Mathematical Problems in Engineering

Impulsive Differential Equations of Fractional Order

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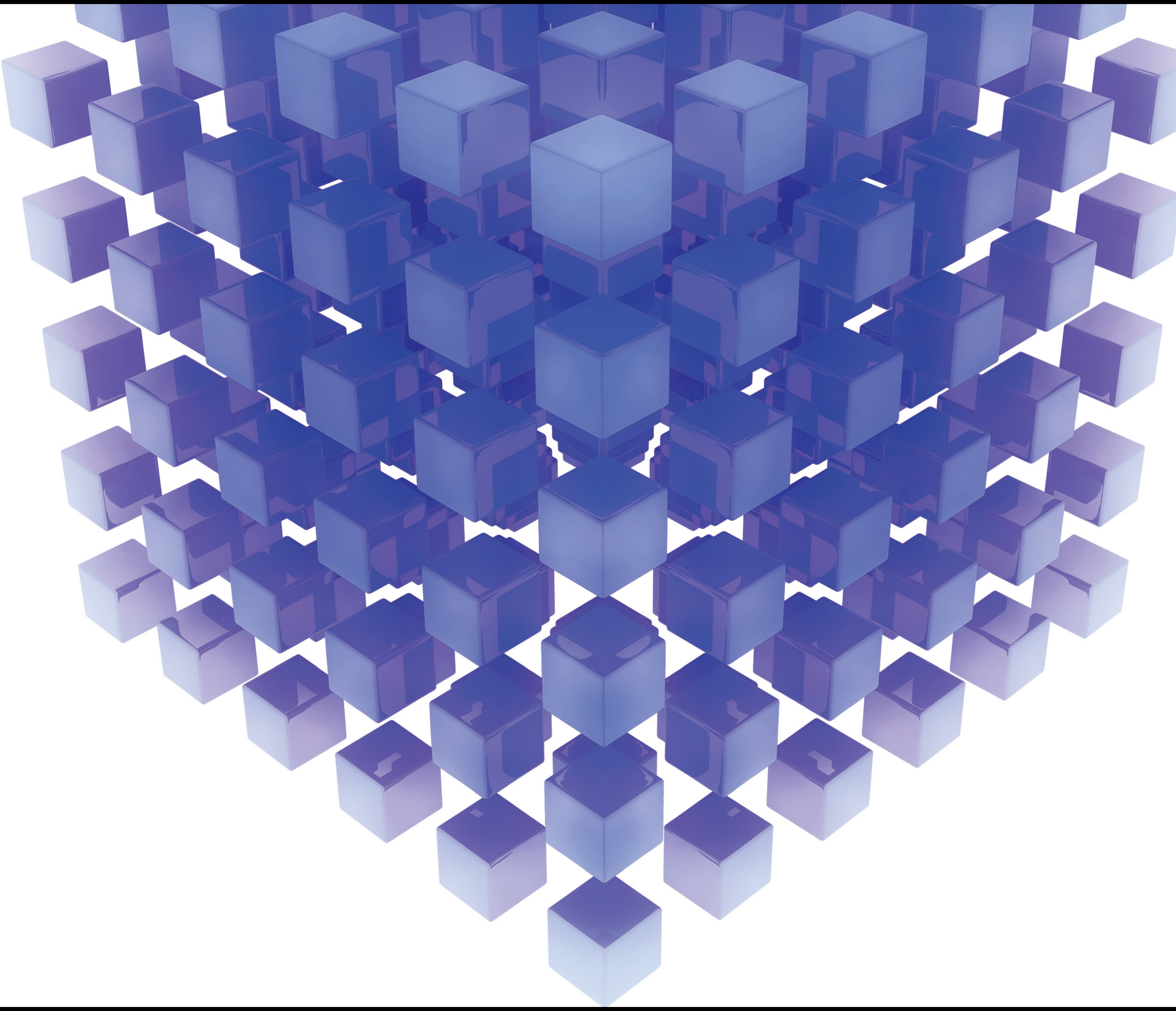
Guest Editors: José Francisco Gómez Aguilar and Kamal Shah



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Research Article

Displaying the Structure of the Solutions for Some Fifth-Order Systems of Recursive Equations

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This paper presents the solutions to the following nonlinear systems of rational difference equations: $x_{n+1} = ((x_{n-3}y_{n-4})/(y_n(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4})))$, $y_{n+1} = ((y_{n-3}x_{n-4})/(x_n(\pm 1 \pm y_{n-1}x_{n-2}y_{n-3}x_{n-4})))$ where initial conditions $x_{-\delta}, y_{-\delta}(\delta = 4, 3, \ldots, 0)$ are nonnegative real numbers. Finally some numerical simulations are presented to verify obtained theoretical results.

1. Introduction

The purpose of present study is to solve and deal with the following difference equations systems:

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)},$$
$$y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(\pm 1 \pm y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}, \quad (1)$$

where $x_{-\delta}, y_{-\delta}(\delta = 4, 3, \ldots, 0)$ are arbitrary nonnegative real numbers.

It is anticipated that discrete dynamical systems can be seen as discrete analogous of differential as well as delay differential equations. Moreover, these systems designate certain natural phenomena in economy, physics, biology, and many more. Many scholars and researchers have studied various dynamical properties of difference equations along their systems in recent years. For example, Asiri et al. [1] have investigated the periodicity nature of the following system:

$$x_{n+1} = \frac{y_{n-2}}{-y_{n-2}x_{n-1}y_n + 1},$$
$$y_{n+1} = \frac{x_{n-2}}{\pm x_{n-2}y_{n-1}x_n \pm 1}. \quad (2)$$

Cinar et al. [2] have obtained the solution of the subsequent recursive system:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1},$$
$$y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad (3)$$
$$z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

Metwally and Elsayed [3] have explored the periodicity nature of the following system:

$$x_{n+1} = \frac{y_{n-2}}{-y_{n-2}x_{n-1}y_n - 1},$$
$$y_{n+1} = \frac{x_{n-2}}{\pm x_{n-2}y_{n-1}x_n \pm 1}. \quad (4)$$

Touafek et al. [4] have explored the periodicity nature of the following system:

$$x_{n+1} = \frac{x_{n-3}}{\pm x_{n-3}y_{n-1} \pm 1}, \quad y_{n+1} = \frac{y_{n-3}}{\pm y_{n-3}x_{n-1} \pm 1}. \tag{5}$$

Elsayed [5] has obtained the solution of the following rational system:

$$x_{n+1} = \frac{x_{n-1}}{\pm x_{n-1}y_n \pm 1}, \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n \mp 1}. \tag{6}$$

For more interesting results regarding dynamical properties of difference and differential equations along their systems, we refer the readers to [2, 6–28] and the references cited therein.

2. On 1st System

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)}, \quad y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(1 + y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}. \tag{7}$$

This section is about the investigation of the solution form of the following system:

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)}, \quad y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(1 + y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}. \tag{8}$$

The forms of solutions to (8) are given as Theorem 1.

Theorem 1. *Let $\{x_n, y_n\}$ be a solution to (8), and also let $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, respectively, be $a, b, c, d, e, f, g, h, k, l$. Then one has*

$$x_{4n-4} = \frac{e^n\prod_{\delta=0}^{-1+n}[2\delta acgk + 1]}{a^{n-1}\prod_{\delta=0}^{-1+n}[2\delta cegk + 1]},$$

$$x_{4n-3} = \frac{bf^n\prod_{\delta=0}^{-1+n}[2\delta b\,dhl + 1]}{l^n\prod_{\delta=0}^{-1+n}[(2\delta + 1)b\,df\,h + 1]},$$

$$x_{4n-2} = \frac{ce^n\prod_{\delta=0}^{-1+n}[(1 + 2\delta)acgk + 1]}{a^n\prod_{\delta=0}^{-1+n}[(1 + 2\delta)cegk + 1]},$$

$$x_{4n-1} = \frac{df^n\prod_{i=1}^{-1+n}[(1 + 2\delta)b\,dhl + 1]}{l^n\prod_{\delta=0}^{-1+n}[(2 + 2\delta)b\,df\,h + 1]},$$

$$y_{4n-4} = \frac{l^n\prod_{\delta=0}^{-1+n}[2\delta b\,df\,h + 1]}{f^{-1+n}\prod_{\delta=0}^{-1+n}[2\delta b\,dhl + 1]},$$

$$y_{4n-3} = \frac{ga^n\prod_{\delta=0}^{-1+n}[2\delta cegk + 1]}{e^n\prod_{\delta=0}^{-1+n}[(1 + 2\delta)acgk + 1]},$$

$$y_{4n-2} = \frac{hl^n\prod_{\delta=0}^{-1+n}[(2\delta + 1)b\,df\,h + 1]}{f^n\prod_{\delta=0}^{-1+n}[(1 + 2\delta)b\,dhl + 1]},$$

$$y_{4n-1} = \frac{ka^n\prod_{\delta=0}^{-1+n}[(2\delta + 1)cegk + 1]}{e^n\prod_{\delta=0}^{-1+n}[(2\delta + 2)acgk + 1]}. \tag{9}$$

Proof. Obviously results true if $n = 0$. Assuming that for $n-1$ and $n-2$ results hold, that is

$$x_{4n-9} = \frac{df^{-2+n}\prod_{\delta=0}^{n-3}[(2\delta + 1)b\,dhl + 1]}{l^{n-2}\prod_{\delta=0}^{n-3}[(2 + 2\delta)b\,df\,h + 1]},$$

$$x_{4n-8} = \frac{e^{n-1}\prod_{\delta=0}^{-2+n}[2\delta acgki + 1]}{a^{n-2}\prod_{\delta=0}^{-2+n}[2\delta cegk + 1]},$$

$$x_{4n-7} = \frac{bf^{-1+n}\prod_{\delta=0}^{-2+n}[2\delta b\,dhl + 1]}{l^{n-1}\prod_{\delta=0}^{-2+n}[(1 + 2\delta)b\,df\,h + 1]},$$

$$x_{4n-6} = \frac{ce^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)acgk + 1]}{a^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)cegk + 1]},$$

$$x_{4n-5} = \frac{d \times f^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta + 1)b\,dhl + 1]}{l^{n-1}\prod_{\delta=0}^{-2+n}[(2 + 2\delta)b\,df\,h + 1]},$$

$$y_{4n-9} = \frac{ka^{n-2}\prod_{\delta=0}^{n-3}[(2\delta + 1)cegk + 1]}{e^{n-2}\prod_{\delta=0}^{n-3}[(2 + 2\delta)acgk + 1]},$$

$$y_{4n-8} = \frac{l^{n-1}\prod_{\delta=0}^{-2+n}[2\delta b\,df\,h + 1]}{f^{-2+n}\prod_{\delta=0}^{-2+n}[2\delta b\,dhl + 1]},$$

$$y_{4n-7} = \frac{ga^{n-1}\prod_{\delta=0}^{-2+n}[2\delta cegk + 1]}{e^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)acgk + 1]},$$

$$y_{4n-6} = \frac{hl^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)b\,df\,h + 1]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[(1 + 2\delta)b\,dhl + 1]},$$

$$y_{4n-5} = \frac{ka^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)cegk + 1]}{e^{n-1}\prod_{\delta=0}^{-2+n}[(2 + 2\delta)acgk + 1]}. \tag{10}$$

Now from (8), one has

$$
\begin{aligned}
x_{4n-4} &= \frac{x_{4n-8}y_{4n-9}}{y_{4n-5}\left(1+x_{4n-6}y_{4n-7}x_{4n-8}y_{4n-9}\right)} \\
&= \frac{\left(\left(e^{n-1}\prod_{\delta=0}^{-2+n}[2\delta acgk+1]\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[2\delta cegk+1]\right)\right)\left(\left(ka^{n-2}\prod_{\delta=0}^{n-3}[(1+2\delta)cegk+1]\right)/\left(e^{n-2}\prod_{\delta=0}^{n-3}[(2\delta+2)acgk+1]\right)\right)}{\left(\left(ka^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+2)acgk+1]\right)\right)} \\
&\qquad \left(1+\left(\left(ce^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1]\right)/\left(a^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)cegk+1]\right)\right)\right. \\
&\qquad \left(\left(ga^{n-1}\prod_{\delta=0}^{n-2}[2\delta cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{n-2}[(2\delta+1)acgk+1]\right)\right) \\
&\qquad \left(\left(e^{n-1}\prod_{\delta=0}^{-2+n}[2\delta acgk+1]\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)\right) \\
&\qquad \left.\left(\left(ka^{n-2}\prod_{\delta=0}^{n-3}[(2\delta+1)cegk+1]\right)/\left(e^{n-2}\prod_{\delta=0}^{n-3}[(2\delta+2)acgk+1]\right)\right)\right) \\
&= \frac{\left(\left(ek\prod_{\delta=0}^{n-3}[(2\delta+1)cegk+1]\right)/\left(\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)\right)}{\left(\left(ka^{n-1}\prod_{\delta=0}^{n-2}[(1+2\delta)cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{-2+n}[(2+2\delta)acgk+1]\right)\right)\left(1+\left(\left(cegk\prod_{\delta=0}^{n-3}[(1+2\delta)cegk+1]\right)/\left(\prod_{\delta=0}^{n-2}[(2\delta+1)cegk+1]\right)\right)\right)} \\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[(2+2\delta)acgk+1]}{\left(a^{n-1}[(2n-3)cegk+1]\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)\left(1+\left(cegk/[(2n-3)cegk+1]\right)\right)} \\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[(2\delta+2)acgk+1]}{\left(a^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)\left([(2n-3)cegk+1]+cegk\right)} \\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[(2+2\delta)acgk+1]}{\left(a^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)[(-2+2n)cegk+1]} \\
&= \frac{e^{n}\prod_{\delta=0}^{-1+n}[(2\delta)acgk+1]}{a^{n-1}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]}, \\
y_{4n-4} &= \frac{y_{4n-8}x_{4n-9}}{x_{4n-5}\left(1+y_{4n-6}x_{4n-7}y_{4n-8}x_{4n-9}\right)} \\
&= \frac{\left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}[2\delta bdfh+1]\right)/\left(f^{-2+n}\prod_{\delta=0}^{-2+n}[2\delta bdhl+1]\right)\right)\left(\left(df^{-2+n}\prod_{\delta=0}^{n-3}[(1+2\delta)bdhl+1]\right)/\left(l^{n-2}\prod_{\delta=0}^{n-3}[(2+2\delta)bdfh+1]\right)\right)}{\left(\left(df^{-1+n}\prod_{\delta=0}^{-2+n}[(1+2\delta)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2+2\delta)bdfh+1]\right)\right)} \\
&\qquad \left(1+\left(\left(hl^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1]\right)/\left(f^{-1+n}\prod_{\delta=0}^{-2+n}[(1+2\delta)bdhl+1]\right)\right)\right. \\
&\qquad \left(\left(bf^{-1+n}\prod_{\delta=0}^{n-2}[(2\delta)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{n-2}[(1+2\delta)bdfh+1]\right)\right) \\
&\qquad \left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)bdfh+1]\right)/\left(f^{-2+n}\prod_{\delta=0}^{-2+n}[(2\delta)bdhl+1]\right)\right) \\
&\qquad \left.\left(\left(df^{-2+n}\prod_{\delta=0}^{n-3}[(1+2\delta)bdhl+1]\right)/l^{n-2}\prod_{\delta=0}^{n-3}[(2+2\delta)bdfh+1]\right)\right) \\
&= \frac{\left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)bdfh+1]\right)/\left(f^{-2+n}\prod_{\delta=0}^{-2+n}[2\delta bdhl+1]\right)\right)\left(\left(df^{-2+n}\prod_{\delta=0}^{n-3}[(2\delta+1)bdhl+1]\right)/\left(l^{n-2}\prod_{\delta=0}^{n-3}[(2\delta+2)bdfh+1]\right)\right)}{\left(\left(df^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+2)bdfh+1]\right)\right)\left(1+\left(\left(bdhl\prod_{\delta=0}^{n-3}[(2\delta+1)bdhl+1]\right)/\left(\prod_{\delta=0}^{n-2}[(2\delta+1)bdhl+1]\right)\right)\right)} \\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[(2\delta+2)bdfh+1]}{f^{-1+n}[(2n-3)bdhl+1]\prod_{\delta=0}^{-2+n}[2\delta bdhl+1]\left(1+\left(bdhl/[(2n-3)bdhl+1]\right)\right)} \\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[(2+2\delta)bdfh+1]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[2\delta bdhl+1]\left([(2n-3)bdhl+1]+bdhl\right)} \\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[(2\delta+2)bdfh+1]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta)bdhl+1][(2n-2)bdhl]} \\
&= \frac{l^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdfh+1]}{f^{-1+n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}.
\end{aligned}
\tag{11}
$$

Again, from system (8), we obtain

$$x_{4n-3} = \frac{x_{4n-7}y_{4n-8}}{y_{4n-4}\left(1 + x_{4n-5}y_{4n-6}x_{4n-7}y_{4n-8}\right)}$$

$$= \frac{\left(\left(bf^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1]\right)\right)\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[(2\delta)bdfh+1]\right)/\left(f^{-2+n}\prod_{\delta=0}^{n-2}[(2\delta)bdhl+1]\right)\right)}{\left(\left(l^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdfh+1]\right)/\left(f^{-1+n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]\right)\right)\left(1+\left(\left(df^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+2)bdfh+1]\right)\right)\left(\left(hl^{n-1}\prod_{\delta=0}^{n-2}[(2\delta+1)bdfh+1]\right)/\left(f^{-1+n}\prod_{\delta=0}^{n-2}[(2\delta+1)bdhl+1]\right)\right)\left(\left(bf^{-1+n}\prod_{\delta=0}^{-2+n}[(2\delta)bdhl+1]\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1]\right)\right)\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[(2\delta)bdfh+1]\right)/\left(f^{-2+n}\prod_{\delta=0}^{n-2}[(2\delta)bdhl+1]\right)\right)\right)}$$

$$= \frac{bf^{n}\prod_{\delta=0}^{-2+n}[(2\delta)bdfh+1]\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}{l^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdfh+1]\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1]\left(1+\left(\left(bdfh\prod_{\delta=0}^{n-2}[(2\delta)bdfh+1]\right)/\left(\prod_{\delta=0}^{n-2}[(2\delta+2)bdfh+1]\right)\right)\right)}$$

$$= \frac{bf^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}{l^{n}[(2n-2)bdfh+1]\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1](1+(bdfh/[(2n-2)bdfh+1]))}$$

$$= \frac{bf^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}{l^{n}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1]([(2n-2)bdfh+1]+bdfh)}$$

$$= \frac{bf^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}{l^{n}\prod_{\delta=0}^{-2+n}[(2\delta+1)bdfh+1][(-1+2n)bdfh+1]}$$

$$= \frac{bf^{n}\prod_{\delta=0}^{-1+n}[(2\delta)bdhl+1]}{l^{n}\prod_{\delta=0}^{-1+n}[(2\delta+1)bdfh+1]},$$

$$y_{4n-3} = \frac{y_{4n-7}x_{4n-8}}{x_{4n-4}\left(1 + y_{4n-5}x_{4n-6}y_{4n-7}x_{4n-8}\right)}$$

$$= \frac{\left(\left(ga^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1]\right)\right)\left(\left(e^{n-1}\prod_{\delta=0}^{n-2}[(2\delta)acgk+1]\right)/\left(a^{n-2}\prod_{\delta=0}^{n-2}[(2\delta)cegk+1]\right)\right)}{\left(\left(e^{n}\prod_{\delta=0}^{-1+n}[(2\delta)acgk+1]\right)/\left(a^{n-1}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]\right)\right)\left(1+\left(\left(ka^{n-1}\prod_{\delta=0}^{-2+n}[(1+2\delta)cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+2)acgk+1]\right)\right)\left(\left(ce^{n-1}\prod_{\delta=0}^{n-2}[(1+2\delta)acgk+1]\right)/a^{n-1}\prod_{\delta=0}^{n-2}[(2\delta+1)cegk+1]\right)\left(\left(ga^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta)cegk+1]\right)/\left(e^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1]\right)\right)\left(\left(e^{n-1}\prod_{\delta=0}^{n-2}[(2\delta)acgk+1]\right)/\left(a^{n-2}\prod_{\delta=0}^{n-2}[(2\delta)cegk+1]\right)\right)\right)}$$

$$= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]}{e^{n}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1][(2n-2)acgk+1](1+(acgk/[(2n-2)acgk+1]))}$$

$$= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]}{e^{n}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1]([(2n-2)acgk+1]+acgk)}$$

$$= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]}{e^{n}\prod_{\delta=0}^{-2+n}[(2\delta+1)acgk+1][(-1+2n)acgk+1]}$$

$$= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[(2\delta)cegk+1]}{e^{n}\prod_{\delta=0}^{-1+n}[(1+2\delta)acgk+1]}. \tag{12}$$

