Discrete Dynamics in Nature and Society

Theory and Applications of Periodic Solutions and Almost Periodic Solutions

Guest Editors: Yonghui Xia, Youssef N. Raffoul, Patricia J. Y. Wong, Zhengkun Huang, Tailei Zhang, and Junli Liu

Theory and Applications of Periodic Solutions and Almost Periodic Solutions

Discrete Dynamics in Nature and Society

Theory and Applications of Periodic Solutions and Almost Periodic Solutions

Guest Editors: Yonghui Xia, Youssef N. Raffoul, Patricia J. Y. Wong, Zhengkun Huang, Tailei Zhang, and Junli Liu

Editorial Board

Contents

Theory and Applications of Periodic Solutions and Almost Periodic Solutions, Yonghui Xia
Volume 2013, Article ID 170948, 2 pages

Complex Dynamical Behaviors in a Predator-Prey System with Generalized Group Defense and Impulsive Control Strategy, Shunyi Li
Volume 2013, Article ID 358930, 15 pages

Almost Automorphic Mild Solutions to Neutral Parabolic Nonautonomous Evolution Equations with Nondense Domain, Zhanrong Hu and Zhen Jin
Volume 2013, Article ID 183420, 10 pages

Permanence, Extinction, and Almost Periodic Solution of a Nicholson's Blowflies Model with Feedback Control and Time Delay, Haihui Wu and Shengbin Yu
Volume 2013, Article ID 798961, 9 pages

Persistence Property and Asymptotic Description for DGH Equation with Strong Dissipation, Ke-chuang Wang
Volume 2013, Article ID 163070, 8 pages

Pattern Formation in a Diffusive Ratio-Dependent Holling-Tanner Predator-Prey Model with Smith Growth, Bo Yang
Volume 2013, Article ID 454209, 8 pages

Qualitative Analysis of a Diffusive Ratio-Dependent Holling-Tanner Predator-Prey Model with Smith Growth, Zongmin Yue and Wenjuan Wang
Volume 2013, Article ID 267173, 9 pages

Effect of Delay on Selection Dynamics in Long-Term Sphere Culture of Cancer Stem Cells, Peng Tang, Aijun Fan, Jianquan Li, Jun Jiang, and Kaifa Wang
Volume 2013, Article ID 606250, 5 pages

The Atom-Bond Connectivity Index of Catacondensed Polyomino Graphs, Jinsong Chen, Jianping Liu, and Qiaoliang Li
Volume 2013, Article ID 598517, 7 pages

Monotonicity of Eventually Positive Solutions for a Second Order Nonlinear Difference Equation, Huiqin Chen, Zhen Jin, and Shugui Kang
Volume 2013, Article ID 635149, 5 pages

Almost Periodic Solutions for Wilson-Cowan Type Model with Time-Varying Delays, Shasha Xie and Zhenkun Huang
Volume 2013, Article ID 683091, 7 pages

Dynamics of a Diffusive Predator-Prey Model with Allee Effect on Predator, Xiaoqin Wang, Yongli Cai, and Huihai Ma
Volume 2013, Article ID 984960, 10 pages

Pattern Formation in a Cross-Diffusive Holling-Tanner Model, Weiming Wang, Zhengguang Guo, R. K. Upadhyay, and Yezhi Lin
Volume 2012, Article ID 828219, 12 pages

A Lyapunov Function and Global Stability for a Class of Predator-Prey Models, Xiaoqin Wang and Huihai Ma
Volume 2012, Article ID 218785, 8 pages

Stochastically Perturbed Epidemic Model with Time Delays, Tailei Zhang
Volume 2012, Article ID 454073, 7 pages

Qualitative Analysis for a Predator Prey System with Holling Type III Functional Response and Prey Refuge, Xia Liu and Yepeng Xing
Volume 2012, Article ID 678957, 11 pages

Pattern Formation in a Cross-Diffusive Ratio-Dependent Predator-Prey Model, Xinze Lian, Yanhong Yue, and Hailing Wang
Volume 2012, Article ID 814069, 13 pages

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 170948, 2 pages
http://dx.doi.org/10.1155/2013/170948

Editorial

Theory and Applications of Periodic Solutions and Almost Periodic Solutions

Yonghui Xia

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Correspondence should be addressed to Yonghui Xia; yhxia@zjnu.cn

Received 17 September 2013; Accepted 17 September 2013

Theory of periodic solutions and almost periodic solutions is an important and well-established branch of the modern theory of differential equations concerned, in a broad sense, with the study of periodic (almost periodic) phenomena arising in applied problems in technology, natural, and social sciences. For example, to explore the impact of environmental factors in mathematical biology, the assumption of periodicity of parameters is more realistic and important due to many periodic factors such as seasonal effects of weather, food supplies, mating habits, and harvesting. However, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic. It is well known that existence and nonexistence of periodic (or almost periodic) solutions to a given system are the classical theory of oscillation theory, which is an intrinsic feature of many dynamical systems.

This special issue places its emphasis on the study of periodic or almost periodic solutions of initial/boundary value problems for ordinary differential equations. It includes many topics such as almost automorphic mild solutions, biological systems, neural networks, oscillation criteria, delay differential equation, stochastically epidemic model, initial value problems, Lyapunov function, complex dynamical behaviors, stability and bifurcation, and impulsive differential equation.

All manuscript submitted to this special issue went through a thorough peer-refereeing process. Based on the reviewers' reports, we collect 16 original research articles by more than fifty active international researchers on differential equations.

To be a more comprehensive description for all articles in this special issue, we provide a short editorial note summering each paper.

Z. Hu and Z. Jin establish new existence and uniqueness theorems for almost automorphic mild solutions to neutral parabolic nonautonomous evolution equations with nondense domain. A unified framework is set up to investigate the existence and uniqueness of almost automorphic mild solutions to some classes of parabolic partial differential equations and neutral functional differential equations.

S. Xie et al. studied almost periodic solutions for Wilson-Cowan type model with time-varying delays. Some Su cient conditions for the existence and delay-based exponential stability of a unique almost periodic solution are established.

S. Li showed that the system considered has more complicated dynamics, including (1) high-order quasi-periodic and periodic oscillation, (2) period-doubling and halving bifurcation, (3) nonunique dynamics (meaning that several attractors coexist), and (4) chaos and attractor crisis.

H. Wu et al. studied a Nicholson's blowflies model with feedback control and time delay. By applying the comparison theorem of the differential equation and fluctuation lemma and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence, extinction, and existence of a unique globally attractive positive almost periodic solution of the system are obtained.

W. Wang et al. presented a theoretical analysis of processes of pattern formation that involves organisms distribution and their interaction of spatially distributed population with self- as well as cross-diffusion in a Holling-Tanner predator-prey model; the sufficient conditions for the Turing instability with zero-flux boundary conditions are obtained; Hopf and Turing bifurcation in a spatial domain is presented, too.

T. Zhang investigated a stochastic epidemic model with time delays. By using Lyapunov functionals, he obtained stability conditions for the stochastic stability of endemic equilibrium.

K.-C. Wang concerned with the Dullin-Gottwald-Holm (DGH) equation with strong dissipation. He established a sufficient condition to guarantee global in time solutions.

Z. Yue et al. investigated the dynamics of a diffusive ratio-dependent Holling-Tanner predator-prey model with Smith growth subject to zero-flux boundary condition. Some qualitative properties, including the dissipation, persistence, and local and global stability of positive constant solution are discussed.

B. Yang investigated the spatiotemporal dynamics of a diffusive ratio-dependent Holling-Tanner predator-prey model with Smith growth subject to zero-flux boundary condition.

H. Ma et al. constructed a new Lyapunov function for a class of predation models.

J. Chen studied the atom-bond connectivity index of catacondensed polyomino graphs.

P. Tang quantitatively studied the effect of delay on selection dynamics in long-term sphere culture of cancer stem cells (CSCs); a selection dynamic model with time delay is proposed.

H. Chen et al. derived several sufficient conditions for monotonicity of eventually positive solutions on a class of second order perturbed nonlinear difference equation.

X. Liu et al. considered a predator prey system with Holling III functional response and constant prey refuge. By using the Dulac criterion, we discuss the global stability of the positive equilibrium of the system. By transforming the system to a Lienard system, the conditions for the existence of exactly one limit cycle for the system are given.

Lian et al. presented a theoretical analysis of evolutionary process that involves organisms distribution and their interaction of spatial distribution of the species with self and crossdiffusion in a Holling III ratio-dependent predator-prey model.

X. Wang et al. are concerned with the reaction-diffusion Holling-Tanner prey-predator model considering the Allee effect on predator, under zero-flux boundary conditions.

Acknowledgment

We would also like to thank the editorial board members of this journal, for their support and help throughout the preparation of this special issue.

Yonghui Xia

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 358930, 15 pages
http://dx.doi.org/10.1155/2013/358930

Research Article

Complex Dynamical Behaviors in a Predator-Prey System with Generalized Group Defense and Impulsive Control Strategy

Shunyi Li

Department of Mathematics, Qiannan Normal College for Nationalities, Duyun, Guizhou 558000, China

Correspondence should be addressed to Shunyi Li; lishunyi19820425@163.com

Received 1 November 2012; Accepted 20 May 2013

Academic Editor: Patricia J. Y. Wong

A predator-prey system with generalized group defense and impulsive control strategy is investigated. By using Floquet theorem and small amplitude perturbation skills, a local asymptotically stable prey-eradication periodic solution is obtained when the impulsive period is less than some critical value. Otherwise, the system is permanent if the impulsive period is larger than the critical value. By using bifurcation theory, we show the existence and stability of positive periodic solution when the pest eradication lost its stability. Numerical examples show that the system considered has more complicated dynamics, including (1) high-order quasiperiodic and periodic oscillation, (2) period-doubling and halving bifurcation, (3) nonunique dynamics (meaning that several attractors coexist), and (4) chaos and attractor crisis. Further, the importance of the impulsive period, the released amount of mature predators and the degree of group defense effect are discussed. Finally, the biological implications of the results and the impulsive control strategy are discussed.

1. Introduction

In population dynamics, a functional response of the predator to the prey density refers to the change in the density of prey attached per unit time per predator as the prey density changes and it is assumed to be monotonically increasing in most predator-prey systems. For example, Holling type I, II, and III functional response [1]

$$f_1(x, y) = rx, \qquad f_2(x, y) = \frac{rx}{a + bx}, \qquad f_3(x, y) = \frac{rx^2}{a + bx^2}, \tag{1}$$

and the sigmoidal type response function [2]

$$f_4(x, y) = \frac{rx^2}{(a + x)(b + x)}, \tag{2}$$

and Ivlev type response function [3]

$$f_5(x, y) = r\left(1 - e^{-ax}\right). \tag{3}$$

The previous functional responses are prey dependent. But, both predator and prey densities have an effect on the response, such as Beddington-DeAngelis functional response [4, 5]

$$f_6(x, y) = \frac{rx}{a + bx + cy} \tag{4}$$

and modified Holling type II and type III response functions [6]

$$f_7(x, y) = \frac{rx}{(a + bx)(b + y)}, \qquad f_8(x, y) = \frac{rx^2}{(a + bx^2)(b + y)}. \tag{5}$$

However, some experimental and observational evidence shown that the functional response is not always monotonically increasing, such as Holling type IV [7]

$$f_9(x, y) = \frac{rx}{a + bx + cx^2} \tag{6}$$

and $f_{10}(x, y) = \alpha x e^{-\beta x}$ [8]. Group defense is a term used to describe the phenomenon whereby predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise itself when it exists in enough large numbers [9–11]. The buffalo group defense was modeled using a generalized group defense in [12],

$$f_{11}(x,y) = \frac{\alpha x}{1 + hx^{\beta}}, \tag{7}$$

where β is a positive integer whose value determines the degree of antipredator behavior and group defense.

Recently, it is of great interest to investigate complex dynamics for impulsive perturbations in populations dynamics. In particular, the impulsive prey-predator population models have been investigated by many researchers. The results of studies of the dynamics of a predator-prey model with nonmonotonic functional response, such as Holling type IV functional response with respect to an impulsive control strategy, were presented in [13–24]. To the best of our knowledge, there are few papers studying the group defense predator-prey with impulsive effect, where the antipredator behavior and group defense effect described by nonmonotonic functional response. Zhang et al. [25] considered a predator-prey system with defensive ability of prey by Holling type IV functional response and impulsive perturbations on the predator:

$$\begin{aligned} &\left.\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)\,y(t)}{a_1 + x^2(t)},\\ y'(t) &= y(t)\left(-d + \frac{\mu x(t)}{a_1 + x^2(t)}\right), \end{aligned}\right\} \quad t \neq nT,\\ &\left.\begin{aligned} &x(nT^+) = x(nT), \qquad y(nT^+) = y(nT) + \tau,\\ &X(0^+) = x_0 = \left(x^0, y^0\right)^T, \end{aligned}\right\} \quad t = nT. \end{aligned} \tag{8}$$

The conditions for the local asymptotically stable prey-eradication periodic solution and permanence of the system are obtained; a series of complex phenomena are displayed by numerical simulation. Furthermore, based on this work, Pei et al. [26] investigated a one-prey multi-predator model with defensive ability of the prey by introducing impulsive biological control strategy:

$$\begin{aligned} &\left.\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t)}{k}\right) - \sum_{i=1}^{m}\frac{x(t)\,y(t)}{a_i + x^2(t)},\\ y_i'(t) &= y_i(t)\left(\sum_{i=1}^{m}\frac{\mu_i x(t)}{a_i + x^2(t)} - d_i\right), \end{aligned}\right\} \quad t \neq nT,\\ &\left.\begin{aligned} &x(nT^+) = x(nT), \qquad y_i(nT^+) = y_i(nT) + p_i,\\ &X_0 = \left(x(0^+), y_1(0^+), \ldots, y_m(0^+)\right)^T = \left(x_0, y_{01}, \ldots, y_{0m}\right)^T, \end{aligned}\right\} \quad t = nT. \end{aligned} \tag{9}$$

And it shown that the multi-predator impulsive control strategy is more effective than the classical one and makes the dynamical behaviors of the system more complex. Recently, a predator-prey system with impulsive effect and group defense with the nonmonotone function $f_{10}(x,y) = \alpha x e^{-\beta x}$ was studied by Li et al. [27],

$$\begin{aligned} &\left.\begin{aligned} x'(t) &= x(t)(a - bx(t)) - \alpha x(t)\,y(t)\,e^{-\beta x(t)},\\ y'(t) &= k\alpha x(t)\,y(t)\,e^{-\beta x(t)} - y(t)\,d, \end{aligned}\right\} \quad t \neq nT,\\ &\left.\begin{aligned} \Delta x(t) &= -px(t),\\ \Delta y(t) &= q, \end{aligned}\right\} \quad t = nT. \end{aligned} \tag{10}$$

They proved that there exists a locally stable pest-eradication periodic solution when the impulsive period is less than certain critical values; otherwise, the system is permanent. Some complicated dynamics, such as quasiperiodic oscillation, bifurcation, and attractor crisis, were shown by numerical simulations.

In this paper, we study a predator-prey system with impulsive effect and generalized group defense with the nonmonotone function $f_{11}(x,y) = \alpha x/(1 + hx^{\beta})$:

$$\begin{aligned} &\left.\begin{aligned} x'(t) &= x(t)(a - bx(t)) - \frac{\alpha x(t)}{1 + hx^{\beta}(t)}\,y(t),\\ y'(t) &= k\frac{\alpha x(t)}{1 + hx^{\beta}(t)}\,y(t) - dy(t), \end{aligned}\right\} \quad t \neq nT,\\ &\left.\begin{aligned} \Delta x(t) &= -p_1 x(t),\\ \Delta y(t) &= -p_2 y(t) + q, \end{aligned}\right\} \quad t = nT, \end{aligned} \tag{11}$$

where $x(t)$ and $y(t)$ represent the prey and the predator populations at time t, respectively; a, b, α, h, β, and k are positive. a is the intrinsic rate of increase of the prey and d is the death rate of the predator, a/b is the carrying capacity of the prey, $\beta > 1$ is the degree of anti-predator behavior and group defense, and k $(0 < k < 1)$ is the rate of conversing prey into predator. $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, T is the periodic of the impulse for predator in order to eradicate target pests, protect nontarget pest (or harmless insect) from extinction and drive target pest to extinction, or control target pest at acceptably low level to prevent an increasing pest population from causing an economic loss. $n \in \mathbb{N}_+$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $p_i > 0$ $(i = 1, 2)$ is the proportionality constant which represents the rate of mortality due to the applied pesticide; for example, impulsive reduction of the population is possible by harvesting or by poisoning with chemicals used in agriculture. $q > 0$ is the number of predators released each time, for example, by artificial breeding of the species or release of some species.

The paper is arranged as follows. In Section 2, some notations and Lemmas are given. In Section 3, using the Floquet theory of impulsive equation and small amplitude perturbation skills, we will prove the local stability of prey-eradication periodic solution when the impulsive period is less than some critical value and give the condition of permanence. In Section 4, by using bifurcation theory, the existence and stability of positive periodic solution are studied when T is close to the critical value T_0. In Section 5, the results of numerical examples are shown, and some rich dynamic behaviors are obtained; the effects of the impulsive period, the released amount of mature predators and the coefficient of group defense effect are discussed. Finally, the conclusions are discussed briefly in Section 6.

2. Preliminaries

In this section, we will give some definitions, notations, and lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_+^2 = \{x \in \mathbb{R}^2 \mid x \geq 0\}$. Denote by $f = (f_1, f_2)$ the map defined by the right hand of the first two equations of system (11), and denote by $\mathbb{N}$ the set of all nonnegative integers. Let $V : \mathbb{R}_+ \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, then V is said to belong to class V_0 if

(1) V is continuous in $(t,x) \in (nT,(n+1)T] \times \mathbb{R}_+^2$ and for each $x \in \mathbb{R}_+^2$, $n \in \mathbb{N}$, $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+,x)$ exists,

(2) V is locally Lipschitzian in x.

Definition 1. Let $V \in V_0$; then for $(t,x) \in (nT,(n+1)T] \times \mathbb{R}_+^2$, the upper right derivative of $V(t,x)$ with respect to the impulsive differential system (11) is defined as

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\left[V\left(t+h, x+hf(t,x)\right) - V(t,x)\right]. \tag{12}$$

Definition 2. System (11) is said to be permanent if there exist two positive constants m, M and T_0 such that each positive solution $(x(t), y(t))$ of the system (11) satisfies $m \leq x(t) \leq M$, $m \leq y(t) \leq M$, for all $t > T_0$.

The solution of system (11) is a piecewise continuous function $x : \mathbb{R}_+ \mapsto \mathbb{R}_+^2$, $x(t)$ is continuous on $(nT,(n+1)T]$, $n \in \mathbb{N}$, and $x(nT^+) = \lim_{t\to nT^+} x(t)$ exists; the smoothness properties of f guarantee the global existence and uniqueness of solutions of system (11); for details see [28, 29]. The following lemma is obvious.

Lemma 3. *Let $X(t)$ be a solution of system (11) with $X(0^+) \geq 0$; then $X(t) \geq 0$ for all $t \geq 0$ and further $X(t) > 0$ for all $t \geq 0$ if $X(0^+) > 0$.*

And we will use the following important comparison theorem on impulsive differential equation [29].

Lemma 4. *Suppose $V \in V_0$. Assume that*

$$\begin{gathered} D^+V(t,x) \leq g(t, V(t,x)), \quad t \neq nT, \\ V\left(t, x\left(t^+\right)\right) \leq \psi_n(V(t,x)), \quad t = nT, \end{gathered} \tag{13}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous in $(nT,(n+1)T] \times \mathbb{R}_+$, and for $u \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\lim_{(t,y)\to(nT^+,u)} = g(nT^+,u)$ exists; $\psi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{gathered} u'(t) = g(t, u(t)), \quad t \neq nT, \\ u\left(t^+\right) = \psi_n(u(t)), \quad t = nT, \\ u\left(0^+\right) = u_0, \end{gathered} \tag{14}$$

existing on $[0,\infty)$. Then $V(0^+, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t)$, $t \geq 0$ where $X(t)$ is any solution of system (11).

Finally, we give some basic properties about the following subsystem of system (11):

$$\begin{gathered} y'(t) = -dy(t), \quad t \neq nT, \\ \Delta y(t) = -p_2 y(t) + q, \quad t = nT, \\ y\left(0^+\right) = y_0 \geq 0. \end{gathered} \tag{15}$$

Clearly, when $t \in (nT,(n+1)T]$, $n \in \mathbb{N}$,

$$\begin{gathered} y^*(t) = \frac{q\exp\left[-d(t-nT)\right]}{1-\left(1-p_2\right)\exp(-dT)}, \\ y^*(0) = \frac{q}{1-\left(1-p_2\right)\exp(-dT)}, \end{gathered} \tag{16}$$

is a positive periodic solution of system (15). Since

$$\begin{aligned} y(t) = {} & \left(y\left(0^+\right) - \frac{q}{1-\left(1-p_2\right)\exp(-dT)}\right) \\ & \times \exp(-dt) + y^*(t) \end{aligned} \tag{17}$$

is the solution of system (15) with initial value $y_0 \geq 0$, where $t \in (nT,(n+1)T]$, $n \in \mathbb{N}$; then one can get the following.

Lemma 5. *Let $y^*(t)$ be a positive periodic solution of system; (15) and every solution $y(t)$ of system (15) with $y_0 \geq 0$, one has $|y(t) - y^*(t)| \rightarrow 0$, when $t \rightarrow \infty$.*

Therefore, one obtains the pest-eradication periodic solution

$$\left(0, y^*(t)\right) = \left(0, \frac{q\exp\left[-d(t-nT)\right]}{1-\left(1-p_2\right)\exp(-dT)}\right) \tag{18}$$

for $t \in (nT,(n+1)T]$.

3. Extinction and Permanence

Firstly, we study the stability of prey-eradication periodic solution.

Theorem 6. *Let $X(t) = (x(t), y(t))$ be any solution of system (11); then $X(t) = (0, y^*(t))$ is locally asymptotically stable provided that*

$$aT - \frac{\alpha q\left[1-\exp(-dT)\right]}{d\left[1-\left(1-p_2\right)\exp(-dT)\right]} < \ln\left(\frac{1}{1-p_1}\right). \tag{19}$$

Proof. The local stability of periodic solution $X(t) = (0, y^*(t))$ may be determined by considering the behavior of small amplitude perturbations of the solution. Consider

$$x(t) = u(t), \qquad y(t) = y^*(t) + v(t). \tag{20}$$

There may be written

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \quad 0 \leq t < T, \tag{21}$$

where $\Phi(t)$ satisfies

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}=\begin{pmatrix} a-\alpha y^{*}(t) & 0 \\ k\alpha y^{*}(t) & -d \end{pmatrix}\Phi(t), \tag{22}$$

and $\Phi(0)=I$, the identity matrix. The linearization of the third and fourth equations of system (11) becomes

$$\begin{pmatrix} u(nT^{+}) \\ v(nT^{+}) \end{pmatrix}=\begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix}\begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{23}$$

Hence, if both eigenvalues of

$$M=\begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix}\Phi(t) \tag{24}$$

have absolute values less than one, then the periodic solution $X(t)=(0,y^{*}(t))$ is locally stable. Since all eigenvalues of M are

$$\begin{aligned} \mu_1 &= (1-p_2)\exp(-dT)<1, \\ \mu_2 &= (1-p_1)\exp\left(\int_0^T (a-\alpha y^{*}(t))\,dt\right), \end{aligned} \tag{25}$$

$\mu_2<1$ if and only if

$$aT-\frac{\alpha q\left[1-\exp(-dT)\right]}{d\left[1-(1-p_2)\exp(-dT)\right]}<\ln\left(\frac{1}{1-p_1}\right). \tag{26}$$

According to Floquet theory [28] of impulsive differential equation, the prey-eradication solution $X(t)=(0,y^{*}(t))$ is locally stable. This completes the proof. □

Theorem 7. *There exists a constant $M>0$, such that $x(t)\le M$, $y(t)\le M$ for each solution $X(t)=(x(t),y(t))$ of system* (11) *with all t being large enough.*

Proof. Let $V(t)=kx(t)+y(t)$. It is clear that $V\in V_0$. We calculate the upper right derivative of $V(t,x)$ along a solution of system (11) and get the following impulsive differential equation:

$$\begin{aligned} &D^{+}V(t)\big|_{(11)}+LV(t) \\ &\quad = kx(a+L-bx)-(d-L)y, \quad t\neq nT, \\ &V(t^{+})\le V(t)+q, \quad t=nT. \end{aligned} \tag{27}$$

Let $0<L<d$; then $kx(a+L-bx)-(d-L)y$ is bounded. Select L_0 and M_0 such that

$$\begin{aligned} D^{+}V(t) &\le -L_0V(t)+M_0, \quad t\neq nT, \\ V(t^{+}) &\le V(t)+q, \quad t=nT, \end{aligned} \tag{28}$$

where L_0 and M_0 are two positive constants. According to Lemma 4, we have

$$\begin{aligned} V(t) \le{}& \frac{M_0}{L_0}+\left(V(0^{+})-\frac{M_0}{L_0}\right) \\ &\times\exp(-L_0t)-\frac{q}{1-\exp(-L_0T)}\exp(-L_0t) \\ &+\frac{q}{1-\exp(-L_0T)}\exp(-L_0(t-nT)), \end{aligned} \tag{29}$$

where $t\in(nT,(n+1)T]$. Hence

$$\lim_{t\to\infty}V(t)\le\frac{M_0}{L_0}+\frac{q}{1-\exp(-L_0T)}. \tag{30}$$

Therefore, $V(t,x)$ is ultimately bounded. We obtain that each positive solution of system (11) is uniformly ultimately bounded. This completes the proof. □

In the following, we investigate the permanence of system (11).

Theorem 8. *System* (11) *is permanent if*

$$aT-\frac{\alpha q\left[1-\exp(-dT)\right]}{d\left[1-(1-p_2)\exp(-dT)\right]}>\ln\left(\frac{1}{1-p_1}\right). \tag{31}$$

Proof. Suppose $X(t)=(x(t),y(t))$ is a solution of system (11) with $X_0>0$. From Theorem 7 we may assume that $x(t)\le M$, $y(t)\le M$, and $M>a/b$, $t\ge0$. Let

$$m_2=\frac{q\exp(-dT)}{1-(1-p_2)\exp(-dT)}-\varepsilon_2, \quad \varepsilon_2>0. \tag{32}$$

According to Lemmas 4 and 5, we have $y(t)>m_2$ for all t large enough. In the following, we want to find m_1 such that $x_1(t)\ge m_1$ for all t large enough. We will do it in the following two steps for convenience.

Step 1. Since

$$aT-\frac{\alpha q\left[1-\exp(-dT)\right]}{d\left[1-(1-p_2)\exp(-dT)\right]}>\ln\left(\frac{1}{1-p_1}\right), \tag{33}$$

we can select $m_3>0$, $\varepsilon_1>0$ small enough such that $m_3<a/b$, $\delta=k\alpha m_3<d$, and

$$\begin{aligned} \sigma ={}& (a-bm_3-\alpha\varepsilon_1)T \\ &-\frac{\alpha q\left[1-\exp((-d+\delta)T)\right]}{(d-\delta)\left[1-(1-p_2)\exp((-d+\delta)T)\right]} \\ &-\ln\left(\frac{1}{1-p_1}\right)>0. \end{aligned} \tag{34}$$

We will prove there exists $t_1\in(0,\infty)$ such that $x(t_1)\ge m_3$. Otherwise, according to the above assumption, we get $y'(t)\le y(t)(-d+\delta)$, and by Lemmas 4 and 5, we have $y(t)\le z(t)$ and $z(t)\le z^{*}(t)$, where

$$\begin{aligned} z^{*}(t) &= \frac{q\exp\left[(-d+\delta)(t-nT)\right]}{1-(1-p_2)\exp((-d+\delta)T)}, \\ &\qquad t\in(nT,(n+1)T], \end{aligned} \tag{35}$$

and $z(t)$ is the solution of the following equation:

$$\begin{aligned} z'(t) &= z(t)(-d+\delta), \quad t\neq nT, \\ \Delta z(t) &= -p_2z(t)+q, \quad t=nT, \\ z(0^{+}) &= y_0\ge0. \end{aligned} \tag{36}$$

Therefore, there exists a $T_1 > 0$ such that

$$\begin{aligned} y(t) &\le z(t) \le z^*(t) + \varepsilon_1, \\ x'(t) &\ge x(t)\left(a - bm_3 - \left(z^*(t) + \varepsilon_1\right)\right). \end{aligned} \tag{37}$$

Let $N_1 \in \mathbb{N}$ and let $N_1 T \ge T_1$. We can get

$$\begin{aligned} x'(t) &\ge x(t)\left(a - bm_3 - \left(z^*(t) + \varepsilon_1\right)\right), \quad t \ne nT, \\ \Delta x(t) &= -p_1 x(t), \quad t = nT. \end{aligned} \tag{38}$$

Integrating (38) on $(nT, (n+1)T]$ $(n \ge N_1)$, we have

$$\begin{aligned} &x((n+1)T) \\ &\ge x(nT^+)\exp\left(\int_{nT}^{(n+1)T}\left(a - bm_3 - \alpha\left(z^*(t) + \varepsilon_1\right)\right)dt\right) \\ &= (1-p_1)x(nT) \\ &\quad \times \exp\left[(a - bm_3 - \alpha\varepsilon_1)T - \frac{\alpha q\left[1 - \exp((-d+\delta)T)\right]}{(d-\delta)\left[1 - (1-p_2)\exp((-d+\delta)T)\right]}\right] \\ &= x(nT)\exp(\sigma). \end{aligned} \tag{39}$$

Then $x((N_1 + h)T) \ge x(N_1 T)\exp(h\sigma) \to \infty$ as $h \to \infty$, which is a contradiction to the boundedness of $x(t)$. Hence there exists a $t_1 > 0$ such that $x(t_1) \ge m_3$.

Step 2. If $x(t_1) \ge m_3$ for all $t \ge t_1$, then our aim is obtained.

Hence we only need to consider those solutions which leave the region $R = \{X(t) \in \mathbb{R}_2^+ : x(t) < m_3\}$ and reenter again. Let $t^* = \inf_{t \ge t_1}\{x(t) < m_3\}$. Then t^* is impulsive point or nonimpulsive point.

Case 1. If t^* is impulsive point, there exist a $n_1 \in \mathbb{N}$ such that $t^* = n_1 T$. Then $x(t) \ge m_3$ for $t \in [t_1, t^*]$ and

$$(1-p_1)m_3 \le x(t^{*+}) = (1-p_1)x(t^*) < m_3. \tag{40}$$

Choose $n_2, n_3 \in \mathbb{N}$ such that

$$\begin{aligned} n_2 T > T_2 &= \frac{\ln\left(\varepsilon_1/(M+q)\right)}{(-d+\delta)}, \\ (1-p)^{n_2+1}&\exp\left((n_2+1)\sigma_1 T\right)\exp(n_3\sigma) > 1, \end{aligned} \tag{41}$$

where $\sigma_1 = a - bm_3 - \alpha M < 0$. Let $\overline{T} = (n_2 + n_3)T$. Then, there exists a $t_2 \in [(n_1+1)T, (n_1+1)T + \overline{T}]$ such that $x(t_2) \ge m_3$. Otherwise $x(t) < m_3$, $t \in [(n_1+1)T, (n_1+1)T + \overline{T}]$. Consider (36) with $z((n_1+1)T^+) = y((n_1+1)T^+)$; we have

$$\begin{aligned} z(t) &= \left(z(n_1+1)T^+ - \frac{q}{1 - (1-p_2)\exp((-d+\delta)T)}\right) \\ &\quad \times \exp\left[(-d+\delta)(t - (n_1+1)T)\right] + z^*(t), \end{aligned} \tag{42}$$

where $t \in (nT, (n+1)T]$, $n_1 + 1 \le n \le (n_1+1) + n_2 + n_3$. Then

$$\begin{aligned} \left|z(t) - z^*(t)\right| &< (M+q)\exp\left(-(d-\delta)n_2 T\right) < \varepsilon_1, \\ y(t) &\le z(t) \le z^*(t) + \varepsilon_1 \end{aligned} \tag{43}$$

for $(n_1 + 1 + n_2)T \le t \le (n_1+1)T + \overline{T}$, which implies (39) holds for $(n_1 + 1 + n_2)T \le t \le (n_1+1)T + \overline{T}$. As in step 1, we have

$$x((n_1 + 1 + n_2 + n_3)T) \ge x((n_1 + 1 + n_2)T)\exp(n_3\sigma). \tag{44}$$

The first and third equations of system (11) given

$$\begin{aligned} x'(t) &> x(t)(a - bm_3 - \alpha M) = \sigma_1 x(t), \quad t \ne nT, \\ x(t^+) &= (1-p_1)x(t), \quad t = nT. \end{aligned} \tag{45}$$

Integrating the above equation on $[t^*, (n_1 + 1 + n_2)T]$, we can get

$$\begin{aligned} &x((n_1 + 1 + n_2)T) \\ &\quad \ge (1-p)^{n_2+1} m_3 \exp\left(\sigma_1(n_2+1)T\right), \end{aligned} \tag{46}$$

and thus

$$\begin{aligned} &x((n_1 + 1 + n_2 + n_3)T) \\ &\quad \ge m_3(1-p)^{n_2+1}\exp\left((n_2+1)\sigma_1 T\right)\exp(n_3\sigma) \\ &\quad > m_3, \end{aligned} \tag{47}$$

a contradiction.

Let $\bar{t} = \inf_{t \ge t^*}\{x(t) \ge m_3\}$; then $x(\bar{t}) \ge m_3$. For $t \in [t^*, \bar{t})$, we have

$$x(t) \ge m_3\exp\left(\sigma_1(1 + n_2 + n_3)T\right)(1-p)^{1+n_2+n_3} \triangleq m_1. \tag{48}$$

For $t > \bar{t}$, the same arguments can be continued since $x(\bar{t}) \ge m_3$.

Case 2. If t^* is nonimpulsive point, then $x(t) \ge m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$; suppose $t^* \in (n_1' T, (n_1'+1)T)$, $n_1' \in \mathbb{N}$. There are two possible cases for $t \in (t^*, (n_1'+1)T)$.

Case 2.1. $x(t) \le m_3$ for all $t \in (t^*, (n_1'+1)T)$. As in Step 1, we can prove that there must be a $t_2' \in [(n_1'+1)T, (n_1'+1)T + \overline{T}]$ such that $x(t_2') > m_3$; Let $\bar{t} = \inf_{t \ge t^*}\{x(t) \ge m_3\}$, then $x(\bar{t}) \le m_3$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, we have

$$\begin{aligned} x(t) &\ge m_3\exp\left(\sigma_1(1 + n_2 + n_3)T\right)(1-p)^{1+n_2+n_3} \\ &\triangleq m_1. \end{aligned} \tag{49}$$

For $t > \bar{t}$, the same arguments can be continued since $x(\bar{t}) \ge m_3$.

Case 2.2. There exists a $t \in (t^*, (n_1'+1)\overline{T})$ such that $x(t) > m_3$. Let $\check{t} = \inf_{t > t^*}\{x(t) > m_3\}$; then $x(t) \le m_3$ for $t \in (t^*, \check{t})$ and

$x(\check{t}) = m_3$. For $t \in (t^*, \check{t})$, (45) holds true; integrating (45) on $t \in (t^*, \check{t})$, we have

$$x(t) \geq x(t^*) \exp[\sigma_1 (t - t^*)] \geq m_3 \exp(\sigma_1 T) > m_1. \tag{50}$$

Since $x(\check{t}) \geq m_3$ for $t > \check{t}$, the same arguments can be continued.

Hence $x(t) \geq m_1$ for all $t > t_1$. The proof is completed. □

Remark 9. Let

$$f(T) = aT - \frac{\alpha q [1 - \exp(-dT)]}{d[1 - (1 - p_2)\exp(-dT)]} - \ln\left(\frac{1}{1 - p_1}\right). \tag{51}$$

Since $f(0) = -\ln(1/(1-p_1)) < 0$, $f(T) \to +\infty$ as $T \to +\infty$, and

$$\begin{aligned} f''(T) = {} & (\alpha q p_2 d [[1 - (1 - p_2)\exp(-dT)]\exp(-dT) \\ & +2(1 - p_2)\exp(-2dT)]) \\ & \times \left([1 - (1 - p_2)\exp(-dT)]^3\right)^{-1} \\ & > 0, \end{aligned} \tag{52}$$

so $f(T) = 0$ has a unique positive root, denoted by $T_{\max}$. From Theorems 6 and 8 we know that $T_{\max}$ is a threshold. If $T < T_{\max}$, then pest-eradication periodic solution $(0, y^*(t))$ is asymptotically stable; if $T > T_{\max}$, then system (11) is permanent.

Remark 10. If $p_1 = p_2 = 0$, $q = 0$; that is, there are without taking any pest-management strategy, large numbers of preys (pest) would coexisting with predators (natural enemy). If $q = 0$, $0 < p_1, p_2 < 1$, that is, there is periodic spraying pesticide (or harvesting) only. Thus, we can easily obtain that $T'_{\max} = \ln(1/(1 - p_1))/a < T_{\max}$ is the threshold. If $p_1 = p_2 = 0, q > 0$; that is, there is periodic releasing of predator (natural enemy) only, without periodic spraying pesticide (or harvesting). We can easily get that $T''_{\max} = \alpha q/(ad) < T_{\max}$ is the threshold. Comparing with the classic methods (such as biological control or chemical control), the integrated pest management (IPM) is a better one, since $T_{\max} > T'_{\max}$ and $T_{\max} > T''_{\max}$. Some numerical examples will be given in Section 5.

4. Bifurcation and Existence of Positive Periodic Solution

In this section, we deal with the existence of a nontrivial periodic solution to system (11) near the prey-eradication periodic solution $(0, y^*(t))$ via bifurcation.

Let $x_1(t) = y(t)$, $x_2(t) = x(t)$; system (11) becomes as follows:

$$\begin{aligned} x_1'(t) &= x_1(t)\left(\frac{k\alpha x_2(t)}{1 + hx_2^{\beta}(t)} - d\right) \\ &\triangleq F_1(x_1(t), x_2(t)), \\ x_2'(t) &= x_2(t)\left(a - bx_2(t) - \frac{\alpha x_1(t)}{1 + hx_2^{\beta}(t)}\right) \\ &\triangleq F_2(x_1(t), x_2(t)), \\ & \qquad t \neq nT, \\ x_1(nT^+) &= (1 - p_2)x_1(nT) + q \\ &\triangleq \Theta_1(x_1(nT), x_2(nT)), \\ x_2(nT^+) &= (1 - p_1)x_2(nT) \\ &\triangleq \Theta_2(x_1(nT), x_2(nT)), \\ & \qquad t = nT. \end{aligned} \tag{53}$$

All notations used in this section are the same as those in [30]. Let Φ be the flow associated to (53); we have $x(t) = \Phi(t, x_0)$, $0 < t \leq T$, where $x(t) = (x_1(t), x_2(t))$, $x_0 = x(0^+)$,

$$d_0' = 1 - \left(\frac{\partial \Theta_2}{\partial x_2}\frac{\partial \Phi_2}{\partial x_2}\right)(T_0, x_0), \tag{54}$$

where T_0 is the root of $d_0' = 0$,

$$\begin{aligned} a_0' &= 1 - \left(\frac{\partial \Theta_1}{\partial x_1}\frac{\partial \Phi_1}{\partial x_1}\right)(T_0, x_0), \\ b_0' &= -\left(\frac{\partial \Theta_1}{\partial x_1}\frac{\partial \Phi_1}{\partial x_2} + \frac{\partial \Theta_1}{\partial x_2}\frac{\partial \Phi_2}{\partial x_2}\right)(T_0, x_0), \\ \frac{\partial \Phi_1(t, x_0)}{\partial x_1} &= \exp\left(\int_0^t \frac{\partial F_1(\xi(r))}{\partial x_1} \mathrm{d}r\right), \\ \frac{\partial \Phi_2(t, x_0)}{\partial x_2} &= \exp\left(\int_0^t \frac{\partial F_2(\xi(r))}{\partial x_2} \mathrm{d}r\right), \\ \frac{\partial \Phi_1(t, x_0)}{\partial x_2} &= \int_0^t \exp\left(\int_u^t \frac{\partial F_1(\xi(r))}{\partial x_1} \mathrm{d}r\right)\left(\frac{\partial F_1(\xi(u))}{\partial x_2}\right) \\ & \quad \times \exp\left(\int_0^u \frac{\partial F_2(\xi(r))}{\partial x_2} \mathrm{d}r\right) \mathrm{d}u, \\ \frac{\partial^2 \Phi_2(t, x_0)}{\partial x_1 \partial x_2} &= \int_0^t \exp\left(\int_u^t \frac{\partial F_2(\xi(r))}{\partial x_2} \mathrm{d}r\right)\left(\frac{\partial^2 F_1(\xi(u))}{\partial x_1 \partial x_2}\right) \\ & \quad \times \exp\left(\int_0^u \frac{\partial F_2(\xi(r))}{\partial x_2} \mathrm{d}r\right) \mathrm{d}u, \end{aligned}$$

$$\frac{\partial^2\Phi_2(t,x_0)}{\partial x_2^2} = \int_0^t \exp\left(\int_u^t \frac{\partial F_2(\xi(r))}{\partial x_2}\mathrm{d}r\right)\left(\frac{\partial^2 F_2(\xi(u))}{\partial x_2^2}\right) \times \exp\left(\int_0^u \frac{\partial F_2(\xi(r))}{\partial x_2}\mathrm{d}r\right)\mathrm{d}u + \int_0^t \exp\left(\int_u^t \frac{\partial F_2(\xi(r))}{\partial x_2}\mathrm{d}r\right)\left(\frac{\partial^2 F_2(\xi(u))}{\partial x_1\partial x_2}\right) \times \int_0^u \exp\left(\int_p^u \frac{\partial F_1(\xi(r))}{\partial x_1}\mathrm{d}r\right)\left(\frac{\partial F_1(\xi(u))}{\partial x_2}\right) \times \exp\left(\int_0^p \frac{\partial F_2(\xi(r))}{\partial x_2}\mathrm{d}r\right)\mathrm{d}p\,\mathrm{d}u,$$

$$\frac{\partial^2\Phi_2(t,x_0)}{\partial\overline{T}\partial x_2} = \frac{\partial F_2(\xi(t))}{\partial x_2}\exp\left(\int_0^t \frac{\partial F_2(\xi(r))}{\partial x_2}\mathrm{d}r\right),$$

$$\frac{\partial^2\Phi_1(T_0,x_0)}{\partial T} = y'(T_0),$$

$$B = -\frac{\partial^2\Theta_2}{\partial x_1\partial x_2}\left(\frac{\partial\Phi_1(T_0,X_0)}{\partial\overline{T}} + \frac{\partial\Phi_1(T_0,X_0)}{\partial x_1}\frac{1}{a_0'}\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1(T_0,X_0)}{\partial\overline{T}}\right) \times \frac{\partial\Phi_2(T_0,X_0)}{\partial x_2} - \frac{\partial\Theta_2}{\partial x_2}\left(\frac{\partial^2\Phi_2(t,X_0)}{\partial\overline{T}\partial x_2} + \frac{\partial^2\Phi_2(t,X_0)}{\partial x_1\partial x_2} \times \frac{1}{a_0'}\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1(T_0,X_0)}{\partial\overline{T}}\right),$$

$$C = -2\frac{\partial^2\Theta_2}{\partial x_1\partial x_2}\left(-\frac{b_0'}{a_0'}\frac{\partial\Phi_1(T_0,X_0)}{\partial x_1} + \frac{\partial\Phi_1(T_0,X_0)}{\partial x_2}\right) \times \frac{\partial\Phi_2(T_0,X_0)}{\partial x_2} - \frac{\partial^2\Theta_2}{\partial x_2^2}\left(\frac{\partial\Phi_2(T_0,X_0)}{\partial x_2}\right)^2 + 2\frac{\partial\Theta_2}{\partial x_2}\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2(T_0,X_0)}{\partial x_1\partial x_2} - \frac{\partial\Theta_2}{\partial x_2}\frac{\partial^2\Phi_2(T_0,X_0)}{\partial x_2^2}, \tag{55}$$

where $\xi(t) = (y^*(t), 0)$.

Lemma 11 (see [30]). *If $|1 - a_0'| < 1$ and $d_0' = 0$, then one has the following.*

(a) *If $BC \neq 0$, then one has a bifurcation. Moreover, one has a bifurcation of a nontrivial periodic solution of (53) if $BC < 0$ and a subcritical case if $BC > 0$.*

(b) *If $BC = 0$, then one has an undetermined case.*

In order to apply Lemma 11, we compute the following:

$$d_0' = 1-(1-p_1)\exp\left(\int_0^T (a-\alpha y^*(r))\,\mathrm{d}r\right) = 1-(1-p_1)\exp\left[aT_0 - \frac{\alpha q(1-\exp(-dT))}{d[1-(1-p_2)\exp(-dT)]}\right]. \tag{56}$$

If $d_0' = 0$, this corresponds to T_0 satisfying

$$aT_0 = \frac{\alpha q(1-\exp(-dT_0))}{d[1-(1-p_2)\exp(-dT_0)]} + \ln\left(\frac{1}{1-p_1}\right). \tag{57}$$

Further, we can get

$$a_0' = 1-\exp(-dT_0) > 0,$$
$$b_0' = -k\alpha(1-p_2)\exp\left(\int_0^{T_0} y^*(r)\,\mathrm{d}r\right) < 0,$$
$$\frac{\partial\Phi_1(T_0,x_0)}{\partial x_1} = \exp(-dT_0) > 0,$$
$$\frac{\partial\Phi_2(T_0,x_0)}{\partial x_2} = \exp\left[aT_0 - \frac{\alpha q(1-\exp(-dT))}{d[1-(1-p_2)\exp(-dT)]}\right] = \frac{1}{1-p_1} > 0,$$
$$\frac{\partial\Phi_1(T_0,x_0)}{\partial x_2} = k\alpha\int_0^{T_0}\exp(-d(T_0-u))\,y^*(u) \times \exp\left(\int_0^u (a-\alpha y^*(r))\,\mathrm{d}r\right)\mathrm{d}u > 0,$$
$$\frac{\partial^2\Phi_2(T_0,x_0)}{\partial\overline{T}\partial x_2} = \left(a - \frac{\alpha q(1-\exp(-dT_0))}{d[1-(1-p_2)\exp(-dT_0)]}\right)\frac{1}{1-p_1},$$
$$\frac{\partial^2\Phi_2(T_0,x_0)}{\partial x_1\partial x_2} = -\frac{k\alpha T_0}{1-p_1} < 0,$$
$$\frac{\partial\Phi_1(T_0,x_0)}{\partial T} = y'(T_0) = -\frac{dq\exp(-dT_0)}{1-(1-p_2)\exp(-dT_0)} < 0. \tag{58}$$

Note that

$$\frac{\partial^2 F_2(\xi(u))}{\partial x_2^2} = -(2b+\alpha y^*(u)) < 0,$$
$$\frac{\partial^2 F_2(\xi(u))}{\partial x_1\partial x_2} = -\alpha < 0, \tag{59}$$
$$\frac{\partial F_1(\xi(u))}{\partial x_2} = k\alpha y^*(u) > 0;$$

then

$$\frac{\partial^2\Phi_2(T_0,x_0)}{\partial x_2^2} < 0. \tag{60}$$

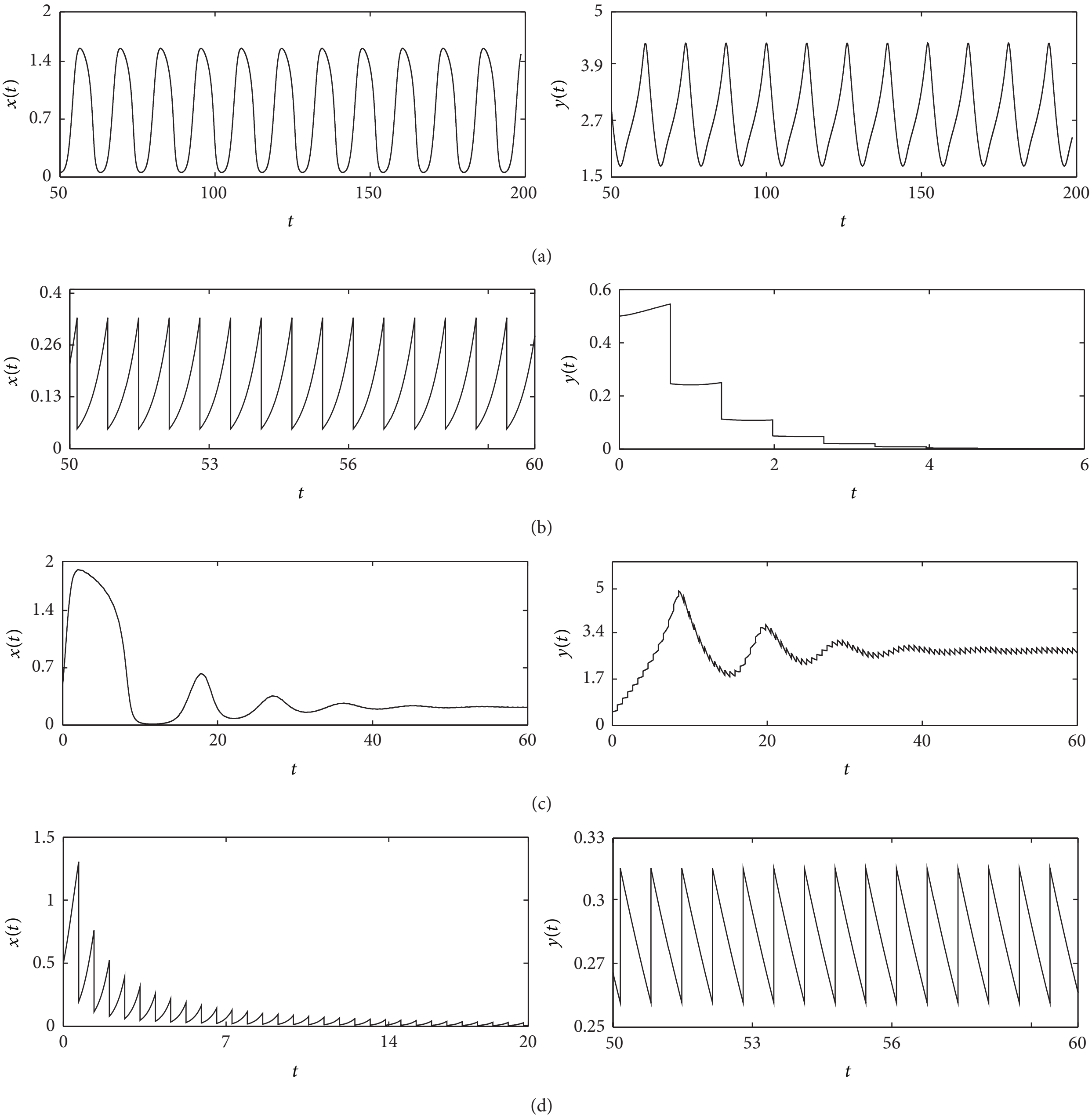


Figure 1: Time series of system (11) when $T = 0.66 < T_{\max} \approx 0.6776$. (a) $p_1 = 0$, $p_2 = 0$, and $q = 0$, without taking any pest-management strategy; (b) $p_1 = 0.85$, $p_2 = 0.55$, and $q = 0$, with spraying pesticide (or harvesting) only; (c) $p_1 = 0$, $p_2 = 0$, and $q = 0.2$, with releasing of predator (natural enemy) only; (d) $p_1 = 0.85$, $p_2 = 0.55$, and $q = 0.2$, with taking integrated pest-management strategy.

Since

$$\frac{\partial \Theta_1}{\partial x_2} = \frac{\partial \Theta_2}{\partial x_1} = 0, \qquad \frac{\partial \Theta_1}{\partial x_1} = 1 - p_2, \qquad \frac{\partial \Theta_2}{\partial x_2} = 1 - p_1, \tag{61}$$

it is easy to verify that $C > 0$ and

$$B = -\left[a - \frac{\alpha q \exp(-dT_0)}{d\left[1 - (1 - p_2)\exp(-dT_0)\right]} + \frac{k\alpha d q T_0 (1 - p_2)\exp(-dT_0)}{\left[1 - (1 - p_2)\exp(-dT_0)\right]^2}\right]. \tag{62}$$

In order to determine the sign of B, let

$$f(t) = a - \frac{\alpha q \exp(-dt)}{1 - (1 - p_2)\exp(-dT_0)}. \tag{63}$$

We have

$$f'(t) = \frac{\alpha q \exp(-dt)}{d\left[1 - (1 - p_2)\exp(-dT_0)\right]} > 0. \tag{64}$$

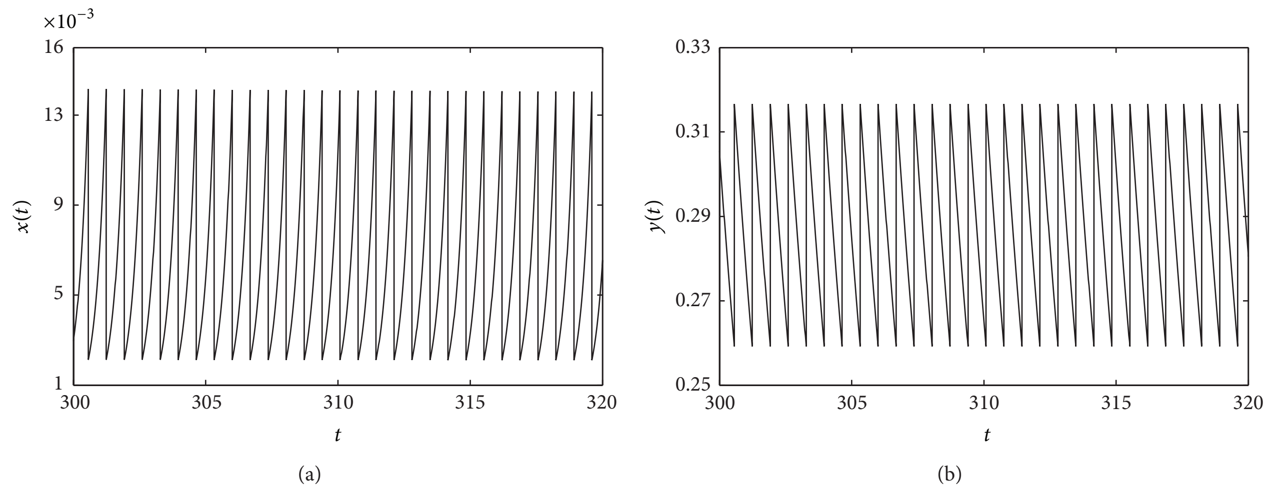


FIGURE 2: Time series of system (11) when $T = 0.68 > T_{\max} \approx 0.6776$; prey population x and predator population y coexist with periodic oscillations.

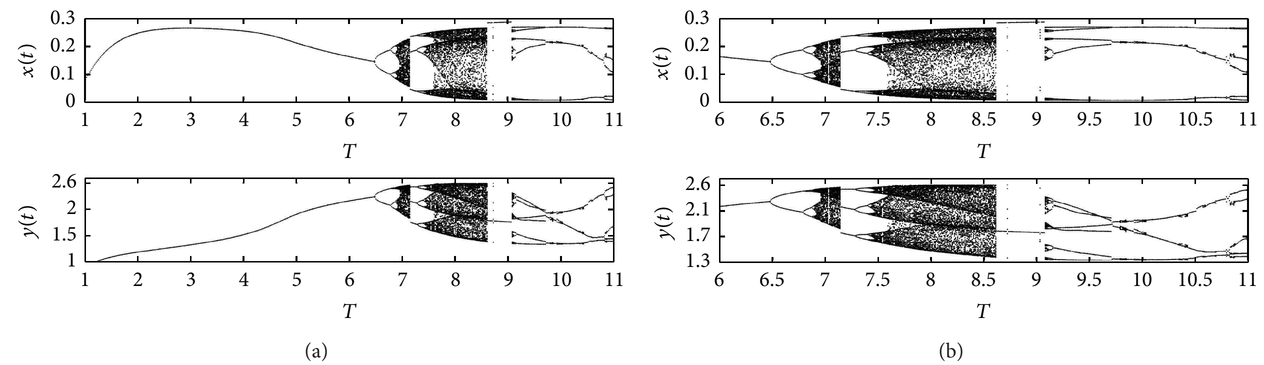


FIGURE 3: Bifurcation diagrams of system (11): (a) prey population x and predator population y with impulsive period T over $[1, 11]$; (b) prey population x and predator population y with impulsive period T over $[6, 11]$.

Thus, we can conclude that $f(T_0) > 0$, since

$$\begin{aligned} \int_0^{T_0} f(t)\,\mathrm{d}t &= aT_0 - \frac{\alpha q \exp(-dt)}{d\left[1-(1-p_2)\exp(-dT_0)\right]} \\ &= \ln\left(\frac{1}{1-p_1}\right) \geq 0, \end{aligned} \tag{65}$$

and $f(t)$ is strictly increasing. Therefore, we have $B < 0$. In view of $T_0 = T_{\max}$ and according to Lemma 11, we obtain the following result.

Theorem 12. *System* (11) *has a positive periodic solution if* $T > T_0$ *and* T *is close to* T_0, *where* T_0 *satisfies*

$$aT_0 = \frac{\alpha q\left(1-\exp(-dT_0)\right)}{d\left[1-(1-p_2)\exp(-dT_0)\right]} + \ln\left(\frac{1}{1-p_1}\right), \tag{66}$$

and the nontrivial periodic solution is supercritical case via bifurcation, which means that the positive periodic solution is stable.

5. Numerical Analysis

In this section, we will study the impulsive effect on system (11) and show that the impulsive perturbations cause complicated dynamical behavior for system (11). The influence of T, q, and β may be documented by stroboscopically sampling one of the variables over a range of their values. Stroboscopic map is a special case of the Poincaré map for periodically forced system or periodically pulsed system. Fixing points of the stroboscopic map correspond to periodic solutions of system (11) having the same period as the pulsing term; periodic points of period k about stroboscopic map correspond to entrained periodic solutions of system (11) having exactly k times the period of the pulsing; invariant circles correspond to quasi-periodic solutions of system (11); system (11) possibly appear chaotic (or strange) attractors.

Example 13. Let $a = 3.1$, $b = 1.5, \alpha = 1.05$, $k = 0.85$, $n = 2.15$, $h = 0.97$, $d = 0.3$, $p_1 = 0.85$, $p_2 = 0.55$, and $q = 0.2$ with initial value $X(0) = (0.5, 0.5)$.

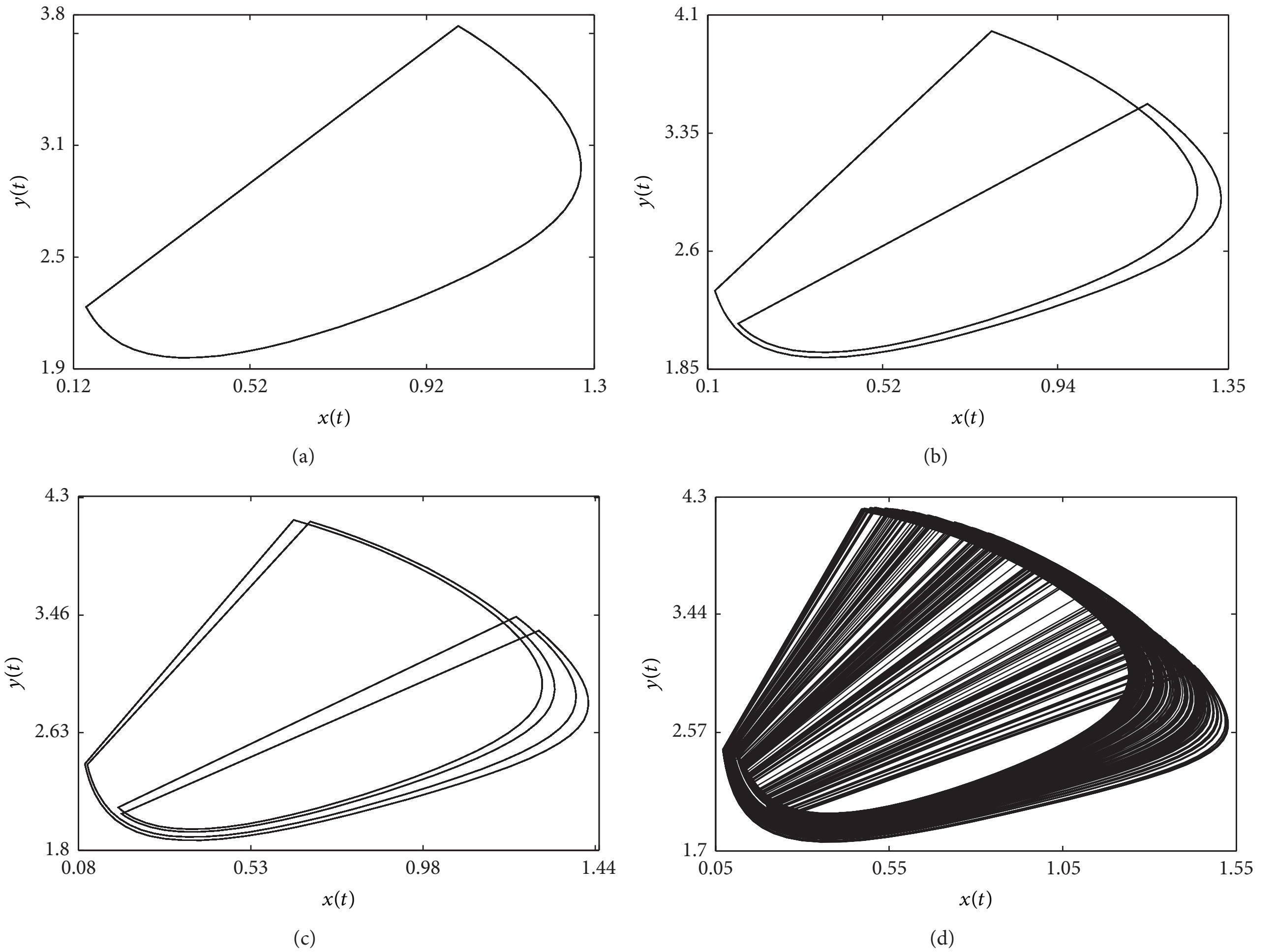


Figure 4: Period doubling bifurcation leads to chaos: T, $2T$, and $4T$ periodic solutions and chaos when $T = 6.4, 6.6, 6.8$, and 7.0, respectively.

From Remark 10, large numbers of preys (pest) could coexist with predators (natural enemy) with periodic oscillations, if we are not taking pest-management strategy ($p_1 = 0$, $p_2 = 0$, $q = 0$) (Figure 1(a)). $q = 0$, $p_1 = 0.85$, and $p_2 = 0.55$; that is, there are periodic spraying pesticide (or harvesting) only, without releasing of predator (natural enemy), large numbers of preys (pest) coexist with periodic oscillation, but predators (natural enemy) rapidly decrease to zero when $T = 0.66 < T_{\max}$ (Figure 1(b)). $p_1 = 0$, $p_2 = 0$, $q = 0.2$; that is, there is periodic releasing of predator (natural enemy) only; without spraying pesticide (or harvesting), a few of preys (pest) coexist with predators (natural enemy) when $T = 0.66 < T_{\max}$ (Figure 1(c)). We cannot make the prey population $x(t)$ eradicate when $T = 0.66$. From Theorem 6, we know that the prey-eradication periodic solution is asymptotically stable provided that $T < T_{\max} \approx 0.6776$. A typical prey-eradication periodic solution of the system (11) is shown in Figure 1(d), where we observe how the variable $y(t)$ oscillates in a stable cycle. In contrast, the prey population $x(t)$ rapidly decreases to zero when $T = 0.66 < T_{\max} \approx 0.6776$. Hence, the integrated pest management (IPM) is better than the classic methods (such as biological control or chemical control).

According to Theorem 12, if the impulsive periodic $T > T_{\max}$ and is close to $T_{\max}$, the prey eradication solution becomes unstable, there is a supercritical bifurcation, then the prey and predator can coexist on a stable positive periodic solution when $T = 0.68 > T_{\max} \approx 0.6776$ (Figure 2). Therefore, in order to control the pest populations, we would choose an appropriate impulsive periodic T. $T > T_{\max}$ and close to $T_{\max}$ would be a better one.

Let $q = 0.55$ and fix other parameter sets of values; we have displayed bifurcation diagrams for the pest population x and the predator population y for impulsive period T over $[1, 11]$ and $[6, 11]$. We find that by increasing the impulsive period T, system (11) undergoes a process of period-doubling cascade $\rightarrow$ chaos $\rightarrow$ crisis and high-order periodic oscillations (Figure 3). When T increases from 6 to 7, there is a cascade of period-doubling bifurcations leading to chaos (Figure 4). When $T = 8.62$, the chaos suddenly disappears and a T-periodic solution appears, then the T-periodic solution abruptly disappears and the chaos abruptly appears again when $T = 9.08$, these constituting several types of crises (Figure 5). However, when $T = 8.62$ and $T = 9.08$, it appears that attractors are nonunique, coexistence of stranger attractor with T-periodic solution (Figure 6).

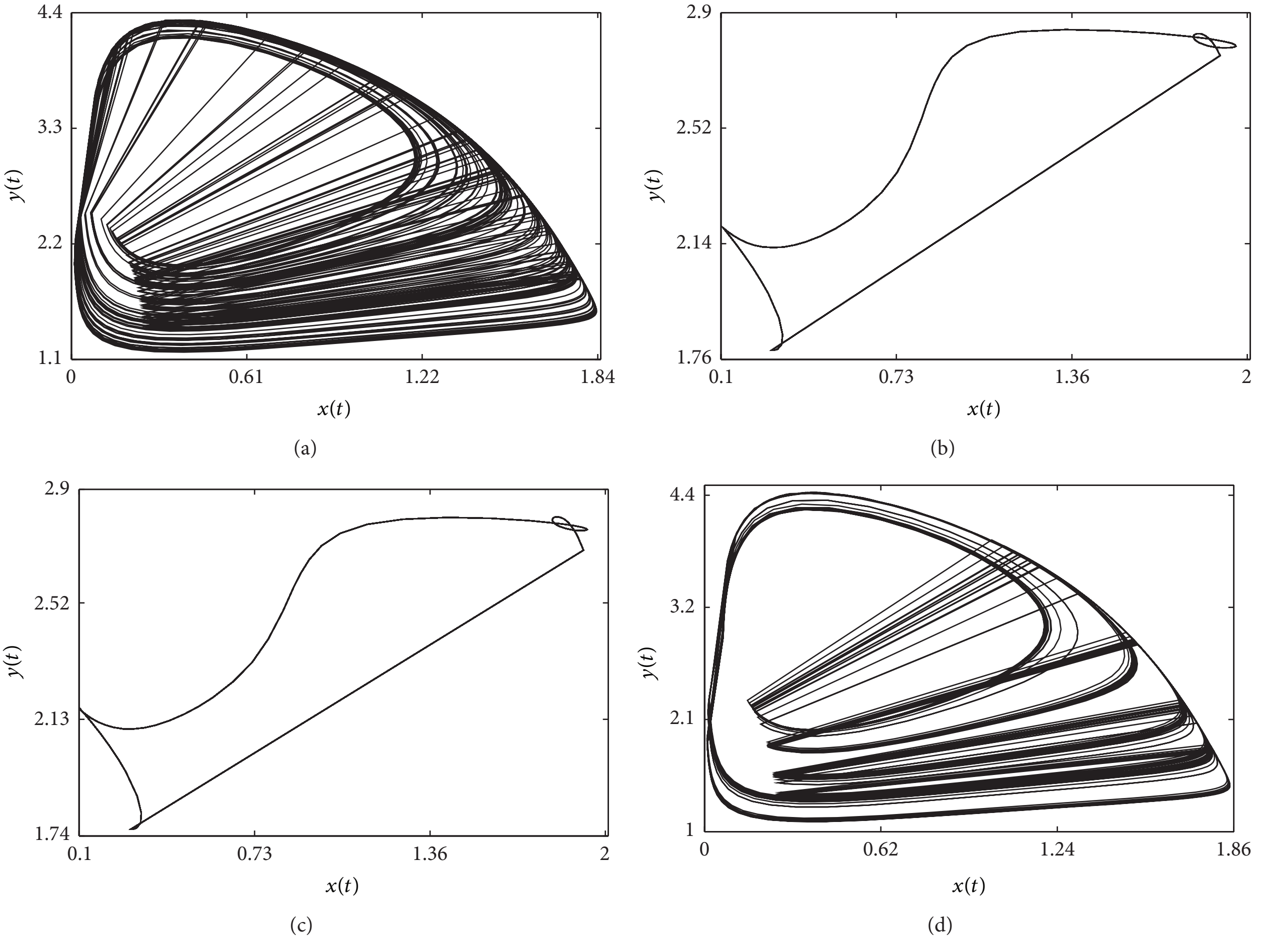


FIGURE 5: Crises are shown. There is a crisis that the chaos suddenly disappears when $T = 8.61, 8.62$, and there is a crisis that the chaos suddenly appears when $T = 9.07, 9.08$.

Obviously, which one of the attractors is reached depends on the initial values.

Example 14. Let $a = 3.1$, $b = 1.5$, $\alpha = 1.05$, $k = 0.85$, $n = 2.15$, $h = 0.97$, $d = 0.3$, $p_1 = 0.85$, $p_2 = 0.55$, and $T = 8$ with initial value $X(0) = (0.5, 0.5)$. We investigate the effect of q on the system (11). Figure 7 showed bifurcation diagrams obtained by stroboscopically sampling the pest population x and the predator population y for q over $[0.1, 3.1]$. The resulting bifurcation diagrams clearly showed that system (11) has rich dynamics, including period-doubling bifurcation, period-halving bifurcation, and chaos. When q increases from 0.9 to 2.2, there is a period-halving bifurcation leading to a T-periodic solution (Figure 8).

Example 15. Let $a = 3.1$, $b = 1.5$, $\alpha = 1.05$, $k = 0.85$, $n = 2.15$, $h = 0.97$, $d = 0.3$, $p_1 = 0.85$, $p_2 = 0.55$, $q = 1.2$, and $T = 8$ with initial value $X(0) = (0.5, 0.5)$. We consider the effect of β on the system (11). The resulting bifurcation diagrams (Figure 9), the pest population x, and the predator population y for β over $[2.0, 6.0]$ clearly showed that system (11) has complex dynamics, such as period-doubling bifurcation, high-order periodic oscillation, and chaos. In Figure 10, the typical high-order oscillation of system (11) is shown: $7T, 12T, 17T$, and $3T$ periodic solutions when $\beta = 2.8, 3.05, 3.4$, and 5.5, respectively. Further, Figure 11 shown the maximin and mean amount of prey population x and predator population y of system (11) with β over $[2.0, 5.5]$.

From bifurcation diagrams in Figures 3, 7, and 9, we can easily see that the dynamical behavior of these three cases is very complicated, which includes (1) high-order quasi-periodic and periodic oscillations, (2) period-doubling bifurcation, (3) period-halving bifurcations, (4) nonunique dynamics (meaning that several attractors coexist), and (5) cries (the phenomenon of “crisis” in chaotic attractors can suddenly appear or disappear, or change size discontinuously as a parameter smoothly varies).

6. Conclusion

In this paper, we have investigated a predator-prey system with generalized group defense and concerning impulsive control strategy for pest control in detail. We have shown that there exists an asymptotically stable pest-eradication periodic

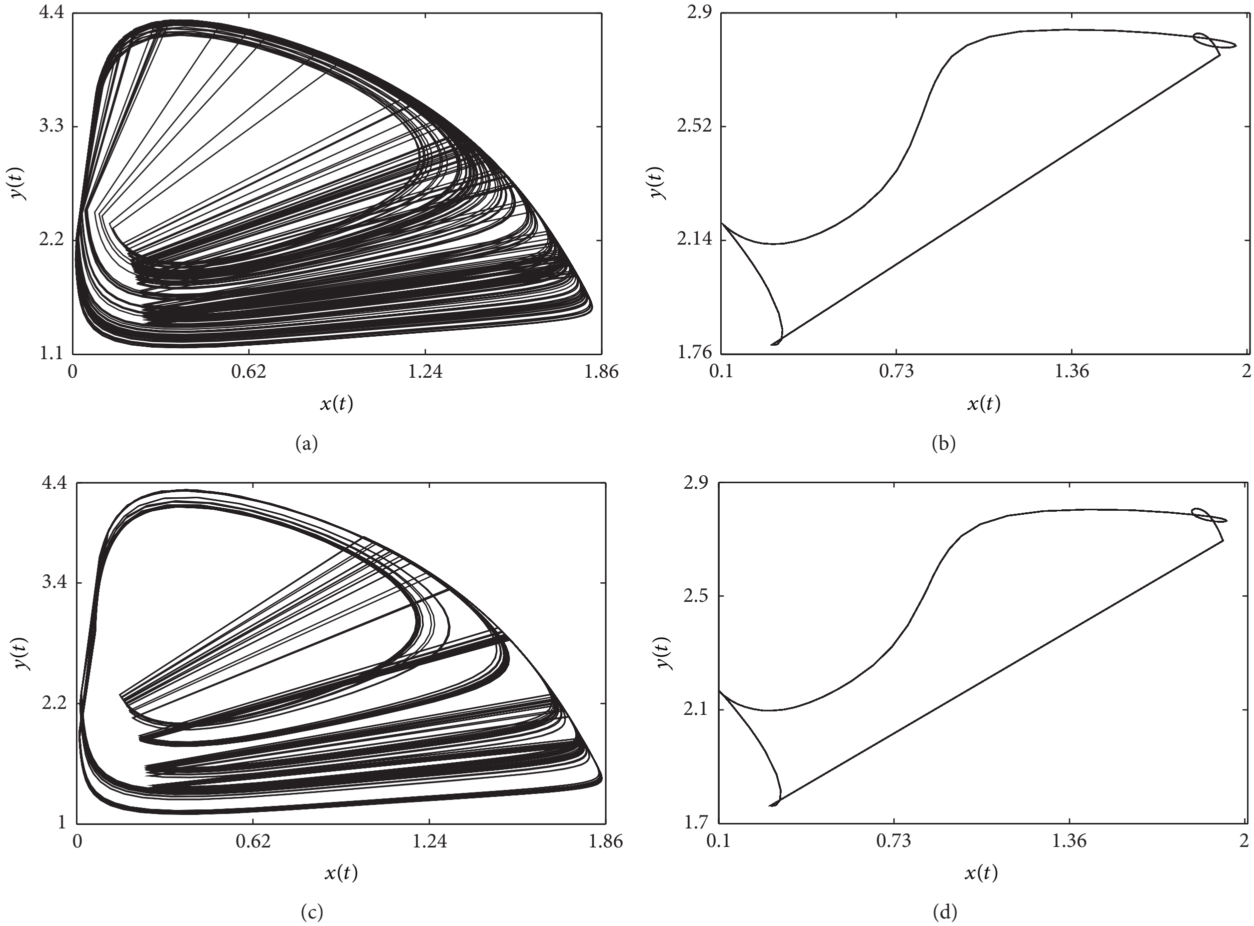


Figure 6: Different attractors' coexistence: solution with initial value $X(0) = (0.5, 2.5)$ will finally tend to strange attractor, solution with $X(0) = (0.6, 2.5)$ will finally tend to T periodic solution when $T = 8.62$, solution with initial value $X(0) = (1.1, 2.5)$ will finally tend to strange attractor, and solution with $X(0) = (1.2, 2.5)$ will finally tend to T periodic solution when $T = 9.08$, respectively.

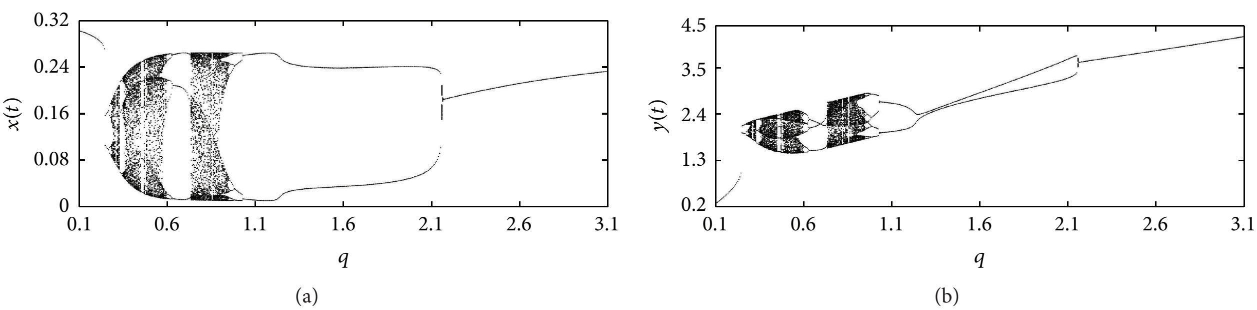


Figure 7: Bifurcation diagrams of system (11): prey population x and predator population y with q over $[0.1, 3.1]$.

solution if the impulsive period is less than the critical value $T_{\max}$. If we choose our impulsive control strategy, in order to drive the pest to extinction, we can determine the impulsive period T according to the effect of the chemical pesticides on the populations and the cost of releasing natural enemies such that $T < T_{\max}$.

But, in a real world, complete eradication of pest populations is generally not possible, nor is it biologically or economically desirable. A good-pest control program should reduce pest population to levels acceptable to the public. When $T > T_{\max}$, the stability of the pest-eradication periodic solution is lost, system (11) is permanent, and there exists a nontrivial periodic solution when T is close to $T_{\max}$. The smaller the period, the fewer the pests. Therefore, we can control the pest population below some economic threshold (ET is defined as the pest population level that produces

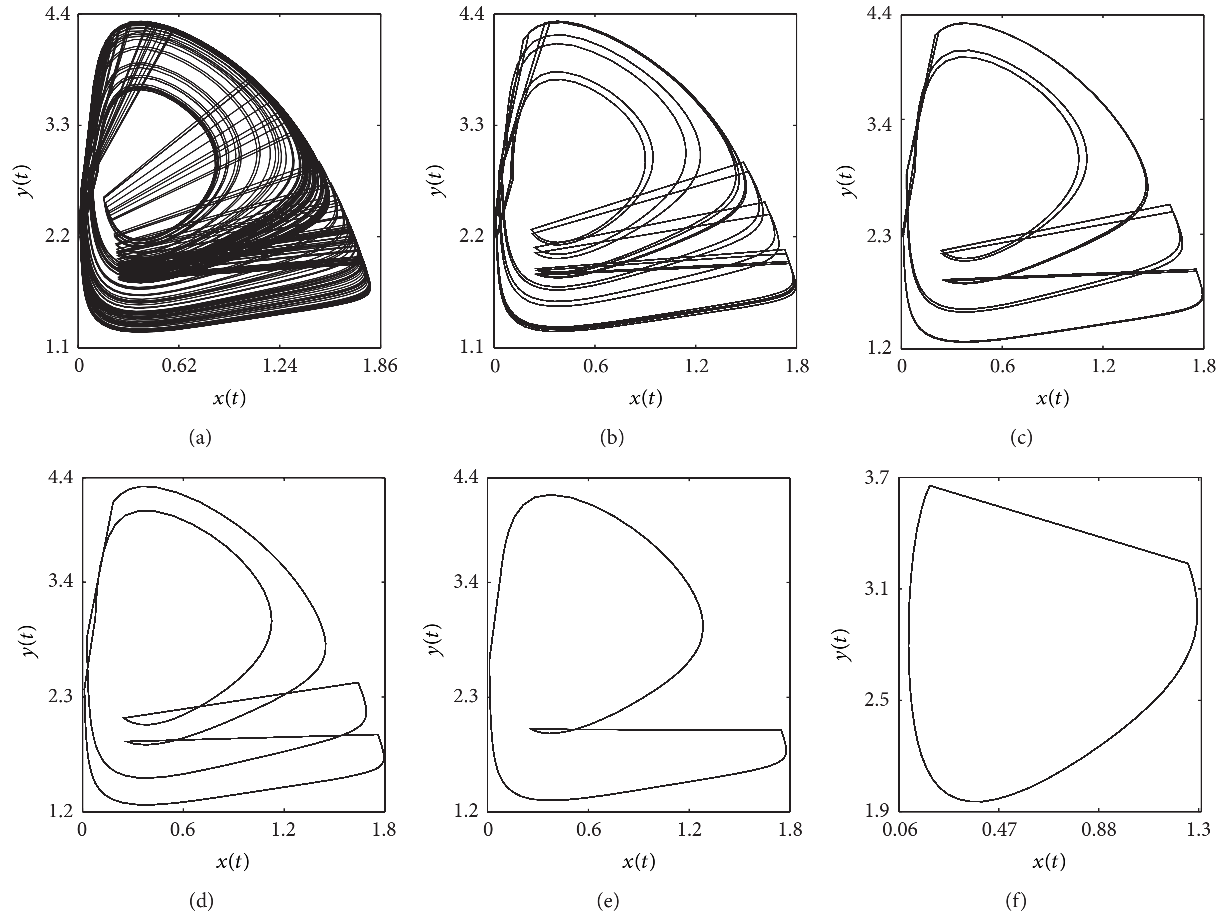


Figure 8: Period-halving bifurcation leads to a T-periodic solution of system (11): chaos and $16T, 8T, 4T, 2T$, and T periodic solutions when $q = 0.9, 0.95, 0.98, 1.0, 1.1$, and 2.2, respectively.

