Journal of Mathematics

Fractional Calculus and Related Inequalities

Lead Guest Editor: Serkan Araci
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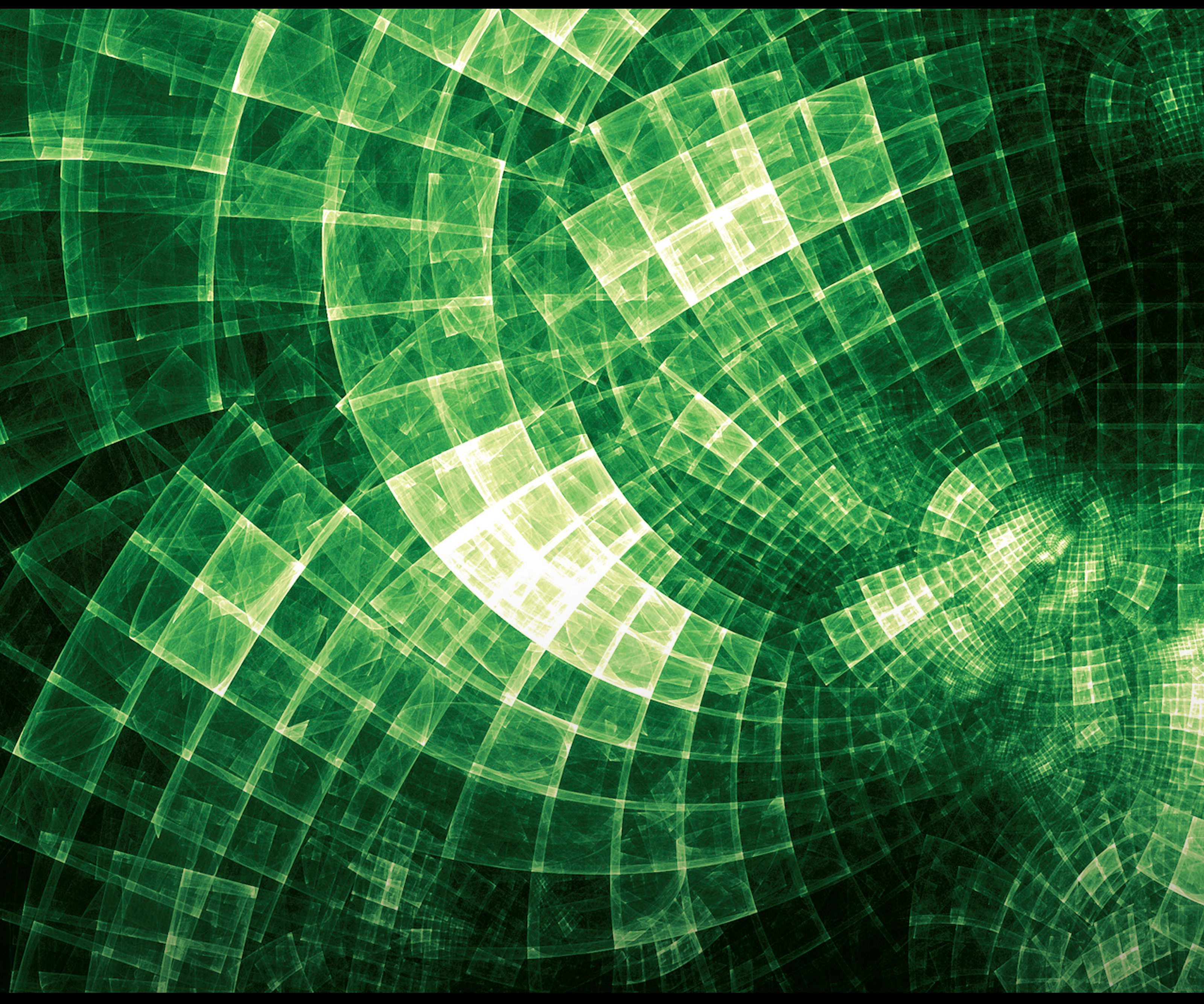

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Research Article

The New Mittag-Leffler Function and Its Applications

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In this paper, we investigate some properties of the Pochhammer (p, s, k)-symbol ${}_p[\xi]_{n,k,s}$ and gamma (p, s, k)-function ${}_p\Gamma_{s,k}(\xi)$. We then prove several identities for newly defined symbol ${}_p[\xi]_{n,k,s}$ and the function ${}_p\Gamma_{s,k}(\xi)$. The integral representations for the gamma (p, s, k)-function and beta (p, s, k)-function are presented. Also, we define a new Mittag-Leffler (p, s, k)-function and study its analytic properties and its transforms.

1. Introduction

The theory of special functions comprises a major part of mathematics. In the last three centuries, the essential of solving the problems taking place in the fields of classical mechanics, hydrodynamics, and control theories motivated the development of the theory of special functions. This field also has wide applications in both pure mathematics and applied mathematics. The interested readers may consult the literature [1–4].

The Mittag-Leffler function takes place naturally similar to that of the exponential function in the solutions of fractional integro-differential equations having the arbitrary order. The Mittag-Leffler functions have to gain more recognition due to its wide applications in diverse fields. We suggest the readers to review the literature [5–16] for more details.

Throughout this article, let $\mathbb{C}, \mathbb{R}^+, \mathbb{Z}^-$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, negative integers, and natural numbers, respectively.

2. Preliminaries

This section contains some basic definitions and mathematical preliminaries. We begin with the well-known Mittag-Leffler function.

In [17], Gosta Mittag-Leffler introduced the following Mittag-Leffler function which is defined by

$$E_\theta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + 1)}, \quad z \in \mathbb{C}, \ \Re(\theta) > 0. \tag{1}$$

In [18], a generalization of Mittag-Leffler function $E_\theta(z)$ (1) is given by

$$E_{\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + \vartheta)}, \quad z, \vartheta \in \mathbb{C}, \ \Re(\theta) > 0. \tag{2}$$

In [6], Prabhakar proposed the following three parameters of Mittag-Leffler function, which is defined by

$$E^{\varrho}_{\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n (\varrho)_n}{\Gamma(\theta n + \vartheta) n!}, \quad z, \vartheta, \varrho \in \mathbb{C}, \ \Re(\theta) > 0, \tag{3}$$

where $(\varrho)_n$ is the well-known Pochhammer's symbol defined as follows (see [19]). For $\varrho \in \mathbb{C}$,

$$\begin{aligned}(\varrho)_n &= \varrho(\varrho+1)(\varrho+2)\cdots(\varrho+n-1), \quad \text{for } n \in \mathbb{N} \\ &= 1, \quad \text{for } n = 0.\end{aligned} \tag{4}$$

Definition 1. In [20], Gehlot introduced the two parameters Pochhammer's symbol and two parameters of gamma (p,k)-function.

Let $k, p > 0$, $\xi \in \mathbb{C}$ with $\Re(\xi) > 0$, and $n \in \mathbb{N}$. Then, the Pochhammer (p,k)-symbol ${}_p(\xi)_{n,k}$ is defined as

$${}_p(\xi)_{n,k} = \left(\frac{\xi p}{k}\right)\left(\frac{\xi p}{k}+p\right)\cdots\left(\frac{\xi p}{k}+(n-1)p\right) \tag{5}$$

and the gamma (p,k)-function is defined by

$${}_p\Gamma_k(\xi) = \frac{1}{k}\lim_{n\longrightarrow\infty}\frac{n!p^{n+1}(np)^{\xi/k}}{p(\xi)_{n+1,k}}. \tag{6}$$

Definition 2. The gamma (p,k)-function is also represented by the following forms:

$${}_p\Gamma_k(\xi) = \int_0^\infty e^{-t^k/p}t^{x-1}\mathrm{d}t, \tag{7}$$

$${}_p\Gamma_k(\xi) = \left(\frac{p}{k}\right)^{\xi/k}\Gamma_k(\xi) = \frac{p^{\xi/k}}{k}\Gamma\left(\frac{\xi}{k}\right). \tag{8}$$

Also, the relationship between the Pochhammer (p,k)-symbol, Pochhammer k-symbol, and the classical Pochhammer's symbol is represented by [4]

$${}_p(\xi)_{n,k} = \left(\frac{p}{k}\right)^n(\xi)_{n,k} = p^n\left(\frac{\xi}{k}\right)_n. \tag{9}$$

The gamma (p,k)-function ${}_p\Gamma_k(\xi)$ satisfies the following relation:

$${}_p(\xi)_{n,k} = \frac{p\Gamma_k(\xi+nk)}{p\Gamma_k(\xi)}, \tag{10}$$

where

$${}_p\Gamma_k(\xi+k) = \left(\frac{\xi p}{k}\right)\Gamma_k(\xi). \tag{11}$$

Definition 3. In [21], the Mittag-Leffler k-function is defined by

$$E^{\varrho}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty}\frac{(\varrho)_{n,k}z^n}{\Gamma_k(\theta n+\vartheta)n!}, \tag{12}$$

where $k \in \mathbb{R}$, z, θ, ϑ, $\varrho \in \mathbb{C}$, $\Re(\theta)$, and $\Re(\vartheta) > 0$.

Recently, Cerutti et al. [22] introduced the following Mittag-Leffler (p,k)-function is defined as follows.

Definition 4. For $p, k \in \mathbb{R}, z, \theta, \vartheta, \varrho \in \mathbb{C}, \Re(\theta) > 0$, and $\Re(\vartheta) > 0$,

$${}_pE^{\varrho}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty}\frac{p(\varrho)_{n,k}z^n}{p\Gamma_k(\theta n+\vartheta)n!}, \tag{13}$$

where ${}_p(\varrho)_{n,k}$ is the Pochhammer (p,k)-symbol given by (5) and ${}_p\Gamma_k(\xi)$ is gamma (p,k)-function given by (6).

In [23], Pochhammer (p,s,k)-symbol ${}_p[\xi]_{n,k,s}$ is defined as follows.

Definition 5. For $p, k, \xi \in \mathbb{R}$, $0 < s < 1$, and $n \in \mathbb{N}$,

$$\begin{aligned}{}_p[\xi]_{n,k,s} &= \left[\frac{\xi p}{k}\right]_s\left[\frac{\xi p}{k}+p\right]_s\cdots\left[\frac{\xi p}{k}+(n-1)p\right]_s \\ &= \prod_{i=0}^{n-1}\left[\frac{\xi p}{k}+ip\right]_s,\end{aligned} \tag{14}$$

where

$$[\xi]_s = \frac{1-s^\xi}{1-s}, \quad \forall \xi \in \mathbb{R}. \tag{15}$$

The following identities are satisfied:

(1) $[\xi+y]_s = [\xi]_s + s^\xi[y]_s \, \forall \xi, y \in \mathbb{R}$
(2) $[1]_s = 1$
(3) $[\xi y]_s = [\xi]_{s^y}[y]_s \, \forall \xi, y \in \mathbb{R}$
(4) $[0]_s = 0$

Definition 6. In [23], the gamma (p,s,k)-function in term of s-series is given by

$${}_p\Gamma_{s,k}(kn) = \left[\frac{p}{k}\right]_s\frac{(s^p;s^p)_{n-1}}{(1-s)^{n-1}}. \tag{16}$$

The relationship between ${}_p[\xi]_{n,k,s}$ and ${}_p\Gamma_{s,k}(\xi)$ is given by [23]

$${}_p[\xi]_{n,k,s} = \frac{p\Gamma_{s_k}(\xi+nk)}{p\Gamma_{s,k}(\xi)}. \tag{17}$$

The beta (p,k)-function is defined by

$${}_pB_k(\xi,y) = \frac{p\Gamma_k(\xi)_p\Gamma_k(y)}{p\Gamma_k(\xi+y)}, \quad \Re(\xi) > 0, \Re(y) > 0. \tag{18}$$

Definition 7. The beta (p,k)-function is represented by the following integral:

$${}_pB_k(\xi,y) = \frac{1}{k}\int_0^1 t^{(\xi/k)-1}(1-t)^{(y/k)-1}e^{-p/t}\mathrm{d}t. \tag{19}$$

Definition 8. The well-known Laplace transform of piecewise continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\mathscr{L}\{f(\theta)\} = \int_0^\infty e^{-s\theta}f(\theta)\mathrm{d}\theta, \Re(\mathbf{s}) > 0. \tag{20}$$

3. Applications and Properties of the Pochhammer (p,s,k)-Symbol ${}_p[\xi]_{n,k,s}$ and Gamma (p,s,k)-Function ${}_p\Gamma_{s,k}(\xi)$

In this section, we define gamma (p,s,k)-function ${}_p\Gamma_{s,k}(\xi)$ in terms of limit function and give its integral representation. Also, we define beta (p,s,k)-function ${}_pB_{s,k}(\xi,y)$ and its integral representation. Furthermore, we prove some identities of the Pochhammer (p,s,k)-symbol ${}_p[\xi]_{n,k,s}$ and the gamma (p,s,k)-function ${}_p\Gamma_{s,k}(\xi)$.