Hence, the proof of other relations is obvious. □

3. On 2nd System

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)}, \quad y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(1 - y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}. \tag{13}$$

In this section, we explore the solutions of the recursive system:

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)}, \quad y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(1 - y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}. \tag{14}$$

Theorem 2. *Let* $\{x_n, y_n\}$ *be a solution to (14), and also let* $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, *respectively, be* $a, b, c, d, e, f, g, h, k, l$*; then*

$$\begin{aligned}
x_{4n-4} &= \frac{e^n\prod_{\delta=0}^{-1+n}[1-(2\delta)acgk]}{a^{n-1}\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]},\\
x_{4n-3} &= \frac{bf^n\prod_{\delta=0}^{-1+n}[1-(2\delta)bdhl]}{l^n\prod_{\delta=0}^{-1+n}[1+(2\delta+1)bdfh]},\\
x_{4n-2} &= \frac{ce^n\prod_{i=1}^{n-1}[1-(2\delta+1)acgk]}{a^n\prod_{\delta=0}^{-1+n}[1+(1+2\delta)cegk]},\\
x_{4n-1} &= \frac{df^n\prod_{\delta=0}^{-1+n}[1-(2\delta+1)bdhl]}{l^n\prod_{\delta=0}^{-1+n}[1+(2\delta+2)bdfh]},\\
y_{4n-4} &= \frac{l^n\prod_{\delta=0}^{-1+n}[1+(2\delta)bdfh]}{f^{-1+n}\prod_{\delta=0}^{-1+n}[1-(2\delta)bdhl]},\\
y_{4n-3} &= \frac{ga^n\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]}{e^n\prod_{\delta=0}^{-1+n}[1-(2\delta+1)acgk]},\\
y_{4n-2} &= \frac{hl^n\prod_{\delta=0}^{-1+n}[1+(2\delta+1)bdfh]}{f^n\prod_{\delta=0}^{-1+n}[1-(2\delta+1)bdhl]},\\
y_{4n-1} &= \frac{ka^n\prod_{\delta=0}^{-1+n}[1+(2\delta+1)cegk]}{e^n\prod_{\delta=0}^{-1+n}[1-(2\delta+2)acgk]}.
\end{aligned} \tag{15}$$

Proof. Obviously results true if $n = 0$. Assuming that for $n - 1$ and $n - 2$ results hold, that is

$$\begin{aligned}
x_{4n-9} &= \frac{df^{-2+n}\prod_{\delta=0}^{n-3}[1-(2\delta+1)bdhl]}{l^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+2)bdfh]},\\
x_{4n-8} &= \frac{e^{n-1}\prod_{\delta=0}^{-2+n}[1-(2\delta)acgk]}{a^{n-2}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]},\\
x_{4n-7} &= \frac{bf^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]}{l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]},\\
x_{4n-6} &= \frac{ce^{n-1}\prod_{i=1}^{n-2}[1-(2\delta+1)acgk]}{a^{n-1}\prod_{\delta=0}^{-2+n}[1+(1+2\delta)cegk]},\\
x_{4n-5} &= \frac{df^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta+1)bdhl]}{l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2+2\delta)bdfh]},\\
y_{4n-9} &= \frac{ka^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+1)cegk]}{e^{n-2}\prod_{\delta=0}^{n-3}[1-(2+2\delta)acgk]},\\
y_{4n-8} &= \frac{l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]}{f^{-2+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]},\\
y_{4n-7} &= \frac{ga^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]}{e^{n-1}\prod_{\delta=0}^{-2+n}[1-(2\delta+1)acgk]},\\
y_{4n-6} &= \frac{hl^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta+1)bdhl]},\\
y_{4n-5} &= \frac{ka^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]}{e^{n-1}\prod_{\delta=0}^{-2+n}[1-(2\delta+2)acgk]}.
\end{aligned} \tag{16}$$

Next, one can obtain from system (14) that

$$\begin{aligned}
x_{4n-4} &= \frac{x_{4n-8}y_{4n-9}}{y_{4n-5}\left(1+x_{4n-6}y_{4n-7}x_{4n-8}y_{4n-9}\right)}\\
&= \Big(\big(e^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1-(2\delta)acgk]\big)/\big(a^{n-2}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\big)\Big)\Big(\big(ka^{n-2}\prod_{\delta=0}^{n-3}[1+(1+2\delta)cegk]\big)/\big(e^{n-2}\prod_{\delta=0}^{n-3}[1-(2+2\delta)acgk]\big)\Big)\\
&\quad \times \Big[\Big(\big(ka^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\big)/\big(e^{n-1}\prod_{\delta=0}^{-2+n}[1-(2\delta+2)acgk]\big)\Big)\\
&\quad\quad \Big(1+\Big(\big(ce^{n-1}\textstyle\prod_{i=1}^{n-2}[1-(2\delta+1)acgk]\big)/\big(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\big)\Big)\\
&\quad\quad \Big(\big(ga^{n-1}\textstyle\prod_{\delta=0}^{n-2}[1+(2\delta)cegk]\big)/\big(e^{n-1}\prod_{\delta=0}^{-2+n}[1-(2\delta+1)acgk]\big)\Big)\\
&\quad\quad \Big(\big(e^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1-(2\delta)acgk]\big)/\big(a^{n-2}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\big)\Big)\\
&\quad\quad \Big(\big(ka^{n-2}\textstyle\prod_{\delta=0}^{n-3}[1+(2\delta+1)cegk]\big)/\big(e^{n-2}\prod_{\delta=0}^{n-3}[1-(2\delta+2)acgk]\big)\Big)\Big)\Big]^{-1}\\
&= \frac{e^{n}\prod_{\delta=0}^{n-3}[1+(1+2\delta)cegk]\prod_{\delta=0}^{-2+n}[1-(2+2\delta)acgk]}{\left(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)\left(1+\left(cegk/[1+(2n-3)cegk]\right)\right)}\\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[1-(2+2\delta)acgk]}{\left(a^{n-1}[1+(2n-3)cegk]\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)\left(1+\left(cegk/[1+(2n-3)cegk]\right)\right)}\\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[1-(2\delta+2)acgk]}{\left(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)\left([1+(2n-3)cegk]+cegk\right)}\\
&= \frac{e^{n}\prod_{\delta=0}^{-2+n}[1-(2+2\delta)acgk]}{\left(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)\left(1+(2n-2)cegk\right)}\\
&= \frac{e^{n}\prod_{\delta=0}^{-1+n}[1-(2\delta)acgk]}{a^{n-1}\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]}.
\end{aligned} \tag{17}$$

Similarly, system (14) gives

$$\begin{aligned}
y_{4n-4} &= \frac{y_{4n-8}x_{4n-9}}{x_{4n-5}\left(1-y_{4n-6}x_{4n-7}y_{4n-8}x_{4n-9}\right)}\\
&= \Big(\big(l^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]\big)/\big(f^{-2+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]\big)\Big)\Big(\big(df^{-2+n}\prod_{\delta=0}^{n-3}[1-(2\delta+1)bdhl]\big)/\big(l^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+2)bdfh]\big)\Big)\\
&\quad \times \Big[\Big(\big(df^{-1+n}\textstyle\prod_{\delta=0}^{-2+n}[1-(2\delta+1)bdhl]\big)/\big(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]\big)\Big)\\
&\quad\quad \Big(1-\Big(\big(hl^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\big)/\big(f^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta+1)bdhl]\big)\Big)\\
&\quad\quad \Big(\big(bf^{-1+n}\textstyle\prod_{\delta=0}^{n-2}[1-(2\delta)bdhl]\big)/\big(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\big)\Big)\\
&\quad\quad \Big(\big(l^{n-1}\textstyle\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]\big)/\big(f^{-2+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]\big)\Big)\\
&\quad\quad \Big(\big(df^{-2+n}\textstyle\prod_{\delta=0}^{n-3}[1-(2\delta+1)bdhl]\big)/\big(l^{n-2}\prod_{\delta=0}^{n-3}[1+(2+2\delta)bdfh]\big)\Big)\Big)\Big]^{-1}\\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]}{f^{-1+n}[1-(2n-3)bdhl]\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]\left(1-\left(bdhl/[1-(2n-3)bdhl]\right)\right)}\\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl]\left([1-(2n-3)bdhl]-bdhl\right)}\\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}[1+(2+2\delta)bdfh]}{f^{-1+n}\prod_{\delta=0}^{-2+n}[1-(2\delta)bdhl][1-(2n-2)bdhl]}\\
&= \frac{l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)bdfh]}{f^{-1+n}\prod_{\delta=0}^{-1+n}[1-(2\delta)bdhl]}.
\end{aligned} \tag{18}$$

Obviously, the subsequent unassertive relations follows, and this complete the proof. □

4. On 3rd System

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4})},$$
$$y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n(-1 - y_{n-1}x_{n-2}y_{n-3}x_{n-4})}. \tag{19}$$

In this section, we will explore the solutions of following rational systems:

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4})},$$
$$y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n(-1 - y_{n-1}x_{n-2}y_{n-3}x_{n-4})}. \tag{20}$$

Theorem 3. *Let $\{x_n, y_n\}$ be a solution of (20), and also let $x_{-\delta}, y_{-\delta}$ $(\delta = 4, 3, \ldots, 0)$, respectively, are $a, b, c, d, e, f, g, h, k, l$. Then one has*

$$x_{4n-4} = \frac{e^n}{a^{n-1}\prod_{\delta=0}^{-1+n}((1 + (2\delta))cegk)},$$
$$x_{4n-3} = \frac{b \times f^n}{l^n\prod_{\delta=0}^{n-1}[1 + (1 + 2\delta)bdfh]},$$
$$x_{4n-2} = \frac{c(-e)^n(acgk + 1)^n}{a^n\prod_{\delta=0}^{n-1}[1 + (1 + 2\delta)cegk]},$$
$$x_{4n-1} = \frac{d(-f)^n(1 + bdhl)^n}{l^n\prod_{\delta=0}^{-1+n}[1 + (2 + 2\delta)bdfh]},$$
$$y_{4n-4} = \frac{l^n\prod_{\delta=0}^{-1+n}[1 + (2\delta)bdfh]}{f^{-1+n}},$$
$$y_{4n-3} = \frac{g(-a)^n\prod_{\delta=0}^{-1+n}[1 + (2\delta)cegk]}{e^n(1 + acgk)^n},$$
$$y_{4n-2} = \frac{h(-l)^n\prod_{\delta=0}^{-1+n}[1 + (2\delta + 1)bdfh]}{f^n(1 + bdhl)^n},$$
$$y_{4n-1} = \frac{ka^n\prod_{\delta=0}^{-1+n}[1 + (2\delta + 1)cegk]}{e^n}. \tag{21}$$

Proof. Obviously results true if $n = 0$. Assuming that for $n - 1$ and $n - 2$ results hold, that is

$$x_{4n-9} = \frac{d(-f)^{n-2}(1 + bdhl)^{n-2}}{l^{n-2}\prod\prod_{\delta=0}^{n-3}[1 + (2\delta + 2)bdfh]},$$
$$x_{4n-8} = \frac{e^{n-1}}{a^{n-2}\prod_{\delta=0}^{-2+n}[(1 + (2\delta))cegk]},$$
$$x_{4n-7} = \frac{b \times f^{-1+n}}{l^{n-1}\prod_{\delta=0}^{n-2}[1 + (2\delta + 1)bdfh]},$$
$$x_{4n-6} = \frac{c(-e)^{n-1}(1 + acgk)^{n-1}}{a^{n-1}\prod_{\delta=0}^{-2+n}[(2\delta + 1)cegk + 1]},$$
$$x_{4n-5} = \frac{d(-f)^{n-1}(1 + bdhl)^{n-1}}{l^{n-1}\prod_{\delta=0}^{-2+n}[(2 + 2\delta)bdfh + 1]},$$
$$y_{4n-9} = \frac{ka^{n-2}\prod_{\delta=0}^{n-3}[1 + (1 + 2\delta)cegk]}{e^{n-2}},$$
$$y_{4n-8} = \frac{l^{n-1}\prod_{\delta=0}^{-2+n}[1 + (2\delta)bdfh]}{f^{-2+n}},$$
$$y_{4n-7} = \frac{g(-a)^{n-1}\prod_{\delta=0}^{-2+n}[1 + (2\delta)cegk]}{e^{n-1}(1 + acgk)^{n-1}},$$
$$y_{4n-6} = \frac{h(-l)^{n-1}\prod_{\delta=0}^{-2+n}[1 + (2\delta + 1)bdfh]}{f^{-n+1}(1 + bdhl)^{n-1}},$$
$$y_{4n-5} = \frac{ka^{n-1}\prod_{\delta=0}^{-2+n}[1 + (2\delta + 1)cegk]}{e^{n-1}}. \tag{22}$$

Next, it can be seen from system (20) that

$$x_{4n-4} = \frac{x_{4n-8}y_{4n-9}}{y_{4n-5}\left(1 + x_{4n-6}y_{4n-7}x_{4n-8}y_{4n-9}\right)}$$

$$= \frac{/e^{n-1}a^{n-2}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk]/ka^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+1)cegk]e^{n-2}}{\left(\left(ka^{n-1}\prod_{\delta=0}^{n-2}[1+(1+2\delta)cegk]\right)/\left(e^{n-1}\right)\right)\left(1+\left(\left(c(-e)^{n-1}(1+acgk)^{n-1}\right)/\left(a^{n-1}\prod_{\delta=0}^{n-2}[1+(1+2\delta)cegk]\right)\right.\right.}$$

$$\left(\left(g(-a)^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)/\left(e^{n-1}(1+acgk)^{n-1}\right)\right)$$

$$\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk]\right)\right)$$

$$\left.\left.\left(\left(ka^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+1)cegk]\right)/\left(e^{n-2}\right)\right)\right)\right)$$

$$= \frac{e^{n}\prod_{\delta=0}^{n-3}[1+(2\delta+1)cegk]}{a^{n-1}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk]\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\left(1+\left(\left(cegk\prod_{\delta=0}^{n-3}[1+(1+2\delta)cegk]\right)/\left(\prod_{\delta=0}^{n-2}[1+(1+2\delta)cegk]\right)\right)\right)}$$

$$= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk][1+(2n-3)cegk](1+(cegk/[1+(2n-3)cegk]))}$$

$$= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}(1+(2\delta))cegk)([1+(2n-3)cegk]+cegk)}$$

$$= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk](1+(2n-2)cegk)}$$

$$= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-1+n}[(2\delta+1)cegk]},$$

$$y_{4n-4} = \frac{y_{4n-8}x_{4n-9}}{x_{4n-5}\left(-1 - y_{4n-6}x_{4n-7}y_{4n-8}x_{4n-9}\right)}$$

$$= \frac{\left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)\left(\left(d(-f)^{n-2}(1+bdhl)^{n-2}\right)/\left(l^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+2)bdfh]\right)\right)}{\left(\left(d(-f)^{n-1}(1+bdhl)^{n-1}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]\right)\right)}$$

$$\left(-1-\left(\left(h(-l)^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)/\left(f^{-1+n}(1+bdhl)^{n-1}\right)\right)\right.$$

$$\left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)$$

$$\left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)$$

$$\left.\left(\left(d(-f)^{n-2}(1+bdhl)^{n-2}\right)/\left(l^{n-2}\prod_{\delta=0}^{n-3}[1+(2\delta+2)bdfh]\right)\right)\right)$$

$$= \frac{(-1)^{n-2}l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]}{(-f)^{n-1}(1+bdhl)(-1+(bdhl/(1+bdhl)))}$$

$$= \frac{-l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]}{f^{-1+n}(1+bdhl)(-1+(bdhl/(1+bdhl)))}$$

$$= \frac{-l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)bdfh]}{f^{-1+n}(-1-bdhl+bdhl)}$$

$$= \frac{l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)bdfh]}{f^{-1+n}}. \tag{23}$$

Now, hereafter, we prove an extra result. It is noted that system (20) leads to

$$\begin{aligned}
x_{4n-3} &= \frac{x_{4n-7}y_{4n-8}}{y_{4n-4}\left(1+x_{4n-5}y_{4n-6}x_{4n-7}y_{4n-8}\right)} \\
&= \frac{\left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)}{\left(\left(l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)bdfh]\right)/\left(f^{-1+n}\right)\right)\left(1+\left(\left(d(-f)^{n-1}(1+bdhl)^{n-1}\right)/\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta+2)bdfh]\right)\right)\right.} \\
&\quad \left(\left(h(-l)^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)/\left(f^{-1+n}(1+bdhl)^{n-1}\right)\right) \\
&\quad \left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right) \\
&\quad \left.\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)\right) \\
&= \frac{\left((bf^{n})/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)}{\left(\left(l^{n}\prod_{\delta=0}^{n-1}[1+(2\delta)bdfh]\right)/\left(\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)\right)\left(1+\left(bdfh\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]/\left(\prod_{\delta=0}^{-2+n}[1+(2+2\delta)bdfh]\right)\right)\right)} \\
&= \frac{\left((bf^{n})/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)}{l^{n}[1+(2n-2)bdfh](1+(bdfh/[1+(2n-2)bdfh]))} \\
&= \frac{bf^{n}}{l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]([1+(2n-2)bdfh]+bdfh)} \\
&= \frac{bf^{n}}{l^{n}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh](1+(2n-1)bdfh)} \\
&= \frac{bf^{n}}{l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta+1)bdfh]}, \\
y_{4n-3} &= \frac{y_{4n-7}x_{4n-8}}{x_{4n-4}\left(-1-y_{4n-5}x_{4n-6}y_{4n-7}x_{4n-8}\right)} \\
&= \frac{\left(\left(g(-a)^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)/\left(e^{n-1}(1+acgk)^{n-1}\right)\right)\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk]\right)\right)}{\left((e^{n})/\left(a^{n-1}\prod_{\delta=0}^{-1+n}[(1+(2\delta))cegk]\right)\right)} \\
&\qquad \left(-1-\left(\left(ka^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\right)/\left(e^{n-1}\right)\right)\right. \\
&\qquad \left(\left(c(-e)^{n-1}(1+acgk)^{n-1}\right)/\left(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\right)\right) \\
&\qquad \left(\left(g(-a)^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)/\left(e^{n-1}(1+acgk)^{n-1}\right)\right) \\
&\qquad \left.\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[(1+(2\delta))cegk]\right)\right)\right) \\
&= \frac{\left(\left(ga(-1)^{n-1}\right)/\left((1+acgk)^{n-1}\right)\right)}{/e^{n}a^{n-1}\prod_{\delta=0}^{-1+n}[(1+(2\delta))cegk](-1-acgk)} \\
&= \frac{g(-1)^{n-1}a^{n}\prod_{\delta=0}^{-1+n}[(1+(2\delta))cegk]}{e^{n}(1+acgk)^{n-1}(-1-acgk)} \\
&= \frac{g(-a)^{n}\prod_{\delta=0}^{-1+n}[1+2\delta cegk]}{e^{n}(1+acgk)^{n}}.
\end{aligned} \tag{24}$$

