \ i = 1,2,3,4,$$
$$\frac{2}{\Gamma(q-n+2)\eta^*}\left(\sum_{k=1}^{n-2}\frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^*}{(n-3)!}L_n + \frac{b^*}{(n-3)!}L_{n+1}\right) \approx 0.1565 < 1. \tag{4.7}$$

that is, condition (H2) holds for $L_1 = L_2 = L_3 = 0, L_4 = 1/12$.

On the other hand, take $\gamma = 1/4$, $\delta = 3/4$. Then $1/4 = \gamma \le \min\{\delta/2, \ 2\delta - 1\} = 3/8$, $2/3 < \delta$, which implies that condition (H3) holds. By (4.2), we have

$$f_k(t, u, v, x, y) \ge \frac{(1+t)^3}{960k^3}(\|u\| + \|v\|)^3, \quad \forall t \in I^*, \ u, v, x, y \in P, \ (k = 1,2,3,\ldots),$$
$$f_k(t, u, v, x, y) \ge \frac{(1+t)^3}{960k^3}\sqrt{\|u\| + \|v\|}, \quad \forall t \in I^*, \ u, v, x, y \in P, \ (k = 1,2,3,\ldots). \tag{4.8}$$

Hence condition (H4) is satisfied for

$$c_1(t) = \frac{(1+t)^3}{960}, \qquad h_{1,k}(u, v) = (\|u\| + \|v\|)^3, \qquad u^* = \left(1, \ldots, \frac{1}{k^3}, \ldots\right), \tag{4.9}$$

in this situation,

$$h_{1,k} = \lim_{\|u\|+\|v\| \to \infty} \frac{(\|u\| + \|v\|)^3}{\|u\| + \|v\|} = \infty. \tag{4.10}$$

And condition (H5) is also satisfied for

$$c_2(t) = \frac{(1+t)^3}{960}, \qquad h_{2,k}(u, v) = \sqrt{\|u\| + \|v\|}, \qquad u_* = \left(1, \ldots, \frac{1}{k^3}, \ldots\right), \tag{4.11}$$

in this situation,

$$h_{2,k} = \lim_{\|u\|+\|v\| \to 0} \frac{\sqrt{\|u\| + \|v\|}}{\|u\| + \|v\|} = \infty. \tag{4.12}$$

Finally, choose $\beta = 1$, it is easy to check that condition (H6) is satisfied. In this case, $M_\beta \approx 0.4162$, and so

$$NM_\beta \int_0^1 G(s)a(s)ds \approx 0.7287 < \beta = 1. \tag{4.13}$$

Hence, our conclusion follows from Theorem 3.6. □

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Research Article

Application of Sumudu Decomposition Method to Solve Nonlinear System of Partial Differential Equations

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We develop a method to obtain approximate solutions of nonlinear system of partial differential equations with the help of Sumudu decomposition method (SDM). The technique is based on the application of Sumudu transform to nonlinear coupled partial differential equations. The nonlinear term can easily be handled with the help of Adomian polynomials. We illustrate this technique with the help of three examples, and results of the present technique have close agreement with approximate solutions obtained with the help of Adomian decomposition method (ADM).

1. Introduction

Most of phenomena in nature are described by nonlinear differential equations. So scientists in different branches of science try to solve them. But because of nonlinear part of these groups of equations, finding an exact solution is not easy. Different analytical methods have been applied to find a solution to them. For example, Adomian has presented and developed a so-called decomposition method for solving algebraic, differential, integrodifferential, differential-delay and partial differential equations. In the nonlinear case for ordinary differential equations and partial differential equations, the method has the advantage of dealing directly with the problem [1, 2]. These equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization, or any unrealistic assumptions [3, 4]. It was suggested in [5] that the noise terms appears always for inhomogeneous equations. Most recently, Wazwaz [6] established a necessary condition that is essentially needed to ensure the appearance of “noise terms” in the inhomogeneous equations. In the present paper, the intimate connection between the Sumudu transform theory and decomposition method arises in the solution of nonlinear partial differential equations is demonstrated.

The Sumudu transform is defined over the set of the functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{t/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty) \right\} \tag{1.1}$$

by the following formula:

$$\begin{aligned} G(u) &= S[f(t); u] \\ &= \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \end{aligned} \tag{1.2}$$

The existence and the uniqueness were discussed in [7], for further details and properties of the Sumudu transform and its derivatives we refer to [8]. In [9], some fundamental properties of the Sumudu transform were established.

In [10], this new transform was applied to the one-dimensional neutron transport equation. In fact one can easily show that there is a strong relationship between double Sumudu and double Laplace transforms, see [7].

Further in [11], the Sumudu transform was extended to the distributions and some of their properties were also studied in [12]. Recently Kılıçman et al. applied this transform to solve the system of differential equations, see [13].

A very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor $n!$. Thus if

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{1.3}$$

then

$$F(u) = \sum_{n=0}^{\infty} n! a_n t^n, \tag{1.4}$$

see [14].

Similarly, the Sumudu transform sends combinations, $C(m,n)$, into permutations, $P(m,n)$ and hence it will be useful in the discrete systems. Further

$$\begin{aligned} S(H(t)) &= \pounds(\delta(t)) = 1, \\ \pounds(H(t)) &= S(\delta(t)) = \frac{1}{u}. \end{aligned} \tag{1.5}$$

Thus we further note that since many practical engineering problems involve mechanical or electrical systems where action is defined by discontinuous or impulsive forcing terms. Then the Sumudu transform can be effectively used to solve ordinary differential equations as well as partial differential equations and engineering problems. Recently, the Sumudu transform was introduced as a new integral transform on a time scale $\mathbb{T}$ to solve a system of dynamic equations, see [15]. Then the results were applied on ordinary differential equations when $\mathbb{T} = \mathbb{R}$, difference equations when $\mathbb{T} = \mathbb{N}_0$, but also, for q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$, where $q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$ for $q > 1$ which has important applications in quantum theory and on different types of time scales like $\mathbb{T} = h\mathbb{N}_0$, $\mathbb{T} = \mathbb{N}_0^2$, and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. During this study we use the following Sumudu transform of derivatives.

Theorem 1.1. *Let $f(t)$ be in A, and let $G^n(u)$ denote the Sumudu transform of the nth derivative, $f^n(t)$ of $f(t)$, then for $n \geq 1$*

$$G^n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}. \tag{1.6}$$

For more details, see [16].

We consider the general inhomogeneous nonlinear equation with initial conditions given below:

$$LU + RU + NU = h(x,t), \tag{1.7}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, NU represents the nonlinear terms and $h(x,t)$ is the source term. First we explain the main idea of SDM: the method consists of applying Sumudu transform

$$S[LU] + S[RU] + S[NU] = S[h(x,t)]. \tag{1.8}$$

Using the differential property of Laplace transform and initial conditions we get

$$\frac{1}{u^n} S[U(x,t)] - \frac{1}{u^n} U(x,0) - \frac{1}{u^{n-1}} U'(x,0) - \cdots - \frac{U^{n-1}(x,0)}{u} + S[RU] + S[NU] = S[h(x,t)]. \tag{1.9}$$

By arrangement we have

$$S[U(x,t)] = U(x,0) + uU'(x,0) + \cdots + u^{n-1} U^{n-1}(x,0) - u^n S[RU] - u^n S[NU] + u^n S[h(x,t)]. \tag{1.10}$$

The second step in Sumudu decomposition method is that we represent solution as an infinite series:

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) \tag{1.11}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{i=0}^{\infty} A_i, \tag{1.12}$$

where A_i are Adomian polynomials [6] of $U_0, U_1, U_2, \ldots, U_n$ and it can be calculated by formula

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\sum_{i=0}^{\infty}\lambda^i U_i\right]_{\lambda=0}, \quad i = 0,1,2,\ldots. \tag{1.13}$$

Substitution of (1.11) and (1.12) into (1.10) yields

$$\begin{aligned} S\left[\sum_{i=0}^{\infty} U_i(x,t)\right] &= U(x,0) + uU'(x,0) + \cdots + u^{n-1}U^{n-1}(x,0) - u^n S[RU(x,t)] \\ &\quad - u^n S\left[\sum_{i=0}^{\infty} A_i\right] + u^n S[h(x,t)]. \end{aligned} \tag{1.14}$$

On comparing both sides of (1.14) and by using standard ADM we have:

$$S[U_0(x,t)] = U(x,0) + uU'(x,0) + \cdots + u^{n-1}U^{n-1}(x,0) + u^n S[h(x,t)] = Y(x,u) \tag{1.15}$$

then it follows that

$$\begin{aligned} S[U_1(x,t)] &= -u^n S[RU_0(x,t)] - u^n S[A_0], \\ S[U_2(x,t)] &= -u^n S[RU_1(x,t)] - u^n S[A_1]. \end{aligned} \tag{1.16}$$

In more general, we have

$$S[U_{i+1}(x,t)] = -u^n S[RU_i(x,t)] - u^n S[A_i], \quad i \geq 0. \tag{1.17}$$

On applying the inverse Sumudu transform to (1.15) and (1.17), we get

$$\begin{aligned} U_0(x,t) &= K(x,t), \\ U_{i+1}(x,t) &= -S^{-1}[u^n S[RU_i(x,t)] + u^n S[A_i]], \quad i \geq 0, \end{aligned} \tag{1.18}$$

where $K(x,t)$ represents the term that is arising from source term and prescribed initial conditions. On using the inverse Sumudu transform to $h(x,t)$ and using the given condition we get

$$\Psi = \Phi + S^{-1}[h(x,t)], \tag{1.19}$$

where the function Ψ, obtained from a term by using the initial condition is given by

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \cdots + \Psi_n, \tag{1.20}$$

the terms $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_n$ appears while applying the inverse Sumudu transform on the source term $h(x,t)$ and using the given conditions. We define

$$U_0 = \Psi_k + \cdots + \Psi_{k+r}, \tag{1.21}$$

where $k = 0, 1, \ldots, n$, $r = 0, 1, \ldots, n-k$. Then we verify that U_0 satisfies the original equation (1.7). We now consider the particular form of inhomogeneous nonlinear partial differential equations:

$$LU + RU + NU = h(x,t) \tag{1.22}$$

with the initial condition

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x), \tag{1.23}$$

where $L = \partial^2/\partial t^2$ is second-order differential operator, NU represents a general non-linear differential operator where as $h(x,t)$ is source term. The methodology consists of applying Sumudu transform first on both sides of (1.10) and (1.23),

$$S[U(x,t)] = f(x) + ug(x) - u^2 S[RU] - u^2 S[NU] + u^2 S[h(x,t)]. \tag{1.24}$$

Then by the second step in Sumudu decomposition method and inverse transform as in the previous we have

$$U(x,t) = f(x) + tg(t) - S^{-1}\left[u^2 S[RU] - u^2 S[NU]\right] + S^{-1}\left[u^2 S[h(x,t)]\right]. \tag{1.25}$$

2. Applications

Now in order to illustrate STDM we consider some examples. Consider a nonlinear partial differential equation

$$U_{tt} + U^2 - U_x^2 = 0, \quad t > 0 \tag{2.1}$$

with initial conditions

$$\begin{aligned} U(x,0) &= 0, \\ U_t(x,0) &= e^x. \end{aligned} \tag{2.2}$$

By taking Sumudu transform for (2.1) and (2.2) we obtain

$$S[U(x,t)] = ue^{x} + u^2 S\left[U_x^2 - U^2\right]. \tag{2.3}$$

By applying the inverse Sumudu transform for (2.3), we get

$$[U(x,t)] = te^{x} + S^{-1}\left[u^2 S\left[U_x^2 - U^2\right]\right] \tag{2.4}$$

which assumes a series solution of the function $U(x,t)$ and is given by

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t). \tag{2.5}$$

Using (2.4) into (2.5) we get

$$\sum_{i=0}^{\infty} U_i(x,t) = te^{x} + S^{-1}\left[u^2 S\left[\sum_{i=0}^{\infty} A_i(U) - \sum_{i=0}^{\infty} B_i(U)\right]\right]. \tag{2.6}$$

In (2.6) $A_i(u)$ and $B_i(u)$ are Adomian polynomials that represents nonlinear terms. So Adomian polynomials are given as follows:

$$\begin{aligned} \sum_{i=0}^{\infty} A_i(U) &= U_x^2, \\ \sum_{i=0}^{\infty} A_i(U) &= U^2. \end{aligned} \tag{2.7}$$

The few components of the Adomian polynomials are given as follows:

$$\begin{aligned} &A_0(U) = U_{0x}^2, \qquad A_1(U) = 2U_{0x}U_{1x}, \qquad A_i(U) = \sum_{r=0}^{i} U_{rx}U_{i-rx}, \\ &B_0(U) = U_0^2, \qquad B_1(U) = 2U_0U_1, \qquad B_i(U)\sum_{r=0}^{i} U_r U_{i-r}. \end{aligned} \tag{2.8}$$

From the above equations we obtain

$$\begin{aligned} U_0(x,t) &= te^{x}, \\ U_{i+1}(x,t) &= S^{-1}\left[S\left[\sum_{i=0}^{\infty} A_i(U) - \sum_{i=0}^{\infty} B_i(U)\right]\right], \quad n \geq 0. \end{aligned} \tag{2.9}$$

Then the first few terms of $U_i(x,t)$ follow immediately upon setting

$$\begin{aligned}
U_1(x,t) &= S^{-1}\left[u^2 S\left[\sum_{i=0}^{\infty} A_0(U) - \sum_{i=0}^{\infty} B_0(U)\right]\right] \\
&= S^{-1}\left[u^2 S\left[U_{0x}^2 - U_0^2\right]\right] = S^{-1}\left[u^2 S\left[t^2 e^{2x} - t^2 e^{2x}\right]\right] \\
&= S^{-1}\left[u^2 S[0]\right] = 0.
\end{aligned} \tag{2.10}$$

Therefore the solution obtained by LDM is given as follows:

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) = te^x. \tag{2.11}$$

Example 2.1. Consider the system of nonlinear coupled partial differential equation

$$\begin{aligned}
U_t(x,y,t) - V_x W_y &= 1, \\
V_t(x,y,t) - W_x U_y &= 5, \\
W_t(x,y,t) - U_x V_y &= 5
\end{aligned} \tag{2.12}$$

with initial conditions

$$\begin{aligned}
U(x,y,0) &= x + 2y, \\
V(x,y,0) &= x - 2y, \\
W(x,y,0) &= -x + 2y.
\end{aligned} \tag{2.13}$$

Applying the Sumudu transform (denoted by S) we have

$$\begin{aligned}
U(x,y,u) &= x + 2y + u + uS[V_x W_y], \\
V(x,y,u) &= x - 2y + 5u + uS[W_x U_y], \\
W(x,y,u) &= -x + 2y + 5u + uS[U_x V_y].
\end{aligned} \tag{2.14}$$

On using inverse Sumudu transform in (2.14), our required recursive relation is given by

$$\begin{aligned}
U(x,y,t) &= x + 2y + t + S^{-1}[uS[V_x W_y]], \\
V(x,y,t) &= x - 2y + 5t + S^{-1}[uS[W_x U_y]], \\
U(x,y,t) &= -x + 2y + 5t + S^{-1}[uS[U_x V_y]].
\end{aligned} \tag{2.15}$$

The recursive relations are

$$\begin{aligned}
U_0(x,y,t) &= t + x + 2y, \\
U_{i+1}(x,y,t) &= S^{-1}\left[uS\left[\sum_{i=0}^{\infty} C_i(V,W)\right]\right], \quad i \geq 0, \\
V_0(x,y,t) &= 5t + x - 2y, \\
V_{i+1}(x,y,t) &= S^{-1}\left[uS\left[\sum_{i=0}^{\infty} D_i(U,W)\right]\right], \quad i \geq 0, \\
W_0(x,y,t) &= 5t - x + 2y, \\
W_{i+1}(x,y,t) &= S^{-1}\left[uS\left[\sum_{i=0}^{\infty} E_i(U,V)\right]\right], \quad i \geq 0,
\end{aligned} \tag{2.16}$$

where $C_i(V,W)$, $D_i(U,W)$, and $E_i(U,V)$ are Adomian polynomials representing the nonlinear terms [1] in above equations. The few components of Adomian polynomials are given as follows

$$\begin{aligned}
C_0(V,W) &= V_{0x}W_{0y}, \\
C_1(V,W) &= V_{1x}W_{0y} + V_{0x}W_{1y}, \\
&\vdots \\
C_i(V,W) &= \sum_{r=0}^{i} V_{rx}W_{i-ry}, \\
D_0(U,W) &= U_{0y}W_{0x}, \\
D_1(U,W) &= U_{1y}W_{0x} + W_{1x}U_{0y}, \\
&\vdots \\
D_i(U,W) &= \sum_{r=0}^{i} W_{rx}U_{i-ry}, \\
E_0(U,V) &= U_{0x}V_{0y}, \\
E_1(U,V) &= U_{1x}V_{0y} + U_{0x}V_{1y}, \\
&\vdots \\
E_i(V,W) &= \sum_{r=0}^{i} U_{rx}V_{i-ry}.
\end{aligned} \tag{2.17}$$

By this recursive relation we can find other components of the solution

$$\begin{aligned}
U_1(x,y,t) &= S^{-1}[uS[C_0(V,W)]] = S^{-1}\left[uS\left[V_{0x}W_{0y}\right]\right] = S^{-1}[uS[(1)(2)]] = 2t,\\
V_1(x,y,t) &= S^{-1}[uS[D_0(U,W)]] = S^{-1}\left[uS\left[W_{0x}U_{0y}\right]\right] = S^{-1}[uS[(-1)(2)]] = -2t,\\
W_1(x,y,t) &= S^{-1}[uS[E_0(U,V)]] = S^{-1}\left[uS\left[U_{0x}V_{0y}\right]\right] = S^{-1}[uS[(1)(-2)]] = -2t,\\
U_2(x,y,t) &= S^{-1}[uS[C_1(V,W)]] = S^{-1}\left[uS\left[V_{1x}W_{0y} + V_{0x}W_{1y}\right]\right] = 0,\\
V_2(x,y,t) &= S^{-1}[uS[D_1(U,W)]] = S^{-1}\left[uS\left[U_{0y}W_{1x} + U_{1y}W_{0x}\right]\right] = 0,\\
W_2(x,y,t) &= S^{-1}[uS[D_1(U,V)]] = S^{-1}\left[uS\left[U_{1x}V_{0y} + U_{0x}V_{1y}\right]\right] = 0.
\end{aligned} \tag{2.18}$$

The solution of above system is given by

$$\begin{aligned}
U(x,y,t) &= \sum_{i=0}^{\infty} U_i(x,y,t) = x + 2y + 3t,\\
V(x,y,t) &= \sum_{i=0}^{\infty} V_i(x,y,t) = x - 2y + 3t,\\
W(x,y,t) &= \sum_{i=0}^{\infty} W_i(x,y,t) = -x + 2y + 3t.
\end{aligned} \tag{2.19}$$

Example 2.2. Consider the following homogeneous linear system of PDEs:

$$\begin{aligned}
U_t(x,t) - V_x(x,t) - (U - V) &= 2,\\
V_t(x,t) + U_x(x,t) - (U - V) &= 2,
\end{aligned} \tag{2.20}$$

with initial conditions

$$U(x,0) = 1 + e^x, \qquad V(x,0) = -1 + e^x. \tag{2.21}$$

Taking the Sumudu transform on both sides of (2.20), then by using the differentiation property of Sumudu transform and initial conditions, (2.21) gives

$$\begin{aligned}
S[U(x,t)] &= 1 + e^x - 2u + uS[V_x] + uS[U - V],\\
S[V(x,t)] &= -1 + e^x - 2u - uS[U_x] + uS[U - V],
\end{aligned} \tag{2.22}$$

$$U_x(x,t) = \sum_{i=0}^{\infty} U_{xi}(x,t), \qquad V_x(x,t) = \sum_{i=0}^{\infty} V_{xi}(x,t). \tag{2.23}$$