Definition 9. Suppose that k, s, $p>0$, $\xi\in\mathbb{C}/k\mathbb{Z}^-$ with $\Re(\xi)>0$, and $n\in\mathbb{N}$. Then, the gamma (p,s,k)-function ${}_p\Gamma_{s,k}(\xi)$ is given as

$$ {}_p\Gamma_{s,k}(\xi)=\frac{s}{k}\lim_{n\longrightarrow\infty}\frac{n!p^{n+1}(snp)^{(\xi/k)-1}}{p(\xi)_{n,k}}. \tag{21}$$

Theorem 1. *Suppose that $k,p,s,r,q>0$, $\xi\in\mathbb{C}/k\mathbb{Z}^-$ with $\Re(\xi)>0$, and $n\in\mathbb{N}$. Then, the following identities hold:*

$$(1).\ {}_p[\xi]_{n,r,s}={}_p\left[\frac{q\xi}{r}\right]_{n,q,s} \tag{22}$$

$$(2).\ {}_p[\xi]_{n,r,s}=\left(\frac{p}{r}\right)^n {}_r\left[\frac{q\xi}{r}\right]_{n,q,s} \tag{23}$$

$$(3).\ {}_p[\xi]_{n,k,s}={}_s[\xi]_{n,k,p} \tag{24}$$

$$(4).\ {}_p\Gamma_{s,k}(\xi)=\left(\frac{q}{k}\right){}_p\Gamma_{s,q}\left(\frac{q\xi}{k}\right) \tag{25}$$

$$(5).\ {}_r\Gamma_{s,q}(\xi)=\left(\frac{k}{q}\right)\left(\frac{r}{p}\right)^{\xi/s}{}_p\Gamma_{s,k}\left(\frac{k\xi}{s}\right) \tag{26}$$

$$(6).\ {}_r\Gamma_{k,q}(\xi)=\left(\frac{r}{p}\right)^{\xi/q}{}_p\Gamma_{k,q}(\xi) \tag{27}$$

Proof. The properties (22), (23), and (24), respectively, follow from definition (11) and equations (2.8), (2.9), and (2.10) of [16]. The properties (25), (26), and (27), respectively, follow by using equation (21) and (2.11), (2.12), and (2.13) of [16]. □

Theorem 2. *Let $k,p,s,r>0$, $\xi\in\mathbb{C}/k\mathbb{Z}^-$ with $\Re(\xi)>0$, and $n\in\mathbb{N}$. Then, the following integral representation of gamma (p,s,k)-function is defined by*

$$ {}_p\Gamma_{s,k}(\xi)=\int_0^\infty e^{-t^k/sp}t^{\xi-1}dt. \tag{28}$$

Proof. Consider the right hand side of (28). By applying ([8], page 2) Tannery theorem and using (21), we have

$$\int_0^\infty e^{-t^k/sp}t^{\xi-1}dt=\lim_{n\longrightarrow\infty}\int_0^{(snp)^{1/k}}\left(1-\frac{t^k}{nsp}\right)^n t^{\xi-1}dt. \tag{29}$$

Let $C_{n,i}(\xi), i=0,1,\ldots,n$, be given by

$$C_{n,i}(\xi)=\lim_{n\longrightarrow\infty}\int_0^{(snp)^{1/k}}\left(1-\frac{t^k}{nsp}\right)^i t^{\xi-1}dt. \tag{30}$$

After integrating by parts, we obtain

$$C_{n,i}(\xi)=\frac{ki}{nspx}C_{n,i-1}(\xi+k),\quad \forall i=1,2,\ldots,n. \tag{31}$$

Also,

$$C_{n,0}(\xi)=\frac{(snp)^{\xi/k}}{\xi}. \tag{32}$$

Therefore,

$$C_{n,n}(\xi)=\frac{s}{k}\frac{n!p^{n+1}(snp)^{(\xi/k)-1}}{{}_p(\xi)_{n,k}(1+(\xi/kn))},$$

$$ {}_p\Gamma_{s,k}(\xi)=\lim_{n\longrightarrow\infty}C_{n,n}(\xi)=\frac{s}{k}\lim_{n\longrightarrow\infty}\frac{n!p^{n+1}(snp)^{(\xi/k)-1}}{{}_p(\xi)_{n,k}}. \tag{33}$$

□

Definition 10. Let ξ, $y\in\mathbb{C}/k\mathbb{Z}^-$, $k,p,s,r\in\mathbb{R}^+-0$, and $\Re(\xi)>0, n\in\mathbb{N}$. Then, the integral representation of ${}_pB_{s,k}(\xi,y)$ is given by

$$ {}_pB_{s,k}(\xi,y)=\frac{1}{k}\int_0^1 t^{(\xi/k)-1}(1-t)^{(y/k)-1}e^{-(p/t)-(s/(1-t))}dt, \tag{34}$$

where $\Re(p)>0, \Re(s)>0, \Re(\xi)>0$, and $\Re(y)>0$.

Theorem 3. *The relation between three parameters, two parameters, and the classical Pochhammer's symbol is given by*

$$ {}_p[\xi]_{n,k,s}=s^n{}_p(\xi)_{n,k}=\left(\frac{sp}{k}\right)^n(\xi)_{n,k}=(sp)^n\left(\frac{\xi}{k}\right)_n. \tag{35}$$

Proof. Using (14) and (9), we get the desired result. □

Theorem 4. *The relation between gamma (p,s,k)-function, gamma (p,k)-function, gamma k-function, and classic gamma function is given by*

$$ {}_p\Gamma_{s,k}(\xi)=(s)^{\xi/k}{}_p\Gamma_k(\xi)=\left(\frac{sp}{k}\right)^{\xi/k}\Gamma_k(\xi)=\frac{(sp)^{\xi/k}}{k}\Gamma\left(\frac{\xi}{k}\right). \tag{36}$$

Proof. Using (21) and (8), we get the desired result. □

Theorem 5. *Given* $\xi \in \mathbb{C}kZ^{-}, k, s, p \in \mathbb{R}^{+} - \{0\}, \Re(\xi) > 0,$ *and* $n \in \mathbb{N}$, *the recurrence relation for Pochhammer* (p, s, k)*-symbol is given by*

$$\begin{aligned} {}_p[\xi]_{n+j,k,s} &= {}_p[\xi]_{j,k,s} \times {}_p[\xi + jk]_{n,k,s}, \\ (ns)_p[\xi]_{n-1,k,s} &= {}_p[\xi]_{n,k,s} - {}_p[\xi - k]_{n,k,s}. \end{aligned} \tag{37}$$

Proof. Using the definition of Pochhammer (p, s, k)-symbol, we get the desired result. □

4. Definition and Convergence Condition of the Mittag-Leffler (p, s, k)-Function ${}_pE^{\varrho,s}_{k,\theta,\vartheta}(z)$

In this section, we define a new generalization of the Mittag-Leffler (p, s, k)-function. Also, we check the convergence of the Mittag-Leffler (p, s, k)-function.

Definition 11. Suppose that $p, k \in \mathbb{R}, \theta, \vartheta, \varrho \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta)$, and $\Re(\varrho) > 0$. Then, Mittag-Leffler (p, s, k)-function is defined by

$$ {}_pE^{\varrho,s}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^n}{{}_p\Gamma_{s,k}(\theta n + \vartheta) n!}, \tag{38}$$

where ${}_p[\varrho]_{n,k,s}$ is Pochhammer (p, s, k)-symbol defined in (14), and ${}_p\Gamma_{s,k}(\xi)$ is defined in (21).The recurrence relation of gamma (p, s, k)-function ${}_p\Gamma_{s,k}(\xi)$ given in [23] is

$$ {}_p\Gamma_{s,k}(x + k) = \left[\frac{\xi p}{k}\right]_s {}_p\Gamma_{s,k}(\xi). \tag{39}$$

Now, some characteristics of the Mittag-Leffler (p, s, k)-function are presented. We show that the M-L (p, s, k)-function is an entire function. Also, its order and type are given.

Theorem 6. *The Mittag-Leffler* (p, s, k)*-function, defined in (38), is an entire function of order* ρ *and type* σ *given by*

$$\begin{aligned} \rho &= \frac{k}{\operatorname{Re}(\theta)}, \\ \sigma &= \left[\rho p e^{\operatorname{Re}((\theta/k)\ln(\theta/k))\rho}\right]^{-1}. \end{aligned} \tag{40}$$

Proof. Let R denotes the radius of convergence of the Mittag-Leffler (p, s, k)-function. By considering the properties (5) and (8) and using the asymptotic expansions for the gamma function [1] and the asymptotic Stirling's formula, we have

$$\begin{aligned} \Gamma(z) &= (2\pi)^{1/2} z^{z-(1/2)} e^{z} [1 + O(z - 1)] \\ &\cdot (|\arg(z)| < \pi; |z| \longrightarrow \infty). \end{aligned} \tag{41}$$

In particular,

$$ n! = (2\pi n)^{1/2} n^n e^{-n} \left[1 + O\left(n^{-1}\right)\right] (n \in N; n \longrightarrow \infty), \tag{42}$$

and the following quotient expansion of two gamma functions at infinity is given as

$$ \frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}\left[1 + O\left(z^{-1}\right)\right] (|\arg(z) + a| < \pi; |z| \longrightarrow \infty). \tag{43}$$

Series (38) can be written in the following forms:

$$\begin{aligned} {}_pE^{\varrho,s}_{k,\theta,\vartheta}(z) &= \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^n}{{}_p\Gamma_{s,k}(\theta n + \vartheta) n!} \\ &= \sum_{n=0}^{\infty} c_n z^n, \end{aligned} \tag{44}$$

since

$$ R = \lim_{n\longrightarrow\infty} \sup \left|\frac{c_n}{c_{n+1}}\right|. \tag{45}$$

In view of the properties (35) and (36) and using (III.9) of Theorem (1) in [22], we get

$$\begin{aligned} \left|\frac{c_n}{c_{n+1}}\right| &= \left|\frac{{}_p[\varrho]_{n,k,s}\,{}_p\Gamma_{s,k}(\theta(n + 1) + \vartheta)(n + 1)!}{{}_p[\varrho]_{n+1,k,s}\,{}_p\Gamma_{s,k}(\theta n + \vartheta) n!}\right| \\ &\simeq \left|\frac{s^n\,{}_p[\varrho]_{n,k} s^{(\theta n+\theta+\vartheta)/k}\,{}_p\Gamma_k(\theta(n + 1) + \vartheta)(n + 1)!}{s^{n+1}\,{}_p[\varrho]_{n+1,k} s_p^{(\theta n+\vartheta)/k}\Gamma_k(\theta n + \vartheta) n!}\right| \\ &\simeq (n + 1) s^{\vartheta/k} p^{(\theta/k)-1}\left(\frac{\theta n}{k}\right)^{\theta/k} \longrightarrow \infty. \end{aligned} \tag{46}$$

Thus, the Mittag-Leffler (p, s, k)-function is an entire function.

To obtain the order ρ and the type σ, we apply the following definitions of ρ and σ, respectively:

$$ \rho = \lim_{n\longrightarrow\infty} \sup \frac{n \ln n}{\ln\left(1/\left|c_n\right|\right)}, \tag{47}$$

$$ e^{\rho}\sigma = \lim_{n\longrightarrow\infty} \sup\left(n\left|c_n\right|^{\rho/n}\right). \tag{48}$$

Consider

$$\begin{aligned} \frac{1}{\left|c_n\right|} &= \left|\frac{{}_p\Gamma_{s,k}(\varrho)\,{}_p\Gamma_{s,k}(\theta n + \vartheta) n!}{{}_p\Gamma_{s,k}(\varrho + nk)}\right| \\ &= s^{(\theta(n-1)+\vartheta)/k}\left|\frac{{}_p\Gamma_k(\varrho)\,{}_p\Gamma_k(\theta n + \vartheta) n!}{p\Gamma_k(\varrho + nk)}\right|. \end{aligned} \tag{49}$$

By using Theorem (1) equation (III.12) ([22]) and definition of ρ (47), we get

$$ \rho = \frac{k}{\operatorname{Re}(\theta)}. \tag{50}$$

Similarly, by putting the value of $|c_n|$ in the definition of σ (48) and simplify as the same in Theorem (1) in equation (III.14) ([22]), we obtain

$$ \sigma = \left[\rho p e^{\Re((\theta/k)\ln(\theta/k))\rho}\right]^{-1}. \tag{51}$$

□

5. Applications and Properties of the Mittag-Leffler (p, s, k)-Function ${}_pE^{\varrho,s}_{k,\theta,\vartheta}(z)$

Some basic properties of Mittag-Leffler (p, s, k)-function ${}_pE^{\varrho,s}_{k,\theta,\vartheta}(z)$ are presented in this section.

Theorem 7. *Suppose that* $p, k, s \in \mathbb{N}, \theta, \vartheta, \varrho \in \mathbb{C}$ *with* $\Re(\theta) > 0, \Re(\vartheta)$ *and* $\Re(\varrho) > 0$, *then*

(1)

$$\left(\frac{d}{dz}\right)^m {}_pE^{\varrho,s}_{k,\theta,\vartheta}(z) = {}_p[\varrho]_{m,k,s}\, {}_pE^{\varrho+mk,s}_{k,\theta,\theta m+\vartheta}(z). \tag{52}$$

(2)

$$\left(\frac{d}{dz}\right)^m \left(z^{\vartheta/k}\right) {}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(z^{\theta/k}\right) = z^{((\theta n/k)-m+(\vartheta/k)-1)} \cdot {}_p[\varrho]_{m,k,s}\, {}_pE^{\varrho+mk,s}_{k,\theta,\theta m+\vartheta}\left(z^{\theta/k}\right). \tag{53}$$

Proof. (1) Taking L.H.S of (42),

$$\begin{aligned}\left(\frac{d}{dz}\right)^m {}_pE^{\varrho,s}_{k,\theta,\vartheta}(z) &= \left(\frac{d}{dz}\right)^m \left(\sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^n}{{}_p\Gamma_{s,k}(\theta n+\vartheta) n!}\right)\\ &= \sum_{n=m}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^{n-m}}{{}_p\Gamma_{s,k}(\theta n+\vartheta)(n-m)!}\\ &= \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n+m,k,s} z^n}{{}_p\Gamma_{s,k}(\theta(n+m)+\vartheta) n!}.\end{aligned} \tag{54}$$

By using

$${}_p[\xi]_{n+j,k,s} = {}_p[\xi]_{j,k,s} \times {}_p[\xi + jk]_{n,k,s}, \tag{55}$$

in the above equation, we get the desired result (52).