In a similar way, one can establish other formulas. □

5. On 4th System

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1 + x_{n-1}y_{n-3}\right)},$$
$$y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1 + y_{n-1}x_{n-3}\right)}. \quad (25)$$

This section determines and studies the formulas for solutions of the following nonlinear system:

$$x_{n+1} = \frac{x_{n-3}y_{n-4}}{y_n\left(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4}\right)},$$
$$y_{n+1} = \frac{y_{n-3}x_{n-4}}{x_n\left(-1 + y_{n-1}x_{n-2}y_{n-3}x_{n-4}\right)}. \quad (26)$$

Theorem 4. *Let $\{x_n, y_n\}$ be a solution to (26), and also let $x_{-\delta}, y_{-\delta}$ $(\delta = 4, 3, \ldots, 0)$, respectively, are $a, b, c, d, e, f, g, h, k, l$. Then one has*

$$x_{4n-4} = \frac{e^n}{a^{n-1}\prod_{\delta=0}^{-1+n}\left[1 + (2\delta)cegk\right]},$$
$$x_{4n-3} = \frac{b \times f^n}{l^n\prod_{\delta=0}^{n-1}\left[1 + (2\delta + 1)bdfh\right]},$$
$$x_{4n-2} = \frac{ce^n\left[-1 + acgk\right]^n}{a^n\prod_{\delta=0}^{n-1}\left[1 + (2\delta + 1)cegk\right]},$$
$$x_{4n-1} = \frac{df^n\left[-1 + bdhl\right]^n}{l^n\prod_{\delta=0}^{-1+n}\left[1 + (2\delta + 2)bdfh\right]},$$
$$y_{4n-4} = \frac{l^n\prod_{\delta=0}^{-1+n}\left[1 + (2\delta)bdfh\right]}{f^{-1+n}},$$
$$y_{4n-3} = \frac{ga^n\prod_{\delta=0}^{n-1}\left[1 + (2\delta)cegk\right]}{e^n\left(-1 + acgk\right)^n},$$
$$y_{4n-2} = \frac{hl^n\prod_{\delta=0}^{-1+n}\left[1 + (2\delta + 1)bdfh\right]}{f^n\left(-1 + bdhl\right)^n},$$
$$y_{4n-1} = \frac{ka^n\prod_{\delta=0}^{-1+n}\left[1 + (2\delta + 1)cegk\right]}{e^n}. \quad (27)$$

Proof. Obviously results true if $n = 0$. Assuming that for $n - 1$ and $n - 2$ results hold, that is

$$x_{4n-9} = \frac{df^{-2+n}\left[-1 + bdhl\right]^{n-2}}{l^{n-2}\prod_{\delta=0}^{n-3}\left[1 + (2\delta + 2)bdfh\right]},$$
$$x_{4n-8} = \frac{e^{n-1}}{a^{n-2}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta)cegk\right]},$$
$$x_{4n-7} = \frac{bf^{-1+n}}{l^{n-1}\prod_{\delta=0}^{n-2}\left[1 + (2\delta + 1)bdfh\right]},$$
$$x_{4n-6} = \frac{ce^{n-1}\left[-1 + acgk\right]^{n-1}}{a^{n-1}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta + 1)cegk\right]},$$
$$x_{4n-5} = \frac{df^{-1+n}\left[-1 + bdhl\right]^{n-1}}{l^{n-1}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta + 2)bdfh\right]},$$
$$y_{4n-9} = \frac{ka^{n-2}\prod_{\delta=0}^{n-3}\left[1 + (2\delta + 1)cegk\right]}{e^{n-2}},$$
$$y_{4n-8} = \frac{l^{n-1}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta)bdfh\right]}{f^{-2+n}},$$
$$y_{4n-7} = \frac{ga^{n-1}\prod_{\delta=0}^{n-2}\left[1 + (2\delta)cegk\right]}{e^{n-1}\left(-1 + acgk\right)^{n-1}},$$
$$y_{4n-6} = \frac{hl^{n-1}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta + 1)bdfh\right]}{f^{-1+n}\left(-1 + bdhl\right)^{n-1}},$$
$$y_{4n-5} = \frac{ka^{n-1}\prod_{\delta=0}^{-2+n}\left[1 + (2\delta + 1)cegk\right]}{e^{n-1}}. \quad (28)$$

Next, one can obtain from system (26) that

$$
\begin{aligned}
x_{4n-4} &= \frac{x_{4n-8}y_{4n-9}}{y_{4n-5}\left(1+x_{4n-6}y_{4n-7}x_{4n-8}y_{4n-9}\right)} \\
&= \frac{\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\right)\right)\left(\left(ka^{n-2}\prod_{\delta=0}^{n-3}\left[1+(2\delta+1)cegk\right]\right)/\left(e^{n-2}\right)\right)}{\left(\left(ka^{n-1}\prod_{\delta=0}^{n-2}\left[1+(2\delta+1)cegk\right]\right)/\left(e^{n-1}\right)\right)\left(1+\left(\left(ce^{n-1}\left[-1+acgk\right]^{n-1}\right)/\left(a^{n-1}\prod_{\delta=0}^{n-2}\left[1+(2\delta+1)cegk\right]\right)\right)\right.} \\
&\quad \left(\left(ga^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\right)/\left(e^{n-1}\left(-1+acgk\right)^{n-1}\right)\right) \\
&\quad \left.\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\right)\right)\left(\left(ka^{n-2}\prod_{\delta=0}^{n-3}\left[1+(2\delta+1)cegk\right]\right)/\left(e^{n-2}\right)\right)\right) \\
&= \frac{\left(\left(e^{n}\prod_{\delta=0}^{n-3}\left[1+(2\delta+1)cegk\right]\right)/\left(\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\right)\right)}{a^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+1)cegk\right]\left(1+\left(\left(cegk\prod_{\delta=0}^{n-3}\left[1+(2\delta+1)cegk\right]\right)/\left(\prod_{\delta=0}^{-2+n}\left[1+(2\delta+1)cegk\right]\right)\right)\right)} \\
&= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\left[1+(2n-3)cegk\right]\left(1+\left((cegk)/\left(1+(2n-3)cegk\right)\right)\right)} \\
&= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\left(1+(2n-3)cegk+cegk\right)} \\
&= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)cegk\right]\left(1+(2n-2)cegk\right)} \\
&= \frac{e^{n}}{a^{n-1}\prod_{\delta=0}^{-1+n}\left[1+(2\delta)cegk\right]}, \\
y_{4n-4} &= \frac{y_{4n-8}x_{4n-9}}{x_{4n-5}\left(-1+y_{4n-6}x_{4n-7}y_{4n-8}x_{4n-9}\right)} \\
&= \frac{/l^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)bdfh\right]f^{-2+n}/df^{-2+n}\left[-1+bdhl\right]^{n-2}l^{n-2}\prod_{\delta=0}^{n-3}\left[1+(2\delta+2)bdfh\right]}{\left(\left(df^{-1+n}\left[-1+bdhl\right]^{n-1}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+2)bdfh\right]\right)\right)} \\
&\quad \left(-1+\left(\left(hl^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+1)bdfh\right]\right)/\left(f^{-1+n}\left(-1+bdhl\right)^{n-1}\right)\right)\left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+1)bdfh\right]\right)\right)\right. \\
&\quad \left.\left(\left(l^{n-1}\prod_{\delta=0}^{-2+n}\left[1+(2\delta)bdfh\right]\right)/\left(f^{-2+n}\right)\right)\left(\left(df^{-2+n}\left[-1+bdhl\right]^{n-2}\right)/\left(l^{n-2}\prod_{\delta=0}^{n-3}\left[1+(2\delta+2)bdfh\right]\right)\right)\right) \\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+2)bdfh\right]}{f^{-1+n}\left[-1+bdhl\right]\left(-1+\left((bdhl)/(-1+bdhl)\right)\right)} \\
&= \frac{l^{n}\prod_{\delta=0}^{-2+n}\left[1+(2\delta+2)bdfh\right]}{f^{-1+n}\left(1-bdhl+bdhl\right)} \\
&= \frac{l^{n}\prod_{\delta=0}^{-1+n}\left[1+(2\delta)bdfh\right]}{f^{-1+n}}.
\end{aligned}
\tag{29}
$$

Also from system (26) one gets:

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-7}y_{4n-8}}{y_{4n-4}\left(1+x_{4n-5}y_{4n-6}x_{4n-7}y_{4n-8}\right)} \\
&= \frac{\left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)}{\left(\left(l^{n}\prod_{\delta=0}^{n-1}[1+(2\delta)bdfh]\right)/\left(f^{-1+n}\right)\right)\left(1+\left(\left(df^{-1+n}[-1+bdhl]^{n-1}\right)/\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta+2)bdfh]\right)\right)\right.} \\
&\quad \left(\left(hl^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)/\left(f^{-1+n}(-1+bdhl)^{n-1}\right)\right) \\
&\quad \left.\left(\left(bf^{-1+n}\right)/\left(l^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)\left(\left(l^{n-1}\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)/\left(f^{-2+n}\right)\right)\right) \\
&= \frac{\left(\left(bf^{n}\right)/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)}{\left(\left(l^{n}\prod_{\delta=0}^{n-1}[1+(2\delta)bdfh]\right)/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta)bdfh]\right)\right)\left(1+\left(\left(bdfh\prod_{\delta=0}^{n-2}[1+(2\delta)bdfh]\right)/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta+2)bdfh]\right)\right)\right)} \\
&= \frac{\left(\left(bf^{n}\right)/\left(\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]\right)\right)}{l^{n}[1+(2n-2)bdfh](1+((bdfh)/[1+(2n-2)bdfh]))} \\
&= \frac{bf^{n}}{l^{n}([1+(2n-2)bdfh]+bdfh)\prod_{\delta=0}^{-2+n}[1+(2\delta+1)bdfh]} \\
&= \frac{bf^{n}}{l^{n}[1+(2n-1)bdfh]\prod_{\delta=0}^{n-2}[1+(2\delta+1)bdfh]} \\
&= \frac{bf^{n}}{l^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta+1)bdfh]}, \\
y_{4n-3} &= \frac{y_{4n-7}x_{4n-8}}{x_{4n-4}\left(-1+y_{4n-5}x_{4n-6}y_{4n-7}x_{4n-8}\right)} \\
&= \frac{\left(\left(ga^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)/\left(e^{n-1}(-1+acgk)^{n-1}\right)\right)\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)\right)}{\left(\left(e^{n}\right)/\left(a^{n-1}\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]\right)\right)\left(-1+\left(\left(ka^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\right)/\left(e^{n-1}\right)\right)\left(\left(ce^{n-1}[-1+acgk]^{n-1}\right)/\left(a^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta+1)cegk]\right)\right)\right.} \\
&\quad \left.\left(\left(ga^{n-1}\prod_{\delta=0}^{-2+n}[1+(2\delta)cegk]\right)/\left(e^{n-1}(-1+acgk)^{n-1}\right)\right)\left(\left(e^{n-1}\right)/\left(a^{n-2}\prod_{\delta=0}^{n-2}[1+(2\delta)cegk]\right)\right)\right) \\
&= \frac{\left(\left(ga^{n}\right)/\left((-1+acgk)^{n-1}\right)\right)}{\left(\left(e^{n}\right)/\left(\prod_{\delta=0}^{n-1}[1+(2\delta)cegk]\right)\right)(-1+acgk)} \\
&= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]}{e^{n}(-1+acgk)^{n-1}(-1+acgk)} \\
&= \frac{ga^{n}\prod_{\delta=0}^{-1+n}[1+(2\delta)cegk]}{e^{n}(-1+acgk)^{n}}.
\end{aligned} \tag{30}
$$

Subsequent results follow from the above assertion. □

6. Numerical Simulation

In order to verify our theoretical results, we consider four interesting numerical examples by fixing suitable initial values. These simulation shows the solutions of under consideration discrete-time systems, which are depicted in (8), (14), (20), and (26).

Example 1. This example presents that the solutions of discrete-time system (8) with $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, respectively, are $1, 0.2, 0.1, 0.02, 2, 1.5, 0.3, 1, 0.4, 4$ as shown in Figure 1.

Example 2. This example shows the behavior of system (14). The relevant plot is given in Figure 2 under the initial conditions: $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, respectively, are $0.5, 3.5, 0.01, 2, 0.2, 1.5, 3, 0.1, 0.7, 0.3$.

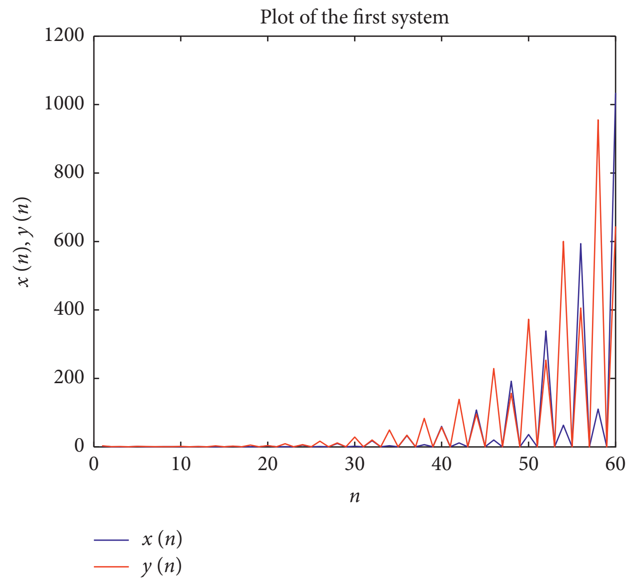

FIGURE 1: Plot of the solutions of 1st system.

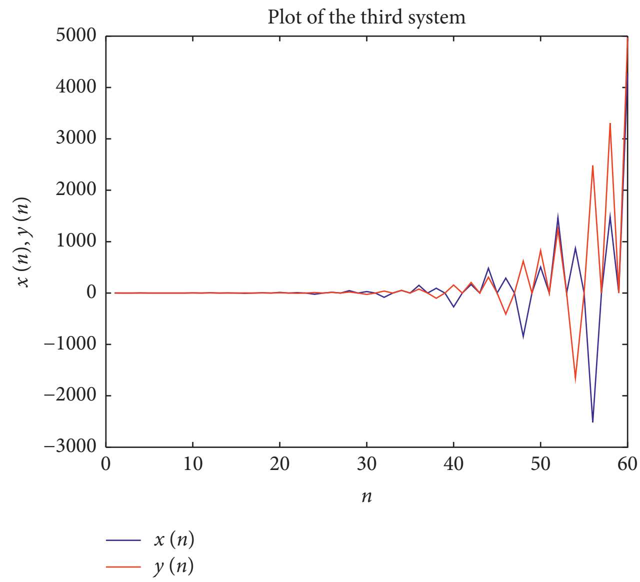

FIGURE 3: Plot of the solutions of 3rd system.

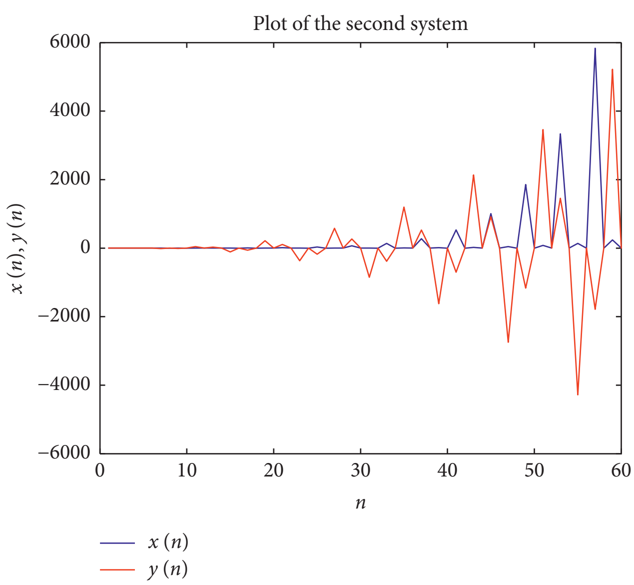

FIGURE 2: Plot of the solutions of 2nd system.

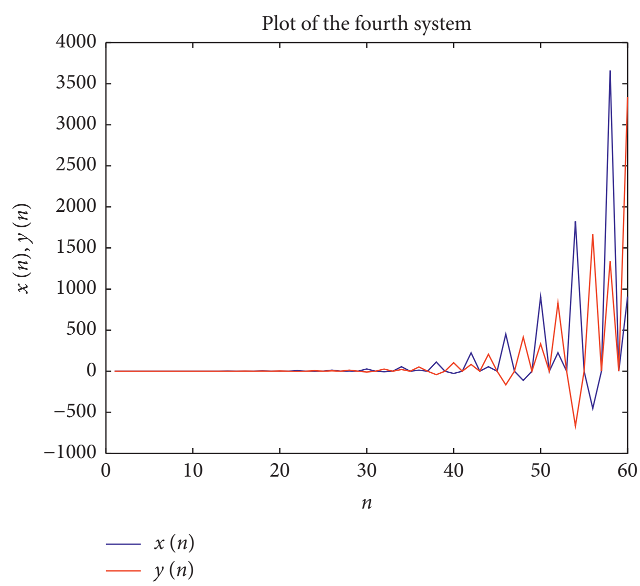

FIGURE 4: Plot of the solutions of 4th system.

Example 3. Figure 3 depicts the dynamics of solutions to (20) when we randomly consider the values as follows: $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, respectively, are $0.1, 0.9, 0.01, 0.3, 0.2, 0.4, 0.3, 0.03, 0.2, 0.8$.

Example 4. The behavior of system (26) are plotted in Figure 4 with $x_{-\delta}, y_{-\delta}\,(\delta = 4, 3, \ldots, 0)$, respectively, are $0.1, 0.05, 0.06, 0.2, 0.2, 0.4, 0.2, 0.04, 0.08, 0.81$.

7. Conclusion

In this paper, we deal with the form of the solutions of four cases of the nonlinear systems of difference equations $x_{n+1} = ((x_{n-3}y_{n-4})/(y_n(1 + x_{n-1}y_{n-2}x_{n-3}y_{n-4})))$, $y_{n+1} = ((y_{n-3}x_{n-4})/(x_n(\pm 1 \pm y_{n-1}x_{n-2}y_{n-3}x_{n-4})))$. Finally some numerical examples are giving by fixing suitable initial values to show the qualitative behavior of under consideration systems.

Data Availability

All the data used in this article have been included, and the sources were they where adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

Study of Fractional Integral Operators Containing Mittag-Leffler Functions via Strongly (α, m)-Convex Functions

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The main aim of this paper is to give refinement of bounds of fractional integral operators involving extended generalized Mittag-Leffler functions. A new definition, namely, strongly (α, m)-convex function is introduced to obtain improvements of bounds of fractional integral operators for convex, m-convex, and (α, m)-convex functions. The results of this paper will provide simultaneous generalizations as well as refinements of various published results.

