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Fractional Problems with Variable-Order or Variable Exponents 2021

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Contents

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Research Article

Hardy-Leindler-Type Inequalities via Conformable Delta Fractional Calculus

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In this article, some fractional Hardy-Leindler-type inequalities will be illustrated by utilizing the chain law, Hölder's inequality, and integration by parts on fractional time scales. As a result of this, some classical integral inequalities will be obtained. Also, we would have a variety of well-known dynamic inequalities as special cases from our outcomes when $\alpha = 1$.

1. Introduction

The Hardy discrete inequality is known as (see [1])

$$\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{j=1}^{r} l(j)\right)^{\mu} \leq \left(\frac{\mu}{\mu-1}\right)^{\mu}\sum_{r=1}^{\infty} l^{\mu}(r), \quad \mu > 1. \tag{1}$$

where $l(r) > 0$ for all $r \geq 1$.

In [2], Hardy employed the calculus of variations and exemplified the continuous version for (1) as follows:

$$\int_0^{\infty}\left(\frac{1}{y}\int_0^{y} g(s)ds\right)^{\mu} dy \leq \left(\frac{\mu}{\mu-1}\right)^{\mu}\int_0^{\infty} g^{\mu}(y)dy, \quad \mu > 1, \tag{2}$$

where $\mu \geq 0$ is integrable over any finite interval $(0, y)$,g^{μ} is convergent and integrable over $(0, \infty)$, and $(\mu/(\mu-1))^{\mu}$ is a sharp constant in (1) and (2).

Leindler in [3] exemplified that if $\mu > 1$ and $\lambda(r)$,$f(r) > 0$, then

$$\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{r} f(s)\right)^{\mu} \leq \mu^{\mu}\left(\sum_{r=1}^{\infty}\lambda^{1-\mu}(r)\left(\sum_{s=r}^{\infty}\lambda(s)\right)^{\mu} f^{\mu}(r)\right), \tag{3}$$

$$\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{k=r}^{\infty} f(k)\right)^{\mu} \leq \mu^{\mu}\left(\sum_{r=1}^{\infty}\lambda^{1-\mu}(r)\left(\sum_{s=1}^{r}\lambda(s)\right)^{\mu} f^{\mu}(r)\right). \tag{4}$$

The converses of (3) and (4) are exemplified by Leindler in [4]. Precisely, he established that if $0 < \mu \leq 1$, then

$$\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{r}f(s)\right)^{\mu}\geq\mu^{\mu}\left(\sum_{r=1}^{\infty}\lambda^{1-\mu}(r)\left(\sum_{s=r}^{\infty}\lambda(s)\right)^{\mu}f^{\mu}(r)\right),\tag{5}$$

$$\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{k=r}^{\infty}f(k)\right)^{\mu}\geq\mu^{\mu}\left(\sum_{r=1}^{\infty}\lambda^{1-\mu}(r)\left(\sum_{s=1}^{r}\lambda(s)\right)^{\mu}f^{\mu}(r)\right).\tag{6}$$

Saker [5] exemplified the time scale version of (3) and (4), respectively, as follows: suppose that $\mathbb{T}$ be a time scale and $\mu>1$. If $\Lambda(\tau)\coloneqq\int_{\tau}^{\infty}\lambda(s)\Delta s$,$\Phi(\tau)\coloneqq\int_{l}^{\tau}f(s)\Delta s$, for any $\tau\in l,\infty)_{\mathbb{T}}$, then

$$\int_{l}^{\infty}\lambda(\tau)(\Phi^{\sigma}(\tau))^{\mu}\Delta\tau\leq\mu^{\mu}\left(\int_{l}^{\infty}\lambda^{1-\mu}(\tau)\Lambda^{\mu}(\tau)f^{\mu}(\tau)\Delta\tau\right).\tag{7}$$

Also, if $\bar{\Lambda}(\tau)\coloneqq\int_{l}^{\tau}\lambda(s)\Delta s$ and $\bar{\Phi}(\tau)\coloneqq\int_{\tau}^{\infty}f(s)\Delta s$, for any $\tau\in l,\infty)_{\mathbb{T}}$, then

$$\int_{l}^{\infty}\lambda(\tau)\left(\bar{\Phi}(\tau)\right)^{\mu}\Delta\tau\leq\mu^{\mu}\left(\int_{l}^{\infty}\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^{\sigma}(\tau)\right)^{\mu}f^{\mu}(\tau)\Delta\tau\right).\tag{8}$$

The converses of (7) and (8) are established by Saker [5]. Precisely, he exemplified that, if $\mathbb{T}$ is a time scale, $0<\mu\leq1$, $\Omega(\tau)=\int_{\tau}^{\infty}\lambda(s)\Delta s$, and $\Psi(\tau)=\int_{l}^{\tau}f(s)\Delta s$, for any $\tau\in l,\infty)_{\mathbb{T}}$, then

$$\int_{l}^{\infty}\lambda(\tau)(\Psi^{\sigma}(\tau))^{\mu}\Delta\tau\geq\mu^{\mu}\left(\int_{l}^{\infty}f^{\mu}(\tau)\Omega^{\mu}(\tau)\lambda^{1-\mu}(\tau)\Delta\tau\right).\tag{9}$$

Also, if $\bar{\Omega}(\tau)=\int_{l}^{\tau}\lambda(s)\Delta s$ and $\bar{\Psi}(\tau)=\int_{\tau}^{\infty}f(s)\Delta s$, for any $\tau\in l,\infty)_{\mathbb{T}}$, then

$$\int_{l}^{\infty}\lambda(\tau)\left(\bar{\Psi}(\tau)\right)^{\mu}\Delta\tau\geq\mu^{\mu}\left(\int_{l}^{\infty}f^{\mu}(\tau)\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}\lambda^{1-\mu}(\tau)\Delta\tau\right),\tag{10}$$

which are the time scale version for (5) and (6), respectively. For developing dynamic inequalities, see the papers ([6–11]).

Our target in this article is proving some fractional dynamic inequalities for Hardy-Leindler's type, and it is reversed with employing conformable calculus on time scales. This article is structured as follows: In Section 2, we discuss the preliminaries of conformable fractional on time scale calculus which will be required in proving our main outcomes. In Section 3, we will exemplify the major consequences.

2. Basic Concepts

In this part, we introduce the essentials of conformable fractional integral and derivative of order $\alpha\in[0,1]$ on time scales that will be used in this article (see [12–15]). For a time scale $\mathbb{T}$, we define the operator $\sigma:\mathbb{T}\longrightarrow\mathbb{T}$, as

$$\sigma(\tau)\coloneqq\inf\{s\in\mathbb{T}:s>\tau\}.\tag{11}$$

Also, we define the function $\mu:\mathbb{T}\longrightarrow 0,\infty)$ by

$$\mu(\tau)\coloneqq\sigma(\tau)-\tau.\tag{12}$$

Finally, for any $\tau\in\mathbb{T}$, we refer to the notation $\xi^{\sigma}(\tau)$ by $\xi(\sigma(\tau))$, i.e., $\xi^{\sigma}=\xi\circ\sigma$. In the following, we define conformable α-fractional derivative and α-fractional integral on $\mathbb{T}$.

Definition 1 (see [16], Definition 3.1). Suppose that $\xi:\mathbb{T}\longrightarrow\mathbb{R}$ and $\alpha\in(0,1]$. Then, for $\tau>0$, we define $D_{\alpha}(\xi^{\Delta})(\tau)$ to be the number with the property that, for any $\varepsilon>0$, there is a neighborhood V of τ s.t. $\forall\tau\in V$, we have

$$\left|\left(\xi^{\sigma}(\tau)-\xi(s)\right)\sigma^{1-\alpha}(\tau)-D_{\alpha}\left(\xi^{\Delta}\right)(\tau)(\sigma(\tau)-s)\right|\leq\varepsilon|\sigma(\tau)-s)|.\tag{13}$$

The conformable α-fractional derivative on $\mathbb{T}$ at 0 is

$$D_{\alpha}\left(\xi^{\Delta}\right)(0)=\lim_{\tau\longrightarrow0}D_{\alpha}\left(\xi^{\Delta}\right)(\tau).\tag{14}$$

Theorem 2 (see [16], Theorem 3.6). *Assume $0<\alpha\leq1$ and $\nu,\xi:\mathbb{T}\longrightarrow\mathbb{R}$ are conformable α-fractional derivatives at $\tau\in\mathbb{T}^{k}$. Then, we have the following.*

(i) The sum $\nu+\xi$ is a conformable α-fractional derivative and

$$D_{\alpha}\left((\nu+\xi)^{\Delta}\right)(\tau)=D_{\alpha}\left(\nu^{\Delta}\right)(\tau)+D_{\alpha}\left(\xi^{\Delta}\right)(\tau)\tag{15}$$

(ii) The product $\nu\xi:\mathbb{T}\longrightarrow\mathbb{R}$ is a conformable α-fractional derivative with

$$\begin{aligned}D_{\alpha}\left((\nu\xi)^{\Delta}\right)(\tau)&=D_{\alpha}\left(\nu^{\Delta}\right)(\tau)\xi(\tau)+\nu(\sigma(\tau))D_{\alpha}\left(\xi^{\Delta}\right)(\tau)\\&=\nu(\tau)D_{\alpha}\left(\xi^{\Delta}\right)(\tau)+D_{\alpha}\left(\nu^{\Delta}\right)(\tau)\xi(\sigma(\tau))\end{aligned}\tag{16}$$

(iii) If $\xi(\tau)\xi(\sigma(\tau))\neq0$, then ν/ξ is a conformable α-fractional derivative with

$$D_{\alpha}\left(\left(\frac{\nu}{\xi}\right)^{\Delta}\right)(\tau)=\frac{D_{\alpha}\left(\nu^{\Delta}\right)(\tau)\xi(\tau)-\nu(\tau)D_{\alpha}\left(\xi^{\Delta}\right)(\tau)}{\xi(\tau)\xi(\sigma(\tau))}\tag{17}$$

Lemma 3 (Chain rule). *Suppose that $\xi:\mathbb{T}\longrightarrow\mathbb{R}$ is continuous and α-fractional differentiable at $\tau\in\mathbb{T}$, for $\alpha\in(0,1]$ and $\nu:\mathbb{R}\longrightarrow\mathbb{R}$ is continuously differentiable. Then, $(\nu\circ\xi)$: $\mathbb{T}\longrightarrow\mathbb{R}$ is α-fractional differentiable and*

$$D_\alpha\left((\nu\circ\zeta)^\Delta\right)(\tau)=\nu'(\xi(d))D_\alpha\left(\xi^\Delta\right)(\tau),\ where\ d\in[\tau,\sigma(\tau)]. \tag{18}$$

Definition 4 (see [16], Definition 4.1). For $0<\alpha\le 1$, then the α-conformable Δ fractional integral of ξ is defined as

$$I_\alpha\left(\xi^\Delta\right)(s)=\int\xi(s)\Delta_\alpha s=\int\xi(s)\sigma^{\alpha-1}(s)\Delta s,\ \text{for all}\ s\in\mathbb{T}^k. \tag{19}$$

Theorem 5 (see [16], Theorem 4.3). *Let $l,m,n\in\mathbb{T},\beta\in\mathbb{R}$, $\alpha\in(0,1]$, and $\nu,\xi:\mathbb{T}\longrightarrow\mathbb{R}$ be rd-continuous functions. Then,*

(i) $\int_l^m[\nu(s)+\xi(s)]\Delta_\alpha s=\int_l^m\nu(s)\Delta_\alpha s+\int_l^m\xi(s)\Delta_\alpha s$

(ii) $\int_l^m\beta\nu(s)\Delta_\alpha s=\beta\int_l^m\nu(s)\Delta_\alpha s$

(iii) $\int_l^m\nu(s)\Delta_\alpha s=-\int_m^l\nu(s)\Delta_\alpha s$

(iv) $\int_l^m\nu(s)\Delta_\alpha s=\int_l^n\nu(s)\Delta_\alpha s+\int_n^m\nu(s)\Delta_\alpha s$

(v) $\int_l^l\nu(s)\Delta_\alpha s=0$

Lemma 6 (Integration by parts formula [16], Theorem 4.3). *Suppose that $l,m\in\mathbb{T}$ where $m>l$. If ν,ξ are rd-continuous functions and $\alpha\in(0,1]$, then*

$$(i)\int_l^m\nu(s)D_\alpha\left(\xi^\Delta\right)(s)\Delta_\alpha s=[\nu(s)\xi(s)]_l^m-\int_l^m\xi^\sigma(s)D_\alpha\left(\nu^\Delta\right)(s)\Delta_\alpha s. \tag{20}$$

$$(ii)\int_l^m\nu^\sigma(s)D_\alpha\left(\xi^\Delta\right)(s)\Delta_\alpha s=[\nu(s)\xi(s)]_l^m-\int_l^m\xi(s)D_\alpha\left(\nu^\Delta\right)(s)\Delta_\alpha s. \tag{21}$$

Lemma 7 (Hölder's inequality). *Let $l,m\in\mathbb{T}$ where $m>l$. If $\alpha\in(0,1]$ and $F,G:\mathbb{T}\longrightarrow\mathbb{R}$, then*

$$\int_l^m|F(s)G(s)|\Delta_\alpha s\le\left(\int_l^m|F(s)|^\beta\Delta_\alpha s\right)^{1/\beta}\left(\int_l^m|G(s)|^\mu\Delta_\alpha s\right)^{1/\mu}, \tag{22}$$

where $\beta>1$ and $1/\beta+1/\mu=1$.

Through our paper, we will consider the integrals are given exist (are finite, i.e., convergent).

3. Main Results

Here, we will exemplify our major results in this article. In the pursuing theorem, we will exemplify Leindler's inequality (7) for fractional time scales as follows.

Theorem 8. *Suppose that $\mathbb{T}$ be a time scale and $0<\alpha\le 1$. If $\mu>1,\Lambda(\tau)\coloneqq\int_\tau^\infty\lambda(s)\Delta_\alpha s$ and $\Phi(\tau)\coloneqq\int_l^\tau f(s)\Delta_\alpha s$, for any $\tau\in[l,\infty)_\mathbb{T}$, then*

$$\begin{aligned}&\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\\&\le(\mu-\alpha+1)\left(\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\Delta_\alpha\tau\right)^{1/\mu}\\&\quad\times\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu(\mu-\alpha)/\mu-1}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}.\end{aligned} \tag{23}$$

Proof. By utilizing (20) on

$$\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau, \tag{24}$$

with $\zeta^\sigma(\tau)=(\Phi^\sigma(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\nu^\Delta)(\tau)=\lambda(\tau)$, we have

$$\begin{aligned}&\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\\&=\nu(\tau)\Phi^{\mu-\alpha+1}(\tau)\big|_l^\infty\\&\quad+\int_l^\infty(-\nu(\tau))D_\alpha\left(\left(\Phi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau,\end{aligned} \tag{25}$$

where

$$\nu(\tau)=-\int_\tau^\infty\lambda(s)\Delta_\alpha s=-\Lambda(\tau). \tag{26}$$

Substituting (26) into (25), we get

$$\begin{aligned}&\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\\&=\nu(\tau)\Phi^{\mu-\alpha+1}(\tau)\big|_l^\infty+\int_l^\infty\Lambda(\tau)D_\alpha\left(\left(\Phi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau.\end{aligned} \tag{27}$$

Using $\Phi(l)=0$ and $\Lambda(\infty)=0$ in (27), we have

$$\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau=\int_l^\infty\Lambda(\tau)D_\alpha\left(\left(\Phi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau. \tag{28}$$

Utilizing the chain rule (18), we get

$$\begin{aligned}D_\alpha\left(\left(\Phi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)&=(\mu-\alpha+1)\Phi^{\mu-\alpha}(d)D_\alpha\left(\Phi^\Delta\right)(\tau)\\&\le(\mu-\alpha+1)D_\alpha\left(\Phi^\Delta\right)(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}.\end{aligned} \tag{29}$$

Since $D_\alpha(\Phi^\Delta)(\tau)=f(\tau)$, we get

$$D_\alpha\left(\left(\Phi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\le(\mu-\alpha+1)f(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}. \tag{30}$$

Substituting (30) into (28) yields

$$\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\le(\mu-\alpha+1)\int_l^\infty\Lambda(\tau)f(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}\Delta_\alpha\tau. \tag{31}$$

Inequality (31) can be written as

$$\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau \le (\mu-\alpha+1)\int_l^\infty \frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}\Delta_\alpha\tau. \tag{32}$$

Implementing Hölder's inequality on the R.H.S of (32) with indices μ, $\mu/(\mu-1)$, we get

$$\begin{aligned}&\int_l^\infty \frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}\Delta_\alpha\tau\\ &\le\left(\int_l^\infty\left(\frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\right)^\mu\Delta_\alpha\tau\right)^{1/\mu}\left(\int_l^\infty\left(\lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha}\right)^{\mu/(\mu-1)}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}\\ &=\left(\int_l^\infty\frac{\Lambda^\mu(\tau)f^\mu(\tau)}{\lambda^{\mu-1}(\tau)}\Delta_\alpha\tau\right)^{1/\mu}\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{(\mu(\mu-\alpha))/(\mu-1)}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}.\end{aligned} \tag{33}$$

By substituting (33) into (32), we get

$$\begin{aligned}\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau &\le (\mu-\alpha+1)\left(\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\Delta_\alpha\tau\right)^{1/\mu}\\ &\quad\times\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^{\mu(\mu-\alpha)/\mu-1}\Delta_\alpha\tau\right)^{(\mu-1)/\mu},\end{aligned} \tag{34}$$

which is (23). □

Corollary 9. *At $\alpha = 1$ in Theorem 8, then*

$$\begin{aligned}&\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu\Delta\tau\\ &\le\mu\left(\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\Delta\tau\right)^{1/\mu}\\ &\quad\times\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^\mu\Delta\tau\right)^{(\mu-1)/\mu},\end{aligned} \tag{35}$$

where $>1, \Lambda(\tau) := \int_\tau^\infty \lambda(s)\Delta s$, and $\Phi(\tau) := \int_l^\tau f(s)\Delta s$, for any $\tau \in l, \infty)_{\mathbb{T}}$.

Remark 10. In Corollary 9, if we divide both sides of (35) by the factor

$$\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^\mu\Delta\tau\right)^{(\mu-1)/\mu}, \tag{36}$$

and using the fact that $1-(\mu-1)/\mu = 1/\mu$, then

$$\left(\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^\mu\Delta\tau\right)^{1/\mu}\le\mu\left(\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\Delta\tau\right)^{1/\mu}. \tag{37}$$

Elevating the last inequality to the μth power, we get

$$\int_l^\infty\lambda(\tau)(\Phi^\sigma(\tau))^\mu\Delta\tau\le\mu^\mu\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\Delta\tau, \tag{38}$$

which is (7) in Introduction.

Remark 11. If we put $\mathbb{T}=\mathbb{R}$ (i.e., $\sigma(\tau)=\tau$) in Theorem 5, then

$$\begin{aligned}\int_l^\infty\lambda(\tau)\Phi^{\mu-\alpha+1}(\tau)\tau^{\alpha-1}d\tau &\le(\mu-\alpha+1)\left(\int_l^\infty\lambda^{1-\mu}(\tau)\Lambda^\mu(\tau)f^\mu(\tau)\tau^{\alpha-1}d\tau\right)^{1/\mu}\\ &\quad\times\left(\int_l^\infty\lambda(\tau)(\Phi(\tau))^{\mu(\mu-\alpha)/\mu-1}\tau^{\alpha-1}d\tau\right)^{(\mu-1)/\mu},\end{aligned} \tag{39}$$

where $\alpha\in(0,1]$, $\mu>1$, $\Lambda(\tau) := \int_\tau^\infty\lambda(s)s^{\alpha-1}ds$, and $\Phi(\tau) := \int_l^\tau f(s)s^{\alpha-1}ds$, for any $\tau\in[l,\infty)$.

Remark 12. Clearly, for $\alpha = 1$ and $l = 1$, Remark 12 coincides with Remark 10 in [5].

Remark 13. When $\mathbb{T}=\mathbb{Z}$ (i.e., $\sigma(\tau)=\tau+1$), $\mu>1$, and $l=1$ in (23), then we get

$$\begin{aligned}&\sum_{r=1}^\infty\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)(s+1)^{\alpha-1}\right)^{\mu-\alpha+1}(r+1)^{\alpha-1}\\ &\le(\mu-\alpha+1)\left(\sum_{r=1}^\infty\lambda^{1-\mu}(r)\left(\sum_{s=r}^\infty\lambda(s)(s+1)^{\alpha-1}\right)^\mu f^\mu(r)(r+1)^{\alpha-1}\right)^{1/\mu}\\ &\quad\times\left(\sum_{r=1}^\infty\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)(s+1)^{\alpha-1}\right)^{(\mu(\mu-\alpha))/(\mu-1)}(r+1)^{\alpha-1}\right)^{(\mu-1)/\mu}.\end{aligned} \tag{40}$$

If $\alpha = 1$, then (40) becomes

$$\begin{aligned}&\sum_{r=1}^\infty\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)\right)^\mu\\ &\le\mu\left(\sum_{r=1}^\infty\lambda^{1-\mu}(r)\left(\sum_{s=r}^\infty\lambda(s)\right)^\mu f^\mu(r)\right)^{1/\mu}\\ &\quad\times\left(\sum_{r=1}^\infty\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)\right)^\mu\right)^{\mu-1/\mu},\end{aligned} \tag{41}$$

which is Remark 11 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (8) on fractional time scales as follows.

Theorem 14. *Suppose that $\mathbb{T}$ be a time scale and $0<\alpha\le 1$. If $>1, \bar{\Lambda}(\tau) := \int_l^\tau\lambda(s)\Delta_\alpha s$, and $\Psi(\tau) := \int_\tau^\infty f(s)\Delta_\alpha s$, for any $\tau\in l, \infty)_{\mathbb{T}}$, then*

$$\begin{aligned}\int_l^\infty\lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau&\le(\mu-\alpha+1)\left(\int_l^\infty\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Delta_\alpha\tau\right)^{1/\mu}\\ &\quad\times\left(\int_l^\infty\lambda(\tau)(\Psi(\tau))^{(\mu(\mu-\alpha))/(\mu-1)}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}.\end{aligned} \tag{42}$$

Proof. By utilizing (20) on

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau, \tag{43}$$

with $\nu(\tau)=(\Psi(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\zeta^\Delta)(\tau)=\lambda(\tau)$, we have

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau=\zeta(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Big|_l^\infty+\int_l^\infty \zeta^\sigma(\tau)\left(-D_\alpha\left(\left(\Psi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau, \tag{44}$$

where

$$\zeta(\tau)=\int_l^\tau \lambda(s)\Delta_\alpha s=\bar{\Lambda}(\tau). \tag{45}$$

Substituting (45) into (44), we get

$$\begin{aligned}\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau&=\bar{\Lambda}(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Big|_l^\infty\\&\quad+\int_l^\infty \left(\bar{\Lambda}(\tau)\right)^\sigma\left(-D_\alpha\left(\left(\Psi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau.\end{aligned} \tag{46}$$

Using the fact that $\Psi(\infty)=0$ and $\bar{\Lambda}(l)=0$, (46) became

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau=\int_l^\infty \left(\bar{\Lambda}(\tau)\right)^\sigma\left(-D_\alpha\left(\left(\Psi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau. \tag{47}$$

Utilizing chain rule (18), we get

$$\begin{aligned}-D_\alpha\left(\left(\Psi^\Delta\right)^{\mu-\alpha+1}\right)(\tau)&=-(\mu-\alpha+1)\Psi^{\mu-\alpha}(d)D_\alpha\left(\Psi^\Delta\right)(\tau)\\&=-(\mu-\alpha+1)(-f(\tau))\Psi^{\mu-\alpha}(d)\\&\le(\mu-\alpha+1)f(\tau)\Psi^{\mu-\alpha}(\tau).\end{aligned} \tag{48}$$

By substituting (48) into (47), we get

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\le(\mu-\alpha+1)\int_l^\infty \left(\bar{\Lambda}(\tau)\right)^\sigma f(\tau)\Psi^{\mu-\alpha}(\tau)\Delta_\alpha\tau. \tag{49}$$

Inequality (49) can be written as

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\le(\mu-\alpha+1)\int_l^\infty \frac{\left(\bar{\Lambda}(\tau)\right)^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau)\Delta_\alpha\tau. \tag{50}$$

Implementing Hölder's inequality on the R.H.S of (50) with indices μ, $\mu/(\mu-1)$, we get

$$\begin{aligned}&\int_l^\infty \frac{\left(\bar{\Lambda}(\tau)\right)^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau)\Delta_\alpha\tau\\&\quad\le\left(\int_l^\infty\left(\frac{\left(\bar{\Lambda}(\tau)\right)^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)}\right)^\mu\Delta_\alpha\tau\right)^{1/\mu}\left(\int_l^\infty\left(\lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau)\right)^{\mu/(\mu-1)}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}\\&\quad=\left(\int_l^\infty\frac{\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)}{\lambda^{\mu-1}(\tau)}\Delta_\alpha\tau\right)^{1/\mu}\left(\int_l^\infty\lambda(\tau)(\Psi(\tau))^{(\mu(\mu-\alpha))/(\mu-1)}\Delta_\alpha\tau\right)^{(\mu-1)/\mu}.\end{aligned} \tag{51}$$

By substituting (51) into (50), we get

$$\begin{aligned}\int_l^\infty \lambda(\tau)\Psi^{\mu-\alpha+1}(\tau)\Delta_\alpha\tau&\le(\mu-\alpha+1)\left(\int_l^\infty\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Delta_\alpha\tau\right)^{1/\mu}\\&\quad\times\left(\int_l^\infty\lambda(\tau)(\Psi(\tau))^{\mu(\mu-\alpha)/\mu-1}\Delta_\alpha\tau\right)^{(\mu-1)/\mu},\end{aligned} \tag{52}$$

which is (42). □

Corollary 15. *At $\alpha=1$ in Theorem 14, then*

$$\begin{aligned}\int_l^\infty \lambda(\tau)(\Psi(\tau))^\mu\Delta\tau&\le\mu\left(\int_l^\infty\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Delta\tau\right)^{1/\mu}\\&\quad\times\left(\int_l^\infty\lambda(\tau)\left(\bar{\Phi}(\tau)\right)^\mu\Delta\tau\right)^{(\mu-1)/\mu},\end{aligned} \tag{53}$$

where $\mu>1$, $\bar{\Lambda}(\tau):=\int_l^\tau\lambda(s)\Delta s$, and $\Psi(\tau):=\int_\tau^\infty f(s)\Delta s$, for any $\tau\in l,\infty)_{\mathbb{T}}$.

Remark 16. In Corollary 15, if we divide both sides of (53) by the factor

$$\left(\int_l^\infty\lambda(\tau)(\Psi(\tau))^\mu\Delta\tau\right)^{(\mu-1)/\mu}, \tag{54}$$

and using the fact that $1-(\mu-1)/\mu=1/\mu$, then

$$\left(\int_l^\infty\lambda(\tau)(\Psi(\tau))^\mu\Delta\tau\right)^{1/\mu}\le\mu\left(\int_l^\infty\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Delta\tau\right)^{1/\mu}. \tag{55}$$

Elevating the last inequality to the μth power, we get

$$\int_l^\infty\lambda(\tau)(\Psi(\tau))^\mu\Delta\tau\le\mu^\mu\int_l^\infty\lambda^{1-\mu}(\tau)\left(\bar{\Lambda}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Delta\tau, \tag{56}$$

which is (8) in Introduction.

Remark 17. As a result, if $\mathbb{T}=\mathbb{R}$ (i.e., $\sigma(\tau)=\tau$) in Theorem 14, then

$$\int_l^\infty \lambda(\tau)\Psi^{\mu-\alpha+1}(\tau)\tau^{\alpha-1}d\tau \le (\mu-\alpha+1)\left(\int_l^\infty \lambda^{1-\mu}(\tau)\left(\bar{\Lambda}(\tau)\right)^\mu f^\mu(\tau)\tau^{\alpha-1}d\tau\right)^{1/\mu} \times\left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu(\mu-\alpha)/(\mu-1)}\tau^{\alpha-1}d\tau\right)^{(\mu-1)/\mu}, \tag{57}$$

where $\alpha\in(0,1]$,$\mu>1$,$\bar{\Lambda}(\tau)\coloneqq\int_l^\tau \lambda(s)s^{\alpha-1}ds$, and $\Psi(\tau)\coloneqq\int_\tau^\infty f(s)s^{\alpha-1}ds$, for any $\tau\in[l,\infty)$.

Remark 18. Clearly, for $\alpha=1$ and $l=1$, Remark 17 coincides with Remark 12 in [5].

Remark 19. When $\mathbb{T}=\mathbb{Z}$ (i.e., $\sigma(\tau)=\tau+1$), $\mu>1$, and $l=1$ in (42), we get

$$\begin{aligned}&\sum_{r=1}^\infty \lambda(r)\left(\sum_{s=r}^\infty f(s)(s+1)^{\alpha-1}\right)^{\mu-\alpha+1}(r+1)^{\alpha-1}\\ &\quad\le(\mu-\alpha+1)\left(\sum_{r=1}^\infty \lambda^{1-\mu}(r)\left(\sum_{k=1}^{r-1}\lambda(k)(k+1)^{\alpha-1}\right)^\mu f^\mu(r)(r+1)^{\alpha-1}\right)^{1/\mu}\\ &\quad\times\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{s=r}^\infty f(s)(s+1)^{\alpha-1}\right)^{\mu(\mu-\alpha)/(\mu-1)}(r+1)^{\alpha-1}\right)^{(\mu-1)/\mu}.\end{aligned} \tag{58}$$

If $\alpha=1$, then (58) becomes

$$\sum_{r=1}^\infty \lambda(r)\left(\sum_{s=r}^\infty f(s)\right)^\mu \le \mu\left(\sum_{r=1}^\infty \lambda^{1-\mu}(r)\left(\sum_{k=1}^r \lambda(k)\right)^\mu f^\mu(r)\right)^{1/\mu} \times\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{s=r}^\infty f(s)\right)^\mu\right)^{(\mu-1)/\mu}, \tag{59}$$

which is Remark 13 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (9) for fractional time scales as follows.

Theorem 20. *Suppose that $\mathbb{T}$ be a time scale and $\alpha\in(0,1]$. If $0<\mu\le 1$,$\Omega(\tau)=\int_\tau^\infty \lambda(s)\Delta_\alpha s$ and $F(\tau)=\int_l^\tau f(s)\Delta_\alpha s$, for any $\tau\in l,\infty)_{\mathbb{T}}$, then*

$$\left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^\mu \ge(\mu-\alpha+1)^\mu\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right) \times\left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}. \tag{60}$$

Proof. By applying (20) on

$$\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau, \tag{61}$$

with $\zeta^\sigma(\tau)=(\mathrm{F}^\sigma(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\nu^\Delta)(\tau)=\lambda(\tau)$, we have

$$\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau = \nu(\tau)F^{\mu-\alpha+1}(\tau)\Big|_\infty^l+\int_l^\infty(-\nu(\tau))D_\alpha\left(\left(F^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau, \tag{62}$$

where

$$\nu(\tau)=-\int_\tau^\infty \lambda(s)\Delta_\alpha s=-\Omega(\tau). \tag{63}$$

Substituting (63) into (62) yields

$$\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau = -\Omega(\tau)F^{\mu-\alpha+1}(\tau)\Big|_\infty^l+\int_l^\infty \Omega(\tau)D_\alpha\left(\left(F^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau. \tag{64}$$

Using the fact that $\nu(\infty)=0$ and $\mathrm{F}(l)=0$, (64) became

$$\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau=\int_l^\infty \Omega(\tau)D_\alpha\left(\left(F^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\Delta_\alpha\tau. \tag{65}$$

Utilizing chain rule (18), we get

$$\begin{aligned}&D_\alpha\left(\left(F^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\\ &\quad=(\mu-\alpha+1)\mathrm{F}^{\mu-\alpha}(d)D_\alpha\left(F^\Delta\right)(\tau)\\ &\quad\ge(\mu-\alpha+1)D_\alpha\left(F^\Delta\right)(\tau)(F^\sigma(\tau))^{\mu-\alpha}.\end{aligned} \tag{66}$$

Since $D_\alpha(F^\Delta)(\tau)=f(\tau)$, we obtain

$$D_\alpha\left(\left(F^\Delta\right)^{\mu-\alpha+1}\right)(\tau)\ge(\mu-\alpha+1)f(\tau)(F^\sigma(\tau))^{\mu-\alpha}. \tag{67}$$

By substituting (67) into (65), we have

$$\begin{aligned}&\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\\ &\quad\ge(\mu-\alpha+1)\int_l^\infty \Omega(\tau)f(\tau)(F^\sigma(\tau))^{\mu-\alpha}\Delta_\alpha\tau\\ &\quad=(\mu-\alpha+1)\int_l^\infty\left[\Omega^\mu(\tau)f^\mu(\tau)(F^\sigma(\tau))^{\mu(\mu-\alpha)}\right]^{1/\mu}\Delta_\alpha\tau.\end{aligned} \tag{68}$$

Raises (68) to the factor μ, we get

$$\begin{aligned}&\left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^\mu\\&\quad\geq(\mu-\alpha+1)^\mu\left(\int_l^\infty \lambda\left[\Omega^\mu(\tau)f^\mu(\tau)(F^\sigma(\tau))^{\mu(\mu-\alpha)}\right]^{1/\mu}\Delta_\alpha\tau\right)^\mu.\end{aligned}\tag{69}$$

By applying Hölder's inequality on

$$\left(\int_l^\infty\left[\Omega^\mu(\tau)f^\mu(\tau)(F^\sigma(\tau))^{\mu(\mu-\alpha)}\right]^{1/\mu}\Delta_\alpha\tau\right)^\mu,\tag{70}$$

with indices $1/\mu$, $1/(1-\mu)$, and

$$\begin{aligned}F(\tau)&=\frac{f^\mu(\tau)\Omega^\mu(\tau)}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}},\\G(\tau)&=\lambda^{1-\mu}(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)},\end{aligned}\tag{71}$$

we see that

$$\begin{aligned}&\left(\int_l^\infty F^{1/\mu}(\tau)\Delta_\alpha\tau\right)^\mu\\&\quad=\left(\int_l^\infty\left(\frac{f^\mu(\tau)\Omega^\mu(\tau)}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}}\right)^{1/\mu}\Delta_\alpha\tau\right)^\mu\\&\quad\geq\frac{\int_l^\infty F(\tau)G(\tau)\Delta_\alpha\tau}{\left(\int_l^\infty G^{1/(1-\mu)}(\tau)\Delta_\alpha\tau\right)^{1-\mu}}\\&\quad=\left(\int_l^\infty\frac{f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)})}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}}\Delta_\alpha\tau\right)\\&\qquad\times\left(\int_l^\infty\left(\lambda^{1-\mu}(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)}\right)^{1/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}\\&\quad=\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right)\\&\qquad\cdot\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}.\end{aligned}\tag{72}$$

This implies that

$$\begin{aligned}&\left(\int_l^\infty\Omega(\tau)f(\tau)(F^\sigma(\tau))^{\mu-\alpha}\Delta_\alpha\tau\right)^\mu\\&\quad\geq\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right)\\&\quad\times\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}.\end{aligned}\tag{73}$$

By substituting (73) into (69), we get

$$\begin{aligned}&\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^\mu\\&\quad\geq(\mu-\alpha+1)^\mu\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right)\\&\quad\times\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1},\end{aligned}\tag{74}$$

which is (60). □

Corollary 21. *At $\alpha=1$ in Theorem 20, then*

$$\begin{aligned}&\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^\mu\Delta\tau\right)^\mu\\&\quad\geq\mu^\mu\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta\tau\right)\\&\quad\times\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^\mu\Delta\tau\right)^{\mu-1},\end{aligned}\tag{75}$$

where $0<\mu\leq1$, $\Omega(\tau)=\int_\tau^\infty\lambda(s)\Delta s$, and $F(\tau)=\int_l^\tau f(s)\Delta s$, for any $\tau\in[l,\infty)$.

Remark 22. In Corollary 21, if we divide both sides of (75) by the factor

$$\left(\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^\mu(\tau)\Delta\tau\right)^{\mu-1},\tag{76}$$

then (75) can be written as

$$\int_l^\infty\lambda(\tau)(F^\sigma(\tau))^\mu\Delta\tau\geq\mu^\mu\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\Delta\tau\right),\tag{77}$$

which is (9) in Introduction.

Remark 23. As a result, if $\mathbb{T}=\mathbb{R}$ (i.e., $\sigma(\tau)=\tau$) in Theorem 20, then

$$\begin{aligned}&\left(\int_l^\infty\lambda(\tau)(F(\tau))^{\mu-\alpha+1}\tau^{\alpha-1}d\tau\right)^\mu\\&\quad\geq(\mu-\alpha+1)^\mu\left(\int_l^\infty f^\mu(\tau)\Omega^\mu(\tau)\lambda^{1-\mu}(\tau)\tau^{\alpha-1}d\tau\right)\\&\quad\times\left(\int_l^\infty\lambda(\tau)(F(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\tau^{\alpha-1}d\tau\right)^{\mu-1},\end{aligned}\tag{78}$$

where $\alpha\in(0,1]$, $0<\mu\leq1$, $\Omega(\tau)=\int_\tau^\infty\lambda(s)s^{\alpha-1}ds$, and $F(\tau)=\int_l^\tau f(s)s^{\alpha-1}ds$, for any $\tau\in[l,\infty)$.

Remark 24. Clearly, for $\alpha=1$ and $l=1$, Remark 23 coincides with Remark 16 in [5].

Remark 25. When $\mathbb{T}=\mathbb{Z}$ (i.e., $\sigma(\tau)=\tau+1$), $\mu\le 1$, and $l=1$ in (60), then we get

$$\left(\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)(s+1)^{\alpha-1}\right)^{\mu-\alpha+1}(r+1)^{\alpha-1}\right)^{\mu}$$
$$\ge(\mu-\alpha+1)^{\mu}\left(\sum_{r=1}^{\infty}f^{\mu}(r)\left(\sum_{s=r}^{\infty}\lambda(s)(s+1)^{\alpha-1}\right)^{\mu}\lambda^{1-\mu}(r)(r+1)^{\alpha-1}\right)$$
$$\times\left(\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)(s+1)^{\alpha-1}\right)^{(\mu-\alpha+1)/(1-\mu)}(r+1)^{\alpha-1}\right)^{\mu-1}. \quad (79)$$

If $\alpha=1$, then (79) becomes

$$\left(\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)\right)^{\mu}\right)^{\mu}\ge\mu^{\mu}\left(\sum_{r=1}^{\infty}f^{\mu}(r)\left(\sum_{s=r}^{\infty}\lambda(s)\right)^{\mu}\lambda^{1-\mu}(r)\right)$$
$$\times\left(\sum_{r=1}^{\infty}\lambda(r)\left(\sum_{s=1}^{\sigma(r)-1}f(s)\right)^{\mu}\right)^{\mu-1}, \quad (80)$$

which is Remark 17 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (10) for fractional time scales as follows.

Theorem 26. *Suppose that $\mathbb{T}$ be a time scale and $\alpha\in(0,1]$. If $0<\mu\le 1$, $\bar{\Omega}(\tau)=\int_l^{\tau}\lambda(s)\Delta_\alpha s$, and $\Gamma(\tau)=\int_{\tau}^{\infty}f(s)\Delta_\alpha s$, for any $\tau\in[l,\infty)_{\mathbb{T}}$, then*

$$\left(\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^{\mu}$$
$$\ge(\mu-\alpha+1)^{\mu}\left(\int_l^{\infty}f^{\mu}(\tau)\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right) \quad (81)$$
$$\times\left(\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu(\alpha-1)/1-\mu}\Delta_\alpha\tau\right)^{\mu-1}.$$

Proof. By applying (20) on

$$\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau, \quad (82)$$

with $\nu(\tau)=(\Gamma(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\zeta^{\Delta})(\tau)=\lambda(\tau)$, we have

$$\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau$$
$$=\zeta(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Big|_l^{\infty}+\int_l^{\infty}\zeta^{\sigma}(\tau)\left(-D_\alpha\left(\left(\Gamma^{\Delta}\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau, \quad (83)$$

where

$$\zeta(\tau)=\int_l^{\tau}\lambda(s)\Delta_\alpha s=\bar{\Omega}(\tau). \quad (84)$$

Substituting (84) into (83), we get

$$\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau$$
$$=\bar{\Omega}(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Big|_l^{\infty}+\int_l^{\infty}\bar{\Omega}^{\sigma}(\tau)\left(-D_\alpha\left(\left(\Gamma^{\Delta}\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau. \quad (85)$$

Using the fact that $\Gamma(\infty)=0$ and $\bar{\Omega}(l)=0$, (85) became

$$\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau=\int_l^{\infty}\bar{\Omega}^{\sigma}(\tau)\left(-D_\alpha\left(\left(\Gamma^{\Delta}\right)^{\mu-\alpha+1}\right)(\tau)\right)\Delta_\alpha\tau. \quad (86)$$

Utilizing chain rule (18), we have

$$\begin{aligned}&-D_\alpha\left(\left(\Gamma^{\Delta}\right)^{\mu-\alpha+1}\right)(\tau)\\&=-(\mu-\alpha+1)\Gamma^{\mu-\alpha}(d)D_\alpha\left(\Gamma^{\Delta}\right)(\tau)\\&\ge-(\mu-\alpha+1)\Gamma^{\mu-\alpha}(\tau)D_\alpha\left(\Gamma^{\Delta}\right)(\tau).\end{aligned} \quad (87)$$

Since $D_\alpha(\Gamma^{\Delta})(\tau)=-f(\tau)$, we get

$$-D_\alpha\left(\left(\Gamma^{\Delta}\right)^{\mu-\alpha+1}\right)(\tau)\ge(\mu-\alpha+1)f(\tau)\Gamma^{\mu-\alpha}(\tau). \quad (88)$$

By substituting (88) into (86), we get

$$\begin{aligned}&\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\\&\ge(\mu-\alpha+1)\int_l^{\infty}\bar{\Omega}^{\sigma}(\tau)f(\tau)\Gamma^{\mu-\alpha}(\tau)\Delta_\alpha\tau,\\&=\int_l^{\infty}\left(\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}f^{\mu}(\tau)\Gamma^{\mu(\mu-\alpha)}(\tau)\right)^{1/\mu}\Delta_\alpha\tau.\end{aligned} \quad (89)$$

Raises (89) to the factor μ, we have

$$\left(\int_l^{\infty}\lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^{\mu}$$
$$\ge(\mu-\alpha+1)^{\mu}\left(\int_l^{\infty}\left(\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}f^{\mu}(\tau)\Gamma^{\mu(\mu-\alpha)}(\tau)\right)^{1/\mu}\Delta_\alpha\tau\right)^{\mu}. \quad (90)$$

By applying Hölder's inequality on

$$\left(\int_l^{\infty}\left(\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}f^{\mu}(\tau)\Gamma^{\mu(\mu-\alpha)}(\tau)\right)^{1/\mu}\Delta_\alpha\tau\right)^{\mu}, \quad (91)$$

with indices $1/\mu, 1/(1-\mu)$, and

$$\begin{aligned}F(\tau)&=\frac{f^{\mu}(\tau)\left(\bar{\Omega}^{\sigma}(\tau)\right)^{\mu}}{\Gamma^{\mu(\alpha-\mu)}(\tau)},\\G(\tau)&=\lambda^{1-\mu}(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)},\end{aligned} \quad (92)$$

we see that

$$\begin{aligned}
&\left(\int_l^\infty F^{1/\mu}(\tau)\Delta_\alpha\tau\right)^\mu \\
&= \left(\int_l^\infty \left(\frac{f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu}{\Gamma^{\mu(\alpha-\mu)}(\tau)}\right)^{1/\mu}\Delta_\alpha\tau\right)^\mu \\
&\geq \frac{\int_l^\infty F(\tau)G(\tau)\Delta_\alpha\tau}{\left(\int_l^\infty G^{1/1-\mu}(\tau)\Delta_\alpha\tau\right)^{1-\mu}} \\
&= \left(\int_l^\infty \frac{f^\mu(\tau)\lambda^{1-\mu}(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu(\Gamma(\tau))^{\mu(\alpha-\mu)}}{\Gamma^{\mu(\alpha-\mu)}(\tau)}\Delta_\alpha\tau\right) \\
&\quad\times\left(\int_l^\infty \left(\lambda^{1-\mu}(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)}\right)^{1/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1} \\
&= \left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right) \\
&\quad\times\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}.
\end{aligned} \tag{93}$$

This implies that

$$\begin{aligned}
\left(\int_l^\infty \left(\left(\bar{\Omega}^\sigma(\tau)\right)^\mu f^\mu(\tau)\Gamma^{\mu(\mu-\alpha)}(\tau)\right)^{1/\mu}\Delta_\alpha\tau\right)^\mu &\geq \left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right) \\
&\quad\times\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1}.
\end{aligned} \tag{94}$$

By substituting (94) into (90), we get

$$\begin{aligned}
&\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\Delta_\alpha\tau\right)^\mu \\
&\quad\geq(\mu-\alpha+1)^\mu\left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\Delta_\alpha\tau\right) \\
&\quad\times\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)}\Delta_\alpha\tau\right)^{\mu-1},
\end{aligned} \tag{95}$$

which is (81). □

Corollary 27. *At $\alpha = 1$ in Theorem 26, then*

$$\begin{aligned}
&\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^\mu\Delta\tau\right)^\mu \\
&\quad\geq\mu^\mu\left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\Delta\tau\right) \\
&\quad\times\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^\mu\Delta\tau\right)^{\mu-1},
\end{aligned} \tag{96}$$

where $0<\mu\leq 1$, $\bar{\Omega}(\tau) = \int_l^\tau \lambda(s)\Delta s$, and $\Gamma(\tau) = \int_\tau^\infty f(s)\Delta s$, for any $\tau\in[l,\infty)_{\mathbb{T}}$.

Remark 28. In Corollary 27, if we divide both sides of (96) by the factor

$$\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^\mu\Delta\tau\right)^{\mu-1}, \tag{97}$$

then (96) can be written as

$$\int_l^\infty \lambda(\tau)(\Gamma(\tau))^\mu\Delta\tau \geq \mu^\mu\left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}^\sigma(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\Delta\tau\right), \tag{98}$$

which is (10) in Introduction.

Remark 29. As a result, if $\mathbb{T}=\mathbb{R}$ (i.e., $\sigma(\tau)=\tau$) in Theorem 26, then

$$\begin{aligned}
&\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu-\alpha+1}\tau^{\alpha-1}d\tau\right)^\mu \\
&\quad\geq(\mu-\alpha+1)^\mu\left(\int_l^\infty f^\mu(\tau)\left(\bar{\Omega}(\tau)\right)^\mu\lambda^{1-\mu}(\tau)\tau^{\alpha-1}d\tau\right) \\
&\quad\times\left(\int_l^\infty \lambda(\tau)(\Gamma(\tau))^{\mu(\alpha-\mu)/1-\mu}\tau^{\alpha-1}d\tau\right)^{\mu-1},
\end{aligned} \tag{99}$$

where $\alpha\in(0,1]$, $0<\mu\leq 1$, $\bar{\Omega}(\tau)=\int_l^\tau \lambda(s)s^{\alpha-1}ds$, and $\Gamma(\tau)=\int_\tau^\infty f(s)s^{\alpha-1}ds$, for any $\tau\in[l,\infty)$.

Remark 30. Clearly, for $\alpha=1$ and $l=1$, Remark 29 coincides with Remark 18 in [5].

Remark 31. When $\mathbb{T}=\mathbb{Z}$ (i.e., $\sigma(\tau)=\tau+1$), $\mu\leq 1$, and $l=1$ in (81), then we get

$$\begin{aligned}
&\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{k=r}^\infty f(k)(k+1)^{\alpha-1}\right)^{\mu-\alpha+1}(r+1)^{\alpha-1}\right)^\mu \\
&\quad\geq(\mu-\alpha+1)^\mu\left(\sum_{r=1}^\infty \lambda^{1-\mu}(r)\left(\sum_{k=1}^{r-1}\lambda(k)(k+1)^{\alpha-1}\right)^\mu f^\mu(r)(r+1)^{\alpha-1}\right) \\
&\quad\times\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{k=r}^\infty f(k)(k+1)^{\alpha-1}\right)^{\mu(\alpha-\mu)/1-\mu}(r+1)^{\alpha-1}\right)^{\mu-1}.
\end{aligned} \tag{100}$$

If $\alpha=1$, then (100) becomes

$$\begin{aligned}
\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{k=r}^\infty f(k)\right)^\mu\right)^\mu &\geq(\mu)^\mu\left(\sum_{r=1}^\infty \lambda^{1-\mu}(r)\left(\sum_{k=1}^{r-1}\lambda(k)\right)^\mu f^\mu(r)\right) \\
&\quad\times\left(\sum_{r=1}^\infty \lambda(r)\left(\sum_{k=r}^\infty f(k)\right)^\mu\right)^{\mu-1},
\end{aligned} \tag{101}$$

which is Remark 19 in [5].

4. Conclusions and Future Work

In this article, we explore new generalizations of the integral Hardy-Leindler-type inequalities by the utilization of the delta conformable calculus on time scales which are used in various problems involving symmetry. We generalize a number of those inequalities to a general time scale measure space. In addition to this, in order to obtain some new inequalities as special cases, we also extend our inequalities to a discrete and continuous calculus. In future work, we will continue to generalize more fractional dynamic inequalities by using Specht's ratio, Kantorovich's ratio, and n-tuple fractional integral.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

General Decay of a Nonlinear Viscoelastic Wave Equation with Balakrishnân-Taylor Damping and a Delay Involving Variable Exponents

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This paper was aimed at investigating the stability of the following viscoelastic problem with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term $u_{tt} - \mathscr{M}(\|\nabla u\|_2^2)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)ds + \mu_1 |u_t|^{p(.)-2}u_t + \mu_2 |u_t(t-\tau)|^{p(.)-2}u_t(t-\tau) = 0$ in $\Omega \times \mathbb{R}^+$, where Ω is a bounded domain of $\mathbb{R}^n$, $p(.)$: $\bar{\Omega} \longrightarrow \mathbb{R}$ is a measurable function, $g > 0$ is a memory kernel that decays exponentially, $\alpha \geq 0$ is the potential, and $\mathscr{M}(\|\nabla u\|_2^2) = a + b\|\nabla u(t)\|_2^2 + \sigma\int_\Omega \nabla u \nabla u_t dx$ for some constants $a > 0$, $b \geq 0$, and $\sigma > 0$. Under some assumptions on the relaxation function, we use some suitable Lyapunov functionals to derive the general decay estimate for the energy. The problem considered is novel and meaningful because of the presence of the flutter panel equation and the spillover problem including memory and variable-exponent time delay control. Our result generalizes and improves previous conclusion in the literature.

1. Introduction

In recent years, much attention has been paid to the study systems with variable exponents of nonlinearities which are models of hyperbolic, parabolic, and elliptic equations. These models may be nonlinear over the gradient of unknown solutions and have nonlinear variable exponents. Researches of these systems usually use the imbedding of Lebesgue and Sobolev spaces with variable exponents (see, e.g., [1, 2]). Or see [3–14] and the references therein for more details of relevant problems.

In this paper, we concentrate on the asymptotic behavior of weak solutions for the following weakly damped viscoelastic wave equation with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term

$$\begin{cases} u_{tt} - \mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)ds + \mu_1|u_t|^{p(x)-2}u_t + \mu_2|u_t(t-\tau)|^{p(x)-2}u_t(t-\tau) = 0, & \text{in } \Omega\times(0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,t) = j_0(x,t-\tau), & \text{in } \Omega\times(0,\tau), \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1}$$

where $u : \bar{\Omega} \times [0,\infty) \longrightarrow \mathbb{R}$ is unknown function, $\mu_1 \geq 0$,μ_2 is a real number, $\tau > 0$ is the time delay, $g > 0$ is a memory kernel, and $\alpha \geq 0$ is the potential.

Much attention has been paid to the simulation of phenomena such as the vibration of elastic strings and elastic plates, when $g = 0$, and $\mu_1 = \mu_2 = 0$; equation (1)) degrades into the Kirchhoff's original equation

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, 0 \leq x \leq L, t \geq 0, \tag{2}$$

which was first introduced to study the oscillations of stretched strings and plates in [15]. In addition, equation (2) is also said to be the wave equation of Kirchhoff type, where the unknown function $u = u(x, t)$ represents lateral deflection and E, ρ, h, L, p_0, and f, respectively, denote Young's modulus, mass density, cross-section area, length, initial axial tension, and external force. The Kirchhoff equation has been investigated in a lot of articles due to its abundant physical background. At the present paper, we try to mention some considerable efforts on this topic.

There are many important results, such as the local solutions in time, well-posedness, and solvability; for the Kirchhoff type, equation (2) in general dimensions and domains has been obtained in lots of articles (see, e.g., [16–24] and the references therein).

When $p > 1$ identically equals to a constant, problem (1) with the Balakrishnân-Taylor damping term $(\sigma > 0)$ is related to the flutter panel equation and the spillover problem involving time delay term. Balakrishnân and Taylor in [25] and Bass and Zes in [26] introduced Balakrishnân-Taylor damping, which arises from a wind tunnel experiment at supersonic speeds (see, e.g., [22, 27–32]).

On damping terms, we point out several excellent works: Lian and Xu in [33] studied a class of nonlinear wave equations with weak and strong damping terms, and they established the existence of weak solutions and related blow-up results under three different initial energy levels and different conditions. Yang et al. [34] investigated the exponential stability of a system with locally distributed damping. Lian et al. [35] were interested in a fourth-order wave equation with strong and weak damping terms; they obtained the local solution, the global existence, asymptotic behavior, and blow-up of solutions under some condition.

Time delays are common phenomena in many physical, chemical, biological, thermal, and so on (see [36–38] for more details). Several authors have investigated existence and stability of the solutions to the viscoelastic wave equation involving delay term under some appropriate conditions on μ_1, μ_2, and g (see, e.g., [39]). For other related problems, one can also refer to [40–44]. The terminology variable exponents mean that $p(.)$ is a measurable function and not a constant. This term $\mu_1 |u_t|^{p(.)-2} u_t + \mu_2 |u_t(t-\tau)|^{p(.)-2} u_t(t-\tau)$ is a generalization of $\mu_1 u_t + \mu_2 u_t(t-\tau)$, which corresponds to $p(.) > 1$. In fact, (1) is also an extension of the second-order viscoelastic wave equation under variable growth conditions

$$\begin{aligned} &u_{tt} - \mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u \\ &\quad + \alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s + \mu_1 u_t + \mu_2 u_t(t-\tau) \\ &\quad = 0 \text{ in } \Omega \times \mathbb{R}^+, \end{aligned} \tag{3}$$

which is obtained when considering $\mu_1 |u_t|^{p(.)-2} u_t + \mu_2 |u_t(t-\tau)|^{p(.)-2} u_t(t-\tau)$. Equation (3) is a well-known electrorheological fluid model that appears in fluid dynamic treatment (see in [45]). However, the researches related to the viscoelastic wave equation possessing delay terms, Balakrishnân-Taylor damping, and variable growth conditions are not sufficient, and the results about these equations are relatively rare (see [46]). In particular, in [40], the authors considered this class of equations under some suitable assumptions; they use suitable Lyapunov functionals to derive general energy decay results, and one see similar work in [44]. Mingione and Rădulescu [47] were concerned with the regularity theory of elliptic variational problems under nonstandard growth conditions.

This paper devotes to generalize some previous results. In particular, in this case, we will use the relaxation function, the specified initial data, and a special Lyapunov functional, which depends on the behavior of the relation function and is not necessary to decay in some polynomial or exponential form, to get a general decay estimate of the energy.

In addition to the introduction, this paper is divided into two parts. In Section 2, we review some basic definitions about Lebesgue and Sobolev spaces with variable exponentials and give some related properties. At the end of this section, we present our main results. In Section 3, we prove our results, showing that a solution of (1) possesses a general decay with small initial values (u_0, u_1).

2. Functional Setting and Main Results

In this section, we will give some preliminaries and our main results.

Without loss of generality, hereinafter, we suppose $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary Γ. Moreover, let $p : \bar{\Omega} \longrightarrow (1,+\infty)$ be a measurable function and denote

$$\begin{aligned} p^- &:= \operatorname*{essinf}_{x\in\Omega} [p(x)], \\ p^+ &:= \operatorname*{esssup}_{x\in\Omega} [p(x)]. \end{aligned} \tag{4}$$

As in [1, 48, 49], we define the following variable-exponent Lebesgue spaces and Sobolev spaces. The first one is the variable-exponent space $L^{p(.)}(\Omega)$:

$$L^{p(.)}(\Omega) = \left\{ \psi : \Omega \longrightarrow \mathbb{R} \text{ measurable} | \varrho_{p(.),\Omega}(\psi) := \int_\Omega |\psi(x)|^{p(x)} \mathrm{d}x < +\infty \right\}, \tag{5}$$

and it is obvious a Banach space with the following

Luxemburg norm

$$\|\psi\|_{p(.),\Omega} := \inf\left\{\nu > 0 | \int_\Omega \left|\frac{u(x)}{\nu}\right|^{p(x)} dx \le 1\right\}. \quad (6)$$

Actually, in many respects, variable-exponent Lebesgue spaces are very similar to classical Lebesgue spaces (see [49]). In particular, from the above definition of the norm, we can directly get the following results:

$$\min\left(\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right) \le \varrho_{p(.),\Omega}(u) \le \max\left(\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right). \quad (7)$$

For any measurable function $p : \bar{\Omega} \longrightarrow [p^-, p^+] \subset (2,\infty)$, where $p^\pm$ are constants, we define the second space and the variable-exponent Lebesgue space

$$L^{p(.)}(\Omega) = \left\{\phi : \Omega \longrightarrow \mathbb{R} : \phi \text{ is measurable function on } \Omega, \int_\Omega |\phi(x)|^{p(x)} dx < \infty\right\}, \quad (8)$$

which is a Banach space with the following Luxemburg norm:

$$\|u\|_{p(.)} = \inf\left\{\nu > 0, \int_\Omega \left|\frac{u}{\nu}\right|^{p(x)} dx \le 1\right\}. \quad (9)$$

We also assume that p satisfies the following Zhikov-Fan condition for the local uniform continuity: there exist a constant $M > 0$ such that for all points x,y in Ω with $|x-y| < 1/2$, we have the inequality

$$|p(x) - p(y)| \le \frac{M}{|\log|x-y||}. \quad (10)$$

In addition, $\|.\|_q$ and $\|.\|_{H^1(\Omega)}$ denote the usual $L^q(\Omega)$ norm and $H^1(\Omega)$ norm.

In order to obtain the main results, we give the following lemma firstly.

Lemma 1 (see [1]).

(1) If

$$2 \le p^- := ess\inf_{x\in\Omega} p(x) \le p(x) \le p^+ := ess\sup_{x\in\Omega} p(x) < \infty, \quad (11)$$

then

$$\min\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \le \int_\Omega |u(.)|^{p(x)} dx \le \max\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \quad (12)$$

for any $u \in L^{p(.)}(\Omega)$

(2) Assume that $m, n, p : \bar{\Omega} \longrightarrow (1,+\infty)$ *are measurable functions satisfying*

$$\frac{1}{m(.)} = \frac{1}{p(.)} + \frac{1}{n(.)} \quad (13)$$

Then, for all functions $u \in L^{p(.)}(\Omega)$ *and* $v \in L^{n(.)}(\Omega)$, *we have* $uv \in L^{m(.)}(\Omega)$ *with*

$$\|uv\|_{m(.)} \le \mathscr{C}\|u\|_{p(.)}\|v\|_{n(.)}. \quad (14)$$

Lemma 2. *Suppose that* $p : \Omega \longrightarrow [p^-, p^+] \subset [1,+\infty)$ *is a measurable function satisfying*

$$ess\sup_{x\in\Omega} p(x) < p_* \le \frac{2n}{n-2} \text{ with } p_* = \frac{np(x)}{ess\sup_{x\in\Omega}(n-p(x))}. \quad (15)$$

Then, the embedding $H_0^1(\Omega) = W_0^{1,2}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact, and there is a constant $c_* = c_*(\Omega, p^\pm)$ such that

$$\|\phi\|_{p(.)} \le c_* \|\nabla\phi\|_2 \text{ for } \phi \in H_0^1(\Omega). \quad (16)$$

We assume that the relaxation function g and the potential α satisfy the following assumptions:

Hypothesis g, α: $g, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are nonincreasing differentiable functions such that

$$g(s) \ge 0, l_0 = \int_0^\infty g(s) ds < \infty, \alpha(t) > 0, a - \alpha(t)\int_0^t g(s) ds \ge l > 0 \quad (17)$$

Hypothesis ξ: there exist a positive differentiable functions ξ satisfying

$$g'(t) \le -\xi(t) g(t), \text{ for } t \ge 0, \lim_{t\longrightarrow\infty} \frac{-\alpha'(t)}{\xi(t)\alpha(t)} = 0 \quad (18)$$

Hypothesis $p(.)$: the function $p(.)$ satisfies

$$p^- \ge 2, \text{ if } n = 1, 2, 2 < p^- \le p(x) \le p^+ < \frac{n+2}{n-2} \text{ if } n \ge 3 \quad (19)$$

Hypothesis μ_1 and μ_2: the constants μ_1 and μ_2 satisfy

$$|\mu_2| < p^- \mu_1 \quad (20)$$

Calculating $(d/dt)\alpha(t)(g \circ u)(t)$ with respect to t, it shows that

$$
\begin{aligned}
&\alpha(t)\int_0^t g(t-s)\int_\Omega u(s)\mathrm{d}s u_t(t)\mathrm{d}x = \\
&-\frac{\alpha(t)}{2}g(t)\|u(t)\|_2^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\alpha(t)}{2}(g\circ u)(t) - \frac{\alpha(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s\right] \\
&+\frac{\alpha(t)}{2}\left(g'\circ u\right)(t) + \frac{\alpha'(t)}{2}(g\circ u)(t) - \frac{\alpha'(t)}{2}\|u(t)\|_2^2\int_0^t g(s)\mathrm{d}s,
\end{aligned}
\tag{21}
$$

where

$$
(g\circ u)(t) = \int_0^t g(t-s)\|u(t)-u(s)\|_2^2\mathrm{d}s. \tag{22}
$$

As in [38, 43], we present a new time-dependent variable to deal with the time delay term:

$$
z(x,\rho,t) = u_t(x, t-\tau\rho), x\in\Omega, \rho\in(0,1), t>0. \tag{23}
$$

Consequently, we have

$$
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \text{in } \Omega\times(0,1)\times(0,\infty). \tag{24}
$$

Therefore, problem (1) can be transformed into

$$
\begin{gathered}
u_{tt} - \mathscr{M}\left(\|\nabla u\|_2^2\right)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s + \mu_1|u_t|^{p(x)-2}u_t + \mu_2|z(1,t)|^{p(x)-2}z(1,t) = 0, \text{in } \Omega\times\mathbb{R}^+, \\
\tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \text{in } (0,1)\times(0,\infty), \\
z(0,t) = u_t, \text{in } (0,+\infty), \\
z(\rho,0) = j_0(-\rho(\tau+1)), \text{in } (0,1), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), x\in\Omega.
\end{gathered}
\tag{25}
$$

By the standard methods as in Section 3 of [50], we can easily prove the well-posedness of problem (1) presented as follows.