(2) Taking L.H.S of (43),

$$\begin{aligned}\left(\frac{d}{dz}\right)^m \left(z_p^{\vartheta/k} E^{\varrho,s}_{k,\theta,\vartheta}\left(z^{\theta/k}\right)\right) &= \left(\frac{d}{dz}\right)^m \left(\sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^{(\theta n/k)+(\vartheta/k)-1}}{{}_p\Gamma_{s,k}(\theta n+\vartheta) n!}\right)\\ &= \sum_{n=m}^{\infty} \frac{{}_p[\varrho]_{n,k,s} z^{(\theta n/k)+(\vartheta/k)-m-1}}{{}_p\Gamma_{s,k}(\theta n+\vartheta)(n-m)!}\\ &= \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n+m,k,s} z^{\theta/k)(n+m+(\vartheta/k)-m-1}}{{}_p\Gamma_{q,k}(\theta(n-m)+\vartheta) n!}.\end{aligned} \tag{56}$$

By using

$${}_p[\xi]_{n+j,k,s} = {}_p[\xi]_{j,k,s} \times {}_p[\xi + jk]_{n,k,s}, \tag{57}$$

in the above equation, we get the desired result (53). □

Theorem 8. *Suppose that* $k > 0$ *and* θ, ϑ *and* $\varrho \in \mathbb{C}$ *with* $\Re(\theta) > 0, \Re(\vartheta) > 0$ *and* $\Re(\varrho) > 0$, *then*

$$\int_0^z t^{(\vartheta/k)-1}\, {}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(at^{\theta/k}\right) dt = s^{(\theta n+\vartheta)/k} p z^{\vartheta/k}\, {}_pE^{\varrho,s}_{k,\theta,(\vartheta+k)}\left(az^{\theta/k}\right). \tag{58}$$

Proof. Consider the L.H.S.

$$\int_0^z t^{(\vartheta/k)-1}\, {}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(at^{\theta/k}\right) dt = \int_0^z t^{(\vartheta/k)-1} \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s}\left(at^{\theta/k}\right)^n}{{}_p\Gamma_{s,k}(\theta n+\vartheta) n!}. \tag{59}$$

By changing integration and summation orders in the above equation, we have

$$\begin{aligned}\int_0^z t^{(\vartheta/k)-1}\, {}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(at^{\theta/k}\right) dt &= \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s}(a)^n}{{}_p\Gamma_{s,k}(\theta n+\vartheta) n!} \cdot \int_0^z t^{(\theta n/k)+(\vartheta/k)-1} dt\\ &= \sum_{n=0}^{\infty} \frac{{}_p[\varrho]_{n,k,s}(a)^n z^{(\theta n/k)+(\vartheta/k)}}{((\theta n+\vartheta)/k)\, {}_p\Gamma_{s,k}(\theta n+\vartheta) n!}.\end{aligned} \tag{60}$$

By using the recurrence relation of ${}_p\Gamma_{s,k}(\xi)$ and ${}_p\Gamma_k(\xi)$ and equation (III.31) in [22], we get

$$\left(\frac{\theta n+\vartheta}{k}\right) {}_p\Gamma_{s,k}(\theta n+\vartheta) = \frac{s^{(\theta n+\vartheta)/k}}{p}\, {}_p\Gamma_{s,k}(\theta n+\vartheta+k). \tag{61}$$

By using the above result in equation (60), we have

$$\int_0^z t^{(\vartheta/k)-1}\, {}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(at^{\theta/k}\right) dt = s^{(\theta n+\vartheta)/k} p z_p^{\vartheta/k} E^{\varrho,s}_{k,\theta,\vartheta+k}\left(az^{\theta/k}\right). \tag{62}$$

□

6. The Euler-Beta Transform of the Mittag-Leffler (p, s, k)-Function ${}_pE^{\varrho,s}_{k,\theta,\vartheta}(z)$

The well-known Euler-beta transform is defined by

$$\mathbf{B}\{f(\theta); a; b\} = \int_0^1 \theta^{a-1}(1-\theta)^{b-1} f(\theta) d\theta, \tag{63}$$

where $a, b \in \mathbb{R}$ and $\min\{\Re(\theta), \Re(\vartheta)\} > 0$.

Next, we define the beta transform of newly defined M-L function.

Theorem 9. *Suppose that* $p, s, k > 0$ *and* $\theta, \vartheta, \varrho, a, b \in \mathbb{C}$ *with* $\Re(\theta) > 0, \Re(\vartheta) > 0$, $\Re(\varrho) > 0, \Re(a) > 0$, *and* $\Re(b) > 0$, *then*

$$\mathbf{B}\left({}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\lambda z^{\theta/k}\right);\frac{\vartheta}{k};\frac{b}{k}\right)=k\,{}_p\Gamma_{s,k}(b)\,{}_pE^{\varrho,s}_{k,\theta,\vartheta+b}(\lambda). \tag{64}$$

Proof.

$$\mathbf{B}\left({}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\lambda z^{\theta/k}\right);\frac{\vartheta}{k};\frac{b}{k}\right)=\int_0^1 z^{(\vartheta/k)-1}(1-z)^{(b/k)-1}\cdot\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}\lambda^n z^{\theta n/k}}{{}_p\Gamma_{s,k}(\theta n+\vartheta)n!}dz. \tag{65}$$

After interchanging the integration and summation orders of the above equation and using the relation between the gamma (p,s,k)-function and the classical gamma function given by (3.4), we have

$$\begin{aligned}\mathbf{B}\left({}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\lambda z^{\theta/k}\right);\frac{\vartheta}{k};\frac{b}{k}\right)&=\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}\lambda^n}{{}_p\Gamma_{s,k}(\theta n+\vartheta)n!}\int_0^1 z^{(\theta n/k)+(\vartheta/k)-1}(1-z)^{(b/k)-1}dz\\&=\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}\lambda^n}{{}_p\Gamma_{s,k}(\theta n+\vartheta)n!}\frac{\Gamma((\theta n+\vartheta)/k)\Gamma(b/k)}{\Gamma((\theta n+\vartheta+b)/k)}\\&=\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}\lambda^n}{s^{(\theta n+\vartheta)/k}\Gamma((\theta n+\vartheta)/k)n!}\frac{\Gamma((\theta n+\vartheta)/k)\Gamma(b/k)}{\Gamma((\theta n+\vartheta+b)/k)}.\end{aligned} \tag{66}$$

After simplification, we obtain the desired result:

$$\mathbf{B}\left({}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(z^{\theta/k}\right);\frac{\vartheta}{k};\frac{b}{k}\right)=k\,{}_p\Gamma_{s,k}(b)\,{}_pE^{\varrho,s}_{k,\theta,\vartheta+b}(\lambda). \tag{67}$$

□

Remark 1.

(i) If $s=1$, then equation (64) coincides with the result of [22], sec (IV.2)

(ii) If $s=1$ and $p=k$, then equation (64) coincides with the formula (11, 37) in [21]

(iii) If $s=p=k=1$, then equation (64) coincides with the formula (2.2.14) in [24]

7. The Laplace Transform of the Mittag-Leffler (p,s,k)-Function ${}_pE^{\varrho,s}_{k,\theta,\vartheta}(z)$

Theorem 10. *Let* $p,s,k,c>0$ *and* $\theta,\vartheta,\varrho\in\mathbb{C}$ *with* $\Re(\theta)>0,\Re(\vartheta)>0$, *and* $\Re(\varrho)>0$. *Then, the Laplace transform of* ${}_pE^{\varrho,s}_{k,\theta,\vartheta}$ *is given by*

$$\mathscr{L}\left\{z^{(\vartheta/k)-1}\,{}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\pm(cz)^{\theta/k}\right)\right\}=k(sp\mathbf{s})^{-\vartheta/k}\left[1\mp p\left(\frac{c}{sk\mathbf{s}}\right)^{\theta/k}\right]^{-\varrho/k}. \tag{68}$$

Proof. Applying the Laplace transform on the left hand side of (68), the Laplace transform of the potential function (see [1], equation (1.4.58)) and the generalized binomial formula is given by

$$(1-kw)^{-\varrho/k}=\sum_{k=0}^{\infty}\frac{(\varrho)_{n,k}w^n}{n!}. \tag{69}$$

We have

$$\begin{aligned}\mathscr{L}\left\{z^{(\vartheta/k)-1}\,{}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\pm(cz)^{\theta/k}\right)\right\}&=\int_0^{\infty}e^{-sz}z^{(\vartheta/k)-1}\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}}{{}_p\Gamma_{s,k}(\theta n+\vartheta)n!}(\pm1)^n\left((cz)^{\theta n/k}\right)dz\\&=\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}(\pm1)^n(c)(\theta n/k)}{{}_p\Gamma_{s,k}(\theta n+\vartheta)n!}\int_0^{\infty}e^{-sz}z^{(\theta n/k)+(\vartheta/k)-1}dz\\&=\frac{k}{(sp\mathbf{s})^{\vartheta/k}}\sum_{n=0}^{\infty}\frac{{}_p[\varrho]_{n,k,s}}{n!(c/sp\mathbf{s})^{\theta n/k}}.\end{aligned} \tag{70}$$

Now, using the relation between ${}_p[\varrho]_{n,k,s}$ and ${}_p(\varrho)_{n,k}$, we have

$$\begin{aligned}\mathscr{L}\left\{z^{(\vartheta/k)-1}\,{}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\pm(cz)^{\theta/k}\right)\right\} &= \frac{k}{(sp\mathbf{s})^{\vartheta/k}}\sum_{n=0}^{\infty}\frac{s^n_p[\varrho]_{n,k}}{n!}\left(\frac{c}{sp\mathbf{s}}\right)^{\theta n/k}\\ &= \frac{k}{(sp\mathbf{s})^{\vartheta/k}}\sum_{n=0}^{\infty}\frac{s^n(p/k)^n(\varrho)_{n,k}}{n!}\left(\frac{c}{sp\mathbf{s}}\right)^{\theta n/k}\\ &= \frac{k}{(sp\mathbf{s})^{\vartheta/k}}\sum_{n=0}^{\infty}\frac{(\varrho)_{n,k}}{n!}\left[\left(\frac{c}{sp\mathbf{s}}\right)^{\theta/k}\frac{sp}{k}\right]^n.\end{aligned}\tag{71}$$

Writing the series in compact form, we have

$$\mathscr{L}\left\{z^{(\vartheta/k)-1}\,{}_pE^{\varrho,s}_{k,\theta,\vartheta}\left(\pm(cz)^{\theta/k}\right)\right\} = \frac{k}{(sp\mathbf{s})^{\vartheta/k}}\left(1\mp p\mathbf{s}\left(\frac{c}{ks}\right)^{\theta/k}\right)^{-\varrho/k}.\tag{72}$$

□

Remark 2. (a) If $s=1$ in the above theorem, then the result coincides with the formula of [22]

(b) If $p=s=k=c=1$ in Theorem 10, then

$$\mathscr{L}\left\{z^{\vartheta-1}\,{}_1E^{\varrho,1}_{1,\theta,\vartheta}\left(z^{\theta}\right)\right\} = \mathbf{s}^{\vartheta}\left(1-\mathbf{s}^{-\theta}\right)^{-\varrho},\tag{73}$$

which coincides with formula (11.13) of [25].

8. Conclusion

In this paper, we established a new extension of the Mittag-Leffler function and investigated some of its properties. We concluded as follows:

(1) If $s=1$, then we get the results of the Mittag-Leffler (p,s,k)-function defined in [22]

(2) If $p=1$ and $s=1$, then we get the results of the Mittag-Leffler k-function defined in [21]

(3) If $k=1$, $p=1$, and $s=1$, then we get the results of the Mittag-Leffler defined in [6]

(4) If $p=1, k=1, s=1$, and $\varrho=1$, then we get the results of the Mittag-Leffler defined in [18]

(5) If $p=1, k=1, s=1, \vartheta=1$, and $\varrho=1$, then we get the results of the Mittag-Leffler k-function defined in [17]

(6) If $p=1, k=1, s=1, \theta=1, \vartheta=1$, and $\varrho=1$, then we get the exponential function

Recently, the Mittag-Leffler function is used to construct the fractional operators with nonsingular kernels [26, 27]. In [28, 29], the authors introduced the generalized fractional integrals and differential operators, which contain the Mittag-Leffler k-function in the kernels, and proved their various properties. Recently, Samraiz et al. [30] introduced the Hilfer Prabhakar $(k;s)$-fractional derivative by using the Mittag-Leffler k-function. They discussed its various properties and the generalized Laplace transform of the said operator. They also discussed the applications of Hilfer Prabhakar's $(k;s)$-derivative in mathematical physics. In this study, we defined further generalization of the Mittag-Leffler and proved its various basic properties. Hence, it would be of great interest that the Mittag-Leffler function studied in this article will be utilized to generalize such classes of fractional and differential operators.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

U.M. gave the main idea, and all other authors contributed equally to improve the final manuscript.

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Research Article

Some Existence Results for a System of Nonlinear Fractional Differential Equations

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In this paper, we show that a sequence satisfying a Suzuki-type JS-rational contraction or a generalized Suzuki-type Ćirić JS-contraction, under some conditions, is a Cauchy sequence. This paper presents some common fixed point theorems and an application to resolve a system of nonlinear fractional differential equations. Some examples and consequences are also given.

1. Introduction

The application of Banach contraction principle (BCP) [1] is wide spread. Recently, fixed point theory is being applied to show the existence of solutions of different mathematical models expressed in the forms of differential, integral, functional, fractional differential, and matrix equations (both linear and nonlinear). There are several common fixed point theorems, in the literature, which generalize BCP and have been applied to show the existence of solutions of different mathematical models involving two or more functions (see [2–19], for details).

Suzuki [20] presented a new generalization of BCP, so called a Suzuki-contraction, and established an existence theorem which characterized the metric completeness. Piri and Kumam [21, 22] investigated results in [20] under the F-contraction structure both in metric and b-metric spaces, respectively. Further generalizations of Suzuki contractions have been studied by Aydi et al. [23] who introduced a Suzuki-type multivalued contraction to obtain fixed point theorems in the setting of weak partial metric spaces. Abbas et al. [2] discussed generalized Suzuki-type multivalued contractions under the effect of a binary relation and obtained some fixed point results.