1. Introduction

Convexity is one of the most important and key concept in mathematics, and many researchers have extended, generalized, and refined it in different ways. Numerous generalizations and extensions have been produced in recent past, for example, in generalization and extension point of views, m-convexity, (α, m)-convexity, s-convexity, (s, m)-convexity, h-convexity, and (h, m)-convexity are remarkable, and in refinement point of view, the strongly convexity is the tremendous notion. In this paper, we have introduced the notion of strongly (α, m)-convex function. By utilizing this refined form of convex function, we obtain refinements of the bounds of fractional integral operators involving Mittag-Leffler functions in their kernels. Therefore, the results of this paper are refinements of all the results proved in [1]. First, we give definitions of convex, strongly convex, and (α, m)-convex functions.

Definition 1 (see [2]). Let I be an interval on real line. A function $f: I \longrightarrow \mathbb{R}$ is said to be convex function if the following inequality holds:

$$f(tu_1 + (1-t)u_2) \leq t f(u_1) + (1-t) f(u_2), \quad (1)$$

for all $u_1, u_2 \in I$ and $t \in [0,1]$.

Definition 2 (see [3]). Let I be an interval on real line. A real-valued function f is said to be strongly convex with modulus $\lambda \geq 0$ on I if, for each $u_1, u_2 \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} f(tu_1 + (1-t)u_2) &\leq t f(u_1) + (1-t) f(u_2) \\ &\quad - \lambda t(1-t)\left|u_2 - u_1\right|^2. \end{aligned} \quad (2)$$

Definition 3 (see [4]). A function $f: [0,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be (α, m)-convex function, where $(\alpha, m) \in [0,1]^2$ and $b > 0$ if, for every $u_1, u_2 \in [0,b]$ and $t \in [0,1]$, we have

$$f(tu_1 + m(1-t)u_2) \leq t^\alpha f(u_1) + m(1-t^\alpha) f(u_2). \quad (3)$$

The well-known Mittag-Leffler function is denoted by $E_\xi(.)$ and defined as follows (see [5]):

$$E_\xi(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\xi n + 1)}, \quad (4)$$

where $t, \xi \in \mathbb{C}, \mathfrak{R}(\xi) > 0$, and $\Gamma(.)$ is the gamma function. It is a natural extension of exponential, hyperbolic, and trigonometric functions. This function and its extensions appear as solution of fractional integral equations and fractional differential equations. For a detailed study of Mittag-Leffler function and its extensions, see [6–10]. The following extended generalized Mittag-Leffler function is introduced by Andrić et al.

Definition 4 (see [11]). Let $\mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then, the Mittag-Leffler function $E_{\mu,\xi,l}^{\gamma,\delta,k,c}(t;p)$ is defined by

$$E_{\mu,\xi,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \quad (5)$$

where β_p is defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-p/t(1-t)} \mathrm{d}t, \quad (6)$$

and. $(c)_{nk} = \Gamma(c + nk)/\Gamma(c)$.

A derivative formula of the extended generalized Mittag-Leffler function is given in the following lemma.

Lemma 1 (see [11]). *If* $m \in \mathbb{N}, \omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0$, *and* $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$, *and* $0 < k < \delta + \Re(\mu)$, *then*

$$\begin{aligned}&\left(\frac{d}{\mathrm{d}t}\right)^m \left[t^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega t^\mu; p)\right] \\ &= t^{\xi-m-1} E_{\mu,\xi-m,l}^{\gamma,\delta,k,c}(\omega t^\mu; p), \quad \Re(\xi) > m.\end{aligned} \quad (7)$$

Remark 1. The extended generalized Mittag-Leffler function (5) produces the related functions defined in [8–10, 12, 13], see Remark 1.3 in [14].

Next, we have the definition of the generalized fractional integral operator containing the extended generalized Mittag-Leffler function (5).

Definition 5 (see [11]). Let $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then, the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$\left(\epsilon_{\mu,\xi,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) = \int_a^x (x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x-t)^\mu; p) f(t) \mathrm{d}t \quad (8)$$

and

$$\left(\epsilon_{\mu,\xi,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x;p) = \int_x^b (t-x)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(t-x)^\mu; p) f(t) \mathrm{d}t. \quad (9)$$

Remark 2. The operators defined in (8) and (9) produce several kinds of known fractional integral operators, see Remark 1.4 in [14].

The classical Riemann–Liouville fractional integral operator is defined as follows.

Definition 6 (see [13]). Let $f \in L_1[a,b]$. Then, Riemann–Liouville fractional integral operators of order $\xi > 0$ are defined by

$$I_{a^+}^{\xi} f(x) = \frac{1}{\Gamma(\xi)} \int_x^b (x-t)^{\xi-1} f(t) \mathrm{d}t, \quad x > a, \quad (10)$$

$$I_{b^-}^{\xi} f(x) = \frac{1}{\Gamma(\xi)} \int_a^x (t-x)^{\xi-1} f(t) \mathrm{d}t, \quad x < b. \quad (11)$$

It can be noted that $(\epsilon_{\mu,\xi,l,0,a^+}^{\gamma,\delta,k,c} f)(x;0) = I_{a^+}^{\xi} f(x)$ and $(\epsilon_{\mu,\xi,l,0,b^-}^{\gamma,\delta,k,c} f)(x;0) = I_{b^-}^{\xi} f(x)$. From fractional integral operators (8) and (9), we can have

$$\begin{aligned}J_{\xi,a^+}(x;p) &:= \left(\epsilon_{\mu,\xi,l,\omega,a^+}^{\gamma,\delta,k,c} 1\right)(x;p) \\ &= (x-a)^{\xi} E_{\mu,\xi+1,l}^{\gamma,\delta,k,c}(w(x-a)^\mu; p),\end{aligned} \quad (12)$$

$$\begin{aligned}J_{\eta,b^-}(x;p) &:= \left(\epsilon_{\mu,\eta,l,\omega,b^-}^{\gamma,\delta,k,c} 1\right)(x;p) \\ &= (b-x)^{\eta} E_{\mu,\eta+1,l}^{\gamma,\delta,k,c}(w(b-x)^\mu; p).\end{aligned} \quad (13)$$

In view of wide applications of Riemann–Liouville fractional integrals and derivatives, the problems which involve this integral operator are studied extensively by many authors. The aim of this paper is to provide fractional integral inequalities which are generalizations of Riemann–Liouville fractional integral inequalities. These inequalities also give associated inequalities for fractional integral operators containing Mittag-Leffler functions with different parameters.

The bounds of fractional integrals (8) and (9) for (α,m)-convex functions are given in the following theorems.

Theorem 1 (see [1]). *Let* $f\colon [a,b] \longrightarrow \mathbb{R}$ *be a real-valued function. If* f *is positive* (α,m)*-convex, then for* $\xi, \eta \geq 1$, *the following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x_0;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,b^-} f\right)(x_0;p) \le \left(\frac{f(a) + m\alpha f(x_0/m)}{\alpha+1}\right)(x_0 - a) J_{\xi-1,a^+}(x_0;p) + \left(\frac{f(b) + m\alpha f(x_0/m)}{\alpha+1}\right)(b - x_0) J_{\eta-1,b^-}(x_0;p), \quad x_0 \in [a,b]. \tag{14}$$

Theorem 2 (see [1]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is (α,m)-convex, then, for $\xi,\eta \ge 1$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(x_0;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(x_0;p) - \left(J_{\xi-1,a^+}(x_0;p) f(a) + J_{\eta-1,b^-}(x_0;p) f(b)\right)\right| \le \left(\frac{|f'(a)| + m\alpha|f'(x_0/m)|}{\alpha+1}\right)(x_0 - a) J_{\xi-1,a^+}(x_0;p) + \left(\frac{|f'(b)| + m\alpha|f'(x_0/m)|}{\alpha+1}\right)(b - x_0) J_{\eta-1,b^-}(x_0;p), \quad x_0 \in [a,b]. \tag{15}$$

Theorem 3 (see [1]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$, $a > b$, be a real-valued function. If f is positive, (α,m)-convex, and $f(a + mb - x_0/m) = f(x_0)$, then, for $\xi,\eta > 0$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\frac{1}{1/2^\alpha + m(1 - 1/2^\alpha)}\left(f\left(\frac{a+mb}{2}\right)\left(J_{\eta+1,b^-}(a;p) + J_{\xi+1,a^+}(b;p)\right)\right) \le \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(b;p) \le \left(J_{\eta-1,b^-}(a;p) + J_{\xi-1,a^+}(b;p)\right)(b-a)^2\left(\frac{f(b) + m\alpha f(a/m)}{\alpha+1}\right). \tag{16}$$

In Section 2, by using definition of strongly (α,m)-convex function, we establish new refinements of the bounds of generalized fractional integral operators. Also, the refinements of bounds of these operators are presented in the form of Hadamard-like inequality by using strongly (α,m)-convex functions. The results of this paper are connected with several well-known inequalities.

2. Main Results

First, we define strongly (α,m)-convex function.

Definition 7. A function $f\colon [0,+\infty) \longrightarrow \mathbb{R}$ is said to be strongly (α,m)-convex function, with modulus $\lambda \ge 0$, for $(\alpha,m) \in [0,1]^2$, if

$$f(tu_1 + m(1-t)u_2) \le t^\alpha f(u_1) + m(1 - t^\alpha) f(u_2) - \lambda m t^\alpha (1 - t^\alpha)|u_2 - u_1|^2 \tag{17}$$

holds, for all $u_1, u_2 \in [0,+\infty)$ and $t \in [0,1]$.

Remark 3

(i) By setting $\alpha = 1$ in (17), strongly m-convex function can be obtained [15]

(ii) By setting $\lambda = 0$ in (17), (α,m)-convex function can be obtained

(iii) By setting $\alpha = m = 1$ and $\lambda = 0$ in (17), convex function can be obtained

(iv) By setting $\alpha = 1$ and $\lambda = 0$ in (17), m-convex function can be obtained

(v) By setting $\alpha = m = 1$ in (17), strongly convex function can be obtained

In the following, by using strongly (α,m)-convex functions the refinement of already proved results are given.

Theorem 4. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a real-valued function. If f is positive and strongly (α,m)-convex, then, for $\xi,\eta \ge 1$, the* *following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+}f\right)(x_0;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,b^-}f\right)(x_0;p)\\&\leq\left(\frac{f(a)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)}\right)(x_0-a)J_{\xi-1,a^+}(x_0;p)\\&\quad+\left(\frac{f(b)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right)(b-x_0)J_{\eta-1,b^-}(x_0;p),\\&x_0\in[a,b].\end{aligned}\tag{18}$$

Proof. Let $x_0\in[a,b]$. Then, for $t\in[a,x_0)$ and $\xi\geq 1$, one can have the following inequality:

$$(x_0-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-t)^{\mu};p)\leq(x_0-a)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\cdot(\omega(x_0-a)^{\mu};p).\tag{19}$$

The function f is strongly (α,m)-convex; therefore, one can obtain

$$f(t)\leq\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}f(a)+m\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)f\left(\frac{x_0}{m}\right)-\lambda m\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left(\frac{x_0-ma}{m}\right)^2.\tag{20}$$

By multiplying (19) and (20) and then integrating over $[a,x_0]$, we obtain

$$\begin{aligned}&\int_a^{x_0}(x_0-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-t)^{\mu};p)f(t)dt\\&\leq(x_0-a)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-a)^{\mu};p)\left(f(a)\int_a^{x_0}\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}dt\right.\\&\quad+mf\left(\frac{x_0}{m}\right)\int_a^{x_0}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)dt\\&\quad\left.-\frac{\lambda}{m}(x_0-ma)^2\int_a^{x_0}\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)dt\right).\end{aligned}\tag{21}$$

From which we have that the left integral operator satisfies the following inequality:

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+}f\right)(x_0;p)\\&\leq(x_0-a)J_{\xi-1,a^+}(x_0;p)\left(\frac{f(a)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)}\right).\end{aligned}\tag{22}$$

Now, on the contrary, for $t\in(x_0,b]$ and $\eta\geq 1$, one can have the following inequality:

$$(t-x_0)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(t-x_0)^{\mu};p)\leq(b-x_0)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}\cdot(\omega(b-x_0)^{\mu};p).\tag{23}$$

Again from strongly (α,m)-convexity of f, we have

$$f(t)\leq\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}f(b)+m\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)f\left(\frac{x_0}{m}\right)-\lambda m\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)\left(\frac{mb-x_0}{m}\right)^2.\tag{24}$$

By multiplying (23) and (24) and then integrating over $[x_0,b]$, we have

$$\begin{aligned}&\int_{x_0}^{b}(t-x_0)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(t-x_0)^{\mu};p)f(t)dt\\&\leq(b-x_0)^{\eta-1}E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(b-x_0)^{\mu};p)\\&\quad\times\left(f(a)\int_{x_0}^{b}\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}dt+mf\left(\frac{x_0}{m}\right)\right.\\&\quad\cdot\int_{x_0}^{b}\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)dt\\&\quad\left.-\frac{\lambda}{m}(mb-x_0)^2\int_{x_0}^{b}\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)dt\right).\end{aligned}\tag{25}$$

The right integral operator satisfies the following inequality:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,b^-} f\right)(x_0;p) \leq (b-x_0)J_{\eta-1,b^-}(x_0;p)\left(\frac{f(b)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right). \tag{26}$$

By adding (22) and (26), the required inequality (18) is established. □

Remark 4

(i) Inequality (18) provides the refinement of Theorem 2.1 in [1]

(ii) If $\alpha = 1$ in (18), then result for strongly m-convex will be obtained

(iii) If $\alpha = m = 1$ and $\lambda = 0$ in (18), then Corollary 1 [16] is obtained

(iv) If $\alpha = m = 1$, $\lambda = 0$, and $\omega = p = 0$ in (18), then Theorem 1 in [17] is obtained

(v) If $\alpha = 1$ and $t = t^s$ in (18), then Theorem 3 in [18] is obtained

Corollary 1. *If we set $\xi = \eta$ in (18), then the following inequality is obtained:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x_0;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-} f\right)(x_0;p)\\ &\leq\left(\frac{f(a)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)}\right)(x_0-a)J_{\xi-1,a^+}(x_0;p)\\ &+\left(\frac{f(b)+m\alpha f(x_0/m)}{\alpha+1}-\frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right)(b-x_0)J_{\xi-1,b^-}(x_0;p),\\ &x_0\in[a,b].\end{aligned} \tag{27}$$

Remark 5. Inequality (27) provides the refinement of Corollary 2.1 in [1].

Corollary 2. *Along with assumptions of Theorem 1, if $f \in L_\infty[a,b]$, then the following inequality is obtained:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x_0;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,b^-} f\right)(x_0;p)\\ &\leq\frac{\|f\|_\infty(1+m\alpha)}{\alpha+1}\left[(x_0-a)J_{\xi-1,a^+}(x_0;p)+(b-x_0)J_{\eta-1,b^-}(x_0;p)\right]\\ &-\left[\frac{\lambda\alpha(x_0-ma)^2(x_0-a)}{m(\alpha+1)(2\alpha+1)}J_{\xi-1,a^+}(x_0;p)+\frac{\lambda\alpha(mb-x_0)^2(b-x_0)}{m(\alpha+1)(2\alpha+1)}J_{\eta-1,b^-}(x_0;p)\right].\end{aligned} \tag{28}$$

Corollary 3. *For $\xi = \eta$ in (28), we get the following result:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x_0;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-} f\right)(x_0;p)\\ &\leq\frac{\|f\|_\infty(1+m\alpha)}{\alpha+1}\left((x_0-a)J_{\xi-1,a^+}(x_0;p)+(b-x_0)J_{\xi-1,b^-}(x_0;p)\right)\\ &-\left(\frac{\lambda\alpha(x_0-ma)^2(x_0-a)}{m(\alpha+1)(2\alpha+1)}J_{\xi-1,a^+}(x_0;p)+\frac{\lambda\alpha(mb-x_0)^2(b-x_0)}{m(\alpha+1)(2\alpha+1)}J_{\xi-1,b^-}(x_0;p)\right).\end{aligned} \tag{29}$$

Theorem 5. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is strongly (α,m)-convex, then for $\xi,\eta \geq 1$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(x_0;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(x_0;p) - \left(J_{\xi-1,a^+}(x_0;p) f(a) + J_{\eta-1,b^-}(x_0;p) f(b)\right)\right|$$
$$\leq \left(\frac{|f'(a)| + m\alpha|f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)}\right)(x_0-a)J_{\xi-1,a^+}(x_0;p)$$
$$+\left(\frac{|f'(b)| + m\alpha|f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right)(b-x_0)J_{\eta-1,b^-}(x_0;p), \tag{30}$$

$x_0 \in [a,b]$.

Proof. As $x_0 \in [a,b]$ and $t \in [a, x_0)$, by using strongly (α, m)-convexity of $|f'|$, we have

$$\left|f'(t)\right| \leq \left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left|f'(a)\right| + m\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left|f'\left(\frac{x_0}{m}\right)\right|$$
$$-\lambda m\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left(\frac{x_0-ma}{m}\right)^2. \tag{31}$$

From (31), one can have

$$f'(t) \leq \left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left|f'(a)\right| + m\left(1-\left(\frac{x_0-t}{x_0-ma}\right)^{\alpha}\right)\left|f'\left(\frac{x_0}{m}\right)\right|$$
$$-\lambda m\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left(\frac{x_0-a}{m}\right)^2. \tag{32}$$

The product of (19) and (32) gives the following inequality:

$$(x_0-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-t)^{\mu};p)f'(t)\mathrm{d}t$$
$$\leq (x_0-a)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-a)^{\mu};p)$$
$$\left(\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left|f'(a)\right| + m\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left|f'\left(\frac{x_0}{m}\right)\right| - \lambda m\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left(\frac{x_0-ma}{m}\right)^2\right). \tag{33}$$

After integrating above inequality over $[a, x_0]$, we obtain

$$\int_a^{x_0}(x_0-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-t)^{\mu};p)f'(t)\mathrm{d}t$$
$$\leq (x_0-ta)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-a)^{\mu};p)\left(\left|f'(a)\right|\int_a^{x_0}\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\mathrm{d}t\right.$$
$$+m\left|f'\left(\frac{x_0}{m}\right)\right|\int_a^{x_0}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\mathrm{d}t$$
$$\left.-\frac{\lambda}{m}(x_0-ma)^2\int_a^{x_0}\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\mathrm{d}t\right)$$
$$= \frac{(x_0-a)^{\xi}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-a)^{\mu};p)}{\alpha+1}\left(\left|f'(a)\right| + m\alpha\left|f'\left(\frac{x_0}{m}\right)\right| - \frac{\lambda\alpha(x_0-ma)^2}{m(2\alpha+1)}\right). \tag{34}$$

The left-hand side of (34) is calculated as follows:

$$\int_a^{x_0}(x_0-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(x_0-t)^{\mu};p)f'(t)\mathrm{d}t. \tag{35}$$

Put $x_0 - t = z$, that is, $t = x_0 - z$; also using the derivative property (7) of Mittag-Leffler function, we have

$$\begin{aligned}&\int_0^{x_0-a} z^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega z^{\mu};p\right)f'\left(x_0-z\right)\mathrm{d}z\\&=\left(x_0-a\right)^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega\left(x_0-a\right)^{\mu};p\right)f\left(a\right)-\int_0^{x_0-a} z^{\xi-2}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega z^{\mu};p\right)f\left(x_0-z\right)\mathrm{d}z.\end{aligned}\tag{36}$$

Now, put $x_0 - z = t$ in the second term of the right-hand side of the above equation, and then using (8), we obtain

$$\begin{aligned}&\int_0^{x_0-a} z^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega z^{\mu};p\right)f'\left(x_0-z\right)\mathrm{d}z\\&=\left(x_0-a\right)^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega\left(x_0-a\right)^{\mu};p\right)f\left(a\right)\\&\quad-\left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)\left(x_0;p\right).\end{aligned}\tag{37}$$