Using the decomposition series (2.23) for the linear terms $U(x,t)$, $V(x,t)$ and U_x, V_x, we obtain

$$S\left[\sum_{i=0}^{\infty}U_i(x,t)\right] = 1 + e^x - 2u + uS\left[\sum_{i=0}^{\infty}V_{ix}\right] + uS\left[\sum_{i=0}^{\infty}U_i - \sum_{i=0}^{\infty}V_i\right],$$
$$S\left[\sum_{i=0}^{\infty}V_i(x,t)\right] = -1 + e^x - 2u - uS\left[\sum_{i=0}^{\infty}U_{ix}\right] + uS\left[\sum_{i=0}^{\infty}U_i - \sum_{i=0}^{\infty}V_i\right]. \tag{2.24}$$

The SADM presents the recursive relations

$$\begin{aligned} S[U_0(x,t)] &= 1 + e^x - 2u, \\ S[V_0(x,t)] &= -1 + e^x - 2u, \\ S[U_{i+1}] &= uS[V_{ix}] + uS[U_i - V_i], \quad i \geq 0, \\ S[V_{i+1}] &= -uS[U_{ix}] + uS[U_i - V_i], \quad i \geq 0. \end{aligned} \tag{2.25}$$

Taking the inverse Sumudu transform of both sides of (2.25) we have

$$\begin{aligned} U_0(x,t) &= 1 + e^x - 2t, \\ V_0(x,t) &= -1 + e^x - 2t, \\ U_1 = S^{-1}[uS[V_{0x}] + uS[U_0 - V_0]] &= S^{-1}[ue^x + 2u] = te^x + 2t, \\ V_1 = S^{-1}[-uS[U_{0x}] + uS[U_0 - V_0]] &= S^{-1}[-ue^x + 2u] = -te^x + 2t, \\ U_2 = S^{-1}\left[u^2e^x\right] &= \frac{t^2}{2!}e^x, \\ V_2 = S^{-1}\left[u^2e^x\right] &= \frac{t^2}{2!}e^x, \end{aligned} \tag{2.26}$$

and so on for other components. Using (1.11), the series solutions are given by

$$\begin{aligned} U(x,t) &= 1 + e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\cdots\right), \\ V(x,t) &= -1 + e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\cdots\right). \end{aligned} \tag{2.27}$$

Then the solutions follows

$$\begin{aligned} U(x,t) &= 1 + e^{x+t}, \\ V(x,t) &= -1 + e^{x-t}. \end{aligned} \tag{2.28}$$

Example 2.3. Consider the system of nonlinear partial differential equations

$$\begin{aligned} U_t + VU_x + U &= 1, \\ V_t - UV_x - V &= 1 \end{aligned} \tag{2.29}$$

with initial conditions

$$U(x,0) = e^x, \qquad V(x,0) = e^{-x}. \tag{2.30}$$

On using Sumudu transform on both sides of (2.29), and by taking Sumudu transform for the initial conditions of (2.30) we get

$$\begin{aligned} S[U(x,t)] &= e^x + u - uS[VU_x] - uS[U], \\ S[V(x,t)] &= e^x + u + uS[UV_x] + uS[V]. \end{aligned} \tag{2.31}$$

Similar to the previous example, we rewrite $U(x,t)$ and $V(x,t)$ by the infinite series (1.11), then inserting these series into both sides of (2.31) yields

$$\begin{aligned} S\left[\sum_{i=0}^{\infty} U_i(x,t)\right] &= e^x + u - uS\left[\sum_{i=0}^{\infty} A_i\right] - uS\left[\sum_{i=0}^{\infty} U_i\right], \\ S\left[\sum_{i=0}^{\infty} V_i(x,t)\right] &= e^{-x} + u + uS\left[\sum_{i=0}^{\infty} B_i\right] - uS\left[\sum_{i=0}^{\infty} V_i\right], \end{aligned} \tag{2.32}$$

where the terms A_i and B_i are handled with the help of Adomian polynomials by (1.12) that represent the nonlinear terms VU_x and UV_x, respectively. We have a few terms of the Adomian polynomials for VU_x and UV_x which are given by

$$\begin{gathered} A_0 = U_{0x}V_0, \qquad A_1 = U_{0x}V_1 + U_{1x}V_0, \\ A_2 = U_{0x}V_2 + U_{1x}V_1 + U_{2x}V_0, \\ \vdots \\ B_0 = V_{0x}U_0, \qquad B_1 = V_{0x}U_1 + V_{1x}U_0, \\ B_2 = V_{0x}U_2 + V_{1x}U_1 + V_{2x}U_0, \\ \vdots \end{gathered} \tag{2.33}$$

By taking the inverse Sumudu transform we have

$$\begin{aligned} U_0 &= e^{x} + t, \\ V_0 &= e^{-x} + t, \end{aligned} \tag{2.34}$$

$$\begin{aligned} U_{i+1} &= S^{-1}[u] - S^{-1}[uS[A_i]] - S^{-1}[uS[U_i]], \\ V_{i+1} &= S^{-1}[u] + S^{-1}[uS[B_i]] + S^{-1}[uS[V_i]]. \end{aligned} \tag{2.35}$$

Using the inverse Sumudu transform on (2.35) we have

$$\begin{aligned} U_1 &= -t - \frac{t^2}{2!} - te^{x} - \frac{t^2}{2!}e^{x}, \\ V_1 &= -t - \frac{t^2}{2!} + te^{-x} - \frac{t^2}{2!}e^{-x}, \\ U_2 &= \frac{t^2}{2!} + \frac{t^2}{2!}e^{x} \cdots, \\ V_2 &= \frac{t^2}{2!} + \frac{t^2}{2!}e^{-x} \cdots. \end{aligned} \tag{2.36}$$

The rest terms can be determined in the same way. Therefore, the series solutions are given by

$$\begin{aligned} U(x,t) &= e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots\right), \\ V(x,t) &= e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots\right). \end{aligned} \tag{2.37}$$

Then the solution for the above system is as follows:

$$U(x,t) = e^{x-t}, \qquad V(x,t) = e^{-x+t}. \tag{2.38}$$

3. Conclusion

The Sumudu transform-Adomian decomposition method has been applied to linear and nonlinear systems of partial differential equations. Three examples have been presented, this method shows that it is very useful and reliable for any nonlinear partial differential equation systems. Therefore, this method can be applied to many complicated linear and nonlinear PDEs.

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Review Article

Infinite System of Differential Equations in Some *BK* Spaces

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The first measure of noncompactness was defined by Kuratowski in 1930 and later the Hausdorff measure of noncompactness was introduced in 1957 by Goldenštein et al. These measures of noncompactness have various applications in several areas of analysis, for example, in operator theory, fixed point theory, and in differential and integral equations. In particular, the Hausdorff measure of noncompactness has been extensively used in the characterizations of compact operators between the infinite-dimensional Banach spaces. In this paper, we present a brief survey on the applications of measures of noncompactness to the theory of infinite system of differential equations in some *BK* spaces $c_0, c, \ell_p (1 \leq p \leq \infty)$ and $n(\phi)$.

1. *FK* and *BK* Spaces

In this section, we give some basic definitions and notations about FK and BK spaces for which we refer to [1–3].

We will write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let φ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent, and null sequences, respectively, and cs be the set of all convergent series. We write $\ell_p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section.

Note that ℓ_∞, c and c_0 are Banach spaces with the norm $\|x\|_\infty = \sup_{k \geq 0} |x_k|$, and $\ell_p (1 \leq p < \infty)$ are Banach spaces with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. A sequence space X with a linear topology is called a $K space$ if each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$

is continuous for all $i \in \mathbb{N}$. A K space is called an FK *space* if X is complete linear metric space; a BK *space* is a normed FK space. An FK space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x = \lim_{n\to\infty} x^{[n]}$.

A linear space X equipped with a translation invariant metric d is called a *linear metric space* if the algebraic operations on X are continuous functions with respect to d. A complete linear metric space is called a *Fréchet space*. If X and Y are linear metric spaces over the same field, then we write $\mathcal{B}(X, Y)$ for the class of all continuous linear operators from X to Y. Further, if X and Y are normed spaces then $\mathcal{B}(X, Y)$ consists of all bounded linear operators $L : X \to Y$, which is a normed space with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$, where S_X denotes the unit sphere in X, that is, $S_X := \{x \in X : \|x\| = 1\}$. Also, we write $\overline{B}_X := \{x \in X : \|x\| \le 1\}$ for the closed unit ball in a normed space X. In particular, if $Y = \mathbb{C}$ then we write X^* for the set of all continuous linear functionals on X with the norm $\|f\| = \sup_{x\in S_X} |f(x)|$.

The *theory of FK spaces* is the most powerful and widely used tool in the characterization of matrix mappings between sequence spaces, and the most important result was that matrix mappings between FK spaces are continuous.

A sequence space X is called an *FK space* if it is a locally convex Fréchet space with continuous coordinates $p_n : X \to \mathbb{C}(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A normed FK space is called a *BK space*, that is, a *BK space* is a Banach sequence space with continuous coordinates.

The famous example of an FK space which is not a BK space is the space (w, d_w), where

$$d_w(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k}\left(\frac{|x_k - y_k|}{1 + |x_k - y_k|}\right); \quad (x, y \in w). \tag{1.1}$$

On the other hand, the classical sequence spaces are BK spaces with their natural norms. More precisely, the spaces ℓ_∞, c, and c_0 are BK spaces with the sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$. Also, the space ℓ_p $(1 \le p < \infty)$ is a BK space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$. Further, the spaces bs, cs, and cs_0 are BK spaces with the same norm given by $\|x\|_{bs} = \sup_n \sum_{k=0}^{n} |x_k|$, and bv is a BK space with $\|x\|_{bv} = \sum_k |x_k - x_{k-1}|$.

An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $\lim_{n\to\infty}(\sum_{k=0}^{n} x_k e^{(k)}) = x$. This means that $(e^{(k)})_{k=0}^{\infty}$ is a Schauder basis for any FK space with AK such that every sequence, in an FK space with AK, coincides with its sequence of coefficients with respect to this basis.

Although the space ℓ_∞ has no Schauder basis, the spaces w, c_0, c, and ℓ_p all have Schauder bases. Moreover, the spaces w, c_0, and ℓ_p have AK, where $1 \le p < \infty$.

There are following BK spaces which are closely related to the spaces ℓ_p $(1 \le p \le \infty)$.

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further

$$\mathcal{C}_s = \left\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \le s\right\}, \tag{1.2}$$

that is, $\mathcal{C}_s$ is the set of those σ whose support has cardinality at most s, and define

$$\Phi = \{\phi = (\phi_k) \in w : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1},\ (n+1)\phi_n \geq n\phi_{n+1}\}. \tag{1.3}$$

For $\phi \in \Phi$, the following sequence spaces were introduced by Sargent [4] and further studied in [5–8]:

$$\begin{aligned} m(\phi) &= \left\{ x = (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| \right) < \infty \right\}, \\ n(\phi) &= \left\{ x = (x_k) \in w : \|x\|_{n(\phi)} = \sup_{u\in S(x)} \left(\sum_{k=1}^{\infty} |u_k| \Delta\phi_k \right) < \infty \right\}, \end{aligned} \tag{1.4}$$

where $S(x)$ denotes the set of all sequences that are rearrangements of x.

Remark 1.1. (i) The spaces $m(\phi)$ and $n(\phi)$ are BK spaces with their respective norms.

(ii) If $\phi_n = 1$ for all $n \in \mathbb{N}$, then $m(\phi) = l_1$, $n(\phi) = l_\infty$, and if $\phi_n = n$ for all $n \in \mathbb{N}$, then $m(\phi) = l_\infty$, $n(\phi) = l_1$.

(iii) $l_1 \subseteq m(\phi) \subseteq l_\infty [l_\infty \supseteq n(\phi) \supseteq l_1]$ for all ϕ of Φ.

2. Measures of Noncompactness

The first measure of noncompactness was defined and studied by Kuratowski [9] in 1930. The Hausdorff measure of noncompactness was introduced by Goldenštein et al. [10] in 1957, and later studied by Goldenštein and Markus [11] in 1965.

Here, we shall only consider the Hausdorff measure of noncompactness; it is the most suitable one for our purposes. The basic properties of measures of noncompactness can be found in [12–14].

Let S and M be subsets of a metric space (X, d) and let $\epsilon > 0$. Then S is called an *ϵ-net* of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x, s) < \epsilon$. Further, if the set S is finite, then the ϵ-net S of M is called a *finite ϵ-net* of M, and we say that M has a finite ϵ-net in X. A subset M of a metric space X is said to be *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$ and is said to be *relatively compact* if its closure $\overline{M}$ is a compact set. Moreover, if the metric space X is complete, then M is totally bounded if and only if M is relatively compact.

Throughout, we shall write $\mathcal{M}_X$ for the collection of all bounded subsets of a metric space (X, d). If $Q \in \mathcal{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$ and centers in X. This can equivalently be redefined as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{-net}\}. \tag{2.1}$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness*.

If Q, Q_1, and Q_2 are bounded subsets of a metric space X, then we have

$$\chi(Q) = 0 \quad \text{if and only if } Q \text{ is totally bounded},$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2). \tag{2.2}$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, for example,

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$
$$\chi(\alpha Q) = |\alpha|\ \chi(Q), \quad \forall \alpha \in \mathbb{C}. \tag{2.3}$$

Let X and Y be Banach spaces and χ_1 and χ_2 be the Hausdorff measures of noncompactness on X and Y, respectively. An operator $L : X \to Y$ is said to be (χ_1,χ_2)-*bounded* if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exist a constant $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X$. If an operator L is (χ_1,χ_2)-bounded then the number $\|L\|_{(\chi_1,\chi_2)} := \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}_X\}$ is called the (χ_1,χ_2)-*measure of noncompactness of* L. If $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi_1,\chi_2)} = \|L\|_\chi$.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows: let X and Y be Banach spaces and $L \in B(X, Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, can be determined by

$$\|L\|_\chi = \chi(L(S_X)), \tag{2.4}$$

and we have that L is compact if and only if

$$\|L\|_\chi = 0. \tag{2.5}$$

Furthermore, the function χ is more applicable when X is a Banach space. In fact, there are many formulae which are useful to evaluate the Hausdorff measures of noncompactness of bounded sets in some particular Banach spaces. For example, we have the following result of Goldenštein et al. [10, Theorem 1] which gives an estimate for the Hausdorff measure of noncompactness in Banach spaces with Schauder bases. Before that, let us recall that if $(b_k)_{k=0}^{\infty}$ is a Schauder basis for a Banach space X, then every element $x \in X$ has a unique representation $x = \sum_{k=0}^{\infty} \phi_k(x) b_k$, where ϕ_k $(k \in \mathbb{N})$ are called the basis functionals. Moreover, the operator $P_r : X \to X$, defined for each $r \in \mathbb{N}$ by $P_r(x) = \sum_{k=0}^{r} \phi_k(x) b_k (x \in X)$, is called the projector onto the linear span of $\{b_0, b_1, \ldots, b_r\}$. Besides, all operators P_r and $I - P_r$ are equibounded, where I denotes the identity operator on X.

Theorem 2.1. *Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $E \in \mathcal{M}_X$ and $P_n : X \to X(n \in \mathbb{N})$ be the projector onto the linear span of $\{b_0, b_1, \ldots, b_n\}$. Then, one has*

$$\frac{1}{a} \cdot \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right) \le \chi(E) \le \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right), \tag{2.6}$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$.

In particular, the following result shows how to compute the Hausdorff measure of noncompactness in the spaces c_0 and ℓ_p $(1 \le p < \infty)$ which are BK spaces with AK.

Theorem 2.2. *Let E be a bounded subset of the normed space X, where X is ℓ_p for $1 \le p < \infty$ or c_0. If $P_n : X \to X(n \in \mathbb{N})$ is the operator defined by $P_n(x) = x^{[n]} = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ for all $x = (x_k)_{k=0}^{\infty} \in X$, then one has*

$$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right). \tag{2.7}$$

It is easy to see that for $E \in \mathcal{M}_{\ell_p}$

$$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in Q}\sum_{k\ge n}|x_k|^p\right). \tag{2.8}$$

Also, it is known that $(e, e^{(0)}, e^{(1)}, \ldots)$ is a Schauder basis for the space c and every sequence $z = (z_n)_{n=0}^{\infty} \in c$ has a unique representation $z = \overline{z}e + \sum_{n=0}^{\infty}(z_n - \overline{z})e^{(n)}$, where $\overline{z} = \lim_{n\to\infty} z_n$. Thus, we define the projector $P_r : c \to c\,(r \in \mathbb{N})$, onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$, by

$$P_r(z) = \overline{z}e + \sum_{n=0}^{r}(z_n - \overline{z})e^{(n)}; \quad (r \in \mathbb{N}), \tag{2.9}$$

for all $z = (z_n) \in c$ with $\overline{z} = \lim_{n\to\infty} z_n$. In this situation, we have the following.

Theorem 2.3. *Let $Q \in \mathcal{M}_c$ and $P_r : c \to c\,(r \in \mathbb{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$. Then, one has*

$$\frac{1}{2} \cdot \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right) \le \chi(Q) \le \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right), \tag{2.10}$$

where I is the identity operator on c.

Theorem 2.4. *Let Q be a bounded subset of $n(\phi)$. Then*

$$\chi(Q) = \lim_{k\to\infty}\sup_{x\in Q}\left(\sup_{u\in S(x)}\left(\sum_{n=k}^{\infty}|u_n|\Delta\phi_n\right)\right). \tag{2.11}$$

The idea of *compact operators* between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows.

Let X and Y be complex Banach spaces. Then, a linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X, that is, $D(L) = X$, and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. Equivalently, we say that L is compact if its domain is all of X and $L(Q)$ is relatively compact in Y for every $Q \in \mathcal{M}_X$.

Further, we write $\mathcal{C}(X, Y)$ for the class of all compact operators from X to Y. Let us remark that every compact operator in $\mathcal{C}(X, Y)$ is bounded, that is, $\mathcal{C}(X, Y) \subset \mathcal{B}(X, Y)$. More precisely, the class $\mathcal{C}(X, Y)$ is a closed subspace of the Banach space $\mathcal{B}(X, Y)$ with the operator norm.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows.

Let X and Y be Banach spaces and $L \in \mathcal{B}(X; Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, can be given by

$$\|L\|_\chi = \chi(L(S_X)), \tag{2.12}$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{2.13}$$

Since matrix mappings between BK spaces define bounded linear operators between these spaces which are Banach spaces, it is natural to use the Hausdorff measure of noncompactness to obtain necessary and sufficient conditions for matrix operators between BK spaces to be compact operators. This technique has recently been used by several authors in many research papers (see, for instance, [5, 6, 15–32]).

3. Applications to Infinite Systems of Differential Equations

This section is mainly based on the work of Banaś and Lecko [33], Mursaleen and Mohiuddine [26], and Mursaleen [32]. In this section, we apply the technique of measures of noncompactness to the theory of infinite systems of differential equations in some Banach sequence spaces c_0, c, $\ell_p (1 \le p < \infty)$, and $n(\phi)$.

Infinite systems of ordinary differential equations describe numerous world real problems which can be encountered in the theory of branching processes, the theory of neural nets, the theory of dissociation of polymers, and so on (cf. [34–38], e.g.). Let us also mention that several problems investigated in mechanics lead to infinite systems of differential equations [39–41]. Moreover, infinite systems of differential equations can be also used in solving some problems for parabolic differential equations investigated via semidiscretization [42, 43]. The theory of infinite systems of ordinary differential equation seems not to be developed satisfactorily up to now. Indeed, the existence results concerning systems of such a kind were formulated mostly by imposing the Lipschitz condition on right-hand sides of those systems (cf. [10, 11, 39, 40, 44–50]). Obviously, the assumptions formulated in terms of the Lipschitz condition are rather restrictive and not very useful in applications. On the other hand, the infinite systems of ordinary differential equations can be considered as a particular case of ordinary differential equations in Banach spaces. Until now several existence results have been obtained concerning the Cauchy problem for ordinary differential equations in Banach spaces [33, 35, 51–53]. A considerable number of those results were formulated in terms of measures of noncompactness. The results of such a type have a concise form and give the possibility to formulate more general assumptions than those requiring the Lipschitz continuity. But in general those results are not immediately applicable in concrete situations, especially in the theory of infinite systems of ordinary differential equations.