On the contrary, Jleli et al. [24, 25] introduced another generalization of BCP, known as a JS-contraction (also known as a θ-contraction) and Li and Jiang [26] modified the JS-contraction to JS-quasi contractions in order to obtain more general fixed point results, as compared to [24]. Altun et al. [27] obtained some fixed point theorems for JS-contraction type mappings via a binary operation. Several honorable researchers have published their valuable investigations on JS-contractions in well ranked journals (see [28–31]).

In this paper, we structure two common fixed point theorems comprising four self-mappings involving generalized Suzuki-type θ-rational contractions and generalized

Suzuki-type Ćirić θ-contractions. The existence of the solution of a mathematical model in terms of fractional differential equations is shown through a common fixed point theorem.

2. Advances on θ-Contractions

Let $\Theta := \{\theta\colon (0,\infty) \longrightarrow (1,\infty)$ such that θ satisfies $\Theta_1, \Theta_2, \Theta_3\}$, where

(Θ_1) θ is nondecreasing.

(Θ_2) For each sequence $\{\breve{t}_n\} \subset (0,\infty)$,

$$\lim_{n\longrightarrow\infty} \theta\left(\breve{t}_n\right) = 1, \quad \text{if and only if } \lim_{n\longrightarrow\infty} \breve{t}_n = 0. \tag{1}$$

(Θ_3) There exist $q \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{\breve{t}\longrightarrow 0^+} \theta(\breve{t}) - 1/\breve{t}^q = \ell$.

Jleli and Samet [24] introduced the following.

Definition 1. Let (χ,ρ) be a metric space. The mapping $\breve{T}\colon \chi \longrightarrow \chi$ is called a θ-contraction (or a JS-contraction) if there exist a constant $k \in [0,1)$ and $\theta \in \Theta$ such that

$$r, j \in \chi, \quad \rho(\breve{T}r, \breve{T}j) \neq 0 \Rightarrow \theta(\rho(\breve{T}r, \breve{T}j)) \leq [\theta(\rho(r,j))]^k. \tag{2}$$

The following examples show that the set Θ is not empty.

Example 1. Let $\Phi, \psi\colon (0,\infty) \longrightarrow (1,\infty)$ be defined by, for all $t \in (0,\infty)$,

$$\begin{aligned} \Phi(t) &= e^{\sqrt{t}}, \\ \psi(t) &= e^{\sqrt{te^t}}. \end{aligned} \tag{3}$$

Then, Φ and ψ are in Θ.

Jleli and Samet [24] established the following fixed point theorem.

Theorem 1 (see [24]). *Let (χ,ρ) be a complete metric space and $\breve{T}\colon \chi \longrightarrow \chi$ be a θ-contraction. Then, T has a unique fixed point.*

Consistent with [28], let $\mu\colon = \{\theta\colon (0,\infty) \longrightarrow (1,\infty) | \theta$ satisfy $(\Theta_1), (\Theta_2), (\Theta_3')\}$, where

($\Theta'3$) θ is continuous on $(0,\infty)$.

Note that (Θ3) and (Θ'3) are independent of each other. For this, see examples in [28]. The set of functions μ is not empty, as shown in the following.

Example 2 (see [28]). Let $\phi, \varphi, \psi, F, G, H\colon (0,\infty) \longrightarrow (1,\infty)$ be defined by, for all $t \in (0,\infty)$,

$$\begin{aligned} \phi(t) &= e^t, \\ \varphi(t) &= \cos ht, \\ \psi(t) &= e^{\sqrt{te^t}}, \\ F(t) &= 1 + \ln(t+1), \\ G(t) &= e^{\sqrt{t}}, \\ H(t) &= e^{te^t}. \end{aligned} \tag{4}$$

Here, $\phi, \varphi, \psi, F, G, H \in \mu$.

Hussain et al. [29] modified the class Θ of mappings as follows:

$$\text{Let } \Psi = \{\theta\colon [0,\infty) \longrightarrow [1,\infty)\colon \theta \text{ satisfies } (\Psi_1) - (\Psi_5)\}, \tag{5}$$

where

(Ψ_1) θ is non decreasing.

(Ψ_2) $\theta(\breve{t}) = 1$ if and only if $\breve{t} = 0$.

(Ψ_3) For each sequence $\{\breve{t}_n\} \subset (0,\infty)$:

$$\lim_{n\longrightarrow\infty} \theta\left(\breve{t}_n\right) = 1, \quad \text{if and only if } \lim_{n\longrightarrow\infty} \breve{t}_n = 0. \tag{6}$$

(Ψ_4) There exist $q \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{\breve{t}\longrightarrow 0^+} \theta(\breve{t}) - 1/\breve{t}^q = \ell$.

(Ψ_5)$\theta(\breve{t}_1 + \breve{t}_2) \leq \theta(\breve{t}_1)\theta(\breve{t}_2)$.

The functions $f(t) = e^{\sqrt{t}}$ and $g(t) = 5^{\sqrt{t}}$ (for all $t \in [0,\infty)$) are in Ψ. (E^*): for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 0$, there exists $k \in [0,1)$ such that $E(t_1, t_2, t_3, t_4) = k$.

Let Δ_E denote the set of all functions $E\colon \mathbb{R}^{+4} \longrightarrow \mathbb{R}^+$ satisfying the condition (E^*).

Example 3 (see [31]). If $E(t_1, t_2, t_3, t_4) = \lambda e^{\upsilon \min\{t_1, t_2, t_3, t_4\}}$, where $\upsilon \in \mathbb{R}^+$ and $\lambda \in [0,1)$, then $E \in \Delta_E$.

Example 4 (see [31]). If $E(t_1, t_2, t_3, t_4) = m \cdot \min\{t_1, t_2, t_3, t_4\} + \lambda$, where $m \in \mathbb{R}^+$ and $\lambda \in [0,1)$, then $E \in \Delta_E$.

The following class of mappings was considered in [32] to develop some fixed point theorems. We also use this class of mappings in our investigations:

$$\Phi = \left\{\Upsilon\colon \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R} \text{ and } \Upsilon(s,t) = \frac{1}{2}s - t\right\}. \tag{7}$$

Definition 2 (see [5]). Let (χ,ρ) be a metric space. The pair $\{f, g\}$ is said to be compatible if and only if

$$\lim_{n\longrightarrow\infty} \rho(f(g(r_n)), g(f(r_n))) = 0, \tag{8}$$

whenever $\{r_n\}$ is a sequence in χ so that

$$\lim_{n\longrightarrow\infty} f(r_n) = \lim_{n\longrightarrow\infty} g(r_n) = t, \quad \text{for some } t \in \chi. \tag{9}$$

The following lemma is important in the sequel.

Lemma 1 (see [5]). *Let (χ, ρ) be a metric space. If there exist two sequences $\{r_n\}$ and $\{u_n\}$ such that*

$$\lim_{n\longrightarrow\infty} \rho(r_n, u_n) = 0, \qquad \lim_{n\longrightarrow\infty} r_n = t, \quad \text{for some } t \in \chi, \tag{10}$$

then

$$\lim_{n\longrightarrow\infty} u_n = t. \tag{11}$$

3. Generalized Suzuki-Type Contractions and Fixed Point Results

Jleli and Samet [24, 25] have employed θ-contractions to obtain some fixed point theorems. Suzuki [20] extended Edelstein contraction to develop a new generalization of Banach contraction called Suzuki contraction. The contractions developed in [20, 24] were then followed by Liu et al. [33] to introduce a Suzuki-type θ-contraction. In this section, we introduce more general forms of Suzuki-type θ-contractions involving four self-mappings. We construct some conditions under which a sequence, whose terms satisfy generalized Suzuki-type θ-rational contractions or generalized Suzuki-type Ćirić θ-contractions, is a Cauchy sequence. We obtain two common fixed point theorems, which then are applied to show the existence of the solution of a system of fractional differential equations. We start with the following definition.

Definition 3. Let (χ, ρ) be a metric space, and $f, g, \widehat{S}, \breve{T}\colon \chi \longrightarrow \chi$ be four single-valued maps. These maps form a new generalized Suzuki-type (θ, E)-rational contraction if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu$, $E \in \Delta_E$, and $\Upsilon \in \Phi$,

$$\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0 \Rightarrow \theta(\rho(fr, gj)) \le [\theta(A(r, j))]^{E(G(r,j))}, \tag{12}$$

where

$$A(r, j) = \max\left\{\begin{array}{c} \rho(\widehat{S}r, \breve{T}j), \rho(fr, \widehat{S}r), \rho(gj, \breve{T}j), \dfrac{\rho(fr, \breve{T}j) + \rho(gj, \widehat{S}r)}{2}, \\ \dfrac{(1 + \rho(fr, \widehat{S}r))\rho(gj, \breve{T}j)}{1 + \rho(\widehat{S}r, \breve{T}j)} \end{array}\right\},$$
$$G(r, j) = \left\{\rho(fr, \widehat{S}r), \rho(gj, \breve{T}j), \rho(fr, \breve{T}j), \rho(gj, \widehat{S}r)\right\}. \tag{13}$$

Our first result is as follows.

Theorem 2. *Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \breve{T}\colon \chi \longrightarrow \chi$ be four single-valued maps. Suppose that the mappings $f, g, \widehat{S}, \breve{T}$ form a new generalized Suzuki-type (θ, E)-rational contraction with $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \breve{T})$ are compatible pair and $\breve{T}$ and $\widehat{S}$ are continuous on (χ, ρ), then $f, g, \widehat{S}$, and $\breve{T}$ have a unique common fixed point in χ.*

Proof. Let $r_0 \in \chi$ be an arbitrary point. As $f(\chi) \subseteq \breve{T}(\chi)$, there exists $r_0 \in \chi$ such that

$$f(r_0) = \breve{T}(r_1). \tag{14}$$

Since $g(r_1) \in \widehat{S}(\chi)$, we can choose $r_2 \in \chi$ such that $g(r_1) = \widehat{S}(r_2)$. In general r_{2n+1} and r_{2n+2} are chosen in χ such that

$$f(r_{2n}) = \breve{T}(r_{2n+1}), \qquad g(r_{2n+1}) = \widehat{S}(r_{2n+2}). \tag{15}$$

We construct a sequence $\{\mathfrak{R}_n\}$ in χ such that

$$\mathfrak{R}_{2n} = f(r_{2n}) = \breve{T}(r_{2n+1}), \qquad \mathfrak{R}_{2n+1} = g(r_{2n+1}) = \widehat{S}(r_{2n+2}), \tag{16}$$

for $n = 0, 1, 2, \ldots$. Assume that $\rho(f(r_{2n}), g(r_{2n+1})) > 0$ and since

$$\frac{1}{2}\rho\Big(f(r_{2n}), \widehat{S}(r_{2n})\Big) = \frac{1}{2}\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) < \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) = \rho\Big(\widehat{S}(r_{2n}), \breve{T}(r_{2n+1})\Big). \tag{17}$$

We have

$$\Upsilon\Big(\rho\Big(f(r_{2n}), \widehat{S}(r_{2n})\Big), \rho\Big(\widehat{S}(r_{2n}), \breve{T}(r_{2n+1})\Big)\Big) < 0. \tag{18}$$

Hence, from the contractive condition (12),

$$1 < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) = \theta(\rho(f(r_{2n}), g(r_{2n+1}))) \le [\theta(A(r_{2n}, r_{2n+1}))]^{E(G(r_{2n}, r_{2n+1}))}. \tag{19}$$

For every $n \ge 1$, where

$$\begin{aligned} G(r_{2n}, r_{2n+1}) &= \left\{ \begin{array}{c} \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ 0, \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \end{array} \right\}. \end{aligned} \tag{20}$$

As $\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) \cdot \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}) \cdot \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}) \cdot 0 = 0$, so by (E^*), there exists $z \in [0,1)$ such that

$$\begin{aligned} E(G(r_{2n}, r_{2n+1})) = E(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), 0, \rho \\ \cdot (\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})) = z, \end{aligned} \tag{21}$$

$$\begin{aligned} A(r_{2n}, r_{2n+1}) &= \max \left\{ \begin{array}{c} \rho\left(\widehat{S}r_{2n}, \breve{T}r_{2n+1}\right), \rho\left(fr_{2n}, \widehat{S}r_{2n}\right), \rho\left(gr_{2n+1}, \breve{T}r_{2n+1}\right), \\ \dfrac{\rho\left(fr_{2n}, \breve{T}r_{2n+1}\right) + \rho\left(gr_{2n+1}, \widehat{S}r_{2n}\right)}{2}, \dfrac{\left(1 + \rho\left(fr_{2n}, \widehat{S}r_{2n}\right)\right)\rho\left(gr_{2n+1}, \breve{T}r_{2n+1}\right)}{1 + \rho\left(\widehat{S}r_{2n}, \breve{T}r_{2n+1}\right)}, \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \\ \dfrac{\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2}, \dfrac{(1 + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}))\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})}{1 + \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n})}, \end{array} \right\} \\ &= \max \left\{ \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}), \frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2} \right\}. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} \frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1})}{2} &\leq \frac{\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1})}{2} + \frac{\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})}{2} \\ &\leq \max\{\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})\}. \end{aligned} \tag{23}$$

one writes

$$A(r_{2n}, r_{2n+1}) = \max\{\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}), \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n})\}. \tag{24}$$

If, for some n, $A(r_{2n}, r_{2n+1}) = \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})$, then from (19), (21), and (24), we have

$$\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^z < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})), \tag{25}$$

a contradiction. Therefore,

$$\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^z, \quad \text{for } n \geq 1. \tag{26}$$

Thus, for all $n \in \mathbb{N}$, we have

$$\theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^z, \quad \text{for } n \geq 1. \tag{27}$$

It implies that

$$\begin{aligned} 1 &< \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) \\ &\leq [\theta(\rho(\mathfrak{R}_{n-1}, \mathfrak{R}_n))]^{z^2} \\ &\vdots \\ &\leq [\theta(\rho(\mathfrak{R}_0, \mathfrak{R}_1))]^{z^n}, \end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ in (28) and using the fact that $\theta \in \mu$, we have

$$\lim_{n \longrightarrow \infty} \theta(\rho(\mathfrak{R}_n, \mathfrak{R}_{n+1})) = 1. \tag{29}$$