Therefore, (34) takes the following form:

$$\begin{aligned}&\left(J_{\xi-1,a^+}\left(x_0;p\right)\right)f\left(a\right)-\left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)\left(x_0;p\right)\\&\le\left(x_0-a\right)J_{\xi-1,a^+}\left(x_0;p\right)\left(\frac{\left|f'\left(a\right)\right|+m\alpha\left|f'\left(x_0/m\right)\right|}{\alpha+1}-\frac{\lambda\alpha\left(x_0-ma\right)^2}{m\left(\alpha+1\right)\left(2\alpha+1\right)}\right).\end{aligned}\tag{38}$$

Also, from (31), one can have

$$\begin{aligned}f'\left(t\right)\ge&-\left(\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left|f'\left(a\right)\right|+m\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left|f'\left(\frac{x_0}{m}\right)\right|\right.\\&\left.-\lambda m\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\left(1-\left(\frac{x_0-t}{x_0-a}\right)^{\alpha}\right)\left(\frac{x_0-ma}{m}\right)^2\right).\end{aligned}\tag{39}$$

Following the same procedure as we did for (32), one can obtain

$$\begin{aligned}&\left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)\left(x_0;p\right)-J_{\xi-1,a^+}\left(x_0;p\right)f\left(a\right)\\&\le\left(x_0-a\right)J_{\xi-1,a^+}\left(x_0;p\right)\left(\frac{\left|f'\left(a\right)\right|+m\alpha\left|f'\left(x_0/m\right)\right|}{\alpha+1}-\frac{\lambda\alpha\left(x_0-ma\right)^2}{m\left(\alpha+1\right)\left(2\alpha+1\right)}\right).\end{aligned}\tag{40}$$

From (38) and (40), we obtain

$$\begin{aligned}&\left|\left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)\left(x_0;p\right)-J_{\xi-1,a^+}\left(x_0;p\right)f\left(a\right)\right|\\&\le\left(x_0-a\right)J_{\xi-1,a^+}\left(x_0;p\right)\left(\frac{\left|f'\left(a\right)\right|+m\alpha\left|f'\left(x_0/m\right)\right|}{\alpha+1}-\frac{\lambda\alpha\left(x_0-ma\right)^2}{m\left(\alpha+1\right)\left(2\alpha+1\right)}\right).\end{aligned}\tag{41}$$

Now, we let $x_0 \in [a,b]$ and $t \in (x_0, b]$. Then, by using strongly (α, m)-convexity of $|f'|$, we have

$$\left|f'(t)\right| \leq \left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\left|f'(b)\right| + m\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)\left|f'\left(\frac{x_0}{m}\right)\right| - \lambda m\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\left(1-\left(\frac{t-x_0}{b-x_0}\right)^{\alpha}\right)\left(\frac{mb-x_0}{m}\right)^2. \quad (42)$$

On the same lines as we have done for (19), (32), and (39), one can get, from (23) and (12), the following inequality:

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(x_0;p) - J_{\eta-1,b^-}(x_0;p) f(b)\right| \leq (b-x_0) J_{\eta-1,b^-}(x_0;p)\left(\frac{\left|f'(b)\right| + m\alpha\left|f'(x_0/m)\right|}{\alpha+1} - \frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right). \quad (43)$$

From inequalities (41) and (43), (30) is obtained. □

Remark 6

(i) Inequality (30) provides the refinement of Theorem 2.2 [1]

(ii) If $\alpha = 1$ in (30), then result for strongly m-convex will be obtained

(iii) If $\alpha = m = 1$ and $\lambda = 0$ in (30), then Corollary 2 [16] is obtained

(iv) If $\alpha = m = 1$, $\lambda = 0$, and $\omega = p = 0$ in (30), then Theorem 2 [17] is obtained

(v) If $\alpha = 1$ and $t = t^s$ in (30), then Theorem 5 in [18] is obtained

Corollary 4. *If we put $\xi = \eta$ in (30), then the following inequality is obtained:*

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(x_0;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,b^-} f\right)(x_0;p) - \left(J_{\xi-1,a^+}(x_0;p) f(a) + J_{\xi-1,b^-}(x_0;p) f(b)\right)\right| \leq \left(\frac{\left|f'(a)\right| + m\alpha\left|f'(x_0/m)\right|}{\alpha+1} - \frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)}\right)(x_0-a) J_{\xi-1,a^+}(x_0;p) + \left(\frac{\left|f'(b)\right| + m\alpha\left|f'(x_0/m)\right|}{\alpha+1} - \frac{\lambda\alpha(mb-x_0)^2}{m(\alpha+1)(2\alpha+1)}\right)(b-x_0) J_{\xi-1,b^-}(x_0;p), \quad (44)$$

$x_0 \in [a,b]$.

Remark 7. Inequality (44) provides the refinement of Corollary 2.2 in [8].

To prove our next result, we consider following lemma.

Lemma 2. *Let $f: [a,b] \longrightarrow \mathbb{R}, a < mb$, be strongly (α, m)-convex function. If $f(a + mb - x_0/m) = f(x_0), m \neq 0$ and $(\alpha, m) \in [0,1]^2$; then, the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq f(x_0)\left(\frac{1}{2^{\alpha}} + m\left(1-\frac{1}{2^{\alpha}}\right)\right) - \frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right) \times (a+mb-x_0-mx_0)^2. \quad (45)$$

Proof. As f is strongly (α, m)-convex function, we have

$$f\left(\frac{a+mb}{2}\right) \leq \frac{f((1-t)a+mtb)}{2^{\alpha}} + m\left(1-\frac{1}{2^{\alpha}}\right) f\left(\frac{ta+m(1-t)b}{m}\right) - \lambda m\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)\left(\frac{ta+m(1-t)b}{m} - ((1-t)a+mtb)\right)^2. \quad (46)$$

Let $x_0 = a(1-t) + mtb$, and we have $a + mb - x_0 = ta + m(1-t)b$.

$$f\left(\frac{a+mb}{2}\right) \leq \frac{f(x_0)}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{a+mb-x_0}{m}\right) - \frac{\lambda}{m}\left(\frac{2^\alpha - 1}{2^{2\alpha}}\right)(a + mb - x_0 - mx_0)^2. \tag{47}$$

Hence, by using $f(a + mb - x_0/m) = f(x_0), m \neq 0$, inequality (45) can be obtained. □

Theorem 6. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$, $a > b$, be a real-valued function. If f is positive, strongly (α, m)-convex, and $f(a + mb - x_0/m) = f(x_0), m \neq 0$, then, for $\xi, \eta > 0$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:*

$$\begin{aligned}
&\frac{1}{1/2^\alpha + m(1 - 1/2^\alpha)}\left(f\left(\frac{a+mb}{2}\right)\left(J_{\eta+1,b^-}(a;p) + J_{\xi+1,a^+}(b;p)\right) + \frac{\lambda}{m}\left(\left(\frac{2^\alpha - 1}{2^{2\alpha}}\right)(M1 + M2)\right)\right.\\
&\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(b;p)\\
&\leq \left(J_{\eta-1,b^-}(a;p) + J_{\xi-1,a^+}(b;p)\right)(b-a)^2\left(\frac{f(b) + m\alpha f(a/m)}{\alpha+1} - \frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
M1 &= (b-a)^{\eta+2} J_{\eta+1,b^-}(a;p) + 2(1+m)(b-a)^{\eta+1} J_{\eta+2,b^-}(a;p) + 2(1+m)^2 J_{\eta+3,b^-}(a;p),\\
M2 &= (b-a)^{\xi+2} J_{\xi+1,a^+}(b;p) + 2(1+m)(b-a)^{\xi+1} J_{\xi+2,a^+}(b;p) + 2(1+m)^2 J_{\xi+3,a^+}(b;p).
\end{aligned} \tag{49}$$

Proof. For $x_0 \in [a,b]$, we have

$$(x_0 - a)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(x_0 - a)^\mu; p\right) \leq (b-a)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l} \cdot \left(\omega(b-a)^\mu; p\right), \eta > 0. \tag{50}$$

As f is strongly (α, m)-convex, so, for $x_0 \in [a,b]$, we have

$$f(x_0) \leq \left(\frac{x_0 - a}{b-a}\right)^\alpha f(b) + m\left(1 - \left(\frac{x_0 - a}{b-a}\right)^\alpha\right) f\left(\frac{a}{m}\right) - \lambda m\left(\frac{x_0 - a}{b-a}\right)^\alpha\left(1 - \left(\frac{x_0 - a}{b-a}\right)^\alpha\right)\left(\frac{a - mb}{m}\right)^2. \tag{51}$$

By multiplying (50) and (51) and then integrating over $[a,b]$, we obtain

$$\begin{aligned}
&\int_a^b (x_0 - a)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(x_0 - a)^\mu; p\right) f(x_0)\mathrm{d}x_0\\
&\leq (b-a)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(b-a)^\mu; p\right)\left(f(b)\int_a^b \left(\frac{x_0 - a}{b-a}\right)^\alpha \mathrm{d}x_0\right.\\
&\quad + mf\left(\frac{a}{m}\right)\int_a^b\left(1 - \left(\frac{x_0 - a}{b-a}\right)^\alpha\right)\mathrm{d}x_0 - \frac{\lambda}{m}(a - mb)^2\\
&\quad \left.\times \int_a^b \left(\frac{x_0 - a}{b-a}\right)^\alpha\left(1 - \left(\frac{x_0 - a}{b-a}\right)^\alpha\right)\mathrm{d}x_0\right).
\end{aligned} \tag{52}$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(a;p) \leq (b-a)^{\eta+1} E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(b-a)^\mu; p\right) \times \left(\frac{f(b) + m\alpha f(a/m)}{\alpha+1} - \frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right), \tag{53}$$

that is,

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-} f\right)(a;p) \leq (b-a)^2 J_{\eta-1,b^-}(a;p) \times\left(\frac{f(b)+m\alpha f(a/m)}{\alpha+1} - \frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right). \tag{54}$$

Now, on the contrary, for $x_0 \in [a,b]$, we have

$$(b-x_0)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(b-x_0)^{\mu};p) \leq (b-a)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l} \cdot (\omega(b-a)^{\mu};p), \xi > 0. \tag{55}$$

By multiplying (51) and (55) and then integrating over $[a,b]$, we obtain

$$\begin{aligned}&\int_a^b (b-x_0)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(b-x_0)^{\mu};p) f(x_0)\mathrm{d}x_0 \\ &\leq (b-a)^{\xi} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(b-a)^{\mu};p)\left(f(b)\int_a^b \left(\frac{x_0-a}{b-a}\right)^{\alpha}\mathrm{d}x_0\right. \\ &\quad + mf\left(\frac{a}{m}\right)\int_a^b\left(1-\left(\frac{x_0-a}{b-a}\right)^{\alpha}\right)\mathrm{d}x_0 \\ &\quad \left. - \frac{\lambda}{m}(a-mb)^2\int_a^b\left(\frac{x_0-a}{b-a}\right)^{\alpha}\left(1-\left(\frac{x_0-a}{b-a}\right)^{\alpha}\right)\mathrm{d}x_0\right).\end{aligned} \tag{56}$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(b;p) \leq (b-a)^{\xi+1} E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(b-a)^{\mu};p) \times\left(\frac{f(b)+m\alpha f(a/m)}{\alpha+1} - \frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right), \tag{57}$$

that is,

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+} f\right)(b;p) \leq (b-a)^2 J_{\xi-1,a^+}(b;p) \times\left(\frac{f(b)+m\alpha f(a/m)}{\alpha+1} - \frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right). \tag{58}$$

Adding (54) and (58), we get the second inequality of (48). Multiplying (45) with $(x_0-a)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x_0-a)^{\mu};p)$ and integrating over $[a,b]$, we obtain

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\int_a^b (x_0-a)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x_0-a)^{\mu};p)\mathrm{d}x_0 \leq \left(\frac{1}{2^{\alpha}} + m\left(1-\frac{1}{2^{\alpha}}\right)\right) \\ &\int_a^b (x_0-a)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x_0-a)^{\mu};p) f(x_0)\mathrm{d}x_0 - \frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right) \\ &\times\int_a^b (x_0-a)^{\eta} E^{\gamma,\delta,k,c}_{\mu,\eta,l}(\omega(x_0-a)^{\mu};p)(a+mb-x_0-mx_0)^2\mathrm{d}x_0.\end{aligned} \tag{59}$$

By using (9) and (12), we obtain

$$f\left(\frac{a+mb}{2}\right)J_{\eta+1,b^-}(a;p)\le\left(\frac{1}{2^{\alpha}}+m\left(1-\frac{1}{2^{\alpha}}\right)\right)\times\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-}f\right)(a;p)-\frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)\times\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-}(a+mb-x_0-mx_0)^2\right)(a;p). \quad (60)$$

The integral operator appearing in the last term is computed as follows:

$$\begin{aligned}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-}(a+mb-x_0-mx_0)^2\right)(a;p)&=\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\\&\quad\times\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{1}{(l)_{n\delta}}\int_b^a(x_0-a)^{\eta+\mu n}(a+mb-x_0-mx_0)^2dx_0\\&=\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{1}{(l)_{n\delta}}\left(\frac{(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)}\right.\\&\quad\left.+\frac{2(1+m)(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)(\eta+\mu n+2)}+\frac{2(1+m)^2(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)(\eta+\mu n+2)(\eta+\mu n+3)}\right)\\&=(b-a)^{\eta+2}J_{\eta+1,b^-}(a;p)+2(1+m)(b-a)^{\eta+1}J_{\eta+2,b^-}(a;p)\\&\quad+2(1+m)^2J_{\eta+3,b^-}(a;p).\end{aligned} \quad (61)$$

Now, inequality (43) becomes

$$\begin{aligned}f\left(\frac{a+mb}{2}\right)J_{\eta+1,b^-}(a;p)&\le\left(\frac{1}{2^{\alpha}}+m\left(1-\frac{1}{2^{\alpha}}\right)\right)\\&\quad\times\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,b^-}f\right)(a;p)-\frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)\left((b-a)^{\eta+2}J_{\eta+1,b^-}(a;p)\right.\\&\quad\left.+2(1+m)(b-a)^{\eta+1}J_{\eta+2,b^-}(a;p)+2(1+m)^2J_{\eta+3,b^-}(a;p)\right).\end{aligned} \quad (62)$$

By multiplying (45) with $(b-x_0)^{\xi}E^{\gamma,\delta,k,c}_{\mu,\xi,l}(\omega(b-x_0)^{\mu};p)$ and integrating over $[a,b]$, also using (8) and (12), we obtain

$$\begin{aligned}f\left(\frac{a+mb}{2}\right)J_{\xi+1,a^+}(b;p)&\le\left(\frac{1}{2^{\alpha}}+m\left(1-\frac{1}{2^{\alpha}}\right)\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+}f\right)(b;p)\\&\quad-\frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+}(a+mb-x_0-mx_0)^2\right)(b;p).\end{aligned} \quad (63)$$

The integral operator appearing in the last term is computed as follows:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+}(a+mb-x_0-mx_0)^2\right)(b;p)$$
$$=(b-a)^{\xi+2}J_{\xi+1,a^+}(b;p)+2(1+m)(b-a)^{\xi+1}J_{\xi+2,a^+}(b;p)$$
$$+2(1+m)^2J_{\xi+3,a^+}(b;p). \tag{64}$$

Now, inequality (63) becomes

$$f\left(\frac{a+mb}{2}\right)J_{\xi+1,a^+}(b;p)\le\left(\frac{1}{2^\alpha}+m\left(1-\frac{1}{2^\alpha}\right)\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+}f\right)(b;p)$$
$$-\frac{\lambda}{m}\left(\frac{2^\alpha}{2^{2\alpha}-1}\right)\Big((b-a)^{\xi+2}J_{\xi+1,a^+}(b;p) \tag{65}$$
$$+2(1+m)(b-a)^{\xi+1}J_{\xi+2,a^+}(b;p)+2(1+m)^2J_{\xi+3,a^+}(b;p)\Big).$$

By adding (62) and (65), first inequality of (48) can be obtained. □

Remark 8

(i) Inequality (48) provides the refinement of Theorem 2.3 in [1]

(ii) If $\alpha=1$ in (48), then result for strongly m-convex will be obtained

(iii) If $\alpha=m=1$ and $\lambda=0$ in (48), then Corollary 2 in [16] is obtained

(iv) If $\alpha=m=1$, $\lambda=0$, and $\omega=p=0$ in (48), then Theorem 2 in [17] is obtained

(v) If $\alpha=1$ and $t=t^s$ in (48), then Theorem 7 in [18] is obtained

Corollary 5. *If we put $\xi=\eta$ in (48), then the following inequality is obtained:*

$$\frac{1}{1/2^\alpha+m(1-1/2^\alpha)}\left(f\left(\frac{a+mb}{2}\right)\left(J_{\xi+1,b^-}(a;p)+J_{\xi+1,a^+}(b;p)\right)\right.$$
$$\left.+\left(\frac{\lambda}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)(M1+M2)\right)\right)\le\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,a^+}f\right)(b;p) \tag{66}$$
$$\le\left(J_{\xi-1,b^-}(a;p)+J_{\xi-1,a^+}(b;p)\right)(b-a)^2\left(\frac{f(b)+m\alpha f(a/m)}{\alpha+1}-\frac{\lambda\alpha(a-mb)^2}{m(\alpha+1)(2\alpha+1)}\right).$$

Remark 9. Inequality (66) provides the refinement of Corollary 2.3 in [1].

3. Concluding Remarks

The presented results are the refinements of the bounds of generalized fractional integral operators given in (8) and (9) for strongly (α,m)-convex functions. From the presented results, one can obtain already proved results for convex, m-convex, and (α,m)-convex functions. Moreover, the refinements of some known fractional versions of the Hadamard inequality are also given.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Discrete-Time Predator-Prey Model with Bifurcations and Chaos

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In this paper, local dynamics, bifurcations and chaos control in a discrete-time predator-prey model have been explored in $\mathbb{R}_+^2$. It is proved that the model has a trivial fixed point for all parametric values and the unique positive fixed point under definite parametric conditions. By the existing linear stability theory, we studied the topological classifications at fixed points. It is explored that at trivial fixed point model does not undergo the flip bifurcation, but flip bifurcation occurs at the unique positive fixed point, and no other bifurcations occur at this point. Numerical simulations are performed not only to demonstrate obtained theoretical results but also to tell the complex behaviors in orbits of period-4, period-6, period-8, period-12, period-17, and period-18. We have computed the Maximum Lyapunov exponents as well as fractal dimension numerically to demonstrate the appearance of chaotic behaviors in the considered model. Further feedback control method is employed to stabilize chaos existing in the model. Finally, existence of periodic points at fixed points for the model is also explored.

1. Introduction

In the mid-1920s, an Italian Biologist U. D'Ancona studied the population variations in different species of fish that were interacting with one another. In his investigation, he found some percentage data of total catch of different fish species during the World War I which was brought into various Mediterranean ports. In precise, the data determine the percentage (%) of total catch of selachians, which are not adorable as food fish, and during the years 1914–1923, the percentage data for the port of Fiume, Italy, is given in Table 1.