Theorem 3. *Let (17)–(20) be in force and $(u_0, u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $j_0\in L^2((\Omega)\times(0,1))$. Then, problem (1) possesses a unique local solution u such that*

$$
u\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right), u_t\in C\left([0,T];H_0^1(\Omega)\right)\cap L^2([0,T]\times(\Omega)). \tag{26}
$$

3. Main Asymptotic Theorem

Next, we will give the proof of Theorem 4.

The functional E of problem (25) is as follows:

$$
\begin{aligned}
E(t) = &\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(a - \alpha(t)\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\
&+ \frac{b}{4}\|\nabla u\|_2^4 + \xi\int_\Omega\frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x \\
&+ \frac{1}{2}\alpha(t)(g\circ\nabla(u))(t),
\end{aligned}
\tag{27}
$$

where ξ and λ are positive constants and they satisfy

$$
\mu_1 p^- - |\mu_2| > \xi > |\mu_2|p^+\frac{p^+-1}{p^-}, \lambda < \frac{1}{\tau}\left|\ln\frac{\mu_2 p^+(p^+-1)}{\xi p^-}\right|. \tag{28}
$$

The most important key to solve problem (1) is to obtain a result that concerns the asymptotic stability of solutions.

The main result is as follows.

Theorem 4. *Suppose (17)–(20) and (28) hold. Then, there exists positive constants C_0, C, and $t_1 > 0$ such that*

$$
E(t) \le C_0 e^{-C\int_{t_1}^t \nu}, \textit{for } t\ge t_1. \tag{29}
$$

To prove this theorem, the following technical lemmas are necessary.

Lemma 5. *If u is a solution of problem (25). Then,*

$$
\begin{aligned}
E'(t) \le & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{1}{2}\alpha(t)\left(g'\circ\nabla u\right)(t) \\
& - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 + \frac{1}{2}\alpha'(t)(g\circ\nabla u)(t) \\
& - \left(\mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}dx \\
& - \left(\frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}dx \\
& - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}dsdx.
\end{aligned} \tag{30}
$$

Proof. Using the same idea as in [50], multiply the first equation in (25) by u_t and then integrate in Ω. Similarly, multiply the second equation in (25) by $\xi z e^{-\lambda\tau\rho}$ and integrate in $(0,1)\times\Omega$. Summarizing the above, we can obtain

$$
\begin{aligned}
E'(t) = & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t) \\
& - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s - \frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2 \\
& + \frac{\alpha'(t)}{2}(g\circ\nabla u)(t) - \mu_1\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\
& - \xi\int_\Omega \frac{1}{p(x)}e^{-\lambda\tau}|u_t(x,t-\tau)|^{p(x)}\mathrm{d}x \\
& - \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x \\
& + \xi\int_\Omega \boxtimes\frac{1}{p(x)}|u_t(x,t)|^{p(x)}\mathrm{d}x \\
& - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{31}
$$

By $z(1,t) = u_t(t-\tau)$ and the Young inequality, we get

$$
\begin{aligned}
& -\mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u_t\mathrm{d}x \\
& \quad \le |\mu_2|\frac{p^+-1}{p^-}\int_\Omega |z(1,t)|^{p(x)}dx + \frac{|\mu_2|}{p^-}\int_\Omega |u_t|^{p(x)}\mathrm{d}x.
\end{aligned} \tag{32}
$$

From (23), we have

$$
\begin{aligned}
& -\xi\int_\Omega \frac{1}{p(x)}e^{-\lambda\tau}|u_t(x,t-\tau)|^{p(x)}\mathrm{d}x \\
& \quad \le -\frac{\xi}{p^+}e^{-\lambda\tau}\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x.
\end{aligned} \tag{33}
$$

Comparing (31) and (32), we obtain

$$
\begin{aligned}
E'(t) \le & -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{\alpha(t)}{2}\left(g'\circ\nabla u\right)(t) \\
& - \frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s - \frac{\alpha(t)}{2}g(t)\|\nabla u\|_2^2 \\
& + \frac{\alpha'(t)}{2}(g\circ\nabla u)(t) - \left(\mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}\right)\int_\Omega |u_t|^{p(x)}\mathrm{d}x \\
& - \left(\frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-}\right)\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x \\
& - \lambda\xi\int_\Omega \frac{1}{p(x)}\int_{t-\tau}^t e^{\lambda(s-t)}|u_t(x,s)|^{p(x)}\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{34}
$$

Setting

$$
\begin{aligned}
c_0 &= \mu_1 - \frac{\xi}{p^-} - \frac{|\mu_2|}{p^-}, \\
c_1 &= \frac{\xi}{p^+}e^{-\lambda\tau} - |\mu_2|\frac{p^+-1}{p^-},
\end{aligned} \tag{35}
$$

by condition (28), we derived the desired inequality (30). □

Remark 6. If

$$
-\frac{1}{2}\alpha'(t)\|\nabla u(t)\|_2^2\int_0^t g(s)\mathrm{d}s \ge 0 \tag{36}
$$

holds, $E(t)$ may not be nonincreasing.

Lemma 7. *Assume that u be a solution of problem (25). Then,*

$$
\|\nabla u\|_2^2 \le \frac{2E(0)}{l}e^{(l_0/l)\alpha(0)}, t \ge 0, \tag{37}
$$

where l_0 and l as in (17).

Proof. From (27) and (30), we have

$$
E'(t) \le -\frac{1}{2}\alpha'(t)\|\nabla u\|_2^2\int_0^t g(s)ds \le -\frac{1}{2}l_0\alpha'(t)\|\nabla u\|_2^2 \le -\frac{l_0}{l}\alpha'(t)E(t). \tag{38}
$$

Integrating the above inequality in $(0,t)$, we get

$$
E(t) \le E(0)e^{-(l_0/l)\alpha(t)+(l_0/l)\alpha(0)} \le E(0)e^{(l_0/l)\alpha(0)}. \tag{39}
$$

From (27), we see that

$$
\|\nabla u\|_2^2 \le \frac{2}{l}E(t). \tag{40}
$$

Combining it with (39), it gives (37).

Now, we give a modified functional:

$$\mathrm{L}(t) = NE(t) + \varepsilon_1 \alpha(t)\varphi(t) + \varepsilon_2 \alpha(t)\psi(t), \tag{41}$$

$$\varphi(t) = \int_\Omega u(t)u_t(t)\mathrm{d}x + \frac{\sigma}{4}\|\nabla u\|_2^4, \tag{42}$$

$$\psi(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x, \tag{43}$$

where ε_1, ε_2, and N are positive constants. In fact, L is equivalent to E by the following lemma. □

Lemma 8. *There exists $C_1, C_2 > 0$ such that*

$$C_1 E(t) \le L(t) \le C_2 E(t), t \ge 0. \tag{44}$$

Proof. By the Poincaré theorem and Young inequality, we have the following results through integrating by parts:

$$\begin{aligned} |\mathrm{L}(t) - NE(t)| &= \left|\varepsilon_1\alpha(t)\int_\Omega u(t)u_t(t)\mathrm{d}x + \varepsilon_1\alpha(t)\frac{\sigma}{4}\|\nabla u\|_2^4 + \varepsilon_2\alpha(t)\psi(t)\right| \\ &\le \varepsilon_1|\alpha(t)|\int_\Omega |u(t)||u_t(t)|\mathrm{d}x + \varepsilon_1\frac{\sigma}{4}|\alpha(t)|\|\nabla u\|_2^4 + \varepsilon_2\frac{1}{2}|\alpha(t)|\|u_t\|_2^2 \\ &+ \varepsilon_2\frac{1}{2}|\alpha(t)|c_*^2(a-l)(g\circ\nabla(u))(t) \le \varepsilon_1\frac{\alpha(0)}{2}c_*^2\|\nabla u\|_2^2 + \varepsilon_1\frac{\alpha(0)}{2}\|u_t\|_2^2 \\ &+ \varepsilon_1\sigma\frac{\alpha(0)}{4}\|\nabla u\|_2^4 + \varepsilon_2\frac{1}{2}\alpha(0)\|u_t\|_2^2 + \varepsilon_2\frac{1}{2}\alpha(0)c_*^2(a-l)(g\circ\nabla(u))(t) \\ &\le C(\varepsilon_1+\varepsilon_2)E(t), \end{aligned} \tag{45}$$

where c_* as in Lemma 1, taking $C_1 = N - C(\varepsilon_1 + \varepsilon_2)$ and $C_2 = N + C(\varepsilon_1 + \varepsilon_2)$, provided ε_1 and ε_2 are sufficiently small, and the proof is completed. □

Lemma 9. *There exists $c_\varepsilon, C_\varepsilon > 0$ fulfilling*

$$\begin{aligned} \varphi'(t) \le \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\frac{a}{2l}(g\circ\nabla u)(t) \\ + c_\varepsilon\left(\int_\Omega |u_t|^{p(x)}dx + \int_\Omega |z(1,t)|^{p(x)}dx\right) \\ + C_\varepsilon\int_\Omega |u|^{p(x)}dx. \end{aligned} \tag{46}$$

Proof. By the first equation of (25), we differentiate (42), and then we have

$$\begin{aligned} \varphi'(t) &= \|u_t\|_2^2 + \int_\Omega v + \sigma\|\nabla u\|_2^2\int_\Omega \nabla u\nabla u_t\mathrm{d}x \\ &= \|u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u(t)\mathrm{d}x \\ &- \mu_1\int_\Omega |u_t|^{p(x)-2}u_t u\mathrm{d}x - \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)u\mathrm{d}x = \|u_t\|_2^2 \\ &- a\|\nabla u\|_2^2 - b\mathrm{v}_2^4 + I_1 + I_2 + I_3. \end{aligned} \tag{47}$$

By the Hölder inequality, Sobolev-Poincaré inequalities, and (17), we estimate the second part of the right-hand side in (47).

$$\begin{aligned} I_1 &= \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u(t)\mathrm{d}x \\ &\le \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega\left|\int_0^t ⊠g(t-s)\nabla u(s)\mathrm{d}s\right|^2\mathrm{d}x\right)^{1/2} \\ &\le \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega\int_0^t g(s)\mathrm{d}s\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{1/2} \\ &\le \alpha(t)\left(\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s\right)^{1/2}\left(\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x\right)^{1/2\,2}\mathrm{d}s\mathrm{d}x. \\ &\le \frac{\alpha(t)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s + \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\mathrm{d}s\mathrm{d}x \\ &\le \frac{\alpha(t)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x\int_0^t g(s)\mathrm{d}s + \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s) \\ &-\nabla u(t)+\nabla u(t)| \end{aligned} \tag{48}$$

For every $\eta > 0$, using the Young inequality and (17), we deduce

$$\begin{aligned} &\frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)[\nabla u(s)-\nabla u(t)+\nabla u(t)]^2\mathrm{d}s\mathrm{d}x \\ &\le \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)\left((\nabla u(s)-\nabla u(t))^2 + 2|\nabla u(s)-\nabla u(t)||\nabla u| + |\nabla u|^2\right) \\ &\mathrm{d}s\mathrm{d}x \le \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\mathrm{d}s\mathrm{d}x \\ &+ \frac{\alpha(t)}{2}\int_\Omega\int_0^t g(t-s)|\nabla u|^2\mathrm{d}s\mathrm{d}x \\ &+ \alpha(t)\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)||\nabla u|\mathrm{d}s\mathrm{d}x \\ &\le \frac{\alpha(t)}{2}(g\circ\nabla u)(t) + \frac{\alpha(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x \\ &+ \eta\frac{\alpha(t)}{2}\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2\eta}(g\circ\nabla u)(t) \\ &\le \frac{\alpha(t)}{2}(1+\eta)\int_0^t g(s)\mathrm{d}s\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t) \\ &\le (1+\eta)\frac{(a-l)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t). \end{aligned} \tag{49}$$

Summarizing the above estimates, (48) and (49), we obtain

$$\begin{aligned} \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x &\le \frac{(a-l)}{2}\int_\Omega |\nabla u|^2\mathrm{d}x \\ &+ \frac{(a-l)}{2}(1+\eta)\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t) \\ &= (2+\eta)\frac{(a-l)}{2}\|\nabla u\|^2 + \frac{\alpha(t)}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla u)(t). \end{aligned} \tag{50}$$

Setting $\eta = l/(a-l)$, it is easy to obtain

$$|I_1| \le \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\nabla u\mathrm{d}x \le \left(a-\frac{l}{2}\right)\|\nabla u\|_2^2 + \frac{a}{2l}\alpha(t)(g\circ\nabla u)(t), \tag{51}$$

and by means of the Young inequality, we have

$$|I_2| \le c_\varepsilon\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \varepsilon \max\left(\mu_1^{p^-},\mu_1^{p^+}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x := c_\varepsilon\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \varepsilon c_2\int_\Omega |u|^{p(x)}\mathrm{d}x, \tag{52}$$

$$|I_3| \le c_\varepsilon\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x + \varepsilon \max\left(\mu_1^{p^-},\mu_1^{p^+}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x := c_\varepsilon\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x + \varepsilon c_3\int_\Omega |u|^{p(x)}\mathrm{d}x. \tag{53}$$

Substituting (51)–(53) into (47), we deduce

$$\varphi'(t) \le \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 + C_\varepsilon\int_\Omega |u|^{p(x)}\mathrm{d}x - b\|\nabla u\|_2^4 + \frac{a}{2l}\alpha(t)(g\circ\nabla u)(t) + c_\varepsilon\left(\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x\right), \tag{54}$$

set $C_\varepsilon = \varepsilon(c_2+c_3) > 0$, for ε sufficiently small. □

Lemma 10. *There exists positive constants δ and c_δ satisfying*

$$\psi'(t) \le -\left\{\left(\int_0^t g(s)ds\right\} - \delta\right)\|u_t\|_2^2 + \delta\left\{a + 2(a-l)^2\alpha(t)\right\}\|\nabla u\|_2^2 + \delta b\|\nabla u\|_2^4 + \delta\frac{2\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right)(a-l)\alpha(t)\right\}(g\circ\nabla u)(t) + c_\delta\left(\int_\Omega |u_t|^{p(x)}dx + \int_\Omega |z(1,t)|^{p(x)}dx\right) - \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t). \tag{55}$$

Proof. Similar to Lemma,9 by the first equation (25), we differentiate (43), and it yields

$$\begin{aligned}\psi'(t) &= -\int_\Omega u_{tt}\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\ &= \left(a + b\|\nabla u\|_2^2\right)\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &+ \sigma\int_\Omega \nabla u\nabla u_t\mathrm{d}x\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x \\ &- \alpha(t)\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\right)\mathrm{d}x \\ &+ \mu_1\int_\Omega |u_t|^{p(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &+ \mu_2\int_\Omega |z(1,t)|^{p(x)-2}z(1,t)\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x \\ &- \int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2 \\ &= \sum_{i=1}^6 I_i - \left(\int_0^t g(s)\mathrm{d}s\right)\|u_t\|_2^2.\end{aligned} \tag{56}$$

By the Hölder inequality, Sobolev-Poincaré inequalities, and (17), we estimate the second part of the right-hand side in (56).

$$|I_1| \le \left(a + b\|\nabla u\|_2^2\right)\left\{\delta\|\nabla u\|_2^2 + \frac{l_0}{4\delta}(g\circ\nabla u)(t)\right\} \le \delta a\|\nabla u\|_2^2 + \delta b\|\nabla u\|_2^4 + \left\{\frac{al_0}{4\delta} + \frac{bl_0E(0)}{2\delta l}e^{(l_0/l)\alpha(0)}\right\}(g\circ\nabla u)(t), \tag{57}$$

$$|I_2| \le \delta\sigma\left(\int_\Omega \nabla u\nabla u_t\mathrm{d}x\right)^2\|\nabla u\|_2^2 + \frac{\sigma l_0}{4\delta}(g\circ\nabla u)(t) \le \delta\frac{2\sigma E(0)}{l}e^{(l_0/l)\alpha(0)}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 + \frac{\sigma l_0}{4\delta}(g\circ\nabla u)(t), \tag{58}$$

$$\begin{aligned}|I_3| &\le \delta\alpha(t)\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)| + |\nabla u(t)|)\mathrm{d}s\right)^2\mathrm{d}x + \frac{1}{4\delta}\alpha(t)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\mathrm{d}s\right)^2\mathrm{d}x \\ &\le 2\delta l_0^2\alpha(t)\|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)l_0\alpha(t)(g\circ\nabla u)(t),\end{aligned} \tag{59}$$

$$|I_4| \le c_\delta \int_\Omega |u_t|^{p(x)} \mathrm{d}x + \delta \max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\right)^{p(x)} \mathrm{d}x$$
$$\le c_\delta \int_\Omega |u_t|^{p(x)} \mathrm{d}x + \delta\Big\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left(l_0^{p^+-1}, l_0^{p^--1}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right)$$
$$\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^{p(x)} \mathrm{d}s\Big\}$$
$$\le c_\delta \int_\Omega |u_t|^{p(x)} \mathrm{d}x + \delta\Big\{\max\left(\mu_1^{p^-}, \mu_1^{p^+}\right) \max\left(c_*^{p^+}, c_*^{p^-}\right) \max$$
$$\left(\left(\frac{2E(0)}{l} e^{(l_0/l)\alpha(0)}\right)^{(p^+-2)/2}, \left(\frac{2E(0)}{l} e^{(l_0/l)\alpha(0)}\right)^{(p^--2)/2}\right)$$
$$(g\circ\nabla u)(t)\Big\} \coloneqq c_\delta \int_\Omega |u_t|^{p(x)} \mathrm{d}x + \delta c_4 (g\circ\nabla u)(t). \tag{60}$$

Similarly,

$$|I_5| \le c_\delta \int_\Omega |z(1,t)|^{p(x)} \mathrm{d}x + \delta c_5 (g\circ\nabla u)(t),$$
$$|I_6| \le \delta\|u_t\|_2^2 - \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t). \tag{61}$$

Comparing these above estimates (57)–(61), we have

$$\psi'(t) \le -\left(\int_0^t g(s)\mathrm{d}s - \delta\right)\|u_t\|_2^2 + \delta\left\{a + 2l_0^2\alpha(t)\right\}\|\nabla u\|_2^2$$
$$+ \delta b\|\nabla u\|_2^4 + \delta\frac{2\sigma E(0)}{l} e^{(l_0/l)\alpha(0)}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2$$
$$+ \left\{C_\delta + \left(2\delta + \frac{1}{4\delta}\right) l_0\alpha(t)\right\}(g\circ\nabla u)(t)$$
$$+ c_\delta\left(\int_\Omega |u_t|^{p(x)}\mathrm{d}x + \int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x\right)$$
$$- \frac{g(0)c_*^2}{4\delta}\left(g'\circ\nabla u\right)(t), \tag{62}$$

where $C_\delta = \{al_0/4\delta + (bl_0E(0)/2\delta l)e^{(l_0/l)\alpha(0)} + \sigma l_0/4\delta + \delta(c_4 + c_5)\}$. □

Lemma 11. *There exists positive constants C_3, C_4, and t_0 satisfying*

$$L'(t) \le -C_3\alpha(t)E(t) + C_4\alpha(t)(g\circ\nabla u)(t), t > t_0. \tag{63}$$

Proof. Since $g > 0$ and is continuous, then for any $t \ge t_0 > 0$, we get

$$\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s = g_0 > 0. \tag{64}$$

Differentiate (41), and using Lemmas 9 and 10, we get

$$\mathrm{L}'(t) = NE'(t) + \varepsilon_1\alpha'(t)\varphi(t) + \varepsilon_1\alpha(t)\varphi'(t) + \varepsilon_2\alpha'(t)\psi(t) + \varepsilon_2\alpha(t)\psi'(t)$$
$$\le -\alpha(t)\{\varepsilon_2(g_0-\delta) - \varepsilon_1\}\|u_t\|_2^2$$
$$- \alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a + 2l_0^2\right)\alpha(t)\right\}\|\nabla u\|_2^2$$
$$- \alpha(t)(b(\varepsilon_1 - \varepsilon_2\delta))\|u_t\|_2^4$$
$$- \alpha(t)\left\{\sigma - \varepsilon_2\delta\frac{\sigma E(0)}{l} e^{(l_0/l)\alpha(0)}\right\}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2$$
$$+ \alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta + \frac{1}{4\delta}\right) l_0\alpha(t)\right\}(g\circ\nabla u)(t)$$
$$+ \alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t)$$
$$- \alpha(t)\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\mathrm{d}x$$
$$- \alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x$$
$$- \frac{N\alpha'(t)}{2}\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \varepsilon_1\alpha'(t)\int_\Omega u u_t \mathrm{d}x$$
$$+ \varepsilon_2\alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x. \tag{65}$$

Indeed,

$$\alpha'(t)\int_\Omega u u_t\mathrm{d}x + \alpha'(t)\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x$$
$$\le -\alpha'(t)\frac{c_*^2}{2}\|\nabla u\|_2^2 - \alpha'(t)\|u_t\|_2^2 - \alpha'(t)\frac{c_*^2}{2}\left(\int_0^t g(s)\mathrm{d}s\right)(g\circ\nabla u)(t). \tag{66}$$

Thus,

$$\mathrm{L}'(t) \le -\alpha(t)\left\{\varepsilon_2(g_0-\delta) - \varepsilon_1 + \frac{\alpha'(t)}{\alpha(t)}\right\}\|u_t\|_2^2$$
$$- \alpha(t)\left\{\varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a + 2l_0^2\right)\alpha(0) + \frac{N\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right) + \frac{c_*^2\alpha'(t)}{2\alpha(t)}\right\}$$
$$\|\nabla u\|_2^2 - \alpha(t)b(\varepsilon_1 - \varepsilon_2\delta)\|\nabla u\|_2^4 - \alpha(t)\left\{\sigma\varepsilon_1 - \varepsilon_2\delta\frac{\sigma E(0)}{l} e^{(l_0/l)\alpha(0)}\right\}$$
$$\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2$$
$$+ \alpha(t)\left\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2 C_\delta + \varepsilon_2\left(2\delta + \frac{1}{4\delta}\right) l_0\alpha(t) - \frac{c_*^2\alpha'(t)}{2\alpha(t)}\left(\int_0^t g(s)\mathrm{d}s\right)\right\}$$
$$(g\circ\nabla u)(t) + \alpha(t)\left\{\frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta}\right\}\left(g'\circ\nabla u\right)(t) - \alpha(t)$$
$$\left\{\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}\int_\Omega |u_t|^{p(x)}\mathrm{d}x - \alpha(t)\left\{\frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta\right\}$$
$$\int_\Omega |z(1,t)|^{p(x)}\mathrm{d}x. \tag{67}$$

Fix $\delta > 0$ such that

$$g_0 - \delta > \frac{1}{2}g_0, \frac{\delta}{C_\varepsilon}\left(a + 2l_0^2\right)\alpha(0) < \frac{1}{4}g_0, \tag{68}$$

and take ε_1 and ε_2 small enough to satisfy

$$\begin{aligned}&\frac{g_0}{4}\varepsilon_2 < \varepsilon_1 < \varepsilon_2\frac{g_0}{2},\\ &c_5 = \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0,\\ &c_6 = \varepsilon_1 C_\varepsilon - \varepsilon_2\delta\left(a + 2l_0^2\right)\alpha(0) > 0.\end{aligned} \tag{69}$$

Select ε_1 and ε_2 small enough to make (44) and (67) hold, and moreover

$$\begin{aligned}&b(\varepsilon_1 - \varepsilon_2\delta) > 0, \sigma\varepsilon_1 - \varepsilon_2\delta\frac{\sigma E(0)}{l}e^{(l_0/l)\alpha(0)} > 0, \frac{N}{2} - \varepsilon_2\frac{g(0)c_*^2}{4\delta} > 0\\ &\frac{c_0}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta > 0, \frac{c_1}{\alpha(0)} - \varepsilon_1 c_\varepsilon - \varepsilon_2 c_\delta > 0.\end{aligned} \tag{70}$$

Hence, for a generic positive constant c, (67) is equal to the following results:

$$\begin{aligned}\mathrm{L}'(t) \le{}& -\alpha(t)\left\{c + \frac{\alpha'(t)}{\alpha(t)}\right\}\|u_t\|_2^2\\ &-\alpha(t)\left\{c + \frac{\alpha'(t)}{2\alpha(t)}\left(\left(\int_0^t g(s)\mathrm{d}s\right) + \frac{c_*^2}{2}\right)\right\}\|\nabla u\|_2^2\\ &+\alpha(t)\left\{c - \frac{c_*^2 g_0\alpha'(t)}{2\alpha(t)}\right\}(g\circ\nabla u)(t), \forall t \ge t_0.\end{aligned} \tag{71}$$

Noticing that $\lim_{t\longrightarrow\infty} - \alpha'(t)/\xi(t)\alpha(t) = 0$, so choose $t_1 > t_0$, we see

$$\begin{aligned}\mathrm{L}'(t) &\le -\alpha(t)\left(c\|u_t\|_2^2 + C\|\nabla u\|_2^2\right) + c(g\circ\nabla u)(t)\\ &\le -\mathrm{C}_3\alpha(t)E(t) + \mathrm{C}_4\alpha(t)(g\circ\nabla u)(t), \forall t \ge t_1,\end{aligned} \tag{72}$$

where C_3 and C_4 are positive constants. □

Now, we are in the position to prove Theorem 4.

Proof of Theorem 4. According to Lemma 5, Lemma 11, and (17), we have

$$\begin{aligned}\zeta(t)\mathrm{L}'(t) &\le -\mathrm{C}_3\alpha(t)\zeta(t)E(t) + \mathrm{C}_4\alpha(t)\zeta(t)(g\circ\nabla u)(t)\\ &\le -\mathrm{C}_3\alpha(t)\zeta(t)E(t) - \mathrm{C}_4\alpha(t)\left(g'\circ\nabla u\right)(t)\\ &\le -\mathrm{C}_3\alpha(t)\zeta(t)E(t)\\ &\quad - \mathrm{C}_4\left(2E'(t) + \alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\right).\end{aligned} \tag{73}$$

Since $\zeta(t)$ is nonincreasing, by assumption (17) and the definition of $E(t)$, we get

$$\begin{aligned}&\frac{l}{2}\|\nabla u\|_2^2 \le E(t),\\ &\frac{\mathrm{d}}{\mathrm{d}t}(\zeta(t)\mathrm{L}(t) + 2\mathrm{C}_4E(t)) \le -\mathrm{C}_3\alpha(t)\zeta(t)E(t)\\ &\qquad - \mathrm{C}_4\alpha'(t)\left(\int_0^s g(s)\mathrm{d}s\right)\|\nabla u\|_2^2,\end{aligned} \tag{74}$$

which leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\zeta(t)F(t) + 2\mathrm{C}_4E(t)) \le -\mathrm{C}_3\alpha(t)\zeta(t)E(t)\\ &\quad - \mathrm{C}_4\alpha'(t)\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \le -\mathrm{C}_3\alpha(t)\zeta(t)E(t)\\ &\quad - \frac{2\mathrm{C}_4E(t)}{l}\alpha'(t)\int_0^t g(s)\mathrm{d}s \le -\alpha(t)\zeta(t)\left(\mathrm{C}_3 + \frac{2\mathrm{C}_4l_0\alpha'(t)}{l\alpha(t)\zeta(t)}\right)E(t).\end{aligned} \tag{75}$$

Since $\lim_{t\longrightarrow\infty} - \alpha'(t)/\alpha(t)\zeta(t) = 0$, we can choose $t_1 \ge t_0$ such that $\mathrm{C}_3 + 2\mathrm{C}_4l_0\alpha'(t)/l\alpha(t)\zeta(t) > 0$ for $t \ge t_1$. Hence, if we let

$$\mathscr{L}(t) = \zeta(t)\mathrm{L}(t) + 2\mathrm{C}_4E(t), \tag{76}$$

then it is obvious that $\mathscr{L}(t)$ is equivalent to $E(t)$ and satisfies

$$\mathscr{L}'(t) \le -k\zeta(t)\alpha(t)\mathscr{L}(t) \text{ for } t \ge t_1. \tag{77}$$

Consequently, to integrate (77) over (t_1, t), it yields

$$\mathscr{L}(t) \le \mathscr{L}(t_1)e^{-\mathrm{C}\int_{t_1}^t \zeta(s)\alpha(s)\mathrm{d}s} t \ge t_0. \tag{78}$$

Thus, the desired result yields from the equivalence relations of $\mathscr{L}(t)$,$\mathrm{L}(t)$, and $E(t)$. □

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Darbo Fixed Point Criterion on Solutions of a Hadamard Nonlinear Variable Order Problem and Ulam-Hyers-Rassias Stability

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The existence aspects along with the stability of solutions to a Hadamard variable order fractional boundary value problem are investigated in this research study. Our results are obtained via generalized intervals and piecewise constant functions and the relevant Green function, by converting the existing Hadamard variable order fractional boundary value problem to an equivalent standard Hadamard fractional boundary problem of the fractional constant order. Further, Darbo's fixed point criterion along with Kuratowski's measure of noncompactness is used in this direction. As well as, the Ulam-Hyers-Rassias stability of the proposed Hadamard variable order fractional boundary value problem is established. A numerical example is presented to express our results' validity.

1. Introduction

Fractional calculus is fundamentally established by having arbitrary numbers in the order of derivation operators instead of natural numbers. This idea is considered preliminary and simple. However, it involves remarkable effects and outcomes which describe some physical processes, dynamics, mathematical modelings, control theory, bioengineering, biomedical applications, etc. [1, 2]. The main effectiveness of this field can be found in recent studies. For example, Thabet et al. in [3] simulated a fractional model of pantograph in the Caputo conformable settings. In [4], Khan et al. designed a model of p-Laplacian FBVP in the form of a singular problem, and Matar et al. derived similar results for a new p-Laplacian model via generalized fractional derivative [5]. The fractional Langevin impulsive equations are studied by Rizwan et al. regarding existence property of solutions in [6], and Zada et al. [7] analyzed the Ulam-Hyers stability for an impulsive integro-differential equations. Etemad et al. used a new property entitled approximate endpoint for studying a novel fractional problem via the Caputo-Hadamard operators [8].

Thabet et al. also modeled COVID-19 transmission by Caputo-Fabrizio operators and analyzed its dynamical behavior [9]. In [10], Shah et al. compared the results of two classical and fractional models of COVID-19 and showed the accuracy of fractional operators in simulation of processes. Pratap et al. [11] studied finite-time Mittag-Leffler stability criteria for fractional quaternion-valued memristive neural networks. Along with these, Boulares et al. [12] conducted a theoretical research on the generalized weakly singular integral inequalities and their applications to generalized FBVPs. Naifar et al. [13] studied a global Mittag-Leffler stabilization by output feedback in relation to a class of nonlinear systems of FBVPs.

It is notable that in recent advanced mathematical models, constant fractional orders have not needed effectiveness for describing the specifications of some processes and phenomena, and consequently, some researchers had to model their boundary problems via the fractional operators equipped with orders as a real-valued functions. These operators are known as variable order ones [14, 15]. The investigation of the variable order fractional boundary problems (VOFBVPs) in the field of existence theory is considered as a new and important branch of fractional calculus, which are published limited research works in this regard. In 2018, Yang et al. presented a numerical scheme for a VOFBVP and analyzed their system from numerical point of view [16]. Zhang studied the solutions of a singular two-point VOFBVP for the first time in [17]. In recent years, Zhang et al. [18, 19] derived approximate solutions of two different VOFIVP on the half-axis in 2018 and 2019. In addition, a multiterm FBVP involving the nonlinear fractional differential equation (NFDE) of the variable order type was investigated in detail by Bouazza et al. in [20]. In this paper, motivated by other related works in this regard, we investigate the solutions' existence of the Hadamard nonlinear VOFBVP as follows:

$$ {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)} r(t) + m_1(t, r(t)) = 0, t \in \mathscr{U} \coloneqq [1, \mathscr{T}], \tag{1} $$

via boundary conditions $r(1) = r(\mathscr{T}) = 0$, where $1 < \mathscr{T} < +\infty$, $1 < \vartheta(t) \le 2$, and $m_1 : \mathscr{U} \times \mathscr{S} \longrightarrow \mathscr{S}$.

$\mathscr{S}$ is a continuous function (S is a real (or complex) space) and ${}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)}$ specifies the Hadamard derivative of variable order $\psi(t)$. For the first time, as the novelty of this research, we here consider a FBVP in the variable order Hadamard settings and establish the existence specifications of solutions to mentioned system on the generalized subintervals by combining the existing notions in relation to the Kuratowski's measure of noncompactness (KMNCS) in the context of Darbo fixed point criterion. The piecewise constant functions will play a vital role in our study for converting the Hadamard VOFBVP (1) to the standard Hadamard FBVP. Lastly, another criterion of the behavior of solutions like the Ulam-Hyers-Rassias stability (UHRS) is analyzed, and a numerical illustrative example will complete the consistency of our findings.

2. Essential Preliminaries

Basic definitions are discussed in this section to be used later. Throughout the paper, the set S stands for the real numbers.

The symbol $C(U, S)$ denotes a set that contains all continuous functions $f : \mathscr{U} \longrightarrow \mathscr{S}$. It is a Banach space by defining

$$ \|f\| = \sup \{|f(t)|: t \in \mathscr{U}\}, \quad \mathscr{U} \coloneqq [1, \mathscr{T}]. \tag{2} $$

Definition 1 (see [21, 22]). For $1 \le a_1 < a_2 < +\infty$, we consider the mappings $h_1(t)$: $[a_1, a_2] \longrightarrow (0,+\infty)$ and $q(t)$: $[a_1, a_2] \longrightarrow (n-1, n)$. The Hadamard $(\vartheta(t))^{\text{th}}$ variable order integral of h_1 is

$$ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)} h_1(t) = \frac{1}{\Gamma(\vartheta(t))} \int_{a_1}^{t} \left(\log \frac{t}{s}\right)^{\vartheta(t)-1} \frac{h_1(s)}{s} ds, \quad t > a_1, \tag{3} $$

and the Hadamard $(q(t))^{\text{th}}$ variable order derivative of h_1 is

$$ \left({}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{q(t)} h_1\right)(t) = \frac{1}{\Gamma(n-q(t))} \left(t \frac{d}{dt}\right)^{n} \cdot \int_{a_1}^{t} \left(\log \frac{t}{s}\right)^{n-q(t)-1} \frac{h_1(s)}{s} ds, \quad t > a_1. \tag{4} $$

Obviously, in case of $\vartheta(t)$ and $q(t)$ are constant, then both above Hadamard variable order operators are in coincidence with the usual Hadamard constant order operators (refer to [1, 21, 22]).

Lemma 2 (see [1]). *Assume that $a_1 > 1$, $\gamma_1, \gamma_2 > 0$, $h_1 \in L(a_1, a_2)$, and ${}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\gamma_1} h_1 \in L(a_1, a_2)$. Then, the homogeneous differential equation*

$$ {}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\gamma_1} h_1 = 0 \tag{5} $$

admits the unique solution

$$ \begin{aligned} h_1(t) &= \omega_1 \left(\log \frac{t}{a_1}\right)^{\gamma_1-1} + \omega_2 \left(\log \frac{t}{a_1}\right)^{\gamma_1-2} + \cdots + \omega_n \left(\log \frac{t}{a_1}\right)^{\gamma_1-n}, \\ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\gamma_1} \left({}^{H}D_{a_1^+}^{\gamma_1}\right) h_1(t) &= h_1(t) + \omega_1 \left(\log \frac{t}{a_1}\right)^{\gamma_1-1} + \omega_2 \left(\log \frac{t}{a_1}\right)^{\gamma_1-2} + \cdots + \omega_n \left(\log \frac{t}{a_1}\right)^{\gamma_1-n}, \end{aligned} \tag{6} $$

with $n = [\gamma_1] + 1$, $\omega_j \in \mathbb{R}$, and $j = 1, 2, \cdots, n$.

Moreover, for constants $\alpha_j > 0, j = 1, 2$,

$$ \begin{aligned} &{}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\alpha_1} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1}\right) h_1(t) = h_1(t), \\ &{}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_2}\right) h_1(t) = {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_2} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1}\right) h_1(t) = {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1+\alpha_1} h_1(t). \end{aligned} \tag{7} $$

Remark 3. The semigroup property is not fulfilled for the functions $\vartheta(t)$ and $q(t)$, i.e.,

$$ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{q(t)}\right) h_1(t) \neq {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)+q(t)} h_1(t). \tag{8} $$

Example 1. Let

$$\begin{aligned} \vartheta(t) &= \begin{cases} 1, & t \in [1,2], \\ 2, & t \in]2,4], \end{cases} \\ q(t) &= \begin{cases} 3, & t \in [1,2], \\ 4, & t \in]2,4], \end{cases} \\ h_1(t) &= 2t^2, \\ &t \in [1,4]. \end{aligned} \tag{9}$$

We obtain

$$\begin{aligned} &{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t) = \frac{1}{\Gamma(\vartheta(t))}\int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{\Psi(t)-1} \\ &\cdot \left[\frac{1}{\Gamma(q(s))}\int_1^s \left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right] \\ &\cdot ds = \frac{1}{\Gamma(\Psi(t))}\int_1^2 \frac{1}{s}\left(\log\frac{t}{s}\right)^{\Psi(t)-1}\left[\frac{1}{\Gamma(q(s))}\int_1^s \left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right] \\ &\cdot ds + \frac{1}{\Gamma(\vartheta(t))}\int_2^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{\vartheta(t)-1}\left[\frac{1}{\Gamma(q(s))}\int_1^s \left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right] \\ &\cdot ds = \frac{1}{\Gamma(1)}\int_1^2 \frac{1}{s}\left(\log\frac{t}{s}\right)^0 \int_1^s \frac{1}{\Gamma(3)}\left(\log\frac{s}{\tau}\right)^2 2\tau d\tau ds + \frac{1}{\Gamma(2)}\int_2^t \frac{1}{s}\left(\log\frac{t}{s}\right) \\ &\cdot \left[\frac{1}{\Gamma(3)}\int_1^2 \left(\log\frac{s}{\tau}\right)^2 2\tau d\tau + \frac{1}{\Gamma(4)}\int_2^s \left(\log\frac{s}{\tau}\right)^3 2\tau d\tau\right]ds, \\ &= \int_1^2 \left(\frac{s}{4} - \frac{1}{2s}(\log s)^2 - \frac{1}{2s}(\log s) - \frac{1}{4s}\right)ds + \int_2^t \frac{1}{s}\left(\log\frac{t}{s}\right) \\ &\cdot \left[-\frac{2}{3}\left(\log\frac{s}{2}\right)^3 + \left(\log\frac{s}{2}\right)^2 + \left(\log\frac{s}{2}\right) - \frac{1}{2}(\log s)^2 - \frac{1}{2}(\log s) + \frac{1}{8}s^2 + \frac{1}{4}\right]ds, \end{aligned} \tag{10}$$

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t) = \frac{1}{\Gamma(\vartheta(t)+q(t))}\int_1^t \left(\log\frac{t}{s}\right)^{\vartheta(t)+q(t)-1}\frac{h_1(s)}{s}ds. \tag{11}$$

So,

$$\begin{aligned} &{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t)|_{t=3} = -\frac{1}{30}\left(\log\frac{3}{2}\right)^5 + \frac{1}{24}\left(\log\frac{3}{2}\right)^4 \\ &+ \frac{1}{12}\left(\log\frac{3}{2}\right)^3 + \frac{1}{8}\left(\log\frac{3}{2}\right)^2 - \frac{1}{4}\left(\log\frac{3}{2}\right) \\ &- \frac{1}{6}(\log 2)^2\left(\log\frac{3}{2}\right)^2 - \frac{1}{6}(\log 2)\left(\log\frac{3}{2}\right)^3 - \frac{(\log 2)^3}{6} \\ &- \frac{(\log 2)^2}{4} - \frac{\log 2}{4} - \frac{1}{4}(\log 2)\left(\log\frac{3}{2}\right)^2 + \frac{17}{32} \simeq 0.0522. \end{aligned} \tag{12}$$

On the other side,

$$\begin{aligned} &{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t)|_{t=3} = \int_1^2 \frac{1}{\Gamma(4)}\left(\log\frac{3}{s}\right)^3 2sds + \int_2^3 \frac{1}{\Gamma(6)}\left(\log\frac{3}{s}\right)^5 \\ &\cdot 2sds = -\frac{1}{30}\left(\log\frac{3}{2}\right)^5 - \frac{1}{12}\left(\log\frac{3}{2}\right)^4 + \frac{1}{3}\left(\log\frac{3}{2}\right)^3 + \frac{3}{4}\left(\log\frac{3}{2}\right)^2 \\ &+ \frac{3}{4}\left(\log\frac{3}{2}\right) - \frac{1}{6}(\log 3)^3 - \frac{1}{4}(\log 3)^2 - \frac{1}{4}(\log 3) + \frac{17}{32} \simeq 0.1809. \end{aligned} \tag{13}$$

Therefore, we obtain

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t)|_{t=3} \neq {}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t)|_{t=3}. \tag{14}$$

Lemma 4 (see [23, 24]). *If $\vartheta : \mathscr{U} \longrightarrow (1,2]$ has the continuity property, then for*

$$\begin{aligned} &h_1 \in \mathscr{C}_\beta(\mathscr{U},\mathscr{S}) \\ &= \left\{h_1(t) \in \mathscr{C}(\mathscr{U},\mathscr{S}), (\log t)^\beta h_1(t) \in \mathscr{C}(\mathscr{U},\mathscr{S})\right\}, \quad 0 \le \beta \le 1, \end{aligned} \tag{15}$$

the integral ${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}h_1(t)$ admits a finite value $\forall t \in \mathscr{U}$.

Lemma 5 (see [23, 24]). *Assume that $\vartheta : \mathscr{U} \longrightarrow (1,2]$ has the continuity property. Then,*

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}h_1(t) \in \mathscr{C}(\mathscr{U},\mathscr{S}) \text{for } h_1 \in \mathscr{C}(\mathscr{U},\mathscr{S}). \tag{16}$$

Definition 6 (see [25–27]). $I \subseteq \mathbb{R}$ is termed as a generalized interval if I is either an interval, or $\{a_1\}$, or $\varnothing$. A finite set $\mathscr{F}$ is defined to be a partition of I if every $x \in I$ belongs to exactly one and one generalized interval $\mathbb{I}$ in $\mathscr{F}$. Finally, $w : I \longrightarrow \mathscr{S}$ is piecewise constant w.r.t $\mathscr{F}$ as a partition of I; if $\forall \mathbb{I} \in \mathscr{F}$, w is constant on $\mathbb{I}$.

2.1. Some Properties regarding KMNCS. Here, we regard $\mathscr{S}$ as a Banach space.

Definition 7 (see [28]). Suppose that $\omega_{\mathscr{S}}$ is a bounded set in $\mathscr{S}$. The KMNCS is the function $\Phi : \omega_{\mathscr{S}} \longrightarrow [0,\infty]$ as

$$\Phi(\mathfrak{P}) = \inf\left\{\delta > 0 : \mathfrak{P} \subseteq \cup_{=1}^{n}\mathfrak{P}_j, \boldsymbol{Diam}\left(\mathfrak{P}_j\right) \le \delta, \left(\mathfrak{P} \in \omega_{\mathscr{S}}\right)\right\}, \tag{17}$$

in which

$$\boldsymbol{Diam}\left(\mathfrak{P}_j\right) = \sup\left\{\|x - r\| : x, r \in \mathfrak{P}_j\right\}. \tag{18}$$

The symbol Diam denotes the diameter of the given set. Some valid properties KMNCS are as follows.

Proposition 8 (see [28, 29]). *Let $\mathfrak{P}, \mathfrak{P}_1, \mathfrak{P}_2$ be bounded in $\mathscr{S}$. Then,*

(a) *$\mathfrak{P}$ is relatively compact if $\Phi(\mathfrak{P}) = 0$*

(b) $\Phi(\varnothing) = 0$

(c) $\Phi(\mathfrak{P}) = \Phi(\bar{\mathfrak{P}})$

(d) $\mathfrak{P}_1 \subset \mathfrak{P}_2 \Rightarrow \Phi(\mathfrak{P}_1) \le \Phi(\mathfrak{P}_2)$

(e) $\Phi(\mathfrak{P}_1 + \mathfrak{P}_2) \le \Phi(\mathfrak{P}_1) + \Phi(\mathfrak{P}_2)$

(f) $\Phi(\lambda\mathfrak{P}) = |\lambda|\Phi(\mathfrak{P}), \lambda \in \mathbb{R}$

(g) $\Phi(\mathfrak{P}_1 \cup \mathfrak{P}_2) = \max\{\Phi(\mathfrak{P}_1), \Phi(\mathfrak{P}_2)\}$

(h) $\Phi(\mathfrak{P}_1 \cap \mathfrak{P}_2) = \min\{\Phi(\mathfrak{P}_1), \Phi(\mathfrak{P}_2)\}$

(i) $\Phi(\mathfrak{P}+x_0)=\Phi(\mathfrak{P})$ for any $x_0\in\mathbb{R}$

Lemma 9 (see [30]). *If the bounded set $\mathcal{W}\subset\mathcal{C}(\mathcal{U},\mathcal{S})$ is equicontinuous, then*

(i) $\Phi(\mathcal{W})$ has continuity, and

$$\widehat{\Phi}(\mathcal{W})=\sup_{t\in\mathcal{U}}\Phi(\mathcal{W}(t)) \tag{19}$$

(ii) $\Phi(\int_0^{\mathcal{T}} r(\theta)d\theta : r\in\mathcal{W})\le\int_0^{\mathcal{T}}\Phi(\mathcal{W}(\theta))d\theta$,

where

$$\mathcal{W}(\theta)=\{r(\theta): r\in\mathcal{W}\},\quad \theta\in\mathcal{U} \tag{20}$$

In the next theorem, we point out the Darbo's fixed point criterion.

Theorem 10 (see [28]). *Consider the closed, convex, and bounded set $\Lambda\neq\varnothing$ in $\mathcal{S}$ and the continuous map $\mathrm{F}:\Lambda\longrightarrow\Lambda$ satisfying (k-set contractive property for F).*

$$\Phi(F(V))\le k\Phi(V),\forall\varnothing\neq V\subset\Lambda,k\in[0,1). \tag{21}$$

Then, F *admits at least a fixed point belonging to Λ.*

3. Existence Criterion of Solutions

To achieve the main purpose of this section, some assumptions are proposed as:

(H1) Consider $\mathcal{F}=\{\mathcal{U}_1:=[1,\mathcal{T}_1],\mathcal{U}_2:=(\mathcal{T}_1,\mathcal{T}_2],\mathcal{U}_3:=(\mathcal{T}_2,\mathcal{T}_3],\cdots,\mathcal{U}_n:=(\mathcal{T}_{n-1},\mathcal{T}]\}$ as a partition for the interval $\mathcal{U}$ and $\vartheta(t)$: $\mathcal{U}\longrightarrow(1,2]$ as a piecewise constant function w.r.t $\mathcal{F}$, i.e.,

$$\vartheta(t)=\sum_{i=1}^{n}\vartheta_\jmath\mathcal{J}_\jmath(t)=\begin{cases}\vartheta_1, & \text{if } t\in\mathcal{U}_1, \quad 1<\vartheta_1\le 2,\\ \vartheta_2, & \text{if } t\in\mathcal{U}_2, \quad 1<\vartheta_2\le 2,\\ . & \\ . & \\ . & \\ \vartheta_n, & \text{if } t\in\mathcal{U}_n, \quad 1<\vartheta_n\le 2,\end{cases} \tag{22}$$

in which $J_\jmath$ interprets the indicator of $\mathcal{J}_\jmath$ interprets the indicator of $\mathcal{U}_\jmath:=(\mathcal{T}_{\jmath-1},\mathcal{T}_\jmath]$, $\in\mathbb{N}_1^n$, so that $\mathcal{T}_0=1$ and $\mathcal{T}_n=\mathcal{T}$, and

$$\mathcal{J}_\jmath(t)=\begin{cases}1, & \text{for } t\in\mathcal{U}_\jmath,\\ 0, & \text{for } \textbf{\textit{elsewhere}}.\end{cases} \tag{23}$$

(H2) Let $(\log t)^\beta m_1:\mathcal{U}\times\mathcal{S}\longrightarrow\mathcal{S}$ be continuous, $(0\le\beta\le 1)$, and $\exists K>0$, such that $(\log t)^\beta\|m_1(t,r)-m_1(t,\bar{r})\|\le K\|r-\bar{r}\|$, for any r, $\bar{r}\in\mathcal{S}$ and $t\in\mathcal{U}$.

Remark 11 (see [31]). Note that the inequality

$$\Phi\left((\log t)^\beta\|m_1(t,B_1)\|\right)\le K\Phi(B_1) \tag{24}$$

is equivalent to (H2) for each $B_1\subset\mathcal{S}$ and $t\in\mathcal{U}$, where B_1 is bounded.

Further, for a supposed set $\mathcal{W}$ of all mappings $w:\mathcal{U}\longrightarrow\mathcal{S}$, define

$$\mathcal{W}(t)=\{w(t),w\in\mathcal{W}\},\quad t\in\mathcal{U}, \tag{25}$$

$\mathcal{W}(t)=\{w(t): w\in\mathcal{W},\ t\in\mathcal{U}\}$.

Let us now establish the solutions' existence for the Hadamard VOFBVP (1) via KMNCS and Darbo's criterion (Theorem 10).

Here, $\forall\jmath\in\{1,2,\cdots,n\}$, the symbol $\mathcal{E}_\jmath=\mathcal{C}(\mathcal{U}_\jmath,r)$, indicated as Banach spaces of continuous mappings $r:\mathcal{U}_\jmath\longrightarrow\mathcal{S}$ is furnished with the norm

$$\|r\|_{\mathcal{E}_\jmath}=\sup_{t\in\mathcal{U}_\jmath}|r(t)|, \tag{26}$$

where $\jmath\in\{1,2,\cdots,n\}$.

First, we analyze the Hadamard VOFBVP defined in (1). In the light of (4), the Hadamard VOFBVP (1) can be rewritten by

$$\frac{1}{\Gamma(2-\vartheta(t))}\left(t\frac{d}{dt}\right)^2\int_1^t\left(\log\frac{t}{s}\right)^{1-\vartheta(t)}\frac{r(s)}{s}ds+m_1(t,r(t))=0,\quad t\in\mathcal{U}. \tag{27}$$

From (H1), equation (27) on the interval $\mathcal{U}_\jmath$, $\in\mathbb{N}_1^n$, can be expressed as

$$\left(t\frac{d}{dt}\right)^2\left(\frac{1}{\Gamma(2-\vartheta_1)}\int_1^{\mathcal{T}_1}\left(\log\frac{t}{s}\right)^{1-\vartheta_1}\frac{r(s)}{s}ds+\cdots+\frac{1}{\Gamma(2-\vartheta_\jmath)}\cdot\int_{\mathcal{T}_{\jmath-1}}^t\left(\log\frac{t}{s}\right)^{1-\vartheta_\jmath}\frac{r(s)}{s}ds\right)+m_1(t,r(t))=0,\ t\in\mathcal{U}_\jmath. \tag{28}$$

Definition 12. The Hadamard VOFBVP (1) admits a solution like functions $r_\jmath$, $\jmath=1,2,\cdots,n$, if $r_\jmath\in\mathcal{C}([1,\mathcal{T}_\jmath],\mathcal{S})$ satisfies equation (28) and $r_\jmath(1)=0=r_\jmath(\mathcal{T}_\jmath)$.

From the above, the Hadamard VOFBVP (1) written in (27) can be given as (28) on $\mathcal{U}_\jmath$, $\in\mathbb{N}_1^n$. For $1\le t\le\mathcal{T}_{\jmath-1}$, put $r(t)\equiv 0$. Then, (28) is formulated by

$$D_{\mathcal{T}_{\jmath-1}^+}^{\vartheta_\jmath}r(t)+m_1(t,r(t))=0,t\in\mathcal{U}_\jmath. \tag{29}$$

In this case, we follow our study by considering the standard Hadamard constant-order FBVP (COFBVP) as follows:

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}^{\vartheta_j}_{\mathscr{T}^+_{j-1}} r(t) + m_1(t, r(t)), & t \in \mathscr{U}_j, \\ r(\mathscr{T}_{j-1}) = 0, & r(\mathscr{T}_j) = 0. \end{cases} \quad (30)$$

The fundamental part of our analysis regarding solutions of the Hadamard COFBVP (30) is discussed below.

Lemma 13. *A function $r \in \mathscr{E}_j$ is a solution of the Hadamard COFBVP (30) if r fulfills the integral equation*

$$r(t) = \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s} \mathbb{G}_j(t, s) m_1(s, r(s)) ds,\ t \in \mathscr{U}_j, \quad (31)$$

where $\mathbb{G}_j(t, s)$ stands for the Green function formulated by

$$\mathbb{G}_j(t, s) = \frac{1}{\Gamma(\vartheta_j)} \begin{cases} \left(\log \dfrac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left[\left(\log \dfrac{t}{\mathscr{T}_{j-1}}\right)\left(\log \dfrac{\mathscr{T}_j}{s}\right)\right]^{\vartheta_j - 1} - \left(\log \dfrac{t}{s}\right)^{\vartheta_j - 1}, & \mathscr{T}_{-1} \le s \le t \le \mathscr{T}_j, \\ \left(\log \dfrac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left[\left(\log \dfrac{t}{\mathscr{T}_{j-1}}\right)\left(\log \dfrac{\mathscr{T}_j}{s}\right)\right]^{\vartheta_j - 1}, & \mathscr{T}_{-1} \le t \le s \le \mathscr{T}_j, \end{cases} \quad (32)$$

where $j \in \mathbb{N}_1^n$.

Proof. Suppose that $r \in \mathscr{E}_j$ satisfies the Hadamard COFBVP (30). Let us employ the operator ${}^{\mathscr{H}}\mathscr{I}^{\psi_j}_{\mathscr{T}^+_{j-1}}$ on both sides (30) and using Lemma 2, we get

$$r(t) = \omega_1 \left(\log \frac{t}{\tau_{j-1}}\right)^{\vartheta_j - 1} + \omega_2 \left(\log \frac{t}{\tau_{j-1}}\right)^{\vartheta_j - 2 - \mathscr{H}} \mathscr{I}^{\vartheta_j}_{\tau^+_{j-1}} m_1(t, r(t)), \quad t \in u_j, j \in \mathbb{N}_1^n. \quad (33)$$

From definition of m_1 along with $r(\mathscr{T}_{j-1}) = 0$, we get $\omega_2 = 0$.

Suppose that r satisfies $r(\mathscr{T}_j) = 0$. Hence,

$$\omega_1 = \left(\log \frac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} {}^{\mathscr{H}}_{j}\mathscr{I}^{\vartheta_j}_{\mathscr{T}^+_{j-1}} m_1(\mathscr{T}_j, r(\mathscr{T}_j)). \quad (34)$$

Thus,

$$r(t) = \left(\log \frac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left(\log \frac{t}{\mathscr{T}_{j-1}}\right)^{\vartheta_j - 1 \mathscr{H}} \cdot \mathscr{I}^{\vartheta_j}_{\mathscr{T}^+_{j-1}} m_1(\mathscr{T}_j, r(\mathscr{T}_j)) - {}^{\mathscr{H}}\mathscr{I}^{\vartheta_j}_{\mathscr{T}^+_{j-1}} m_1(t, r(t)), \quad (35)$$

Then, the solution of the Hadamard COFBVP (30) is given by

$$\begin{aligned} r(t) = {} & \left(\log \frac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left(\log \frac{t}{\mathscr{T}_{j-1}}\right)^{\vartheta_j - 1} \frac{1}{\Gamma(\vartheta_j)} \\ & \cdot \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \left(\log \frac{\mathscr{T}_j}{s}\right)^{\vartheta_j - 1} \frac{m_1(s, r(s))}{s} \\ & \cdot ds - \frac{1}{\Gamma(\vartheta_j)} \int_{\mathscr{T}_{j-1}}^{t} \left(\log \frac{t}{s}\right)^{\vartheta_j - 1} \frac{m_1(s, r(s))}{s} \\ & \cdot ds = \frac{1}{\Gamma(\vartheta_j)} \left[\int_{\mathscr{T}_{j-1}}^{t} \left[\left(\log \frac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left(\log \frac{t}{\mathscr{T}_{j-1}}\right)^{\vartheta_i - 1}\right.\right. \\ & \left. \cdot \left(\log \frac{\mathscr{T}_j}{s}\right)^{\vartheta_j - 1} - \left(\log \frac{t}{s}\right)^{\vartheta_j - 1}\right] \frac{m_1(s, r(s))}{s} \\ & \left. \cdot ds + \int_{t}^{\mathscr{T}_j} \left(\log \frac{\mathscr{T}_j}{\mathscr{T}_{j-1}}\right)^{1-\vartheta_j} \left(\log \frac{t}{\mathscr{T}_{j-1}}\right)^{\vartheta_j - 1} \left(\log \frac{\mathscr{T}_j}{s}\right)^{\vartheta_j - 1} \frac{m_1(s, r(s))}{s} ds\right], \end{aligned} \quad (36)$$

and the continuity property of the Green function gives

$$r(t) = \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s} \mathbb{G}_j(t, s) m_1(s, r(s)) ds, \quad t \in \mathscr{U}_j. \quad (37)$$

Conversely, let $r \in \mathscr{E}_j$ be the integral equation's (31) solution. Because of the continuity of $(\log t)^\beta m_1$ and by Lemma 2, it is simply verified that r satisfies the Hadamard COFBVP (30) solution. □

Proposition 14. *Assume that $(\log t)^\beta m_1, (0 \le \beta \le 1)$ belongs to $\mathscr{C}(\mathscr{U} \times \mathscr{S}, \mathscr{S})$ and $\vartheta(t)\colon \mathscr{U} \longrightarrow (1, 2]$ satisfies (H1). Then, $\mathbb{G}_j(t, s)$ given by (32) satisfy the following: ($j \in \mathbb{N}_1^n$)*

(1) $0 \le \mathbb{G}_j(t, s), \forall \mathscr{T}_{j-1} \le t, s \le \mathscr{T}_j$

(2) $\max_{t \in \mathscr{U}_j} \mathbb{G}_j(t, s) = \mathbb{G}_j(s, s), s \in \mathscr{U}_j$

(3) $\mathbb{G}_j(s, s)$ has a maximum value uniquely given by

$$\max_{s\in\mathscr{U}_\jmath}\mathbb{G}_\jmath(s,s)=\frac{1}{\Gamma(\vartheta_\jmath)}\left(\frac{\log\mathscr{T}_\jmath-\log\mathscr{T}_{-1}}{4}\right)^{\vartheta_\jmath-1}, \tag{38}$$

where $\jmath=1,2,\cdots,n$

Proof. Let $\varphi(t,s)=(\log(\mathscr{T}_\jmath/\mathscr{T}_{\jmath-1}))^{1-\vartheta_\jmath}[(\log(t/\mathscr{T}_{\jmath-1}))(\log(\mathscr{T}_\jmath/s))]^{\vartheta_\jmath-1}-(\log(t/s))^{\vartheta_\jmath-1}$. We see that

$$\begin{aligned}\varphi_t(t,s)&=\left(\frac{\vartheta_\jmath-1}{t}\right)\left(\log\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath}\left(\log\frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-1}\left(\log\frac{t}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-2}\\&\quad-\left(\frac{\vartheta_\jmath-1}{t}\right)\left(\log\frac{t}{s}\right)^{\vartheta_\jmath-2}\le\left(\frac{\vartheta_\jmath-1}{t}\right)\left(\log\frac{\mathscr{T}_\jmath}{s}\right)^{1-\vartheta_\jmath}\\&\quad\cdot\left(\log\frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-1}\left(\log\frac{t}{s}\right)^{\vartheta_\jmath-2}-\left(\frac{\vartheta_\jmath-1}{t}\right)\left(\log\frac{t}{s}\right)^{\vartheta_\jmath-2}=0,\end{aligned} \tag{39}$$

which means that $\varphi(t,s)$ is nonincreasing w.r.t t, so $\varphi(t,s)\ge\varphi(\mathscr{T}_\jmath,s)=0$, for $\mathscr{T}_{\jmath-1}\le s\le t\le\mathscr{T}_\jmath$. Thus, $0\le\mathbb{G}_\jmath(t,s)$ for any $\mathscr{T}_{\jmath-1}\le t,s\le\mathscr{T}_\jmath,\jmath=1,\cdots,n$.