By $(\Theta 2)$, we obtain

$$\lim_{n \longrightarrow \infty} \rho(\mathfrak{R}_n, \mathfrak{R}_{n+1}) = 0. \tag{30}$$

Now, we will show that $\{\mathfrak{R}_n\}$ is a Cauchy sequence. We suppose on the contrary that $\{\mathfrak{R}_n\}$ is not a Cauchy sequence; then, there exist $\varepsilon > 0$ and subsequences $\{n(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$ of natural numbers such that, for $m(k) > n(k) > k$, we have $\rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}) \geq \varepsilon$. Then, $\rho(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1}) < \varepsilon$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} \varepsilon &\leq \rho\left(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}\right) \leq \rho\left(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)-1}\right) + \rho\left(\mathfrak{R}_{m(k)-1}, \mathfrak{R}_{m(k)}\right) \\ &< \varepsilon + \rho\left(\mathfrak{R}_{m(k)-1}, \mathfrak{R}_{m(k)}\right). \end{aligned} \tag{31}$$

By taking the upper limit as $k \longrightarrow \infty$ in (31) and using (30), we obtain

$$\lim_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)}\right)=\varepsilon. \tag{32}$$

Using the triangular inequality, we have

$$\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)}\right)\le\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{n(k)+1}\right)+\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right), \tag{33}$$

$$\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right)\le\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{n(k)}\right)+\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)}\right). \tag{34}$$

By taking upper limit as $k\longrightarrow\infty$ in (33) and (34) and applying (30) and (32),

$$\varepsilon\le\lim_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right)\le\varepsilon. \tag{35}$$

Thus,

$$\lim_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right)=\varepsilon. \tag{36}$$

Similarly, we can obtain

$$\lim_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)+1}\right)=\lim_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)+1}\right)=\varepsilon. \tag{37}$$

Assume that $\rho(f(r_{n(k)+1}),g(r_{m(k)+1}))>0$, and since

$$\begin{aligned}\frac{1}{2}\rho\left(f\left(r_{n(k)}\right),\widehat{S}\left(r_{m(k)}\right)\right)&=\frac{1}{2}\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)-1}\right)\\&<\rho\left(\mathfrak{R}_{n(k)},\mathfrak{R}_{m(k)-1}\right)\\&=\rho\left(\widehat{S}\left(r_{m(k)}\right),\breve{T}\left(r_{n(k)+1}\right)\right),\end{aligned} \tag{38}$$

we obatin

$$\Upsilon\left(\rho\left(f\left(r_{n(k)}\right),\widehat{S}\left(r_{m(k)}\right)\right),\rho\left(\widehat{S}\left(r_{n(k)}\right),\breve{T}\left(r_{m(k)+1}\right)\right)\right)<0. \tag{39}$$

Hence, from (12),

$$\begin{aligned}\theta\left(\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)+1}\right)\right)&=\theta\left(\rho\left(f\left(r_{n(k)+1}\right),g\left(r_{m(k)+1}\right)\right)\right)\\&\le\left[\theta\left(A\left(r_{n(k)},r_{m(k)}\right)\right)\right]^{E\left(G\left(r_{n(k)},r_{m(k)}\right)\right)},\end{aligned} \tag{40}$$

where

$$G\left(r_{n(k)},r_{m(k)}\right)=\left\{\begin{array}{l}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{n(k)}\right),\rho\left(\mathfrak{R}_{m(k)+1},\mathfrak{R}_{m(k)}\right),\\\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right),\rho\left(\mathfrak{R}_{m(k)+1},\mathfrak{R}_{n(k)}\right)\end{array}\right\}. \tag{41}$$

Taking limit at $k\longrightarrow\infty$ in (41), we obtain

$$\lim_{k\longrightarrow\infty}G\left(r_{n(k)},r_{m(k)}\right)=\left\{\begin{array}{l}\lim\limits_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{n(k)}\right),\lim\limits_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{m(k)+1},\mathfrak{R}_{m(k)}\right),\\\lim\limits_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{n(k)+1},\mathfrak{R}_{m(k)}\right),\lim\limits_{k\longrightarrow\infty}\rho\left(\mathfrak{R}_{m(k)+1},\mathfrak{R}_{n(k)}\right)\end{array}\right\}=\{0,0,\varepsilon,\varepsilon\}. \tag{42}$$

By E^*, there exists $z\in[0,1)$ such that $E(0,0,\varepsilon,\varepsilon)=z$. Using the continuity of E and (41),

$$E\left(\lim_{k\longrightarrow\infty}G\left(r_{n(k)},r_{m(k)}\right)\right)\le z. \tag{43}$$

Also,

$$A\left(r_{n(k)},r_{m(k)}\right)=\max\left\{\begin{array}{c}\rho\left(\widehat{S}\left(r_{n(k)+1}\right),\breve{T}\left(r_{m(k)+1}\right)\right),\rho\left(f\left(r_{n(k)+1}\right),\widehat{S}\left(r_{n(k)+1}\right)\right),\\[2ex]\rho\left(g\left(r_{m(k)+1}\right),\breve{T}\left(r_{m(k)+1}\right)\right),\\[2ex]\dfrac{\rho\left(f\left(r_{n(k)+1}\right),\breve{T}\left(r_{m(k)+1}\right)\right)+\rho\left(g\left(r_{m(k)+1}\right),\widehat{S}\left(r_{n(k)+1}\right)\right)}{2},\\[2ex]\dfrac{\left(1+\rho\left(f\left(r_{n(k)+1}\right),\widehat{S}\left(r_{n(k)+1}\right)\right)\right)\rho\left(g\left(r_{m(k)+1}\right),\breve{T}\left(r_{m(k)+1}\right)\right)}{1+\rho\left(\widehat{S}\left(r_{n(k)+1}\right),\breve{T}\left(r_{m(k)+1}\right)\right)}\end{array}\right\},$$

$$= \max \left\{ \begin{array}{c} \rho\left(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}\right), \rho\left(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}\right), \\ \rho\left(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)}\right), \\ \dfrac{\rho\left(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)}\right) + \rho\left(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{n(k)}\right)}{2}, \\ \dfrac{\left(1 + \rho\left(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{n(k)}\right)\right)\rho\left(\mathfrak{R}_{m(k)+1}, \mathfrak{R}_{m(k)}\right)}{1 + \rho\left(\mathfrak{R}_{n(k)}, \mathfrak{R}_{m(k)}\right)} \end{array} \right\}. \tag{44}$$

Taking $k \longrightarrow \infty$ and using (30), (32), (36), and (37), we obtain

$$\lim_{k \longrightarrow \infty} A\left(r_{n(k)}, r_{m(k)}\right) \leq \varepsilon. \tag{45}$$

Thus, from $(\Theta' 3)$, (37), (40), (43), and (45), we have

$$\begin{aligned} \theta(\varepsilon) &= \theta\left(\lim_{k \longrightarrow \infty} \rho\left(\mathfrak{R}_{n(k)+1}, \mathfrak{R}_{m(k)+1}\right)\right) \\ &\leq \left[\theta\left(\lim_{k \longrightarrow \infty} A\left(r_{n(k)}, r_{m(k)}\right)\right)\right]^{E\left(\lim_{k \longrightarrow \infty} G\left(r_{n(k)}, r_{m(k)}\right)\right)} \\ &\leq [\theta(\varepsilon)]^{z} < \theta(\varepsilon), \end{aligned} \tag{46}$$

which is a contradiction. Thus, $\{\mathfrak{R}_n\}$ is a Cauchy sequence. Since χ is a complete metric space, there exists $r^* \in \chi$ such that $\lim_{n \longrightarrow \infty} \rho(\mathfrak{R}_n, r^*) = 0$ and

$$\begin{aligned} \lim_{n \longrightarrow \infty} f\left(r_{2n}\right) &= \lim_{n \longrightarrow \infty} \breve{T}\left(r_{2n+1}\right) = \lim_{n \longrightarrow \infty} g\left(r_{2n+1}\right) \\ &= \lim_{n \longrightarrow \infty} \widehat{S}\left(r_{2n+2}\right) = r^*. \end{aligned} \tag{47}$$

Now, we shall prove that $\mathfrak{R}^*$ is a common fixed point of $f, g, \widehat{S}$, and $\breve{T}$. As $\widehat{S}$ is continuous, so

$$\lim_{n \longrightarrow \infty} \widehat{S}\left(f\left(r_{2n}\right)\right) = \widehat{S}\left(r^*\right) = \lim_{n \longrightarrow \infty} \widehat{S}\left(\widehat{S}\left(r_{2n+2}\right)\right). \tag{48}$$

Since $(f, \widehat{S})$ is a compatible pair,

$$\lim_{n \longrightarrow \infty} \rho\left(g\left(\breve{T}\left(r_{2n+1}\right)\right), \breve{T}\left(g\left(r_{2n+1}\right)\right)\right) = 0. \tag{49}$$

From Lemma 1, we have

$$\lim_{n \longrightarrow \infty} f\left(\widehat{S}\left(r_{2n}\right)\right) = \widehat{S}\left(r^*\right). \tag{50}$$

Put $r_1 = \widehat{S}(r_{2n})$ and $r_2 = r_{2n+1}$ in (12) and if $\rho(\widehat{S}(r^*), r^*) > 0$, we obtain

$$\frac{1}{2}\rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), \widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right)\right) < \rho\left(\widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right), \breve{T}\left(r_{2n+1}\right)\right). \tag{51}$$

Hence,

$$\Upsilon\left(\rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), \widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right)\right), \rho\left(\widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right), \breve{T}\left(r_{2n+1}\right)\right)\right) < 0, \tag{52}$$

and by (12), we obtain

$$\theta\left(\rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), g\left(r_{2n+1}\right)\right)\right) \leq \left[\theta\left(A\left(r_{2n}, r_{2n+1}\right)\right)\right]^{E\left(G\left(r_{2n}, r_{2n+1}\right)\right)}, \tag{53}$$

where

$$A\left(r_{2n}, r_{2n+1}\right) = \max \left\{ \begin{array}{c} \rho\left(\widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right), \breve{T}\left(r_{2n+1}\right)\right), \rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), \widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right)\right), \\ \rho\left(g\left(r_{2n+1}\right), \breve{T}\left(r_{2n+1}\right)\right), \\ \dfrac{\rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), \breve{T}\left(r_{2n+1}\right)\right) + \rho\left(g\left(r_{2n+1}\right), \widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right)\right)}{2}, \\ \dfrac{\left(1 + \rho\left(f\left(\widehat{S}\left(r_{2n}\right)\right), \widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right)\right)\right)\rho\left(g\left(r_{2n+1}\right), \breve{T}\left(r_{2n+1}\right)\right)}{1 + \rho\left(\widehat{S}\left(\widehat{S}\left(r_{2n}\right)\right), \breve{T}\left(r_{2n+1}\right)\right)} \end{array} \right\},$$

$$G(r_{2n}, r_{2n+1}) = \left\{ \begin{array}{l} \rho\left(f\left(\widehat{S}(r_{2n})\right), \widehat{S}\left(\widehat{S}(r_{2n})\right)\right),\ \rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right), \\ \rho\left(f\left(\widehat{S}(r_{2n})\right), \breve{T}(r_{2n+1})\right),\ \rho\left(g(r_{2n+1}), \widehat{S}\left(\widehat{S}(r_{2n})\right)\right) \end{array} \right\}. \tag{54}$$

Setting the upper limit in (53), we obtain

$$\theta\left(\rho\left(\widehat{S}(r^*), r^*\right)\right) \leq \left[\theta\left(\rho\left(\widehat{S}(r^*), r^*\right)\right)\right]^z < \theta\left(\rho\left(\widehat{S}(r^*), r^*\right)\right), \tag{55}$$

a contradiction. Thus, $\rho(\widehat{S}(r^*), r^*) = 0$ and $\widehat{S}(r^*) = r^*$. Again since $\breve{T}$ is continuous,

$$\lim_{n\longrightarrow\infty} \breve{T}(g(r_{2n+1})) = \breve{T}(r^*) = \lim_{n\longrightarrow\infty} \breve{T}\left(\breve{T}(r_{2n+1})\right). \tag{56}$$

Since $(g, \breve{T})$ is a compatible pair,

$$\lim_{n\longrightarrow\infty} \rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}(g(r_{2n+1}))\right) = 0. \tag{57}$$

From Lemma 1, we have

$$\lim_{n\longrightarrow\infty} g\left(\breve{T}(r_{2n+1})\right) = \breve{T}(r^*). \tag{58}$$

Put $r_1 = r_{2n}$ and $r_2 = \breve{T}(r_{2n+1})$ in (12) and if $\rho(r^*, \breve{T}(r^*)) > 0$, we obtain

$$\frac{1}{2}\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right) < \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right). \tag{59}$$

Therefore,

$$\Upsilon\left(\begin{array}{c} \rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right), \\ \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right) \end{array} \right) < 0. \tag{60}$$

By (12), one obtains

$$\theta\left(\rho\left(f(r_{2n}), g\left(\breve{T}(r_{2n+1})\right)\right)\right) \leq \left[\theta(A(r_{2n}, r_{2n+1}))\right]^{E(G(r_{2n}, r_{2n+1}))}, \tag{61}$$

where

$$A(r_{2n}, r_{2n+1}) = \max\left\{ \begin{array}{c} \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right), \rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right), \\ \rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right), \\ \dfrac{\rho\left(f(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right) + \rho\left(g\left(\breve{T}(r_{2n+1})\right), \widehat{S}(r_{2n})\right)}{2}, \\ \dfrac{\left(1 + \rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right)\right)\rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right)}{1 + \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right)} \end{array} \right\},$$

$$G(r_{2n}, r_{2n+1}) = \left\{ \begin{array}{l} \rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right), \rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right), \\ \rho\left(f(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right), \rho\left(g\left(\breve{T}(r_{2n+1})\right), \widehat{S}(r_{2n})\right) \end{array} \right\}. \tag{62}$$