In this section, we adopt the technique of measures of noncompactness to the theory of infinite systems of differential equations. Particularly, we are going to present a few existence results for infinite systems of differential equations formulated with the help of convenient and handy conditions.

Consider the ordinary differential equation

$$x' = f(t, x), \tag{3.1}$$

with the initial condition

$$x(0) = x_0. \tag{3.2}$$

Then the following result for the existence of the Cauchy problem (3.1)-(3.2) was given in [34] which is a slight modification of the result proved in [33].

Assume that X is a real Banach space with the norm $\|\cdot\|$. Let us take an interval $I = [0, T]$, $T > 0$ and $B(x_0, r)$ the closed ball in X centered at x_0 with radius r.

Theorem A (see [34]). *Assume that $f(t, x)$ is a function defined on $I \times X$ with values in X such that*

$$\|f(t, x)\| \leq Q + R\|x\|, \tag{3.3}$$

for any $x \in X$, where Q and R are nonnegative constants. Further, let f be uniformly continuous on the set $I_1 \times B(x_0, r)$, where $r = (QT_1 + RT_1\|x_0\|)/(1 - RT_1)$ and $I_1 = [0, T_1] \subset I$, $RT_1 < 1$. Moreover, assume that for any nonempty set $Y \subset B(x_0, s)$ and for almost all $t \in I$ the following inequality holds:

$$\mu\big(f(t, Y)\big) \leq q(t)\mu(Y), \tag{3.4}$$

with a sublinear measure of noncompactness μ such that $\{x_0\} \in \ker\mu$. Then problem (3.1)-(3.2) *has a solution x such that $\{x(t)\} \in \ker\mu$ for $t \in I_1$, where $q(t)$ is an integrable function on I, and* $\ker\mu = \{E \in \mathfrak{M}_X : \mu(E) = 0\}$ *is the kernel of the measure μ.*

Remark 3.1. In the case when $\mu = \chi$ (the Hausdorff measure of noncompactness), the assumption of the uniform continuity on f can be replaced by the weaker one requiring only the continuity of f.

Results of Sections 3.1 and 3.2 are from [33], Section 3.3 from [26], and Section 3.4 from [32].

3.1. Infinite Systems of Differential Equations in the Space c_0

From now on, our discussion is exactly same as in Section 3 of [33].

In this section, we study the solvability of the infinite systems of differential equations in the Banach sequence space c_0. It is known that in the space c_0 the Hausdorff measure of noncompactness can be expressed by the following formula [33]:

$$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in E}\left\{\sup_{k\geq n}|x_k|\right\}\right), \tag{3.5}$$

where $E \in \mathfrak{M}_{c_0}$.

We will be interested in the existence of solutions $x(t) = (x_i(t))$ of the infinite systems of differential equations

$$x_i' = f_i(t, x_0, x_1, x_2, \ldots), \tag{3.6}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{3.7}$$

$(i = 0, 1, 2, \ldots)$ which are defined on the interval $I = [0, T]$ and such that $x(t) \in c_0$ for each $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space c_0 can be formulated by making the following assumptions.

Assume that the functions f_i $(i = 1, 2, \ldots)$ are defined on the set $I \times \mathbb{R}^\infty$ and take real values. Moreover, we assume the following hypotheses:

(i) $x_0 = (x_i^0) \in c_0$,

(ii) the map $f = (f_1, f_2, \ldots)$ acts from the set $I \times c_0$ into c_0 and is continuous,

(iii) there exists an increasing sequence (k_n) of natural numbers (obviously $k_n \to \infty$ as $n \to \infty$) such that for any $t \in I$, $x = (x_i) \in c_0$ and $n = 1, 2, \ldots$ the following inequality holds:

$$\left|f_n(t, x_1, x_2, \ldots)\right| \leq p_n(t) + q_n(t)\left(\sup_{i\geq k_n}|x_i|\right), \tag{3.8}$$

where $(p_i(t))$ and $(q_i(t))$ are real functions defined and continuous on I such that the sequence $(p_i(t))$ converges uniformly on I to the function vanishing identically and the sequence $(q_i(t))$ is equibounded on I.

Now, let us denote

$$q(t) = \sup_{n \geq 1} q_n(t),$$
$$Q = \sup_{t \in I} q(t), \tag{3.9}$$
$$P = \sup\{p_n(t) : t \in I, n = 1, 2, \ldots\}.$$

Then we have the following result.

Theorem 3.2 (see [33]). *Under the assumptions (i)–(iii), initial value problem* (3.6)-(3.7) *has at least one solution* $x = x(t) = (x_i(t))$ *defined on the interval* $I_1 = [0, T_1]$ *whenever* $T_1 < T$ *and* $QT_1 < 1$. *Moreover,* $x(t) \in c_0$ *for any* $t \in I_1$.

Proof. Let $x = (x_i)$ be any arbitrary sequence in c_0. Then, by (i)–(iii), for any $t \in I$ and for a fixed $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |f_n(t,x)| = |f_n(t, x_1, x_2, \ldots)| &\leq p_n(t) + q_n(t)\left(\sup_{i \geq k_n}|x_i|\right) \\ &\leq P + Q\sup_{i \geq k_n}|x_i| \leq P + Q\|x\|_\infty. \end{aligned} \tag{3.10}$$

Hence, we get

$$\|f(t,x)\| \leq P + Q\|x\|_\infty. \tag{3.11}$$

In what follows, let us take the ball $B(x_0, r)$, where r is chosen according to Theorem A. Then, for a subset Y of $B(x_0, r)$ and for $t \in I_1$, we obtain

$$\begin{aligned} \chi(f(t,Y)) &= \lim_{n \to \infty} \sup_{x \in Y}\left(\sup_{i \geq n}|f_i(t,x)|\right), \\ \chi(f(t,Y)) &= \lim_{n \to \infty} \sup_{x \in Y}\left(\sup_{i \geq n}|f_i(t, x_1, x_2, \ldots)|\right) \\ &\leq \lim_{n \to \infty} \sup_{x \in Y}\left(\sup_{i \geq n}\left\{p_i(t) + q_i(t)\sup_{j \geq k_n}|x_j|\right\}\right) \\ &\leq \lim_{n \to \infty} \sup_{i \geq n} p_i(t) + q(t) \lim_{n \to \infty}\left\{\sup_{x \in Y}\left(\sup_{i \geq n}\left\{\sup_{j \geq k_n}|x_j|\right\}\right)\right\}. \end{aligned} \tag{3.12}$$

Hence, by assumptions, we get

$$\chi(f(t,Y)) \leq q(t)\chi(Y). \tag{3.13}$$

Now, using our assumptions and inequalities (3.11) and (3.13), in view of Theorem A and Remark 3.1 we deduce that there exists a solution $x = x(t)$ of the Cauchy problem (3.6)-(3.7) such that $x(t) \in c_0$ for any $t \in I_1$.

This completes the proof of the theorem. □

We illustrate the above result by the following examples.

Example 3.3 (see [33]). Let $\{k_n\}$ be an increasing sequence of natural numbers. Consider the infinite system of differential equations of the form

$$x_i' = f_i(t, x_1, x_2, \ldots) + \sum_{j=k_i+1}^{\infty} a_{ij}(t)x_j, \tag{3.14}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{3.15}$$

$(i = 1, 2, \ldots;\ t \in I = [0, T])$.

We will investigate problem (3.14)-(3.15) under the following assumptions:

(i) $x_0 = (x_0) \in c_0$,

(ii) the functions $f_i : I \times \mathbb{R}^{k_i} \to \mathbb{R}\ (i = 1, 2, \ldots)$ are uniformly equicontinuous and there exists a function sequence $(p_i(t))$ such that $p_i(t)$ is continuous on I for any $i \in \mathbb{N}$ and $(p_i(t))$ converges uniformly on I to the function vanishing identically. Moreover, the following inequality holds:

$$\left| f_i(t, x_1, x_2, \ldots x_{k_i}) \right| \leq p_i(t), \tag{3.16}$$

for $t \in I, (x_1, x_2, \ldots x_{k_i}) \in \mathbb{R}^{k_i}$ and $i \in \mathbb{N}$,

(iii) the functions $a_{ij}(t)$ are defined and continuous on I and the function series $\sum_{j=k_i+1}^{\infty} a_{ij}(t)$ converges absolutely and uniformly on I (to a function $a_i(t)$) for any $i = 1, 2, \ldots$,

(iv) the sequence $(a_i(t))$ is equibounded on I,

(v) $QT < 1$, where $Q = \sup\{a_i(t) : i = 1, 2, \ldots; t \in I\}$.

It can be easily seen that the assumptions of Theorem 3.2 are satisfied under assumptions (i)–(v). This implies that problem (3.14)-(3.15) has a solution $x(t) = (x_i(t))$ on the interval I belonging to the space c_0 for any fixed $t \in I$.

As mentioned in [33], problem (3.14)-(3.15) considered above contains as a special case the infinite system of differential equations occurring in the theory of dissociation of polymers [35]. That system was investigated in [37] in the sequence space ℓ_∞ under very strong assumptions. The existing result proved in [35] requires also rather restrictive assumptions. Thus, the above result is more general than those quoted above.

Moreover, the choice of the space c_0 for the study of the problem (3.14)-(3.15) enables us to obtain partial characterization of solutions of this problem since we have that $x_n(t) \to 0$ when $n \to \infty$, for any fixed $t \in [0, T]$.

On the other hand, let us observe that in the study of the heat conduction problem via the method of semidiscretization we can obtain the infinite systems of form (3.14) (see [42] for details).

Example 3.4 (see [33]). In this example, we will consider some special cases of problem (3.14)-(3.15). Namely, assume that $k_i = i$ for $i = 1, 2, \ldots$ and $a_{ij} \equiv 0$ on I for all i, j. Then system (3.14) has the form

$$\begin{gathered} x_1' = f_1(t, x_1), \qquad x_2' = f_2(t, x_1, x_2), \ldots, \\ x_i' = f_i(t, x_1, x_2, \ldots, x_i), \ldots, \end{gathered} \tag{3.17}$$

and is called a row-finite system [35].

Suppose that there are satisfied assumptions from Example 3.3, that is, $x_0 = (x_i^0) \in c_0$ and the functions f_i act from $I \times \mathbb{R}^i$ into $\mathbb{R}$ $(i = 1, 2, \ldots)$ and are uniformly equicontinuous on their domains. Moreover, there exist continuous functions $p_i(t)$ $(t \in I)$ such that

$$\left| f_i(t, x_1, x_2, \ldots x_{k_i}) \right| \leq p_i(t), \tag{3.18}$$

for $t \in I$ and $x_1, x_2, \ldots, x_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$. We assume also that the sequence $p_i(t)$ converges uniformly on I to the function vanishing identically.

Further, let $|\cdot|_i$ denote the maximum norm in $\mathbb{R}^i (i = 1, 2, \ldots)$. Take $f^i = (f_1, f_2, \ldots, f_i)$. Then we have

$$\begin{aligned} \left| f^i(t, x) \right|_i &= \max\{ \left| f_1(t, x_1) \right|, \left| f_2(t, x_1, x_2) \right|, \ldots, \left| f_i(t, x_1, x_2, \ldots, x_i) \right| \} \\ &\leq \max\{ p_1(t), p_2(t), \ldots, p_i(t) \}. \end{aligned} \tag{3.19}$$

Taking $P_i(t) = \max\{p_1(t), p_2(t), \ldots, p_i(t)\}$, then the above estimate can be written in the form

$$\left| f^i(t, x) \right|_i \leq P_i(t). \tag{3.20}$$

Observe that from our assumptions it follows that the initial value problem $u' = P_i(t)$, $u(0) = x_i^0$ has a unique solution on the interval I. Hence, applying a result from [33], we infer that Cauchy problem (3.17)-(3.15) has a solution on the interval I. Obviously from the result contained in Theorem 3.2 and Example 3.3, we deduce additionally that the mentioned solution belongs to the space c_0.

Finally, it is noticed [33] that the result described above for row-finite systems of the type (3.17) can be obtained under more general assumptions.

In fact, instead of inequality (3.18), we may assume that the following estimate holds to be true:

$$\left| f_i(t, x_1, x_2, \ldots, x_i) \right| \leq p_i(t) + q_i(t) \max\{|x_1|, |x_2|, \ldots, |x_i|\}, \tag{3.21}$$

where the functions $p_i(t)$ and $q_i(t)$ $(i = 1, 2, \ldots)$ satisfy the hypotheses analogous to those assumed in Theorem 3.2.

Remark 3.5 (see [33]). Note that in the birth process one can obtain a special case of the infinite system (3.17) which is lower diagonal linear infinite system [35, 43]. Thus, the result proved above generalizes that from [35, 37].

3.2. Infinite Systems of Differential Equations in the Space c

Now, we will study the solvability of the following perturbed diagonal system of differential equations

$$x_i' = a_i(t)x_i + g_i(t, x_1, x_2, \ldots), \tag{3.22}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{3.23}$$

$(i = 1, 2, \ldots)$, where $t \in I = [0, T]$.

From now, we are going exactly the same as in Section 4 of [33].

We consider the following measures μ of noncompactness in c which is more convenient, regular, and even equivalent to the Hausdorff measure of noncompactness [33].

For $E \in \mathfrak{M}_c$

$$\begin{aligned} \mu(E) &= \lim_{p \to \infty} \left\{ \sup_{x \in E} \left\{ \sup_{n,m \geq p} |x_n - x_m| \right\} \right\}, \\ \chi(E) &= \lim_{n \to \infty} \left(\sup_{x \in Q} \left\{ \sup_{k \geq n} |x_k| \right\} \right). \end{aligned} \tag{3.24}$$

Let us formulate the hypotheses under which the solvability of problem (3.22)-(3.23) will be investigated in the space c. Assume that the following conditions are satisfied.

Assume that the functions f_i $(i = 1, 2, \ldots)$ are defined on the set $I \times \mathbb{R}^\infty$ and take real values. Moreover, we assume the following hypotheses:

(i) $x_0 = (x_i^0) \in c$,

(ii) the map $g = (g_1, g_2, \ldots)$ acts from the set $I \times c$ into c and is uniformly continuous on $I \times c$,

(iii) there exists sequence $(b_i) \in c_0$ such that for any $t \in I$, $x = (x_i) \in c$ and $n = 1, 2, \ldots$ the following inequality holds:

$$\left| g_n(t, x_1, x_2, \ldots) \right| \leq b_i, \tag{3.25}$$

(iv) the functions $a_i(t)$ are continuous on I such that the sequence $(a_i(t))$ converges uniformly on I.

Further, let us denote

$$a(t) = \sup_{i\geq 1} a_i(t),$$
$$Q = \sup_{t\in I} a(t). \tag{3.26}$$

Observe that in view of our assumptions, it follows that the function $a(t)$ is continuous on I. Hence, $Q < \infty$.

Then we have the following result which is more general than Theorem 3.2.

Theorem 3.6 (see [33]). *Let assumptions (i)–(iv) be satisfied. If $QT < 1$, then the initial value problem* (3.12)-(3.13) *has a solution $x(t) = (x_i(t))$ on the interval I such that $x(t) \in c$ for each $t \in I$.*

Proof. Let $t \in I$ and $x = (x_i) \in c$ and

$$f_i(t,x) = a_i(t)x_i + g_i(t;x),$$
$$f(t,x) = \left(f_1(t,x), f_2(t;x), \dots\right) = \left(f_i(t,x)\right). \tag{3.27}$$

Then, for arbitrarily fixed natural numbers n, m we get

$$\begin{aligned}
\left|f_n(t,x) - f_m(t,x)\right| &= \left|a_n(t)x_n + g_n(t;x) - a_m(t)x_m - g_m(t;x)\right| \\
&\leq \left|a_n(t)x_n + g_n(t;x)\right| - \left|a_m(t)x_m + g_m(t;x)\right| \\
&\leq |a_n(t)x_n - a_n(t)x_m| + |a_n(t)x_m - a_m(t)x_m| + b_n + b_m \\
&\leq |a_n(t)||x_n - x_m| + |a_n(t) - a_m(t)||x_m| + b_n + b_m \\
&\leq |a_n(t)||x_n - x_m| + \|x\|_\infty |a_n(t) - a_m(t)|\ b_n + b_m.
\end{aligned} \tag{3.28}$$

By assumptions (iii) and (iv), from the above estimate we deduce that $(f_i(t,x))$ is a real Cauchy sequence. This implies that $(f_i(t,x)) \in c$.

Also we obtain the following estimate:

$$\begin{aligned}
\left|f_i(t,x)\right| &\leq |a_i(t)||x_i| + \left|g_i(t,x)\right| \\
&\leq Q|x_i| + b_i \leq Q\|x\|_\infty + B,
\end{aligned} \tag{3.29}$$

where $B = \sup_{i\geq 1} b_i$. Hence,

$$\left\|f(t,x)\right\| \leq Q\|x\|_\infty + B. \tag{3.30}$$

In what follows, let us consider the mapping $f(t,x)$ on the set $I \times B(x_0, r)$, where r is taken according to the assumptions of Theorem A, that is, $r = (BT + QT\|x_0\|_\infty)/(1 - QT)$.

Further, fix arbitrarily $t, s \in I$ and $x, y \in B(x_0, r)$. Then, by our assumptions, for a fixed i, we obtain

$$\begin{aligned}\left|f_i(t,x) - f_i(s,y)\right| &= \left|a_i(t)x_i + g_i(t,x) - a_i(s)y_i - g_i(s,y)\right| \\ &\leq \left|a_i(t)x_i - a_i(s)y_i\right| + \left|g_i(t,x) - g_i(s,y)\right| \\ &\leq |a_i(t) - a_i(s)||x_i| + |a_i(s)|\left|x_i - y_i\right| + \left|g_i(t,x) - g_i(s,y)\right|.\end{aligned} \tag{3.31}$$

Then,

$$\begin{aligned}\left\|f(t,x) - f(s,y)\right\| &= \sup_{i\geq 1}\left|f_i(t,x) - f_i(s,y)\right| \\ &\leq (r + \|x_0\|_\infty) \cdot \sup_{i\geq 1}|a_i(t) - a_i(s)| \\ &\quad + Q\left\|x - y\right\|_\infty + \left\|g(t,x) - g(s,y)\right\|.\end{aligned} \tag{3.32}$$

Hence, taking into account that the sequence $(a_i(t))$ is equicontinuous on the interval I and g is uniformly continuous on $I \times c$, we conclude that the operator $f(t,x)$ is uniformly continuous on the set $I \times B(x_0, r)$.

In the sequel, let us take a nonempty subset E of the ball $B(x_0, r)$ and fix $t \in I$, $x \in X$. Then, for arbitrarily fixed natural numbers n, m we have

$$\begin{aligned}\left|f_n(t,x) - f_m(t,x)\right| &\leq |a_n(t)||x_n - x_m| + |x_m||a_n(t) - a_m(t)| + \left|g_n(t,x)\right| + \left|g_m(t,x)\right| \\ &\leq a(t)|x_n - x_m| + (r + \|x_0\|_\infty)|a_n(t) - a_m(t)| + b_n + b_m.\end{aligned} \tag{3.33}$$

Hence, we infer the following inequality:

$$\mu(f(t,E)) \leq a(t)\mu(E). \tag{3.34}$$

Finally, combining (3.30), (3.34) and the fact (proved above) that f is uniformly continuous on $I \times B(x_0, r)$, in view of Theorem A, we infer that problem (3.22)-(3.23) is solvable in the space c.

This completes the proof of the theorem. □

Remark 3.7 (see [33, Remark 4]). The infinite systems of differential equations (3.22)-(3.23) considered above contain as special cases the systems studied in the theory of neural sets (cf. [35, pages 86-87] and [37], e.g.). It is easy to notice that the existence results proved in [35, 37] are obtained under stronger and more restrictive assumptions than our one.