Since $\varphi(t,s)$ is nonincreasing w.r.t t, then $\varphi(t,s)\le\varphi(s,s)$ for $\mathscr{T}_{\jmath-1}\le s\le t\le\mathscr{T}_\jmath$.

On the other hand, for $\mathscr{T}_{\jmath-1}\le t\le s\le\mathscr{T}_\jmath$, we get

$$\begin{aligned}&\left(\log\left(\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)\right)^{1-\vartheta_\jmath}\left(\log\left(\frac{t}{\mathscr{T}_{\jmath-1}}\right)\log\left(\frac{\mathscr{T}_\jmath}{s}\right)\right)^{\vartheta_\jmath-1}\\&\le\left(\log\left(\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)\right)^{1-\vartheta_\jmath}\left(\log\left(\frac{s}{\mathscr{T}_{\jmath-1}}\right)\log\left(\frac{\mathscr{T}_\jmath}{s}\right)\right)^{\vartheta_\jmath-1}.\end{aligned} \tag{40}$$

These confirm that $\max_{t\in[\mathscr{T}_{\jmath-1},\mathscr{T}_\jmath]}\mathbb{G}_\jmath(t,s)=\mathbb{G}_\jmath(s,s),s\in[\mathscr{T}_{\jmath-1},\mathscr{T}_\jmath],\jmath=1,\cdots,n$.

Further, we verify (3) of Proposition 14. Clearly, the maximum points of $\mathbb{G}_\jmath(s,s)$ are not $\mathscr{T}_{\jmath-1}$ and $\mathscr{T}_\jmath$, $\jmath=\in\mathbb{N}_1^n$. For $s\in[\mathscr{T}_{\jmath-1},\mathscr{T}_\jmath],\jmath=1,\cdots,n$, we have

$$\begin{aligned}\frac{d\mathbb{G}_\jmath(s,s)}{ds}&=\left(\frac{\vartheta_\jmath-1}{s}\right)\left(\log\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath}\left(\log\frac{s}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-2}\\&\quad\cdot\left(\log\frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-2}\left[\left(\log\frac{\mathscr{T}_\jmath}{s}\right)-\left(\log\frac{s}{\mathscr{T}_{\jmath-1}}\right)\right],\\&=\left(\frac{\vartheta_\jmath-1}{s}\right)\left(\log\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath}\left(\log\frac{s}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-2}\\&\quad\cdot\left(\log\frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-2}\left[\log\left(\mathscr{T}_\jmath\mathscr{T}_{\jmath-1}\right)-\log\left(s^2\right)\right],\end{aligned} \tag{41}$$

which indicates that the maximum points of $\mathbb{G}_\jmath(s,s)$ is $s=\sqrt{\mathscr{T}_\jmath\mathscr{T}_{\jmath-1}},\jmath=1,\cdots,n$.

Hence, for $\jmath=1,\cdots,n$,

$$\begin{aligned}\max_{s\in[\mathscr{T}_{\jmath-1},\mathscr{T}_\jmath]}\mathbb{G}_\jmath(s,s)&=\mathbb{G}_\jmath\left(\sqrt{\mathscr{T}_\jmath\mathscr{T}_{\jmath-1}},\sqrt{\mathscr{T}_\jmath\mathscr{T}_{\jmath-1}}\right)\\&=\frac{1}{\Gamma(\vartheta_\jmath)}\left(\frac{1}{4}\log\frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-1}\\&=\frac{1}{\Gamma(\vartheta_\jmath)}\left(\frac{\log\mathscr{T}_\jmath-\log\mathscr{T}_{\jmath-1}}{4}\right)^{\vartheta_\jmath-1}.\end{aligned} \tag{42}$$

This shows the completion of the proof. □

The existence criterion of solutions for the Hadamard VOFBVP (1) in this work depends on the hypotheses of Theorem 10 which we investigate them in this position.

Theorem 15. *Suppose that both (H1) and (H2) hold, and*

$$\frac{K\left(\left(\log\mathscr{T}_\jmath\right)^{1-\beta}-\left(\log\mathscr{T}_{\jmath-1}\right)^{1-\beta}\right)\left(\log\mathscr{T}_\jmath-\log\mathscr{T}_{\jmath-1}\right)^{\vartheta_\jmath-1}}{4^{\vartheta_\jmath-1}(1-\beta)\Gamma(\vartheta_\jmath)}<1. \tag{43}$$

Then, there is a solution to the Hadamard VOFBVP (1) on $\mathscr{U}$.

Proof. We construct the operator

$$\mathscr{Z}:\mathscr{E}_\jmath\longrightarrow\mathscr{E}_\jmath \tag{44}$$

by

$$\mathscr{Z}r(t)=\int_{\mathscr{T}_{\jmath-1}}^{\mathscr{T}_\jmath}\frac{1}{s}\mathbb{G}_\jmath(t,s)m(s,r(s))ds,t\in\mathscr{U}_\jmath. \tag{45}$$

Some properties of fractional integrals along with the continuity for the function $(\log)^\beta m_1$ imply that the operator $\mathscr{Z}:\mathscr{E}_\jmath\longrightarrow\mathscr{E}_\jmath$ defined in (45) is well-defined.

Let $\exists R_\jmath>0$ so that

$$R_\jmath\ge\frac{m^\star\left(\log\mathscr{T}_\jmath-\log\mathscr{T}_{\jmath-1}\right)^{\vartheta_\jmath}/4^{\vartheta_\jmath-1}\Gamma(\vartheta_\jmath)}{1-\left(K\left(\left(\log\mathscr{T}_\jmath\right)^{1-\beta}-\left(\log\mathscr{T}_{\jmath-1}\right)^{1-\beta}\right)\left(\log\mathscr{T}_\jmath-\log\mathscr{T}_{\jmath-1}\right)^{\vartheta_\jmath-1}/4^{\vartheta_\jmath-1}(1-\beta)\Gamma(\vartheta_\jmath)\right)}, \tag{46}$$

with

$$m^\star=\sup_{t\in\mathscr{U}_\jmath}\|m_1(t,0)\|. \tag{47}$$

Let us consider the following set:

$$B_{R_\jmath}=\left\{r\in\mathscr{E}_\jmath,\|r\|_{\mathscr{E}_\jmath}\le R_\jmath\right\}. \tag{48}$$

Clearly $B_{R_J} \neq \varnothing$ contains all three properties of the convexity, boundedness, and closedness.

We shall show that $\mathscr{Z}$ satisfies Theorem 10 in four stages.

Step A. Claim: $\mathscr{Z}(B_{R_J}) \subseteq (B_{R_J})$. For $r \in B_{R_J}$, by Proposition 14 and (H2), we get

$$\begin{aligned}
\|\mathscr{Z}r(t)\| &= \left\| \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t,x)m_1(s,r(s))ds \right\| \\
&\leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t,s)\|m_1(s,r(s))\| \\
&\quad \cdot ds \leq \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1} \\
&\quad \cdot \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\|m_1(s,r(s))\|ds \leq \frac{1}{\Gamma(\vartheta_J)} \\
&\quad \cdot \left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1} \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\|m_1(s,r(s)) \\
&\quad - m_1(s,0)\|ds + \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1} \\
&\quad \cdot \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\|m_1(s,0)\|ds \leq \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1} \\
&\quad \cdot \int_{\mathscr{T}_{-1}}^{\mathscr{T}_J} \frac{1}{s}(\log s)^{-\beta}(K\|r(s)\|)ds + \frac{m^\star(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J}}{4^{\vartheta_J-1}\Gamma(\vartheta_J)} \\
&\leq \frac{K}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1}\|r\|_{\mathscr{E}_J}\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}(\log s)^{-\beta} \\
&\quad \cdot ds + \frac{m^\star(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J}}{4^{\vartheta_J-1}\Gamma(\vartheta_J)} \leq \frac{K}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1} \\
&\quad \cdot R_J\left(\frac{(\log \mathscr{T}_J)^{1-\beta} - (\log \mathscr{T}_{J-1})^{1-\beta}}{1-\beta}\right) + \frac{m^\star(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J}}{4^{\vartheta_J-1}\Gamma(\vartheta_J)} \\
&\leq \frac{K\left((\log \mathscr{T}_J)^{1-\beta} - (\log \mathscr{T}_{J-1})^{1-\beta}\right)(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J-1}}{4^{\vartheta_J-1}(1-\beta)\Gamma(\vartheta_J)} \\
&\quad \cdot R_J + \frac{m^\star(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J}}{4^{\vartheta_J-1}\Gamma(\vartheta_J)} \leq R_J,
\end{aligned} \tag{49}$$

which means that $\mathscr{Z}(B_{R_J}) \subseteq B_{R_J}$.

Step B. Claim: $\mathscr{Z}$ is continuous.

The sequence (r_n) is supposed to be convergent to r in $\mathscr{E}_J$ and $t \in \mathscr{U}_J$. Then,

$$\begin{aligned}
\|(\mathscr{Z}r_n)(t) - (\mathscr{Z}r)(t)\| &\leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t,s)\|m_1(s,r_n(s)) - m_1(s,r(s))\|ds \\
&\leq \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1}\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\|m_1(s,r_n(s)) - m_1(s,r(s))\|ds \\
&\leq \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1}\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}(\log s)^{-\beta}(K\|r_n(s) - r(s)\|)ds \\
&\leq \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1}\left(K\|r_n - y\|_{\mathscr{E}_J}\right)\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}(\log s)^{-\beta}ds \\
&\leq \frac{K\left((\log \mathscr{T}_J)^{1-\beta} - (\log \mathscr{T}_{J-1})^{1-\beta}\right)(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J-1}}{4^{\vartheta_J-1}(1-\beta)\Gamma(\vartheta_J)}\|r_n - r\|_{\mathscr{E}_J},
\end{aligned} \tag{50}$$

i.e., we get

$$\|(\mathscr{Z}r_n) - (\mathscr{Z}r)\|_{\mathscr{E}_J} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{51}$$

and the correctness of the claim in this step is confirmed for Z.

Step C. Claim: $\mathscr{Z}$ is bounded and equicontinuous.

From A, $\mathscr{Z}(B_{R_J}) = \{\mathscr{Z}(r): r \in B_{R_J}\} \subset B_{R_J}$; thus, for each $r \in B_{R_J}$, we get $\|\mathscr{Z}(r)\|_{\mathscr{E}_J} \leq R_J$; in other ways, it means that $\mathscr{Z}(B_{R_J})$ is bounded. It remains to check the equicontinuity of $\mathscr{Z}(B_{R_J})$.

Now, $\forall t_1 < t_2 \in \mathscr{U}_J, t_1 < t_2$ and $r \in B_{R_J}$, we write

$$\begin{aligned}
\|(\mathscr{Z}r)(t_2) - (\mathscr{Z}r)(t_1)\| &= \left\| \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t_2,s)m_1(s,r(s)) \right. \\
&\quad \left. \cdot ds - \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t_1,s)m_1(s,r(s))ds \right\| \leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \\
&\quad \cdot \frac{1}{s}\left\|\left(\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right)m_1(s,r(s))\right\| \\
&\quad \cdot ds \leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right\|\|m_1(s,r(s))\| \\
&\quad \cdot ds \leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_i(t_1,s)\right\|(\|m_1(s,r(s)) - m_1(s,0)\| \\
&\quad + \|m_1(s,0)\|)ds \leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right\| \\
&\quad \cdot \left[(\log s)^{-\beta}(K\|r(s)\|) + m^\star\right]ds \leq \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right\| \\
&\quad \cdot \left[\frac{1}{s}(\log s)^{-\beta}\left(K\|r\|_{\mathscr{E}_J}\right) + \frac{1}{s}m^\star\right]ds \leq \frac{K(\log \mathscr{T}_{J-1})^{-\beta}}{\mathscr{T}_{J-1}}\|r\|_{\mathscr{E}_J} \\
&\quad \cdot \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right\|ds + \frac{m^\star}{\mathscr{T}_{J-1}}\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \left\|\mathbb{G}_J(t_2,s) - \mathbb{G}_J(t_1,s)\right\|ds,
\end{aligned} \tag{52}$$

using the continuity of G. Hence, $\|(\mathscr{Z}r)(t_2) - (\mathscr{Z}r)(t_1)\|_{\mathscr{E}_J} \longrightarrow 0$ as $|t_2 - t_1| \longrightarrow 0$. It yields that $\mathscr{Z}(B_{R_J})$ is equicontinuous.

Step D. Claim: $\mathscr{Z}$ is k-set contraction.

This time, let $\mathscr{W} \in B_{R_J}$ and $t \in \mathscr{U}_J$. So,

$$\begin{aligned}
\Phi(\mathscr{Z}(\mathscr{W})(t)) &= \Phi((\mathscr{Z}r)(t), r \in \mathscr{W}) \\
&\leq \left\{ \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t,s)\Phi m_1(s,r(s))ds \quad r \in \mathscr{W} \right\}.
\end{aligned} \tag{53}$$

Remark 11 indicates that

$$\begin{aligned}
\Phi(\mathscr{Z}(\mathscr{W})(t)) &\leq \left\{ \int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}\mathbb{G}_J(t,s)[K\Phi(\{r(s), r \in \mathscr{W}\})] \right\} \\
&\leq \left\{ \frac{1}{\Gamma(\vartheta_J)}\left(\frac{\log \mathscr{T}_J - \log \mathscr{T}_{J-1}}{4}\right)^{\vartheta_J-1}\left[K\widehat{\Phi}(\mathscr{W})\int_{\mathscr{T}_{J-1}}^{\mathscr{T}_J} \frac{1}{s}(\boldsymbol{logs})^{-\beta}ds\right], r \in \mathscr{W} \right\} \\
&\leq \frac{K\left(\left(\log \mathscr{T}_J{}^{1-\beta}\right) - \left(\log \mathscr{T}_{J-1}{}^{1-\beta}\right)\right)(\log \mathscr{T}_J - \log \mathscr{T}_{J-1})^{\vartheta_J-1}}{4^{\vartheta_J-1}(1-\beta)\Gamma(\vartheta_J)}\widehat{\Phi}(\mathscr{W}),
\end{aligned} \tag{54}$$

for any $s \in \mathscr{U}_J$.

Therefore,

$$\widehat{\Phi}(\mathcal{Z}\mathcal{W}) \leq \frac{K\left(\left(\log \mathcal{T}_j{}^{1-\beta}\right) - \left(\log \mathcal{T}_{j-1}{}^{1-\beta}\right)\right)\left(\log \mathcal{T}_j - \log \mathcal{T}_{j-1}\right)^{\vartheta_j-1}}{4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)} \widehat{\Phi}(\mathcal{W}). \tag{55}$$

Consequently by (43), we deduce that $\mathcal{Z}$ admits a set contraction.

The conclusion of Theorem 10 gives this result that the Hadamard COFBVP (30) involves at least a solution $\widetilde{r_j}$ in B_{R_j}.

Assume that

$$r_j = \begin{cases} 0, & t \in \left[1, \mathcal{T}_{j-1}\right], \\ \tilde{r}_j, & t \in \mathcal{U}_j. \end{cases} \tag{56}$$

We know that $r_j \in \mathscr{C}([1, \mathcal{T}_j], \mathcal{S})$ defined by (56) satisfies equation

$$\frac{d^2}{dt^2}\left(\int_1^{\mathcal{T}_1} \frac{(t-s)^{1-\vartheta_1}}{\Gamma(2-\vartheta_1)} r_j(s)ds + \cdots + \int_{\mathcal{T}_{j-1}}^t \frac{(t-s)^{1-\vartheta_j}}{\Gamma(2-\vartheta_j)} r_j(s)ds\right) + m_1\left(s, r_j(s)\right) = 0, \tag{57}$$

for $t \in \mathcal{U}_j$, which implies that r_j is regarded as a solution for (28) along with $r_j(1) = 0$ and $r_j(\mathcal{T}_j) = \tilde{r}_j(\mathcal{T}_j) = 0$.

Then,

$$r(t) \begin{cases} r_1(t), t \in \mathcal{U}_1, \\ r_2(t) = \begin{cases} 0, & t \in \mathcal{U}_1, \\ \tilde{x}_2, & t \in \mathcal{U}_{2,} \end{cases} \\ . \\ . \\ . \\ r_j(t) = \begin{cases} 0, & t \in \left[1, \mathcal{T}_{j-1}\right], \\ \tilde{r}_j, & t \in \mathcal{U}_j \end{cases} \end{cases} \tag{58}$$

gives the solution for the Hadamard VOFBVP (1) and this completes the argument. □

4. Ulam-Hyers-Rassias Stability

The stability issue has gained substantially important attention in several research fields through applications. There are many kinds of stability; one of them is the stability introduced by Ulam in 1940. Since then, the problem is known as Ulam-Hyers stability or simply Ulam stability. Later, other generalizations of this notion were introduced by other researchers. Its applications for many types of equations have been investigated by many mathematicians. Ben Makhlouf [32] derived sufficient conditions of different types of stability such as uniform stability, Mittag-Leffler stability, and asymptotic uniform stability for a nonlinear Caputo fraction BVP via a method with respect to Lyapunov-like functions. Ahmad et al. [33] investigated the notion of stability for a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian operator. Now, we aim to accomplish an argument regarding the UHRS stability of the given Hadamard VOFBVP (1) in the framework of Theorem 17.

Definition 16 (see [34]). Let $\varrho \in \mathscr{C}(\mathcal{U}, \mathcal{S})$. Then, the Hadamard VOFBVP (1) is Ulam-Hyers-Rassias stable (UHRS) w.r.t ϱ if $\exists c_m > 0$, so that $\forall \epsilon > 0$ and for every solution $\varsigma \in \mathscr{C}(\mathcal{U}, \mathcal{S})$ of

$$\left\| {}^{\mathcal{H}}\mathfrak{D}_{1^+}^{\vartheta(t)} \varsigma(t) - (-m_1(t, \varsigma(t)))\right\| \leq \epsilon\varrho(t), \quad t \in \mathcal{U}, \tag{59}$$

$\exists$ a solution $r \in \mathscr{C}(\mathcal{U}, \mathcal{S})$ of (1) with

$$\|\varsigma(t) - r(t)\| \leq c_m \epsilon \varrho(t). \tag{60}$$

Theorem 17. *Let both (H1) and (H2) along with (43) hold. Also,*

(H3) Let $\varrho \in \mathscr{C}(\mathcal{U}_j, \mathcal{S})$ is a increasing function and $\exists$ $\lambda_\varrho > 0$ provided

$${}^{\mathcal{H}}\mathcal{F}_{\mathcal{T}_{j-1}^+}^{\vartheta_j} \varrho(t) \leq \lambda_{\varrho(t)} \varrho(t), \textit{for any } t \in \mathcal{U}_j. \tag{61}$$

Then, the Hadamard VOFBVP (1) is UHRS stable w.r.t the function ϱ.

Proof. Assume that $\epsilon > 0$ is chosen arbitrarily and ς from $\varsigma \in \mathscr{C}(\mathcal{U}_j, \mathbb{R})$ satisfies (59). Now, $\forall j \in \mathbb{N}_1^n$, the following are defined: $\varsigma_1(t) \equiv \varsigma(t), t \in [1, \mathcal{T}_1]$ and for $j = 2, 3, \cdots, n$:

$$\varsigma_j(t) = \begin{cases} 0, & t \in \left[0, \mathcal{T}_{j-1}\right], \\ \varsigma(t), & t \in \mathcal{U}_j. \end{cases} \tag{62}$$

Taking ${}^{\mathcal{H}}\mathcal{F}_{\mathcal{T}_{j-1}^+}^{\vartheta_j}$ on both sides (59), we get

$$\begin{aligned} &\left\| \varsigma(t) - \left(-\int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s} \mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \varsigma(s)) ds\right)\right\| \\ &\leq \frac{\epsilon}{\Gamma(\vartheta_{\mathcal{J}})} \int_{\mathcal{T}_{\mathcal{J}-1}}^t \frac{1}{s}\left(\log \frac{t}{s}\right)^{\vartheta_{\mathcal{J}}-1} \varrho(s) ds \leq \epsilon \lambda_{\varrho(t)} \varrho(t). \end{aligned} \tag{63}$$

According to the argument above, the Hadamard VOFBVP (1) involves a solution $r \in \mathscr{C}(\mathcal{U}, \mathbb{R})$ formulated as $r(t) = r_{\mathcal{J}}(t)$ *for* $t \in \mathcal{U}_{\mathcal{J}}, \in \mathbb{N}_1^n$, in which

$$r_j = \left\{ \begin{array}{ll} 0, & t \in \left[0, \mathcal{T}_{\mathcal{J}-1}\right], \\ \tilde{r}_j, & t \in \mathcal{U}_{\mathcal{J}}, \end{array} \right\} \tag{64}$$

and $\tilde{r}_{\mathcal{J}} \in \mathscr{E}_{\mathcal{J}}$ is a solution of the Hadamard COFBVP (30).

In accordance with Lemma 13, we have

$$\tilde{r}_{\mathcal{J}}(t) = -\frac{(\mathcal{T}_{\mathcal{J}} - \mathcal{T}_{\mathcal{J}-1})^{-1}(t - \mathcal{T}_{\mathcal{J}-1})}{\Gamma(\vartheta_{\mathcal{J}})} \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} (\mathcal{T}_{\mathcal{J}} - s)^{\vartheta_{\mathcal{J}-1}} m_1(s, \tilde{r}_{\mathcal{J}}(s))ds + \frac{1}{\Gamma(\vartheta_{\mathcal{J}})} \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{t} (t-s)^{\vartheta_{\mathcal{J}-1}} m_1(s, \tilde{r}_{\mathcal{J}}(s))ds. \tag{65}$$

Suppose that $t \in \mathcal{U}_{\mathcal{J}}, = 1, 2, \cdots, n$. Then, by equations (64) and (65), we get

$$\begin{aligned}
&\|\varsigma(t) - r(t)\| = \|\varsigma(t) - r_{\mathcal{J}}(t)\| = \|\varsigma_{\mathcal{J}}(t) - \tilde{r}_{\mathcal{J}}(t)\| \\
&= \left\|\varsigma_{\mathcal{J}}(t) - \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \tilde{r}_{\mathcal{J}}(s))ds\right\| \le \left\|\varsigma_{\mathcal{J}}(t) - \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \right. \\
&\quad \cdot \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \varsigma_{\mathcal{J}}(s))ds\| + \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s)\|m_1(s, \varsigma_{\mathcal{J}}(s)) \\
&\quad - m_1(s, \tilde{r}_{\mathcal{J}}(s))\|ds \le \left\|\varsigma_{\mathcal{J}}(t) + \int_{\mathcal{T}_{-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \varsigma_{\mathcal{J}}(s))\right. \\
&\quad \cdot ds\| + \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s)\left\|m_1(s, \varsigma_{\mathcal{J}}(s)) - m_1\left(s, \tilde{\tilde{r}}_{\mathcal{J}}(s)\right)\right\| \\
&\quad \cdot ds \le \lambda_{\varrho(t)}\epsilon\varrho(t) + \frac{1}{\Gamma(\vartheta_{\mathcal{J}})}\left(\frac{\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1}}{4}\right)^{\vartheta_{\mathcal{J}}-1} \\
&\quad \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} (\log s)^{-\beta} \frac{K\|\varsigma_{\mathcal{J}}(s) - \tilde{r}_{\mathcal{J}}(s)\|}{s} ds \le \lambda_{\varrho(t)}\epsilon\varrho(t) \\
&\quad + \frac{K}{\Gamma(\vartheta_{\mathcal{J}})}\left(\frac{\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1}}{4}\right)^{\vartheta_{\mathcal{J}}-1} \|\varsigma_{\mathcal{J}} - \tilde{r}_{\mathcal{J}}\|_{\mathcal{G}_{\mathcal{J}}} \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \\
&\quad \cdot \frac{1}{s}(\log s)^{-\beta} ds \le \lambda_{\varrho(t)}\epsilon\varrho(t) \\
&\quad + \frac{K\left((\log \mathcal{T}_{\mathcal{J}})^{1-\beta} - (\log \mathcal{T}_{\mathcal{J}-1})^{1-\beta}\right)(\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1})^{\vartheta_{\mathcal{J}}-1}}{(1-\beta)4^{\vartheta_{\mathcal{J}}-1}\Gamma(\vartheta_{\mathcal{J}})} \\
&\quad \cdot \|\varsigma_{\mathcal{J}} - \tilde{r}_{\mathcal{J}}\|_{\mathcal{G}_{\mathcal{J}}} \le \lambda_{\varrho(t)}\epsilon\varrho(t) + \mu\|\varsigma - r\|,
\end{aligned} \tag{66}$$

where

$$\mu = \max_{\mathcal{J}=1,2,\cdots,n} \frac{K\left((\log \mathcal{T}_{\mathcal{J}})^{1-\beta} - (\log \mathcal{T}_{\mathcal{J}-1})^{1-\beta}\right)(\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1})^{\vartheta_{\mathcal{J}}-1}}{(1-\beta)4^{\vartheta_{\mathcal{J}}-1}\Gamma(\vartheta_{\mathcal{J}})}. \tag{67}$$

Then,

$$\|\varsigma - r\|(1-\mu) \le \lambda_{\varrho(t)}\epsilon\varrho(t), \tag{68}$$

and so by assuming $c_{m_1} \coloneqq \lambda_{\varrho(t)}/(1-\mu)$,

$$\|\varsigma(t) - r(t)\| \le \frac{\lambda_{\varrho(t)}\varrho(t)}{(1-\mu)}\epsilon \coloneqq c_{m_1}\epsilon\varrho(t). \tag{69}$$

Therefore, the Hadamard VOFBVP (1) is UHRS stable w.r.t ϱ. This result completes the proof. □

5. Numerical Illustrative Example

Example 2. Consider the Hadamard VOFBVP (based on the VOFBVP (1)) as follows:

$$\begin{aligned}
&{}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)} r(t) + \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t) = 0, \\
&t \in \mathcal{U} \coloneqq [1, e], r(1) = 0, r(e) = 0.
\end{aligned} \tag{70}$$

Hence, $\mathcal{T} = e$ and

$$m_1(t, r) = \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t), \quad (t, r) \in [1, e] \times [0, +\infty), \tag{71}$$

$$\vartheta(t) \begin{cases} 1.2, & t \in \mathcal{U}_1 \coloneqq [1, 2], \\ 1.6, & t \in \mathcal{U}_2 \coloneqq]2, e]. \end{cases} \tag{72}$$

Then, we get

$$\begin{aligned}
&(\log t)^{1/4}|m(t, r) - m(t, \bar{r})| \\
&= \left|(\log t)^{1/4}\frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4}r(t) - (\log t)^{1/4}\frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} - \frac{1}{4}\bar{r}(t)\right| \\
&\le \frac{1}{4}|r(t) - \bar{r}(t)|.
\end{aligned} \tag{73}$$

(H2) holds with $\beta = 1/4$ and $K = 1/4$.

From (72), the Hadamard VOFBVP (70) is classified into the following:

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{1.2} r(t) + \dfrac{(\log t)^{1.2}}{\sqrt{\pi}} + \dfrac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_1, \\ {}^{\mathscr{H}}\mathfrak{D}_{2^+}^{1.6} r(t) + \dfrac{(\log t)^{1.6}}{\sqrt{\pi}} + \dfrac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_2. \end{cases} \tag{74}$$

For $t \in \mathcal{U}_1$, the Hadamard VOFBVP (70) is equivalent to the Hadamard COFBVP

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{1.2} r(t) + \dfrac{(\log t)^{1.2}}{\sqrt{\pi}} + \dfrac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_1, \\ r(1) = 0, & r(2) = 0. \end{cases} \tag{75}$$

Let us now show that condition (43) is satisfied. Clearly, the following value is obtained

$$\frac{K\left((\log \mathcal{T}_1)^{1-\beta} - (\log \mathcal{T}_0)^{1-\beta}\right)(\log \mathcal{T}_1 - \log \mathcal{T}_0)^{\vartheta_1 - 1}}{4^{\vartheta_1 - 1}(1-\beta)\Gamma(\vartheta_1)} = \frac{1/4(\log 2)^{3/4}(\log 2)^{0.2}}{\left(4^{0.2}\right)3/4\Gamma(1.2)} \simeq 0.1941 < 1. \tag{76}$$

On the other side, let $\varrho(t) = (\log t)^{1/2}$. In this case,

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\vartheta_1}\varrho(t) &= \frac{1}{\Gamma(1.2)}\int_1^2 \left(\log \frac{t}{s}\right)^{1.2-1} \frac{(\log t)^{1/2}}{s} \\ &\cdot ds \le \frac{(\log t)^{1/2}}{\Gamma(1.2)}\int_1^2 \left(\log \frac{2}{s}\right)^{0.2} \\ &\cdot \frac{ds}{s} \le \frac{(\log 2)^{1.2}}{\Gamma(2.2)}(\log t)^{1/2} \coloneqq \lambda_{\varrho(t)}\varrho(t). \end{aligned} \tag{77}$$

As a result, (H3) is fulfilled with $\varrho(t) = \sqrt{(\log t)}$ and $\lambda_{\varrho(t)} = (\log 2)^{1.2}/\Gamma(2.2) \in \mathbb{R}$.

Theorem 15 guarantees the existence of a solution for the Hadamard COFBVP (75) like $r_1 \in \mathcal{E}_1$, and from Theorem 17, the Hadamard constant-order system (75) is UHRS stable. For $t \in \mathcal{U}_2$, the Hadamard VOFBVP (70) can be written as the following COFBVP, i.e.,

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_{2^+}^{1.6} r(t) + \dfrac{(\log t)^{1.6}}{\sqrt{\pi}} + \dfrac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_2, \\ r(2) = 0, & r(e) = 0. \end{cases} \tag{78}$$

We see that

$$\frac{K\left((\log \mathcal{T}_2)^{1-\beta} - (\log \mathcal{T}_1)^{1-\beta}\right)(\log \mathcal{T}_2 - \log \mathcal{T}_1)^{\vartheta_2 - 1}}{4^{\vartheta_2 - 1}(1-\beta)\Gamma(\vartheta_2)} = \frac{1/4\left(1 - (\log 2)^{3/4}\right)(1 - \log 2)^{0.6}}{\left(4^{0.6}\right)(3/4)\Gamma(1.6)} \simeq 0.0191 < 1. \tag{79}$$

Accordingly, condition (43) is achieved on the subinterval $\mathcal{U}_2$. Further,

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{I}_{2^+}^{\vartheta_1}\varrho(t) &= \frac{1}{\Gamma(1.6)}\int_2^e \left(\log \frac{t}{s}\right)^{1.6-1} \frac{(\log t)^{1/2}}{s} \\ &\cdot ds \le \frac{(\log t)^{1/2}}{\Gamma(1.6)}\int_2^e \left(\log \frac{e}{s}\right)^{0.6} \\ &\cdot \frac{ds}{s} \le \frac{(\log (e/2))^{1.6}}{\Gamma(2.6)}(\log t)^{1/2} \coloneqq \lambda_{\varrho(t)}\varrho(t). \end{aligned} \tag{80}$$

As a result, (H3) is also valid with $\varrho(t) = \sqrt{(\log t)}$ and $\lambda_{\varrho(t)} = (\log (e/2))^{1.6}/\Gamma(2.6) \in \mathbb{R}$.

On account of Theorem 15, the Hadamard COFBVP (78) possesses a solution $\tilde{x}_2 \in \mathcal{E}_2$. Further, Theorem 17 yields that the mentioned Hadamard system (78) is UHRS stable. It is known that

$$r_2(t) = \begin{cases} 0, & t \in \mathcal{U}_1, \\ \tilde{r}_2(t), & t \in \mathcal{U}_2. \end{cases} \tag{81}$$

Consequently, the Hadamard VOFBVP (70) has a solution

$$r(t) = \begin{cases} r_1(t), & t \in \mathcal{U}_1, \\ r_2(t) = \begin{cases} 0, & t \in \mathcal{U}_1, \\ \tilde{r}_2(t), & t \in \mathcal{U}_2. \end{cases} \end{cases} \tag{82}$$

From Theorem 17, the Hadamard VOFBVP given by (70) is UHRS stable.

6. Conclusions

In this paper, the nonlinear Hadamard VOFBVP (1) was considered in which we established some theorems regarding existence and stability of solutions of it by following a new method based on the generalized subintervals and piecewise constant functions. By applying such notions, we converted the given Hadamard VOFBVP (1) to the standard Hadamard COFBVP (30). After investigating some specifications of the Green function, we focused on the solutions' existence via a combined technique in terms of KMNCS in the context of Darbo's fixed point criterion. The UHRS stability of the proposed Hadamard VOFBVP was also studied. At the end, a numerical example has been discussed to validate the applicability of our results. In future works, our results can be extended to other fractional mathematical models equipped with variable orders such as studies on simulations and dynamical behaviors of COVID-19.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Qualitative Analyses of Fractional Integrodifferential Equations with a Variable Order under the Mittag-Leffler Power Law

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This research paper intends to study some qualitative analyses for a nonlinear fractional integrodifferential equation with a variable order in the frame of a Mittag-Leffler power law. At first, we convert the considered problem of variable order into an equivalent standard problem of constant order using generalized intervals and piecewise constant functions. Next, we prove the existence and uniqueness of analytic results by application of Krasnoselskii's and Banach's fixed point theorems. Besides, the guarantee of the existence of solutions is shown by different types of Ulam-Hyer's stability. Then, we investigate sufficient conditions of positive solutions for the proposed problem. In the end, we discuss an example to illustrate the applicability of our obtained results.

1. Introduction

Fractional calculus and its applications [1, 2] has recently gained in popularity due to their applicability in modeling many complex phenomena in a wide range of science and engineering disciplines. Several biological models [3] and optimal control problems [4] have been presented in the literature through the development of fractional calculus. In order to describe the dynamics of real-world problems, new methods and techniques have been discovered. In particular, Caputo and Fabrizio in [5] investigated a new type of fractional derivatives (FDs) in the exponential kernel. There are some interesting properties of Caputo and Fabrizio that were discussed by Losada and Nieto in [6]. In [7], Atangana and Baleanu, investigated a new type and interesting FD with a Mittag-Leffler (ML) kernel. Atangana-Baleanu (AB) FD was extended to higher arbitrary order by Abdeljawad in [8], along with their associated integral operators. For some theoretical works on AB fractional operator, we refer the reader to a series of papers [9–13].

Variable order fractional operators can be seen as a natural analytical extension of constant order fractional operators. In recent years, variable order fractional operators have been designed and formalized mathematically only. After that, the applications of this effect rolled rapidly. In this regard, Lorenzo and Hartley [14] studied the behaviors of a fractional diffusion problem with fractional operators in the variable order. Afterward, different applications of variable order spaces of a fractional kind have shown up in striking and fascinating points of interest, see [15–20]. For instance, Sun et al. [21] introduced a comparative study on constant and variable order models to describe the memory identification of certain systems. The authors in [22] have formulated a nonlinear model of alcoholism in the frame of FDEs with variable order and discussed the solutions of such a model numerically and analytically. The authors in [23] analyzed a variable order mathematical fuzzy model through a computational approach for nonlinear fuzzy partial FDEs.

The probability of formulating evolutionary control equations has led to the effective application of these operators to model complex real-world problems going from mechanics to transition and control processes to theory and biology. For available applications of variable order fractional operators in the overall area of engineering and scientific modeling, see [24, 25]. Such broad and various applications promptly require a progression of systematic studies on the qualitative analyses of solutions of FDEs with variable order such as existence, uniqueness, and stability.

Recently, Li et al. [26], by a new numerical approach, have studied the following fractional problem

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) + a(\varkappa)\vartheta(\varkappa) = \mathscr{K}(\varkappa, \vartheta(\varkappa)), \quad \varkappa \in [0, 1], \\ B(\vartheta) = 0, \end{cases} \tag{1}$$

where ${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ is the Atangana-Baleanu FD with a variable order $\varrho(\varkappa)$ and $B(\vartheta)$ is the linear boundary condition.

Bouazza et al. [27] established the existence and stability results for a multiterm fractional BVP with a variable order of the form:

$$\begin{cases} {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), \mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in [0, b], \\ \vartheta(0) = 0, \vartheta(b) = 0, \end{cases} \tag{2}$$

where ${}^{C}\mathbf{D}_{0^+}^{\varrho(t)}, \mathbf{I}_{0^+}^{\varrho(t)}$ are Caputo's and Riemann-Liouville's operators of variable order $\varrho(t)$. The existence and Ulam-Hyers stability results of a Caputo-type problem (2) have been obtained by Benkerrouche et al. [28]. Kaabar et al. [29] investigated some qualitative analyses of solutions for the following implicit FDE with variable order

$$\begin{cases} {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in [0, b], \\ \vartheta(0) = 0, \vartheta(b) = 0, \end{cases} \tag{3}$$

where ${}^{C}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ is the Caputo FD of variable order $\varrho(\varkappa)$.

Cauchy's type of nonlocal problems can be used to explain differential laws in the development of systems, which is remarkable. Nonnegative quantities, such as the concentration of a species or the distribution of mass or temperature, are often described using these types of equations. In this regard, the first question to ask before analyzing any system or model of a real-world phenomenon is whether or not the problem actually exists. The answer to this question is given by the fixed point theory. We refer here to some results that dealt with the stability approach in the concept of Ulam-Hyers and others related to fixed point techniques, see [30–35].

Motivated by the argumentations above, we intend to analyze and investigate the sufficient conditions of solution for ML-type nonlinear fractional integrodifferential equations with a variable order of the form

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa), {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right), \varkappa \in \mathscr{E} \coloneqq (0, b], \\ \vartheta(0) = 0, \vartheta(b) = \sum_{j=1}^{n}\tau_j\vartheta(\kappa_j), \kappa_j \in (0, b), \end{cases} \tag{4}$$

where ${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}$ and ${}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}$ are the ML-type derivative and the ML-type integral of fractional variable order $\varrho(\varkappa) > 0$,, respectively, $\tau_j \in \mathbb{R}$, κ_j,$j = 1, 2, \cdots, n$ are prefixed points satisfying $0 < \kappa_1 \le \kappa_2 \le \cdots \le \kappa_i < b$ and $\mathscr{K} : \mathscr{E} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function fulfilling some later-specified assumptions.

Let $C(\mathscr{E}, \mathbb{R})$ be a Banach space of continuous functions $\vartheta : \mathscr{E} \longrightarrow \mathbb{R}$ equipped with the norm $\|\vartheta\| = \sup\{|\vartheta(\varkappa)|: \varkappa \in \mathscr{E}\}$.

Definition 1 (see [36]). Let $\varrho(\varkappa) \in (n-1, n]$,$\vartheta \in H^1(\mathscr{E})$. Then, the ML-type FD of a variable order $\varrho(\varkappa)$ for a function ϑ with the lower limit zero is defined by

$${}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \frac{\Upsilon(\varrho\varkappa)}{1-\varrho(\varkappa)}\int_0^{\varkappa} E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho}\right)\vartheta'(\theta)d\theta, \varkappa > 0, \tag{5}$$

respectively. The normalization function $\Upsilon(\varrho(\varkappa))$ satisfies $\Upsilon(0) = \Upsilon(1) = 1$, where $E_{\varrho(\varkappa)}$ is the ML function defined by

$$\mathbf{E}_{\varrho(\varkappa)}(\vartheta) = \sum_{i=0}^{\infty}\frac{\vartheta^i}{\Gamma(i\varrho(\varkappa)+1)}, \operatorname{Re}(\varrho(\varkappa)) > 0, \vartheta \in \mathbb{C}. \tag{6}$$

The correspondent ML fractional integral is given by

$${}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \frac{1-\varrho(\varkappa)}{\Upsilon(\varrho(\varkappa))}\vartheta(\varkappa) + \frac{\varrho(\varkappa)}{\Upsilon(\varrho(\varkappa))\Gamma(\varrho(\varkappa))}\int_0^{\varkappa}(\varkappa-s)^{\varrho(\varkappa)-1}\vartheta(s)ds. \tag{7}$$

For an integer $n \in \mathbb{N}$ and $\mathscr{B}$ is a partition of the interval $\mathscr{E}$ defined as

$$\mathscr{B} = \{\mathscr{E}_1 = [0, b_1], \mathscr{E}_2 = [b_1, b_2], \mathscr{E}_3 = [b_2, b_3], \ldots \cdots, \mathscr{E}_n = [b_{n-1}, b_n]\}. \tag{8}$$

Let $\varrho(\varkappa)$: $\mathscr{E} \longrightarrow (1, 2]$ be a piecewise constant function with respect to $\mathscr{B}$ such that

$$\varrho(\varkappa)=\sum_{l=1}^{n}\varrho_l Q_l(\varkappa)=\begin{cases}\varrho_1, & \text{if } \varkappa\in\mathscr{E}_1\\ \varrho_2, & \text{if } \varkappa\in\mathscr{E}_2\\ \quad . & \\ \quad . & \\ \quad . & \\ \varrho_n, & \text{if } \varkappa\in\mathscr{E}_n,\end{cases} \tag{9}$$

where $\varrho_l\in(1,2]$ are constants, and Q_l is the indicator of the interval $\mathscr{E}_l:=(b_{l-1},b_l]$, $l=1,2,\cdots,n$ (with $b_0=0$, $b_n=b$) such that

$$Q_l(\varkappa)=\begin{cases}1, & \text{if } \varkappa\in\mathscr{E}_l\\ 0, & \text{for } elsewhere.\end{cases} \tag{10}$$

Let $C(\mathscr{E}_l,\mathbb{R})$, $l\in\{1,2,\cdots,n\}$ be a Banach space of continuous functions $\vartheta:\mathscr{E}_l\longrightarrow\mathbb{R}$ equipped with the norm $\|\vartheta\|=\sup\{|\vartheta(\varkappa)|:\varkappa\in\mathscr{E}_l\}$. Then, for $\varkappa\in\mathscr{E}_l, l=1,2,\cdots,n$ the ML-type FD of variable order $\varrho(\varkappa)$ for a function $\vartheta\in C(\mathscr{E},\mathbb{R})$ defined by (5) can be written as a sum of left ML-type FD of constant orders ϱ_l, $l=1,2,\cdots,n$ as follows.

$$\begin{aligned}{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)&=\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho(\varkappa)}\right)\\&\quad\cdot\vartheta'(\theta)d\theta+.\cdots\cdots+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\\&\quad\cdot\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta.\end{aligned} \tag{11}$$

Thus, according to (11), the BVP (4) can be written for any $\varkappa\in\mathscr{E}_l, l=1,2,\cdots,n$ in the form

$$\begin{aligned}&\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b_1-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta+.\cdots\cdots\\&\quad+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b-\theta)^{\varrho(\varkappa)}\right)\vartheta'(\theta)d\theta\\&=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right).\end{aligned} \tag{12}$$

Let the function $\vartheta\in C(\mathscr{E}_l,\mathbb{R})$ be such that $\vartheta(\varkappa)=0$ on $\varkappa\in[0,b_{l-1}]$ and such that it solves the integral equation (12). Then, (12) is reduced to

$$\begin{cases}{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_1}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_1,\\ {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_2}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_2,\\ .\\ .\\ {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_n}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_n.\end{cases} \tag{13}$$

In our forthcoming analysis, we shall deal with the following BVP:

$$\begin{cases}{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa)=\mathscr{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa),{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa)\right),\varkappa\in\mathscr{E}_l\\ \vartheta(b_{l-1})=0,\vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j),\kappa_j\in(0,b).\end{cases} \tag{14}$$

Observe that, problem (4) is more general of problems (1), (2), and (3). In addition, it is assumed that it should be noted that due to the complexity of the computations and the division of the underlying time interval, not many papers can be found in the literature in which the authors focused on the existence and stability results of fractional variable order integrodifferential equations. To fill this gap, we investigate some qualitative analyses for the fractional variable order problem (14) in the frame of ML-type fractional operators. More precisely, we convert the ML-type fractional variable order problem into an equivalent standard ML-type fractional constant order problem using generalized intervals and piecewise constant functions. Then, we prove the existence and uniqueness of solutions for problem (14) via Krasnoselskii's and Banach's fixed point techniques. Also, we discuss the Ulam-Hyers stability result to the proposed problem. Further, we establish the sufficient conditions of positive solutions for problem (14).

The major contribution of this work is to develop the nonlocal fractional calculus with respect to variable order and learn more properties for the proposed ML-type fractional problems, which makes use of nonsingular kernel derivatives with fractional variable order. Already significant amount of work on constant fractional order for different operators has been done in literature. But to the best of our information, variable order problems have not been well studied so for fractional calculus. There is a waste gap between constant and variable fractional order problems in literature, the first one has got tremendous attention as compared to the second one. Very recently, the area of variable order has started attention to be investigated. In line with these developments, a new approach is used in this work to discuss some qualitative properties of solution for the considered problem. Multiple terms can be solved using this approach. To the best of our understanding, this is the first work dealing with the ML-type derivative with fractional variable order. The results of this work will therefore make a useful contribution to the existing literature on this subject.

The outline of our work is as follows. Some basic notions and axiom results are presented in Section 2. Our main results are obtained in Sections 3, 4, and 5 based on Krasnoselskii's and Banach's fixed point theorems. An illustrative example is fitted in Section 6. Concluding remarks about our results are in the final section.

2. Auxiliary Results

Definition 2. Problem (14) has a solution in $C(\mathscr{E},\mathbb{R})$, if there are functions $\vartheta_l, l=1,2,\cdots,n$, so that $\vartheta_l\in C(\mathscr{E}_l,\mathbb{R})$, satisfying Equation (12), and $\vartheta_l(0)=0, \vartheta_l(b)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j)$.

Lemma 3 (see [8]). *Let $\vartheta(\varkappa)$ be a function defined on $[0,b]$ and $n<\varrho\le n+1$, for some $n\in\mathbb{N}_0$, we have*

$$\left({}^{ML}\mathbf{I}_{0^+}^{\varrho\,ML}\mathbf{D}_{0^+}^{\varrho}\vartheta\right)(\varkappa)=\vartheta(\varkappa)-\sum_{i=0}^{n}\frac{\vartheta^{(i)}(0)}{i!}\varkappa^i. \tag{15}$$

Theorem 4 (see [37]). *Let $\mathscr{S}$ be closed subspace from a Banach space $\mathscr{K}$, and let $\Pi:\mathscr{S}\longrightarrow\mathscr{S}$ be a strict contraction such that*

$$\|\Pi(x)-\Pi(y)\|\le\rho\|x-y\|, \tag{16}$$

for some $0<\rho<1$, and for all $x,y\in\mathscr{S}$. Then, Π has a fixed point in $\mathscr{S}$.

Theorem 5 (see [38]). *Let K be a nonempty, closed, convex, and bounded subset of the Banach space $\mathfrak{X}$. If there are two operators Φ^1,Φ^2 such that*

(1) $\Phi^1u+\Phi^2v\in\mathfrak{X}$, for all $u,v\in\mathfrak{X}$,

(2) Φ^1 is compact and continuous

(3) Φ^2 is a contraction mapping

Then, there exists a function $z\in K$ such that $z=\Phi^1z+\Phi^2z$..

Remark 6. Let $u(\varkappa), v(\varkappa)\in C(\mathscr{E},\mathbb{R})$ be two functions. We notice that the semigroup property is not valid, meaning that

$${}^{ML}\mathbf{I}_{0^+}^{u(\varkappa)\,ML}\mathbf{I}_{0^+}^{v(\varkappa)}\vartheta(\varkappa)\neq{}^{ML}\mathbf{I}_{0^+}^{u(\varkappa)+v(\varkappa)}\vartheta(\varkappa). \tag{17}$$

Lemma 7 (see [8]). *Let $\varrho\in(1,2]$ and $\hbar\in C(\mathscr{E},\mathbb{R})$. Then, the following ML-type linear problem,*

$$\begin{cases}{}^{ML}\mathbf{D}_{a^+}^{\varrho}\vartheta(\varkappa)=\hbar(\varkappa), \vartheta(a)=c_1,\\ \vartheta'(a)=c_2,\end{cases} \tag{18}$$

is equivalent to the following integral equation

$$\vartheta(\varkappa)=c_1+c_2(\varkappa-a)+{}^{ML}\mathbf{I}_{a^+}^{\varrho}\hbar(\varkappa), \tag{19}$$

where

$${}^{ML}\mathbf{I}_{a^+}^{\varrho}\hbar(\varkappa)=\frac{2-\varrho}{\Upsilon(\varrho-1)}\int_a^{\varkappa}\hbar(s)ds+\frac{\varrho-1}{\Upsilon(\varrho-1)\Gamma(\varrho)}\int_a^{\varkappa}(\varkappa-s)^{\varrho-1}\hbar(s)ds. \tag{20}$$

Lemma 8. *Let $\varrho_l\in(1,2], l=1,2,\cdots,n$ and $\hbar\in C(\mathscr{E}_l,\mathbb{R})$ and let $\Theta=(b_l-b_{l-1})-\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})\neq0, \tau_j\in\mathbb{R}, \kappa_j\in(b_{l-1},b_l)$, with $b_0=0, b_n=b, j=1,2,\cdots,n$. Then, the following ML-type linear problem,*

$$\begin{cases}{}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta(\varkappa)=\hbar(\varkappa),\\ \vartheta(b_{l-1})=0, \vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j),\end{cases} \tag{21}$$

is equivalent to the following integral equation

$$\vartheta(\varkappa)=\frac{(\varkappa-b_{l-1})}{\Theta}\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\kappa_j)-{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(b_l)\right]+{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa), \tag{22}$$

where

$${}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa)=\frac{2-\varrho_l}{\Upsilon(\varrho_l-1)}\int_{b_{l-1}}^{\varkappa}\hbar(s)ds+\frac{\varrho_l-1}{\Upsilon(\varrho_l-1)\Gamma(\varrho_l)}\cdot\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\hbar(s)ds. \tag{23}$$

Proof. Suppose that $\vartheta\in C(\mathscr{E}_l,\mathbb{R})$ is a solution of problem (21). Applying the operator ${}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}$ to both sides of (21), we find

$$\vartheta(\varkappa)=c_1+c_2(\varkappa-b_{l-1})+{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa). \tag{24}$$

By the condition $\vartheta(b_{l-1})=0$, we get $c_1=0$.. Hence, Equation (24) reduces to

$$\vartheta(\varkappa)=c_2(\varkappa-b_{l-1})+{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\varkappa). \tag{25}$$

As per condition $\vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j)$, we obtain

$$c_2=\frac{1}{\Theta}\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(\kappa_j)-{}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}\hbar(b_l)\right]. \tag{26}$$

Substitute the values of c_1, c_2 into Equation (24), we get Equation (22). Conversely, we assume that ϑ satisfies Equation (22). Then, by applying the operator ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}$ on both sides of Equation (22) and using the fact ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}(\varkappa-b_{l-1})=0$,, we have

$$
\begin{aligned}
{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa) &= {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\frac{(\varkappa-b_{l-1})}{\Theta} \\
&\cdot\left[\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
&+{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\varkappa)\right)=\hbar(\varkappa).
\end{aligned}
\tag{27}
$$

As $\varkappa \longrightarrow \kappa_j$ in (25) and multiply by τ_j, we get

$$
\begin{aligned}
\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j) &= \frac{\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})}{\Theta} \\
&\cdot\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
&+\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)=\frac{(b_l-b_{l-1})-\Theta}{\Theta} \\
&\cdot\left[\sum_{j=1}^{n}\tau_j^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)-{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(b_l)\right] \\
&+\sum_{j=1}^{n}\tau_j\left({}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\hbar(\kappa_j)\right)=\vartheta(b_l).
\end{aligned}
\tag{28}
$$

□

Thus, nonlocal conditions are satisfied.

Theorem 9. *Let* $\varrho_l\in(1,2], l=1,2,\cdots,n$ *and* $\mathcal{K}:\mathcal{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ *be continuous function and* $\Theta=(b_l-b_{l-1})-\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})\neq 0, \tau_j\in\mathbb{R}, \kappa_j\in(b_{l-1},b_l)$*, with* $b_0=0, b_n=b, j=1,2,\cdots,n$*. If* $\vartheta\in C(\mathcal{E}_l,\mathbb{R})$ *is a solution of the following ML-type problem*

$$
\begin{cases}
{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)=\mathcal{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa),{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)\right), \varkappa\in\mathcal{E}_l \\
\vartheta(b_{l-1})=0, \vartheta(b_l)=\sum_{j=1}^{n}\tau_j\vartheta(\kappa_j),
\end{cases}
\tag{29}
$$

then, ϑ *satisfies the following fractional integral equation*

$$
\begin{aligned}
\vartheta(\varkappa) &= P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathcal{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathcal{K}_\vartheta(s)ds\right) \\
&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathcal{K}_\vartheta(s)ds\right. \\
&\left.-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathcal{K}_\vartheta(s)ds\right)+P_3\int_{b_{l-1}}^{\varkappa}|\mathcal{K}_\vartheta(s)| \\
&\cdot ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathcal{K}_\vartheta(s)ds,
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
\mathcal{K}_\vartheta(\varkappa) &= \mathcal{K}\left(\varkappa,\vartheta(\varkappa),{}^{ML}\mathbf{I}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa),{}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}}\vartheta(\varkappa)\right), \\
P_1 &= \frac{2-\varrho_l}{\Theta\Upsilon(\varrho_l-1)}, P_2=\frac{\varrho_l-1}{\Theta\Upsilon(\varrho_l-1)}, \\
P_3 &= \frac{2-\varrho_l}{\Upsilon(\varrho_l-1)}, P_4=\frac{\varrho_l-1}{\Upsilon(\varrho_l-1)\Gamma(\varrho_l)}.
\end{aligned}
\tag{31}
$$

3. Existence and Uniqueness of Solutions

This section is devoted to proving the existence and uniqueness theorems for the ML-type problem (14). For simplicity, we set

$$
\begin{aligned}
\mathcal{M} &= \left[\frac{(2-\varrho_l)(b_l-b_{l-1})}{\Upsilon(\varrho_l)}+\frac{(\varrho_l-1)(b_l-b_{l-1})}{\Upsilon(\varrho_l-1)\Gamma(\varrho_l+1)}\right], \\
\mathcal{R}_P &= P_1(b_l-b_{l-1})\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})+(b_l-b_{l-1})\right) \\
&+\frac{P_2(b_l-b_{l-1})}{\Gamma(\varrho_l+1)}\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})^{\varrho_l}+(b_l-b_{l-1})^{\varrho_l}\right) \\
&+\left(P_3(b_l-b_{l-1})+\frac{P_4}{\Gamma(\varrho_l+1)}(b_l-b_{l-1})^{\varrho_l}\right), \\
\Omega &= \mathcal{R}_P\frac{\mathfrak{N}_f(1+\mathcal{M})}{1-\mathfrak{N}_f}.
\end{aligned}
\tag{32}
$$

Theorem 10. *Suppose that*

$$
\begin{aligned}
&(H_1): |\mathcal{K}(\varkappa,x,y,z)-\mathcal{K}(\varkappa,\bar{x},\bar{y},\bar{z})| \\
&\leq\mathfrak{N}_f(|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|), \mathfrak{N}_f>0,
\end{aligned}
\tag{33}
$$

for all $x,y,z,\bar{x},\bar{y},\bar{z}\in C(\mathcal{E}_l,\mathbb{R})$*. Then, problem (14) has a unique solution provided that* $\Omega<1$*.*

Proof. As per Theorem 9, we define the operator $\mathbf{\Pi}:C(\mathcal{E}_l,\mathbb{R})\longrightarrow C(\mathcal{E}_l,\mathbb{R})$

$$
\begin{aligned}
(\mathbf{\Pi}\vartheta)(\varkappa) &= P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathcal{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathcal{K}_\vartheta(s)ds\right) \\
&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathcal{K}_\vartheta(s)\right. \\
&\left.\cdot ds-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathcal{K}_\vartheta(s)ds\right) \\
&+P_3\int_{b_{l-1}}^{\varkappa}|\mathcal{K}_\vartheta(s)|ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}|\mathcal{K}_\vartheta(s)|ds.
\end{aligned}
\tag{34}
$$

Let us consider a closed ball Π_φ as

$$\mathbf{K}_{\eta_l} = \{\vartheta \in C(\mathscr{E}_l, \mathbb{R}) \colon \|\vartheta\| \leq \eta_l\}, \tag{35}$$

with the radius $\eta_l \geq \Omega_1/1 - \Omega$, where

$$\Omega = \mathscr{R}_P \omega_f, \tag{36}$$

and $\omega_f = \max_{\varkappa \in \mathscr{E}_l} |\mathscr{K}_\vartheta(0)|$. Now, we show that $\mathbf{\Pi K}_{\eta_l} \subset \mathbf{K}_{\eta_l}$. For all $\vartheta \in \mathbf{K}_{\eta_l}$ and $\varkappa \in \mathscr{E}_l$, we have

$$\begin{aligned} |(\mathbf{\Pi}\vartheta)(\varkappa)| &= P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} |\mathscr{K}_\vartheta(s)| ds + \int_{b_{l-1}}^{b_l} |\mathscr{K}_\vartheta(s)| ds \right) \\ &\quad + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| \right. \\ &\quad \left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \right) \\ &\quad + P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds. \end{aligned} \tag{37}$$

By (H_1) and definition of ${}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l}$ in the case of $\varrho_l \in (1, 2]$ defined as Equation (23), we have

$$\begin{aligned} &\left| \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l} \vartheta(\varkappa), {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l} \vartheta(\varkappa)\right) \right| \\ &= \left| \mathscr{K}\left(\varkappa, \vartheta(\varkappa), {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l} \vartheta(\varkappa), {}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l} \vartheta(\varkappa)\right) - f(\varkappa, 0, 0, 0) \right| \\ &\quad + |f(\varkappa, 0, 0, 0)| \leq \frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f} |\vartheta(\varkappa)| + \omega_f. \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \|\mathbf{\Pi}\vartheta\| &\leq P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} |\mathscr{K}_\vartheta(s)| ds + \int_{b_{l-1}}^{b_l} |\mathscr{K}_\vartheta(s)| ds \right) \\ &\quad + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| \right. \\ &\quad \left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \right) \\ &\quad + P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} |\mathscr{K}_\vartheta(s)| ds \\ &\leq \Omega \eta_l + \Omega_1 \leq \eta_l. \end{aligned} \tag{39}$$

Thus $\mathbf{\Pi}\vartheta \in D_l$. Now, we need to prove that $\mathbf{\Pi}$ is a contraction map. Let $\vartheta, \widehat{\vartheta} \in D_l$ and $\varkappa \in \mathscr{E}_l$ .Then, we have

$$\begin{aligned} &\left| (\mathbf{\Pi}\vartheta)(\varkappa) - \left(\mathbf{\Pi}\widehat{\vartheta}\right)(\varkappa) \right| \leq P_1(\varkappa - b_{l-1}) \\ &\quad \cdot \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds + \int_{b_{l-1}}^{b_l} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds \right) \\ &\quad + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \right. \\ &\quad \left. \cdot ds + \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds \right) + P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s) \\ &\quad - \mathscr{K}_{\widehat{\vartheta}}(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| ds. \end{aligned} \tag{40}$$

From our assumption, we obtain

$$\begin{aligned} &\left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \leq \mathfrak{N}_f \left(\left| \vartheta(s) - \widehat{\vartheta}(s) \right| + \left| {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l} \vartheta(\varkappa) - {}^{ML}\mathbf{I}_{b_{l-1}^+}^{\varrho_l} \widehat{\vartheta}(\varkappa) \right| \right. \\ &\quad \left. + \left| \mathscr{K}_\vartheta(s) - \mathscr{K}_{\widehat{\vartheta}}(s) \right| \right) \leq \frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f} \left\| \vartheta - \widehat{\vartheta} \right\|. \end{aligned} \tag{41}$$

Hence,

$$\left\| \mathbf{\Pi}\vartheta - \mathbf{\Pi}\widehat{\vartheta} \right\| \leq \Omega \left\| \vartheta - \widehat{\vartheta} \right\|. \tag{42}$$

Due to $\Omega < 1$, we conclude that $\mathbf{\Pi}$ is a contraction mapping. Hence, by the Banach fixed point Theorem 4, $\mathbf{\Pi}$ has a unique fixed point. □

Theorem 11. *Under the hypotheses of Theorem 10, the ML-type problem (14) has at least one solution.*

Proof. Let us consider the operator $\mathbf{\Pi}$ defined by Theorem 10. Now, we will divided the operator $\mathbf{\Pi}$ into two operators $\mathbf{\Pi}_1, \mathbf{\Pi}_2$ such that $(\mathbf{\Pi}\vartheta)(\varkappa) = (\mathbf{\Pi}_1\vartheta)(\varkappa) + (\mathbf{\Pi}_2\vartheta)(\varkappa)$, where

$$\begin{aligned} (\mathbf{\Pi}_1\vartheta)(\varkappa) &= P_1(\varkappa - b_{l-1}) \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} \mathscr{K}_\vartheta(s) ds - \int_{b_{l-1}}^{b_l} \mathscr{K}_\vartheta(s) ds \right) + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \\ &\quad \cdot \left(\sum_{j=1}^n \tau_j \int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds - \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds \right), \\ (\mathbf{\Pi}_2\vartheta)(\varkappa) &= P_3 \int_{b_{l-1}}^{\varkappa} |\mathscr{K}_\vartheta(s)| ds + \frac{P_4}{\Gamma(\varrho_l)} \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l - 1} \mathscr{K}_\vartheta(s) ds. \end{aligned} \tag{43}$$

□

Consider a closed ball $\mathbf{K}_{\eta_l}$ defined as in Theorem 10. In order to fulfill the conditions in Theorem 5, we split the proof into the following steps:

Step 1. $\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$ for all $\vartheta,\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$. First, in order to operator $\mathbf{\Pi}_1$. For $\vartheta\in\mathbf{K}_{\eta_l}$ and $\varkappa\in\mathscr{E}_l$, we have

$$|(\mathbf{\Pi}_1\vartheta)(\varkappa)|\leq P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}|\mathscr{K}_\vartheta(s)|ds\right)$$
$$+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\right). \quad (44)$$

By Equation (38), we have

$$\|\mathbf{\Pi}_1\vartheta\|\leq\left[P_1(b_l-b_{l-1})\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})+(b_l-b_{l-1})\right)+\frac{P_2(b_l-b_{l-1})}{\Gamma(\varrho_l+1)}\left(\sum_{j=1}^{n}\tau_j(\kappa_j-b_{l-1})^{\varrho_l}+(b_l-b_{l-1})^{\varrho_l}\right)\right]\cdot\left(\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}\eta_l+\omega_f\right). \quad (45)$$

Next, for the operator $\mathbf{\Pi}_2$, we have

$$\|\mathbf{\Pi}_2\vartheta\|\leq\left(P_3(b_l-b_{l-1})+\frac{P_4}{\Gamma(\varrho_l+1)}(b_l-b_{l-1})^{\varrho_l}\right)\left(\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}\eta_l+\omega_f\right). \quad (46)$$

Inequalities (45) and (46) give

$$\|\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\vartheta\|\leq\|\mathbf{\Pi}_1\vartheta\|+\|\mathbf{\Pi}_2\vartheta\|<\eta_l. \quad (47)$$

Thus, $\mathbf{\Pi}_1\vartheta+\mathbf{\Pi}_2\widehat{\vartheta}\in\mathbf{K}_{\eta_l}$.