Setting the upper limit, we obtain

$$\theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right) \leq \left[\theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right)\right]^z < \theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right), \tag{63}$$

a contradiction. Thus, $\rho(r^*, \breve{T}(r^*)) = 0$ and so $r^* = \breve{T}(r^*)$. Suppose $\rho(f(r^*), r^*) > 0$, we obtain

$$\frac{1}{2}\rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right) < \rho\left(f(r^*), \widehat{S}(r^*)\right). \tag{64}$$

Hence,

$$\Upsilon\left(\begin{array}{c} \rho\left(f(r^*), \widehat{S}(r^*)\right), \\ \rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right) \end{array} \right) < 0, \tag{65}$$

and by (12), we obtain

$$\theta(\rho(f(r^*), g(r_{2n+1}))) \leq [\theta(A(r^*, r_{2n+1}))]^{E(G(r^*, r_{2n+1}))}, \tag{66}$$

where

$$A(r^*, r_{2n+1}) = \max\left\{\begin{array}{c} \rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right), \rho\left(f(r^*), \widehat{S}(r^*)\right), \\ \rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right), \dfrac{\rho\left(f(r^*), \breve{T}(r_{2n+1})\right) + \rho\left(g(r_{2n+1}), \widehat{S}(r^*)\right)}{2}, \\ \dfrac{\left(1 + \rho\left(f(r^*), \widehat{S}(r^*)\right)\right)\rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right)}{1 + \rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right)} \end{array}\right\}, \tag{67}$$

$$G(r^*, r_{2n+1}) = \left\{\begin{array}{c} \rho\left(f(r^*), \widehat{S}(r^*)\right), \rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right), \\ \rho\left(f(r^*), \breve{T}(r_{2n+1})\right), \rho\left(g(r_{2n+1}), \widehat{S}(r^*)\right) \end{array}\right\}.$$

Taking the upper limit in (66), and as $r^* = \breve{T}(r^*) = \widehat{S}(r^*)$, so we obtain

$$\theta(\rho(f(r^*), r^*)) \leq [\theta(\rho(f(r^*), r^*))]^z < \theta(\rho(f(r^*), r^*)), \tag{68}$$

a contradiction. Thus, $\rho(f(r^*), r^*) = 0$ and $r^* = f(r^*)$. Finally, suppose $\rho(r^*, g(r^*)) > 0$, and as $r^* = \breve{T}(r^*) = \widehat{S}(r^*) = f(r^*)$, so, we obtain

$$\Upsilon(0, \rho(r^*, g(r^*))) < 0. \tag{69}$$

Thus, from (12)

$$\begin{aligned} \theta(\rho(r^*, g(r^*))) = \theta(\rho(f(r^*), g(r^*))) &\leq [\theta(\rho(r^*, g(r^*)))]^z \\ &< \theta(\rho(r^*, g(r^*))), \end{aligned} \tag{70}$$

a contradiction. Thus, $\rho(r^*, g(r^*)) = 0$ and $r^* = g(r^*)$. Therefore, r^* is a common of the four mappings $\widehat{S}, g, f$ and $\breve{T}$. Next, assume that $v^*$is another common of the four mappings $\widehat{S}, g, f$, and $\breve{T}$ such that $r^* \neq v^*$, one obtains

$$\Upsilon\left(0, \rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right)\right) < 0. \tag{71}$$

Then, by (12), we obtain

$$\theta(\rho(r^*, v^*)) = \theta(\rho(f(r^*), g(v^*))) \leq [\theta(A(r^*, v^*))]^{E(G(r^*, v^*))}, \tag{72}$$

where

$$A(r^*, v^*) = \max\left\{\begin{array}{c} \rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right), \rho\left(f(r^*), \widehat{S}(r^*)\right), \rho\left(g(v^*), \breve{T}(v^*)\right), \\ \dfrac{\rho\left(f(r^*), \breve{T}(v^*)\right) + \rho\left(g(v^*), \widehat{S}(r^*)\right)}{2}, \dfrac{\left(1 + \rho\left(f(r^*), \widehat{S}(r^*)\right)\right)\rho\left(g(v^*), \breve{T}(v^*)\right)}{1 + \rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right)} \end{array}\right\}, \tag{73}$$

$$G(r^*, v^*) = \left\{\begin{array}{c} \rho\left(f(r^*), \widehat{S}(r^*)\right), \rho\left(g(v^*), \breve{T}(v^*)\right), \\ \rho\left(f(r^*), \breve{T}(v^*)\right), \rho\left(g(v^*), \widehat{S}(r^*)\right) \end{array}\right\}.$$

It implies that

$$\theta(\rho(r^*, v^*)) \le [\theta(\rho(r^*, v^*))]^z < \theta(\rho(r^*, v^*)), \tag{74}$$

a contradiction. Thus, $r^* = v^*$. Therefore, r^* is the unique common fixed point of $\widehat{S}, g, f$, and $\breve{T}$. □

Corollary 1. *Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \breve{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that if, for all $r, j \in \chi$ with $\rho(fr, gj) > 0$ for some $\theta \in \mu$, $z \in [0, 1)$ and $\Upsilon \in \Phi$,*

$$\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0 \Longrightarrow \theta(\rho(fr, gj)) \le [\theta(A(r, j))]^z, \tag{75}$$

where

$$A(r, j) = \max\left\{\begin{array}{c}\rho(\widehat{S}r, \breve{T}j), \rho(fr, \widehat{S}r), \rho(gj, \breve{T}j), \dfrac{\rho(fr, \breve{T}j) + \rho(gj, \widehat{S}r)}{2}, \\[2ex] \dfrac{(1 + \rho(fr, \widehat{S}r))\rho(gj, \breve{T}j)}{1 + \rho(\widehat{S}r, \breve{T}j)}\end{array}\right\}, \tag{76}$$

with $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \breve{T})$ are compatible pairs and $\breve{T}$ and $\widehat{S}$ are continuous on (χ, ρ), then $f, g, \widehat{S}$, and $\breve{T}$ have a unique common fixed point in χ.

Example 5. Let $\chi = [0, 1]$ and define the function $\rho: \chi \times \chi \longrightarrow [0, +\infty)$ by $\rho(r, j) = |r - j|$. Clearly, (χ, ρ) is a complete metric space. Let $\theta(t) = e^t$, $t > 0$, and $\Upsilon(s, t) = (s/2) - t$, then $\theta \in \mu$ and $\Upsilon \in \Phi$. Define the mappings $\widehat{S}, g, f, \breve{T}: \chi \longrightarrow \chi$ by

$$\begin{aligned}\widehat{S}(r) &= \left(\frac{r}{3}\right)^8, \\ g(r) &= \left(\frac{r}{3}\right)^8, \\ f(r) &= \left(\frac{r}{3}\right)^{16}, \\ \breve{T}(r) &= \left(\frac{r}{3}\right)^4.\end{aligned} \tag{77}$$

Clearly, $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$, if $\{r_n\}$ is a sequence in χ such that

$$\lim_{n\longrightarrow\infty} f(r_n) = \lim_{n\longrightarrow\infty} \widehat{S}(r_n) = t, \quad \text{for some } t \in \chi, \tag{78}$$

then

$$\lim_{n\longrightarrow\infty} |f(r_n) - t| = \lim_{n\longrightarrow\infty} |\widehat{S}(k_n) - t| = 0, \tag{79}$$

and equivalently,

$$\lim_{n\longrightarrow\infty} \left|\left[\frac{r_n}{3}\right]^{16} - t\right| = \lim_{n\longrightarrow\infty} \left|\left[\frac{r_n}{3}\right]^8 - t\right| = 0. \tag{80}$$

Thus,

$$\lim_{n\longrightarrow\infty} \left|[r_n]^{16} - 3^{16}t\right| = \lim_{n\longrightarrow\infty} \left|[r_n]^8 - 3^8 t\right| = 0. \tag{81}$$

We conclude $t^{1/16} = t^{1/8}$ (by uniqueness of limit); hence, $t \in \{0, 1\}$. Using continuity of f and $\widehat{S}$, one obtains

$$\begin{aligned}\lim_{n\longrightarrow\infty} \rho\left(f\widehat{S}(r_n), \widehat{S}f(r_n)\right) &= \lim_{n\longrightarrow\infty} \left|f\widehat{S}(r_n) - \widehat{S}f(r_n)\right| \\ &= |f(t) - \widehat{S}(t)| = |0 - 0| = 0,\end{aligned} \tag{82}$$

for $t = 0 \in \chi$. Hence, the pair $(f, \widehat{S})$ is compatible. Similarly, the pair $(g, \breve{T})$ is compatible. Define $E: \mathbb{R}^4 \longrightarrow \mathbb{R}^+$ as $E(s_1, s_2, s_3, s_4) = (9/10)$. Next, for all $r, j \in \chi$ with $\rho(fr, gj) > 0$,

$$\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0, \tag{83}$$

and so

$$\theta(\rho(fr, gj)) \le [\theta(A(r, j))]^{E(G(r,j))}. \tag{84}$$

Hence, all hypotheses of Theorem 2 are satisfied. Thus, $f, g, \widehat{S}$, and $\breve{T}$ have a unique common fixed point, which is $r^* = 0$.

We can easily obtain the following results by chasing the proof of Theorem 2.

Corollary 2. *Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \breve{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu$, $\Upsilon \in \Phi$ and $E: \mathbb{R}^{+4} \longrightarrow \mathbb{R}^+$,*

$$\begin{aligned}\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0 \Longrightarrow \theta(\rho(fr, gj)) \\ \le [\theta(\max\left\{\rho(fr, \widehat{S}r), \rho(gj, \breve{T}j)\right\})]^{E(G(r,j))},\end{aligned} \tag{85}$$

where

$$G(r, j) = \left\{\rho(fr, \widehat{S}r), \rho(gj, \breve{T}j), \rho(fr, \breve{T}j), \rho(gj, \widehat{S}r)\right\}, \tag{86}$$

with $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \breve{T})$ are compatible pairs, and $\breve{T}$ and $\widehat{S}$ are continuous on (χ, ρ), then $f, g, \widehat{S}$, and $\breve{T}$ have a unique common fixed point in χ.

The following corollary generalizes the result introduced by Sehgal [34].

Corollary 3. *Let (χ, ρ) be a complete metric space, and $f, g, \widehat{S}, \breve{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that if, for all $r, j \in \chi$, with $\rho(fr, gj) > 0$, for some $\theta \in \mu$, $\Upsilon \in \Phi$ and $E: \mathbb{R}^{+4} \longrightarrow \mathbb{R}^+$, $\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0$ implies that*

$$\theta(\rho(fr, gj)) \leq [\theta(\max\left\{\rho(\widehat{S}r, \breve{T}j), \rho(fr, \widehat{S}r), \rho(gj, \breve{T}j)\right\})]^{E(G(r,j))}, \tag{87}$$

where

$$G(r, j) = \left\{\rho(fr, \widehat{S}r), \rho(gj, \breve{T}j), \rho(fr, \breve{T}j), \rho(gj, \widehat{S}r)\right\}, \tag{88}$$

with $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \breve{T})$ are compatible pairs, and $\breve{T}$ and $\widehat{S}$ are continuous on (χ, ρ), then f, g, $\widehat{S}$, and $\breve{T}$ have a unique common fixed point in χ.

4. On Suzuki-Type Ćirić JS-Contractions

We start this section with the following definition.

Definition 4. Let (χ, ρ) be a metric space and $f, g, \widehat{S}, \breve{T}: \chi \longrightarrow \chi$ be four single-valued maps. These maps form a new generalized Suzuki-type Ćirić JS-contraction if, for all $r, j \in \chi$, for some $\theta \in \Psi$, $\Upsilon \in \Phi$, $\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0$ implies that

$$\begin{aligned}\theta(\rho(fr, gj)) \leq{}& [\theta(\rho(\widehat{S}r, \breve{T}j))]^a \cdot [\theta(\rho(fr, \widehat{S}r))]^b \\ &\cdot [\theta(\rho(gj, \breve{T}j))]^c \cdot [\theta(\rho(fr, \breve{T}j) + \rho(gj, \widehat{S}r))]^d,\end{aligned} \tag{89}$$

where $a, b, c, d \geq 0$ with $a + b + c + 2d < 1$.

Our second result is as follows.