3.3. *Infinite Systems of Differential Equations in the Space ℓ_p*

In this section, we study the solvability of the infinite systems of differential equations (3.6)-(3.7) in the Banach sequence space ℓ_p $(1 \leq p < \infty)$ such that $x(t) \in \ell_p$ for each $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space ℓ_p can be formulated by making the following assumptions:

(i) $x_0 = (x_i^0) \in \ell_p$,

(ii) $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ $(i = 0, 1, 2, \ldots)$ maps continuously the set $I \times \ell_p$ into ℓ_p,

(iii) there exist nonnegative functions $q_i(t)$ and $r_i(t)$ defined on I such that

$$\left|f_i(t,x)\right|^p = \left|f_i(t, x_0, x_1, x_2, \ldots)\right|^p \leq q_i(t) + r_i(t)|x_i|^p, \tag{3.35}$$

for $t \in I; x = (x_i) \in \ell_p$ and $i = 0, 1, 2, \ldots$,

(iv) the functions $q_i(t)$ are continuous on I and the function series $\sum_{i=0}^{\infty} q_i(t)$ converges uniformly on I,

(v) the sequence $(r_i(t))$ is equibounded on the interval I and the function $r(t) = \limsup_{i\to\infty} r_i(t)$ is integrable on I.

Now, we prove the following result.

Theorem 3.8 (see [26]). *Under the assumptions (i)–(v), problem* (3.6)-(3.7) *has a solution* $x(t) = (x_i(t))$ *defined on the interval* $I = [0, T]$ *whenever* $RT < 1$, *where* R *is defined as the number*

$$R = \sup\{r_i(t) : t \in I,\ i = 0, 1, 2, \ldots\}. \tag{3.36}$$

Moreover, $x(t) \in \ell_p$ *for any* $t \in I$.

Proof. For any $x(t) \in \ell_p$ and $t \in I$, under the above assumptions, we have

$$\begin{aligned}
\|f(t,x)\|_p^p &= \sum_{i=0}^{\infty}\left|f_i(t,x)\right|^p \\
&\leq \sum_{i=0}^{\infty}\left[q_i(t) + r_i(t)|x_i|^p\right] \\
&\leq \sum_{i=0}^{\infty} q_i(t) + \left(\sup_{i\geq 0} r_i(t)\right)\left(\sum_{i=0}^{\infty}|x_i|^p\right) \\
&\leq Q + R\|x\|_p^p,
\end{aligned} \tag{3.37}$$

where $Q = \sup_{t\in I}(\sum_{i=0}^{\infty} q_i(t))$.

Now, choose the number $s = (QT + RT\|x_0\|_p^p)/(1 - RT)$ as defined in Theorem A. Consider the operator $f = (f_i)$ on the set $I \times B(x_0; s)$. Let us take a set $Y \in \mathcal{M}_{\ell_p}$. Then by using (2.8), we get

$$\begin{aligned}
\chi(f(t,Y)) &= \lim_{n\to\infty}\sup_{x\in Y}\left(\sum_{i\geq n}\left|f_i(t, x_1, x_2, \ldots)\right|^p\right) \\
&\leq \lim_{n\to\infty}\left(\sum_{i\geq n} q_i(t) + \left(\sup_{i\geq n} r_i(t)\right)\left(\sum_{i\geq n}|x_i|^p\right)\right).
\end{aligned} \tag{3.38}$$

Hence, by assumptions (iv)-(v), we get

$$\chi\big(f(t,Y)\big) \leq r(t)\chi(Y), \tag{3.39}$$

that is, the operator f satisfies condition (3.4) of Theorem A. Hence, by Theorem A and Remark 3.1 we conclude that there exists a solution $x = x(t)$ of problem (3.6)-(3.7) such that $x(t) \in \ell_p$ for any $t \in I$.

This completes the proof of the theorem. □

Remark 3.9. (I) For $p = 1$, we get Theorem 5 of [33].

(II) It is easy to notice that the existence results proved in [44] are obtained under stronger and more restrictive assumptions than our one.

(III) We observe that the above theorem can be applied to the perturbed diagonal infinite system of differential equations of the form

$$x_i' = a_i(t)x_i + g_i(t, x_0, x_1, x_2, \ldots), \tag{3.40}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{3.41}$$

$(i = 0, 1, 2, \ldots)$ where $t \in I$.

An existence theorem for problem (3.6)-(3.7) in the space ℓ_p can be formulated by making the following assumptions:

(i) $x_0 = (x_i^0) \in \ell_p$,

(ii) the sequence $(|a_i(t)|)$ is defined and equibounded on the interval $I = [0, T]$. Moreover, the function $a(t) = \limsup_{i\to\infty} \sup|a_i(t)|$ is integrable on I,

(iii) the mapping $g = (g_i)$ maps continuously the set $I \times \ell_p$ into ℓ_p,

(iv) there exist nonnegative functions $b_i(t)$ such that

$$\big|f_i(t, x_0, x_1, x_2, \ldots)\big|^p \leq b_i(t), \tag{3.42}$$

for $t \in I$; $x = (x_i) \in \ell_p$ and $i = 0, 1, 2, \ldots$,

(v) the functions $b_i(t)$ are continuous on I and the function series $\sum_{i=0}^{\infty} b_i(t)$ converges uniformly on I.

3.4. Infinite Systems of Differential Equations in the Space $n(\phi)$

An existence theorem for problem (3.6)-(3.7) in the space $n(\phi)$ can be formulated by making the following assumptions:

(i) $x_0 = (x_i^0) \in n(\phi)$,

(ii) $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ $(i = 1, 2, \ldots)$ maps continuously the set $I \times n(\phi)$ into $n(\phi)$,

(iii) there exist nonnegative functions $p_i(t)$ and $q_i(t)$ defined on I such that

$$\left|f_i(t,u)\right| = \left|f_i(t,u_1,u_2,\ldots)\right| \le p_i(t) + q_i(t)|u_i|, \tag{3.43}$$

for $t \in I; x = (x_i) \in n(\phi)$ and $i = 1, 2, \ldots$, where $u = (u_i)$ is a sequence of rearrangement of $x = (x_i)$,

(iv) the functions $p_i(t)$ are continuous on I and the function series $\sum_{i=1}^{\infty} p_i(t)\Delta\phi_i$ converges uniformly on I,

(v) the sequence $(q_i(t))$ is equibounded on the interval I and the function $q(t) = \limsup_{i\to\infty} q_i(t)$ is integrable on I.

Now, we prove the following result.

Theorem 3.10 (see [32]). *Under the assumptions (i)–(v), problem* (3.6)-(3.7) *has a solution* $x(t) = (x_i(t))$ *defined on the interval* $I = [0, T]$ *whenever* $QT < 1$, *where* Q *is defined as the number*

$$Q = \sup\left\{q_i(t) : t \in I,\ i = 1, 2, \ldots\right\}. \tag{3.44}$$

Moreover, $x(t) \in n(\phi)$ *for any* $t \in I$.

Proof. For any $x(t) \in n(\phi)$ and $t \in I$, under the above assumptions, we have

$$\begin{aligned}
\|f(t,x)\|_{n(\phi)} &= \sup_{u\in S(x)} \sum_{i=1}^{\infty}\left|f_i(t,u)\right|\Delta\phi_i \\
&\le \sup_{u\in S(x)} \sum_{i=1}^{\infty}\left[p_i(t) + q_i(t)|u_i|\right]\Delta\phi_i \\
&\le \sum_{i=1}^{\infty} p_i(t)\Delta\phi_i + \left(\sup_i q_i(t)\right)\left(\sup_{u\in S(x)}\sum_{i=1}^{\infty}|u_i|\Delta\phi_i\right) \\
&\le P + Q\|x\|_{n(\phi)},
\end{aligned} \tag{3.45}$$

where $P = \sup_{t\in I}\sum_{i=1}^{\infty} p_i(t)\Delta\phi_i$.

Now, choose the number r defined according to Theorem A, that is, $r = (PT + QT\|x_0\|_{n(\phi)})/(1 - QT)$. Consider the operator $f = (f_i)$ on the set $I \times B(x_0; r)$. Let us take a set $X \in \mathcal{M}_{n(\phi)}$. Then by using Theorem 2.4, we get

$$\begin{aligned}
\chi(f(t,X)) &= \lim_{k\to\infty}\sup_{x\in X}\left(\sup_{u\in S(x)}\left(\sum_{n=k}^{\infty}\left|f_n(t,u_1,u_2,\ldots)\right|\Delta\phi_n\right)\right) \\
&\le \lim_{k\to\infty}\left(\sum_{n=k}^{\infty} p_n(t)\Delta\phi_n + \left(\sup_{n\ge k} q_n(t)\right)\left(\sup_{u\in S(x)}\sum_{n=k}^{\infty}|u_n|\Delta\phi_n\right)\right).
\end{aligned} \tag{3.46}$$

Hence, by assumptions (iv)-(v), we get

$$\chi\big(f(t, X)\big) \leq q(t)\chi(X), \tag{3.47}$$

that is, the operator f satisfies condition (3.4) of Theorem A. Hence, the problem (3.6)-(3.7) has a solution $x(t) = (x_i(t))$.

This completes the proof of the theorem. □

Remark 3.11. Similarly, we can establish such type of result for the space $m(\phi)$.

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Research Article

Some Existence Results for Impulsive Nonlinear Fractional Differential Equations with Closed Boundary Conditions

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We investigate some existence results for the solutions to impulsive fractional differential equations having closed boundary conditions. Our results are based on contracting mapping principle and Burton-Kirk fixed point theorem.

1. Introduction

This paper considers the existence and uniqueness of the solutions to the closed boundary value problem (BVP), for the following impulsive fractional differential equation:

$$\begin{gathered} {}^{C}D^{\alpha}x(t) = f(t, x(t)), \quad t \in J := [0, T],\ t \neq t_k,\ 1 < \alpha \leq 2, \\ \Delta x(t_k) = I_k\left(x\left(t_k^-\right)\right), \quad \Delta x'(t_k) = I_k^*\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, p, \\ x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0), \end{gathered} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is Caputo fractional derivative, $f \in C(J \times R, R)$, $I_k, I_k^* \in C(R, R)$,

$$\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right), \tag{1.2}$$

with

$$x(t_k^+) = \lim_{h\to 0^+} x(t_k + h), \qquad x(t_k^-) = \lim_{h\to 0^-} x(t_k + h), \tag{1.3}$$

and $\Delta x'(t_k)$ has a similar meaning for $x'(t)$, where

$$0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T, \tag{1.4}$$

a, b, c, and d are real constants with $\Delta := c(1-b) + (1-a)(1-d) \neq 0$.

The boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during last decades. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Actually, fractional differential equations arise in many engineering and scientific disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media (see [1–7]). For some recent development, see, for example, [8–14].

On the other hand, theory of impulsive differential equations for integer order has become important and found its extensive applications in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a noticeable development in impulsive theory. For instance, for the general theory and applications of impulsive differential equations we refer the readers to [15–17].

Moreover, boundary value problems for impulsive fractional differential equations have been studied by some authors (see [18–20] and references therein). However, to the best of our knowledge, there is no study considering closed boundary value problems for impulsive fractional differential equations.

Here, we notice that the closed boundary conditions in (1.1) include quasi-periodic boundary conditions ($b = c = 0$) and interpolate between periodic ($a = d = 1, b = c = 0$) and antiperiodic ($a = d = -1, b = c = 0$) boundary conditions.

Motivated by the mentioned recent work above, in this study, we investigate the existence and uniqueness of solutions to the closed boundary value problem for impulsive fractional differential equation (1.1). Throughout this paper, in Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following sections. In Section 3, we discuss some existence and uniqueness results for solutions of BVP (1.1), that is, the first one is based on Banach's fixed point theorem, the second one is based on the Burton-Kirk fixed point theorem. At the end, we give an illustrative example for our results.

2. Preliminaries

Let us set $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_{k-1} = (t_{k-1}, t_k]$, $J_k = (t_k, t_{k+1}]$, $J' := [0, T] \setminus \{t_1, t_2, \ldots, t_p\}$ and introduce the set of functions:

$PC(J, R) = \{x : J \to R : x \in C((t_k, t_{k+1}], R)$, $k = 0, 1, 2, \ldots, p$ and there exist $x(t_k^+)$ and $x(t_k^-)$, $k = 1, 2, \ldots, p$ with $x(t_k^-) = x(t_k)\}$ and

$PC^1(J, R) = \{x \in PC(J, R)$, $x' \in C((t_k, t_{k+1}], R)$, $k = 0, 1, 2, \ldots, p$ and there exist $x'(t_k^+)$ and $x'(t_k^-)$, $k = 1, 2, \ldots, p$ with $x'(t_k^-) = x'(t_k)\}$ which is a Banach space with the norm $\|x\| = \sup_{t\in J}\{\|x\|_{PC}, \|x'\|_{PC}\}$ where $\|x\|_{PC} := \sup\{|x(t)| : t \in J\}$.

The following definitions and lemmas were given in [4].

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $\alpha \in R_+$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. For a function h given on the interval J, Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^C D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds, \quad n = [\alpha] + 1, \tag{2.2}$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3. *Let $\alpha > 0$, then the differential equation*

$${}^C D^\alpha h(t) = 0 \tag{2.3}$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, \ i = 0, 1, 2, \ldots, n-1, \ n = [\alpha] + 1. \tag{2.4}$$

The following lemma was given in [4, 10].

Lemma 2.4. *Let $\alpha > 0$, then*

$$I^{\alpha C} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

The following theorem is known as Burton-Kirk fixed point theorem and proved in [21].

Theorem 2.5. *Let X be a Banach space and $A, D : X \to X$ two operators satisfying:*

(a) *A is a contraction, and*

(b) *D is completely continuous.*

Then either

(i) *the operator equation $x = A(x) + D(x)$ has a solution, or*

(ii) *the set $\varepsilon = \{x \in X : x = \lambda A(x/\lambda) + \lambda D(x)\}$ is unbounded for $\lambda \in (0, 1)$.*

Theorem 2.6 (see [22], Banach's fixed point theorem). *Let S be a nonempty closed subset of a Banach space X, then any contraction mapping T of S into itself has a unique fixed point.*

Next we prove the following lemma.

Lemma 2.7. *Let $1 < \alpha \leq 2$ and let $h : J \to R$ be continuous. A function $x(t)$ is a solution of the fractional integral equation:*

$$x(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \Omega_1(t)\Lambda_1 + \Omega_2(t)\Lambda_2, \quad t \in J_0 \\ \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + [1+\Omega_1(t)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \quad +[(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \\ \quad +[1+\Omega_1(t)] \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \Omega_1(t) \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ \quad +\Omega_2(t) \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + [1+\Omega_1(t)] \left[\sum_{i=1}^{k} I_i(x(t_i^-)) + I_k^*(x(t_k^-)) \right] \\ \quad +\Omega_1(t) \sum_{i=1}^{k-1} (T-t_i) I_i^*(x(t_i^-)) + \Omega_2(t) \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) \\ \quad + \sum_{i=1}^{k-1} (t-t_i) I_i^*(x(t_i^-)), \quad t \in J_k,\ k = 1,2,\ldots,p \end{cases} \tag{2.6}$$

if and only if $x(t)$ is a solution of the fractional BVP

$$\begin{gathered} {}^C D^\alpha x(t) = h(t), \quad t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = I_k^*(x(t_k^-)), \\ x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0), \end{gathered} \tag{2.7}$$

where

$$\begin{aligned} \Lambda_1 := &\int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &+ \sum_{i=1}^{k} (T-t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} (t_k-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds \\ &+ \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k-1} (T-t_i) I_i^*(x(t_i^-)) + I_k^*(x(t_k^-)), \end{aligned}$$

$$\Lambda_2 := \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} I_i^*\left(x\left(t_i^-\right)\right),$$
$$\Omega_1(t) := -\frac{(1-d)}{\Delta} - \frac{ct}{T\Delta},$$
$$\Omega_2(t) := \frac{(1-b)T}{\Delta} - \frac{(1-a)t}{\Delta}. \tag{2.8}$$

Proof. Let x be the solution of (2.7). If $t \in J_0$, then Lemma 2.4 implies that

$$x(t) = I^{\alpha}h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 - c_1 t,$$
$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1, \tag{2.9}$$

for some $c_0, c_1 \in R$.

If $t \in J_1$, then Lemma 2.4 implies that

$$x(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - d_0 - d_1(t-t_1),$$
$$x'(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - d_1, \tag{2.10}$$

for some $d_0, d_1 \in R$. Thus we have

$$x\left(t_1^-\right) = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 - c_1 t_1, \qquad x\left(t_1^+\right) = -d_0,$$
$$x'\left(t_1^-\right) = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1, \qquad x'\left(t_1^+\right) = -d_1. \tag{2.11}$$

Observing that

$$\Delta x(t_1) = x\left(t_1^+\right) - x\left(t_1^-\right) = I_1\left(x\left(t_1^-\right)\right), \qquad \Delta x'(t_1) = x'\left(t_1^+\right) - x'\left(t_1^-\right) = I_1^*\left(x\left(t_1^-\right)\right), \tag{2.12}$$

then we have

$$-d_0 = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - c_0 - c_1 t_1 + I_1\left(x\left(t_1^-\right)\right),$$
$$-d_1 = \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - c_1 + I_1^*\left(x\left(t_1^-\right)\right), \tag{2.13}$$

hence, for $t \in (t_1, t_2]$,

$$x(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$
$$+ (t-t_1)\int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1\left(x\left(t_1^-\right)\right) + (t-t_1) I_1^*\left(x\left(t_1^-\right)\right) - c_0 - c_1 t, \tag{2.14}$$
$$x'(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1^*\left(x\left(t_1^-\right)\right) - c_1.$$

If $t \in J_2$, then Lemma 2.4 implies that

$$x(t) = \int_{t_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - e_0 - e_1(t-t_2),$$
$$x'(t) = \int_{t_2}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - e_1, \tag{2.15}$$

for some $e_0, e_1 \in R$. Thus we have

$$x\left(t_2^-\right) = \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$
$$+ (t_2-t_1)\int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1\left(x\left(t_1^-\right)\right) + (t_2-t_1) I_1^*\left(x\left(t_1^-\right)\right) - c_0 - c_1 t_2,$$
$$x\left(t_2^+\right) = -e_0, \tag{2.16}$$
$$x'\left(t_2^-\right) = \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + I_1^*\left(x\left(t_1^-\right)\right) - c_1,$$
$$x'\left(t_2^+\right) = -e_1.$$

Similarly we observe that

$$\Delta x(t_2) = x\left(t_2^+\right) - x\left(t_2^-\right) = I_2\left(x\left(t_2^-\right)\right), \qquad \Delta x'(t_2) = x'\left(t_2^+\right) - x'\left(t_2^-\right) = I_2^*\left(x\left(t_2^-\right)\right), \tag{2.17}$$

thus we have

$$\begin{aligned}
-e_0 &= x(t_2^-) + I_2(x(t_2^-)),\\
-e_1 &= x'(t_2^-) + I_2^*(x(t_2^-)).
\end{aligned} \tag{2.18}$$

Hence, for $t \in (t_2, t_3]$,

$$\begin{aligned}
x(t) = {} & \int_{t_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\\
& + (t_2 - t_1) \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds\\
& + (t - t_2)\left[\int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha-1)} h(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha-1)} h(s)ds\right]\\
& + I_1(x(t_1^-)) + I_2(x(t_2^-)) + (t - t_1)I_1^*(x(t_1^-)) + I_2^*(x(t_2^-)) - c_0 - c_1 t.
\end{aligned} \tag{2.19}$$

By a similar process, if $t \in J_k$, then again from Lemma 2.4 we get

$$\begin{aligned}
x(t) &= \begin{cases}
\displaystyle\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\\
\displaystyle\quad + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds\\
\displaystyle\quad + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k-1} (t - t_i) I_i^*(x(t_i^-)) + I_k^*(x(t_k^-)) - c_0 - c_1 t,
\end{cases}\\
x'(t) &= \left\{ \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \sum_{i=1}^{k-1} I_i^*(x(t_i^-)) - c_1. \right.
\end{aligned} \tag{2.20}$$

Now if we apply the conditions:

$$x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0), \tag{2.21}$$

we have

$$\begin{aligned}
\Lambda_1 &= (1-a)c_0 + T(1-b)c_1,\\
\Lambda_2 &= -\frac{c}{T}c_0 + (1-d)c_1,\\
-c_0 &= -\frac{(1-d)\Lambda_1}{\Delta} + \frac{(1-b)T\Lambda_2}{\Delta},\\
-c_1 &= -\frac{c\Lambda_1}{T\Delta} - \frac{(1-a)\Lambda_2}{\Delta}.
\end{aligned} \tag{2.22}$$

In view of the relations (2.8), when the values of $-c_0$ and $-c_1$ are replaced in (2.9) and (2.20), the integral equation (2.7) is obtained.