D'Ancona was surprised by a very large increase in the percentage of selachians during the World War I. He reasoned the increase in the percentage of selachians was because of decline in fishing during this period. But how does the intensity of fishing affect the fish populations? The answer was obviously the struggle for existence between competing species, which was of great concern to D'Ancona and also to the fishing industry. Naturally, selachians depend on food fish for their survival as selachians are predators and food fish are prey. D'Ancona thought that this accounted for the large increase in number of selachians during the war period. Since the level of fishing was less, there were more food fish available to selachians, which therefore increased rapidly. D'Ancona just shows the increase in number of selachians when fishing's level is reduced. D'Ancona did not explain why a declining level of fishing is more helpful to predators than to their prey. Later, paying all possible biological clarifications to this phenomenon, D'Ancona twisted to his coworker, the well-known mathematician Vito Volterra. With hope, Volterra would come up with a mathematical model of the growth of the selachians and food fish, their prey, and his model would provide the answer to D'Ancona's question. Volterra started his analysis on this problem by separating all the fish into the prey population $x(t)$ and the predator population $y(t)$. Then, he reasoned that the food fish do not compete very fast among themselves for their food supply since this is very plentiful, and the density of fish population is not very much. Hence, in the absence of predators, their prey would grow according to the Malthusian law of population growth: $(dx/dt) = rx$ for positive constant r. Next, Volterra reasoned the number of contacts per unit time between selachians (predators) and

Table 1: Percentage data for the port of Fiume, Italy, during the years 1914–1923.

Number of years	% values
1914	11.9
1915	21.4
1916	22.1
1917	21.2
1918	36.4
1919	27.3
1920	16.0
1921	15.9
1922	14.8
1923	10.7

food fish (prey) is bxy for positive constant b. Therefore, $(\mathrm{d}x/\mathrm{d}t) = rx - bxy$, and so Volterra reasoned that predators have a characteristic pace of lessening $-\mathrm{d}y$ relative to their present number and that they likewise increment at a rate cxy relative to their present number y and their nourishment supply x, and hence, one has [1–5]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx - bxy,\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cxy - dy. \tag{1}$$

Now include the effects of fishing in (1). It is observed that fishing decreases the population of food fish at a rate $\varepsilon x(t)$ and decreases the population of selachians at a rate $\varepsilon y(t)$, and the parameter ϵ reflects the intensity of fishing. So, by the effect of fishing, the continuous-time model which is depicted in (1) becomes as follows [1]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (r-\varepsilon)x - bxy,\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cxy - (d+\varepsilon)y. \tag{2}$$

It is important here to note that generally many biological models are directed by continuous as well as discrete-time systems, and in recent years, many authors gave great contribution towards discrete models [6–15]. The reasons are that, for nonoverlapping generation, discrete models are much convincing than continuous models, and moreover, these models provide more effective computational results for numerical simulations [6, 16–21]. Due to scientific computation, in the present study, our aim is to explore the qualitative behavior of the discrete-time model corresponding to (2). It is predicted that the discrete model is dynamically consistent with the continuous-time model. It is also mentioned in [15] that if population has a nonoverlapping generation, as stated above, it is essential to write a discrete system that corresponds to model (2). So, applying the forward Euler scheme, (2) becomes as follows:

$$\frac{x_{t+1}-x_t}{h} = (r-\varepsilon)x_t - bx_t y_t,\qquad \frac{y_{t+1}-y_t}{h} = cx_t y_t - (d+\varepsilon)y_t. \tag{3}$$

After some manipulations, from (3), one gets

$$x_{t+1} = (1+h(r-\varepsilon))x_t - bhx_t y_t,\qquad y_{t+1} = (1-h(d+\varepsilon))y_t + chx_t y_t. \tag{4}$$

This paper is organized as follows: Section 2 is about the topological classifications at fixed points of the model (4), whereas existence of possible bifurcations at respective fixed points is given in Section 3. The comprehensive bifurcation analysis at a positive fixed point is investigated in Section 4. In Section 5, some simulations are performed to demonstrate obtained theoretical results, and this is also about the study of fractal demission that characterized the strange attractors. In Section 6, we investigated the chaos control by the feedback control method, whereas existence of periodic points is studied in Section 7. The conclusion of the paper is given in Section 8.

2. Topological Classifications at Fixed Points

We will study the topological classifications at fixed points of the model (4) in this section. In order to determine fixed points, one need to solve the following system, where $(\widehat{x},\widehat{y})$ is the fixed point of (4):

$$\widehat{x} = (1+h(r-\varepsilon))\widehat{x} - bh\widehat{x}\widehat{y},\qquad \widehat{y} = (1-h(d+\varepsilon))\widehat{y} + ch\widehat{x}\widehat{y}. \tag{5}$$

It is noted that $(\widehat{x},\widehat{y}) = (0,0)$, satisfying (5) obviously, and so $E_{\text{trival}} = (0,0)$ is a trivial fixed point of (4). For a positive fixed point, system (5) reduces into the following form:

$$h(r-\varepsilon) - bh\widehat{y} = 0,\qquad -h(d+\varepsilon) + ch\widehat{x} = 0. \tag{6}$$

From (6), one gets $x = ((d+\varepsilon)/c)$ and $y = ((r-\varepsilon)/b)$. Therefore, (4) has the unique positive fixed point: $E_{\text{positive}} = (((d+\varepsilon)/c), ((r-\varepsilon)/b))$ if $r > \epsilon$.

Hereafter, we will study the topological classifications at E_{trival} and E_{positive} of the model (4) by method of linearization. So, the Jacobian matrix $\Omega\big|_{E_{\widehat{(x,y)}}}$ evaluated at $E_{(\widehat{x},\widehat{y})}$ becomes as follows:

$$\Omega\big|_{E_{\widehat{(x,y)}}} := \begin{pmatrix} 1+h(r-\varepsilon)-bh\widehat{y} & -bh\widehat{x} \\ ch\widehat{y} & 1-h(d+\varepsilon)+ch\widehat{x} \end{pmatrix}. \tag{7}$$

Now, the auxiliary equation of $\Omega\big|_{E_{\widehat{(x,y)}}}$ at $E_{(\widehat{x},\widehat{y})}$ is as follows:

$$\zeta^2 - \mathcal{T}\left(E_{(\widehat{x},\widehat{y})}\right)\zeta + \mathcal{D}\left(E_{(\widehat{x},\widehat{y})}\right) = 0, \tag{8}$$

where

$$\mathcal{T}\left(E_{(\widehat{x},\widehat{y})}\right) = 2 + hr - 2h\varepsilon - bh\widehat{y} - hd + ch\widehat{x},$$
$$\mathcal{D}\left(E_{(\widehat{x},\widehat{y})}\right) = (1+h(r-\varepsilon)-bh\widehat{y})(1-h(d+\varepsilon)+ch\widehat{x}) + bch^2\widehat{x}\widehat{y}. \tag{9}$$

Now, in the rest of the section, we will give topological classification at equilibria: E_{trival} and E_{positive} as follows:

2.1. Topological Classifications at E_{trival}. It is noted that, at E_{trival}, (7) takes the following form:

$$\Omega\Big|_{E_{\text{trival}}} := \begin{pmatrix} 1+h(r-\varepsilon) & 0 \\ 0 & 1-h(d+\varepsilon) \end{pmatrix}, \quad (10)$$

whose characteristic roots are $\zeta_1 = 1 + h(r-\varepsilon)$ and $\zeta_2 = 1 - h(d+\varepsilon)$. So based on stability theory, one can conclude the topological classifications at E_{trival} as the following result.

Proposition 1. *For E_{trival}, the following classifications hold:*

(I) If

$$0 < h < \min\left\{\frac{2}{\varepsilon - r}, \frac{2}{d+\varepsilon}\right\}, \quad \text{where } \varepsilon > r, \quad (11)$$

then E_{trival} is a sink.

(II) If

$$h > \max\left\{\frac{2}{\varepsilon - r}, \frac{2}{d+\varepsilon}\right\}, \quad \text{where } \varepsilon > r, \quad (12)$$

then E_{trival} is a source.

(III) If

$$\frac{2}{\varepsilon - r} < h < \frac{2}{d+\varepsilon}, \quad (13)$$

then E_{trival} is a saddle.

(IV) If

$$h = \frac{2}{\varepsilon - r} \quad (14)$$

or

$$h = \frac{2}{d+\varepsilon}, \quad (15)$$

then E_{trival} is nonhyperbolic.

2.2. Topological Classifications at $E_{positive}$. Now, we will give topological classifications at E_{positive} for the considered system. After some manipulations, at E_{positive}, (7) takes the following form:

$$\Omega\Big|_{E_{\text{positive}}} := \begin{pmatrix} 1 & -\frac{bh}{c}(d+\varepsilon) \\ \frac{ch}{b}(r-\varepsilon) & 1 \end{pmatrix}. \quad (16)$$

Furthermore, at E_{positive}, (8) becomes

$$\zeta^2 - \mathcal{T}\left(E_{\text{positive}}\right)\zeta + \mathcal{D}\left(E_{\text{positive}}\right) = 0, \quad (17)$$

where

$$\begin{aligned} \mathcal{T}\left(E_{\text{positive}}\right) &= 2, \\ \mathcal{D}\left(E_{\text{positive}}\right) &= 1 + h^2(r-\varepsilon)(d+\varepsilon). \end{aligned} \quad (18)$$

Finally, from (17) one gets

$$\zeta_{1,2} = \frac{2 \pm \sqrt{\text{Disc}}}{2}, \quad (19)$$

where

$$\begin{aligned} \text{Disc} &:= \left(\mathcal{T}\left(E_{\text{positive}}\right)\right)^2 - 4\mathcal{D}\left(E_{\text{positive}}\right) \\ &= 4h^2(\varepsilon - r)(d+\varepsilon). \end{aligned} \quad (20)$$

Hereafter, at equilibrium: E_{positive}, we will summarize the topological classifications by allowing sign of discriminant quantity, i.e., $\text{Disc} = 4h^2(\varepsilon - r)(d+\varepsilon) < 0$ (respectively ≥ 0), as a following proposition.

Proposition 2. *If $\text{Disc} = 4h^2(\varepsilon - r)(d+\varepsilon) < 0$, then for $E_{positive}$, the following classifications hold:*

(I) $E_{positive}$ is never stable focus.

(II) If

$$r > \varepsilon, \quad (21)$$

then $E_{positive}$ is unstable focus.

(III) If

$$r = \varepsilon, \quad (22)$$

then $E_{positive}$ is nonhyperbolic.

Proposition 3. *If $\text{Disc} = 4h^2(\varepsilon - r)(d+\varepsilon) \geq 0$, then for $E_{positive}$, the following classifications hold:*

(I) If

$$r > \frac{\varepsilon h^2(d+\varepsilon) - 4}{h^2(d+\varepsilon)} \quad (23)$$

with

$$d > \frac{4 - \varepsilon^2 h^2}{\varepsilon h^2}, \tag{24}$$

then $E_{positive}$ is a stable node.

(II) If

$$r < \frac{\varepsilon h^2 (d + \varepsilon) - 4}{h^2 (d + \varepsilon)}, \tag{25}$$

along with (24) holds, then $E_{positive}$ is an unstable node.

(III) If

$$r = \frac{\varepsilon h^2 (d + \varepsilon) - 4}{h^2 (d + \varepsilon)}, \tag{26}$$

then $E_{positive}$ is nonhyperbolic.

3. Existence of Possible Bifurcations at Fixed Points: E_{trival} and E_{positive}

In view of obtained results in Section 2 regarding topological classifications at equilibria E_{trival} and E_{positive}, we will study the existence of possible bifurcations in this section, as follows:

(I) Recall that, at E_{trival}, $\Omega\big|_{E_{\text{trival}}}$ has two characteristic roots in which $\zeta_1\big|_{(14)} = -1$, but $\zeta_2\big|_{(14)} = 1 - (2/\varepsilon - r)(d + \varepsilon) \neq -1$ or 1. This implies that (4) may undergo the flip bifurcation if $(b, c, d, h, r, \varepsilon)$ locate in the set:

$$\text{FB}\Big|_{E_{\text{trival}}} = \left\{(b, c, d, h, r, \varepsilon), h = \frac{2}{\varepsilon - r}\right\}, \tag{27}$$

or

Again under the nonhyperbolic condition, which is depicted in (15), one can obtain that $\zeta_1\big|_{(15)} = 1 + (2/d + \varepsilon)(r - \varepsilon) \neq -1$ or 1, but $\zeta_2\big|_{(15)} = -1$. This implies that model (4) may undergo the flip bifurcation if $(b, c, d, h, r, \varepsilon)$ locate in the set:

$$\text{FB}\Big|_{E_{\text{trival}}} = \left\{(b, c, d, h, r, \varepsilon), h = \frac{2}{d + \varepsilon}\right\}. \tag{28}$$

(II) Under the hypothesis of Proposition 2 and obtained nonhyperbolic condition, which is depicted in (22), one gets $\left|\zeta_{1,2}\right|_{(22)} = 1$. Hence, model (4) may undergo a Neimark–Sacker bifurcation if parameters $(b, c, d, h, r, \varepsilon)$ locate in the set:

$$\text{HB}\Big|_{E_{\text{positive}}} = \{(b, c, d, h, r, \varepsilon), r = \varepsilon\}. \tag{29}$$

(III) Under the hypothesis of Proposition 3 and obtained nonhyperbolic condition, which is depicted in (26), one gets $\zeta_1|_{(26)} = -1$, but $\zeta_2|_{(26)} = 3 \neq -1$ or 1. So model (4) may undergo the flip bifurcation if $(b, c, d, h, r, \varepsilon)$ locate in the set:

$$\text{FB}\Big|_{E_{\text{positive}}} = \left\{(b, c, d, h, r, \varepsilon), r = \frac{\varepsilon h^2 (d + \varepsilon) - 4}{h^2 (d + \varepsilon)}\right\}. \tag{30}$$

Note: for case I, it is easy to see that model does not undergo flip bifurcation if $(b, c, d, h, r, \varepsilon) \in \text{FB}|E_{\text{trival}}$, and hence, E_{trival} is degenerated with a higher codimension.

In the subsequent section, we will present comprehensive N-S and flip bifurcations analysis when parameters, respectively, $(b, c, d, h, r, \varepsilon) \in \text{HB}|_{E_{\text{positive}}}$ and $(b, c, d, h, r, \varepsilon) \in \text{FB}|_{E_{\text{positive}}}$.

4. Comprehensive Bifurcation Analysis at E_{positive}

4.1. N-S Bifurcation at $E_{positive}$. In the subsequent section, we will discuss N-S bifurcation of model (4) at E_{positive} if $(b, c, d, h, r, \varepsilon) \in HB|_{E_{\text{positive}}}$, and hence, the result can be stated as the following theorem.

Theorem 1. *If $(b, c, d, h, r, \varepsilon) \in HB|_{E_{positive}}$, then at $E_{positive}$, model (4) does not undergo N-S bifurcation.*

Proof. Since $(b, c, d, h, r, \varepsilon) \in HB|_{E_{\text{positive}}}$ and hence if bifurcation parameter r varies in a small neighborhood of r^*, that is, $r = r^* + \tau$, where $\tau \ll 1$, then (4) becomes as follows:

$$\begin{aligned} x_{t+1} &= (1 + h(r^* + \tau - \varepsilon))x_t - bhx_t y_t, \\ y_{t+1} &= (1 - h(d + \varepsilon))y_t + chx_t y_t, \end{aligned} \tag{31}$$

with $E_{\text{positive}}(\tau) = ((d + \varepsilon/c), (r^* + \tau - \varepsilon/b))$ if $r^* + \tau > \varepsilon$. The auxiliary equation of $\Omega|_{E_{\text{positive}}}(\tau)$ at $E_{\text{positive}}(\tau)$ becomes

$$\zeta^2 - \mathcal{T}\left(E_{\text{positive}}(\tau)\right)\zeta + \mathcal{D}\left(E_{\text{positive}}(\tau)\right) = 0, \tag{32}$$

where

$$\begin{aligned} \mathcal{T}\left(E_{\text{positive}}(\tau)\right) &= 2, \\ \mathcal{D}\left(E_{\text{positive}}(\tau)\right) &= 1 + h^2 (r^* + \tau - \varepsilon)(d + \varepsilon). \end{aligned} \tag{33}$$

Finally, from (32), one gets

$$\zeta_{1,2}(\tau) = 1 \pm h\sqrt{(r^* + \tau - \varepsilon)(d + \varepsilon)}\iota. \tag{34}$$

From (34), we obtain

$$\left|\zeta_{1,2}\right| := \sqrt{\mathscr{D}\left(E_{\text{positive}}(\tau)\right)},$$
$$\left.\frac{d\left|\zeta_{1,2}\right|}{d\tau}\right|_{\tau=0} := \frac{h^2}{2}(d+\varepsilon) > 0. \tag{35}$$

Hereafter by using the following transformation

$$\Phi_t = x_t - \frac{d+\varepsilon}{c},$$
$$\Psi_t = y_t - \frac{r^* + \tau - \varepsilon}{b}, \tag{36}$$

transform $E_{\text{positive}}(\tau)$ in origin. In view of (36), from (31), one gets

$$\Phi_{t+1} = \left(1 + h\left(r^* + \tau - \varepsilon\right)\right)\left(\Phi_t + \frac{d+\varepsilon}{c}\right) - bh\left(\Phi_t + \frac{d+\varepsilon}{c}\right)\left(\Psi_t + \frac{r^* + \tau - \varepsilon}{b}\right) - \frac{d+\varepsilon}{c},$$
$$\Psi_{t+1} = \left(1 - h(d+\varepsilon)\right)\left(\Psi_t + \frac{r^* + \tau - \varepsilon}{b}\right) + ch\left(\Phi_t + \frac{d+\varepsilon}{c}\right)\left(\Psi_t + \frac{r^* + \tau - \varepsilon}{b}\right) - \frac{r^* + \tau - \varepsilon}{b}. \tag{37}$$

Now, we will study normal form of (37) if $\tau = 0$. After Taylor series expansion about $(\Phi_t, \Psi_t) = (0,0)$, one gets

$$\Phi_{t+1} = \Phi_t - \frac{bh}{c}(d+\varepsilon)\Psi_t - bh\Phi_t\Psi_t,$$
$$\Psi_{t+1} = \frac{ch}{b}(r-\varepsilon)\Phi_t + \Psi_t + ch\Phi_t\Psi_t. \tag{38}$$

In order to transform liner part of (38) into canonical form, we use the following transformation that can be easily constructed by computation:

$$\begin{pmatrix}\Phi_t \\ \Psi_t\end{pmatrix} = \begin{pmatrix} -\frac{bh}{c}(d+\varepsilon) & 0 \\ 0 & h\sqrt{(r-\varepsilon)(d+\varepsilon)}\end{pmatrix}\begin{pmatrix}X_t \\ Y_t\end{pmatrix}. \tag{39}$$

In view of (39) and (38),

$$\begin{pmatrix}X_{t+1} \\ Y_{t+1}\end{pmatrix} = \begin{pmatrix} 1 & h\sqrt{(r-\varepsilon)(d+\varepsilon)} \\ -h\sqrt{(r-\varepsilon)(d+\varepsilon)} & 1\end{pmatrix}\begin{pmatrix}X_t \\ Y_t\end{pmatrix} + \begin{pmatrix}F(X_t, Y_t) \\ G(X_t, Y_t)\end{pmatrix}, \tag{40}$$

where

$$F(X_t, Y_t) = -bh^2\sqrt{(r-\varepsilon)(d+\varepsilon)}X_tY_t,$$
$$G(X_t, Y_t) = -bh^2(d+\varepsilon)X_tY_t. \tag{41}$$