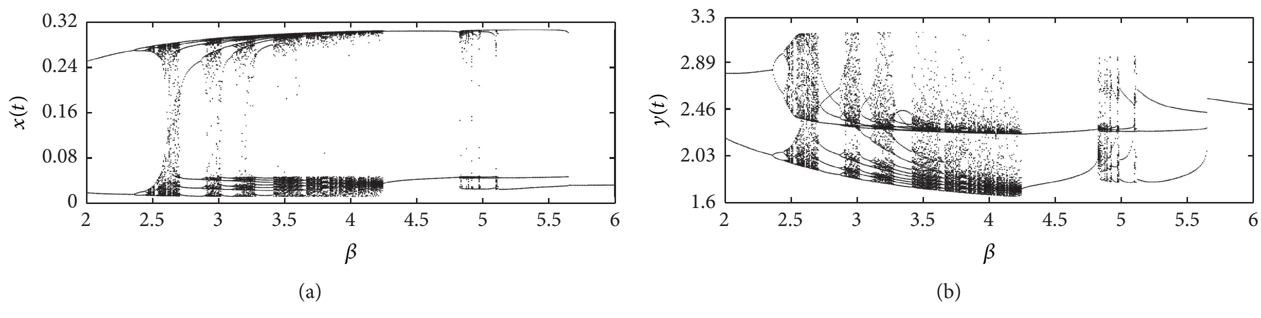


Figure 9: Bifurcation diagrams of system (11): prey population x and predator population y with β over $[2.0, 6.0]$.

damage equal to the costs of preventing damage) by choosing appropriate impulsive period T and the number of mature predator released q, according to the degree of antipredator behavior and group defense β, making an integrated pest-management strategy every period T. Then, the periodic releasing of natural enemies and spraying pesticides change the properties of the system without impulses and our results suggest an effective approach in the pest control.

Numerical results show that system (11) can take on various kinds of periodic fluctuations and several types of attractor coexistence and is dominated by high-order periodic oscillations, quasi-periodic oscillations, and chaotic oscillations. These results imply that the presence of pulses destroys equilibria, initiates multiple attractors, quasi-periodic oscillations, and chaos, and makes the dynamical behaviors more complex.

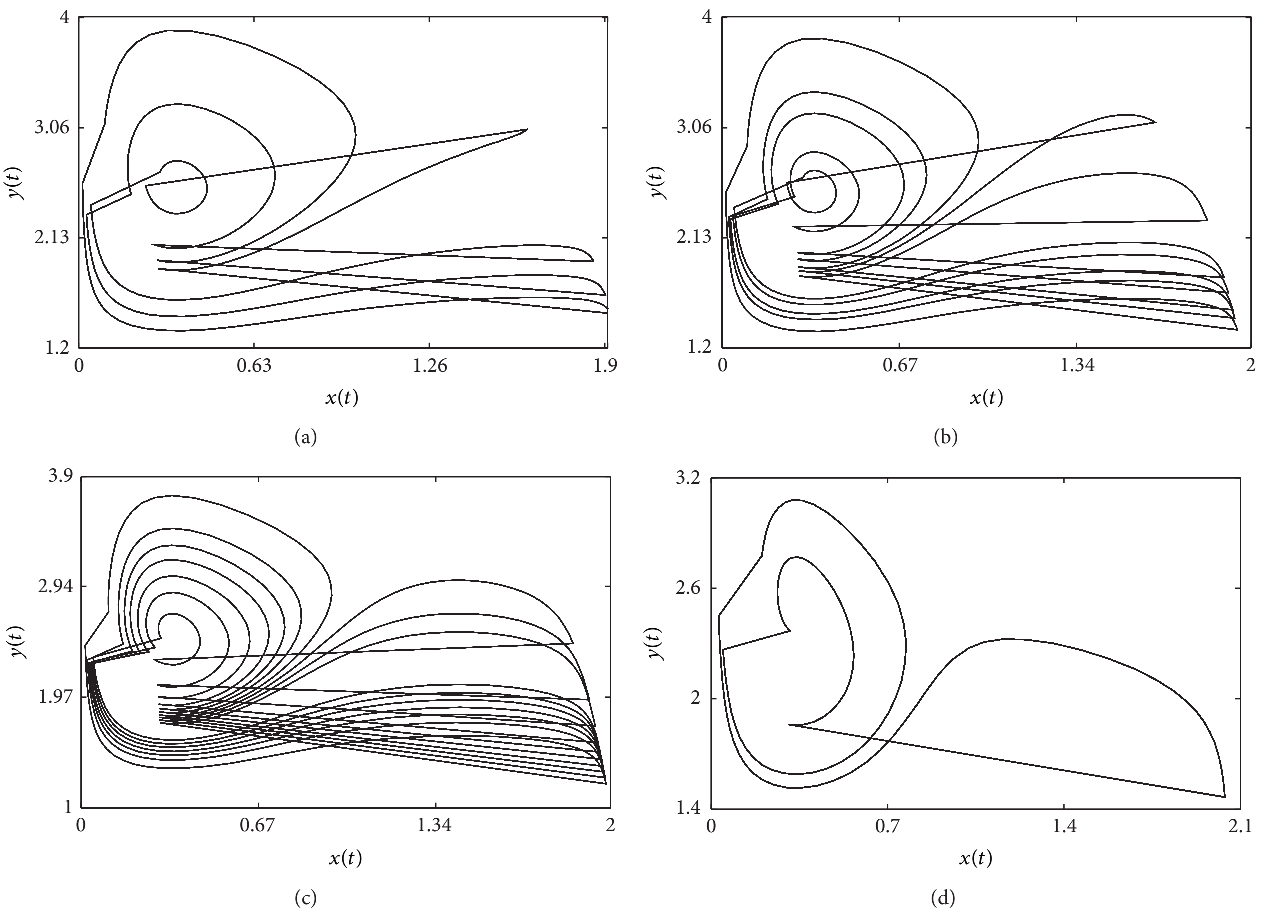


Figure 10: High-order oscillations of system (11): $7T, 12T, 17T$, and $3T$ periodic solutions when $\beta = 2.8, 3.05, 3.4$, and 5.5, respectively.

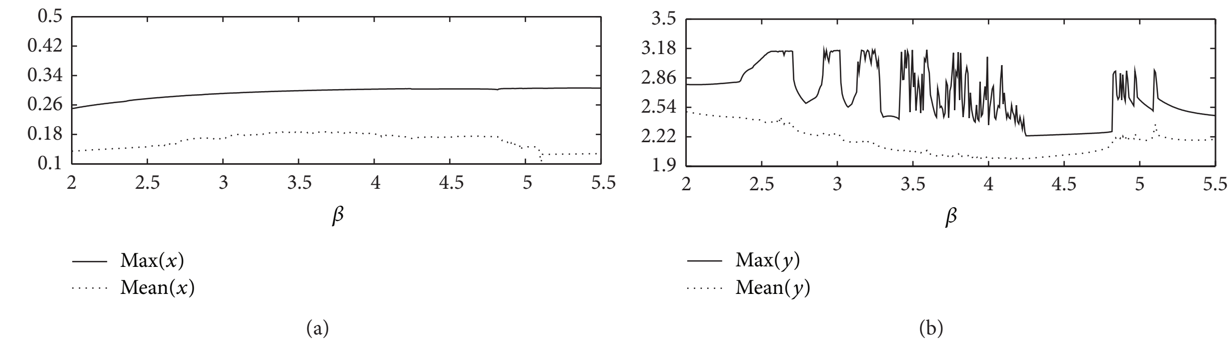


Figure 11: The maximin and mean amount of prey population x and predator population y of system (11) with β over [2.0, 5.5], respectively.

Acknowledgment

This work was supported by the Science and Research Project Foundation of Educational Department of Guizhou Province (no. 2010096).

References

[1] C. S. Holling, "The functional response of predator to prey density and its role in mimicry and population regulation," *Memoirs of the Entomological Society of Canada*, vol. 1, no. 45, pp. 1–60, 1965.

[2] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, vol. 57 of *Monograph Textbooks Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[3] V. S. Ivlev, *Experimental Ecology of the Feeding of Fishes*, Yale University Press, New Haven, Conn, USA, 1961.

[4] D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, "A Model for trophic interaction," *Ecology*, vol. 56, pp. 881–892, 1975.

[5] J. R. Beddington, "Mutual interference between parasites or predators and its effect on searching efficiency," *Journal of Animal Ecology*, vol. 44, pp. 331–340, 1975.

[6] D. A. Miller, J. B. Grand, T. F. Fondell, and M. Anthony, "Predator functional response and prey survival: direct and indirect

interactions affecting a marked prey population," *Journal of Animal Ecology*, vol. 75, no. 1, pp. 101–110, 2006.

[7] J. F. Andrews, "A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates," *Biotechnology and Bioengineering*, vol. 10, pp. 707–723, 1968.

[8] H. I. Freedman and S. G. Ruan, "Hopf bifurcation in three-species food chain models with group defense," *Mathematical Biosciences*, vol. 111, no. 1, pp. 73–87, 1992.

[9] H. I. Freedman and G. S. K. Wolkowicz, "Predator-prey systems with group defence: the paradox of enrichment revisited," *Bulletin of Mathematical Biology*, vol. 48, no. 5-6, pp. 493–508, 1986.

[10] H. I. Freedman and H. S. Quan, "Interactions leading to persistence in predator-prey systems with group defence," *Bulletin of Mathematical Biology*, vol. 50, no. 5, pp. 517–530, 1988.

[11] D. M. Xiao and S. G. Ruan, "Codimension two bifurcations in a predator-prey system with group defense," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 11, no. 8, pp. 2123–2131, 2001.

[12] H. Ddumba, J. Y. T. Mugisha, J. W. Gonsalves, and G. I. H. Kerley, "The role of predator fertility and prey threshold bounds on the global and local dynamics of a predator-prey model with a prey out-flux dilution effect," *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9169–9186, 2012.

[13] S. W. Zhang, F. Y. Wang, and L. S. Chen, "A food chain model with impulsive perturbations and Holling IV functional response," *Chaos, Solitons and Fractals*, vol. 26, no. 3, pp. 855–866, 2005.

[14] Z. L. Xiong, Y. Xue, and S. Y. Li, "A food chain system with Holling IV functional responses and impulsive effect," *International Journal of Biomathematics*, vol. 1, no. 3, pp. 361–375, 2008.

[15] H. Baek, "A food chain system with Holling type IV functional response and impulsive perturbations," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1152–1163, 2010.

[16] J. J. Jiao, X. Z. Meng, and L. S. Chen, "A stage-structured Holling mass defence predator-prey model with impulsive perturbations on predators," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1448–1458, 2007.

[17] S. W. Zhang, D. J. Tan, and L. S. Chen, "Chaos in periodically forced Holling type IV predator-prey system with impulsive perturbations," *Chaos, Solitons and Fractals*, vol. 27, no. 4, pp. 980–990, 2006.

[18] Z. J. Liu and R. H. Tan, "Impulsive harvesting and stocking in a Monod-Haldane functional response predator-prey system," *Chaos, Solitons and Fractals*, vol. 34, no. 2, pp. 454–464, 2007.

[19] Q. Wang, B. X. Dai, and Y. M. Chen, "Multiple periodic solutions of an impulsive predator-prey model with Holling-type IV functional response," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1829–1836, 2009.

[20] H. Baek and Y. Do, "Stability for a Holling type IV food chain system with impulsive perturbations," *Kyungpook Mathematical Journal*, vol. 48, no. 3, pp. 515–527, 2008.

[21] C.-Y. Huang, Y.-J. Li, and H.-F. Huo, "The dynamics of a stage-structured predator-prey system with impulsive effect and Holling mass defence," *Applied Mathematical Modelling*, vol. 36, no. 1, pp. 87–96, 2012.

[22] Y. Z. Wang and M. Zhao, "Dynamic analysis of an impulsively controlled predator-prey model with Holling type IV functional response," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 141272, 18 pages, 2012.

[23] H.-F. Huo and W.-T. Li, "Dynamics of a nonautonomous semiratio-dependent predator-prey system with nonmonotonic functional responses," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 70656, 19 pages, 2006.

[24] X. P. Li and W. S. Yang, "Permanence of a semi-ratio-dependent predator-prey system with nonmonotonic functional response and time delay," *Abstract and Applied Analysis*, vol. 2009, Article ID 960823, 6 pages, 2009.

[25] S. W. Zhang, L. Z. Dong, and L. S. Chen, "The study of predator-prey system with defensive ability of prey and impulsive perturbations on the predator," *Chaos, Solitons & Fractals*, vol. 23, no. 2, pp. 631–643, 2005.

[26] Y. Z. Pei, C. G. Li, L. S. Chen, and C. Wang, "Complex dynamics of one-prey multi-predator system with defensive ability of prey and impulsive biological control on predators," *Advances in Complex Systems*, vol. 8, no. 4, pp. 483–495, 2005.

[27] S. Y. Li, Z. L. Xiong, and X. Wang, "The study of a predator-prey system with group defense and impulsive control strategy," *Applied Mathematical Modelling*, vol. 34, no. 9, pp. 2546–2561, 2010.

[28] D. D. Bainov and D. D. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman UK, 1993.

[29] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6, World Scientific Publishing, Teaneck, NJ, USA, 1989.

[30] A. Lakmeche and O. Arino, "Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 7, no. 2, pp. 265–287, 2000.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 183420, 10 pages
http://dx.doi.org/10.1155/2013/183420

Research Article

Almost Automorphic Mild Solutions to Neutral Parabolic Nonautonomous Evolution Equations with Nondense Domain

Zhanrong Hu[1,2] and Zhen Jin[1,2]

[1] *Department of Mathematics, North University of China, Taiyuan 030051, China*
[2] *School of Mechatronic Engineering, North University of China, Taiyuan 030051, China*

Correspondence should be addressed to Zhanrong Hu; huzhanrong22@126.com

Received 5 March 2013; Accepted 22 April 2013

Academic Editor: Yonghui Xia

Combining the exponential dichotomy of evolution family, composition theorems for almost automorphic functions with Banach fixed point theorem, we establish new existence and uniqueness theorems for almost automorphic mild solutions to neutral parabolic nonautonomous evolution equations with nondense domain. A unified framework is set up to investigate the existence and uniqueness of almost automorphic mild solutions to some classes of parabolic partial differential equations and neutral functional differential equations.

1. Introduction

In this paper, we are interested in the existence and uniqueness of almost automorphic mild solutions to the following neutral parabolic evolution equations in Banach space $\mathbb{X}$:

$$\frac{d}{dt}\left[u(t)+f(t,u(t))\right] = A(t)\left[u(t)+f(t,u(t))\right] + g(t,u(t)), \quad t\in\mathbb{R}, \tag{1}$$

$$\frac{d}{dt}\left[u(t)+f(t,Bu(t))\right] = A(t)\left[u(t)+f(t,Bu(t))\right] + g(t,Cu(t)), \quad t\in\mathbb{R}, \tag{2}$$

where sectorial operators $A(t) : D(A(t)) \subset \mathbb{X} \to \mathbb{X}$ have a domain $D(A(t))$ not necessarily dense in $\mathbb{X}$ and satisfy "Acquistapace-Terreni" conditions, $f, g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ are almost automorphic in the first argument and Lipschitz in the second argument, and $B, C : \mathbb{X} \mapsto \mathbb{X}$ are bounded linear operators.

Bochner has shown in the seminal work [1] that in certain situations it is possible to establish the almost periodicity of an object by first establishing its almost automorphy and then invoking auxiliary conditions which, when coupled with almost automorphy, give almost periodicity. From then on, automorphy has been widely investigated. Fundamental properties of almost automorphic functions on groups and abstract almost automorphic minimal flows were studied by Veech [2, 3] and others. Afterwards, Zaki [4] extended the notion of scalar-valued almost automorphy to the one of vector-valued almost automorphic functions, paving the road to many applications to differential equations and dynamical systems. Among other things, Shen and Yi [5] showed that almost automorphy is essential and fundamental in the qualitative study of almost periodic differential equations in the sense that almost automorphic solutions are the right class for almost periodic systems. We refer the readers to the monographs [6, 7] by N'Guérékata for more information on this topic.

In the autonomous case, namely $A(t) = A$, the existence and uniqueness of almost automorphic mild solutions to evolution equation (1) with $f = 0$ have been successfully investigated in [6–13] in the framework of semigroups of bounded linear operators. In [13], N'Guérékata studied the existence and uniqueness of almost automorphic solutions for semilinear evolution equation

$$\frac{d}{dt}u(t) = Au(t) + g(t,u(t)), \quad t\in\mathbb{R}, \tag{3}$$

where A generates an exponentially stable semigroup on Banach space $\mathbb{X}$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic. The author proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}$ of (3) is almost automorphic. In [8], Boulite et al. studied the existence and uniqueness of almost automorphic solutions for evolution equation (3), assuming that A generates a hyperbolic semigroup on Banach space $\mathbb{X}$ and $g : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}$ is almost automorphic, where $\mathbb{X}_\alpha$ is an intermediate space between $D(A)$ and $\mathbb{X}$. The authors proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}_\alpha$ of (3) is almost automorphic. Cieutat and Ezzinbi [9] studied the existence of bounded and compact almost automorphic solutions for semilinear evolution equation (3). The main methods are through the minimizing of some subvariant functionals. They gave sufficient conditions ensuring the existence of an almost automorphic mild solution when there is at least one bounded mild solution on $\mathbb{R}^+$.

In the nonautonomous case, almost automorphic mild solutions to evolution equation (1) with $f = 0$ have been successfully investigated in [14–16] in the framework of evolution family. Among others, Baroun et al. [14] generalized the main results of [8] to the nonautonomous case. The authors proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}_\alpha$ of the semilinear evolution equation

$$\frac{d}{dt} u(t) = A(t) u(t) + g(t, u(t)), \quad t \in \mathbb{R}, \tag{4}$$

is almost automorphic, assuming that the evolution family $\{U(t,s)\}_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy and $g : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}$ is almost automorphic. Ding et al. [15] established the existence and uniqueness theorem of almost automorphic mild solutions to the evolution equation (4), where the evolution family $\{U(t,s)\}_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic. Liu and Song [16] proved the existence and uniqueness of an almost automorphic or a weighted pseudo almost automorphic mild solution to (4), assuming that the evolution family $\{U(t,s)\}_{t\geq s}$ generated by $A(t)$ is exponentially stable and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic or weighted pseudo almost automorphic.

A rich source of the literature exists on almost automorphic mild solutions to linear and semilinear evolution equations. However, to the best of our knowledge, there are few results available on the existence and uniqueness of almost automorphic mild solutions to neutral parabolic nonautonomous evolution equations (1) and (2), especially in the case of not necessarily dense domain and bounded perturbations. Nondensity occurs in many situations, from restrictions made on the space where the equation is considered or from boundary conditions. For example, the space $C^2[0,\pi]$ of twice continuously differential functions with null value on the boundary is nondense in $C[0,\pi]$, the space of continuous functions. One can refer for this to [17–19] or Section 5 for more details. We further remark that our first main result (Theorem 18) recovers partly Theorem 2.2 in [15] and Theorem 3.2 in [16] in the parabolic case. Moreover, a unified framework is set up in the second main result (Theorem 21) to study the existence and uniqueness of almost automorphic mild solutions to some classes of parabolic partial differential equations and neutral functional differential equations. As one will see, the additional neutral term $f(t, u(t))$ greatly widens the applications of the main result since (2) is general enough to incorporate some classes of parabolic partial differential equations and neutral functional differential equations as special cases.

As a preparation, in Section 2 we fix our notation and collect some basic facts on evolution family and almost automorphy. Section 3 deals with the proof of the existence and uniqueness theorem of almost automorphic mild solutions to evolution equation (1). In Section 4, we study the existence and uniqueness of almost automorphic mild solutions to evolution equation (2) with bounded perturbations. Finally, the abstract results are applied to some classes of parabolic partial differential equations and neutral functional differential equations.

2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the sets of positive integer, integer, real, and complex numbers, and $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space. If $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is another Banach space, the space $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with the uniform operator topology. The resolvent operator $R(\lambda, A)$ is defined by $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$, the resolvent set of a linear operator A.

2.1. Evolution Family and Exponential Dichotomy

Definition 1 (see [20, 21]). A family of bounded linear operators $\{U(t,s)\}_{t\geq s}$ on a Banach space $\mathbb{X}$ is called an evolution family if

(1) $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$;

(2) the map $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in \mathbb{X}$, $t > s$, and $t, s \in \mathbb{R}$.

Definition 2 (see [20, 21]). An evolution family $\{U(t,s)\}_{t\geq s}$ on a Banach space $\mathbb{X}$ has an exponential dichotomy (or is called hyperbolic) if there exist projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t and constants $M > 0$, $\delta > 0$ such that

(1) $U(t,s)P(s) = P(t)U(t,s)$ for $t \geq s$ and $t, s \in \mathbb{R}$;

(2) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible for $t \geq s$ (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$);

(3) $\|U(t,s)P(s)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)}$, $\|U_Q(s,t)Q(t)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

Here and below we set $Q := I - P$.

Remark 3. Exponential dichotomy is a classical concept in the study of the long-term behavior of evolution equations, combining forward exponential stability on some subspaces with backward exponential stability on their complements. Its importance relies in particular on the robustness; that is, exponential dichotomy persists under small linear or nonlinear perturbations (see, e.g., [20–24]).

Definition 4 (see [20, 21]). Given an evolution family $\{U(t,s)\}_{t\geq s}$ with an exponential dichotomy, one defines its Green's function by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}. \end{cases} \tag{5}$$

2.2. Almost Automorphy and Bi-Almost Automorphy. Let $C(\mathbb{R},\mathbb{X})$ denote the collection of continuous functions from $\mathbb{R}$ into $\mathbb{X}$. Let $BC(\mathbb{R},\mathbb{X})$ denote the Banach space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ equipped with the sup norm $\|u\|_\infty := \sup_{t\in\mathbb{R}}\|u(t)\|$. Similarly, $C(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the collection of all jointly continuous functions from $\mathbb{R}\times\mathbb{X}$ into $\mathbb{Y}$, and $BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the collection of all bounded and jointly continuous functions $f:\mathbb{R}\times\mathbb{X}\to\mathbb{Y}$.

Definition 5 (Bochner). A function $f \in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for any sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m) = f(t) \tag{6}$$

pointwise for each $t\in\mathbb{R}$. This limit means that

$$g(t) = \lim_{n\to\infty} f(t+s_n) \tag{7}$$

is well defined for each $t\in\mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t-s_n) \tag{8}$$

for each $t\in\mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Example 6 (Levitan). The function $f(t) = \sin(1/(2+\cos t+\cos\pi t))$, $t\in\mathbb{R}$, is almost automorphic but not almost periodic.

Remark 7. An almost automorphic function may not be uniformly continuous, while an almost periodic function must be uniformly continuous.

Lemma 8 (see [6, 7]). *Assume that $f,g:\mathbb{R}\to\mathbb{X}$ are almost automorphic and λ is any scalar. Then the following holds true:*

(1) *$f+g$, λf are almost automorphic;*

(2) *the range R_f of f is precompact, so f is bounded;*

(3) *f_τ defined by $f_\tau(t) = f(t+\tau)$, $\tau\in\mathbb{R}$, is almost automorphic.*

Lemma 9 (see [6, 7]). *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n\to f$ $(n\to\infty)$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

Lemma 10 (see [6]). *The space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty = \sup_{t\in\mathbb{R}}\|u(t)\|$ is a Banach space.*

Definition 11 (see [25]). A function $f\in C(\mathbb{R}\times\mathbb{X},\mathbb{X})$ is said to be almost automorphic if f is almost automorphic in $t\in\mathbb{R}$ for each $u\in\mathbb{X}$. That is to say, for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m,u) = f(t,u) \tag{9}$$

pointwise on $\mathbb{R}$ for each $u\in\mathbb{X}$. Denote by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the collection of all such functions.

Lemma 12 (see [6, Theorem 2.2.6]). *Assume that $f\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and there exists a constant $L_f>0$ such that for all $t\in\mathbb{R}$ and $u,v\in\mathbb{X}$,*

$$\|f(t,u)-f(t,v)\| \leq L_f\|u-v\|. \tag{10}$$

If $\phi(\cdot)\in AA(\mathbb{X})$, then $f(\cdot,\phi(\cdot))\in AA(\mathbb{X})$.

Corollary 13 (see [6, Corollary 2.1.6]). *Assume that $u\in AA(\mathbb{X})$ and $B\in B(\mathbb{X})$. If for each $t\in\mathbb{R}$, $v(t)=Bu(t)$, then $v\in AA(\mathbb{X})$.*

Definition 14 (see [26]). A function $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{X})$ is called bi-almost automorphic if for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, one can extract a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t,s) = \lim_{n\to\infty} f(t+s_n,s+s_n) \tag{11}$$

is well defined for each $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n,s-s_n) = f(t,s) \tag{12}$$

for each $t,s\in\mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

In other words, a function $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{X})$ is said to be bi-almost automorphic if for any sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m,s+s_n-s_m) = f(t,s) \tag{13}$$

pointwise for each $t,s\in\mathbb{R}$.

3. Neutral Parabolic Nonautonomous Evolution Equation

In this section, we will establish the existence and uniqueness theorem of almost automorphic mild solutions to neutral parabolic nonautonomous evolution equation (1) under assumptions (H1)–(H5) listed below:

(H1) there exist constants $\lambda_0\geq 0$, $\theta\in(\pi/2,\pi)$, L_0, $K_0\geq 0$, and α, $\beta\in(0,1]$ with $\alpha+\beta>1$ such that

$$\begin{gathered} \Sigma_\theta\cup\{0\}\subset\rho(A(t)-\lambda_0), \\ \|R(\lambda,A(t)-\lambda_0)\|_{B(\mathbb{X})} \leq \frac{K_0}{1+|\lambda|}, \\ \|(A(t)-\lambda_0)R(\lambda,A(t)-\lambda_0) \\ \times[R(\lambda_0,A(t))-R(\lambda_0,A(s))]\|_{B(\mathbb{X})} \\ \leq L_0|t-s|^\alpha|\lambda|^{-\beta} \end{gathered} \tag{14}$$

for $t,s\in\mathbb{R}$, $\lambda\in\Sigma_\theta := \{\lambda\in\mathbb{C}\setminus\{0\} : |\arg\lambda|\leq\theta\}$, (H2) the evolution family $\{U(t,s)\}_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy with dichotomy constants $M>0$, $\delta>0$, dichotomy projections $P(t)$, $t\in\mathbb{R}$, and Green's function $\Gamma(t,s)$,

(H3) $\Gamma(t,s)x\in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$ for each $x\in\mathbb{X}$,

(H4) $f\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a constant $L_f>0$ such that for all $t\in\mathbb{R}$ and $u,v\in\mathbb{X}$,

$$\left\|f(t,u)-f(t,v)\right\|\leq L_f\|u-v\|, \tag{15}$$

(H5) $g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a constant $L_g>0$ such that for all $t\in\mathbb{R}$ and $u,v\in\mathbb{X}$,

$$\left\|g(t,u)-g(t,v)\right\|\leq L_g\|u-v\|. \tag{16}$$

Remark 15. Assumption (H1) is usually called "Acquistapace-Terreni" conditions, which was first introduced in [27] for $\lambda_0=0$. If (H1) holds, then there exists a unique evolution family $\{U(t,s)\}_{t\geq s}$ on $\mathbb{X}$ such that $(t,s)\mapsto U(t,s)\in B(\mathbb{X})$ is strongly continuous for $t>s$, $U(\cdot,s)\in C^1((s,\infty),B(\mathbb{X}))$, $\partial_t U(t,s)=A(t)U(t,s)$ for $t>s$. These assertions are established in Theorem 2.3 of [28]. See also [27, 29, 30].

Definition 16. A mild solution to (1) is a continuous function $u:\mathbb{R}\to\mathbb{X}$ satisfying integral equation

$$\begin{aligned} u(t) =& -f(t,u(t))+U(t,s)\left[u(s)+f(s,u(s))\right]\\ &+\int_s^t U(t,\sigma)\,g(\sigma,u(\sigma))\,d\sigma, \end{aligned} \tag{17}$$

for all $t\geq s$ and all $s\in\mathbb{R}$.

Lemma 17. *Assume that (H1)–(H3) and (H5) hold. Define nonlinear operator Λ on $AA(\mathbb{X})$ by*

$$\begin{aligned} (\Lambda u)(t) =& \int_{-\infty}^t U(t,s)P(s)\,g(s,u(s))\,ds\\ &-\int_t^{+\infty} U_Q(t,s)Q(s)\,g(s,u(s))\,ds, \quad t\in\mathbb{R}. \end{aligned} \tag{18}$$

Then Λ maps $AA(\mathbb{X})$ into itself.

Proof. Combining the ideas from Theorem 4.28 in [22], the technique of exponential dichotomy, composition theorem of almost automorphic functions, and the Lebesgue dominated convergence theorem, we strive for a more self-contained proof. Let $u\in AA(\mathbb{X})$. Then it follows from Lemma 12 [6, Theorem 2.2.6] that $h:=g(\cdot,u(\cdot))\in AA(\mathbb{X})$, in view of (H5). Hence, $(\Lambda u)(t)$ can be rewritten as

$$\begin{aligned} (\Lambda u)(t) =& \int_{-\infty}^t U(t,s)P(s)h(s)\,ds\\ &-\int_t^{+\infty} U_Q(t,s)Q(s)h(s)\,ds, \quad t\in\mathbb{R}. \end{aligned} \tag{19}$$

From triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, it follows that

$$\begin{aligned} \|(\Lambda u)(t)\| \leq& M\int_{-\infty}^t e^{-\delta(t-s)}\|h(s)\|\,ds\\ &+M\int_t^{+\infty} e^{-\delta(s-t)}\|h(s)\|\,ds\\ \leq& M\|h\|_\infty\int_{-\infty}^t e^{-\delta(t-s)}ds\\ &+M\|h\|_\infty\int_t^{+\infty} e^{-\delta(s-t)}ds\\ \leq& \frac{2M}{\delta}\|h\|_\infty. \end{aligned} \tag{20}$$

Hence, Λ is well defined for each $t\in\mathbb{R}$. To show $\Lambda u\in C(\mathbb{R},\mathbb{X})$, we will verify that

$$\lim_{\Delta t\to 0}\|(\Lambda u)(t+\Delta t)-(\Lambda u)(t)\|=0, \quad \text{for each } t\in\mathbb{R}. \tag{21}$$

By $h\in C(\mathbb{R},\mathbb{X})$ and the strong continuity of $\Gamma(t,s)$, we have for each $t\in\mathbb{R}$, $x\in\mathbb{X}$, $\sigma>0$,

$$\begin{aligned} &\lim_{\Delta t\to 0}\|h(t+\Delta t)-h(t)\|=0,\\ &\lim_{\Delta t\to 0}\|U(t+\Delta t,t+\Delta t-\sigma)P(t+\Delta t-\sigma)x\\ &\qquad -U(t,t-\sigma)P(t-\sigma)x\|=0,\\ &\lim_{\Delta t\to 0}\left\|U_Q(t+\Delta t,t+\Delta t+\sigma)Q(t+\Delta t+\sigma)x\right.\\ &\qquad \left.-U_Q(t,t+\sigma)Q(t+\sigma)x\right\|=0. \end{aligned} \tag{22}$$

Transforming (19) into another form, we have, for $\sigma>0$,

$$\begin{aligned} (\Lambda u)(t) =& \int_0^{+\infty} U(t,t-\sigma)P(t-\sigma)h(t-\sigma)\,d\sigma\\ &-\int_0^{+\infty} U_Q(t,t+\sigma)Q(t+\sigma)h(t+\sigma)\,d\sigma,\\ &t\in\mathbb{R}. \end{aligned} \tag{23}$$

In view of (23), triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, we have

$$\begin{aligned} &\|(\Lambda u)(t+\Delta t)-(\Lambda u)(t)\|\\ &\leq \left\|\int_0^{+\infty} U(t+\Delta t,t+\Delta t-\sigma)P(t+\Delta t-\sigma)\right.\\ &\qquad \times h(t+\Delta t-\sigma)\,d\sigma\\ &\qquad \left.-\int_0^{+\infty} U(t,t-\sigma)P(t-\sigma)h(t-\sigma)\,d\sigma\right\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{+\infty} U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma)\right. \\
&\qquad \times h(t+\Delta t+\sigma)\, d\sigma \\
&\qquad \left. -\int_0^{+\infty} U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\, d\sigma\right\| \\
&\leq \left\|\int_0^{+\infty} U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma)\right. \\
&\qquad \left.\times [h(t+\Delta t-\sigma)-h(t-\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma)\right. \\
&\qquad \left. -U(t, t-\sigma) P(t-\sigma)]\, h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma)\right. \\
&\qquad \left.\times [h(t+\Delta t+\sigma)-h(t+\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma)\right. \\
&\qquad \left. -U_Q(t, t+\sigma) Q(t+\sigma)]\, h(t+\sigma)\, d\sigma\right\| \\
&\leq M\int_0^{+\infty} e^{-\delta\sigma}\|h(t+\Delta t-\sigma)-h(t-\sigma)\|\, d\sigma \\
&+\int_0^{+\infty}\|U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma) h(t-\sigma) \\
&\qquad -U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\|\, d\sigma \\
&+M\int_0^{+\infty} e^{-\delta\sigma}\|h(t+\Delta t+\sigma)-h(t+\sigma)\|\, d\sigma \\
&+\int_0^{+\infty}\|U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma) h(t+\sigma) \\
&\qquad -U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\|\, d\sigma.
\end{aligned} \tag{24}
$$

Thus, in the Lebesgue dominated convergence theorem, (22) leads to (21) and therefore to $\Lambda u \in C(\mathbb{R}, \mathbb{X})$.

To show $\Lambda u \in AA(\mathbb{X})$, let us take a sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ and show that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$
\lim_{m\to\infty}\lim_{n\to\infty}\left\|(\Lambda u)(t+s_n-s_m)-(\Lambda u)(t)\right\| = 0, \qquad \text{for each } t\in\mathbb{R}. \tag{25}
$$

By $h \in AA(\mathbb{X})$ and (H3), there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that for each $t\in\mathbb{R}$, $x\in\mathbb{X}$, $\sigma>0$,

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\|h(t+s_n-s_m)-h(t)\| = 0, \\
&\lim_{m\to\infty}\lim_{n\to\infty}\|U(t+s_n-s_m, t+s_n-s_m-\sigma) \\
&\qquad \times P(t+s_n-s_m-\sigma)\, x \\
&\qquad -U(t, t-\sigma) P(t-\sigma)\, x\| = 0, \\
&\lim_{m\to\infty}\lim_{n\to\infty}\|U_Q(t+s_n-s_m, t+s_n-s_m+\sigma) \\
&\qquad \times Q(t+s_n-s_m+\sigma)\, x \\
&\qquad -U_Q(t, t+\sigma) P(t+\sigma)\, x\| = 0.
\end{aligned} \tag{26}
$$

Again, in view of (23), triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, we obtain

$$
\begin{aligned}
&\left\|(\Lambda u)(t+s_n-s_m)-(\Lambda u)(t)\right\| \\
&\leq \left\|\int_0^{+\infty} U(t+s_n-s_m, t+s_n-s_m-\sigma)\right. \\
&\qquad \times P(t+s_n-s_m-\sigma) h(t+s_n-s_m-\sigma)\, d\sigma \\
&\qquad \left. -\int_0^{+\infty} U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+s_n-s_m, t+s_n-s_m+\sigma)\right. \\
&\qquad \times Q(t+s_n-s_m+\sigma) h(t+s_n-s_m+\sigma)\, d\sigma \\
&\qquad \left. -\int_0^{+\infty} U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\, d\sigma\right\| \\
&\leq \left\|\int_0^{+\infty} U(t+s_n-s_m, t+s_n-s_m-\sigma)\right. \\
&\qquad \times P(t+s_n-s_m-\sigma) \\
&\qquad \left. \cdot [h(t+s_n-s_m-\sigma)-h(t-\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U(t+s_n-s_m, t+s_n-s_m-\sigma)\right. \\
&\qquad \times P(t+s_n-s_m-\sigma) \\
&\qquad \left. - U(t, t-\sigma) P(t-\sigma)]\, h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+s_n-s_m, t+s_n-s_m+\sigma)\right. \\
&\qquad \times Q(t+s_n-s_m+\sigma) \\
&\qquad \left. \cdot [h(t+s_n-s_m+\sigma)-h(t+\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U_Q(t+s_n-s_m, t+s_n-s_m+\sigma)\right. \\
&\qquad \times Q(t+s_n-s_m+\sigma) \\
&\qquad \left. - U_Q(t, t+\sigma) Q(t+\sigma)]\, h(t+\sigma)\, d\sigma\right\| \\
&\leq M\int_0^{+\infty} e^{-\delta\sigma}\|h(t+s_n-s_m-\sigma)-h(t-\sigma)\|\, d\sigma \\
&+\int_0^{+\infty}\|U(t+s_n-s_m, t+s_n-s_m-\sigma) \\
&\qquad \times P(t+s_n-s_m-\sigma) h(t-\sigma) \\
&\qquad -U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\|\, d\sigma
\end{aligned}
$$

$$+ M \int_0^{+\infty} e^{-\delta\sigma} \left\| h\left(t + s_n - s_m + \sigma\right) - h\left(t + \sigma\right) \right\| d\sigma$$
$$+ \int_0^{+\infty} \left\| U_Q\left(t + s_n - s_m, t + s_n - s_m + \sigma\right) \right.$$
$$\times Q\left(t + s_n - s_m + \sigma\right) h\left(t + \sigma\right)$$
$$\left. - U_Q\left(t, t + \sigma\right) Q\left(t + \sigma\right) h\left(t + \sigma\right) \right\| d\sigma. \tag{27}$$

Thus, in the Lebesgue dominated convergence theorem, (26) leads to (25) and therefore to $\Lambda u \in AA(\mathbb{X})$. Here we used the translation invariance of almost automorphic functions, which is collected in Lemma 8(3). The proof is complete. □

Now we are in a position to state and prove the first main result of this paper.

Theorem 18. *Suppose that (H1)–(H5) hold. If $\Theta = L_f + (2ML_g/\delta) < 1$, then there exists a unique mild solution $u \in AA(\mathbb{X})$ to* (1) *such that*

$$u\left(t\right) = -f\left(t, u\left(t\right)\right) + \int_{-\infty}^{t} U\left(t, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_t^{+\infty} U_Q\left(t, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma. \tag{28}$$

Proof. Firstly, define nonlinear operator Γ on $AA(\mathbb{X})$ by

$$\left(\Gamma u\right)\left(t\right) = -f\left(t, u\left(t\right)\right) + \int_{-\infty}^{t} U\left(t, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_t^{+\infty} U_Q\left(t, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma. \tag{29}$$

Let $u \in AA(\mathbb{X})$, then it follows from Lemma 12 [6, Theorem 2.2.6] that $f(\cdot, u(\cdot)) \in AA(\mathbb{X})$, in view of (H4). Together with Lemma 17, we deduce that the operator Γ is well defined and maps $AA(\mathbb{X})$ into itself.

Secondly, we will prove that Γ is a strict contraction on $AA(\mathbb{X})$. Let $v, w \in AA(\mathbb{X})$. By (H2), (H4), and (H5), we have

$$\left\| \left(\Gamma v\right)\left(t\right) - \left(\Gamma w\right)\left(t\right) \right\|$$
$$\leq L_f \left\| v\left(t\right) - w\left(t\right) \right\|$$
$$+ M \int_{-\infty}^{t} e^{-\delta(t-s)} \left\| g\left(s, v\left(s\right)\right) - g\left(s, w\left(s\right)\right) \right\| ds$$
$$+ M \int_t^{+\infty} e^{-\delta(s-t)} \left\| g\left(s, v\left(s\right)\right) - g\left(s, w\left(s\right)\right) \right\| ds$$
$$\leq L_f \|v - w\|_\infty + ML_g \int_{-\infty}^{t} e^{-\delta(t-s)} \left\| v\left(s\right) - w\left(s\right) \right\| ds$$
$$+ ML_g \int_t^{+\infty} e^{-\delta(s-t)} \left\| v\left(s\right) - w\left(s\right) \right\| ds$$
$$\leq \left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty. \tag{30}$$

Hence,

$$\|\Gamma v - \Gamma w\|_\infty \leq \left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty. \tag{31}$$

If $\Theta = L_f + (2ML_g/\delta) < 1$, then the operator Γ becomes a strict contraction on $AA(\mathbb{X})$. Since the space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$ is a Banach space by Lemma 10, an application of Banach fixed point theorem shows that there exists a unique $u \in AA(\mathbb{X})$ such that (28) holds.

Finally, to prove that u satisfies (17) for all $t \geq s$, all $s \in \mathbb{R}$. For this, we let

$$u\left(s\right) = -f\left(s, u\left(s\right)\right) + \int_{-\infty}^{s} U\left(s, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_s^{+\infty} U_Q\left(s, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma. \tag{32}$$

Multiplying both sides of (32) by $U(t, s)$ for all $t \geq s$, we have

$$U\left(t, s\right) u\left(s\right) = -U\left(t, s\right) f\left(s, u\left(s\right)\right)$$
$$+ \int_{-\infty}^{s} U\left(t, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_s^{+\infty} U_Q\left(t, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$= -U\left(t, s\right) f\left(s, u\left(s\right)\right)$$
$$+ \int_{-\infty}^{t} U\left(t, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_s^{t} U\left(t, \sigma\right) P\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_t^{+\infty} U_Q\left(t, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$- \int_s^{t} U_Q\left(t, \sigma\right) Q\left(\sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma$$
$$= -U\left(t, s\right) f\left(s, u\left(s\right)\right) + u\left(t\right) + f\left(t, u\left(t\right)\right)$$
$$- \int_s^{t} U\left(t, \sigma\right) g\left(\sigma, u\left(\sigma\right)\right) d\sigma. \tag{33}$$

Hence, $u \in AA(\mathbb{X})$ is a unique mild solution to (1). The proof is complete. □

4. Bounded Perturbations

In this section, we consider neutral parabolic nonautonomous evolution equation (2). For this, we need assumptions (H1)–(H5) listed in the previous section and the following assumption:

(H6) $B, C \in B(\mathbb{X})$ with $\max\{\|B\|_{B(\mathbb{X})}, \|C\|_{B(\mathbb{X})}\} = K$.

Definition 19. A mild solution to (2) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying integral equation

$$u(t) = -f(t, Bu(t)) + U(t,s)[u(s) + f(s, Bu(s))] + \int_s^t U(t,\sigma) g(\sigma, Cu(\sigma)) d\sigma \tag{34}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Lemma 20. *Let assumptions (H1)–(H3), (H5), and (H6) hold. Define nonlinear operator Λ_1 on $AA(\mathbb{X})$ by*

$$(\Lambda_1 u)(t) = \int_{-\infty}^t U(t,s) P(s) g(s, Cu(s)) ds - \int_t^{+\infty} U_Q(t,s) Q(s) g(s, Cu(s)) ds, \quad t \in \mathbb{R}. \tag{35}$$

Then Λ_1 maps $AA(\mathbb{X})$ into itself.

Proof. Let $u(\cdot) \in AA(\mathbb{X})$. By (H6) and Corollary 13, we obtain $Cu(\cdot) \in AA(\mathbb{X})$. Then it follows from Lemma 12 [6, Theorem 2.2.6] that $h_1 := g(\cdot, Cu(\cdot)) \in AA(\mathbb{X})$, in view of (H5). The left is almost same as the proof of Lemma 17, remembering to replace Λ, h by Λ_1, and h_1, respectively. This ends the proof. □

Now we are in a position to state and prove the second main result of this paper.

Theorem 21. *Suppose that (H1)–(H6) hold. If $\Theta_1 = K(L_f + (2ML_g/\delta)) < 1$, then there exists a unique mild solution $u \in AA(\mathbb{X})$ to (2) such that*

$$u(t) = -f(t, Bu(t)) + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \tag{36}$$

Proof. Firstly, define nonlinear operator Γ_1 on $AA(\mathbb{X})$ by

$$(\Gamma_1 u)(t) = -f(t, Bu(t)) + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \tag{37}$$

Let $u \in AA(\mathbb{X})$. By (H6) and Corollary 13, we obtain $Bu \in AA(\mathbb{X})$. Hence, it follows from Lemma 12 [6, Theorem 2.2.6] that $f(\cdot, Bu(\cdot)) \in AA(\mathbb{X})$, in view of (H4). Together with Lemma 20, we deduce that the operator Γ_1 is well defined and maps $AA(\mathbb{X})$ into itself.

Secondly, we will prove that Γ_1 is a strict contraction on $AA(\mathbb{X})$ and apply Banach fixed point theorem. Let $v, w \in AA(\mathbb{X})$. Then it follows from (H2), (H4), (H5), and (H6) that

$$\begin{aligned}
&\|(\Gamma_1 v)(t) - (\Gamma_1 w)(t)\| \\
&\leq KL_f \|v(t) - w(t)\| \\
&\quad + M \int_{-\infty}^t e^{-\delta(t-s)} \|g(s, Cv(s)) - g(s, Cw(s))\| ds \\
&\quad + M \int_t^{+\infty} e^{-\delta(s-t)} \|g(s, Cv(s)) - g(s, Cw(s))\| ds \\
&\leq KL_f \|v - w\|_\infty + MKL_g \int_{-\infty}^t e^{-\delta(t-s)} \|v(s) - w(s)\| ds \\
&\quad + MKL_g \int_t^{+\infty} e^{-\delta(s-t)} \|v(s) - w(s)\| ds \\
&\leq K\left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty.
\end{aligned} \tag{38}$$

Hence,

$$\|\Gamma_1 v - \Gamma_1 w\|_\infty \leq K\left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty. \tag{39}$$

If $\Theta_1 = K(L_f + (2ML_g/\delta)) < 1$, then the operator Γ_1 becomes a strict contraction on $AA(\mathbb{X})$. Since the space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$ is a Banach space by Lemma 10, an application of Banach fixed point theorem shows that there exists a unique $u \in AA(\mathbb{X})$ such that (36) holds.

Finally, to prove that u satisfies (34) for all $t \geq s$, all $s \in \mathbb{R}$. For this, we let

$$u(s) = -f(s, Bu(s)) + \int_{-\infty}^s U(s,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_s^{+\infty} U_Q(s,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \tag{40}$$

Multiplying both sides of (40) by $U(t,s)$ for all $t \geq s$, then

$$\begin{aligned}
U(t,s) u(s) &= -U(t,s) f(s, Bu(s)) \\
&\quad + \int_{-\infty}^s U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&\quad - \int_s^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&= -U(t,s) f(s, Bu(s)) \\
&\quad + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&\quad - \int_s^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) \, d\sigma \\
&- \int_s^t U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) \, d\sigma \\
= &- U(t,s) f(s, Bu(s)) + u(t) + f(t, Bu(t)) \\
&- \int_s^t U(t,\sigma) g(\sigma, Cu(\sigma)) \, d\sigma.
\end{aligned}
\tag{41}
$$

Hence, $u \in AA(\mathbb{X})$ is a unique mild solution to (2). The proof is complete. □

5. Applications to Parabolic Partial Differential Equations and Neutral Functional Differential Equations

In this section, two examples are given to illustrate the effectiveness and flexibility of Theorem 21. By a mild solution to a partial or neutral functional differential equation, we mean a mild solution to the corresponding evolution equation.

Example 22. Consider the following parabolic partial differential equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t} [u(t,x) + f(t, q(x) u(t,x))] \\
&\quad = \frac{\partial^2}{\partial x^2} [u(t,x) + f(t, q(x) u(t,x))] \\
&\qquad + (-3 + \sin t + \sin \pi t) [u(t,x) + f(t, q(x) u(t,x))] \\
&\qquad + g(t, q(x) u(t,x)), \quad t \in \mathbb{R}, \ x \in [0,\pi], \\
&[u(t,x) + f(t, q(x) u(t,x))]|_{x=0} \\
&\quad = [u(t,x) + f(t, q(x) u(t,x))]|_{x=\pi} = 0, \quad t \in \mathbb{R},
\end{aligned}
\tag{42}
$$

where q is continuous on $[0,\pi]$.

Let $\mathbb{X} := C[0,\pi]$ denote the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the sup norm and define the operator A by

$$
\begin{aligned}
D(A) &:= \{\varphi \in C^2[0,\pi] : \varphi(0) = \varphi(\pi) = 0\}, \\
A\varphi &:= \varphi'', \ \varphi \in D(A).
\end{aligned}
\tag{43}
$$

It is known (see [18, Example 14.4]) that A is sectorial, $D(A)$ is not dense in $C[0,\pi]$, and A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ not strongly continuous at 0. Since the spectrum of A consists of the sequence of eigenvalues $\lambda_n = -n^2$, $n \in \mathbb{N}$, it can be easily checked that $\|T(t)\| \leq e^{-t}$ for $t \geq 0$, remembering that the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ of A coincides with its growth bound

$$
\omega_A = \inf\{\gamma \in \mathbb{R} : \exists M > 0 \text{ s.t. } \|T(t)\| \leq M e^{\gamma t}, t \geq 0\}.
\tag{44}
$$

Define a family of linear operators $A(t)$, $t \in \mathbb{R}$ by

$$
\begin{aligned}
D(A(t)) &= D(A(0)) = D(A), \\
A(t)\varphi &= (A - 3 + \sin t + \sin \pi t)\varphi, \quad \forall \varphi \in D(A).
\end{aligned}
\tag{45}
$$

In the case that $A(t) : D(A(t)) \subset \mathbb{X} \to \mathbb{X}$ have a constant domain $D(A(t)) \equiv D(A(0))$ and $\lambda_0 = 0$, it is known that [31, 32] assumption (H1) can be replaced by the following assumption (ST).

(ST) There exist constants $\theta \in (\pi/2, \pi)$, $L_0, K_0 \geq 0$, and $\alpha \in (0,1]$ such that

$$
\begin{aligned}
\Sigma_\theta \cup \{0\} \subset \rho(A(t)), \qquad & \|R(\lambda, A(t))\|_{B(\mathbb{X})} \leq \frac{K_0}{1 + |\lambda|}, \\
\left\|(A(t) - A(s)) A(r)^{-1}\right\|_{B(\mathbb{X})} &\leq L_0 |t - s|^\alpha
\end{aligned}
\tag{46}
$$

for $t, s, r \in \mathbb{R}, \lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \theta\}$.

Now, it is not hard to verify that $A(t)$ satisfy (H1). By Theorem 2.3 of [28], $A(t)$ generate an evolution family $\{U(t,s)\}_{t\geq s}$ that is strongly continuous for $t > s$. Furthermore,

$$
U(t,s)\varphi = T(t-s) e^{\int_s^t (-3 + \sin\tau + \sin \pi\tau) d\tau} \varphi.
\tag{47}
$$

Hence,

$$
\|U(t,s)\| \leq e^{-2(t-s)} \quad \text{for } t \geq s,
\tag{48}
$$

and (H2) is satisfied with $M = 1, \delta = 2, P(s) = I$.

As for (H3), it is obvious that $U(t,s)\varphi \in C(\mathbb{R}\times\mathbb{R}, C[0,\pi])$ for each $\varphi \in C[0,\pi]$.

To show that $U(t,s)\varphi \in bAA(\mathbb{R} \times \mathbb{R}, C[0,\pi])$ for each $\varphi \in C[0,\pi]$, let us take a sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ and show that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$
\lim_{m\to\infty} \lim_{n\to\infty} U(t + s_n - s_m, s + s_n - s_m)\varphi = U(t,s)\varphi
\tag{49}
$$

pointwise for each $t, s \in \mathbb{R}$.

By $-3 + \sin\tau + \sin\pi\tau \in AA(\mathbb{R})$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that pointwise for each $\tau \in \mathbb{R}$,

$$
\begin{aligned}
&\lim_{m\to\infty} \lim_{n\to\infty} (-3 + \sin(\tau + s_n - s_m) + \sin\pi(\tau + s_n - s_m)) \\
&\quad = -3 + \sin\tau + \sin\pi\tau.
\end{aligned}
\tag{50}
$$

In view of (47) and (50), an application of the Lebesgue dominated convergence theorem shows that

$$
\begin{aligned}
&\lim_{m\to\infty} \lim_{n\to\infty} U(t + s_n - s_m, s + s_n - s_m)\varphi \\
&\quad = \lim_{m\to\infty} \lim_{n\to\infty} T(t-s) e^{\int_{s+s_n-s_m}^{t+s_n-s_m} (-3 + \sin\tau + \sin\pi\tau) d\tau} \varphi \\
&\quad = T(t-s) \\
&\qquad \times e^{\lim_{m\to\infty}\lim_{n\to\infty} \int_s^t (-3 + \sin(\tau + s_n - s_m) + \sin\pi(\tau + s_n - s_m)) d\tau} \varphi \\
&\quad = T(t-s) \\
&\qquad \times e^{\int_s^t \lim_{m\to\infty}\lim_{n\to\infty} (-3 + \sin(\tau + s_n - s_m) + \sin\pi(\tau + s_n - s_m)) d\tau} \varphi \\
&\quad = T(t-s) e^{\int_s^t (-3 + \sin\tau + \sin\pi\tau) d\tau} \varphi = U(t,s)\varphi
\end{aligned}
\tag{51}
$$

pointwise for each $t, s \in \mathbb{R}$. Hence, (H3) is satisfied.

Define the operators $B,\ C$ by

$$D(B) = D(C) = C[0,\pi],$$
$$B\varphi = C\varphi = q(\xi)\varphi, \quad \xi \in [0,\pi],\ \varphi \in C[0,\pi], \tag{52}$$

then $\|B\|_{B(C[0,\pi])} = \|C\|_{B(C[0,\pi])} = \|q\|_\infty := \max_{\xi\in[0,\pi]}\{q(\xi)\}$.

In view of the above, (42) can be transformed into the abstract form (2), and assumptions (H1)–(H3) and (H6) are satisfied.

We add the following assumptions:

(H4a) $f : \mathbb{R} \times C[0,\pi] \to C[0,\pi], (t,u) \mapsto f(t,u)$ is almost automorphic, and there exists a constant $L_f > 0$ such that for all $t \in \mathbb{R}$, $u(t,\cdot), v(t,\cdot) \in C[0,\pi]$,

$$\|f(t,u(t,\cdot)) - f(t,v(t,\cdot))\|_{C[0,\pi]} \le L_f \|u(t,\cdot) - v(t,\cdot)\|_{C[0,\pi]}, \tag{53}$$

(H5a) $g : \mathbb{R} \times C[0,\pi] \to C[0,\pi], (t,u) \mapsto g(t,u)$ is almost automorphic, and there exists a constant $L_g > 0$ such that for all $t \in \mathbb{R}$, $u(t,\cdot), v(t,\cdot) \in C[0,\pi]$,

$$\|g(t,u(t,\cdot)) - g(t,v(t,\cdot))\|_{C[0,\pi]} \le L_g \|u(t,\cdot) - v(t,\cdot)\|_{C[0,\pi]}. \tag{54}$$

Now, the following proposition is an immediate consequence of Theorem 21.

Proposition 23. *Under assumptions (H4a) and (H5a), parabolic partial differential equation* (42) *admits a unique almost automorphic mild solution if*

$$\|q\|_\infty \left(L_f + L_g\right) < 1. \tag{55}$$

Furthermore, if one takes

$$f(t,u) = u \sin \frac{1}{-3 + \sin t + \sin \pi t},$$
$$t \in \mathbb{R},\ u \in C[0,\pi],$$
$$g(t,u) = \frac{1}{8} u \left(1 + \sin \frac{1}{-3 + \sin t + \sin \pi t}\right),$$
$$t \in \mathbb{R},\ u \in C[0,\pi]. \tag{56}$$

A simple computation shows that $f, g \in AA(\mathbb{R} \times C[0,\pi], C[0,\pi])$, (H4a), and (H5a) are satisfied with $L_f = 1$, $L_g = 1/4$. By Proposition 23, (42) *admits a unique almost automorphic mild solution whenever $\|q\|_\infty < 4/5$.*

Example 24. Consider neutral functional differential equation

$$\frac{\partial}{\partial t}[u(t,x) + f(t,u(t-\tau,x))]$$
$$= \frac{\partial^2}{\partial x^2}[u(t,x) + f(t,u(t-\tau,x))]$$
$$+ (-3 + \sin t + \sin \pi t)[u(t,x) + f(t,u(t-\tau,x))]$$
$$+ g(t,u(t-\tau,x)), \quad t \in \mathbb{R},\ x \in [0,\pi],$$
$$[u(t,x) + f(t,u(t-\tau,x))]\big|_{x=0}$$
$$= [u(t,x) + f(t,u(t-\tau,x))]\big|_{x=\pi} = 0, \quad t \in \mathbb{R}, \tag{57}$$

where $\tau \in \mathbb{R}$ is a fixed constant.

Take $\mathbb{X}$, A, $A(t)$, $t \in \mathbb{R}$, (H4a), and (H5a) as in Example 22. Define the operators $B,\ C$ by

$$D(B) = D(C) = C[0,\pi],$$
$$B\varphi(\cdot) = C\varphi(\cdot) = \varphi(\cdot - \tau), \quad \varphi(\cdot) \in C[0,\pi], \tag{58}$$

then $\|B\|_{B(C[0,\pi])} = \|C\|_{B(C[0,\pi])} = 1$.

Now, (57) can be transformed into the abstract form (2) and assumptions (H1)–(H3) and (H6) are satisfied. Hence, Theorem 21 leads also to the following proposition.

Proposition 25. *Under assumptions (H4a) and (H5a), neutral functional differential equation* (57) *admits a unique almost automorphic mild solution if*

$$L_f + L_g < 1. \tag{59}$$

Furthermore, if one takes

$$f(t,u) = \frac{1}{2} u \sin \frac{1}{-3 + \sin t + \sin \pi t},$$
$$t \in \mathbb{R},\ u \in C[0,\pi],$$
$$g(t,u) = \frac{1}{8} u \left(1 + \sin \frac{1}{-3 + \sin t + \sin \pi t}\right),$$
$$t \in \mathbb{R},\ u \in C[0,\pi]. \tag{60}$$

A simple computation shows that $f, g \in AA(\mathbb{R} \times C[0,\pi], C[0,\pi])$, (H4a), and (H5a) are satisfied with $L_f = 1/2$, $L_g = 1/4$. By Proposition 25, (57) *admits a unique almost automorphic mild solution.*

Acknowledgments

The authors would like to thank the referees for their careful reading of this paper. This work is supported by the National Science Foundation of China (11171314).

References

[1] S. Bochner, "A new approach to almost periodicity," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 48, pp. 2039–2043, 1962.

[2] W. A. Veech, "Almost automorphic functions on groups," *American Journal of Mathematics*, vol. 87, pp. 719–751, 1965.

[3] W. A. Veech, "On a theorem of Bochner," *Annals of Mathematics*, vol. 86, pp. 117–137, 1967.

[4] M. Zaki, "Almost automorphic solutions of certain abstract differential equations," *Annali di Matematica Pura ed Applicata*, vol. 101, pp. 91–114, 1974.

[5] W. Shen and Y. Yi, "Almost automorphic and almost periodic dynamics in skew-product semi ows," *Memoirs of the American Mathematical Society*, vol. 136, no. 647, 1998.

[6] G. M. N'Guérékata, *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum, New York, NY, USA, 2001.

[7] G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, NY, USA, 2005.

[8] S. Boulite, L. Maniar, and G. M. N'Guérékata, "Almost automorphic solutions for hyperbolic semilinear evolution equations," *Semigroup Forum*, vol. 71, no. 2, pp. 231–240, 2005.

[9] P. Cieutat and K. Ezzinbi, "Almost automorphic solutions for some evolution equations through the minimizing for some subvariant functional, applications to heat and wave equations with nonlinearities," *Journal of Functional Analysis*, vol. 260, no. 9, pp. 2598–2634, 2011.

[10] T. Diagana, G. Nguerekata, and Nguyen Van Minh, "Almost automorphic solutions of evolution equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 11, pp. 3289–3298, 2004.

[11] J. A. Goldstein and G. M. N'Guérékata, "Almost automorphic solutions of semilinear evolution equations," *Proceedings of the American Mathematical Society*, vol. 133, no. 8, pp. 2401–2408, 2005.

[12] G. M. N'guerekata, "Almost automorphy, almost periodicity and stability of motions in Banach spaces," *Forum Mathematicum*, vol. 13, no. 4, pp. 581–588, 2001.

[13] G. M. N'Guérékata, "Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations," *Semigroup Forum*, vol. 69, no. 1, pp. 80–86, 2004.

[14] M. Baroun, S. Boulite, G. M. N'Guérékata, and L. Maniar, "Almost automorphy of semilinear parabolic evolution equations," *Electronic Journal of Differential Equations*, vol. 2008, pp. 1–9, 2008.

[15] H.-S. Ding, W. Long, and G. M. N'Guérékata, "Almost automorphic solutions of nonautonomous evolution equations," *Nonlinear Analysis. Theory, Methods & Applications A: Theory and Methods*, vol. 70, no. 12, pp. 4158–4164, 2009.

[16] J.-H. Liu and X.-Q. Song, "Almost automorphic and weighted pseudo almost automorphic solutions of semilinear evolution equations," *Journal of Functional Analysis*, vol. 258, no. 1, pp. 196–207, 2010.

[17] M. Adimy and K. Ezzinbi, "A class of linear partial neutral functional-differential equations with nondense domain," *Journal of Differential Equations*, vol. 147, no. 2, pp. 285–332, 1998.

[18] G. Da Prato and E. Sinestrari, "Differential operators with nondense domain," *Annali della Scuola Normale Superiore di Pisa*, vol. 14, no. 2, pp. 285–344, 1987.

[19] E. Sinestrari, "On the abstract Cauchy problem of parabolic type in spaces of continuous functions," *Journal of Mathematical Analysis and Applications*, vol. 107, no. 1, pp. 16–66, 1985.

[20] L. Maniar and R. Schnaubelt, "Almost periodicity of inhomogeneous parabolic evolution equations," in *Lecture Notes in Pure and Applied Mathematics*, vol. 234, pp. 299–318, Dekker, New York, NY, USA, 2003.

[21] R. Schnaubelt, "Asymptotic behaviour of parabolic nonautonomous evolution equations," in *Functional Analytic Methods for Evolution Equations*, vol. 1855 of *Lecture Notes in Mathematics*, pp. 401–472, Springer, Berlin, Germany, 2004.

[22] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical systems and Differential Equations*, American Mathematical Society, Providence, RI, USA, 1999.

[23] W. A. Coppel, *Dichotomies in Stability Theory*, Springer, Berlin, Germany, 1978.

[24] R. J. Sacker and G. R. Sell, "Dichotomies for linear evolutionary equations in Banach spaces," *Journal of Differential Equations*, vol. 113, no. 1, pp. 17–67, 1994.

[25] H.-S. Ding, J. Liang, and T.-J. Xiao, "Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications Series A: Theory and Methods*, vol. 73, no. 5, pp. 1426–1438, 2010.

[26] T.-J. Xiao, X.-X. Zhu, and J. Liang, "Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications," *Nonlinear Analysis. Theory, Methods & Applications. A: Theory and Methods*, vol. 70, no. 11, pp. 4079–4085, 2009.

[27] P. Acquistapace and B. Terreni, "A unified approach to abstract linear nonautonomous parabolic equations," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 78, pp. 47–107, 1987.

[28] P. Acquistapace, "Evolution operators and strong solutions of abstract linear parabolic equations," *Differential and Integral Equations*, vol. 1, no. 4, pp. 433–457, 1988.

[29] A. Yagi, "Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups. II," *Funkcialaj Ekvacioj*, vol. 33, no. 1, pp. 139–150, 1990.

[30] A. Yagi, "Abstract quasilinear evolution equations of parabolic type in Banach spaces," *Unione Matematica Italiana B*, vol. 5, no. 2, pp. 341–368, 1991.

[31] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume 1: Abstract Linear Theory*, Birkhäuser Boston, Boston, Mass, USA, 1995.

[32] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, Switzerland, 1995.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 798961, 9 pages
http://dx.doi.org/10.1155/2013/798961

Research Article

Permanence, Extinction, and Almost Periodic Solution of a Nicholson's Blowflies Model with Feedback Control and Time Delay

Haihui Wu and Shengbin Yu

Sunshine College, Fuzhou University, Fuzhou, Fujian 350015, China

Correspondence should be addressed to Haihui Wu; whh3346@sina.com

Received 20 February 2013; Accepted 7 April 2013

Academic Editor: Yonghui Xia

A Nicholson's blowflies model with feedback control and time delay is studied. By applying the comparison theorem of the differential equation and fluctuation lemma and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence, extinction, and existence of a unique globally attractive positive almost periodic solution of the system are obtained. It is proved that the feedback control variable and time delay have no influence on the permanence and extinction of the system.

1. Introduction

Let $f(t)$ be any continuous bounded function defined on $[0, +\infty)$; we set

$$f^l = \inf_{t\geq 0} f(t), \qquad f^u = \sup_{t\geq 0} f(t). \tag{1}$$

In order to describe the dynamics of Nicholson's blowflies, Gurney et al. [1] proposed the following mathematical model in 1980:

$$\dot{N}(t) = -\delta N(t) + PN(t-\tau)e^{-aN(t-\tau)}, \tag{2}$$

where $N(t)$ is the size of the population at time t, P is the maximum per capita daily egg production rate, $(1/a)$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. Kulenović and Ladas [2], Győri and Ladas [3], and Győri and Trofimchuk [4] investigated the oscillatory behaviors of the solutions of (2). For the attractivity, Kulenović et al. [5] and So and Yu [6] have shown that, when $P > \delta$, every positive solution $N(t)$ of (2) tends to a positive equilibrium $N^* = (1/a)\ln(P/\delta)$ as $t \to \infty$ if

$$\left(e^{\delta\tau} - 1\right)\left(\frac{P}{\delta} - 1\right) < 1. \tag{3}$$

Reference [5] further showed that, for $P \leq \delta$, every nonnegative solution of (2) tends to zero as $t \to \infty$, and for $P > \delta$, (2) is uniformly persistent. Furthermore, Li and Fan [7] considered the following nonautonomous equation:

$$\dot{x}(t) = x(t)\left(-\delta(t) + p(t)\exp\{-\alpha(t)x(t)\}\right), \tag{4}$$

where $\alpha(t)$, $\delta(t)$, and $p(t)$ are all positive ω-periodic functions. The authors show that (4) has a unique globally attractive ω-periodic positive solution if

$$p(t) > \delta(t) \quad \text{for } t \in [0, \omega]. \tag{5}$$

Their results improved the results of Saker and Agarwal [8] who considered system (4) with $\alpha(t) = a$ (a is a constant).

Recently, Wang and Fan [9] proposed the following discrete Nicholson's blowflies model with feedback control:

$$\begin{aligned} x(n+1) &= x(n)\exp\{-\delta(n) + p(n)\exp\{-\alpha(n)x(n)\} \\ &\qquad -c(n)\mu(n)\}, \\ \Delta\mu(n) &= -a(n)\mu(t) + b(n)x(n-m). \end{aligned} \tag{6}$$

Sufficient conditions are established for the permanence and the extinction of the system (6). They show that the bounded feedback terms do not have any influence on the permanence or extinction of (6). The authors in [9] also proposed the following continuous model:

$$\dot{x}(t) = x(t)\left(-\delta(t) + p(t)\exp\{-\alpha(t)x(t)\} - c(t)\mu(t)\right),$$
$$\dot{\mu}(t) = -a(t)\mu(t) + b(t)x(t-\tau); \tag{7}$$

however, they did not discuss the dynamic behaviors of the system (7). Considering that continuous models can excellently show the dynamic behaviors of those populations who have a long life cycle, overlapping generations, and large quantity, sufficient conditions for the permanence, global attractivity, and the existence of a unique, globally attractive, strictly positive almost periodic solution of the system (7) with $\tau = 0$ are obtained by Yu [10]. As pointed out by Nindjin et al. [11], time delay plays an important role in many biological dynamical systems, being particularly relevant in ecology and a model with time delay is a more realistic approach to the understanding of dynamics. Hence, it is necessary to study the model (7) which contains time delay.

In the following discussion, we always assume that $\delta(t)$, $p(t)$, $\alpha(t)$, $c(t)$, $a(t)$, $b(t)$ are all continuous, positive almost periodic functions. Also, from the viewpoint of mathematical biology, we consider (7) together with the following initial conditions:

$$\begin{aligned} x(\theta) &= \varphi(\theta) \ge 0, \quad \theta \in [-\tau, 0] \ \ \varphi(0) > 0, \\ \mu(\theta) &= \psi(\theta) \ge 0, \quad \theta \in [-\tau, 0] \ \ \psi(0) > 0, \end{aligned} \tag{8}$$

where $\varphi(s)$ and $\psi(s)$ are continuous on $[-\tau, 0]$. It is not difficult to see that solutions of (7) and (8) are well defined for all $t \ge 0$ and satisfy

$$x(t) > 0, \quad \mu(t) > 0, \quad \text{for } t \ge 0. \tag{9}$$

The aim of this paper is, by constructing a suitable Lyapunov functional and applying the analysis technique of Feng et al. [12], to obtain sufficient conditions for the existence of a unique globally attractive positive almost periodic solution of the system (7) with initial condition (8).

This paper is organized as follows. In Section 2, by applying the analysis technique of [13, 14] and Fluctuation lemma [15, 16], we present the permanence and the extinction of model (7) and (8). In Section 3, by constructing a suitable Lyapunov functional, a sufficient conditions for the existence of a unique globally attractive positive almost periodic solution of the system (7) and (8). Examples together with their numeric simulations are stated in Section 4. For more works on almost periodic solutions of the ecosystem with feedback control, one could refer to [17–23] and the references cited therein.

2. Permanence and Extinction

Now let us state several lemmas which will be useful in proving the main result of this section.

Lemma 1 (see [13]). *Assume that $a > 0$, $b(t) > 0$ is a boundedness continuous function and $x(0) > 0$. Further suppose that*

(i)

$$\dot{x}(t) \le -ax(t) + b(t) \tag{10}$$

then for all $t \ge s$,

$$x(t) \le x(t-s)\exp\{-as\} + \int_{t-s}^{t} b(\tau)\exp\{a(\tau - t)\}\,d\tau. \tag{11}$$

Particularly, if $b(t)$ is bounded above with respect to M, then

$$\limsup_{t\to+\infty} x(t) \le \frac{M}{a}. \tag{12}$$

(ii) *Also suppose that*

$$\dot{x}(t) \ge -ax(t) + b(t); \tag{13}$$

then for all $t \ge s$,

$$x(t) \ge x(t-s)\exp\{-as\} + \int_{t-s}^{t} b(\tau)\exp\{a(\tau - t)\}\,d\tau. \tag{14}$$

Particularly, if $b(t)$ is bounded above with respect to m, then

$$\liminf_{t\to+\infty} x(t) \ge \frac{m}{a}. \tag{15}$$

Lemma 2 (see [20]). *If $a > 0$, $b > 0$ and $\dot{x} \ge x(b - ax)$, when $t \ge 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \ge \frac{b}{a}. \tag{16}$$

If $a > 0$, $b > 0$, and $\dot{x} \le x(b - ax)$, when $t \ge 0$ and $x(0) > 0$, one has

$$\limsup_{t\to+\infty} x(t) \le \frac{b}{a}. \tag{17}$$

Lemma 3. *Let $(x(t), \mu(t))^T$ be any solution of system (7) with initial condition (8); there exists positive numbers M_1 and M_2, which are independent of the solution of the system, such that*

$$\limsup_{t\to+\infty} x(t) \le M_1, \qquad \limsup_{t\to+\infty} \mu(t) \le M_2. \tag{18}$$

Proof. Let $(x(t), \mu(t))^T$ be any solution of system (7) satisfying initial condition (8). Since $x\exp\{-\alpha^l x(t)\} \le (1/\alpha^l e)$ for $x > 0$, according to the positivity of solution and the first equation of system (7), for $t \ge 0$,

$$\begin{aligned} \dot{x}(t) &\le x(t)\left(-\delta^l + p^u \exp\left\{-\alpha^l x(t)\right\}\right) \\ &\le -\delta^l x(t) + \frac{p^u}{\alpha^l e}, \end{aligned} \tag{19}$$

where e is the mathematical constant.

By applying Lemma 1(i) to (19), we have

$$\limsup_{t\to+\infty} x(t) \le \frac{p^u}{\delta^l \alpha^l e} \triangleq M_1. \tag{20}$$

Hence, there exists $T_1 > 0$ such that

$$x(t) \le 2M_1, \quad \forall t \ge T_1. \tag{21}$$

Equation (21) together with the second equation of (7) leads to

$$\dot{\mu}(t) \le -a^l \mu(t) + 2b^u M_1, \quad \forall t \ge T_1 + \tau. \tag{22}$$

Using Lemma 1(i) again, one has

$$\limsup_{t\to+\infty} \mu(t) \le \frac{2b^u M_1}{a^l} \triangleq M_2. \tag{23}$$

Obviously, $M_i (i = 1, 2)$ are independent of the solution of system (7). Equations (20) and (23) show that the conclusion of Lemma 3 holds. The proof is completed. □

Lemma 4. *Assume that*

(H_1)

$$p(t) > \delta(t), \quad t \ge 0, \tag{24}$$

holds. Then there exists positive constants m_1 and m_2, which are independent of the solution of system (7)*, such that*

$$\liminf_{t\to+\infty} x(t) \ge m_1, \qquad \liminf_{t\to+\infty} \mu(t) \ge m_2. \tag{25}$$

Proof. Let $(x(t), \mu(t))^T$ be any solution of system (7) satisfying initial condition (8). From Lemma 3, there exists a $T_2 > T_1 + \tau$ such that for all $t \ge T_2$, $x(t) \le M$, $\mu(t) \le M$, where $M = 2\max\{M_1, M_2\}$. According to the first equation of system (7) and the positivity of solution, for $t \ge T_2$,

$$\begin{aligned}\dot{x}(t) &= x(t)\left(-\delta(t) + p(t)\exp\{-\alpha(t)x(t)\} - c(t)\mu(t)\right) \\ &\ge x(t)\left(-\delta(t) + p(t)\exp\{-\alpha(t)M\} - c(t)M\right) \\ &\triangleq -Q(t)x(t),\end{aligned} \tag{26}$$

where $Q(t) = \delta(t) - p(t)\exp\{-\alpha(t)M\} + c(t)M$.

Integrating both sides of (26) from η $(\eta \le t)$ to t leads to

$$\frac{x(t)}{x(\eta)} \ge \exp\left\{-\int_\eta^t Q(s)\,ds\right\}, \tag{27}$$

or

$$x(\eta) \le x(t)\exp\left\{\int_\eta^t Q(s)\,ds\right\}. \tag{28}$$

Particularly, taking $\eta = t - \tau$, one can get

$$x(t-\tau) \le x(t)\exp\left\{\int_{t-\tau}^t Q(s)\,ds\right\}. \tag{29}$$

Substituting (29) into the second equation of system (7) leads to

$$\dot{\mu}(t) \le -a^l \mu(t) + b^u x(t)\exp\left\{\int_{t-\tau}^t Q(s)\,ds\right\}. \tag{30}$$

Applying Lemma 1(i) to the above differential inequality, for $0 \le s \le t$, one has

$$\begin{aligned}\mu(t) &\le \mu(t-s)\exp\{-a^l s\} \\ &\quad + \int_{t-s}^t b^u x(\eta)\exp\left\{\int_{\eta-\tau}^\eta Q(u)\,du\right\}\exp\{a^l(\eta - t)\}\,d\eta \\ &\overset{\text{from (28)}}{\le} \mu(t-s)\exp\{-a^l s\} \\ &\quad + \int_{t-s}^t b^u x(t)\exp\left\{\int_\eta^t Q(u)\,du\right\}\exp\left\{\int_{\eta-\tau}^\eta Q(u)\,du\right\} \\ &\quad \times \exp\{a^l(\eta-t)\}\,d\eta \\ &\le \mu(t-s)\exp\{-a^l s\} + b^u x(t)\int_{t-s}^t \exp\left\{\int_\eta^t Q(u)\,du\right\} \\ &\quad \times \exp\left\{\int_{\eta-\tau}^\eta Q(u)\,du\right\}d\eta,\end{aligned} \tag{31}$$

where we used the fact $\max_{\eta\in[t-s,t]} \exp\{a^l(\eta - t)\} = \exp\{0\} = 1$.

Note that there exists a K, such that $2c^u M\exp\{-a^l s\} < (\beta/2)$, as $s \ge K$, where $\beta = \inf_{t\ge 0}(p(t) - \delta(t))$. In fact, we can choose $K > (1/a^l)\ln(4c^u M/\beta)$. And so, fixing K, combined with (31), we can obtain

$$\begin{aligned}\mu(t) &\le M\exp\{-a^l K\} + b^u x(t)\int_{t-K}^t \exp\left\{\int_\eta^t Q(u)\,du\right\} \\ &\quad \times \exp\left\{\int_{\eta-\tau}^\eta Q(u)\,du\right\}d\eta \\ &\le M\exp\{-a^l K\} + Dx(t),\end{aligned} \tag{32}$$

for all $t > T_2 + K$, where $D = \sup_{t\ge T_3}(b^u \int_{t-K}^t \exp\{\int_\eta^t Q(u)du\}\exp\{\int_{\eta-\tau}^\eta Q(u)du\}d\eta) > 0$.