Conversely, assume that x satisfies the impulsive fractional integral equation (2.6), then by direct computation, it can be seen that the solution given by (2.6) satisfies (2.7). The proof is complete. □

3. Main Results

Definition 3.1. A function $x \in PC^1(J, R)$ with its α-derivative existing on J' is said to be a solution of (1.1), if x satisfies the equation ${}^C D^\alpha x(t) = f(t, x(t))$ on J' and satisfies the conditions:

$$\begin{gathered} \Delta x(t_k) = I_k\left(x\left(t_k^-\right)\right), \qquad \Delta x'(t_k) = I_k^*\left(x\left(t_k^-\right)\right), \\ x(T) = ax(0) + bTx'(0), \qquad Tx'(T) = cx(0) + dTx'(0). \end{gathered} \tag{3.1}$$

For the sake of convenience, we define

$$\Omega_1^* = \sup_{t\in J}|\Omega_1(t)|, \qquad \Omega_2^* = \sup_{t\in J}|\Omega_2(t)|, \qquad \Omega_1^{**} = \sup_{t\in J}\left|\Omega_1'(t)\right|, \qquad \Omega_2^{**} = \sup_{t\in J}\left|\Omega_2'(t)\right|. \tag{3.2}$$

The followings are main results of this paper.

Theorem 3.2. *Assume that*

(A1) *the function $f : J \times R \to R$ is continuous and there exists a constant $L_1 > 0$ such that $\|f(t,u) - f(t,v)\| \le L_1\|u - v\|$, for all $t \in J$, and $u, v \in R$,*

(A2) *I_k, $I_k^* : R \to R$ are continuous, and there exist constants $L_2 > 0$ and $L_3 > 0$ such that $\|I_k(u) - I_k(v)\| \le L_2\|u - v\|$, $\|I_k^*(u) - I_k^*(v)\| \le L_3\|u - v\|$ for each $u, v \in R$ and $k = 1, 2, \ldots, p$.*

Moreover, consider the following:

$$\begin{aligned} &\left[\frac{L_1 T^\alpha}{\Gamma(\alpha+1)}\left(1+\Omega_1^*\right)(1+p+2p\alpha) + \frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)}\Omega_2^*(1+p)\right. \\ &\qquad \left. +\left(1+\Omega_1^*\right)(pL_2 + L_3) + \left(\Omega_1^* T + \Omega_2^* + T\right)pL_3\right] < 1. \end{aligned} \tag{3.3}$$

Then, BVP (1.1) *has a unique solution on J.*

Proof. Define an operator $F : PC^1(J, R) \to PC^1(J, R)$ by

$$\begin{aligned} (Fx)(t) = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds + [1 + \Omega_1(t)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds \\ &+ [(T - t_k)\Omega_1(t) + \Omega_2(t) + (t - t_k)] \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f(s, x(s))ds \end{aligned}$$

$$
\begin{aligned}
&+ [1+\Omega_1(t)] \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds \\
&+ \Omega_1(t) \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&+ \Omega_2(t) \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds + [1+\Omega_1(t)] \left[\sum_{i=1}^{k} I_i\left(x\left(t_i^-\right)\right) + I_k^*\left(x\left(t_k^-\right)\right) \right] \\
&+ \Omega_1(t) \sum_{i=1}^{k-1} (T-t_i) I_i^*\left(x\left(t_i^-\right)\right) + \Omega_2(t) \sum_{i=1}^{k-1} I_i^*\left(x\left(t_i^-\right)\right) + \sum_{i=1}^{k-1} (t-t_i) I_i^*\left(x\left(t_i^-\right)\right).
\end{aligned}
\tag{3.4}
$$

Now, for $x, y \in PC(J,R)$ and for each $t \in J$, we obtain

$$
\begin{aligned}
\left|(Fx)(t) - (Fy)(t)\right| \le{}& \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |1+\Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)| \\
&\times \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |1+\Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) - f(s,y(s)) \right| ds \\
&+ |1+\Omega_1(t)| \left[\sum_{i=1}^{k} \left| I_i\left(x\left(t_i^-\right)\right) - I_i\left(y\left(t_i^-\right)\right) \right| + \left| I_k^*\left(x\left(t_k^-\right)\right) - I_k^*\left(y\left(t_k^-\right)\right) \right| \right] \\
&+ |\Omega_1(t)| \sum_{i=1}^{k-1} (T-t_i) \left| I_i^*\left(x\left(t_i^-\right)\right) - I_i^*\left(y\left(t_i^-\right)\right) \right| \\
&+ |\Omega_2(t)| \sum_{i=1}^{k-1} \left| I_i^*\left(x\left(t_i^-\right)\right) - I_i^*\left(y\left(t_i^-\right)\right) \right| \\
&+ \sum_{i=1}^{k-1} |t-t_i| \left| I_i^*\left(x\left(t_i^-\right)\right) - I_i^*\left(y\left(t_i^-\right)\right) \right|,
\end{aligned}
$$

$$\|(Fx)(t) - (Fy)(t)\| \leq \left[\frac{L_1 T^{\alpha}}{\Gamma(\alpha+1)}(1+\Omega_1^*)(1+p+2p\alpha) + \frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)}\Omega_2^*(1+p) \right. \\ \left. +(1+\Omega_1^*)(pL_2+L_3) + (\Omega_1^* T + \Omega_2^* + T)pL_3\right]\|x(s)-y(s)\|. \tag{3.5}$$

Therefore, by (3.3), the operator F is a contraction mapping. In a consequence of Banach's fixed theorem, the BVP (1.1) has a unique solution. Now, our second result relies on the Burton-Kirk fixed point theorem. □

Theorem 3.3. *Assume that (A1)-(A2) hold, and*

(A3) *there exist constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$ such that $\|f(t,u)\| \leq M_1$, $\|I_k(u)\| \leq M_2$, $\|I_k^*(u)\| \leq M_3$ for each $u, v \in R$ and $k = 1, 2, \ldots, p$.*

Then the BVP (1.1) *has at least one solution on J.*

Proof. We define the operators $A, D : PC^1(J,R) \to PC^1(J,R)$ by

$$\begin{aligned}
(Ax)(t) &= [1+\Omega_1(t)]\left[\sum_{i=1}^{k} I_i(x(t_i^-)) + I_k^*(x(t_k^-))\right] \\
&\quad + \Omega_1(t)\sum_{i=1}^{k-1}(T-t_i)I_i^*(x(t_i^-)) + \Omega_2(t)\sum_{i=1}^{k-1} I_i^*(x(t_i^-)) + \sum_{i=1}^{k-1}(t-t_i)I_i^*(x(t_i^-)), \\
(Dx)(t) &= \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds + [1+\Omega_1(t)]\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds \\
&\quad + [(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)]\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds \\
&\quad + [1+\Omega_1(t)]\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds \\
&\quad + \Omega_1(t)\int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds + \Omega_2(t)\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds.
\end{aligned} \tag{3.6}$$

It is obvious that A is contraction mapping for

$$(1+\Omega_1^*)(pL_2+L_3) + (\Omega_1^* T + \Omega_2^* + T)pL_3 < 1. \tag{3.7}$$

Now, in order to check that D is completely continuous, let us follow the sequence of the following steps.

Step 1 (D is continuous). Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J,R)$. Then for $t \in J$, we have

$$\begin{aligned}
|(Dx_n)(t) - (Dx)(t)| \le & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \\
& + |1+\Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \\
& + |(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)| \\
& \times \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \\
& + |1+\Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \\
& + |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \\
& + |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x_n(s)) - f(s,x(s)) \right| ds.
\end{aligned} \tag{3.8}$$

Since f is continuous function, we get

$$\|(Dx_n)(t) - (Dx)(t)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Step 2 (D maps bounded sets into bounded sets in $PC(J,R)$). Indeed, it is enough to show that for any $r > 0$, there exists a positive constant l such that for each $x \in B_r = \{x \in PC(J,R) : \|x\| \le r\}$, we have $\|D(x)\| \le l$. By (A3), we have for each $t \in J$,

$$\begin{aligned}
|(Dx)(t)| \le & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds \\
& + |1+\Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds \\
& + |(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
& + |1+\Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds
\end{aligned}$$

$$
\begin{aligned}
&+ |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds + |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds, \\
\|D(x)\| \leq \frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} \left(1+\Omega_1^*\right)(1+p+2p\alpha) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^* (1+p) := l.
\end{aligned}
\tag{3.10}
$$

Step 3 (D maps bounded sets into equicontinuous sets in $PC^1(J,R)$). Let $\tau_1, \tau_2 \in J_k, 0 \leq k \leq p$ with $\tau_1 < \tau_2$ and let B_r be a bounded set of $PC^1(J,R)$ as in Step 2, and let $x \in B_r$. Then

$$
\left| (Dy)(\tau_2) - (D)(\tau_1) \right| \leq \int_{\tau_1}^{\tau_2} \left| (Dy)'(s) \right| ds \leq L(\tau_2 - \tau_1),
\tag{3.11}
$$

where

$$
\begin{aligned}
\left| (Dx)'(t) \right| \leq &\int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds + \left| \Omega_1'(t) \right| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds \\
&+ \left| (T-t_k)\Omega_1'(t) + \Omega_2'(t) + 1 \right| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
&+ \left| \Omega_1'(t) \right| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
&+ \left| \Omega_1'(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds + \left| \Omega_2'(t) \right| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds, \\
\leq &\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} \Omega_1^* (1+p+2p\alpha) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \left(1+\Omega_2^*\right)(1+p) := L.
\end{aligned}
\tag{3.12}
$$

This implies that A is equicontinuous on all the subintervals J_k, $k = 0,1,2,\ldots,p$. Therefore, by the Arzela-Ascoli Theorem, the operator $D : PC^1(J,R) \to PC^1(J,R)$ is completely continuous. □

To conclude the existence of a fixed point of the operator $A + D$, it remains to show that the set

$$
\varepsilon = \left\{ x \in X : x = \lambda A\left(\frac{x}{\lambda}\right) + \lambda D(x) \text{ for some } \lambda \in (0,1) \right\}
\tag{3.13}
$$

is bounded.

Let $x \in \varepsilon$, for each $t \in J$,

$$
x(t) = \lambda D(x)(t) + \lambda A\left(\frac{x}{\lambda}\right)(t).
\tag{3.14}
$$

Hence, from (A3), we have

$$
\begin{aligned}
|x(t)| \leq{}& \lambda \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds + \lambda |1+\Omega_1(t)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds \\
&+ \lambda |(T-t_k)\Omega_1(t) + \Omega_2(t) + (t-t_k)| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
&+ \lambda |1+\Omega_1(t)| \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
&+ \lambda |\Omega_1(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,x(s)) \right| ds + \lambda |\Omega_2(t)| \int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f(s,x(s)) \right| ds \\
&+ \lambda |1+\Omega_1(t)| \left[\sum_{i=1}^{k} \left| I_i\left(\frac{x(t_i^-)}{\lambda} \right) \right| + \left| I_k^*\left(\frac{x(t_k^-)}{\lambda} \right) \right| \right] \\
&+ \lambda |\Omega_1(t)| \sum_{i=1}^{k-1} (T-t_i) \left| I_i^*\left(\frac{x(t_i^-)}{\lambda} \right) \right| \\
&+ \lambda |\Omega_2(t)| \sum_{i=1}^{k-1} \left| I_i^*\left(\frac{x(t_i^-)}{\lambda} \right) \right| + \lambda \sum_{i=1}^{k-1} (t-t_i) \left| I_i^*\left(\frac{x(t_i^-)}{\lambda} \right) \right|, \\
\|x(t)\| \leq{}& \left[\frac{M_1 T^{\alpha}}{\Gamma(\alpha+1)} (1+\Omega_1^*)(1+p+2p\alpha) + \frac{M_1 T^{\alpha-1}}{\Gamma(\alpha)} \Omega_2^* (1+p) \right. \\
&\left. + (1+\Omega_1^*)(pM_2 + M_3) + (\Omega_1^* T + \Omega_2^* + T) p M_3 \right].
\end{aligned}
\tag{3.15}
$$

Consequently, we conclude the result of our theorem based on the Burton-Kirk fixed point theorem.

4. An Example

Consider the following impulsive fractional boundary value problem:

$$
\begin{gathered}
{}^{C}D^{3/2} x(t) = \frac{\sin 2t |x(t)|}{(t+5)^2 (1+|x(t)|)}, \quad t \in [0,1], \; t \neq \frac{1}{3}, \\
\Delta x\left(\frac{1}{3} \right) = \frac{|x(1^-/3)|}{15 + |x(1^-/3)|}, \qquad \Delta x'\left(\frac{1}{3} \right) = \frac{|x'(1^-/3)|}{10 + |x'(1^-/3)|}, \\
x(1) = 5x(0) - 2x'(0), \qquad x'(1) = x(0) + 4x'(0).
\end{gathered}
\tag{4.1}
$$

Here, $a = 5$, $b = -2$, $c = 1$, $d = 4$, $\alpha = 3/2$, $T = 1$, $p = 1$. Obviously, $L_1 = 1/25$, $L_2 = 1/15$, $L_3 = 1/10$, $\Omega_1^* = 4/15$, $\Omega_2^* = 7/15$. Further,

$$\begin{aligned}&\left[\frac{L_1 T^\alpha}{\Gamma(\alpha+1)}\left(1+\Omega_1^*\right)\left(1+p(1+L_2)+2p\alpha+L_3\right)+\frac{L_1 T^{\alpha-1}}{\Gamma(\alpha)}\Omega_2^*(1+p)+\left(\Omega_1^* T+\Omega_2^*+T\right)pL_3\right]\\&\quad=\frac{464}{1125\sqrt{\pi}}+\frac{173}{450}<1.\end{aligned} \tag{4.2}$$

Since the assumptions of Theorem 3.2 are satisfied, the closed boundary value problem (4.1) has a unique solution on $[0,1]$. Moreover, it is easy to check the conclusion of Theorem 3.3.

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Research Article

Some New Estimates for the Error of Simpson Integration Rule

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We obtain some new estimates for the error of Simpson integration rule, which develop available results in the literature. Indeed, we introduce three main estimates for the residue of Simpson integration rule in $L^1[a,b]$ and $L^\infty[a,b]$ spaces where the compactness of the interval $[a,b]$ plays a crucial role.

1. Introduction

A general $(n+1)$-point-weighted quadrature formula is denoted by

$$\int_a^b w(x)f(x)dx = \sum_{k=0}^{n} w_k f(x_k) + R_{n+1}[f], \tag{1.1}$$

where $w(x)$ is a positive weight function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are, respectively, nodes and weight coefficients, and $R_{n+1}[f]$ is the corresponding error [1].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most d. The quadrature formula (1.1) has degree of exactness d if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p] = 0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (1.1) has precise degree of exactness d.

The convergence order of quadrature rule (1.1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomia

$$\Psi_{n+1}(x) = \prod_{k=0}^{n}(x - x_k), \tag{1.2}$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)L(x; x_k) + r_{n+1}(f; x), \tag{1.3}$$

where

$$L(x; x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \quad (k = 0, 1, \dots, n), \tag{1.4}$$

we obtain (1.1), with

$$\begin{gathered} w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)w(x)}{x - x_k} dx \quad (k = 0, 1, \dots, n), \\ R_{n+1}[f] = \int_a^b r_{n+1}(f; x)w(x)dx. \end{gathered} \tag{1.5}$$

Note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f; x) = 0$, and therefore $R_{n+1}[f] = 0$.

Quadrature formulae obtained in this way are known as interpolatory. Usually the simplest interpolatory quadrature formula of type (1.1) with predetermined nodes $\{x_k\}_{k=0}^n \in [a, b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a + kh\}_{k=0}^n$ with $h = (b - a)/n$, the classical Newton-Cotes formulas are derived. One of the important cases of the classical Newton-Cotes formulas is the well-known Simpson's rule:

$$\int_a^b f(t)dt = \frac{b - a}{6}\left(f(a) + 4f\left(\frac{a + b}{2}\right) + f(b)\right) + E(f). \tag{1.6}$$

In this direction, Simpson inequality [2–7] gives an error bound for the above quadrature rule. There are few known ways to estimate the residue value in (1.6). The main aim of this paper is to give three new estimations for $E(f)$ in $L^1[a, b]$ and $L^\infty[a, b]$ spaces.

Let $L^p[a, b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a, b]$ with the standard norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \tag{1.7}$$

and $L^\infty[a, b]$ the space of all essentially bounded functions on $[a, b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)|. \tag{1.8}$$

If $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$, then the following inequality is well known:

$$\left|\int_a^b f(x)g(x)dx\right| \le \|f\|_1 \|g\|_\infty. \tag{1.9}$$

Recently in [8], a main inequality has been introduced, which can estimate the error of Simpson quadrature rule too.