Theorem 3. *Let (χ, ρ) be a complete metric space and $f, g, \widehat{S}, \breve{T}: \chi \longrightarrow \chi$ be four single-valued maps. Suppose that the mappings $f, g, \widehat{S}, \breve{T}$ form a generalized Suzuki-type Ćirić JS-contraction with $f(\chi) \subseteq \breve{T}(\chi)$ and $g(\chi) \subseteq \widehat{S}(\chi)$. If $(f, \widehat{S})$ and $(g, \breve{T})$ are compatible pairs and $\breve{T}$ and $\widehat{S}$ are continuous on (χ, ρ), then f, g, $\widehat{S}$, and $\breve{T}$ have a unique common fixed point in χ.*

Proof. Let $r_0 \in \chi$ be an arbitrary point. As $f(\chi) \subseteq \breve{T}(\chi)$, there exists $r_0 \in \chi$ such that

$$f(r_0) = \breve{T}(r_1). \tag{90}$$

Since $g(r_1) \in \widehat{S}(\chi)$, we can choose $r_2 \in \chi$ such that $g(r_1) = \widehat{S}(r_2)$. In general, r_{2n+1} and r_{2n+2} are chosen in χ such that

$$\begin{aligned}f(r_{2n}) &= \breve{T}(r_{2n+1}), \\ g(r_{2n+1}) &= \widehat{S}(r_{2n+2}).\end{aligned} \tag{91}$$

We construct a sequence $\{\mathfrak{R}_n\}$ in χ such that

$$\begin{aligned}\mathfrak{R}_{2n} &= f(r_{2n}) = \breve{T}(r_{2n+1}), \\ \mathfrak{R}_{2n+1} &= g(r_{2n+1}) = \widehat{S}(r_{2n+2}),\end{aligned} \tag{92}$$

for $n = 0, 1, 2, \ldots$. Since

$$\begin{aligned}\frac{1}{2}\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right) &= \frac{1}{2}\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}) < \rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) \\ &= \rho\left(\widehat{S}(r_{2n}), \breve{T}(r_{2n+1})\right),\end{aligned} \tag{93}$$

we have

$$\Upsilon\left(\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right), \rho\left(\widehat{S}(r_{2n}), \breve{T}(r_{2n+1})\right)\right) < 0. \tag{94}$$

Then, from (89),

$$\begin{aligned}\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) = \theta(\rho(f(r_{2n}), g(r_{2n+1}))) \leq{}& \left[\theta\left(\rho\left(\widehat{S}(r_{2n}), \breve{T}(r_{2n+1})\right)\right)\right]^a \cdot \left[\theta\left(\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right)\right)\right]^b \\ &\cdot \left[\theta\left(\rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right)\right)\right]^c \cdot \left[\theta\left(\rho\left(f(r_{2n}), \breve{T}(r_{2n+1})\right) + \rho\left(g(r_{2n+1}), \widehat{S}(r_{2n})\right)\right)\right]^d \\ ={}& [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^a \cdot [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n-1}))]^b \cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^c \cdot [\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n-1}))]^d \\ ={}& [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b} \cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^c \cdot [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^d.\end{aligned} \tag{95}$$

By (Ψ_5), we have, for every $n \in \mathbb{N}$,

$$\begin{aligned}\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}) + \rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq{}& \theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n})) \\ &\cdot \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})).\end{aligned} \tag{96}$$

Hence, (95) becomes

$$\begin{aligned}1 < \theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1})) \leq{}& [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b+d} \\ &\cdot [\theta(\rho(\mathfrak{R}_{2n+1}, \mathfrak{R}_{2n}))]^{c+d}.\end{aligned} \tag{97}$$

It implies that

$$[\theta(\rho(\mathfrak{R}_{2n}, \mathfrak{R}_{2n+1}))]^{1-c-d} \leq [\theta(\rho(\mathfrak{R}_{2n-1}, \mathfrak{R}_{2n}))]^{a+b+d}. \tag{98}$$

Therefore,

$$\theta(\rho(\mathfrak{R}_{2n},\mathfrak{R}_{2n+1}))\leq[\theta(\rho(\mathfrak{R}_{2n-1},\mathfrak{R}_{2n}))]^{(a+b+d)/(1-c-d)},\tag{99}$$

and so

$$\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))\leq[\theta(\rho(\mathfrak{R}_{n-1},\mathfrak{R}_{n}))]^{(a+b+d)/(1-c-d)}.\tag{100}$$

It implies that

$$\begin{aligned}1&<\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))\\&\leq[\theta(\rho(\mathfrak{R}_{n-1},\mathfrak{R}_{n}))]^{((a+b+d)/(1-c-d))^2}\\&\ \vdots\\&\leq[\theta(\rho(\mathfrak{R}_{0},\mathfrak{R}_{1}))]^{((a+b+d)/(1-c-d))^n},\end{aligned}\tag{101}$$

for all $n\in\mathbb{N}$. Taking the limit as $n\longrightarrow\infty$ in (101) and knowing that $\theta\in\Psi$, we have

$$\lim_{n\longrightarrow\infty}\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))=1.\tag{102}$$

By (Ψ_3), we obtain

$$\lim_{n\longrightarrow\infty}\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})=0.\tag{103}$$

From condition (Ψ_4), there exist $q\in(0,1)$ and $\ell\in(0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1}{[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q}=\ell.\tag{104}$$

Suppose that $\ell<\infty$. Let $V=(\ell/2)>0$. From the definition of the limit, there exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1}{[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q}-\ell\right|\leq V,\quad\text{for all } n\geq n_0.\tag{105}$$

This implies

$$\frac{\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1}{[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q}\geq\ell-V=V,\quad\text{for all } n\geq n_0.\tag{106}$$

Then,

$$n[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q\leq \mathrm{Pn}[\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1],\quad\text{for all } n\geq n_0,\tag{107}$$

where $P=(1/V)$. Suppose now that $\ell=\infty$. Let $V>0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0\geq 1$ such that

$$\frac{\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1}{[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q}\geq V,\quad\text{for all } n\geq n_0,\tag{108}$$

which implies

$$n[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q\leq Pn[\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1],\quad n\geq n_0,\tag{109}$$

where $P=(1/V)$. Thus, in all cases, there exist $P>0$ and $n_0\geq 1$ such that

$$n[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q\leq Pn[\theta(\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1}))-1],\quad n\geq n_0.\tag{110}$$

By using (101), we obtain

$$n[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q\leq Pn\Big([\theta(\rho(\mathfrak{R}_{0},\mathfrak{R}_{1}))]^{((a+b+d)/(1-c-d))^n}-1\Big),\quad\text{for all } n\geq n_0.\tag{111}$$

Setting $n\longrightarrow\infty$ in inequality (111), we obtain

$$\lim_{n\longrightarrow\infty}n[\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})]^q=0.\tag{112}$$

Thus, there exists $n_1\in\mathbb{N}$ such that

$$\rho(\mathfrak{R}_{n},\mathfrak{R}_{n+1})\leq\frac{1}{n^{(1/q)}},\quad\text{for all } n\geq n_1.\tag{113}$$

To prove $\{\mathfrak{R}_n\}$ is a Cauchy sequence, we use (113) and for $m>n\geq n_1$,

$$\rho(\mathfrak{R}_{n},\mathfrak{R}_{m})\leq\sum_{i=n}^{m-1}\rho(\mathfrak{R}_{i},\mathfrak{R}_{i+1})\leq\sum_{i=n}^{\infty}\rho(\mathfrak{R}_{i},\mathfrak{R}_{i+1})\leq\sum_{i=n}^{\infty}\frac{1}{i^{(1/q)}}.\tag{114}$$

The convergence of the series $\sum_{i=n}^{\infty}1/i^{(1/q)}$ entails $\lim_{n,m\longrightarrow\infty}\rho(\mathfrak{R}_{n},\mathfrak{R}_{m})=0$. Thus, $\{\mathfrak{R}_n\}$ is a Cauchy sequence. Since χ is a complete metric space, there exists $r^*\in\chi$ such that $\lim_{n\longrightarrow\infty}d(\mathfrak{R}_n,r^*)=0$ and

$$\begin{aligned}\lim_{n\longrightarrow\infty}f(r_{2n})&=\lim_{n\longrightarrow\infty}\breve{T}(r_{2n+1})=\lim_{n\longrightarrow\infty}g(r_{2n+1})\\&=\lim_{n\longrightarrow\infty}\widehat{S}(r_{2n+2})=r^*.\end{aligned}\tag{115}$$

Now, we shall prove that $\mathfrak{R}^*$ is a common fixed point of $f,g,\widehat{S}$, and $\breve{T}$. As $\widehat{S}$ is continuous, so

$$\lim_{n\longrightarrow\infty}\widehat{S}(f(r_{2n}))=\widehat{S}(r^*)=\lim_{n\longrightarrow\infty}\widehat{S}\big(\widehat{S}(r_{2n+2})\big).\tag{116}$$

Since $(f,\widehat{S})$ is a compatible pair,

$$\lim_{n\longrightarrow\infty}\rho\big(f\big(\widehat{S}(r_{2n})\big),\widehat{S}(f(r_{2n}))\big)=0.\tag{117}$$

From Lemma 1, we have

$$\lim_{n\longrightarrow\infty}f\big(\widehat{S}(r_{2n})\big)=\widehat{S}(r^*).\tag{118}$$

Put $r=\widehat{S}(r_{2n})$ and $j=r_{2n+1}$ in (89) and if $\rho(\widehat{S}(r^*),r^*)>0$, we obtain

$$\frac{1}{2}\rho\big(f\big(\widehat{S}(r_{2n})\big),\widehat{S}\big(\widehat{S}(r_{2n})\big)\big)<\rho\Big(\widehat{S}\big(\widehat{S}(r_{2n})\big),\breve{T}(r_{2n+1})\Big).\tag{119}$$

Hence,

$$\Upsilon\begin{pmatrix}\rho\big(f\big(\widehat{S}(r_{2n})\big),\widehat{S}\big(\widehat{S}(r_{2n})\big)\big),\\ \rho\Big(\widehat{S}\big(\widehat{S}(r_{2n})\big),\breve{T}(r_{2n+1})\Big)\end{pmatrix}<0.\tag{120}$$

Then, from (89), we obtain

$$\begin{aligned}\theta\left(\rho\left(f\left(\widehat{S}(r_{2n})\right), g(r_{2n+1})\right)\right) \leq & \left[\theta\left(\rho\left(\widehat{S}\left(\widehat{S}(r_{2n})\right), \breve{T}(r_{2n+1})\right)\right)\right]^a \\ & \cdot \left[\theta\left(\rho\left(f\left(\widehat{S}(r_{2n})\right), \widehat{S}\left(\widehat{S}(r_{2n})\right)\right)\right)\right]^b \\ & \cdot \left[\theta\left(\rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right)\right)\right]^c \\ & \cdot \left[\theta\left(\rho\left(f\left(\widehat{S}(r_{2n})\right), \breve{T}(r_{2n+1})\right) + \rho\left(g(r_{2n+1}), \widehat{S}\left(\widehat{S}(r_{2n})\right)\right)\right)\right]^d.\end{aligned} \tag{121}$$

Setting the upper limit, we obtain

$$\begin{aligned}\theta\left(\rho\left(\widehat{S}(r^*), r^*\right)\right) \leq & \left[\theta\left(\rho\left(\widehat{S}r^*, r^*\right)\right)\right]^a \\ & \cdot \left[\theta\left(\rho\left(\widehat{S}r^*, r^*\right) + \rho\left(r^*, \widehat{S}r^*\right)\right)\right]^d \\ \leq & \left[\theta\left(\rho\left(\widehat{S}r^*, r^*\right)\right)\right]^{a+2d} < \theta\left(\rho\left(\widehat{S}(r^*), r^*\right)\right),\end{aligned} \tag{122}$$

a contradiction. Thus, $\rho(\widehat{S}(r^*), r^*) = 0$, and so $\widehat{S}(r^*) = r^*$. Again, since $\breve{T}$ is continuous,

$$\lim_{n\longrightarrow\infty} \breve{T}(g(r_{2n+1})) = \breve{T}(r^*) = \lim_{n\longrightarrow\infty} \breve{T}\left(\breve{T}(r_{2n+1})\right). \tag{123}$$

Since $(g, \breve{T})$ is a compatible pair,

$$\lim_{n\longrightarrow\infty} \rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}(g(r_{2n+1}))\right) = 0. \tag{124}$$

From Lemma 1, we have

$$\lim_{n\longrightarrow\infty} g\left(\breve{T}(r_{2n+1})\right) = \breve{T}(r^*). \tag{125}$$

Put $r_1 = r_{2n}$ and $r_2 = \breve{T}(r_{2n+1})$ in (89) and if $\rho(r^*, \breve{T}(r^*)) > 0$, we obtain

$$\frac{1}{2}\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right) < \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right). \tag{126}$$

Therefore,

$$\Upsilon\begin{pmatrix}\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right), \\ \rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right)\end{pmatrix} < 0, \tag{127}$$

and from (89), one obtains

$$\begin{aligned}\theta\left(\rho\left(f(r_{2n}), g\left(\breve{T}(r_{2n+1})\right)\right)\right) \leq & \left[\theta\left(\rho\left(\widehat{S}(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right)\right)\right]^a \cdot \left[\theta\left(\rho\left(f(r_{2n}), \widehat{S}(r_{2n})\right)\right)\right]^b \\ & \cdot \left[\theta\left(\rho\left(g\left(\breve{T}(r_{2n+1})\right), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right)\right)\right]^c \\ & \cdot \left[\theta\left(\rho\left(f(r_{2n}), \breve{T}\left(\breve{T}(r_{2n+1})\right)\right) + \rho\left(g\left(\breve{T}(r_{2n+1})\right), \widehat{S}(r_{2n})\right)\right)\right]^d.\end{aligned} \tag{128}$$

Passing to the upper limit in (128), we obtain

$$\begin{aligned}\theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right) \leq & \left[\theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right)\right]^{a+2d} \\ & < \theta\left(\rho\left(r^*, \breve{T}(r^*)\right)\right),\end{aligned} \tag{129}$$

a contradiction. Thus, $\rho(r^*, \breve{T}(r^*)) = 0$ and hence $r^* = \breve{T}(r^*)$. Suppose that $\rho(f(r^*), r^*) > 0$, we obtain

$$\frac{1}{2}\rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right) < \rho\left(f(r^*), \widehat{S}(r^*)\right), \quad \text{for all } n \in \mathbb{N}, \tag{130}$$

and so

$$\Upsilon\begin{pmatrix}\rho\left(f(r^*), \widehat{S}(r^*)\right), \\ \rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right)\end{pmatrix} < 0. \tag{131}$$

By (89), we obtain

$$\begin{aligned}\theta(\rho(f(r^*), g(r_{2n+1}))) \leq & \left[\theta\left(\rho\left(\widehat{S}(r^*), \breve{T}(r_{2n+1})\right)\right)\right]^a \cdot \left[\theta\left(\rho\left(f(r^*), \widehat{S}(r^*)\right)\right)\right]^b \\ & \cdot \left[\theta\left(\rho\left(g(r_{2n+1}), \breve{T}(r_{2n+1})\right)\right)\right]^c \cdot \left[\theta\left(\rho\left(f(r^*), \breve{T}(r_{2n+1})\right) + \rho\left(g(r_{2n+1}), \widehat{S}(r^*)\right)\right)\right]^d.\end{aligned} \tag{132}$$