Considering that $e^{-x} \ge 1 - x$, for $x > 0$, from the first equation of system (7) and the positivity of the solution, for $t > T_2 + K$, we can get

$$\begin{aligned}\dot{x}(t) &= x(t)\left(-\delta(t) + p(t)\exp\{-\alpha(t)x(t)\} - c(t)\mu(t)\right) \\ &\ge x(t)\Big(-\delta(t) + p(t) - p(t)\alpha(t)x(t) \\ &\qquad -2c^u M\exp\{-a^l K\} - c^u Dx(t)\Big) \\ &\ge x(t)\left(\frac{\beta}{2} - (p^u\alpha^u + c^u D)x(t)\right).\end{aligned} \tag{33}$$

By Lemma 2, we have

$$\liminf_{t\to+\infty} x(t) \geq \frac{\beta}{2(p^u\alpha^u + c^u D)} \triangleq m_1. \tag{34}$$

Thus, there exists $T_3 > T_2 + K$ such that for all $t > T_3$,

$$x(t) \geq \frac{m_1}{2}. \tag{35}$$

Equation (35) together with the second equation of (7) leads to

$$\dot{\mu}(t) \geq -a^u \mu(t) + b^l \frac{m_1}{2}, \quad \forall t > T_3. \tag{36}$$

By applying Lemma 1(ii) to the above differential inequality, we have

$$\liminf_{t\to+\infty} \mu(t) \geq \frac{b^l m_1}{2a^u} \triangleq m_2. \tag{37}$$

Obviously, m_i $(i = 1, 2)$ are independent of the solution of system (7). Equations (34) and (37) show that the conclusion of Lemma 4 holds. The proof is completed. □

From Lemmas 3–4 and the definition of permanence, we can obtain the following conclusion.

Theorem 5. *Assume that* (H_1) *holds; then system* (7) *with initial condition* (8) *is permanent.*

As a direct corollary of Theorem 2 in [24], from Theorem 5, we have the following.

Corollary 6. *Suppose that* (H_1) *holds; then system* (7) *admits at least one positive ω-periodic solution if* $\delta(t)$, $p(t)$, $\alpha(t)$, $c(t)$, $a(t)$, $b(t)$ *are all continuous positive ω-periodic functions.*

Theorem 7. *Let* $(x(t), \mu(t))^T$ *be any positive solution of the system* (7) *with initial condition* (8)*. Assume that*

$$\int_0^\infty (p(t) - \delta(t))\,dt = -\infty \quad \text{or} \quad p(t) \leq \delta(t), \quad t \geq 0 \tag{38}$$

holds; then

$$\lim_{t\to+\infty} x(t) = 0, \qquad \lim_{t\to+\infty} \mu(t) = 0. \tag{39}$$

Proof. Firstly, from the the first equation of (7),

$$\dot{x}(t) \leq x(t)(p(t) - \delta(t)). \tag{40}$$

If the former case of (38) holds, then

$$0 < x(t) \leq x(0) \exp\left[\int_0^t (p(t) - \delta(t))\,dt\right] \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \tag{41}$$

which shows that $\lim_{t\to+\infty} x(t) = 0$.

If the latter case of (38) holds, from (40) we have $\dot{x}(t) < 0$ or $x(t)$ is decreasing; therefore, $\lim_{t\to+\infty} x(t) = q \in [0, +\infty)$. Hence $\limsup_{t\to+\infty} = \liminf_{t\to+\infty} x(t) = q$. We only need to show that $q = 0$. Otherwise, if $q > 0$, then there exists a $T_4 > 0$, such that $x(t) > (q/2)$ for $t \geq T_4$. According to the Fluctuation lemma, there exists a sequence $\xi_n \to \infty$ as $n \to \infty$ such that $\dot{x}(\xi_n) \to 0$, $x(\xi_n) \to \limsup_{t\to\infty} = q$, as $n \to \infty$. We can choose a large enough number N such that $\xi_n > T_4$ for $n > N$; hence, $x(\xi_n) > (q/2)$ for all $n > N$.

For $n > N$, $p(t) \leq \delta(t)$ together with the first equation of (7) leads to

$$\begin{aligned}\dot{x}(\xi_n) &\leq x(\xi_n)(-\delta(\xi_n) + p(\xi_n)\exp\{-\alpha(\xi_n)x(\xi_n)\}) \\ &\leq x(\xi_n)\left(-\delta(\xi_n) + \delta(\xi_n)\exp\left\{-\alpha(\xi_n)\frac{q}{2}\right\}\right) \\ &\leq x(\xi_n)\left(-1 + \exp\left\{-\alpha^l \frac{q}{2}\right\}\right)\delta^l.\end{aligned} \tag{42}$$

Let $n \to \infty$; we obtain that $0 \leq q(-1 + \exp\{-\alpha^l(q/2)\})\delta^l$ or $\exp\{-\alpha^l(q/2))\} > 1$ which is impossible. Hence, $q = 0$ or

$$\lim_{t\to+\infty} x(t) = 0. \tag{43}$$

Now, we come to prove that

$$\lim_{t\to+\infty} \mu(t) = 0. \tag{44}$$

For any $\epsilon > 0$, according to (43), there exists a $T_5 > 0$, such that

$$x(t) < \epsilon \quad \forall t > T_5. \tag{45}$$

Then, for $t > T_5 + \tau$,

$$\dot{\mu}(t) \leq -a^l \mu(t) + b^u \epsilon. \tag{46}$$

Thus, by applying Lemma 1(i) to the above differential inequality, we have

$$0 < \mu(t) \leq \frac{b^u \epsilon}{a^l} \tag{47}$$

which implies that

$$\lim_{t\to+\infty} \mu(t) = 0. \tag{48}$$

The proof is complete. □

3. Existence of a Unique Almost Periodic Solution

Now, we give the definition of the almost periodic function.

Definition 8 (see [25, 26]). A function $f(t, x)$, where f is an m-vector, t is a real scalar, and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset R^n$, if $f(t, x)$ is continuous in $t \in R$ and $x \in X$, and if for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists τ such that the inequality

$$\|f(t+\tau) - f(t)\| = \sum_{i=1}^{m} \left| f_i(t+\tau, x) - f_i(t, x) \right| < \varepsilon \quad (49)$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in X$. The number τ is called an ε-translation number of $f(t, x)$.

Definition 9 (see [25, 26]). A function $f : R \to R$ is said to be an asymptotically almost periodic function if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that

$$f(t) = q(t) + r(t), \quad t \in R, \ r(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty. \quad (50)$$

We denote by $S(E)$ the set of all solutions $z(t) = (x(t), \mu(t))^T$ of system (7) satisfying $m_1 \le x(t) \le M_1$, $m_2 \le \mu(t) \le M_2$ for all $t \in R$.

Lemma 10. *One has* $S(E) \neq \emptyset$.

Proof. Since $\delta(t)$, $p(t)$, $\alpha(t)$, $c(t)$, $a(t)$, $b(t)$ are almost periodic functions, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned} \delta(t+t_n) &\longrightarrow \delta(t), & p(t+t_n) &\longrightarrow \delta(t), \\ \alpha(t+t_n) &\longrightarrow \delta(t), & c(t+t_n) &\longrightarrow \delta(t), \\ a(t+t_n) &\longrightarrow \delta(t), & b(t+t_n) &\longrightarrow \delta(t), \end{aligned} \quad (51)$$

as $n \to \infty$ uniformly on R. Suppose $z(t) = (x(t), \mu(t))^T$ is a solution of (7) satisfying $m_1 \le x(t) \le M_1$, $m_2 \le \mu(t) \le M_2$ for $t > T$. Obviously, the sequence $(z(t+t_n))$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore, by the Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, which we still denote by $\{t_n\}$, such that $x(t+t_n) \to m(t)$, $\mu(t+t_n) \to n(t)$, as $n \to \infty$ uniformly on each bounded subset of R. For any $T_1 \in R$, we may assume that $t_n + T_1 \ge T$ for all n. For $t \ge 0$, we have

$$\begin{aligned} & x(t+t_n+T_1) - x(t_n+T_1) \\ &= \int_{T_1}^{t+T_1} x(s+t_n)(-\delta(s+t_n) + p(s+t_n) \\ &\qquad \times \exp\{-\alpha(s+t_n)x(s+t_n)\} \\ &\qquad -c(s+t_n)\mu(s))\,ds, \\ & \mu(t+t_n+T_1) - \mu(t_n+T_1) \\ &= \int_{T_1}^{t+T_1} (-a(s+t_n)\mu(s+t_n) + b(s+t_n)x(s+t_n-\tau))\,ds. \end{aligned} \quad (52)$$

Applying Lebesgue's dominated convergence theorem and letting $n \to \infty$ in to previous equations, we obtain

$$\begin{aligned} & m(t+T_1) - m(T_1) \\ &= \int_{T_1}^{t+T_1} m(s)(-\delta(s) + p(s) \\ &\qquad \times \exp\{-\alpha(s)m(s)\} - c(s)n(s))\,ds, \\ & n(t+T_1) - n(T_1) = \int_{T_1}^{t+T_1} (-a(s)n(s) + b(s)m(s-\tau))\,ds, \end{aligned} \quad (53)$$

for all $t \ge 0$. Since $T_1 \in R$ is arbitrarily given, $(m(t), n(t))^T$ is a solution of system (7) on R. It is clear that $m_1 \le m(t) \le M_1$, $m_2 \le n(t) \le M_2$ for $t \in R$. That is to say, $(m(t), n(t))^T \in S(E)$. This completes the proof. □

Lemma 11 (see [27]). *Let f be a nonnegative function defined on $[0, +\infty)$ such that f is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then,* $\lim_{t\to+\infty} f(t) = 0$.

Theorem 12. *In addition to* (H_1)*, further suppose that*

(H_2) *there exists a* $h > 0$*, such that*

$$p^l \alpha^l \exp(\alpha^u M_1) - b^u > h, \qquad a^l - c^u > h, \quad (54)$$

where M_1 is defined in (23); then system (7) with initial conditions (8) is globally attractive. That is to say, for any two positive solutions, one has

$$\lim_{t\to+\infty} \left| x(t) - x^*(t) \right| = 0, \qquad \lim_{t\to+\infty} \left| \mu(t) - \mu^*(t) \right| = 0. \quad (55)$$

Proof. Let $(x^*(t), \mu^*(t))^T$ and $(x(t), \mu(t))^T$ be any two positive solutions of system (7)-(8). Theorem 5 implies there exist positive constants T, m_i, and M_i $(i = 1, 2)$ such that for $t \ge T$

$$\begin{aligned} m_1 &\le x(t) \le M_1, & m_1 &\le x^*(t) \le M_1, \\ m_2 &\le \mu(t) \le M_2, & m_2 &\le \mu^*(t) \le M_2, \end{aligned} \quad (56)$$

where m_i and M_i $(i = 1, 2)$ are defined in Lemma 3 and Lemma 4. Set $V(t) = V_1(t) + V_2(t)$, where

$$\begin{aligned} V_1(t) &= \left| \ln x(t) - \ln x^*(t) \right|, \\ V_2(t) &= \left| \mu(t) - \mu^*(t) \right| + b^u \int_{t-\tau}^{t} \left| x(u) - x^*(u) \right| du. \end{aligned} \quad (57)$$

Calculating the upper right derivatives of $V_1(t)$ along the solution of (7) leads to

$$\begin{aligned} D^+V_1(t) &= \operatorname{sgn}[x(t)-x^*(t)] \\ &\quad \times \left(p(t)\left(\exp\{-\alpha(t)x(t)\}-\exp\{-\alpha(t)x^*(t)\}\right)\right. \\ &\qquad \left. + c(t)\left(\mu^*(t)-\mu(t)\right)\right) \\ &= \operatorname{sgn}[x(t)-x^*(t)] \\ &\quad \times \left(p(t)\left(-\alpha(t)\exp\{-\xi(t)\}\left(x(t)-x^*(t)\right)\right)\right. \\ &\qquad \left. + c(t)\left(\mu^*(t)-\mu(t)\right)\right) \\ &\le -p(t)\alpha(t)\left(\exp\{-\xi(t)\}\left|x(t)-x^*(t)\right|\right) \\ &\quad + c(t)\left|\mu(t)-\mu^*(t)\right|, \end{aligned} \tag{58}$$

where we used the elementary mean value theorem of differential calculus and $\xi(t)$ lies between $\alpha(t)x(t)$ and $\alpha(t)x^*(t)$. Then, for $t \ge T$, we have

$$\alpha^l m_1 \le \xi(t) \le \alpha^u M_1. \tag{59}$$

Hence, by (58), we can have

$$\begin{aligned} D^+V_1(t) &\le -p^l\alpha^l \exp\left(\alpha^u M_1\right)\left|x(t)-x^*(t)\right| \\ &\quad + c^u\left|\mu(t)-\mu^*(t)\right|. \end{aligned} \tag{60}$$

Calculating the upper right derivatives of $V_2(t)$ along the solution of (7), one has

$$\begin{aligned} D^+V_2(t) &= \operatorname{sgn}[\mu(t)-\mu^*(t)] \\ &\quad \times \left(-a(t)\left(\mu(t)-\mu^*(t)\right)\right. \\ &\qquad \left. + b(t)\left(x(t-\tau)-x^*(t-\tau)\right)\right) \\ &\qquad + b^u\left(\left|x(t)-x^*(t)\right| - \left|x(t-\tau)-x^*(t-\tau)\right|\right) \\ &\le -a^l\left|\mu(t)-\mu^*(t)\right| + b^u\left|x(t)-x^*(t)\right|. \end{aligned} \tag{61}$$

According to (58), (66), and condition (H_2), we can obtain

$$\begin{aligned} D^+V(t) &\le \left(b^u - p^l\alpha^l\exp\left(\alpha^u M_1\right)\right)\left|x(t)-x^*(t)\right| \\ &\quad + \left(c^u - a^l\right)\left|\mu(t)-\mu^*(t)\right| \\ &< h\left[\left|\mu(t)-\mu^*(t)\right| + \left|x(t)-x^*(t)\right|\right]. \end{aligned} \tag{62}$$

Integrating both sides of (62) from T to t leads to

$$\begin{aligned} V(t) &+ h\int_T^t \left[\left|\mu(s)-\mu^*(s)\right| + \left|x(s)-x^*(s)\right|\right] ds \\ &< V(T) < +\infty, \quad t \ge T. \end{aligned} \tag{63}$$

Then,

$$\int_T^t \left[\left|\mu(s)-\mu^*(s)\right| + \left|x(s)-x^*(s)\right|\right] ds < \frac{V(T)}{h} < +\infty, \quad t \ge T. \tag{64}$$

Hence, $|\mu(t)-\mu^*(t)| + |x(t)-x^*(t)| \in L^1([T,+\infty))$. By system (7) and Theorem 5, we get $\mu(t)$, $\mu^*(t)$, $x(t)$, $x^*(t)$, and their derivatives are bounded on $[T,+\infty)$, which implies that $|\mu(t)-\mu^*(t)| + |x(t)-x^*(t)|$ is uniformly continuous on $[T,+\infty)$. By Lemma 11, we obtain

$$\lim_{t\to+\infty}\left|x(t)-x^*(t)\right| = 0, \qquad \lim_{t\to+\infty}\left|\mu(t)-\mu^*(t)\right| = 0. \tag{65}$$

The proof of Theorem 12 is complete. □

Theorem 13. *Suppose all conditions of Theorem 12 hold; then there exists a unique almost periodic solution of systems* (7) *and* (8).

Proof. According to Lemma 10, there exists a bounded positive solution $u(t) = (u_1(t), u_2(t))^T$ of (7) with initial condition (8). Then there exists a sequence $\{t'_k\}$, $\{t'_k\} \to \infty$ as $k \to \infty$, such that $(u_1(t+t'_k), u_2(t+t'_k))^T$ is a solution of the following system:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(-\delta\left(t+t'_k\right) + p\left(t+t'_k\right)\exp\left\{-\alpha\left(t+t'_k\right)x(t)\right\}\right. \\ &\qquad \left. - c\left(t+t'_k\right)\mu(t)\right), \\ \dot{\mu}(t) &= -a\left(t+t'_k\right)\mu(t) + b\left(t+t'_k\right)x(t-\tau). \end{aligned} \tag{66}$$

According to Theorem 5 and the fact that $\delta(t)$, $p(t)$, $\alpha(t)$, $c(t)$, $a(t)$, $b(t)$ are all continuous, positive almost periodic functions, we know that both $\{u_i(t+t'_k)\}$ $(i=1,2)$ and its derivative function $\{\dot{u}(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded; thus, $\{u_i(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded and equi-continuous. By Ascoli's theorem, there exists a uniformly convergent subsequence $\{u_i(t+t_k)\} \subseteq \{u_i(t+t'_k)\}$ such that for any $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ with the property that if $m, k \ge K(\varepsilon)$, then

$$\left|u_i(t+t_m) - u_i(t+t_k)\right| < \varepsilon, \quad i = 1,2. \tag{67}$$

That is to say, $u_i(t)$ $(i = 1,2)$ are asymptotically almost periodic functions. Hence there exists two almost periodic functions $r_i(t+t_k)$ $(i=1,2)$ and two continuous functions $s_i(t+t_k)$ $(i=1,2)$ such that

$$u_i(t+t_k) = r_i(t+t_k) + s_i(t+t_k), \quad i = 1,2, \tag{68}$$

where

$$\lim_{k\to+\infty} r_i(t+t_k) = r_i(t), \quad \lim_{k\to+\infty} s_i(t+t_k) = 0, \quad i = 1,2, \tag{69}$$

$r_i(t)$ $(i = 1,2)$ are also almost periodic functions.

Therefore,

$$\lim_{k\to+\infty} u_i(t+t_k) = r_i(t), \quad i = 1,2. \tag{70}$$

On the other hand,

$$\begin{aligned} \lim_{k\to+\infty} \dot{u}_i(t+t_k) &= \lim_{k\to+\infty}\lim_{h\to 0} \frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ &= \lim_{h\to 0}\lim_{k\to+\infty} \frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ &= \lim_{h\to 0} \frac{r_i(t+h)-r_i(t)}{h}. \end{aligned} \tag{71}$$

So $\dot{r}_i(t)$ $(i = 1,2)$ exist. Moreover,

$$\begin{aligned} \dot{r}_1(t) &= \lim_{k\to+\infty} \dot{u}_1(t+t_k) \\ &= \lim_{k\to+\infty} \{u_1(t+t_k) \\ &\quad \times(-\delta(t+t_k) - c(t+t_k)u_2(t+t_k) + p(t+t_k) \\ &\quad \times\exp\{-\alpha(t+t_k)u_1(t+t_k)\})\} \\ &= r_1(t)(-\delta(t) + p(t)\exp\{-\alpha(t)r_1(t)\} - c(t)r_2(t)), \\ \dot{r}_2(t) &= \lim_{k\to+\infty} \dot{u}_2(t+t_k) \\ &= \lim_{k\to+\infty} \{-a(t+t_k)r_2(t+t_k) \\ &\qquad + b(t+t_k)r_1(t+t_k-\tau)\} \\ &= -a(t)r_2(t) + b(t)r_1(t-\tau). \end{aligned} \tag{72}$$

These show that $(r_1(t), r_2(t))^T$ satisfied system (7). Hence, $(r_1(t), r_2(t))^T$ is a positive almost periodic solution of (7). Then, it follows from Theorem 12 that system (7) has a unique positive almost periodic solution. The proof is completed. □

Without the feedback terms, that is $a(t) = 0$, $b(t) = 0$, $c(t) = 0$, and (7) becomes the following equation:

$$\dot{x}(t) = x(t)(-\delta(t) + p(t)\exp\{-\alpha(t)x(t)\}). \tag{73}$$

Equation (73) with periodic coefficients has been studied by Li and Fan [7] and Saker and Agarwal [8] with $\alpha(t) = a$. Since the periodic case is a special case of almost periodic, hence, as a direct corollary of Theorem 13, we have the following.

Corollary 14. *Suppose $\delta(t)$, $p(t)$, $\alpha(t)$, $c(t)$, $a(t)$, $b(t)$ are all continuous positive ω-periodic functions and $p(t) > \delta(t)$ for $t \in [0,\omega]$; then* (73) *has a unique globally attractive ω-periodic positive solution.*

Remark 15. Li and Fan in [7] show that (73) has a unique globally attractive ω-periodic positive solution if $p(t) > \delta(t)$ for $t \in [0,\omega]$, which is the same as Corollary 14. Thus, Theorem 13 supplements and generalizes results in [7, 8].

4. Examples and Numeric Simulations

Now we give several examples together with their numeric simulations to show the feasibility of our main results.

Example 16. Consider the following example:

$$\begin{aligned} \dot{x}(t) &= x(t)\Big(-5 - \sin\left(\sqrt{5}t\right) \\ &\qquad + 10\exp\left\{-\left(30+\cos\left(\sqrt{11}t\right)\right)x(t)\right\} \\ &\qquad - \left(1 + 0.5\sin\left(\sqrt{7}t\right)\right)\mu(t)\Big), \\ \dot{\mu}(t) &= -\left(2.5 + 0.5\cos\left(\sqrt{7}t\right)\right)\mu(t) \\ &\qquad + \left(5.8 + 0.2\sin\left(\sqrt{3}t\right)\right)x(t-2). \end{aligned} \tag{74}$$

In this case, corresponding to system (7), we have $\delta(t) = 5 + \sin(\sqrt{5}t)$, $p(t) = 10$, $\alpha(t) = 30 + \cos(\sqrt{11}t)$, $a(t) = 2.5 + 0.5\cos(\sqrt{7}t)$, $b(t) = 5.8 + 0.2\sin(\sqrt{3}t)$, $c(t) = 1 + 0.5\sin(\sqrt{7}t)$, $\tau = 2$. According to the proof of Lemmas 3 and 4, one has

$$\begin{aligned} M_1 &= \frac{3p^u}{2\delta^l\alpha^l e} = 0.04757, \qquad M_2 = \frac{3b^u M_1}{2a^l} = 0.214066, \\ m_1 &= \frac{p^l - \delta^u - c^u M_2}{2p^l\alpha^u} = 0.005934, \\ m_2 &= \frac{b^l m_1}{2a^u} = 0.0055384. \end{aligned} \tag{75}$$

Hence,

$$\begin{aligned} p(t) > \delta(t), \qquad & p^l\alpha^l\exp(\alpha^u M_1) - b^u \cong 1261.193896 > 0, \\ & a^l - c^u = 0.5 > 0. \end{aligned} \tag{76}$$

Thus, all the conditions of Theorem 13 are satisfied, and so, there exists a unique almost periodic solution of systems (74). Figure 1 shows this property.

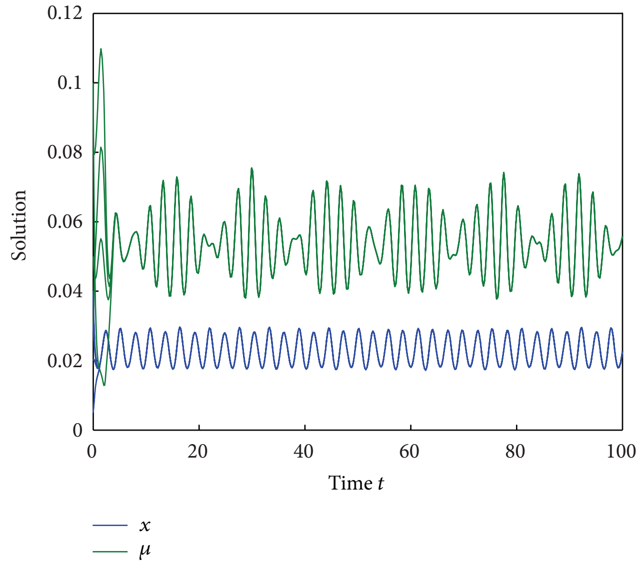


FIGURE 1: Dynamics of the $x(t)$) and $\mu(t)$ of system (74) with the initial values $(x(0), \mu(0))^T = (0.02, 0.05)^T, (0.03, 0.03)^T, (0.04, 0.08)^T, (0.005, 0.1)^T$; here $t \in [0, 100]$.

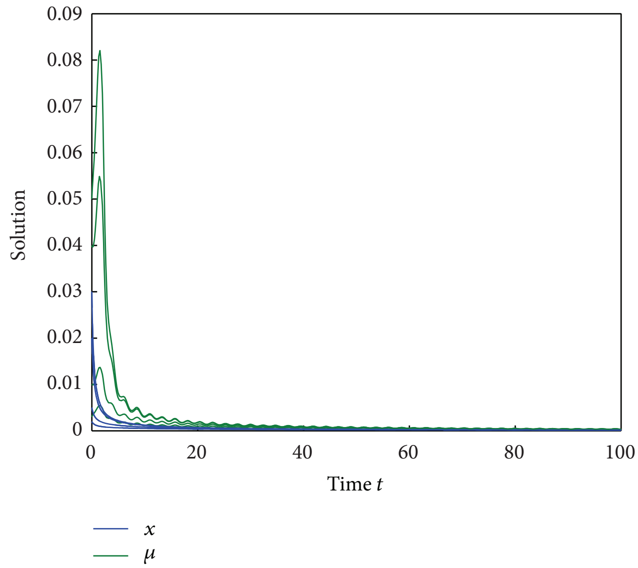


FIGURE 2: The dynamic behavior of system (77) with initial condition $(x(0), \mu(0))^T = (0.03, 0.05)^T, (0.02, 0.04)^T, (0.005, 0.01)^T, (0.002, 0.003)^T$; here $t \in [0, 100]$.

Example 17. Consider the following example:

$$\begin{aligned}\dot{x}(t) = x(t)\Bigg(&-2 - \frac{1}{(t+1)^2} \\ &+ 2\exp\left\{-\left(30 + \cos\left(\sqrt{11}t\right)\right)x(t)\right\} \\ &- \left(1 + 0.5\sin\left(\sqrt{7}t\right)\right)\mu(t)\Bigg), \\ \dot{\mu}(t) = &-\left(2.5 + 0.5\cos\left(\sqrt{7}t\right)\right)\mu(t) \\ &+ \left(5.8 + 0.2\sin\left(\sqrt{3}t\right)\right)x(t-2).\end{aligned} \tag{77}$$

In this case, we have

$$p(t) < \delta(t). \tag{78}$$

Hence, By Theorem 7, we know that any positive solution of system (77) satisfies $\lim_{t\to+\infty} x(t) = 0$, $\lim_{t\to+\infty}\mu(t) = 0$. Numerical simulation also confirms our result (see Figure 2).

5. Conclusion

In this paper, we consider a Nicholson's blowflies model with feedback control and time delay. It is shown that feedback control variable and time delay have no influence on the permanence and extinction of the system. Also, by constructing a suitable Lyapunov functional, a set of sufficient conditions which ensure the existence of a unique globally attractive positive almost periodic solution of the system is established. Moreover, compared with the main result of the relative discrete model (see [9]), we can see that the continuous and discrete models have similar results on permanence and the extinction of the Nicholson's blowflies model with feedback control and time delay. At the end of this paper, two examples together with their numerical simulations show the verification of our main results. Our results supplement and generalize the results in [7, 8].

References

[1] W. S. C. Gurney, S. P. Blythe, and R. M. Nisbet, "Nicholson's blowflies revisited," *Nature*, vol. 287, no. 5777, pp. 17–21, 1980.

[2] M. R. S. Kulenović and G. Ladas, "Linearized oscillations in population dynamics," *Bulletin of Mathematical Biology*, vol. 49, no. 5, pp. 615–627, 1987.

[3] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, The Clarendon Press, New York, NY, USA, 1991.

[4] I. Győri and S. I. Trofimchuk, "On the existence of rapidly oscillatory solutions in the Nicholson blowflies equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 48, no. 7, pp. 1033–1042, 2002.

[5] M. R. S. Kulenović, G. Ladas, and Y. G. Sficas, "Global attractivity in Nicholson's blowflies," *Applicable Analysis*, vol. 43, no. 1-2, pp. 109–124, 1992.

[6] J. W.-H. So and J. S. Yu, "Global attractivity and uniform persistence in Nicholson's blowflies," *Differential Equations and Dynamical Systems*, vol. 2, no. 1, pp. 11–18, 1994.

[7] W.-T. Li and Y.-H. Fan, "Existence and global attractivity of positive periodic solutions for the impulsive delay Nicholson's blowflies model," *Journal of Computational and Applied Mathematics*, vol. 201, no. 1, pp. 55–68, 2007.

[8] S. H. Saker and S. Agarwal, "Oscillation and global attractivity in a periodic Nicholson's blowflies model," *Mathematical and Computer Modelling*, vol. 35, no. 7-8, pp. 719–731, 2002.

[9] L.-L. Wang and Y.-H. Fan, "Permanence for a discrete Nicholson's blowflies model with feedback control and delay," *International Journal of Biomathematics*, vol. 1, no. 4, pp. 433–442, 2008.

[10] S. B. Yu, "Almost periodic solution of Nicholson's blowflies model with feedback control," *Journal of Fuzhou University. Natural Science Edition*, vol. 38, no. 4, pp. 481–485, 2010 (Chinese).

[11] A. F. Nindjin, M. A. Aziz-Alaoui, and M. Cadivel, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1104–1118, 2006.

[12] C. H. Feng, Y. J. Liu, and W. G. Ge, "Almost periodic solutions for delay Lotka-Volterra competitive systems," *Acta Mathematicae Applicatae Sinica*, vol. 28, no. 3, pp. 458–465, 2005 (Chinese).

[13] F. Chen, J. Yang, and L. Chen, "Note on the persistent property of a feedback control system with delays," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 2, pp. 1061–1066, 2010.

[14] F. Chen, J. Yang, L. Chen, and X. Xie, "On a mutualism model with feedback controls," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 581–587, 2009.

[15] A. Tineo, "Asymptotic behaviour of positive solutions of the nonautonomous Lotka-Volterra competition equations," *Differential and Integral Equations*, vol. 6, no. 2, pp. 449–457, 1993.

[16] W. M. Hirsch, H. Hanisch, and J.-P. Gabriel, "Differential equation models of some parasitic infections: methods for the study of asymptotic behavior," *Communications on Pure and Applied Mathematics*, vol. 38, no. 6, pp. 733–753, 1985.

[17] Z. J. Du and Y. S. Lv, "Permanence and almost periodic solution of a Lotka-Volterra model with mutual interference and time delays," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1054–1068, 2012.

[18] X. Lin and F. Chen, "Almost periodic solution for a Volterra model with mutual interference and Beddington-DeAngelis functional response," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 548–556, 2009.

[19] X. X. Chen and F. D. Chen, "Almost-periodic solutions of a delay population equation with feedback control," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 4, pp. 559–571, 2006.

[20] F. Chen, Z. Li, and Y. Huang, "Note on the permanence of a competitive system with infinite delay and feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 8, no. 2, pp. 680–687, 2007.

[21] W. Qi and B. Dai, "Almost periodic solution for n-species Lotka-Volterra competitive system with delay and feedback controls," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 133–146, 2008.

[22] Y. Xia, J. Cao, H. Zhang, and F. Chen, "Almost periodic solutions of n-species competitive system with feedback controls," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 2, pp. 503–522, 2004.

[23] F. Chen and X. Cao, "Existence of almost periodic solution in a ratio-dependent Leslie system with feedback controls," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1399–1412, 2008.

[24] Z. D. Teng, "The almost periodic Kolmogorov competitive systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 7, pp. 1221–1230, 2000.

[25] A. M. Fink, *Almost Periodic Differential Equations*, vol. 377 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1974.

[26] C. Y. He, *Almost Periodic Differential Equations*, Higher Education Publishing House, Beijing, China, 1992.

[27] I. Barbălat, "Systems d'equations differential d'oscillations nonlinearies," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 4, pp. 267–270, 1959.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 163070, 8 pages
http://dx.doi.org/10.1155/2013/163070

Research Article

Persistence Property and Asymptotic Description for DGH Equation with Strong Dissipation

Ke-chuang Wang

Department of Basic, Zhejiang Dongfang Vocational Technical College, Wenzhou, Zhejiang 325011, China

Correspondence should be addressed to Ke-chuang Wang; wangkechuang@126.com

Received 9 February 2013; Accepted 10 March 2013

Academic Editor: Yonghui Xia

The present work is mainly concerned with the Dullin-Gottwald-Holm (DGH) equation with strong dissipation. We establish a sufficient condition to guarantee global-in-time solutions, then present persistence property for the Cauchy problem, and describe the asymptotic behavior of solutions for compactly supported initial data.

1. Introduction

Dullin et al. [1] derived a new equation describing the unidirectional propagation of surface waves in a shallow water regime:

$$\begin{aligned} u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} \\ = \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right), \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1}$$

where the constants α^2 and γ/c_0 are squares of length scales and the constant $c_0 > 0$ is the critical shallow water wave speed for undisturbed water at rest at spatial infinity. Since this equation is derived by Dullin, Gottwald, and Holm, in what follows, we call this new integrable shallow water equation (1) DGH equation.

If $\alpha = 0$, (1) becomes the well-known KdV equation, whose solutions are global as long as the initial data is square integrable. This is proved by Bourgain [2]. If $\gamma = 0$ and $\alpha = 1$, (1) reduces to the Camassa-Holm equation which was derived physically by Camassa and Holm in [3] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime, where $u(x,t)$ represents the free surface above a flat bottom. The properties about the well-posedness, blow-up, global existence, and propagation speed have already been studied in recent works [4–13], and the generalized version of a family of dispersive equations related to Camassa-Holm equation was discussed in [14].

It is very interesting that (1) preserves the bi-Hamiltonian structure and has the following two conserved quantities:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx, \\ F(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \alpha^3 uu_x^2 + c_0 u^2 - \gamma u_x^2\right) dx. \end{aligned} \tag{2}$$

Recently, in [15], local well-posedness of strong solutions to (1) was established by applying Kato's theory [16], and some sufficient conditions were found to guarantee finite time blow-up phenomenon. Moreover, Zhou [17] found the best constants for two convolution problems on the unit circle via variational method and applied the best constants on (1) to give some blow-up criteria. Later, Zhou and Guo improved the results and got some new criteria for wave breaking [18].

In general, it is quite difficult to avoid energy dissipation mechanism in the real world. Ghidaglia [19] studied the long time behavior of solutions to the weakly dissipative KdV equation as a finite dimensional dynamic system. Moreover, some results on blow-up criteria and the global existence condition for the weakly dissipative Camassa-Holm equation are presented in [20], and very related work can be found in

[21, 22]. In this work, we are interested in the following model, which can be viewed as the DGH equation with dissipation

$$u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} + \lambda\left(1 - \alpha^2\partial_x^2\right)u = \alpha^2\left(2u_x u_{xx} + uu_{xxx}\right), \tag{3}$$

where $x \in \mathbb{R}$, $t > 0$, $\lambda(1 - \alpha^2\partial_x^2)u$ is the weakly dissipative term and λ is a positive dissipation parameter. Set $Q = (1 - \alpha^2\partial_x^2)^{1/2}$, then the operator Q^{-2} can be expressed by

$$Q^{-2} f = G * f = \int_{\mathbb{R}} G(x-y) f(y)\, dy, \tag{4}$$

for all $f \in L^2(\mathbb{R})$ with $G(x) = (1/2\alpha)e^{-|x|/\alpha}$. With this in hand, we can rewrite (3) as a quasilinear equation of hyperbolic type

$$u_t + \left(u - \frac{\gamma}{\alpha^2}\right)u_x + \partial_x G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \left(c_0 + \frac{\gamma}{\alpha^2}\right)u\right) + \lambda u = 0. \tag{5}$$

It is the dissipative term that causes the previous conserved quantities $E(u)$ and $F(u)$ to be no longer conserved for (3), and this model could also be regarded as a model of a type of a certain rate-dependent continuum material called a compressible second grade fluid [23]. Our consideration is based on this fact. Furthermore, we will show how the dissipation term affects the behavior of solutions in our forthcoming paper. As a whole, the current dissipation model is of great importance mathematically and physically, and it is worthy of being considered. In what follows, we assume that $c_0 + \gamma/\alpha^2 = 0$ and $\alpha > 0$ just for simplicity. Since $u(x,t)$ is bounded by its H^1-norm, a general case with $c_0 + \gamma/\alpha^2 \neq 0$ does not change our results essentially, but it would lead to unnecessary technical complications. So the above equation is reduced to a simpler form as follows:

$$u_t + (u + c_0)u_x + \partial_x G * F(u) + \lambda u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{6}$$

where

$$F(u) = u^2 + \frac{\alpha^2}{2}u_x^2. \tag{7}$$

The rest of this paper is organized as follows. In Section 2, we list the local well-posedness theorem for (6) with initial datum $u_0 \in H^s$, $s > 3/2$ and collect some auxiliary results. In Section 3, we establish the condition for global existence in view of the initial potential. Persistence properties of the strong solutions are explored in Section 4. Finally, in Section 5, we give a detailed description of the corresponding solution with compactly supported initial data.

2. Preliminaries

In this section, we make some preparations for our consideration. Firstly, the local well-posedness of the Cauchy problem of (6) with initial data $u_0 \in H^s$ with $s > 3/2$ can be obtained by applying Kato's theorem [16]. More precisely, we have the following local well-posedness result.

Theorem 1. *Given that $u_0(x) \in H^s$, $s > 3/2$, there exist $T = T(\lambda, \|u_0\|_{H^s}) > 0$ and a unique solution u to (6), such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}). \tag{8}$$

Moreover, the solution depends continuously on the initial data; that is, the mapping $u_0 \to u(\cdot, u_0) : H^s \to C([0,T); H^s) \cap C^1([0,T); H^{s-1})$ is continuous, and the maximal time of existence $T > 0$ is independent of s.

Proof. Set $A(u) = (u + c_0)\partial_x$, $f(u) = -\partial_x(1 - \alpha^2\partial_x^2)^{-1}F(u) - \lambda u$, $Y = H^s$, $X = H^{s-1}$, $s > 3/2$, and $Q = (1 - \alpha^2\partial_x^2)^{1/2}$. Applying Kato's theory for abstract quasilinear evolution equation of hyperbolic type, we can obtain the local well-posedness of (6) in H^s, $s > 3/2$ and $u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$. □

The maximal value of T in Theorem 1 is called the lifespan of the solution in general. If $T < \infty$, that is $\limsup_{t \to T} \|u(\cdot,t)\|_{H^s} = \infty$, we say that the solution blows up in finite time, otherwise, the solution exists globally in time. Next, we show that the solution blows up if and only if its first-order derivative blows up.

Lemma 2. *Given that $u_0 \in H^s$, $s > 3/2$, the solution $u = u(\cdot, u_0)$ of (3) blows up in finite time $T < +\infty$ if and only if*

$$\liminf_{t \to T}\left\{\inf_{x \in \mathbb{R}}\left[u_x(x,t)\right]\right\} = -\infty. \tag{9}$$

Proof. We first assume that $u_0 \in H^s$ for some $s \in \mathbb{N}$, $s \geq 4$. Equation (6) can be written into the following form in terms of $y = (1 - \alpha^2\partial_x^2)u$

$$y_t + (yu)_x + \frac{1}{2}\left(u^2 - \alpha^2 u_x^2\right)_x + c_0 y_x + \lambda y = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{10}$$

Multiplying (10) by $y = (1 - \alpha^2\partial_x^2)u$, and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx = 2\int_{\mathbb{R}} yy_t dx = -3\int_{\mathbb{R}} u_x y^2 dx - 2\lambda\int_{\mathbb{R}} y^2 dx. \tag{11}$$

Differentiating (10) with respect to the spatial variable x, then multiplying by $y_x = (1 - \alpha^2\partial_x^2)u_x$, and integrating by parts again, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} y_x^2 dx = 2\int_{\mathbb{R}} y_x y_{xt} dx = -5\int_{\mathbb{R}} u_x y_x^2 dx + \frac{2}{\alpha^2}\int_{\mathbb{R}} u_x y^2 dx - 2\lambda\int_{\mathbb{R}} y_x^2 dx. \tag{12}$$

Summarizing (11) and (12), we obtain

$$\frac{d}{dt}\left(\int_{\mathbb{R}}\left(y^2+y_x^2\right)dx\right) = -\left(3-\frac{2}{\alpha^2}\right)\int_{\mathbb{R}} u_x y^2 dx - 5\int_{\mathbb{R}} u_x y_x^2 dx - 2\lambda\left(\int_{\mathbb{R}}\left(y^2+y_x^2\right)dx\right). \quad (13)$$

If u_x is bounded from below on $[0,T)$, for example, $u_x \geq -C$, C is a positive constant, then we get by (13) and Gronwall's inequality the following:

$$\|y\|_{H^1}^2 \leq \exp\{(KC-2\lambda)t\}\|y_0\|_{H^1}^2, \quad (14)$$

where $K=\max\{5,(3-2/\alpha^2)\}$. Therefore, the H^3-norm of the solution to (10) does not blow up in finite time. Furthermore, similar argument shows that the H^k-norm with $k\geq 4$ does not blow up either in finite time. Consequently, this theorem can be proved by Theorem 1 and simple density argument for all $s>3/2$. □

Lemma 3. *Let $u_0 \in H_\alpha^1$, then as long as the solution $u(x,t)$ given by Theorem 1 exists, for any $t\in[0,T)$, one has*

$$\|u\|_{H_\alpha^1}^2 = \exp(-2\lambda t)\|u_0\|_{H_\alpha^1}^2, \quad (15)$$

where the norm is defined as

$$\|u\|_{H_\alpha^1}^2 = \int_{\mathbb{R}}\left(u^2+\alpha^2u_x^2\right)dx. \quad (16)$$

Proof. Multiplying both sides of (10) by u and integrating by parts on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} uy_t\,dx + \int_{\mathbb{R}}(yu)_x u\,dx + \int_{\mathbb{R}}\frac{1}{2}\left(u^2-\alpha^2u_x^2\right)_x u\,dx + \int_{\mathbb{R}} c_0 y_x u\,dx + \int_{\mathbb{R}}\lambda yu\,dx = 0. \quad (17)$$

Note that

$$\int_{\mathbb{R}}(yu)_x u\,dx + \int_{\mathbb{R}}\frac{1}{2}\left(u^2-\alpha^2u_x^2\right)_x u\,dx = 0, \qquad \int_{\mathbb{R}} c_0 y_x u\,dx = 0. \quad (18)$$

Then, we have

$$\int_{\mathbb{R}} u\left(u_t-\alpha^2u_{xxt}\right)dx + \int_{\mathbb{R}}\lambda\left(u^2-\alpha^2uu_{xx}\right)dx = 0. \quad (19)$$

Hence,

$$\int_{\mathbb{R}} uu_t dx - \alpha^2\int_{\mathbb{R}} uu_{xxt}\,dx + \lambda\int_{\mathbb{R}} u^2dx - \lambda\alpha^2\int_{\mathbb{R}} uu_{xx}\,dx = 0. \quad (20)$$

Thus, we easily get

$$\int_{\mathbb{R}}\left(uu_t+\alpha^2u_xu_{xt}\right)dx + \lambda\int_{\mathbb{R}}\left(u^2+\alpha^2u_x^2\right)dx = 0, \quad (21)$$

and, therefore,

$$\frac{d}{dt}\|u\|_{H_\alpha^1}^2 + 2\lambda\|u\|_{H_\alpha^1}^2 = 0. \quad (22)$$

By integration from 0 to t, we get

$$\|u\|_{H_\alpha^1}^2 = \exp(-2\lambda t)\|u_0\|_{H_\alpha^1}^2, \quad \text{for any } t\in[0,\mathrm{T}). \quad (23)$$

Hence, the lemma is proved. □

We also need to introduce the standard particle trajectory method for later use. Consider now the following initial value problem as follows:

$$q_t = u(t,q)+c_0, \quad t\in[0,T), \qquad q(0,x)=x, \quad x\in\mathbb{R}, \quad (24)$$

where $u\in C^1([0,T),H^{s-1})$ is the solution to (6) with initial data $u_0\in H^s$, $(s>3/2)$ and $T>0$ is the maximal time of existence. By direct computation, we have

$$q_{tx}(t,x) = u_x(t,q(t,x))\,q_x(t,x). \quad (25)$$

Then,

$$q_x(t,x) = \exp\left(\int_0^t u_x(\tau,q(\tau,x))\,d\tau\right) > 0, \quad t>0,\ x\in\mathbb{R}, \quad (26)$$

which means that $q(t,\cdot):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism of the line for every $t\in[0,T)$. Consequently, the L^∞-norm of any function $v(t,\cdot)$ is preserved under the family of the diffeomorphism $q(t,\cdot)$, that is,

$$\|v(t,\cdot)\|_{L^\infty} = \|v(t,q(t,\cdot))\|_{L^\infty}, \quad t\in[0,T). \quad (27)$$

Similarly,

$$\inf_{x\in\mathbb{R}} v(t,x) = \inf_{x\in\mathbb{R}} v(t,q(t,x)), \quad t\in[0,T), \qquad \sup_{x\in\mathbb{R}} v(t,x) = \sup_{x\in\mathbb{R}} v(t,q(t,x)), \quad t\in[0,T). \quad (28)$$

Moreover, one can verify the following important identity for the strong solution in its lifespan:

$$\frac{d}{dt}\left(y(q(x,t),t)\,q_x^2(x,t)\right) = -\lambda y(q(x,t),t)\,q_x^2(x,t). \quad (29)$$

We get that

$$y(q(x,t),t)\,q_x^2(x,t) = y_0(x)\exp(-\lambda t), \quad (30)$$

where $y(x,t)$ is defined by $y(x,t)=(1-\alpha^2\partial_x^2)u(x,t)$, for $t\geq 0$ in its lifespan.

From the expression of $u(x,t)$ in terms of $y(x,t)$, for all $t \in [0,T)$, $x \in \mathbb{R}$, we can rewrite $u(x,t)$ and $u_x(x,t)$ as follows:

$$\begin{aligned} u(x,t) = & \frac{1}{2\alpha} e^{-x/\alpha} \int_{-\infty}^{x} e^{\xi/\alpha} y(\xi,t)\, d\xi \\ & + \frac{1}{2\alpha} e^{x/\alpha} \int_{x}^{\infty} e^{-\xi/\alpha} y(\xi,t)\, d\xi, \end{aligned} \tag{31}$$

from which we get that

$$\begin{aligned} u_x(x,t) = & -\frac{1}{2\alpha^2} e^{-x/\alpha} \int_{-\infty}^{x} e^{\xi/\alpha} y(\xi,t)\, d\xi \\ & + \frac{1}{2\alpha^2} e^{x/\alpha} \int_{x}^{\infty} e^{-\xi/\alpha} y(\xi,t)\, d\xi. \end{aligned} \tag{32}$$

3. Global Existence

It is shown that it is the sign of initial potential not the size of it that can guarantee the global existence of strong solutions.

Theorem 4. *Assume that $u_0 \in H^s$, $s > 3/2$, and $y_0 = u_0 - \alpha^2 u_{0xx}$ satisfies*

$$\begin{aligned} y_0(x) \le 0, \quad x \in (-\infty, x_0), \\ y_0(x) \ge 0, \quad x \in (x_0, \infty), \end{aligned} \tag{33}$$

for some point $x_0 \in \mathbb{R}$. Then, the solution $u(x,t)$ to (6) exists globally in time.

Proof. From the hypothesis and (30), we obtain that $y(x,t) \ge 0$, $q(x_0,t) \le x < \infty$; $y(x,t) \le 0$, $-\infty < x \le q(x_0,t)$. According to (31) and (32), one can get that when $x > x_0$,

$$\begin{aligned} & u(q(x,t),t) + \alpha u_x(q(x,t),t) \\ & \quad = \frac{1}{\alpha} e^{q(x,t)/\alpha} \int_{q(x,t)}^{\infty} e^{-\xi/\alpha} y(\xi,t)\, d\xi \ge 0, \end{aligned} \tag{34}$$

it follows that

$$\begin{aligned} -\alpha u_x(q(x,t),t) & \le u(q(x,t),t) \le \|u\|_{L^\infty} \\ & \le \frac{\exp(-\lambda t)}{\sqrt{2\alpha}} \|u_0\|_{H^1_\alpha} \le \frac{1}{\sqrt{2\alpha}} \|u_0\|_{H^1_\alpha}, \end{aligned} \tag{35}$$

that is, $u_x(x,t)$ is bounded below. Similarly, when $x < x_0$,

$$\begin{aligned} & u(q(x,t),t) - \alpha u_x(q(x,t),t) \\ & \quad = \frac{1}{\alpha} e^{-q(x,t)/\alpha} \int_{-\infty}^{q(x,t)} e^{\xi/\alpha} y(\xi,t)\, d\xi \le 0, \end{aligned} \tag{36}$$

so $-\alpha u_x(q(x,t),t) \le -u(q(x,t),t)$. We also get the bounded below result as above. Therefore, the theorem is proved by Lemma 2. □

Corollary 5. *Assume that $u_0 \in H^s$, $s > 3/2$, and $y_0 = u_0 - \alpha^2 u_{0xx}$ is of one sign, then the corresponding solution $u(x,t)$ to (6) exists globally.*

In fact, if x_0 is regarded as $\pm\infty$, we prove this corollary immediately from Theorem 4.

4. Persistence Properties

In this section, we will investigate the following property for the strong solutions to (6) in L^∞-space which behave algebraically at infinity as their initial profiles do. The main idea comes from the recent work of Himonas and his collaborators [7].

Theorem 6. *Assume that for some $T > 0$ and $s > 3/2$, $u \in C([0,T]; H^s)$ is a strong solution of the initial value problem associated to (6), and that $u_0(x) = u(x,0)$ satisfies*

$$|u_0(x)|, \quad |u_{0x}(x)| \sim O\left(x^{-\theta/\alpha}\right) \quad x \uparrow \infty, \tag{37}$$

for some $\theta \in (0,1)$ and $\alpha \ge 1$. Then,

$$|u(x,t)|, \quad |u_x(x,t)| \sim O\left(x^{-\theta/\alpha}\right) \quad x \uparrow \infty, \tag{38}$$

uniformly in the time interval $[0,T]$.

Proof. The proof is organized as follows. Firstly, we will estimate $\|u(x,t)\|_{L^\infty}$ and $\|u_x(x,t)\|_{L^\infty}$. Then, we apply the weight function to obtain the desired result. In the following proof, we denote some constants by c; they may be different from instance to instance, changing even within the same line.

Multiplying (6) by u^{2n-1} with $n \in \mathbb{Z}^+$, then integrating both sides with respect to x variable, we can get

$$\begin{aligned} & \int_{\mathbb{R}} u^{2n-1} u_t dx + \int_{\mathbb{R}} u^{2n-1} (u + c_0) u_x dx \\ & \quad + \int_{\mathbb{R}} u^{2n-1} \partial_x G * F(u)\, dx = -\lambda \int_{\mathbb{R}} u^{2n} dx. \end{aligned} \tag{39}$$

The first term of the above identity is

$$\begin{aligned} \int_{\mathbb{R}} u^{2n-1} u_t dx & = \frac{1}{2n} \frac{d}{dt} \|u(t)\|_{L^{2n}}^{2n} \\ & = \|u(t)\|_{L^{2n}}^{2n-1} \frac{d}{dt} \|u(t)\|_{L^{2n}}, \end{aligned} \tag{40}$$

and the estimates of the second term is

$$\begin{aligned} \int_{\mathbb{R}} u^{2n-1} u u_x dx & \le \|u_x(t)\|_{L^\infty} \|u(t)\|_{L^{2n}}^{2n}, \\ c_0 \int_{\mathbb{R}} u^{2n-1} u_x dx & = c_0 \int_{\mathbb{R}} \left(\frac{u^{2n}}{2n}\right)_x dx = 0. \end{aligned} \tag{41}$$

In view of Hölder's inequality, we can obtain the following estimate for the third term in (39)

$$\left| \int_{\mathbb{R}} u^{2n-1} \partial_x G * F(u)\, dx \right| \le \|u(t)\|_{L^{2n}}^{2n-1} \left\|\partial_x G * F(u)\right\|_{L^{2n}}. \tag{42}$$

For the last term

$$\left| \int_{\mathbb{R}} u^{2n-1} \lambda u\, dx \right| \le \lambda \|u(t)\|_{L^{2n}}^{2n}, \tag{43}$$

putting all the inequalities above into (39) yields

$$\frac{d}{dt}\|u(t)\|_{L^{2n}} \leq \left(\|u_x(t)\|_{L^{2n}} + \lambda\right)\|u(t)\|_{L^{2n}} + \left\|\partial_x G * F(u)\right\|_{L^{2n}}. \tag{44}$$

Using the Sobolev embedding theorem, there exists a constant

$$M = \sup_{t\in[0,T]} \|u(x,t)\|_{H^s}, \tag{45}$$

such that we have by applying Gronwall's inequality

$$\|u(t)\|_{L^{2n}} \leq ce^{Mt}\left(\|u(0)\|_{L^{2n}} + \int_0^t \left\|\partial_x G * F(u)\right\|_{L^{2n}} d\tau\right). \tag{46}$$

For any $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we know that

$$\lim_{q\uparrow\infty}\|f\|_{L^q} = \|f\|_{L^\infty}. \tag{47}$$

Taking the limits in (46) (notice that $G \in L^1$ and $F(u) \in L^1 \cap L^\infty$) from (47), we get

$$\|u(t)\|_{L^\infty} \leq ce^{Mt}\left(\|u(0)\|_{L^\infty} + \int_0^t \left\|\partial_x G * F(u)\right\|_{L^\infty} d\tau\right). \tag{48}$$

Then, differentiating (6) with respect to variable x produces the following equation:

$$u_{xt} + uu_{xx} + c_0 u_{xx} + u_x^2 + \partial_x^2 G * F(u) + \lambda u_x = 0. \tag{49}$$

Again, multiplying (49) by u_x^{2n-1} with $n \in \mathbb{Z}^+$, integrating the result in x variable, and considering the second term and the third term in the above identity with integration by parts, one gets

$$\begin{aligned}\int_{\mathbb{R}} uu_{xx}u_x^{2n-1}dx &= \int_{\mathbb{R}} u\left(\frac{u_x^{2n-1}}{2n}\right)_x dx \\ &= -\frac{1}{2n}\int_{\mathbb{R}} u_x u_x^{2n} dx,\end{aligned} \tag{50}$$

$$c_0\int_{\mathbb{R}} u_{xx}u_x^{2n-1}dx = c_0\int_{\mathbb{R}}\left(\frac{u_x^{2n-1}}{2n}\right)_x dx = 0,$$

so, we have

$$\begin{aligned}&\int_{\mathbb{R}} u_{xt}u_x^{2n-1}dx - \frac{1}{2n}\int_{\mathbb{R}} u_x u_x^{2n}dx + \int_{\mathbb{R}} u_x^{2n+1}dx \\ &\quad = -\int_{\mathbb{R}} u_x^{2n-1}\partial_x^2 G * F(u)\,dx - \lambda\int_{\mathbb{R}} u_x^{2n-1}u_x dx.\end{aligned} \tag{51}$$

Similarly, the following inequality holds

$$\frac{d}{dt}\|u_x(t)\|_{L^{2n}} \leq \left(2\|u_x(t)\|_{L^\infty} + \lambda\right)\|u_x(t)\|_{L^{2n}} + \left\|\partial_x^2 G * F(u)(t)\right\|_{L^{2n}}, \tag{52}$$

and therefore as before, we obtain

$$\|u_x(t)\|_{L^{2n}} \leq ce^{2Mt}\left(\|u_x(0)\|_{L^{2n}} + \int_0^t \left\|\partial_x^2 G * F(u)\right\|_{L^{2n}} d\tau\right). \tag{53}$$

Taking the limits in (53), we obtain

$$\|u_x(t)\|_{L^\infty} \leq ce^{2Mt}\left(\|u_x(0)\|_{L^\infty} + \int_0^t \left\|\partial_x^2 G * F(u)\right\|_{L^\infty} d\tau\right). \tag{54}$$

Next, we will introduce the weight function to get our desired result. This function $\varphi_N(x)$ with $N \in \mathbb{Z}^+$ is independent of t as the following:

$$\varphi_N(x) = \begin{cases} 1, & x \leq 1, \\ x^{\theta/\alpha}, & x \in (1, N), \\ N^{\theta/\alpha}, & x \geq N. \end{cases} \tag{55}$$

From (6) and (49), we get the following two equations:

$$\begin{gathered}\varphi_N u_t + \varphi_N uu_x + \varphi_N c_0 u_x + \varphi_N \partial_x G * F(u) + \lambda\varphi_N u = 0, \\ \varphi_N u_{xt} + \varphi_N uu_{xx} + \varphi_N c_0 u_{xx} + \varphi_N u_x^2 \\ + \varphi_N \partial_x^2 G * F(u) + \lambda\varphi_N u_x = 0.\end{gathered} \tag{56}$$

We need some tricks to deal with the following term as in [18]:

$$\begin{aligned}&\int_{\mathbb{R}} (\varphi_N)^{2n-1}u^{2n-1}\varphi_N u_x dx \\ &\quad = \int_{\mathbb{R}} (\varphi_N u)^{2n-1}\left[(u\varphi_N)_x - u(\varphi_N)_x\right]dx \\ &\quad = \int_{\mathbb{R}} (\varphi_N u)^{2n-1} d(\varphi_N u) - \int_{\mathbb{R}} (\varphi_N u)^{2n-1} u(\varphi_N)_x dx \\ &\quad \leq \int_{\mathbb{R}} (\varphi_N u)^{2n} dx,\end{aligned} \tag{57}$$

where we have used the fact $0 \leq \varphi_N'(x) \leq \varphi_N(x)$, a.e. $x \in \mathbb{R}$. Similar technique is used for the term $\int_{\mathbb{R}}(\varphi_N)^{2n-1}u_x^{2n-1}\varphi_N u_{xx}dx$. Hence, as in the weightless case, we get the following inequality in view of (48) and (54) as follows:

$$\begin{aligned}&\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} \\ &\quad \leq ce^{2Mt}\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right) + ce^{2Mt} \\ &\qquad \times\left(\int_0^t \left(\|\varphi_N\partial_x G * F(u)\|_{L^\infty} + \left\|\varphi_N\partial_x^2 G * F(u)\right\|_{L^\infty}\right)d\tau\right).\end{aligned} \tag{58}$$

On the other hand, a simple calculation shows that there exists $C > 0$, depending only on α and θ such that for any $N \in \mathbb{Z}^+$,

$$\varphi_N(x)\int_{\mathbb{R}} e^{-|x-y|/\alpha}\frac{1}{\varphi_N(y)}dy \leq C. \tag{59}$$

Therefore, for any appropriate function g, one obtains that

$$\begin{aligned}
&\left|\varphi_N \partial_x G * g^2(x)\right| \\
&\quad = \left|\frac{1}{2\alpha}\varphi_N(x)\int_{\mathbb{R}} e^{-|x-y|/\alpha} g^2(y)\,dy\right| \\
&\quad \leq \frac{1}{2\alpha}\varphi_N(x)\int_{\mathbb{R}} e^{-|x-y|/\alpha}\frac{1}{\varphi_N(y)}\varphi_N(y)\,g(y)\,g(y)\,dy \\
&\quad \leq \frac{1}{2\alpha}\left(\varphi_N(x)\int_{\mathbb{R}} e^{-|x-y|/\alpha}\frac{1}{\varphi_N(y)}dy\right)\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty} \\
&\quad \leq \frac{C}{\alpha}\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty},
\end{aligned} \tag{60}$$

and similarly, $|\varphi_N \partial_x^2 G * g^2(x)| \leq (C/\alpha)\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty}$. Using the same method, we can estimate the following two terms

$$\begin{aligned}
\left|\varphi_N G * g(x)\right| &\leq \frac{C}{\alpha}\|g\varphi_N\|_{L^\infty}, \\
\left|\varphi_N \partial_x G * g(x)\right| &\leq \frac{C}{\alpha}\|g\varphi_N\|_{L^\infty}.
\end{aligned} \tag{61}$$

Therefore, it follows that there exists a constant $C_1(M, T, \alpha, \lambda) > 0$ such that

$$\begin{aligned}
&\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} \\
&\quad \leq C_1\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right) \\
&\qquad + C_1\int_0^t \left(\left(\|u(\tau)\|_{L^\infty} + \|u_x(\tau)\|_{L^\infty}\right)\right. \\
&\qquad\qquad \left.\cdot\left(\|\varphi_N u(\tau)\|_{L^\infty} + \|\varphi_N u_x(\tau)\|_{L^\infty}\right)\right) d\tau \\
&\quad \leq C_1\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right. \\
&\qquad \left. + \int_0^t \left(\|\varphi_N u(\tau)\|_{L^\infty} + \|\varphi_N u_x(\tau)\|_{L^\infty}\right) d\tau\right).
\end{aligned} \tag{62}$$

Hence, the following inequality is obtained for any $N \in \mathbb{Z}^+$ and any $t \in [0, T]$:

$$\begin{aligned}
&\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} \\
&\quad \leq C_1\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right) \\
&\quad \leq C_1\left(\left\|u(0)\max\left(1, x^{\theta/\alpha}\right)\right\|_{L^\infty}\right. \\
&\qquad \left. + \left\|u_x(0)\max\left(1, x^{\theta/\alpha}\right)\right\|_{L^\infty}\right).
\end{aligned} \tag{63}$$

Finally, taking the limit as N goes to infinity in the above inequality, we can find that for any $t \in [0, T]$,

$$\begin{aligned}
&\left(\left|u(x,t)\,x^{\theta/\alpha}\right| + \left|u_x(x,t)\,x^{\theta/\alpha}\right|\right) \\
&\quad \leq C_1\left(\left\|u(0)\max\left(1, x^{\theta/\alpha}\right)\right\|_{L^\infty}\right. \\
&\qquad \left. + \left\|u_x(0)\max\left(1, x^{\theta/\alpha}\right)\right\|_{L^\infty}\right),
\end{aligned} \tag{64}$$

which completes the proof of Theorem 6. □

5. Asymptotic Description

The following result is to give a detailed description on the corresponding strong solution $u(x, t)$ in its lifespan with $u_0(x)$ being compactly supported.

Theorem 7. *Assume that the initial datum $0 \not\equiv u_0(x) \in H^s$ with $s > 5/2$ is compactly supported in $[a, c]$, then the corresponding solution $u(x, t) \in C([0, T); H^s)$ to (6) has the following property: for any $t \in (0, T)$,*

$$\begin{aligned}
u(x,t) &= L(t)\,e^{-x/\alpha} \quad \text{as } x > q(c,t), \\
u(x,t) &= l(t)\,e^{x/\alpha} \quad \text{as } x < q(a,t),
\end{aligned} \tag{65}$$

where $q(x, t)$ is defined by (24) and T is its lifespan. Furthermore, $L(t)$ and $l(t)$ denote continuous nonvanishing functions, with $L(t) > 0$ and $l(t) < 0$ for $t \in (0, T)$. Moreover, $L(t)$ is a strictly increasing function, while $l(t)$ is strictly decreasing.

Remark 8. This is an interesting phenomenon for our model; it implies that the strong solution does not have compact x-support for any $t > 0$ in its lifespan anymore, although the corresponding $u_0(x)$ is compactly supported. No matter that the initial profile $u_0(x)$ is (no matter it is positive or negative), for any $t > 0$ in its lifespan, the nontrivial solution $u(x; t)$ is always positive at infinity and negative at negative infinity. Moreover, we found that the dissipative coefficient does not affect this behavior.

Proof. First, since $u_0(x)$ has a compact support, so does $y_0(x) = (1 - \alpha^2\partial_x^2)u_0(x)$. Equation (30) tells us that $y = (1 - \alpha^2\partial_x^2)u(x, t) = ((1 - \alpha^2\partial_x^2)u_0(q^{-1}(x, t))\exp(-\lambda t))/(\partial_x q^{-1}((x, t), t))^2$ is compactly supported in $[q(a, t), q(c, t)]$ in its lifespan. Hence, the following functions are well defined

$$E(t) = \int_{\mathbb{R}} e^{\xi/\alpha} y(\xi, t)\,d\xi, \qquad F(t) = \int_{\mathbb{R}} e^{-\xi/\alpha} y(\xi, t)\,d\xi, \tag{66}$$

with

$$\begin{aligned}
E(0) &= \int_{\mathbb{R}} e^{\xi/\alpha} y_0(\xi)\,d\xi \\
&= \int_{\mathbb{R}} e^{\xi/\alpha} u_0(\xi)\,d\xi - \alpha^2 \int_{\mathbb{R}} e^{\xi/\alpha} u_{0xx}(\xi)\,d\xi = 0.
\end{aligned} \tag{67}$$

And $F(0) = 0$ by integration by parts.

Then, for $x > q(c, t)$, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\alpha} e^{-|x|/\alpha} * y(x,t) \\
&= \frac{1}{2\alpha} e^{-x/\alpha} \int_{q(a,t)}^{q(c,t)} e^{\xi/\alpha} y(\xi, t)\,d\xi = \frac{1}{2\alpha} e^{-x/\alpha} E(t),
\end{aligned} \tag{68}$$

where (66) is used.

Similarly, when $x < q(a,t)$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2\alpha} e^{-|x|/\alpha} * y(x,t) \\ &= \frac{1}{2\alpha} e^{x/\alpha} \int_{q(a,t)}^{q(c,t)} e^{-\xi/\alpha} y(\xi,t)\, d\xi = \frac{1}{2\alpha} e^{x/\alpha} F(t). \end{aligned} \tag{69}$$

Because $y(x,t)$ has a compact support in x in the interval $[q(a,t), q(c,t)]$ for any $t \in [0,T]$, we get $y(x,t) = u(x,t) - \alpha^2 u_{xx}(x,t) = 0$, for $x > q(c,t)$ or $x < q(a,t)$. Hence, as consequences of (68) and (69), we have

$$\begin{aligned} u(x,t) &= -\alpha u_x(x,t) = \alpha^2 u_{xx}(x,t) \\ &= \frac{1}{2\alpha} e^{-x/\alpha} E(t), \quad \text{as } x > q(c,t), \\ u(x,t) &= \alpha u_x(x,t) = \alpha^2 u_{xx}(x,t) \\ &= \frac{1}{2\alpha} e^{x/\alpha} F(t), \quad \text{as } x < q(a,t). \end{aligned} \tag{70}$$

On the other hand,

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^{\xi/\alpha} y_t(\xi,t)\, dx. \tag{71}$$

Substituting the identity (10) into $dE(t)/dt$, we obtain

$$\begin{aligned} \frac{dE(t)}{dt} &= -\int_{\mathbb{R}} e^{\xi/\alpha} \left[(yu)_x + \frac{1}{2}\left(u^2 - \alpha^2 u_x^2\right)_x + c_0 y_x + \lambda y \right] d\xi \\ &= \frac{1}{\alpha} \int_{\mathbb{R}} e^{\xi/\alpha} yu\, d\xi + \frac{1}{2\alpha} \int_{\mathbb{R}} e^{\xi/\alpha} \left(u^2 - \alpha^2 u_x^2\right) d\xi \\ &\quad + \frac{c_0}{\alpha} \int_{\mathbb{R}} e^{\xi/\alpha} y\, d\xi + \alpha^2 \int_{\mathbb{R}} e^{\xi/\alpha} \lambda u_{xx}\, d\xi \\ &\quad - \int_{\mathbb{R}} e^{\xi/\alpha} \lambda u\, d\xi = \frac{3}{2\alpha} \int_{\mathbb{R}} e^{\xi/\alpha} u^2 d\xi + \frac{\alpha}{2} \int_{\mathbb{R}} e^{\xi/\alpha} u_x^2\, d\xi \\ &\quad + \int_{\mathbb{R}} e^{\xi/\alpha} u u_x\, d\xi = \int_{\mathbb{R}} e^{\xi/\alpha} \left(\frac{1}{\alpha} u^2 + \frac{\alpha}{2} u_x^2 \right) d\xi > 0, \end{aligned} \tag{72}$$

where we used (70). Therefore, in the lifespan of the solution, we have that $E(t)$ is an increasing function with $E(0) = 0$; thus, it follows that $E(t) > 0$ for $t \in (0,T]$; that is,

$$E(t) = \int_0^t \int_{\mathbb{R}} e^{\xi/\alpha} \left(\frac{1}{\alpha} u^2 + \frac{\alpha}{2} u_x^2 \right)(\xi,\tau)\, d\xi\, d\tau > 0. \tag{73}$$

By similar argument, one can verify that the following identity for $F(t)$ is true:

$$F(t) = -\int_0^t \int_{\mathbb{R}} e^{-\xi/\alpha} \left(\frac{1}{\alpha} u^2 + \frac{\alpha}{2} u_x^2 \right)(\xi,\tau)\, d\xi\, d\tau < 0. \tag{74}$$

In order to finish the proof, it is sufficient to let $L(t) = (1/2\alpha)E(t)$, and to let $l(t) = (1/2\alpha)F(t)$, respectively. □

References

[1] H. R. Dullin, G. A. Gottwald, and D. D. Holm, "An integrable shallow water equation with linear and nonlinear dispersion," *Physical Review Letters*, vol. 87, no. 19, Article ID 194501, 4 pages, 2001.

[2] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II. The KdV equation," *Geometric and Functional Analysis*, vol. 3, no. 3, pp. 209–262, 1993.

[3] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[4] A. Constantin, "Finite propagation speed for the Camassa-Holm equation," *Journal of Mathematical Physics*, vol. 46, no. 2, Article ID 023506, 4 pages, 2005.

[5] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.

[6] A. Constantin and J. Escher, "Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation," *Communications on Pure and Applied Mathematics*, vol. 51, no. 5, pp. 475–504, 1998.

[7] A. A. Himonas, G. Misiołek, G. Ponce, and Y. Zhou, "Persistence properties and unique continuation of solutions of the Camassa-Holm equation," *Communications in Mathematical Physics*, vol. 271, no. 2, pp. 511–522, 2007.

[8] H. P. McKean, "Breakdown of the Camassa-Holm equation," *Communications on Pure and Applied Mathematics*, vol. 57, no. 3, pp. 416–418, 2004.

[9] Z. Xin and P. Zhang, "On the weak solutions to a shallow water equation," *Communications on Pure and Applied Mathematics*, vol. 53, no. 11, pp. 1411–1433, 2000.

[10] Y. Zhou, "Wave breaking for a shallow water equation," *Nonlinear Analysis: Theory, Methods & Application*, vol. 57, no. 1, pp. 137–152, 2004.

[11] Y. Zhou, "Wave breaking for a periodic shallow water equation," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 2, pp. 591–604, 2004.

[12] Y. Zhou, "Stability of solitary waves for a rod equation," *Chaos, Solitons and Fractals*, vol. 21, no. 4, pp. 977–981, 2004.

[13] Y. Zhou, "Local well-posedness and blow-up criteria of solutions for a rod equation," *Mathematische Nachrichten*, vol. 278, no. 14, pp. 1726–1739, 2005.

[14] L. Ni and Y. Zhou, "Wave breaking and propagation speed for a class of nonlocal dispersive θ-equations," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 592–600, 2011.

[15] L. Tian, G. Gui, and Y. Liu, "On the well-posedness problem and the scattering problem for the Dullin-Gottwald-Holm equation," *Communications in Mathematical Physics*, vol. 257, no. 3, pp. 667–701, 2005.

[16] T. Kato, "Quasi-linear equations of evolution, with applications to partial differential equations," in *Spectral Theory and Differential Equations: Proceedings of the Symposium Held at Dundee, Scotland, 1–19 July 1974*, vol. 448 of *Lecture Notes in Mathetics*, pp. 25–70, Springer, Berlin, Germany, 1975.

[17] Y. Zhou, "Blow-up of solutions to the DGH equation," *Journal of Functional Analysis*, vol. 250, no. 1, pp. 227–248, 2007.

[18] Y. Zhou and Z. Guo, "Blow up and propagation speed of solutions to the DGH equation," *Discrete and Continuous Dynamical Systems B*, vol. 12, no. 3, pp. 657–670, 2009.

[19] J.-M. Ghidaglia, "Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time," *Journal of Differential Equations*, vol. 74, no. 2, pp. 369–390, 1988.

[20] Z. Guo, "Blow up, global existence, and infinite propagation speed for the weakly dissipative Camassa-Holm equation," *Journal of Mathematical Physics*, vol. 49, no. 3, Article ID 033516, 9 pages, 2008.

[21] Z. Guo and L. Ni, "Wave breaking for the periodic weakly dissipative Dullin-Gottwald-Holm equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 3, pp. 965–973, 2011.

[22] S. Wu and Z. Yin, "Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation," *Journal of Differential Equations*, vol. 246, no. 11, pp. 4309–4321, 2009.

[23] A. V. Busuioc and T. S. Ratiu, "The second grade fluid and averaged Euler equations with Navier-slip boundary conditions," *Nonlinearity*, vol. 16, no. 3, pp. 1119–1149, 2003.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 454209, 8 pages
http://dx.doi.org/10.1155/2013/454209

Research Article

Pattern Formation in a Diffusive Ratio-Dependent Holling-Tanner Predator-Prey Model with Smith Growth

Bo Yang

School of Electronic and Information Engineering, Gansu Lianhe University, Lanzhou 730000, China

Correspondence should be addressed to Bo Yang; yangbo_20130202@163.com

Received 3 February 2013; Accepted 4 March 2013

Academic Editor: Yonghui Xia

The spatiotemporal dynamics of a diffusive ratio-dependent Holling-Tanner predator-prey model with Smith growth subject to zero-flux boundary condition are investigated analytically and numerically. The asymptotic stability of the positive equilibrium and the existence of Hopf bifurcation around the positive equilibrium are shown; the conditions of Turing instability are obtained. And with the help of numerical simulations, it is found that the model exhibits complex pattern replication: stripes, spots-stripes mixtures, and spots Turing patterns.

1. Introduction

The problem of pattern formation is, perhaps, the most challenging in modern ecology, biology, chemistry, and many other fields of science [1]. Patterns generated in abiotically homogeneous environments are particularly interesting because they require an explanation based on the individual behavior of organisms. They are commonly called “emergent patterns,” because they emerge from interactions in spatial scales that are much larger than the characteristic scale of individuals [2].

Turing [3] showed how the coupling of reaction and diffusion can induce instability and pattern formation. Turing’s revolutionary idea was that the passive diffusion could interact with chemical reaction in such a way that even if the reaction by itself has no symmetry-breaking capabilities, diffusion can destabilize the symmetry so that the system with diffusion can have them. Segel and Jackson [4] first used reaction-diffusion system to explain pattern formation in ecological context based upon the seminal work by Turing [3]. Since then, a lot of studies have been devoted to spatiotemporal patterns which were produced by reaction-diffusion predator-prey, models with either a prey-dependent or a ratio-dependent predator functional response, for example, [1, 2, 5–20] and references cited therein.

Recently, there is a growing explicit biological and physiological evidence [21–23] that in many situations, especially, when the predator has to search for food (and therefore has to share or compete for food), a more suitable general predator-prey theory should be based on the so-called ratio-dependent function which can be roughly stated as that the per capital predator growth rate should be a function of the ratio of prey to predator abundance, and so would be the so-called predator functional responses [24]. This is supported by numerous fields and laboratory experiments and observations [25, 26]. In [24], the authors investigated the effect of time delays on the stability of the model and discussed the local asymptotic stability and the Hopf bifurcation. Liang and Pan [27] have studied the local and global asymptotic stability of the coexisting equilibrium point and obtained the conditions for Poincare-Andronov-Hopf-bifurcating periodic solution. M. Banerjee and S. Banerjee [28] have studied the local asymptotic stability of the equilibrium point and obtained the conditions for the occurrence of Turing-Hopf instability for reaction-diffusion model. It is shown that prey and predator populations exhibit spatiotemporal patterns resulting from temporal oscillation of both the population and spatial instability.

Besides, in [29], Smith has shown that the logistic equation is not realistic for a food-limited population under the effects of environmental toxicants and established a new growth function—Smith growth function. And it has been proposed by several authors [29–34] for the dynamics of a population where the growth limitations are based on the proportion of available resources not utilized. However, pattern formation in the case of Holling-Tanner type predator-prey models with ratio-dependent functional response and Smith growth still remains an interesting area of research.

In this present work, we will focus on the ratio-dependent Holling-Tanner model with Smith growth for predator-prey interaction where random movement of both species is taken into account. The rest of the paper is organized as follows. In Section 2, we establish the ratio-dependent Holling-Tanner predator-prey model with Smith growth and study the local asymptotic stability of the positive equilibrium, existence of Hopf bifurcation around the positive equilibrium, and the conditions for the occurrence of Turing instability. In Section 3, we present and discuss the results of pattern formation via numerical simulation, which is followed by the last section, that is, conclusions and discussions.

2. The Model and the Linear Stability Analysis

2.1. The Model. In this paper, we rigorously consider the radio-dependent Holling-Tanner predator-prey model with Smith growth taking the form:

$$\begin{aligned} \frac{du}{dt} &= \frac{ru(K-u)}{K+cu} - \frac{muv}{u+av}, \\ \frac{dv}{dt} &= sv\left(1-\frac{hv}{u}\right), \end{aligned} \tag{1}$$

where $u(t)$ and $v(t)$ stand for prey and predator population (density) at any instant of time t. r, K, m, a, s, h are positive constants that stand for prey intrinsic growth rate, carrying capacity, capturing rate, half capturing saturation constant, predator intrinsic growth rate, conversion rate of prey into predators biomass, respectively. And r/c is the replacement of mass in the population at K. The model with Smith growth takes into account both environmental and food chain effects of toxicant stress.

From the standpoint of biology, we are interested only in the dynamics of model (1) in the closed first quadrant $\mathbb{R}_+^2 = \{(u,v) : u \geq 0, v \geq 0\}$. Thus, we consider only the biologically meaningful initial conditions

$$u(0) > 0, \qquad v(0) > 0, \tag{2}$$

which are continuous functions due to their biological sense.

Straightforward computation shows that model (1) is continuous and Lipschitzian in $\mathbb{R}_+^2$ if we redefine

$$\frac{du}{dt} = \frac{dv}{dt} = 0, \quad \text{if } (u,v) = (0,0). \tag{3}$$

Hence, the solution of model (1) with positive initial conditions exists and is unique.

Also considering the spatial dispersal and environmental heterogeneity, in this paper we study the diffusive Holling-Tanner model obtained from the temporal model (1) by incorporating diffusion terms as follows:

$$\begin{aligned} \frac{\partial u}{dt} &= d_1 \Delta u + ru\frac{K-u}{K+cu} - \frac{muv}{u+av}, \\ \frac{\partial v}{dt} &= d_2 \Delta v + sv\left(1-\frac{hv}{u}\right), \end{aligned} \tag{4}$$

where the nonnegative constants d_1 and d_2 are the diffusion coefficients of u and v, respectively. $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, the usual Laplacian operator in two-dimensional space, is used to describe the Brownian random motion.

Model (4) is to be analyzed under the following nonzero initial conditions:

$$u(x,y,0) > 0, \quad v(x,y,0) > 0, \quad (x,y) \in \Omega, \tag{5}$$

and zero-flux boundary conditions:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad (x,y) \in \partial\Omega, \; t > 0, \tag{6}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$ and ν is the outward unit normal vector on $\partial\Omega$. The zero-flux boundary condition indicates that predator-prey system is self-contained with zero population flux across the boundary.

2.2. The Stability of the Nonspatial Model (1). In this subsection, we restrict ourselves to the stability analysis of the nonspatial model (1). It is easy to verify that model (1) has a trivial equilibrium point $E_0 = (K, 0)$. Simple computation shows that if $m < r(a+h)$, model (1) possess a unique positive equilibrium, denoted by $E^* = (u^*, v^*)$, where

$$u^* = \frac{K(ar+hr-m)}{ar+cm+hr}, \qquad v^* = \frac{1}{h}u^*. \tag{7}$$

The Jacobian matrix at $E_0 = (K, 0)$ is

$$J_0 = \begin{pmatrix} -\dfrac{r}{1+c} & -m \\ 0 & s \end{pmatrix}. \tag{8}$$

Clearly, $E_0 = (K, 0)$ is a saddle point.