Step 2. $\mathbf{\Pi}_2$ is a contraction map. Due to the operator $\mathbf{\Pi}$ is a contraction map, we conclude that $\mathbf{\Pi}_1$ is contraction too.

Step 3. $\mathbf{\Pi}_1$ is continuous and compact. Since $\mathscr{K}_\vartheta$ is continuous, $\mathbf{\Pi}_1$ is continuous too. Also, by Equation (45), $\mathbf{\Pi}_1$ is uniformly bounded on $\mathbf{K}_{\eta_l}$. Now, we show that $\mathbf{\Pi}_1(\mathbf{K}_{\eta_l})$ is an equicontinuous. For this purpose, let $\vartheta\in\mathbf{K}_{\eta_l}, a\leq\varkappa_1<\varkappa_2\leq b$. Then, we have

$$|(\mathbf{\Pi}_1\vartheta)(\varkappa_2)-(\mathbf{\Pi}_1\vartheta)(\varkappa_1)|\leq P_1[(\varkappa_2-b_{l-1})-(\varkappa_1-b_{l-1})]\cdot\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}\mathscr{K}_\vartheta(s)ds\right)+\frac{P_2(\varkappa_2-b_{l-1})-(\varkappa_1-b_{l-1})}{\Gamma(\varrho_l)}\cdot\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds-\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_\vartheta(s)ds\right). \quad (48)$$

Thus,

$$\|(\mathbf{\Pi}_1\vartheta)(\varkappa_2)-(\mathbf{\Pi}_1\vartheta)(\varkappa_1)\|\longrightarrow 0\ as\,\varkappa_2\longrightarrow\varkappa_1. \quad (49)$$

In view of the previous steps along with the theorem of Arzela-Ascoli, we deduce that $(\mathbf{\Pi}_1\mathbf{K}_{\eta_l})$ is relatively compact. Consequently, $\mathbf{\Pi}_1$ is completely continuous. Hence, by Theorem 5, there exists a solution ϑ_l of problem (14). For $l\in\{1,2,\cdots,n\}$, we define the function

$$\vartheta_l=\begin{cases}0, & \varkappa\in[0,b_{l-1}],\\ \tilde{\vartheta}_l, & \varkappa\in\mathscr{E}_l.\end{cases} \quad (50)$$

As a result of this, it is well known that $\vartheta_l\in C(\mathscr{E}_l,\mathbb{R})$ given by Equation (50) satisfies the following problem

$$\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\int_0^{b_1}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(\varkappa_1-\theta)^{\varrho(\varkappa)}\right)\vartheta_l'(\theta)d\theta+\cdots\cdots+\frac{\Upsilon(\varrho(\varkappa))}{1-\varrho(\varkappa)}\cdot\int_{b_{l-1}}^{\varkappa}E_{\varrho(\varkappa)}\left(\frac{\varrho(\varkappa)}{\varrho(\varkappa)-1}(b-\theta)^{\varrho(\varkappa)}\right)\vartheta_l'(\theta)d\theta$$
$$=\mathscr{K}\left(\varrho,\vartheta_l(\varrho),{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta_l(\varrho),{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta_l(\varrho)\right), \quad (51)$$

where ϑ_l is a solution to Equation (12) equipped with $\vartheta_l(0)=\vartheta_l(b_l)=\tilde{\vartheta}_l(b_l)=0$. Then,

$$\vartheta(\varkappa)=\begin{cases}\vartheta_1(\varkappa), & \varkappa\in\mathscr{E}_1,\\ \vartheta_2(\varkappa)= & \begin{cases}0, & \varkappa\in\mathscr{E}_1,\\ \tilde{\vartheta}_2, & \varkappa\in\mathscr{E}_2,\end{cases}\\ \vdots & \\ \vartheta_n(\varkappa)= & \begin{cases}0, & \varkappa\in[0,b_{l-1}],\\ \tilde{\vartheta}_n, & \varkappa\in\mathscr{E}_n,\end{cases}\end{cases} \quad (52)$$

is the solution of problem (14).

4. Stability Results for ML-Type Problem (14)

In this section, we discuss Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) stability results for problem (14). Let $\varepsilon>0$ and ϑ be a function such that $\vartheta(\varkappa)\in C(\mathscr{E}_l,\mathbb{R})$ satisfies the following inequations:

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa)-\mathscr{K}_\vartheta(\varkappa)\right|\leq\varepsilon,\varkappa\in\mathscr{E}. \quad (53)$$

Define the functions $\vartheta_l(\varkappa)$, $\widehat{\vartheta}_l(\varkappa)$, $\Bbbk_l(\varkappa)$,$\varkappa\in\mathscr{E}_l$ as follows.

$$\vartheta_l(\varkappa)=\begin{cases}0, & \text{if}\,\varkappa\in[0,b_{l-1}],\\ \vartheta(\varkappa) & \text{if}\,\varkappa\in\mathscr{E}_l,\end{cases} \quad (54)$$

$$\widehat{\vartheta}_l(\varkappa) = \begin{cases} 0, & \text{if } \varkappa \in [0, b_{l-1}], \\ \widehat{\vartheta}(\varkappa) & \text{if } \varkappa \in \mathscr{E}_l, \end{cases} \tag{55}$$

$$\Bbbk_l(\varkappa) = \begin{cases} 0, & \text{if } \varkappa \in [0, b_{l-1}], \\ \Bbbk(\varkappa) & \text{if } \varkappa \in \mathscr{E}_l. \end{cases} \tag{56}$$

Definition 12. Problem (14) is UH stable if there exists a real number $C_{\mathscr{K}} > 0$ such that, for each $\varepsilon > 0$ and for each solution $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ of inequality (53), there exists a unique solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of problem (14) with

$$\left|\widehat{\vartheta}(\varkappa) - \vartheta(\varkappa)\right| \leq C_{\mathscr{K}}\varepsilon, \tag{57}$$

where ϑ and $\widehat{\vartheta}$ are defined by Equation (54) and Equation (55), respectively.

Remark 13. Let $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ be the solution of inequality (53) if and only if we have a function $\Bbbk \in C(\mathscr{E}_l, \mathbb{R})$ which depends on ϑ such that

(i) $|\Bbbk(\varkappa)| \leq \varepsilon$ for all $\varkappa \in \mathscr{E}_l$

(ii) ${}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}} \widehat{\vartheta}(\varkappa) = \mathscr{K}_{\widehat{\vartheta}}(\varkappa) + \Bbbk(\varkappa)$, for all $\varkappa \in \mathscr{E}$.

Lemma 14. *If $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ is a solution of inequality (53), then ϑ satisfies the following inequality*

$$\left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right| \leq \varepsilon\mathscr{R}_P, \tag{58}$$

where

$$\begin{aligned}\Psi_{\widehat{\vartheta}} &= P_1(\varkappa - b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j} \mathscr{K}_{\widehat{\vartheta}}(s)ds - \int_{b_{l-1}}^{b_l} \mathscr{K}_{\widehat{\vartheta}}(s)ds\right) \\ &\quad + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds - \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right).\end{aligned} \tag{59}$$

Proof. As per Remark 13, we have

$$\begin{cases} {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}} \widehat{\vartheta}(\varkappa) = \mathscr{K}_{\widehat{\vartheta}}(\varkappa) + \Bbbk(\varkappa), \varkappa \in \mathscr{E}_l \\ \widehat{\vartheta}(b_{l-1}) = \widehat{\vartheta}(b_l) = 0. \end{cases} \tag{60}$$

Then, by Lemma 8, we get

$$\begin{aligned}\widehat{\vartheta}(\varkappa) &= \Psi_{\widehat{\vartheta}} + P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\widehat{\vartheta}}(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s) \\ &\quad \cdot ds + P_1(\varkappa - b_{l-1})\left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j} \Bbbk(s)ds - \int_{b_{l-1}}^{b_l} \Bbbk(s)ds\right) + \frac{P_2(\varkappa - b_{l-1})}{\Gamma(\varrho_l)} \\ &\quad \cdot \left(\sum_{j=1}^{n}\tau_j\int_{b_{l-1}}^{\kappa_j} (\kappa_j - s)^{\varrho_l-1}\Bbbk(s)ds - \int_{b_{l-1}}^{b_l} (b_l - s)^{\varrho_l-1}\Bbbk(s)ds\right) \\ &\quad + P_3\int_{b_{l-1}}^{\varkappa} \Bbbk(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\Bbbk(s)ds,\end{aligned} \tag{61}$$

which implies

$$\left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\right| \leq \varepsilon\mathscr{R}_P. \tag{62}$$

□

Theorem 15. *Suppose that (H_1) holds. If*

$$\left(P_3(b_l - b_{l-1}) + \frac{P_4(b_l - b_{l-1})^{\varrho_l}}{\Gamma(\varrho_l + 1)}\right)\frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f} < 1, \tag{63}$$

then the ML-type problem (14) is UH and GUH stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ be a function that satisfies the inequality (53), and let $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ be the unique solution of the following problem.

$$\begin{cases} {}^{ML}\mathbf{D}^{\varrho_l}_{b^+_{l-1}} \vartheta(\varkappa) = \mathscr{K}_{\vartheta}(\varkappa), \varkappa \in \mathscr{E}_l \\ \vartheta(b_{l-1}) = \widehat{\vartheta}(b_{l-1}) = 0 \\ \vartheta(b_l) = \widehat{\vartheta}(b_l) = 0. \end{cases} \tag{64}$$

Then, by Lemma 8, the solution of Equation (64) is given by

$$\vartheta(\varkappa) = \Psi_{\widehat{\vartheta}} + P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\vartheta}(s)ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\vartheta}(s)ds. \tag{65}$$

Hence, by Lemma 14, we have

$$\begin{aligned}\left|\widehat{\vartheta}(\varkappa) - \vartheta(\varkappa)\right| &\leq \left|\widehat{\vartheta}(\varkappa) - \Psi_{\widehat{\vartheta}} - P_3\int_{b_{l-1}}^{\varkappa} \mathscr{K}_{\widehat{\vartheta}}(s)ds - \frac{P_4}{\Gamma(\varrho_l)}\right. \\ &\quad \cdot \int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\mathscr{K}_{\widehat{\vartheta}}(s)ds\| + P_3\int_{b_{l-1}}^{\varkappa} \left|\mathscr{K}_{\widehat{\vartheta}}(s) - \mathscr{K}_{\vartheta}(s)\right| \\ &\quad \cdot ds + \frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa} (\varkappa - s)^{\varrho_l-1}\left|\mathscr{K}_{\widehat{\vartheta}}(s) - \mathscr{K}_{\vartheta}(s)\right|ds \leq \varepsilon\mathscr{R}_P \\ &\quad + \left(P_3(b_l - b_{l-1}) + \frac{P_4}{\Gamma(\varrho_l + 1)}(b_l - b_{l-1})^{\varrho_l}\right)\frac{\mathfrak{N}_f(1 + \mathscr{M})}{1 - \mathfrak{N}_f}\left\|\widehat{\vartheta} - \vartheta\right\|.\end{aligned} \tag{66}$$

Thus,

$$\left\|\widehat{\vartheta}-\vartheta\right\| \leq C_{\mathscr{K}}\varepsilon, \tag{67}$$

where

$$C_{\mathscr{K}}=\frac{\mathscr{R}_P}{1-\left(P_3(b_l-b_{l-1})+\left(P_4/\Gamma(\varrho_l+1)\right)(b_l-b_{l-1})^{\varrho_l}\right)\left(\mathfrak{N}_f(1+\mathscr{M})/1-\mathfrak{N}_f\right)}>0. \tag{68}$$

Therefore, the ML-type problem (14) is UH stable. Finally, by choosing $C_{\mathscr{K}}(\varepsilon)=C_{\mathscr{K}}\varepsilon$ such that $C_{\mathscr{K}}(0)=0$, then the ML-type problem (14) has GUH stability. □

5. Existence of Positive Solution for ML-Type Problem (14)

In this section, we extend and develop the sufficient conditions of the existence and uniqueness of positive solution for problem (14). For the forthcoming analysis, the following assumptions must be satisfied:

(V_1): $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is continuous function.

(V_2): There exists constants numbers $n_1, n_1>0, n_1\neq n_2$ such that

$$n_1\leq\mathscr{K}_\vartheta(\varkappa)\leq n_2.$$

$$(V_3):\ \Omega=\mathscr{R}_P\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}. \tag{69}$$

Define the cone $\mathscr{P}\subset C(\mathscr{E}_l,\mathbb{R})$ as

$$\mathscr{P}=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}):\ \vartheta(\varkappa)\geq 0, \varkappa\in[0,b]\}. \tag{70}$$

Lemma 16. *Assume that (V_1)-(V_2) hold. Then, $\mathbf{\Pi}:\mathscr{P}\longrightarrow\mathscr{P}$ is completely continuous.*

Proof. By Theorem 11, we conclude $\mathbf{\Pi}:\mathscr{P}\longrightarrow\mathscr{P}$ is completely continuous due to $\mathbf{\Pi}:C(\mathscr{E}_l,\mathbb{R})\longrightarrow C(\mathscr{E}_l,\mathbb{R})$ is completely continuous, since $\mathscr{P}\subset C(\mathscr{E}_l,\mathbb{R})$. □

Theorem 17. *Assume that (V_1)-(V_3) hold. Then, (14) has at least one positive solution.*

Proof. First, we have $\mathbf{\Pi}$ is compact due to Lemma 16. Next, we define two sets $\mathscr{A}_1,\mathscr{A}_2$ such that $\mathscr{A}_1=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}):\ \|\vartheta\|\leq n_1\Omega\}$ and $\mathscr{A}_2=\{\vartheta\in C(\mathscr{E}_l,\mathbb{R}):\ \|\vartheta\|\leq n_2\Omega\}$. Now, for $\vartheta\in\mathscr{P}\cap\partial\mathscr{A}_2$, we have $0\leq\vartheta(\varkappa)\leq n_2\Omega, \varkappa\in\mathscr{E}_l$. Since $\mathscr{K}_\vartheta(\varkappa)\leq n_2$, we have

$$\begin{aligned}|(\mathbf{\Pi}\vartheta)(\varkappa)|&\leq P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}|\mathscr{K}_\vartheta(s)|ds\right)\\&+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}|\mathscr{K}_\vartheta(s)|ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}|\mathscr{K}_\vartheta(s)|ds\leq\mathscr{R}_P\frac{\mathfrak{N}_f(1+\mathscr{M})}{1-\mathfrak{N}_f}n_2\leq\Omega n_2.\end{aligned} \tag{71}$$

Hence, $\|\mathbf{\Pi}\vartheta\|\leq\Omega n_2$. Next, for $\vartheta\in\mathscr{P}\cap\partial\mathscr{A}_1$, we have $0\leq\vartheta(\varkappa)\leq n_1\Omega, \varkappa\in[0,b]$. Since $\mathscr{K}_\vartheta(\varkappa)\geq n_1$, we have $\|\mathbf{\Pi}\vartheta\|\geq\Omega n_1$. Thus, the operator $\mathbf{\Pi}$ has a fixed point in $\mathscr{P}\cap(\bar{\mathscr{A}}_2\setminus\mathscr{A}_1)$. ,which implies that the ML-problem (14) has a positive solution. □

Theorem 18. *Let $\varrho_l\in(1,2], l=1,2,3,\cdots,n$ and $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is nondecreasing continuous function for each $\varkappa\in\mathscr{E}_l$ and let $\vartheta^*,\vartheta_*\in\mathscr{P}$ such that $0<\vartheta_*<\vartheta^*<b$, $\varkappa\in\mathscr{E}_l$, satisfying ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta^*(\varkappa)\leq\vartheta^*$ and ${}^{ML}\mathbf{D}_{b_{l-1}^+}^{\varrho_l}\vartheta_*(\varkappa)\geq\vartheta_*$. Then, problem (14) has a positive solution.*

Proof. Let $\vartheta^*,\vartheta_*\in\mathscr{P}$ such that $0<\vartheta_*<\vartheta^*<b$. Then, we have

$$\begin{aligned}(\mathbf{\Pi}\vartheta_*)(\varkappa)&=P_1(\varkappa-b_{l-1})\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_{\vartheta_*}(s)ds+\int_{b_{l-1}}^{b_l}\mathscr{K}_{\vartheta_*}(s)ds\right)+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_{\vartheta_*}(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_{\vartheta_*}(s)ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\vartheta_*}(s)ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathscr{K}_{\vartheta_*}(s)ds\leq P_1(\varkappa-b_{l-1})\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}\mathscr{K}_{\vartheta^*}(s)ds+\int_{b_{l-1}}^{b_l}\mathscr{K}_{\vartheta^*}(s)ds\right)+\frac{P_2(\varkappa-b_{l-1})}{\Gamma(\varrho_l)}\\&\cdot\left(\sum_{j=1}^n\tau_j\int_{b_{l-1}}^{\kappa_j}(\kappa_j-s)^{\varrho_l-1}|\mathscr{K}_{\vartheta^*}(s)|ds+\int_{b_{l-1}}^{b_l}(b_l-s)^{\varrho_l-1}\mathscr{K}_{\vartheta^*}(s)ds\right)\\&+P_3\int_{b_{l-1}}^{\varkappa}\mathscr{K}_{\vartheta^*}(s)ds+\frac{P_4}{\Gamma(\varrho_l)}\int_{b_{l-1}}^{\varkappa}(\varkappa-s)^{\varrho_l-1}\mathscr{K}_{\vartheta^*}(s)ds=(\mathbf{\Pi}\vartheta^*)(\varkappa).\end{aligned} \tag{72}$$

Thus, $(\mathbf{\Pi}\vartheta_*)(\varkappa)\leq(\mathbf{\Pi}\vartheta^*)(\varkappa)$. According to Theorem 1.3 in [39], the operator $\mathbf{\Pi}$ is compact and hence $\mathbf{\Pi}$ has a fixed point in the ordered Banach space $\langle\vartheta_*,\vartheta^*\rangle$. Thus, $\mathbf{\Pi}:\langle\vartheta_*,\vartheta^*\rangle\longrightarrow\langle\vartheta_*,\vartheta^*\rangle$ is compact. Accordingly, $\mathbf{\Pi}$ has a fixed point $\vartheta\in\langle\vartheta_*,\vartheta^*\rangle$. Thus, problem (14) has at least one positive solution. □

Corollary 19. *Let $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ be nondecreasing continuous function in $\mathscr{E}_l$. If*

$$0<\lim_{\vartheta\longrightarrow\infty}\mathscr{K}_\vartheta(\varkappa)<\infty, \varkappa\in\mathscr{E}_l, \tag{73}$$

then problem (14) has at least one positive solution.

Corollary 20. *Let $\mathscr{K}:\mathscr{E}_l\times\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ be nondecreasing continuous function in $\mathscr{E}_l$. If*

$$0<\lim_{\|\vartheta\|\longrightarrow\infty}\frac{\mathscr{K}_\vartheta(\varkappa)}{\|\vartheta\|}<\infty, \varkappa\in\mathscr{E}_l, \tag{74}$$

then problem (14) has at least one positive solution.

Corollary 21. *If there exist constants $m_1,m_2>0,\rho\in(0,1]$ such that $\mathscr{K}_\vartheta(\varkappa)=m_1\vartheta(\varkappa)+m_1^\rho$, then problem (14) has at least one positive solution.*

Corollary 22. *If the function $\mathscr{K} : \mathscr{E}_l \times \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ is continuous and there is a constant $\mathfrak{N}_f > 0$ and*

$$|\mathscr{K}(\varkappa, x, y, z) - \mathscr{K}(\varkappa, \bar{x}, \bar{y}, \bar{z})| \le \mathfrak{N}_f(|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|), \mathfrak{N}_f > 0, \tag{75}$$

for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in C(\mathscr{E}_l, \mathbb{R}^+)$ such that $\mathscr{R}_P(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f) < 1$, then problem (14) has a positive solution (Theorem 10).

Corollary 23. *Assume that there exists two continuous functions f_1, f_2 such that $0 < f_1(\varkappa) \le \mathscr{K}_\vartheta(\varkappa) \le f_2(\varkappa), \varkappa \in \mathscr{E}_l$. Then, problem (14) has at least one positive solution $\vartheta(\varkappa) \in C(\mathscr{E}_l, \mathbb{R})$.*

6. An Example

Example 24. Let $\varrho \in (1, 2]$ and let us consider the following ML-type problem.

$$\begin{cases} {}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^{\varkappa}}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)\right|}{1+{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)} + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}\right) \\ \vartheta(0) = 0, \vartheta(1) = 0. \end{cases} \tag{76}$$

Here, $a = 0, b = 2$ and

$$\mathscr{K}_\vartheta(\varkappa) = \frac{\varkappa^2}{10e^{\varkappa}}\left(e^{-\varkappa} + \frac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \frac{{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)} + \frac{\%{}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}{1+{}^{ML}D_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa)}\right). \tag{77}$$

Let $\varkappa \in [0, 2]$, and $\vartheta, \bar{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$. Then,

$$\begin{aligned} |\mathscr{K}_\vartheta(\varkappa) - \mathscr{K}_{\bar{\vartheta}}(\varkappa)| \le{}& \frac{1}{20}\left(\left|\vartheta(\varkappa) - \bar{\vartheta}(\varkappa)\right|\right. \\ &+ \left|{}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) - {}^{ML}\mathbf{I}_{0^+}^{\varrho(\varkappa)}\bar{\vartheta}(\varkappa)\right| + \left.\left|{}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\vartheta(\varkappa) - {}^{ML}\mathbf{D}_{0^+}^{\varrho(\varkappa)}\bar{\vartheta}(\varkappa)\right|\right). \end{aligned} \tag{78}$$

Therefore, (H_1) holds with $\mathfrak{N}_f = 1/20$. Here,

$$\varrho(\varkappa) = \begin{cases} \dfrac{3}{2}, & \text{if } \varkappa \in (0, 1], \\ \dfrac{5}{2} & \text{if } \varkappa \in (1, 2]. \end{cases} \tag{79}$$

For $l = 1$, we have

$$\begin{cases} {}^{ML}D_{0^+}^{\frac{3}{2}}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^{\varkappa}}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{3/2}\vartheta(\varkappa)\right|}{} 1 + {}^{ML}\mathbf{I}_{0^+}^{\frac{3}{2}}\vartheta(\varkappa) + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{3/2}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{D}_{0^+}^{3/2}\vartheta(\varkappa)}\right), \varkappa \in (0, 1] \\ \vartheta(0) = 0, \vartheta(1) = 0 \end{cases} \tag{80}$$

Also, $\Omega = 0.68 < 1$. Thus, all conditions of Theorem 10 are satisfied, and hence, the ML-type problem (14) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ satisfies

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa) - \mathscr{K}_\vartheta(\varkappa)\right| \le \varepsilon. \tag{81}$$

There exists a solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of the ML-type problem (14) with

$$\left\|\widehat{\vartheta} - \vartheta\right\| \le C_{\mathscr{K}}\varepsilon, \tag{82}$$

where

$$C_{\mathscr{K}} = \frac{\mathscr{R}_P}{1 - \left(P_3(b_l - b_{l-1}) + \left(P_4(b_l - b_{l-1})^{\varrho_l}/\Gamma(\varrho_l + 1)\right)\right)\left(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f\right)} > 0. \tag{83}$$

Therefore, all conditions in Theorem 15 are satisfied, and hence, the ML-problem (14) is UH stable.

Next, for $l = 2$,, we have

$$\begin{cases} {}^{ML}\mathbf{D}_{0^+}^{\frac{5}{2}}\vartheta(\varkappa) = \dfrac{\varkappa^2}{20e^{\varkappa}}\left(e^{-\varkappa} + \dfrac{|\vartheta(\varkappa)|}{1+|\vartheta(\varkappa)|} + \dfrac{\left|{}^{ML}\mathbf{I}_{0^+}^{5/2}\vartheta(\varkappa)\right|}{1+{}^{ML}\mathbf{I}_{0^+}^{5/2}\vartheta(\varkappa)} + \dfrac{{}^{ML}\mathbf{D}_{0^+}^{5/2}\vartheta(\varkappa)}{1+{}^{ML}\mathbf{D}_{0^+}^{5/2}\vartheta(\varkappa)}\right), \varkappa \in (1, 2] \\ \vartheta(0) = 0, \vartheta(1) = 0, \end{cases} \tag{84}$$

and $\Omega = 0.55 < 1$. Thus, all conditions of Theorem 10 are satisfied, and hence, the ML-type problem (14) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(\mathscr{E}_l, \mathbb{R})$ satisfies

$$\left|{}^{ML}\mathbf{D}_0^{\varrho(\varkappa)}\vartheta(\varkappa) - \mathscr{K}_\vartheta(\varkappa)\right| \le \varepsilon. \tag{85}$$

There exists a solution $\vartheta \in C(\mathscr{E}_l, \mathbb{R})$ of the ML-type problem (14) with

$$\left\|\widehat{\vartheta} - \vartheta\right\| \le C_{\mathscr{K}}\varepsilon, \tag{86}$$

where

$$C_{\mathscr{K}} = \frac{\mathscr{R}_P}{1 - \left(P_3(b_l - b_{l-1}) + \left(P_4(b_l - b_{l-1})^{\varrho_l}/\Gamma(\varrho_l + 1)\right)\right)\left(\mathfrak{N}_f(1+\mathscr{M})/1 - \mathfrak{N}_f\right)} > 0. \tag{87}$$

7. Conclusion Remarks

AB fractional operators are very fertile and interesting topic of research recently; thus, there are some researchers who studied and developed some qualitative properties of solutions of FDEs involving such operators. Already significant amount of work on fractional constant order for various operators has been done in literature. But to the best of our information, fractional variable order problems have not been well studied so for fractional calculus. There is a waste gap between constant and variable fractional order problems in literature, the first one has got tremendous attention as compared to the second one. Very recently, the area of variable order has started attention to be investigated. In line with these developments, we developed and investigated sufficient conditions of the existence and uniqueness of solutions for fractional variable order integro-differential equations in the frame of a ML power law. Our approach was based on the reduction of the proposed problem into the fractional integral equation and using some standard fixed point theorems as per the Banach-type and Krasnoselskii-type. Furthermore, through mathematical analysis techniques, we have analyzed the stability results in UH and GUH sense. An example has been provided to justify the main results. Due to the wide recent investigations and applications of the ML power law, we believe that acquired results here will be interesting for future investigations on the theory of fractional calculus.

In future studies, it would be interesting to study the current problem using a Mittag-Leffler power law with respect to another function introduced by Fernandez and Baleanu [40].

Data Availability

Data are available upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

The Analysis of the Fractional-Order Navier-Stokes Equations by a Novel Approach

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This article introduces modified semianalytical methods, namely, the Shehu decomposition method and q-homotopy analysis transform method, a combination of decomposition method, the q-homotopy analysis method, and the Shehu transform method to provide an approximate method analytical solution to fractional-order Navier-Stokes equations. Navier-Stokes equations are widely applied as models for spatial effects in biology, ecology, and applied sciences. A good agreement between the exact and obtained solutions shows the accuracy and efficiency of the present techniques. These results reveal that the suggested methods are straightforward and effective for engineering sciences models.

1. Introduction

Many investigations have provided unique solutions that meet the needs of various fields, making fractional differential equation resolution lucrative mathematical models. Caputo and Riemann-Liouville derivatives were the most acceptable method to model the different natural processes that required noninteger derivatives. However, their limits led to the search for additional derivatives. Singular kernels exist in both fractional derivatives of Riemann-Liouville and Caputo. Furthermore, the constant's Riemann-Liouville derivative does not equal zero. To outcome the problem of the unique kernel, Caputo and Fabrizio [1] suggested without a singular kernel of the fractional derivative operator. Through several applications, Caputo Fabrizio has proven to be an efficient operator [2–5]. By using the Mittag-Leffler function, the authors suggested an actual new fractional derivative in [6]. The Atangana-Baleanu Riemann derivative (Riemann-Liouville sense) is one, whereas the Atangana-Baleanu Caputo derivative is the other (Caputo sense). The Atangana-Baleanu Caputo and Riemann derivatives feature all fractional derivative properties except the semigroup property, additionally to the fact that the kernel is nonsingular and nonlocal. However, these new fractional operators have recently been identified in a newly constructed fractional operator categorization [7–10].

Fluid mechanics, mathematical biology, viscoelasticity, electrochemistry, life sciences, and physics all use fractional-order partial differential equations (PDEs) to explain various nonlinear complex systems [11–14]. For example, fractional derivatives can be used to describe nonlinear seismic oscillations [15], and fractional derivatives can be used to overcome the assumption's weakness in the fluid-dynamic traffic model [16]. Furthermore, based on actual results, [17, 18] offer fractional partial differential equations for the propagation of shallow-water waves and seepage flow in porous media. In most circumstances, obtaining the right behavior of fractional differential equations is extremely difficult. Much work has gone into developing strategies for calculating the approximate behavior of these kinds of equations. The fractional model is the most potential candidate in nanohydrodynamics, where the continuum assumption fails miserably. The homotopy perturbation Sumudu transform method [19], homotopy perturbation method [20], Adomian decomposition method [21], homotopy analysis method [22], fractional reduced differential transform method [23, 24], and variation iteration method [25] have all been proposed in recent years for solving fractional PDEs.

The Navier-Stokes equation, which depicts the viscous fluid's flow motion, was developed by Navier in 1822. They used numerous physical processes as examples of fluid movements, such as blood flow, ocean currents, liquid flow in pipelines, and airflow around airplane arms. Due to its nonlinear character, it can only be solved exactly in a few circumstances. In these situations, we must consider a simple flow pattern configuration and make assumptions about the fluid's state. The central equation of viscous fluid flow movement, known as the Navier-Stokes equation, was published in 1822 [26]. This equation depicts a few projections, including sea streams, fluid flow in channels, blood flow, and wind current around an airship's wings. There are numerous methods for solving fractional-order Navier-Stokes equations in the literature. El-Shahed and Salem published the first fractional version of the Navier-Stokes equation in 2005 [27]. Kumar et al. [28] used a mixture of homotopy perturbation method and the Laplace transform to solve a nonlinear fractional Navier-Stokes problem analytically. Ragab et al. [29] and Ganji et al. [30] used the homotopy analysis method to solve the same Navier-Stokes equation. For the solution of fractional Navier-Stokes equations, Birajdar [31] and Maitama [32] used the Adomian decomposition method. Kumar et al. [33] used the Adomian decomposition method and Laplace transform algorithms to obtain the analytical answer of the fractional Navier-Stokes problem. In contrast, Jena et al. [34] used the Laplace transform and finite Hankel transform algorithm to solve the same equation. The current paper uses the homotopy perturbation transform method to provide a precise or approximate solution to the stated problem.

Hashim et al. were the first to introduce the homotopy analysis method (HAM) [35, 36]. A continuous mapping is created in HAM by constructing it from a preliminary calculation to get close to the right solution of the considered equation. To make such a continuous mapping, an auxiliary linear operator is chosen, and the convergence of the series solution is ensured through an auxiliary parameter. The effect of higher-order wave dispersion can be investigated using time-fractional Korteweg-De Vries equations. The Korteweg-De Vries-Burgers equation describes the waves on shallow water surfaces. The nonlocal property of the fractional Korteweg-De Vries [37, 38] equation is its strength.

The Adomian decomposition technique [39, 40] is a well-known systematic technique for solving deterministic or stochastic operator equations, such as integral equations, integro-differential equations, ordinary differential equations, and partial differential equations. In real-world implementations in engineering and applied sciences, the Adomian decomposition method is vital for approximation analytic solutions and numeric simulations. It enables us to solve nonlinear initial value problems and boundary value problems without relying on nonphysical assumptions such as linearization, perturbation, and beliefs, estimating the starting function or a set of fundamental terms [41, 42]. Furthermore, because Green's functions are difficult to determine, the Adomian decomposition method does not necessitate their use, which would complicate such analytic calculations in most cases. The accuracy of the approximate analytical answers obtained can be verified using direct substitution [43] emphasized the ADM's benefits over Picard's iterated method. More benefits of the Adomian decomposition method over the variational iteration approach were highlighted in [44, 45]. Adomian polynomials, which are tailored to the specific nonlinearity to solve nonlinear operator equations, are essential ideas.

2. Preliminaries Concepts

Definition 1. The Sumudu transformation is achieved across the function set [46, 47]

$$A = \left\{ \nu(\mathfrak{I}) \colon \exists N, \tau_1, \tau_2 > 0, |\nu(\mathfrak{I})| < N e^{(|\mathfrak{I}|/\tau_i)}, \text{if} \, \mathfrak{I} \in (-1)^i \times [0,\infty) \right\}, \tag{1}$$

by

$$S[f(\mathfrak{I})] = G(u) = \int_0^\infty f(u\mathfrak{I}) e^{(-\mathfrak{I})} \mathrm{d}\mathfrak{I}, u \in (-\tau_1, \tau_2). \tag{2}$$

Many researchers have identified and applied this transform to models in various scientific areas and [46, 48]. The link between Laplace and Sumudu transformations is demonstrated in the next theorem.

Theorem 2. *Let G and F be the Laplace and the Sumudu transformations of $f(\mathfrak{I}) \in A$. Then [49],*

$$G(u) = \frac{F(1/u)}{u}. \tag{3}$$

The Shehu transformation was developed in [50] generalizes the Laplace and the Sumudu integral transformations; they have applied it to the analysis of ODEs and PDEs.

Definition 3. The Shehu transformation is achieved over the set A by the following [50]:

$$\mathbb{H}[f(\mathfrak{I})] = V(s,u) = \int_0^\infty e^{(-s\mathfrak{I}/u)} f(\mathfrak{I}) d\mathfrak{I}. \tag{4}$$

It is evident that the Shehu transformation is linear as the Laplace and Sumudu transforms. The function Mittag-Leffler $E\delta(\mathfrak{I})$ is a straightforward generalized form of the exponential series. For $\delta = 1$, we get $E_\delta(\mathfrak{I}) = e^{(\mathfrak{I})}$. It is expressed as [51]

$$E_\delta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + 1)}, \delta \in \mathbb{C}, \text{Re}\,(\delta) > 0. \tag{5}$$

Definition 4. Let $f \in H^1(a,b), b > a$; then, for $\delta \in (0,1)$, the fractional derivative of Atangana-Baleanu in the sense

Caputo is define as [6]

$$ {}^{ABC}_{a}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F})) = \frac{B(\delta)}{1-\delta}\int_a^{\mathfrak{F}} f'(x)E_\delta\left(-\delta\frac{(\mathfrak{F}-x)^\delta}{1-\delta}\right)dx. \quad (6) $$

Definition 5. Let $f \in H^1(a,b), b > a$; then, for $\delta \in (0,1)$, the fractional derivative of Atangana-Baleanu in the sense Riemann-Liouville is expressed as [6]

$$ {}^{ABR}_{a}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F})) = \frac{B(\delta)}{1-\delta}\frac{d}{d\mathfrak{F}}\int_a^{\mathfrak{F}} f(x)E_\delta\left(-\delta\frac{(\mathfrak{F}-x)^\delta}{1-\delta}\right)dx. \quad (7) $$

Under the constraints $B(0) = B(1) = 1$, $B(\delta)$ is a normalising function.

Theorem 6. *The Laplace transformation of fractional derivative Atangana-Baleanu in sense of Caputo is define as [6]*

$$ \mathbb{L}\left\{{}^{ABC}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right\}(s) = \frac{B(\delta)}{1-\delta}\frac{s^\delta F(s) - s^{\delta-1}f(0)}{s^\delta + \delta/1-\delta}, \quad (8) $$

and the Laplace transformation of the fractional derivative Atangana-Baleanu in sense of Riemann-Liouville is define as

$$ \mathbb{L}\left\{{}^{ABC}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right\}(s) = \frac{B(\delta)}{1-\delta}\frac{s^\delta F(s)}{s^\delta + \delta/1-\delta}. \quad (9) $$

3. Main Solutions

In what follows, we suppose that $f \in H^1(a,b), b > a, \delta \in (0,1)$ and $f(\mathfrak{F}) \in A$.

Theorem 7. *The fractional derivative of Atangana-Baleanu Sumudu transformation in Caputo sense is define as [8]*

$$ S\left({}^{ABC}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right) = \frac{B(\delta)}{1-\delta+\delta u^\delta}(G(u) - f(0)). \quad (10) $$

Proof. Applying (8) and (3), we achieve

$$ \begin{aligned} S\left({}^{ABC}_{0}D^{\delta}_{t}(f(\mathfrak{F}))\right) &= \frac{1}{u}\left(\frac{B(\delta)}{1-\delta}\frac{(1/u)^\delta F(1/u) - (1/u)^{\delta-1}f(0)}{(1/u)^\delta + \delta/1-\delta}\right) \\ &= \left(\frac{1}{u}\right)^\delta \frac{B(\delta)}{1-\delta}\frac{G(u) - f(0)}{(1/u)^\delta + \delta/1-\delta}. \end{aligned} \quad (11) $$

Then, we achieve the desired outcome

$$ S\left({}^{ABC}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right) = \frac{B(\delta)}{1-\delta+\delta u^\delta}(G(u) - f(0)). \quad (12) $$

□

Theorem 8. *The Sumudu transformation of fractional derivative Atangana-Baleanu in sense of Riemann-Liouville is define as [8]*

$$ S\left({}^{ABR}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right) = \frac{B(\delta)}{1-\delta+\delta u^\delta}G(u). \quad (13) $$

Proof. Using (9) and (3), we obtain

$$ \begin{aligned} S\left\{{}^{ABR}_{\delta}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right\}(s) &= \frac{1}{u}\frac{B(\delta)}{1-\delta}\frac{(1/u)^\delta F(1/u)}{(1/u)^\delta + \delta/1-\delta} \\ &= \frac{1}{u}\frac{B(\delta)}{1-\delta}\frac{(1/u)^\delta uG(u)}{(1/u)^\delta + \delta/1-\delta} \\ &= \frac{B(\delta)}{1-\delta}\frac{G(u)}{1+\delta/1-\delta u^\delta}. \end{aligned} \quad (14) $$

Then, we achieve the desired outcome

$$ S\left({}^{ABR}_{0}D^{\delta}_{\mathfrak{F}}(f(\mathfrak{F}))\right) = \frac{B(\delta)}{1-\delta+\delta u^\delta}G(u). \quad (15) $$

The Sumudu and Shehu transformations are demonstrated in the following theorem. □

Theorem 9. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{F}) \in A$. Then [8],*

$$ V(s,u) = \frac{u}{s}G\left(\frac{u}{s}\right). \quad (16) $$

Proof. If $f(\mathfrak{F}) \in A$, then

$$ \mathrm{V}(\mathrm{s},\mathrm{u}) = \int_0^\infty e^{(-\mathrm{s}\mathfrak{F}/\mathrm{u})}\mathrm{f}(\mathfrak{F})\mathrm{d}\mathfrak{F}. \quad (17) $$

If we set $\tau = s\mathfrak{F}/u(\mathfrak{F} = u\tau/s)$, then the right hand side can be represent as

$$ \mathrm{V}(\mathrm{s},\mathrm{u}) = \int_0^\infty e^{(-\tau)}\mathrm{f}\left(\frac{\mathrm{u}}{\mathrm{s}}\tau\right)\frac{\mathrm{u}}{\mathrm{s}}\mathrm{d}\tau = \frac{\mathrm{u}}{\mathrm{s}}\int_0^\infty e^{(-\tau)}\mathrm{f}\left(\frac{\mathrm{u}}{\mathrm{s}}\tau\right)\mathrm{d}\tau. \quad (18) $$

The right side integral is obviously $G(u/s)$, thus obtaining (16). It is obvious that

$$ \mathrm{V}(\mathrm{s},1) = \frac{1}{\mathrm{s}}\mathrm{G}\left(\frac{1}{\mathrm{s}}\right) = \mathrm{F}(\mathrm{s}), \quad (19) $$

where $F(s)$ is the Laplace transformation of $f(\mathfrak{F})$. □

The following significant properties are achieved by applying the relationship among Sumudu and Shehu transformations (16).

Theorem 10. *The Shehu transformation of $\mathfrak{F}^{x-1}$ is [8]*

$$ V(s,u) = \Gamma(x)\left(\frac{u}{s}\right)^x, x > 0. \quad (20) $$

Proof. When $x > 0$, the Sumudu of $\mathfrak{T}^{x-1}$ is define as

$$\mathrm{G(u)} = \Gamma(\mathrm{x})\mathrm{u}^{\mathrm{x}-1}, \tag{21}$$

where $\Gamma(x)$ is the Gamma function expressed as

$$\Gamma(x) = \int_0^\infty \mathfrak{T}^{x-1} e^{-\mathfrak{T}} d\mathfrak{T}. \tag{22}$$

Then, by applying (16), we get the achieved solution. □

Theorem 11. *Let $\delta, \omega \in \mathbb{C}$, with* Re $(\delta) > 0$. *Shehu transform of $E_\delta(\omega\mathfrak{T}^\delta)$ is given by [8]*

$$H\left(E_\delta\left(\omega\mathfrak{T}^\delta\right)\right) = \frac{u}{s}\left(1 - \omega\left(\frac{u}{s}\right)^\delta\right)^{-1}. \tag{23}$$

Proof. Based on the reference [52], we get

$$S\left(E_\delta\left(\omega\mathfrak{T}^\delta\right)\right) = \left(1 - \omega u^\delta\right)^{-1}, \tag{24}$$

and then, by applying (16), we get

$$H\left(E_\delta\left(\omega\mathfrak{T}^\delta\right)\right) = \left(\frac{u}{s}\right)\left(1 - \omega\left(\frac{u}{s}\right)^\delta\right)^{-1}. \tag{25}$$

□

Theorem 12. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{T}) \in A$. Then, the fractional derivative of Atangana-Baleanu Shehu transform in sense of Caputo is define as* $\mathbb{H}$

$$H\left({}_0^{ABC}D_{\mathfrak{T}}^\delta(f(\mathfrak{T}))\right) = \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \tag{26}$$

Proof. Applying (10) and the relationship among Shehu and Sumudu transformations, we achieve

$$\begin{aligned} H\left({}_0^{ABC}D_{\mathfrak{T}}^\delta(f(\mathfrak{T}))\right) &= \frac{u}{s}\frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(G\left(\frac{u}{s}\right) - f(0)\right) \\ &= \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \end{aligned} \tag{27}$$

□

Theorem 13. *Let $G(u)$ and $V(s,u)$ be the Shehu and the Sumudu transformations of $f(\mathfrak{T}) \in A$. Then, the fractional derivative of Atangana-Baleanu Shehu transform in sense of Riemann-Liouville is define as*

$$H\left({}_0^{ABR}D_{\mathfrak{T}}^\delta(f(\mathfrak{T}))\right) = \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}\left(V(s,u) - \frac{u}{s}f(0)\right). \tag{28}$$

Proof. Applying (13) and the relationship among Shehu and Sumudu transformations (16), we achieve

$$\begin{aligned} H\left({}_0^{ABR}D_{\mathfrak{T}}^\delta(f(\mathfrak{T}))\right) &= \frac{u}{s}\frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}G\left(\frac{u}{s}\right) \\ &= \frac{B(\delta)}{1 - \delta + \delta(u/s)^\delta}V(s,u). \end{aligned} \tag{29}$$

□

4. The Procedure of SDM

In this section, we describe the SDM procedure for fractional PDEs.

$$\begin{gathered} D_{\mathfrak{T}}^\delta\mu(\chi,\mathfrak{T}) + \mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{P}_1(\chi,\mathfrak{T}) = 0, \\ D_{\mathfrak{T}}^\delta\nu(\chi,\mathfrak{T}) + \mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{P}_2(\chi,\mathfrak{T}) = 0, 0 < \delta \le 1, \end{gathered} \tag{30}$$

with initial condition

$$\mu(\delta,0) = g_1(\chi), \quad \nu(\delta,0) = g_2(\chi), \tag{31}$$

where $D_{\mathfrak{T}}^\delta = \partial^\delta/\partial\mathfrak{T}^\delta$ is the Caputo derivative of fractional-order δ, $\mathscr{R}_1$ and $\mathscr{R}_2$ and $\mathscr{N}_1$ and $\mathscr{N}_2$ are linear and nonlinear terms, respectively, and $\mathscr{P}_1$ and $\mathscr{P}_2$ are source functions.

Applying the Shehu transform to Equation (30),

$$\begin{aligned} S\left[D_{\mathfrak{T}}^\delta\mu(\chi,\mathfrak{T})\right] + S[\mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu) - \mathscr{P}_1(\chi,\mathfrak{T})] = 0, \\ S\left[D_{\mathfrak{T}}^\delta\nu(\chi,\mathfrak{T})\right] + S[\mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu) - \mathscr{P}_2(\chi,\mathfrak{T})] = 0. \end{aligned} \tag{32}$$

Applying the Shehu transformation of differentiation property, we have

$$\begin{aligned} S[\mu(\chi,\mathfrak{T})] &= \frac{u}{s}\mu(\chi,0) + \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S[\mathscr{P}_1(\chi,\mathfrak{T})] \\ &\quad - \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S\{\mathscr{R}_1(\mu,\nu) + \mathscr{N}_1(\mu,\nu)\}], \\ S[\nu(\chi,\mathfrak{T})] &= \frac{u}{s}\nu(\chi,0) + \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S[\mathscr{P}_2(\chi,\mathfrak{T})] \\ &\quad - \frac{1 - \delta + \delta(u/s)^\delta}{B(\delta)}S\{\mathscr{R}_2(\mu,\nu) + \mathscr{N}_2(\mu,\nu)\}]. \end{aligned} \tag{33}$$

SDM defines the result of infinite series $\mu(\chi, \mathfrak{J})$ and $\nu(\chi, \mathfrak{J})$.

$$\mu(\chi, \mathfrak{J}) = \sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{J}), \quad \nu(\chi, \mathfrak{J}) = \sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{J}). \tag{34}$$

The nonlinear functions defined by Adomian polynomials $\mathcal{N}_1$ and $\mathcal{N}_2$ are expressed as

$$\mathcal{N}_1(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{A}_m, \mathcal{N}_2(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{B}_m. \tag{35}$$

The Adomian polynomials can be expressed as

$$\begin{aligned} \mathcal{A}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial \lambda^m}\left\{\mathcal{N}_1\left(\sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k\right)\right\}\right]_{\lambda=0}, \\ \mathcal{B}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial \lambda^m}\left\{\mathcal{N}_2\left(\sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k\right)\right\}\right]_{\lambda=0}. \end{aligned} \tag{36}$$

Putting Equations (34) and (36) into (33) gives

$$\begin{aligned} S\left[\sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{J})\right] &= \frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{J})\} \\ &\quad - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \\ &\quad \cdot S\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{A}_m\right\}, \\ S\left[\sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{J})\right] &= \frac{u}{s}\nu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{J})\} \\ &\quad - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \\ &\quad \cdot S\left\{\mathcal{R}_2\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{B}_m\right\}. \end{aligned} \tag{37}$$

Using the inverse Shehu transform of Equation (37),

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\chi, \mathfrak{J}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{J})\}\right] \\ &\quad - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{A}_m\right\}\right], \\ \sum_{m=0}^{\infty} \nu_m(\chi, \mathfrak{J}) &= S^{-1}\left[\frac{u}{s}\nu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{J})\}\right] \\ &\quad - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\left\{\mathcal{R}_2\left(\sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m\right) + \sum_{m=0}^{\infty} \mathcal{B}_m\right\}\right], \end{aligned} \tag{38}$$

and we expressed the following terms:

$$\begin{aligned} \mu_0(\chi, \mathfrak{J}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_1(\chi, \mathfrak{J})\}\right], \\ \nu_0(\chi, \mathfrak{J}) &= S^{-1}\left[\frac{u}{s}\mu(\chi, 0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{P}_2(\chi, \mathfrak{J})\}\right], \\ \mu_1(\chi, \mathfrak{J}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{R}_1(\mu_0, \nu_0) + \mathcal{A}_0\}\right], \\ \nu_1(\chi, \mathfrak{J}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{R}_2(\mu_0, \nu_0) + \mathcal{B}_0\}\right]. \end{aligned} \tag{39}$$

The general for $m \geq 1$ is given by

$$\begin{aligned} \mu_{m+1}(\chi, \mathfrak{J}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{R}_1(\mu_m, \nu_m) + \mathcal{A}_m\}\right], \\ \nu_{m+1}(\chi, \mathfrak{J}) &= -S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\{\mathcal{R}_2(\mu_m, \nu_m) + \mathcal{B}_m\}\right], \end{aligned} \tag{40}$$

5. Solution of SDM

Example 1. Consider the two dimensional fractional-order Navier-Stokes equation

$$\begin{aligned} D_{\mathfrak{J}}^{\delta}(\mu) + \mu\frac{\partial \mu}{\partial \chi} + \mu\frac{\partial \mu}{\partial \xi} &= \rho\left[\frac{\partial^2 \mu}{\partial \chi^2} + \frac{\partial^2 \mu}{\partial \xi^2}\right] + q, \\ D_{\mathfrak{J}}^{\delta}(\nu) + \mu\frac{\partial \nu}{\partial \chi} + \nu\frac{\partial \nu}{\partial \xi} &= \rho\left[\frac{\partial^2 \nu}{\partial \chi^2} + \frac{\partial^2 \nu}{\partial \xi^2}\right] - q, \end{aligned} \tag{41}$$

with initial conditions

$$\begin{cases} \mu(\chi, \xi, 0) = -\sin(\chi+\xi), \\ \nu(\chi, \xi, 0) = \sin(\chi+\xi). \end{cases} \tag{42}$$

Using Shehu transform of Equation (41), we have

$$S\left\{\frac{\partial^{\delta}\mu(\chi, \xi, \mathfrak{J})}{\partial \mathfrak{J}^{\delta}}\right\} = -S\left[\mu\frac{\partial \mu}{\partial \chi} + \mu\frac{\partial \mu}{\partial \xi} - \rho\left[\frac{\partial^2 \mu}{\partial \chi^2} + \frac{\partial^2 \mu}{\partial \xi^2}\right] + q\right],$$

$$S\left\{\frac{\partial^{\delta}\nu(\chi, \xi, \mathfrak{J})}{\partial \mathfrak{J}^{\delta}}\right\} = -S\left[\mu\frac{\partial \nu}{\partial \chi} + \nu\frac{\partial \nu}{\partial \xi} - \rho\left[\frac{\partial^2 \nu}{\partial \chi^2} + \frac{\partial^2 \nu}{\partial \xi^2}\right] - q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\mu(\chi,\xi,\mathfrak{I})-\frac{u}{s}\mu(\chi,\xi,0)\right\} = -S\left[\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\nu(\chi,\xi,\mathfrak{I})-\frac{u}{s}\nu(\chi,\xi,0)\right\} = -S\left[\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q\right]. \tag{43}$$

The above equations can be written as

$$S\{\mu(\chi,\xi,\mathfrak{I})\} = \frac{u}{s}\{\mu(\chi,\xi,0)\} - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q\right],$$

$$S\{\nu(\chi,\xi,\mathfrak{I})\} = \frac{u}{s}\{\nu(\chi,\xi,0)\} - \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q\right]. \tag{44}$$

Using inverse Shehu transform, we have

$$\mu(\chi,\xi,\mathfrak{I}) = \mu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S[q]\right] - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\mu}{\partial\chi}+\mu\frac{\partial\mu}{\partial\xi}-\rho\left(\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right)\right\}\right],$$

$$\nu(\chi,\xi,\mathfrak{I}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S[q]\right] - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left\{\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho\left(\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right)\right\}\right]. \tag{45}$$

Suppose that the unknown functions $\mu(\chi,\xi,\mathfrak{I})$ and $\nu(\chi,\xi,\mathfrak{I})$ infinite series solution are as follows:

$$\mu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{I}),$$
$$\nu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{I}). \tag{46}$$

Note that $\mu\mu_\chi = \sum m = 0^{\infty}\mathscr{A}_m$, $\nu\mu_\xi = \sum m = 0^{\infty}\mathscr{B}_m$, $\mu\nu_\chi = \sum m = 0^{\infty}\mathscr{C}m$, and $\nu\nu_\xi = \sum_{m=0}^{\infty}\mathscr{D}_m$ are the Adomian polynomials, and the nonlinear terms were described. Applying such terms, Equation (45) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{I}) = \mu(\chi,\xi,0) + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right) + \rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{I}) = \nu(\chi,\xi,0) - S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right) + \rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{47}$$

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\xi^2}\right\}\right]\right]. \tag{48}$$

According to Equation (36), the Adomian polynomials can be expressed as

$$\mathscr{A}_0 = \mu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{A}_1 = \mu_0\frac{\partial\mu_1}{\partial\chi}+\mu_1\frac{\partial\mu_0}{\partial\chi},$$
$$\mathscr{B}_0 = \nu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{B}_1 = \nu_0\frac{\partial\mu_1}{\partial\xi}+\nu_1\frac{\partial\mu_0}{\partial\xi},$$

$$\mathscr{C}_0 = \mu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{C}_1 = \mu_0\frac{\partial\nu_1}{\partial\chi}+\mu_1\frac{\partial\nu_0}{\partial\chi},$$
$$\mathscr{D}_0 = \nu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_1 = \nu_0\frac{\partial\nu_1}{\partial\chi}+\nu_1\frac{\partial\nu_0}{\partial\chi}. \tag{49}$$

Thus, we can easy achieve the recursive relationship Equation (48).

$$\mu_0(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_0(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right]. \tag{50}$$

For $m=0$,

$$\mu_1(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi)\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_1(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi)\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right]. \tag{51}$$

For $m=1$,

$$\mu_2(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right],$$
$$\nu_2(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right]. \tag{52}$$

For $m=2$,

$$\mu_3(\chi,\xi,\mathfrak{T}) = \sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2(1-\delta)\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}\right],$$

$$\nu_3(\chi,\xi,\mathfrak{T}) = -\sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2(1-\delta)\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}\right] \vdots \tag{53}$$

In the same method, the remaining μ_m and $\nu_m (m \geq 3)$ components of the SDM solution can be obtained seamlessly. Consequently, we describe the series of alternative solutions as

$$\mu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi) = \mu_0(\chi,\xi)+\mu_1(\chi,\xi) + \mu_2(\chi,\xi)+\mu_3(\chi,\xi)+\cdots,$$

$$\nu(\chi,\xi,\mathfrak{T}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi) = \nu_0(\chi,\xi)+\nu_1(\chi,\xi) + \nu_2(\chi,\xi)+\nu_3(\chi,\xi)+\cdots,$$

$$\begin{aligned}\mu(\chi,\xi,\mathfrak{T}) = {} & -\sin(\chi+\xi) + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ & + \sin(\chi+\xi)\frac{2\rho}{\mathrm{B}(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ & - \sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right] \\ & + \sin(\chi+\xi)\frac{(2\rho)^3}{(\mathrm{B}(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2(1-\delta)\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}\right]-\cdots,\end{aligned}$$

$$\begin{aligned}\nu(\chi,\xi,\mathfrak{T}) = {} & \sin(\chi+\xi) - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ & - \sin(\chi+\xi)\frac{2\rho}{\mathrm{B}(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] \\ & + \sin(\chi+\xi)\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}\right] \\ & - \sin(\chi+\xi)\frac{(2\rho)^3}{(B(\delta))^3}\left[(1-\delta)^3+\frac{3\delta(1-\delta)^2\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2(1-\delta)\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}+\frac{2\delta^2(1-\delta)\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)}+\frac{\delta^3\mathfrak{T}^{2\delta+1}}{\Gamma(2\delta+2)}\right]+\cdots.\end{aligned} \tag{54}$$

The exact result of Equation (41) at $\delta = 1$ and $q = 0$ is as follows:

$$\mu(\chi,\xi,\mathfrak{T}) = -e^{-2\rho\mathfrak{T}}\sin(\chi+\xi),$$
$$\nu(\chi,\xi,\mathfrak{T}) = e^{-2\rho\mathfrak{T}}\sin(\chi+\xi). \tag{55}$$

Example 2. Consider the two dimensional fractional-order Navier-Stokes equation

$$D^{\delta}_{\mathfrak{T}}(\mu) + \mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi} = \rho\left[\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right]+q,$$
$$D^{\delta}_{\mathfrak{T}}(\nu) + \mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi} = \rho\left[\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right]-q, \tag{56}$$

with the initial conditions

$$\begin{cases}\mu(\chi,\xi,0) = -e^{\chi+\xi}, \\ \nu(\chi,\xi,0) = e^{\chi+\xi}.\end{cases} \tag{57}$$

Using Shehu transform of Equation (56), we have

$$S\left\{\frac{\partial^{\delta}\mu}{\partial\mathfrak{T}^{\delta}}\right\} = S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$S\left\{\frac{\partial^{\delta}\nu}{\partial\mathfrak{T}^{\delta}}\right\} = S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\mu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\mu(\chi,\xi,0)\right\} = S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$\frac{B(\delta)}{1-\delta+\delta(u/s)^{\delta}} S\left\{\nu(\chi,\xi,\mathfrak{T})-\frac{u}{s}\nu(\chi,\xi,0)\right\} = S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right]. \tag{58}$$

The above equations can be written as

$$S\{\mu(\chi,\xi,\mathfrak{T})\}=\frac{u}{s}\{\mu(\chi,\xi,0)\}+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}+q\right],$$

$$S\{\nu(\chi,\xi,\mathfrak{T})\}=\frac{u}{s}\{\nu(\chi,\xi,0)\}=\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right)+\rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}-q\right]. \tag{59}$$

Applying inverse Shehu transformation, we get

$$\mu(\chi,\xi,\mathfrak{T})=\mu(\chi,\xi,0)+S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}\right) + \rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$

$$\nu(\chi,\xi,\mathfrak{T})=\nu(\chi,\xi,0)-S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}\right) + \rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{60}$$

Suppose that the unknown functions $\mu(\chi,\xi,\mathfrak{T})$ and $\nu(\chi,\xi,\mathfrak{T})$ infinite series result as follows:

$$\mu(\chi,\xi,\mathfrak{T})=\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T}), \quad \nu(\chi,\xi,\mathfrak{T})=\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T}). \tag{61}$$

Note that $\mu\mu_\chi=\sum m=0^{\infty}\mathscr{A}_m$, $\nu\mu_\xi=\sum m=0^{\infty}\mathscr{B}_m$, $\mu\nu_\chi=\sum m=0^{\infty}\mathscr{C}m$, and $\nu\nu_\xi=\sum_{m=0}^{\infty}\mathscr{D}_m$ are the Adomian polynomials, and the nonlinear terms were described. Applying such terms, Equation (60) can be rewritten in the form

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T})=\mu(\chi,\xi,0)+S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right) + \rho\left\{\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T})=\nu(\chi,\xi,0)-S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\{q\}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right) + \rho\left\{\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right\}\right]\right]. \tag{62}$$

$$\sum_{m=0}^{\infty}\mu_m(\chi,\xi,\mathfrak{T})=-\sin(\chi+\xi)+\frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{A}_m+\sum_{m=0}^{\infty}\mathscr{B}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\mu_m}{\partial\xi^2}\right\}\right]\right],$$

$$\sum_{m=0}^{\infty}\nu_m(\chi,\xi,\mathfrak{T})=\sin(\chi+\xi)-\frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[-\left(\sum_{m=0}^{\infty}\mathscr{C}_m+\sum_{m=0}^{\infty}\mathscr{D}_m\right)\right]\right] + S^{-1}\left[\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}S\left[\rho\left\{\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\chi^2}+\sum_{m=0}^{\infty}\frac{\partial^2\nu_m}{\partial\xi^2}\right\}\right]\right]. \tag{63}$$

According to Equation (36), the Adomian polynomials can be expressed as

$$\mathscr{A}_0=\mu_0\frac{\partial\mu_0}{\partial\chi}, \mathscr{A}_1=\mu_0\frac{\partial\mu_1}{\partial\chi}+\mu_1\frac{\partial\mu_0}{\partial\chi}, \mathscr{B}_0=\nu_0\frac{\partial\mu_0}{\partial\beta}, \mathscr{B}_1=\nu_0\frac{\partial\mu_1}{\partial\beta}+\nu_1\frac{\partial\mu_0}{\partial\beta},$$
$$\mathscr{C}_0=\mu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{C}_1=\mu_0\frac{\partial\nu_1}{\partial\chi}+\mu_1\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_0=\nu_0\frac{\partial\nu_0}{\partial\chi}, \mathscr{D}_1=\nu_0\frac{\partial\nu_1}{\partial\chi}+\nu_1\frac{\partial\nu_0}{\partial\chi}. \tag{64}$$

Thus, we can quickly achieve the recursive relationship

Equation (63)

$$\mu_0(\chi,\xi,\mathfrak{I}) = -e^{\chi+\xi} + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_0(\chi,\xi,\mathfrak{I}) = e^{\chi+\xi} - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]. \tag{65}$$

For $m=0$,

$$\mu_1(\chi,\xi,\mathfrak{I}) = e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_1(\chi,\xi,\mathfrak{I}) = -e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]. \tag{66}$$

For $m=1$,

$$\mu_2(\chi,\xi,\mathfrak{I}) = -e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right],$$
$$\nu_2(\chi,\xi,\mathfrak{I}) = e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]. \tag{67}$$

For $m=2$,

$$\mu_3(\chi,\xi,\mathfrak{I}) = e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{I}^{3\delta}}{\Gamma(3\delta+1)},\ \nu_3(\chi,\xi,\mathfrak{I}) = -e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{I}^{3\delta}}{\Gamma(3\delta+1)}.$$
$$\vdots \tag{68}$$

In same method, the remaining μ_m and $\nu_m (m \geq 3)$ components of the SDM solution can be obtained seamlessly. Consequently, we describe the series of alternative solutions as

$$\mu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\mu_m(\chi,\xi) = \mu_0(\chi,\xi)+\mu_1(\chi,\xi)+\mu_2(\chi,\xi)+\mu_3(\chi,\xi)+\cdots,$$

$$\nu(\chi,\xi,\mathfrak{I}) = \sum_{m=0}^{\infty}\nu_m(\chi,\xi) = \nu_0(\chi,\xi)+\nu_1(\chi,\xi)+\nu_2(\chi,\xi)+\nu_3(\chi,\xi)+\cdots,$$

$$\begin{aligned}\mu(\chi,\xi,\mathfrak{I}) = &-e^{\chi+\xi} + \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]\\ &+ e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]\\ &- e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]\\ &+ e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{I}^{3\delta}}{\Gamma(3\delta+1)}-\cdots,\end{aligned}$$

$$\begin{aligned}\nu(\chi,\xi,\mathfrak{I}) = &e^{\chi+\xi} - \frac{q}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]\\ &- e^{\chi+\xi}\frac{2\rho}{B(\delta)}\left[1-\delta+\frac{\delta\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}\right]\\ &+ e^{\chi+\xi}\frac{(2\rho)^2}{(B(\delta))^2}\left[(1-\delta)^2+\frac{2\delta(1-\delta)\mathfrak{I}^{\delta}}{\Gamma(\delta+1)}+\frac{\delta^2\mathfrak{I}^{2\delta}}{\Gamma(2\delta+1)}\right]\\ &- e^{\chi+\xi}\frac{(2\rho)^3\mathfrak{I}^{3\delta}}{\Gamma(3\delta+1)}+\cdots.\end{aligned} \tag{69}$$

The exact result of Equation (56) at $\delta=1$ and $q=0$ is as follows:

$$\mu(\chi,\xi,\mathfrak{I}) = -e^{\chi+\xi+2\rho\mathfrak{I}},$$
$$\nu(\chi,\xi,\mathfrak{I}) = e^{\chi+\xi+2\rho\mathfrak{I}}. \tag{70}$$

6. The Methodology of q-HATM

Consider a nonlinear nonhomogeneous fractional partial differential equation:

$$D_{\mathfrak{I}}^{\delta}\mu(\chi,\xi,\mathfrak{I}) + R\mu(\chi,\xi,\mathfrak{I}) + N\mu(\chi,\xi,\mathfrak{I}) = f(\chi,\xi,\mathfrak{I}),\quad n-1<\delta\leq n. \tag{71}$$

Here, $D_{\mathfrak{I}}^{\delta}\mu$ is the Caputo derivative, and R and N are linear and nonlinear functions, respectively. $f(\chi,\xi,\mathfrak{I})$ is the source operator.