From (41), one gets

$$\left.F_{X_tX_t}\right|_{E_{\text{trival}}} = \left.F_{Y_tY_t}\right|_{E_{\text{trival}}} = 0,$$
$$\left.F_{X_tY_t}\right|_{E_{trival}} = -bh^2\sqrt{(r-\varepsilon)(d+\varepsilon)},$$
$$\left.G_{X_tX_t}\right|_{E_{\text{trival}}} = \left.G_{Y_tY_t}\right|_{E_{\text{trival}}} = 0,$$
$$\left.G_{X_tY_t}\right|_{E_{\text{trival}}} = -bh^2(d+\varepsilon),$$
$$\left.F_{X_tX_tX_t}\right|_{E_{\text{trival}}} = \left.F_{X_tX_tY_t}\right|_{E_{\text{trival}}} = \left.F_{X_tY_tY_t}\right|_{E_{\text{trival}}} = \left.F_{Y_tY_tY_t}\right|_{E_{\text{trival}}} = 0,$$
$$\left.G_{X_tX_tX_t}\right|_{E_{\text{trival}}} = \left.G_{X_tX_tY_t}\right|_{E_{\text{trival}}} = \left.G_{X_tY_tY_t}\right|_{E_{\text{trival}}} = \left.G_{Y_tY_tY_t}\right|_{E_{\text{trival}}} = 0. \tag{42}$$

In order to undergo the said bifurcation, the following quantity should be nonzero [22–29]:

$$\Gamma = -\Re\left(\frac{(1-2\bar{\zeta})\bar{\zeta}^2}{1-\zeta}\delta_{11}\delta_{20}\right) - \frac{1}{2}\|\delta_{11}\|^2 - \|\delta_{02}\|^2 + \Re\left(\bar{\zeta}\delta_{21}\right), \quad (43)$$

where

$$\begin{aligned}
\delta_{02} &= \frac{1}{8}\left[F_{X_tX_t} - F_{Y_tY_t} + 2G_{X_tY_t} + \iota\left(G_{X_tX_t} - G_{Y_tY_t} + 2F_{X_tY_t}\right)\right]\Big|_{E_{\text{trival}}},\\
\delta_{11} &= \frac{1}{4}\left[F_{X_tX_t} + F_{Y_tY_t} + \iota\left(G_{X_tX_t} + G_{Y_tY_t}\right)\right]\Big|_{E_{\text{trival}}},\\
\delta_{20} &= \frac{1}{8}\left[F_{X_tX_t} - F_{Y_tY_t} + 2G_{X_tY_t} + \iota\left(G_{X_tX_t} - G_{Y_tY_t} - 2F_{X_tY_t}\right)\right]\Big|_{E_{\text{trival}}},\\
\delta_{21} &= \frac{1}{16}\left[F_{X_tX_tX_t} + F_{X_tY_tY_t} + G_{X_tX_tY_t} + G_{Y_tY_tY_t} + \iota\left(G_{X_tX_tX_t} + G_{X_tY_tY_t} - F_{X_tX_tY_t} - F_{Y_tY_tY_t}\right)\right]_{E_{\text{trival}}}.
\end{aligned} \quad (44)$$

Utilizing (42) in (44), one gets

$$\begin{aligned}
\delta_{02} &= -\frac{bh^2}{4}\left(d+\varepsilon+\sqrt{(r-\varepsilon)(d+\varepsilon)}\,\iota\right),\\
\delta_{11} &= \delta_{21} = 0,\\
\delta_{20} &= \frac{bh^2}{4}\left(-d-\varepsilon+\sqrt{(r-\varepsilon)(d+\varepsilon)}\iota\right).
\end{aligned} \quad (45)$$

Using (45) in (43), one gets $\Omega = -(bh^4/16)(d+\varepsilon)^2 < 0$. Finally, the model considered undergoing N-S bifurcation requires that $\zeta_{1,2}^m \neq 1$, $m = 1,2,3,4$ if $\tau = 0$ which corresponds to $\mathcal{T}(E_{\text{positive}}(0)) \neq -2,0,1,2$. However, $\mathcal{T}(E_{\text{positive}}(0)) = 2$ which contradicts to the fact that $\mathcal{T}(E_{\text{positive}}(0)) \neq -2,0,1,2$, and hence, eigenvalues $\zeta_{1,2}$ of fixed point $(0,0)$ lay in the interaction of the unit circle with the coordinate axes when $\tau = 0$. Therefore, model (4) does not undergo N-S bifurcation.

4.2. Flip Bifurcation at $E_{positive}$. In the subsequent section, we will discuss flip bifurcation of model (4) at E_{positive} if $(b,c,d,h,r,\varepsilon) \in FB|_{E_{\text{positive}}}$, and hence, the result can be stated as the following theorem.

Theorem 2. *If $(b,c,d,h,r,\varepsilon) \in FB|_{E_{positive}}$, then at $E_{positive}$, model (4) undergoes the flip bifurcation if p varies in a small neighborhood of the origin.*

Proof. It is noted that if r in a small *nbhd* of r^*, *i.e*, $r = r^* + p$, where $p \ll 1$, then model (4) becomes

$$\begin{aligned}
x_{t+1} &= \left(1 + h\left(r^* + p - \varepsilon\right)\right)x_t - bhx_ty_t,\\
y_{t+1} &= \left(1 - h(d+\varepsilon)\right)y_t + chx_ty_t.
\end{aligned} \quad (46)$$

Now, one can transform $E_{\text{positive}}(p)$ in origin by using the following transformations:

$$\widetilde{\Phi_t} = x_t - \frac{d+\varepsilon}{c}, \tilde{\Psi}_t = y_t - \frac{r^*+p-\varepsilon}{b}. \quad (47)$$

From (47), (46) takes the following form:

$$\begin{aligned}
\widetilde{\Phi_{t+1}} &= \left(1 + h\left(r^* + p - \varepsilon\right)\right)\left(\widetilde{\Phi_t} + \frac{d+\varepsilon}{c}\right) - bh\left(\widetilde{\Phi_t} + \frac{d+\varepsilon}{c}\right)\left(\tilde{\Psi}_t + \frac{r^*+p-\varepsilon}{b}\right) - \frac{d+\varepsilon}{c}, \widetilde{\Psi_{t+1}} = \left(1 - h(d+\varepsilon)\right)\left(\tilde{\Psi}_t + \frac{r^*+p-\varepsilon}{b}\right)\\
&\quad + ch\left(\widetilde{\Phi_t} + \frac{d+\varepsilon}{c}\right)\left(\tilde{\Psi}_t + \frac{r^*+p-\varepsilon}{b}\right) - \frac{r^*+p-\varepsilon}{b}.
\end{aligned} \quad (48)$$

By Taylor series about $(\widetilde{\Phi_t}, \tilde{\Psi}_t) = (0,0)$, one gets

$$\widetilde{\Phi_{t+1}} = \widetilde{\Phi_t} - \frac{bh}{c}(d+\varepsilon)\tilde{\Psi}_t - bh\widetilde{\Phi_t}\tilde{\Psi}_t + hp\widetilde{\Phi_t}, \widetilde{\Psi_{t+1}} = \frac{ch}{b}(r-\varepsilon)\widetilde{\Phi_t} + \tilde{\Psi}_t + ch\widetilde{\Phi_t}\tilde{\Psi}_t. \quad (49)$$

Now by using following transformation

$$\begin{pmatrix} \widetilde{\Phi}_t \\ \widetilde{\Psi}_t \end{pmatrix} := \begin{pmatrix} -\frac{b}{c(r-\varepsilon)}\sqrt{(\varepsilon-r)(d+\varepsilon)} & \frac{b}{c(r-\varepsilon)}\sqrt{(\varepsilon-r)(d+\varepsilon)} \\ 1 & 1 \end{pmatrix} (\widetilde{X}_t \widetilde{Y}_t), \tag{50}$$

linear part of (49) transforms into canonical form. In view of (50), (49) becomes

$$\begin{pmatrix} \widetilde{X_{t+1}} \\ \widetilde{Y_{t+1}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} \widetilde{F}(\widetilde{X_t}, \widetilde{Y}_t, p) \\ \widetilde{G}(\widetilde{X_t}, \widetilde{Y}_t, p) \end{pmatrix}, \tag{51}$$

where

$$\widetilde{F}(\widetilde{X_t}, \widetilde{Y}_t, p) = -\frac{1}{2}bh\widetilde{X_t}^2 + \frac{1}{2}bh\widetilde{Y}_t^{\,2} - \frac{1}{2}hp\widetilde{X_t}, \widetilde{G}(\widetilde{X_t}, \widetilde{Y}_t, p) = \frac{1}{2}ch\widetilde{X_t}^2 - \frac{1}{2}ch\widetilde{Y}_t^{\,2}. \tag{52}$$

Now, for (51), center manifold $M^c E_{\text{trival}}$ at E_{trival} is explored in a small *nbhd* of p. Therefore, $M^c E_{\text{trival}}$ can be expressed as follows:

$$M^c E_{\text{trival}} = \left\{ (\widetilde{X_t}, \widetilde{Y}_t) \colon \widetilde{Y}_t = c_0 p + c_1 \widetilde{X_t}^2 + c_2 \widetilde{X_t} p + c_3 p^3 + O\left(\left(|\widetilde{X_t}| + |p| \right)^3 \right) \right\}. \tag{53}$$

After some manipulations, one gets

$$\begin{aligned} c_0 &= c_2 = c_3 = 0, \\ c_1 &= -\frac{1}{4}ch. \end{aligned} \tag{54}$$

Finally, the map (51) restricting to $M^c E_{\text{trival}}$ is

$$\widetilde{F}(\widetilde{X_t}) = -\widetilde{X_t} + h_1 \widetilde{X_t}^2 + h_2 \widetilde{X_t} p + h_3 \widetilde{X_t}^2 p + h_4 \widetilde{X_t} p^2 + h_5 \widetilde{X_t}^3 + O\left(\left(|\widetilde{X_t}| + |p| \right)^4 \right), \tag{55}$$

with

$$\begin{aligned} h_1 &= -\frac{1}{2}bh, \\ h_2 &= -\frac{h}{2}, \\ h_3 &= h_4 = h_5 = 0. \end{aligned} \tag{56}$$

In order for the map (51) to undergo flip bifurcation, the following discriminatory quantities are required to be nonzero [28, 29]:

$$\begin{aligned} \underset{1}{\wedge} &= \left(\frac{\partial^2 \widetilde{F}}{\partial \widetilde{X_t} \partial p} + \frac{1}{2} \frac{\partial \widetilde{F}}{\partial p} \frac{\partial^2 \widetilde{F}}{\partial \widetilde{X_t}^2} \right) \Big|_{(\widetilde{X}_t, p)} = (0,0), \\ \underset{2}{\wedge} &= \left(\frac{1}{6} \frac{\partial^3 \widetilde{F}}{\partial \widetilde{X_t}^3} + \left(\frac{1}{2} \frac{\partial^2 \widetilde{F}}{\partial \widetilde{X_t}^2} \right)^2 \right) \Big|_{(\widetilde{X}_t, p)} = (0,0). \end{aligned} \tag{57}$$

On computation, one gets $\underset{1}{\wedge} = -(h/2) \neq 0$, but $\underset{2}{\wedge} = (1/6)b^2h^2 > 0$, which shows that model (4) undergoes flip bifurcation if $(b, c, d, h, r, \varepsilon) \in FB|_{E_{\text{positive}}}$ and, in particular, stable period-2 points bifurcating from E_{positive}.

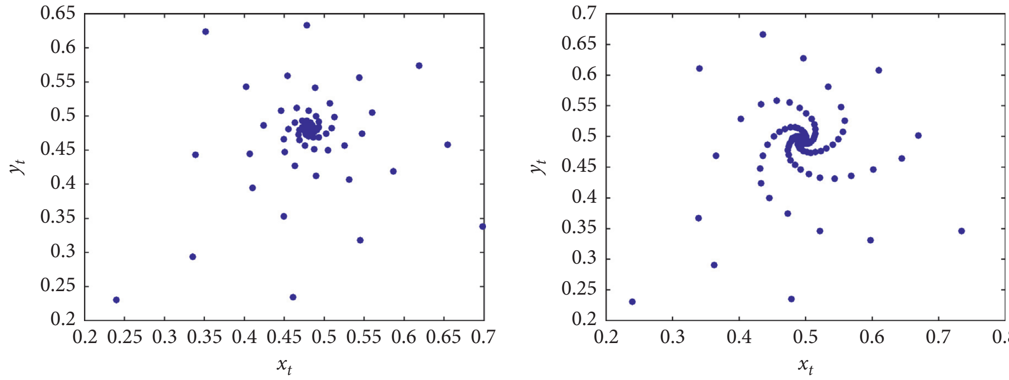

Figure 1: Stable node of model (4). (a) $r = 2.1$ with (0.24, 0.23). (b) $r = 2.3$ with (0.4, 0.5).

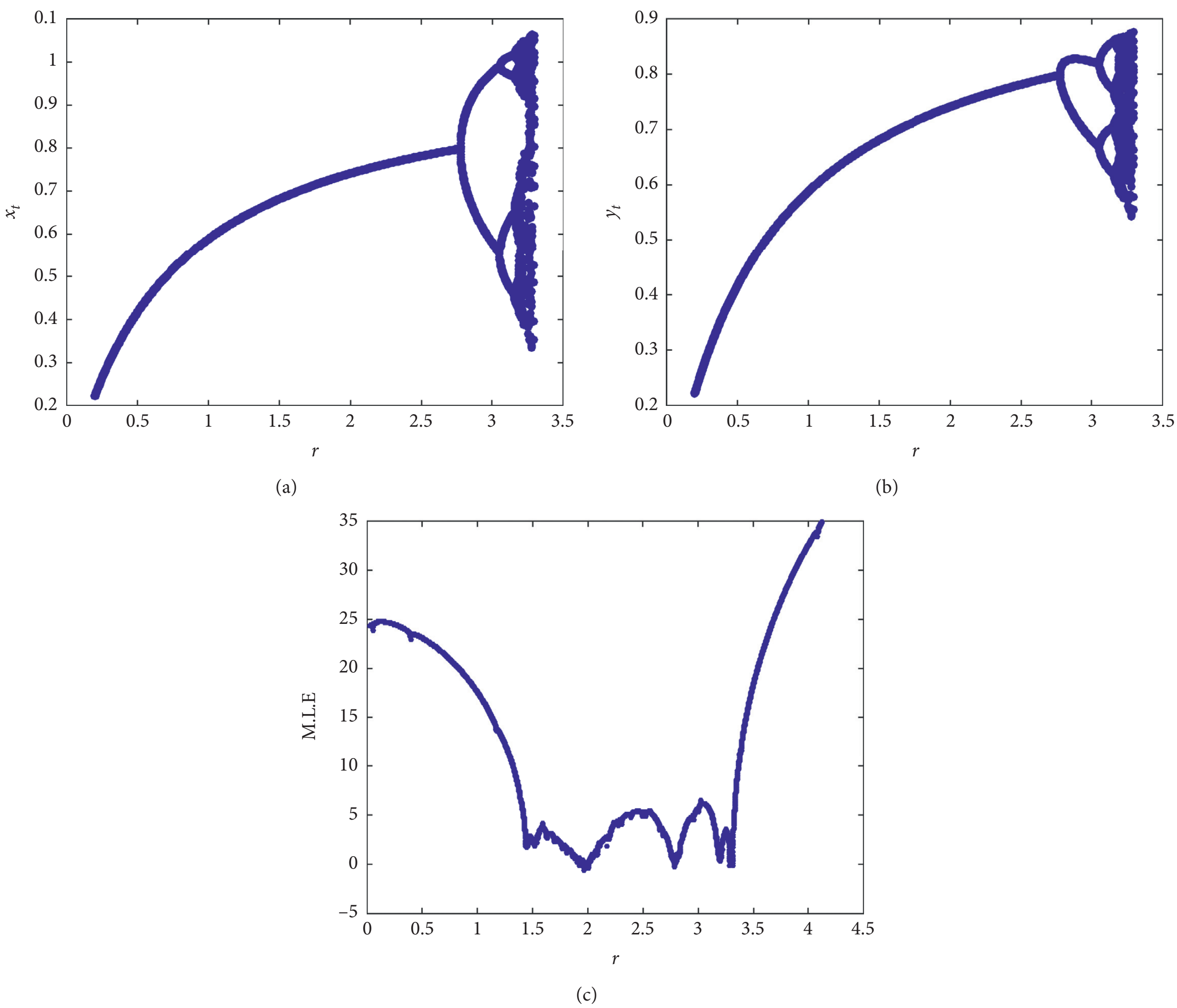

Figure 2: (a, b) Flip bifurcation diagram with $r \in [0.2, 3.95]$ and $(0.24, 0.25)$. (c) M.L.E corresponding to (a, b).

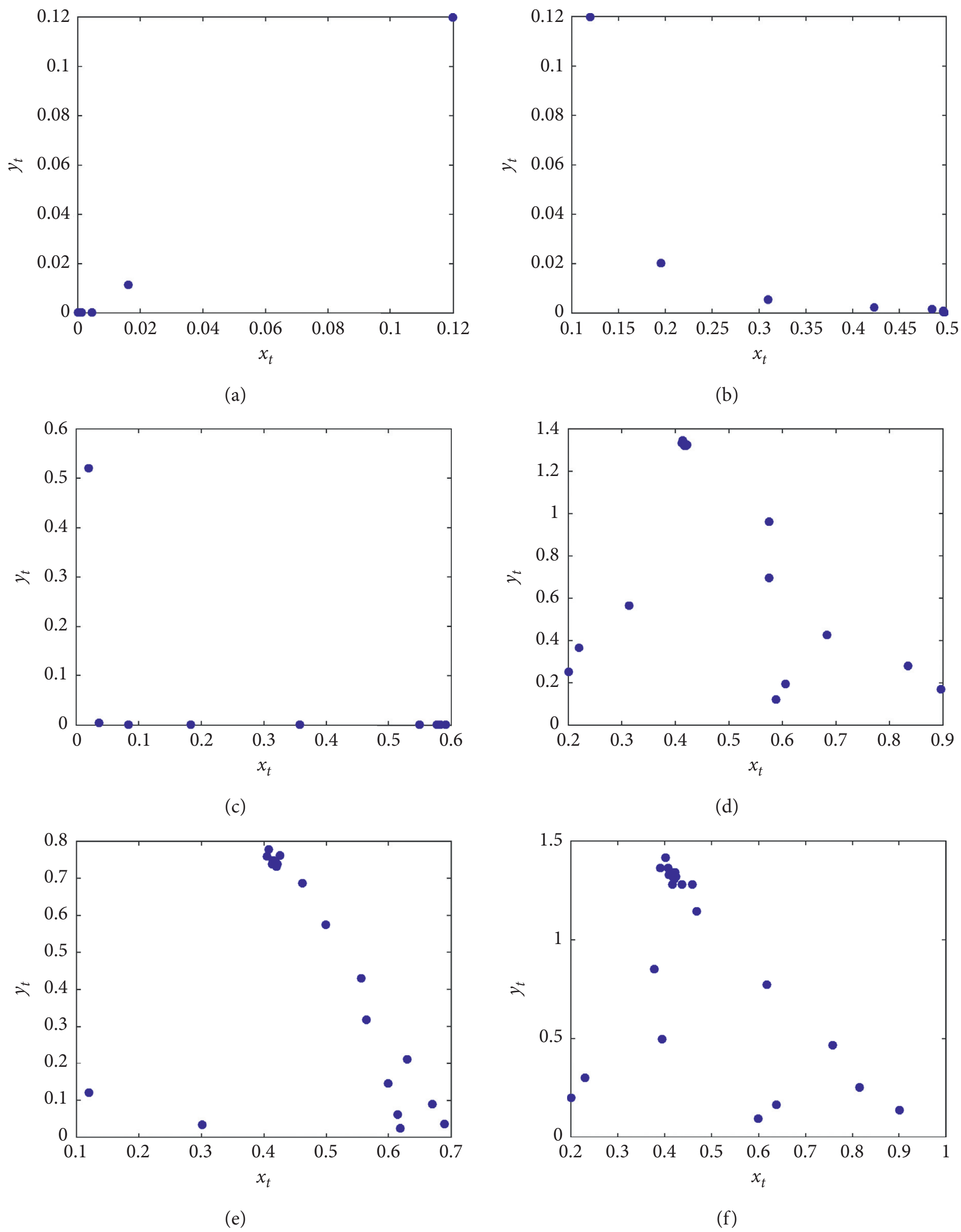

FIGURE 3: Complex dynamics of model (4). (a) $r = 0.01$ with (0.4, 0.5). (b) $r = 0.12$ with (0.4, 0.5). (c) $r = 0.193$ with (0.04, 0.02). (d) $r = 0.2112$ with (0.61, 0.61). (e) $r = 0.21$ with (0.0023, 0.6). (f) $r = 0.221$ with (0.24, 0.22).