In the following, we will discuss the stability of the positive equilibrium E^* of model (1). The Jacobian matrix at E^* is given by

$$J^* = \begin{pmatrix} a_1 & a_2 \\ \dfrac{s}{h} & -s \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} a_1 &= \frac{m(ar+cm+2hr-acr) - r^2(a+h)^2}{r(1+c)(a+h)^2}, \\ a_2 &= -\frac{mh^2}{(a+h)^2}. \end{aligned} \tag{10}$$

Then we can get

$$\det(J^*) = s\left(\frac{a_2}{h} - a_1\right) = \frac{s(ar + hr - m)(ar + hr + cm)}{r(1+c)(a+h)^2} > 0,$$
$$\operatorname{tr}(J^*) = a_1 - s = \frac{m(ar + cm + 2hr - acr) - r(a+h)^2(r + s(1+c))}{r(1+c)(a+h)^2}. \quad (11)$$

Theorem 1. (i) *The positive equilibrium $E^* = (u^*, v^*)$ is locally asymptotically stable if and only if*

$$m(ar + cm + 2hr - acr) < r(a+h)^2(r + s(1+c)). \quad (12)$$

(ii) *The positive equilibrium $E^* = (u^*, v^*)$ is unstable if and only if*

$$m(ar + cm + 2hr - acr) > r(a+h)^2(r + s(1+c)). \quad (13)$$

(iii) *The model enters into a Hopf-bifurcation around $E^* = (u^*, v^*)$ at $s = s^*$, where s^* satisfies the equality*

$$s^* = \frac{m(ar + cm + 2hr - acr) - r^2(a+h)^2}{r(1+c)(a+h)^2}. \quad (14)$$

Proof. (i) If $m(ar + cm + 2hr - acr) < r(a+h)^2(r + s(1+c))$, then $\operatorname{tr}(J^*) < 0$. Thus, the equilibrium point E^* is locally asymptotically stable, similar to the proof of (ii).

(iii) A Hopf bifurcation occurs if and only if there exists a $s = s^*$ such that

$$\operatorname{tr}(J^*) = a_1 - s^* = 0, \qquad \frac{d}{ds}\operatorname{Re}(\lambda(s))_{s=s^*} \neq 0, \quad (15)$$

where λ is a root of the characteristic equation of J^*:

$$\lambda^2 - \operatorname{tr}(J^*)\lambda + \det(J^*) = 0. \quad (16)$$

The condition $s^* = a_1$ gives $\operatorname{tr}(J^*) = 0$. Thus for $s = s^*$, both eigenvalues will be purely imaginary and there are no other eigenvalues with negative real part. Now we verify the transversality condition $(d/ds)\operatorname{Re}(\lambda(s))|_{s=s^*} \neq 0$.

Substituting $\lambda = \alpha + i\beta$ into the equation $\lambda^2 - \operatorname{tr}(J^*)\lambda + \det(J^*) = 0$ and separating real and imaginary parts we obtain

$$\alpha^2 - \beta^2 - \alpha\operatorname{tr}(J^*) + \det(J^*) = 0,$$
$$2\alpha\beta - \beta\operatorname{tr}(J^*) = 0. \quad (17)$$

Differentiating (17) both sides with respect to s, we get

$$\varphi\frac{d\alpha}{ds} - 2\beta\frac{d\beta}{ds} = \gamma,$$
$$2\beta\frac{d\alpha}{ds} + \varphi\frac{d\beta}{ds} = \phi, \quad (18)$$

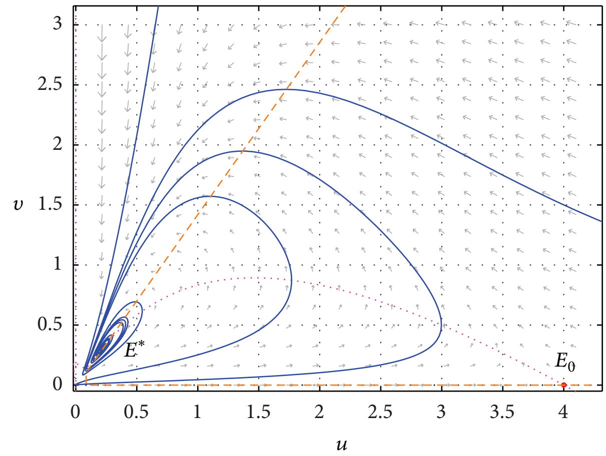


Figure 1: Phase portraits of model (1). The parameters are taken as $r = 1$, $m = 1$, $a = 0.4$, $K = 4$, $h = 0.7$, $c = 0.8$, $s = 0.55$. $E_0 = (K, 0)$ is a saddle point. $E^* = (0.21053, 0.30075)$ is locally asymptotically stable. The dashed curve is the u-nullcline, and the dotted vertical line is the v-nullcline.

where $\varphi = 2\alpha - \operatorname{tr}(J^*)$, $\phi = \alpha(d(\operatorname{tr}(J^*))/ds) - (d(\det(J^*))/ds)$, $\gamma = \beta(d(\det(J^*))/ds)$. Thus, we obtain

$$\left.\frac{d}{ds}\operatorname{Re}(\lambda(s))\right|_{s=s^*} = \left.\frac{\gamma\varphi + 2\beta\phi}{\varphi^2 + 4\beta^2}\right|_{s=s^*} \neq 0, \quad (19)$$

which verify the transversality condition. Hence, the system undergoes a Hopf bifurcation at E^* as s passes through the value s^*. This ends the proof. □

In Figure 1, we show the phase portraits of (1) with $r = 1$, $m = 1$, $a = 0.4$, $K = 4$, $h = 0.7$, $c = 0.8$, and $s = 0.55$. The horizontal axis is the prey population u, and the vertical axis is the predator population v. The dashed curve is the u-nullcline, and the dotted vertical line is the v-nullcline. It is easy to see that the equilibrium $E_0 = (4, 0)$ is a saddle and $E^* = (0.21053, 0.30075)$ is locally asymptotically stable.

Figure 2 illustrates a Hopf-bifurcation situation of the model around $E^* = (0.21053, 0.30075)$ for $s = s^* = 0.4912764003$. In this case, limit cycle arising through Hopf bifurcation is a stable limit cycle which attracts all trajectories starting from a point in the interior of first quadrant.

2.3. The Stability of the Spatial Model (4). In this subsection, we will focus on the effect of diffusion on the model system about the positive equilibrium.

Now, we study the nonlinear evolution of a perturbation

$$U(x,t) = u(x,t) - u^*, \qquad V(x,t) = v(x,t) - v^* \quad (20)$$

around $E^* = (u^*, v^*)$. The corresponding linearized model (4) then takes the form

$$\frac{\partial U}{dt} = d_1\Delta U + a_1 U + a_2 V,$$
$$\frac{\partial V}{dt} = d_2\Delta V + \frac{s}{h}U - sV, \quad (21)$$

where a_1, a_2 are defined the same as (10).

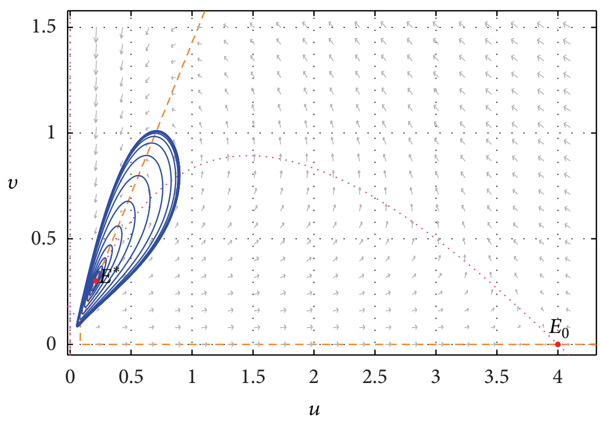


FIGURE 2: Phase portraits of model (1). The parameters are taken as $r = 1$, $m = 1$, $a = 0.4$, $K = 4$, $h = 0.7$, $c = 0.8$, $s = 0.4912764003$. Model (1) enters into Hopf bifurcation around $E^* = (0.21053, 0.30075)$, and there is a limit cycle. The dashed curve is the u-nullcline, and the dotted vertical line is the v-nullcline.

We use $[\cdot,\cdot]$ to denote a column vector, and let

$$\mathbf{w}(x,t) = [U(x,t), V(x,t)]. \quad (22)$$

Let $\mathbf{q} = (q_1, q_2) \in \Omega$ and

$$e_{\mathbf{q}}(x) = \prod_{i=1}^{2} \cos(q_i x_i). \quad (23)$$

Then $\{e_{\mathbf{q}}(x)\}_{\mathbf{q}\in\Omega}$ forms a basis of the space of functions in $\mathbb{R}^2$ that satisfy zero-flux boundary conditions. We look for a normal mode corresponding to model (21) as following form:

$$\mathbf{w}(x,t) = \mathbf{r}_{\mathbf{q}} \exp(\lambda_{\mathbf{q}} t)\, e_{\mathbf{q}}(x), \quad (24)$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on $\mathbf{q}$. Plugging (24) into model (21) yields

$$\lambda_{\mathbf{q}} \mathbf{r}_{\mathbf{q}} = \begin{pmatrix} -d_1 q^2 + a_1 & a_2 \\ \dfrac{s}{h} & -d_2 q^2 - s \end{pmatrix} \mathbf{r}_{\mathbf{q}}, \quad (25)$$

where $q^2 = q_1^2 + q_2^2$. A nontrivial normal mode can be obtained by setting

$$\det \begin{pmatrix} \lambda_{\mathbf{q}} + d_1 q^2 - a_1 & -a_2 \\ -\dfrac{s}{h} & \lambda_{\mathbf{q}} + d_2 q^2 + s \end{pmatrix} = 0. \quad (26)$$

This leads to the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$\lambda_{\mathbf{q}}^2 + \rho_1 \lambda_{\mathbf{q}} + \rho_2 = 0, \quad (27)$$

where

$$\begin{aligned} \rho_1 &= (d_1 + d_2) q^2 - a_1 + s, \\ \rho_2 &= d_1 d_2 q^4 + (d_1 s - d_2 a_1) q^2 + s\left(\frac{a_2}{h} - a_1\right). \end{aligned} \quad (28)$$

Mathematically speaking, a positive equilibrium E^* of model (4) is Turing unstable, which means that it is an asymptotically stable steady-state solution of the model (1) without diffusion but is unstable with respect to the solutions of the model (4) with diffusion.

Therefore, the Turing instability sets in when at least one of the following conditions is violated:

$$\rho_1 < 0, \qquad \rho_2 > 0. \quad (29)$$

But it is evident that $\rho_1 < 0$ is not violated if $s - a_1 < 0$. Hence only the violation of condition $\rho_2 > 0$ gives rise to diffusion instability. As a consequence, a necessary condition is

$$d_1 s < d_2 a_1. \quad (30)$$

Otherwise $\rho_2 > 0$ for all $q > 0$. For instability we must have $\rho_2 < 0$ for some $q > 0$, and we notice that ρ_2 achieves its minimum:

$$\min_{p\in\mathbb{R}^+} \rho_2 = -\frac{(d_1 s - d_2 a_1)^2}{4 d_1 d_2} + s\left(\frac{a_2}{h} - a_1\right), \quad (31)$$

at the critical value $p_c^2 > 0$ when

$$p_c^2 = \frac{d_2 a_1 - d_1 s}{2 d_1 d_2} > 0. \quad (32)$$

Summarizing the previous calculation, we conclude the following theorem.

Theorem 2. *Assume that*

(A1) $m(ar + cm + 2hr - acr) < r(a+h)^2(r + s + cs)$,

(A2) $r(a+h)^2(d_1 s(1+c) + d_2 r) + 2\sqrt{d_1 d_2 r s(1+c)(a+h)^2(ar+hr-m)(ar+hr+cm)} < d_2 m(ar + cm + 2hr - acr)$.

Then the positive equilibrium E^* of model (4) is Turing unstable.

From Theorem 2, we can know that there is Turing instability in model (4) if conditions (A1) and (A2) hold. In this situation, the solutions to model (4) may be unstable and Turing patterns can emerge in the model.

3. Turing Pattern Formation

In this section, we perform extensive numerical simulations of the spatially extended model (4) in two-dimensional space, and the qualitative results are shown here. All our numerical simulations employ the zero-flux boundary conditions with a system size of 100×100. Other parameters are set as

$$\begin{gathered} r = 1, \quad m = 1, \quad a = 0.4, \quad K = 4, \\ h = 0.7, \quad c = 0.8, \quad d_1 = 0.025, \quad d_2 = 1. \end{gathered} \quad (33)$$

The numerical integration of model (4) is performed by using a finite difference approximation for the spatial derivatives and an explicit Euler method for the time integration

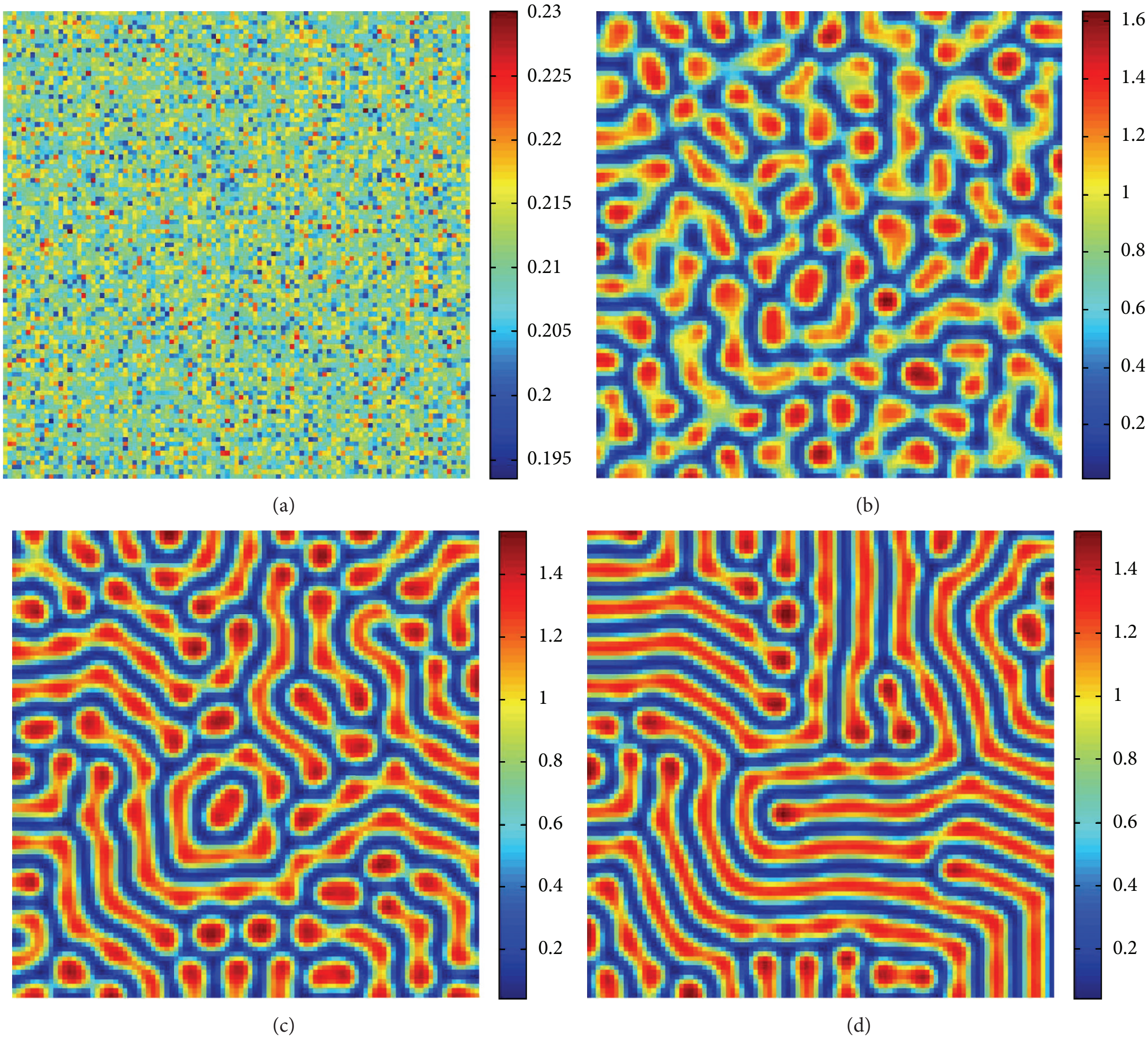


FIGURE 3: Stripes pattern formation for model (4) by taking $s = 1$. Other parameters are fixed as (33). Times: (a) 0; (b) 50; (c) 250; (d) 2500.

[35] with a time step size of 1/100. The initial condition is always a small amplitude random perturbation around the positive constant steady-state solution $E^* = (u^*, v^*)$. After the initial period during which the perturbation spread, the model either goes to a time-dependent state or to an essentially steady-state solution (time independent).

We use the standard five-point approximation [36] for the 2D Laplacian with the zero-flux boundary conditions. More precisely, the concentrations $(u_{i,j}^{n+1}, v_{i,j}^{n+1})$ at the moment $(n+1)\tau$ at the mesh position (x_i, y_j) are given by

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \tau d_1 \Delta_e u_{i,j}^{n} + \tau f\left(u_{i,j}^{n}, v_{i,j}^{n}\right),$$
$$v_{i,j}^{n+1} = v_{i,j}^{n} + \tau d_2 \Delta_e v_{i,j}^{n} + \tau g\left(u_{i,j}^{n}, v_{i,j}^{n}\right), \tag{34}$$

with the Laplacian defined by

$$\Delta_e u_{i,j}^{n} = \frac{u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}}{e^2}, \tag{35}$$

where $f(u,v) = (ru(K-u)/(K+cu)) - (muv/(u+av))$, $g(u,v) = sv(1-(hv/u))$, and the space step size $e = 1/3$.

In the numerical simulations, different types of dynamics are observed and it is found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show the distribution of prey u, for instance. We have taken some snapshots with red (blue) corresponding to the high (low) value of prey u.

Now, we show the Turing patterns for the different values of the control parameter s. Via numerical simulations, one can see that the model dynamics exhibits spatiotemporal complexity of pattern formation, including stripes, stripes-spots mixtures, and spots Turing patterns.

In Figure 3, with $s = 1$, starting with a homogeneous state $E^* = (0.21, 0.3)$ (cf. Figure 3(a)), the random perturbations lead to the formation to stripes spots (cf. Figure 3(c)), and the latter random perturbations make these spots decay, ending with the time-independent stripes pattern (cf. Figure 3(d)).

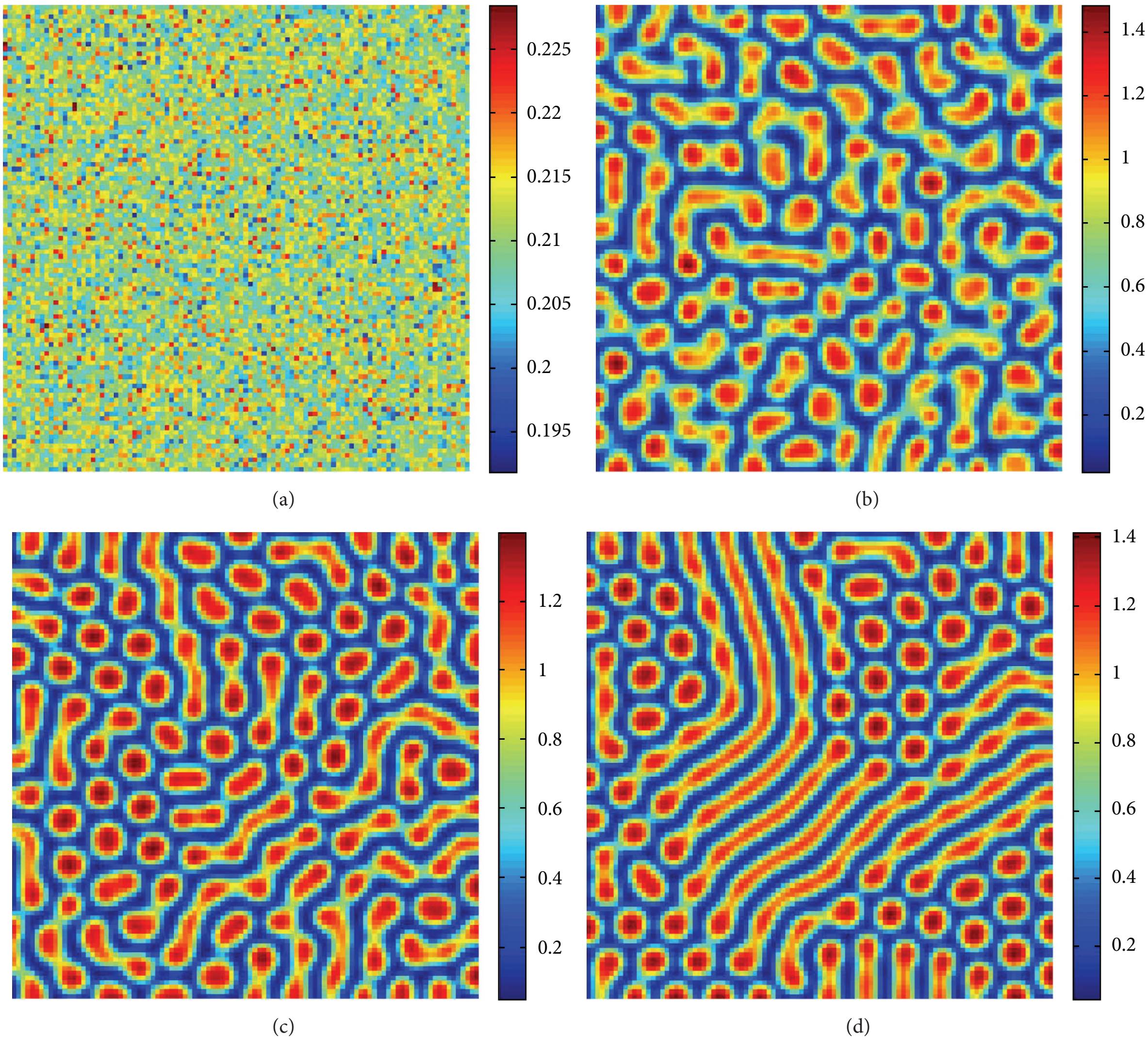


Figure 4: Stripes-spots mixtures pattern formation for model (4) by taking $s = 1.375$. Other parameters are fixed as (33). Times: (a) 0; (b) 50; (c) 250; (d) 2500.

In Figure 4, with $s = 1.375$, we show the stripes-spots mixtures pattern for model (4).

Figure 5 shows the time process of spots pattern formation of prey u for $s = 2.5$. In this case, the pattern takes a long time to settle down, starting with a homogeneous state $E^* = (0.21, 0.3)$ (cf. Figure 5(a)), and the random perturbations lead to the formation of stripes and spots (cf. Figure 5(b)), ending with spots only (cf. Figure 5(d))—the prey u is isolated zones with high population density, and the remainder region is of low density.

From Figure 3 to Figure 5, we can see that, on increasing the control parameter s from 1 to 2.5, the pattern sequence "stripe $\rightarrow$ stripes-spots mixtures $\rightarrow$ spots" can be observed.

4. Conclusions and Remarks

In summary, in this paper, we have investigated the spatiotemporal dynamics of a diffusive predator-prey model where the interaction between prey and predator follows Holling-Tanner formulation with ratio-dependent functional response and Smith growth. The value of this study is threefold. First, it presents the conditions for the stability of the equilibrium and the existence of Hopf bifurcation for the nonspatial model. Second, it rigorously proves Turing instability by linear stability analysis for the spatial model. Third, it illustrates the Turing pattern formation via numerical simulations, which shows that the spatial model dynamics exhibits complex pattern replication.

By a series of numerical simulations, we find that the spatial model (4) has rich Turing pattern replications, such as stripes, stripes-spots mixtures, and spots patterns. In the viewpoint of population ecology, in the case of stripe pattern (cf. Figure 3), the prey u is the isolated "stripes-like region" with high density, and the remainder stripes-like region is of low density. And in the case of spots pattern (cf. Figure 5), the prey u is the isolated "cycle region" with high density, and the

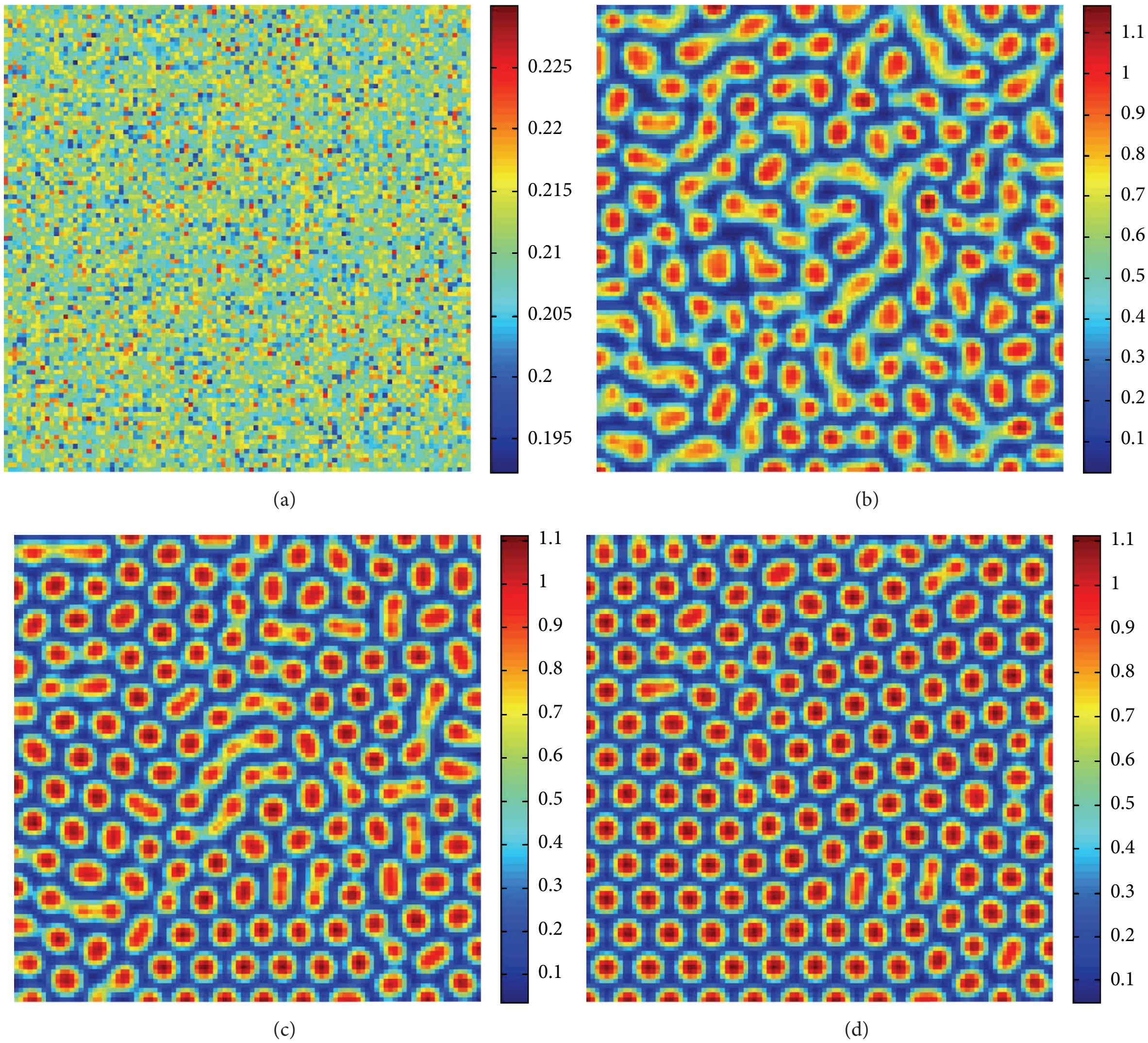


FIGURE 5: Spots pattern formation for model (4) by taking $s = 2.5$. Other parameters are fixed as (33). Times: (a) 0; (b) 50; (c) 250; (d) 2500.

remainder region is of low density, which is larger than the "spots" region.

References

[1] S. V. Petrovskii and H. Malchow, "A minimal model of pattern formation in a prey-predator system," *Mathematical and Computer Modelling*, vol. 29, no. 8, pp. 49–63, 1999.

[2] D. Alonso, F. Bartumeus, and J. Catalan, "Mutual interference between predators can give rise to turing spatial patterns," *Ecology*, vol. 83, no. 1, pp. 28–34, 2002.

[3] A. M. Turing, "The chemical basis of morphogenesis," *Philosactions Transactions of the Royal Society of London. Series B*, vol. 237, no. 641, pp. 37–72, 1952.

[4] L. A. Segel and J. L. Jackson, "Dissipative structure: an explanation and an ecological example," *Journal of Theoretical Biology*, vol. 37, no. 3, pp. 545–559, 1972.

[5] L. S. Luckinbill, "The effects of space and enrichment on a predator-prey system," *Ecology*, vol. 55, no. 5, pp. 1142–1147, 1974.

[6] S. A. Levin, "The problem of pattern and scale in ecology: the Robert H. MacArthur award lecture," *Ecology*, vol. 73, no. 6, pp. 1943–1967, 1992.

[7] M. Pascual, "Diffusion-induced chaos in a spatial predator-prey system," *Proceedings of the Royal Society B*, vol. 251, no. 1330, pp. 1–7, 1993.

[8] A. B. Medvinsky, S. V. Petrovskii, I. A. Tikhonova, H. Malchow, and B.-L. Li, "Spatiotemporal complexity of plankton and fish dynamics," *SIAM Review*, vol. 44, no. 3, pp. 311–370, 2002.

[9] J. D. Murray, *Mathematical Biology. II: Spatial Models and Biomedical Applications*, vol. 18 of *Interdisciplinary Applied Mathematics*, Springer, New York, NY, USA, 3rd edition, 2003.

[10] A. B. Peet, P. A. Deutsch, and E. Peacock-López, "Complex dynamics in a three-level trophic system with intraspecies interaction," *Journal of Theoretical Biology*, vol. 232, no. 4, pp. 491–503, 2005.

[11] D. A. Griffith and P. R. Peres-Neto, "Spatial modeling in ecology: the flexibility of eigenfunction spatial analyses," *Ecology*, vol. 87, no. 10, pp. 2603–2613, 2006.

[12] I. G. Pearce, M. A. J. Chaplain, P. G. Schofield, A. R. A. Anderson, and S. F. Hubbard, "Modelling the spatio-temporal dynamics of multi-species host-parasitoid interactions: heterogeneous patterns and ecological implications," *Journal of Theoretical Biology*, vol. 241, no. 4, pp. 876–886, 2006.

[13] W. Wang, Q.-X. Liu, and Z. Jin, "Spatiotemporal complexity of a ratio-dependent predator-prey system," *Physical Review E*, vol. 75, no. 5, Article ID 051913, 9 pages, 2007.

[14] W. Wang, L. Zhang, H. Wang, and Z. Li, "Pattern formation of a predator-prey system with Ivlev-type functional response," *Ecological Modelling*, vol. 221, no. 2, pp. 131–140, 2010.

[15] W. Wang, Y. Lin, F. Rao, L. Zhang, and Y. Tan, "Pattern selection in a ratio-dependent predator-prey model," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2010, Article ID P11036, 2010.

[16] W. Wang, Y. Lin, L. Zhang, F. Rao, and Y. Tan, "Complex patterns in a predator-prey model with self and cross-diffusion," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2006–2015, 2011.

[17] X. Guan, W. Wang, and Y. Cai, "Spatiotemporal dynamics of a Leslie-Gower predator-prey model incorporating a prey refuge," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 2385–2395, 2011.

[18] Y. Zhu, Y. Cai, S. Yan, and W. Wang, "Dynamical analysis of a delayed reaction-diffusion predator-prey system," *Abstract and Applied Analysis*, vol. 2012, Article ID 323186, 23 pages, 2012.

[19] Y. Cai, W. Wang, and J. Wang, "Dynamics of a diffusive predator-prey model with additive Allee effect," *International Journal of Biomathematics*, vol. 5, no. 2, Article ID 1250023, 11 pages, 2012.

[20] W. Wang, Z. Guo, R. K. Upadhyay, and Y. Lin, "Pattern formation in a cross-diffusive Holling-Tanner model," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 828219, 12 pages, 2012.

[21] R. Arditi, L. R. Ginzburg, and H. R. Akcakaya, "Variation in plankton densities among lakes: a case for ratio-dependent models," *The American Naturalist*, vol. 138, no. 5, pp. 287–1296, 1991.

[22] R. Arditi and H. Saiah, "Empirical evidence of the role of heterogeneity in ratio-dependent consumption," *Ecology*, vol. 73, no. 5, pp. 1544–1551, 1992.

[23] A. P. Gutierrez, "Physiological basis of ratio-dependent predator-prey theory: the metabolic pool model as a paradigm," *Ecology*, vol. 73, no. 5, pp. 1552–1563, 1992.

[24] T. Saha and C. Chakrabarti, "Dynamical analysis of a delayed ratio-dependent Holling-Tanner predator-prey model," *Journal of Mathematical Analysis and Applications*, vol. 358, no. 2, pp. 389–402, 2009.

[25] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.

[26] R. Arditi, N. Perrin, and H. Saiah, "Functional responses and heterogeneities: an experimental test with cladocerans," *Oikos*, vol. 60, no. 1, pp. 69–75, 1991.

[27] Z. Liang and H. Pan, "Qualitative analysis of a ratio-dependent Holling-Tanner model," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 954–964, 2007.

[28] M. Banerjee and S. Banerjee, "Turing instabilities and spatio-temporal chaos in ratio-dependent Holling-Tanner model," *Mathematical Biosciences*, vol. 236, no. 1, pp. 64–76, 2012.

[29] F. E. Smith, "Population dynamics in Daphnia Magna and a new model for population growth," *Ecology*, vol. 44, pp. 651–663, 1963.

[30] E. C. Pielou, *An Introduction to Mathematical Ecology*, John Wiley & Sons, New York, NY, USA, 1969.

[31] T. G. Hallam and J. T. Deluna, "Effects of toxicants on populations: a qualitative approach III," *Journal of Theoretical Biology*, vol. 109, no. 3, pp. 411–429, 1984.

[32] K. Gopalsamy, M. R. S. Kulenović, and G. Ladas, "Environmental periodicity and time delays in a "food-limited" population model," *Journal of Mathematical Analysis and Applications*, vol. 147, no. 2, pp. 545–555, 1990.

[33] W. Feng and X. Lu, "On diffusive population models with toxicants and time delays," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 373–386, 1999.

[34] M. Fan and K. Wang, "Periodicity in a "food-limited" population model with toxicants and time delays," *Acta Mathematicae Applicatae Sinica*, vol. 18, no. 2, pp. 309–314, 2002.

[35] M. R. Garvie, "Finite-difference schemes for reaction-diffusion equations modeling predator-prey interactions in MATLAB," *Bulletin of Mathematical Biology*, vol. 69, no. 3, pp. 931–956, 2007.

[36] A. Munteanu and R. V. Solé, "Pattern formation in noisy self-replicating spots," *International Journal of Bifurcation and Chaos*, vol. 16, no. 12, pp. 3679–3685, 2006.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 267173, 9 pages
http://dx.doi.org/10.1155/2013/267173

Research Article

Qualitative Analysis of a Diffusive Ratio-Dependent Holling-Tanner Predator-Prey Model with Smith Growth

Zongmin Yue[1] and Wenjuan Wang[2]

[1] *Faculty of Science, Shaanxi University of Science and Technology, Xi'an 710021, China*
[2] *Department of Biostatistics and Computational Biology, University of Rochester Medical Center, 601 Elmwood Avenue, P.O. Box 630, Rochester, NY 14642, USA*

Correspondence should be addressed to Zongmin Yue; joanna_yzm@163.com

Received 28 January 2013; Accepted 15 February 2013

Academic Editor: Yonghui Xia

We investigated the dynamics of a diffusive ratio-dependent Holling-Tanner predator-prey model with Smith growth subject to zero-flux boundary condition. Some qualitative properties, including the dissipation, persistence, and local and global stability of positive constant solution, are discussed. Moreover, we give the refined a priori estimates of positive solutions and derive some results for the existence and nonexistence of nonconstant positive steady state.

1. Introduction

In order to precisely describe the real ecological interactions between species such as mite and spider mite, lynx and hare, sparrow and sparrow hawk, and some other species [1, 2], Robert May developed a prey-predator model of Holling-type functional response [3, 4] to describe the predation rate and Leslie's formulation [5, 6] to describe predator dynamics. This model is known as Holling-Tanner model for prey-predator interaction, which takes the form of

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - \frac{mNP}{a + N}, \\ \frac{dP}{dt} &= sP\left(1 - \frac{hP}{N}\right), \end{aligned} \tag{1}$$

where $N(t)$ and $P(t)$ stand for prey and predator population (density) at any instant of time t. r, K, m, a, s, h are positive constants that stand for prey intrinsic growth rate, carrying capacity, capturing rate, half capturing saturation constant, predator intrinsic growth rate, and conversion rate of prey into predators biomass, respectively.

The dynamics of model (1) has been considered in many articles. For example, Hsu and Huang [7] obtained some results on the global stability of the positive equilibrium, more precisely, under the conditions which local stability of the positive equilibrium implies its global stability. Gasull and coworkers [8] investigated the conditions of the asymptotic stability of the positive equilibrium which does not imply global stability. Sáez and González-Olivares [9] showed the asymptotic stability of a positive equilibrium and gave a qualitative description of the bifurcation curve.

Recently, there is a growing explicit biological and physiological evidence [10–12] that in many situations, especially, when the predator has to search for food (and therefore has to share or compete for food), a more suitable general predator-prey theory should be based on the so-called radio-dependent theory which can be roughly stated as that the per capital predator growth rate should be a function of the ratio of prey to predator abundance, and so would be the so-called predator functional responses [13]. This is supported by numerous fields and laboratory experiments and observations [14, 15]. Generally, a ratio-dependent Holling-Tanner predator-prey model takes the form of

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - \frac{mNP}{N + aP}, \\ \frac{dP}{dt} &= sP\left(1 - \frac{hP}{N}\right). \end{aligned} \tag{2}$$

For model (2), in [13], the authors investigated the effect of time delays on the stability of the model and discussed the local asymptotic stability and the Hopf-bifurcation. Liang and Pan [16] have studied the local and global asymptotic stability of the coexisting equilibrium point and obtained the conditions for the Poincaré-Andronov-Hopf-bifurcating periodic solution. M. Banerjee and S. Banerjee [17] have studied the local asymptotic stability of the equilibrium point and obtained the conditions for the occurrence of the Turing-Hopf instability for PDE model. It is shown that prey and predator populations exhibit spatiotemporal chaos resulting from temporal oscillation of both the population and spatial instability.

On the other hand, an implicit assumption contained in the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{3}$$

is that the average growth rate $N'(t)/N$ is a linear function of the density $N(t)$. It has been shown that this assumption is not realistic for a food-limited population under the effects of environmental toxicants. The following alternative model has been proposed by several authors [18–23] for the dynamics of a population where the growth limitations are based upon the proportion of available resources not utilized:

$$\frac{dN}{dt} = rN\frac{K - N}{K + cN}, \tag{4}$$

where r/c is the replacement of mass in the population at K. Equation (4) takes into account both environmental and food chain effects of toxicant stress.

Based on the above discussions, in this paper, we rigorously consider the radio-dependent Holling-Tanner model with Smith growth that takes the form of

$$\begin{aligned} \frac{dN}{dt} &= rN\frac{K - N}{K + cN} - \frac{mNP}{aP + N}, \\ \frac{dP}{dt} &= sP\left(1 - \frac{hP}{N}\right). \end{aligned} \tag{5}$$

Also considering the spatial dispersal and environmental heterogeneity, in this paper, we study the following generalized reaction-diffusion system for model (5):

$$\begin{aligned} \frac{\partial N}{dt} &= rN\frac{K - N}{K + cN} - \frac{mNP}{aP + N} + d_1\Delta N, \quad x \in \Omega,\ t > 0, \\ \frac{\partial P}{dt} &= sP\left(1 - \frac{hP}{N}\right) + d_2\Delta P, \quad x \in \Omega,\ t > 0, \\ \frac{\partial N}{\partial \nu} &= \frac{\partial P}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \end{aligned} \tag{6}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$ and ν is the outward unit normal vector on $\partial\Omega$. The nonnegative constants d_1 and d_2 are the diffusion coefficients of N and P, respectively. The zero-flux boundary condition indicates that predator-prey system is self-contained with zero population flux across the boundary. From the standpoint of biology, we are interested only in the dynamics of model (6) in the closed first quadrant $\mathbb{R}^2_+ = \{(N, P) : N \geq 0, P \geq 0\}$. Thus, we consider only the biologically meaningful initial conditions

$$N(x, 0) = N_0(x) > 0, \quad P(x, 0) = P_0(x) > 0, \quad x \in \Omega, \tag{7}$$

which are continuous functions due to its biological sense. Straightforward computation shows that model (6) are continuous and Lipschizian in $\mathbb{R}^2_+$ if we redefine that when

$$\frac{\partial N}{\partial t} = \frac{\partial P}{\partial t} = 0, \quad \text{if } (N, P) = (0, 0). \tag{8}$$

Hence, the solution of model (6) with positive initial conditions exists and is unique.

The stationary problem of model (6), which may display the dynamical behavior of solutions to model (6) as time goes to infinity, satisfies the following elliptic system:

$$\begin{aligned} -d_1\Delta N &= rN\frac{K - N}{K + cN} - \frac{mNP}{N + aP}, \quad x \in \Omega,\ t > 0, \\ -d_2\Delta P &= sP\left(1 - \frac{hP}{N}\right), \quad x \in \Omega,\ t > 0, \\ \frac{\partial N}{\partial \nu} &= \frac{\partial P}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ N(x, 0) &= N_0(x) > 0, \quad P(x, 0) = P_0(x) > 0, \quad x \in \Omega. \end{aligned} \tag{9}$$

Simple computation shows that if $m < r(a + h)$, then model (6) and (9) possess a unique positive constant solution, denoted by $E^* = (N^*, P^*)$, where

$$N^* = \frac{K(ar + hr - m)}{ar + cm + hr}, \qquad P^* = \frac{1}{h}N^*. \tag{10}$$

In addition, $(K, 0)$ is the second nonnegative constant steady state of model (6) and (9).

The rest of the paper is organized as follows. In Section 2, we investigate the lager time behavior of model (6), including the dissipation, persistence property, and local and global stability of positive constant solution E^*. In Section 3, we first give a priori upper and lower bounds for positive solutions of model (9), and then we deal with existence and nonexistence of nonconstant positive solutions of model (9), which imply some certain conditions under which the pattern happens or not.

2. Large Time Behavior of Solution to Model (6)

In this section, the dissipation and persistence properties are studied for solution of model (6). Moreover, the local and global asymptotic stability of positive constant solution $E^* = (N^*, P^*)$ are investigated.

2.1. *The Properties of Dissipation and Persistence of Solution to Model* (6)

Theorem 1. *All the solutions of model* (6) *are nonnegative and defined for all* $t > 0$. *Furthermore, the nonnegative solution* (N, P) *of model* (6) *satisfies*

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} N(\cdot,t) \le K, \qquad \limsup_{t\to\infty} \max_{\overline{\Omega}} P(\cdot,t) \le \frac{K}{h}. \tag{11}$$

Proof. The nonnegativity of the solution of model (6) is clear since the initial value is nonnegative. We only consider the latter of the theorem.

Note that N satisfies

$$\begin{aligned} &\frac{\partial N}{\partial t} - d_1 \Delta N \le \frac{rN(K-N)}{K+cN}, \quad x\in\Omega,\ t>0, \\ &\frac{\partial N}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t>0, \\ &N(x,0) = N_0(x), \quad x\in\Omega. \end{aligned} \tag{12}$$

Let $z(t)$ be a solution of the ordinary differential equation:

$$\begin{aligned} &\dot{z}(t) = \frac{rN(K-N)}{K+cN}, \quad x\in\Omega,\ t>0, \\ &z(0) = \max_{\overline{\Omega}} N(x,0) > 0. \end{aligned} \tag{13}$$

Then, $\lim_{t\to\infty} z(t) = K$. From the comparison principle, one can get $N(x,t) \le z(t)$; hence,

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} N(x,t) \le K. \tag{14}$$

As a result, for any $\varepsilon > 0$, there exists $t_0 > 0$, such that $N(x,t) \le K+\varepsilon$ for all $x\in\overline{\Omega}$ and $t \ge t_0$. Hence, $P(x,t)$ is a lower solution

$$\begin{aligned} &\frac{\partial z}{\partial t} - d_2\Delta z = sz\left(1-\frac{hw}{K+\varepsilon}\right), \quad x\in\Omega,\ t>t_0, \\ &\frac{\partial z}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>t_0, \\ &z(x,t_0) = P(x,t_0). \end{aligned} \tag{15}$$

Let $P(t)$ be the unique positive solution of problem

$$\begin{aligned} &\dot{w}(t) = sw\left(1-\frac{hw}{K+\varepsilon}\right), \quad t>t_0, \\ &w(t_0) = \max_{\overline{\Omega}} P(x,t_0). \end{aligned} \tag{16}$$

Then, $P(t)$ is an upper solution of (15). As $\lim_{t\to\infty} P(t) = (K+\varepsilon)/h$, we get from the comparison principle that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} P(x,t) \le \frac{K+\varepsilon}{h}, \tag{17}$$

which implies the second assertion by the arbitrariness of $\varepsilon > 0$. This ends the proof. □

Definition 2 (see [24]). The spatial model (6) is said to have the persistence property if for any nonnegative initial data $(N_0(x), P_0(x))$, there exists a positive constant $\varepsilon = \varepsilon(N_0, P_0)$, such that the corresponding solution (N, P) of model (6) satisfies

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} N(x,t) \ge \varepsilon, \qquad \liminf_{t\to\infty} \min_{\overline{\Omega}} P(x,t) \ge \varepsilon. \tag{18}$$

Theorem 3. *If* $m(1+c) < ar$, *then model* (6) *has the persistence property.*

Proof. Let $N(x,t)$ be an upper solution of the following problem:

$$\begin{aligned} &\frac{\partial z}{\partial t} - d_1\Delta z = z\left(\frac{r}{1+c} - \frac{m}{a} - \frac{rz}{K(1+c)}\right), \quad x\in\Omega,\ t>T, \\ &\frac{\partial z}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>T, \\ &z(x,T) = N_0(x,T) \ge 0, \quad x\in\overline{\Omega}. \end{aligned} \tag{19}$$

Let $N(t)$ be the unique positive solution to the following problem:

$$\begin{aligned} &\frac{dw}{dt} = w\left(\frac{r}{1+c} - \frac{m}{a} - \frac{rw}{K(1+c)}\right), \quad t>T, \\ &w(T) = \max_{\overline{\Omega}} N_0(x,T) \ge 0. \end{aligned} \tag{20}$$

Due to $m(1+c) < ar$, we have that $\lim_{t\to\infty} w(t) = K(ar - m(1+c))/ar$. By comparison, it follows that

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} N(x,t) \ge \frac{K(ar-m(1+c))}{ar} \triangleq \eta. \tag{21}$$

Hence, $N(x,t) > \eta - \varepsilon$ for $t > T$ and $x\in\overline{\Omega}$.

Similarly, by the second equation of model (6), we have that $P(x,t)$ is an upper solution of problem

$$\begin{aligned} &\frac{\partial z}{\partial t} - d_2\Delta z = sz\left(1-\frac{hz}{\eta-\varepsilon}\right), \quad x\in\Omega,\ t>T, \\ &\frac{\partial z}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>T, \\ &z(x,T) = P_0(x,T) \ge 0, \quad x\in\overline{\Omega}. \end{aligned} \tag{22}$$

Let $P(t)$ be the unique positive solution to the following problem:

$$\begin{aligned} &\frac{dw}{dt} = sz\left(1-\frac{hz}{\eta-\varepsilon}\right), \quad t>T, \\ &w(T) = \max_{\overline{\Omega}} P_0(x,T) \ge 0. \end{aligned} \tag{23}$$

Then, $\lim_{t\to\infty} w(t) = \eta/h$ for the arbitrariness of ε, and an application of the comparison principle gives

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} P(x,t) \ge \frac{\eta}{h}. \tag{24}$$

The proof is complete. □

2.2. The Local Stability of the Constant Steady State. In this subsection, we shall analyze the asymptotical stability of the positive constant solution E^* for model (6). Before developing our argument, let us set up the following notations.

(i) Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots \to \infty$ be the eigenvalues of the operator $-\Delta$ on Ω with the zero-flux boundary condition;

(ii) Let $E(\mu) = \{\phi \mid -\Delta\phi = \mu\phi \text{ in } \Omega, \partial_\nu \phi = 0 \text{ on } \partial\Omega\}$ with $\mu \in \mathbb{R}^1$;

(iii) Let $\{\phi_{ij} \mid j = 1, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$;

(iv) Let

$$\mathbf{X} = \left\{(N, P) \in \left[H^2(\Omega)\right]^2 \mid \partial_\nu N = \partial_\nu P = 0 \text{ on } \partial\Omega\right\}, \quad (25)$$

then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i, \quad (26)$$

where $\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}$.

Theorem 4. *Assume that*

$$m(ar + cm + 2hr - acr) < r(a+h)^2(r + s + cs) \quad (27)$$

and the first eigenvalues μ_1 of the Dirichlet operator subject to zero-flux boundary conditions satisfy

$$\mu_1 > \max\left\{\frac{m(ar + cm + 2hr - acr) - r^2(a+h)^2}{d_1 r(1+c)(a+h)^2} - \frac{s}{d_2}, 0\right\}. \quad (28)$$

Then the positive constant solution E^* of model (6) *is locally asymptotically stable.*

Proof. Define $\mathscr{L} : \mathbf{X} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ by

$$\mathscr{L} = \begin{pmatrix} d_1\Delta + J_1 & -\dfrac{mh^2}{(a+h)^2} \\ -\dfrac{s}{h} & d_2\Delta - s \end{pmatrix}, \quad (29)$$

where $J_1 = (m(ar+cm+2hr-acr)-r^2(a+h)^2)/r(1+c)(a+h)^2$.

For each $i = 0, 1, 2, \dots, \mathbf{X}_i$ is invariant under the operator $\mathscr{L}$, and λ is an eigenvalue of this operator on $\mathbf{X}_i$ if and only if it is an eigenvalue of the following matrix:

$$A_i = \begin{pmatrix} -d_1\mu_i + J_1 & -\dfrac{mh^2}{(a+h)^2} \\ -\dfrac{s}{h} & -d_2\mu_i - s \end{pmatrix}. \quad (30)$$

Moreover,

$$\det(\lambda I - A_i) = \lambda^2 - \operatorname{tr}(A_i)\lambda + \det(A_i), \quad (31)$$

where

$$\det(A_i) = d_1 d_2 \mu_i^2 + (d_1 s - d_2 J_1)\mu_i + \frac{s(ar + hr - m)(ar + hr + cm)}{r(1+c)(a+h)^2}, \quad (32)$$
$$\operatorname{tr}(A_i) = -(d_1 + d_2)\mu_i + J_1 - s.$$

In view of (27) and (28), we have $\det(A_i) > 0 > \operatorname{tr}(A_i)$ for any $i \geq 0$. Therefore, the eigenvalues of the matrix A_i have negative real parts.

In the following, we prove that there exists $\delta > 0$ such that

$$\operatorname{Re}\{\lambda_{i1}\} \leq -\delta, \qquad \operatorname{Re}\{\lambda_{i2}\} \leq -\delta. \quad (33)$$

Let $\lambda = \mu_i \xi$, then

$$\widetilde{\varphi}_i(\lambda) \triangleq \mu_i^2\xi^2 - \operatorname{tr}(A_i)\mu_i\xi + \det(A_i). \quad (34)$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty} \frac{\widetilde{\varphi}_i(\lambda)}{\mu_i^2} = \xi^2 + (d_1 + d_2)\xi + d_1 d_2. \quad (35)$$

By the Routh-Hurwitz criterion, it follows that the two roots ξ_1, ξ_2 of $\widetilde{\varphi}_i(\lambda) = 0$ all have negative real parts. Thus, let $\widetilde{d} = \min\{d_1, d_2\}$, we have that $\operatorname{Re}\{\xi_1\}, \operatorname{Re}\{\xi_2\} \leq -\widetilde{d}$. By continuity, we see that there exists i_0 such that the two roots ξ_{i1}, ξ_{i2} of $\widetilde{\varphi}_i(\lambda) = 0$ satisfy $\operatorname{Re}\{\xi_{i1}\} \leq -\widetilde{d}/2$, $\operatorname{Re}\{\xi_{i2}\} \leq -\widetilde{d}/2$, for all $i \geq i_0$. In turn, $\operatorname{Re}\{\lambda_{i1}\}, \operatorname{Re}\{\lambda_{i2}\} \leq -\mu_i\widetilde{d}/2 \leq -\widetilde{d}/2$, for all $i \geq i_0$.

Let $-\widetilde{\delta} = \max_{1\leq i\leq i_0}\{\operatorname{Re}\{\lambda_{i1}\}, \operatorname{Re}\{\lambda_{i2}\}\}$, then $\widetilde{\delta} > 0$ and (33) hold for $\delta = \min\{\widetilde{\delta}, \widetilde{d}/2\}$. Consequently, the spectrum of $\mathscr{L}$ which consists of eigenvalues, lies in $\{\operatorname{Re}\lambda \leq -\delta\}$. In the sense of [25], we obtain that the positive constant solution $E^* = (N^*, P^*)$ of model (6) is uniformly asymptotically stable. This ends the proof. □

2.3. The Global Stability of the Constant Solution. This subsection is devoted to the global stability of the constant solution E^* for model (6).

Theorem 5. *Assume that the following hold:*

(A1) $m(1+c) < ar$;

(A2) $h(K+cN^*)(N^*+2P^*)+(K+cN^*)(N^*+aP^*)(a+h) \leq 2\eta r(N^* + aP^*)(a+h)$;

(A3) $hKN^* \leq \eta(2h-1)(N^* + aP^*)(a+h)$,

where $\eta = K(ar - m(1+c))/ar$. Then the constant solution E^* is globally asymptotically stable.

Proof. In order to give the proof, we need to construct a Lyapunov function. Define

$$V(N, P) = \int_{N^*}^{N} \frac{\xi - N^*}{\xi} d\xi + \int_{P^*}^{P} \frac{\zeta - P^*}{\zeta} d\zeta,$$
$$E(t) = \int_\Omega V(N(x,t), P(x,t))\, dx. \quad (36)$$

We note that $E(t)$ is nonnegative, $E(t) = 0$ if and only if $(N(x,t), P(x,t)) = (N^*, P^*)$. Furthermore, by simple computations, it follows that

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_\Omega \left(V_N(N(x,t),P(x,t))N_t\right.\\&\quad\left.+V_P(N(x,t),P(x,t))P_t\right)dx\\&= \int_\Omega\left(\frac{d_1(N-N^*)}{N}\Delta N+\frac{d_2(P-P^*)}{P}\Delta P\right)dx\\&\quad+\int_\Omega\left((N-N^*)\left(\frac{r(K-N)}{K+cN}-\frac{mP}{N+aP}\right)\right.\\&\quad\left.+(P-P^*)\left(1-\frac{hP}{N}\right)\right)dx\\&= -\int_\Omega\left(\frac{d_1N^*}{N^2}|\nabla N|^2+\frac{d_2P^*}{P^2}|\nabla P|^2\right)dx+I(t),\end{aligned}\tag{37}$$

where

$$\begin{aligned}I(t) = &\int_\Omega\left((N-N^*)^2\left(-\frac{rK(1+c)}{(K+cN^*)(K+cN)}\right.\right.\\&\left.\left.+\frac{P^*}{(N^*+aP^*)(N+aP)}\right)\right)dx\\&+\int_\Omega\left((N-N^*)(P-P^*)\right.\\&\quad\times\left(\frac{1}{N}-\frac{N^*}{(N^*+aP^*)(N+aP)}\right)\\&\quad\left.-\frac{h}{N}(P-P^*)^2\right)dx.\end{aligned}\tag{38}$$

Set $\varphi = N-N^*, \phi = P-P^*$. We have

$$\begin{aligned}I(t) = &-\int_\Omega\left(\frac{rK(1+c)}{(K+cN^*)(K+cN)}\right.\\&\left.-\frac{N^*+2P^*}{2(N^*+aP^*)(N+aP)}-\frac{1}{2N}\right)\varphi^2dx\\&-\int_\Omega\left(\frac{2h-1}{2N}-\frac{N^*}{2(N^*+aP^*)(N+aP)}\right)\phi^2dx\\&-\int_\Omega\left(\frac{1}{2N}(\varphi-\phi)^2+\frac{N^*}{2(N^*+aP^*)(N+aP)}\right.\\&\left.\times(\varphi+\phi)^2\right)dx\\&\leq -\int_\Omega\left(\frac{rK(1+c)}{(K+cN^*)(K+cN)}\right.\\&\left.-\frac{N^*+2P^*}{2(N^*+aP^*)(N+aP)}-\frac{1}{2N}\right)\varphi^2dx\\&-\int_\Omega\left(\frac{2h-1}{2N}-\frac{N^*}{2(N^*+aP^*)(N+aP)}\right)\phi^2dx.\end{aligned}\tag{39}$$

By virtue of Theorems 1 and 3 and under the assumption of Theorem, we have

$$\begin{aligned}&\frac{rK(1+c)}{(K+cN^*)(K+cN)}-\frac{N^*+2P^*}{2(N^*+aP^*)(N+aP)}-\frac{1}{2N}\\&\quad\geq\frac{r}{K+cN^*}-\frac{h(N^*+2P^*)}{2\eta(N^*+aP^*)(a+h)}-\frac{1}{2\eta}\\&\quad=\left(2\eta r(N^*+aP^*)(a+h)\right.\\&\qquad-h(K+cN^*)(N^*+2P^*)\\&\qquad\left.-(K+cN^*)(N^*+aP^*)(a+h)\right)\\&\qquad\times\left(2\eta(K+cN^*)(N^*+aP^*)(a+h)\right)^{-1}\\&\quad\geq 0,\\&\frac{2h-1}{2N}-\frac{N^*}{2(N^*+aP^*)(N+aP)}\\&\quad\geq\frac{2h-1}{2K}-\frac{hN^*}{2\eta(N^*+aP^*)(a+h)}\\&\quad=\frac{\eta(2h-1)(N^*+aP^*)(a+h)-hKN^*}{2K\eta(N^*+aP^*)(a+h)}\\&\quad\geq 0.\end{aligned}\tag{40}$$

As a result, we have $I(t) \leq 0$. Thus $dE(t)/dt \leq 0$, which implies the desired assertion. The proof is completed. □

3. A Priori Estimates and Existence of Nonconstant Positive Solution

In this section, we will deduce a priori estimates of positive upper and lower bounds for positive solution of model (9). Then, based on a priori estimates, we discuss the existence of nonconstant positive solution of model (9) for certain parameter ranges.

3.1. A Priori Estimates. In order to obtain the desired bound, we recall the following two lemmas which are due to Lin et al. [26] and Lou and Ni [27], respectively.

Lemma 6 (Harnack's inequality [26]). *Assume that $c \in C(\overline{\Omega})$ and let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to*

$$\Delta w(x) + c(x) w(x) = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{41}$$

Then there exists a positive constant $C^* = C^*(\|c\|_\infty)$ such that

$$\max_{\overline{\Omega}} w \le C^* \min_{\overline{\Omega}} w. \tag{42}$$

Lemma 7 (maximum principle [27]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and $g \in C(\overline{\Omega} \times \mathbb{R})$.*

(a) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \ge 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu} \le 0 \quad \text{on } \partial\Omega. \tag{43}$$

If $w(x_0) = \max_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \ge 0$.

(b) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta w(x) + g(x, w(x)) \le 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu} \ge 0 \quad \text{on } \partial\Omega. \tag{44}$$

If $w(x_0) = \min_{\overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \le 0$.

For convenience, let us denote the constants a, c, h, m, K, r, s collectively by Λ. The positive constants C, $\underline{C}$, $\overline{C}$, and so forth will depend only on the domain Ω and Λ. Now, we can state the main result which will play a critical role in Section 3.3.

Theorem 8. *For any positive solution (N, P) of model (9),*

$$\max_{\overline{\Omega}} N(x) \le K, \qquad \max_{\overline{\Omega}} P(x) \le \frac{K}{h}. \tag{45}$$

Proof. Assume that (N, P) is a positive solution of model (9). Set

$$N(x_1) = \max_{\overline{\Omega}} N(x). \tag{46}$$

Then, by Lemma 7, it follows from the first equation of (9) that

$$\frac{rN(x_1)(K - N(x_1))}{K + cN(x_1)} \ge \frac{mN(x_1) P(x_1)}{N(x_1) + aP(x_1)} > 0. \tag{47}$$

This clearly gives $N(x_1) < K$.

Since $0 < P(x) \le (1/h)\|N(x)\|_\infty$, we have $P(x) \le K/h$ in $\overline{\Omega}$. □

Theorem 9. *Let d be a fix positive constant. Then there exists positive constant $\underline{C} = \underline{C}(\Lambda, d)$ such that if $d_1, d_2 > d$, any positive solution (N, P) of model (6) satisfies*

$$\min_{\overline{\Omega}} N(x) \ge \underline{C}, \qquad \min_{\overline{\Omega}} P(x) \ge \underline{C}. \tag{48}$$

Proof. Let

$$N(x_0) = \min_{\overline{\Omega}} N(x), \quad P(y_0) = \min_{\overline{\Omega}} P(x), \quad P(y_1) = \max_{\overline{\Omega}} P(x). \tag{49}$$

By Lemma 7, it is clear that

$$\frac{r(K - N(x_0))}{K + cN(x_0)} - \frac{mP(x_0)}{N(x_0) + aP(x_0)} \le 0, \quad 1 - \frac{hP(y_0)}{N(y_0)} \le 0, \quad 1 - \frac{hP(y_1)}{N(y_1)} \ge 0. \tag{50}$$

So, we have

$$\frac{1}{h}N(x_0) \le \frac{1}{h}N(y_0) \le P(y_0), \tag{51}$$

$$P(y_1) \le \frac{1}{h}N(y_1) \le \frac{1}{h}\max_{\overline{\Omega}} N(x). \tag{52}$$

Since $m(K + cN(x_0))/(N(x_0) + aP(x_0)) \le C$ with $C > 0$, then, by virtue of (52), we derive

$$K - N(x_0) \le \frac{mP(x_0)(K + cN(x_0))}{r(N(x_0) + aP(x_0))} \le \frac{C}{r}P(x_0) \le \frac{C}{r}P(y_1) \le \frac{C}{hr}\max_{\overline{\Omega}} N(x), \tag{53}$$

which implies that

$$K \le \min_{\overline{\Omega}} N(x) + \frac{C}{hr}\max_{\overline{\Omega}} N(x). \tag{54}$$

Define $c(x) = d_1^{-1}(r(K - N)/(K + cN) - mP/(N + aP))$, then N satisfies

$$\Delta N(x) + c(x) N(x) = 0 \quad \text{in } \Omega, \qquad \frac{\partial N}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{55}$$

Therefore, we have

$$\max_{\overline{\Omega}} N(x) \le C^* \min_{\overline{\Omega}} N(x), \tag{56}$$

herein a positive constant $C^* = C^*(\|c\|_\infty)$. Hence, we obtain

$$\min_{\overline{\Omega}} N(x) \ge \frac{hKr}{hr + CC^*}. \tag{57}$$

It follows from (51) that

$$\min_{\overline{\Omega}} P(x) \ge \frac{Kr}{hr + CC^*}. \tag{58}$$

The proof is completed. □

3.2. Nonexistence of the Nonconstant Positive Solutions. Note that μ_1 is the smallest positive eigenvalues of the operator $-\Delta$ in Ω subject to the zero-flux boundary condition. Now, using the energy estimates, we can claim the following results.

Theorem 10. *Let $D > s/\mu_1$ be a fixed positive constant. Then there exists a positive constant $d^* = d^*(\Lambda, D)$ such that model (9) has no positive nonconstant solution provided that $d_1 > d^*$ and $d_2 > D$.*

Proof. Let (N, P) be any positive solution of model (9) and denote $\overline{g} = |\Omega|^{-1}\int_\Omega g\,dx$. Then

$$\int_\Omega \left(N - \overline{N}\right) dx = \int_\Omega \left(P - \overline{P}\right) dx = 0. \tag{59}$$

Then, multiplying the first equation of model (9) by $(N - \overline{N})$, integrating over Ω, we have that

$$\begin{aligned} d_1 \int_\Omega \left|\nabla\left(N - \overline{N}\right)\right|^2 dx &= \int_\Omega \frac{rN(K-N)\left(N - \overline{N}\right)}{K + cN} dx - \int_\Omega \frac{mNP\left(N - \overline{N}\right)}{N + aP} dx \\ &\le \int_\Omega rN\left(1 - \frac{N}{K}\right)\left(N - \overline{N}\right) dx - \int_\Omega \frac{am\overline{P}P\left(N - \overline{N}\right)^2}{(N + aP)\left(\overline{N} + a\overline{P}\right)} dx \\ &\quad - \int_\Omega \frac{m\overline{N}N\left(N - \overline{N}\right)\left(P - \overline{P}\right)}{(N + aP)\left(\overline{N} + a\overline{P}\right)} dx \\ &\le \int_\Omega \left(r\left(N - \overline{N}\right)^2 + m\left|N - \overline{N}\right|\left|P - \overline{P}\right|\right) dx. \end{aligned} \tag{60}$$

In a similar manner, we multiply the second equation in model (9) by $(P - \overline{P})$ to have

$$\begin{aligned} d_2 \int_\Omega \left|\nabla\left(P - \overline{P}\right)\right|^2 dx &= \int_\Omega sP\left(1 - \frac{hP}{N}\right)\left(P - \overline{P}\right) dx \\ &= \int_\Omega s\left(P - \overline{P}\right) \times \left(P - \overline{P} - \frac{h\left(P + \overline{P}\right)\left(P - \overline{P}\right)}{N} + \frac{h\overline{P}^2\left(N - \overline{N}\right)}{\overline{N}N}\right) dx \\ &\le \int_\Omega s\left(\left(P - \overline{P}\right)^2 + \frac{h\overline{P}^2}{\overline{N}N}\left|N - \overline{N}\right|\left|P - \overline{P}\right|\right) dx. \end{aligned} \tag{61}$$

By the ε-Young inequality and the Poincaré inequality, we obtain that

$$\begin{aligned} &\int_\Omega \left(d_1\left|\nabla\left(N - \overline{N}\right)\right|^2 + d_2\left|\nabla\left(P - \overline{P}\right)\right|^2\right) dx \\ &\quad \le \int_\Omega \left(r\left(N - \overline{N}\right)^2 + 2M\left(N - \overline{N}\right)\left(P - \overline{P}\right) + s\left(P - \overline{P}\right)^2\right) dx \\ &\quad \le \frac{1}{\mu_1}\int_\Omega \left(\left|\nabla\left(N - \overline{N}\right)\right|^2\left(r + \frac{M}{\epsilon}\right) + \left|\nabla\left(P - \overline{P}\right)\right|^2 (s + \epsilon M)\right) dx \end{aligned} \tag{62}$$

for some positive constant M and an arbitrary small positive constant ϵ.

In view of $d_2 > D > s/\mu_1$, we can find a sufficiently small $\epsilon_0 > 0$ such that $d_2\mu_1 \ge s + \epsilon_0 M$. Let $d^* = (1/\mu_1)(r + M/\epsilon)$, then

$$\nabla\left(N - \overline{N}\right) = \nabla\left(P - \overline{P}\right) = 0 \tag{63}$$

and (N, P) must be a constant solution. This completes the proof. □

3.3. Existence of the Nonconstant Positive Solutions. In this subsection, we shall discuss the existence of the positive nonconstant solution of model (9).

Unless otherwise specified, in this subsection, we always require that $m < r(a + h)$ holds, which guarantees that model (9) has the unique positive constant solution $E^* = (N^*, P^*)$. From now on, we denote $\mathbf{w} = (N, P)^T$ and $\mathbf{w}_0 = E^*$.

Let $\mathbf{X}$ be the space defined in (25) and let

$$\mathbf{X}^+ = \left\{(N, P) \in \mathbf{X} \mid N, P > 0 \text{ on } C\left(\overline{\Omega}\right)\right\}. \tag{64}$$

We write model (9) in the following form:

$$\begin{aligned} -\Delta\mathbf{w} &= \mathcal{G}(\mathbf{w}), \quad \mathbf{w} \in \mathbf{X}^+, \\ \partial_\nu \mathbf{w} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{65}$$

where

$$\mathcal{G}(\mathbf{w}) = \begin{pmatrix} \dfrac{N}{d_1}\left(\dfrac{r(K-N)}{K + cN} - \dfrac{mP}{N + aP}\right) \\ \dfrac{sP}{d_2}\left(1 - \dfrac{hP}{N}\right) \end{pmatrix}. \tag{66}$$

Then $\mathbf{w}$ is a positive solution of model (65) if and only if $\mathbf{w}$ satisfies

$$\mathcal{F}(\mathbf{w}) = \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\{\mathcal{G}(\mathbf{w} + \mathbf{w})\} = 0, \quad \text{in } \mathbf{X}^{+}, \tag{67}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse operator of $\mathbf{I} - \Delta$ subject to the zero-flux boundary condition. Then

$$\nabla\mathcal{F}(\mathbf{w}_0) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\mathbf{I} + \mathcal{A}), \tag{68}$$

where

$$\mathcal{A} \triangleq \nabla\mathcal{G}(\mathbf{w}_0) = \begin{pmatrix} \dfrac{J_1}{d_1} & -\dfrac{mh^2}{d_1(a+h)^2} \\ -\dfrac{s}{d_2 h} & -\dfrac{s}{d_2} \end{pmatrix}. \tag{69}$$

If $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible, by Theorem 2.8.1 of [28], the index of $\mathcal{F}$ at $\mathbf{w}_0$ is given by

$$\operatorname{index}(\mathcal{F}, \mathbf{w}_0) = (-1)^{\gamma}, \tag{70}$$

where γ is the multiplicity of negative eigenvalues of $\nabla\mathcal{F}(\mathbf{w}_0)$.

On the other hand, using the decomposition (26), we have that $\mathbf{X}_i$ is an invariant space under $\nabla\mathcal{F}(\mathbf{w}_0)$ and $\xi \in \mathbb{R}$ is an eigenvalue of $\nabla\mathcal{F}(\mathbf{w}_0)$ in $\mathbf{X}_i$, if and only if, ξ is an eigenvalue of $(\mu_i + 1)^{-1}(\mu_i\mathbf{I} - \mathcal{A})$. Therefore, $\nabla\mathcal{F}(\mathbf{w}_0)$ is invertible, if and only if, for any $i \geq 0$ the matrix $\mu_i\mathbf{I} - \mathcal{A}$ is invertible.

Let $m(\mu_i)$ be the multiplicity of μ_i. For the sake of convenience, we denote

$$H(\mu) = \det(\mu\mathbf{I} - \mathcal{A}). \tag{71}$$

Then, if $\mu_i\mathbf{I} - \mathcal{A}$ is invertible for any $i \geq 0$, with the same arguments as in [29], we can assert the following conclusion.

Lemma 11. *Assume that, for all $i \geq 0$, the matrix $\mu_i\mathbf{I} - \mathcal{A}$ is nonsingular, then*

$$\operatorname{index}(\mathcal{F}, \mathbf{w}_0) = (-1)^{\gamma}, \quad \textit{where } \gamma = \sum_{i\geq 0, H(\mu_i)<0} m(\mu_i). \tag{72}$$

To compute $\operatorname{index}(\mathcal{F}, \mathbf{w}_0)$, we have to consider the sign of $H(\mu)$. A straightforward computation yields

$$H(\mu) = \mu^2 - \theta_1\mu + \theta_2, \tag{73}$$

where $\theta_1 = (d_2 J_1 - d_1 s)/d_1 d_2$, $\theta_2 = s(ar + hr - m)(ar + hr + cm)/d_1 d_2 r(1 + c)(a + h)^2$.

If $\theta_1^2 - 4\theta_2 > 0$, then $H(\mu) = 0$ has two positive solutions $\mu^{\pm}$ given by

$$\mu^{\pm} = \frac{1}{2}\left(\theta_1 \pm \sqrt{\theta_1^2 - 4\theta_2}\right). \tag{74}$$

Theorem 12. *Assume that $m(ar + cm + 2hr - acr) > r^2(a+h)^2$ and $\theta_1^2 - 4\theta_2 > 0$. If $\mu^- \in (\mu_i, \mu_{i+1})$ and $\mu^+ \in (\mu_j, \mu_{j+1})$ for some $0 \leq i < j$, and $\sum_{k=i+1}^{j} m(\mu_k)$ is odd, then model (9) has at least one nonconstant solution.*

Proof. By Theorem 10, we can fix $\overline{d}_1 > d_1$ and $\overline{d}_2 > d_2$ such that model (9) with diffusion coefficients $\overline{d}_1$ and $\overline{d}_2$ has no nonconstant solutions.

By virtue of Theorems 8 and 9, there exists a positive constant $\underline{C}$, $\overline{C}$ such that $\underline{C} < N, P < \overline{C}$.

Set

$$\mathcal{M} = \left\{(N,P) \in C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) : \underline{C} < N, \ P < \overline{C} \text{ in } \overline{\Omega}\right\}, \tag{75}$$

and define

$$\Psi : \mathcal{M} \times [0,1] \longrightarrow C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \tag{76}$$

by

$$\Psi(\mathbf{w}, t) = (\mathbf{I} - \Delta)^{-1}\{\mathbf{G}(\mathbf{w}, t) + \mathbf{w}\}, \tag{77}$$

where

$$\mathbf{G}(\mathbf{w}, t) = \begin{pmatrix} \left(td_1 + (1-t)\overline{d}_1\right)^{-1}\left(\dfrac{rN(K-N)}{K+cN} - \dfrac{mNP}{N+aP}\right) \\ \left(td_2 + (1-t)\overline{d}_2\right)^{-1}\left(sP\left(1 - \dfrac{hP}{N}\right)\right) \end{pmatrix}. \tag{78}$$

It is clear that finding the positive solution of model (9) becomes equivalent to finding the positive solution of $\Psi(\mathbf{w}, 1) = 0$ in $\mathcal{M}$. Further, by virtue of the definition of $\mathcal{M}$, we have that $\Psi(\mathbf{w}, t) = 0$ has no positive solution in $\partial\mathcal{M}$ for all $0 \leq t \leq 1$.

Since $\Psi(\mathbf{w}, t)$ is compact, the Leray-Schauder topological degree $\deg(\mathbf{I} - \Psi(\mathbf{w}, t), \mathcal{M}, 0)$ is well defined. From the invariance of Leray-Schauder degree at the homotopy, we deduce

$$\deg(\mathbf{I} - \Psi(\mathbf{w}, 1), \mathcal{M}, 0) = \deg(\mathbf{I} - \Psi(\mathbf{w}, 0), \mathcal{M}, 0). \tag{79}$$

Clearly, $\mathbf{I} - \Psi(\mathbf{w}, 1) = \mathcal{F}$. Thus, if model (9) has no other solutions except the constant one $\mathbf{w}_0$, then Lemma 11 shows that

$$\begin{aligned} \deg(\mathbf{I} - \Psi(\mathbf{w}, 1), \mathcal{M}, 0) &= \operatorname{index}(\mathcal{F}, \mathbf{w}_0) \\ &= (-1)^{\sum_{k=i+1}^{j} m(\mu_k)} = -1. \end{aligned} \tag{80}$$

On the contrary, by the choice of $\overline{d}_1$ and $\overline{d}_2$, we have that $\mathbf{w}_0$ is the only solution of $\Psi(\mathbf{w}, 0) = 0$. Furthermore, we have

$$\deg(\mathbf{I} - \Psi(\mathbf{w}, 0), \mathcal{M}, 0) = \operatorname{index}(\mathbf{I} - \Psi(\mathbf{w}, 0), \mathbf{w}_0) = 1. \tag{81}$$

From (79)–(81), we get a contradiction, and the proof is completed. □

Acknowledgments

The authors would like to thank the anonymous referee for the very helpful suggestions and comments which led to improvements of our original paper. And this work is supported by the Cooperative Project of Yulin City (2011).

References

[1] J. T. Tanner, "The stability and the intrinsic growth rates of prey and predator populations," *Ecology*, vol. 56, pp. 855–886, 1975.

[2] D. J. Wollkind, J. B. Collings, and J. A. Logan, "Metastability in a temperature-dependent model system for predator-prey mite outbreak interactions on fruit trees," *Bulletin of Mathematical Biology*, vol. 50, no. 4, pp. 379–409, 1988.

[3] C. S. Holling, "The functional response of predators to prey density and its role in mimicry and population regulation," *Memoirs of the Entomological Society of Canada*, vol. 45, supplement 45, pp. 5–60, 1965.

[4] M. P. Hassell, *The Dynamics of Arthropod Predator–Prey Systems*, Princeton University Press, Princeton, NJ, USA, 1978.

[5] P. H. Leslie and J. C. Gower, "The properties of a stochastic model for the predator-prey type of interaction between two species," *Biometrika*, vol. 47, pp. 219–234, 1960.

[6] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, USA, 1973.

[7] S. B. Hsu and T. W. Huang, "Global stability for a class of predator-prey systems," *SIAM Journal on Applied Mathematics*, vol. 55, no. 3, pp. 763–783, 1995.

[8] A. Gasull, R. E. Kooij, and J. Torregrosa, "Limit cycles in the Holling-Tanner model," *Publicacions Matemàtiques*, vol. 41, no. 1, pp. 149–167, 1997.

[9] E. Sáez and E. González-Olivares, "Dynamics of a predator-prey model," *SIAM Journal on Applied Mathematics*, vol. 59, no. 5, pp. 1867–1878, 1999.

[10] R. Arditi, L. R. Ginzburg, and H. R. Akcakaya, "Variation in plankton densities among lakes: a case for ratio-dependent models," *The American Naturalist*, vol. 138, no. 5, pp. 287–296, 1991.

[11] R. Arditi and H. Saiah, "Empirical evidence of the role of heterogeneity in ratio-dependent consumption," *Ecology*, vol. 73, pp. 1544–1551, 1992.

[12] A. P. Gutierrez, "The physiological basis of ratio-dependent predator-prey theory: a metabolic pool model of Nicholson's blowflies as an example," *Ecology*, vol. 73, pp. 1552–1563, 1992.

[13] T. Saha and C. Chakrabarti, "Dynamical analysis of a delayed ratio-dependent Holling-Tanner predator-prey model," *Journal of Mathematical Analysis and Applications*, vol. 358, no. 2, pp. 389–402, 2009.

[14] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: Ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.

[15] R. Arditi, N. Perrin, and H. Saiah, "Functional responses and heterogeneities: an experimental test with cladocerans," *Oikos*, vol. 60, no. 1, pp. 69–75, 1991.

[16] Z. Liang and H. Pan, "Qualitative analysis of a ratio-dependent Holling-Tanner model," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 954–964, 2007.

[17] M. Banerjee and S. Banerjee, "Turing instabilities and spatio-temporal chaos in ratio-dependent Holling-Tanner model," *Mathematical Biosciences*, vol. 236, no. 1, pp. 64–76, 2012.

[18] F. E. Smith, "Population dynamics in Daphnia Magna and a new model for population growth," *Ecology*, vol. 44, pp. 651–663, 1963.

[19] E. C. Pielou, *An Introduction to Mathematical Ecology*, Wiley, New York, NY, USA, 1969.

[20] T. G. Hallam and J. T. Deluna, "Effects of toxicants on populations: a qualitative approach III," *Journal of Theoretical Biology*, vol. 109, no. 3, pp. 411–429, 1984.

[21] K. Gopalsamy, M. R. S. Kulenović, and G. Ladas, "Environmental periodicity and time delays in a "food-limited" population model," *Journal of Mathematical Analysis and Applications*, vol. 147, no. 2, pp. 545–555, 1990.

[22] W. Feng and X. Lu, "On diffusive population models with toxicants and time delays," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 373–386, 1999.