Theorem A. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a differentiable function in the interior $\mathbf{I}^0$of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If α_0, β_0 are two real constants such that $\alpha_0 \le f'(t) \le \beta_0$ for all $t \in [a,b]$, then for any $\lambda \in [1/2,1]$ and all $x \in [(a+(2\lambda-1)b)/2\lambda, (b+(2\lambda-1)a)/2\lambda] \subseteq [a,b]$ we have*

$$\begin{aligned}
&\left|f(x) - \frac{1}{\lambda(b-a)}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}x + \frac{(2\lambda-1)a+b}{2\lambda(b-a)}f(b) - \frac{a+(2\lambda-1)b}{2\lambda(b-a)}f(a)\right| \\
&\quad \le \frac{\beta_0-\alpha_0}{4(b-a)}\frac{\lambda^2+(1-\lambda)^2}{\lambda}\left((x-a)^2+(b-x)^2\right).
\end{aligned} \tag{1.10}$$

As is observed, replacing $x = (a+b)/2$ and $\lambda = 2/3$ in (1.10) gives an error bound for the Simpson rule as

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \le \frac{5}{72}(b-a)^2(\beta_0-\alpha_0). \tag{1.11}$$

To introduce three new error bounds for the Simpson quadrature rule in $L^1[a,b]$ and $L^\infty[a,b]$ spaces we first consider the following kernel on $[a,b]$:

$$K(t) = \begin{cases} t - \dfrac{5a+b}{6}, & t \in \left[a, \dfrac{a+b}{2}\right], \\ t - \dfrac{a+5b}{6}, & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases} \tag{1.12}$$

After some calculations, it can be directly concluded that

$$\int_a^b f'(t)K(t)dt = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt, \tag{1.13}$$

$$\max_{t\in[a,b]}|K(t)| = \frac{1}{3}(b-a). \tag{1.14}$$

2. Main Results

Theorem 2.1. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If $\alpha(x) \leq f'(x) \leq \beta(x)$ for any $\alpha, \beta \in C[a,b]$ and $x \in [a,b]$, then the following inequality holds:*

$$\begin{aligned}
m_1 = &\int_a^{(5a+b)/6}\left(t-\frac{5a+b}{6}\right)\beta(t)dt + \int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt \\
&+ \int_{(a+b)/2}^{(a+5b)/6}\left(t-\frac{a+5b}{6}\right)\beta(t)dt + \int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)\alpha(t)dt \\
\leq &\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt \leq \\
M_1 = &\int_a^{(5a+b)/6}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt + \int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt \\
&+ \int_{(a+b)/2}^{(a+5b)/6}\left(t-\frac{a+5b}{6}\right)\alpha(t)dt + \int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)\beta(t)dt.
\end{aligned} \tag{2.1}$$

Proof. By referring to the kernel (1.12) and identity (1.13) we first have

$$\begin{aligned}
&\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt \\
&= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt - \frac{1}{2}\left(\int_a^b K(t)(\alpha(t)+\beta(t))dt\right) \\
&= \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \int_a^b f(t)dt \\
&\quad - \frac{1}{2}\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)(\alpha(t)+\beta(t))dt + \int_{(a+b)/2}^{b}\left(t-\frac{a+5b}{6}\right)(\alpha(t)+\beta(t))dt\right).
\end{aligned} \tag{2.2}$$

On the other hand, the given assumption $\alpha(t) \leq f'(t) \leq \beta(t)$ results in

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right| \leq \frac{\beta(t)-\alpha(t)}{2}. \tag{2.3}$$

Therefore, one can conclude from (2.2) and (2.3) that

$$\begin{aligned}
&\left|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\right.\\
&\qquad\left.-\frac{1}{2}\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)(\alpha(t)+\beta(t))dt+\int_{(a+b)/2}^{b}\left(t-\frac{a+5b}{6}\right)(\alpha(t)+\beta(t))dt\right)\right|\\
&=\left|\int_a^b K(t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\right|\leq\int_a^b|K(t)|\ \frac{\beta(t)-\alpha(t)}{2}dt\\
&=\frac{1}{2}\left(\int_a^{(a+b)/2}\left|\ t-\frac{5a+b}{6}\ \right|(\beta(t)-\alpha(t))dt+\int_{(a+b)/2}^{b}\left|\ t-\frac{a+5b}{6}\right|(\beta(t)-\alpha(t))dt\right).
\end{aligned}\tag{2.4}$$

After rearranging (2.4) we obtain

$$\begin{aligned}
m_1&=\int_a^{(a+b)/2}\left(\left(t-\frac{5a+b}{6}-\left|\ t-\frac{5a+b}{6}\ \right|\right)\frac{\beta(t)}{2}+\left(t-\frac{5a+b}{6}+\left|\ t-\frac{5a+b}{6}\ \right|\right)\frac{\alpha(t)}{2}\right)dt\\
&\quad+\int_{(a+b)/2}^{b}\left(\left(t-\frac{a+5b}{6}-\left|\ t-\frac{a+5b}{6}\ \right|\right)\frac{\beta(t)}{2}+\left(t-\frac{a+5b}{6}+\left|\ t-\frac{a+5b}{6}\ \right|\right)\frac{\alpha(t)}{2}\right)dt\\
&=\int_a^{(5a+b)/6}\left(x-\frac{5a+b}{6}\right)\beta(x)dx+\int_{(5a+b)/6}^{(a+b)/2}\left(x-\frac{5a+b}{6}\right)\alpha(x)dx\\
&\quad+\int_{(a+b)/2}^{(a+5b)/6}\left(x-\frac{a+5b}{6}\right)\beta(x)dx+\int_{(a+5b)/6}^{b}\left(x-\frac{a+5b}{6}\right)\alpha(x)dx,\\
M_1&=\int_a^{(a+b)/2}\left(\left(t-\frac{5a+b}{6}-\left|t-\frac{5a+b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{5a+b}{6}+\left|t-\frac{5a+b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt\\
&\quad+\int_{(a+b)/2}^{b}\left(\left(t-\frac{a+5b}{6}-\left|t-\frac{a+5b}{6}\right|\right)\frac{\alpha(t)}{2}+\left(t-\frac{a+5b}{6}+\left|t-\frac{a+5b}{6}\right|\right)\frac{\beta(t)}{2}\right)dt\\
&=\int_a^{(5a+b)/6}\left(x-\frac{5a+b}{6}\right)\alpha(x)dx+\int_{(5a+b)/6}^{(a+b)/2}\left(x-\frac{5a+b}{6}\right)\beta(x)dx\\
&\quad+\int_{(a+b)/2}^{(a+5b)/6}\left(x-\frac{a+5b}{6}\right)\alpha(x)dx+\int_{(a+5b)/6}^{b}\left(x-\frac{a+5b}{6}\right)\beta(x)dx.
\end{aligned}\tag{2.5}$$

□

The advantage of Theorem 2.1 is that necessary computations in bounds m_1 and M_1 are just in terms of the preassigned functions $\alpha(t)$, $\beta(t)$ (not f').

Special Case 1

Substituting $\alpha(x) = \alpha_1 x + \alpha_0 \neq 0$ and $\beta(x) = \beta_1 x + \beta_0 \neq 0$ in (2.1) gives

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{5(b-a)^2}{144}\left((\beta_1 - \alpha_1)(a+b) + 2(\beta_0 - \alpha_0)\right). \tag{2.6}$$

In particular, replacing $\alpha_1 = \beta_1 = 0$ in above inequality leads to one of the results of [9] as

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq \frac{5}{72}(b-a)^2(\beta_0 - \alpha_0). \tag{2.7}$$

Remark 2.2. Although $\alpha(x) \leq f'(x) \leq \beta(x)$ is a straightforward condition in Theorem 2.1, however, sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2.3. *Let $f : \mathbf{I} \to \mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b] \subset \mathbf{I}^0$. If $\alpha(x) \leq f'(x)$ for any $\alpha \in C[a,b]$ and $x \in [a,b]$ then*

$$\begin{aligned}
&\int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt - \frac{b-a}{3}\left(f(b) - f(a) - \int_a^b \alpha(t)dt\right) \\
&\quad \leq \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)\ dt \\
&\quad \leq \int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt \\
&\qquad + \frac{b-a}{3}\left(f(b) - f(a) - \int_a^b \alpha(t)dt\right).
\end{aligned} \tag{2.8}$$

Proof. Since

$$\begin{aligned}
\int_a^b K(t)(f'(t) - \alpha(t))dt &= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt - \left(\int_a^b K(t)\alpha(t)dt\right) \\
&= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \int_a^b f(t)dt \\
&\quad - \left(\int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\alpha(t)dt + \int_{(a+b)/2}^b\left(t - \frac{a+5b}{6}\right)\alpha(t)dt\right),
\end{aligned} \tag{2.9}$$

so we have

$$\begin{aligned}
&\left|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\right.\\
&\quad\left.-\left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\alpha(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\alpha(t)dt\right)\right|\\
&\qquad=\left|\int_a^b K(t)\left(f'(t)-\alpha(t)\right)dt\right|\leq\int_a^b|K(t)|\left(f'(t)-\alpha(t)\right)dt\\
&\qquad\leq\max_{t\in[a,b]}|K(t)|\int_a^b\left(f'(t)-\alpha(t)\right)dt=\frac{b-a}{3}\left(f(b)-f(a)-\int_a^b\alpha(t)dt\right).
\end{aligned}\tag{2.10}$$

After rearranging (2.10), the main inequality (2.8) will be derived. □

Special Case 2

If $\alpha(x)=\alpha_1x+\alpha_0\neq 0$, then (2.8) becomes

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\leq\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\left(\alpha_0+\frac{a+b}{2}\alpha_1\right)\right)\tag{2.11}$$

if and only if $\alpha_1x+\alpha_0\leq f'(x)$ for all $x\in[a,b]$. In particular, replacing $\alpha_1=0$ in above inequality leads to [10, Theorem 1, relation (4)] as follows:

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\leq\frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right).\tag{2.12}$$

Theorem 2.4. *Let $f:\mathbf{I}\to\mathbf{R}$, where $\mathbf{I}$ is an interval, be a function differentiable in the interior $\mathbf{I}^0$ of $\mathbf{I}$, and let $[a,b]\subset\mathbf{I}^0$. If $f'(x)\leq\beta(x)$ for any $\beta\in C[a,b]$ and $x\in[a,b]$ then*

$$\begin{aligned}
&\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt-\frac{b-a}{3}\left(\int_a^b\beta(t)dt-f(b)+f(a)\right)\\
&\leq\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt\\
&\leq\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt+\frac{b-a}{3}\left(\int_a^b\beta(t)dt-f(b)+f(a)\right).
\end{aligned}\tag{2.13}$$

Proof. Since

$$\begin{aligned}
&\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt \\
&\quad = \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt-\left(\int_a^b K(t)\beta(t)dt\right) \\
&\quad = \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt \\
&\qquad - \left(\int_a^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt\right),
\end{aligned} \tag{2.14}$$

so we have

$$\begin{aligned}
&\Bigg|\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\int_a^b f(t)dt \\
&\qquad -\left(\int_a^{(a+b)/2}\int_a\left(t-\frac{5a+b}{6}\right)\beta(t)dt+\int_{(a+b)/2}^b\left(t-\frac{a+5b}{6}\right)\beta(t)dt\right)\Bigg| \\
&\quad = \left|\int_a^b K(t)\left(f'(t)-\beta(t)\right)dt\right| \le \int_a^b |K(t)|\left(\beta(t)-f'(t)\right)dt \\
&\quad \le \max_{t\in[a,b]}|K(t)|\int_a^b\left(\beta(t)-f'(t)\right)dt = \frac{b-a}{3}\left(\int_a^b \beta(t)dt-f(b)+f(a)\right).
\end{aligned} \tag{2.15}$$

After rearranging (2.15), the main inequality (2.13) will be derived. □

Special Case 3

If $\beta(x)=\beta_1 x+\beta_0\neq 0$ in (2.13), then

$$\left|\int_a^b f(t)\ dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right| \le \frac{(b-a)^2}{3}\left(\beta_0+\frac{a+b}{2}\ \beta_1-\frac{f(b)-f(a)}{b-a}\right) \tag{2.16}$$

if and only if $f'(x)\le\beta_1 x+\beta_0$, for all $x\in[a,b]$. In particular, replacing $\beta_1=0$ in above inequality leads to [10, Theorem 1, relation (5)] as follows:

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right| \le \frac{(b-a)^2}{3}\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right). \tag{2.17}$$

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Research Article

Travelling Wave Solutions of the Schrödinger-Boussinesq System

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We establish exact solutions for the Schrödinger-Boussinesq System $iu_t + u_{xx} - auv = 0$, $v_{tt} - v_{xx} + v_{xxxx} - b(|u|^2)_{xx} = 0$, where a and b are real constants. The (G'/G)-expansion method is used to construct exact periodic and soliton solutions of this equation. Our work is motivated by the fact that the (G'/G)-expansion method provides not only more general forms of solutions but also periodic and solitary waves. As a result, hyperbolic function solutions and trigonometric function solutions with parameters are obtained. These solutions may be important and of significance for the explanation of some practical physical problems.

1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, the modulation of monochromatic waves, propagation of Langmuir waves in plasmas, and so forth. The nonlinear Schrödinger equations play an important role in many areas of applied physics, such as nonrelativistic quantum mechanics, laser beam propagation, Bose-Einstein condensates, and so on (see [1]). Some properties of solutions for the nonlinear Schrödinger equations on $\mathbb{R}^n$ have been extensively studied in the last two decades (e.g., see [2]).

The Boussinesq-type equations are essentially a class of models appearing in physics and fluid mechanics. The so-called Boussinesq equation was originally derived by Boussinesq [3] to describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel. It also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice waves. The study on the soliton solutions for various generalizations of the Boussinesq equation has recently attracted considerable attention from many mathematicians and physicists (see [4]). We should remark that it was the first equation proposed in the literature to describe this kind of physical phenomena. This equation was also used by Zakharov [5] as a model of nonlinear string and by Falk et al. [6] in their study of shape-memory alloys.

In the laser and plasma physics, the Schrödinger-Boussinesq system (hereafter referred to as the SB-system) has been raised. Consider the SB-system

$$\begin{aligned} &iu_t + u_{xx} - auv = 0, \\ &v_{tt} - v_{xx} + v_{xxxx} - b\left(|u|^2\right)_{xx} = 0, \end{aligned} \tag{1.1}$$

where $t > 0$, $x \in [0, L]$, for some $L > 0$, and a, b are real constants. Here u and v are, respectively, a complex-valued and a real-valued function defined in space-time $[0, L] \times \mathbb{R}$. The SB-system is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [7] and diatomic lattice system [8]. The short wave term $u(x,t) : [0, L] \times \mathbb{R} \to \mathbb{C}$ is described by a Schrödinger type equation with a potential $v(x,t) : [0, L] \times \mathbb{R} \to \mathbb{R}$ satisfying some sort of Boussinesq equation and representing the intermediate long wave. The SB-system also appears in the study of interaction of solitons in optics. The solitary wave solutions and integrability of nonlinear SB-system has been considered by several authors (see [7, 8]) and the references therein.

In the literature, there is a wide variety of approaches to nonlinear problems for constructing travelling wave solutions, such as the inverse scattering method [9], Bcklund transformation [10], Hirota bilinear method [11], Painlevé expansion methods [12], and the Wronskian determinant technique [13].

With the help of the computer software, most of mentioned methods are improved and many other algebraic method, proposed, such as the tanh/coth method [14], the Exp-function method [15], and first integral method [16]. But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of nonlinearity increases.

Recently, the (G'/G)-expansion method, firstly introduced by Wang et al. [17], has become widely used to search for various exact solutions of NLEEs [17–19].

The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by a polynomial in (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = kx + \omega t$ and k, ω are arbitrary constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the non-linear terms appearing in the given non-linear equations.

Our first interest in the present work is in implementing the (G'/G)-expansion method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact traveling wave solutions for the SB-system (1.1).

2. Description of the (G'/G)-Expansion Method

The objective of this section is to outline the use of the (G'/G)-expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_x, u_t, u_{xx}, u_{x,t}, u_{tt}, \ldots) = 0, \tag{2.1}$$

where $u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The main steps of the (G'/G)-expansion method are the following:

Step 1. Combining the independent variables x and t into one variable $\xi = kx+\omega t$, we suppose that

$$u(x,t) = U(\xi), \qquad \xi = kx + \omega t. \tag{2.2}$$

The travelling wave variable (2.2) permits us to reduce (2.1) to an ODE for $u(x,t) = U(\xi)$, namely,

$$P\Big(U, kU', \omega U', k^2U'', k\omega U'', \omega^2 U'', \ldots\Big) = 0, \tag{2.3}$$

where prime denotes derivative with respect to ξ.

Step 2. We assume that the solution of (2.3) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=1}^{m} \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0, \tag{2.4}$$

where m is called the balance number, α_0, and α_i, are constants to be determined later, $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \tag{2.5}$$

where λ and μ are arbitrary constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (2.3).

Step 3. By substituting (2.4) into (2.3) and using the second order linear ODE (2.5), collecting all terms with the same order of (G'/G) together, the left-hand side of (2.3) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $k, \omega, \lambda, \mu, \alpha_0, \alpha_1, \ldots, \alpha_m$.

Step 4. Assuming that the constants $k, \omega, \lambda, \mu, \alpha_0, \alpha_1, \ldots, \alpha_m$ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order linear ODE (2.5) is well known for us, then substituting $k, \omega, \lambda, \mu, \alpha_0, \ldots, \alpha_m$ and the general solutions of (2.5) into (2.4) we have more travelling wave solutions of the nonlinear evolution (2.1).

In the subsequent section, we will illustrate the validity and reliability of this method in detail with complex model of Schrödinger-Boussinesq System (1.1).

3. Application

To look for the traveling wave solution of the Schrödinger-Boussinesq System (1.1), we use the gauge transformation:

$$\begin{aligned} u(x,t) &= U(\xi)e^{i\eta}, \\ v(x,t) &= V(\xi), \end{aligned} \tag{3.1}$$

where $\xi = kx + \omega t$, $\eta = px + qt$, and p, q, ω are constants and $i = \sqrt{-1}$. We substitute (3.1) into (1.1) to obtain nonlinear ordinary differential equation

$$k^2U'' - i(2kp + \omega)U' - aUV - \left(p^2 + q\right)U = 0, \tag{3.2}$$

$$\left(\omega^2 - k^2\right)V'' + k^4V^{(4)} - bk^2\left(U^2\right)'' = 0. \tag{3.3}$$

In order to simplify, integrating (3.3) twice and taking integration constant to zero, the system (3.2)-(3.3) reduces to the following system:

$$\begin{aligned} k^2U'' - i(2kp + \omega)U' - aUV - \left(p^2 + q\right)U &= 0, \\ \left(\omega^2 - k^2\right)V + k^4V'' - bk^2U^2 &= 0. \end{aligned} \tag{3.4}$$

Suppose that the solution of the nonlinear ordinary differential system (3.4) can be expressed by a polynomial in (G'/G) as follows:

$$\begin{aligned} U(\xi) &= \sum_{i=1}^{m}\alpha_i\left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0, \\ V(\xi) &= \sum_{i=1}^{n}\beta_i\left(\frac{G'}{G}\right)^i + \beta_0, \quad \beta_n \neq 0, \end{aligned} \tag{3.5}$$

where m, n are called the balance number, α_i, $(i = 0, 1, \ldots, m)$ and β_j, $(j = 0, 1, \ldots, n)$ are constants to be determined later, $G(\xi)$ satisfies a second order linear ordinary differential equation (2.5). The integers m, n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in nonlinear ordinary differential system (3.4) as follow:

$$\begin{aligned} m+n &= m+2, \\ 2m &= n+2, \end{aligned} \tag{3.6}$$

so that $m = n = 2$. We then suppose that (3.4) has the following formal solutions:

$$\begin{aligned} U(\xi) &= \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0, \\ V(\xi) &= \beta_2\left(\frac{G'}{G}\right)^2 + \beta_1\left(\frac{G'}{G}\right) + \beta_0, \quad \beta_2 \neq 0. \end{aligned} \tag{3.7}$$

Substituting (3.7) along with (2.5) into (3.4) and collecting all the terms with the same power of (G'/G) together, equating each coefficient to zero, yields a set of simultaneous algebraic equations for k, ω, λ, μ, α_j, and β_j, $(j = 0,1,2)$, as follows:

$$\begin{gathered} 2k^2\alpha_2\mu^2 + i\omega\alpha_1\mu + k^2\alpha_1\lambda\mu - a\alpha_0\beta_0 - p^2\alpha_0 - q\alpha_0 + 2ikp\alpha_1\mu = 0, \\ \omega^2\beta_0 + 2k^4\beta_2\mu^2 - k^2\left(-\mu\lambda\beta_1 k^2 + b{\alpha_0}^2 + \beta_0\right) = 0, \\ k^2\alpha_1\lambda^2 - p^2\alpha_1 + (4ik\alpha_2\mu + 2ik\alpha_1\lambda)p + (2i\alpha_2\mu + i\alpha_1\lambda)\omega - q\alpha_1 \\ - a\alpha_0\beta_1 + 2k^2\alpha_1\mu - a\alpha_1\beta_0 + 6k^2\alpha_2\lambda\mu = 0, \\ \omega^2\beta_1 + 6k^4\beta_2\lambda\mu - k^2\left(-k^2\lambda^2\beta_1 - 2k^2\mu\beta_1 + \beta_1 + 2b\alpha_1\alpha_0\right) = 0, \\ (4ik\alpha_2\lambda + 2ik\alpha_1)p + (2i\alpha_2\lambda + i\alpha_1)\omega - p^2\alpha_2 + \left(4\lambda^2k^2 - q - a\beta_0 + 8\mu k^2\right)\alpha_2 \\ + 3k^2\alpha_1\lambda - a\alpha_1\beta_1 - a\alpha_0\beta_2 = 0, \\ \omega^2\beta_2 + k^2\left(4\lambda^2k^2 + 8\mu k^2 - 1\right)\beta_2 - k^2\left(2b\alpha_2\alpha_0 + b{\alpha_1}^2 - 3\lambda\beta_1 k^2\right) = 0, \\ 4ikp\alpha_2 + 2i\omega\alpha_2 + \left(10\lambda k^2 - a\beta_1\right)\alpha_2 - \alpha_1\left(-2k^2 + a\beta_2\right) = 0, \\ 10k^4\beta_2\lambda - 2k^2\left(-k^2\beta_1 + b\alpha_2\alpha_1\right) = 0, \\ \left(6k^2 - a\beta_2\right)\alpha_2 = 0, \\ 6k^4\beta_2 - bk^2{\alpha_2}^2 = 0. \end{gathered} \tag{3.8}$$

Then, explicit and exact wave solutions can be constructed through our ansatz (3.7) via the associated solutions of (2.5).

In the process of constructing exact solutions to the Schrödinger-Boussinesq system (1.1), (2.5) is often viewed as a key auxiliary equation, and the types of its solutions determine the solutions for the original system (1.1) indirectly. In order to seek more new solutions to (1.1), we here combine the solutions to (2.5) which were listed in [18, 19]. Our computation results show that this combination is an efficient way to obtain more diverse families of explicit exact solutions.