Now, using the Shehu transformation on Equation (71), we get

$$\begin{aligned}S[\mu(\chi,\xi,\mathfrak{I})] - \frac{u}{s}\mu(\chi,\xi,0) + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\\ \cdot\{S[R\mu(\chi,\xi,\mathfrak{I})] + S[NL(\chi,\xi,\mathfrak{I})] - S[f(\chi,\xi,\mathfrak{I})]\} = 0.\end{aligned} \tag{72}$$

The nonlinear function is

$$\begin{aligned}N[\phi(\chi,\xi,\mathfrak{I};\bar{q})] = &S[\phi(\chi,\xi,\mathfrak{I};\bar{q})] - \frac{u}{s}\phi(\chi,\xi,\mathfrak{I};\bar{q})(0^+)\\ &+ \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\{S[R\phi(\chi,\xi,\mathfrak{I};\bar{q})]\\ &+ S[N\phi(\chi,\xi,\mathfrak{I};\bar{q})] - S[f(\chi,\xi,\mathfrak{I})]\}.\end{aligned} \tag{73}$$

Here, $\phi(\chi,\xi,\mathfrak{I};\bar{q})$ is an unknown term, and $\bar{q}\in[0,1/4]$ is the embedding parameter, $n\geq 1$. Construct a homotopy as

$$\begin{aligned}(1-n\bar{q})S[\phi(\chi,\xi,\mathfrak{I};\bar{q}) - \mu_0(\chi,\xi,\mathfrak{I})]\\ = \hbar q H(\chi,\xi,\mathfrak{I})N[\phi(\chi,\xi,\mathfrak{I};\bar{q})],\end{aligned} \tag{74}$$

where μ_0 is an initial condition and $\hbar\neq 0$ is an auxiliary parameter. The following solutions hold for $=0, 1/n=$

$$\phi(\chi,\xi,\mathfrak{J};0)=\mu_0(\chi,\xi,\mathfrak{J}),\quad \phi\left(\chi,\xi,\mathfrak{J};\frac{1}{n}\right)=\mu(\chi,\xi,\mathfrak{J}). \tag{75}$$

Elevating q,ϕ converges from U_0 to U. Intensifying ϕ about q by Taylor's theorem, we get

$$\phi(\chi,\xi,\mathfrak{J};\bar{q})=U_0+\sum_{m=1}^{\infty}\mu_m(\chi,\xi,\mathfrak{J})\bar{q}^m, \tag{76}$$

where

$$\mu_m=\frac{1}{m!}\frac{\partial^m\phi(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\bar{q}^m}\bigg|_{\bar{q}=0}. \tag{77}$$

By an appropriate selection of auxiliary linear operator, $U_0, n, \hbar$ and H, series (76) converges at $\bar{q}=1/n$, thereby providing a result

$$\mu(\chi,\xi,\mathfrak{J})=\mu_0+\sum_{m=1}^{\infty}\mu_m(\chi,\xi,\mathfrak{J})\left(\frac{1}{n}\right)^m. \tag{78}$$

Now, differential Equation (74) m times, divide by $m!$ and taking $\bar{q}=0$,

$$S[\mu_m(\chi,\xi,\mathfrak{J})-k_m\mu_{m-1}(\chi,\xi,\mathfrak{J})], \tag{79}$$

where the vector is described as

$$\vec{\mu}_m=\{\mu_0(\chi,\xi,\mathfrak{J}),\mu_1(\chi,\xi,\mathfrak{J}),\cdots,\mu_m(\chi,\xi,\mathfrak{J})\}. \tag{80}$$

Applying the inverse transform on Equation (80),

$$\mu_m(\chi,\xi,I)=\quad k_m\mu_{m-1}(\chi,\xi,I)+\hbar S^{-1}\left[H(\chi,\xi,I)R_m\left(\vec{\mu}_{m-1}\right)\right]. \tag{81}$$

Here,

$$\mathfrak{R}_m\left(\vec{\mu}_{m-1}\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(\chi,\xi,\mathfrak{J};\bar{q})]}{\partial\bar{q}^{m-1}}\bigg|_{\bar{q}=0},$$

$$k_r=\begin{cases}0, & r\le 1,\\ n, & r>1.\end{cases} \tag{82}$$

Lastly, by solving Equation (81), the elements of the q-HATM result are readily available.

Example 3. Consider the two dimensional fractional-order Navier-Stokes equation

$$\begin{cases} D_{\mathfrak{J}}^{\delta}\mu+\mu\dfrac{\partial\mu}{\partial\chi}+\nu\dfrac{\partial\mu}{\partial\xi}=\rho_0\left(\dfrac{\partial^2\mu}{\partial\chi^2}+\dfrac{\partial^2\mu}{\partial\xi^2}\right)+g, \\ D_{\mathfrak{J}}^{\delta}\nu+\mu\dfrac{\partial\nu}{\partial\chi}+\nu\dfrac{\partial\nu}{\partial\xi}=\rho_0\left(\dfrac{\partial^2\nu}{\partial\chi^2}+\dfrac{\partial^2\nu}{\partial\xi^2}\right)-g, \end{cases}\quad 0<\delta\le 1, \tag{83}$$

with initial conditions

$$\nu(\chi,\xi,0)=\sin(\chi+\xi),\quad \mu(\chi,\xi,0)=-\sin(\chi+\xi). \tag{84}$$

Using the Shehu transformation on Equation (83) and applying Equation (84), we have

$$S[\mu(\chi,\xi,\mathfrak{J})]+\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\mu\frac{\partial\mu}{\partial\chi}+\nu\frac{\partial\mu}{\partial\xi}-\rho_0\left(\frac{\partial^2\mu}{\partial\chi^2}+\frac{\partial^2\mu}{\partial\xi^2}\right)-g\right\}=0,$$

$$S[\nu(\chi,\xi,\mathfrak{J})]-\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\cdot S\left\{\mu\frac{\partial\nu}{\partial\chi}+\nu\frac{\partial\nu}{\partial\xi}-\rho_0\left(\frac{\partial^2\nu}{\partial\chi^2}+\frac{\partial^2\nu}{\partial\xi^2}\right)+g\right\}=0. \tag{85}$$

Define the nonlinear operators

$$\begin{aligned} &N^1[\phi_1(\chi,\xi,\mathfrak{J};\bar{q}),\phi_2(\chi,\xi,\mathfrak{J};\bar{q})]\\ &=S[\phi_1(\chi,\xi,\mathfrak{J};\bar{q})]+\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\\ &\cdot S\left\{\phi_1(\chi,\xi,\mathfrak{J};\bar{q})\frac{\partial\phi_1(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\chi}+\phi_2(\chi,\xi,\mathfrak{J};\bar{q})\right.\\ &\left.\cdot\frac{\partial\phi_1(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\xi}-\rho_0\left(\frac{\partial^2\phi_1(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\chi^2}\frac{\partial^2\phi_1(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\xi^2}\right)-g\right\}, \end{aligned}$$

$$\begin{aligned} &N^2[\phi_1(\chi,\xi,\mathfrak{J};\bar{q}),\phi_2(\chi,\xi,\mathfrak{J};\bar{q})]=S[\phi_2(\chi,\xi,\mathfrak{J};\bar{q})]\\ &-\frac{u}{s}\sin(\chi+\xi)+\frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)}\\ &\cdot S\left\{\phi_1(\chi,\xi,\mathfrak{J};\bar{q})\frac{\partial\phi_2(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\chi}+\phi_2(\chi,\xi,\mathfrak{J};\bar{q})\right.\\ &\left.\cdot\frac{\partial\phi_2(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\xi}-\rho_0\left(\frac{\partial^2\phi_2(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\chi^2}+\frac{\partial^2\phi_2(\chi,\xi,\mathfrak{J};\bar{q})}{\partial\xi^2}\right)+g\right\}, \end{aligned} \tag{86}$$

and the Shehu operators as

$$S[\mu_m(\chi,\xi,\mathfrak{I}) - k_m\mu_{m-1}(\chi,\xi,\mathfrak{I})] = \hbar R_{1,m}\left[\vec{\mu}_{m-1}, \vec{v}_{m-1}\right],$$
$$S[v_m(\chi,\xi,\mathfrak{I}) - k_m v_{m-1}(\chi,\xi,\mathfrak{I})] = \hbar R_{2,m}\left[\vec{\mu}_{m-1}, \vec{v}_{m-1}\right], \tag{87}$$

$$R_{1,m}\left[\vec{\mu}_{m-1}, \vec{v}_{m-1}\right] = S[\mu_{m-1}(\chi,\xi,\mathfrak{I})] + \left(1-\frac{k_m}{n}\right)\frac{u}{s}\sin(\chi+\xi) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\mu_m - 1 - i}{\partial\chi} + \sum_{i=0}^{m-1} v\frac{\partial\mu_{m-1} - i}{\partial\xi} - \rho_0\left(\frac{\partial^2\mu_{m-1}}{\partial\chi^2} + \frac{\partial^2\mu_{m-1}}{\partial\xi^2}\right) - g\right\}, \tag{88}$$

$$R_{2,m}\left[\vec{\mu}_{m-1}, \vec{v}_{m-1}\right] = S[v_{m-1}(\chi,\xi,\mathfrak{I})] - \left(1-\frac{k_m}{n}\right)\frac{u}{s}\sin(\chi+\xi) + \frac{1-\delta+\delta(u/s)^\delta}{B(\delta)} S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial v_{m-1} - i}{\partial\chi} + \sum_{i=0}^{m-1} v_i\frac{\partial v_{m-1} - i}{\partial\xi} - \rho_0\left(\frac{\partial^2 v_{m-1}}{\partial\chi^2} + \frac{\partial^2 v_{m-1}}{\partial\xi^2}\right) + g\right\}. \tag{89}$$

Using the inverse Shehu transformation on Equation (87), we have

$$\mu_m(\chi,\xi,I) = k_m\mu_{m-1} + \hbar S^{-1}\left\{R_{1,m}\left[\vec{\mu}_{m-1}, \vec{v}_{m-1}\right]\right\},$$
$$v_m(\chi,\xi,I) = k_m v_{m-1} + \hbar S^{-1}\left\{R_{2,m}\left[\vec{\mu}_{m-1}, \bar{v}_{m-1}\right]\right\}. \tag{90}$$

Using μ_0 and v_0 in Equation (90), we get

$$\mu_1 = -\frac{2\rho_0\hbar\sin(\chi+\xi)}{B(\delta)}\left[1-\delta+\frac{\delta I^\delta}{\Gamma(\delta+1)}\right],$$

$$v_1 = \frac{2\rho_0\hbar\sin(\chi+\xi)}{B(\delta)}\left[1-\delta+\frac{\delta I^\delta}{\Gamma(\delta+1)}\right],$$

$$\mu_2 = -\frac{2(n+n)\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]} - \frac{4\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right],$$

$$v_2 = \frac{2(n+h)\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]} + \frac{4\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right],$$

$$\mu_3 = -\frac{2(n+\hbar)^2\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]} - \frac{8(n+\hbar)\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right] - \frac{8\rho_0^3\hbar^3\sin(\chi+\xi)I^{3\delta}}{\Gamma[3\delta+1]},$$

$$v_3 = \frac{2(n+\hbar)^2\rho_0\hbar\sin(\chi+\xi)I^\delta}{\Gamma[\delta+1]} + \frac{8(n+\hbar)\rho_0^2\hbar^2\sin(\chi+\xi)}{(B(\delta))^2}\cdot\left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^\delta}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right] + \frac{8\rho_0^3\hbar^3\sin(\chi+\xi)I^{3\delta}}{\Gamma[3\delta+1]}, \tag{91}$$

and so forth. The rest of the components are discovered in the same way. The q-HATM result of Equation (83) is then determined:

$$\mu(\chi,\xi,\mathfrak{I}) = \mu_0 + \sum_{m=1}^{\infty}\mu_m\left(\frac{1}{n}\right)^m,$$
$$v(\chi,\xi,\mathfrak{I}) = 2_0 + \sum_{w=1}^{\infty} v_m\left(\frac{1}{n}\right)^m. \tag{92}$$

For $\delta = 1$, $n = -1$, $n = 1$ and $g = 0$, solutions $\sum_{m=1}^N \mu_m(\chi,\xi,\mathfrak{I})(1/n)^m$ and $\sum_{m=1}^N v_m(\chi,\xi,\mathfrak{I})(1/n)^m$ are convergent to exact solutions as $N \longrightarrow \infty$:

$$\mu(\chi,\xi,\mathfrak{I}) = -\sin(\chi+\xi)\left[1 - \frac{2\rho_0\mathfrak{I}}{1!} + \frac{(2\rho_0\mathfrak{I})^2}{2!} - \frac{(2\rho_0\mathfrak{I})^3}{3!} + \cdots\right] = -e^{-2\rho 0\mathfrak{I}}\sin(\chi+\xi),$$

$$v(\chi,\xi,\mathfrak{I}) = \sin(\chi+\xi)\left[1 - \frac{2\rho_0\mathfrak{I}}{1!} + \frac{(2\rho_0\mathfrak{I})^2}{2!} - \frac{(2\rho_0\mathfrak{I})^3}{3!} + \cdots\right] = e^{-2\rho_0\mathfrak{I}}\sin(\chi+\xi). \tag{93}$$

Example 4. In Equation (83), we take

$$v(\chi,\xi,0) = e^{\chi+\xi}, \mu(\chi,\xi,0) = -e^{\chi+\xi}. \tag{94}$$

Using the Shehu transformation on Equation (83) and applying Equation (94), we have

$$S[\mu] + \frac{u}{s} e^{\chi+\xi} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\left\{\mu\frac{\partial\mu}{\partial\chi} + \nu\frac{\partial\mu}{\partial\xi} - \rho_0\left(\frac{\partial^2\mu}{\partial\chi^2} + \frac{\partial^2\mu}{\partial\xi^2}\right) - g\right\} = 0,$$
$$S[\nu] - \frac{u}{s} e^{\chi+\xi} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} S\left\{\mu\frac{\partial\nu}{\partial\chi} + \nu\frac{\partial\nu}{\partial\xi} - \rho_0\left(\frac{\partial^2\nu}{\partial\chi^2} + \frac{\partial^2\nu}{\partial\xi^2}\right) + g\right\} = 0. \quad (95)$$

Define nonlinear operators as

$$N^1[\phi_1, \phi_2] = S[\phi_1] + \frac{u}{s} e^{\chi+\xi} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left\{\phi_1\frac{\partial\phi_1}{\partial\chi} + \phi_2\frac{\partial\phi_1}{\partial\xi} - \rho_0\left(\frac{\partial^2\phi_1}{\partial\chi^2} + \frac{\partial^2\phi_1}{\partial\xi^2}\right) - g\right\},$$

$$N^2[\phi_1, \phi_2] = S[\phi_2] - \frac{u}{s} e^{\chi+\xi} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left\{\phi_1\frac{\partial\phi_2}{\partial\chi} + \phi_2\frac{\partial\phi_2}{\partial\xi} - \rho_0\left(\frac{\partial^2\phi_2}{\partial\chi^2} + \frac{\partial^2\phi_2}{\partial\xi^2}\right) + g\right\}, \quad (96)$$

and Shehu operators as

$$S[\mu_m(\chi,\xi,I) - k_m\mu_{m-1}(\chi,\xi,I)] = \hbar R_{1,m}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right],$$
$$S[\nu_m(\chi,\xi,I) - k_m\nu_{m-1}(\chi,\xi,I)] = \hbar R_{2,m}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right], \quad (97)$$

where

$$R_{1,m}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right] = S[\mu_{m-1}] + \left(1 - \frac{k_m}{n}\right)\frac{e^{\chi+\xi}}{s} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\mu_m - 1 - i}{\partial\chi} + \sum_{i=0}^{m-1}\nu_i\frac{\partial\mu_{m-1} - i}{\partial\xi} - \rho_0\left(\frac{\partial^2\mu_{m-1}}{\partial\chi^2} + \frac{\partial^2\mu_{m-1}}{\partial\xi^2}\right) - g\right\},$$

$$R_{2,\mu}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right] = S[\nu_{m-1}] - \left(1 - \frac{k_m}{n}\right)\frac{u}{s} e^{\chi+\xi} + \frac{1-\delta+\delta(u/s)^{\delta}}{B(\delta)} \cdot S\left\{\sum_{i=0}^{m-1}\mu_i\frac{\partial\nu_{m-1-i}}{\partial\chi} + \sum_{i=0}^{m-1}\nu_i\frac{\partial\nu_{m-1-i}}{\partial\xi} - \rho_0\left(\frac{\partial^2\nu_m - 1}{\partial\chi^2} + \frac{\partial^2\nu_m - 1}{\partial\xi^2}\right) + g\right\}. \quad (98)$$

By the inverse Shehu transformation on Equation (97), we get

$$\mu_m(\chi,\xi,I) = k_m\mu_{m-1} + \hbar S^{-1}\left\{R_{1,m}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right]\right\},$$
$$\nu_m(\chi,\xi,I) = k_m\nu_{m-1} + \hbar S^{-1}\left\{R_{2,m}\left[\vec{\mu}_{m-1}, \vec{\nu}_{m-1}\right]\right\}. \quad (99)$$

Using μ_0 and ν_0, we get from Equation (99),

$$\mu_1 = \frac{2\rho_0\hbar e^{\chi+\xi}\mathfrak{J}^{\delta}}{B(\delta)}\left[1 - \delta + \frac{\delta\mathfrak{J}^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\nu_1 = -\frac{2\rho_0\hbar e^{\chi+\xi}\mathfrak{J}^{\delta}}{B(\delta)}\left[1 - \delta + \frac{\delta\mathfrak{J}^{\delta}}{\Gamma(\delta+1)}\right],$$

$$\mu_2 = \frac{2(n+\hbar)\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1 - \delta + \frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right] - \frac{4\rho_0^2\hbar^2 e^{\chi+\xi}}{(B(\delta))^2} \cdot \left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right], \nu_2$$
$$= -\frac{2(n+\hbar)\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1 - \delta + \frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right] + \frac{4\rho_0^2\hbar^2 e^{\chi+\xi}}{(B(\delta))^2} \cdot \left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right], \mu_3$$
$$= \frac{2(n+\hbar)^2\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1 - \delta + \frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right] - \frac{8(n+\hbar)\rho_0^2\hbar^2 e^{\chi+\xi}}{(B(\delta))^2} \cdot \left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right] + \frac{8\rho_0^3\hbar^3 e^{\chi+\xi} I^{3\delta}}{\Gamma[3\delta+1]},$$
$$\nu_3 = -\frac{2(n+\hbar)^2\rho_0\hbar e^{\chi+\xi}}{B(\delta)}\left[1 - \delta + \frac{\delta I^{\delta}}{\Gamma(\delta+1)}\right] + \frac{8(n+\hbar)\rho_0^2\hbar^2 e^{\chi+\xi}}{(B(\delta))^2}\left[(1-\delta)^2 + \frac{2\delta(1-\delta)I^{\delta}}{\Gamma(\delta+1)} + \frac{\delta^2 I^{2\delta}}{\Gamma(2\delta+1)}\right] - \frac{8\rho_0^3\hbar^3 e^{\chi+\xi} I^{3\delta}}{\Gamma[3\delta+1]}, \quad (100)$$

and so on. Accordingly, rest of the components are identified. The q-HATM result of Equation (42) is

$$\mu(\chi,\xi,\mathfrak{J}) = \mu_0 + \sum_{m=1}^{\infty}\mu_m\left(\frac{1}{n}\right)^m,$$
$$\nu(\chi,\xi,\mathfrak{J}) = \nu_0 + \sum_{i=1}^{\infty}\nu_m\left(\frac{1}{n}\right)^m. \quad (101)$$

For $\delta = 1 = n, \hbar = -1$ and $g = 0$, solutions $\sum_{m=1}^{N}\mu_m(1/n)^m$ and $\sum_{m=1}^{N}\nu_m(1/n)^m$ are convergent to exact results as $N \longrightarrow \infty$

$$\mu(\chi,\xi,\mathfrak{J}) = -e^{\chi+\xi}\left[1 + \frac{2\rho_0\mathfrak{J}}{1!} + \frac{(2\rho_0\mathfrak{J})^2}{2!} + \frac{(2\rho_0\mathfrak{J})^3}{3!} + \cdots\right] = -e^{\chi+\xi+2\rho_0\mathfrak{J}},$$
$$\nu(\chi,\xi,\mathfrak{J}) = e^{\chi+\xi}\left[1 + \frac{2\rho_0\mathfrak{J}}{1!} + \frac{(2\rho_0\mathfrak{J})^2}{2!} + \frac{(2\rho_0\mathfrak{J})^3}{3!} + \cdots\right] = e^{\chi+\xi+2\rho_0\mathfrak{J}}. \quad (102)$$

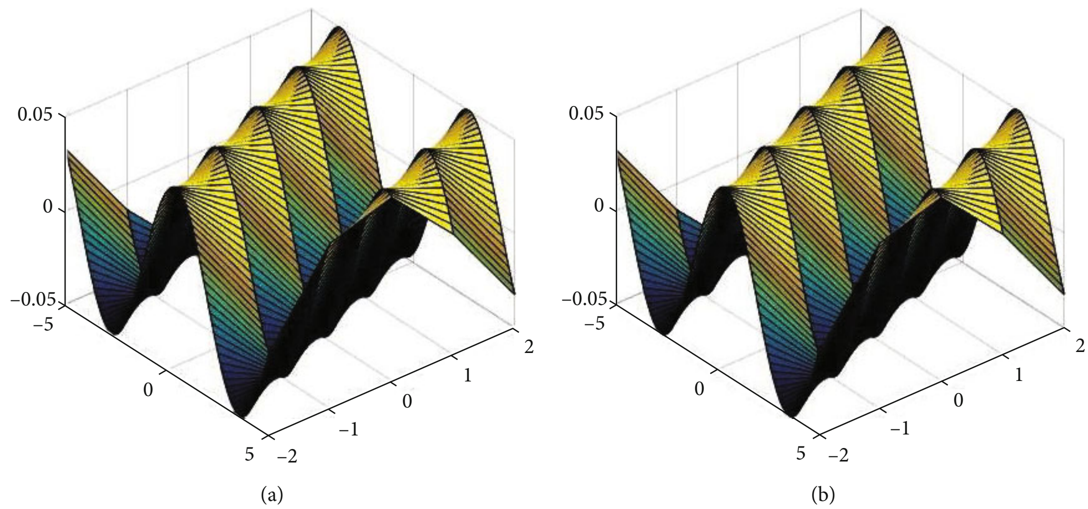


FIGURE 1: (a) Actual and (b) SDM/q-HATM result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.

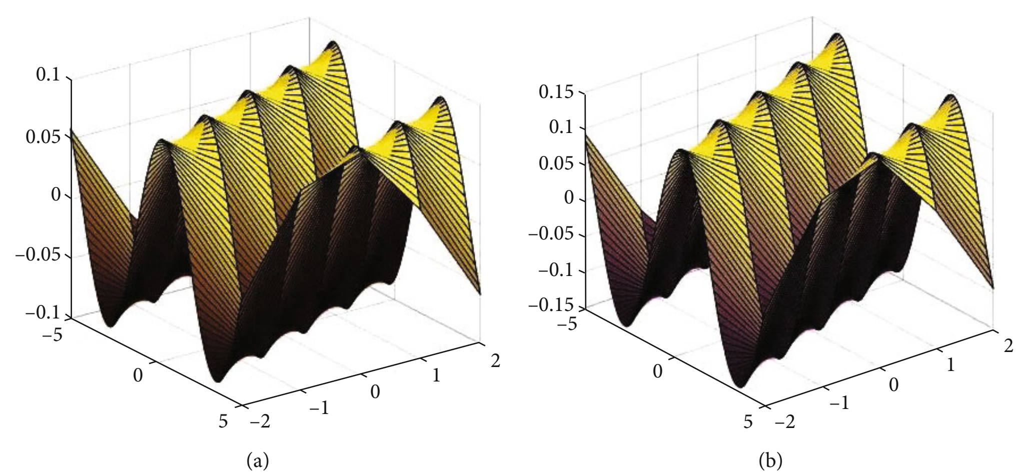


FIGURE 2: The different fractional-order result of $\mu(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

7. Results and Discussion

In this section, we analyze the solution-figures of the problem, which have been investigated by applying the q-homotopy analysis transform method and Adomian decomposition transform method in the sense of the Atangana-Baleanu operator. Figure 1 represents the three-dimensional solution-figures for variable μ of Example 1 at fractional-order $\delta = 1$, respectively, Figure 2 shows different fractional order of $\delta =$ 0.8 and 0.6, and Figure 3 shows that $\delta = 0.4$. It is observed that the q-homotopy analysis transform method and Adomian decomposition transform method solution-figures are identical and in close contact with each other. In the same way, Figures 4–6 show the different fractional-order graphs of δ at ν of Example 1. In similar way, Figure 7 represents the three-dimensional solution-figures for variable μ of Example 2 at fractional-order $\delta = 1$, respectively, Figure 8 shows different fractional-order of $\delta = 0.8$ and 0.6, and Figure 9 shows that $\delta = 0.4$. In the same way, Figures 4–6 show the different fractional-order graphs of δ at ν of Example 2. The same graphs of the suggested methods are attained and confirm

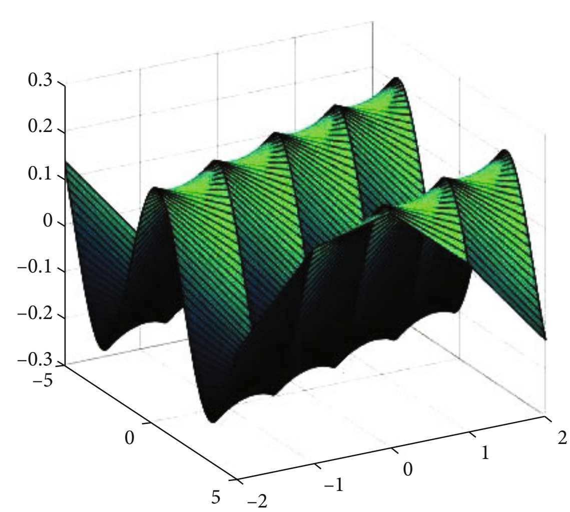
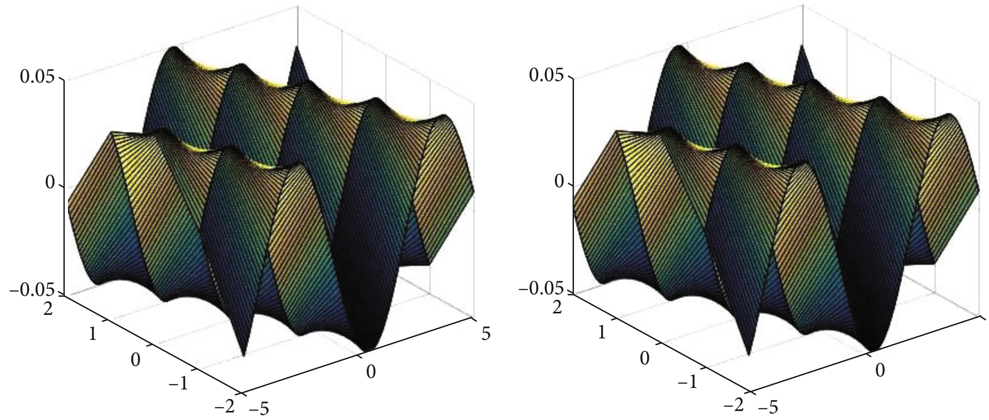
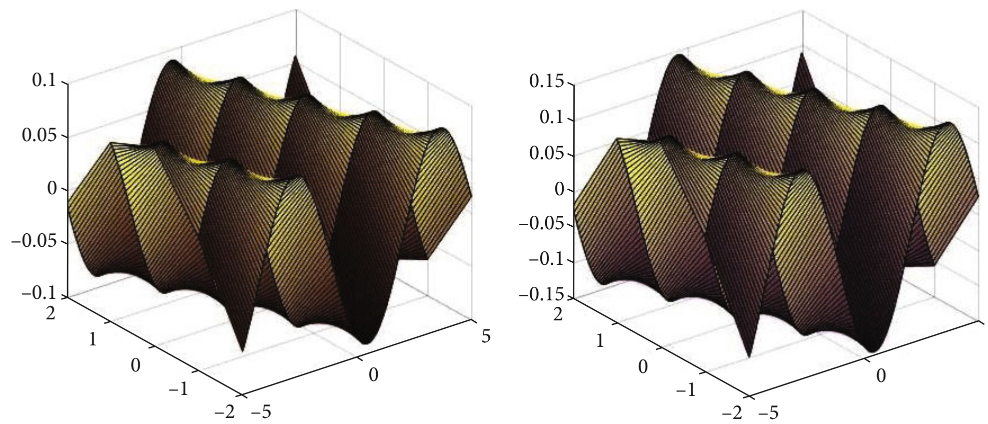


FIGURE 3: Analytical result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.



(a) (b)

Figure 4: (a) Actual and (b) SDM/q-HATM result of $\nu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.



(a) (b)

Figure 5: The different fractional-order result of $\nu(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

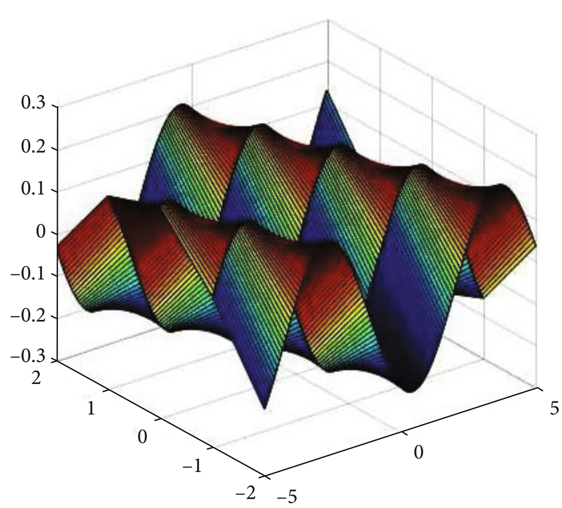
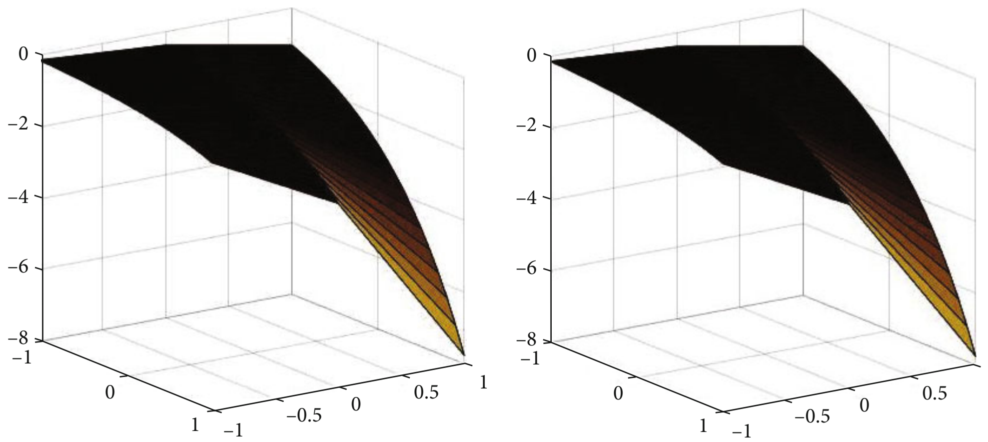


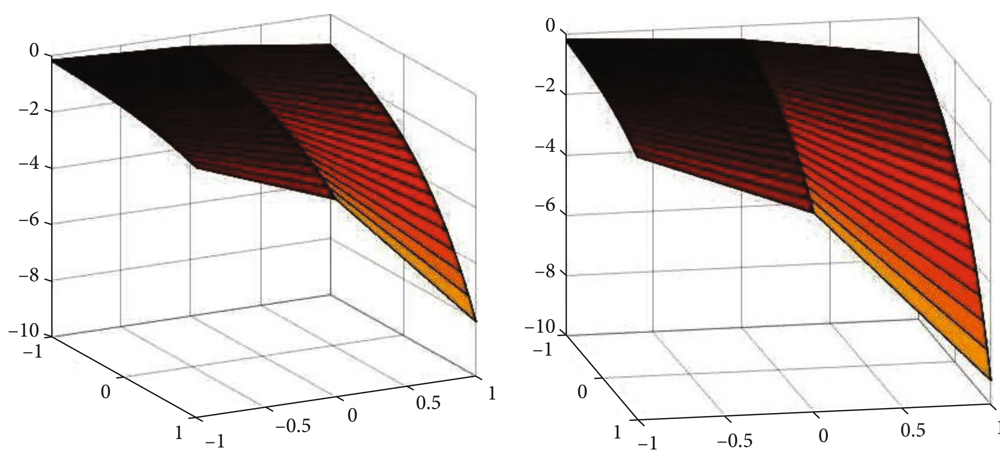
Figure 6: Analytical result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.

the applicability of the present techniques. In Figures 7–9, the q-homotopy analysis transform method and Adomian decomposition transform method solutions are plotted in three dimensional at fractional-order $\delta = 1$, 0.8, 0.6, and 0.4 of Example 2. Similarly Figures 10–12 show the exact and different fractional-order behavior of analytical solutions. The



(a) (b)

Figure 7: (a) Actual and (b) SDM/q-HATM result of $\mu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.



(a) (b)

Figure 8: The different fractional-order result of $\mu(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

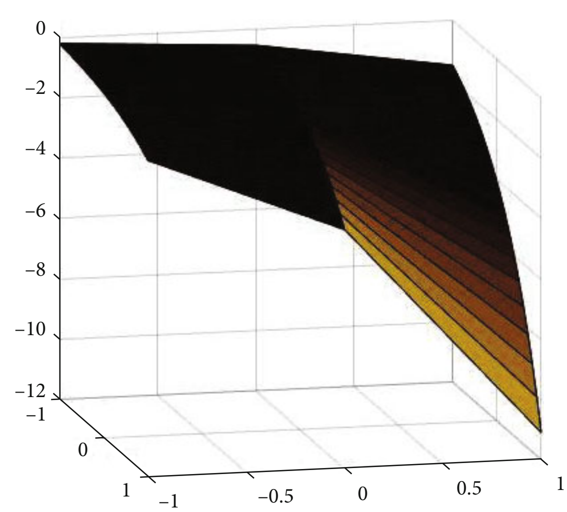


Figure 9: (a) Actual and (b) SDM/q-HATM result of $\nu(\chi, \xi, \mathfrak{I})$ at $\delta = 1$.

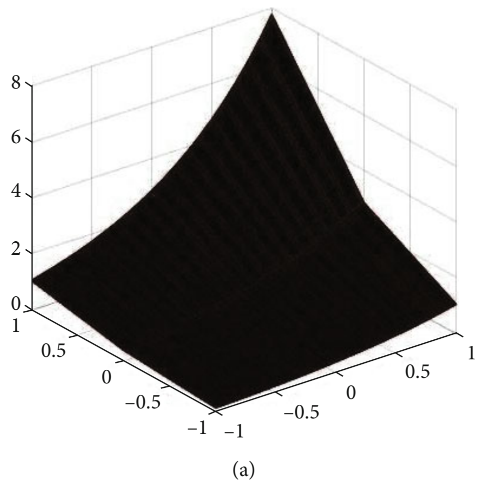


Figure 10: The different fractional-order result of $v(\chi, \xi, \mathfrak{I})$ at δ = (a) 0.8 and (b) 0.6 of example.

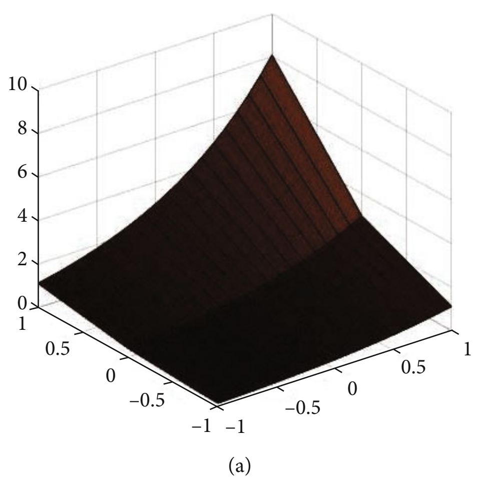


Figure 11: The graph of different approximate solution of $\Psi(\chi, \varphi, \eta)$ at Υ = (a) 0.8 and (b) 0.6 of example 2.

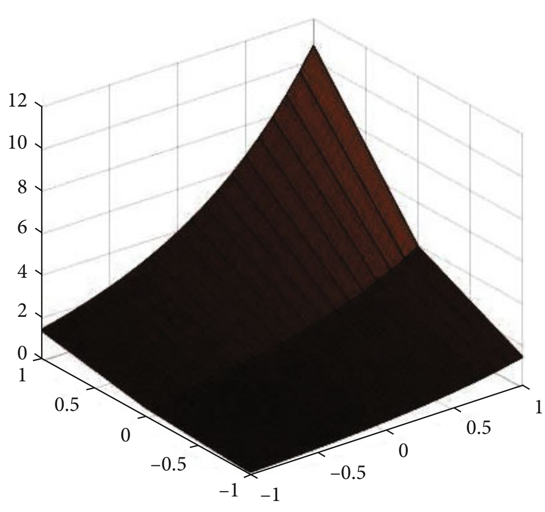


Figure 12: Analytical result of $v(\chi, \xi, \mathfrak{I})$ at $\delta = 0.4$ of example.

convergence phenomenon of the fractional solutions towards integer-solution is observed. The same accuracy is achieved by using the present techniques.

8. Conclusion

This article calculates a result of the fractional system of Navier-Stokes equations determined numerical solution using the suggested q-homotopy analysis transform method and Shehu decomposition method. The result is obtained in quick convergent series. The test samples provided demonstrate the approach's strength and efficacy. The proposed algorithm includes a parameter $\hbar$ that allows us to control the series solution's convergence region. As q-homotopy analysis transform method and Shehu decomposition method do not necessarily require small perturbation linearization or discretisation, it decreases computations significantly. In comparison with other techniques, q-homotopy analysis transform method and Shehu decomposition method are competent tools to obtain mathematical result of system nonlinear fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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Research Article

The Boundedness of Doob's Maximal and Fractional Integral Operators for Generalized Grand Morrey-Martingale Spaces

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In this paper, we introduce the generalized grand Morrey spaces in the framework of probability space setting in the spirit of the martingale theory and grand Morrey spaces. The Doob maximal inequalities on the generalized grand Morrey spaces are provided. Moreover, we present the boundedness of fractional integral operators for regular martingales in this new framework.

1. Introduction

A real-valued function f is said to belong to the Morrey space $\mathscr{L}_{p,\lambda}(\mathbb{R}^N)$ on the N-dimensional Euclidean space $\mathbb{R}^N$ provided the following norm is finite:

$$\|f\|_{\mathscr{L}_{p,\lambda}(\mathbb{R}^N)} = \left(\sup_{(x,r)\in\left(\mathbb{R}^N\times\mathbb{R}_+\right)} r^{\lambda-N}\int_{B(x,r)} |f(y)|^p dy \right)^{1/p}. \quad (1)$$

Here $1 \le p < \infty$, $0 \le \lambda \le N$, $\mathbb{R}_+ = (0,\infty)$, and $B(x, r)$ are a ball in $\mathbb{R}^N$ centered at x of radius r. This class of functions was first introduced by Morrey [1] in order to study regularity problem arising in Calculus of Variations, describe local regularity more precisely than Lebesgue spaces. In the past, Morrey spaces have been studied heavily, such as the maximal operators, fractional integral operators, and singular operators. The results are extensively applied not only in partial differential equations but also in harmonic analysis. We refer the readers to [2, 3] and the references therein.

The Morrey spaces on Euclidean spaces have been developed to the generalization versions, for example, the generalized Morrey spaces [4, 5], the Orlicz-Morrey spaces [6, 7], the Triebel-Lizorkin-Morrey spaces [8], and the variable exponent Morrey spaces [9]. Especially, Meskhi [10] introduced the grand Morrey spaces and established the boundedness of the Hardy-Littlewood maximal, Calderón-Zygmund, and potential operators in these spaces. The generalized grand Morrey spaces in a general setting of the quasi-metric measure spaces are studied by Kokilashvili et al. [11, 12].

Moreover, in probability theory, Nakai and Sadasue [13] introduced Morrey spaces of martingales as the following:

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\{\mathscr{F}_n\}_{n\ge0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\cup_{n\ge0} \mathscr{F}_n)$.

We assume that every σ-algebra $\mathscr{F}_n$ is generated by countable atoms, where $B \in \mathscr{F}_n$ is called an atom, if any $A \subset B$ with $A \in \mathscr{F}_n$ satisfies $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = \mathbb{P}(B)$. Denote by $A(\mathscr{F}_n)$ the set of all atoms in $\mathscr{F}_n$. For $p \in [1,\infty)$ and $\mu \in (-\infty, \infty)$, martingale Morrey space $L_{p,\mu}(\Omega)$ consists of all $f \in L_1(\Omega)$ having the finite norm

$$\|f\|_{L_{p,\mu}(\Omega)} = \sup_{n\ge0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{\mathbb{P}(B)^\mu} \left(\frac{1}{\mathbb{P}(B)} \int_B |f|^p d\mathbb{P} \right)^{1/p}. \quad (2)$$

They introduced some basic properties of the martingale Morrey spaces. Furthermore, the Doob maximal inequality was established, and the mapping properties for the fractional integral operators were investigated on these spaces. Two generalized versions of them introduced in [14, 15]. Ho [16] presented atomic decompositions of martingale Hardy-Morrey spaces. Later on, he [17] introduced a version of martingale Morrey spaces equipping with Banach function spaces. Jiao et al. [18] studied the maximal operator, atom decompositions, and fractional integral operators on martingale Morrey spaces with variable exponents.

Recently, Deng and Li [19] studied the Doob maximal operator and fractional integral operator in the framework of grand Morrey-martingale spaces associated with an almost decreasing function. Moreover, compared with classical martingale spaces, the grand martingale spaces have not of absolutely continuous norm based on [20]. Consequently, we need a further research about grand martingale spaces. Motivated by the works of this and [11], the paper is to investigate the generalized grand Morrey space theory for the martingale setting. More precisely, we first introduce the generalized grand Morrey-martingale spaces and then establish the Doob maximal inequality in this new framework. As an application, we discuss the boundedness of fractional integral operators for regular martingales in the generalized grand Morrey-martingale spaces.

2. Preliminaries

Now we recall some standard notations from martingale theory. Refer to [21, 22] for more information on martingale theory. The expectation is denoted by E with respect to $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that the conditional expectation operator relative to $\mathscr{F}_n$ is denoted by E_n, i.e., $E(f|\mathscr{F}_n) = E_n(f)$. A sequence of measurable functions $f = (f_n)_{n\geq 0} \subset L_1(\Omega)$ is called a martingale with respect to $(\mathscr{F}_n)_{n\geq 0}$ if $E_n(f_{n+1}) = f_n$ for every $n \geq 0$. Let $\mathscr{M}$ be the set of all martingale $f = (f_n)_{n\geq 0}$ relative to $(\mathscr{F}_n)_{n\geq 0}$ such that $f_0 = 0$. For $f \in \mathscr{M}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ $(n \geq 0$, with convention $d_0 f = 0)$.

The maximal function of $f \in \mathscr{M}$ is defined by

$$M_m f = \sup_{n\leq m} |f_n|, Mf = \sup_{n\geq 0} |f_n|. \tag{3}$$

For $p > 1$ and $f \in L_p(\mathscr{M})$, we have

$$\|Mf\|_{L_p} \leq \frac{p}{p-1} \|f\|_{L_p}, \tag{4}$$

which is well known in the literature as the Doob maximal inequality (see [22]).

Hence, it follows from the above inequality that if $p \in (1,\infty)$, then L_p-bounded martingale converges in L_p. Moreover, if $p \in [1,\infty)$, then, for any $f \in L_p$, its corresponding martingale $(f_n)_{n\geq 0}$ with $f_n = E_n f$ is an L_p-bounded martingale and converges to f in L_p (see [21]). For this reason, a function $f \in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ will be denoted by the same symbol f.

It is convenient for us to state the generalized grand Morrey-martingale spaces, we first need to recall the definition of martingale Morrey spaces $\mathscr{L}_{p,\lambda} = \mathscr{L}_{p,\lambda}(\Omega)$ as follows.

Definition 1. For $p \in [1,\infty)$ and $\lambda \in (-\infty, \infty)$, let

$$\mathscr{L}_{p,\lambda} = \left\{ f \in \mathscr{M} : \|f\|_{\mathscr{L}_{p,\lambda}} < \infty \right\}, \tag{5}$$

where

$$\|f\|_{\mathscr{L}_{p,\lambda}} = \sup_{n\geq 0} \sup_{B\in A(\mathscr{F}_n)} \left(\frac{1}{\mathbb{P}(B)^{\lambda}} \int_B |f|^p d\mathbb{P} \right)^{1/p}. \tag{6}$$

Remark 2. If $\lambda = pu + 1$, the above definition of $\|\cdot\|_{\mathscr{L}_{p,\lambda}}$ is equivalent to $\|\cdot\|_{L_{p,\mu}}$ (see (2)), which introduced by Nakai and Sadasue [13].

If $\lambda = 0$ and $\mathscr{F}_0 = \{\varnothing, \Omega\}$, then the Morrey-martingale space $\mathscr{L}_{p,\lambda}$ is L_p by the above definition.

Now we introduce a new type Morrey-martingale spaces as follows.

Definition 3. Let $1 < p < \infty$, $0 \leq \lambda < 1$, φ be a nondecreasing real-valued nonnegative function defined on $(0, p-1]$ with $\lim_{x\longrightarrow 0^+} \varphi(x) = 0$, and δ be a positive number. The generalized grand Morrey-martingale space $\mathscr{L}^{\delta,\varphi}_{p),\lambda)}(\Omega)$ consists of $f \in \mathscr{M}$ such that

$$\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} = \sup_{\substack{0<\varepsilon\leq s \\ s=\min\{p-1,\alpha\}}} \varepsilon^{\delta/(p-\varepsilon)} \sup_{n\geq 0} \sup_{B\in A(\mathscr{F}_n)} \cdot \left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\varepsilon)}} \int_B |f|^{p-\varepsilon} d\mathbb{P} \right)^{1/(p-\varepsilon)} \tag{7}$$

is finite, where $\alpha = \sup\{x > 0 : \varphi(x) \leq \lambda\}$.

Notice that, in the above condition, $\|\cdot\|_{\mathscr{L}^{\delta,\varphi}_{p)\lambda)}}$ is a norm and can be expressed as

$$\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} = \sup_{\substack{0<\varepsilon\leq s \\ s=\min\{p-1,\alpha\}}} \varepsilon^{\delta/(p-\varepsilon)} \|f\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}. \tag{8}$$

Remark 4. If $\lambda > 0$ and $\varphi \equiv 0$, then $\mathscr{L}^{\delta,\varphi}_{p),\lambda)}(\Omega)$ is called grand Morrey-martingale space, which was introduced in [19]. If $\lambda = 0$, $\delta = 1$, and $\varphi \equiv 0$, we recover the grand Lebesgue spaces for martingales introduced in [23]. In this case, if consider $\Omega = [0, 1)$, we have grand Lebesgue spaces introduced in [24]. We mention that there exists a martingale $f = (f_n)_{n\geq 0}$ such that it does not converge in $L_{p)}([0, 1))$. Indeed, $L_{p)}([0, 1))$ is a rearrangement-invariant Banach function space, $L_{p)}([0, 1)) \neq L_1([0, 1))$, but is not of absolutely continuous norm from [20]. According to Theorem 3.3 in [25], there exists a martingale $f = (f_n)_{n\geq 0}$ such that it does not converge in $L_{p)}([0, 1))$.

The stochastic basis $\{\mathscr{F}_n\}_{n\geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \tag{9}$$

holds for all nonnegative martingale $(f_n)_{n\geq 0}$ adapted to $\{\mathscr{F}_n\}_{n\geq 0}$.

For regular stochastic basis, there has the following property, proved in [13].

Lemma 5. *Let $\{\mathscr{F}_n\}_{n\geq 0}$ be regular. Then, for every sequence*

$$B_n \subset B_{n-1} \subset \cdots \subset B_k \subset \cdots \subset B_0, B_k \in A(\mathscr{F}_k), \tag{10}$$

we have

$$B_k = B_{k-1} \text{ or } \left(1 + \frac{1}{R}\right)\mathbb{P}(B_k) \leq \mathbb{P}(B_{k-1}) \leq R\mathbb{P}(B_k) (1 \leq k \leq n), \tag{11}$$

where R is the positive constant in (9).

3. The Doob Maximal Operator

In this section, we present the boundedness of Doob's maximal operator on generalized grand Morrey-martingale spaces.

Theorem 6. *Let $1 < p < \infty$, $\delta > 0$, and $0 \leq \lambda < 1$. Then,*

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} \leq C\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}, \tag{12}$$

where the constant C satisfies

$$C = \inf_{0<\theta<s} s^{\delta/(p-\varepsilon)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1} + 1\right), \tag{13}$$

which only depends on the parameters $p, \lambda, \delta, \varphi$ for $s = \min\{p-1, \alpha\}$ and $\alpha = \sup\{x > 0 : \varphi(x) \leq \lambda\}$.

In order to prove Theorem 6, we need the following useful lemma:

Lemma 7. *Let $f = (f_n)_{n\geq 0} \in L_1$, $1 < p < \infty$, $0 \leq \lambda < 1$. Then,*

$$\|Mf\|_{\mathscr{L}_{p,\lambda}} \leq \left(\frac{p}{p-1} + 1\right)\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{14}$$

Proof. For any $B \in A(\mathscr{F}_m)$ and $m \geq 0$, suppose that $f = g + h$ and $g = f\chi_B$.

Then, according to the well-known Doob's maximal inequality, that is,

$$\|Mf\|_{L_p} \leq \frac{p}{p-1}\|f\|_{L_p}, \tag{15}$$

we have

$$\begin{aligned}\int_B |Mg|^p d\mathbb{P} &\leq \int_\Omega |Mg|^p d\mathbb{P} \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |g|^p d\mathbb{P} \\ &= \left(\frac{p}{p-1}\right)^p \int_B |f|^p d\mathbb{P}.\end{aligned} \tag{16}$$

Hence,

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B |Mg|^p d\mathbb{P}\right)^{1/p} \leq \frac{p}{p-1}\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{17}$$

Next, take $B_n \in A(\mathscr{F}_n)$, $n = 0, 1, \cdots, m$, such that $B = B_m \subset B_{m-1} \subset \cdots \subset B_0$. Then, for a.e. $\omega \in B$,

$$E_n h(\omega) = \begin{cases} 0, & \text{if } n \geq m, \\ \dfrac{1}{\mathbb{P}(B_n)}\displaystyle\int_{B_n} h d\mathbb{P}, & \text{if } n < m. \end{cases} \tag{18}$$

If $n < m$, according to Jensen's inequality, then

$$\begin{aligned}|E_n h(\omega)| &\leq \left(\frac{1}{\mathbb{P}(B_n)}\int_{B_n} |h|^p d\mathbb{P}\right)^{1/p} \leq \mathbb{P}(B_n)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}} \\ &\leq \mathbb{P}(B)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}},\end{aligned} \tag{19}$$

where the last inequality dues to $0 \leq \lambda < 1$ and $\mathbb{P}(B) \leq \mathbb{P}(B_n)$. This means

$$(Mh)(\omega) \leq \mathbb{P}(B)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}} \text{ for any } \omega \in B. \tag{20}$$

Then, we obtain

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B |Mh|^p d\mathbb{P}\right)^{1/p} \leq \|f\|_{\mathscr{L}_{p,\lambda}}. \tag{21}$$

Combining inequalities (17) and (21) and $Mf \leq Mg + Mh$, we can get

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B (Mf)^p d\mathbb{P}\right)^{1/p} \leq \left(\frac{p}{p-1} + 1\right)\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{22}$$

The proof is complete. □

Note that Nakai and Sadasue [13] proved that, for $1 < p < \infty$ and $-1/p \leq \mu < 0$,

$$\|Mf\|_{L_{p,\mu}} \leq C_p\|f\|_{L_{p,\mu}}. \tag{23}$$

The proof of Lemma 7 is devoted to determination of the constant C_p. Now we prove Theorem 6:

Proof. Let $0 < \theta < s$, and we have

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} = \sup_{\substack{0<\varepsilon\le s\\ s=\min\{p-1,\alpha\}}} \varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}} = \max\left\{\sup_{0<\varepsilon<\theta}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}, \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}\right\}, \tag{24}$$

where $\alpha = \sup\{x > 0 : \varphi(x) \le \lambda\}$. Let

$$I = \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}} \varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}. \tag{25}$$

Note that the function $h(\varepsilon) \coloneqq \varepsilon^{\delta/(p-\varepsilon)}$ is increasing in $0 < \varepsilon < p$, which means

$$\begin{aligned}
I &\le s^{\delta/(p-s)} \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}} \sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)} \cdot \left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\varepsilon)}}\int_B |Mf|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)} \\
&\le s^{\delta/(p-s)} \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}} \sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)} \cdot \frac{1}{\mathbb{P}(B)^{(\lambda-\varphi(\varepsilon)-1)/(p-\varepsilon)}}\left(\frac{1}{\mathbb{P}(B)}\int_B |Mf|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)} \\
&\le s^{\delta/(p-s)} \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}} \sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)} \cdot \frac{\theta^{\delta/(p-\theta)}\theta^{-\delta/(p-\theta)}}{\mathbb{P}(B)^{\Delta(\varepsilon,\theta)}}\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\theta)}}\int_B |Mf|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)},
\end{aligned} \tag{26}$$

where

$$\Delta(\varepsilon,\theta) \coloneqq \frac{\lambda-\varphi(\varepsilon)-1}{p-\varepsilon} - \frac{\lambda-\varphi(\theta)-1}{p-\theta}. \tag{27}$$

Note that for $\theta \le \varepsilon$,

$$\begin{aligned}
\Delta(\varepsilon,\theta) &= \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\theta)(p-\varepsilon)-\varphi(\varepsilon)(p-\theta)}{(p-\varepsilon)(p-\theta)} \\
&\le \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\varepsilon)(p-\varepsilon)-\varphi(\varepsilon)(p-\theta)}{(p-\varepsilon)(p-\theta)} \\
&\le \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\varepsilon)(\theta-\varepsilon)}{(p-\varepsilon)(p-\theta)} \le 0.
\end{aligned} \tag{28}$$

Then, $0 < 1/\mathbb{P}(B)^{\Delta(\varepsilon,\theta)} \le 1$, and we obtain $I \le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}(\theta^{\delta/(p-\theta)}\|Mf\|_{\mathscr{L}_{p-\theta,\lambda-\varphi(\theta)}})$. Obviously, $s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)} > 1$ as $0 < \theta < s$. Thus, according to Lemma 7, we deduce that

$$\begin{aligned}
\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} &\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)} \sup_{0<\varepsilon\le\theta}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}} \\
&\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)} \sup_{0<\varepsilon\le\theta}\varepsilon^{\delta/(p-\varepsilon)}\left(\frac{p-\varepsilon}{p-\varepsilon-1}+1\right)\|f\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}} \\
&\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1}+1\right)\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}.
\end{aligned} \tag{29}$$

Taking the infimum over all θ, we obtain that

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} \le C\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}, \tag{30}$$

where

$$C = \inf_{\substack{0<\theta<s\\ s=\min\{p-1,\alpha\}\\ \alpha=\sup\{x>0\,:\,\varphi(x)\le\lambda\}}} s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1}+1\right) \in (1,\infty). \tag{31}$$

□

4. The Fractional Integral Operator

In this section, we present the boundedness of the fractional integral operator in the new type grand Morrey-martingale spaces. In martingale theory, Chao and Ombe [26] introduced the fractional integrals for dyadic martingales. The fractional integrals in this section are defined for more general martingale setting as in [13, 14] (see also [15, 27–32]).

Definition 8. Let $f = (f_n)_{n\ge0} \in \mathscr{M}$ and $\iota > 0$, and the fractional integral $I^\iota f = ((I^\iota f)_n)_{n\ge0}$ of martingale f is defined by

$$(I^\iota f)_n = \sum_{k=0}^{n} b_{k-1}^\iota d_k f, \tag{32}$$

where b_k is an $\mathscr{F}_k$-measurable function such that

$$b_k(\omega) = \sum_{B\in A(\mathscr{F}_k)} \mathbb{P}(B)\chi_B(\omega), \quad \omega\in\Omega. \tag{33}$$

Remark 9. Obviously, b_k is bounded in above definition; there $I^\iota f = ((I^\iota f)_n)_{n\ge0}$ is a martingale transform of f.

The following lemma was shown in [13]. Here we focus on more accurate upper boundedness.

Lemma 10. *Suppose that $\{\mathscr{F}_n\}_{n\ge0}$ is regular. Let $1 < p < \infty$, $1 \le q \le (\nu/u)p$, $-1/p \le \nu < 0$, and $u = \nu + \iota < 0$. Then, for $f \in L_1$,*

$$\|M(I^\iota f)\|_{L_{q,u}} \le C_{q,u,p,\nu}\|f\|_{L_{p,\nu}}, \tag{34}$$

where $C_{q,u,p,v} = [(1+(1/R)^v)/(1-(1+1/R)^u)+2](p/(p-1)+1)^{p/q}$ *and R is the constant in formula (9).*

Proof. Since $\|f\|_{L_{q_1,\lambda}} \le \|f\|_{L_{q_2,\lambda}}$ for $q_1 \le q_2$ by Hölder's inequality, it is enough to prove it in the case where $q=(v/u)p$. Without loss of generality, we let $\|f\|_{L_{p,v}} \ne 0$.

First, we prove the following inequality holds for any $n \ge 1$, and any $B_n \in A(\mathscr{F}_n)$,

$$\left|(I^\iota f)_n(\omega)\right| \le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u}+2\right)(Mf(\omega))^{u/v}\|f\|_{L_{p,v}}^{-\iota/v}, \quad \omega \in B_n. \tag{35}$$

Choose $B_k \in A(\mathscr{F}_k)$ and $0 \le k < n$, such that $B_n \subset B_{n-1} \subset \cdots \subset B_0$, and let

$$K=\{k : 0<k\le n, B_k \ne B_{k-1}\} = \{k_1, k_2, \cdots, k_h\}, \tag{36}$$

where $0=k_0<k_1<k_2<\cdots<k_h$.