[23] M. Fan and K. Wang, "Periodicity in a "food-limited" population model with toxicants and time delays," *Acta Mathematicae Applicatae Sinica*, vol. 18, no. 2, pp. 309–314, 2002.

[24] W. Chen and M. Wang, "Qualitative analysis of predator-prey models with Beddington-DeAngelis functional response and diffusion," *Mathematical and Computer Modelling*, vol. 42, no. 1-2, pp. 31–44, 2005.

[25] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 61 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1981.

[26] C.-S. Lin, W.-M. Ni, and I. Takagi, "Large amplitude stationary solutions to a chemotaxis system," *Journal of Differential Equations*, vol. 72, no. 1, pp. 1–27, 1988.

[27] Y. Lou and W.-M. Ni, "Diffusion, self-diffusion and cross-diffusion," *Journal of Differential Equations*, vol. 131, no. 1, pp. 79–131, 1996.

[28] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute, 2nd edition, 2001.

[29] R. Peng, J. Shi, and M. Wang, "On stationary patterns of a reaction-diffusion model with autocatalysis and saturation law," *Nonlinearity*, vol. 21, no. 7, pp. 1471–1488, 2008.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 606250, 5 pages
http://dx.doi.org/10.1155/2013/606250

Research Article

Effect of Delay on Selection Dynamics in Long-Term Sphere Culture of Cancer Stem Cells

Peng Tang,[1] Aijun Fan,[2] Jianquan Li,[3] Jun Jiang,[1] and Kaifa Wang[4]

[1] *Breast Disease Center, Southwest Hospital, Third Military Medical University, Chongqing 400038, China*
[2] *Chongqing Academy of Science & Technology, Chongqing 401123, China*
[3] *Department of Applied Mathematics and Physics, Air Force Engineering University, Xi'an 710051, China*
[4] *School of Biomedical Engineering, Third Military Medical University, Chongqing 400038, China*

Correspondence should be addressed to Kaifa Wang; kaifawang@yahoo.com.cn

Received 26 January 2013; Accepted 15 February 2013

Academic Editor: Yonghui Xia

To quantitatively study the effect of delay on selection dynamics in long-term sphere culture of cancer stem cells (CSCs), a selection dynamic model with time delay is proposed. Theoretical results show that the ubiquitous time delay in cell proliferation may be one of the important factors to induce fluctuation, and numerical simulations indicate that the proposed selection dynamical model with time delay can provide a better fitting effect for the experiment of a long-term sphere culture of CSCs. Thus, it is valuable to consider the delay effect in the future study on the dynamics of nongenetic heterogeneity of clonal cell populations.

1. Introduction

In the past years research on cancer stem cells (CSCs) has become a focus of cancer research, because CSCs have self-renewing and multidirectional differentiation capability and may result in tumors [1–6]. Recently, cell state dynamics due to non-genetic heterogeneity of clonal cell populations also has received more and more attention [7–10].

In order to expand CSCs, sphere culture is performed by experimental cell biologists [11, 12]. However, whether long-term sphere culture can maintain a high ratio of CSCs is unclear. For this question, it is interesting that [13, 14] obtain a similar quantitative result through different mathematical model; that is, the ratio of CSCs will towards an apparent equilibrium state in a long-term sphere culture. Concretely, [13] proposed a kinetic model using ordinary differential equations that considered the symmetric and asymmetric division of CSCs, as well as the proliferation and transformation of differentiated cancer cells (DCCs). And [14] puts forward a Markov model in which cells transition stochastically between states. However, the time delay due to the maturation of individual cells has been ignored in [13, 14].

In the present paper, based on the kinetic model in [13], we further explore the effect of time delay on selection dynamics in long-term sphere culture. The results show that the ubiquitous time delay in cell proliferation may be one of the important factors to induce fluctuation, and the proposed selection dynamical model with time delay can provide a better fitting effect for the experiment of a long-term sphere culture of CSCs in [13]. The organization of the paper is as follows. In Section 2, we formulate the selection dynamical model with time delay in long-term sphere culture of CSCs. Section 3 first gives the analytic analysis on our proposed model and then presents numerical simulations to compare the effect of time delay. Finally, some predictive conclusions with biological implications are given in Section 4.

2. Model Description

Let $x(t)$ and $y(t)$ denote the population sizes of CSCs and DCCs at time t in long-term sphere culture, respectively. Our previous work [13] proposed the following mathematical model to describe the interactive growth of the CSCs and DCCs in a long-term sphere culture:

$$\frac{dx(t)}{dt} = b_x(1-\beta_x)x(t) + b_y\beta_y y(t) \triangleq x(t) f_1(x(t), y(t)),$$
$$\frac{dy(t)}{dt} = b_x\beta_x x(t) + b_y(1-\beta_y)y(t) \triangleq y(t) f_2(x(t), y(t)). \quad (1)$$

Here the constants b_x, b_y are called the net birth rate or intrinsic growth rate of population x, y, respectively. β_x denotes the conversion rate from CSCs to DCCs in the process of CSCs proliferation, and β_y denotes the conversion rate from DCCs to CSCs in the process of DCCs proliferation.

Note that time delay may play an important role in many biological models. As shown in [15], the maturation of individual cells may need a period of time τ; that is, the number of these cells at time t may depend on the population at a previous time $t-\tau$. Under the assumption of equal discrete retarded cell proliferation, model (1) can be modified to

$$\frac{dx(t)}{dt} = x(t-\tau) f_1(x(t-\tau), y(t-\tau)),$$
$$\frac{dy(t)}{dt} = y(t-\tau) f_2(x(t-\tau), y(t-\tau)). \quad (2)$$

Here τ is the time delay due to maturation time.

For (2), inspired by [16], we give the following average fitness of the population:

$$\phi = b_x x(t-\tau) + b_y y(t-\tau). \quad (3)$$

Thus, the selection dynamics in long-term sphere culture of CSCs can be written as

$$\frac{dx(t)}{dt} = x(t-\tau)(f_1(x(t-\tau), y(t-\tau)) - \phi),$$
$$\frac{dy(t)}{dt} = y(t-\tau)(f_2(x(t-\tau), y(t-\tau)) - \phi). \quad (4)$$

For (4), let $N(t) = x(t) + y(t)$. We have

$$\frac{dN(t)}{dt} = (1 - N(t-\tau))\phi. \quad (5)$$

Therefore, $N(t) \to 1$ as $t \to \infty$; that is, the total population size remains constant. Hence $x(t)$ and $y(t)$ in (4) can be understood as the frequency of CSCs and DCCs, respectively. Furthermore, since $y(t)$ can be replaced by $1-x(t)$, system (4) describes only a single differential equation; that is,

$$\frac{dx(t)}{dt} = a_1 + a_2 x(t-\tau) + a_3 x^2(t-\tau) \triangleq f(x(t-\tau)), \quad (6)$$

in which $a_1 = b_y\beta_y, a_2 = b_x(1-\beta_x) - b_y(1+\beta_y)$, and $a_3 = b_y - b_x$. Note that (6) is the final selection dynamical model with time delay in long-term sphere culture of CSCs.

3. Results

3.1. Dynamic Analysis. The objective of this subsection is to analyze the dynamical behavior of (6). In order to explore the effect of the delay, we split this into two cases.

3.1.1. Case of $\tau = 0$. In this case, we focus on the dynamic analysis if the delay is nonexistent; that is, $\tau = 0$ in (6). We start by studying the existence of nonnegative equilibria in the interval $[0, 1]$. Let $f(x) = a_1 + a_2 x + a_3 x^2 = 0$. Clearly, the discriminant of the quadratic equation is

$$\Delta = a_2^2 - 4a_1a_3 = (b_y - b_x + b_x\beta_x - b_y\beta_y)^2 + 4b_xb_y\beta_x\beta_y > 0. \quad (7)$$

Hence there are two different real roots for $f(x) = 0$ if $a_3 \neq 0$. Furthermore, since

$$f(1) = a_1 + a_2 + a_3 = -b_x\beta_x < 0, \quad (8)$$

we know that there is a unique positive equilibrium $x^* \in (0,1)$ for (6) if $a_3 \neq 0$ (see Figures 1(a) and 1(b)). When $a_3 = 0$ (see Figure 1(c)), it is clear that there is only one positive equilibrium

$$x^* = -\frac{a_1}{a_2} = \frac{\beta_x}{\beta_x + \beta_y} < 1. \quad (9)$$

The combination of the above results and the phase diagram (see Figure 1(d)) of system (6) yields the following result.

Proposition 1. *For system* (6)*, when* $\tau = 0$*, a unique positive equilibrium* $x^* \in (0,1)$ *always exists, and it is globally asymptotically stable.*

3.1.2. Case of $\tau > 0$. In this case, we focus on the dynamic analysis if the delay is existent; that is, $\tau > 0$ in (6). Clearly, the unique positive equilibrium $x^* \in (0,1)$ still remains for (6) in spite of the delay. To study the stability of the equilibrium x^*, we first translate x^* to the origin. Let

$$\tilde{x} = x - x^*. \quad (10)$$

Then (6) becomes, after replacing $\tilde{x}$ by x again,

$$\frac{dx(t)}{dt} = (a_2 + 2a_3x^*)x(t-\tau) + a_3x^2(t-\tau). \quad (11)$$

The variational system of (11) at the origin is given by

$$\frac{dx(t)}{dt} = (a_2 + 2a_3x^*)x(t-\tau). \quad (12)$$

The characteristic equation of linear system (12) is given by

$$\lambda - (a_2 + 2a_3x^*)e^{-\tau\lambda} = 0. \quad (13)$$

If we let $\lambda = \alpha + i\beta$, then (13) becomes

$$\alpha - (a_2 + 2a_3x^*)e^{-\tau\alpha}\cos\beta\tau = 0,$$
$$\beta + (a_2 + 2a_3x^*)e^{-\tau\alpha}\sin\beta\tau = 0. \quad (14)$$

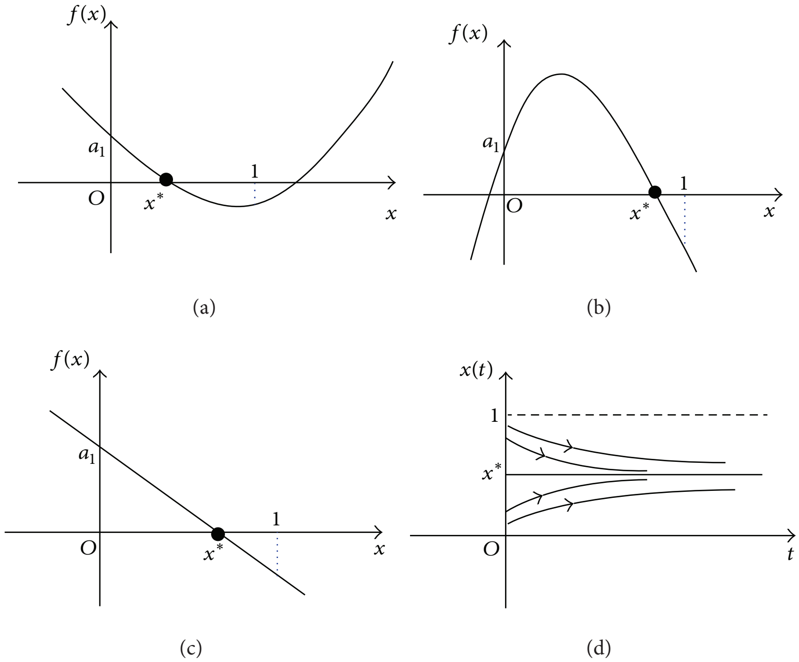


Figure 1: Illustrations of function image of $f(x) = 0$ under different cases ((a), (b), and (c)), and the phase diagram (d) of system (6). Here (a) $a_3 > 0$, (b) $a_3 < 0$, and (c) $a_3 = 0$.

By setting $\alpha = 0$ in (14), we have

$$\cos \beta\tau = 0,$$
$$\sin \beta\tau = -\frac{\beta}{a_2 + 2a_3 x^*}. \tag{15}$$

Solving the first algebraic equation in (15), we have

$$\beta\tau = \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, \ldots. \tag{16}$$

Substituting (16) into the second equation of (15), we have

$$\beta = -\left(a_2 + 2a_3 x^*\right). \tag{17}$$

Now, substituting (17) into (16), we have

$$\tau_n = -\frac{\pi}{2\left(a_2 + 2a_3 x^*\right)} - \frac{2n\pi}{a_2 + 2a_3 x^*}, \quad n = 0, 1, 2, \ldots. \tag{18}$$

Next, we compute $\alpha'(\tau_n)$. Differentiating (13) with respect to τ, we have

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = -\frac{\left(a_2 + 2a_3 x^*\right)\lambda e^{-\tau\lambda}}{1 + \tau\left(a_2 + 2a_3 x^*\right)e^{-\tau\lambda}}, \tag{19}$$

thus,

$$\alpha'(\tau_n) = \operatorname{Re}\left(\left.\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau_n}\right) = \frac{\beta^2}{1 + \tau_n^2\beta^2} > 0. \tag{20}$$

Therefore, similar to [17], according to the results in [18, 19] or [15, Theorem 2.2], we can obtain the following results on (6).

Proposition 2. *Suppose $\tau > 0$. Then system (6) has a Hopf bifurcation at*

$$\tau = \tau_n = -\frac{\pi}{2\left(a_2 + 2a_3 x^*\right)} - \frac{2n\pi}{a_2 + 2a_3 x^*}, \quad n = 0, 1, 2, \ldots. \tag{21}$$

Furthermore, according to the results in [15, 20, 21], we have the following.

Proposition 3. *Suppose $\tau > 0$ in (6). Then the unique positive equilibrium $x^* \in (0, 1)$ is stable if $0 < \tau < \tau_0$ and unstable if $\tau > \tau_0$.*

3.2. Numerical Simulations. For model (1), we designed a long-term sphere culture of human breast cancer MCF-7 stem cells [13]. Based on the experimental data, using an adaptive Metropolis-Hastings (M-H) algorithm to carry out an extensive Markov-chain Monte-Carlo (MCMC) simulation, we obtained the estimated parameter values as follows:

$$b_x = 2.7506 \times 10^{-1}, \quad b_y = 3.2635 \times 10^{-1},$$
$$\beta_x = 1.4407 \times 10^{-2}, \quad \beta_y = 2.3288 \times 10^{-3}. \tag{22}$$

When retarded cell proliferation was considered, based on the induced selection dynamic model (6) and the experimental data in [13], using extensive MCMC simulation again, we can obtain the estimated delay $\tau = 5.1401$ (Figure 2).

Using the estimated values, we can plot the best-fit solution by fitting model (1) and (6) to the experimental data,

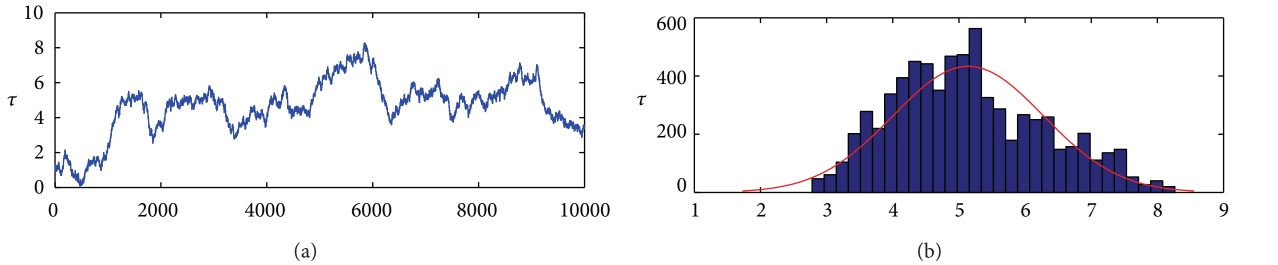


Figure 2: MCMC analysis of parameter τ based on (6), (22), and the experimental data in [13]. (a) is the random series, and (b) is its histogram. The algorithm ran for 10^4 iterations with a burn-in of 3000 iterations. The initial conditions were $\tau = 0.9$.

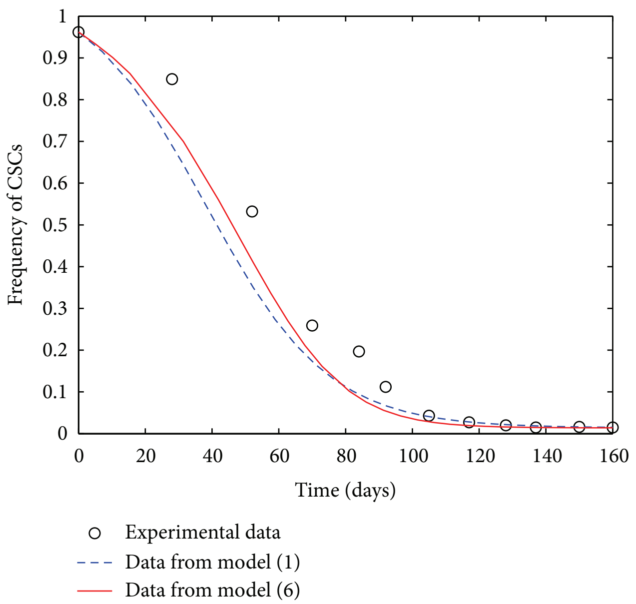


Figure 3: Simulations of the dynamical behaviors of the frequency of CSCs. Experimental data are represented by open circles. The blue dashed line denotes the best fit of model (1), and the red solid line denotes the best fit of model (6).

respectively, (Figure 3). From Figure 3, we find that there is a better simulation effect in model (6) than that in model (1). In fact, the sum of squares of the deviations (SSD) in (1) is $\mathrm{SSD}_{(1)} = 7.2943$, whereas $\mathrm{SSD}_{(6)} = 4.6387 \times 10^{-2}$ in (6). Note that $\mathrm{SSD}_{(6)}$ is far less than $\mathrm{SSD}_{(1)}$. These results quantitatively confirmed that the induced selection dynamic model (6) with time delay can provide a better fitting effect in long-term sphere culture of CSCs.

4. Conclusions

In order to demonstrate the interesting facts about the structural heterogeneity of cancer (the stable ratio between CSCs and DCCs), many studies have been reported because it is helpful for the cancer community to elucidate the controversy about the CSC hypothesis and the clone evolution theory of cancer [10, 13, 14] and references cited therein. In the present paper, a selection dynamic model with time delay is proposed, and its dynamical behavior is studied. Based on the theoretical analysis and numerical simulations, we can conclude the following predictive conclusions.

(i) The maturation of individual cells may produce a significant effect on the dynamic behavior of the selection dynamics. When the delay is nonexistent, the frequency of CSCs will tend to a stable size because the unique positive equilibrium is globally asymptotically stable (Proposition 1). Conversely, if the delay is existent, the unique positive equilibrium may not always maintain its stability and a Hopf bifurcation may be induced (Propositions 2 and 3); that is, an oscillated phenomenon may be induced by the maturation of individual cells.

(ii) Since the induced selection dynamic model (6) with time delay can provide a better fitting effect in long-term sphere culture of CSCs (Figure 3), it is reasonable to consider the delay effect in the future study on the dynamics of non-genetic heterogeneity of clonal cell populations.

Since mathematical models can be at best approximate the behavior of real biological process, the results presented here may extend those studies on the structural heterogeneity of cancer. Note that distributed delay may be more tractable and realistic than discrete delay in the applications of biology. Hence it is a worthwhile study in future work to better understand these topics based on the idea of [15].

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (nos. 30872517 and 11271369) and the Natural Science Foundation Project of CQ CSTC (2010BB5020).

References

[1] T. Reya, S. J. Morrison, M. F. Clarke, and I. L. Weissman, “Stem cells, cancer, and cancer stem cells,” *Nature*, vol. 414, no. 6859, pp. 105–111, 2001.

[2] J. Marx, “Mutant stem cells may seed cancer,” *Science*, vol. 301, no. 5638, pp. 1308–1310, 2003.

[3] M. Al-Hajj and M. F. Clarke, “Self-renewal and solid tumor stem cells,” *Oncogene*, vol. 23, no. 43, pp. 7274–7282, 2004.

[4] B. K. Abraham, P. Fritz, M. McClellan, P. Hauptvogel, M. Athelogou, and H. Brauch, “Prevalence of CD44+/CD24-$^{/low}$ cells in breast cancer may not be associated with clinical

outcome but may favor distant metastasis," *Clinical Cancer Research*, vol. 11, no. 3, pp. 1154–1159, 2005.

[5] M. Balic, H. Lin, L. Young et al., "Most early disseminated cancer cells detected in bone marrow of breast cancer patients have a putative breast cancer stem cell phenotype," *Clinical Cancer Research*, vol. 12, no. 19, pp. 5615–5621, 2006.

[6] M. S. Wicha, "Cancer stem cells and metastasis: lethal seeds," *Clinical Cancer Research*, vol. 12, no. 19, pp. 5606–5607, 2006.

[7] S. Huang, "Non-genetic heterogeneity of cells in development: more than just noise," *Development*, vol. 136, no. 23, pp. 3853–3862, 2009.

[8] T. Quinn and Z. Sinkala, "Dynamics of prostate cancer stem cells with diffusion and organism response," *BioSystems*, vol. 96, no. 1, pp. 69–79, 2009.

[9] S. J. Altschuler and L. F. Wu, "Cellular heterogeneity: do differences make a difference?" *Cell*, vol. 141, no. 4, pp. 559–563, 2010.

[10] S. Huang, "Tumor progression: chance and necessity in Darwinian and Lamarckian somatic (mutationless) evolution," *Progress in Biophysics and Molecular Biology*, vol. 110, no. 1, pp. 69–86, 2012.

[11] P. Cammareri, Y. Lombardo, M. G. Francipane, S. Bonventre, M. Todaro, and G. Stassi, "Isolation and culture of colon Cancer stem cells," *Methods in Cell Biology*, vol. 86, pp. 311–324, 2008.

[12] Y. Zhong, K. Guan, S. Guo et al., "Spheres derived from the human SK-RC-42 renal cell carcinoma cell line are enriched in cancer stem cells," *Cancer Letters*, vol. 299, no. 2, pp. 150–160, 2010.

[13] T. Peng, M. Qinghua, T. Zhenning, W. Kaifa, and J. Jun, "Long-term sphere culture cannot maintain a high ratio of Cancer stem cells: a mathematical model and experiment," *PLoS ONE*, vol. 6, no. 11, Article ID e25518, 2011.

[14] P. B. Gupta, C. M. Fillmore, G. Jiang et al., "Stochastic state transitions give rise to phenotypic equilibrium in populations of cancer cells," *Cell*, vol. 146, no. 4, pp. 633–644, 2011.

[15] Y. Yuan and J. Bélair, "Stability and Hopf bifurcation analysis for functional differential equation with distributed delay," *SIAM Journal on Applied Dynamical Systems*, vol. 10, no. 2, pp. 551–581, 2011.

[16] M. A. Nowak, *Evolutionary Dynamics: Exploring the Equations of Life*, The Belknap Press of Harvard University Press, Cambridge, Mass, USA, 2006.

[17] K. Wang, N. Zhang, and D. Niu, "Periodic oscillations in a spatially explicit model with delay effect for vegetation dynamics in freshwater marshes," *Journal of Biological Systems*, vol. 19, no. 2, pp. 131–147, 2011.

[18] J. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 1977.

[19] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, Mass, USA, 1993.

[20] H. I. Freedman, J. W.-H. So, and P. Waltman, "Coexistence in a model of competition in the chemostat incorporating discrete delays," *SIAM Journal on Applied Mathematics*, vol. 49, no. 3, pp. 859–870, 1989.

[21] T. Zhao, "Global periodic solutions for a differential delay system modeling a microbial population in the chemostat," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 329–352, 1995.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 598517, 7 pages
http://dx.doi.org/10.1155/2013/598517

Research Article

The Atom-Bond Connectivity Index of Catacondensed Polyomino Graphs

Jinsong Chen,[1,2] Jianping Liu,[1] and Qiaoliang Li[2]

[1] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350108, China*
[2] *College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China*

Correspondence should be addressed to Jinsong Chen; fzu_cjs@126.com

Received 3 January 2013; Accepted 4 February 2013

Academic Editor: Zhengkun Huan

Let $G = (V, E)$ be a graph. The atom-bond connectivity (ABC) index is defined as the sum of weights $((d_u + d_v - 2)/d_u d_v)^{1/2}$ over all edges uv of G, where d_u denotes the degree of a vertex u of G. In this paper, we give the atom-bond connectivity index of the zigzag chain polyomino graphs. Meanwhile, we obtain the sharp upper bound on the atom-bond connectivity index of catacondensed polyomino graphs with h squares and determine the corresponding extremal graphs.

1. Introduction

One of the most active fields of research in contemporary chemical graph theory is the study of topological indices (graph topological invariants) that can be used for describing and predicting physicochemical and pharmacological properties of organic compounds. In chemistry and for chemical graphs, these invariant numbers are known as the topological indices. There are many publications on the topological indices, see [1–6].

Let $G = (V, E)$ be a simple graph of order n. A few years ago, Estrada et al. [7] introduced a further vertex-degree-based graph invariant, known as the atom-bond connectivity (ABC) index. It is defined as:

$$\mathrm{ABC}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}. \quad (1)$$

The ABC index keeps the spirit of the Randić index, and it provides a good model for the stability of linear and branched alkanes as well as the strain energy of cycloalkanes [7]. Recently, the study of the ABC index attracts some research attention [6, 8–12].

Polyomino graphs [13], also called chessboards [14] or square-cell configurations [15] have attracted some mathematicians' considerable attention because many interesting combinatorial subjects are yielded from them such as domination problem and modeling problems of surface chemistry. A polyomino graph [16] is a connected geometric graph obtained by arranging congruent regular squares of side length 1 (called a cell) in a plane such that two squares are either disjoint or have a common edge. The polyomino graph has received considerable attentions.

Next, we introduce some graph definitions used in this paper.

Definition 1 (see [4]). Let G be a polyomino graph. If all vertices of G lie on its perimeter, then G is said to be catacondensed polyomino graph or tree-like polyomino graph. (see Figure 1).

Definition 2 (see [16]). Let G be a chain polyomino graph with h squares. If the subgraph obtained from G by deleting all the vertices of degree 2 and all the edges adjacent to the vertices is a path, then G is said to be the zigzag chain polyomino graph, denoted by Z_h (see Figure 1).

In this paper, we give the ABC indices of the zigzag chain polyomino graphs with h squares and obtain the sharp upper bound on the ABC indices of catacondensed polyomino graphs with h squares and determine the corresponding extremal graphs.

2. The ABC Indices of Catacondensed Polyomino Graphs

Let

$$
\begin{aligned}
S_1 = &\left\{\sqrt{2}+\frac{2}{3}, \sqrt{2}+\frac{\sqrt{15}}{3}-\frac{2}{3}, 2\right\},\\
S_2 = &\left\{2, \frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{3}-2,\right.\\
&\frac{4}{3}+\frac{\sqrt{15}}{6}, \frac{2}{3}+\frac{\sqrt{15}}{3}, \sqrt{2}+\frac{\sqrt{15}}{3}-\frac{2}{3},\\
&\sqrt{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{6}-\frac{2}{3}, \sqrt{2}+\frac{\sqrt{6}}{4},\\
&\sqrt{2}+\frac{\sqrt{6}}{2}-\frac{\sqrt{15}}{6}, \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{2}{3},\\
&\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{6}, \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}+\frac{2}{3}-\frac{\sqrt{15}}{6},\\
&\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{3}-\frac{2}{3}, \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2},\\
&\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{2}-\frac{8}{3}, \frac{3\sqrt{2}}{2}+\frac{2\sqrt{6}}{4}+\frac{\sqrt{15}}{6}-2,\\
&\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{4\sqrt{15}}{6}-\frac{10}{3},\\
&\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}+\frac{2\sqrt{15}}{6}-\frac{8}{3}, \frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-2,\\
&\frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-\frac{\sqrt{15}}{3}-\frac{2}{3}, \frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-\frac{\sqrt{15}}{6}-\frac{4}{3},\\
&\frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{\sqrt{15}}{2}-\frac{2}{3}, \frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{2\sqrt{15}}{6}-\frac{4}{3},\\
&\frac{3\sqrt{2}}{2}+\frac{5\sqrt{6}}{4}-\frac{2\sqrt{15}}{3}-\frac{2}{3},\\
&\left.\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}-\frac{4}{3}, \frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}+\frac{\sqrt{15}}{6}-\frac{8}{3}\right\},\\
S = &S_1 \bigcup S_2.
\end{aligned}
\tag{2}
$$

We call $\sqrt{(d_u+d_v-2)/d_u d_v}$ the weight of the edge uv, denoted by W_{uv}.

Note that for any catacondensed polyomino graph H^* with h squares, it can be obtained by gluing a new square s to some catacondensed polyomino graph H with $h-1$ squares. So, we have the following lemma.

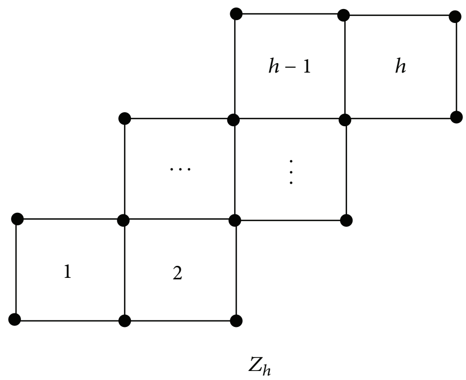


Figure 1: The zigzag chain polyomino graph Z_h.

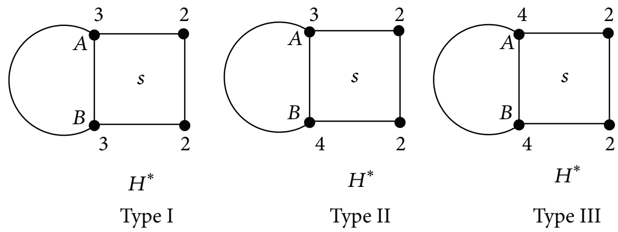


Figure 2

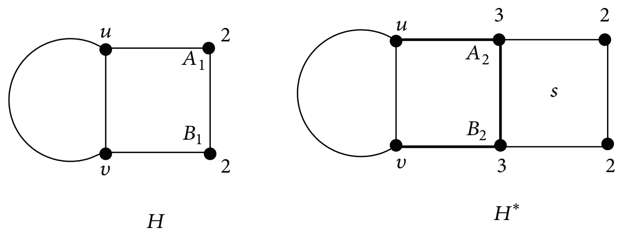


Figure 3

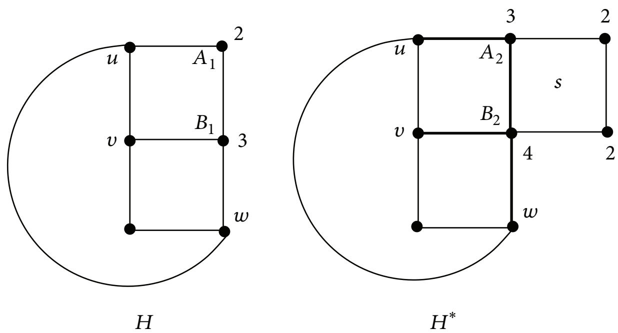


Figure 4

Lemma 3. *Let H^* be a catacondensed polyomino graph with h squares which is obtained by gluing a new square s to some graph H, where H is a catacondensed polyomino graph with $h-1$ squares. One has*

(i) *If* $2 \le h \le 3$, *then* $ABC(H^*) - ABC(H) \in S_1$,

(ii) *if* $h \ge 4$, *then* $ABC(H^*) - ABC(H) \in S_2$.

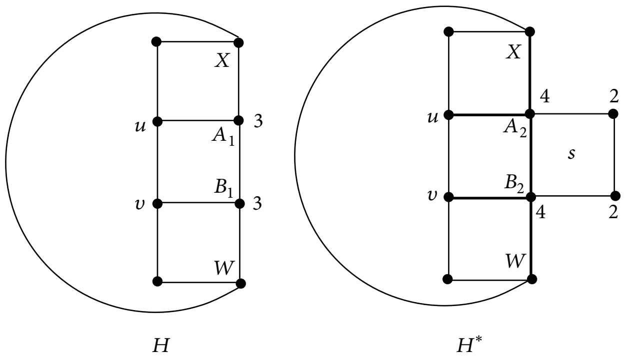


Figure 5

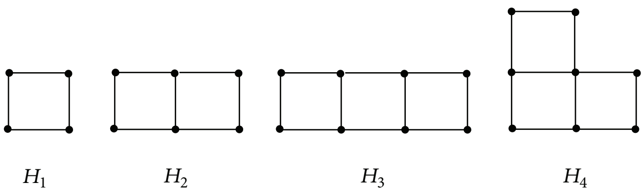


Figure 6: A catacondensed polyomino graph with h ($h \le 3$) squares.

Proof. Consider the following: (i) if $2 \le h \le 3$, by directly calculating, we have $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) \in S_1$,

(ii) now, let $h \ge 4$. Without the loss of generality, let square s be adjacent to the edge AB in H (see Figure 2). In the following, if the weights of some edges of H have been changed when s is adjacent to the edge AB in H, then we marked these edges with thick lines in H^*. Let $D_i = \mathrm{ABC}(H^*) - \mathrm{ABC}(H)$ $(i = 1, 2, \ldots, 35)$. Note that except the edge AB of s, the summation of the weights of the remaining three edges is always $(3/2)\sqrt{2}$ in H^*. There are exactly three types of formations (see Figure 2).

Case 1. In Type I, $d_{A_1} = d_{B_1} = 2$ and $d_{A_2} = d_{B_2} = 3$ (see Figure 3).

By the definition of ABC index, we have $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) = (3/2)\sqrt{2} + (W_{uA_2} - W_{uA_1}) + (W_{A_2B_2} - W_{A_1B_1}) + (W_{vB_2} - W_{vB_1}) = D_i$ $(i = 1, 2, 3)$.

If $d_u = 3$ and $d_v = 3$, then $D_1 = 2$.

If $d_u = 3$ and $d_v = 4$ or $d_u = 4$ and $d_v = 3$, then $D_2 = 4/3 + \sqrt{15}/6$.

If $d_u = 4$ and $d_v = 4$, then $D_3 = 2/3 + \sqrt{15}/3$.

Case 2. In Type II, $d_{A_1} = 2$, $d_{B_1} = 3$, $d_{A_2} = 3$, and $d_{B_2} = 4$. (see Figure 4).

Let u adjacent to A and v, w adjacent to B (see Figure 4). Then $d_u \in \{2, 3, 4\}$, $d_v \in \{3, 4\}$, and $d_w \in \{2, 3, 4\}$. If $d_u = 2$, $d_v = 3$, and $d_w = 2$, which is in contradiction with $h \ge 4$; if $d_u = 3$ and $d_v = 3$, which is in contradiction with $d_{B_1} = 3$; if $d_u = 4$ and $d_v = 3$, which is in contradiction with $d_{A_1} = 2$ $(A \in V(H))$.

By the definition of ABC index, we have $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) = (3/2)\sqrt{2} + (W_{uA_2} - W_{uA_1}) + (W_{A_2B_2} - W_{A_1B_1}) + (W_{vB_2} - W_{vB_1}) + (W_{wB_2} - W_{wB_1}) = D_i$ $(i = 4, 5, \ldots, 14)$.

If $d_u = 2$, $d_v = 3$, and $d_w = 3$, then $D_4 = \sqrt{2} + \sqrt{15}/2 - 4/3$.

If $d_u = 2$, $d_v = 3$, and $d_w = 4$, then $D_5 = \sqrt{2} + \sqrt{6}/4 + \sqrt{15}/6 - 2/3$.

If $d_u = 2$, $d_v = 4$, and $d_w = 2$, then $D_6 = \sqrt{2} + \sqrt{6}/4$.

If $d_u = 2$, $d_v = 4$, and $d_w = 3$, then $D_7 = \sqrt{2} + \sqrt{6}/4 + \sqrt{15}/6 - 2/3$.

If $d_u = 2$, $d_v = 4$, and $d_w = 4$, then $D_8 = \sqrt{2} + \sqrt{6}/2 - \sqrt{15}/6$.

If $d_u = 3$, $d_v = 4$, and $d_w = 2$, then $D_9 = \sqrt{2}/2 + \sqrt{6}/4 + 2/3$.

If $d_u = 3$, $d_v = 4$, and $d_w = 3$, then $D_{10} = \sqrt{2}/2 + \sqrt{6}/4 + \sqrt{5/12}$.

If $d_u = 3$, $d_v = 4$, and $d_w = 4$, then $D_{11} = \sqrt{2}/2 + \sqrt{6}/2 + 2/3 - \sqrt{5/12}$.

If $d_u = 4$, $d_v = 4$, and $d_w = 2$, then $D_{12} = \sqrt{2}/2 + \sqrt{6}/4 + \sqrt{5/12}$.

If $d_u = 4$, $d_v = 4$, and $d_w = 3$, then $D_{13} = \sqrt{2}/2 + \sqrt{6}/4 + 2\sqrt{5/12} - 2/3$.

If $d_u = 4$, $d_v = 4$, and $d_w = 4$, then $D_{14} = \sqrt{2}/2 + \sqrt{6}/2$.

Case 3. In Type III, $d_{A_1} = d_{B_1} = 3$ and $d_{A_2} = d_{B_2} = 4$ (see Figure 5).

Let u, x adjacent to A and v, w adjacent to B (see Figure 5). Then, $d_u \in \{3, 4\}$, $d_v \in \{3, 4\}$, $d_w \in \{2, 3, 4\}$, and $d_x \in \{2, 3, 4\}$. Since the case $d_u = 3$, $d_v = 4$, $d_x = y_1$, and $d_w = y_2$ is the same as $d_u = 4$, $d_v = 3$, $d_x = y_2$, and $d_w = y_1$, where $y_1, y_2 \in \{2, 3, 4\}$. And note that if $d_u = d_v = 3$ or $d_u = d_v = 4$, the vertices x and w are symmetric.

By the definition of *ABC* index, we have $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) = (3/2)\sqrt{2} + (W_{xA_2} - W_{xA_1}) + (W_{uA_2} - W_{uA_1}) + (W_{A_2B_2} - W_{A_1B_1}) + (W_{vB_2} - W_{vB_1}) + (W_{wB_2} - W_{wB_1}) = D_i$ $(i = 15, 16, \ldots, 35)$.

If $d_u = d_v = 3$, $d_x = 2$, and $d_w = 3$, then $D_{15} = 3\sqrt{2}/2 + \sqrt{6}/4 + \sqrt{15}/2 - 8/3$.

If $d_u = d_v = 3$, $d_x = 2$, and $d_w = 4$, then $D_{16} = 3\sqrt{2}/2 + \sqrt{6}/2 + \sqrt{15}/6 - 2$.

If $d_u = d_v = 3$, $d_x = 3$, and $d_w = 3$, then $D_{17} = 3\sqrt{2}/2 + \sqrt{6}/4 + 2\sqrt{15}/3 - 10/3$.

If $d_u = d_v = 3$, $d_x = 3$, and $d_w = 4$, then $D_{18} = 3\sqrt{2}/2 + \sqrt{6}/2 + \sqrt{15}/3 - 8/3$.

If $d_u = d_v = 3$, $d_x = 4$, and $d_w = 4$, then $D_{19} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - 2$.

If $d_u = d_v = 4$, $d_x = 2$, and $d_w = 2$, then $D_{20} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - \sqrt{15}/3 - 2/3$.

If $d_u = d_v = 4$, $d_x = 2$, and $d_w = 3$, then $D_{21} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - \sqrt{15}/6 - 4/3$.

If $d_u = d_v = 4$, $d_x = 2$, and $d_w = 4$, then $D_{22} = 3\sqrt{2}/2 + \sqrt{6} - \sqrt{15}/2 - 2/3$.

If $d_u = d_v = 4$, $d_x = 3$, and $d_w = 3$, then $D_{23} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - 2$.

If $d_u = d_v = 4$, $d_x = 3$, and $d_w = 4$, then $D_{24} = 3\sqrt{2}/2 + \sqrt{6} - \sqrt{15}/3 - 4/3$.

If $d_u = d_v = 4$, $d_x = 4$, and $d_w = 4$, then $D_{25} = 3\sqrt{2}/2 + 5\sqrt{6}/4 - 2\sqrt{15}/3 - 2/3$.

If $d_u = 3$, $d_v = 4$, $d_x = 2$, and $d_w = 2$, then $D_{26} = 3\sqrt{2}/2 + \sqrt{6}/2 - 4/3$.

If $d_u = 3$, $d_v = 4$, $d_x = 2$, and $d_w = 3$, then $D_{27} = 3\sqrt{2}/2 + \sqrt{6}/2 + \sqrt{15}/6 - 2$.

If $d_u = 3$, $d_v = 4$, $d_x = 2$, and $d_w = 4$, then $D_{28} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - \sqrt{15}/6 - 4/3$.

If $d_u = 3$, $d_v = 4$, $d_x = 3$, and $d_w = 2$, then $D_{29} = 3\sqrt{2}/2 + \sqrt{6}/2 + \sqrt{15}/6 - 2$.

If $d_u = 3$, $d_v = 4$, $d_x = 3$, and $d_w = 3$, then $D_{30} = 3\sqrt{2}/2 + \sqrt{6}/2 + \sqrt{15}/3 - 8/3$.

If $d_u = 3$, $d_v = 4$, $d_x = 3$, and $d_w = 4$, then $D_{31} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - 2$.

If $d_u = 3$, $d_v = 4$, $d_x = 4$, and $d_w = 2$, then $D_{32} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - \sqrt{15}/6 - 4/3$.

If $d_u = 3$, $d_v = 4$, $d_x = 4$, and $d_w = 3$, then $D_{33} = 3\sqrt{2}/2 + 3\sqrt{6}/4 - 2$.

If $d_u = 3$, $d_v = 4$, $d_x = 4$, and $d_w = 4$, then $D_{34} = 3\sqrt{2}/2 + \sqrt{6} - \sqrt{15}/3 - 4/3$.

If $d_u = 3$, $d_v = 3$, $d_x = 2$, and $d_w = 2$, then $D_{35} = 3\sqrt{2}/2 + \sqrt{6}/4 + \sqrt{15}/3 - 2$.

By directly calculating, we have $D_5 = D_7$, $D_{10} = D_{12}$, $D_{16} = D_{27} = D_{29}$, $D_{18} = D_{30}$, $D_{19} = D_{23} = D_{31} = D_{33}$, $D_{21} = D_{28} = D_{32}$, $D_{24} = D_{34}$, and $D_6 = \max_{1\le i\le 35} D_i$, $D_{14} = \min_{1\le i\le 35} D_i$. So $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) \in S_2$, where $h \ge 4$.

Therefore, $\mathrm{ABC}(H^*) - \mathrm{ABC}(H) \in S$. □

By Lemma 3, we have the following theorem.

Theorem 4. *Let G be a catacondensed polyomino graph with h ($h \ge 2$) squares, then*

$$\begin{aligned}
ABC(G) = {} & 3\sqrt{2} + \frac{2}{3} + \left(\sqrt{2} + \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_1 \\
& + 2a_2 + \left(\frac{4}{3} + \frac{\sqrt{15}}{6}\right) a_3 \\
& + \left(\frac{2}{3} + \frac{\sqrt{15}}{3}\right) a_4 \\
& + \left(\sqrt{2} + \frac{\sqrt{15}}{2} - \frac{4}{3}\right) a_5 \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{6} - \frac{2}{3}\right) a_6 \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{4}\right) a_7 \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{2} - \frac{\sqrt{15}}{6}\right) a_8 \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{2}{3}\right) a_9 \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{6}\right) a_{10} \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{2}{3} - \frac{\sqrt{15}}{6}\right) a_{11} \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_{12} \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\right) a_{13} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{2} - \frac{8}{3}\right) a_{14} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{15}}{6} - 2\right) a_{15} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{2\sqrt{15}}{3} - \frac{10}{3}\right) a_{16} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{15}}{3} - \frac{8}{3}\right) a_{17} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{4} - 2\right) a_{18} \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{4} - \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_{19}
\end{aligned}$$

$$+\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-\frac{\sqrt{15}}{6}-\frac{4}{3}\right)a_{20}$$
$$+\left(\frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{\sqrt{15}}{2}-\frac{2}{3}\right)a_{21}$$
$$+\left(\frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{\sqrt{15}}{3}-\frac{4}{3}\right)a_{22}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{5\sqrt{6}}{4}-\frac{2\sqrt{15}}{3}-\frac{2}{3}\right)a_{23}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}-\frac{4}{3}\right)a_{24}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{3}-2\right)a_{25}, \tag{3}$$

where a_i is a nonnegative integer for $i = 1, 2, \ldots, 25$ and $h = 2 + \sum_{i=1}^{25} a_i$.

Proof. We prove Theorem 4 by the induction on h. If $h = 2$, by directly calculating, we have $\mathrm{ABC}(G) = 3\sqrt{2} + 2/3$, where $a_i = 0$ ($i = 1, 2, \ldots, 25$). So, Theorem 4 holds for $h = 2$.

Assume that Theorem 4 holds for all catacondensed polyomino graphs with $h-1$ ($h-1 \geq 2$) squares, that is,

$$\mathrm{ABC}(G) = 3\sqrt{2}+\frac{2}{3}+\left(\sqrt{2}+\frac{\sqrt{15}}{3}-\frac{2}{3}\right)a_1$$
$$+2a_2+\left(\frac{4}{3}+\frac{\sqrt{15}}{6}\right)a_3$$
$$+\left(\frac{2}{3}+\frac{\sqrt{15}}{3}\right)a_4$$
$$+\left(\sqrt{2}+\frac{\sqrt{15}}{2}-\frac{4}{3}\right)a_5$$
$$+\left(\sqrt{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{6}-\frac{2}{3}\right)a_6$$
$$+\left(\sqrt{2}+\frac{\sqrt{6}}{4}\right)a_7$$
$$+\left(\sqrt{2}+\frac{\sqrt{6}}{2}-\frac{\sqrt{15}}{6}\right)a_8$$
$$+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{2}{3}\right)a_9$$
$$+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{6}\right)a_{10}$$
$$+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}+\frac{2}{3}-\frac{\sqrt{15}}{6}\right)a_{11}$$
$$+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{3}-\frac{2}{3}\right)a_{12}$$
$$+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)a_{13}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{2}-\frac{8}{3}\right)a_{14}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}+\frac{\sqrt{15}}{6}-2\right)a_{15}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{2\sqrt{15}}{3}-\frac{10}{3}\right)a_{16}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}+\frac{\sqrt{15}}{3}-\frac{8}{3}\right)a_{17}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-2\right)a_{18}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-\frac{\sqrt{15}}{3}-\frac{2}{3}\right)a_{19}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{6}}{4}-\frac{\sqrt{15}}{6}-\frac{4}{3}\right)a_{20}$$
$$+\left(\frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{\sqrt{15}}{2}-\frac{2}{3}\right)a_{21}$$
$$+\left(\frac{3\sqrt{2}}{2}+\sqrt{6}-\frac{\sqrt{15}}{3}-\frac{4}{3}\right)a_{22}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{5\sqrt{6}}{4}-\frac{2\sqrt{15}}{3}-\frac{2}{3}\right)a_{23}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}-\frac{4}{3}\right)a_{24}$$
$$+\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{15}}{3}-2\right)a_{25}, \tag{4}$$

where a_i is a nonnegative integer for $i = 1, 2, \ldots, 25$ and $h-1 = 2 + \sum_{i=1}^{25} a_i$.

We will prove that Theorem 4 holds for h in the following. Let G^* be a catacondensed polyomino graph with h squares. Without the loss of generality, G^* can be obtained from some catacondensed polyomino graph G with $h-1$ squares by gluing a new square s to G. By Lemma 3, we have $\mathrm{ABC}(G^*) - \mathrm{ABC}(G) \in S$. It means that $\mathrm{ABC}(G^*) = \mathrm{ABC}(G) + a$, where

$a \in S$. By the induction assumption and direct computation, we have

$$\begin{aligned}
\mathrm{ABC}(G^*) = {} & 3\sqrt{2} + \frac{2}{3} + \left(\sqrt{2} + \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_1^* \\
& + 2a_2^* + \left(\frac{4}{3} + \frac{\sqrt{15}}{6}\right) a_3^* \\
& + \left(\frac{2}{3} + \frac{\sqrt{15}}{3}\right) a_4^* \\
& + \left(\sqrt{2} + \frac{\sqrt{15}}{2} - \frac{4}{3}\right) a_5^* \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{6} - \frac{2}{3}\right) a_6^* \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{4}\right) a_7^* \\
& + \left(\sqrt{2} + \frac{\sqrt{6}}{2} - \frac{\sqrt{15}}{6}\right) a_8^* \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{2}{3}\right) a_9^* \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{6}\right) a_{10}^* \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{2}{3} - \frac{\sqrt{15}}{6}\right) a_{11}^* \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_{12}^* \\
& + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\right) a_{13}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{2} - \frac{8}{3}\right) a_{14}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{15}}{6} - 2\right) a_{15}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{2\sqrt{15}}{3} - \frac{10}{3}\right) a_{16}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{15}}{3} - \frac{8}{3}\right) a_{17}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{4} - 2\right) a_{18}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{4} - \frac{\sqrt{15}}{3} - \frac{2}{3}\right) a_{19}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{4} - \frac{\sqrt{15}}{6} - \frac{4}{3}\right) a_{20}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \sqrt{6} - \frac{\sqrt{15}}{2} - \frac{2}{3}\right) a_{21}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \sqrt{6} - \frac{\sqrt{15}}{3} - \frac{4}{3}\right) a_{22}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{5\sqrt{6}}{4} - \frac{2\sqrt{15}}{3} - \frac{2}{3}\right) a_{23}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2} - \frac{4}{3}\right) a_{24}^* \\
& + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{3} - 2\right) a_{25}^* .
\end{aligned} \tag{5}$$

There exists some $l \in \{1, 2, \ldots, 25\}$ such that $a_l^* = a_l + 1$ and $a_j^* = a_j$ for $j \neq l$ ($j \in \{1, 2, \ldots, 25\}$). Obviously, a_i^* is a nonnegative integer for $i = 1, 2, \ldots, 25$ and $2 + \sum_{i=1}^{25} a_i^* = 2 + 1 + \sum_{i=1}^{25} a_i = h$. □

Lemma 5. *Let H be a catacondensed polyomino graph with h squares. If $h \leq 3$, there are exactly four nonisomorphism catacondensed polyomino graphs (see Figure 6), where $ABC(H_1) = 2\sqrt{2}$, $ABC(H_2) = 3\sqrt{2} + 2/3$, $ABC(H_3) = 3\sqrt{2} + 8/3$, $ABC(H_4) = 4\sqrt{2} + \sqrt{15}/3$.*

Theorem 6. *Let Z_h be a zigzag chain polyomino graph with h squares, then*

$$ABC(Z_h) = \begin{cases} 2\sqrt{2}, & h = 1, \\ 3\sqrt{2} + \dfrac{2}{3}, & h = 2, \\ (h+1)\sqrt{2} + (h-3) \cdot \dfrac{\sqrt{6}}{4} + \dfrac{\sqrt{15}}{3}, & h \geq 3. \end{cases} \tag{6}$$

Proof. Obviously, Z_h can be obtained by gluing a new square s_h to Z_{h-1}. Let s_{h-1} be the square adjacent to s_h (see Figure 1). We will prove Theorem 6 by the induction on h.

If $h = 1, 2, 3$, then Theorem 6 holds (by Lemma 5). Assume that $\mathrm{ABC}(Z_{h-1}) = (h-1+1)\sqrt{2} + (h-1-3)\cdot(\sqrt{6}/4) + (\sqrt{15}/3) = h\sqrt{2} + (h-4)\cdot(\sqrt{6}/4) + (\sqrt{15}/3)$ for $h - 1 \geq 3$. By the induction assumption and the D_6 in Lemma 3, we have

$$\begin{aligned}
\mathrm{ABC}(Z_h) &= \mathrm{ABC}(Z_{h-1}) + D_6 \\
&= h\sqrt{2} + (h-4) \cdot \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{3} + \left(\sqrt{2} + \frac{\sqrt{6}}{4}\right) \\
&= (h+1)\sqrt{2} + (h-3) \cdot \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{3}.
\end{aligned} \tag{7}$$

So, Theorem 6 holds. □

Note that $D_6 = \max_{1\leq i\leq 35} D_i$ for $h \geq 4$ and by Lemma 5, we obtain the following Theorem 7.

Theorem 7. *Let G be a catacondensed polyomino graph with h squares, then $ABC(G) \leq (h+1)\sqrt{2}+(h-3)\cdot(\sqrt{6}/4)+(\sqrt{15}/3)$, with the equality if and only if $G \cong Z_h$.*

Acknowledgments

The project was supported by NSFC (no. 11071272; 11101086; 11101087) and the Foundation of Fuzhou University (no. 2012-XQ-30).

References

[1] A. Behtoei, M. Jannesari, and B. Taeri, "Maximum Zagreb index, minimum hyper-Wiener index and graph connectivity," *Applied Mathematics Letters*, vol. 22, no. 10, pp. 1571–1576, 2009.

[2] K. C. Das, I. Gutman, and B. Furtula, "Survey on geometric-arithmetic indices of graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, no. 3, pp. 595–644, 2011.

[3] B. Liu and I. Gutman, "On general Randić indices," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 58, no. 1, pp. 147–154, 2007.

[4] B. Liu and Z. You, "A survey on comparing Zagreb indices," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, no. 3, pp. 581–593, 2011.

[5] J. Liu and B. Liu, "E-L equienergetic graphs," *Communications in Mathematical and in Computer Chemistry*, vol. 66, no. 3, pp. 971–976, 2011.

[6] B. Zhou and N. Trinajstic, "Further results on atom-bond connectivity index of trees," *Chemical Physics Letters*, vol. 455, pp. 120–123, 2008.

[7] E. Estrada, L. Torres, L. Rodriguez, and I. Gutman, "An atom-bond connectivity index: modelling the enthalpy of formation of alkanes," *Indian Journal of Chemistry A*, vol. 37, pp. 849–855, 1998.

[8] J. Chen and X. Guo, "Extreme atom-bond connectivity index of graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, no. 3, pp. 713–722, 2011.

[9] J. Chen and X. Guo, "The atom-bond connectivity index of chemical bicyclic graphs," *Applied Mathematics B*, vol. 27, no. 2, pp. 243–252, 2012.

[10] J. Chen, J. Liu, and X. Guo, "Some upper bounds for the atom-bond connectivity index of graphs," *Applied Mathematics Letters*, vol. 25, no. 7, pp. 1077–1081, 2012.

[11] B. Furtula, A. Graovac, and D. Vukičević, "Atom-bond connectivity index of trees," *Discrete Applied Mathematics*, vol. 157, no. 13, pp. 2828–2835, 2009.

[12] R. Xing, B. Zhou, and Z. Du, "Further results on atom-bond connectivity index of trees," *Discrete Applied Mathematics*, vol. 158, no. 14, pp. 1536–1545, 2010.

[13] C. Berge, C. C. Chen, V. Chvátal, and C. S. Seow, "Combinatorial properties of polyominoes," *Combinatorica*, vol. 1, no. 3, pp. 217–224, 1981.

[14] E. J. Cockayne, "Chessboard domination problems," *Discrete Mathematics*, vol. 86, no. 1–3, pp. 13–20, 1990.

[15] F. Harary and P. Mezey, "Cell-shedding transformations, equivalence relations, and similarity measures for square-cell configurations," *International Journal of Quantum Chemistry*, vol. 62, pp. 353–361, 1997.

[16] R. Chen, "Perfect matchings of generalized polyomino graphs," *Graphs and Combinatorics*, vol. 21, no. 4, pp. 515–529, 2005.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 635149, 5 pages
http://dx.doi.org/10.1155/2013/635149

Research Article

Monotonicity of Eventually Positive Solutions for a Second Order Nonlinear Difference Equation

Huiqin Chen,[1] Zhen Jin,[2] and Shugui Kang[1]

[1] *School of Mathematics and Computer Sciences, Datong University, Datong, Shanxi 037009, China*
[2] *Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China*

Correspondence should be addressed to Shugui Kang; dtkangshugui@126.com

Received 7 October 2012; Revised 27 December 2012; Accepted 19 January 2013

Academic Editor: Tailei Zhang

We derive several sufficient conditions for monotonicity of eventually positive solutions on a class of second order perturbed nonlinear difference equation. Furthermore, we obtain a few nonexistence criteria for eventually positive monotone solutions of this equation. Examples are provided to illustrate our main results.

1. Introduction

The theory of difference equations and their applications have received intensive attention. In the last few years, new research achievements kept emerging (see [1–7]). Among them, in [3], Saker considered the second order nonlinear delay difference equation

$$\Delta\left(p_n \Delta x_n\right) + q_n f\left(x_{n-\sigma}\right) = 0, \quad n \geqslant 0. \tag{1}$$

Saker used the Riccati transformation technique to obtain several sufficient conditions which guarantee that every solution of (1) oscillates or converges to zero. In [4], Rath et al. considered the more general second order equations

$$\begin{aligned} &\Delta\left(r_n \Delta\left(y_n - p_n y_{n-m}\right)\right) + q_n G\left(y_{n-k}\right) = 0, \quad n \geqslant 0, \\ &\Delta\left(r_n \Delta\left(y_n - p_n y_{n-m}\right)\right) + q_n G\left(y_{n-k}\right) = f_n, \quad n \geqslant 0. \end{aligned} \tag{2}$$

They found necessary conditions for the solutions of the above equations to be oscillatory or tend to zero. Following this trend, this paper is concerned with the second order perturbed nonlinear difference equation

$$\Delta\left(a_n \Delta x_n\right) + P\left(n, x_n, x_{n+1}\right) = Q\left(n, x_n, \Delta x_n\right), \quad n \geqslant 0, \tag{3}$$

where $\{a_n\}$ is a positive sequence, $P, Q : N \times R^2 \rightarrow R$ are two continuous functions, and Δ is the forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$.

In [8], Li and Cheng considered the special case of (3)

$$\Delta\left(p_{n-1} \Delta x_{n-1}\right) + q_n f\left(x_n\right) = 0, \quad n \geqslant 0. \tag{4}$$

They got the sufficient conditions for asymptotically monotone solutions of (4). Enlightened by [8, 9], in this paper, we derive several sufficient conditions for monotonicity of eventually positive solutions on (3) and obtain a few nonexistence criteria for eventually positive monotone solutions of (3). Our results improve and generalize results in [8]. We also provide examples to illustrate our main results.

For convenience, these essential conditions used in main results are listed as follows:

(H_1) there exists a continuous function $f : R \rightarrow R$ such that $xf(x) > 0$ for all $x \neq 0$;

(H_2) f is a derivable function and $f'(x) \geqslant 0$ for $x \neq 0$;

(H_3) there exist two sequences $\{p_n\}$ and $\{q_n\}$, such that $P(n, x_n, x_{n+1})/f(x_{n+1}) \geqslant p_n$ and $Q(n, x_n, \Delta x_n)/f(x_{n+1}) \leqslant q_n$ for $x_n \neq 0$;

(H_4) $\sum_{n=n_0}^{\infty} 1/a_n = +\infty$, n_0 is a positive integral number,

where a_n, $P(n, x_n, x_{n+1})$ and $Q(n, x_n, \Delta x_{n+1})$ are all in (3).

2. Main Results

We first state a result which relates a positive sequence and a positive nondecreasing function. Its proof can be found in [8].

Lemma 1 (see [8]). *Let $f(x)$ be a positive nondecreasing function defined for $x > 0$. Let $\{x_k\}$ be a real sequence such that $x_k > 0$ for $i \leqslant k \leqslant j+1$. Then*

$$\sum_{k=i}^{j} \frac{\Delta x_k}{f(x_{k+1})} \leqslant \int_{x_i}^{x_{j+1}} \frac{du}{f(u)} \leqslant \sum_{k=i}^{j} \frac{\Delta x_k}{f(x_k)}. \tag{5}$$

Theorem 2. *Suppose that conditions (H_1)–(H_4) hold, p_n and q_n satisfy the following conditions:*

(H_5) $\sum_{s=n_0}^{\infty}(p_s - q_s) < +\infty$;

(H_6) $\liminf_{n\to\infty} \sum_{s=n_0}^{n}(p_s - q_s) \geqslant 0$

for all n_0. Then eventually positive solutions of (3) *are eventually monotone increasing.*

Proof. Suppose that $\{x_n\}$ is a positive solution of (3), say $x_n > 0$ for $n > N > n_0$. If conclusion cannot hold, without any loss of generality, assume $\Delta x_N \leqslant 0$, in view of (3) and conditions, we have

$$\begin{aligned}\Delta\left(\frac{a_n \Delta x_n}{f(x_n)}\right) &= \frac{\Delta(a_n \Delta x_n)}{f(x_{n+1})} - \frac{a_n (\Delta x_n)^2 f'(x_n + \theta \Delta x_n)}{f(x_n) f(x_{n+1})} \\ &\leqslant q_n - p_n \quad (0 < \theta < 1),\end{aligned} \tag{6}$$

by summing (6) from N to $n-1$, then

$$\frac{a_n \Delta x_n}{f(x_n)} \leqslant \frac{a_N \Delta x_N}{f(x_N)} - \sum_{s=N}^{n-1}(p_s - q_s). \tag{7}$$

Making use of condition (H_6), we know $\Delta x_n < 0$ for $n \geqslant N$. Summing (3) and using (H_3), we have

$$\begin{aligned} a_n \Delta x_n &\leqslant a_N \Delta x_N - \sum_{s=N}^{n-1} f(x_{s+1})(p_s - q_s) \\ &= a_N \Delta x_N - f(x_{n+1}) \sum_{s=N}^{n-1}(p_s - q_s) \\ &\quad + \sum_{s=N}^{n-1} \Delta f(x_s) \left(\sum_{t=N}^{s-1}(p_t - q_t) \right) \\ &\leqslant a_N \Delta x_N. \end{aligned} \tag{8}$$

By summing (8), we then see that

$$x_{n+1} \leqslant x_N + a_N \Delta x_N \sum_{s=N}^{n} \frac{1}{a_s} \longrightarrow -\infty \quad (\text{as } n \longrightarrow \infty), \tag{9}$$

which contradicts the fact $x_n > 0$. The proof is complete. □

Example 3. Consider the difference equation

$$\begin{aligned}\Delta\left(\frac{\Delta x_n}{n^2}\right) + x_{n+1}\left(r(n, x_n) + \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^3}\right) \\ = x_{n+1} r(n, x_n), \quad n \geqslant 0,\end{aligned} \tag{10}$$

where $r(n, x_n)$ is any function of n and x_n. By taking $f(x) = x$, we have

$$\begin{aligned}\frac{P(n, x_n, x_{n+1})}{f(x_{n+1})} &= r(n, x_n) + \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^3} = p_n, \\ \frac{Q(n, x_n, \Delta x_n)}{f(x_{n+1})} &= r(n, x_n) = q_n.\end{aligned} \tag{11}$$

So conditions of Theorem 2 hold. By Theorem 2, (10) has a positive monotone increasing solution $\{x_n\} = \{n\}$.

Theorem 4. *If conditions (H_1)–(H_4) hold, there exist $M > 0$ and $j > n_0$ for $n_0 \geqslant M$ such that*

$$\limsup_{n\to\infty} \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=n_0}^{k-1}(p_s - q_s) > 0. \tag{12}$$

Then eventually positive solutions $\{x_n\}$ of (3) *are eventually monotone increasing or* $\liminf_{n\to\infty} x_n = 0$.

Proof. Suppose $\{x_n\}$ is a positive solution of (3), there exists $N > n_0$ such that $x_n > 0$ for $n > N$. Let $\Delta x_N \leqslant 0$, and

$$\limsup_{n\to\infty} \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=N}^{k-1}(p_s - q_s) > 0, \quad j > N. \tag{13}$$

If $\liminf_{n\to\infty} x_n \neq 0$, then there exist $T \geqslant N$ and a number $\alpha > 0$ such that $x_n > \alpha > 0$ for $n \geqslant T$; in view of (7), we get

$$\frac{\Delta x_n}{f(x_n)} \leqslant \frac{a_N \Delta x_N}{f(x_N)} \cdot \frac{1}{a_n} - \frac{1}{a_n} \sum_{s=N}^{n-1}(p_s - q_s). \tag{14}$$

Summing (14) and making use of Lemma 1, we know

$$\begin{aligned}\int_{x_j}^{\alpha} \frac{du}{f(u)} &\leqslant \int_{x_j}^{x_{n+1}} \frac{du}{f(u)} \\ &\leqslant \frac{a_N \Delta x_N}{f(x_N)} \sum_{k=j}^{n} \frac{1}{a_k} - \sum_{k=j}^{n} \frac{1}{a_k} \sum_{s=N}^{k-1}(p_s - q_s).\end{aligned} \tag{15}$$

By (H_4), the right side of (15) tends to $-\infty$ as $n \to \infty$, whereas the left side is finite. This contradiction completes our proof. □

Example 5. Consider the difference equation

$$\Delta\left(\frac{\Delta x_n}{\sqrt{n}}\right) + \frac{x_{n+1}}{\sqrt{n}} = \frac{\sqrt{n+2}}{n+1} x_{n+1}, \quad n \geqslant 0. \tag{16}$$

By taking $f(x) = x$, we have $P(n, x_n, x_{n+1})/f(x_{n+1}) = 1/\sqrt{n} = p_n, Q(n, x_n, \Delta x_n)/f(x_{n+1}) = \sqrt{n+2}/n+1 = q_n$. So conditions of Theorem 4 hold. By Theorem 4, (16) has a positive monotone increasing solution $\{x_n\} = \{\sqrt{n}\}$.

Theorem 6. *If conditions* (H_1)–(H_3) *hold, and*

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{s=n_0}^{n-1} (p_s - q_s) > 0 \tag{17}$$

holds for all n_0. *Then eventually positive solutions* $\{x_n\}$ *of* (3) *are eventually monotone increasing or eventually monotone decreasing and* $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose $\{x_n\}$ is a positive solution of (3), there exists $N > n_0$ such that $x_n > 0$ for $n > N$. Let $\Delta x_N \leqslant 0$ and $\liminf_{n\to\infty} 1/a_n \sum_{s=N}^{n-1} (p_s - q_s) > 0$, then there exists $\beta > 0$ such that

$$\frac{1}{a_n} \sum_{s=N}^{n-1} (p_s - q_s) \geqslant \beta > 0, \quad n \geqslant N. \tag{18}$$

From (7), we have

$$\Delta x_n \leqslant -f(x_n) \cdot \frac{1}{a_n} \sum_{s=N}^{n-1} (p_s - q_s) \leqslant -\beta f(x_n) < 0, \quad n > N. \tag{19}$$

If $\lim_{n\to\infty} x_n \neq 0$, then there exists $c > 0$ such that $x_n \geqslant c > 0$. There is no harm in assumption $x_n \geqslant c$ for $n \geqslant N$. Summing (19), we obtain

$$c \leqslant x_{n+1} \leqslant x_N - (n + 1 - N) \beta f(c) \longrightarrow -\infty \quad (n \longrightarrow \infty), \tag{20}$$

which is a contrary. The proof is complete. □

Example 7. Consider the difference equation

$$\Delta\left(n^2 \Delta x_n\right) + x_{n+1}\left(r(n, x_n) + \frac{1}{n+2}\right) = x_{n+1} r(n, x_n), \tag{21}$$

where $r(n, x_n)$ is any function of n and x_n. By taking $f(x) = x$, we have

$$\begin{gathered} \frac{P(n, x_n, x_{n+1})}{f(x_{n+1})} = r(n, x_n) + \frac{1}{n+2} = p_n, \\ \frac{Q(n, x_n, \Delta x_n)}{f(x_{n+1})} = r(n, x_n) = q_n. \end{gathered} \tag{22}$$

So conditions of Theorem 6 hold. By Theorem 6, (21) has a monotone decreasing positive solution $\{x_n\} = \{1/n\}$.

Theorem 8. *If conditions* (H_1)–(H_3) *hold and*

(H_7) $\limsup_{n\to\infty} \sum_{s=k}^{n} 1/a_s \sum_{t=n_0}^{s-1} (p_t - q_t) = +\infty$;

(H_8) *for all* $\varepsilon > 0$, $\int_0^{\varepsilon} du/f(u) < +\infty$.

Then eventually positive solutions of (3) *are eventually monotone increasing.*

Proof. Suppose $x_n > 0$ for $n > N > n_0$, $\{x_n\}$ is a solution of (3), and $\limsup_{n\to\infty} \sum_{s=k}^{n} 1/a_s \sum_{t=N}^{s-1} (p_t - q_t) = \infty$. If the result does not hold, without any loss of generality, assume $\Delta x_N \leqslant 0$. In view of (7), we see that

$$\begin{aligned} \frac{\Delta x_n}{f(x_n)} &\leqslant \frac{a_N \Delta x_N}{f(x_N)} \cdot \frac{1}{a_n} - \frac{1}{a_n} \sum_{t=N}^{n-1} (p_t - q_t) \\ &\leqslant -\frac{1}{a_n} \sum_{t=N}^{n-1} (p_t - q_t). \end{aligned} \tag{23}$$

Summing (23) and using Lemma 1, we know

$$\int_{x_k}^{x_{n+1}} \frac{du}{f(u)} \leqslant \sum_{s=k}^{n} \frac{\Delta x_s}{f(x_s)} \leqslant -\sum_{s=k}^{n} \frac{1}{a_s} \sum_{t=N}^{s-1} (p_t - q_t). \tag{24}$$

This is a contradiction. The proof is complete. □

Remark 9. In Theorems 2 and 4, condition (H_4) is essential; that is, the series with positive terms $\sum_{n=n_0}^{\infty} 1/a_n$ is divergent, but it is not required in Theorems 6 and 8.

Remark 10. The eventually positive solutions in Theorems 4 and 8 are increasing it is not necessarily so in Theorem 6.

Next, we will derive several nonexistence criteria for eventually positive monotone solutions of (3).

Theorem 11. *If conditions* (H_1)–(H_3) *hold and*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} (p_k - q_k) = +\infty. \tag{25}$$

Then, (3) *cannot have any eventually positive monotone increasing solutions.*

Proof of Theorem 11 is obvious. If $x_n > 0$ is an eventually positive increasing solution, by means of conditions, (7) is a contrary.

Theorem 12. *If conditions* (H_1)–(H_3) *hold, and there is a nonnegative and nondegenerate sequence* $\{\varphi_n\}$ *such that*

$$\limsup_{n\to\infty} \frac{\sum_{k=n_0}^{n} \varphi_{k+1}/a_k \sum_{s=n_0}^{k-1} (p_s - q_s)}{\sum_{k=n_0}^{n} \varphi_{k+1}/a_k} = \infty \tag{26}$$

holds for all n_0. *Then,* (3) *cannot have any eventually positive nondecreasing solutions.*

Proof. Suppose that $\{x_n\}$ is a positive solution of (3), there exists $N > n_0$ such that $x_n > 0$ and $\Delta x_n \geqslant 0$ for $n > N$. Multiplying (7) by φ_{n+1}/a_n, we have

$$\frac{\varphi_{n+1} \Delta x_n}{f(x_n)} + \frac{\varphi_{n+1}}{a_n} \sum_{s=N}^{n-1} (p_s - q_s) \leqslant \frac{\varphi_{n+1}}{a_n} \cdot \frac{a_N \Delta x_N}{f(x_N)}. \tag{27}$$

So we obtain

$$\sum_{k=N}^{n} \frac{\varphi_{k+1}}{a_k} \sum_{s=N}^{k-1} (p_s - q_s) \leqslant \frac{a_N \Delta x_N}{f(x_N)} \sum_{k=N}^{n} \frac{\varphi_{k+1}}{a_k}. \tag{28}$$

This is contrary to our condition. The proof is complete. □

Theorem 13. *If (H_1) and (H_3) hold, $\{a_n\}$ is a nondecreasing sequence, $f(x)$ is a nondecreasing function, and there is a nonnegative sequence $\{\varphi_n\}$, where $\{\Delta\varphi_n\}$ is bounded, and*

(H_9) $\lim_{n\to\infty}\sum_{s=n_0}^{n-1}\varphi_{s+1}(p_s-q_s)/a_{s+1}=+\infty$ *for all* n_0;

(H_{10}) $0<\int_{\varepsilon}^{+\infty}du/f(u)<+\infty,\ \varepsilon>0$.

Then, (3) cannot have any eventually positive monotone increasing solutions.

Proof. Suppose that $\{x_n\}$ is a solution of (3), and there exists $N>n_0$ such that $x_n>0$ and $\Delta x_n>0$ for $n>N$. Multiplying (3) by $\varphi_{n+1}/a_{n+1}f(x_{n+1})$ and summing from N to $n-1$ again, we have

$$\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}f(x_{s+1})}\Delta\left(a_s\Delta x_s\right)\leqslant\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}\left(q_s-p_s\right). \tag{29}$$

Namely,

$$\begin{aligned}&\frac{\varphi_n\Delta x_n}{f(x_n)}-\frac{\varphi_N\Delta x_N}{f(x_N)}-\sum_{s=N}^{n-1}a_s\Delta x_s\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)\\&\quad\leqslant\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}\left(q_s-p_s\right).\end{aligned} \tag{30}$$

As $\{a_n\}$ is a nondecreasing sequence, we get

$$\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)=\frac{\varphi_{s+1}}{a_{s+1}f(x_{s+1})}-\frac{\varphi_s}{a_sf(x_s)}\leqslant\frac{\Delta\varphi_s}{a_{s+1}f(x_{s+1})}. \tag{31}$$

Thus

$$a_s\Delta x_s\cdot\Delta\left(\frac{\varphi_s}{a_sf(x_s)}\right)\leqslant\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}. \tag{32}$$

From (30), we obtain

$$\frac{\varphi_n\Delta x_n}{f(x_n)}-\frac{\varphi_N\Delta x_N}{f(x_N)}+\sum_{s=N}^{n-1}\frac{\varphi_{s+1}}{a_{s+1}}\left(p_s-q_s\right)\leqslant\sum_{s=N}^{n-1}\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}, \tag{33}$$

using Lemma 1 and conditions, we have

$$\sum_{s=N}^{n-1}\frac{\Delta x_s\Delta\varphi_s}{f(x_{s+1})}\leqslant M\sum_{s=N}^{n-1}\frac{\Delta x_s}{f(x_{s+1})}\leqslant M\int_{x_N}^{+\infty}\frac{du}{f(u)},\qquad M>0. \tag{34}$$

By letting $n\to\infty$, we see that the left-hand side of (33) is bounded, this is contrary to our condition (H_9). The proof is complete. □

By means of proof of Theorem 13, we get

Corollary 14. *If (H_1), (H_3), and (H_9) hold, $\{a_n\}$ is a nondecreasing sequence, $f(x)$ is a nondecreasing function, and there is a nonnegative sequence $\{\varphi_n\}$, $\{\Delta\varphi_n\}$ is bounded, and $0<\int_0^{\varepsilon}du/f(u)<+\infty$ for $\varepsilon>0$. Then, (3) cannot have any eventually positive nondecreasing bounded solutions.*

Corollary 15. *Suppose (H_1), (H_3), and (H_9) hold, $\{a_n\}$ is a nondecreasing sequence, $f(x)$ is a nondecreasing function, and there is a nonnegative nonincreasing sequence $\{\varphi_n\}$. Then, (3) cannot have any eventually positive monotone increasing solutions.*

Theorem 16. *Suppose (H_1), (H_3), and (H_5) hold, $f(x)$ is a nondecreasing function, and*

(H_{11}) $\limsup_{n\to\infty}\sum_{k=n_0}^{n}1/a_k\sum_{s=k}^{\infty}(p_s-q_s)=+\infty$ *for all* n_0;

(H_{12}) $0<\int_{\varepsilon}^{+\infty}du/f(u)<+\infty,\ \varepsilon>0$.

Then, (3) cannot have any eventually positive nondecreasing solutions.

Proof. Assume to the contrary that there exists $N>n_0$ such that $x_n>0$ and $\Delta x_n>0$ for $n>N$. $\{x_n\}$ is a solution of (3). By means of (3) and (H_3), we get

$$\frac{\Delta\left(a_n\Delta x_n\right)}{f(x_{n+1})}\leqslant\left(q_n-p_n\right), \tag{35}$$

by summing (35) from N to $n-1$, thus

$$\begin{aligned}&\frac{a_n\Delta x_n}{f(x_{n+1})}-\frac{a_N\Delta x_N}{f(x_{N+1})}-\sum_{s=N}^{n-1}a_{s+1}\Delta x_{s+1}\Delta\left(\frac{1}{f(x_{s+1})}\right)\\&\quad\leqslant\sum_{s=N}^{n-1}\left(q_s-p_s\right).\end{aligned} \tag{36}$$

As $f(x)$ is a nondecreasing function, we know $\Delta x_{s+1}\Delta(1/f(x_{s+1}))\leqslant 0$, so

$$\sum_{s=N}^{\infty}\left(p_s-q_s\right)\leqslant\frac{a_N\Delta x_N}{f(x_{N+1})}. \tag{37}$$

In view of Lemma 1, we see that

$$\sum_{N=T}^{n}\frac{1}{a_N}\sum_{s=N}^{\infty}\left(p_s-q_s\right)\leqslant\sum_{N=T}^{n}\frac{\Delta x_N}{f(x_{N+1})}\leqslant\int_{x_T}^{x_{n+1}}\frac{du}{f(u)}. \tag{38}$$

This contradiction establishes our assertion. □

By means of proof of Theorem 16, we obtain the following.

Corollary 17. *Suppose (H_1), (H_3), (H_5), and (H_{11}) hold, $f(x)$ is a nondecreasing function. Then (3) cannot have any eventually positive nondecreasing bounded solutions.*

Acknowledgments

The authors are very grateful to the referee for her/his valuable suggestions. This work is supported by National Science Foundation of China (11271235), Shanxi Province (2008011002-1), Shanxi Datong University (2009-Y-15,2010-B-1), Sci-tech Research and Development Projects in Institutions of Higher Education of Shanxi Province (20111117, 20111020), and the Program for international cooperation of Shanxi Province (2010081005).

References

[1] A. Andruch-Sobiło and A. Drozdowicz, "Asymptotic behaviour of solutions of third order nonlinear difference equations of neutral type," *Mathematica Bohemica*, vol. 133, no. 3, pp. 247–258, 2008.

[2] R. N. Rath, N. Misra, and S. K. Rath, "Sufficient conditions for oscillatory behaviour of a first order neutral difference equation with oscillating coefficients," *Acta Mathematica*, vol. 25, no. 1, pp. 55–63, 2009.

[3] S. H. Saker, "Oscillation of second order nonlinear delay difference equations," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 3, pp. 489–501, 2003.

[4] R. N. Rath, J. G. Dix, B. L. S. Barik, and B. Dihudi, "Necessary conditions for the solutions of second order non-linear neutral delay difference equations to be oscillatory or tend to zero," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 60907, 16 pages, 2007.

[5] B. Karpuz, R. Rath, and S. K. Rath, "On oscillation and asymptotic behaviour of a higher order functional difference equation of neutral type," *International Journal of Difference Equations*, vol. 4, no. 1, pp. 69–96, 2009.

[6] S. S. Cheng and G. Zhang, "Monotone solutions of a higher-order neutral difference equation," *Georgian Mathematical Journal*, vol. 5, no. 1, pp. 49–54, 1998.

[7] E. Thandapani, Z. Liu, R. Arul, and P. S. Raja, "Oscillation and asymptotic behavior of second order difference equations with nonlinear neutral terms," *Applied Mathematics E-Notes*, vol. 4, pp. 59–67, 2004.

[8] H. J. Li and S. S. Cheng, "Asymptotically monotone solutions of a nonlinear difference equation," *Tamkang Journal of Mathematics*, vol. 24, no. 3, pp. 269–282, 1993.