Solving (3.8) by use of Maple, we get the following reliable results:

$$\begin{aligned}
&\Big\{\lambda = \pm\frac{1}{3}\frac{\sqrt{3\omega^2-3k^2}}{k^2},\ \mu = \frac{1}{6}\frac{k^2-\omega^2}{k^4},\ p = -\frac{1}{2}\frac{\omega}{k},\\
&q = \frac{1}{24}\frac{24k^4+30\omega^4-54k^2\omega^2}{k^2(k^2-\omega^2)},\ \alpha_0 = 0,\\
&\alpha_1 = \pm 2\sqrt{\frac{1}{ab}}\sqrt{3\omega^2-3k^2},\ \alpha_2 = 6k^2\sqrt{\frac{1}{ab}},\\
&\beta_0 = 0,\ \beta_1 = \pm 2\frac{\sqrt{3\omega^2-3k^2}}{a},\ \beta_2 = \frac{6k^2}{a}\Big\},
\end{aligned} \tag{3.9}$$

where k and ω are free constant parameters and $k \neq \pm\omega$. Therefore, substitute the above case in (3.7), we get

$$\begin{aligned}
U(\xi) &= 6k^2\sqrt{\frac{1}{ab}}\left(\frac{G'}{G}\right)^2 \pm 2\sqrt{\frac{1}{ab}}\sqrt{3\omega^2-3k^2}\left(\frac{G'}{G}\right),\\
V(\xi) &= \frac{6k^2}{a}\left(\frac{G'}{G}\right)^2 \pm 2\frac{\sqrt{3\omega^2-3k^2}}{a}\left(\frac{G'}{G}\right).
\end{aligned} \tag{3.10}$$

Substituting the general solutions of ordinary differential equation (2.5) into (3.10), we obtain two types of traveling wave solutions of (1.1) in view of the positive and negative of $\lambda^2 - 4\mu$.

When $\mathfrak{D} = \lambda^2 - 4\mu = (\omega^2 - k^2)/k^4 > 0$, using the general solutions of ordinary differential equation (2.5) and relationships (3.10), we obtain hyperbolic function solutions $u_{\mathscr{H}}(x,t)$ and $v_{\mathscr{H}}(x,t)$ of the Schrödinger-Boussinesq system (1.1) as follows:

$$\begin{aligned}
u_{\mathscr{H}}(x,t) = 2\sqrt{\frac{1}{ab}}\Bigg[3k^2&\left(\frac{\sqrt{\mathfrak{D}}}{2}\left(\frac{C_1\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}{C_1\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}\right)\right.\\
&\left.\mp\frac{1}{6}\frac{\sqrt{3\omega^2-3k^2}}{k^2}\right)^2
\end{aligned}$$

$$\pm\sqrt{3\omega^2-3k^2}\left(\frac{\sqrt{\mathfrak{D}}}{2}\left(\frac{C_1\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}{C_1\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}\right)\right.$$

$$\left.\left.\mp\frac{1}{6}\frac{\sqrt{3\omega^2-3k^2}}{k^2}\right)\right]e^{i\eta},$$

$$v_{\mathscr{H}}(x,t)=\frac{6k^2}{a}\left(\frac{\sqrt{\mathfrak{D}}}{2}\left(\frac{C_1\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}{C_1\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}\right)\mp\frac{1}{6}\frac{\sqrt{3\omega^2-3k^2}}{k^2}\right)^2$$

$$\pm 2\frac{\sqrt{3\omega^2-3k^2}}{a}\left(\frac{\sqrt{\mathfrak{D}}}{2}\left(\frac{C_1\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}{C_1\cosh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)+C_2\sinh\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}\right)\right.$$

$$\left.\mp\frac{1}{6}\frac{\sqrt{3\omega^2-3k^2}}{k^2}\right),\tag{3.11}$$

where $\mathfrak{D}=(\omega^2-k^2)/k^4>0$, $\xi=kx+\omega t$, $\eta=-(1/2)(\omega x/k)+(1/24)((24k^4+30\omega^4-54k^2\omega^2)/k^2(k^2-\omega^2))t$, and k,ω,C_1, and C_2 are arbitrary constants and $k\neq\pm\omega$.

It is easy to see that the hyperbolic solutions (3.11) can be rewritten at $C_1^2>C_2^2$, as follows:

$$u_{\mathscr{H}}(x,t)=\frac{1}{2}\sqrt{\frac{1}{ab}\frac{\omega^2-k^2}{k^2}}\left(3\tanh^2\left(\frac{1}{2}\sqrt{\frac{\omega^2-k^2}{k^4}}\,\xi+\rho_{\mathscr{H}}\right)-1\right)e^{i\eta},\tag{3.12a}$$

$$v_{\mathscr{H}}(x,t)=\frac{1}{2}\frac{\omega^2-k^2}{ak^2}\left(3\tanh^2\left(\frac{1}{2}\sqrt{\frac{\omega^2-k^2}{k^4}}\,\xi+\rho_{\mathscr{H}}\right)-1\right),\tag{3.12b}$$

while at $C_1^2<C_2^2$, one can obtain

$$u_{\mathscr{H}}(x,t)=\frac{1}{2}\sqrt{\frac{1}{ab}\frac{\omega^2-k^2}{k^2}}\left(3\coth^2\left(\frac{1}{2}\sqrt{\frac{\omega^2-k^2}{k^4}}\,\xi+\rho_{\mathscr{H}}\right)-1\right)e^{i\eta},\tag{3.12c}$$

$$v_{\mathscr{H}}(x,t)=\frac{1}{2}\frac{\omega^2-k^2}{ak^2}\left(3\coth^2\left(\frac{1}{2}\sqrt{\frac{\omega^2-k^2}{k^4}}\,\xi+\rho_{\mathscr{H}}\right)-1\right),\tag{3.12d}$$

where $\xi=kx+\omega t$, $\eta=-(1/2)(\omega x/k)+(1/24)((24k^4+30\omega^4-54k^2\omega^2)/k^2(k^2-\omega^2))t$, $\rho_{\mathscr{H}}=\tanh^{-1}(C_1/C_2)$, and k,ω are arbitrary constants.

Now, when $\mathfrak{D} = \lambda^2 - 4\mu = (\omega^2 - k^2)/k^4 < 0$, we obtain trigonometric function solutions u_τ and v_τ of Schrödinger-Boussinesq system (1.1) as follows:

$$
\begin{aligned}
u_\tau(x,t) = 2\sqrt{\frac{1}{ab}}\Bigg[& 3k^2\left(\frac{\sqrt{-\mathfrak{D}}}{2}\left(\frac{-C_1\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}{C_1\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}\right)\right. \\
& \left. \mp\frac{1}{6}\frac{\sqrt{3\omega^2 - 3k^2}}{k^2}\right)^2 \\
& \pm\sqrt{3\omega^2 - 3k^2}\left(\frac{\sqrt{-\mathfrak{D}}}{2}\left(\frac{-C_1\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\cos\left(\left(\sqrt{\mathfrak{D}}/2\right)\xi\right)}{C_1\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}\right)\right. \\
& \left.\mp\frac{1}{6}\frac{\sqrt{3\omega^2 - 3k^2}}{k^2}\right)\Bigg]e^{i\eta}, \\
v_\tau(x,t) = \frac{6k^2}{a} & \left(\frac{\sqrt{-\mathfrak{D}}}{2}\left(\frac{-C_1\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}{C_1\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}\right) \mp \frac{1}{6}\frac{\sqrt{3\omega^2 - 3k^2}}{k^2}\right)^2 \\
& \pm 2\frac{\sqrt{3\omega^2 - 3k^2}}{a}\left(\frac{\sqrt{-\mathfrak{D}}}{2}\left(\frac{-C_1\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}{C_1\cos\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right) + C_2\sin\left(\left(\sqrt{-\mathfrak{D}}/2\right)\xi\right)}\right)\right. \\
& \left.\mp\frac{1}{6}\frac{\sqrt{3\omega^2 - 3k^2}}{k^2}\right),
\end{aligned}
\tag{3.13}
$$

where $\mathfrak{D} = (\omega^2 - k^2)/k^4 < 0$, $\xi = kx + \omega t$, $\eta = -(1/2)(\omega x/k) + (1/24)((24k^4 + 30\omega^4 - 54k^2\omega^2)/k^2(k^2 - \omega^2))t$, and k, ω, C_1, and C_2 are arbitrary constants and $k \neq \pm\omega$. Similarity, the trigonometric solutions (3.13) can be rewritten at $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows:

$$u_\tau(x,t) = \frac{1}{2}\sqrt{\frac{1}{ab}}\frac{k^2 - \omega^2}{k^2}\left(3\tan^2\left(\frac{1}{2}\sqrt{\frac{k^2 - \omega^2}{k^4}}\,\xi + \rho\tau\right) + 1\right)e^{i\eta}, \tag{3.14a}$$

$$v_\tau(x,t) = \frac{1}{2}\frac{k^2 - \omega^2}{ak^2}\left(3\tan^2\left(\frac{1}{2}\sqrt{\frac{k^2 - \omega^2}{k^4}}\,\xi + \rho\tau\right) + 1\right), \tag{3.14b}$$

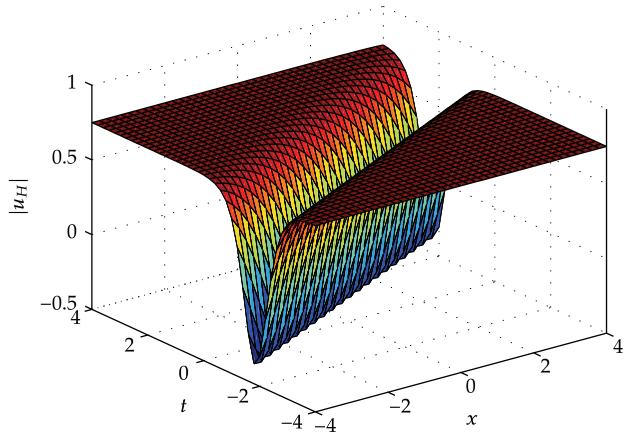

Figure 1: Soliton solution $|u_{\mathscr{H}}(x,t)|$ of the Schrödinger-Boussinesq system, (3.12a), for $\alpha = 1$, $\beta = 16$, $k = 1$, $\omega = -2$, and $\rho_{\mathscr{H}} = \tanh^{-1}(1/4)$.

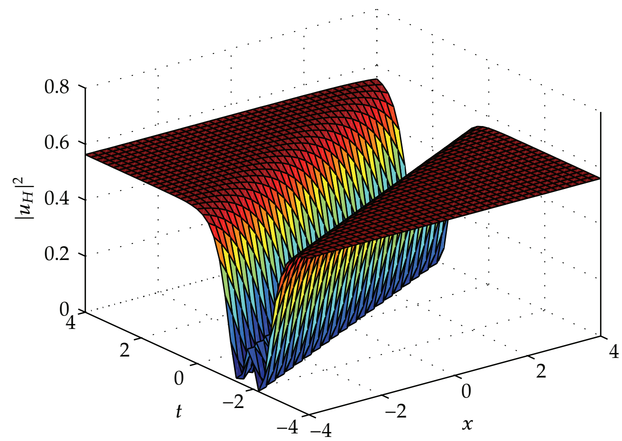

Figure 2: Soliton solution $|u_{\mathscr{H}}(x,t)|^2$ of the Schrödinger-Boussinesq system, (3.12a), for $\alpha = 1$, $\beta = 16$, $k = 1$, $\omega = -2$, and $\rho_{\mathscr{H}} = \tanh^{-1}(1/4)$.

$$u_{\mathcal{T}}(x,t) = \frac{1}{2}\sqrt{\frac{1}{ab}}\frac{k^2-\omega^2}{k^2}\left(3\cot^2\left(\frac{1}{2}\sqrt{\frac{k^2-\omega^2}{k^4}}\ \xi + \rho_{\mathcal{T}}\right)+1\right)e^{i\eta}, \tag{3.14c}$$

$$v_{\mathcal{T}}(x,t) = \frac{1}{2}\frac{k^2-\omega^2}{ak^2}\left(3\cot^2\left(\frac{1}{2}\sqrt{\frac{k^2-\omega^2}{k^4}}\ \xi + \rho_{\mathcal{T}}\right)+1\right), \tag{3.14d}$$

where $\xi = kx + \omega t$, $\eta = -(1/2)(\omega x/k) + (1/24)((24k^4 + 30\omega^4 - 54k^2\omega^2)/k^2(k^2-\omega^2))t$, $\rho_{\mathcal{T}} = \tan^{-1}(C_1/C_2)$, and k, ω are arbitrary constants.

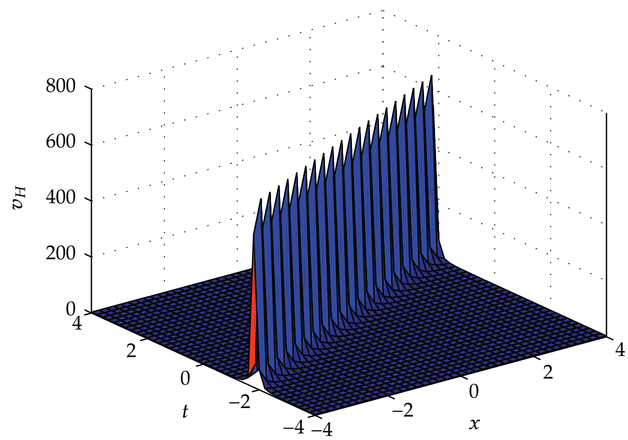

Figure 3: Soliton solution $|v_{\mathscr{H}}(x,t)|$ of the Schrödinger-Boussinesq system, (3.12b), for $\alpha = 1$, $\beta = 16$, $k = 1$, $\omega = -2$, and $\rho_{\mathscr{H}} = \tanh^{-1}(1/4)$.

4. Conclusions

This study shows that the (G'/G)-expansion method is quite efficient and practically well suited for use in finding exact solutions for the Schrödinger-Boussinesq system. With the aid of `Maple`, we have assured the correctness of the obtained solutions by putting them back into the original equation. To illustrate the obtained solutions, the hyperbolic type of obtained solutions, (3.12a) and (3.12b), are attached as Figures 1, 2, and 3. We hope that they will be useful for further studies in applied sciences.

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Research Article

Existence of Solutions for Nonlinear Mixed Type Integrodifferential Functional Evolution Equations with Nonlocal Conditions

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Using Mönch fixed point theorem, this paper proves the existence and controllability of mild solutions for nonlinear mixed type integrodifferential functional evolution equations with nonlocal conditions in Banach spaces, some restricted conditions on a priori estimation and measure of noncompactness estimation have been deleted, our results extend and improve many known results. As an application, we have given a controllability result of the system.

1. Introduction

This paper related to the existence and controllability of mild solutions for the following nonlinear mixed type integrodifferential functional evolution equations with nonlocal conditions in Banach space X:

$$\begin{gathered} x'(t) = A(t)x(t) + f\left(t, x_t, \int_0^t K(t,s,x_s)ds, \int_0^b H(t,s,x_s)ds\right), \quad t \in J, \\ x_0 = \phi + g(x), \quad t \in [-q, 0], \end{gathered} \tag{1.1}$$

where $q > 0$, $J = [0, b]$, $A(t)$ is closed linear operator on X with a dense domain $D(A)$ which is independent of t, $x_0 \in X$, and $x_t : [-q, 0] \to X$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-q, 0]$ and $x \in C(J, X)$, $f : J \times C([-q, 0], X) \times X \times X \to X$, $K : \Delta \times C([-q, 0], X) \to X$, $\Delta = \{(t,s) \in J \times J : s \le t\}$, $H : J \times J \times C([-q, 0], X) \to X$, $g : C([0, b], X) \to C([-q, 0], X)$, $\phi : [-q, 0] \to X$ are given functions.

For the existence and controllability of solutions of nonlinear integrodifferential functional evolution equations in abstract spaces, there are many research results, see [1–13] and their references. However, in order to obtain existence and controllability of mild solutions in these study papers, usually, some restricted conditions on a priori estimation and compactness conditions of evolution operator are used. Recently, Xu [6] studied existence of mild solutions of the following nonlinear integrodifferential evolution system with equicontinuous semigroup:

$$(Ex(t))' + Ax(t) = f\left(t, x(\sigma_1(t)), \int_0^t k(t,s)h(s, x(\sigma_2(s))ds)\right), \quad t \in [0,b],$$
$$x(0) + g(x) = x_0. \tag{1.2}$$

Some restricted conditions on a priori estimation and measure of noncompactness estimation:

$$\frac{(1-\alpha\beta Mc)N}{\alpha\beta M(d + \|x_0\|) + \alpha M\|\theta_1\|_{L^1}\Omega_1(N + K\|\theta_2\|_{L^1}\Omega_2(N))} > 1,$$
$$2\alpha\|\theta_1\|_{L^1}M(1 + 2K\|\theta_2\|_{L^1}) < 1 \tag{1.3}$$

are used, and some similar restricted conditions are used in [14, 15]. But estimations (3.15) and (3.21) in [15] seem to be incorrect. Since spectral radius $\sigma(B) = 0$ of linear Volterra integral operator $(Bx)(t) = \int_0^t k(t,s)x(s)ds$, in order to obtain the existence of solutions for nonlinear Volterra integrodifferential equations in abstract spaces by using fixed point theory, usually, some restricted conditions on a priori estimation and measure of noncompactness estimation will not be used even if the infinitesimal generator $A = 0$.

In this paper, using Mönch fixed point theorem, we investigate the existence and controllability of mild solution of nonlinear Volterra-Fredholm integrodifferential system (1.1), some restricted conditions on a priori estimation and measure of noncompactness estimation have been deleted, our results extend and improve the corresponding results in papers [2–20].

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $C([a,b], X)$ be a Banach space of all continuous X-valued functions defined on $[a,b]$ with norm $\|x\|_{[a,b]} = \sup_{t\in[a,b]}\|x(t)\|$ for $x \in C([a,b], X)$. $B(X)$ denotes the Banach space of bounded linear operators from X into itself.

Definition 2.1. The family of linear bounded operators $\{R(t,s) : 0 \le s \le t < +\infty\}$ on X is said an evolution system, if the following properties are satisfied:

(i) $R(t,t) = I$, where I is the identity operator in X;

(ii) $R(t,s)R(s,\tau) = R(t,\tau)$ for $0 \le s \le t < +\infty$;

(iii) $R(t,s) \in B(X)$ the space of bounded linear operator on X, where for every $(t,s) \in \{(t,s) : 0 \le s \le t < +\infty\}$ and for each $x \in X$, the mapping $(t,s) \to R(t,s)x$ is continuous.

The evolution system $R(t,s)$ is said to be equicontinuous if for all bounded set $Q \subset X$, $\{s \to R(t,s)x : x \in Q\}$ is equicontinuous for $t > 0$. $x \in C([-q,b],X)$ is said to be a mild solution of the nonlocal problem (1.1), if $x(t) = \phi(t) + g(x)(t)$ for $t \in [-q,0]$, and, for $t \in J$, it satisfies the following integral equation:

$$x(t) = R(t,0)\left[\phi(0) + g(x)(0)\right] + \int_0^t R(t,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds. \tag{2.1}$$

The following lemma is obvious.