Since $\{\mathscr{F}_n\}_{n\ge 0}$ is regular, according to Lemma 5, we have

$$\left(1+\frac{1}{R}\right)b_{k_j} \le b_{k_{j-1}} \le R\, b_{k_j} \text{ on } B_n. \tag{37}$$

So, for $k \notin K$, we have $b_k = b_{k-1}$ and $d_k f = 0$. Hence, we obtain

$$(I^\iota f)_n = \sum_{0<k_j\le n} b^\iota_{k_{j-1}} d_{k_j} f = \sum_{j=1}^{h} b^\iota_{k_{j-1}} d_{k_j} f \text{ on } B_n. \tag{38}$$

For $\omega \in B_n$,

$$\begin{aligned}
\left|d_{k_j} f(\omega)\right| &= \left|f_{k_j}(\omega) - f_{k_{j-1}}(\omega)\right| \le \left|f_{k_j}(\omega)\right| + \left|f_{k_{j-1}}(\omega)\right| \\
&= \left|\frac{1}{\mathbb{P}\left(B_{k_j}\right)}\int_{B_{k_j}} f\,d\mathbb{P}\right| + \left|\frac{1}{\mathbb{P}\left(B_{k_{j-1}}\right)}\int_{B_{k_{j-1}}} f\,d\mathbb{P}\right| \\
&\le \left(\frac{1}{\mathbb{P}\left(B_{k_j}\right)}\int_{B_{k_j}} |f|^p d\mathbb{P}\right)^{1/p} + \left(\frac{1}{\mathbb{P}\left(B_{k_{j-1}}\right)}\int_{B_{k_{j-1}}} |f|^p d\mathbb{P}\right)^{1/p} \\
&\le \left(\mathbb{P}\left(B_{k_j}\right)^v + \mathbb{P}\left(B_{k_{j-1}}\right)^v\right)\|f\|_{L_{p,v}} \\
&\le \left(1+\left(\frac{1}{R}\right)^v\right) b_{k_{j-1}}(\omega)^v \|f\|_{L_{p,v}}.
\end{aligned} \tag{39}$$

Then, for $\omega \in B_n$ and when $0<k\le m$ where $m \le n$,

$$\begin{aligned}
\left|\sum_{k=0}^{m} b_{k-1}(\omega)^\iota d_k f(\omega)\right| &\le \left(1+\left(\frac{1}{R}\right)^v\right)\sum_{0<k_j\le m} b_{k_{j-1}}(\omega)^{v+\iota}\|f\|_{L_{p,v}} \\
&= \left(1+\left(\frac{1}{R}\right)^v\right)\sum_{0<k_j\le m} b_{k_{j-1}}(\omega)^{u}\|f\|_{L_{p,v}} \\
&\le \frac{1+(1/R)^v}{1-(1+1/R)^u} b_m(\omega)^u \|f\|_{L_{p,v}}.
\end{aligned} \tag{40}$$

For $\omega \in B_n$ and when $m+1 \le k < n$, let $j(k) = \min\{j : k < k_j\}$, and we have

$$\begin{aligned}
&\left|\sum_{k=m+1}^{n} b_{k-1}(\omega)^\iota d_k f(\omega)\right| \\
&= \left|\sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^\iota d_{k_j} f(\omega)\right| \\
&= \left|\sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^\iota f_{k_j}(\omega) - \sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^\iota f_{k_{j-1}}(\omega)\right| \\
&= \left|b_{k_{h-1}}(\omega)^\iota f_{k_h}(\omega) + \sum_{j=j(k)}^{h-1}\left(b_{k_{j-1}}(\omega)^\iota - b_{k_j}(\omega)^\iota\right) f_{k_j}(\omega) - b_{k_{j(k)-1}}(\omega)^\iota f_{k_{j(k)-1}}(\omega)\right| \\
&\le b_{k_{h-1}}(\omega)^\iota Mf(\omega) + \sum_{j=j(k)}^{h-1}\left|b_{k_{j-1}}(\omega)^\iota - b_{k_j}(\omega)^\iota\right| Mf(\omega) + b_{k_{j(k)-1}}(\omega)^\iota Mf(\omega) \\
&\le 2 b_{k_{j(k)-1}}(\omega)^\iota Mf(\omega) = 2 b_m(\omega)^\iota Mf(\omega).
\end{aligned} \tag{41}$$

Now let

$$\Lambda_1 = \left\{\omega \in \Omega : \left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{1/v} \le b_0(\omega)\right\} \text{ and } \Lambda_2 = \Omega \setminus \Lambda_1. \tag{42}$$

Next we estimate $(I^\iota f)_n$ from the following cases. For the first case, if $\omega \in \Lambda_1 \cap B_n$ and

$$\left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{1/v} \le b_n(\omega), \tag{43}$$

then, by formula (40) and $u = v + \iota < 0$, we have

$$\begin{aligned}
\left|(I^\iota f)_n(\omega)\right| &\le \frac{1+(1/R)^v}{1-(1+1/R)^u} b_n(\omega)^u \|f\|_{L_{p,v}} \\
&\le \frac{1+(1/R)^v}{1-(1+1/R)^u}\left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{u/v}\|f\|_{L_{p,v}} \\
&= \frac{1+(1/R)^v}{1-(1+1/R)^u}(Mf(\omega))^{u/v}\|f\|_{L_{p,v}}^{-\iota/v}.
\end{aligned} \tag{44}$$

For the second case, if $\omega \in \Lambda_1 \cap B_n$ and

$$b_n(\omega) < \left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{1/v}, \tag{45}$$

then there exists m such that

$$\frac{1}{R} b_m(\omega) < \left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{1/v} \le b_m(\omega). \tag{46}$$

Combining (46) with formulas (40) and (41), we have

$$\begin{aligned} \left|(I^\iota f)_n(\omega)\right| &\le \frac{1+(1/R)^v}{1-(1+1/R)^u} b_m(\omega)^u \|f\|_{L_{p,v}} + 2b_m(\omega)^\iota Mf(\omega) \\ &\le \frac{1+(1/R)^v}{1-(1+1/R)^u}\left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{u/v}\|f\|_{L_{p,v}} + 2\left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{\iota/v} Mf(\omega) \\ &\le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)(Mf(\omega))^{u/v}\|f\|_{L_{p,v}}^{-\iota/v}. \end{aligned} \tag{47}$$

For the third case, if $\omega \in \Lambda_2 \cap B_n$, then by (41), we have

$$\begin{aligned} \left|(I^\iota f)_n(\omega)\right| &\le 2b_0(\omega)^\iota Mf(\omega) \\ &\le 2\left(\frac{Mf(\omega)}{\|f\|_{L_{p,v}}}\right)^{\iota/v} Mf(\omega) = 2(Mf(\omega))^{u/v}\|f\|_{L_{p,v}}^{-\iota/v}. \end{aligned} \tag{48}$$

Formulas (44), (47), and (48) give that

$$\left|(I^\iota f)_n(\omega)\right| \le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)(Mf(\omega))^{u/v}\|f\|_{L_{p,v}}^{-\iota/v}, \omega \in B_n, \tag{49}$$

which implies that

$$\begin{aligned} &\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f(\omega))|^q d\mathbb{P}\right)^{1/q} \\ &\quad\le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)\left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^{qu/v} d\mathbb{P}\right)^{1/q}\|f\|_{L_{p,v}}^{-\iota/v} \\ &\quad= \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)\left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^p d\mathbb{P}\right)^{(1/p)(p/q)}\|f\|_{L_{p,v}}^{1-p/q}. \end{aligned} \tag{50}$$

Moreover, by Lemma 7, we have

$$\begin{aligned} \left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^p d\mathbb{P}\right)^{(1/p)(p/q)} &\le \left(\mathbb{P}(B)^v \|Mf\|_{L_{p,v}}\right)^{p/q} \\ &\le \left(\frac{p}{p-1}+1\right)^{p/q}\mathbb{P}(B)^u \|f\|_{L_{p,v}}^{p/q}. \end{aligned} \tag{51}$$

It follows from the above inequality and (50) that

$$\begin{aligned} &\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f(\omega))|^q d\mathbb{P}\right)^{1/q} \\ &\quad\le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)\left(\frac{p}{p-1}+1\right)^{p/q}\mathbb{P}(B)^u\|f\|_{L_{p,v}}, \end{aligned} \tag{52}$$

that is to say,

$$\|M(I^\iota f)\|_{L_{q,u}} \le \left(\frac{1+(1/R)^v}{1-(1+1/R)^u} + 2\right)\left(\frac{p}{p-1}+1\right)^{p/q}\|f\|_{L_{p,v}}. \tag{53}$$

The proof is complete. □

Theorem 11. *Let $1 < q < \infty$, $0 \le \lambda < 1$, $0 < \iota < (1-\lambda)/q$, $1/q - 1/p = \iota/(1-\lambda)$, $\delta_2 > 0$, and $\delta_1 \ge \delta_2(1+\iota p/(1-\lambda))$. Suppose that φ_1 and φ_2 are continuous nonnegative and nondecreasing real-valued functions on $(0, p-1]$ and $(0, q-1]$, respectively, satisfying*

(i) $\varphi_1 \in C^1(0,\kappa]$ for some positive $\kappa > 0$

(ii) $\lim_{x\longrightarrow 0^+}\varphi_1(x) = 0$

(iii) $0 \le \lim_{x\longrightarrow 0^+} d\varphi_1(x)/dx < (1-\lambda)^2/(\iota p^2)$

(iv) $\varphi_2(\eta) = \varphi_1(\phi^{-1}(\eta))$, where ϕ^{-1} is the inverse of ϕ on $(0,\kappa]$ for $\kappa > 0$, and $\phi(x) = q-(p-x)(1-\lambda+\varphi_1(x))/[1-\lambda+\varphi_1(x)+\iota(p-x)]$. Then, for $f \in \mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}$,

$$\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}} \le C(p,\delta_1,\delta_2,\varphi_1,\lambda)\|f\|_{\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}}, \tag{54}$$

where $C(p,\delta_1,\delta_2,\varphi_1,\lambda)$ only depends on p, δ_1, δ_2, φ_1, and λ.

Proof. The equation $1/q - 1/p = \iota/(1-\lambda)$ and $\lim_{x\longrightarrow 0^+}\varphi_1(x) = 0$ give that $\lim_{x\longrightarrow 0^+}\phi(x) = 0$. The condition (iii) ensures that $\lim_{x\longrightarrow 0^+} d\phi(x)/dx > 0$. Then, there exists small positive number $\epsilon < \min\{1,\kappa\}$ such that ϕ is increasing in $(0,\epsilon]$ and $\phi(\epsilon) < (q-1)/2$. Now fix

$$\theta \in (0, \min\{s,\epsilon\}), \tag{55}$$

where $s = \min\{p-1,\alpha\}$ and $\alpha = \sup\{x > 0 : \varphi_1(x) \le \lambda\}$.

Firstly, we consider the case of $\varepsilon \in (\theta, s)$. In this situation, let

$$I(\varepsilon) := \varepsilon^{\delta_1/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(\varepsilon)}}\int_B |M(I^\iota f)|^{p-\varepsilon} d\mathbb{P}\right)^{1/(p-\varepsilon)}. \tag{56}$$

Since $p-\varepsilon<p-\theta$, then it follows from Jensen's inequality that

$$\begin{aligned} I(\varepsilon) &= \varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\varepsilon)+1-\lambda)/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f)|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)} \\ &\le \varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\varepsilon)+1-\lambda)/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f)|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)}. \end{aligned} \tag{57}$$

Note that $[\varphi_1(x)+1-\lambda]/(p-x)$ is a nonnegative and nondecreasing function on $(\theta, s]$; hence,

$$\begin{aligned} I(\varepsilon) &\le \varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\theta)+1-\lambda)/(p-\theta)}\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f)|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)} \\ &\le s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\sup_{n\ge 0}\sup_{B\in A(\mathcal{F}_n)} \\ &\quad\cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B |M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)}. \end{aligned} \tag{58}$$

Since $s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}>1$ for $\theta<s$, then the following inequality holds

$$\begin{aligned} \|M(I^\iota f)\|_{\mathcal{L}^{\delta_1,\varphi_1}_{p),\lambda)}} &\le s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\sup_{n\ge 0}\sup_{B\in A(\mathcal{F}_n)} \\ &\quad\cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B |M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)}. \end{aligned} \tag{59}$$

Next, we consider $t\in(0,\theta]$ in the following discussion. Since $1/q-1/p=\iota/(1-\lambda)$, we can choose η and t satisfying

$$\frac{1}{q-\eta}-\frac{1}{p-t}=\frac{\iota}{1-\lambda+\varphi_1(t)}. \tag{60}$$

Obviously we know that $t\longrightarrow 0$ if and only if $\eta\longrightarrow 0$, and we obtain η with respect to t as follows:

$$\eta = q-\frac{(p-t)(1-\lambda+\varphi_1(t))}{1-\lambda+\varphi_1(t)+\iota(p-t)}=\phi(t). \tag{61}$$

Let

$$\begin{gathered} \tilde{p}=q-\eta, \tilde{q}=p-t, \tilde{v}=\frac{\lambda-\varphi_1(t)-1}{q-\eta}, \\ \tilde{u}=\frac{\lambda-\varphi_1(t)-1}{p-t}. \end{gathered} \tag{62}$$

It is not hard to see that $1\le\tilde{q}=(\tilde{v}/\tilde{u})\tilde{p}$, $-1/\tilde{p}\le\tilde{v}<0$, $\tilde{v}+\iota=\tilde{u}\le-(1-\lambda)/(p-t)<0$, and

$$q-\eta=q-\phi(t)\ge q-\phi(\epsilon)\ge\frac{q+1}{2}>1. \tag{63}$$

Moreover,

$$\begin{aligned} C_{\tilde{q},\tilde{u},\tilde{p},\tilde{v}} &= \left(\frac{1+(1/R)^{\tilde{v}}}{1-(1+1/R)^{\tilde{u}}}+2\right)\left(\frac{\tilde{p}}{\tilde{p}-1}+1\right)^{\tilde{p}/\tilde{q}} \\ &\le \left(\frac{1+R^{1+\varphi_1(\theta)}}{1-(R/(R+1))^{(1-\lambda)/p}}+2\right)\left(\frac{q-\phi(\varepsilon)}{q-\phi(\varepsilon)-1}+1\right)^q \\ &\le \left(\frac{1+R^{1+\varphi_1(\theta)}}{1-(R/(R+1))^{(1-\lambda)/p}}+2\right)\left(\frac{q-\phi(\theta)}{q-\phi(\theta)-1}+1\right)^q \\ &=: C(\theta). \end{aligned} \tag{64}$$

Notice that $\varphi_1(\theta)\le\varphi_1(s)$ and $q-\phi(\theta)-1\ge q-\phi(\epsilon)-1\ge(q-1)/2$. This implies that $C(\theta)<\infty$.

Thus, according to the inequalities (59) and (64) and Lemma 10, we have

$$\begin{aligned} \|M(I^\iota f)\|_{\mathcal{L}^{\delta_1,\varphi_1}_{p),\lambda)}} &\le s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\sup_{n\ge 0}\sup_{B\in A(\mathcal{F}_n)} \\ &\quad\cdot\frac{1}{\mathbb{P}(B)^{(\lambda-\varphi_1(t)-1)/(p-t)}}\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)} \\ &= s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\|M(I^\iota f)\|_{L_{\tilde{q},\tilde{u}}} \\ &\le C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\|f\|_{L_{\tilde{p},\tilde{v}}} \\ &= C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\sup_{n\ge 0}\sup_{B\in A(\mathcal{F}_n)} \\ &\quad\cdot\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)-1/q-\eta}}\left(\frac{1}{\mathbb{P}(B)}\int_B |f|^{q-\eta}d\mathbb{P}\right)^{1/(q-\eta)} \\ &= C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\sup_{n\ge 0}\sup_{B\in A(\mathcal{F}_n)} \\ &\quad\cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B |f|^{q-\eta}d\mathbb{P}\right)^{1/q-\eta} \\ &\le C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)}\|f\|_{\mathcal{L}^{\delta_2,\varphi_2}_{q),\lambda)}}, \end{aligned} \tag{65}$$

where the last inequality holds because of $\varphi_2(\eta)=\varphi_1(\phi^{-1}(\eta))=\varphi_1(t)$.

Finally, we shall show that $\sup_{0<t\le\theta} t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)}$ is bounded. Since $\lim_{x\longrightarrow 0^+}\phi(x)=0$, by l'Hospital's rule, we have

$$\lim_{x\longrightarrow 0^+}\frac{\phi(x)}{x}=\lim_{x\longrightarrow 0^+}\frac{\phi'(x)}{x'}=\lim_{x\longrightarrow 0^+}\frac{(1-\lambda)^2-\iota p^2\varphi_1'(x)}{(1-\lambda+\iota p)^2}. \tag{66}$$

Combining with the condition (iii), we have $\phi(x)\sim x$ as $x\longrightarrow 0^+$. This implies

$$\eta^{-\delta_2/(q-\eta)}\sim t^{-\delta_2/(q-\eta)}, \text{as} \quad t\longrightarrow 0^+. \tag{67}$$

Moreover, using $\delta_1 \geq \delta_2(1 + \iota p/(1-\lambda))$, $0 < t \leq \theta < 1$, and formula (60), we obtain

$$t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)} \leq Ct^{\delta_2(1+\iota p/(1-\lambda))/(p-t)}t^{-\delta_2(1/(p-t)+\iota/(1-\lambda+\varphi_1(t)))} = Ct^{\iota\delta_2[p/((1-\lambda)(p-t))-1/(1-\lambda+\varphi_1(t))]}. \tag{68}$$

Obviously, $p/[(1-\lambda)(p-t)] - 1/[1-\lambda+\varphi_1(t)] > 0$, which implies that $\sup_{0<t\leq\theta} t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)} \leq C$ is bounded.

To sum up, we have

$$\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}} \leq C(p,\delta_1,\delta_2,\varphi_1,\lambda)\|f\|_{\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}}, \tag{69}$$

where

$$C(p,\delta_1,\delta_2,\varphi_1,\lambda) = Cs^{\delta_1/(p-s)} \inf_{\theta\in(0,\min\{s,\epsilon\})} C(\theta)\theta^{-\delta_1/(p-\theta)}. \tag{70}$$

□

Remark 12. Recently, new results concerning the grand variable exponent Lebesgue spaces for martingales have emerged (see [33]).

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Qualitative Analysis of a Hyperchaotic Lorenz-Stenflo Mathematical Model via the Caputo Fractional Operator

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The Lorenz-Stenflo mathematical model describes a complex dynamical behavior related to atmospheric acoustic-gravity waves. In this study, qualitative analysis of the four-dimensional hyperchaotic Lorenz-Stenflo system via the Caputo fractional derivative is implemented. By using the Matignon stability criterion, the local stability analysis of the system showed that all the equilibrium points of the system are locally unstable. Calculation of the Lyapunov exponents along with the relevant bifurcation diagrams with respect to different fractional orders exposed the hyperchaotic dynamical behavior for the system. Bifurcation diagrams for all the four parameters in the system also showed the hyperchaotic nature of the Lorenz-Stenflo system. Different phase attractors of the system corresponding to different fractional derivatives and parameters are presented to specify the dynamical nature of the system. The Lorenz-Stenflo system showed sensitivity to initial conditions. The master and slave systems showed a strong correlation among themselves, as verified by graphs of time series solutions of the two systems.

1. Introduction

In recent years, fractional operators have been applied to develop mathematical models for which we can investigate different dynamical systems in some areas such as mathematical biology, epidemiology, and engineering [1].

Different researchers introduced several concepts of fractional derivatives. Some of them are the Caputo fractional-order derivative [2–4], the Riemann Liouville fractional-order derivative [4], the Caputo-Fabrizio fractional-order derivative [5], and the Atangana-Baleanu fractional derivatives [6].

In the mathematical models with integer derivatives, the derivative orders give the instantaneous rate of change of the function. On the other hand, in the case of mathematical models with fractional derivatives, the parameters of dynamical systems represent the memory index of variation of the function [7].

Thus, the advantage of using the memory index of a dynamical system with fractional derivative made fractional derivatives advantageously applied in the fields of chaotic dynamics, epidemiological modeling, and several other fields. In particular, one can see these applications in the modeling of COVID-19 based on real information from Pakistan [8], designing the SEIR model of COVID-19 [9], the modeling of the memristor-based hyperchaotic circuit via nonsingular operator [10, 11], the thermostat model via Bernstein polynomials [12], the modeling of coronavirus by the Caputo operator [13], the optimal control of nonsingular tumor-immune surveillance [14], the SEIRA model [15], designing a bank data with fractal-fractional operators [16], the nonlinear modeling

Table 1: LEs for different fractional orders η.

η	0.856	0.857	0.866	0.868	0.888	0.9	0.956	0.98	0.988	0.999	1
LE1	14.1773	14.1214	13.6282	13.5211	12.4969	11.9217	9.5811	8.7302	8.4644	8.1127	8.0815
LE2	1.0824	1.0788	1.0461	1.039	0.9704	0.9313	0.7682	0.707	0.6877	0.662	0.6597
LE3	−5.5673	−5.553	−5.4255	−5.3973	−5.1187	−4.9549	−4.2295	−3.9409	−3.8478	−3.7225	−3.7113
LE4	−9.6713	−9.6712	−9.6592	−9.654	−9.5574	−9.4648	−8.7868	−8.4139	−8.2829	−8.0987	−8.0818
Sum	**0.0211**	**−0.024**	**−0.4104**	**−0.4912**	**−1.2088**	**−1.5667**	**−2.667**	**−2.9176**	**−2.9786**	**−3.0465**	**−3.0519**

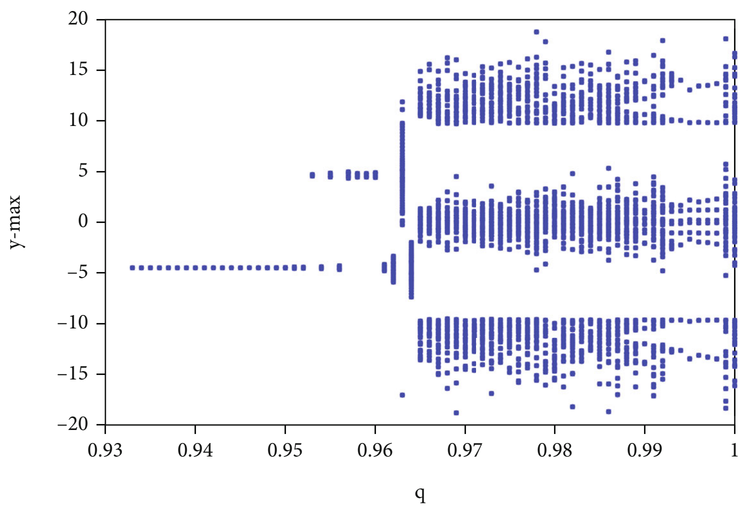


Figure 1: Bifurcation diagram of (5) due to orders from 0.93 to 1.

of the Navier system [17], analyzing an Atangana–Baleanu SEAIR model [18], compartmental disease modeling [19], and the references therein.

Lorenz developed the famous three-dimensional mathematical model to study the dynamics and behaviors of the atmosphere in 1963 [20–22]. Later on, in 1996, Lennart Stenflo modified the Lorenz system, adding a fourth parameter to consider the evolution of finite-amplitude acoustic-gravity waves in a rotating atmosphere, and developed a new four-dimensional Lorenz-Stenflo mathematical model. Since then, many studies have been conducted on the complex dynamical behaviors of the system by applying both integer and fractional derivatives. For further information, see [21] and the references therein.

Some of the studies conducted on the complex dynamics of Lorenz-Stenflo systems are reviewed as follows: Zhang et al. [22] investigated the qualitative properties of the higher integer-order Lorenz-Stenflo Chaotic system which appeared in mathematical physics. The authors proved that the higher-order Lorenz-Stenflo system is globally stable. A globally attractive set of Lorenz-Stenflo systems indicating the evolution of a finite amplitude of acoustic gravity waves contained in the rotating atmosphere is investigated by Zhang et al. [20]. Adaptive control synchronization and circuit implementation of the ordinary Lorenz-Stenflo system with an integer order was established by Yang and Wu [23]. An analysis on the dynamics of the Lorenz-Stenflo system via fractional operators along with sets of parameters is done through the spectra of the Lyapunov exponent type, bifurcation graphs, and 0-1 test.

The complex dynamics of Lorenz-Stenflo dynamical systems are analyzed with Lyapunov exponents, bifurcation diagrams, and the 0-1 test by using the Adomian decomposition numerical scheme and fractional-order representation of the model in the sense of the Riemann-Liouville integral [24]. A robust chaos suppression control is designed and applied via sliding mode control to the integer-order Lorenz-Stenflo system constrained to uncertainties and nonlinearities [25].

There is only limited literature devoted to studying the complex dynamics of the Lorenz-Stenflo systems by using integer-order derivatives and fractional-order derivatives. Moreover, to the best of the authors' knowledge, no study is conducted on the qualitative analysis of the Lorenz-Stenflo mathematical model in the sense of the Caputo fractional derivative and by using a numerical scheme developed by Garrappa [26], which is the main focus of this study.

Among the several notions of fractional derivatives presented above, the Caputo fractional derivative gives the opportunity of including the classical initial conditions in a mathematical model and the Caputo fractional derivative of a constant is zero, which is not the case for instance in the Riemann-Liouville fractional derivative. Moreover, it has a MATLAB code that can obtain phase portraits and time series

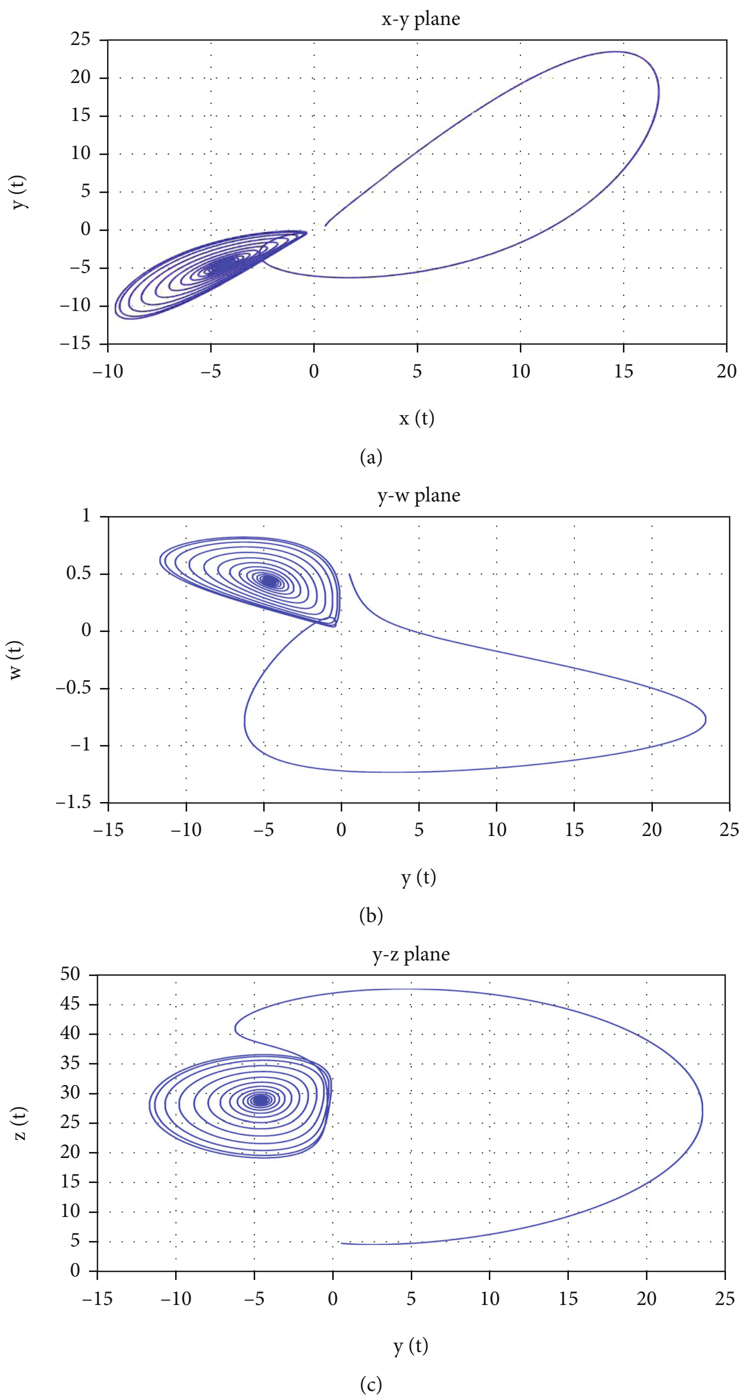


FIGURE 2: Phase portraits of (5) for $\eta = 0.950$ projected on three planes: (a) x-y, (b) y-w, and (c) y-z.

solutions of chaotic or hyperchaotic systems using a numerical scheme named the predictor-corrector method reported by Garrappa and applied in several studies on chaotic systems [27–29].

The manuscript is organized as follows: Section 2 involves the statement of the problem. In this section, the fractional representation of the Lorenz-Stenflo system (LS) is designed via the Caputo derivatives after recalling some critical preliminary definitions of fractional operators. Section 3 is devoted to analyzing the local stability of the mentioned LS system. In Section 4, the numerical scheme for the fractional-order LS system is developed based on a research reported by Garrappa [26]. The broader part of the manuscript is devoted to Lyapunov exponents, bifurcation, and chaos of the LS system in Section 5, followed by sensitivity analysis to initial conditions in Section 6. In Section 7, the development of the master and slave system to create a strong relationship between the systems using a coupling function is considered. Finally, a conclusion and a reference list are provided.

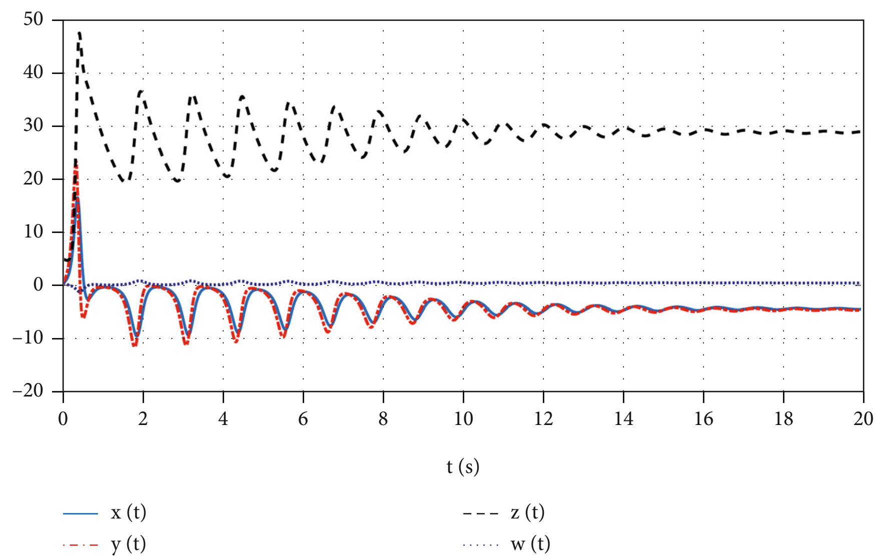


Figure 3: Time series trajectories of the LS system (5) for the fractional order $\eta = 0.950$.

2. Statement of the Problem

The Lorenz-Stenflo (LS) system of differential equations is given by [30, 31].

$$\begin{aligned} \dot{x} &= a(y-x)+cw, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -kz + xy, \\ \dot{w} &= -x - aw. \end{aligned} \tag{1}$$

The variable x is the intensity of motion of the fluid; y and z denote the horizontal and vertical direction temperature variations of the atmosphere. Parameters a, r, and k are all positive real numbers and are the Prandtl number, c is the rotation number, r is the Rayleigh number, and k is a geometric parameter. Parameters a and k depend on the material and geometrical properties of the layer of the fluid. Parameter r is proportional to the difference in temperature.

We used fractional operators to find the hidden properties of the nature of solution of the LS system that are not observable via integer-order derivatives.

2.1. The Formulation of the LS System via the Caputo Fractional Derivative. Here, we shortly give several definitions of fractional operators pertinent to our study.

Definition 1 (see [2, 3]). The Riemann-Liouville fractional integral for a continuous function $f:[0,+\infty) \longrightarrow \mathbb{R}$ is defined by

$${}_0^{RL}I_t^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} f(\tau) d\tau, \quad \eta \in (0,1), t > 0. \tag{2}$$

Definition 2 (see [2, 3]). The Riemann-Liouville fractional derivative for a continuous function $f:[0,+\infty) \longrightarrow \mathbb{R}$ is defined by

$${}_0^{RL}D_t^{\eta} f(t) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\eta} f(\tau) d\tau, \quad \eta \in (0,1), t > 0. \tag{3}$$

Definition 3 (see [2, 3]). The Caputo fractional derivative for a continuous function $f:[0,+\infty) \longrightarrow \mathbb{R}$ is defined by

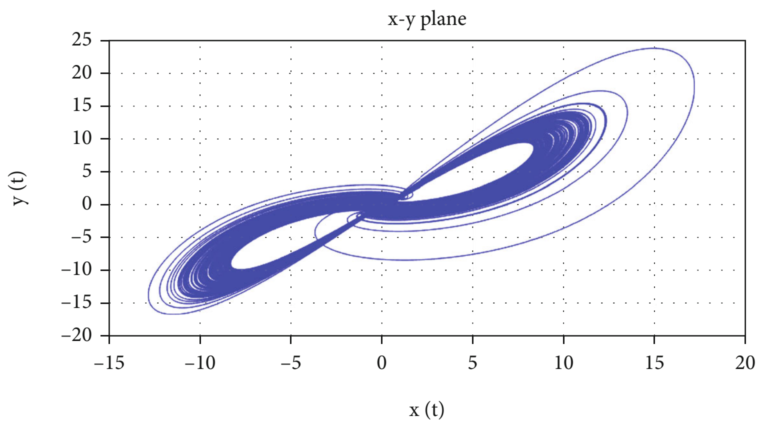
$${}_0^{C}D_t^{\eta} f = \frac{1}{\Gamma(1-\eta)} \int_0^t (t-\tau)^{-\eta} \frac{d}{d\tau} f(\tau) d\tau, \quad \eta \in (0,1), t > 0. \tag{4}$$

This section develops the fractional order representation of the LS system (1). The Caputo fractional derivative is used because it can incorporate customary initial conditions in the model, unlike the Riemann-Liouville fractional derivative. Furthermore, the Caputo fractional derivative of a constant is zero, which is not the case in the Riemann-Liouville fractional derivative. Accordingly, the Caputo fractional representation of the LS system (1) is given by

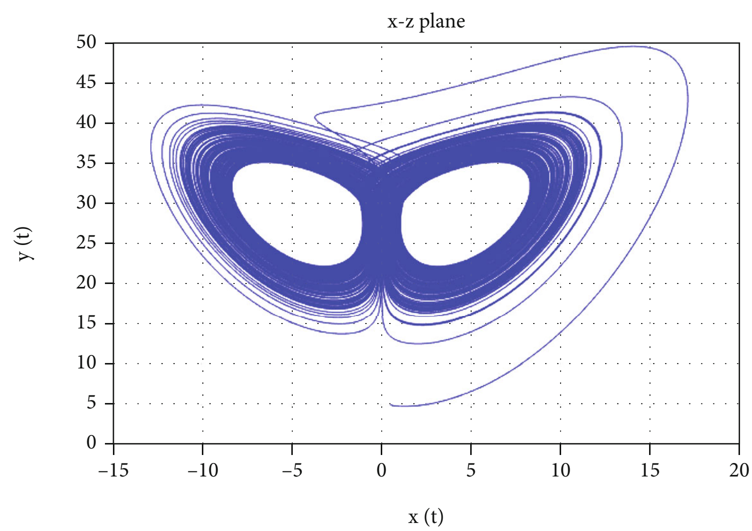
$$\begin{aligned} {}_0^{C}D_t^{\eta} x &= L_1(x,y,z,w), \\ {}_0^{C}D_t^{\eta} y &= L_2(x,y,z,w), \\ {}_0^{C}D_t^{\eta} z &= L_3(x,y,z,w), \\ {}_0^{C}D_t^{\eta} w &= L_4(x,y,z,w), \end{aligned} \tag{5}$$

where

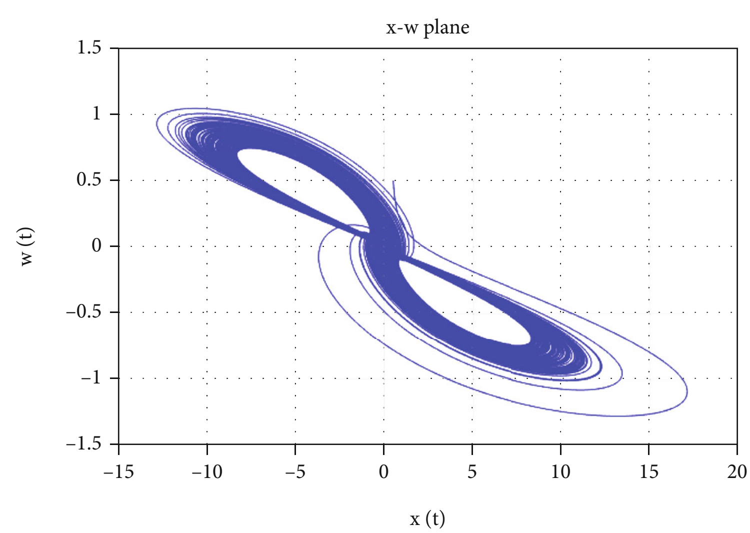
$$\begin{aligned} L_1(t,x,y,z,w) &= a(y-x)+cw, \\ L_2(t,x,y,z,w) &= rx - y - xz, \\ L_3(t,x,y,z,w) &= -kz + xy, \\ L_4(t,x,y,z,w) &= -x - aw, \end{aligned} \tag{6}$$



(a)



(b)



(c)

Figure 4: Continued.

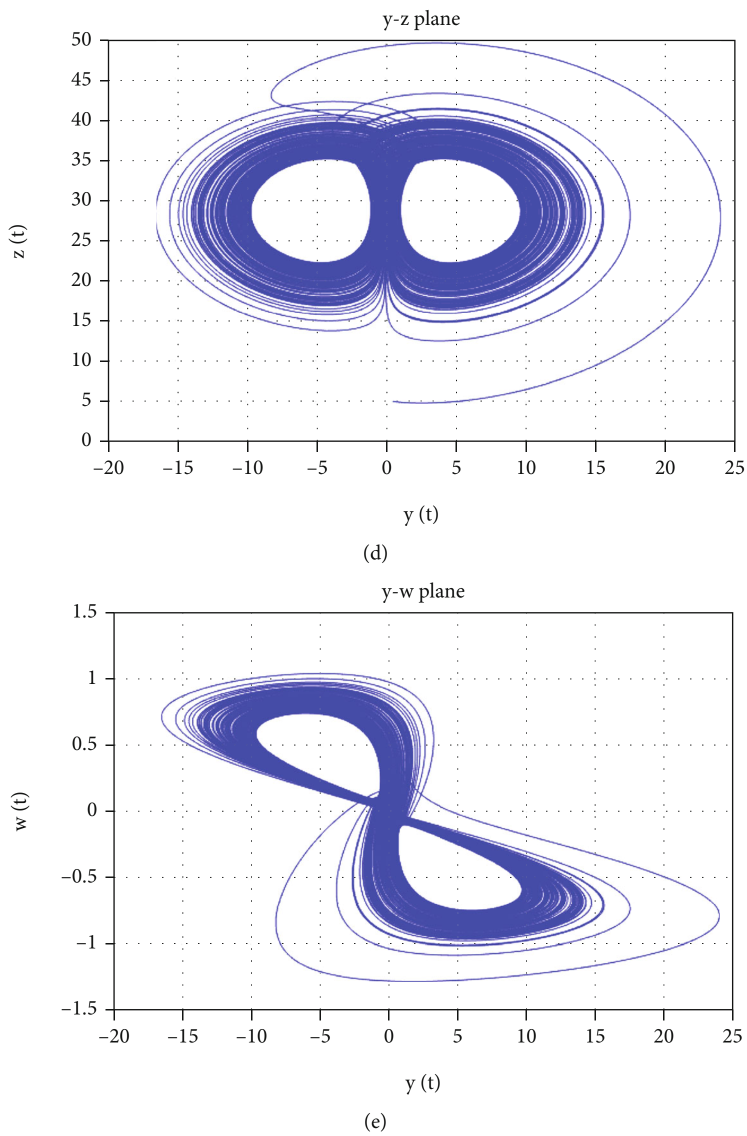


Figure 4: Attractors of the hyperchaotic LS system (5) projected on the planes: (a) x-y, (b) x-z, (c) x-w, (d) y-z, and (e) y-w.

with the initial value conditions $x(0) = 0.5, y(0) = 0.5, z(0) = 5$, and $w(0) = 0.5$.

The fractional-order derivative $\eta \in (0, 1]$ and the parameter values are $a = 10$, $r = 30$, $k = 0.7$, and $c = 4$.

The specific objectives of this study are to analyze the local stability of system (5), to investigate the chaotic behavior of system (5) via Lyapunov exponents, and to plot the bifurcation diagrams, attractors, and time series trajectories of the system by variation of fractional orders and four parameters of system (5). In addition, sensitivity to initial conditions and synchronization of the LS fractional system are considered.

3. Stability Analysis of the LS System (5)

In this section of the manuscript, analysis of the local stability of the mentioned system (5) is conducted. There are several methods for performing local stability analysis of systems. Some of these methods are the Laplace transform techniques and the Matignon criterion. For the local stability analysis of the LS system (5), we used the Matignon method, because it is most commonly used in literature [30, 31]. The Matignon condition for fractional stability is given by

$$|\arg \lambda(J)| > \frac{\eta\pi}{2}, \tag{7}$$

where J is the Jacobian matrix of the system, $\lambda(J)$ denotes the class of all eigenvalues of J, and $\eta \in (0, 1]$ is the order of the LS system (5). The LS system (5) is said to be locally asymptotically stable whenever all the eigenvalues of J satisfy the Matignon criterion. Firstly, we considered the equilibrium points of (5), given by $E_0 = (0, 0, 0, 0)$, and constant values of the parameters given by $k = 0.7$, $r = 30$, $c = 4$, and $a = 10$.

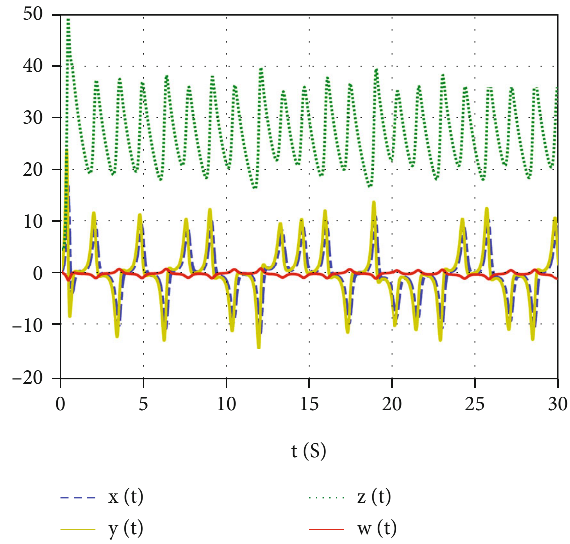


Figure 5: Time series trajectories of system (5) for the fractional order $\eta = 0.974$.

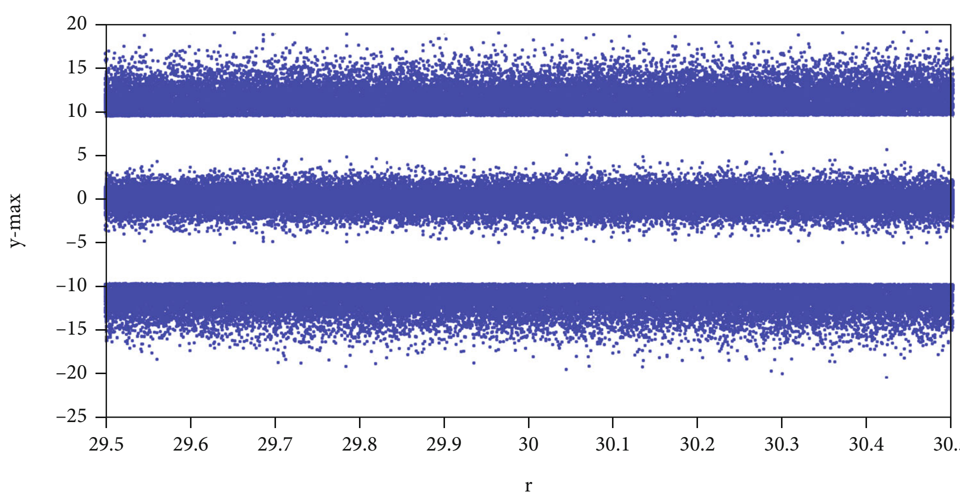


Figure 6: Bifurcation diagram for $r \in [29.5,30.5]$ and $\eta = 0.974$.

The matrix J at $E_0 = (0, 0, 0, 0)$ is given by

$$J_0 = \begin{bmatrix} -a & a & 0 & c \\ r & -1 & 0 & 0 \\ 0 & 0 & -k & 0 \\ -1 & 0 & 0 & -a \end{bmatrix}. \tag{8}$$

The eigenvalues of the matrix J_0 are $\lambda_1 = 12.3287$, $\lambda_2 = -23.2066$, $\lambda_3 = -10.1221$, and $\lambda_4 = -0.7000$, and the corresponding arguments of the eigenvalues are 0 for λ_1 and π for the remaining eigenvalues. It is easy to conclude that E_0 is locally unstable, because the absolute value of the arguments does not satisfy the Matignon criteria (7). The presence of an eigenvalue with a positive real part, $\lambda_1 = 12.3287$, is necessary for the SL system to exhibit a double-scroll attractor [31].

The general equilibrium points of the system in terms of the parameters are given by

$$\left(\mp a\theta, \mp\left(\frac{a^2+c}{a}\right)\theta, \frac{a^2 r - (a^2+c)}{a^2}, \pm\theta\right), \quad \theta = \sqrt{\frac{k(a^2 r - (a^2+c))}{a^2(a^2+c)}}. \tag{9}$$

Substituting the parameter values $k = 0.7$, $r = 30$, $c = 4$, and $a = 10$, the equilibrium points are given by $E_1 = (-4.4150, -4.5916, 28.9600, +0.4415)$ and $E_2 = (+4.4150, +4.5916, 28.9600, -0.4415)$.

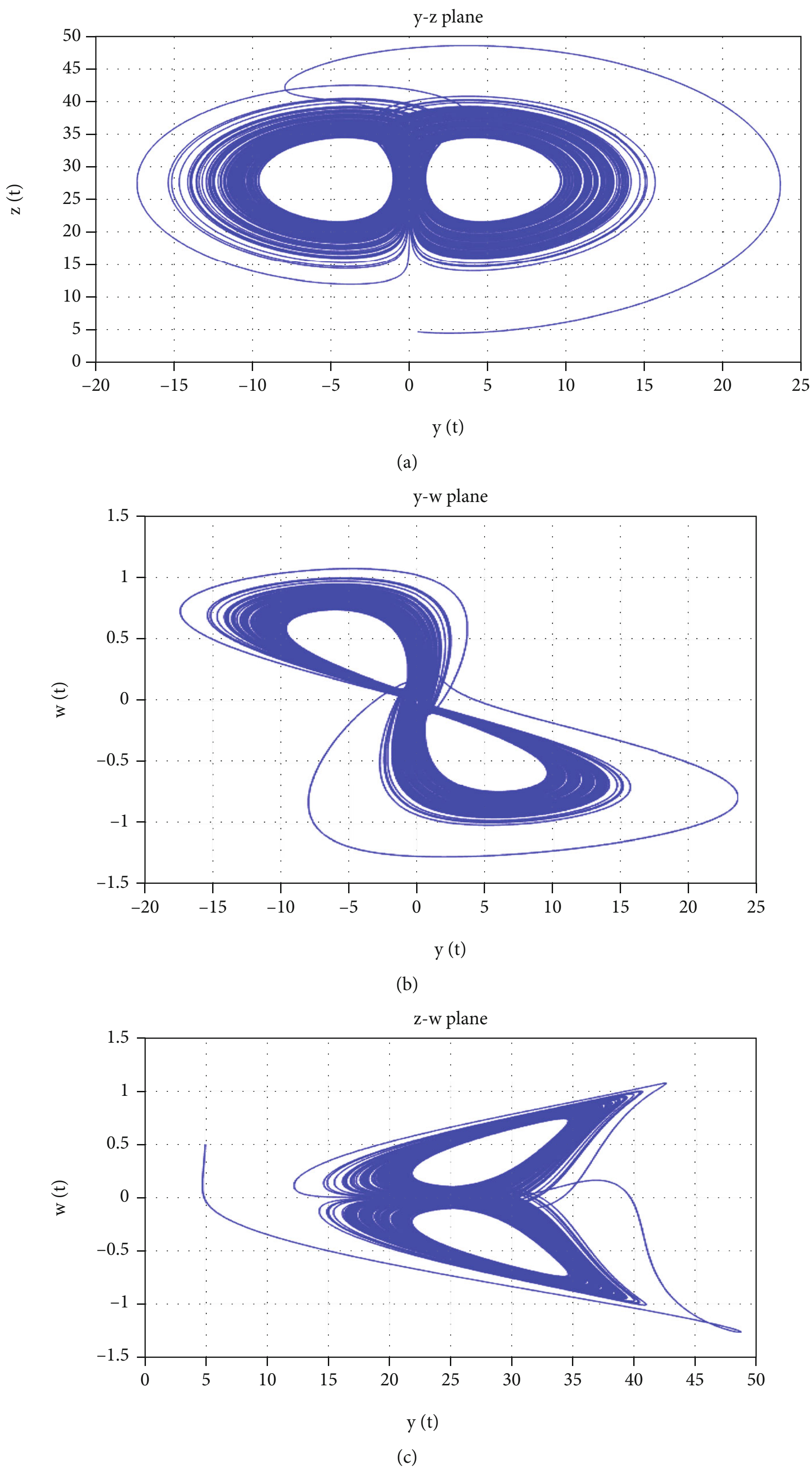


FIGURE 7: Attractors of the LS system (5) for $r = 29.6$ projected on the planes: (a) z-y, (b) y-w, and (c) z-w.

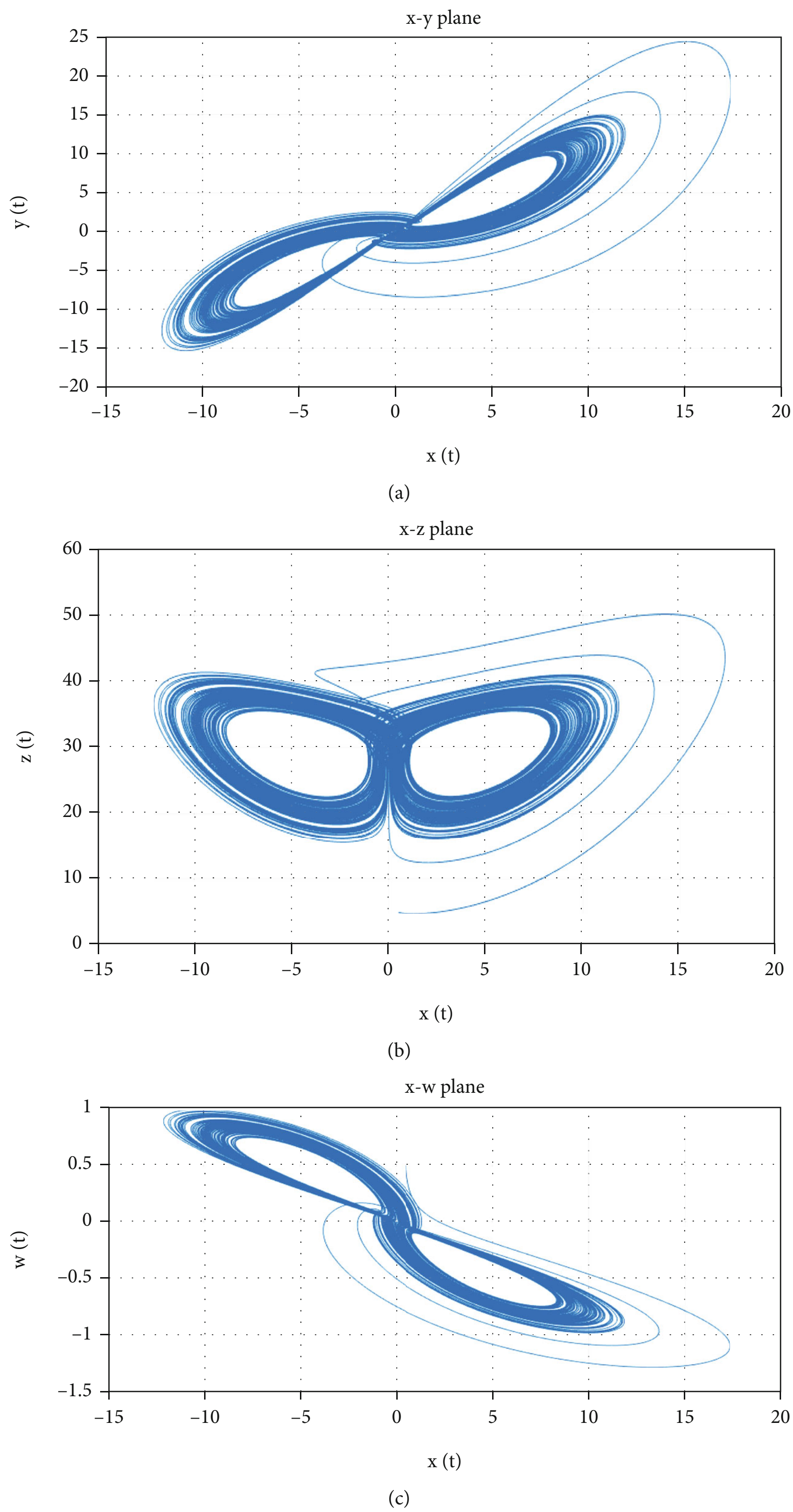


Figure 8: Attractors of the LS system (5) for $r = 30.4$ projected on the planes: (a) x-y, (b) x-z, and (c) x-w.

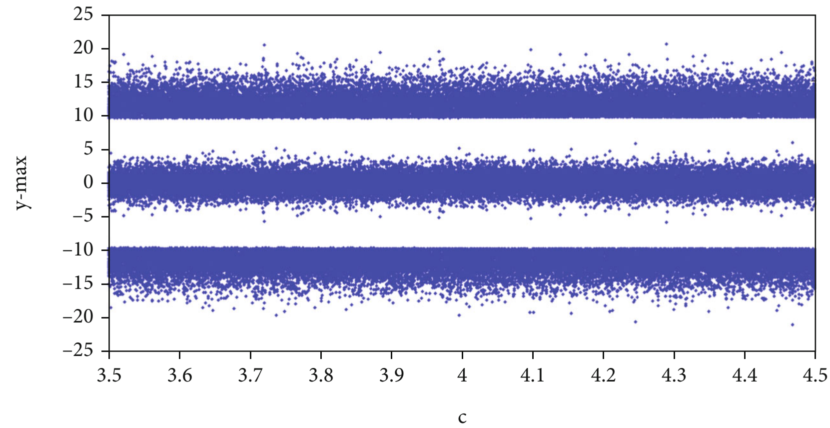


FIGURE 9: Bifurcation diagram for $c \in [3.5, 4.5]$ and $\eta = 0.974$.

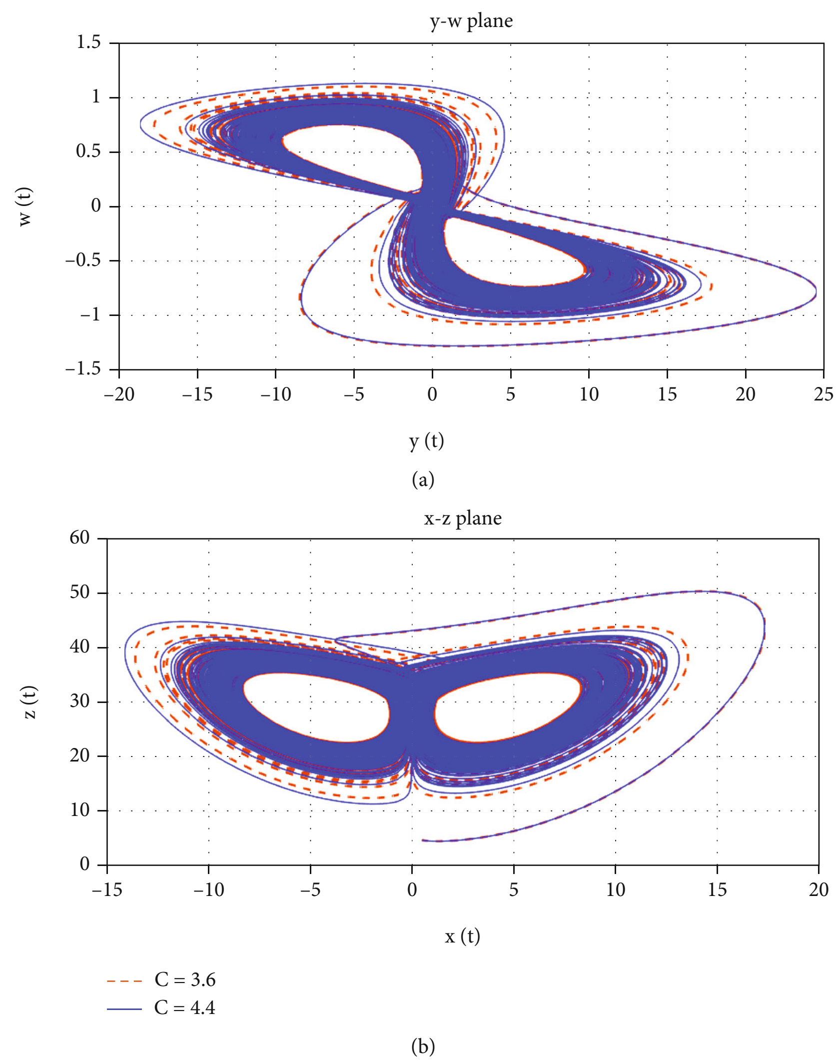


FIGURE 10: Attractors of the LS system (5) for $c = 3.6$ and 4.4 projected on the planes: (a) y-w, (b) x-z, and (c) x-w.

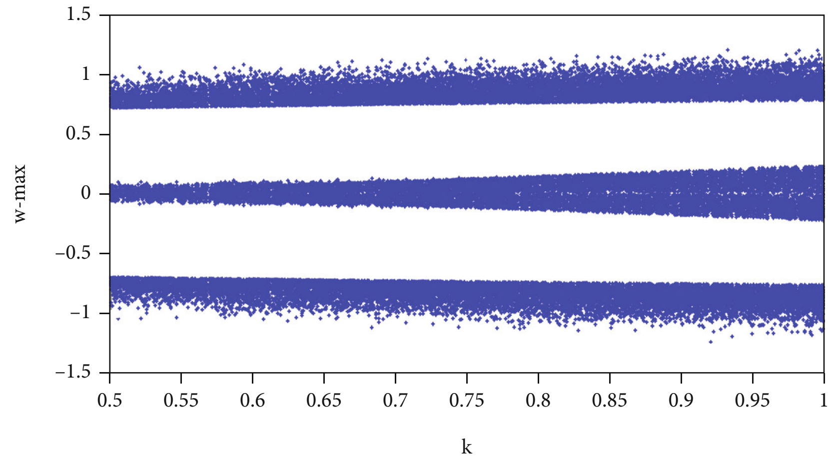


FIGURE 11: Bifurcation diagram of the LS system (5) for parameter k in the interval [0.5, 1].

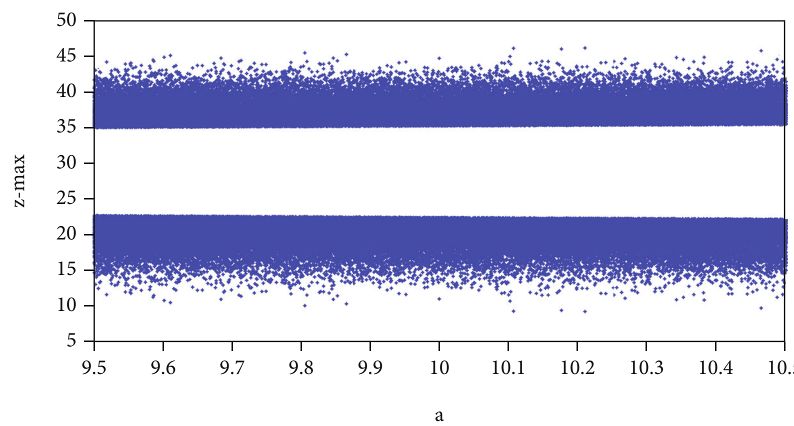


FIGURE 12: Bifurcation diagram of the LS system (5), for the values of a in [0.5,1].