[9] H. J. Li and S. S. Cheng, "An oscillation theorem for a second order nonlinear difference equation," *Utilitas Mathematica*, vol. 44, pp. 177–181, 1993.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 683091, 7 pages
http://dx.doi.org/10.1155/2013/683091

Research Article

Almost Periodic Solutions for Wilson-Cowan Type Model with Time-Varying Delays

Shasha Xie and Zhenkun Huang

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Zhenkun Huang; hzk974226@jmu.edu.cn

Received 9 October 2012; Accepted 26 December 2012

Academic Editor: Junli Liu

Wilson-Cowan model of neuronal population with time-varying delays is considered in this paper. Some sufficient conditions for the existence and delay-based exponential stability of a unique almost periodic solution are established. The approaches are based on constructing Lyapunov functionals and the well-known Banach contraction mapping principle. The results are new, easily checkable, and complement existing periodic ones.

1. Introduction

Consider a well-known Wilson-Cowan type model [1, 2] with time-varying delays

$$\begin{aligned}\frac{dX_P(t)}{dt} &= -X_P(t) + \left[k_P - r_P X_P(t)\right] \\ &\quad\times G\left[w_P^1 X_P(t-\tau_P(t))\right. \\ &\qquad \left. - w_N^1 X_N(t-\tau_N(t)) + I_P(t)\right], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + \left[k_N - r_N X_N(t)\right] \\ &\quad\times G\left[w_P^2 X_P(t-\tau_P(t))\right. \\ &\qquad \left. - w_N^2 X_N(t-\tau_N(t)) + I_N(t)\right],\end{aligned} \tag{1}$$

where $X_P(t), X_N(t)$ represent the proportion of excitatory and inhibitory neurons firing per unit time at the instant t, respectively. r_P and r_N are related to the duration of the refractory period, k_P and k_N are constants. w_P^1, w_N^1, w_P^2, and w_N^2 are the strengths of connections between the populations. $I_P(t), I_N(t)$ are the external inputs to the excitatory and the inhibitory populations. $G(\cdot)$ is the response function of neuronal activity and it is always assumed to be sigmoid type. $\tau_P(t)$, $\tau_N(t)$ correspond to the transmission time-varying delays.

It is interesting to revisit Wilson-Cowan system on the following points.

(i) The Wilson-Cowan model has a realistic biological background which describes interactions between excitatory and inhibitory populations of neurons [1–3]. It has extensive application such as pattern analysis and image processing [4, 5].

(ii) There exists rich dynamical behavior in Wilson-Cowan model. Theoretical results about stable limit cycles, equilibria, chaos, and oscillatory activity have been reported in [2, 3, 6–9]. Recently, Decker and Noonburg [8] reported new results about the existence of three periodic solutions when each neuron was stimulated by periodical inputs. However, under time-varying (periodic or almost periodic) inputs, Wilson-Cowan model can have more complex state space and coexistence of divergent solutions and local stable solutions which could not be easily estimated by its boundary. To see this, we can refer to Figure 1 for the phase portrait of solutions of the following

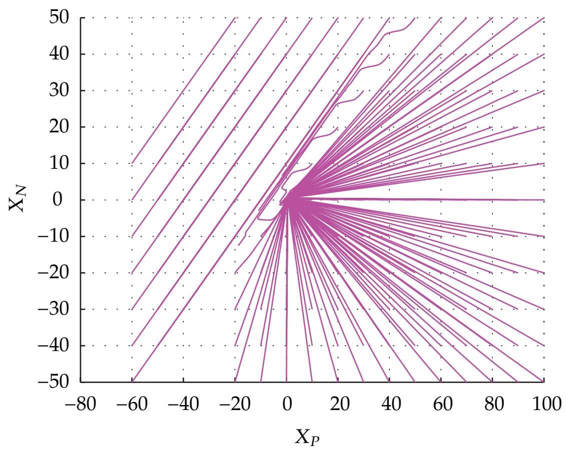


FIGURE 1: Coexistence of divergent and local stable solutions of (2) with almost periodic inputs.

Wilson-Cowan type model with $G(z) = \tanh(z)$ and almost periodic inputs [10–12]:

$$\begin{aligned}\frac{dX_P(t)}{dt} &= -X_P(t) + [1 - X_P(t)] \\ &\quad \times G\left[X_P(t) - X_N(t) + 5\sin\sqrt{2}t\right], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + [1 - X_N(t)] \\ &\quad \times G[X_P(t) - X_N(t) + 0.5\cos t \sin t].\end{aligned} \tag{2}$$

(iii) Few works reported almost periodicity of Wilson-Cowan type model in the literature. Under almost periodic inputs, whether there exists a unique almost periodic solution of (1) which is stable? How to estimate its located boundary? Revealing these results can give a significant insight into the complex dynamical structure of Wilson-Cowan type model.

Throughout this paper, we always assume that k_P, k_N, r_P, r_N, w_P^1, w_P^2, w_N^1, and w_N^2 are positive constants, $\tau_P(t)$, $\tau_N(t)$, $I_P(t)$, and $I_N(t)$ are almost periodic functions [12], and set

$$\begin{aligned}\tau_P^{\top} &= \sup_{t\in\mathbb{R}} \tau_P(t), \qquad \tau_N^{\top} = \sup_{t\in\mathbb{R}} \tau_N(t), \\ I_P^{\top} &= \sup_{t\in\mathbb{R}} |I_P(t)|, \qquad I_N^{\top} = \sup_{t\in\mathbb{R}} |I_N(t)|.\end{aligned} \tag{3}$$

Moreover, we need some basic assumptions in this paper.

(H_1) $G(0) = 0$, $\sup_{v\in\mathbb{R}}|G(v)| \leqslant B_s$ and there exists an $L > 0$ such that

$$|G(u) - G(v)| \leqslant L|u - v| \quad \text{for } \forall u, v \in \mathbb{R}. \tag{4}$$

(H_2) The quadratic equation $(K + Rx)(I + Wx) = x$ has a positive solution δ, where

$$\begin{gathered} W = \max\left\{L\left(w_P^1 + w_N^1\right), L\left(w_P^2 + w_N^2\right)\right\}, \\ K = \max\{k_P, k_N\}, \qquad R = \max\{r_P, r_N\}, \\ I = \max\{LI_P^{\top}, LI_N^{\top}\}. \end{gathered} \tag{5}$$

(H_3) $\tau_P(t), \tau_N(t)$ are bounded and continuously differentiable with $0 \leq \tau_P(t) \leq \tau_P^{\top}$,

$$\begin{gathered} 0 \leq \tau_N(t) \leq \tau_N^{\top}, \qquad 1 - \dot{\tau}_P(t) > 0, \\ 1 - \dot{\tau}_N(t) > 0 \quad \text{for } t \in R. \end{gathered} \tag{6}$$

For for all $u = (u_1, u_2) \in \mathbb{R}^2$, we define the norm $\|u\| = \max\{|u_1|, |u_2|\}$. Let $B := \{\psi \mid \psi = (\psi_1, \psi_1)\}$, where ψ is an almost periodic function on $\mathbb{R}^2$. For all $\psi \in B$, if we define induced nodule $\|\psi\|_B = \sup_{t\in\mathbb{R}}\|\psi(t)\|$, then B is a Banach space. The initial conditions of system (1) are of the form

$$X_P(s) = \psi_P(s), \qquad X_N(s) = \psi_N(s), \quad s \in [-\tau, 0], \tag{7}$$

where $\tau = \max\{\tau_P^{\top}, \tau_N^{\top}\}$ and $\psi = (\psi_P, \psi_N) \in B$.

Definition 1 (see [12]). Let $u(t) : \mathbb{R} \to \mathbb{R}^n$ be continuous. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, and for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}$.

The remaining part of this paper is organized as follows. In Section 2, we will derive sufficient conditions for checking the existence of almost periodic solutions. In Section 3, we present delay-based exponential stability of the unique almost periodic solution of system (1). In Section 4, we will give an example to illustrate our results obtained in the preceding sections. Concluding remarks are given in Section 5.

2. Existence of Almost Periodic Solutions

Theorem 2. *Suppose that (H_1) and (H_2) hold. If $KW + R(B_s + \delta W) < 1$, then there exists a unique almost periodic solution of system (1) in the region*

$$B^* = \{\psi \mid \psi \in B, \|\psi\|_B \leq \delta\}. \tag{8}$$

Proof. For for all $\psi = (\psi_P, \psi_N) \in B^*$, we consider the almost periodic solution $X^{\psi}(t) = (X_P^{\psi}(t), X_N^{\psi}(t))$ of the following almost periodic differential equations:

$$\begin{aligned}\frac{dX_P(t)}{dt} &= -X_P(t) + [k_P - r_P\psi_P(t)] \\ &\quad \times G\big[w_P^1\psi_P(t-\tau_P(t)) \\ &\quad\quad -w_N^1\psi_N(t-\tau_N(t)) + I_P(t)\big], \\ \frac{dX_N(t)}{dt} &= -X_N(t) + [k_N - r_N\psi_N(t)] \\ &\quad \times G\big[w_P^2\psi_P(t-\tau_P(t)) \\ &\quad\quad -w_N^2\psi_N(t-\tau_N(t)) + I_N(t)\big].\end{aligned} \tag{9}$$

By almost periodicity of $\tau_P(t)$, $\tau_N(t)$, $I_P(t)$, and $I_N(t)$ and Theorem 3.4 in [12] or [10], (9) has a unique almost periodic solution

$$\begin{aligned}X_P^{\psi}(t) &= \int_{-\infty}^{t} \exp(-(t-s))[k_P - r_P\psi_P(s)] \\ &\quad \times G\big[w_P^1\psi_P(s-\tau_P(s)) \\ &\quad\quad -w_N^1\psi_N(s-\tau_N(s)) + I_P(s)\big]\,ds,\end{aligned} \tag{10}$$

$$\begin{aligned}X_N^{\psi}(t) &= \int_{-\infty}^{t} \exp(-(t-s))[k_N - r_N\psi_N(s)] \\ &\quad \times G\big[w_P^2\psi_P(s-\tau_P(s)) \\ &\quad\quad -w_N^2\psi_N(s-\tau_N(s)) + I_N(s)\big]\,ds.\end{aligned}$$

Define a mapping $F : B^* \to B$ by setting $F(\psi)(t) = (F_P(\psi)(t), F_N(\psi)(t)) = X^{\psi}(t)$, for all $\psi \in B^*$. Now, we prove that F is a self-mapping from B^* to B^*. From (10) and (H_1), we obtain

$$\begin{aligned}&|F_P(\psi)(t)| \\ &\le \sup_{t\in\mathbb{R}} \Big|\int_{-\infty}^{t} \exp(-(t-s))[k_P - r_P\psi_P(s)] \\ &\quad \times G\big[w_P^1\psi_P(s-\tau_P(s)) \\ &\quad\quad -w_N^1\psi_N(s-\tau_N(s)) + I_P(s)\big]\,ds\Big| \\ &\le \sup_{t\in\mathbb{R}} [k_P + r_P|\psi_P(t)|] \\ &\quad \times G\big|\big[w_P^1\psi_P(t-\tau_P(t)) \\ &\quad\quad -w_N^1\psi_N(t-\tau_N(t)) + I_P(t)\big]\big| \\ &\le (k_P + r_P\|\psi\|_B) \\ &\quad \times L\left(w_P^1\|\psi\|_B + w_N^1\|\psi\|_B + I_P^{\top}\right) \\ &\le (k_P + r_P\|\psi\|_B) \\ &\quad \times \left(L\left(w_P^1 + w_N^1\right)\|\psi\|_B + LI_P^{\top}\right).\end{aligned} \tag{11}$$

By similar estimation, we can get

$$\begin{aligned}|F_N(\psi)| &\le (k_N + r_N\|\psi\|_B) \\ &\quad \times \left(L\left(w_P^2 + w_N^2\right)\|\psi\|_B + LI_N^{\top}\right).\end{aligned} \tag{12}$$

Therefore, by the above estimations and (H_2), we get

$$\begin{aligned}\|F(\psi)\|_B &= \sup_{t\in\mathbb{R}} \{|F_P(\psi)|, |F_N(\psi)|\} \\ &\le (K + R\|\psi\|_B)(W\|\psi\|_B + I) \\ &\le (K + R\delta)(W\delta + I) = \delta,\end{aligned} \tag{13}$$

which implies that $F(\psi) \in B^*$. So, the mapping F is self-mapping from B^* to B^*. Next, we prove that F is a contraction mapping in the region B^*. For all $\psi, \phi \in B^*$, by (10), we have

$$\begin{aligned}&|F_P(\psi)(t) - F_P(\phi)(t)| \\ &\le \sup_{t\in\mathbb{R}} \Big|\int_{-\infty}^{t} \exp(-(t-s))[k_P - r_P\psi_P(s)] \\ &\quad \times G\big[w_P^1\psi_P(s-\tau_P(s)) \\ &\quad\quad -w_N^1\psi_N(s-\tau_N(s)) + I_P(s)\big]\,ds \\ &\quad - \int_{-\infty}^{t} \exp(-(t-s))[k_P - r_P\phi_P(s)] \\ &\quad \times G\big[w_P^1\phi_P(s-\tau_P(s)) \\ &\quad\quad -w_N^1\phi_N(s-\tau_N(s)) + I_P(s)\big]\,ds\Big|,\end{aligned} \tag{14}$$

which leads to

$$\begin{aligned}
&\left|F_P(\psi)(t) - F_P(\phi)(t)\right| \\
&\le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \exp(-(t-s)) \\
&\qquad \times \Big[k_P L \big| w_P^1 \psi_P(s-\tau_P(s)) - w_N^1 \psi_N(s-\tau_N(s)) \\
&\qquad\qquad - w_P^1 \phi_P(s-\tau_P(s)) + w_N^1 \phi_N(s-\tau_N(s))\big| \\
&\qquad\qquad + r_P \Big| \psi_P(s) G\big[w_P^1 \psi_P(s-\tau_P(s)) \\
&\qquad\qquad\qquad - w_N^1 \psi_N(s-\tau_N(s)) \\
&\qquad\qquad\qquad + I_P(s)\big] \\
&\qquad\qquad - \phi_P(s) G\big[w_P^1 \phi_P(s-\tau_P(s)) \\
&\qquad\qquad\qquad - w_N^1 \phi_N(s-\tau_N(s)) \\
&\qquad\qquad\qquad + I_P(s)\big]\Big|\Big] ds \\
&\le k_P L \left(w_P^1 \sup_{t\in\mathbb{R}} \left|\psi_P(t) - \phi_P(t)\right| + w_N^1 \sup_{t\in\mathbb{R}} \left|\psi_N(t) - \phi_N(t)\right| \right) \\
&\quad + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \exp(-(t-s)) r_P \\
&\qquad \times \Big[\Big| \psi_P(s) G\big(w_P^1 \psi_P(s-\tau_P(s)) \\
&\qquad\qquad - w_N^1 \psi_N(s-\tau_N(s)) + I_P(s)\big) \\
&\qquad\qquad - \psi_P(s) G\big(w_P^1 \phi_P(s-\tau_P(s)) \\
&\qquad\qquad\qquad - w_N^1 \phi_N(s-\tau_N(s)) \\
&\qquad\qquad\qquad + I_P(s)\big)\Big| \\
&\qquad + \Big| \psi_P(s) G\big(w_P^1 \phi_P(s-\tau_P(s)) \\
&\qquad\qquad - w_N^1 \phi_N(s-\tau_N(s)) + I_P(s)\big) \\
&\qquad\qquad - \phi_P(s) G\big(w_P^1 \phi_P(s-\tau_P(s)) \\
&\qquad\qquad\qquad - w_N^1 \phi_N(s) + I_P(s)\big)\Big|\Big] ds \\
&\le k_P L \left(w_P^1 + w_N^1\right) \|\psi - \phi\|_B \\
&\quad + r_P \left(B_s \|\psi - \phi\|_B + \delta L \left(w_P^1 + w_N^1\right) \|\psi - \phi\|_B \right).
\end{aligned} \tag{15}$$

By similar argument, we can get

$$\begin{aligned}
&\left|F_N(\psi)(t) - F_N(\psi)(t)\right| \\
&\quad \le k_N L \left(w_P^2 + w_N^2\right) \|\psi - \phi\|_B \\
&\qquad + r_N \left(B_s \|\psi - \phi\|_B + \delta L \left(w_P^2 + w_N^2\right) \|\psi - \phi\|_B \right).
\end{aligned} \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}
&\|F(\psi) - F(\psi)\|_B \\
&\quad = \sup_{t\in\mathbb{R}} \left\{ \left|F_P(\psi) - F_P(\psi)\right|, \left|F_N(\psi) - F_N(\psi)\right| \right\} \\
&\quad \le KW \|\psi - \phi\| \\
&\qquad + R\left(B_s \|\psi - \phi\| + \delta W \|\psi - \phi\|\right) \\
&\quad = \left(KW + RB_s + R\delta W\right) \|\psi - \phi\|.
\end{aligned} \tag{17}$$

Since $KW + RB_s + R\delta W \in (0,1)$, it is clear that the mapping F is a contraction. Therefore the mapping F possesses a unique fixed point $X^* \in B^*$ such that $FX^* = X^*$. By (9), X^* is an almost periodic solution of system (1) in B^*. The proof is complete. □

Remark 3. Obviously, quadratic curve $\mathscr{C}(v) := RWv^2 + (KW + RI - 1)v + KI$ satisfies with $\mathscr{C}(0) \ge 0$. So, $\Delta := (KW + RI - 1)^2 - 4RWKI > 0$ and $KW + RI < 1$ guarantees the existence of δ in (H_2) and δ lies in the following interval:

$$\left[\frac{1 - KW - RI - \sqrt{\Delta}}{2RW}, \frac{1 - KW - RI + \sqrt{\Delta}}{2RW} \right]. \tag{18}$$

By Theorem 2, we know that the unique almost periodic solution depends on $KW + R(B_s + \delta W) < 1$. Choosing $\delta = (1 - (KW + RI))/2RW$, we get a simple assumption as follows:

$(\widehat{H}_2)$ $KW + RI < 1 - 2\sqrt{KWRI}$, $RB_s < (1 - (KW - RI))/2$,

and hence it leads to a parameter-based result.

Corollary 4. *Suppose that (H_1) and $(\widehat{H}_2)$ hold. Then there exists a unique almost periodic solution of system (1) in the region $B^* = \{\psi \mid \psi \in B,\ \|\psi\|_B \le (1 - (KW + RI))/2RW\}$.*

3. Delay-Based Stability of the Almost Periodic Solution

In this section, we establish locally exponential stability of the unique almost periodic solution of system (1) in the region B^*, which is delay dependent.

Theorem 5. *Suppose that (H_1)–(H_3) hold. If $KW + R(B_s + \delta W) < 1$ and there exist constants $\ell_1 > 0$, $\ell_2 > 0$ such that*

$$\begin{aligned}
\left(1 - r_P B_s\right) \ell_1 &> \sup_{t\in\mathbb{R}} \frac{C_P}{1 - \dot{\tau}_P\left(v_P^{-1}(t)\right)}, \\
\left(1 - r_N B_s\right) \ell_2 &> \sup_{t\in\mathbb{R}} \frac{C_N}{1 - \dot{\tau}_N\left(v_N^{-1}(t)\right)},
\end{aligned} \tag{19}$$

where $C_P := L[(k_P + r_P\delta)\ell_1 w_P^1 + (k_N + r_N\delta)\ell_2 w_P^2]$ and $C_N := L[(k_P + r_P\delta)\ell_1 w_N^1 + (k_N + r_N\delta)\ell_2 w_N^2]$, $v_P^{-1}(t)$ and $v_N^{-1}(t)$ are the inverse functions of $v_P(t) = t - \tau_P(t)$ and $v_N(t) = t - \tau_N(t)$, then system (1) has exactly one almost periodic solution $X^*(t)$ in the region B^* which is locally exponentially stable.

Proof. From Theorem 2, system (1) has a unique almost periodic solution $X^*(t) \in B^*$. Let $X(t) = (X_P(t), X_N(t))$ be an arbitrary solution of system (1) with initial value $\psi = (\psi_P, \psi_N) \in B^*$. Set $X(t) = X_P(t) - X_P^*(t)$, $Y(t) = X_N(t) - X_N^*(t)$. By system (1), we get

$$\begin{aligned}
\frac{dX(t)}{dt} &= -X(t) \\
&\quad + \left[k_P - r_P\left(X(t) + X_P^*(t)\right)\right] \\
&\quad \times G\left[w_P^1\left(X\left(t-\tau_P(t)\right) + X_P^*\left(t-\tau_P(t)\right)\right)\right. \\
&\qquad \left. - w_N^1\left(Y\left(t-\tau_N(t)\right) + X_N^*\left(t-\tau_N(t)\right)\right) I_P(t)\right] \\
&\quad - \left[k_P - r_P X_P^*(t)\right] \\
&\quad \times G\left[w_P^1 X_P^*\left(t-\tau_P(t)\right)\right. \\
&\qquad \left. - w_N^1 X_N^*\left(t-\tau_N(t)\right) + I_P(t)\right], \\
\frac{dY(t)}{dt} &= -Y(t) \\
&\quad + \left[k_N - r_N\left(Y(t) + X_N^*(t)\right)\right] \\
&\quad \times G\left[w_P^2\left(X\left(t-\tau_P(t)\right) + X_P^*\left(t-\tau_P(t)\right)\right)\right. \\
&\qquad \left. - w_N^2\left(Y\left(t-\tau_N(t)\right) + X_N^*\left(t-\tau_N(t)\right)\right) + I_N(t)\right] \\
&\quad - \left[k_N - r_N X_N^*(t)\right] \\
&\quad \times G\left[w_P^2 X_P^*\left(t-\tau_P(t)\right)\right. \\
&\qquad \left. - w_N^2 X_N^*\left(t-\tau_N(t)\right) + I_N(t)\right].
\end{aligned} \tag{20}$$

Construct the auxiliary functions $F_P(u)$, $F_N(u)$ defined on $[0, +\infty)$ as follows:

$$\begin{aligned}
F_P(u) &:= \left(u - 1 + r_P B_s\right)\ell_1 + \sup_{t\in\mathbb{R}} \frac{C_P e^{u\tau_P^\top}}{1 - \dot{\tau}_P\left(v_P^{-1}(t)\right)}, \\
F_N(u) &:= \left(u - 1 + r_N B_s\right)\ell_2 + \sup_{t\in\mathbb{R}} \frac{C_N e^{u\tau_N^\top}}{1 - \dot{\tau}_N\left(v_N^{-1}(t)\right)}.
\end{aligned} \tag{21}$$

One can easily show that $F_P(u)$, $F_N(u)$ are well defined and continuous. Assumption (19) implies that $F_P(0) < 0$, $F_P(u) \to +\infty$ as $u \to +\infty$ and $F_N(0) < 0$, $F_N(u) \to +\infty$ as $u \to +\infty$. It follows that there exists a common $\lambda > 0$ such that $F_P(\lambda) < 0$ and $F_N(\lambda) < 0$.

Consider the Lyapunov functional

$$\begin{aligned}
V(t) &= \left[\ell_1 |X(t)| + \ell_2 |Y(t)|\right] e^{\lambda t} \\
&\quad + \int_{t-\tau_P(t)}^{t} \frac{C_P}{1 - \dot{\tau}_P\left(v_P^{-1}(s)\right)} |X(s)| e^{\lambda(s+\tau_P^\top)} ds \\
&\quad + \int_{t-\tau_N(t)}^{t} \frac{C_N}{1 - \dot{\tau}_N\left(v_N^{-1}(s)\right)} |Y(s)| e^{\lambda(s+\tau_N^\top)} ds.
\end{aligned} \tag{22}$$

Calculating the upper right derivative of $V(t)$ along system (1), one has

$$\begin{aligned}
&D^+V(t) \\
&\le \lambda\left[\ell_1 |X(t)| + \ell_2 |Y(t)|\right] e^{\lambda t} \\
&\quad + e^{\lambda t}\ell_1 \left[-|X(t)| \right. \\
&\qquad + \left| \left[k_P - r_P\left(X(t) + X_P^*(t)\right)\right] \right. \\
&\qquad\quad \times G\left[w_P^1\left(X\left(t-\tau_P(t)\right) + X_P^*\left(t-\tau_P(t)\right)\right)\right. \\
&\qquad\qquad - w_N^1\left(Y\left(t-\tau_N(t)\right) + X_N^*\left(t-\tau_N(t)\right)\right) \\
&\qquad\qquad \left. + I_P(t)\right] \\
&\qquad - \left[k_P - r_P X_P^*(t)\right] \\
&\qquad \times G\left[w_P^1 X_P^*\left(t-\tau_P(t)\right)\right. \\
&\qquad\qquad \left.\left.\left. - w_N^1 X_N^*\left(t-\tau_N(t)\right) + I_P(t)\right]\right|\right] \\
&\quad + e^{\lambda t}\ell_2 \left[-|Y(t)| \right. \\
&\qquad + \left| \left[k_N - r_N\left(Y(t) + X_N^*(t)\right)\right] \right. \\
&\qquad\quad \times G\left[w_P^2\left(X\left(t-\tau_P(t)\right) + X_P^*\left(t-\tau_P(t)\right)\right)\right. \\
&\qquad\qquad - w_N^2\left(Y\left(t-\tau_N(t)\right) + X_N^*\left(t-\tau_N(t)\right)\right) \\
&\qquad\qquad \left. + I_N(t)\right] \\
&\qquad - \left[k_N - r_N X_N^*(t)\right] \\
&\qquad \times G\left[w_P^2 X_P^*\left(t-\tau_P(t)\right)\right. \\
&\qquad\qquad \left.\left.\left. - w_N^2 X_N^*\left(t-\tau_N(t)\right) + I_N(t)\right]\right|\right] \\
&\quad + \frac{C_P}{1 - \dot{\tau}_P\left(v_P^{-1}(t)\right)} |X(t)| e^{\lambda(t+\tau_P^\top)} \\
&\quad - C_P \left|X\left(t-\tau_P(t)\right)\right| \exp\left[\lambda\left(t - \tau_P(t) + \tau_P^\top\right)\right] \\
&\quad + \frac{C_N}{1 - \dot{\tau}_N\left(v_N^{-1}(t)\right)} |Y(t)| e^{\lambda(t+\tau_N^\top)} \\
&\quad - C_N \left|Y\left(t-\tau_N(t)\right)\right| \exp\left[\lambda\left(t - \tau_N(t) + \tau_N^\top\right)\right],
\end{aligned} \tag{23}$$

which leads to

$$\begin{aligned}
&D^+V(t) \\
&\le \left[(\lambda - 1)\ell_1 e^{\lambda t} |X(t)| + (\lambda - 1)\ell_2 e^{\lambda t} |Y(t)|\right] \\
&\quad + e^{\lambda t}\ell_1 \left[k_P L\left(w_P^1 \left|X\left(t-\tau_P(t)\right)\right|\right.\right. \\
&\qquad\qquad \left. + w_N^1 \left|Y\left(t-\tau_N(t)\right)\right|\right) + r_P |X(t)| B_s \\
&\qquad + r_P \delta L\left(w_P^1 \left|X\left(t-\tau_P(t)\right)\right|\right. \\
&\qquad\qquad \left.\left. + w_N^1 \left|Y\left(t-\tau_N(t)\right)\right|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ e^{\lambda t}\ell_2\left[k_N L\left(w_P^2\left|X\left(t-\tau_P(t)\right)\right|\right.\right.\\
&\qquad\left.+\,w_N^2\left|Y\left(t-\tau_N(t)\right)\right|\right)\\
&\qquad+ r_N\left|Y(t)\right|B_s + r_N\delta L\left(w_P^2\left|X\left(t-\tau_P(t)\right)\right|\right.\\
&\qquad\left.\left.+w_N^2\left|Y\left(t-\tau_N(t)\right)\right|\right)\right]\\
&+\frac{C_P}{1-\dot{\tau}_P\left(v_P^{-1}(t)\right)}\left|X(t)\right|e^{\lambda(t+\tau_P^{\top})}\\
&-C_P\left|X\left(t-\tau_P(t)\right)\right|e^{\lambda t}\\
&+\frac{C_N}{1-\dot{\tau}_N\left(v_N^{-1}(t)\right)}\left|Y(t)\right|e^{\lambda(t+\tau_N^{\top})}\\
&-C_N\left|Y\left(t-\tau_N(t)\right)\right|e^{\lambda t}\\
\le&\left[\left(\lambda-1+r_PB_s\right)\ell_1+\sup_{t\in\mathbb{R}}\frac{C_Pe^{\lambda\tau_P^{\top}}}{1-\dot{\tau}_P\left(v_P^{-1}(t)\right)}\right]\\
&\times e^{\lambda t}\left|X(t)\right|\\
&+\left[\left(\lambda-1+r_NB_s\right)\ell_2+\sup_{t\in\mathbb{R}}\frac{C_Ne^{\lambda\tau_N^{\top}}}{1-\dot{\tau}_N\left(v_N^{-1}(t)\right)}\right]\\
&\times e^{\lambda t}\left|Y(t)\right|\\
\le& -c_1\left[\left|X(t)\right|+\left|Y(t)\right|\right]e^{\lambda t},
\end{aligned}
\tag{24}
$$

where $c_1 := -(1/2)\max\{F_P(\lambda), F_N(\lambda)\} > 0$. We have from the above that $V(t) \le V(t_0)$ and

$$
\min\left\{\ell_1,\ell_2\right\}e^{\lambda t}\left(\left|X(t)\right|+\left|Y(t)\right|\right)\le V(t)\le V(0),\quad t\ge 0.
\tag{25}
$$

Note that

$$
\begin{aligned}
V(0)=&\left[\left(\ell_1+\ell_2\right)+\tau_P^{\top}\sup_{s\in[-\tau_P^{\top},0]}\frac{C_P}{1-\dot{\tau}_P\left(v_P^{-1}(s)\right)}e^{\lambda\tau_P^{\top}}\right.\\
&\left.+\tau_N^{\top}\sup_{s\in[-\tau_N^{\top},0]}\frac{C_N}{1-\dot{\tau}_N\left(v_N^{-1}(s)\right)}e^{\lambda\tau_N^{\top}}\right]\left\|\psi-\psi^*\right\|,
\end{aligned}
\tag{26}
$$

where $\psi^*(s) = X^*(s)$, $s \in [-\tau, 0]$. Then there exists a positive constant $M > 1$ such that

$$
\begin{aligned}
\left|X_P(t)-X_P^*(t)\right|&\le M\left\|\psi-\psi^*\right\|e^{-\lambda t},\\
\left|X_N(t)-X_N^*(t)\right|&\le M\left\|\psi-\psi^*\right\|e^{-\lambda t}.
\end{aligned}
\tag{27}
$$

The proof is complete. □

Set $\delta = (1-(KW+RI))/2RW$. It follows from Corollary 4 and Theorem 5 the following.

Corollary 6. *Suppose that (H_1), $(\widehat{H}_2)$, and (H_3) hold. If there exist constants $\ell_1 > 0$, $\ell_2 > 0$ such that*

$$
\begin{aligned}
\left(1-r_PB_s\right)\ell_1&>\sup_{t\in\mathbb{R}}\frac{L\left(\alpha_P\ell_1w_P^1+\alpha_N\ell_2w_P^2\right)}{1-\dot{\tau}_P\left(v_P^{-1}(t)\right)},\\
\left(1-r_NB_s\right)\ell_2&>\sup_{t\in\mathbb{R}}\frac{L\left(\alpha_P\ell_1w_N^1+\alpha_N\ell_2w_N^2\right)}{1-\dot{\tau}_N\left(v_N^{-1}(t)\right)},
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\alpha_P&:=k_P+r_P\frac{1-(KW+RI)}{2RW},\\
\alpha_N&:=k_N+r_N\frac{1-(KW+RI)}{2RW},
\end{aligned}
\tag{29}
$$

$v_P^{-1}(t)$ and $v_N^{-1}(t)$ are defined as Theorem 5, then system (1) has exactly one almost periodic solution $X^*(t)$ in the region

$$
B^*=\left\{\psi\mid\psi\in B,\ \|\psi\|_B\le\frac{1-(KW+RI)}{2RW}\right\},
\tag{30}
$$

which is locally exponentially stable.

4. An Example

In this section, we give an example to demonstrate the results obtained in previous sections. Consider a Wilson-Cowan type model with time-varying delays as follows:

$$
\begin{aligned}
\frac{dX_P(t)}{dt}=&-X_P(t)+\left[k_P-r_PX_P(t)\right]\\
&\times G\left[w_P^1X_P\left(t-\tau_P(t)\right)\right.\\
&\left.-w_N^1X_N\left(t-\tau_N(t)\right)+I_P(t)\right],\\
\frac{dX_N(t)}{dt}=&-X_N(t)+\left[k_N-r_NX_N(t)\right]\\
&\times G\left[w_P^2X_P\left(t-\tau_P(t)\right)\right.\\
&\left.-w_N^2X_N\left(t-\tau_N(t)\right)+I_N(t)\right],
\end{aligned}
\tag{31}
$$

where $k_P = k_N = 1$, $r_P = r_N = 0.01$, $w_P^1 = w_P^2 = w_N^1 = w_N^2 = 0.1$, $\tau_P(t) = \tau_N(t) = 0.1t + 10$, $I_P(t) = 7\sin\sqrt{7}t$, $G(v) = \tanh(v)$, and $I_N(t) = 7\cos\sqrt{2}t$. It is easy to calculate that

$$
\begin{gathered}
K=1,\qquad W=0.2,\qquad R=0.01,\qquad I=7,\\
v_P^{-1}(t)=v_N^{-1}(t)=\frac{(t+10)}{0.9},\\
L=B_s=1,\qquad\delta=182.5.
\end{gathered}
\tag{32}
$$

It is easy to check that (H_1) and $(\widehat{H}_2)$ hold. By Corollary 4, (31) has a unique almost periodic solution $X^*(t)$ in region $B^* = \{\psi \mid \psi \in B,\ \|\psi\|_B \le 182.5\}$. Setting $\ell_1 = \ell_2 = 0.5$, we can check that (H_3) and (28) hold and hence $X^*(t)$ is exponentially stable in B^*. Figure 2 shows the transient behavior of the unique almost periodic solution $(X_P(t), X_N(t))$ in B^*. Phase portrait of attractivity of X_P and X_N is illustrated in Figure 3.

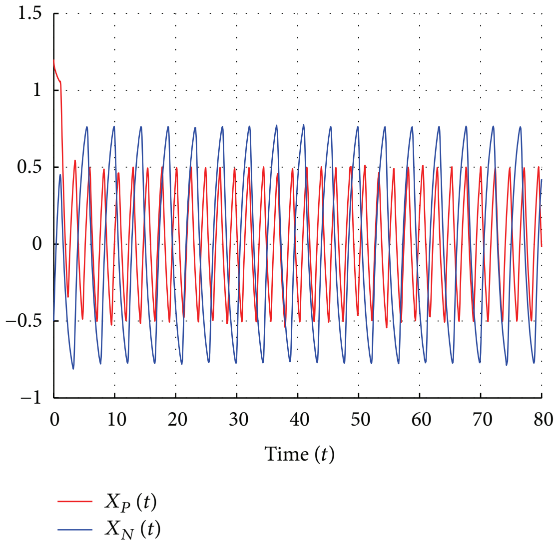


Figure 2: Transient behavior of the almost periodic solution of (31) located in B^*.

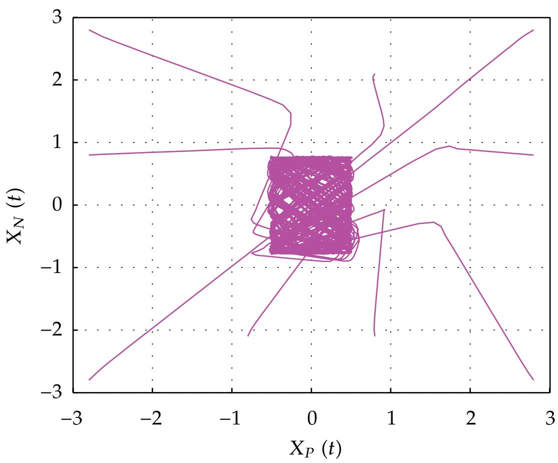


Figure 3: Phase portrait of attractivity of the unique almost periodic solution of (31) located in B^*.

5. Concluding Remarks

In this paper, we investigate Wilson-Cowan type model and obtain the existence of a unique almost periodic solution and its delay-based local stability in a convex subset. Our results are new and can reduce to periodic case, hence, complement existing periodic ones [7, 8]. We point out that there will exist multiple periodic (almost periodic) solution for system (1) under suitable parameter configuration. However, it is difficult to analyze its multistability of almost periodic solution by the existing method [10, 11, 13, 14]. We leave it for interested readers.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187, 61273021), NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), the Foundation of Fujian High Education (JA10184, JA11154), and the Foundation for Young Professors of Jimei University, China.

References

[1] H. R. Wilson and J. D. Cowan, "Excitatory and inhibitory interactions in localized populations of model neurons," *Biophysical Journal*, vol. 12, no. 1, pp. 1–24, 1972.

[2] H. R. Wilson and J. D. Cowan, "A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue," *Kybernetik*, vol. 13, no. 2, pp. 55–80, 1973.

[3] C. van Vreeswijk and H. Sompolinsky, "Chaos in neuronal networks with balanced excitatory and inhibitory activity," *Science*, vol. 274, no. 5293, pp. 1724–1726, 1996.

[4] S. E. Folias and G. B. Ermentrout, "New patterns of activity in a pair of interacting excitatory-inhibitory neural fields," *Physical Review Letters*, vol. 107, no. 22, Article ID 228103, 2011.

[5] K. Mantere, J. Parkkinen, T. Jaaskelainen, and M. M. Gupta, "Wilson-Cowan neural-network model in image processing," *Journal of Mathematical Imaging and Vision*, vol. 2, no. 2-3, pp. 251–259, 1992.

[6] L. H. A. Monteiro, M. A. Bussab, and J. G. C. Berlinck, "Analytical results on a Wilson-Cowan neuronal network modified model," *Journal of Theoretical Biology*, vol. 219, no. 1, pp. 83–91, 2002.

[7] B. Pollina, D. Benardete, and V. W. Noonburg, "A periodically forced Wilson-Cowan system," *SIAM Journal on Applied Mathematics*, vol. 63, no. 5, pp. 1585–1603, 2003.

[8] R. Decker and V. W. Noonburg, "A periodically forced Wilson-Cowan system with multiple attractors," *SIAM Journal on Mathematical Analysis*, vol. 44, no. 2, pp. 887–905, 2012.

[9] P. L. William, "Equilibrium and stability of wilson and cowan's time coarse graining model," in *Proceedings of the 20th International Symposium on Mathematical Theory of Networks and Systems (MTNS '12)*, Melbourne, Australia, July 2012.

[10] Z. Huang, S. Mohamad, X. Wang, and C. Feng, "Convergence analysis of general neural networks under almost periodic stimuli," *International Journal of Circuit Theory and Applications*, vol. 37, no. 6, pp. 723–750, 2009.

[11] Y. Liu and Z. You, "Multi-stability and almost periodic solutions of a class of recurrent neural networks," *Chaos, Solitons and Fractals*, vol. 33, no. 2, pp. 554–563, 2007.

[12] C. Y. He, *Almost Periodic Differential Equation*, Higher Education, Beijing, China, 1992.

[13] H. Zhenkun, F. Chunhua, and S. Mohamad, "Multistability analysis for a general class of delayed Cohen-Grossberg neural networks," *Information Sciences*, vol. 187, pp. 233–244, 2012.

[14] H. Zhenkun and Y. N. Raffoul, "Biperiodicity in neutral-type delayed difference neural networks," *Advances in Difference Equations*, vol. 2012, article 5, 2012.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 984960, 10 pages
http://dx.doi.org/10.1155/2013/984960

Research Article

Dynamics of a Diffusive Predator-Prey Model with Allee Effect on Predator

Xiaoqin Wang,[1] Yongli Cai,[2] and Huihai Ma[3]

[1] *Faculty of Science, Shaanxi University of Science and Technology, Xi'an, Shaanxi 710021, China*
[2] *School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, Guangdong 510275, China*
[3] *College of Electrical and Information Engineering, Shaanxi University of Science and Technology, Xi'an, Shaanxi 710021, China*

Correspondence should be addressed to Xiaoqin Wang; wxiqn@163.com

Received 29 November 2012; Accepted 17 December 2012

Academic Editor: Junli Liu

The reaction-diffusion Holling-Tanner prey-predator model considering the Allee effect on predator, under zero-flux boundary conditions, is discussed. Some properties of the solutions, such as dissipation and persistence, are obtained. Local and global stability of the positive equilibrium and Turing instability are studied. With the help of the numerical simulations, the rich Turing patterns, including holes, stripes, and spots patterns, are obtained.

1. Introduction

The Holling-Tanner prey-predator model is an important and interesting predator-prey model in both biological and mathematical sense [1–4]. The reaction-diffusion Holling-Tanner prey-predator model takes the following form:

$$\begin{aligned}\frac{\partial u}{\partial t} &= ru\left(1-\frac{u}{K}\right)-\frac{m_1uv}{u+a_1}+D_1\Delta u, \quad x=(\xi,\eta)\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t} &= s_1v\left(1-\frac{hv}{u}\right)+D_2\Delta v, \quad x=(\xi,\eta)\in\Omega,\ t>0,\end{aligned} \tag{1}$$

where u and v represent population density of prey and predator at time t, respectively. The parameters r, K, m_1, a_1, s_1, and h are all positive. r stands for the intrinsic growth rate of prey, K is the prey carrying capacity, m_1 is the maximum predation rate, a_1 is the self-saturation prey density, s_1 is predator intrinsic growth rate, and h is conversion rate of prey into predator biomass. D_1 and D_2 are the diffusion coefficients of u and v, respectively, and we always assume that $D_1>0$, $D_2>0$. $\Delta=\partial^2/\partial x^2=\partial^2/\partial\xi^2+\partial^2/\partial\eta^2$ is the usual Laplacian operator in 2-dimensional space. $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$.

Set

$$(u,v,t)=\left(K\tilde{u},K\tilde{v},\frac{\tilde{t}}{r}\right). \tag{2}$$

For the sake of convenience, we still use variables u, v instead of $\tilde{u}$, $\tilde{v}$. Thus, model (1) is converted into

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(1-u)-\frac{muv}{a+u}+d_1\Delta u, \quad x=(\xi,\eta)\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t} &= sv\left(1-\frac{hv}{u}\right)+d_2\Delta v, \quad x=(\xi,\eta)\in\Omega,\ t>0,\end{aligned} \tag{3}$$

where the new parameters are

$$\begin{gathered}m=\frac{Km_1}{r},\qquad a=\frac{a_1}{r},\qquad s=\frac{s_1}{r},\\ d_1=\frac{D_1}{r},\qquad d_2=\frac{D_2}{r}.\end{gathered} \tag{4}$$

The dynamics of the reaction-diffusion Holling-Tanner prey-predator model has proven quite interesting and received intensive study by both ecologists and mathematicians in many articles, see, for example, [5–10] and the references therein. Peng and Wang [5, 6] analyzed the global stability of the unique positive constant steady state and established the results for the existence and nonexistence of positive nonconstant steady states; Wang et al. [7] studied positive steady-state solutions and investigated the appearance of sharp spatial patterns arising from the model. Shi and coworkers [8] studied the global attractor and persistence property, local and global asymptotic stability of the unique positive constant equilibrium, and the existence and nonexistence of nonconstant positive steady states; Li et al. [9] considered the Turing and Hopf bifurcations of the equilibrium solutions; Liu and Xue [10] found the model exhibits the spotted, black-eye, and labyrinthine patterns.

On the other hand, in population dynamics, Allee effect [11] is an ecological phenomenon caused by any mechanism leading to a positive relationship between a component of individual fitness and either the number or density of conspecific [12–15]. In ecological studies, the understanding of the influence of Allee effect plays a central role since the Allee effect can greatly increase the likelihood of local and global extinctions [16, 17]. As a result both ecologists and mathematicians are interested in Allee effect in the predator-prey model, and much progress has been seen in the study of Allee effect, see [14, 18–27], and many more investigations were done in recent years. But there have been few papers discussing the impact of the Allee effect on predator in the predator-prey models.

Based on the previous discussion, in the present paper we adopt the reaction-diffusion Holling-Tanner prey-predator model with Allee effect on predator.

If we assume that the predator population is subject to an Allee effect, taking into account zero-flux boundary conditions, model (3) can be rewritten as

$$\begin{gathered}\frac{\partial u}{\partial t}=u(1-u)-\frac{muv}{a+u}+d_1\Delta u,\quad x=(\xi,\eta)\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t}=sv\left(\frac{v}{v+b}-\frac{hv}{u}\right)+d_2\Delta v,\quad x=(\xi,\eta)\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0,\quad x=(\xi,\eta)\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x)\ge 0,\qquad v(x,0)=v_0(x)\ge 0,\\ x=(\xi,\eta)\in\Omega,\end{gathered}\tag{5}$$

where $v/(v+b)$ is the term for the Allee effect, and b can be defined as the Allee effect constant. The per capita growth rate of the predator is reduced from s to $sv/(v+b)$ due to the Allee effect [17, 28]. ν is the outward unit normal vector on $\partial\Omega$, and the zero-flux boundary conditions mean that model (5) is self-contained and has no population flux across the boundary $\partial\Omega$ [29, 30]. The initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$.

The corresponding kinetic equation to model (5) is

$$\begin{aligned}\dot u&=u(1-u)-\frac{muv}{a+u}\triangleq f(u,v),\\ \dot v&=sv\left(\frac{v}{v+b}-\frac{hv}{u}\right)\triangleq g(u,v).\end{aligned}\tag{6}$$

The plan of the paper is as follows. Section 2 is dedicated to furnish some properties of the solutions concerning the mathematical model used. In Section 3, the local and global stability of the positive equilibrium of the model is considered. Section 4 is devoted to the diffusion-driven instability (Turing effect) and illustrates the different Turing patterns by using the numerical simulations. Finally, in Section 5, some conclusions and discussions are given.

2. Large Time Behavior of Solution to Model (5)

In this section, we give some properties of the solutions, and these results will be often used later.

2.1. Dissipation

Theorem 1. *For any solution $(u(x,t),v(x,t))$ of model (5) with nonnegative initial conditions, then*

$$\limsup_{t\to\infty}\max_{\overline{\Omega}} u(x,t)\le 1,\qquad \limsup_{t\to\infty}\max_{\overline{\Omega}} v(x,t)\le\frac{1}{h}.\tag{7}$$

Hence, for any $\varepsilon>0$, the rectangle $[0,1+\varepsilon]\times[0,1/h+\varepsilon]$ is a global attractor of model (5) in $\mathbb{R}^+$.

Proof. u satisfies

$$\begin{gathered}\frac{\partial u}{\partial t}-d_1\Delta u\le u(1-u),\quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu}=0,\quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad x\in\Omega.\end{gathered}\tag{8}$$

Let $z(t)$ be a solution of the ordinary differential equation,

$$\begin{gathered}\dot z(t)=z(1-u),\quad t\ge 0,\\ z(0)=\max_{\overline{\Omega}} u(x,0)>0.\end{gathered}\tag{9}$$

Then, $\lim_{t\to\infty}z(t)=1$. From the comparison principle, one can get $u(x,t)\le z(t)$; hence,

$$\limsup_{t\to\infty}\max_{\overline{\Omega}} u(x,t)\le 1.\tag{10}$$

As a result, for any $\varepsilon>0$, there exists a $T>0$, such that $u(x,t)\le 1+\varepsilon$ for all $x\in\overline{\Omega}$ and $t\ge T$. Similarly, v satisfies

$$\begin{gathered}\frac{\partial v}{\partial t}-d_2\Delta v\le sv\left(1-\frac{hv}{1+\varepsilon}\right),\quad x\in\Omega,\ t>0,\\ \frac{\partial v}{\partial \nu}=0,\quad x\in\partial\Omega,\ t>0,\\ v(x,0)=v_0(x),\quad x\in\Omega.\end{gathered}\tag{11}$$

Thus, $\limsup_{t\to\infty}\max_{\overline{\Omega}} v(x,t)\le(1+\varepsilon)/h$. From the arbitrariness of $\varepsilon>0$, we can get that

$$\limsup_{t\to\infty}\max_{\overline{\Omega}} v(x,t)\le\frac{1}{h}.\tag{12}$$

This ends the proof. □

2.2. Persistence

Definition 2 (see [31]). Model (5) is said to have the persistence property if for any nonnegative initial data $(u_0(x), v_0(x))$, there exists a positive constant $\varepsilon = \varepsilon(u_0, v_0)$, such that the corresponding solution $(u(x,t), v(x,t))$ of model (5) satisfies

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} u(x,t) \geq \varepsilon, \qquad \liminf_{t\to\infty} \min_{\overline{\Omega}} v(x,t) \geq \varepsilon. \tag{13}$$

In the following, we will show that model (5) is persistent. From the viewpoint of biology, this implies that the two species of prey and predator will always coexist at any time and any location of the inhabit domain, no matter what their diffusion coefficients are, under certain conditions on parameters.

Theorem 3. *If $m < ah$ and $2bh < 1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})}$, then model (5) has the persistence property.*

Proof. The proof is based on comparison principles. From (12), for $0 < \varepsilon \ll 1$, it is clear that there exists a $t \gg 1$, such that $v(x,t) < 1/h + \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq t_0$. Hence, $u(x,t)$ is an upper solution of the following problem:

$$\begin{gathered}\frac{\partial z}{\partial t} - d_1 \Delta z = z \frac{-z^2 + (1-a) z + a - m\left(h^{-1} + \varepsilon\right)}{z + a}, \\ x \in \Omega, \quad t > t_0, \\ \frac{\partial z}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > t_0, \\ z(x, t_0) = u_0(x, t_0) \geq 0, \quad x \in \overline{\Omega}.\end{gathered} \tag{14}$$

Let $\omega(t)$ be the unique positive solution to the following problem:

$$\begin{gathered}\frac{dw}{dt} = w \frac{-w^2 + (1-a) w + a - m\left(h^{-1} + \varepsilon\right)}{w + a}, \quad t > t_0, \\ w(t_0) = \min_{\overline{\Omega}} u_0(x, t_0) \geq 0.\end{gathered} \tag{15}$$

Since $m < ah$, then $\lim_{t\to\infty} w(t) = (1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})})/2$. By comparison, it follows that $\lim_{t\to\infty} z(x,t) = (1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})})/2$. This implies that

$$\liminf_{t\to\infty} \min_{\overline{\Omega}} u(x,t) \geq \frac{1 - a + \sqrt{(1-a)^2 + 4\left(a - mh^{-1}\right)}}{2} \triangleq \alpha. \tag{16}$$

Hence, $u(x,t) > \alpha - \varepsilon$ for $t > t_0$ and $x \in \overline{\Omega}$.

Similarly, by the second equation in model (5), we have that $v(x,t)$ is an upper solution of problem

$$\begin{gathered}\frac{\partial z}{\partial t} - d_2 \Delta z = sz \frac{-hz^2 + (\alpha - \varepsilon - bh) z}{(\alpha - \varepsilon)(v + b)}, \quad x \in \Omega,\ t > t_0, \\ \frac{\partial z}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > t_0, \\ z(x, t_0) = I_0(x) \geq 0, \quad x \in \overline{\Omega}.\end{gathered} \tag{17}$$

Let $v(t)$ be the unique positive solution to the following problem:

$$\begin{gathered}\frac{dw}{dt} = sw \frac{-hw^2 + (\alpha - \varepsilon - bh) w}{(\alpha - \varepsilon)(w + b)}, \quad t > t_0, \\ w(t_0) = \min_{\overline{\Omega}} v_0(x, t_0) \geq 0.\end{gathered} \tag{18}$$

Since $2bh < 1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})}$, there exists a $\varepsilon > 0$, such that $bh < \alpha - \varepsilon$. Hence, we have $\lim_{t\to\infty} w(t) = (1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})} - 2bh)/2h$ for the arbitrariness of ε, and an application of the comparison principle gives

$$\begin{aligned}\liminf_{t\to\infty} \min_{\overline{\Omega}} v(x,t) &\geq \frac{1 - a + \sqrt{(1-a)^2 + 4\left(a - mh^{-1}\right)} - 2bh}{2h} \\ &\triangleq \beta.\end{aligned} \tag{19}$$

The proof is complete. □

3. Stability

In this section, we will devote consideration to the stability of the positive equilibrium for model (5).

Clearly, model (5) has a unique positive equilibrium $E^* = (u^*, v^*)$, where $u^* = h(b + v^*)$ and

$$\begin{aligned}v^* = {} & \frac{h - m - ah - 2bh^2}{2h^2} \\ & + \frac{\sqrt{(h - m - ah - 2bh^2)^2 + 4h^2(a + bh)(1 - bh)}}{2h^2},\end{aligned} \tag{20}$$

with $bh < 1$.

For the sake of simplicity, we rewrite model (5) as the vectorial form

$$\begin{gathered}\mathbf{w}_t = D\Delta \mathbf{w} + \mathcal{H}(\mathbf{w}), \quad x \in \Omega,\ t > 0, \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ \mathbf{w}(x, 0) = (u_0(x), v_0(x))^T, \quad x \in \Omega,\end{gathered} \tag{21}$$

where $\mathbf{w} = (u, v)^T$, $D = \text{diag}(d_1, d_2)$, and

$$\mathscr{H}(\mathbf{w}) = \begin{pmatrix} u(1-u) - \dfrac{muv}{u+a} \\ sv\left(\dfrac{v}{v+b} - \dfrac{hv}{u}\right) \end{pmatrix}. \tag{22}$$

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator Δ on Ω with the zero-flux boundary conditions. And set

$$\begin{gathered} \mathbf{X} = \left\{\mathbf{w} \in \left[H^2(\Omega)\right]^2 \mid \partial_\nu \mathbf{w} = 0 \text{ on } \partial\Omega\right\}, \\ E(\mu) = \{\phi \mid -\Delta\phi = \mu\phi \text{ in } \Omega, \partial_\nu\phi = 0 \text{ on } \partial\Omega\}, \\ \text{with } \mu \in \mathbb{R}^1, \end{gathered} \tag{23}$$

$\{\phi_{ij} \mid j = 1, \ldots, \dim E(\mu_i)\}$ is an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$, then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i, \tag{24}$$

where $\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}$.

The linearization of model (5) at the positive equilibrium $E^* = (u^*, v^*)$ can be expressed by

$$\mathbf{w}_t = \pounds(\mathbf{w}) = D\Delta\mathbf{w} + J\mathbf{w}, \tag{25}$$

where

$$J = \begin{pmatrix} -u^* + \dfrac{mu^*v^*}{(a+u^*)^2} & -\dfrac{mu^*}{a+u^*} \\ \dfrac{shv^{*2}}{u^{*2}} & -\dfrac{sv^{*2}}{(v^*+b)^2} \end{pmatrix} \triangleq \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}. \tag{26}$$

From [32], it is known that if all the eigenvalues of the operator $\pounds$ have negative real parts, then $E^* = (u^*, v^*)$ is asymptotically stable; if there is an eigenvalue with positive real part, then $E^* = (u^*, v^*)$ is unstable; if all the eigenvalues have nonpositive real parts while some eigenvalues have zero real part, then the stability of $E^* = (u^*, v^*)$ cannot be determined by the linearization.

For each $i \geq 0$, $\mathbf{X}_i$ is invariant under the operator $\pounds$ and λ is an eigenvalue of $\pounds$ if and only if λ is an eigenvalue of the matrix $A_i = -\mu_i D + J_{(u,v)}$ for some $i \geq 0$.

So, the local stability of the positive equilibrium $E^* = (u^*, v^*)$ can be analyzed as follows.

Theorem 4. *Assume that $s > u^*(v^* + b)^2(mv^* - (a + u^*)^2)/v^{*2}(a + u^*)^2$ and the first eigenvalue μ_1 subject to the zero-flux boundary conditions satisfies*

$$\mu_1 > \max\left\{0, \frac{u^*\left(mv^* - (a+u^*)^2\right)}{d_1(a+u^*)^2} - \frac{sv^{*2}}{d_2(v^*+b)^2}\right\}. \tag{27}$$

Then, the positive equilibrium $E^* = (u^*, v^*)$ is uniformly asymptotically stable.

Proof. The stability of the positive equilibrium $E^* = (u^*, v^*)$ is reduced to consider the characteristic equation

$$\det(\lambda I - A_i) = \lambda^2 - \text{tr}(A_i)\lambda + \det(A_i), \tag{28}$$

with

$$\begin{gathered} \text{tr}(A_i) = -\mu_i(d_1 + d_2) + \text{tr}(J), \\ \det(A_i) = d_1 d_2 \mu_i^2 - (J_{11}d_2 + J_{22}d_1)\mu_i + \det(J). \end{gathered} \tag{29}$$

In view of $s > u^*(v^* + b)^2(mv^* - (a + u^*)^2)/v^*(a + u^*)^2$, it follows that

$$\text{tr}(J) = \frac{mu^*v^*}{(a+u^*)^2} - u^* - \frac{sv^{*2}}{(v^*+b)^2} < 0. \tag{30}$$

Remark that for any $i \geq 0$, we have $\text{tr}(A_i) < 0$.

In view of the relation $u^* = h(v^* + b)$, one can calculate that

$$\begin{aligned} \det(J) &= -\frac{sv^{*2}}{(v^*+b)^2}\left(-u^* + \frac{mu^*v^*}{(a+u^*)^2}\right) + \frac{mu^*}{a+u^*}\frac{shv^{*2}}{u^{*2}} \\ &= \left(sv^{*2}\left(h^3v^{*2} + \left(2ah^2 + 2bh^3\right)v^* + b^2h^3\right.\right. \\ &\qquad \left.\left. + 2abh^2 + a^2h + bhm + am\right)\right) \\ &\quad \times \left((v^*+b)(a + hv^* + h)^2\right)^{-1} \\ &> 0. \end{aligned} \tag{31}$$

Recall that $\mu_1 > \max\{0, u^*(mv^* - (a + u^*)^2)/d_1(a + u^*)^2 - sv^{*2}/d_2(v^* + b)^2\}$, we conclude that

$$\begin{aligned} \det(A_i) &= \mu_i(d_1d_2\mu_i - (J_{11}d_2 + J_{22}d_1)) + \det(J) \\ &> \mu_i(d_1d_2\mu_1 - (J_{11}d_2 + J_{22}d_1)) + \det(J) \\ &> 0 \end{aligned} \tag{32}$$

for all $i \geq 0$.

Therefore, the eigenvalues of the matrix $-\mu_i D + J$ have negative real parts. It thus follows from the Routh-Hurwitz criterion that, for each $i \geq 0$, the two roots λ_{i1} and λ_{i2} of $\det(\lambda I - A_i) = 0$ all have negative real parts.

In the following, we prove that there exists $\delta > 0$ such that

$$\text{Re}\{\lambda_{i1}\} \leq -\delta, \qquad \text{Re}\{\lambda_{i2}\} \leq -\delta. \tag{33}$$

Let $\lambda = \mu_i\xi$, then

$$\widetilde{\varphi}_i(\lambda) \triangleq \mu_i^2\xi^2 - \text{tr}(A_i)\mu_i\xi + \det(A_i). \tag{34}$$

Since $\mu_i \to \infty$ as $i \to \infty$, it follows that

$$\lim_{i\to\infty} \frac{\widetilde{\varphi}_i(\lambda)}{\mu_i^2} = \xi^2 + (d_1 + d_2)\xi + d_1d_2. \tag{35}$$

By the Routh-Hurwitz criterion, it follows that the two roots ξ_1, ξ_2 of $\widetilde{\varphi}_i(\lambda) = 0$ all have negative real parts. Thus, there exists a positive constant $\tilde{d} = \min\{d_1, d_2\}$, such that $\mathrm{Re}\{\xi_1\}, \mathrm{Re}\{\xi_2\} \le -\tilde{d}$. By continuity, we see that there exists i_0 such that the two roots ξ_{i1}, ξ_{i2} of $\tilde{\varphi}_i(\lambda) = 0$ satisfy $\mathrm{Re}\{\xi_{i1}\} \le -\tilde{d}/2$, $\mathrm{Re}\{\xi_{i2}\} \le -\tilde{d}/2$, for all $i \ge i_0$. In turn, $\mathrm{Re}\{\lambda_{i1}\}, \mathrm{Re}\{\lambda_{i2}\} \le -\mu_i \tilde{d}/2 \le -\tilde{d}/2$, for all $i \ge i_0$. Let

$$-\tilde{\delta} = \max_{1 \le i \le i_0} \{\mathrm{Re}\{\lambda_{i1}\}, \mathrm{Re}\{\lambda_{i2}\}\}. \tag{36}$$

Then $\tilde{\delta} > 0$ and (33) holds for $\delta = \min\{\tilde{\delta}, \tilde{d}/2\}$.

Consequently, the spectrum of £, which consists of eigenvalues, lies in $\{\mathrm{Re}\,\lambda \le -\delta\}$. In the sense of [32], we obtain that the positive constant steady-state solution $E^* = (u^*, u^* + k_2)$ of model (5) is uniformly asymptotically stable. This ends the proof. □

In the following, we shall prove that the positive equilibrium $E^* = (u^*, v^*)$ of model (5) is globally asymptotically stable.

Theorem 5. *Suppose that $bh < 1$, $m < ah$, and $2bh < 1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})}$. The positive equilibrium $E^* = (u^*, v^*)$ of model (5) is globally asymptotically stable, if*

(a1) $mh^{-1} < (a + u^*)(a + \alpha)$,

(a2) $m^2 + 1/\alpha^2 < (4u^*/hv^*)(h(a + u^*) - b(a + u^*)/(v^* + b)(\beta + b) - m/\alpha(a + \alpha))$,

where

$$\alpha = \frac{1 - a + \sqrt{(1-a)^2 + 4(a - mh^{-1})}}{2},$$
$$\beta = \frac{1 - a + \sqrt{(1-a)^2 + 4(a - mph^{-1})} - 2bh}{2h}. \tag{37}$$

Proof. We adopt the Lyapunov function

$$V(t) = \int_\Omega [V_1(u(x,t)) + V_2(v(x,t))]\,dx, \tag{38}$$

where $V_1(u) = (u^* + a)\int_{u^*}^{u}((\xi - u^*)/\xi)d\xi$, $V_2(v) = (u^*/hsv^*)\int_{v^*}^{v}((\eta - v^*)/\eta)d\eta$. It can be easily verified that the function $V(t)$ is zero at the positive equilibrium $E^* = (u^*, v^*)$ and is positive for all other positive values of u and v.

Then,

$$\begin{aligned}
\frac{dV}{dt} &= \int_\Omega \left(\frac{(u^* + a)(u - u^*)}{u} \frac{\partial u}{\partial t} + \frac{u^*(v - v^*)}{hsv^* v} \frac{\partial v}{\partial t} \right) dx \\
&= \int_\Omega \left((u^* + a)(u - u^*)\left(1 - u - \frac{mv}{a + u}\right) \right. \\
&\qquad \left. + \frac{u^*(v - v^*)}{hv^*}\left(\frac{v}{v + b} - \frac{hv}{u}\right)\right) dx \\
&\quad + \int_\Omega \left(\frac{d_1(u^* + a)(u - u^*)}{u}\Delta u + \frac{d_2 u^*(v - v^*)}{hsv^* v}\Delta v \right) dx \\
&= -\int_\Omega \left(a + u^* - \frac{mv}{a + u}\right)(u - u^*)^2 dx \\
&\quad - \int_\Omega \frac{u^*}{hv^*}\left(\frac{h}{u} - \frac{b}{(v^* + b)(v + b)}\right)(v - v^*)^2 dx \\
&\quad + \int_\Omega \left(\frac{1}{u} - m\right)(u - u^*)(v - v^*)\,dx \\
&\quad - \int_\Omega \left(\frac{d_1 u^*(a + u^*)}{u^2}|\nabla u|^2 + \frac{d_2 u^*}{hsv^2}|\nabla v|^2 \right) dx \\
&= M - \int_\Omega \left(\frac{d_1 u^*(a + u^*)}{u^2}|\nabla u|^2 + \frac{d_2 u^*}{hsv^2}|\nabla v|^2 \right) dx,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
M(u,v) = &-\int_\Omega \left(a + u^* - \frac{mv}{a + u}\right)(u - u^*)^2 dx \\
&- \int_\Omega \frac{u^*}{hv^*}\left(\frac{h}{u} - \frac{b}{(v^* + b)(v + b)}\right)(v - v^*)^2 dx \\
&+ \int_\Omega \left(\frac{1}{u} - m\right)(u - u^*)(v - v^*)\,dx.
\end{aligned} \tag{40}$$

It is obvious that $dV/dt < 0$ if $M(u,v)$ is negative definite.

$M(u,v)$ can be expressed in a quadratic form $-XBX^T$, where

$$X = (u - u^*, v - v^*),$$
$$B = \begin{pmatrix} a + u^* - \dfrac{mv}{a + u} & \dfrac{m}{2} - \dfrac{1}{2u} \\ \dfrac{m}{2} - \dfrac{1}{2u} & \dfrac{u^*}{hv^*}\left(\dfrac{h}{u} - \dfrac{b}{(v^* + b)(v + b)}\right) \end{pmatrix}. \tag{41}$$

$M(u,v)$ is negative definite if the symmetric matrices B is positive. It can be easily shown that the symmetric matrix B is positive definite if the following conditions are true:

(i) $a + u^* - mv/(a + u) > 0$,

(ii) $\Phi(u,v) \triangleq (u^*/hv^*)(a + u^* - mv/(a + u))(h/u - b/(v^* + b)(v + b)) - (1/4)(m - 1/u)^2 > 0$.

Proof of (i). Applying Theorems 1 and 3, we get

$$\begin{aligned}
a + u^* - \frac{mv}{a + u} &> a + u^* - \frac{mh^{-1}}{a + \alpha} \\
&= \frac{(a + u^*)(a + \alpha) - mh^{-1}}{a + \alpha}.
\end{aligned} \tag{42}$$

Therefore, if (a1) holds, then $a + u^* - mv/(a + u) > 0$ for all $t \ge 0$.

Proof of (ii). Consider

$$\begin{aligned}\Phi(u,v) &= \frac{u^*}{hv^*}\left(a+u^*-\frac{mv}{a+u}\right)\left(\frac{h}{u}-\frac{b}{(v^*+b)(v+b)}\right) \\ &\quad -\frac{1}{4}\left(m-\frac{1}{u}\right)^2 \\ &> \frac{u^*}{hv^*}\left(\frac{h(a+u^*)}{u}-\frac{b(a+u^*)}{(v^*+b)(v+b)}-\frac{hmv}{u(a+u)}\right) \\ &\quad -\frac{m^2}{4}-\frac{1}{4u^2} \\ &> \frac{u^*}{hv^*}\left(h(a+u^*)-\frac{b(a+u^*)}{(v^*+b)(\beta+b)}-\frac{m}{\alpha(a+\alpha)}\right) \\ &\quad -\frac{m^2}{4}-\frac{1}{4\alpha^2}.\end{aligned} \tag{43}$$

Consequently, if (a2) holds, $\Phi(u,v)>0$.

Hence, E^* is globally asymptotically stable for model (5) following the well-known theorem of Lyapunov stability. □

4. Diffusion-Driven Instability: Turing Effect

In this section, we will investigate Turing instability and bifurcation for our model problem. We will also study pattern formation of the predator-prey solutions.

4.1. Turing Instability. Mathematically speaking, an equilibrium is Turing instability (diffusion-driven instability) means that it is an asymptotically stable equilibrium of model (6) but is unstable with respect to the solutions of diffusion model (5). In this subsection, we mainly focus on the emergency of the Turing instability of the positive equilibrium $E^*=(u^*,v^*)$.

Now, the conditions for the positive equilibrium to be stable for the ODE are given by

$$\det(J)=J_{11}J_{22}-J_{12}J_{21}>0, \qquad \operatorname{tr}(J)=J_{11}+J_{22}<0. \tag{44}$$

Hence, A_i (the matrix $A_i=-\mu_i D+J$) has an eigenvalue with a positive real part, then it must be a real value and the other eigenvalue must be a negative real one. A necessary condition for the Turing instability of model (5) is

$$d_2J_{11}+d_1J_{22}>0, \tag{45}$$

Otherwise, $\det(A_i)>0$ for all eigenvalues μ_i of the operator Δ since $\det(J)>0$. For the Turing instability, we must have $\det(A_i)<0$ for some μ_i. And we notice that $\det(A_i)$ achieves its minimum

$$\min_{\mu_i}\det(A_i)=\frac{4d_1d_2\det(J)-(d_2J_{11}+d_1J_{22})^2}{4d_1d_2} \tag{46}$$

at the critical value $\mu^*>0$ when

$$\mu^*=\frac{d_2J_{11}+d_1J_{22}}{2d_1d_2}. \tag{47}$$

However, the inequality $\min_{\mu_i}\det(A_i)<0$ is necessary but not sufficient for the Turing instability in the bounded domain Ω. The possible eigenvalues μ_i are discrete. In this case, $\det(A_i)=0$ has two positive roots k_1 and k_2

$$\begin{aligned}k_1 &= \frac{d_2J_{11}+d_1J_{22}-\sqrt{(d_2J_{11}+d_1J_{22})^2-4d_1d_2\det(J)}}{2d_1d_2}, \\ k_2 &= \frac{d_2J_{11}+d_1J_{22}+\sqrt{(d_2J_{11}+d_1J_{22})^2-4d_1d_2\det(J)}}{2d_1d_2},\end{aligned} \tag{48}$$

so if we can find some μ_i such that $k_1<\mu_i<k_2$, then $\det(A_i)<0$, and the positive equilibrium $E^*=(u^*,v^*)$ of model (5) is unstable.

Summarizing the previous analysis and calculations, we have the following results.

Theorem 6. *Assume that the positive equilibrium $E^*=(u^*,v^*)$ exists. If the following conditions are true:*

(i) $J_{11}+J_{22}<0$, *that is,*

$$s>\frac{u^*(v^*+b)^2\left(mv^*-(a+u^*)^2\right)}{v^{*2}(a+u^*)^2}, \tag{49}$$

(ii) $d_2J_{11}+d_1J_{22}>0$, *that is,*

$$s<\frac{d_2u^*(v^*+b)^2\left(mv^*-(a+u^*)^2\right)}{d_1v^{*2}(a+u^*)^2}, \tag{50}$$

(iii) $d_2J_{11}+d_1J_{22}>2\sqrt{\det(D)\det(J)}$, *that is,*

$$\begin{aligned}&\frac{d_2u^*\left(mv^*-(a+u^*)^2\right)}{(a+u^*)^2}-\frac{d_1sv^{*2}}{(v^*+b)^2} \\ &\quad >2\sqrt{d_1d_2\det(J)},\end{aligned} \tag{51}$$

then the positive equilibrium E^* of model (5) *is Turing unstable if* $0<k_1<\mu_i<k_2$ *for some* μ_i.

In Figure 1, we show the Turing bifurcation diagram for model (5) with parameters $m=0.1$, $a=0.003$, $h=0.066$, $d_1=0.015$, and $d_2=1$ in b-s parameters plane. The Turing bifurcation breaks spatial symmetry, leading to the formation of patterns that are stationary in time and oscillatory in space [33]. Below the Turing bifurcation curve, the solution of the model is unstable, and Turing instability emerges, that is, Turing patterns emerge. This domain is called the "Turing space." We will focus on the Turing pattern formation in this domain.

4.2. Pattern Formation. In this section, we perform extensive numerical simulations of the spatially extended model (5) in two dimensional space, and the qualitative results are shown here. All our numerical simulations employ the zero-flux

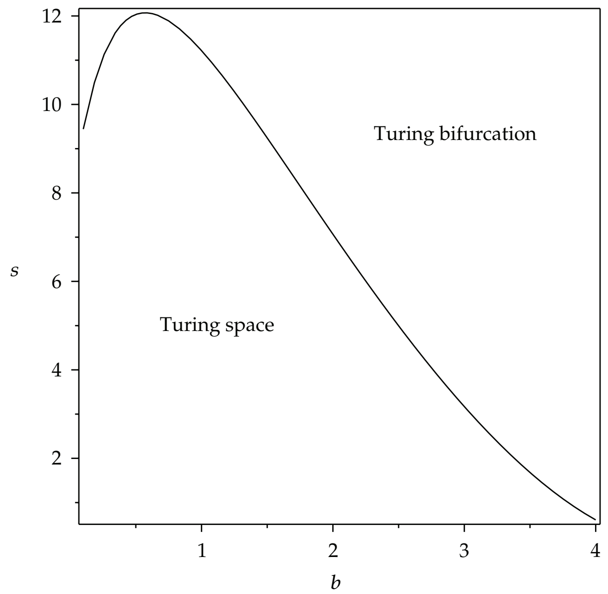


Figure 1: Turing bifurcation diagram for model (5) using b and s as parameters. Other parameters are taken as: $m = 0.1$, $a = 0.003$, $h = 0.066$, $d_1 = 0.015$, $d_2 = 1$.

boundary conditions with a system size of 100×100. Other parameters are set as $a = 0.03$, $m = 0.1$, $h = 0.066$, $d_1 = 0.015$, and $d_1 = 1$.

The numerical integration of model (5) is performed by using a finite difference approximation for the spatial derivatives and an explicit Euler method for the time integration [34, 35] with a time stepsize of $1/1000$ and the space stepsize of $h = 1/10$. The initial condition is always a small amplitude random perturbation around the positive equilibrium $E^* = (u^*, v^*)$. After the initial period during which the perturbation spreads, either the model goes into a time dependent state, or to an essentially steady-state solution (time independent).

In the numerical simulations, different types of dynamics are observed, and it is found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show the distribution of prey u, for instance. We have taken some snapshots with red (blue) corresponding to the high (low) value of prey u.

Figure 2 shows the evolution process of the holes pattern of prey for the parameters $(b, s) = (3.6, 0.9)$ at $0, 1\times10^5, 2\times10^5$ and 3×10^5 iterations. In this case, one can see that for model (5), the random perturbations lead to the formation of stripes holes (cf. Figure 2(b)), and the later random perturbations make these stripes decay ending with the holes pattern (cf. Figure 2(d))—the prey are isolated zones with low population density.

Figure 3 shows the process of spatial pattern formation of prey for the parameters $(b, s) = (0.8, 2)$ at $0, 0.5\times10^5, 1.5\times10^5$, and 3×10^5 iterations. The random perturbations lead to the formation of stripe-holes patterns (cf. Figure 3(b)), and the later random perturbations make these holes decay, ending with a time-independent stripe pattern (cf. Figure 3(d))—the prey are interlaced stripes of high and low population densities.

Figure 4 shows the process of spot pattern formation of prey for $(b, s) = (0.8, 10)$ at $0, 2 \times 10^5, 5 \times 10^5$, and 1×10^6 iterations. There is a competition exhibited between stripes and spots. The pattern takes a long time to settle down, starting with a homogeneous state E^*(cf. Figure 4(a)), and the random perturbations lead to the formation of stripes and spots (cf. Figure 4(b)), ending with spots only (cf. Figure 4(d))—the prey are isolated zones with high population density.

Ecologically speaking, spots pattern shows that the prey population is driven by predators to a high level in those regions, while holes pattern shows that the prey population is driven by predators to a very low level in those regions.

5. Conclusions and Remarks

In this paper, we have studied the dynamics of a reaction-diffusion Holling-Tanner prey-predator model where the predator population is subject to Allee effect under the zero-flux boundary conditions. The value of this study lies in threefolds. First, it investigates qualitative properties of solutions to this reaction-diffusion model. Second, it gives local and global stability of the positive equilibrium of the model. Third, it rigorously proves the Turing instability and illustrates three categories of Turing patterns close to the onset Turing bifurcation, which shows that the model dynamics exhibits complex pattern replication.

It is seen that if Allee effect constant b is low, then the persistence of the model is guaranteed. It is interesting to notice that, from the result of Theorem 5, the condition for global stability of $E^* = (u^*, v^*)$ is independent of the diffusion coefficients d_i $(i = 1, 2)$. So, it can be said that

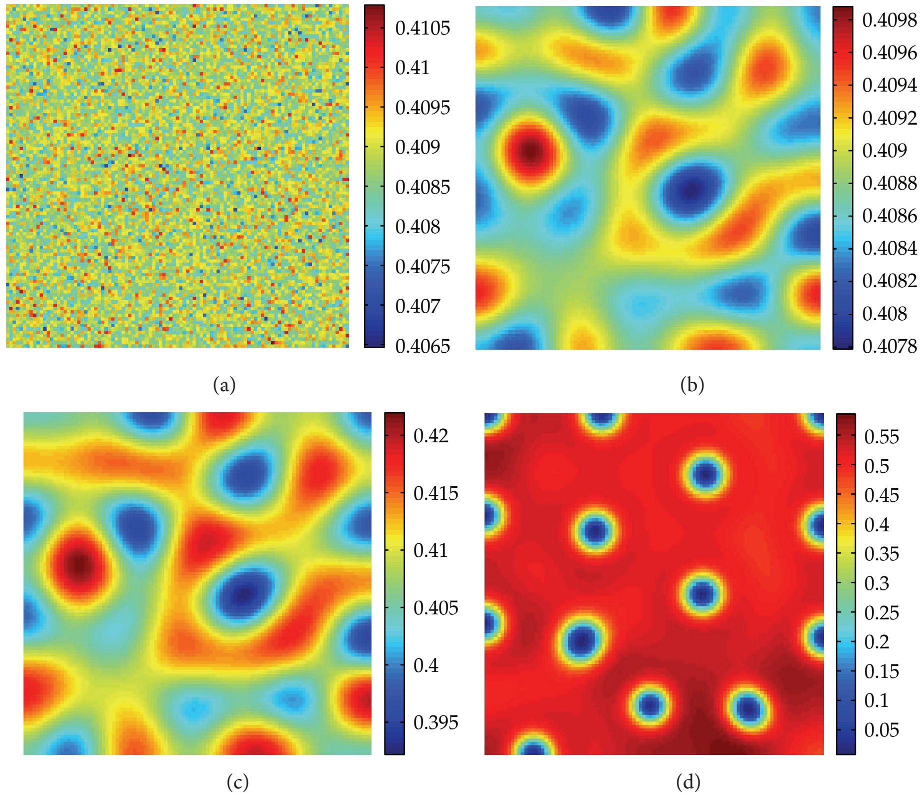


Figure 2: Holes pattern formation for model (5) by taking $(b, s) = (3.6, 0.9)$. Iterations: (a) 0; (b) 1×10^5; (c) 2×10^5; (d) 3×10^5.

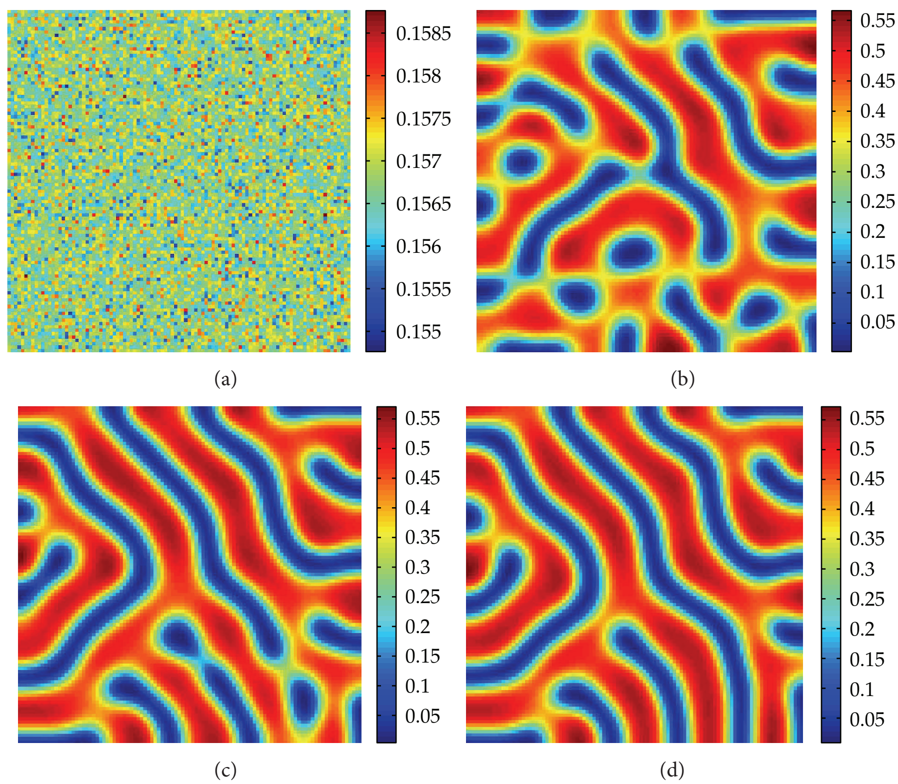


Figure 3: Stripes pattern formation for model (5) by taking $(b, s) = (0.8, 2)$. Iterations: (a) 0; (b) 0.5×10^5; (c) 1.5×10^5; (d) 3×10^5.