5. Numerical Simulations

Here, some simulations will be presented in order to verify the obtained results in Sections 2 and 4. For instance, choose $\varepsilon = 1.9, h = 1$, then from (24), one gets $d > 0.2052631578947369$. Furthermore, if $d = 0.5 > 0.2052631578947369$, then from (23), one gets $r > 0.23333333333333317$. But for the existence of unique positive fixed point, it is required that $r > \varepsilon = 1.9$. So, for these numerical values and the condition on parameter r where unique positive fixed point is stable focus is $r > \max\{0.23333333333333317, 1.9\}$. Hence, if one choose parametric value $r = 2.1$ which satisfies $r > \max\{0.23333333333333317, 1.9\}$ and $b = 0.4, c = 0.5$, then it is clear from Figure 1(a) that $E_{positive} = (4.8, 0.5)$ is stable focus. Similarly, Figure 1(b) also shows that E_{positive} of (4) is stable focus. Now, from (25), one can say that E_{positive} of (4) is unstable focus if $r < 0.23333333333333317$, and hence, flip bifurcation takes place if $r < 0.23333333333333317$. For instance, if $r = 0.09 < 0.23333333333333317$, then

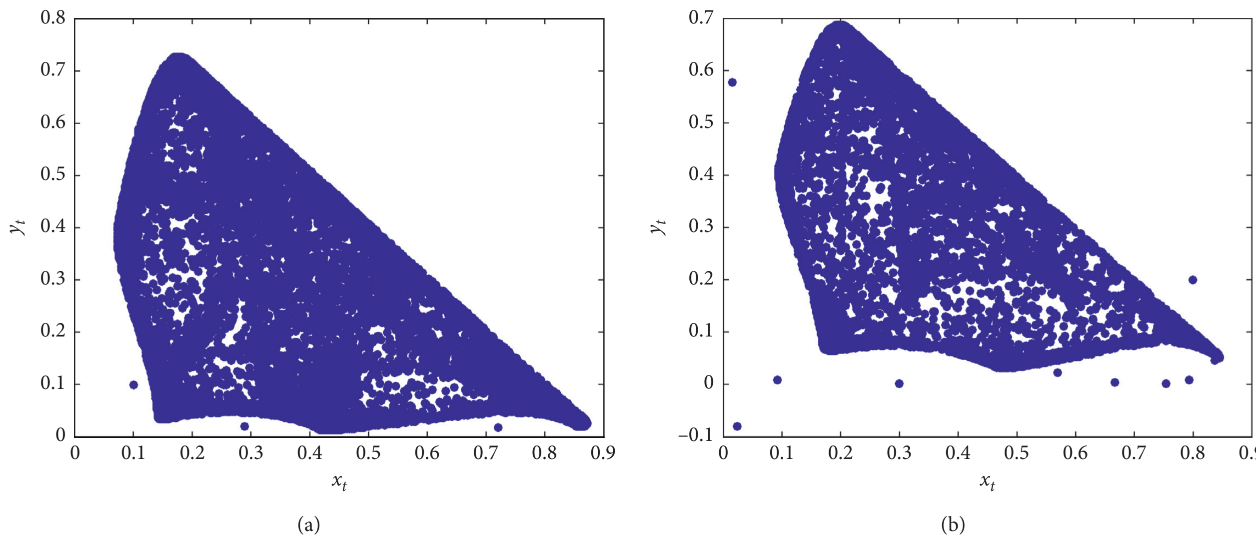

FIGURE 4: Strange attractors if $r = 0.18$ (resp., $r = 0.192$) with $(0.8, 0.2)$.

$\zeta_1 = -1.0842264752180841$, but $\zeta_2 = 3.084226475218084 \neq 1$ or -1. Moreover, $\wedge_1 = -0.5 \neq 0$ but $\wedge_2 = 0.026666666666666672 > 0$ which implies that stable period-2 points bifurcating from E_{positive}. Hence, this simulation agrees with theoretical results in Section 4. The $2D$ bifurcation diagrams with corresponding maximum Lyapunov exponents for said parametric values are plotted in Figure 2. Finally, the trajectories associated with Figures 2(a) and 2(b) are also plotted in Figures 3(a)–3(f) that indicates (4) exhibits complex dynamics having orbits of period-4, period-6, period-8, period-12, period-17, and period-18.

5.1. Fractal Dimension. It is defined by using Lyapunov exponents as follows [30, 31]:

$$D_L = \text{J} + \frac{\sum_{i=1}^{\text{J}} \zeta_{\text{J}}}{|\zeta_{\text{J}}|}, \tag{58}$$

with $\zeta_1, \ldots, \zeta_n$ being Lyapunov exponents, where J is the largest integer s.t. $\sum_{i=1}^{\text{J}} \zeta_{\text{J}} \geq 0$ and $\sum_{i=1}^{\text{J}+1} \zeta_{\text{J}} < 0$. The fractal dimension for considered model (4) becomes

$$D_L = 1 + \frac{\zeta_1}{|\zeta_2|}. \tag{59}$$

Now for values of d, ε, h, c, b, and r, two Lyapunov exponents are numerically computed. If $d = 0.5$, $\varepsilon = 1.9, h = 1, c = 0.5, b = 0.4$, then $\zeta_1 = 3.031748015872047$ (resp., $\zeta_1 = 3.024648117575002$) and $\zeta_2 = -1.0317480158720471$ (resp., $\zeta_2 = -1.0246481175750022$ for $r = 0.18$ (resp., $r = 0.192$). So fractal dimension for the model (4) is

$$\begin{aligned} d_L &= 1 + \frac{3.031748015872047}{|-1.0317480158720471|} \\ &= 3.938457810659876 \text{ for } r = 0.18, \\ d_L &= 1 + \frac{3.024648117575002}{|-1.0246481175750022|} \\ &= 3.951889595750518 \text{ for } r = 0.192. \end{aligned} \tag{60}$$

For above chosen parametric values, strange attractors are also plotted in Figures 4(a) and 4(b) that demonstrate (4) has a complex dynamical behavior.

6. Chaos Control

By a state feedback control method, we will stabilize chaotic orbits at an unstable fixed point motivated from existing literature [32, 33]. By adding control force C_t to model (4), then

$$\begin{aligned} x_{t+1} &= (1 + h(r-\varepsilon))x_t - bhx_t y_t + C_t, \\ y_{t+1} &= (1 - h(d+\varepsilon))y_t + chx_t y_t, \\ C_t &= -g_1\left(x_t - \frac{d+\varepsilon}{c}\right) - g_2\left(y_t - \frac{r-\varepsilon}{b}\right), \end{aligned} \tag{61}$$

with g_1 and g_2 being feedback gains. Now, $\Omega^C|_{E_{\text{positive}}}$ at E_{positive} for the controlled system (61) is

$$\Omega^C|_{E_{\text{positive}}} := \begin{pmatrix} 1 - g_1 & -\frac{bh}{c}(d+\varepsilon) - g_2 \\ \frac{ch}{b}(r-\varepsilon) & 1 \end{pmatrix}, \tag{62}$$

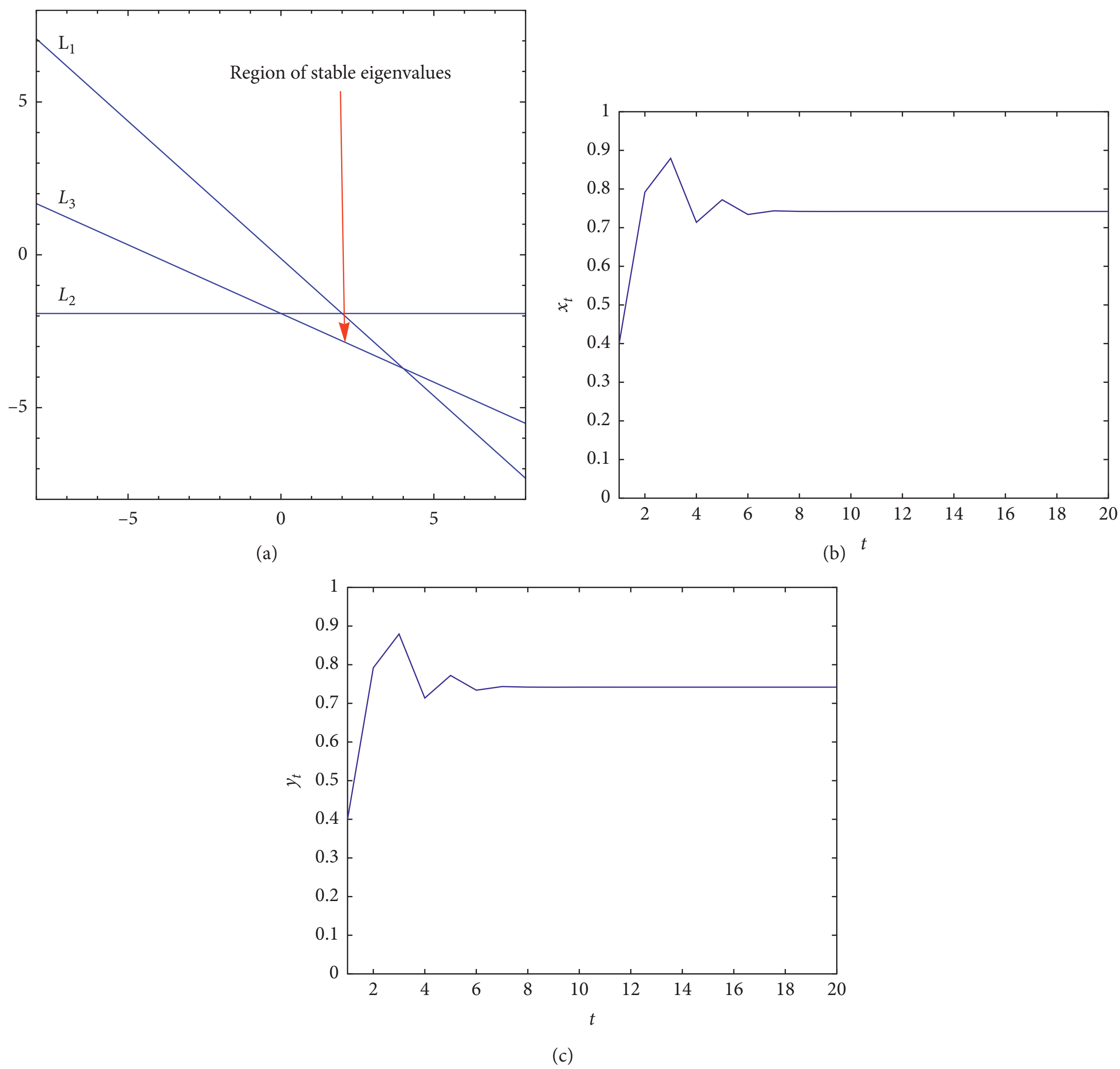

FIGURE 5: Control of chaotic trajectories of (61) for $d = 0.5, \varepsilon = 1.9, h = 1, c = 0.5, b = 0.4$, and $r = 0.12$ with $(0.4, 0.5)$: (a) stability region in (g_1, g_2)-plane; (b, c) dynamics for x_t and y_t, respectively.

whose auxiliary equation is

$$\zeta_C^2 - \mathcal{T}\left(E_{\text{positive}}\right)\zeta_C + \mathcal{D}\left(E_{\text{positive}}\right) = 0, \quad (63)$$

with

$$\mathcal{T}\left(E_{\text{positive}}\right) = 2 - g_1,$$

$$\mathcal{D}\left(E_{\text{positive}}\right) = 1 - g_1 + h^2(r-\varepsilon)(d+\varepsilon) + \frac{ch}{b}(r-\varepsilon)g_2. \quad (64)$$

If $\zeta_{1,2}$ are roots of (63), then

$$\zeta_1 + \zeta_2 = 2 - g_1, \quad (65)$$

$$\zeta_1\zeta_2 = 1 - g_1 + h^2(r-\varepsilon)(d+\varepsilon) + \frac{ch}{b}(r-\varepsilon)g_2. \quad (66)$$

Here, it is noted that solution of the following equations

$$\zeta_1 = 1, \quad (67)$$

$$\zeta_1 = -1, \quad (68)$$

$$\zeta_1\zeta_2 = 1, \quad (69)$$

gives the lines of marginal stability, which further grantee the fact that $|\zeta_{1,2}| < 1$. From (66) and (69), one gets

$$L_1: \; -g_1 + \frac{ch}{b}(r-\varepsilon)g_2 + h^2(r-\varepsilon)(d+\varepsilon) = 0. \quad (70)$$

From (65), (66), and (67), one gets

$$L_2: \frac{ch}{b}(r-\varepsilon)g_2 + h^2(r-\varepsilon)(d+\varepsilon) = 0. \quad (71)$$

From (65), (66), and (68), one gets

Table 2: Behavior of model (4) at E_{trival} and E_{positive}

Fixed points	Respective behavior
E_{trival}	Sink if $0<h<\min\{(2/\varepsilon-r);\ (2/d+\varepsilon)\}$, where $\varepsilon>r$; source if $h>\max\{(2/\varepsilon-r),\ (2/d+\varepsilon)\}$, where $\varepsilon>r$ Saddle if $(2/\varepsilon-r)<h<(2/d+\varepsilon)$; nonhyperbolic if $h=(2/\varepsilon-r)$ or $h=(2/d+\varepsilon)$
E_{positive}	Never stable focus; unstable focus if $r>\varepsilon$; nonhyperbolic if $r=\varepsilon$ Stable node if $r>(\varepsilon h^2(d+\varepsilon)-4/h^2(d+\varepsilon))$; unstable node if $r<(\varepsilon h^2(d+\varepsilon)-4/h^2(d+\varepsilon))$ Nonhyperbolic if $r=(\varepsilon h^2(d+\varepsilon)-4/h^2(d+\varepsilon))$

$$L_2\colon\ 2g_1-\frac{ch}{b}(r-\varepsilon)g_2-h^2(r-\varepsilon)(d+\varepsilon)=4. \tag{72}$$

Thus, L_1, L_2, and L_3 in (g_1,g_2)-plane determines the triangular region that gives $|\zeta_{1,2}|<1$ (see Figure 5(a)), whereas Figures 5(b) and 5(c) tell about E_{positive} that the chaotic trajectories are stabilized. For the qualitative behavior of continuous dynamical systems, we refer the interested readers to [34–36] and references cited therein.

7. Periodic Points of Model (4)

Existence of periodic points at E_{trival} and E_{positive} is investigated in this section.

Theorem 3. *E_{trival} is a periodic point having prime period-1.*

Proof. From (4), we have

$$F(x,y)\coloneqq(f(x,y),g(x,y)), \tag{73}$$

where

$$\begin{aligned}f(x,y)&=(1+h(r-\varepsilon))x-bhxy,\\ g(x,y)&=(1-h(d+\varepsilon))y+chxy.\end{aligned} \tag{74}$$

From (73) along with (74), one gets the required results:

$$F|_{E_{\text{trival}}}=E_{\text{trival}}. \tag{75}$$

Theorem 4. *$E_{positive}$ is a periodic point having prime period-1.*

Proof. From (73) along with (74), one gets the required results:

$$F|_{E_{\text{positive}}}=E_{\text{positive}}. \tag{76}$$

Theorem 5. *E_{trival} is a periodic point of period-$2,3,\ldots,n$.*

Proof. From (73), one gets

$$\begin{aligned}F^2(x,y)&=((1+h(r-\varepsilon))f(x,y)-bhf(x,y)g(x,y),\\ (1-h(d+\varepsilon))g(x,y)+chf(x,y)g(x,y))&\Longrightarrow F^2|_{E_{\text{positive}}}=E_{\text{positive}},\\ F^3(x,y)&=\big((1+h(r-\varepsilon))f^2(x,y)-bhf^2(x,y)g^2(x,y),\\ (1-h(d+\varepsilon))g^2(x,y)+chf^2(x,y)g^2(x,y)\big)&\Longrightarrow F^3|_{E_{\text{positive}}}=E_{\text{positive}},\\ &\vdots\\ F^i(x,y)&=\big((1+h(r-\varepsilon))f^{i-1}(x,y)-bhf^{i-1}(x,y)g^{i-1}(x,y),\\ (1-h(d+\varepsilon))g^{i-1}(x,y)+chf^{i-1}(x,y)g^{i-1}(x,y)\big)&\Longrightarrow F^i|_{E_{\text{positive}}}=E_{\text{positive}}.\end{aligned} \tag{77}$$

So, from (77), one gets the required statement. □

Theorem 6. *$E_{positive}$ is a periodic point of period-$2,3,\ldots,n$.*

Proof. From (77), one gets the required statement:

$$\begin{aligned}F^2|_{E_{\text{positive}}}&=E_{\text{positive}},\\ F^3|_{E_{\text{positive}}}&=E_{\text{positive}},\\ &\vdots\\ F^i|_{E_{\text{positive}}}&=E_{\text{positive}}.\end{aligned} \tag{78}$$

8. Conclusion

In this paper, we have investigated the topological classifications at fixed points, bifurcation analysis, and chaos in a model, which is depicted in (4). It is examined that $\forall d,\varepsilon,h,c,b,r$ model has a trivial fixed point: E_{trival} and the unique positive fixed point: E_{positive} if $r>\varepsilon$. By existing linear theory of stability, we have studied the topological classifications at E_{trival} and E_{positive}, and conclusion is reported in Table 2. Furthermore, we have examined the existence of possible bifurcations at E_{trival} and E_{positive} and proved that model (4) does not undergo flip bifurcation if $(b,c,d,h,r,\varepsilon)\in \text{FB}|_{E_{\text{trival}}}$, and hence, E_{trival} is degenerated with a higher codimension. We have also examined that if $(b, c, d, h, r, \varepsilon) \in \text{HB}|_{E_{\text{positive}}}$, then at E_{positive}, model (4) does not undergo N-S bifurcation. Moreover, at fixed point E_{positive}, we have proved that model (4) undergoes the flip bifurcation if $(b, c, d, h, r, \varepsilon) \in \text{FB}|_{E_{\text{positive}}}$. Some numerical simulations are performed not only to demonstrate obtained theoretical results but also to tell the complex behaviors in orbits of period-4, period-6, period-8, period-12, period-17, and period-18. We have also computed maximum Lyapunov exponents numerically. By the feedback control method, we have stabilized chaos existing in the considered model. Finally, existence of periodic points at E_{trival} and E_{positive} of model (4) is explored.

Data Availability

All the data utilized in this article have been included, and the sources adopted are cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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