The Jacobian matrix corresponding to the equilibrium point $E_1 = (-4.4150, -4.5916, 28.9600, +0.4415)$ is given by

$$J_1 = \begin{bmatrix} -10 & 10 & 0 & 4 \\ 1.04 & -1 & 4.415 & 0 \\ -4.5916 & -4.415 & -0.7 & 0 \\ -1 & 0 & 0 & -10 \end{bmatrix}. \quad (10)$$

The eigenvalues of J_1 are $\lambda_{1,2} = 0.2026 \pm 5.7129i$ and $\lambda_{3,4} = -11.0526 \pm 1.3824i$. The arguments of the eigenvalues are ± 1.5353 for $\lambda_{1,2}$ and ± 3.0172 for $\lambda_{3,4}$. It can then be inferred that the Matignon criterion of local stability is satisfied for $\lambda_{3,4}$, since $|\pm 3.0172| > \eta\pi/2, \forall \eta \in (0, 1]$. On the other hand, the Matignon criterion is not satisfied for $\lambda_{1,2}$ since $|\pm 1.5353| > \eta\pi/2$ only for $\eta < 0.9774$. That is, the equilibrium point E_1 is unstable for $\eta \geq 0.9774$.

The Jacobian matrix corresponding to the equilibrium point $E_2 = (+4.4150, +4.5916, 28.9600, -0.4415)$ is given by

$$J_2 = \begin{bmatrix} -10 & 10 & 0 & 4 \\ 1.04 & -1 & -4.415 & 0 \\ 4.5916 & 4.415 & -0.7 & 0 \\ -1 & 0 & 0 & -10 \end{bmatrix}. \quad (11)$$

The eigenvalues and their corresponding arguments of Jacobian matrix J_2 are identical to those of Jacobian matrix J_1. Hence, the conclusion in relation to the local stability of equilibrium point E_2 is the same as that in equilibrium point E_1.

In summary, the equilibria of the fractional-order system (5) are unstable for the fractional order of $\eta \in [0.9774, 1]$, $k = 0.7$, $r = 30$, $c = 4$, and $a = 10$.

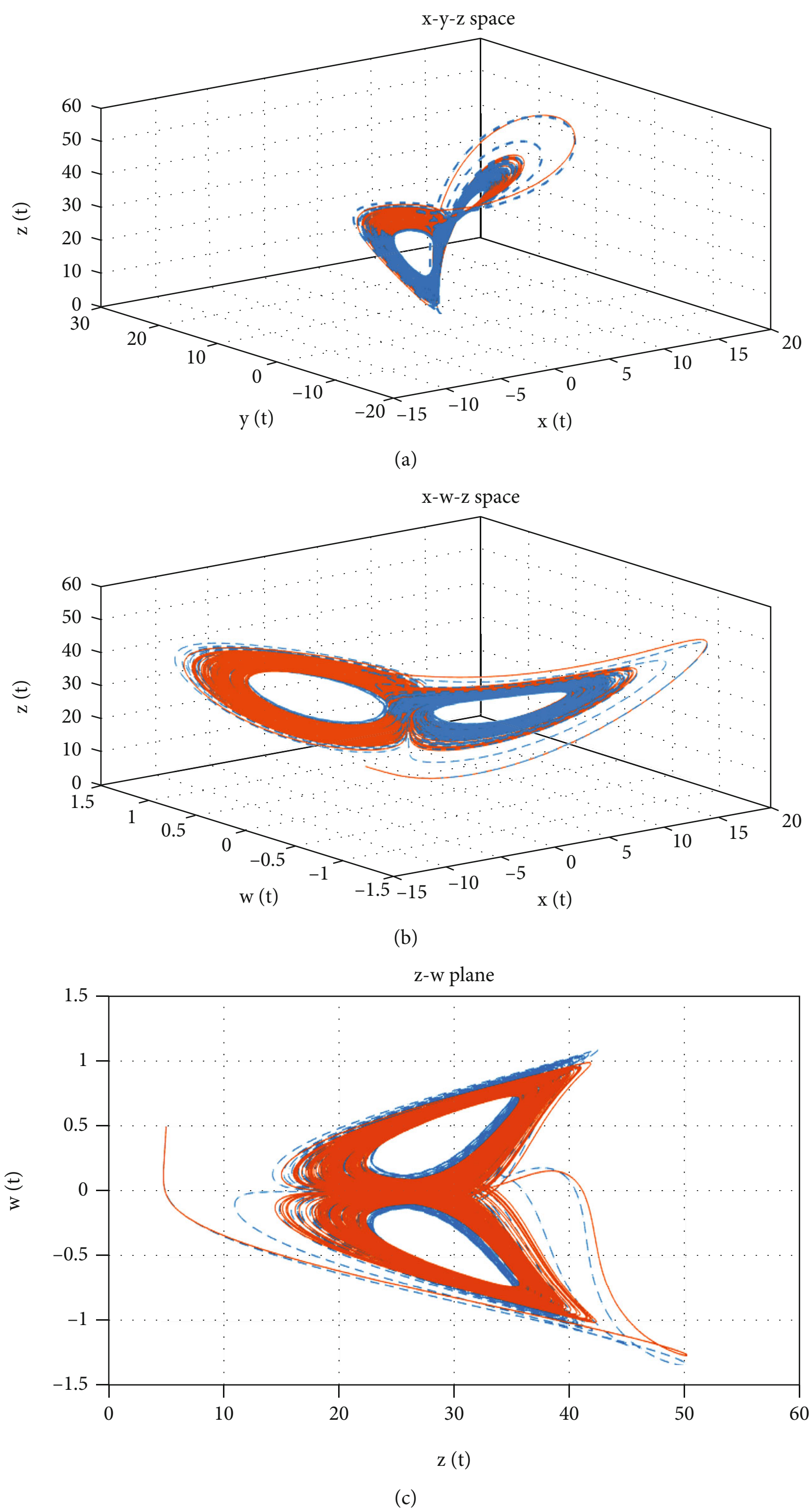
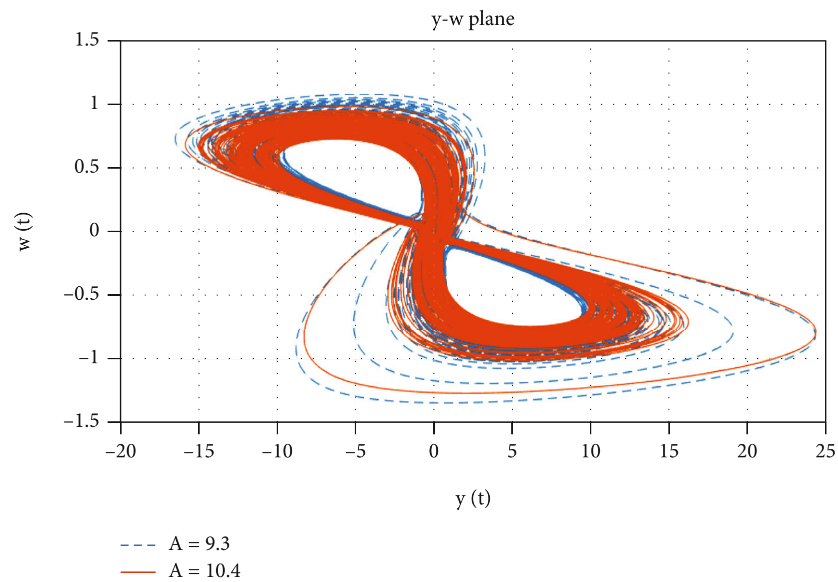


Figure 13: Continued.



(d)

Figure 13: Attractors of the LS system (5) for different values of a in [0.5, 1] projected on: (a) x-y-z space, (b) x-w-z space, (c) z-w plane, and (d) y-w plane.

4. Numerical Solution

This section presents the numerical method used to obtain the phase portraits and time series solutions of the fractional-order LS system (5). To solve fractional-order systems, there are several numerical and analytical methods such as the predictor-corrector method, Homotopy method, and Adomian decomposition method. In this study, we use the predicator-corrector method, because it has a MATLAB code that can be used to obtain phase portraits and time series solutions of chaotic or hyperchaotic systems.

The numerical method in [26] is applied to our system as follows: by starting from the first equation of (5) and applying the Reimann-Liouville fractional integral given by (2), we obtain the following system

$$
\begin{aligned}
x(t) &= x(0) + {}_{0}^{RL}I_t^{\eta} L_1(t, x, y, z, w),\\
y(t) &= y(0) + {}_{0}^{RL}I_t^{\eta} L_2(t, x, y, z, w),\\
z(t) &= z(0) + {}_{0}^{RL}I_t^{\eta} L_3(t, x, y, z, w),\\
w(t) &= w(0) + {}_{0}^{RL}I_t^{\eta} L_4(t, x, y, z, w).
\end{aligned}
\tag{12}
$$

Applying the predictor-corrector method to (12), the integral terms are approximated and it becomes

$$
\begin{aligned}
x(t) &= x(0) + h^{\eta}\left[\kappa_m^{\eta} L_1(0) + \sum_{j=1}^{m-1} \psi_{m-j}^{\eta} L_1\left(t_j, x_j, y_j, z_j, w_j\right) + \psi_0^{\eta} L_1\left(t_j, x_m^p, y_m^p, z_m^p, w_m^p\right)\right],\\
y(t) &= y(0) + h^{\eta}\left[\kappa_m^{\eta} L_2(0) + \sum_{j=1}^{m-1} \psi_{m-j}^{\eta} L_2\left(t_j, x_j, y_j, z_j, w_j\right) + \psi_0^{\eta} L_2\left(t_j, x_m^p, y_m^p, z_m^p, w_m^p\right)\right],\\
z(t) &= z(0) + h^{\eta}\left[\kappa_m^{\eta} L_3(0) + \sum_{j=1}^{m-1} \psi_{n-j}^{\eta} L_3\left(t_j, x_j, y_j, z_j, w_j\right) + \psi_0^{\eta} L_3\left(t_j, x_m^p, y_m^p, z_m^p, w_m^p\right)\right],\\
w(t) &= w(0) + h^{\eta}\left[\kappa_m^{\eta} L_4(0) + \sum_{j=1}^{m-1} \psi_{m-j}^{\eta} L_4\left(t_j, x_j, y_j, z_j, w_j\right) + \psi_0^{\eta} L_4\left(t_j, x_m^p, y_m^p, z_m^p, w_m^p\right)\right],
\end{aligned}
\tag{13}
$$

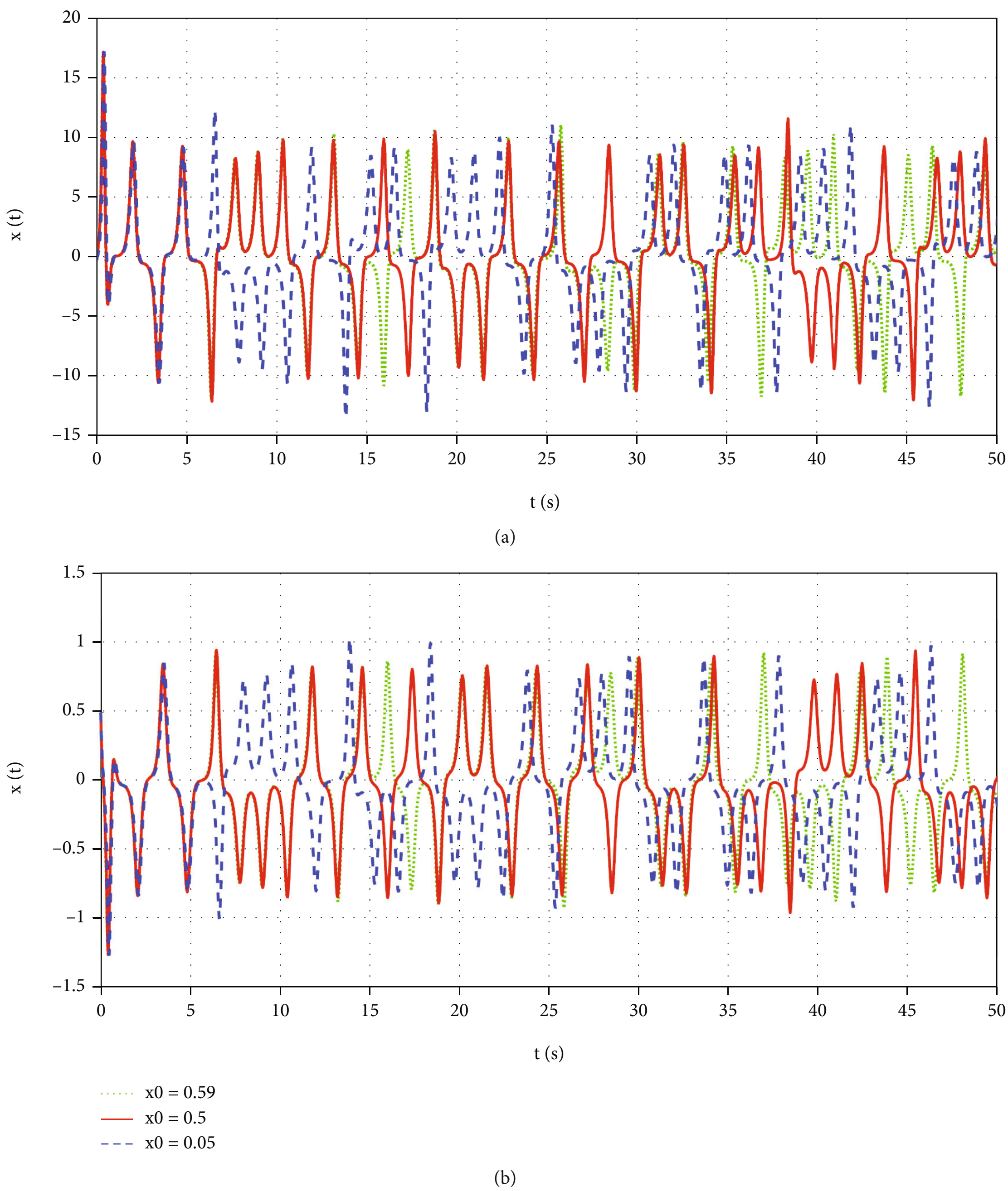


Figure 14: The time series solution of LS (5) corresponding to different initial conditions due to different values of x_0.

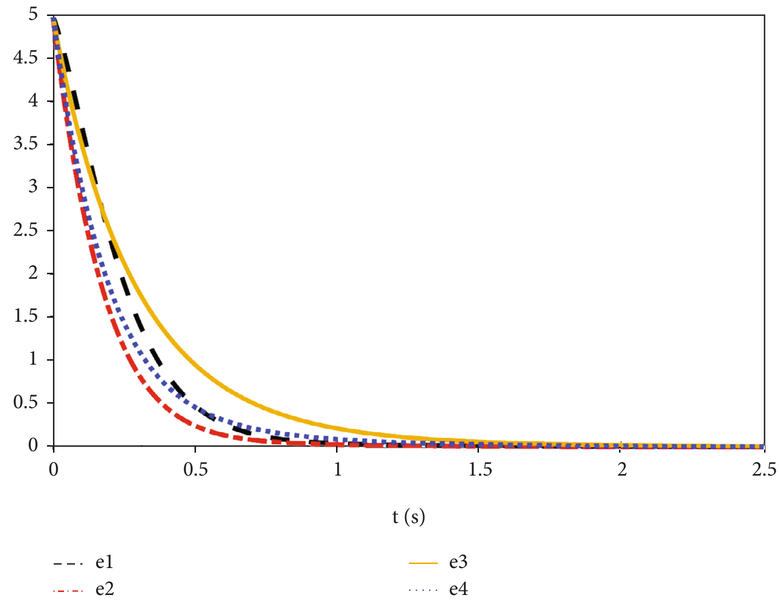


FIGURE 15: Graphs of the error dynamics (27).

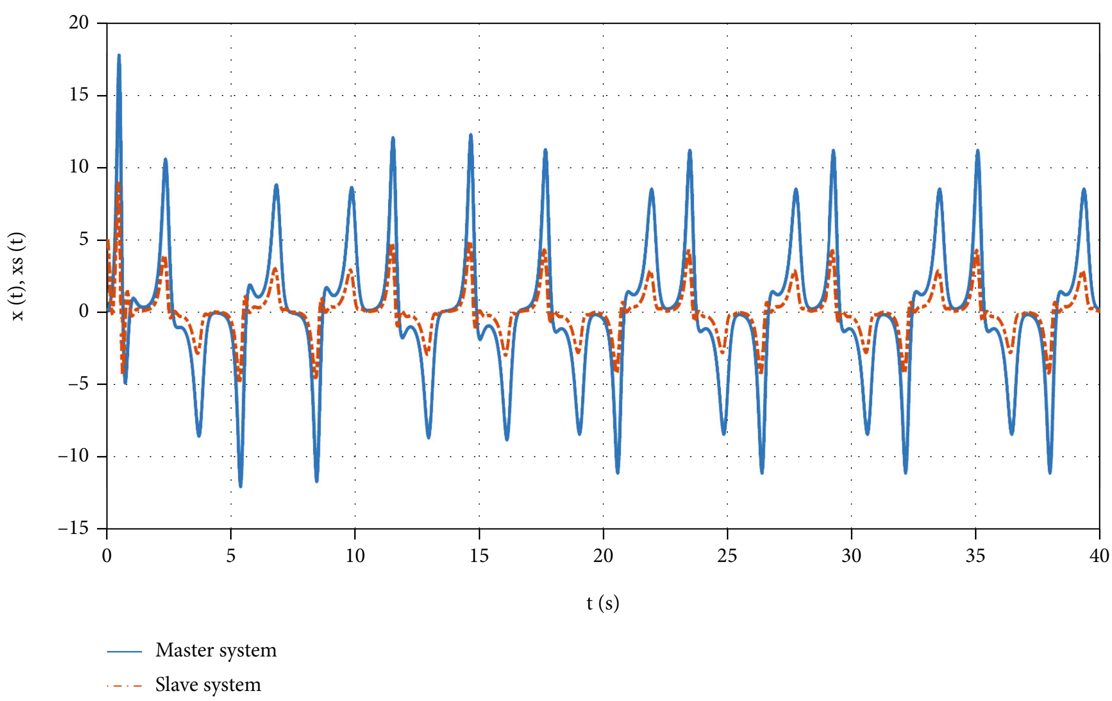


FIGURE 16: Time series trajectories of $x(t)$ and $x_s(t)$.

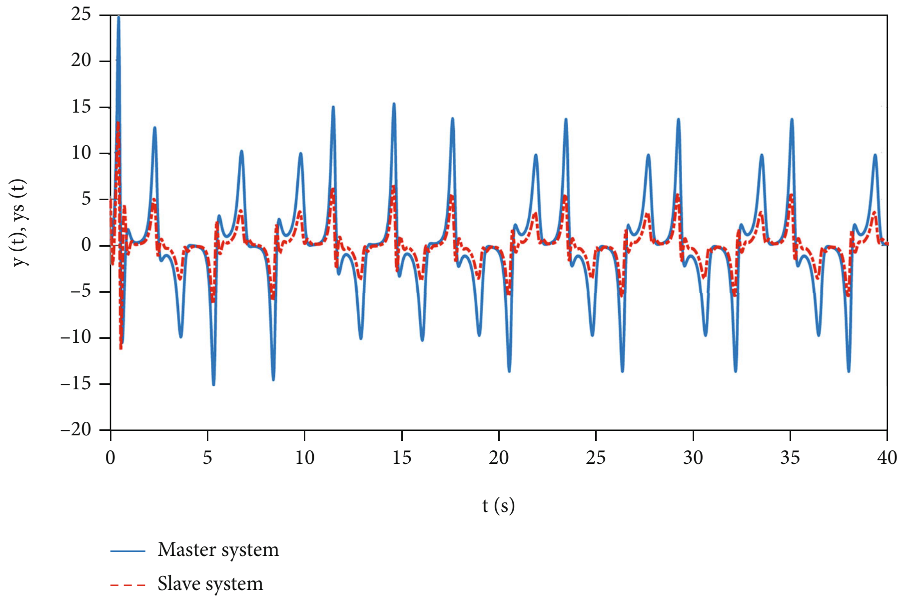


FIGURE 17: Time series orbits of $y(t)$ and $y_s(t)$.

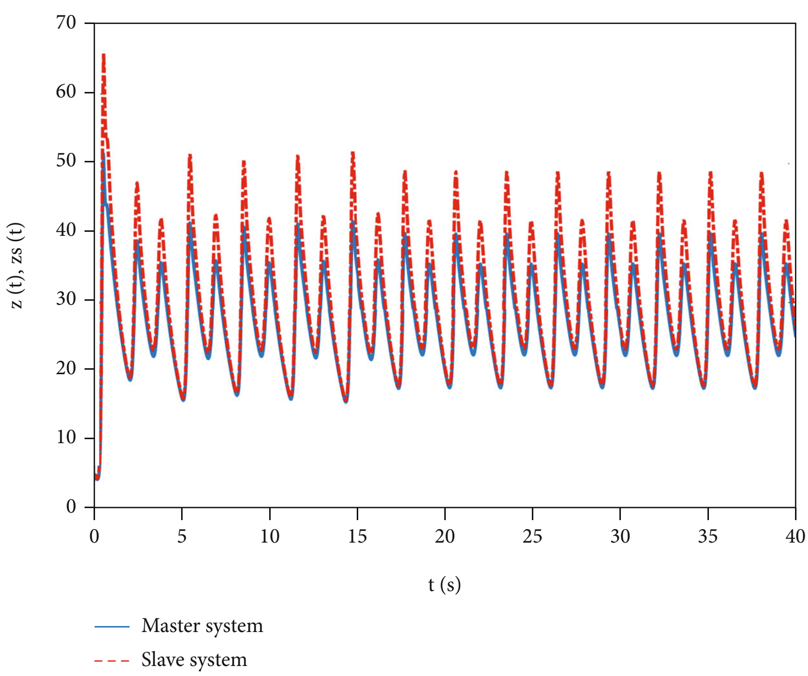


FIGURE 18: Time series orbits of $z(t)$ and $z_s(t)$.

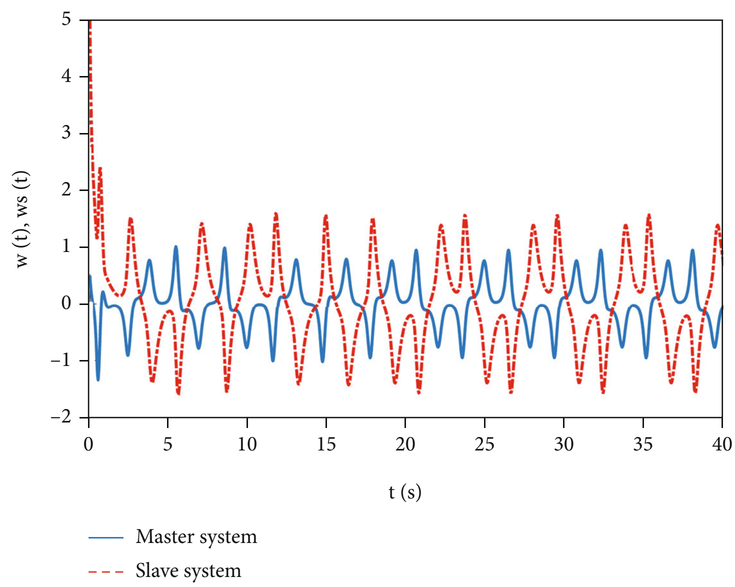


Figure 19: Time series orbits of $w(t)$ and $w_s(t)$.

where L_1, L_2, L_3, and L_4 are defined by (5)

$$L_i(0) = L_i(x(0), y(0), z(0), w(0)), \quad i = 1, 2, 3, \tag{14}$$

and

$$\begin{aligned} \kappa_m^\eta &= \frac{(m-1)^\eta - m^\eta(m-\eta-1)}{\Gamma(2+\eta)}, \\ \psi_m^\eta &= \frac{(m-1)^{\eta+1} - 2m^{\eta+1} + m^\eta(m+1)}{\Gamma(2+\eta)}, \quad m = 1, 2, 3, \cdots, \\ \psi_0^\eta &= \frac{1}{\Gamma(2+\eta)}. \end{aligned} \tag{15}$$

Moreover, the predictors are given as follows:

$$\begin{aligned} x^p(t_m) &= x(0) + h^\eta \sum_{j=1}^{m-1} \psi_{m-j-1}^\eta L_1\left(t_j, x_j, y_j, z_j, w_j\right), \\ y^p(t_m) &= y(0) + h^\eta \sum_{j=1}^{m-1} \psi_{m-j-1}^\eta L_2\left(t_j, x_j, y_j, z_j, w_j\right), \\ z^p(t_m) &= z(0) + h^\eta \sum_{j=1}^{m-1} \psi_{m-j-1}^\eta L_3\left(t_j, x_j, y_j, z_j, w_j\right), \\ w^p(t_m) &= w(0) + h^\eta \sum_{j=1}^{m-1} \psi_{m-j-1}^\eta L_4\left(t_j, x_j, y_j, z_j, w_j\right). \end{aligned} \tag{16}$$

By Garrappa [26], the discretization method applied in this study is stable and convergent. Thus, the advantages of this discretization method are that it is both stable and convergent, and of course, it has a MATLAB code that can be used for plotting the attractors of the LS system (5). Several other advantages of the method are described in [27].

As mentioned above, in this research, the predictor-corrector technique is utilized to obtain the attractors and time series trajectories of the LS system (5). Moreover, the Lyapunov exponents are obtained using an algorithm by Danca et al. [32]. The code is developed to determine all Lyapunov exponents of a class fractional-order system modeled by the Caputo derivative. The predictor-corrector Adams-Bashforth-Moulton numerical method is the underlying numerical method used in this code.

5. Lyapunov Exponents, Bifurcation Diagrams, and Hyperchaotic Behavior of the LS System

This section is devoted to investigating the chaotic or hyperchaotic nature of the LS system (5). The magnitude of the chaos is quantified via the Lyapunov exponents (LE). Furthermore, bifurcation diagrams caused by the variation of the parameters of (5) are portrayed using the values of the parameters and the initial value condition presented in (5).

5.1. Lyapunov Exponents for the Fractional Order η. Based on the Danca algorithm mentioned above, some of the Lyapunov exponents corresponding to different fractional orders of the LS system (5) are shown in Table 1. The simulation is made to run for 300 s.

Based on the LE values in Table 1, it is followed that the dynamical system (5) possibly exhibits hyperchaotic behavior for $\eta \in [0.857, 1]$, because in each column of Table 1, there are two positive LEs. The positive Lyapunov exponent ensures the sensitive dependence on the choice of initial conditions (local instability of the system in the state space), which is only one property of a chaotic system. Crudely, for a given dynamical system to be chaotic, it must have the following properties: sensitivity to initial values, topological transitivity, and also dense periodic orbits.

System (5) is dissipative for $\eta \in [0.857,1]$ as the sum of the LEs in every column of Table 1 is negative. The dissipativity of the system ensures the existence of an attractor. From the experimental result in Table 1, it is observed that for values of $\eta \in (0,0.857)$, the system is not dissipative. For instance, the sum of LEs for the fractional order $\eta = 0.856$ is 0.0211. Therefore, to get a dissipative hyperchaotic system in this study, we considered the fractional order $\eta \in [0.857,1]$.

The Kaplan-Yorke dimension denoted by dim (LE) of system (5) for $\eta \in [0.857,1]$ can be calculated. For instance, for the fractional order $\eta = 0.856$, the Kaplan-Yorke dimension is

$$\dim(\mathrm{LE}) = 3 + \frac{14.1214 + 1.0788 - 5.5530}{|-9.6712|} = 3.9975. \tag{17}$$

The Kaplan-Yorke dimension corresponding to $\eta = 0.888$ is

$$\dim(\mathrm{LE}) = 3 + \frac{12.4969 + 0.9704 - 5.1187}{|-9.5574|} = 3.8735. \tag{18}$$

The Kaplan-Yorke dimension corresponding to $\eta = 0.988$ is

$$\dim(\mathrm{LE}) = 3 + \frac{8.4644 + 0.6620 - 3.7225}{|-8.2829|} = 3.6524. \tag{19}$$

The Kaplan-Yorke dimension corresponding to $\eta = 1$ is

$$\dim(\mathrm{LE}) = 3 + \frac{8.0815 + 0.6597 - 3.7113}{|-8.0818|} = 3.6224. \tag{20}$$

From the above dim (LE)s, it can be said that the Kaplan-Yorke dimension is decreased from 3.9975 to 3.6224, as the fractional order is increased from 0.857 to 1. Note that the Kaplan-York dimension determines the upper bound of the Hausdorff dimensions. In this study, the Hausdorff dimensions of the attractors corresponding to different fractional orders are nonintegers and system (5) has strange attractors with fractal structures. Moreover, in Table 1 and the Kaplan-Yorke dimensions calculated above, one can find that the significance of the hyperchaotic attractors decreases as the fractional order increases from 0.857 to 1. The loss of significance is described by decreasing the sum of the positive LEs in Table 1 and also by decreasing the dimensions as the fractional order increases from 0.857 to 1. The loss of significance is also observable from the bifurcation diagram of the fractional orders shown in Figure 1 in the next section.

5.2. Bifurcation for the Fractional Order η. The bifurcation diagram of the fractional order η is obtained by varying its value in the interval $(0.857,1.00)$ with a time step of $h = 0.001$. The bifurcation diagram due to the variations of the order is illustrated in Figure 1. According to this figure, for values of $\eta \in (0.933,0.953)$, the given system exhibits oscillation with stability and the system excites a chaotic behavior for fractional values $\eta \geq 0.953$.

The simulation results corresponding to different fractional orders are portrayed in Figures 2–4 to verify the conclusion made in Figure 1.

It can be seen in Figures 2(a)–2(c) that the system exhibits oscillation with stability for the fractional order $\eta = 0.95 \in (0.933,0.953)$, which is in agreement with the conclusion in Figure 1. The time series solution of system (5) is shown for $\eta = 0.95$ in Figure 3, where the solution trajectories have the property of oscillation and stability as claimed from the bifurcation diagram shown in Figure 1. Moreover, Figures 4 and 5 characterize that system (5) exhibits a hyperchaotic behavior at $\eta = 0.974$ confirming what is predicted by the bifurcation diagram in Figure 1.

5.3. Bifurcation for Parameter r. To get the bifurcation diagram of parameter r of the hyperchaotic system (5), the fractional order used is $\eta = 0.97$ and $k = 0.7$, $c = 4$, $a = 10$, and $r \in [29.5,30.5]$ with a time step of $h = 0.001$ and the initial value condition is $y_0 = [0.5, 0.5, 5, 0.5]$. The simulation is made to run for 100 s. The bifurcation graph due to the variation of r is illustrated in Figure 6.

It can be inferred in Figure 6 that system (5) exhibits a hyperchaotic behavior for the parameters considered in the simulation for $r \in [29.5,30.5]$. The phase portraits of system (5) shown in Figures 7 and 8 for $r = 29.6$ and $r = 30.4$ verify the conclusion from the bifurcation diagram of Figure 6.

5.4. Bifurcation for Parameter c. For the bifurcation diagram of parameter c of the hyperchaotic system (5), the fractional order used is $\eta = 0.974$, $k = 0.7$, and $a = 10$ and the bifurcation parameter is made to vary in the interval [3.5, 4.5] with a time step of $h = 0.001$, with initial value condition $y_0 = [0.5 ; 0.5 ; 5 ; 0.5]$, and the simulation is made to run for 100 s. The bifurcation diagram due to the variation of parameter c is illustrated in Figure 9.

It is observable in Figure 5 that system (5) shows hyperchaotic dynamics for parameter c in [3.5, 4.5]. Moreover, the phase portraits of system (5) for $c = 3.6$ and 4.4 shown in Figure 10 verify the conclusion drawn in Figure 9.

5.5. Bifurcation for Parameter k. To obtain the bifurcation diagram due to parameter k of the hyperchaotic system (5), the fractional order used is $\eta = 0.974$. The values of the remaining parameters are $k = 0.7$ and $a = 10$, and the bifurcation parameter k is made to vary in the interval [0.5, 1] with a time step of $h = 0.001$. The initial value condition is $y_0 = [0.5 ; 0.5 ; 5 ; 0.5]$ and the simulation is made to run for 100 s. The bifurcation graph due to the variation of k is illustrated in Figure 11.

It is clear in Figure 11 that system (5) shows hyperchaotic dynamics for parameter k of the interval [0.5, 1], the given values of the parameters, and fractional orders. Moreover, the attractors of system (5) for values of parameter k in [0.5, 1] are approximately similar to the figures shown in Figure 10.

5.6. Bifurcation for Parameter a. To obtain the bifurcation diagram due to parameter a of the hyperchaotic system (5), the fractional derivative, the initial condition, the time, and the values of the remaining parameters are the same as those used in Section 5.5. The bifurcation diagram due to the variation of parameter a is illustrated in Figure 12.

It is clear in Figure 12 that system (5) shows the hyperchaotic dynamics for parameter a in [9.5,10.5], the given parameters, and the fractional orders. Moreover, the attractors of system (5) for different values of a are shown in Figure 13, verifying the hyperchaotic nature of system (5).

6. Sensitivity to Initial Conditions

This section examines the effect of different initial values on the dynamics of the hyperchaotic system (5). Since sensitivity to initial conditions is one of the important properties of chaotic or hyperchaotic systems, it is necessary to examine this property for the LS system (5) using different initial conditions. In this study, to examine sensitivity to initial conditions, we used the parameter values indicated in (5) and $\eta = 0.974$. The initial condition considered is $(x_0, 0.5,5, 0.5)$. The different values of x_0 are shown in Figure 14.

It can be observed in Figure 14 that all the trajectories coincided for about the first 5.1 seconds and they were divided into two for about 10.1 seconds. At about 15.1 seconds, they divided into three, diverging from each other. The values of x_0 of two trajectories that overlapped for the first 15.1 seconds are closer to each other than the value of x_0 of the third trajectory.

7. Synchronization of the Lorenz-Stenflo System

In this part of the study, we developed a master-slave system of the dynamic system (5). Firstly, two identical copies of system (5) are associated via a coupling function. Then, we performed a simulation of the coupled systems using different initial conditions. Finally, the error dynamics showed asymptotic stability and the phase portraits of the master and slave system showed a strong correlation.

Let the master hyperchaotic model be given by

$$\begin{aligned}
{}_0^C D_t^q x(t) &= a(y-x) + cw,\\
{}_0^C D_t^q y(t) &= rx - y - xz,\\
{}_0^C D_t^q z(t) &= -kz + xy,\\
{}_0^C D_t^q w(t) &= -x - aw,
\end{aligned} \tag{21}$$

and the slave system be given by

$$\begin{cases}
{}_0^C D_t^q x_s(t) = a(y_s - x_s) + cw_s,\\
{}_0^C D_t^q y_s(t) = rx_s - y_s - x_s z_s,\\
{}_0^C D_t^q z_s(t) = -kz_s + x_s y_s,\\
{}_0^C D_t^q w_s(t) = -x_s - aw_s,
\end{cases} + \left(H - \frac{\partial F}{\partial X}\right)(X_s - X), \tag{22}$$

where the coupling function is $(H - (\partial F/\partial X))(X_s - X)$, $\partial F/\partial X$ is the Jacobian matrix of the system, H is a Hermitian matrix of appropriate size, $X = (x, y, z, w)$, and $X_s = (x_s, y_s, z_s, w_s)$. Moreover, the coupling function is given by

$$\left(H - \frac{\partial F}{\partial X}\right)(X_s - X) = \begin{bmatrix} 0 & 0 & 0 & -5.5 \\ z-32 & -2 & x & 0 \\ -y & -x & -2.6 & 0 \\ -0.5 & 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_s - x \\ y_s - y \\ z_s - z \\ w_s - w \end{bmatrix}, \tag{23}$$

where a Hermitian matrix H is chosen to be

$$H = \begin{bmatrix} -10 & 10 & 0 & -1.5 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & -3.3 & 0 \\ -1.5 & 0 & 0 & -3 \end{bmatrix}. \tag{24}$$

It can be shown that all the real parts of the eigenvalues of H are negative. Thus, the Matignon criterion is satisfied; H is a Hermitian matrix.

Now, considering the coupling function (23), the slave system becomes

$$\begin{aligned}
{}_0^C D_t^q x_s(t) &= a(y_s - x_s) + cw_s - 5.5(w_s - w),\\
{}_0^C D_t^q y_s(t) &= rx_s - y_s - x_s z_s + (-32 + z)(x_s - x) - 2(y_s - y) + x(z_s - z),\\
{}_0^C D_t^q z_s(t) &= -kz_s + x_s y_s - y(x_s - x) - x(y_s - y) - 2.6(z_s - z),\\
{}_0^C D_t^q w_s(t) &= -x_s - aw_s - 0.5(x_s - x) + 7(w_s - w).
\end{aligned} \tag{25}$$

Defining the error as

$$\begin{aligned}
e_1 &= x_s - x,\\
e_2 &= y_s - y,\\
e_3 &= z_s - z,\\
e_4 &= w_s - w,
\end{aligned} \tag{26}$$

the error dynamics is shown in

$$\begin{pmatrix} {}_0^C D_t^q e_1 \\ {}_0^C D_t^q e_2 \\ {}_0^C D_t^q e_3 \\ {}_0^C D_t^q e_4 \end{pmatrix} = He = \begin{pmatrix} -10 & 10 & 0 & -1.5 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & -3.3 & 0 \\ -1.5 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{27}$$

To verify if the master and slave systems in equations (21) and (25) are correlated, the time series trajectories of the two systems, including the graphs of the dynamics of the error, are illustrated in Figures 15–19. The values of the parameters used are $k = 0.7, r = 30, c = 4,$ and $a = 10$ together with the fractional order $\eta = 0.978$. Moreover, the initial value conditions used for the master and the slave system are $y_{0m} = [0.5, 0.5, 5, 0.5]$ and $y_{0s} = [5, 5, 5, 5]$, respectively.

It can be observed in Figure 15 that there is a robust correlation between the master and slave systems because the error graphs converge to zero approximately in the first 2.5 seconds, as shown in Figure 15.

8. Conclusion

This research study implemented a qualitative analysis on the four-dimensional Lorenz-Stenflo mathematical model in the sense of the Caputo operator. The Matignon local stability criterion showed that the equilibrium points of the LS system are locally unstable, implying that the LS system led to hyperchaotic dynamical behavior. The scheme developed by Garrappa approximated the numerical solution, and the corresponding MATLAB code was used to obtain all the figures in the study. The authors believe that the qualitative analysis made in this manuscript, the displayed figures, and the synchronization systems developed have revealed complex dynamical behaviors of the Lorenz-Stenflo system that are not obtained earlier by other studies on the system. It is also worth mentioning that this study have been done by including other concepts of fractional derivatives and comparing the results to the results obtained due to the Caputo fractional derivative, to be treated in the future by the authors or interested researchers in the area.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Chernet Tuge Deressa did the actualization, methodology, formal analysis, validation, investigation, and initial draft. Sina Etemad did the actualization, validation, methodology, formal analysis, investigation, and initial draft. Mohammed K. A. Kaabar did the actualization, methodology, formal analysis, validation, investigation, initial draft, and supervision of the original draft and editing. Shahram Rezapour did the actualization, validation, methodology, formal analysis, investigation, and initial draft. All authors read and approved the final version.

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Research Article

Atomic Decompositions and John-Nirenberg Theorem of Grand Martingale Hardy Spaces with Variable Exponents

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Let $\theta \geq 0$ and $p(\cdot)$ be a variable exponent, and we introduce a new class of function spaces $L_{p(\cdot),\theta}$ in a probabilistic setting which unifies and generalizes the variable Lebesgue spaces with $\theta = 0$ and grand Lebesgue spaces with $p(\cdot) \equiv p$ and $\theta = 1$. Based on the new spaces, we introduce a kind of Hardy-type spaces, grand martingale Hardy spaces with variable exponents, via the martingale operators. The atomic decompositions and John-Nirenberg theorem shall be discussed in these new Hardy spaces.

1. Introduction

The martingale theory is widely studied in the field of mathematical physics, stochastic analysis, and probability. Weisz [1] presented the atomic decomposition theorem for martingale Hardy spaces. Herz [2] established the John-Nirenberg theorem for martingales. Since then, the study of martingale Hardy spaces associated with various functional spaces has attracted a steadily increasing interest. For instance, martingale Orlicz-type Hardy spaces were investigated in [3–6], martingale Lorentz Hardy spaces were studied in [7–9], and variable martingale Hardy spaces were developed in [10–14].

Let $1 < p < \infty$, and the grand Lebesgue space $L_{p)}(E)$ introduced by Iwaniec and Sbordone [15] is defined as the Banach function space of the measurable functions f on finite E such that

$$\|f\|_{L_{p)}} = \sup_{0<\eta<p-1} \left(\eta \frac{1}{|E|} \int_E |f(x)|^{p-\eta} dx \right)^{1/(p-\eta)} < \infty. \quad (1)$$

Such spaces can be used to integrate the Jacobian under minimal hypotheses [15]. The grand Lebesgue spaces as a kind of Banach function space were investigated in the papers of Capone et al. [16, 17], Fiorenza et al. [18–21], Kokilashvili et al. [22, 23], and so forth. In particular, grand Lebesgue spaces with variable exponents were studied in [24, 25].

We find that the framework of grand Lebesgue spaces with variable exponents has not yet been studied in martingale theory. This paper is aimed at discussing the variable grand Hardy spaces defined on the probabilistic setting and showing the atomic decompositions and John-Nirenberg theorem in these new Hardy spaces. More precisely, we first present the atomic characterization of grand Hardy martingale spaces with variable exponents. To do so, we introduce the following new notations of atom.

Definition 1. Let $p(\cdot)$ be a variable exponent and $\theta \geq 0$. A measurable function a is called a $(1, p(\cdot), \theta)$ -atom (resp. $(2, p(\cdot), \theta)$ -atom, $(3, p(\cdot), \theta)$ -atom) if there exists a stopping time τ such that

$(a1)\mathbb{E}_n a = 0, \forall\, n \leq \tau$

$(a2)\|s(a)\|_{L_\infty} (\text{resp.} \|S(a)\|_{L_\infty}, \|Ma\|_{L_\infty}) \leq \|\chi_{\{\tau<\infty\}}\|^{-1}_{L_{p(\cdot),\theta}}$.

See Section 2 for the notation $L_{p(\cdot),\theta}$. Denote by $\mathbb{A}_s(p(\cdot), \infty)$ (resp. $\mathbb{A}_S(p(\cdot), \infty)$, $\mathbb{A}_M(p(\cdot), \infty)$) the collection of all sequences of triplet (a^k, τ_k, μ_k), where a^k are $(1, p(\cdot), \theta)$ -atoms (resp. $(2, p(\cdot), \theta)$-atoms, $(3, p(\cdot), \theta)$-atoms), τ_k are stopping times satisfying $(a1)$ and $(a2)$ in Definition 1, and μ_k are nonnegative numbers and also

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} < \infty. \tag{2}$$

Under these definitions, we show the atomic decompositions of the grand Hardy martingale spaces with variable exponents (see Section 3). To be precise, we prove that for any $f=(f_n)_{n\geq 0}$, $f\in H^s_{p(\cdot),\theta}$ (resp. $Q_{p(\cdot),\theta}$, $D_{p(\cdot),\theta}$) iff there exists a sequence of triplet $(a^k, \tau_k, \mu_k)\in \mathbb{A}_s(p(\cdot),\infty)$ (resp. $\mathbb{A}_S(p(\cdot),\infty)$, $\mathbb{A}_M(p(\cdot),\infty)$) so that for each $n\geq 0$,

$$\begin{aligned} f_n &= \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \ a.e., \\ &\left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} \\ &\approx \|f\|_{H^s_{p(\cdot),\theta}} \left(\text{resp.} \|f\|_{Q_{p(\cdot),\theta}}, \|f\|_{D_{p(\cdot),\theta}}\right). \end{aligned} \tag{3}$$

Moreover, we extend the classical John-Nirenberg theorem to the grand variable Hardy martingale spaces. To be precise, under suitable conditions, we present the following one:

$$\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} \approx \|f\|_{\mathrm{BMO}_1}. \tag{4}$$

See Theorem 11 for the details. This conclusion improves the recent results [12, 26], respectively.

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ denote the integer set, nonnegative integer set, and complex numbers set, respectively. We denote by C the absolute positive constant, which can vary from line to line. The symbol $A\lesssim B$ stands for the inequality $A\leq CB$. If we write $A\approx B$, then it stands for $A\lesssim B\lesssim A$.

2. Preliminaries

2.1. Grand Lebesgue Spaces with Variable Exponents. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. An $\mathcal{F}$-measurable function $p(\cdot)$: $\Omega\longrightarrow(0,\infty)$ which is called a variable exponent. For convenience, we denote

$$\begin{aligned} p_- &:= \operatorname{essinf}\{p(\omega)\colon \omega\in\Omega\}, p_+ := \operatorname{esssup}\{p(\omega)\colon \omega\in\Omega\}, \\ p_-(B) &= \operatorname{essinf}\{p(\omega)\colon \omega\in B\} \text{ and} \\ p_+(B) &= \operatorname{esssup}\{p(\omega)\colon \omega\in B\}. \end{aligned} \tag{5}$$

Denote by $\mathcal{P}=\mathcal{P}(\Omega)$ the collection of all variable exponents $p(\cdot)$ satisfying with $1<p_-\leq p_+<\infty$. The variable Lebesgue space $L_{p(\cdot)}=L_{p(\cdot)}(\Omega)$ consists of all $\mathcal{F}$-measurable functions f such that for some $\lambda>0$,

$$\rho\left(\frac{f}{\lambda}\right) = \int_\Omega \left(\frac{|f(\omega)|}{\lambda}\right)^{p(\omega)} d\mathbb{P} < \infty. \tag{6}$$

This leads to a Banach function space under the Luxemburg-Nakano norm

$$\|f\|_{L_{p(\cdot)}} \equiv \inf\left\{\lambda>0 : \rho\left(\frac{f}{\lambda}\right)\leq 1\right\}. \tag{7}$$

Based on this, we begin with the definition of the grand Lebesgue space with variable exponent.

Definition 2. Suppose that $p(\cdot)\in\mathcal{P}$ and $\theta\geq 0$. We define the grand Lebesgue space with variable exponent $L_{p(\cdot),\theta}=L_{p(\cdot),\theta}(\Omega)$ as the set of all $\mathcal{F}$-measurable functions f satisfying

$$\|f\|_{L_{p(\cdot),\theta}} := \sup_{0<\eta<p_- -1} \eta^{\theta/(p_- -\eta)} \|f\|_{L_{p(\cdot)-\eta}} < \infty. \tag{8}$$

The Grand Lebesgue space with variable exponent can unify and generalize the various function spaces. To be precise, if $\theta=0$, $L_{p(\cdot),\theta}$ degenerates to the variable Lebesgue space $L_{p(\cdot)}$. If $\theta=1$ and $p(\cdot)\equiv p$, $L_{p(\cdot),\theta}$ becomes the grand Lebesgue space $L_{p)}$.

2.2. Martingale Grand Hardy Spaces via Variable Exponents. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ sets with $\mathcal{F}=\sigma(\bigcup_{n\geq 0}\mathcal{F}_n)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f=(f_n)_{n\geq 0}$ of random variables is said to be a martingale, if f_n is $\mathcal{F}_n$-measurable, $\mathbb{E}(|f_n|)<\infty$, and $\mathbb{E}_n(f_{n+1})=f_n$ for every $n\geq 0$. Denote $\mathcal{M}$ to be the set of all martingales $f=(f_n)_{n\geq 0}$ with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $f_0=0$. For $f\in\mathcal{M}$, write its martingale difference by $d_n f=f_n-f_{n-1}(n\geq 0, f_{-1}=0)$. Define the maximal function, the square function, and the conditional square function of f, respectively, as follows:

$$\begin{aligned} M_m f &= \sup_{n\leq m}|f_n|, Mf=\sup_{n\geq 0}|f_n|, \\ S_m(f) &= \left(\sum_{n=0}^m |df_n|^2\right)^{1/2}, S(f)=\left(\sum_{n=0}^\infty |df_n|^2\right)^{1/2}, \\ s_m(f) &= \left(\sum_{n=0}^m \mathbb{E}_{n-1}|df_n|^2\right)^{1/2}, s(f)=\left(\sum_{n=0}^\infty \mathbb{E}_{n-1}|df_n|^2\right)^{1/2}. \end{aligned} \tag{9}$$

Let Γ be the set of all sequences $(\lambda_n)_{n\geq 0}$ of nondecreasing, nonnegative, and adapted functions, and $\lambda_\infty := \lim_{n\longrightarrow\infty}\lambda_n$. For $f\in\mathcal{M}$, $p(\cdot)\in\mathcal{P}$, and $\theta\geq 0$, denote

$$\begin{aligned} \Gamma\left[Q_{p(\cdot),\theta}\right](f) &= \left\{(\lambda_n)_{n\geq 0}\in\Gamma : S_n(f)\leq\lambda_{n-1}(n\geq 1), \lambda_\infty\in L_{p(\cdot),\theta}\right\}, \\ \Gamma\left[D_{p(\cdot),\theta}\right](f) &= \left\{(\lambda_n)_{n\geq 0}\in\Gamma : |f_n|\leq\lambda_{n-1}(n\geq 1), \lambda_\infty\in L_{p(\cdot),\theta}\right\}. \end{aligned} \tag{10}$$

Now we introduce the grand martingale Hardy spaces associated with variable exponents as follows:

$$
\begin{aligned}
H^*_{p(\cdot),\theta} &= \left\{ f \in \mathscr{M} : Mf \in L_{p(\cdot),\theta} \right\}, \|f\|_{H^*_{p(\cdot),\theta}} = \|Mf\|_{L_{p(\cdot),\theta}},\\
H^S_{p(\cdot),\theta} &= \left\{ f \in \mathscr{M} : S(f) \in L_{p(\cdot),\theta)} \right\}, \|f\|_{H^S_{p(\cdot),\theta}} = \|S(f)\|_{L_{p(\cdot),\theta}},\\
H^s_{p(\cdot),\theta} &= \left\{ f \in \mathscr{M} : s(f) \in L_{p(\cdot),\theta} \right\}, \|f\|_{H^s_{p(\cdot),\theta}} = \|s(f)\|_{L_{p(\cdot),\theta}},\\
Q_{p(\cdot),\theta} &= \left\{ f \in \mathscr{M} : \|f\|_{Q_{p(\cdot),\theta}} < \infty \right\}, \|f\|_{Q_{p(\cdot),\theta}} = \inf_{(\lambda_n)_{n\geq 0} \in \Gamma[Q_{p(\cdot),\theta}](f)} \|\lambda_\infty\|_{L_{p(\cdot),\theta}},\\
D_{p(\cdot),\theta} &= \left\{ f \in \mathscr{M} : \|f\|_{D_{p(\cdot),\theta}} < \infty \right\}, \|f\|_{D_{p(\cdot),\theta}} = \inf_{(\lambda_n)_{n\geq 0} \in \Gamma[D_{p(\cdot),\theta}](f)} \|\lambda_\infty\|_{L_{p(\cdot),\theta}}.
\end{aligned}
\tag{11}
$$

The bounded $L_{p(\cdot),\theta}$-martingale spaces

$$
L_{p(\cdot),\theta} = \left\{ f = (f_n)_{n\geq 0} : \sup_{n\geq 0} \|f_n\|_{L_{p(\cdot),\theta}} < \infty \right\}, \tag{12}
$$

where

$$
\|f\|_{L_{p(\cdot),\theta}} = \sup_{n\geq 0} \|f_n\|_{L_{p(\cdot),\theta}}. \tag{13}
$$

Remark 3. If $\theta = 0$, then we obtain the definitions of $H^*_{p(\cdot)}$, $H^S_{p(\cdot)}$, $H^s_{p(\cdot)}$, $Q_{p(\cdot)}$, and $D_{p(\cdot)}$, respectively (see [10, 12, 27]). If we consider the special case $\theta = 1$ and $p(\cdot) \equiv p$ with the notations above, we obtain the definitions of $H^*_{p)}$, $H^S_{p)}$, $H^s_{p)}$, $Q_{p)}$, and $D_{p)}$, respectively (see [26]). In addition, if $p(\cdot) \equiv p$ and $\theta = 0$, we obtain the martingale Hardy spaces H^*_q, H^S_q, H^s_q, Q_q, and D_q, respectively (see [28]).

Refer to [29, 30] for more information on martingale theory.

3. Atomic Characterization

The method of atomic characterization plays an useful tool in martingale theory (see for instance [1, 4, 6, 31–33]). We shall construct the atomic characterizations for grand Hardy martingale spaces with variable exponents in this section.

Theorem 4. *Let $p(\cdot) \in \mathscr{P}$ and $\theta \geq 0$. If the martingale $f \in H^s_{p(\cdot),\theta}$, then there exists a sequence of triplet $(a^k, \tau_k, \mu_k) \in \mathbb{A}_s(p(\cdot),\infty)$ so that for each $n \geq 0$,*

$$
\sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k = f_n, \text{ a.e.}, \tag{14}
$$

$$
\left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}} \lesssim \|f\|_{H^s_{p(\cdot),\theta}}. \tag{15}
$$

Conversely, if the martingale f has a decomposition of (14), then

$$
\|f\|_{H^s_{p(\cdot),\theta}} \lesssim \inf \left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}} \right\|_{L_{p(\cdot),\theta}}, \tag{16}
$$

where the infimum is taken over all the admissible representations of (14).

Proof. Let $f \in H^s_{p(\cdot),\theta}$. Now consider the stopping time for each $k \in \mathbb{Z}$:

$$
\tau_k = \inf \left\{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \right\}. \tag{17}
$$

It is easy to see that the sequence of these stopping times is nondecreasing. For each stopping time τ, denote $f^\tau_n = f_{n\wedge\tau}$. It is easy to write that

$$
f_n = \sum_{k\in\mathbb{Z}} \left(f^{\tau_{k+1}}_n - f^{\tau_k}_n \right). \tag{18}
$$

For each $k \in \mathbb{Z}$, let $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}$. If $\mu_k \neq 0$, we set

$$
a^k_n = \frac{f^{\tau_{k+1}}_n - f^{\tau_k}_n}{\mu_k}, n \in \mathbb{N}. \tag{19}
$$

If $\mu_k = 0$, we set $a^k_n = 0$ for each $n \in \mathbb{N}$. For each fixed $k \in \mathbb{Z}$, $(a^k_n)_{n\geq 0}$ is a martingale. Since $s(f^{\tau_k}) = s_{\tau_k}(f) \leq 2^k$, we get

$$
s\left(a^k_n\right) \leq \frac{s(f^{\tau_{k+1}}) + s(f^{\tau_k})}{\mu_k} \leq \left\| \chi_{\{\tau_k<\infty\}} \right\|^{-1}_{L_{p(\cdot),\theta}}. \tag{20}
$$

We can easily check that $(a^k_n)_{n\geq 0}$ is a bounded L_2-martingale. Hence, there exists an element $a^k \in L_2$ such that $\mathbb{E}_n a^k = a^k_n$. If $n \leq \tau_k$, then $a^k_n = 0$, and $s(a^k) \leq \|\chi_{\{\tau_k<\infty\}}\|^{-1}_{L_{p(\cdot),\theta}}$. Consequently, it implies that a^k is really a $(1, p(\cdot), \theta)$-atom.

Denote $\Lambda_k \coloneqq \{\tau_k < \infty\}$. Knowing that $\{\tau_k < \infty\} = \{s(f) > 2^k\}$ and τ_k is nondecreasing for each $k \in \mathbb{Z}$, we obtain $\Lambda_{k+1} \subseteq \Lambda_k$. Now, we point out that

$$
\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x) = 2 \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k \setminus \Lambda_{k+1}}(x). \tag{21}
$$

Indeed, for a fixed $x_0 \in \Omega$, there is $k_0 \in \mathbb{Z}$ so that $x_0 \in \Lambda_{k_0}$ and $x_0 \in \Lambda_{k_0+1}$, then we have

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) &= \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) = \sum_{k=-\infty}^{k_0} 3 \cdot 2^k = 3 \cdot 2^{k_0+1},\\
\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k \setminus \Lambda_{k+1}}(x_0) &= \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\Lambda_k \setminus \Lambda_{k+1}}(x_0) = 3 \cdot 2^{k_0}.
\end{aligned}
\tag{22}
$$

This means

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}} &= \left\|\sum_{k\in\mathbb{Z}}3\cdot 2^k\chi_{\{\tau_k<\infty\}}\right\|_{L_{p(\cdot),\theta}} = 6\left\|\sum_{k\in\mathbb{Z}}2^k\chi_{\Lambda_k\setminus\Lambda_{k+1}}\right\|_{L_{p(\cdot),\theta}}\\ &= 6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\int_\Omega\left(\sum_{k\in\mathbb{Z}}\frac{2^k\chi_{\Lambda_k\setminus\Lambda_{k+1}}(x)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\ &= 6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\sum_{k\in\mathbb{Z}}\int_{\Lambda_k\setminus\Lambda_{k+1}}\left(\frac{2^k}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\ &= 6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\sum_{k\in\mathbb{Z}}\int_{\Lambda_k\setminus\Lambda_{k+1}}\left(\frac{s(f)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\ &\le 6\sup_{0<\eta<p_--1}\left[\eta^{\theta/(p_--\eta)}\inf\left\{\lambda>0:\int_\Omega\left(\frac{s(f)}{\lambda}\right)^{p(x)-\eta}d\mathbb{P}\le 1\right\}\right]\\ &= 6\sup_{0<\eta<p_--1}\eta^{\theta/(p_--\eta)}\|s(f)\|_{L_{p(\cdot)-\eta}} = 6\|f\|_{H^s_{p(\cdot),\theta}}.\end{aligned} \tag{23}$$

For the converse part, according to the definition of $(1,p(\cdot),\theta)$-atom, we easily conclude

$$s\left(a^k\right)=s\left(a^k\right)\chi_{\{\tau_k<\infty\}}\le\left\|s\left(a^k\right)\right\|_{L_\infty}\chi_{\{\tau_k<\infty\}}\le\left\|\chi_{\{\tau_k<\infty\}}\right\|^{-1}_{L_{p(\cdot),\theta}}\chi_{\{\tau_k<\infty\}}, \tag{24}$$

where a^k is the $(1,p(\cdot),\theta)$-atom and τ_k is the stopping time associated with a^k which, when combined with the subadditivity of the operator s, yields

$$s(f)\le\sum_{k\in\mathbb{Z}}\mu_k s\left(a^k\right)\le\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}. \tag{25}$$

This implies

$$\|f\|_{H^s_{p(\cdot),\theta}}=\|s(f)\|_{L_{p(\cdot),\theta}}\le\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}. \tag{26}$$

Taking over all the admissible representations of (14) for f, we obtain the desired result.

Next, we will characterize $Q_{p(\cdot),\theta}$ and $D_{p(\cdot),\theta}$ by atoms, respectively. The proof is similar to the proof of Theorem 4. For the completeness of this paper, we provide some details. □

Theorem 5. *Suppose $p(\cdot)\in\mathscr{P}$ and $\theta\ge 0$. If the martingale $f=(f_n)_{n\ge 0}\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$), then there exists a sequence of triplet $(a^k,\tau_k,\mu_k)\in\mathbb{A}_S(p(\cdot),\infty)$ (resp. $\mathbb{A}_M(p(\cdot),\infty)$) so that for each $n\in\mathbb{N}$,*

$$f_n=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n a^k, \tag{27}$$

$$\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}\lesssim\|f\|_{Q_{p(\cdot),\theta}}\left(resp.\|f\|_{D_{p(\cdot),\theta}}\right). \tag{28}$$

Conversely, if the martingale $f=(f_n)_{n\ge 0}$ has admissible representation (27), then $f\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$) and

$$\|f\|_{Q_{p(\cdot),\theta}}\left(resp.\|f\|_{D_{p(\cdot),\theta}}\right)\lesssim\inf\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{L_{p(\cdot),\theta}}}\right\|_{L_{p(\cdot),\theta}}, \tag{29}$$

where the infimum is taken over all the admissible representations of (27).

Proof. The proof follows the ideas in Theorem 4, so we omit some details. Suppose $f=(f_n)_{n\ge 0}\in Q_{p(\cdot),\theta}$ (resp. $D_{p(\cdot),\theta}$). We define stopping times as follows:

$$\tau_k=\inf\left\{n\in\mathbb{N}:\lambda_n>2^k\right\},\ \inf\varnothing=\infty, \tag{30}$$

where $(\lambda_n)_{n\ge 0}$ is an adapted, nondecreasing sequence such that almost everywhere $|S_n(f)|\le\lambda_{n-1}$ (resp.$|f_n|\le\lambda_{n-1}$) and $\lambda_\infty\in L_{p(\cdot),\theta}$.

Let $(a^k)_{k\in\mathbb{Z}}$ and $(\mu_k)_{k\in\mathbb{Z}}$ be defined as in the proof of Theorem 4. And replace $\Lambda_k=\{\tau_k<\infty\}=\{s(f)>2^k\}$ by the $\Lambda_k=\{\tau_k<\infty\}=\{\lambda_\infty>2^k\}$. Then, we obtain that $f_n=\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n a^k$ and (28) still hold.

For the converse part, write

$$\lambda_n=\sum_{k\in\mathbb{Z}}\mu_k\|S\left(a^k\right)\|_{L_\infty}\chi_{\{\tau_k\le n\}}\left(resp.\lambda_n=\sum_{k\in\mathbb{Z}}\mu_k\|M\left(a^k\right)\|_{L_\infty}\chi_{\{\tau_k\le n\}}\right). \tag{31}$$

Clearly, $(\lambda_n)_{n\ge 0}$ is a nonnegative, nondecreasing, and adapted sequence with $S_{n+1}(f)\le\lambda_n$ (resp.$|f_n|\le\lambda_n$). Thus, we get

$$\|f\|_{Q_{p(\cdot),\theta}}\left(\text{resp.}\|f\|_{D_{p(\cdot),\theta}}\right)=\|\lambda_\infty\|_{L_{p(\cdot),\theta}}\lesssim\left\|\sum_{k\in\mathbb{Z}}\mu_k\frac{\chi_{\{\tau<\infty\}}}{\|\chi_{\{\tau<\infty\}}\|_{p(\cdot),\theta}}\right\|_{p(\cdot),\theta}. \tag{32}$$

□

Taking over all the admissible representations of (27) for f, we obtain the desired result.