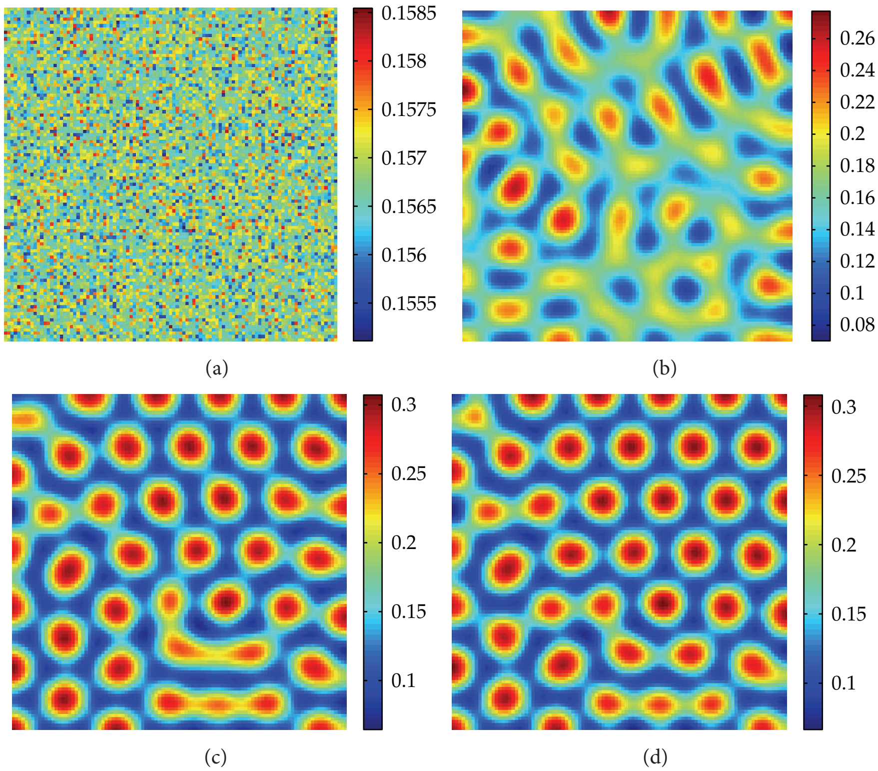


FIGURE 4: Spots pattern formation for model (5) by taking $(b, s) = (0.8, 10)$. Iterations: (a) 0; (b) 2×10^5; (c) 5×10^5; (d) 1×10^6.

when the conditions on the parameters are satisfied, $E^* = (u^*, v^*)$ is stabilized under arbitrary spatially inhomogeneous perturbation. A very interesting observation can be made from the result of the numerical simulations. It indicates that the spatial model dynamics exhibits a diffusion-controlled formation growth not only to holes (cf. Figure 1) and stripes (cf. Figure 2) but also to spots replication (cf. Figure 3).

Comparing Figures 1 and 3, we can conclude that the model exhibits holes and spots Turing patterns due to Allee effect constant b. It is believed that the observations made in this investigation related to Allee effect on predator population remind us of the importance of the Allee effect. Therefore, if the prey or the predator to be protected is subject to an Allee effect, the measures taken should take this into account.

References

[1] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, USA, 1974.

[2] J. T. Tanner, "The stability and the intrinsic growth rates of prey and predator populations," *Ecology*, vol. 56, no. 4, pp. 855–867, 1975.

[3] J. D. Murray, *Mathematical Biology*, Springer, Berlin, Germany, 1989.

[4] S. B. Hsu and T. W. Hwang, "Hopf bifurcation analysis for a predator-prey system of Holling and Leslie type," *Taiwanese Journal of Mathematics*, vol. 3, no. 1, pp. 35–53, 1999.

[5] R. Peng and M. Wang, "Positive steady states of the Holling-Tanner prey-predator model with diffusion," *Proceedings of the Royal Society of Edinburgh A*, vol. 135, no. 1, pp. 149–164, 2005.

[6] R. Peng and M. Wang, "Global stability of the equilibrium of a diffusive Holling-Tanner prey-predator model," *Applied Mathematics Letters*, vol. 20, no. 6, pp. 664–670, 2007.

[7] M. Wang, P. Y. H. Pang, and W. Chen, "Sharp spatial patterns of the diffusive Holling-Tanner prey-predator model in heterogeneous environment," *IMA Journal of Applied Mathematics*, vol. 73, no. 5, pp. 815–835, 2008.

[8] H. B. Shi, W. T. Li, and G. Lin, "Positive steady states of a diffusive predator-prey system with modified Holling-Tanner functional response," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3711–3721, 2010.

[9] X. Li, W. Jiang, and J. Shi, "Hopf bifurcation and Turing instability in the reaction-diffusion Holling-Tanner predator-prey model," *IMA Journal of Applied Mathematics*. In press.

[10] P. P. Liu and Y. Xue, "Spatiotemporal dynamics of a predator-prey model," *Nonlinear Dynamics*, vol. 69, no. 1-2, pp. 71–77, 2012.

[11] W. C. Allee, *Animal Aggregations: A Study in General Sociology*, University of Chicago Press, 1931.

[12] F. Courchamp, T. Clutton-Brock, and B. Grenfell, "Inverse density dependence and the Allee effect," *Trends in Ecology & Evolution*, vol. 14, no. 10, pp. 405–410, 1999.

[13] P. A. Stephens, W. J. Sutherland, and R. P. Freckleton, "What is the Allee effect?" *Oikos*, vol. 87, no. 1, pp. 185–190, 1999.

[14] P. A. Stephens and W. J. Sutherland, "Consequences of the Allee effect for behaviour, ecology and conservation," *Trends in Ecology & Evolution*, vol. 14, no. 10, pp. 401–405, 1999.

[15] F. Courchamp, L. Berec, and J. Gascoigne, *Allee Effects in Ecology and Conservation*, Oxford University Press, 2008.

[16] B. Dennis, "Allee effects: population growth, critical density, and the chance of extinction," *Natural Resource Modeling*, vol. 3, no. 4, pp. 481–538, 1989.

[17] S. R. Zhou, Y. F. Liu, and G. Wang, "The stability of predator-prey systems subject to the Allee effects," *Theoretical Population Biology*, vol. 67, no. 1, pp. 23–31, 2005.

[18] I. Scheuring, "Allee effect increases the dynamical stability of populations," *Journal of Theoretical Biology*, vol. 199, no. 4, pp. 407–414, 1999.

[19] M. H. Wang and M. Kot, "Speeds of invasion in a model with strong or weak Allee effects," *Mathematical Biosciences*, vol. 171, no. 1, pp. 83–97, 2001.

[20] A. Kent, C. Patrick Doncaster, and T. Sluckin, "Consequences for predators of rescue and Allee effects on prey," *Ecological Modelling*, vol. 162, no. 3, pp. 233–245, 2003.

[21] J. C. Gascoigne and R. N. Lipcius, "Allee effects driven by predation," *Journal of Applied Ecology*, vol. 41, no. 5, pp. 801–810, 2004.

[22] A. Morozov, S. Petrovskii, and B. L. Li, "Bifurcations and chaos in a predator-prey system with the Allee effect," *Proceedings of the Royal Society B*, vol. 271, no. 1546, pp. 1407–1414, 2004.

[23] D. S. Boukal, M. W. Sabelis, and L. Berec, "How predator functional responses and Allee effects in prey affect the paradox of enrichment and population collapses," *Theoretical Population Biology*, vol. 72, no. 1, pp. 136–147, 2007.

[24] Y. Du and J. Shi, "Allee effect and bistability in a spatially heterogeneous predator-prey model," *Transactions of the American Mathematical Society*, vol. 359, no. 9, pp. 4557–4593, 2007.

[25] J. Wang, J. Shi, and J. Wei, "Predator-prey system with strong Allee effect in prey," *Journal of Mathematical Biology*, vol. 62, no. 3, pp. 291–331, 2011.

[26] J. Wang, J. Shi, and J. Wei, "Dynamics and pattern formation in a diffusive predator-prey system with strong Allee effect in prey," *Journal of Differential Equations*, vol. 251, no. 4-5, pp. 1276–1304, 2011.

[27] E. González-Olivares, J. Mena-Lorca, A. Rojas-Palma, and J. D. Flores, "Dynamical complexities in the Leslie-Gower predator-prey model as consequences of the Allee effect on prey," *Applied Mathematical Modelling*, vol. 35, no. 1, pp. 366–381, 2011.

[28] G. Wang, X. G. Liang, and F. Z. Wang, "The competitive dynamics of populations subject to an Allee effect," *Ecological Modelling*, vol. 124, no. 2-3, pp. 183–192, 1999.

[29] J. D. Murray, "Discussion: Turing's theory of morphogenesis-its influence on modelling biological pattern and form," *Bulletin of Mathematical Biology*, vol. 52, no. 1, pp. 117–152, 1990.

[30] J. Chattopadhyay and P. K. Tapaswi, "Effect of cross-diffusion on pattern formation—a nonlinear analysis," *Acta Applicandae Mathematicae*, vol. 48, no. 1, pp. 1–12, 1997.

[31] W. Chen and M. Wang, "Qualitative analysis of predator-prey models with Beddington-DeAngelis functional response and diffusion," *Mathematical and Computer Modelling*, vol. 42, no. 1-2, pp. 31–44, 2005.

[32] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1981.

[33] W. Wang, Q. Liu, and Z. Jin, "Spatiotemporal complexity of a ratio-dependent predator-prey system," *Physical Review E*, vol. 75, no. 5, Article ID 051913, 9 pages, 2007.

[34] M. R. Garvie, "Finite-difference schemes for reaction-diffusion equations modeling predator-prey interactions in MATLAB," *Bulletin of Mathematical Biology*, vol. 69, no. 3, pp. 931–956, 2007.

[35] A. Munteanu and R. Sole, "Pattern formation in noisy self-replicating spots," *International Journal of Bifurcation and Chaos*, vol. 16, no. 12, pp. 3679–3683, 2007.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 828219, 12 pages
doi:10.1155/2012/828219

Research Article

Pattern Formation in a Cross-Diffusive Holling-Tanner Model

Weiming Wang,[1] Zhengguang Guo,[1] R. K. Upadhyay,[2] and Yezhi Lin[1]

[1] *College of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang 325035, China*
[2] *Department of Applied Mathematics, Indian School of Mines, Dhanbad, Jharkhand 826004, India*

Correspondence should be addressed to Weiming Wang, weimingwang2003@163.com

Received 29 November 2012; Accepted 13 December 2012

Academic Editor: Yonghui Xia

We present a theoretical analysis of the processes of pattern formation that involves organisms distribution and their interaction of spatially distributed population with self- as well as cross-diffusion in a Holling-Tanner predator-prey model; the sufficient conditions for the Turing instability with zero-flux boundary conditions are obtained; Hopf and Turing bifurcation in a spatial domain is presented, too. Furthermore, we present novel numerical evidence of time evolution of patterns controlled by self- as well as cross-diffusion in the model, and find that the model dynamics exhibits a cross-diffusion controlled formation growth not only to spots, but also to strips, holes, and stripes-spots replication. And the methods and results in the present paper may be useful for the research of the pattern formation in the cross-diffusive model.

1. Introduction

A fundamental goal of theoretical ecology is to understand the interactions of individual organisms with each other and with the environment and to determine the distribution of populations and the structure of communities. Empirical evidence suggests that the spatial scale and structure of environment can influence population interactions [1]. The study of complex population dynamics is nearly as old as population ecology, starting with the pioneering work of Lotka and Volterra, a simple model of interacting species that still bears their joint names [2, 3].

And the predator-prey system models such a phenomenon, pursuit-evasion, predators pursuing prey and prey escaping from the predators [4]. In other words, in nature, there is a tendency that the preys would keep away from predators and the escape velocity of the preys may be taken as proportional to the dispersive velocity of the predators. In the same manner, there is a tendency that the predators would get closer to the preys and the chase velocity of predators may be considered to be proportional to the dispersive velocity of the preys [5]. Keeping these in view, cross-diffusion arises, which was proposed first by Kerner [6] and first applied in competitive population system by Shigesada et al. [7]. From the pioneering work of Turing [8], spatially continuous models formulated as reaction-diffusion equations have been intensively used to describe spatiotemporal dynamics and to investigate mechanisms for pattern formation [9]. And the appearance and evolution of these patterns have been a focus of recent research activity across several disciplines [10].

In recent years, there has been considerable interest to investigate the stability behavior of a predator-prey system by taking into account the effect of self- as well as cross-diffusion [11–25]. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species.

Furthermore, in [23], the authors gave a numerical study of pattern formation in the Holling-Tanner model with self- and cross-diffusion as the form

$$\begin{aligned}\frac{\partial N}{\partial t} &= N(1-N) - \frac{aNP}{N+b} + \nabla^2 N + D_{12}\nabla^2 P \triangleq f(N,P) + \nabla^2 N + D_{12}\nabla^2 P,\\ \frac{\partial P}{\partial t} &= rP\left(1-\frac{P}{N}\right) + D_{21}\nabla^2 N + \nabla^2 P \triangleq g(N,P) + D_{21}\nabla^2 N + \nabla^2 P,\end{aligned} \tag{1.1}$$

with the following nonzero initial conditions:

$$N(x,y,0) > 0, \quad P(x,y,0) > 0, \quad (x,y) \in \Omega = [0,Lx] \times [0,Ly] \tag{1.2}$$

and zero-flux boundary conditions:

$$\frac{\partial N}{\partial n} = \frac{\partial P}{\partial n} = 0, \qquad (x,y) \in \partial\Omega, \tag{1.3}$$

where $N(t)$, $P(t)$ represent population densities of prey and predator, respectively, r is the intrinsic growth rate or biotic potential of the prey, a is the maximal predator per capita consumption rate, that is, the maximum number of prey that can be eaten by a predator in each time unit, and b is the number of preys necessary to achieve one-half of the maximum rate a [26]. And the nonnegative constants, D_{12} and D_{21}, called cross-diffusion coefficients, express the respective population fluxes of the prey and predators resulting from the presence of the other species, respectively. Lx and Ly give the size of the system in the directions of x and y, respectively. In (1.3), n is the outward unit normal vector of the boundary $\partial\Omega$, which we will assume is smooth. The main reason for choosing such boundary conditions is that we are interested in the self-organization of pattern; zero-flux conditions imply no external input [11, 27].

Biologically, cross diffusion implies countertransport [11]. And the induced cross-diffusion rate D_{12} in (1.1) represents the tendency of the prey N to keep away from its predators P and D_{21} represents the tendency of the predator to chase its prey. The cross-diffusion coefficients, D_{12} and D_{21}, may be positive or negative. Positive cross-diffusion coefficient denotes that one species tends to move in the direction of lower concentration of another species while negative cross diffusion expresses the population fluxes of one species in the direction of higher concentration of the other species [13].

In [23], the authors found that, for the equal self-diffusion coefficients (i.e., the coefficients of $\nabla^2 N$ and $\nabla^2 P$ in (1.1) are both equal to 1), only spots pattern could be obtained. In addition, they indicated that cross diffusion may have an effect on the distribution of the species, that is, it may lead the species to be isolated.

There comes a question: besides spots, does model (1.1) exhibit any other pattern replication controlled by cross-diffusion?

In this paper, based on the results of [23], we mainly focus on the effect of self- as well as cross-diffusion on pattern formation in the two-species Holling-Tanner predator-prey model. In the next section, we give the sufficient conditions for the Turing instability with zero-flux boundary conditions. And, by using the bifurcation theory, we give the Hopf and Turing bifurcation analysis of the model. Then, we present and discuss the results of complex, not simple, pattern formations via numerical simulations, which are followed by the last section, that is, concluding remarks.

2. Linear Stability Analysis

In the absence of diffusion, model (1.1) has two equilibrium solutions in the positive quadrant. One equilibrium point is given by $(N_0, P_0) = (1, 0)$. In the (N, P) phase plane, $(1, 0)$ is a saddle [28]. Another equilibrium point $E^* = (N^*, P^*)$ depends on the parameters a and b and is given by

$$N^* = P^* = \frac{1 - a - b + M}{2}, \tag{2.1}$$

where $M = \sqrt{(1 - a - b)^2 + 4b}$.

And in the presence of diffusion, we will introduce small perturbations $U_1 = N - N^*$ and $U_2 = P - P^*$, where $|U_1|, |U_2| \ll 1$. The zero-flux boundary conditions (1.3) imply that no external input is imposed from outside. To study the effect of diffusion on the model system, we have considered the linearized form of the system as follows:

$$\frac{\partial U_1}{\partial t} = J_{11}U_1 + J_{12}U_2 + \nabla^2 U_1 + D_{12}\nabla^2 U_2, \qquad \frac{\partial U_2}{\partial t} = J_{21}U_1 + J_{22}U_2 + D_{21}\nabla^2 U_1 + \nabla^2 U_2, \tag{2.2}$$

where

$$\begin{aligned} J_{11} &= \left.\frac{\partial f}{\partial N}\right|_{(N^*,P^*)} \\ &= \frac{\left(M^3 + (2 - 3a + b)M^2 + \left(1 - 4a + 3a^2 - b^2\right)M - a^3 + (2 - b)a^2 + \left(-1 - b^2 + 2b\right)a - b - 2b^2 - b^3\right)}{(1 - a - b + M)^2}, \end{aligned}$$

$$J_{12} = \left.\frac{\partial f}{\partial P}\right|_{(N^*,P^*)} = \frac{a(-1+a+b-M)}{(1-a+b+M)},$$
$$J_{21} = \left.\frac{\partial g}{\partial N}\right|_{(N^*,P^*)} = r, \qquad J_{22} = \left.\frac{\partial g}{\partial P}\right|_{(N^*,P^*)} = -r. \tag{2.3}$$

Set $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$. And define $D = \begin{pmatrix} 1 & D_{12} \\ D_{21} & 1 \end{pmatrix}$ as the diffusion matrix, and the determinant is $\det(D) = 1 - D_{12}D_{21}$.

Following Malchow et al. [29], we can know that any solution of the system (2.2) can be expanded into a Fourier series so that

$$\begin{aligned} U_1(\mathbf{r},t) &= \sum_{n,m=0}^{\infty} u_{nm}(\mathbf{r},t) = \sum_{n,m=0}^{\infty} \alpha_{nm}(t)\sin \mathbf{kr}, \\ U_2(\mathbf{r},t) &= \sum_{n,m=0}^{\infty} v_{nm}(\mathbf{r},t) = \sum_{n,m=0}^{\infty} \beta_{nm}(t)\sin \mathbf{kr}, \end{aligned} \tag{2.4}$$

where $\mathbf{r} = (x,y)$ and $0 < x < Lx$, $0 < y < Ly$. $\mathbf{k} = (k_n, k_m)$ and $k_n = n\pi/Lx$, $k_m = m\pi/Ly$ are the corresponding wavenumbers.

Having substituted u_{nm} and v_{nm} with (2.2), we obtain

$$\begin{aligned} \frac{d\alpha_{nm}}{dt} &= \left(J_{11} - k^2\right)\alpha_{nm} + \left(J_{12} - D_{12}k^2\right)\beta_{nm}, \\ \frac{d\beta_{nm}}{dt} &= \left(J_{21} - D_{21}k^2\right)\alpha_{nm} + \left(J_{22} - k^2\right)\beta_{nm}, \end{aligned} \tag{2.5}$$

where $k^2 = k_n^2 + k_m^2$.

A general solution of (2.5) has the form $C_1\exp(\lambda_1 t) + C_2\exp(\lambda_2 t)$, where the constants C_1 and C_2 are determined by the initial conditions (1.2) and the exponents λ_1 and λ_2 are the eigenvalues of the following matrix:

$$\widetilde{D} = \begin{pmatrix} J_{11} - k^2 & J_{12} - D_{12}k^2 \\ J_{21} - D_{21}k^2 & J_{22} - k^2 \end{pmatrix}. \tag{2.6}$$

Correspondingly, λ_1 and λ_2 arise as the solution of the following equation:

$$\lambda^2 - \operatorname{tr}\left(\widetilde{D}\right)\lambda + \det\left(\widetilde{D}\right) = 0, \tag{2.7}$$

where

$$\operatorname{tr}\left(\widetilde{D}\right) = J_{11} + J_{22} - 2k^2,$$
$$\det\left(\widetilde{D}\right) = (1 - D_{12}D_{21})k^4 + (-J_{11} - J_{22} + D_{12}J_{21} + D_{21}J_{12})k^2 + \det(J). \tag{2.8}$$

Theorem 2.1. *If* $0 < D_{12}, D_{21} < 1$, *and* $b \geq a/(1+a)$, *then the uniform steady state* E^* *of model* (1.1) *is globally asymptotically stable.*

Proof. For global stability of nonspatial model of (1.1), we select a Liapunov function:

$$V(N,P) = \int_{N^*}^{N} \frac{\xi - N^*}{\xi\phi(\xi)} d\xi + \frac{1}{r}\int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta, \tag{2.9}$$

where $\phi(N) = aN/(N+b)$. Then

$$\frac{dV}{dt}(N,P) = \frac{N - N^*}{N\phi(N)}\frac{dN}{dt} + \frac{P - P^*}{rP}\frac{dP}{dt}. \tag{2.10}$$

Substituting the value of dN/dt and dP/dt from the nonspatial model of (1.1), we obtained

$$\frac{dV}{dt} = \frac{N - N^*}{N}\left(\frac{N(1-N)}{\phi(N)} - P^*\right) - \frac{1}{N}(P - P^*)^2. \tag{2.11}$$

Noting that $P^* = (1/a)(1 - N^*)(N^* + b)$, one can obtain

$$\frac{dV}{dt} = -\frac{(N - N^*)^2}{aN}(N + N^* + b - 1) - \frac{1}{N}(P - P^*)^2. \tag{2.12}$$

If $b \geq a/(1+a)$ holds, $dV/dt \leq 0$.

Next, we select the Liapunov function for model (1.1) (two dimensional with diffusion case):

$$V_2(t) = \iint_{\Omega} V(N,P)\,dx\,dy, \tag{2.13}$$

so

$$\frac{dV_2}{dt} = \iint_{\Omega} \frac{dV}{dt} dx\,dy + \iint_{\Omega}\left(\nabla^2 N + D_{12}\nabla^2 P\right)\frac{\partial V}{\partial N} dx\,dy + \iint_{\Omega}\left(D_{21}\nabla^2 N + \nabla^2 P\right)\frac{\partial V}{\partial P} dx\,dy. \tag{2.14}$$

Using Green's first identity in the plane

$$\iint_{\Omega} F\nabla^2 G\,dx\,dy = \int_{\partial\Omega} F\frac{\partial G}{\partial n} ds - \iint_{\Omega}(\nabla F \cdot \nabla G)\,dx\,dy. \tag{2.15}$$

And considering the zero-flux boundary conditions (1.3), one can show that

$$\begin{aligned}
&\iint_{\Omega}\left(\nabla^2 N + D_{12}\nabla^2 P\right)\frac{\partial V}{\partial N}dx\,dy \\
&\qquad = -\iint_{\Omega}\frac{\partial^2 V}{\partial N^2}\left[\left(\frac{\partial N}{\partial x}\right)^2 + \left(\frac{\partial N}{\partial y}\right)^2 + D_{12}\left(\frac{\partial N}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial P}{\partial y}\right)\right]dx\,dy, \\
&\iint_{\Omega}\left(D_{21}\nabla^2 N + \nabla^2 P\right)\frac{\partial V}{\partial P}dx\,dy \\
&\qquad = -\iint_{\Omega}\frac{\partial^2 V}{\partial P^2}\left[D_{21}\left(\frac{\partial N}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial P}{\partial y}\right) + \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right]dx\,dy,
\end{aligned} \tag{2.16}$$

then

$$\frac{dV_2}{dt} = I_1 + I_2, \tag{2.17}$$

where

$$\begin{aligned}
I_1 &= \iint_{\Omega}\frac{dV}{dt}dx\,dy, \\
I_2 &= -\iint_{\Omega}\left[\frac{\partial^2 V}{\partial N^2}\left(\left(\frac{\partial N}{\partial x}\right)^2 + \left(\frac{\partial N}{\partial y}\right)^2\right) + \frac{\partial^2 V}{\partial P^2}\left(\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2\right)\right]dx\,dy \\
&\quad - \iint_{\Omega}\left(D_{12}\frac{\partial^2 V}{\partial N^2} + D_{21}\frac{\partial^2 V}{\partial P^2}\right)\left(\frac{\partial N}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial P}{\partial y}\right)dx\,dy.
\end{aligned} \tag{2.18}$$

Obviously, $I_1 \le 0$ and $I_2 \le 0$. The proof is complete. □

And from [19], we know that the equilibrium E^* is Turing unstable if it is an asymptotically stable equilibrium of model (1.1) without self and cross diffusion but is unstable with respect to solutions of model (1.1). Hence, Turing instability sets in when the condition either $\operatorname{tr}(\widetilde{D}) < 0$ or $\det(\widetilde{D}) > 0$ is violated, which subject to the conditions $J_{11} + J_{22} < 0$ and $\det(J) > 0$. It is evident that the condition $\operatorname{tr}(\widetilde{D}) < 0$ is not violated when the requirement $J_{11} + J_{22} < 0$ is met. Hence, only violation of condition $\det(\widetilde{D}) > 0$ gives rise to Turing instability. Then the condition for Turing instability is given by

$$H\left(k^2\right) = \det\left(\widetilde{D}\right) \equiv (1 - D_{12}D_{21})k^4 + (-J_{11} - J_{22} + D_{12}J_{21} + D_{21}J_{12})k^2 + \det(J) < 0. \tag{2.19}$$

Thus, a sufficient condition for Turing instability is that $H(k^2)_{\min}$ is negative. Therefore,

$$H\left(k^2\right)_{\min} = \det(J) - \frac{(J_{11} + J_{22} - D_{12}J_{21} - D_{21}J_{12})^2}{4(1 - D_{12}D_{21})} < 0. \tag{2.20}$$

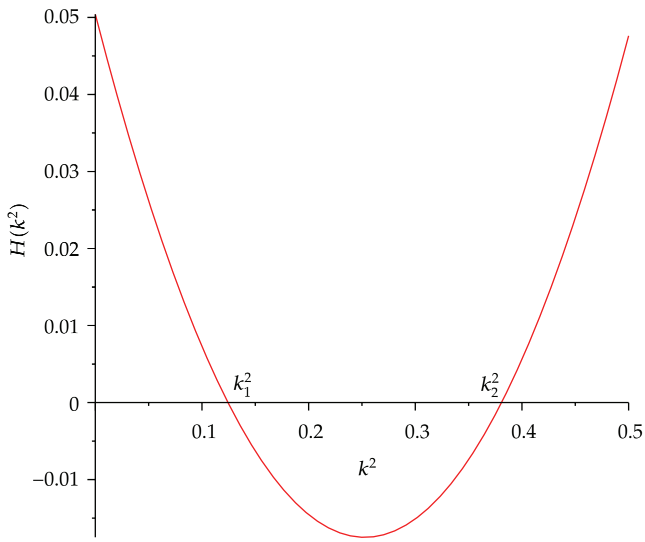


Figure 1: Modes with wavenumbers lying between the zeros, k_1^2, and k_2^2, of $H(k^2)$ grow in the Turing instability. Parameters: $a = 0.8, b = 0.1, r = 0.1, D_{12} = -0.08, D_{21} = 0.8$, hence, $k_1^2 = 0.12440$, $k_2^2 = 0.38086$, and $H_{\min} = -0.01749$.

Equation (2.20) leads to the following final criterion for Turing instability:

$$(J_{11} + J_{22} - D_{12}J_{21} - D_{21}J_{12})^2 > 4\ (1 - D_{12}D_{21})\det(J). \tag{2.21}$$

That is to say, if $H(k^2)_{\min}$ is negative, then a range of modes $k_1^2 < k^2 < k_2^2$ will grow Turing instability (see Figure 1).

Summarizing the previous discussions, we can obtain the following theorem.

Theorem 2.2. *The equilibrium E^* of model* (1.1) *is Turing instability if $J_{11} + J_{22} > D_{12}J_{21} + D_{21}J_{12}$ and $J_{11} + J_{22} - D_{12}J_{21} - D_{21}J_{12} > 2\sqrt{(1 - D_{12}D_{21})\det(J)}$.*

It is easy to see that the minimum of $H(k^2)$ occurs at $k^2 = k_m^2$, where

$$k_m^2 = \frac{J_{11} + J_{22} - D_{12}J_{21} - D_{21}J_{12}}{2(1 - D_{12}D_{21})} > 0. \tag{2.22}$$

The critical wavenumber k_c of the first perturbations to grow is found by evaluating k_m from (2.22).

Figure 2 shows that the linear stability analysis yields the bifurcation diagram with $a = 0.8$, $b = 0.1$, and $D_{21} = 0.8$. Turing and Hopf lines intersect at the Turing-Hopf bifurcation point $(D_{12}, r) = (-0.00489, 0.12572)$ and separate the parametric space into four domains. On domain I, located above all two bifurcation lines, the steady state is the only stable solution of the model. Domain II is the region of pure Turing instability. In domain III, which is located above all two bifurcation lines, both Hopf and Turing instability occur. And domain IV is the region of pure Hopf instability.

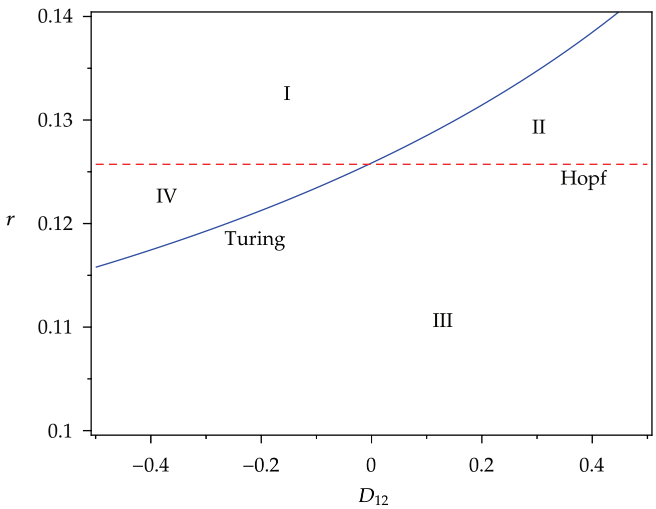


Figure 2: Bifurcation diagram for model (1.1) with parameters $a = 0.8, b = 0.1$, and $D_{21} = 0.8$. The Turing bifurcation line is $r_T = (707983156D_{12}^2 - (7436338235 + 50000A)D_{12} + 8189199110 - 62500A)/(10^9(D_{12} + 1)(4D_{12}^2 - D_{12} - 5))$, where $A = \sqrt{10^9(14.63 - 2.336D_{12}^2 - 8.784D_{12})}$. Hopf bifurcation line is $r_H = 0.12572$. Turing and Hopf lines intersect at the Turing-Hopf bifurcation point $(D_{12}, r) = (-0.00489, 0.12572)$ and separate the parametric space into four domains.

3. Pattern Formation

In this section, we performed extensive numerical simulations of the spatially extended model (1.1) in 2-dimension spaces, and the qualitative results are shown here. All our numerical simulations employ the zero-flux boundary conditions with a system size of $Lx \times Ly$, with $Lx = Ly = 400$ discretized through $x \to (x_0, x_1, x_2, \dots, x_n)$ and $y \to (y_0, y_1, y_2, \dots, y_n)$, with $n = 200$. Other parameters are fixed as $a = 0.8$, $b = 0.1$, and $D_{21} = 0.8$. The numerical integration of (1.1) was performed by means of forward Euler integration, with a time step of $\tau = 0.05$ and spatial resolution $h = 2$ and using the standard five-point approximation for the 2D Laplacian with the zero-flux boundary conditions [30, 31]. More precisely, the concentrations $(N_{i,j}^{n+1}, P_{i,j}^{n+1})$ at the moment $(n+1)\tau$ at the mesh position (i, j) are given by

$$\begin{aligned} N_{i,j}^{n+1} &= N_{i,j}^{n} + \tau\Delta_h N_{i,j}^{n} + \tau D_{12}\Delta_h P_{i,j}^{n} + \tau f\left(N_{i,j}^{n}, P_{i,j}^{n}\right), \\ P_{i,j}^{n+1} &= P_{i,j}^{n} + \tau D_{21}\Delta_h N_{i,j}^{n} + \tau\Delta_h P_{i,j}^{n} + \tau g\left(N_{i,j}^{n}, P_{i,j}^{n}\right), \end{aligned} \tag{3.1}$$

with the Laplacian defined by

$$\Delta_h N_{i,j}^{n} = \frac{N_{i+1,j}^{n} + N_{i-1,j}^{n} + N_{i,j+1}^{n} + N_{i,j-1}^{n} - 4N_{i,j}^{n}}{h^2}. \tag{3.2}$$

Initially, the entire system is placed in the stationary state $(N^*, P^*) = (0.370156, 0.370156)$, and the propagation velocity of the initial perturbation is thus on the order of

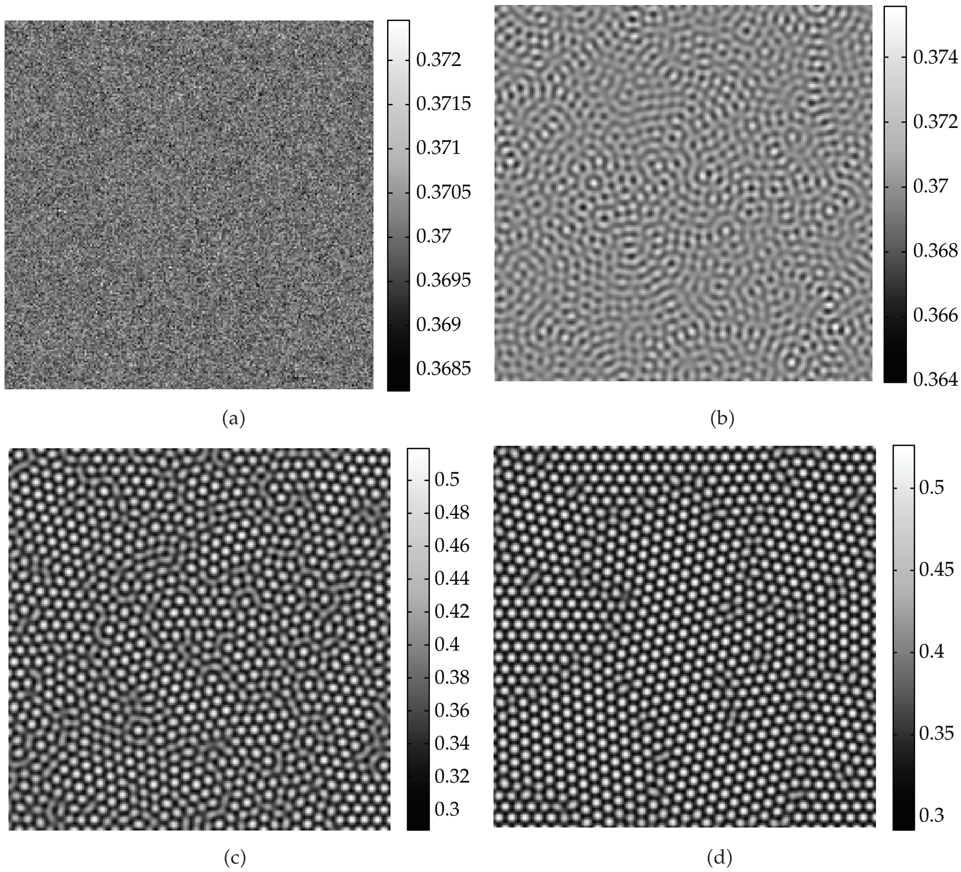


Figure 3: Spots pattern obtained with model (1.1) for $(D_{12}, r) = (0.12, 0.1278)$. Iterations: (a) 0, (b) 10000, (c) 30000, and (d) 300000.

5×10^{-4} space units per time unit. And the system is then integrated for 100000 or 300 000 time steps and some images saved. After the initial period during which the perturbation spreads, the system goes either into a time dependent state, or to an essentially steady state (time independent).

In the numerical simulations, different types of dynamics are observed, and it is found that the distributions of predator and prey are always of the same type. Consequently, we can restrict our analysis of pattern formation to one distribution. In this section, we show the distribution of prey N, for instance.

Firstly, we show the pattern formation for the parameters (D_{12}, r) located in domain II (cf. Figure 2); the region of pure Turing instability occurs while Hopf stability occurs. We have performed a large number of simulations and found that there only exhibits spots pattern in this domain. As an example, we show the time evolution of spots pattern of prey N at 0, 10000, 30000, and 300000 iteration for $(D_{12}, r) = (0.12, 0.1278)$ in Figure 3. In this case, one can see that for model (1.1), the random initial distribution (cf. Figure 3(a)) leads to the formation of regular spots except for apparently stable defects (cf. Figure 3(d)).

Next, we show the pattern for the parameters (D_{12}, r) located in domain III (cf. Figure 2), both Hopf and Turing instability occur. The model dynamics exhibits

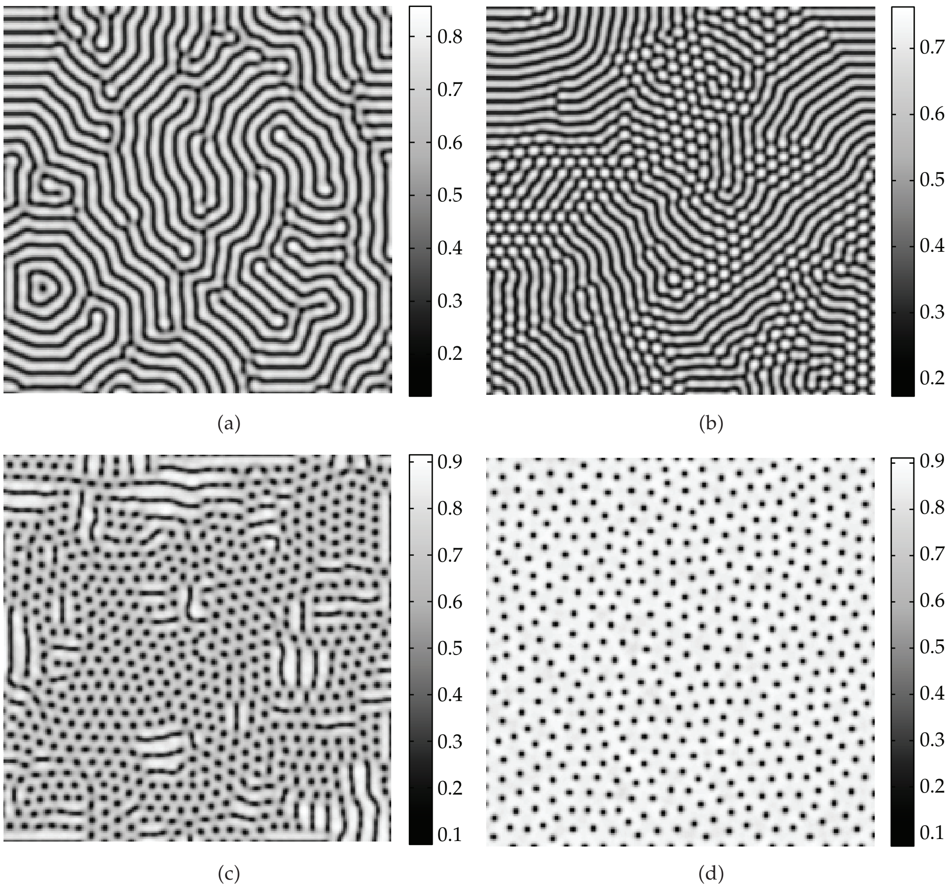


Figure 4: Four types of patterns obtained with model (1.1) for 100000 iterations. (a) Pattern α: $(D_{12}, r) = (-0.04, 0.00574)$. (b) Pattern β: $(D_{12}, r) = (0.152, 0.0882)$. (c) Pattern γ: $(D_{12}, r) = (0.12, 0.03496)$. (d) Pattern δ: $(D_{12}, r) = (0.2, 0.01)$.

spatiotemporal complexity of pattern formation. In Figure 4, we show four typical types of patterns we obtained via numerical simulation.

Pattern α is time independent and consists primarily of stripes. Pattern β is time independent and consists of stripes and spots. Pattern γ is time dependent: a few of the stripes without apparent decay, but the remainder of the spots pattern remains time independent. Pattern δ is time independent and consists of black spots on a white background, that is, isolated zones with low-population densities. Baurmann et al. [32] called this type pattern "cold spots" and von Hardenberg et al. [33] called it "holes." In this paper, we adopt the name "holes."

4. Concluding Remarks

In summary, we have investigated a cross-diffusive Holling-Tanner predator-prey model with equal self-diffusive coefficients. Based on the bifurcation analysis (Hopf and Turing), we give the spatial pattern formation via numerical simulation, that is, the evolution process of the system near the coexistence equilibrium point (N^*, P^*).

In contrast to the results in [23], we find that the model dynamics exhibits a cross-diffusion controlled formation growth not only to spots (in [23], Sun et al. claimed that the spots pattern is the only pattern of the model), but also to stripes, holes, and stripes-spots replication. That is to say, the pattern formation of the Holling-Tanner predator-prey model is not simple, but rich and complex.

On the other hand, in the predator-prey model, predators will tend to gravitate toward higher concentrations of prey while prey will preferentially move toward regions where predators are rare. Models of predator-prey systems with cross diffusion have been extensively analyzed in the literature, though often with respect to their mathematical properties rather than to provide insight into the kinds of patterns that can emerge [24]. And the methods and results in the present paper may be useful for the research of the pattern formation in the cross-diffusive predator-prey model.

Acknowledgments

The authors would like to thank the anonymous referee for the very helpful suggestions and comments, which led to improvements of their original paper. This research was supported by the Natural Science Foundation of Zhejiang Province (LY12A01014 and LQ12A01009), the National Science Foundation of China (11226172), and the National Basic Research Program of China (2012CB426510).

References

[1] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Wiley, Chichester, UK, 2003.
[2] B. E. Kendall, "Nonlinear dynamics and chaos," in *Encyclopedia of Life Sciences*, vol. 13, pp. 255–262, Nature Publishing Group, London, UK, 2001.
[3] R. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, vol. 261, no. 5560, pp. 459–467, 1976.
[4] V. N. Biktashev, J. Brindley, A. V. Holden, and M. A. Tsyganov, "Pursuit-evasion predator-prey waves in two spatial dimensions," *Chaos*, vol. 14, no. 4, pp. 988–994, 2004.
[5] J. B. Shukla and S. Verma, "Effects of convective and dispersive interactions on the stability of two species," *Bulletin of Mathematical Biology*, vol. 43, no. 5, pp. 593–610, 1981.
[6] E. H. Kerner, "Further considerations on the statistical mechanics of biological associations," vol. 21, pp. 217–255, 1959.
[7] N. Shigesada, K. Kawasaki, and E. Teramoto, "Spatial segregation of interacting species," *Journal of Theoretical Biology*, vol. 79, no. 1, pp. 83–99, 1979.
[8] A. Turing, "The chemical basis of morphogenesis," *Philosophical Transactions of the Royal Society of London B*, vol. 237, no. 1, pp. 37–72, 1952.
[9] Y. Huang and O. Diekmann, "Interspecific influence on mobility and Turing instability," *Bulletin of Mathematical Biology*, vol. 65, no. 1, pp. 143–156, 2003.
[10] R. B. Hoyle, *Pattern Formation: An Introduction to Methods*, Cambridge University Press, Cambridge, UK, 2006.
[11] J. Chattopadhyay and P. K. Tapaswi, "Effect of cross-diffusion on pattern formation—a nonlinear analysis," *Acta Applicandae Mathematicae*, vol. 48, no. 1, pp. 1–12, 1997.
[12] L. Chen and A. Jüngel, "Analysis of a parabolic cross-diffusion population model without self-diffusion," *Journal of Differential Equations*, vol. 224, no. 1, pp. 39–59, 2006.
[13] B. Dubey, B. Das, and J. Hussain, "A predator-prey interaction model with self and cross-diffusion," *Ecological Modelling*, vol. 141, no. 1–3, pp. 67–76, 2001.

[14] B. Dubey, N. Kumari, and R. K. Upadhyay, “Spatiotemporal pattern formation in a diffusive predator-prey system: an analytical approach,” *Journal of Applied Mathematics and Computing*, vol. 31, no. 1-2, pp. 413–432, 2009.

[15] M. Iida, M. Mimura, and H. Ninomiya, “Diffusion, cross-diffusion and competitive interaction,” *Journal of Mathematical Biology*, vol. 53, no. 4, pp. 617–641, 2006.

[16] W. Ko and K. Ryu, “On a predator-prey system with cross diffusion representing the tendency of predators in the presence of prey species,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1133–1142, 2008.

[17] W. Ko and K. Ryu, “On a predator-prey system with cross-diffusion representing the tendency of prey to keep away from its predators,” *Applied Mathematics Letters*, vol. 21, no. 11, pp. 1177–1183, 2008.

[18] K. Kuto and Y. Yamada, “Multiple coexistence states for a prey-predator system with cross-diffusion,” *Journal of Differential Equations*, vol. 197, no. 2, pp. 315–348, 2004.

[19] A. Okubo and S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, vol. 14, Springer, New York, NY, USA, 2nd edition, 2001.

[20] P. Y. H. Pang and M. Wang, “Strategy and stationary pattern in a three-species predator-prey model,” *Journal of Differential Equations*, vol. 200, no. 2, pp. 245–273, 2004.

[21] C. V. Pao, “Strongly coupled elliptic systems and applications to Lotka-Volterra models with cross-diffusion,” *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 60, no. 7, pp. 1197–1217, 2005.

[22] S. Raychaudhuri, D. K. Sinha, and J. Chattopadhyay, “Effect of time-varying cross-diffusivity in a two-species Lotka-Volterra competitive system,” *Ecological Modelling*, vol. 92, no. 1, pp. 55–64, 1996.

[23] G. Q. Sun, Z. Jin, Q. X. Liu, and L. Li, “Pattern formation induced by cross-diffusion in a predator-prey system,” *Chinese Physics B*, vol. 17, no. 11, pp. 3936–3941, 2008.

[24] V. K. Vanag and I. R. Epstein, “Cross-diffusion and pattern formation in reaction-diffusion systems,” *Physical Chemistry Chemical Physics*, vol. 11, no. 6, pp. 897–912, 2009.

[25] R. K. Upadhyay, W. Wang, and N. K. Thakur, “Spatiotemporal dynamics in a spatial plankton system,” *Mathematical Modelling of Natural Phenomena*, vol. 5, no. 5, pp. 102–122, 2010.

[26] E. Sáez and E. González-Olivares, “Dynamics of a predator-prey model,” *SIAM Journal on Applied Mathematics*, vol. 59, no. 5, pp. 1867–1878, 1999.

[27] J. D. Murray, “Discussion: Turing’s theory of morphogenesis—its influence on modelling biological pattern and form,” *Bulletin of Mathematical Biology*, vol. 52, no. 1-2, pp. 119–152, 1990.

[28] P. A. Braza, “The bifurcation structure of the Holling-Tanner model for predator-prey interactions using two-timing,” *SIAM Journal on Applied Mathematics*, vol. 63, no. 3, pp. 889–904, 2003.

[29] H. Malchow, S. V. Petrovskii, and E. Venturino, *Spatiotemporal Patterns in Ecology and Epidemiology-Theory, Models, and Simulation*, Mathematical and Computational Biology Series, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2008.

[30] M. R. Garvie, “Finite-difference schemes for reaction-diffusion equations modeling predator-prey interactions in MATLAB,” *Bulletin of Mathematical Biology*, vol. 69, no. 3, pp. 931–956, 2007.

[31] A. Munteanu and R. V. Solé, “Pattern formation in noisy self-replicating spots,” *International Journal of Bifurcation and Chaos*, vol. 16, no. 12, pp. 3679–3683, 2006.

[32] M. Baurmann, T. Gross, and U. Feudel, “Instabilities in spatially extended predator-prey systems: spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations,” *Journal of Theoretical Biology*, vol. 245, no. 2, pp. 220–229, 2007.

[33] J. von Hardenberg, E. Meron, M. Shachak, and Y. Zarmi, “Diversity of vegetation patterns and desertification,” *Physical Review Letters*, vol. 87, no. 19, Article ID 198101, 4 pages, 2001.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 218785, 8 pages
doi:10.1155/2012/218785

Research Article

A Lyapunov Function and Global Stability for a Class of Predator-Prey Models

Xiaoqin Wang[1] and Huihai Ma[2]

[1] *Faculty of Science, Shaanxi University of Science and Technology, Xi'an 710021, China*
[2] *College of Electrical and Information Engineering, Shaanxi University of Science and Technology, Xi'an 710021, China*

Correspondence should be addressed to Xiaoqin Wang, wxiqn@163.com

Received 29 November 2012; Accepted 14 December 2012

Academic Editor: Junli Liu

We construct a new Lyapunov function for a class of predation models. Global stability of the positive equilibrium states of these systems can be established when the Lyapunov function is used.

1. Introduction

The dynamics of predator-prey systems are often described by differential equations, which represent time continuously. A common framework for such a model is [1–3]

$$\begin{aligned} \frac{dN}{dt} &= Nf(N) - Pg(N,P), \\ \frac{dP}{dt} &= h[g(N,P),P]P, \end{aligned} \tag{1.1}$$

where N and P are prey and predator densities, respectively, $f(N)$ is the prey growth rate, $g(N)$ is the functional response, for example, the prey consumption rate by an average single predator, and $h[g(N),P]$ the per capita growth rate of predators (also known as the "predator numerical response"), which obviously increases with the prey consumption rate. The most widely accepted assumption for the numerical response with predator density restricting is as follows:

$$h[g(N,P),P] = \varepsilon g(N,P) - \delta P - \beta, \tag{1.2}$$

where β is a per capita predator death rate, ε the conversion efficiency of food into offspring, δ the density dependent rate [2]. And prototype of the prey growth rate $f(N)$ is the logistic growth

$$f(N) = r\left(1 - \frac{N}{K}\right), \tag{1.3}$$

where $K > 0$ is the carrying capacity of the prey.

When

$$g(N,P) = 1 - e^{-aN}, \tag{1.4}$$

where a is the efficiency of predator capture of prey, model (1.1) is called Ivlev-type predation model, due originally to Ivlev [4]. And Ivlev-type functional response is classified to the prey-dependent; that is, g is independent of predator P [2].

Both ecologists and mathematicians are interested in the Ivlev-type predator-prey model and much progress has been seen in the study of the model [5–13]. Of them, Xiao [8] gave global analysis of the following model:

$$\begin{aligned} \frac{dN}{dt} &= rN(1-N) - \left(1 - e^{-aN}\right)P, \\ \frac{dP}{dt} &= P\left(1 - e^{-aN} - d - \delta P\right). \end{aligned} \tag{1.5}$$

But, in paper [8], the author gave complex process to prove the global asymptotical stability of the positive equilibrium.

In this paper, we will establish a new Lyapunov function to prove the global stability of the positive equilibrium of model (1.5).

Our paper is organized as follows. In the next section, we discuss the existence, uniqueness of the positive equilibrium, and establish a new Lyapunov function to model (1.5). In Section 3, we will give some examples to show the robustness of our Lyapunov function.

2. Main Results

First of all, it is easy to verify that model (1.5) has two trivial equilibria (belonging to the boundary of $\mathbf{R}_+^2$, that is, at which one or more of populations has zero density or is extinct), namely, $E_0 = (0,0)$ and $E_1 = (1,0)$. For the positive equilibrium, set

$$rN(1-N) - \left(1 - e^{-aN}\right)P = 0, \qquad P\left(1 - e^{-aN} - \delta P - d\right) = 0, \tag{2.1}$$

which yields

$$N(1-N)=\frac{1}{r\delta}\left(\left(1-e^{-aN}\right)\left(1-e^{-aN}-d\right)\right). \tag{2.2}$$

We have the following Lemma regarding the existence of the positive equilibrium.

Lemma 2.1 (see [8]). *Suppose* $1-e^{-a}>d$. *Model* (1.5) *has a unique positive equilibrium* $E^*=(N^*,P^*)$ *if either of the following inequalities holds:*

(i) $d\geq 2(1-2e^{-a/2})$;

(ii) $d<2(1-2e^{-a/2})$, $(a/r\delta)(2(1-2e^{-a/2})-d))e^{-a/2}\geq 1+(2/a)\ln(1/2-d/4)$ *and* $(1/r\delta)((1/2+d/4)^2-d(1/2+d/4))<-(1/a)\ln(1/2+d/4)-1/a^2$.

Lemma 2.2. *Let* $1-e^{-a}>d$, *then* $\Omega=\{(N,P)\mid\ 0\leq N\leq 1,\ 0\leq P\leq(1-e^{-a}-d)/\delta\}$ *is a region of attraction for all solutions of model* (1.5) *initiating in the interior of the positive quadrant* $\mathbf{R}_+^2$.

Proof. Let $(N(t),P(t))$ be any solution of model (1.5) with positive initial conditions. Note that $dN/dt\leq N(1-N)$, by a standard comparison argument, we have

$$\limsup_{t\to\infty} N(t)\leq 1. \tag{2.3}$$

Then,

$$\frac{dP}{dt}=P\left(1-e^{-aN}-d-\delta P\right)\leq P(1-e^{-a}-d-\delta P). \tag{2.4}$$

Similarly, since $1-e^{-a}>d$, we have

$$\limsup_{t\to\infty} P(t)\leq\frac{1-e^{-a}-d}{\delta}. \tag{2.5}$$

On the other hand, for all $(N,P)\in\Omega$, we have $dN/dt|_{N=0}=0$ and $dP/dt|_{P=0}=0$. Hence, Ω is a region of attraction. As a consequence, we will focus on the stability of the positive equilibrium E^* only in the region Ω. □

In the following, we devote to the global stability of the positive equilibrium $E^*=(N^*,P^*)$ for model (1.5) by constructing a new Lyapunov function which is motivated by the work of Hsu [3].

Theorem 2.3. *If* $a\leq 2$, *the positive equilibrium* $E^*=(N^*,P^*)$ *of model* (1.5) *is globally asymptotically stable in the region* Ω.

Proof. For model (1.5), we construct a Lyapunov function of the form

$$V(N,P)=\int_{N^*}^{N}\frac{1-e^{-a\xi}-d-\delta P^*}{1-e^{-a\xi}}d\xi+\int_{P^*}^{P}\frac{\eta-P^*}{\eta}d\eta. \tag{2.6}$$

Note that $V(N,P)$ is non-negative, $V(N,P)=0$ if and only if $(N,P)=(N^*,P^*)$. Furthermore, the time derivative of V along the solutions of (1.5) is

$$\frac{dV}{dt}=\frac{1-e^{-aN}-d-\delta P^*}{1-e^{-aN}}\frac{dN}{dt}+\frac{P-P^*}{P}\frac{dP}{dt}. \tag{2.7}$$

Substituting the expressions of dN/dt and dP/dt defined in (1.5) into (2.7), we can obtain

$$\begin{aligned}\frac{dV}{dt}&=\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-P^*\right)-\delta(P-P^*)^2\\&=\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)-\delta(P-P^*)^2.\end{aligned} \tag{2.8}$$

Define

$$\phi(N)=1-e^{-aN}-d-\delta P^*, \tag{2.9}$$

then

$$\begin{aligned}\phi(N^*)&=0,\\ \phi'(N)&=1+ae^{-aN}>0.\end{aligned} \tag{2.10}$$

So,

$$(N-N^*)\left(\phi(N)-\phi(N^*)\right)>0. \tag{2.11}$$

Then we can get

$$\frac{dV}{dt}\le 0. \tag{2.12}$$

If

$$\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)\le 0 \tag{2.13}$$

holds, which is equivalent to

$$(N-N^*)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)\le 0. \tag{2.14}$$

Set $\varphi(N)=rN(1-N)/(1-e^{-aN})$, we obtain $\varphi(0)=0$ and

$$\varphi'(N)=\frac{r\left((1-2N)(1-e^{-aN})-aN(1-N)e^{-aN}\right)}{\left(1-e^{-aN}\right)^2}. \tag{2.15}$$

And set $\psi(N) = (1-2N)(1-e^{-aN}) - aN(1-N)e^{-aN}$, we can get $\psi(0) = 0$ and

$$\begin{aligned}\psi'(N) &= e^{-aN}\left(a^2N(1-N)+2\right)-2,\\ \psi''(N) &= ae^{-aN}\left(a^2N^2-a(a+2)N+a-2\right).\end{aligned} \tag{2.16}$$

In view of $a \le 2$, it follows that $\psi''(N) \le 0$ and $\psi'(N) \le \psi'(0) = 0$ in the region Ω. Then $\psi(N) \le 0$ is always true. It follows that $\varphi'(N) \le 0$, that is, $V' \le 0$. Consequently, the function $V(N,P)$ satisfies the asymptotic stability theorem [14]. Hence, $E^* = (N^*, P^*)$ is globally asymptotically stable. This completes the proof. □

3. Applications

In this paper, we construct a new Lyapunov function for proving the global asymptotical stability of model (1.5). The new Lyapunov function is useful not only to model (1.5), but also to other models.

In this section, we will give some examples to show the robustness of the Lyapunov function (2.6). The parameters of the following models are positive and have the same ecological meanings with those of in model (1.5).

Example 3.1. Considering the following Ivlev predator-prey model incorporating prey refuges (see [9]):

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\left(1-e^{-a(1-m)N}\right)P,\\ \frac{dP}{dt} &= \left(1-e^{-a(1-m)N}-\delta P-d\right)P,\end{aligned} \tag{3.1}$$

where $m \in [0,1)$ is a refuge protecting of the prey. We can choose a Lyapunov functional as follows:

$$V(N,P) = \int_{N^*}^{N} \frac{1-e^{-a(1-m)\xi}-\delta P^*-d}{1-e^{-a(1-m)\xi}}d\xi + \int_{P^*}^{P}\frac{\eta-P^*}{\eta}d\eta. \tag{3.2}$$

The proof is similar to that of the Section 2.

Example 3.2. Considering the following predator-prey model with Rosenzweig functional response (see [10]):

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-bN^{\mu}P,\\ \frac{dP}{dt} &= (cN^{\mu}-\delta N-d)P,\end{aligned} \tag{3.3}$$

where $\mu \in (0,1]$ is the victim's competition constant. We can choose a Lyapunov functional as follows:

$$V(N,P) = \int_{N^*}^{N} \frac{cN^{\xi} - \delta P^* - d}{bN^{\xi}} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.4}$$

We omit the proof here.

Example 3.3. Considering the following model (1.1) with Holling-type functional response (see [11]):

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - bg(N)P, \\ \frac{dP}{dt} &= \left(g(N) - \delta P - d\right)P, \end{aligned} \tag{3.5}$$

where $g(N) = \alpha N/(\beta + N)$ is known as a Holling type-II function, $g(N) = \alpha N^2/(\beta + N^2)$ as a Holling type-III function and $g(N) = \alpha N^2/(\beta + \omega N + N^2)$ as a Holling type-IV function. We choose a Lyapunov function:

$$V(N,P) = \int_{N^*}^{N} \frac{g(\xi) - \delta P^* - d}{bg(\xi)} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.6}$$

For more details, we refer to [12].

Example 3.4. Considering the following diffusive Ivlev-type predator-prey model (see [13]):

$$\begin{aligned} \frac{\partial N}{dt} &= rN(1-N) - \left(1 - e^{-aN}\right)P + d_1 \nabla^2 N, \\ \frac{\partial P}{dt} &= \left(\varepsilon\left(1 - e^{-aN}\right) - d\right)P + d_2 \nabla^2 P, \end{aligned} \tag{3.7}$$

where the nonnegative constants d_1 and d_2 are the diffusion coefficients of N and P, respectively. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the usual Laplacian operator in two-dimensional space, is used to describe the Brownian random motion.

Model (3.7) is to be analyzed under the following non-zero initial conditions

$$N(x,y,0) \geq 0, \quad P(x,y,0) \geq 0, \quad (x,y) \in \Omega = [0,Lx] \times [0,Ly], \tag{3.8}$$

and zero-flux boundary conditions:

$$\frac{\partial N}{\partial n} = \frac{\partial P}{\partial n} = 0. \tag{3.9}$$

In the above, n is the outward unit normal vector of the boundary $\partial\Omega$.

In order to give the proof of the global stability, we construct a Lyapunov function:

$$E(t) = \iint_{\Omega} V(N(t), P(t))\, dx\, dy, \tag{3.10}$$

where

$$V(N, P) = \int_{N^*}^{N} \frac{\varepsilon(1 - e^{-a\xi}) - d}{1 - e^{-a\xi}} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.11}$$

Then, differentiating $E(t)$ with respect to time t along the solutions of model (3.7), we can obtain

$$\frac{dE(t)}{dt} = \iint_{\Omega} \frac{dV}{dt}\, dx\, dy + \iint_{\Omega} \left(\frac{\partial V}{\partial N} d_1 \nabla^2 N + \frac{\partial V}{\partial P} d_2 \nabla^2 P \right) dx\, dy. \tag{3.12}$$

Using Green's first identity in the plane, and considering the zero-flux boundary conditions (3.9), one can show that

$$\begin{aligned} \frac{dE(t)}{dt} &= \iint_{\Omega} \frac{dV}{dt}\, dx\, dy - \frac{d_1 \partial^2 V}{\partial N^2} \iint_{\Omega} \left[\left(\frac{\partial N}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right] dx\, dy \\ &\quad - \frac{d_2 \partial^2 V}{\partial P^2} \iint_{\Omega} \left[\left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right] dx\, dy \\ &\leq \iint_{\Omega} \frac{dV}{dt}\, dx\, dy. \end{aligned} \tag{3.13}$$

The remaining arguments are rather similar as Theorem 2.3.

References

[1] D. Alonso, F. Bartumeus, and J. Catalan, "Mutual interference between predators can give rise to Turing spatial patterns," *Ecology*, vol. 83, no. 1, pp. 28–34, 2002.

[2] P. Abrams and L. Ginzburg, "The nature of predation: prey dependent, ratio dependent or neither?" *Trends in Ecology and Evolution*, vol. 15, no. 8, pp. 337–341, 2000.

[3] S. B. Hsu, "On global stability of a predator-prey system," *Mathematical Biosciences*, vol. 39, no. 1-2, pp. 1–10, 1978.

[4] V. S. Ivlev, *Experimental Ecology of the Feeding of Fishes*, Yale University Press, 1961.

[5] R. E. Kooij and A. Zegeling, "A predator-prey model with Ivlev's functional response," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 2, pp. 473–489, 1996.

[6] J. Sugie, "Two-parameter bifurcation in a predator-prey system of Ivlev type," *Journal of Mathematical Analysis and Applications*, vol. 217, no. 2, pp. 349–371, 1998.

[7] J. Feng and S. Chen, "Global asymptotic behavior for the competing predators of the Ivlev types," *Mathematica Applicata*, vol. 13, no. 4, pp. 85–88, 2000.

[8] H. B. Xiao, "Global analysis of Ivlev's type predator-prey dynamic systems," *Applied Mathematics and Mechanics*, vol. 28, no. 4, pp. 461–470, 2007.

[9] J. B. Collings, "Bifurcation and stability analysis of a temperature-dependent mitepredator-prey interaction model incorporating a prey refuge," *Bulletin of Mathematical Biology*, vol. 57, no. 1, pp. 63–76, 1995.

[10] M. L. Rosenzweig, “Paradox of enrichment: destabilization of exploitation ecosystems in ecological time,” *Science*, vol. 171, no. 3969, pp. 385–387, 1971.
[11] C. S. Holling, “The functional response of predators to prey density and its role in mimicry and population regulation,” *Memoirs of the Entomological Society of Canada*, vol. 97, S45, pp. 5–60, 1965.
[12] S. Ruan and D. Xiao, “Global analysis in a predator-prey system with nonmonotonic functional response,” *SIAM Journal on Applied Mathematics*, vol. 61, no. 4, pp. 1445–1472, 2000/01.
[13] W. Wang, L. Zhang, H. Wang, and Z. Li, “Pattern formation of a predator-prey system with Ivlev-type functional response,” *Ecological Modelling*, vol. 221, no. 2, pp. 131–140, 2010.
[14] D. R. Merkin, F. F. Afagh, and A. L. Smirnov, *Introduction to the Theory of Stability*, vol. 24 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 1997.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 454073, 7 pages
doi:10.1155/2012/454073

Research Article

Stochastically Perturbed Epidemic Model with Time Delays

Tailei Zhang

School of Science, Chang'an University, Xi'an 710064, China

Correspondence should be addressed to Tailei Zhang, t.l.zhang@126.com

Received 3 November 2012; Accepted 4 December 2012

Academic Editor: Junli Liu

We investigate a stochastic epidemic model with time delays. By using Liapunov functionals, we obtain stability conditions for the stochastic stability of endemic equilibrium.

1. Introduction

In [1], Zhen et al. introduced a deterministic SIRS model

$$\begin{aligned}
\dot{S}(t) &= b - \mu S(t) - \beta S(t) \int_0^h f(s) I(t-s) ds + \alpha R(t), \\
\dot{I}(t) &= \beta S(t) \int_0^h f(s) I(t-s) ds - (\mu + c + \lambda) I(t), \\
\dot{R}(t) &= \lambda I(t) - (\mu + \alpha) R(t),
\end{aligned} \tag{1.1}$$

where $S(t)$ is the number of susceptible population, $I(t)$ is the number of infective members and $R(t)$ is the number of recovered members. b is the rate at which population is recruited, μ is the death rate for classes $S(t)$, $I(t)$, and $R(t)$, c is the disease-induced death rate, β is the transmission rate, λ is the recovery rate, and α is the loss of immunity rate. Equation (1.1) represents an SIRS model with epidemics spreading via a vector, whose incubation time period is a distributed parameter over the interval $[0, h]$. $h \in \mathbb{R}^+$ is the limit superior of incubation time periods in the vector population. The $f(s)$ is usually nonnegative and continuous and is the distribution function of incubation time periods among the vectors and $\int_0^h f(s)ds = 1$.

To be more general, the following model is formulated:

$$\begin{aligned}\dot{S}(t) &= b - \mu_1 S(t) - \beta S(t)\int_0^h f(s)I(t-s)ds + \alpha R(t),\\ \dot{I}(t) &= \beta S(t)\int_0^h f(s)I(t-s)ds - (\mu_2+\lambda)I(t),\\ \dot{R}(t) &= \lambda I(t) - (\mu_3+\alpha)R(t).\end{aligned} \tag{1.2}$$

The positive constants μ_1, μ_2, and μ_3 represent the death rates of susceptibles, infectives, and recovered, respectively. It is natural biologically to assume that $\mu_1 < \min\{\mu_2, \mu_3\}$. If $\alpha = 0$, model (1.2) was considered in [2–5]. For $\alpha = 0$ and fixed delay, the global asymptotic stability of (1.2) was considered in [6].

The basic reproduction number for (1.2) is

$$R_0 = \frac{\beta b}{\mu_1(\mu_2+\lambda)}. \tag{1.3}$$

If $R_0 \leq 1$, the system (1.2) has just one disease-free equilibrium $E_0 = (b/\mu_1, 0, 0)$; otherwise, if $R_0 > 1$, the disease-free equilibrium E_0 is still present, but there is also a unique positive endemic equilibrium $E^* = (S^*, I^*, R^*)$, given by $S^* = (\mu_2+\lambda)/\beta$, $I^* = (b(\mu_3+\alpha)(R_0 - 1))/(R_0[\mu_2(\mu_3+\alpha)+\mu_3\lambda])$, $R^* = (\lambda/(\mu_3+\alpha))I^*$.

2. Stability Analysis of the Atochastic Delay Model

Since environmental fluctuations have great influence on all aspects of real life, then it is natural to study how these fluctuations affect the epidemiological model (1.2). We assume that stochastic perturbations are of white noise type and that they are proportional to the distances of S, I, R from S^*, I^*, R^*, respectively. Then the system (1.2) will be reduced to the following form:

$$\begin{aligned}\dot{S}(t) &= b - \mu_1 S(t) - \beta S(t)\int_0^h f(s)I(t-s)ds + \alpha R(t) + \sigma_1(S(t)-S^*)\dot{w}_1(t),\\ \dot{I}(t) &= \beta S(t)\int_0^h f(s)I(t-s)ds - (\mu_2+\lambda)I(t) + \sigma_2(I(t)-I^*)\dot{w}_2(t),\\ \dot{R}(t) &= \lambda I(t) - (\mu_3+\alpha)R(t) + \sigma_3(R(t)-R^*)\dot{w}_3(t).\end{aligned} \tag{2.1}$$

Here, σ_1, σ_2, and σ_3 are constants, and $w(t) = (w_1(t), w_2(t), w_3(t))$ represents a three-dimensional standard Wiener processes.

This system has the same equilibria as system (1.2). We assume that $R_0 > 1$; we discuss the stability of the endemic equilibrium E^* of (2.1). The stochastic system (2.1) can be centered at its endemic equilibrium E^* by the changes of variables $x_1 = S - S^*$, $x_2 = I - I^*$, $x_3 = R - R^*$. By this way, we obtain

$$\begin{aligned}
\dot{x}_1 &= -(\beta I^* + \mu_1)x_1 - \beta x_1 \int_0^h f(s)x_2(t-s)ds - \beta S^* \int_0^h f(s)x_2(t-s)ds + \alpha x_3 + \sigma_1 x_1 \dot{w}_1(t), \\
\dot{x}_2 &= \beta I^* x_1 - \beta S^* x_2 + \beta x_1 \int_0^h f(s)x_2(t-s)ds + \beta S^* \int_0^h f(s)x_2(t-s)ds + \sigma_2 x_2 \dot{w}_2(t), \\
\dot{x}_3 &= \lambda x_2 - (\mu_3 + \alpha)x_3 + \sigma_3 x_3 \dot{w}_3(t).
\end{aligned} \tag{2.2}$$

In order to investigate the stability of endemic equilibrium of system (2.1), we study the stability of the trivial solution of system (2.2).

First, consider the stochastic functional differential equation

$$dy(t) = h(t, y_t)dt + g(t, y_t)dw(t), \quad t \geq 0, \ y_0 = \varphi \in H. \tag{2.3}$$

Let $\{\Omega, \sigma, P\}$ be the probability space, $\{f_t, t \geq 0\}$ the family of σ-algebra, $f_t \in \sigma$, H the space of f_0-adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\| = \sup_{s\leq 0}|\varphi(s)|$, $w(t)$ the m-dimensional f_t-adapted Wiener process, $h(t, y_t)$ the n-dimensional vector, and $g(t, y_t)$ the $n \times m$-dimensional matrix, both defined for $t \geq 0$. We assume that (2.3) has a unique global solution $y(t;\varphi)$ and that $h(t,0) = g(t,0) \equiv 0$. Then, (2.3) has the trivial solution $y(t) \equiv 0$ corresponding to the initial condition $y_0 = 0$.

Definition 2.1. The trivial solution of (2.3) is said to be stochastically stable if, for every $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta > 0$ such that

$$P\{|y(t;\varphi)| > r, \, t \geq 0\} \leq \varepsilon \tag{2.4}$$

for any initial condition $\varphi \in H$ satisfying $P\{\|\varphi\| \leq \delta\} = 1$.

Definition 2.2. The trivial solution of (2.3) is said to be mean square stable if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $E|y(t;\varphi)|^2 < \varepsilon$ for any $t \geq 0$ provided that $\sup_{s\leq 0} E|\varphi(s)|^2 < \delta$.

Definition 2.3. The trivial solution of (2.3) is said to be asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} E|y(t;\varphi)|^2 = 0$.

The differential operator associated to (2.3) is defined by the formula

$$LV(t,\varphi) = \limsup_{\Delta \to 0} \frac{E_{t,\varphi} V(t+\Delta, y_{t+\Delta}) - V(t,\varphi)}{\Delta}, \tag{2.5}$$

where $y(s)$, $s \geq t$ is the solution of (2.3) with initial condition $y_t = \varphi \in H$, and $V(t,\varphi)$ is a functional defined for $t \geq 0$.

If $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s < 0$, we can define the function $V_\varphi(t,y) = V(t,\varphi) = V(t,y_t) = V(t,y,y(t+s))$, $s < 0$, $\varphi = y_t$, $y = \varphi(0) = y(t)$. Let us define $C_{1,2}$ as a class of function $V(t,\varphi)$ so that for almost all $t \geq 0$, the first and second derivatives with respect to y of $V_\varphi(t,y)$ are continuous, and the first derivative with respect to t is continuous and bounded. Then the generating operator L of (2.3) is defined by

$$LV(t,y_t) = \frac{\partial V_\varphi(t,y)}{\partial t} + h^T(t,y_t)\frac{\partial V_\varphi(t,y)}{\partial y} + \frac{1}{2}\mathrm{trace}\left[g^T(t,y_t)\frac{\partial^2 V_\varphi(t,y)}{\partial y^2}g(t,y_t)\right]. \tag{2.6}$$

The following theorems [7] contain conditions under which the trivial solution of (2.3) is asymptotically mean square stable and stochastically stable.

Theorem 2.4. *If there exist a functional $V(t,\varphi) \in C_{1,2}$ such that*

$$c_1E|y(t)|^2 \le EV(t,y_t) \le c_2\sup_{s\le 0}E|y(t+s)|^2, \quad ELV(t,y_t) \le -c_3E|y(t)|^2 \tag{2.7}$$

for $c_i > 0$, $i = 1,2,3$. Then, the trivial solution of (2.3) *is asymptotically mean square stable.*

Theorem 2.5. *Let there exist a functional $V(t,\varphi) \in C_{1,2}$ such that*

$$c_1|y(t)|^2 \le V(t,y_t) \le c_2\sup_{s\le 0}|y(t+s)|^2, \quad LV(t,y_t) \le 0 \tag{2.8}$$

for $c_i > 0$, $i = 1,2$ and for any $\varphi \in H$ such that $P\{\|\varphi\| \le \delta\} = 1$, where $\delta > 0$ is sufficiently small. Then, the trivial solution of (2.3) *is stochastically stable.*

Consider the linear part of (2.2)

$$\begin{aligned}
\dot{y}_1 &= -(\beta I^* + \mu_1)y_1 - \beta S^*\int_0^h f(s)y_2(t-s)ds + \alpha y_3 + \sigma_1 y_1\dot{w}_1(t),\\
\dot{y}_2 &= \beta I^* y_1 - \beta S^* y_2 + \beta S^*\int_0^h f(s)y_2(t-s)ds + \sigma_2 y_2\dot{w}_2(t),\\
\dot{y}_3 &= \lambda y_2 - (\mu_3 + \alpha)y_3 + \sigma_3 y_3\dot{w}_3(t).
\end{aligned} \tag{2.9}$$

Theorem 2.6. *Assume that $R_0 > 1$ and the parameters of system* (2.2) *satisfy conditions*

$$\begin{gathered}
0 \le \sigma_1^2 < 2\mu_1 - \frac{\alpha(1+q)}{q},\\
0 \le \sigma_2^2 < \frac{q(2\beta S^* - \alpha)}{1+q} = \frac{q[2(\mu_2+\lambda) - \alpha]}{1+q},\\
0 \le \sigma_3^2 < 2\mu_3 + \alpha - \lambda,\\
\sqrt{\frac{2\alpha q}{2\mu_3 + \alpha - \lambda - \sigma_3^2}} < \min\left\{\frac{(2\mu_1 - \sigma_1^2)q - \alpha(1+q)}{\beta S^*}, p^*\right\},
\end{gathered} \tag{2.10}$$

where $p^* = (-\beta S^* + \sqrt{((\beta S^*)^2 - 4\lambda[(1+q)\sigma_2^2 - 2q\beta S^* + q\alpha])})/2\lambda$. Then, the trivial solution of system (2.9) *is asymptotically mean square stable.*

Proof. Set

$$V_1 = py_1^2 + y_2^2 + p^2 y_3^2 + q(y_1 + y_2)^2 \tag{2.11}$$

for some $p > 0$ and $q > 0$. Let L be the generating operator of the system (2.9), then

$$\begin{aligned}
LV_1 = {} & \left[-(\beta I^* + \mu_1)y_1 - \beta S^* \int_0^h f(s)y_2(t-s)ds + \alpha y_3\right][2py_1 + 2q(y_1+y_2)] \\
& + \left[\beta I^* y_1 - \beta S^* y_2 + \beta S^* \int_0^h f(s)y_2(t-s)ds\right][2y_2 + 2q(y_1+y_2)] \\
& + 2p^2 y_3[\lambda y_2 - (\mu_3+\alpha)y_3] + (p+q)\sigma_1^2 y_1^2 + 2q\sigma_1\sigma_2 y_1 y_2 + (1+q)\sigma_2^2 y_2^2 + p^2\sigma_3^2 y_3^2 \\
= {} & \left[\left(\sigma_1^2 - 2\mu_1\right)(p+q) - 2p\beta I^*\right]y_1^2 + (1+q)\left(\sigma_2^2 - 2\beta S^*\right)y_2^2 \\
& + p^2\left[\sigma_3^2 - 2(\mu_3+\alpha)\right]y_3^2 + 2\alpha(p+q)y_1y_3 + 2\left(q\alpha + p^2\lambda\right)y_2y_3 \\
& + 2[(\sigma_1\sigma_2 - \beta S^* - \mu_1)q + \beta I^*]y_1y_2 + 2\beta S^*(y_2 - py_1)\int_0^h f(s)y_2(t-s)ds.
\end{aligned} \tag{2.12}$$

Let

$$q = \frac{\beta I^*}{\beta S^* + \mu_1 - \sigma_1\sigma_2}. \tag{2.13}$$

Since $\sigma_1\sigma_2 \le (\sigma_1^2+\sigma_2^2)/2 < \mu_1 + \beta S^*$, it means that $q > 0$. By using the inequality $2|uv| \le u_1^2 + u_2^2$ and $2\alpha p y_1 y_3 \le \alpha p(y_1^2/p + py_3^2) = \alpha y_1^2 + \alpha p^2 y_3^2$, we find that

$$\begin{aligned}
LV_1 \le {} & \left[\left(\sigma_1^2 - 2\mu_1\right)q + \alpha(1+q) + p\beta S^*\right]y_1^2 \\
& + \left[(1+q)\left(\sigma_2^2 - 2\beta S^*\right) + q\alpha + p^2\lambda + \beta S^*\right]y_2^2 \\
& + \left[p^2\left(\sigma_3^2 - 2\mu_3 - \alpha\right) + 2\alpha q + p^2\lambda\right]y_3^2 \\
& + (1+p)\beta S^* \int_0^h f(s)y_2^2(t-s)ds.
\end{aligned} \tag{2.14}$$

We now choose the functional V_2 to eliminate the term with delay

$$V_2 = (1+p)\beta S^* \int_0^h f(s)\int_{t-s}^t y_2^2(\tau)d\tau ds. \tag{2.15}$$

Then for functional $V = V_1 + V_2$, we obtain

$$\begin{aligned} LV \leq & \left[\left(\sigma_1^2 - 2\mu_1\right)q + \alpha(1+q) + p\beta S^*\right]y_1^2 \\ & + \left[p^2\lambda + p\beta S^* + (1+q)\sigma_2^2 - 2q\beta S^* + q\alpha\right]y_2^2 \\ & + \left[p^2\left(\sigma_3^2 - 2\mu_3 - \alpha + \lambda\right) + 2\alpha q\right]y_3^2. \end{aligned} \tag{2.16}$$

If the first condition of (2.10) holds, then $(\sigma_1^2 - 2\mu_1)q + \alpha(1+q) < 0$. Set $F(p) = p^2\lambda + p\beta S^* + (1+q)\sigma_2^2 - 2q\beta S^* + q\alpha$, and if the second condition of (2.10) is true, then $F(0) < 0$, thus $F(p) = 0$ has one positive root $p^* = (-\beta S^* + \sqrt{((\beta S^*)^2 - 4\lambda[(1+q)\sigma_2^2 - 2q\beta S^* + q\alpha])})/2\lambda$, for any $0 < p < p^*$, $F(p) < 0$. From (2.10), there exists a $p > 0$, such that

$$\sqrt{\frac{2\alpha q}{2\mu_3 + \alpha - \lambda - \sigma_3^2}} < p < \min\left\{\frac{(2\mu_1 - \sigma_1^2)q - \alpha(1+q)}{\beta S^*}, p^*\right\}. \tag{2.17}$$

Therefore, there exists a $c > 0$ such that $LV \leq -c|y|^2$, where $y = (y_1, y_2, y_3)$. From Theorem 2.4, we can conclude that the zero solution of system (2.9) is asymptotically mean square stable. The theorem is proved. □

Remark 2.7. If $\alpha = 0$, then the system (2.1) becomes an SIR model, which has been discussed in [8]. The conditions (2.10) of Theorem 2.6 reduce to

$$0 \leq \sigma_1^2 < 2\mu_1, \qquad 0 \leq \sigma_2^2 < \frac{2q(\mu_2 + \lambda)}{1+q}, \qquad 0 \leq \sigma_3^2 < 2\mu_3 - \lambda. \tag{2.18}$$

The constant p in the proof of Theorem 2.6 is $0 < p < \min\{((2\mu_1 - \sigma_1^2)q)/(\beta S^*), p_1^*\}$ with $p_1^* = (-\beta S^* + \sqrt{((\beta S^*)^2 - 4\lambda[(1+q)\sigma_2^2 - 2q\beta S^*])})/2\lambda$. The first two conditions in (2.18) are the same as those in Theorem 7 of [8]. Since for $\alpha > 0$, we use different inequality to zoom up the term $2(q\alpha + p^2\lambda)y_2y_3$, then the third condition in (2.18) is different from that in Theorem 7 of [8].