Lemma 2.2. *Let the evolution system $R(t,s)$ be equicontinuous. If there exists a $\rho \in L^1[J,\mathbb{R}^+]$ such that $\|x(t)\| \le \rho(t)$ for a.e. $t \in J$, then the set $\{\int_0^t R(t,s)x(s)ds\}$ is equicontinuous.*

Lemma 2.3 (see [21]). *Let $V = \{x_n\} \subset L^1([a,b],X)$. If there exists $\sigma \in L^1([a,b],\mathbb{R}^+)$ such that $\|x_n(t)\| \le \sigma(t)$ for any $x_n \in V$ and a.e. $t \in [a,b]$, then $\alpha(V(t)) \in L^1([a,b],\mathbb{R}^+)$ and*

$$\alpha\left(\left\{\int_0^t x_n(s)ds : n \in \mathbb{N}\right\}\right) \le 2\int_0^t \alpha(V(s))ds, \quad t \in [a,b]. \tag{2.2}$$

Lemma 2.4 (see [22]). *Let $V \subset C([a,b],X)$ be an equicontinuous bounded subset. Then $\alpha(V(t)) \in C([a,b],\mathbb{R}^+)$ ($\mathbb{R}^+ = [0,\infty)$), $\alpha(V) = \max_{t\in[a,b]}\alpha(V(t))$.*

Lemma 2.5 (see [23]). *Let X be a Banach space, Ω a closed convex subset in X, and $y_0 \in \Omega$. Suppose that the operator $F : \Omega \to \Omega$ is continuous and has the following property:*

$$V \subset \Omega \text{ countable}, \quad V \subset \overline{\mathrm{co}}\left(\{y_0\} \cup F(V)\right) \Longrightarrow V \text{ is relatively compact}. \tag{2.3}$$

Then F has a fixed point in Ω.

Let $V(t) = \{x(t) : x \in C([-q,b],X)\} \subset X$ $(t \in J)$, $V_t = \{x_t : x \in C([-q,b],X)\} \subset C([-q,0],X)$, $\alpha(\cdot)$ and $\alpha_C(\cdot)$ denote the Kuratowski measure of noncompactness in X and $C([-q,b],X)$, respectively. For details on properties of noncompact measure, see [22].

3. Existence Result

We make the following hypotheses for convenience.

(H_1) $g : C([0,b],X) \to C([-q,0],X)$ is continuous, compact and there exists a constant N such that $\|g(x)\|_{[-q,0]} \le N$.

(H_2) (1) $f : J \times C([-q,0],X) \times X \times X \to X$ satisfies the Carathodory conditions, that is, $f(\cdot,x,y,z)$ is measurable for each $x \in C([-q,b],X), y,z \in X$, $f(t,\cdot,\cdot,\cdot)$ is continuous for a.e. $t \in J$.

(2) There is a bounded measure function $p : J \to \mathbb{R}^+$ such that

$$\left\|f(t,x,y,z)\right\| \le p(t)\left(\|x\|_{[-q,0]} + \left\|y\right\| + \|z\|\right), \quad a.e.\ t \in J,\ x \in C\left([-q,0],X\right),\ y,z \in X. \tag{3.1}$$

(H$_3$) (1) For each $x \in C([-q,0],X), K(\cdot,\cdot,x), H(\cdot,\cdot,x) : J \times J \to X$ are measurable and $K(t,s,\cdot), H(t,s,\cdot) : C([-q,0],X) \to X$ is continuous for a.e. $t,s \in J$.

(2) For each $t \in (0,b]$, there are nonnegative measure functions $k(t,\cdot), h(t,\cdot)$ on $[0,b]$ such that

$$\begin{aligned} \|K(t,s,x)\| &\le k(t,s)\|x\|_{[-q,0]}, \quad (t,s) \in \Delta,\ x \in C([-q,0],X), \\ \|H(t,s,x)\| &\le h(t,s)\|x\|_{[-q,0]}, \quad t,s \in J,\ x \in C([-q,0],X), \end{aligned} \tag{3.2}$$

and $\int_0^t k(t,s)ds, \int_0^b h(t,s)ds$ are bounded on $[0,b]$.

(H$_4$) For any bounded set $V_1 \subset C([-q,0],X), V_2, V_3 \subset X$, there is bounded measure function $l_i \in C[J,\mathbb{R}^+]$ $(i = 1,2,3)$ such that

$$\begin{aligned} \alpha(f(t,V_1,V_2,V_3) &\le l_1(t) \sup_{-q\le\theta\le 0} \alpha(V_1(\theta)) + l_2(t)\alpha(V_2) + l_3(t)\alpha(V_3), \quad a.e.\ t \in J, \\ \alpha\left(\int_0^t K(t,s,V_1)ds\right) &\le k(t,s) \sup_{-q\le\theta\le 0} \alpha(V_1(\theta)), \quad t \in J, \\ \alpha\left(\int_0^b H(t,s,V_1)ds\right) &\le h(t,s) \sup_{-q\le\theta\le 0} \alpha(V_1(\theta)),\ t \in J. \end{aligned} \tag{3.3}$$

(H$_5$) The resolvent operator $R(t,s)$ is equicontinuous and there are positive numbers $M \ge 1$ and

$$w = \max\left\{Mp_0(1 + k_0 + h_0) + 1,\ 2M\left(l_1^0 + 2l_2^0 k_0 + 2l_3^0 h_0\right) + 1\right\}, \tag{3.4}$$

such that $\|R(t,s)\| \le Me^{-w(t-s)}, 0 \le s \le t \le b$, where $k_0 = \sup_{(t,s)\in\Delta} \int_0^t k(t,s)ds$, $h_0 = \sup_{t,s\in J} \int_0^b h(t,s)ds, p_0 = \sup_{t\in J} p(t)$, $l_i^0 = \sup_{t\in J} l_i(t)$ $(i = 1,2,3)$.

Theorem 3.1. *Let conditions* (*H*$_1$)–(*H*$_5$) *be satisfied. Then the nonlocal problem* (1.1) *has at least one mild solution.*

Proof. Define an operator $F : C([-q,b],X) \to C([-q,b],X)$ by

$$(Fx)(t) = \begin{cases} \phi(t) + g(x)(t), & t\in [-q,0], \\ R(t,0)[\phi(0) + g(x)(0)] \\ \quad + \displaystyle\int_0^t R(t,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds, & t \in [0,b]. \end{cases} \tag{3.5}$$

We have by (H_1)–(H_3) and (H_5),

$$\begin{aligned}
\|(Fx)(t)\| &\le \|\phi\|_{[-q,0]} + N \le M\left(\|\phi\|_{[-q,0]} + N\right) =: L, \quad t \in [-q,0], \\
\|(Fx)(t)\| &\le L + M\int_0^t e^{w(s-t)}\left\| f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)\right\| ds \\
&\le L + M\int_0^t e^{w(s-t)} p(s)\left(\|x_s\|_{[-q,0]} + \int_0^s \|K(s,r,x_r)\|dr + \int_0^b \|H(s,r,x_r)\|dr\right)ds \\
&\le L + Mp_0\int_0^t e^{w(s-t)}\left(\|x_s\|_{[-q,0]} + \int_0^s k(s,r)\|x_r\|_{[-q,0]}dr + \int_0^b h(s,r)\|x_r\|_{[-q,0]}dr\right)ds \\
&\le L + Mp_0(1+k_0+h_0)w^{-1}\|x\|_{[-q,b]}, \quad t \in [0,b].
\end{aligned} \tag{3.6}$$

Consequently,

$$\|(Fx)(t)\| \le L + Mp_0(1+k_0+h_0)w^{-1}\|x\|_{[-q,\mathrm{b}]} = L + \eta\|x\|_{[-q,\mathrm{b}]}, \quad t \in [-q,b], \tag{3.7}$$

where $0 < \eta = Mp_0(1+k_0+h_0)w^{-1} < 1$. Taking $R > L(1-\eta)^{-1}$, let

$$B_R = \left\{x \in C([-q,b],X) : \|x\|_{[-q,\mathrm{b}]} \le R\right\}. \tag{3.8}$$

Then B_R is a closed convex subset in $C([-q,b],X)$, $0 \in B_R$ and $F : B_R \to B_R$. Similar to the proof in [14, 24], it is easy to verify that F is a continuous operator from B_R into B_R. For $x \in B_R$, $s \in [0,b]$, (H_2) and (H_3) imply

$$\left\| f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)\right\| \le p(s)(1+k_0+h_0)R. \tag{3.9}$$

We can show that from (H_5), (3.9) and Lemma 2.2 that $F(B_R)$ is an equicontinuous in $C([-q,b],X)$.

Let $V \subset B_R$ be a countable set and

$$V \subset \overline{\mathrm{co}}(\{0\} \cup (FV)). \tag{3.10}$$

From equicontinuity of $F(B_R)$ and (3.10), we know that V is an equicontinuous subset in $C([-q,b],X)$. By (H_1), it is easy to see that $\alpha((FV)(t)) = 0$, $t \in [-q,0]$. By properties of noncompact measure, (H_4) and Lemma 2.3, we have

$$\begin{aligned}
\alpha((FV)(t)) &\le 2\int_0^t \|R(t-s)\|\alpha\left(f\left(s, V_s, \int_0^s K(s,r,V_r)dr, \int_0^b H(s,r,V_r)dr\right)\right)ds \\
&\le 2M\int_0^t e^{w(s-t)}\left[l_1(s)\sup_{-q\le\theta\le 0}\alpha(V_s(\theta)) + l_2(s)\alpha\left(\int_0^s K(s,r,V_r)dr\right)\right. \\
&\qquad\left. + l_3(s)\alpha\left(\int_0^b H(s,r,V_r)dr\right)\right]ds \\
&\le 2M\int_0^t e^{w(s-t)}\left[l_1^0\sup_{-q\le\theta\le 0}\alpha(V(s+\theta)) + 2l_2^0\int_0^s k(s,r)\sup_{-q\le\theta\le 0}\alpha(V(r+\theta))dr\right. \\
&\qquad\left. + 2l_3^0\int_0^b h(s,r)\sup_{-q\le\theta\le 0}\alpha(V(r+\theta))dr\right]ds \\
&\le 2M\left(l_1^0 + 2l_2^0k_0 + 2l_3^0h_0\right)\int_0^t e^{w(s-t)}ds\sup_{-q\le\tau\le b}\alpha(V(\tau)) \\
&\le 2M\left(l_1^0 + 2l_2^0k_0 + 2l_3^0h_0\right)w^{-1}\alpha_C(V), \quad t\in[0,b].
\end{aligned} \tag{3.11}$$

Consequently,

$$\alpha_C(FV) = \sup_{-q\le t\le b}\alpha((FV)(t)) \le 2M\left(l_1^0 + 2l_2^0k_0 + 2l_3^0h_0\right)w^{-1}\alpha_C(V). \tag{3.12}$$

Equations (3.10), (3.12), and Lemma 2.4 imply

$$\alpha_C(V) \le \alpha_C(FV) \le \delta\alpha_C(V), \tag{3.13}$$

where $\delta = 2M(l_1^0 + 2l_2^0k_0 + 2l_3^0h_0)w^{-1} < 1$. Hence $\alpha_C(V) = 0$ and V is relative compact in $C([-q,b],X)$. Lemma 2.5 implies that F has a fixed point in $C([-q,b],X)$, then the system (1.1), (1.2) has at least one mild solution. The proof is completed. □

4. An Example

In this section, we give an example to illustrate Theorem 3.1.

Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following functional integrodifferential equation with nonlocal condition:

$$\begin{aligned} u_t(t,y) &= a_1(t,y)u_{yy}(t,y) \\ &\quad + a_2(t)\left[\sin u(t+\theta,y)ds + \int_0^t\int_{-q}^s a_3(s+\tau)\frac{u(\tau,y)d\tau ds}{(1+t)} + \int_0^b \frac{u(s+\theta,y)ds}{(1+t)(1+s)^2}\right], \\ &\qquad 0 \le t \le b, \\ u(t,y) &= \phi(t,y) + \int_0^\pi\int_0^b F(r,y)\log\left(1+|u(r,s)|^{1/2}\right)dr\,ds, \quad -q \le t \le 0,\ 0 \le y \le \pi, \\ & u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le b, \end{aligned} \tag{4.1}$$

where functions $a_1(t,y)$ is continuous on $[0,b]\times[0,\pi]$ and uniformly Hölder continuous in t, $a_2(t)$ is bounded measure on $[0,b]$, $\phi : [-q,0]\times[0,\pi]$, $a_3 : [-q,b]$, and $F : [0,b]\times[0,\pi]$ are continuous, respectively. Taking $u(t,y) = u(t)(y)$, $\phi(t,y) = \phi(t)(y)$,

$$\begin{aligned} & f\left(t,u_t,\int_0^t K(t,s,u_s)ds,\int_0^b H(t,s,u_s)ds\right)(y) \\ &\qquad = a_2(t)\left[\sin u(t+\theta,y) + \int_0^t\int_{-q}^s a_3(s+\tau)\frac{u(\tau,y)d\tau ds}{(1+t)} + \int_0^b \frac{u(s+\theta,y)ds}{(1+t)(1+s)^2}\right], \\ & K(t,s,u_s)(y) = \int_{-q}^s a_3(s+\tau)\frac{u(\tau,y)d\tau}{(1+t)}, \quad H(t,s,u_s)(y) = \frac{u(s+\theta,y)}{(1+t)(1+s)^2}, \\ & g(u)(y) = \int_0^\pi\int_0^b F(r,y)\log\left(1+|u(r,s)|^{1/2}\right)dr\,ds. \end{aligned} \tag{4.2}$$

The operator A defined by $A(t)w = a_1(t,y)w''$ with the domain

$$D(A) = \{w \in X : w, w' \text{ are absolutely continuous, } w'' \in X,\ w(0) = w(\pi) = 0\}. \tag{4.3}$$

Then $A(t)$ generates an evolution system, and $R(t,s)$ can be deduced from the evolution systems so that $R(t,s)$ is equicontinuous and $\|R(t,s)\| \le Me^{\beta(t-s)}$ for some constants M and β (see [24, 25]). The system (4.1) can be regarded as a form of the system (1.1), (1.2). We have by (4.2)

$$\begin{aligned} & \|f(t,u,v,z)\| \le |a_2(t)|\left(\|u\|_{[-q,0]} + \|v\| + \|z\|\right), \\ & \|K(t,s,u)\| \le \int_{-q}^s |a_3(s+\tau)|d\tau\frac{\|u\|_{[-q,0]}}{(1+t)}, \quad \|H(t,s,u)\| \le \frac{\|u\|_{[-q,0]}}{(1+t)(1+s)} \end{aligned} \tag{4.4}$$

for $u \in C([-q,0],X), v, z \in X$,

$$\|g(u)\|_{[-q,0]} \leq b\pi \max_{(r,y)\in[0,b]\times[0,\pi]} |F(r,y)| \left(\|u\|_{[0,b]} + \sqrt{\pi}\right), \tag{4.5}$$

and $g : C([0,b],X) \rightarrow C([-q,b],X)$ is continuous and compact (see the example in [7]). $w > 0$ and $M \geq 1$ can be chosen such that $\|R(t,s)\| \leq Me^{w(t-s)}, 0 \leq s \leq t \leq b$. In addition, for any bounded set $V_1 \subset C([-q,0],X), V_2, V_3 \subset X$, we can show that by the diagonal method

$$\begin{gathered}
\alpha(f(t,V_1,V_2,V_3)) \leq |a_2(t)| \left(\sup_{-q\leq\theta\leq 0} \alpha(V_1(\theta)) + \alpha(V_2) + \alpha(V_3) \right), \quad t \in J, \\
\alpha(K(t,s,V_1)) \leq \frac{1}{1+t} \sup_{-q\leq\theta\leq 0} \alpha(V_1(\theta)), \quad t,s \in \Delta, \\
\alpha(H(t,s,V_1)) \leq \frac{1}{(1+t)(1+s)^2} \sup_{-q\leq\theta\leq 0} \alpha(V_1(\theta)), \quad t,s \in [0,b].
\end{gathered} \tag{4.6}$$

It is easy to verify that all conditions of Theorem 3.1 are satisfied, so the system (5.1) has at least one mild solution.

5. An Application

As an application of Theorem 3.1, we shall consider the following system with control parameter:

$$\begin{gathered}
x'(t) = A(t)x(t) + f\left(t, x_t, \int_0^t K(t,s,x_s)ds, \int_0^b H(t,s,x_s)ds\right), \quad t \in J, \\
x_0 = \phi + g(x), \quad t \in [-q,0],
\end{gathered} \tag{5.1}$$

where C is a bounded linear operator from a Banach space U to X and $v \in L^2(J,U)$. Then the mild solution of systems (5.1) is given by

$$x(t) = \begin{cases} \phi(t) + g(x)(t), & t \in [-q,0], \\ R(t,0)[\phi(0) + g(x)(0)] + \int_0^t R(t,s)(Cv)(s)ds & \\ \quad + \int_0^t R(t,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds, & t \in [0,b], \end{cases} \tag{5.2}$$

where the resolvent operator $R(t,s) \in B(X)$, f, K, H, g, and ϕ satisfy the conditions stated in Section 3.

Definition 5.1. The system (5.2) is said to be controllable on $J = [0,b]$, if for every initial function $\phi \in C([-q,0],X)$ and $x_1 \in X$ there is a control $v \in L^2(J,U)$ such that the mild solution $x(t)$ of the system (5.1) satisfies $x(b) = x_1$.

To obtain the controllability result, we need the following additional hypotheses.

(H_5') The resolvent operator $R(t,s)$ is equicontinuous and $\|R(t,s)\| \le Me^{-w(t-s)}$, $0 \le s \le t \le b$, for $M \ge 1$ and positive number

$$w = \max\left\{ Mp_0(1+k_0+h_0)(1+MM_1b)+1, 2M\left(l_1^0+2l_2^0k_0+2l_3^0h_0\right)(1+MM_1b)+1 \right\}, \quad (5.3)$$

where k_0, h_0, p_0, l_i^0 $(i=1,2,3)$ are as before.

(H_6) The linear operator W from $L^2(J,U)$ into X, defined by

$$Wv = \int_0^b R(b,s)(Cv)(s)ds, \quad (5.4)$$

has an inverse operator W^{-1}, which takes values in $L^2(J,U)/\ker W$ and there exists a positive constant M_1 such that $\|CW^{-1}\| \le M_1$.

Theorem 5.2. *Let the conditions* (H_1)–(H_4), (H_5') *and* (H_6) *be satisfied. Then the nonlocal problem* (1.1), (1.2) *is controllable.*

Proof. Using hypothesis (H_6), for an arbitrary $x(\cdot)$, define the control

$$\begin{aligned} v(t) = W^{-1}\Bigg(& x_1 - R(b,0)[\phi(0)+g(x)(0)] \\ & + \int_0^b R(b,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds\Bigg)(t), \quad t\in[0,b]. \end{aligned} \quad (5.5)$$

Define the operator $T : C([-q,b],X) \to C([-q,b],X)$ by

$$(Tx)(t) = \begin{cases} \phi(t)+g(x)(t), & t\in[-q,0], \\ R(t,0)[\phi(0)+g(x)(0)] + \displaystyle\int_0^t R(t,s)(Cv)(s)ds & \\ \quad + \displaystyle\int_0^t R(t,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds, & t\in[0,b]. \end{cases} \quad (5.6)$$

Now we show that, when using this control, T has a fixed point. Then this fixed point is a solution of the system (5.1). Substituting $v(t)$ in (5.6), we get

$$(Tx)(t) = \begin{cases} \phi(t)+g(x)(t), & t\in[-q,0], \\ R(t,0)[\phi(0)+g(x)(0)] + \displaystyle\int_0^t R(t,s)CW^{-1}(x_1 - R(b,0)[\phi(0)+g(x)(0)] & \\ \quad + \displaystyle\int_0^b R(b,\tau) f\left(\tau, x_\tau, \int_0^\tau K(\tau,r,x_r)dr, \int_0^b H(\tau,r,x_r)dr\right)d\tau\Bigg)(s)ds & \\ \quad + \displaystyle\int_0^t R(t,s) f\left(s, x_s, \int_0^s K(s,r,x_r)dr, \int_0^b H(s,r,x_r)dr\right)ds, & t\in[0,b]. \end{cases} \quad (5.7)$$

Clearly, $(Tx)(b) = x_1$, which means that the control v steers the system (5.1) from the given initial function ϕ to the origin in time b, provided we can obtain a fixed point of nonlinear operator T. The remaining part of the proof is similar to Theorem 3.1, we omit it. □

Remark 5.3. Since the spectral radius of linear Fredholm type integral operator may be greater than 1, in order to obtain the existence of solutions for nonlinear Volterra-Fredholm type integrodifferential equations in abstract spaces by using fixed point theory, some restricted conditions on a priori estimation and measure of noncompactness estimation will not be used even if the generator $A = 0$. But, these restrictive conditions are not being used in Theorems 3.1 and 5.2.

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