Remark 6. Suppose $p(\cdot)\in\mathscr{P}$ and $\theta\ge 0$. We conclude that the sum $\sum_{k=M}^N\mu_k a^k$ in Theorem 4 converges to f in $H^s_{p(\cdot),\theta}$ as $M\longrightarrow-\infty$, $N\longrightarrow\infty$. Indeed, it follows by the subadditive of the operator s, we get, for any $M,N\in\mathbb{Z}$ with $M<N$,

$$s\left(f-\sum_{k=M}^{N}\mu_k a^k\right)\le s\left(f-f^{\tau_{N+1}}\right)+s\left(f^{\tau_M}\right). \tag{33}$$

Moreover, $s(f-f^{\tau_{N+1}})$ is decreasing and convergent to 0 (a.e.) as $N\longrightarrow\infty$, and $s(f^{\tau_M})$ is decreasing and convergent to 0 (a.e.) as $M\longrightarrow-\infty$. From this and the dominated convergence theorem in $L_{p(\cdot)-\varepsilon}$ for $0<\varepsilon<p_--1$ (see [34], Theorem 2.62), it follows that

$$\begin{aligned}\|f-\sum_{k=M}^{N}\mu_k a^k\|_{H^s_{p(\cdot),\theta}}&\le\|s(f-f^{\tau_{N+1}})+s(f^{\tau_M})\|_{L_{p(\cdot),\theta}}\\&\le\sup_{0<\eta<p_--1}\eta^{\theta/(p_--\eta)}\|s(f-f^{\tau_{N+1}})\|_{L_{p(\cdot)-\eta}}+\sup_{0<\eta<p_--1}\\&\quad\cdot\eta^{\theta/(p_--\eta)}\|s(f^{\tau_M})\|_{L_{p(\cdot)-\eta}}\longrightarrow0\text{ as }M\longrightarrow-\infty,N\longrightarrow\infty.\end{aligned} \tag{34}$$

Furthermore, we can also show the norm convergence of the summation $\sum_{k=M}^{N}\mu_k a^k$ in Theorems 5 as $M\longrightarrow-\infty$, $N\longrightarrow\infty$.

4. The Generalized John-Nirenberg Theorem

In the sequel of this section, we will often suppose that every $\mathcal{F}_n$ is generated by countably many atoms. Recall that $B\in\mathcal{F}_n$ is called an atom, and if for any $A\subseteq B$ with $A\in\mathcal{F}_n$ satisfying $\mathbb{P}(A)<\mathbb{P}(B)$, we have $\mathbb{P}(A)=0$. We denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We shall present the generalized John-Nirenberg theorem on grand Lebesgue spaces with variable exponents. For each $1\le p<\infty$, the Banach space BMO_p (bounded mean oscillation [35]) is defined as

$$\mathrm{BMO}_p=\left\{f\in L_p:\|f\|_{\mathrm{BMO}_p}=\sup_{n\ge1}\left\|\mathbb{E}_n\left(|f-\mathbb{E}_{n-1}f|^p\right)\right\|_{L_\infty}^{1/p}<\infty\right\}. \tag{35}$$

It can be easily shown that the norm of BMO_p is equivalent to

$$\|f\|_{\mathrm{BMO}_p}=\sup_{\tau\in\mathcal{T}}\frac{\|f-f^{\tau-1}\|_{L_p}}{\|\chi_{\{\tau<\infty\}}\|_{L_p}}, \tag{36}$$

where $\mathcal{T}$ consists of all stopping times relative to $\{\mathcal{F}_n\}_{n\ge0}$. It follows immediately from the John-Nirenberg theorem [2, 30] that

$$\mathrm{BMO}_p=\mathrm{BMO}_1,1<p<\infty. \tag{37}$$

What is more, in [2], there has

$$C\cdot p\|f\|_{\mathrm{BMO}_1}\ge\|f\|_{\mathrm{BMO}_p}\ge\|f\|_{\mathrm{BMO}_1}. \tag{38}$$

Definition 7. For $p(\cdot)\in\mathcal{P}$ and $\theta\ge0$, the generalized BMO martingale space is defined by

$$\mathrm{BMO}_{p(\cdot),\theta}=\left\{f\in L_{p(\cdot),\theta}:\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}<\infty\right\}, \tag{39}$$

where

$$\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}=\sup_{\tau\in\mathcal{T}}\frac{\|f-f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot),\theta}}}. \tag{40}$$

Remark 8. If $\theta=0$, $\mathrm{BMO}_{p(\cdot),\theta}$ degenerates to the variable exponent BMO martingale space $\mathrm{BMO}_{p(\cdot)}$ introduced and studied in [12]. If $\theta=1$ and $p(\cdot)\equiv p$, $\mathrm{BMO}_{p(\cdot),\theta}$ becomes the grand BMO martingale space $\mathrm{BMO}_{p)}$ studied in [26].

In order to establish the generalized John-Nirenberg theorem in the framework of $\mathrm{BMO}_{p(\cdot),\theta}$, we need the following lemmas and notations.

Lemma 9 (Hölder's inequality, see [34]). *Let $p(\cdot),q(\cdot),r(\cdot)\in\mathcal{P}$ satisfy*

$$\frac{1}{p(\omega)}+\frac{1}{q(\omega)}=\frac{1}{r(\omega)},a.e.\omega\in\Omega. \tag{41}$$

Then, there exists a constant C such that for all $f\in L_{p(\cdot)}$ and $g\in L_{q(\cdot)}$, we have $fg\in L_{r(\cdot)}$ and

$$\|fg\|_{L_{r(\cdot)}}\le C\|f\|_{L_{p(\cdot)}}\|g\|_{L_{q(\cdot)}}. \tag{42}$$

We mention that if the variable exponent $p(x)$ meets the log-Hölder continuity condition in Euclidean spaces, the inverse Hölder's inequality holds for characteristic functions in $L_{p(\cdot)}(\mathbb{R}^n)$ (see [36]). Compared with Euclidean space $\mathbb{R}^n$, the probability space $(\Omega,\mathbb{P})$ has no natural metric structure. Fortunately, Jiao et al. [11, 27] put forward the following condition: there exists an absolute constant $\kappa\ge1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(B)^{p_-(B)-p_+(B)}\le\kappa,\forall B\in\bigcup_{n\ge0}A(\mathcal{F}_n). \tag{43}$$

Lemma 10 (see [27]). *Suppose $p(\cdot)\in\mathcal{P}$ satisfying (43).*

(1) For each $B\in\bigcup_{n\ge0}A(\mathcal{F}_n)$, we get

$$\mathbb{P}(B)\approx\|\chi_B\|_{L_{p(\cdot)}}\|\chi_B\|_{L_{p'(\cdot)}}. \tag{44}$$

(2) Let $q(\cdot)\in\mathcal{P}$ satisfy (43). If $r(\cdot)$ satisfies

$$\frac{1}{r(\omega)}=\frac{1}{p(\omega)}+\frac{1}{q(\omega)},\ a.e.\omega\in\Omega, \tag{45}$$

then $r(\cdot)$ also satisfies condition (43). Moreover, for each $B\in\bigcup_{n\ge0}A(\mathcal{F}_n)$, we deduce

$$\|\chi_B\|_{L_{r(\cdot)}} \approx \|\chi_B\|_{L_{p(\cdot)}} \|\chi_B\|_{L_{q(\cdot)}}. \tag{46}$$

Theorem 11. *Suppose that $p(\cdot) \in \mathscr{P}$ satisfies (43) and $\theta \geq 0$. Then, for every $f \in BMO_1$, there has*

$$\|f\|_{BMO_1} \lesssim \|f\|_{BMO_{p(\cdot),\theta}} \lesssim \|f\|_{BMO_1}. \tag{47}$$

Proof. If $p(\cdot) \in \mathscr{P}$ satisfies (43), then we clearly get that $p(\cdot) - \eta$ also satisfies (43) for $0 < \eta < p_- - 1$. It follows from Lemmas 9 and 10 that

$$\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \leq \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}} \|\chi_{\{\tau<\infty\}}\|_{L_{(p(\cdot)-\eta)'}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \approx \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot)-\eta}}}, \tag{48}$$

for any $0 < \eta < p_- - 1$. Here, the variable exponent $(p(\cdot) - \eta)'$ is defined by

$$\frac{1}{(p(\omega) - \eta)'} + \frac{1}{p(\omega) - \eta} = 1, \text{ a.e.} \omega \in \Omega. \tag{49}$$

This is equivalent to the following inequality:

$$\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} \cdot \left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot)-\eta}} \lesssim \|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}. \tag{50}$$

Hence, we have

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_{L_1}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}} &= \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \left(\|f - f^{\tau-1}\|_{L_1} / \left\|\chi_{\{\tau<\infty\}}\right\|_{L_1}\right) \cdot \left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \\ &\lesssim \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} = \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}}.\end{aligned} \tag{51}$$

Taking the supremum over all stopping times, we deduce

$$\|f\|_{\mathrm{BMO}_1} \lesssim \|f\|_{\mathrm{BMO}_{p(\cdot),\theta}}. \tag{52}$$

Conversely, from the definition of $L_{p(\cdot),\theta}$, we get

$$\begin{aligned}\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}} &= \frac{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\sup_{0<\eta<p_--1} \eta^{\theta/(p_--\eta)} \|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \\ &\leq \sup_{0<\eta<p_--1} \left\{\frac{\eta^{\theta/(p_--\eta)} \|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\eta^{\theta/(p_--\eta)} \|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}}\right\} \\ &= \sup_{0<\eta<p_--1} \left\{\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}}\right\}.\end{aligned} \tag{53}$$

It follows from Lemma 9 that

$$\frac{\|f - f^{\tau-1}\|_{L_{p(\cdot)-\eta}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \leq \frac{\|f - f^{\tau-1}\|_{L_{2p_+}} \|\chi_{\{\tau<\infty\}}\|_{L_{q(\cdot)}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{p(\cdot)-\eta}}} \approx \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}}, \tag{54}$$

where $q(\cdot)$ satisfies

$$\frac{1}{p(\omega) - \eta} = \frac{1}{2p_+} + \frac{1}{q(\omega)}, \text{ a.e.} \omega \in \Omega. \tag{55}$$

Hence, by (38), we deduce that

$$\begin{aligned}\|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} &= \sup_{\tau\in\mathcal{T}} \frac{\|f - f^{\tau-1}\|_{L_{p(\cdot),\theta}}}{\left\|\chi_{\{\tau<\infty\}}\right\|_{L_{p(\cdot),\theta}}} \lesssim \sup_{\tau\in\mathcal{T}} \sup_{0<\eta<p_--1} \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}} \\ &= \sup_{\tau\in\mathcal{T}} \frac{\|f - f^{\tau-1}\|_{L_{2p_+}}}{\|\chi_{\{\tau<\infty\}}\|_{L_{2p_+}}} = \|f\|_{\mathrm{BMO}_{2p_+}} \leq C \cdot 2p_+ \|f\|_{\mathrm{BMO}_1}.\end{aligned} \tag{56}$$

From what has been discussed above, we draw the conclusion that

$$\|f\|_{\mathrm{BMO}_1} \lesssim \|f\|_{\mathrm{BMO}_{p(\cdot),\theta}} \lesssim \|f\|_{\mathrm{BMO}_1}. \tag{57}$$

Theorem 11 improves the recent results [12, 26], respectively. More precisely, if we consider the case $\theta = 0$, then the following result holds: □

Corollary 12. *If $p(\cdot)$ satisfies (43) with $1 < p_- \leq p_+ < \infty$, then for $f \in BMO_1$,*

$$\|f\|_{BMO_{p(\cdot)}} \approx \|f\|_{BMO_1}. \tag{58}$$

And especially for $\theta = 1$ and $p(\cdot) \equiv p$, we get the conclusion as follows.

Corollary 13 (see [26]). *Suppose $1 < p < \infty$, then for $f \in BMO_1$,*

$$\|f\|_{BMO_{p)}} \approx \|f\|_{BMO_1}. \tag{59}$$

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Numerical Methods for Fractional-Order Fornberg-Whitham Equations in the Sense of Atangana-Baleanu Derivative

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In this paper, we investigate the numerical solution of the Fornberg-Whitham equations with the help of two powerful techniques: the modified decomposition technique and the modified variational iteration technique involving fractional-order derivatives with Mittag-Leffler kernel. To confirm and illustrate the accuracy of the proposed approach, we evaluated in terms of fractional order the projected models. Furthermore, the physical attitude of the results obtained has been acquired for the fractional-order different value graphs. The results demonstrated that the future method is easy to implement, highly methodical, and very effective in analyzing the behavior of complicated fractional-order linear and nonlinear differential equations existing in the related areas of applied science.

1. Introduction

The analysis of the Fornberg-Whitham equation (FWE) is a significant mathematical equation of mathematical physics. The Fornberg-Whitham equation is defined as [1, 2]

$$D_{\mathfrak{Z}}\mu - D_{\zeta\zeta\mathfrak{Z}}\mu + D_{\zeta}\mu = \mu D_{\zeta\zeta\zeta}\mu - \mu D_{\zeta}\mu + 3D_{\zeta}\mu D_{\zeta\zeta}\mu. \quad (1)$$

This model was invented to evaluate the nonlinear breaking dispersive ocean waves. The Fornberg-Whitham equation is shown to yield peakon solutions as a physical equation for waves of restricting height and the occurrences of wave breaking. Fractional calculus is now widely used and accepted, owing to its well-known uses in a variety of fields of seemingly disparate sectors of science and engineering [3, 4]. Many scholars, including Gupta and Singh [5] and Alderremy et al. [6], have examined the fractional of the Fornberg-Whitham equation relevant to the fractional Caputo derivative, Sunthrayuth et al. [7], Singh et al. [8], etc. Because of the singular kernel of the fractional Caputo derivative, its implementations are limited. Caputo and Fabrizio [9] created derivatives of any (real or complex) order with a nonsingular kernel. Caputo and Fabrizio's derivative has been used to a variety of real-world situations, including fractional nonhomogeneous heat models [10], El Nino-Southern fractional oscillations models [11], and arbitrary-order system of smoking models [12]. Atangana and Baleanu [13] devised a novel fractional-order derivative called the Atangana-Baleanu (AB) fractional derivative, which has the kernel of a Mittag-Leffler-type function. Kumar et al. [14] investigated the regularised long-wave equation with a Mittag-Leffler-type kernel incorporating the fractional operator. A Mittag-Leffler-type kernel is used in the chemical kinetics system connected with a fractional derivative which was investigated by Singh et al. [15]. Baleanu et al. [16] recently proposed optimal fractional models with nonsingular Mittag-Leffler kernels. As we all know, the Mittag-Leffler function is more beneficial in expressing physical difficulties than the power function or the exponential function; as a result, the AB fractional derivative is well suited to unravelling material heterogeneities and structures or media with different scales.

George Adomian was introduced to and established a technique for "solving integro-differential, differential equations, delay differential, and partial differential equations" [17, 18]. The result is discovered as an infinite sequence that quickly converges to precise solution. This method has been shown to be effective in solving both linear and nonlinear models. The method for solving a nonlinear operator problem is to use decomposition equations in a series of functions. Each expression's sequence is derived from a polynomial derived from the expansion of an approximate solution into power sequences. The Adomian decomposition method technique is really simple in theory, but the difficulty arises when it comes to determining polynomials and illustrating the convergence of a series of functions [19]. Lesnic [20] analyzed the convergent of the Adomian decomposition method when using heat and wave models for both backward and forward time evolution. Gaber and El-Sayed used the Adomian method of solving fractal-order partial differential equations on a finite domain in [21]. Ghoreishi et al. [22] investigated the Adomian decomposition method's ability to investigate nonlinear wave problems with changing coefficients, demonstrating that the Adomian decomposition method can solve these equations without the need for dissertation, linearization, transformation, or perturbation.

The variational iteration approach [23, 24] was published in the late 1990s to solve a seepage flow with fractional derivatives and a nonlinear oscillator, and it has since been widely utilized as a primary analytical tool for solving a variety of nonlinear problems. It has fully grown into a fully fledged mathematical approach as a result of considerable research by a number of authors, including He [25, 26], Ganji and Sadighi [27], Ozis and Yildirim [28], and Noor and Mohyud-Din [29]. On November 24, 2018, we searched Clarivate's Web of Science for "variational iteration approach" and got 3761 hits. The technique's identification of the Lagrange multiplier necessitates the understanding of variational theory [30], and the technique's sophisticated identification process may limit its implementation to real-world issues.

2. Preliminary Concepts

Definition 1. The Caputo fractional derivative is given as [31]

$$ {}_0^C\mathscr{D}_{\mathfrak{I}}^{\gamma} f(\zeta,\mathfrak{I}) = \frac{1}{\Gamma(n-\gamma)}\int_0^{\mathfrak{I}} (\mathfrak{I}-\theta)^{n-\gamma-1} f^{(n)}(\zeta,\theta)\mathrm{d}\theta, \quad n-1<\gamma\le n. \tag{2} $$

Definition 2. The Laplace transformation connected with fractional Caputo derivative ${}^{LC}D_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\}$ is expressed by [31]

$$ \mathbb{L}\left[{}^{LC}D_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\}\right](s) = s^{\gamma}\mathbb{L}[f(x,\mathfrak{I})](s) - s^{\gamma-1}f(x,0). \tag{3} $$

Definition 3. In Caputo sense, the Atangana-Baleanu derivative is defined as [31]

$$ {}^{ABC}D_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\} = \frac{A(\gamma)}{1-\gamma}\int_a^{\mathfrak{I}} f'(k)E_{\gamma}\left[-\frac{\gamma}{1-\gamma}(1-k)^{\gamma}\right]dk, \tag{4} $$

where $A(\gamma)$ is a normalization function such that $A(0)=A(1)=1, f\in H^1(a,b), b>a, \gamma\in[0,1]$, and E_{γ} represent the Mittag-Leffler function.

Definition 4. The Atangana-Baleanu derivative in the Riemann-Liouville sense is defined as [31]

$$ {}^{ABC}D_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\} = \frac{A(\gamma)}{1-\gamma}\frac{d}{d\mathfrak{I}}\int_a^{\mathfrak{I}} f(k)E_{\gamma}\left[-\frac{\gamma}{1-\gamma}(1-k)^{\gamma}\right]dk. \tag{5} $$

Definition 5. The Laplace transform connected with the Atangana-Baleanu operator is defined as [31]

$$ {}^{AB}D_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\}(s) = \frac{A(\gamma)s^{\gamma}\mathbb{L}\{f(\mathfrak{I})\}(s) - s^{\gamma-1}f(0)}{(1-\gamma)(s^{\gamma}+(\gamma/(1-\gamma)))}. \tag{6} $$

Definition 6. Consider $0<\gamma<1$, and f is a function of γ; then, the fractional-order integral operator of γ is given as [31]

$$ {}^{ABC}I_{\mathfrak{I}}^{\gamma}\{f(\mathfrak{I})\} = \frac{1-\gamma}{A(\gamma)}f(\mathfrak{I}) + \frac{\gamma}{A(\gamma)\Gamma(\gamma)}\int_a^{\mathfrak{I}} f(k)(\mathfrak{I}-k)^{\gamma-1}dk. \tag{7} $$

3. The Methodology of Variational Iteration Method

This section introduces the solution of fractional partial differential equations with the help of the variational iteration method.

$$ {}^{ABC}D_{\mathfrak{I}}^{\gamma}\nu(\zeta,\mathfrak{I}) + \bar{\mathscr{G}}(\zeta,\mathfrak{I}) + \mathscr{N}(\zeta,\mathfrak{I}) - \mathscr{P}(\zeta,\mathfrak{I}) = 0, \quad \phi-1<\gamma\le\phi. \tag{8} $$

The initial condition is

$$ \nu(\zeta,0) = g(\zeta), \tag{9} $$

where ${}^{ABC}D_{\mathfrak{I}}^{\gamma} = \partial^{\gamma}/\partial\mathfrak{I}^{\gamma}$ is the fractional derivative Caputo order γ, $\bar{\mathscr{G}}$ and $\mathscr{N}$ are linear and nonlinear terms, respectively, and $\mathscr{P}$ is the source function.

The Laplace transformation is applied to equation (8); we get

$$ L\left[{}^{ABC}D_{\mathfrak{I}}^{\gamma}\nu(\zeta,\mathfrak{I})\right] + L\left[\bar{\mathscr{G}}(\zeta,\mathfrak{I}) + \mathscr{N}(\zeta,\mathfrak{I}) - \mathscr{P}(\zeta,\mathfrak{I})\right] = 0. \tag{10} $$

The Lagrange multiplier iterative method is

$$L\left[{}^{ABC}D_{\mathfrak{T}}^{\gamma}\nu(\zeta,\mathfrak{T})\right]+L\left[\bar{\mathscr{G}}(\zeta,\mathfrak{T})+\mathscr{N}(\zeta,\mathfrak{T})-\mathscr{P}(\zeta,\mathfrak{T})\right]=0. \tag{11}$$

A Lagrange multiplier is as

$$\lambda(s)=-\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}. \tag{12}$$

Applying inverse Laplace transform L^{-1}, equation (11) can be written as

$$\nu_{\phi+1}(\zeta,\mathfrak{T})=\nu_{\phi}(\zeta,\mathfrak{T})-L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}\left[-L\left\{\bar{\mathscr{G}}(\zeta,\mathfrak{T})+\mathscr{N}(\zeta,\mathfrak{T})\right\}\right]-L[\mathscr{P}(\zeta,\mathfrak{T})]\right]. \tag{13}$$

4. The Conceptualization of MDM

In this section, we discuss the solution of fractional partial differential equations with the help of the modified decomposition method.

$${}^{ABC}D_{\mathfrak{T}}^{\gamma}\nu(\zeta,\mathfrak{T})+\bar{\mathscr{G}}(\zeta,\mathfrak{T})+\mathscr{N}(\zeta,\mathfrak{T})-\mathscr{P}(\zeta,\mathfrak{T})=0,\quad m-1<\gamma\le m. \tag{14}$$

The initial condition is

$$\nu(\zeta,0)=g(\zeta), \tag{15}$$

where ${}^{ABC}D_{\mathfrak{T}}^{\gamma}=\partial^{\gamma}/\partial\mathfrak{T}^{\gamma}$ is the fractional derivative of Caputo order γ, $\bar{\mathscr{G}}$ and $\mathscr{N}$ are linear and nonlinear terms, respectively, and $\mathscr{P}$ is the source term.

Using Laplace transformation to equation (14), we get

$$L\left[{}^{ABC}D_{\mathfrak{T}}^{\gamma}\nu(\zeta,\mathfrak{T})\right]+L\left[\bar{\mathscr{G}}(\zeta,\mathfrak{T})+\mathscr{N}(\zeta,\mathfrak{T})-\mathscr{P}(\zeta,\mathfrak{T})\right]=0. \tag{16}$$

Taking the Laplace transform of differentiation property, we have

$$L[\nu(\zeta,\mathfrak{T})]=\frac{1}{s}\nu(\zeta,0)+\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L[\mathscr{P}(\zeta,\mathfrak{T})]-\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left\{\bar{\mathscr{G}}(\zeta,\mathfrak{T})+\mathscr{N}(\zeta,\mathfrak{T})\right\}\right]. \tag{17}$$

MDM result of infinite series $\nu(\zeta,\mathfrak{T})$,

$$\nu(\zeta,\mathfrak{T})=\sum_{\phi=0}^{\infty}\nu_{\phi}(\zeta,\mathfrak{T}). \tag{18}$$

$\mathscr{N}$ nonlinear function is defined as

$$\mathscr{N}(\zeta,\mathfrak{T})=\sum_{\phi=0}^{\infty}\mathscr{A}_{\phi}. \tag{19}$$

The nonlinear terms can be analyzed with the aid of Adomian polynomials. So the Adomian polynomial formula is expressed as

$$\mathscr{A}_{\phi}=\frac{1}{j!}\left[\frac{\partial^{\phi}}{\partial\lambda^{\phi}}\left\{\mathscr{N}\left(\sum_{\phi=0}^{\infty}\lambda^{\phi}\nu_{\phi}\right)\right\}\right]_{\lambda=0}. \tag{20}$$

Then, put equations (18) and (19) into (17), which gives

$$L\left[\sum_{\phi=0}^{\infty}\nu_{\phi}(\zeta,\mathfrak{T})\right]=\frac{1}{s}\nu(\zeta,0)+\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\{\mathscr{P}(\zeta,\mathfrak{T})\}-\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left\{\bar{\mathscr{G}}\left(\sum_{\phi=0}^{\infty}\nu_{\phi}\right)+\sum_{\phi=0}^{\infty}\mathscr{A}_{\phi}\right\}. \tag{21}$$

Applying the inverse Laplace transformation to equation (21), we get

$$\sum_{\phi=0}^{\infty}\nu_{\phi}(\zeta,\mathfrak{T})=L^{-1}\left[\frac{1}{s}\nu(\zeta,0)+\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\{\mathscr{P}(\zeta,\mathfrak{T})\}-\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left\{\bar{\mathscr{G}}\left(\sum_{\phi=0}^{\infty}\nu_{\phi}\right)+\sum_{\phi=0}^{\infty}\mathscr{A}_{\phi}\right\}\right]. \tag{22}$$

Define the terms as follows:

$$\begin{aligned}\nu_0(\zeta,\mathfrak{T})&=L^{-1}\left[\frac{1}{s}\nu(\zeta,0)+\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\{\mathscr{P}(\zeta,\mathfrak{T})\}\right],\\ \nu_1(\zeta,\mathfrak{T})&=-L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left\{\bar{\mathscr{G}}_1(\nu_0)+\mathscr{A}_0\right\}\right].\end{aligned} \tag{23}$$

In general, $\phi\ge 1$ is defined as

$$\nu_{\phi+1}(\zeta,\mathfrak{T})=-L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left\{\bar{\mathscr{G}}\left(\nu_{\phi}\right)+\mathscr{A}_{\phi}\right\}\right]. \tag{24}$$

5. Application of Techniques

Example 7. Consider the time-fractional nonlinear Fornberg-Whitham equation

$$D_{\mathfrak{T}}^{\gamma}\nu-D_{\zeta\zeta\mathfrak{T}}\nu+D_{\zeta}\nu=\nu D_{\zeta\zeta\zeta}\nu-\nu D_{\zeta}\nu+3D_{\zeta}\nu D_{\zeta\zeta}\nu,\quad 0<\gamma\le 1, \tag{25}$$

with the initial condition

$$\nu(\zeta, 0) = e^{(\zeta/2)}. \tag{26}$$

Taking Laplace transformation of (25),

$$\frac{s^{\gamma}}{(s^{\gamma}(1-\gamma)+\gamma)}\left\{L[\nu(\zeta,\mathfrak{T})]-\frac{1}{s}\nu(\zeta,0)\right\} = L\left[D_{\zeta\zeta\mathfrak{T}}\nu - D_{\zeta}\nu + \nu D_{\zeta\zeta\zeta}\nu - \nu D_{\zeta}\nu + 3D_{\zeta}\nu D_{\zeta\zeta}\nu\right]. \tag{27}$$

Using inverse Laplace transformation

$$\nu(\zeta,\mathfrak{T}) = L^{-1}\left[\frac{\nu(\zeta,0)}{s} - \frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left[D_{\zeta\zeta\mathfrak{T}}\nu - D_{\zeta}\nu + \nu D_{\zeta\zeta\zeta}\nu - \nu D_{\zeta}\nu + 3D_{\zeta}\nu D_{\zeta\zeta}\nu\right]\right]. \tag{28}$$

Applying Adomian procedure, we have

$$\nu_0(\zeta,\mathfrak{T}) = L^{-1}\left[\frac{\nu(\zeta,0)}{s}\right] = L^{-1}\left[\frac{e^{(\zeta/2)}}{s}\right],$$

$$\nu_0(\zeta,\mathfrak{T}) = e^{(\zeta/2)},$$

$$\begin{aligned}
\sum_{\phi=0}^{\infty}\nu_{\phi+1}(\zeta,\mathfrak{T}) &= L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left[\sum_{\phi=0}^{\infty}\left(D_{\zeta\zeta\mathfrak{T}}\nu\right)_{\phi} - \sum_{\phi=0}^{\infty}\left(D_{\zeta}\nu\right)_{\phi} + \sum_{\phi=0}^{\infty}A_{\phi} - \sum_{\phi=0}^{\infty}B_{\phi} + 3\sum_{\phi=0}^{\infty}C_{\phi}\right]\right], \quad \phi = 0, 1, 2, \cdots,\\
A_0\left(\nu D_{\zeta\zeta\zeta}\nu\right) &= \nu_0 D_{\zeta\zeta\zeta}\nu_0,\\
B_0\left(\nu D_{\zeta}\nu\right) &= \nu_0 D_{\zeta}\nu_0,\\
A_1\left(\nu D_{\zeta\zeta\zeta}\nu\right) &= \nu_0 D_{\zeta\zeta\zeta}\nu_1 + \nu_1 D_{\zeta\zeta\zeta}\nu_0,\\
B_1\left(\nu D_{\zeta}\nu\right) &= \nu_0 D_{\zeta}\nu_1 + \nu_1 D_{\zeta}\nu_0,\\
A_2\left(\nu D_{\zeta\zeta\zeta}\nu\right) &= \nu_1 D_{\zeta\zeta\zeta}\nu_2 + \nu_1 D_{\zeta\zeta\zeta}\nu_1 + \nu_2 D_{\zeta\zeta\zeta}\nu_0,\\
B_2\left(\nu D_{\zeta}\nu\right) &= \nu_1 D_{\zeta}\nu_2 + \nu_1 D_{\zeta}\nu_1 + \nu_2 D_{\zeta}\nu_0,\\
C_0\left(D_{\zeta}\nu D_{\zeta\zeta}\nu\right) &= D_{\zeta}\nu_0 D_{\zeta\zeta}\nu_0,\\
C_1\left(D_{\zeta}\nu D_{\zeta\zeta}\nu\right) &= D_{\zeta}\nu_0 D_{\zeta\zeta}\nu_1 + D_{\zeta}\nu_1 D_{\zeta\zeta}\nu_0,\\
C_2\left(D_{\zeta}\nu D_{\zeta\zeta}\nu\right) &= D_{\zeta}\nu_1 D_{\zeta\zeta}\nu_2 + D_{\zeta}\nu_1 D_{\zeta\zeta}\nu_1 + D_{\zeta}\nu_2 D_{\zeta\zeta}\nu_0,
\end{aligned} \tag{29}$$

for $\phi = 1$,

$$\nu_1(\zeta,\mathfrak{T}) = L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left[D_{\zeta\zeta\mathfrak{T}}\nu_0 - D_{\zeta}\nu_0 + A_0 - B_0 + 3C_0\right]\right] = -\frac{1}{2}e^{(\zeta/2)}\left((1-\gamma) + \frac{\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right), \tag{30}$$

for $\phi = 2$,

$$\begin{aligned}
\nu_2(\zeta,\mathfrak{T}) &= L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left[D_{\zeta\zeta\mathfrak{T}}\nu_1 - D_{\zeta}\nu_1 + A_1 - B_1 + 3C_1\right]\right],\\
\nu_2(\zeta,\mathfrak{T}) &= -\frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{T}^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{1}{4}e^{(\zeta/2)}\left((1-\gamma)^2 + \frac{\gamma^2\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2(1-\gamma)\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right),
\end{aligned} \tag{31}$$

for $\phi = 3$,

$$\begin{aligned}
\nu_3(\zeta,\mathfrak{T}) &= L^{-1}\left[\frac{(s^{\gamma}(1-\gamma)+\gamma)}{s^{\gamma}}L\left[D_{\zeta\zeta\mathfrak{T}}\nu_2 - D_{\zeta}\nu_2 + A_2 - B_2 + 3C_2\right]\right],\\
\nu_3(\zeta,\mathfrak{T}) &= -\frac{1}{32}e^{(\zeta/2)}\frac{\mathfrak{T}^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{T}^{3\gamma-1}}{\Gamma(3\gamma)}\\
&\quad - \frac{1}{8}e^{(\zeta/2)}\left\{(1-\gamma)^3 + \gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)} + \frac{3\gamma^2(1-\gamma)\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{T}^{3\gamma}}{\Gamma(3\gamma+1)}\right\}.
\end{aligned} \tag{32}$$

The modified decomposition method solution of example (1) is

$$\begin{aligned}
\nu(\zeta,\mathfrak{T}) &= e^{(\zeta/2)} - \frac{1}{2}e^{(\zeta/2)}\left((1-\gamma) + \frac{\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right) - \frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{T}^{2\gamma-1}}{\Gamma(2\gamma)}\\
&\quad + \frac{1}{4}e^{(\zeta/2)}\left((1-\gamma)^2 + \frac{\gamma^2\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2(1-\gamma)\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right)\\
&\quad - \frac{1}{32}e^{(\zeta/2)}\frac{\mathfrak{T}^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8}e^{(\zeta/2)}\frac{\gamma^{3\gamma-1}}{\Gamma(3\gamma)} - \frac{1}{8}e^{(\zeta/2)}\\
&\quad \cdot\left\{(1-\gamma)^3 + \gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)} + \frac{3\gamma^2(1-\gamma)\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{T}^{3\gamma}}{\Gamma(3\gamma+1)}\right\} - \cdots.
\end{aligned} \tag{33}$$

The simplification of equation (33)

$$\begin{aligned}
\nu(\zeta,\mathfrak{T}) &= e^{(\zeta/2)}\left[1 - 2\left[(1-\gamma) + \frac{\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right] - \frac{1}{8}\frac{\mathfrak{T}^{2\gamma-1}}{\Gamma(2\gamma)}\right.\\
&\quad + \frac{1}{4}\left((1-\gamma)^2 + \frac{\gamma^2\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2(1-\gamma)\gamma\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right)\\
&\quad - \frac{1}{32}\frac{\mathfrak{T}^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8}\frac{\mathfrak{T}^{3\gamma-1}}{\Gamma(3\gamma)}\\
&\quad - \frac{1}{8}\left\{(1-\gamma)^3 + \gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{T}^{\gamma}}{\Gamma(\gamma+1)}\right.\\
&\quad \left.\left. + \frac{3\gamma^2(1-\gamma)\mathfrak{T}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{T}^{3\gamma}}{\Gamma(3\gamma+1)}\right\} + \cdots\right].
\end{aligned} \tag{34}$$

Apply the variational method to obtain series form solution. The iteration formulas for equation (25), we get

$$\nu_{\phi+1}(\zeta,\mathfrak{I})=\nu_j(\zeta,\mathfrak{I})-L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_\phi+D_\zeta\nu_\phi-\nu_\phi D_{\zeta\zeta\zeta}\nu_\phi+\nu_\phi D_\zeta\nu_\phi-3D_\zeta\nu_\phi D_{\zeta\zeta}\nu_\phi\right\}\right], \tag{35}$$

where

$$\nu_0(\zeta,\mathfrak{I})=e^{(\zeta/2)}. \tag{36}$$

For $\phi=0,1,2,\cdots$,

$$\nu_1(\zeta,\mathfrak{I})=\nu_0(\zeta,\mathfrak{I})-L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_0+D_\zeta\nu_0-\nu_0 D_{\zeta\zeta\zeta}\nu_0+\nu_0 D_\zeta\nu_0-3D_\zeta\nu_0 D_{\zeta\zeta}\nu_0\right\}\right],$$

$$\nu_1(\zeta,\mathfrak{I})=e^{(\zeta/2)}-\frac{1}{2}e^{(\zeta/2)}\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right),$$

$$\nu_2(\zeta,\mathfrak{I})=\nu_1(\zeta,\mathfrak{I})-L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_1+D_\zeta\nu_1-\nu_1 D_{\zeta\zeta\zeta}\nu_1+\nu_1 D_\zeta\nu_1-3D_\zeta\nu_1 D_{\zeta\zeta}\nu_1\right\}\right],$$

$$\nu_2(\zeta,\mathfrak{I})=e^{(\zeta/2)}-\frac{1}{2}e^{(\zeta/2)}\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right)-\frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{I}^{2\gamma-1}}{\Gamma(2\gamma)}+\frac{1}{4}e^{(\zeta/2)}\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right),$$

$$\nu_3(\zeta,\mathfrak{I})=\nu_2(\zeta,\mathfrak{I})-L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_2+D_\zeta\nu_2-\nu_2 D_{\zeta\zeta\zeta}\nu_2+\nu_2 D_\zeta\nu_2-3D_\zeta\nu_2 D_{\zeta\zeta}\nu_2\right\}\right],$$

$$\begin{aligned}\nu_3(\zeta,\mathfrak{I})&=e^{(\zeta/2)}-\frac{1}{2}e^{(\zeta/2)}\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right)-\frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{I}^{2\gamma-1}}{\Gamma(2\gamma)}\\&+\frac{1}{4}e^{(\zeta/2)}\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right)\\&-\frac{1}{32}e^{(\zeta/2)}\frac{\mathfrak{I}^{3\gamma-2}}{\Gamma(3\gamma-1)}+\frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{I}^{3\gamma-1}}{\Gamma(3\gamma)}-\frac{1}{8}e^{(\zeta/2)}\\&\cdot\left\{(1-\gamma)^3+\gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}+\frac{3\gamma^2(1-\gamma)\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{I}^{3\gamma}}{\Gamma(3\gamma+1)}\right\},\end{aligned}$$

$$\begin{aligned}\nu(\zeta,\mathfrak{I})&=e^{(\zeta/2)}-\frac{1}{2}e^{(\zeta/2)}\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right)-\frac{1}{8}e^{(\zeta/2)}\frac{\mathfrak{I}^{2\gamma-1}}{\Gamma(2\gamma)}\\&+\frac{1}{4}e^{(\zeta/2)}\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right)\\&-\frac{1}{32}e^{(\zeta/2)}\frac{\mathfrak{I}^{3\gamma-2}}{\Gamma(3\gamma-1)}+\frac{1}{8}e^{(\zeta/2)}\frac{\gamma^{3\gamma-1}}{\Gamma(3\gamma)}-\frac{1}{8}e^{(\zeta/2)}\\&\cdot\left\{(1-\gamma)^3+\gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}+\frac{3\gamma^2(1-\gamma)\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{I}^{3\gamma}}{\Gamma(3\gamma+1)}\right\}-\cdots.\end{aligned} \tag{37}$$

The exact solution of equation (25) at $\gamma=1$,

$$\nu(\zeta,\mathfrak{I})=e^{((\zeta/2)-(2\mathfrak{I}/3))}. \tag{38}$$

In Figure 1, the analytical results of MDM/MVITM example 1 graphs show close contact with each other at $\gamma=1$ and 0.8. It is investigated that analytical results are in close relation with the actual results of example 1. In Figure 2, the results of example 1 at different fractional-order of the derivative are plotted at $\gamma=0.6$ and 0.4. Figure 3 shows the different fractional of two and three dimensional. The graphical representation has shown the convergence phenomena of fractional-order results towards the result at integer-order of example 1.

Example 8. Consider the time-fractional nonlinear Fornberg-Whitham equation is given as

$$D_{\mathfrak{I}}^\gamma\nu-D_{\zeta\zeta\mathfrak{I}}\nu+D_\zeta\nu=\nu D_{\zeta\zeta\zeta}\nu-\nu D_\zeta\nu+3D_\zeta\nu D_{\zeta\zeta}\nu,\quad \mathfrak{I}>0,\ 0<\gamma\le 1. \tag{39}$$

The initial condition is

$$\nu(\zeta,0)=\cosh^2\left(\frac{\zeta}{4}\right). \tag{40}$$

Applying Laplace transformation of (39), we get

$$\frac{s^\gamma}{(s^\gamma(1-\gamma)+\gamma)}\left\{L[\nu(\zeta,\mathfrak{I})]-\frac{1}{s}\nu(\zeta,0)\right\}=L\left[D_{\zeta\zeta\mathfrak{I}}\nu-D_\zeta\nu+\nu D_{\zeta\zeta\zeta}\nu-\nu D_\zeta\nu+3D_\zeta\nu D_{\zeta\zeta}\nu\right]. \tag{41}$$

Using inverse Laplace transformation,

$$\nu(\zeta,\mathfrak{I})=L^{-1}\left[\frac{\nu(\zeta,0)}{s}-\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu-D_\zeta\nu+\nu D_{\zeta\zeta\zeta}\nu-\nu D_\zeta\nu+3D_\zeta\nu D_{\zeta\zeta}\nu\right\}\right]. \tag{42}$$

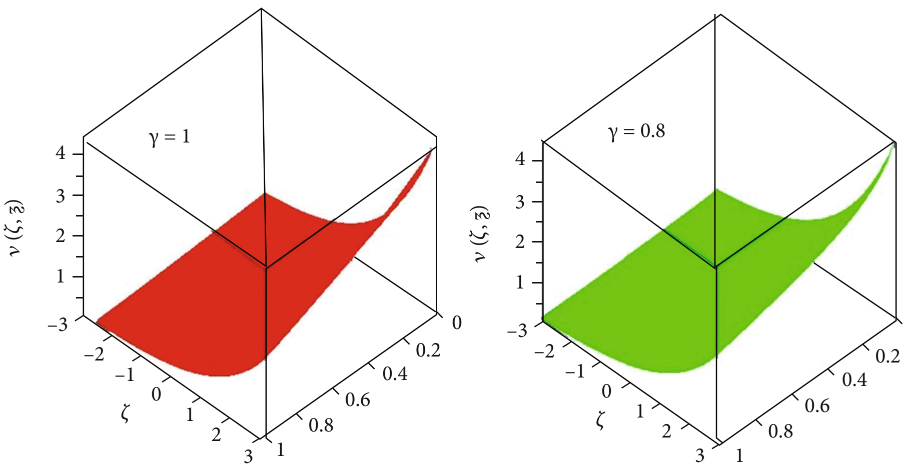

Figure 1: The solution graph of MDM/MVITM at $\gamma = 1$ and 0.8 of example 1.

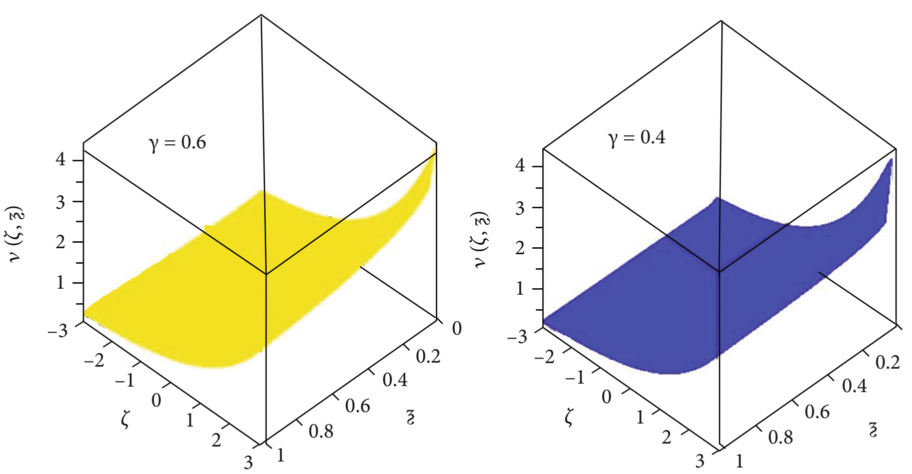

Figure 2: The solutions graph of MDM/MVITM at $\gamma = 0.6$ and 0.4 of example 1.

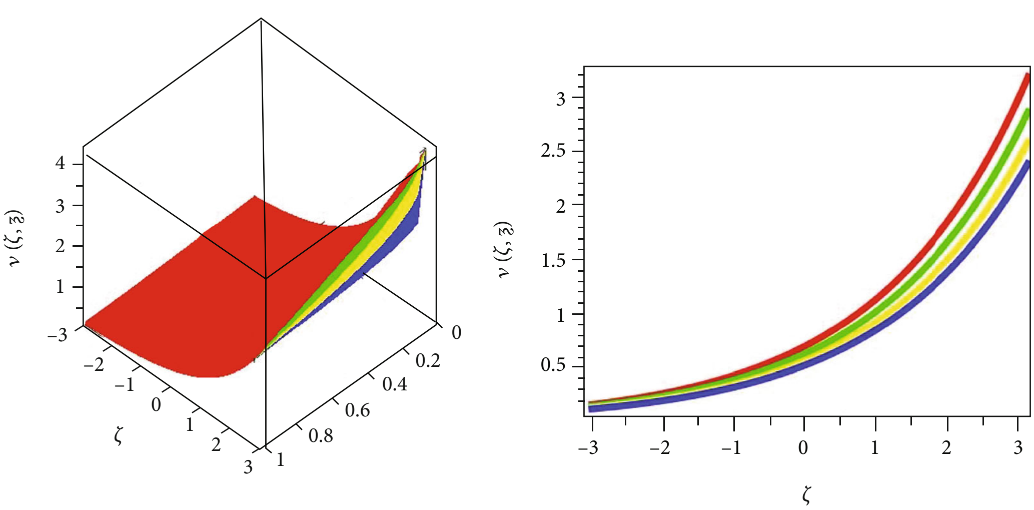

Figure 3: The different fractional-order graph of MDM/MVITM of example 1.

Using ADM procedure, we get

$$\nu_0(\zeta,\mathfrak{I}) = L^{-1}\left[\frac{\nu(\zeta,0)}{s}\right] = L^{-1}\left[\frac{\exp\left(\cosh^2(\zeta/4)\right)}{s}\right],$$

$$\nu_0(\zeta,\mathfrak{I}) = \cosh^2\left(\frac{\zeta}{4}\right),$$

$$\sum_{\phi=0}^{\infty}\nu_{\phi+1}(\zeta,\mathfrak{I}) = L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left[\sum_{\phi=0}^{\infty}\left(D_{\zeta\zeta\mathfrak{I}}\nu\right)_\phi - \sum_{\phi=0}^{\infty}\left(D_\zeta\nu\right)_\phi + \sum_{\phi=0}^{\infty}A_\phi - \sum_{\phi=0}^{\infty}B_\phi + 3\sum_{\phi=0}^{\infty}C_\phi\right]\right],\quad \phi=0,1,2,\cdots, \tag{43}$$

for $\phi=0$,

$$\nu_1(\zeta,\mathfrak{I}) = L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left[D_{\zeta\zeta\mathfrak{I}}\nu_0 - D_\zeta\nu_0 + A_0 - B_0 + 3C_0\right]\right] = -\frac{11}{32}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right), \tag{44}$$

for $\phi=1$,

$$\nu_2(\zeta,\mathfrak{I}) = L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left[D_{\zeta\zeta\mathfrak{I}}\nu_1 - D_\zeta\nu_1 + A_1 - B_1 + 3C_1\right]\right], \nu_2(\zeta,\mathfrak{I}) = -\frac{11}{28}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) + \frac{121}{1024}\cosh\left(\frac{\zeta}{4}\right)\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right), \tag{45}$$

for $\phi=2$,

$$\nu_3(\zeta,\mathfrak{I}) = L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left[D_{\zeta\zeta\mathfrak{I}}\nu_2 - D_\zeta\nu_2 + A_2 - B_2 + 3C_2\right]\right], \nu_3(\zeta,\mathfrak{I}) = -\frac{11}{512}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) + \frac{121}{2048}\cosh\left(\frac{\zeta}{4}\right)\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) - \frac{1331}{49152}\sinh\left(\frac{\zeta}{4}\right)\left\{(1-\gamma)^3+\gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{I}^\gamma}{\Gamma(\gamma+1)} + \frac{3\gamma^2(1-\gamma)\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{I}^{3\gamma}}{\Gamma(3\gamma+1)}\right\}. \tag{46}$$

The MDM solution of example (8) is

$$\nu(\zeta,\mathfrak{I}) = \nu_0(\zeta,\mathfrak{I})+\nu_1(\zeta,\mathfrak{I})+\nu_2(\zeta,\mathfrak{I})+\nu_3(\zeta,\mathfrak{I})+\nu_4(\zeta,\mathfrak{I})+\cdots,$$

$$\nu(\zeta,\mathfrak{I}) = \cosh^2\left(\frac{\zeta}{4}\right) - \frac{11}{32}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) - \frac{11}{28}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) + \frac{121}{1024}\cosh\left(\frac{\zeta}{4}\right)\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) - \frac{11}{512}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) + \frac{121}{2048}\cosh\left(\frac{\zeta}{4}\right)\left((1-\gamma)^2+\frac{\gamma^2\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{2(1-\gamma)\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right) - \frac{1331}{49152}\sinh\left(\frac{\zeta}{4}\right)\left\{(1-\gamma)^3+\gamma(1-\gamma)\left(1+\gamma+2\gamma^2\right)\frac{\mathfrak{I}^\gamma}{\Gamma(\gamma+1)} + \frac{3\gamma^2(1-\gamma)\mathfrak{I}^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{I}^{3\gamma}}{\Gamma(3\gamma+1)}\right\}\cdots. \tag{47}$$

Apply the variational method to find the analytical solution.

The iteration formulas for equation (39), we get

$$\nu_{\phi+1}(\zeta,\mathfrak{I}) = \nu_j(\zeta,\mathfrak{I}) - L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_\phi + D_\zeta\nu_\phi - \nu_\phi D_{\zeta\zeta\zeta}\nu_\phi + \nu_\phi D_\zeta\nu_\phi - 3D_\zeta\nu_\phi D_{\zeta\zeta}\nu_\phi\right\}\right], \tag{48}$$

where

$$\nu_0(\zeta,\mathfrak{I}) = \cosh^2\left(\frac{\zeta}{4}\right), \tag{49}$$

for $\phi=0,1,2,\cdots$,

$$\nu_1(\zeta,\mathfrak{I}) = \nu_0(\zeta,\mathfrak{I}) - L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_0 + D_\zeta\nu_0 - \nu_0 D_{\zeta\zeta\zeta}\nu_0 + \nu_0 D_\zeta\nu_0 - 3D_\zeta\nu_0 D_{\zeta\zeta}\nu_0\right\}\right],$$

$$\nu_1(\zeta,\mathfrak{I}) = \cosh^2\left(\frac{\zeta}{4}\right) - \frac{11}{32}\sinh\left(\frac{\zeta}{4}\right)\left((1-\gamma)+\frac{\gamma\mathfrak{I}^\gamma}{\Gamma(\gamma+1)}\right),$$

$$\nu_2(\zeta,\mathfrak{I}) = \nu_1(\zeta,\mathfrak{I}) - L^{-1}\left[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\left\{D_{\zeta\zeta\mathfrak{I}}\nu_1 + D_\zeta\nu_1 - \nu_1 D_{\zeta\zeta\zeta}\nu_1 + \nu_1 D_\zeta\nu_1 - 3D_\zeta\nu_1 D_{\zeta\zeta}\nu_1\right\}\right], \tag{50}$$

$\nu_2(\zeta,\mathfrak{I}) = \cosh^2(\zeta/4) - 11/32\ \sinh\ (\zeta/4)((1-\gamma)+(\gamma\mathfrak{I}^\gamma/\Gamma(\gamma+1))) - 11/28\ \sinh\ (\zeta/4)((1-\gamma)+(\gamma\mathfrak{I}^\gamma/\Gamma(\gamma+1))) + 121/1024\ \cosh\ (\zeta/4)\ ((1-\gamma)^2+(\gamma^2\mathfrak{I}^{2\gamma}/\Gamma\ (2\gamma+1)) + (2(1-\gamma)\gamma\mathfrak{I}^\gamma/\Gamma\ (\gamma+1)))$,

$$\nu_3(\zeta,\mathfrak{I}) = \nu_2$$

$(\zeta,\mathfrak{I}) - L^{-1}[\frac{(s^\gamma(1-\gamma)+\gamma)}{s^\gamma}L\ \{D_{\zeta\zeta\mathfrak{I}}\nu_2 + D_\zeta\nu_2 - \nu_2 D\ \zeta\zeta\zeta\nu_2 + \nu_2$

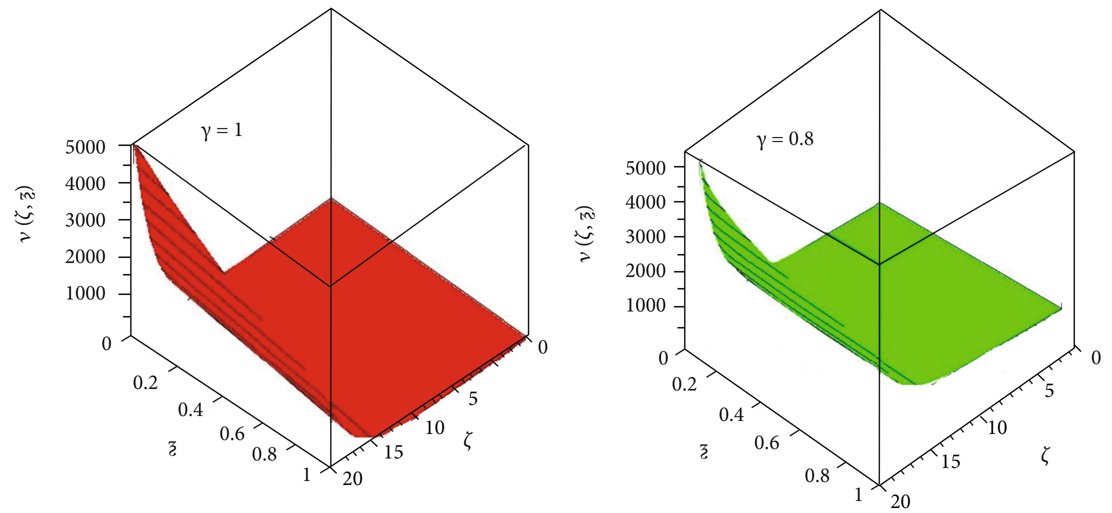

FIGURE 4: The solutions graph of MDM/MVITM at $\gamma = 1$ and 0.8 of example 2.

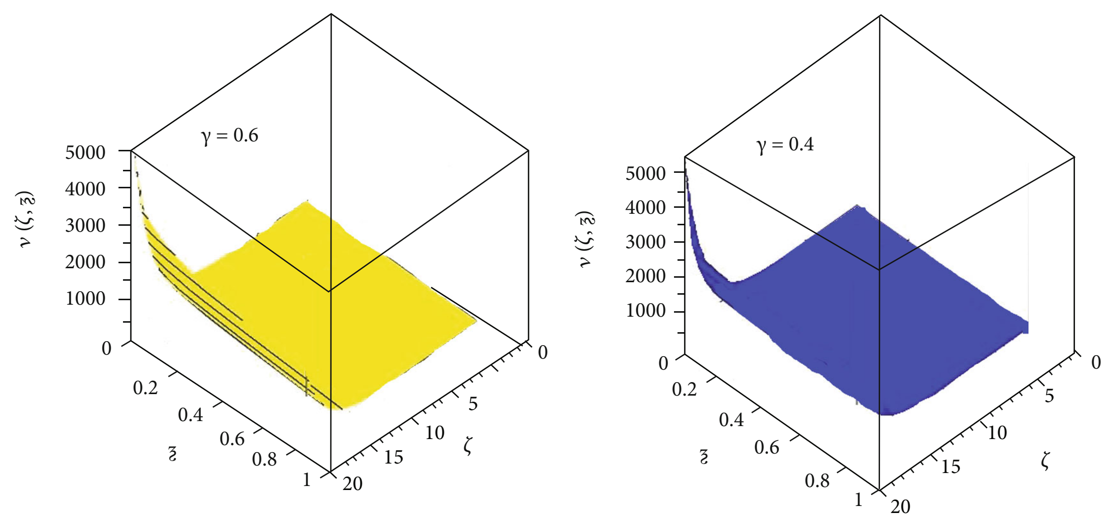

FIGURE 5: The solutions graph of MDM/MVITM at $\gamma = 0.6$ and 0.4 of example 2.

$D_\zeta v_2 - 3 \quad D_\zeta v_2 D_{\zeta\zeta} v_2\}],v_3 \quad (\zeta, \mathfrak{F}) = \cosh^2(\frac{\zeta}{4}) - \frac{11}{32}\sinh(\frac{\zeta}{4})((1 - \gamma) + \frac{\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) - \frac{11}{28}\sinh(\frac{\zeta}{4})((1-\gamma) + \frac{\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) + \frac{121}{1024}\cosh(\frac{\zeta}{4})((1-\gamma)^2 + \frac{\gamma^2\mathfrak{F}^{2\gamma}}{\Gamma}(2\gamma+1) + \frac{2(1-\gamma)\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) - \frac{11}{512}\sinh(\frac{\zeta}{4})((1-\gamma) + \frac{\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) + \frac{121}{2048}\cosh(\frac{\zeta}{4})((1-\gamma)^2 + \frac{\gamma^2\mathfrak{F}^{2\gamma}}{\Gamma}(2\gamma+1) + \frac{2(1-\gamma)\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) - \frac{1331}{49152}\sinh(\frac{\zeta}{4})\{(1-\gamma)^3 + \gamma(1-\gamma)(1+\gamma+2\gamma^2)\frac{\mathfrak{F}^\gamma}{\Gamma(\gamma+1)} + \frac{3\gamma^2(1-\gamma)\mathfrak{F}^{2\gamma}}{\Gamma(2\gamma+1)} + \gamma^3\Gamma(2\gamma+1)\frac{\mathfrak{F}^{3\gamma}}{\Gamma(3\gamma+1)\},v(\zeta,\mathfrak{F}) = \cosh^2(\frac{\zeta}{4}) - \frac{11}{32}\sinh(\frac{\zeta}{4})((1-\gamma) + \frac{\gamma\mathfrak{F}^\gamma}{\Gamma}}(\gamma+1)) - \frac{11}{28}\sinh(\frac{\zeta}{4})((1-\gamma) + \frac{\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) + \frac{121}{1024}\cosh(\frac{\zeta}{4})((1-\gamma)^2 + \frac{\gamma^2\mathfrak{F}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2(1-\gamma)\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) - \frac{11}{512}\sinh(\frac{\zeta}{4})((1-\gamma) + \gamma\mathfrak{F}\frac{\gamma}{\Gamma(\gamma+1)) + \frac{121}{2048}\cosh(\frac{\zeta}{4})((1-\gamma)^2} + \frac{\gamma^2\mathfrak{F}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2(1-\gamma)\gamma\mathfrak{F}^\gamma}{\Gamma(\gamma+1)}) - \frac{1331}{49152}\sinh(\frac{\zeta}{4})\{(1-\gamma)^3 + \gamma(1-\gamma)(1+\gamma+2\gamma^2)\frac{\mathfrak{F}^\gamma}{\Gamma}(\gamma+1) + \frac{3\gamma^2(1-\gamma)\mathfrak{F}^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\gamma^3\Gamma(2\gamma+1)\mathfrak{F}^{3\gamma}}{\Gamma(3\gamma+1)}\} + \cdots$. The exact solution of equation (39) at $\gamma = 1$,

$$v(\zeta, \mathfrak{F}) = \cosh^2\left(\frac{\zeta}{4} - \frac{11\mathfrak{F}}{24}\right). \quad (52)$$

In Figure 4, the analytical results of MDM/MVITM example 2 graphs show that close contact with each other at $\gamma = 1$ and 0.8. It is investigated that analytical results are in close relation with the actual results of example 2. In Figure 5, the results of example 2 at different fractional-order of the derivative are plotted at $\gamma = 0.6$ and 0.4. Figure 6 shows the different fractional of two and three dimensional. The graphical representation has shown the convergence phenomena of fractional-order results towards the result at integer-order of example 2.

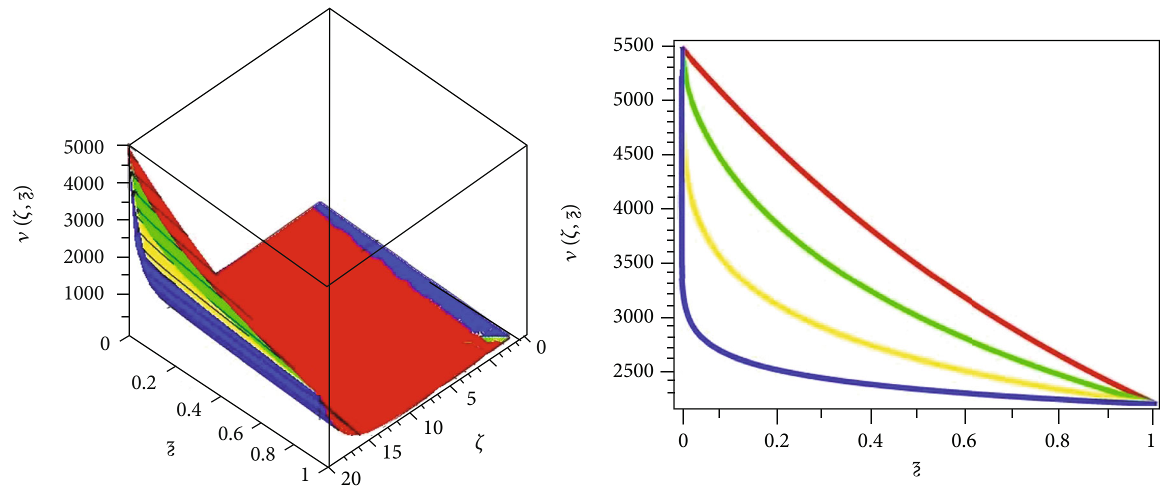

Figure 6: The different fractional-order graph of MDM/MVITM of example 2.

6. Conclusion

In this paper, we have been successfully applied two modified methods to investigate the approximate solutions of fractional Fornberg-Whitham equations. Agreement between numerical results obtained by the modified decomposition method and modified variational iteration method involving fractional-order derivatives with Mittag-Leffler kernel with exact result appears very appreciable by means of illustrative results in figures. The proposed techniques are easy to implement, effective, and suitable for achieving the results of nonlinear fractional Fornberg-Whitham equations. Moreover, both the modified decomposition method and variational iteration method provide the convergent series results with easily calculated components without applying any linearization, perturbation, or limiting assumptions. Finally, we can conclude the suggested methods are more accurate and highly methodical and which can be applied to investigate nonlinear models that arise in applied sciences.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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