Theorem 2.8. *Assume that $R_0 > 1$ and that conditions* (2.10) *are satisfied. Then the trivial solution of system* (2.2) *is stochastically stable.*

The proof is omitted because of the fact that the initial system (2.2) has a nonlinearity order more than one, then the conditions sufficient for asymptotic mean square stability of the trivial solution of the linear part of this system are sufficient for stochastic stability of the trivial solution of the initial system [9, 10]. Thus, if the conditions (2.10) hold, then the trivial solution of system (2.2) is stochastically stable.

3. Conclusions

In this paper, we have extended the well-known SIRS epidemic model with time delays by introducing a white noise term in it. We want to examine how environmental fluctuations affect the stability of system (1.2). By constructing Liapunov functional, we obtain sufficient conditions for the stochastic stability of the endemic equilibrium E^*. Our main results extend the corresponding results in paper [8], which discussed an SIR epidemic model.

Acknowledgments

This research was partially supported by the National Natural Science Foundation of China (nos. 11001215, 11101323) and the Scientific Research Program Funded by Shaanxi Provincial Education Department (no. 12JK0859).

References

[1] J. Zhen, Z. Ma, and M. Han, "Global stability of an SIRS epidemic model with delays," *Acta Mathematica Scientia Series B*, vol. 26, no. 2, pp. 291–306, 2006.
[2] E. Beretta, V. Capasso, and F. Rinaldi, "Global stability results for a generalized Lotka-Volterra system with distributed delays: applications to predator-prey and to epidemic systems," *Journal of Mathematical Biology*, vol. 26, no. 6, pp. 661–688, 1988.
[3] K. L. Cooke, "Stability analysis for a vector disease model," *The Rocky Mountain Journal of Mathematics*, vol. 9, no. 1, pp. 31–42, 1979.
[4] E. Beretta and Y. Takeuchi, "Global stability of an SIR epidemic model with time delays," *Journal of Mathematical Biology*, vol. 33, no. 3, pp. 250–260, 1995.
[5] Y. Takeuchi, W. Ma, and E. Beretta, "Global asymptotic properties of a delay SIR epidemic model with finite incubation times," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 6, pp. 931–947, 2000.
[6] W. Ma, M. Song, and Y. Takeuchi, "Global stability of an SIR epidemic model with time delay," *Applied Mathematics Letters*, vol. 17, no. 10, pp. 1141–1145, 2004.
[7] V. B. Kolmanovskiĭ and V. R. Nosov, *Stability of Functional-Differential Equations*, vol. 180 of *Mathematics in Science and Engineering*, Academic Press, London, UK, 1986.
[8] E. Beretta, V. Kolmanovskii, and L. Shaikhet, "Stability of epidemic model with time delays influenced by stochastic perturbations," *Mathematics and Computers in Simulation*, vol. 45, no. 3-4, pp. 269–277, 1998.
[9] L. E. Shaĭkhet, "Stability in probability of nonlinear stochastic systems with delay," *Mathematical Notes*, vol. 57, no. 1-2, pp. 103–106, 1995.
[10] L. Shaĭkhet, "Stability in probability of nonlinear stochastic hereditary systems," *Dynamic Systems and Applications*, vol. 4, no. 2, pp. 199–204, 1995.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 678957, 11 pages
doi:10.1155/2012/678957

Research Article

Qualitative Analysis for a Predator Prey System with Holling Type III Functional Response and Prey Refuge

Xia Liu[1] and Yepeng Xing[2]

[1] *College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China*
[2] *College of Mathematics and Science, Shanghai Normal University, Shanghai 200234, China*

Correspondence should be addressed to Xia Liu, liuxiapost@163.com

Received 29 September 2012; Accepted 3 November 2012

Academic Editor: Yonghui Xia

A predator prey system with Holling III functional response and constant prey refuge is considered. By using the Dulac criterion, we discuss the global stability of the positive equilibrium of the system. By transforming the system to a Liénard system, the conditions for the existence of exactly one limit cycle for the system are given. Some numerical simulations are presented.

1. Introduction

Recently, the qualitative analysis of predator prey systems with Holling II or III types functional response and prey refuge has been done by several papers, see [1–5]. Their main objective is to discuss under what conditions the positive equilibrium of the corresponding system is stable or unstable and the existence of exactly one limit cycles. In general, the prey refuge has two types, one is the so-called constant proportion prey refuge: $(1 - m)x$, where $m \in (0, 1)$, the other type is called constant prey refuge: $(x - m)$.

In [2], the authors considered the following system with a constant proportion prey refuge:

$$\begin{aligned}\frac{dx}{dt} &= ax - bx^2 - \frac{\alpha(1-m)^2x^2y}{\beta^2 + (1-m)^2x^2},\\ \frac{dy}{dt} &= -cy + \frac{k\alpha(1-m)^2x^2y}{\beta^2 + (1-m)^2x^2},\end{aligned} \tag{1.1}$$

where x and y denote the prey and predator density, respectively, at time t, the parameters $a, b, \alpha, \beta, c, k$ are positive constants, and their biological meanings can be seen in [2]. The main result is that when $0 < m < (1 + 2bc\beta/(a(k\alpha - 2c)))\sqrt{c/(k\alpha - c)}$ system (1.1) admits only one limit cycle which is globally asymptotically stable.

In paper [4], the authors only gave the local stability analysis to the following system with a constant prey refuge:

$$\frac{dx}{dt} = ax - bx^2 - \frac{\alpha(x-m)^2 y}{\beta^2 + (x-m)^2},$$
$$\frac{dy}{dt} = -cy + \frac{k\alpha(x-m)^2 y}{\beta^2 + (x-m)^2}. \tag{1.2}$$

In this paper, we will research under what conditions that the positive equilibrium is globally asymptotically stable and the existence of exactly one stable limit cycle of system (1.2). For ecological reason, we only consider system (1.2) in $\Omega_0 = \{(x, y) \mid x > m, y > 0\}$ or $\overline{\Omega}_0$.

It easy to obtain the following lemma.

Lemma 1.1. *Any solution* $(x(t), y(t))$ *of system* (1.2) *with initial condition* $x(0) > m,\ y(0) > 0$ *is positive and bounded for all* $t \geq 0$.

2. Basic Results

Let $\overline{x} = x - m,\ \overline{y} = \alpha y, dt = (\beta^2 + \overline{x}^2)d\overline{t}$, then system (1.2) changes (still denote $\overline{x}, \overline{y},\ \overline{t}$ as x, y, t)

$$\frac{dx}{dt} = (x+m)(a - b(x+m))\left(\beta^2 + x^2\right) - x^2 y,$$
$$\frac{dy}{dt} = -c\beta^2 y + (k\alpha - c)x^2 y. \tag{2.1}$$

Then Ω_0 transforms to $\Omega = \{(x, y) \mid\ x > 0,\ y > 0\}$ and system (2.1) is bounded.

Clearly, if $(H_1)\ 0 < m < a/b$ holds, system (2.1) has positive boundary equilibrium $E_0((a/b) - m, 0)$; if $(H_2)\ k\alpha > c,\ 0 < m < (a - bx_*)/b$, system (2.1) has a positive equilibrium $E_*(x_*, y_*)$, where

$$x_* = \beta\sqrt{\frac{c}{k\alpha - c}}, \qquad y_* = \frac{k\alpha}{c}(x_* + m)(a - b(x_* + m)). \tag{2.2}$$

It is easy to obtain the following lemma.

Lemma 2.1. *Let* (H_1) *hold. Further assume that* $(H_3)\ k\alpha \leq c$ *and* $(H_4) k\alpha > c,\ m > \max\{0, (a - bx_*)/b\}$. *Then* E_0 *is locally asymptotically stable, if any of* (H_3) *and* (H_4) *holds. When* $k\alpha > c,\ 0 < m < (a - bx_*)/b,\ E_0$ *is unstable, furthermore,* E_0 *is a saddle point.*

About the properties of the positive equilibrium, we have the following theorem.

Theorem 2.2. *Assume $k\alpha > c$. Then*

(I) E_* *is locally asymptotically stable for* $0 < m < (a - bx_*)/b$ *if* $a(2c - k\alpha) \le 2bcx_*$ *holds.*

(II) E_* *is locally asymptotically stable for* $m_1 < m < (a - bx_*)/b$ *and* E_* *is locally unstable for* $0 < m < m_1$ *if* $a(2c - k\alpha) > 2bcx_*$ *holds, where*

$$m_1 = \frac{bx_*^3 + \beta^2(a - bx_*) - \sqrt{\Delta/4}}{2b\beta^2}, \tag{2.3}$$

(III) *system* (2.1) *undergoes Hopf bifurcation at* $m = m_1$ *if* $a(2c - k\alpha) > 2bcx_*$ *holds.*

Proof. The Jacobian matrix of system (2.1) at E_* is

$$J(E_*) = \begin{pmatrix} -\dfrac{P}{x_*} & -x_*^2 \\ 2(k\alpha - c)x_* y_* & 0 \end{pmatrix}, \tag{2.4}$$

where $P = 2bx_*^4 + (2bm - a)x_*^3 + \beta^2(a - 2bm)x_* + 2m\beta^2(a - bm)$. Then $\operatorname{tr}(J(E_*)) = -P/x_* = R(m)/x_*$, where $R(m) = 2b\beta^2 m^2 + 2(b\beta^2 x_* - bx_*^3 - a\beta^2)m - a\beta^2 x_* + ax_*^3 - 2bx_*^4$, the discriminant of $R(m) = 0$ is $\Delta = 4(b^2x_*^6 + 2b^2x_*^4\beta^2 + b^2\beta^4x_*^2 + 4a^2\beta^4) > 0$. Hence, the equation $R(m) = 0$ has two roots m_1 and m_2, where $m_1 = (bx_*^3 + \beta^2\ (a - bx_*) - \sqrt{\Delta/4})\ /2b\beta^2$, $m_2 = (bx_*^3 + \beta^2\ (a - bx_*) + \sqrt{\Delta/4})/2b\beta^2$.

Note that

$$\begin{aligned} \left(bx_*^3 + \beta^2(a - bx_*)\right)^2 - \frac{\Delta}{4} &= 2bx_*\beta^2\left(ax_*^2 - a\beta^2 - 2bx_*^3\right) \\ &= 2bx_*\beta^2\left(a\beta^2\frac{2c - k\alpha}{k\alpha - c} - 2bx_*^3\right) \\ &= 2bx_*\beta^4\frac{a(2c - k\alpha) - 2bcx_*}{k\alpha - c}, \end{aligned} \tag{2.5}$$

and $a(2c - k\alpha) > (\le) 2bcx_*$ implies $m_1 > (\le) 0$. Consider

$$m_2 > \frac{a - bx_*}{2b} + \frac{\beta^2\sqrt{a^2 + b^2x_*^2 - 2abx_* + 2abx_*}}{2b\beta^2} > \frac{a - bx_*}{b}. \tag{2.6}$$

Then

(I) If $a(2c - k\alpha) \le 2bcx_*$ holds, then $m_1 \le 0$, $R(m) < 0$ holds for $m_1 < m < m_2$. Considering (H_2) and $m_2 > (a - bx_*)/b$, for $0 < m < (a - bx_*)/b$, $\operatorname{tr}(J(E_*)) < 0$, which implies E_* is locally asymptotically stable.

(II) If $a(2c - k\alpha) > 2bcx_*$ holds, then $m_1 > 0$, for $m_1 < m < m_2$, $R(m) < 0$, since $m_1 - (a - bx_*)/b = (bx_*^3 - \beta^2(a - bx_*) - \sqrt{\Delta/4})/2b\beta^2$, by $(bx_*^3 - \beta^2(a - bx_*))^2 - (\Delta/4) = -2ab\beta^2x_*(x_*^2 + \beta^2) < 0$, we obtain $m_1 < (a - bx_*)/b$. Together with (H_2), for $m_1 < m < (a - bx_*)/b$, $\operatorname{tr}(J(E_*)) < 0$, which means E_* is locally asymptotically stable. On the other hand, for $0 < m < m_1$, $\operatorname{tr}(J(E_*)) > 0$, E_* is locally unstable.

(III) We have

$$\det(J(E_*)) > 0, \qquad \operatorname{tr}(J(E_*)|_{m_1}) = \frac{R(m_1)}{x_*} = 0, \qquad \left.\frac{\partial tr(J(E_*))}{\partial m}\right|_{m_1} \neq 0, \tag{2.7}$$

these satisfy Liu's Hopf bifurcation criterion (see [6], page 255); hence, the Hopf bifurcation occurs at $m = m_1$. This ends the proof. □

3. Global Stability of the Positive Equilibrium

Denote $m_3 := (9a - 2\sqrt{3}\beta b - \sqrt{81a^2 + 12b^2\beta^2})/18b < 0$, $m_4 := (9a - 2\sqrt{3}\beta b + \sqrt{81a^2 + 12b^2\beta^2})/18b > 0$.

Theorem 3.1. *If $E_*(x_*, y_*)$ is locally stable. Further assume that* $\max\{0, (a - 4b\beta)/2b\} < m < m_4$*, then the positive equilibrium $E_*(x_*, y_*)$ of system* (2.1) *is globally asymptotically stable.*

Proof. Take the Dulac function $B(x, y) = x^{-2}y^{-1}$, for system (2.1) we have

$$T = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = -\frac{\phi(x)}{x^3 y}, \tag{3.1}$$

where

$$\phi(x) = 2bx^4 + (2bm - a)x^3 + \beta^2(a - 2bm)x + 2m\beta^2(a - bm). \tag{3.2}$$

If $a = 2bm, \phi(x) = 2b(x^4 + m^2\beta^2) > 0$ for $x > 0$.
On the other hand, there exist

$$\begin{aligned} \phi'(x) &= 8bx^3 + 3(2bm - a)x^2 + \beta^2(a - 2bm), \\ \phi''(x) &= 24bx^2 + 6(2bm - a)x. \end{aligned} \tag{3.3}$$

The equation $\phi''(x) = 0$ has two roots $x_1 = 0$, $x_2 = (a - 2bm)/4b$.

Case 1. If $a - 2bm > 0$, then for $0 < x < x_2$, $\phi''(x) < 0$; for $x > x_2$, $\phi''(x) > 0$. Hence, $x = x_2$ is the least value of the function $\phi'(x)$. If $\beta > x_2, \phi'(x_2) = (a - 2bm)(\beta - x_2)(\beta + x_2) > 0$, it has $\phi'(x) > 0$ for all $x > 0$, then $\phi(x)$ is increasing for $x > 0$, notice that $\phi(0) > 0$. Therefore, $\phi(x) > 0$ for $x > 0$. Since, $T < 0$ for $x > 0$, system (2.1) does not exist limit cycle.

Case 2. If $bm < a < 2bm$, then $x_2 < 0$, for $x > 0$, $\phi''(x) > 0$, hence, for $x > 0$, $\phi'(x)$ is increasing. Evidently, $\phi'(0) < 0$, $\phi'(\beta/\sqrt{3}) > 0$, then there exists $0 < x_0 < \beta/\sqrt{3}$ such that $\phi'(x_0) = 0$, where $\phi'(x_0) = 8bx_0^3 + 3(2bm - a)x_0^2 + \beta^2(a - 2bm)$, hence, when $0 < x < x_0, \phi'(x) < 0$, when $x > x_0$, $\phi'(x) > 0$. We know that $\phi(x)$ takes the least value at $x = x_0$, that is, $\phi(x) > \phi(x_0)$. According to $\phi'(x_0) = 0$, for $x > 0$ we obtain $\phi(x) > \phi(x_0) = (2bm - a)(x_0^3 - 3\beta^2 x_0 + (8m\beta^2(a - bm))/(2bm - a))$, where $0 < x_0 < \beta/\sqrt{3}$.

To prove $\phi(x) > 0$ for $x > 0$, it suffices to prove $\tilde{\phi}(x) = x^3 - 3\beta^2 x + (8m\beta^2(a - bm))/(2bm - a) > 0$ for $0 < x < \beta/\sqrt{3}$. Clearly, $\tilde{\phi}(x)$ takes the least value at $x = \beta$, and $\tilde{\phi}(x)$ is strictly decreasing at the interval $(0, \beta)$. Hence, for $0 < x < \beta/\sqrt{3}$, $\tilde{\phi}(x) > \tilde{\phi}(\beta/\sqrt{3})$ holds. Since $\tilde{\phi}\beta/\sqrt{3} > 0 \Leftrightarrow (m(a - bm))/(2bm - a) > \beta/3\sqrt{3} \Leftrightarrow -3\sqrt{3}bm^2 + (3\sqrt{3}a - 2\beta b)m + \beta a > 0 \Leftrightarrow m_3 < m < m_4$. Therefore, for $0 < x_0 < \beta/\sqrt{3}$, $\tilde{\phi}(x) > 0$ holds if $m_3 < m < m_4$ holds, then for $x > 0$, $\phi(x) > 0$ holds.

In sum, if one of the following three conditions holds (1) $m = a/2b$; (2) $0 < m < a/2b$, $x_2 < \beta \Rightarrow \max\{0, (a - 4b\beta)/2b\} < m < a/2b$; (3) $a/2b < m < a/b$, $m_3 < m < m_4 \Rightarrow a/2b < m < m_4$, the function T does not change the sign for $x > 0$, then system (2.1) does not exist limit cycle. It is easy to see that the conditions (1), (2), and (3) are equal to $\max\{0, (a - 4b\beta)/2b\} < m < m_4$. The proof is completed. □

4. Existence and Uniqueness of Limit Cycle

Theorem 4.1. *If $a(2c - k\alpha) > 2bcx_*$ holds, for $0 < m < m_1$ system (2.1) admits at least one limit cycle in Ω.*

Proof. We construct a Bendixson loop $\widehat{OABCD}$ which includes E_* of system (2.1). Let $\overline{OA}$ be a length of the line $L_1 : y = 0$, $\overline{AB}$ be a length of line $L_2 : b(x + m) - \mathrm{a} = 0$. Define

$$\begin{aligned} \dot{x} &= x^2(a_0 - y), \\ \dot{y} &= \left(-c\beta^2 + (k\alpha - c)x^2\right)y, \end{aligned} \tag{4.1}$$

where $a_0 = \max_{x_* \le x \le (a/b) - m}\{((x + m)(a - b(x + m))(\beta^2 + x^2))/x^2\}$. The orbit of system (4.1) with initial value $((a/b) - m, a_0)$ intersects with the line $x = x_*$ and the intersection point $C(x_*, y_1)$, we obtain the orbit arc $\widehat{BC}$. Let $\overline{CD}$ be a length of line $L_3 : y = y_1$, $\overline{DO}$ be a length of line $L_4 : x = 0$. Because $\overline{OA}$ is a length of orbit line of system (2.1) and $(dL_2/dt)|_{(2.1)} = -b((a/b) - m)^2 y < 0 (y > 0)$, $(dL_3/dt)|_{(2.1)} = y_1(-c\beta^2 + (k\alpha - c)x^2) < 0$ $(0 < x < x_*)$, $(dL_4/dt)|_{(2.1)} = m\beta^2(a - bm) > 0$, the orbits of system (2.1) tend to the interior of the Bendixson loop from the outer of $\overline{AB}$, $\overline{CD}$, and $\widehat{BC}$, by comparing system (2.1) to system (4.1): $dx/dt|_{(2.1)} < dx/dt|_{(4.1)} < 0$ and $dy/dt|_{(2.1)} = dy/dt|_{(4.1)} > 0$. Then the orbits of system (2.1) tend to the interior of the Bendixson loop from the outer of $\widehat{BC}$. On the other hand, under the condition of Theorem 4.1, $E_*(x_*, y_*)$ is unstable, by Poincaré-Bendixson Theorem, system (2.1) admits at least one limit cycle in the region $\widehat{OABCD} \in \Omega$. This ends the proof. □

Lemma 4.2 (see [7]). *Let $f(x)$, $g(x)$ be continuously differentiable functions on the open interval $(r1, r2)$, and $\varphi(y)$ be continuously differentiable functions on R in*

$$\begin{aligned} \frac{dx}{dt} &= \varphi(y) - \int_{x_0}^{x} f(u)du, \\ \frac{dy}{dt} &= -g(x), \end{aligned} \tag{4.2}$$

such that

(1) $d\varphi(y)/dy > 0$,

(2) *having a unique* $x_0 \in (r_1, r_2)$, *such that* $(x - x_0)g(x - x_0) > 0$ *for* $x \neq x_0$ *and* $g(x_0) = 0$,

(3) $f(x_0)d/dx(f(x)/g(x)) < 0$ *for* $x \neq x_0$,

then system (4.1) *has at most one limit cycle.*

Theorem 4.3. *If* $a(2c - k\alpha) > 2bcx_*$ *holds, for* $0 < m < \min\{m_1, a/2b - (8\sqrt{3}\, x_*^3)/(9(x_*^2 - \beta^2))\}$ *system* (2.1) *exists exactly one limit cycle which is globally asymptotically stable in* Ω.

Proof. Let $u = x,\ v = \ln y,\ \tau = -x^2 t$, still denote u, v, τ, as $x,\ y, t$, then system (2.1) becomes

$$\frac{dx}{dt} = e^y - \frac{(x+m)(a - b(x+m))(\beta^2 + x^2)}{x^2},$$
$$\frac{dy}{dt} = -\frac{(k\alpha - c)x^2 - c\beta^2}{x^2}, \tag{4.3}$$

the positive equilibrium $E_*(x_*, y_*)$ changes $\widetilde{E}_*(x_*, \ln y_*)$.

Let $\overline{x} = x - x_*,\ \overline{y} = y - \ln y_*$, then $\widetilde{E}_*$ transform to the origin $O(0,0)$, still denote $\overline{x},\ \overline{y}$, as $x,\ y$ yield

$$\begin{aligned} \frac{dx}{dt} &= y_* e^y - y_* - \frac{(x + x_* + m)(a - b(x + x_* + m))\left(\beta^2 + (x + x_*)^2\right)}{(x + x_*)^2} + y_* \\ &:= \varphi(y) - F(x), \quad (x > -x_*), \\ \frac{dy}{dt} &= -\frac{(k\alpha - c)(x + x_*)^2 - c\beta^2}{(x + x_*)^2} := -g(x), \end{aligned} \tag{4.4}$$

where $F(x) = ((x + x_* + m)(a - b(x + x_* + m))(\beta^2 + (x + x_*)^2))/(x + x_*)^2 - y_*$.

Clearly, $F(0) = 0$. It is easy to see that the conditions (1) and (2) of Lemma 4.2 for $x_0 = 0$ are satisfied. Consider

$$f(x) = F'(x) = \frac{2b(x + x_*)^4 + (2bm - a)(x + x_*)^3 + \beta^2(a - 2bm)(x + x_*) + 2m\beta^2(a - bm)}{(x + x_*)^3}. \tag{4.5}$$

Note that by the assumption of Theorem 4.3, E_* is unstable equilibrium and

$$\operatorname{tr}(J(E_*)) = -\frac{1}{x_*}\left[2bx_*^4 + (2bm - a)x_*^3 + \beta^2(a - 2bm)x_* + 2m\beta^2(a - bm)\right] > 0, \tag{4.6}$$

then $f(0) = -x_*^{-2}\operatorname{tr}(J(E_*)) < 0$. Consider

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{2\varphi(x)}{(x+x_*)^2\left((k\alpha - c)(x+x_*)^2 - c\beta^2\right)^2}, \tag{4.7}$$

where

$$\begin{aligned}\varphi(x) &= b(k\alpha - c)(x+x_*)^6 - 3bc\beta^2(x+x_*)^4 + \beta^2(2c-k\alpha)(a-2bm)(x+x_*)^3 \\ &\quad - 3m\beta^2(k\alpha - c)(a-bm)(x+x_*)^2 + mc\beta^4(a-bm) \\ &= (k\alpha - c)\widetilde{\varphi}(x),\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}\widetilde{\varphi}(x) &= b(x+x_*)^6 - 3bx_*^2(x+x_*)^4 + \frac{\beta^2(2c-k\alpha)(a-2bm)}{k\alpha - c}(x+x_*)^3 \\ &\quad - 3m\beta^2(a-bm)(x+x_*)^2 + m\beta^2 x_*^2(a-bm).\end{aligned} \tag{4.9}$$

Then, we have

$$\begin{aligned}\widetilde{\varphi}'(x) &= 6b(x+x_*)^5 - 12bx_*^2(x+x_*)^3 + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha - c}(x+x_*)^2 \\ &\quad - 6m\beta^2(a-bm)(x+x_*) = (x+x^*)\widetilde{\phi}(x),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\widetilde{\phi}(x) &= 6b(x+x_*)^4 - 12bx_*^2(x+x_*)^2 \\ &\quad + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha - c}(x+x_*) - 6m\beta^2(a-bm).\end{aligned} \tag{4.11}$$

By a simple computation, we obtain

$$\begin{aligned}\widetilde{\phi}'(x) &= 24b(x+x_*)^3 - 24bx_*^2(x+x_*) + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha - c}, \\ \widetilde{\phi}''(x) &= 24b\left(3(x+x_*)^2 - x_*^2\right).\end{aligned} \tag{4.12}$$

It is easy to verify that $\widetilde{\phi}''(-x_*) < 0$ and $\widetilde{\phi}''(x) = 0$ has two roots $\overline{x}_1$ and $\overline{x}_2$ defined by, respectively,

$$\overline{x}_1 = \left(-1 - \frac{\sqrt{3}}{3}\right)x_*, \qquad \overline{x}_2 = \left(-1 + \frac{\sqrt{3}}{3}\right)x_*. \tag{4.13}$$

Obviously, $\overline{x}_1 < -x_* < \overline{x}_2$. Therefore, $\widetilde{\phi}''(x) < 0$ for $-x_* \leqslant x < \overline{x}_2$ and $\widetilde{\phi}''(x) > 0$ for $x > \overline{x}_2$ which indicates that $\overline{x}_2$ is the minimum point of the function $\widetilde{\phi}'(x)$ when $x \geqslant -x_*$. Substituting $\overline{x}_2$ into $\widetilde{\phi}'(x)$, we obtain

$$\begin{aligned}\min_{x\geqslant -x_*} \widetilde{\phi}'(x) &= \widetilde{\phi}'(\overline{x}_2)\\ &= -\frac{16\sqrt{3}bx_*^3}{3} + \frac{3\beta^2(2c-k\alpha)(a-2bm)}{k\alpha - c}\\ &= -\frac{16\sqrt{3}bx_*^3}{3} + 3\left(x_*^2-\beta^2\right)(a-2bm).\end{aligned} \tag{4.14}$$

It is easy to see that if $0 < m < (a/2b) - (8\sqrt{3}\, x_*^3)/(9(x_*^2-\beta^2))$, then $\widetilde{\phi}'(\overline{x}_2) > 0$, which implies $\widetilde{\phi}'(x) > 0$ for all $x \geqslant -x_*$. That is, the function $\widetilde{\phi}(x)$ is a strictly increasing function for $x \geqslant -x_*$.

Note that $\widetilde{\phi}(-x_*) = -6m\beta^2(a-bm) < 0$ for $0 < m < a/b$ and $\lim_{x\to+\infty}\widetilde{\phi}(x) = +\infty$. It follows from (4.6) that

$$\widetilde{\phi}(0) = -3\left(2bx_*^4 + (2bm-a)x_*^3 + (a-2bm)\beta^2 x_* + 2m\beta^2(a-bm)\right) > 0. \tag{4.15}$$

Hence, there exists a point $-x_* < \widehat{x} < 0$, such that $\widetilde{\phi}(\widehat{x}) = 0$, that is,

$$b(\widehat{x}+x_*)^4 - 2bx_*^2(\widehat{x}+x_*)^2 + \frac{\beta^2(2c-k\alpha)(a-2bm)}{2(k\alpha-c)}(\widehat{x}+x_*) - m\beta^2(a-bm) = 0. \tag{4.16}$$

This, together with the monotonicity of $\widetilde{\phi}(x)$ when $x \geqslant -x_*$, we may conclude that $\widetilde{\varphi}'(x) = (x+x_*)\widetilde{\phi}(x) < 0$ for $x \in (-x_*, \widehat{x})$ and $\widetilde{\varphi}'(x) > 0$ for $x \in (\widehat{x}, \infty)$. Therefore, $\widehat{x}$ is the minimum point of the function $\widetilde{\varphi}(x)$ for $-x_* < x < \infty$.

Together with (4.16), we obtain

$$\begin{aligned}\min_{x\geqslant -x_*} \widetilde{\varphi}(x) = \widetilde{\varphi}(\widehat{x}) &= -bx_*^2(\widehat{x}+x_*)^4 + \frac{\beta^2(2c-k\alpha)(a-2bm)}{2(k\alpha-c)}(\widehat{x}+x_*)^3\\ &\quad - 2m\beta^2(a-2bm)(\widehat{x}+x_*)^2 + m\beta^2(a-bm)x_*^2\\ &= -bx_*^2(\widehat{x}+x_*)^4 + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\widehat{x}+x_*)^3\\ &\quad - 2m\beta^2(a-2bm)(\widehat{x}+x_*)^2 + m\beta^2(a-bm)x_*^2\end{aligned}$$

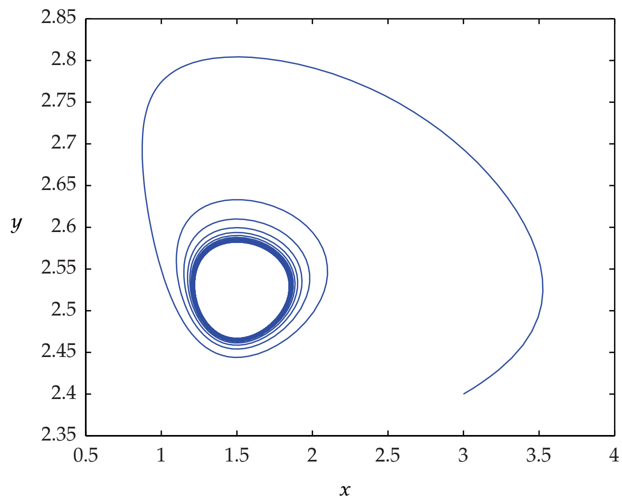


Figure 1: The bifurcated periodic solution is stable.

$$
\begin{aligned}
&= -2bx_*^4(\hat{x}+x_*)^2 + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\left(2x_*^2+2\hat{x}x_*+\hat{x}^2\right) \\
&\quad -2m\beta^2(a-2bm)(\hat{x}+x_*)^2 \\
&= -2bx_*^4(\hat{x}+x_*)^2 + \left(x_*^2-\beta^2\right)(a-2bm)x_*(\hat{x}+x_*)^2 \\
&\quad -2m\beta^2(a-2bm)(\hat{x}+x_*)^2 + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\hat{x}^2 \\
&= -(\hat{x}+x_*)^2\Big(2bx_*^4+(2bm-a)x_*^3+\beta^2(a-2bm)x_* \\
&\quad +2m\beta^2(a-2bm)\Big) + \frac{1}{2}\left(x_*^2-\beta^2\right)(a-2bm)(\hat{x}+x_*)\hat{x}^2.
\end{aligned}
\tag{4.17}
$$

It follows from (4.6), we have $\min_{x\geqslant -x_*}\tilde{\varphi}(x)>0$. This indicates $\tilde{\varphi}(x)>0$ for all $x>-x_*$.

Then all the conditions of Lemma 4.2 are satisfied, considering Theorem 4.1, we obtain the conclusion of this theorem. The proof is completed. □

5. Numerical Simulations

Take $\alpha=0.5$, $k=0.2$, $\beta=0.5$, $a=1$, $b=0.1$, and $c=0.09$. Then $a(2c-k\alpha)-2bcx_*=0.053$, and $m_1\approx 1.986121812$. One can see a Hopf bifurcation occurring at $m=1.955$ and the bifurcated periodic solution is stable in Figure 1.

When taking $m=4.5$, then $x_*=1.5$, $y_*\approx 2.666666667$, $a(2c-k\alpha)-2bcx_*=0.053$, $m_1\approx 1.986121812$,$(a-bx_*)/b=8.5$, $(a-4b\beta)/2b=4, a/2b=5$. Theorem 3.1 is satisfied; the equilibrium E_* of system (2.1) is globally asymptotically stable. See Figure 2.

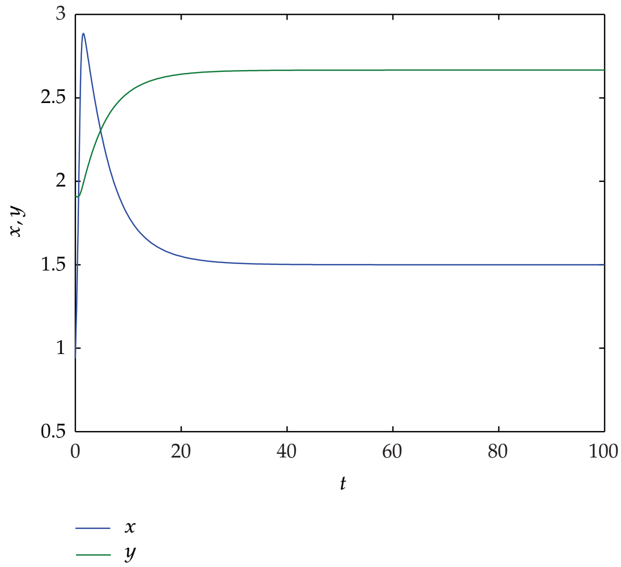
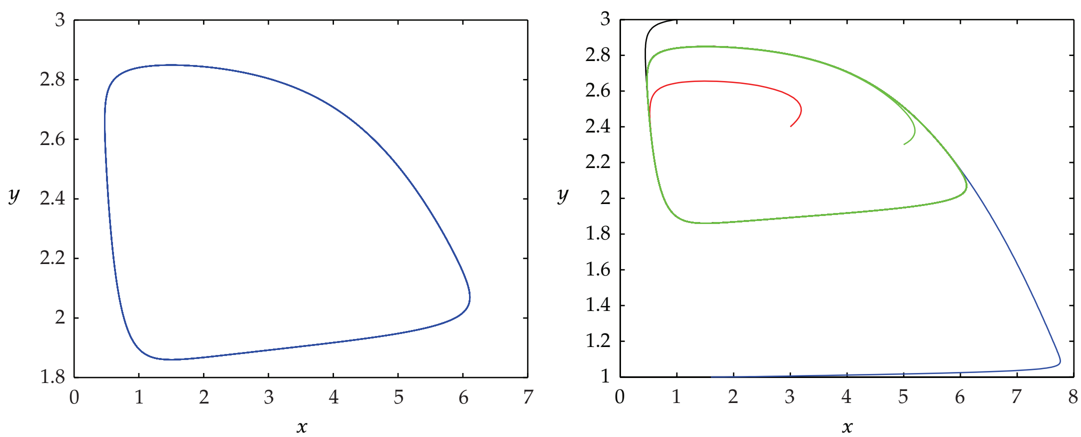


Figure 2: The positive equilibrium E_* of system (2.1) is globally asymptotically stable.



(a) (b)

Figure 3: The dynamical behaviors of system (2.1) when $\alpha = 0.5$, $k = 0.2$, $\beta = 0.5$, $a = 1$, $b = 0.1$, $c = 0.09$, $m = 1$. (a) The existence of unique limit cycle. (b) The global stability of the limit cycle.

Take $m = 1$, we obtain $E_*(1.5, 2.083333333)$, $a(2c - k\alpha) - 2bcx_* = 0.053$, $m_1 \approx 1.986121812$, $(a/2b) - (8\sqrt{3}x_*^3/9(x_*^2 - \beta^2)) \approx 2.401923788$. The conditions in Theorem 4.1 are satisfied; hence, system (2.1) exists exactly one limit cycle which is globally asymptotically stable. One can see Figure 3.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (11226142), Foundation of Henan Educational Committee (2012A110012), Youth Science

Foundation of Henan Normal University (2011QK04), Natural Science Foundation of Shanghai (no. 12ZR1421600), and Shanghai Municipal Educational Committee (no. 10YZ74).

References

[1] X. Liu and M. Han, "Chaos and Hopf bifurcation analysis for a two species predator-prey system with prey refuge and diffusion," *Nonlinear Analysis*, vol. 12, no. 2, pp. 1047–1061, 2011.
[2] Y. Huang, F. Chen, and L. Zhong, "Stability analysis of a prey-predator model with Holling type III response function incorporating a prey refuge," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 672–683, 2006.
[3] L. Chen, F. Chen, and L. Chen, "Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge," *Nonlinear Analysis*, vol. 11, no. 1, pp. 246–252, 2010.
[4] Z. Ma, W. Li, Y. Zhao, W. Wang, H. Zhang, and Z. Li, "Effects of prey refuges on a predator-prey model with a class of functional responses: the role of refuges," *Mathematical Biosciences*, vol. 218, no. 2, pp. 73–79, 2009.
[5] T. K. Kar, "Modelling and analysis of a harvested prey-predator system incorporating a prey refuge," *Journal of Computational and Applied Mathematics*, vol. 185, no. 1, pp. 19–33, 2006.
[6] W. M. Liu, "Criterion of Hopf bifurcations without using eigenvalues," *Journal of Mathematical Analysis and Applications*, vol. 182, no. 1, pp. 250–256, 1994.
[7] W. A. Coppel, "A new class of quadratic systems," *Journal of Differential Equations*, vol. 92, no. 2, pp. 360–372, 1991.

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 814069, 13 pages
doi:10.1155/2012/814069

Research Article

Pattern Formation in a Cross-Diffusive Ratio-Dependent Predator-Prey Model

Xinze Lian,[1] Yanhong Yue,[2] and Hailing Wang[3]

[1] *Chengdu Institute of Computer Application, Chinese Academy of Sciences, Chengdu 610041, China*
[2] *School of Foreign Language, Wenzhou University, Wenzhou 325000, China*
[3] *College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China*

Correspondence should be addressed to Xinze Lian, xinzelian@163.com

Received 29 September 2012; Accepted 3 November 2012

Academic Editor: Yonghui Xia

This paper presents a theoretical analysis of evolutionary process that involves organisms distribution and their interaction of spatial distribution of the species with self- and cross-diffusion in a Holling-III ratio-dependent predator-prey model. The diffusion instability of the positive equilibrium of the model with Neumann boundary conditions is discussed. Furthermore, we present novel numerical evidence of time evolution of patterns controlled by self- and cross-diffusion in the model and find that the model dynamics exhibits a cross-diffusion controlled formation growth to spots, stripes, and spiral wave pattern replication, which show that reaction-diffusion model is useful to reveal the spatial predation dynamics in the real world.

1. Introduction

Pattern formation is a topic in mathematical biology that studies how structures and patterns in nature evolve over time [1–12]. One of the mainstream topics in pattern formation involves the reaction-diffusion mechanisms of two chemicals, originally proposed by Turing [13] in 1952. In 1972, Segel and Jackson [14] called attention to the Turing's ideas that would be also applicable in population dynamics. At the same time, Gierer and Meinhardt [15] gave a biologically justified formulation of a Turing model and studied its properties by numerical simulations. Levin and Segel [11] suggested that the scenario of spatial pattern formation is a possible origin of planktonic patchiness. A significant amount of work has been done using this idea in the field of mathematical biology by Cantrell and Cosner [2], Hoyle [5], Murray [8], Okubo and Levin [16], and others [17–19].

In recent years, many scientists have paid considerable attention to diffusive ratio-dependent predator-prey models, especially those with Holling III functional response

[20–23]. In [23], the author studied the spatial pattern formation of the following ratio-dependent predator-prey model:

$$\begin{aligned} \frac{\partial u}{\partial t} &= ru\left(1 - \frac{u}{K}\right) - \frac{au^2 v}{u^2 + m^2 v^2} + D_{11}\nabla^2 u, \\ \frac{\partial v}{\partial t} &= \frac{bvu^2}{u^2 + m^2 v^2} - dv + D_{22}\nabla^2 v, \end{aligned} \tag{1.1}$$

where u and v are prey and predator density, respectively. r represents the intrinsic growth rate of the prey, K is the carrying capacity of the prey in the absence of predator, a is the maximum consumption, b is the conversion efficiency of food into offspring, m is the predator interference parameter, and d is the per capita predator death rate. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the usual Laplacian operator in two-dimensional space. D_{11} and D_{22} are the self-diffusion coefficients that imply the movement of individuals from a higher to lower concentration region. In addition, the author showed that spots and stripes-spots patterns could be observed in pure Turing instability, and spiral pattern emerged in Hopf and Turing instability [23].

On the other hand, the predator-prey system models such a phenomenon: pursuit-evasion-predators pursuing prey and prey escaping the predators [18, 19, 24, 25]. In other words, in nature, there is a tendency that the preys would keep away from predators and the escape velocity of the preys may be taken as proportional to the dispersive velocity of the predators. In the same manner, there is a tendency that the predators would get closer to the preys, and the chase velocity of predators may be considered to be proportional to the dispersive velocity of the preys. Keeping these in view, cross-diffusion arises, which was proposed first by Kerner [26] and first applied in competitive population system by Shigesada et al. [27].

There has been a considerable interest in investigating the stability behavior of a predator-prey system by taking into account the effect of self- and cross-diffusion [17, 18, 28–35]. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. However, in the studies on spatiotemporal dynamics of the ratio-dependent predator-prey system with functional response, little attention has been paid to study on the effect of cross-diffusion.

In this paper, we mainly focus on the spatiotemporal dynamics of a cross-diffusion ratio-dependent predator-prey model with Holling III functional response. In the next section, we establish the cross-diffusion model and derive the sufficient conditions for Turing instability. Then, we present and discuss the results of pattern formation via numerical simulation in Section 3. Finally, some conclusions are drawn.

2. The Model and Analysis

2.1. The Model

We firstly pay attention to the spatially extended ratio-dependent predator-prey model with self- and cross-diffusion, which is as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= ru\left(1 - \frac{u}{K}\right) - \frac{au^2 v}{u^2 + m^2 v^2} + D_{11}\nabla^2 u + D_{12}\nabla^2 v, \\ \frac{\partial v}{\partial t} &= \frac{bvu^2}{u^2 + m^2 v^2} - dv + D_{21}\nabla^2 u + D_{22}\nabla^2 v, \end{aligned} \tag{2.1}$$

where D_{12} and D_{21} are cross-diffusion coefficients that express population fluxes of the preys and predators resulting from the presence of the other species, respectively.

We consider the model on a square domain Ω. We also add to the reaction-diffusion equation model positive initial conditions:

$$u(x,y,0) > 0, \qquad v(x,y,0) > 0 \quad (x,y) \in \Omega = (0,L) \times (0,L). \tag{2.2}$$

It is natural to assume that nothing enters this model and nothing exits this model. Thus, we will take zero-flux boundary conditions for the flat domain:

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{2.3}$$

In the above, L denotes the size of the system in square domain, and ν is the outward unit normal vector of the boundary $\partial\Omega$.

For simplicity, we nondimensionalize model (2.1) with the following scaling:

$$u \longrightarrow \frac{u}{K}, \qquad v \longrightarrow \frac{mv}{K}, \qquad t \longrightarrow rt. \tag{2.4}$$

Then model (2.1) can be rewritten as

$$\begin{aligned} \frac{\partial u}{\partial t} &= u(1-u) - \frac{\alpha u^2 v}{u^2 + P^2} + d_{11}\nabla^2 u + d_{12}\nabla^2 v, \\ \frac{\partial v}{\partial t} &= \frac{\beta u^2 v}{u^2 + v^2} - \gamma v + d_{21}\nabla^2 u + d_{22}\nabla^2 v, \end{aligned} \tag{2.5}$$

where $\alpha = a/rm$, $\beta = b/r$, $\gamma = d/r$, $d_{11} = D_{11}/r$, $d_{12} = D_{12}/rm$, $d_{21} = D_{21}m/r$, $d_{22} = D_{22}/r$. In addition, we call

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \tag{2.6}$$

the diffusive matrix.

2.2. Summary of the Noncross Diffusion Model

We first consider the case of spatially homogeneous solutions. In this case spatial model (2.5) is equivalent to the ordinary differential equation model

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \frac{\alpha u^2 v}{u^2 + v^2} \triangleq f(u,v), \\ \frac{dv}{dt} &= \frac{\beta u^2 v}{u^2 + v^2} - \gamma v \triangleq g(u,v). \end{aligned} \tag{2.7}$$

It can be seen that model (2.7) has two nonnegative real equilibria as follows.

(i) The equilibrium point $E = (1, 0)$ corresponding to extinction of the predator is a saddle point.

(ii) The equilibrium point $E^* = (u^*, v^*)$ which is corresponding to a nontrivial stationary state coexistence of prey and predator, where

$$u^* = \frac{\beta - \sqrt{\alpha^2\beta\gamma - \alpha^2\gamma^2}}{\beta}, \qquad v^* = \frac{\sqrt{\alpha^2\beta\gamma - \alpha^2\gamma^2}N^*}{\alpha\gamma}. \tag{2.8}$$

It is easy to see that $u^* > 0$ and $v^* > 0$ when $\beta - \gamma > 0$ and $\beta - \sqrt{\alpha^2\beta\gamma - \alpha^2\gamma^2} > 0$ hold.

Besides, Turing instability at the coexistence equilibrium E^* of the model (2.5) has been analysis without cross-diffusion. Here, we only give a summary [23]. The characteristic equation at the steady state E^* of model (2.5) without cross-diffusion is

$$|J_k - \lambda I| = 0, \tag{2.9}$$

where, $J_k = J - \operatorname{diag}(d_1, d_2)k^2$, and J is given by

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix}_{E^*} \triangleq \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} \dfrac{-\beta^2 + 2\sqrt{-\alpha^2\gamma^2(\gamma-\beta)}}{\beta^2} & -\dfrac{\alpha\gamma(2\gamma-\beta)}{\beta^2} \\ -2\dfrac{(\gamma-\beta)\sqrt{-\alpha^2\gamma(\gamma-\beta)}}{\alpha\beta} & 2\dfrac{\gamma(\gamma-\beta)}{\beta} \end{pmatrix}, \tag{2.10}$$

and the trace and determinant of matrix J is as follows:

$$\begin{aligned} \operatorname{tr}(J) &= f_u + f_v, \\ \det(J) &= f_u g_v - f_v g_u. \end{aligned} \tag{2.11}$$

Now (2.9) can be solved, yielding the so-called characteristic polynomial of the original model (2.5) without cross-diffusion:

$$\lambda^2 - \operatorname{tr}(J_k)\lambda + \det(J_k) = 0, \tag{2.12}$$

where

$$\begin{aligned} \operatorname{tr}(J_k) &= \operatorname{tr}(J) - (d_1 + d_2)k^2, \\ \det(J_k) &= d_1 d_2 k^4 - (d_2 f_u + d_1 g_v)k^2 + \det(J). \end{aligned} \tag{2.13}$$

The roots of (2.12) yield the dispersion relation:

$$\lambda_{1,2}(J_k) = \frac{1}{2}\left(\operatorname{tr}(J_k) \pm \sqrt{\operatorname{tr}(J_k)^2 - 4\det(J_k)}\right). \tag{2.14}$$

And an equilibrium is Turing instability means that it is an asymptotically stable equilibrium of nonspatial model (e.g., model (2.7)) but is unstable with respect to solutions of spatial model (e.g., model (2.5)). One can know that the stability of nonspatial model is guaranteed if the following conditions hold

$$\operatorname{tr}(J) = f_u + g_v < 0, \tag{2.15}$$

$$\det(J) = f_u g_v - f_v g_u > 0. \tag{2.16}$$

Then, the Turing instability sets in when at least one of (2.15) or (2.16) the following conditions is violated. However, it is evident that the first condition $\operatorname{tr}(J_k) < 0$ is not violated when the condition $f_u + g_v < 0$ is met. Hence, only the violation of condition $\det(J_k) > 0$ gives rise to diffusion-driven instability. Thus, the condition for Turing instability is given by

$$\det(J_k) = d_1 d_2 k^4 - f_u d_2 k^2 - d_1 g_v k^2 + f_u g_v - f_v g_u < 0. \tag{2.17}$$

In summary, a general linear analysis shows that the necessary conditions for yielding Turing patterns are given by

$$\begin{gathered} f_u + g_v < 0, \\ f_u g_v - f_v g_u > 0, \end{gathered} \tag{2.18}$$

$$\begin{gathered} d_2 f_u + d_1 g_v > 0, \\ \left(d_2 f_u + d_1 g_v\right)^2 > 4 d_1 d_2 \left(f_u g_v - f_v g_u\right). \end{gathered} \tag{2.19}$$

In fact, condition (2.18) enssured, by the definition that the equilibrium (u^*, v^*) is stable for model (2.5) without diffusion model (2.7). (u^*, v^*) becomes unstable for model (2.5) with diffusion if $\operatorname{Re}(\lambda_{1,2}(J_k))$ bifurcate from negative value to positif one. From (2.17), simple algebraic computations lead to (2.19).

2.3. Dynamic Analysis of the Spatial Model

To study the effect of cross-diffusion on the model system, set $u = u^* + \tilde{u}$, $v = v^* + \tilde{v}(|\tilde{u}|, |\tilde{v}| \ll 1)$, we consider the linearized $(\tilde{u}, \tilde{v})$ form of system as follows:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= f_u \tilde{u} + f_v \tilde{v} + d_{11} \nabla^2 \tilde{u} + d_{12} \nabla^2 \tilde{v}, \\ \frac{\partial \tilde{v}}{\partial t} &= g_u \tilde{u} + g_v \tilde{v} + d_{21} \nabla^2 \tilde{u} + d_{22} \nabla^2 \tilde{v}. \end{aligned} \tag{2.20}$$

Following [18], the characteristic equation of the linearized system is given by

$$\lambda^2 - \operatorname{tr}\left(\tilde{J}_k\right)\lambda + \det\left(\tilde{J}_k\right) = 0, \tag{2.21}$$

where $\tilde{J}_k = J - Dk^2$, and

$$\begin{aligned} \operatorname{tr}\left(\tilde{J}_k\right) &= \operatorname{tr}(J) - k^2 \operatorname{tr}(D), \\ \det\left(\tilde{J}_k\right) &= \det(D)k^4 - (d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u)k^2 + \det(J). \end{aligned} \tag{2.22}$$

The Turing instability sets in when at least one of the following conditions is violated:

$$\operatorname{tr}\left(\tilde{J}_k\right) < 0, \qquad \det\left(\tilde{J}_k\right) > 0. \tag{2.23}$$

The first condition $\operatorname{tr}(\tilde{J}_k) = \operatorname{tr}(J_k)$, which is evident that $\operatorname{tr}(J_k)$ is not violated when the condition $\operatorname{tr}(J) = f_u + g_v < 0$ is met. Hence, only the violation of condition $\det(\tilde{J}_k) > 0$ gives rise to diffusion-driven instability. Thus, the condition for diffusion-driven instability occurs when

$$\det\left(\tilde{J}_k\right) = \det(D)k^4 - (d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u)k^2 + \det(J) < 0. \tag{2.24}$$

Based on the above discussions, we can get the following theorem.

Theorem 2.1. *If the following conditions are true:*

$$\begin{gathered} f_u + g_v < 0, \\ d_{11}g_v + d_{22}f_u < 0, \\ d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u > 0, \\ (d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u)^2 > 4(d_{11}d_{12} - d_{21}d_{12})(f_u g_v - f_v g_u), \end{gathered} \tag{2.25}$$

then the positive equilibrium E^* of model (2.5) *is cross-diffusion-driven instability (i.e., Turing instability).*

Proof. In view of $f_u + g_v < 0$ and $d_{11}g_v + d_{22}f_u < 0$, it follows that

$$\operatorname{tr}\left(\tilde{J}_k\right) < 0, \qquad \det\left(\tilde{J}_k\right) > 0 \tag{2.26}$$

when $d_{12} = 0$ and $d_{21} = 0$. This implies the positive equilibrium E^* is asymptotic stable in the absent of cross-diffusion.

A necessary condition for cross-diffusive instability is given by

$$d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u > 0, \tag{2.27}$$

otherwise $\det(\widetilde{J}_k) > 0$ for all $k > 0$ since $\det(D) > 0$ and $\det(J) > 0$.

For the instability, we must have $\det(\widetilde{J}_k) < 0$ for some k. And we notice that $\det(\widetilde{J}_k)$ achieves its minimum:

$$\min_{\mu_i} \sigma_2 = \frac{4\det(D)\det(J) - \left(d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u\right)^2}{4\det(D)} \tag{2.28}$$

at the critical value $k_c^2 > 0$ when

$$k_c^2 = \frac{d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u}{2\det(D)}. \tag{2.29}$$

As a consequence, if $d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u > 0$ and $(d_{11}g_v - d_{12}g_u - d_{21}f_v + d_{22}f_u)^2 > 4(d_{11}d_{12} - d_{21}d_{12})(f_u g_v - f_v g_u)$ hold, then $\det(\widetilde{J}_k) < 0$ is valid. Hence E^* is an unstable equilibrium with respect to model (2.5). This finishes the proof. □

In Figure 1, based on the results of Theorem 2.1, we show the dispersal relation of r with α. The green, red, and blue curves represent Hopf, self-diffusion Turing, and self-cross-diffusion Turing bifurcation curve, respectively. They separate the parametric space into five domains. The domain below the Hopf bifurcation curve is stable, the domain above the self-diffusion Turing bifurcation curve is unstable, and the domain above self-cross-diffusion Turing bifurcation curve is unstable. Hence, among these domains, only the domain (IV) satisfies conditions of Theorem 2.1, and we call domain (IV) as Turing space, where the Turing instability occurs and the Turing patterns may be undergone.

3. Pattern Formation

In this section, we perform extensive numerical simulations of the spatially extended model (2.5) in 2-dimensional (2D) spaces, and the qualitative results are shown here. Our numerical simulations employ the nonzero initial (2.2) and zero-flux boundary conditions (2.3) with a system size of 200×200 by using a finite-difference methods. We use the standard five-point approximation for the 2D Laplacian with the zero-flux boundary conditions. And the time step and the grid width used in the simulations are $\tau = 0.01$ and $\Delta h = 0.25$, respectively. The parameters are fixed as

$$\alpha = 2.5, \qquad \beta = 1.1, \qquad \gamma = 1.05, \qquad d_{11} = 0.2, \qquad d_{22} = 0.2. \tag{3.1}$$

Initially, the entire system is placed in the steady state (u^*, v^*), and the propagation velocity of the initial perturbation is thus on the order of 5×10^{-4} space units per time unit. And the system is then integrated for 1000 000 time steps, and the last images are saved. After the initial period during which the perturbation spreads, either the system goes into a time-dependent state, or to an essentially steady state (time independent).

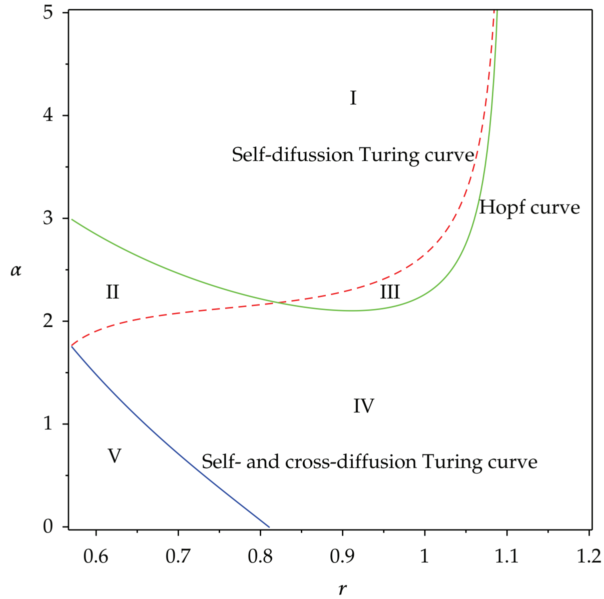


Figure 1: The dispersal relation of r with α. Parameters: $\beta = 1.3$, $d_{11} = 0.2$, $d_{12} = 0.05$, $d_{21} = 0.35$, $d_{22} = 0.2$. The green, red, and blue curves represent Hopf, self-diffusion Turing, and self-cross-diffusion Turing bifurcation curve, respectively. They separate the parametric space into five domains, and domain (IV) is called Turing space.

With parameters (3.1), the positive equilibrium of model (2.5) is $(u^*, v^*) = (0.4793, 0.1046)$. Let $d_{12} = d_{21} = 0$, that is, we first consider Turing instability in the case of self-diffusion model. It is easy to conclude that $\operatorname{tr}(J) < 0$, $\det(J) > 0$, and for all k, $\operatorname{tr}(J_k) < 0$ and $\det(J_k) > 0$. Hence, in this case, there is nonexistence of Turing instability in the self-diffusion model (2.5).

Next, we consider the effect of the cross-diffusion in model (2.5), let $d_{21} = 0.05$, $d_{12} \in (0.2, 0.8)$, and other parameters are fixed as (3.1). It is easy to know that $\operatorname{tr}(\tilde{J}_k) < 0$ for all k, and $\det(\tilde{J}_k) < 0$ for some k. That is to say, in this case, Turing instability can occur. And in Figure 2, we show five typical Turing pattern of prey u in model (2.5) with parameters set (3.1) and d_{12} change from 0.4 to 0.76. From Figure 2, one can see that values for the concention u are represented in a color scale varying from blue to red. And on increasing the control cross-coefficient d_{12}, the sequences "spots patterns (Figure 2(a)) $\longrightarrow$ spot-strips coexist patterns (Figure 2(b)) $\longrightarrow$ strip patterns (c.f., Figure 2(c)) $\longrightarrow$ hole-strips coexist patterns (Figure 2(d)) $\longrightarrow$ holes patterns (Figure 2(e))" can be observed.

For the sake of learning the pattern formation in model (2.5) further, in the following, we select a special perturbed initial condition for investigating the evolutionary process of the infected spatial pattern, the initial condition is introduced as

$$u(x, y, 0) = u^*,$$
$$v(x, y, 0) = \begin{cases} 0.2, & \text{if } (x-100)^2 + (y-200)^2 < 200, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

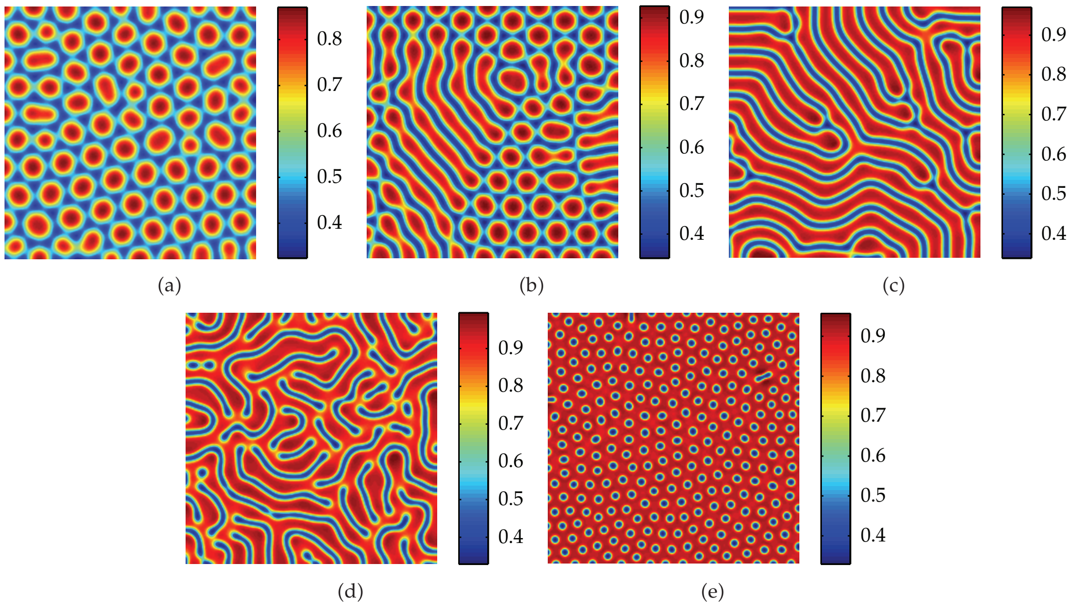


Figure 2: Patterns obtained with model (2.5) for (a) $d_{12} = 0.4$, (b) $d_{12} = 0.55$, (c) $d_{12} = 0.65$, (d) $d_{12} = 0.70$, (e) $d_{12} = 0.76$ and the other parameters are fixed as (3.1). Time: $t = 10000$.

which is a circle in (x, y) plane. The parameters are taken the same as Figure 2(a). Then, we can observe that after the decay of target patterns, the spots pattern prevails over the whole domain finally (c.f. Figure 3(d)).

Besides Turing patterns (c.f., Figures 2 and 3), there exhibits spiral wave pattern self-replication in model (2.5). As an example, in Figure 4, we show spiral patterns with $\alpha = 2.5$, $\beta = 1.3$, $\gamma = 1.1$, $d_{11} = 0.7$, $d_{12} = 0.05$, $d_{21} = 0.01$, $d_{22} = 1$. In this case, the equilibrium is $(u^*, v^*) = (0.0980, 0.0418)$. In order to make the image more clearly, the system size is 400×400 and the grid width Δh is 0.5. One can see the random initial distribution leads to the formation of macroscopic spiral patterns (c.f., Figure 4(a)). In other words, in this case, small random fluctuations will be strongly amplified by diffusion, leading to nonuniform population distributions. For the sake of learning the dynamics of this case further, we show time-series plots (c.f., Figure 4(b)). From Figure 4(b), one can see that the system gives rise to periodic oscillations in time, which is the reason why the spiral pattern emerges.

Thanks to the insightful works of Medvinsky et al. [36] and Upadhyay et al. [37], we have studied the spiral wave pattern for an initial condition discussed in the following equations. In this this case, we employ $\Delta h = 0.5$, and the system size is 400×400. The parameters set are same as Figure 4. The initial condition is given by

$$\begin{aligned} u(x, y, 0) &= u^* - \varepsilon_1 (x - 80)(x - 320), \\ v(x, y, 0) &= v^* - \varepsilon_2 (y - 100) - \varepsilon_2 (y - 300), \end{aligned} \tag{3.3}$$

where $\varepsilon_1 = 3 \times 10^{-7}$ and $\varepsilon_2 = 1 \times 10^{-4}$.

The initial conditions are deliberately chosen to be unsymmetrical in order to make any influence of the corners of the domain more visible. Snapshots of the spatial distribution

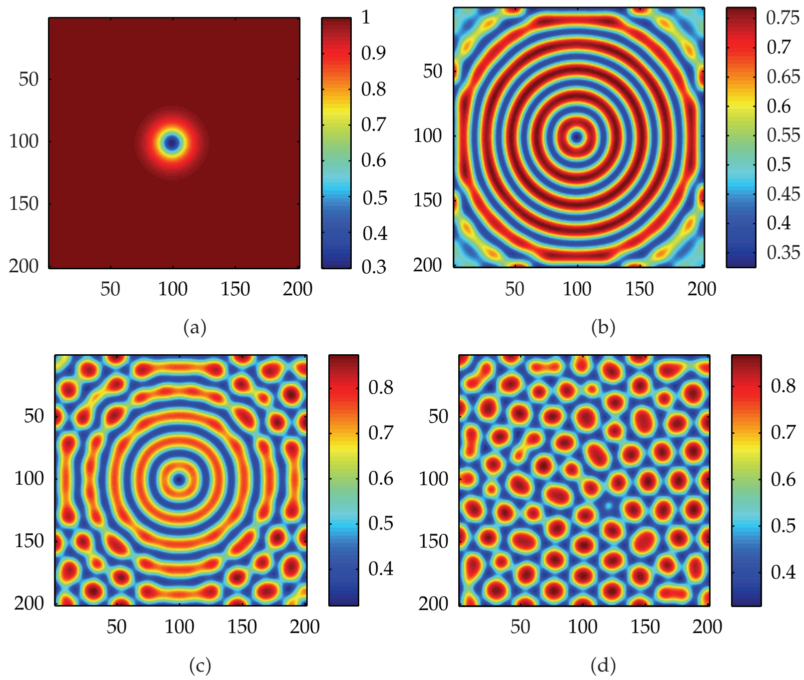


Figure 3: The process of spiral patterns of u for parameters: $\alpha = 2.2$, $\beta = 1.1$, $\gamma = 1.05$, $d_{11} = 0.2$, $d_{12} = 0.05$, $d_{21} = 0.4$, $d_{22} = 0.2$. Time: (a) $t = 50$, (b) $t = 500$, (c) $t = 1000$, (d) $t = 2000$.

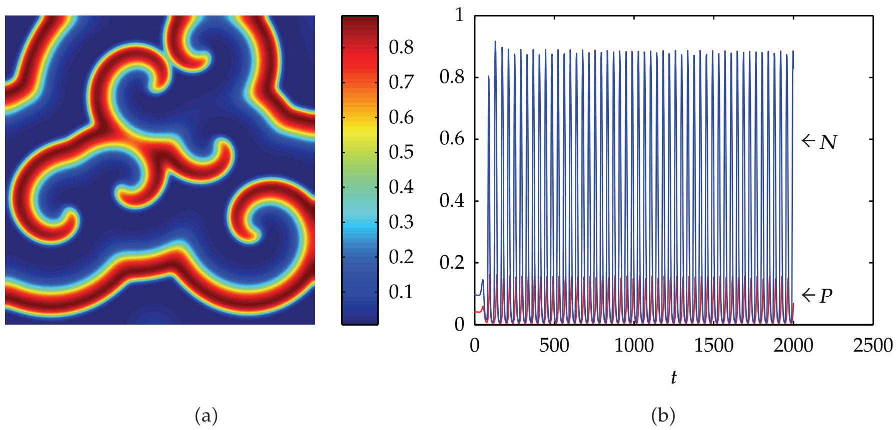
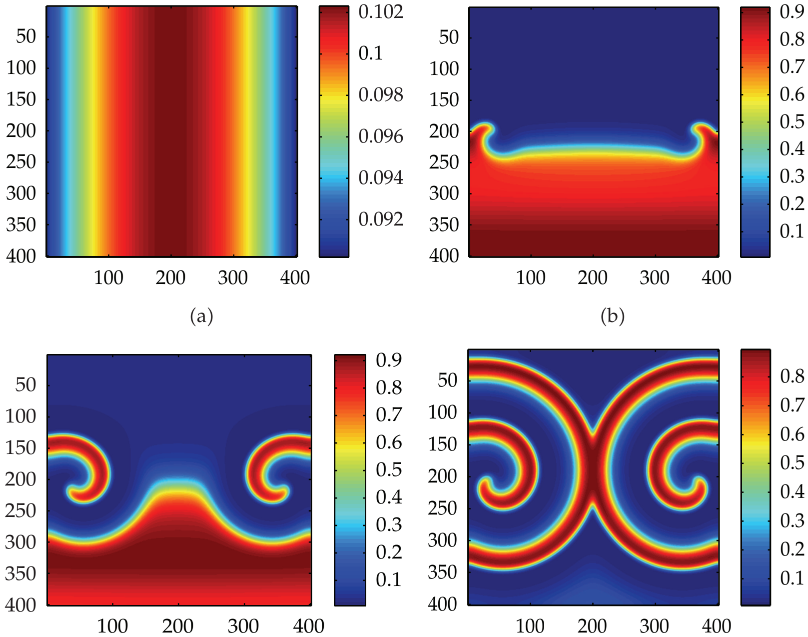


Figure 4: Dynamical behaviors of model (2.5) with the parameters: $\alpha = 2.5$, $\beta = 1.3$, $\gamma = 1.1$, $d_{11} = 0.7$, $d_{12} = 0.05$, $d_{21} = 0.01$, $d_{22} = 1$ at $t = 2000$. (a) Spiral pattern. (b) Time-series plots.

arising from (3.3) are shown in Figure 5 for $t = 0, 100, 250, 500$. Figure 5(a) shows that for the model (2.5) with initial conditions (3.3), the formation of the irregular patchy structure can be preceded by the evolution of a regular spiral spatial pattern. Note that the appearance of the spirals is not induced by the initial conditions. The center of each spiral is situated in a



(a)

(b)

(c)

(d)

Figure 5: The process of spiral patterns of N for the parameters: $\alpha = 2.5$, $\beta = 1.3$, $\gamma = 1.1$, $d_{11} = 0.7$, $d_{12} = 0.05$, $d_{21} = 0.01$, $d_{22} = 1$. Time: (a) $t = 0$, (b) $t = 200$, (c) $t = 250$, (d) $t = 500$.

critical point (x_{cr}, y_{cr}) are (80, 200) and (320, 200), where $u(x_{cr}, y_{cr}) = u^*$, $v(x_{cr}, y_{cr}) = v^*$. The distribution (3.3) contains one point. After the spirals form (Figure 5(b)), they grow slightly for a certain time, their spatial structure becoming more distinct (Figures 5(c) and 5(d)).

4. Conclusions and Discussions

In this paper, we analyzed pattern formation of a cross-diffusion ratio-dependent predator-prey model within two-dimensional space and give the conditions of cross-diffusion-driven driven Turing instability. Then, we use numerical simulations to verify the correctness of the theoretical results and find that the model exhibits complex self-replication.

The results show that model (2.5) has rich spatiotemporal patterns (spots, stripes, holes, and spiral patterns); moreover, the existence of those patterns indicates that the cross-diffusion can induce more complex pattern formation than in the case of self-diffusion.

Compared to the paper of Lin [23], we present the condition of cross-diffusion Turing pattern, while in the case of self-diffusion the solution of the model is stable. We also show that the increasing speed of diffusion d_{12} will decrease the density of the prey. Similar, increasing speed of diffusion d_{21} will decrease density of the predator.

Therefore, we hope that the results presented here will be useful in studying the dynamic complexity of ecosystems or physical systems.

Acknowledgments

This research was supported by Fund Project of Zhejiang Provincial Education Department (Y201223449), Natural Science Foundation of Zhejiang Province (LY12A01014), and Natural Science Foundation of China (10901121).

References

[1] D. Alonso, F. Bartumeus, and J. Catalan, “Mutual interference between predators can give rise to turing spatial patterns,” *Ecology*, vol. 83, no. 1, pp. 28–34, 2002.
[2] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, John Wiley & Sons, Indianapolis, Ind, USA, 2003.
[3] B. Dubey, N. Kumari, and R. K. Upadhyay, “Spatiotemporal pattern formation in a diffusive predator-prey system: an analytical approach,” *Journal of Applied Mathematics and Computing*, vol. 31, no. 1-2, pp. 413–432, 2009.
[4] S. A. Levin, “The problem of pattern and scale in ecology: the Robert H. MacArthur award lecture,” *Ecology*, vol. 73, no. 6, pp. 1943–1967, 1992.
[5] R. B. Hoyle, *Pattern Formation: An Introduction to Methods*, Cambridge University Press, Cambridge, UK, 2006.
[6] E. A. McGehee and E. Peacock-López, “Turing patterns in a modified Lotka-Volterra model,” *Physics Letters A*, vol. 342, no. 1-2, pp. 90–98, 2005.
[7] A. Morozov and S. Petrovskii, “Excitable population dynamics, biological control failure, and spatiotemporal pattern formation in a model ecosystem,” *Bulletin of Mathematical Biology*, vol. 71, no. 4, pp. 863–887, 2009.
[8] J. D. Murray, *Mathematical Biology*, Springer, New York, NY, USA, 2003.
[9] W. Wang, Q.-X. Liu, and Z. Jin, “Spatiotemporal complexity of a ratio-dependent predator-prey system,” *Physical Review E*, vol. 75, no. 5, Article ID 051913, 2007.
[10] Y. Huang and O. Diekmann, “Interspecific influence on mobility and Turing instability,” *Bulletin of Mathematical Biology*, vol. 65, no. 1, pp. 143–156, 2003.
[11] S. A. Levin and L. A. Segel, “Hypothesis for origin of planktonic patchiness,” *Nature*, vol. 259, no. 5545, p. 659, 1976.
[12] J. D. Murray, “Discussion: turing’s theory of morphogenesis—its influence on modelling biological pattern and form,” *Bulletin of Mathematical Biology*, vol. 52, no. 1-2, pp. 117–152, 1990.
[13] A. M. Turing, “The chemical basis of morphogenesis,” *Philosophical Transactions of the Royal Society B*, vol. 237, pp. 37–72, 1952.
[14] L. A. Segel and J. L. Jackson, “Dissipative structure: an explanation and an ecological example,” *Journal of Theoretical Biology*, vol. 37, no. 3, pp. 545–559, 1972.
[15] A. Gierer and H. Meinhardt, “A theory of biological pattern formation,” *Biological Cybernetics*, vol. 12, no. 1, pp. 30–39, 1972.
[16] A. Okubo and S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, Springer, New York, NY, USA, 2001.
[17] W. M. Wang, H. Y. Liu, Y. L. Cai, and Z. Q. Li, “Turing pattern selection in a reaction-diffusion epidemic model,” *Chinese Physics B*, vol. 20, no. 7, Article ID 074702, 2011.
[18] W. Wang, Y. Lin, L. Zhang, F. Rao, and Y. Tan, “Complex patterns in a predator-prey model with self and cross-diffusion,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2006–2015, 2011.
[19] W. M. Wang, W. J. Wang, Y. Z. Lin, and Y. J. Tan, “Pattern selection in a predation model with self and cross diffusion,” *Chinese Physics B*, vol. 20, no. 3, Article ID 034702, 2011.
[20] Y.-H. Fan and W.-T. Li, “Permanence for a delayed discrete ratio-dependent predator-prey system with Holling type functional response,” *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 357–374, 2004.
[21] X. Liu and L. Huang, “Permanence and periodic solutions for a diffusive ratio-dependent predator-prey system,” *Applied Mathematical Modelling*, vol. 33, no. 2, pp. 683–691, 2009.
[22] X. Zeng and Z. Liu, “Nonconstant positive steady states for a ratio-dependent predator-prey system with cross-diffusion,” *Nonlinear Analysis*, vol. 11, no. 1, pp. 372–390, 2010.

[23] W. Lin, "Spatial pattern formation of a ratio-dependent predator prey model," *Chinese Physics B*, vol. 19, no. 9, Article ID 090206, 2010.
[24] V. N. Biktashev, J. Brindley, A. V. Holden, and M. A. Tsyganov, "Pursuit-evasion predator-prey waves in two spatial dimensions," *Chaos*, vol. 14, no. 4, pp. 988–994, 2004.
[25] J. B. Shukla and S. Verma, "Effects of convective and dispersive interactions on the stability of two species," *Bulletin of Mathematical Biology*, vol. 43, no. 5, pp. 593–610, 1981.
[26] E. H. Kerner, "Further considerations on the statistical mechanics of biological associations," *Bulletin of Mathematical Biophysics*, vol. 21, pp. 217–255, 1959.
[27] N. Shigesada, K. Kawasaki, and E. Teramoto, "Spatial segregation of interacting species," *Journal of Theoretical Biology*, vol. 79, no. 1, pp. 83–99, 1979.
[28] J. Chattopadhyay and P. K. Tapaswi, "Effect of cross-diffusion on pattern formation—a nonlinear analysis," *Acta Applicandae Mathematicae*, vol. 48, no. 1, pp. 1–12, 1997.
[29] B. Dubey, B. Das, and J. Hussain, "A predator-prey interaction model with self and cross-diffusion," *Ecological Modelling*, vol. 141, no. 1–3, pp. 67–76, 2001.
[30] M. Iida, M. Mimura, and H. Ninomiya, "Diffusion, cross-diffusion and competitive interaction," *Journal of Mathematical Biology*, vol. 53, no. 4, pp. 617–641, 2006.
[31] K. Kuto and Y. Yamada, "Multiple coexistence states for a prey-predator system with cross-diffusion," *Journal of Differential Equations*, vol. 197, no. 2, pp. 315–348, 2004.
[32] G. Q. Sun, Z. Jin, Q. X. Liu, and L. Li, "Pattern formation induced by cross-diffusion in a predator-prey system," *Chinese Physics B*, vol. 17, no. 11, pp. 3936–3941, 2008.
[33] V. K. Vanag and I. R. Epstein, "Cross-diffusion and pattern formation in reaction-diffusion systems," *Physical Chemistry Chemical Physics*, vol. 11, no. 6, pp. 897–912, 2009.
[34] W. Ko and K. Ryu, "On a predator-prey system with cross diffusion representing the tendency of predators in the presence of prey species," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1133–1142, 2008.
[35] W. Ko and K. Ryu, "On a predator-prey system with cross-diffusion representing the tendency of prey to keep away from its predators," *Applied Mathematics Letters*, vol. 21, no. 11, pp. 1177–1183, 2008.
[36] A. B. Medvinsky, S. V. Petrovskii, I. A. Tikhonova, H. Malchow, and B.-L. Li, "Spatiotemporal complexity of plankton and fish dynamics," *SIAM Review*, vol. 44, no. 3, pp. 311–370, 2002.
[37] R. K. Upadhyay, W. Wang, and N. K. Thakur, "Spatiotemporal dynamics in a spatial plankton system," *Mathematical Modelling of Natural Phenomena*, vol. 5, no. 5, pp. 102–122, 2010.