Taking the upper limit in (132) and as $r^* = \breve{T}(r^*) = \widehat{S}(r^*)$, we obtain

$$\theta(\rho(f(r^*), r^*)) \leq [\theta(\rho(f(r^*), r^*))]^{b+d} < \theta(\rho(f(r^*), r^*)), \tag{133}$$

a contradiction. Thus, $\rho(f(r^*), r^*) = 0$ and hence $r^* = f(r^*)$. Finally, suppose $\rho(r^*, g(r^*)) > 0$ and as $r^* = \breve{T}(r^*) = \widehat{S}(r^*) = f(r^*)$, we have

$$\Upsilon(0, \rho(r^*, g(r^*))) < 0. \tag{134}$$

From (89), we deduce

$$\theta(\rho(r^*, g(r^*))) = \theta(\rho(f(r^*), g(r^*))) < \theta(\rho(r^*, g(r^*))), \tag{135}$$

a contradiction. Thus, $\rho(r^*, g(r^*)) = 0$ and $r^* = g(r^*)$. Therefore, r^* is a common of the four mappings $\widehat{S}, g, f$, and $\breve{T}$. Next, assume that v^* is another common of the four mappings $\widehat{S}, g, f$, and $\breve{T}$ such that $r^* \neq v^*$, one obtains

$$\Upsilon\left(0, \rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right)\right) < 0. \tag{136}$$

Then, by (89),

$$\begin{aligned}\theta(\rho(r^*, v^*)) = \theta\left(\rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right)\right) &\leq \left[\theta\left(\rho\left(\widehat{S}(r^*), \breve{T}(v^*)\right)\right)\right]^a \cdot \left[\theta\left(\rho\left(f(r^*), \widehat{S}(r^*)\right)\right)\right]^b \\ &\cdot \left[\theta\left(\rho\left(g(v^*), \breve{T}(v^*)\right)\right)\right]^c \cdot \left[\theta\left(\rho\left(f(r^*), \breve{T}(v^*)\right) + \rho\left(g(v^*), \widehat{S}(r^*)\right)\right)\right]^d.\end{aligned} \tag{137}$$

It implies that

$$\theta(\rho(r^*, v^*)) \leq [\theta(\rho(r^*, v^*))]^{a+2d} < \theta(\rho(r^*, v^*)), \tag{138}$$

a contradiction. Thus, $r^* = v^*$, that is, r^* is the unique common fixed point of $\widehat{S}, g, f$, and $\breve{T}$. □

Example 6. Let $\chi = [0, \infty)$ and define the function ρ: $\chi \times \chi \longrightarrow [0, +\infty)$ by $\rho(r, j) = |r - j|$. Clearly, (χ, ρ) is a complete metric space. Let $\theta(t) = e^{\sqrt{t}}$, and $\Upsilon(s, t) = (s/2) - t$, then $\theta \in \Psi$ and $\Upsilon \in \Phi$. Define the mappings $\widehat{S}, g, f, \breve{T}$: $\chi \longrightarrow \chi$ by

$$\begin{aligned}\widehat{S}(r) &= e^{7r} - 1, \\ g(r) &= \ln\left(1 + \frac{r}{6}\right), \\ f(r) &= \ln\left(1 + \frac{r}{6}\right), \\ \breve{T}(r) &= e^{6r} - 1.\end{aligned} \tag{139}$$

Clearly, $f(\chi) = \breve{T}(\chi) = g(\chi) = \widehat{S}(\chi)$ if $\{r_n\}$ is a sequence in χ such that

$$\lim_{n \longrightarrow \infty} f(r_n) = \lim_{n \longrightarrow \infty} \widehat{S}(r_n) = t, \quad \text{for some } t \in \chi. \tag{140}$$

Then,

$$\lim_{n \longrightarrow \infty} \left|f(r_n) - t\right| = \lim_{n \longrightarrow \infty} \left|\widehat{S}(r_n) - t\right| = 0, \tag{141}$$

and equivalently,

$$\lim_{n \longrightarrow \infty} \left|\ln\left(1 + \frac{r_n}{6}\right) - t\right| = \lim_{n \longrightarrow \infty} \left|e^{7r_n} - 1 - t\right| = 0. \tag{142}$$

Thus,

$$\lim_{n \longrightarrow \infty} \left|r_n - \left(6e^t - 6\right)\right| = \lim_{n \longrightarrow \infty} \left|r_n - \frac{\ln(1+t)}{7}\right| = 0. \tag{143}$$

It gives that $6e^t - 6 = (\ln(1+t)/7)$ (by uniqueness of limit); hence, $t = 0$. Using continuity of f and $\widehat{S}$, one obtains

$$\begin{aligned}\lim_{n \longrightarrow \infty} \rho\left(f\widehat{S}(r_n), \widehat{S}f(r_n)\right) &= \lim_{n \longrightarrow \infty} \left|f\widehat{S}(r_n) - \widehat{S}f(r_n)\right| \\ &= |f(t) - \widehat{S}(t)| = |0 - 0| = 0,\end{aligned} \tag{144}$$

for $t = 0 \in \chi$. Hence, the pair $(f, \widehat{S})$ is compatible. Similarly, the pair $(g, \breve{T})$ is compatible. Next, for all $r, j \in \chi$ with

$$\Upsilon(\rho(fr, \widehat{S}r), \rho(\widehat{S}r, \breve{T}j)) < 0. \tag{145}$$

So, for $a = b = c = 0.1$ and $d = 0.3$, we have

$$\begin{aligned}\theta(\rho(fr, gj)) &\leq [\theta(\rho(\widehat{S}r, \breve{T}j))]^a \cdot [\theta(\rho(fr, \widehat{S}r))]^b \\ &\cdot [\theta(\rho(gj, \breve{T}j))]^c \\ &\cdot [\theta(\rho(fr, \breve{T}j) + \rho(gj, \widehat{S}r))]^d.\end{aligned} \tag{146}$$

Hence, all hypotheses of Theorem 3 are satisfied. Thus, $f, g, \widehat{S}$, and $\breve{T}$ have a unique common fixed point.

5. An Application

We apply the result given by Theorem 2 to study the existence of a solution for a system of nonlinear fractional differential equations. Let $\chi = C([0, 1], \mathbb{R})$ be the space of all continuous functions on $[0, 1]$. The metric on χ is given by

$$\rho(r, j) = \|r - j\|_\infty = \max_{t \in [0,1]} |r(t) - j(t)|, \quad r, j \in \chi. \tag{147}$$

Then, $\chi = C([0, 1], \mathbb{R})$ is complete metric space.

Consider the following system of fractional differential equations:

$$\left\{ {}^C D^\alpha r(t) = K_1(t, \widehat{S}(r(t))), {}^C D^\alpha j(t) = K_2(t, \breve{T}(j(t))), \right. \tag{148}$$

with boundary conditions

$$\begin{cases} r(0) = 0, Ir(1) = r'(0), \\ j(0) = 0, Qj(1) = j'(0). \end{cases} \tag{149}$$

Note that ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, defined by

$$\left\{ {}^{C}D^{\alpha} K_1(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \left((t-s)^{n-\alpha-1} K_1^n(s) \right) ds, {}^{C}D^{\alpha} K_2(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \left((t-s)^{n-\alpha-1} K_2^n(s) \right) ds, \right. \tag{150}$$

where we consider

$$\begin{aligned} n-1 &< \alpha, \\ \alpha &< 1, \\ n &= [\alpha]+1, \end{aligned} \tag{151}$$

and $I^{\alpha}K_1$ and $I^{\alpha}K_2$ denote the Riemann–Liouville fractional integral of order α of continuous functions K_1 and K_2, given by

$$\begin{cases} I^{\alpha} K_1(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_1(s) ds, & \text{with } \alpha > 0, \\ Q^{\alpha} K_2(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_2(s) ds, & \text{with } \alpha > 0. \end{cases} \tag{152}$$

System (148) can be written in the following integral form:

$$\begin{cases} r(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r(s))) ds + \dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r(u))) du\, ds, \\ j(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_2(s, \breve{T}(j(s))) ds + \dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \breve{T}(j(u))) du\, ds. \end{cases} \tag{153}$$

Define the mappings $f, g \colon \chi \longrightarrow \chi$ by

$$\left\{ f(r(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_1(s, \hat{S}(r(s))) ds \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \hat{S}(r\{u)) du\, ds, \right.$$

$$g(j(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_2(s, \breve{T}(j(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \breve{T}(j(u))) du\, ds. \tag{154}$$

Theorem 4. *Assume that the following conditions hold:*

(i) $K_1, K_2 [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous functions.*

(ii) $K_1(s,.), K_2(s,.)\colon \longrightarrow \mathbb{R}$ *are increasing functions.*

(iii) For all $r, j \in \chi$ *with* $f(r) \leq \breve{T}(j)$ *and* $|f(r(s)) - g(j(s))| > 0$, *we have*

$$\left| K_1(s, \widehat{S}(r(s))) - K_2(s, \breve{T}(j(s))) \right| \leq \frac{e^{-\tau}\Gamma(\alpha+1)}{4} A(r,j), \tag{155}$$

where

$$A(r,j) = \max \left\{ \begin{array}{c} |\widehat{S}(r(s)) - \breve{T}(j(s))|, |f(r(s)) - \widehat{S}(r(s))|, \\ |g(j(s)) - \breve{T}(j(s))|, \dfrac{|f(r(s)) - \breve{T}(j(s))| + |g(j(s)) - \widehat{S}(r(s))|}{2}, \\ \dfrac{(1 + |f(r(s)) - \widehat{S}(r(s))|)|g(j(s)) - \breve{T}(j(s))|}{1 + |\widehat{S}(r(s)) - \breve{T}(j(s))|} \end{array} \right\}. \tag{156}$$

(iv) There exists $r_0, j_0 \in C([0,1],\mathbb{R})$ such that, for all $t \in [0,1]$, we have

$$\begin{cases} r_0(t) \leq \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_1\left(s, \widehat{S}\left(r_0(s)\right)\right) \mathrm{d}s + \dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_1\left(u, \widehat{S}\left(r_0(u)\right)\right) \mathrm{d}u\, \mathrm{d}s, \\ j_0(t) \leq \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_2\left(s, \breve{T}\left(j_0(s)\right)\right) \mathrm{d}s + \dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_2\left(u, \breve{T}\left(j_0(u)\right)\right) \mathrm{d}u\, \mathrm{d}s. \end{cases} \tag{157}$$

(v) If there exists a sequence $\{r_n\}$ in χ such that

$$\begin{aligned} &\lim_{n\longrightarrow\infty} \rho\left(f\left(\widehat{S}(r_n)\right), \widehat{S}\left(f(r_n)\right)\right) = 0, \\ &\lim_{n\longrightarrow\infty} \rho\left(g\left(\breve{T}(r_n)\right), \breve{T}\left(g(r_n)\right)\right) = 0, \end{aligned} \tag{158}$$

whenever

$$\begin{aligned} &\lim_{n\longrightarrow\infty} f(r_n) = \lim_{n\longrightarrow\infty} \widehat{S}(r_n) = t, \\ &\lim_{n\longrightarrow\infty} g(r_n) = \lim_{n\longrightarrow\infty} \breve{T}(r_n) = t, \end{aligned} \tag{159}$$

for some $t \in \chi$.

Then, system (148) has a solution.

Proof. Following assumptions (iii) and (iv), we have

$$\begin{aligned} &|f(r(t)) - g(j(t))| \\ &= \left| \begin{array}{c} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_1(s, \widehat{S}(r(s))) \mathrm{d}s \\ -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} K_2(s, \breve{T}(j(s))) \mathrm{d}s \\ +\dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_1(u, \widehat{S}(r(u))) \mathrm{d}u\mathrm{d}s \\ -\dfrac{2t}{\Gamma(\alpha)} \displaystyle\int_0^1 \int_0^s (s-u)^{\alpha-1} K_2(u, \breve{T}(j(u))) \mathrm{d}u\mathrm{d}s \end{array} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| K_1(s, \widehat{S}(r(s))) - K_2(s, \breve{T}(j(s))) \right| \mathrm{d}s \\ &\quad + \frac{2}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \left| K_1(s, \widehat{S}(r(s))) - K_2(u, \breve{T}(j(u))) \right| \mathrm{d}u\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau}\Gamma(\alpha+1)}{4} \cdot \int_0^t (t-s)^{\alpha-1} A(r,j) \mathrm{d}s \\ &\quad + \frac{2}{\Gamma(\alpha)} \frac{e^{-\tau}\Gamma(\alpha+1)}{4} \cdot \int_0^1 \int_0^s (s-u)^{\alpha-1} A(r,j) \mathrm{d}u\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau}\Gamma(\alpha+1)}{4} \cdot A(r,j) \cdot \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{\Gamma(\alpha)}\frac{e^{-\tau}\Gamma(\alpha+1)}{4}\cdot A(r,j)\cdot\int_0^1\int_0^s (s-u)^{\alpha-1}\mathrm{d}u\mathrm{d}s\\
&\leq\left(\frac{e^{-\tau}\Gamma(\alpha)\cdot\Gamma(\alpha+1)}{4\Gamma(\alpha)\cdot\Gamma(\alpha+1)}\right)\cdot A(r,j)+2e^{-\tau}B(\alpha+1,1)\frac{\Gamma(\alpha)\cdot\Gamma(\alpha+1)}{4\Gamma(\alpha)\cdot\Gamma(\alpha+1)}\cdot A(r,j)\\
&\leq\frac{e^{-\tau}}{4}A(r,j)+\frac{e^{-\tau}}{2}A(r,j),
\end{aligned}\tag{160}
$$

where B is the beta function. From the above inequality, we obtain that

$$\rho(f(r),g(j))\leq\frac{3}{4}e^{-\tau}A(r,j).\tag{161}$$

It implies that

$$
\begin{aligned}
e^{\sqrt{\rho(f(r),g(j))}}&\leq e^{\sqrt{(3/4)e^{-\tau}A(r,j)}}=e^{\sqrt{(3/4)e^{-\tau}}\sqrt{A(r,j)}}\\
&=\left[e^{\sqrt{A(r,j)}}\right]^{\sqrt{(3/4)e^{-\tau}}}=\left[e^{\sqrt{A(r,j)}}\right]^{E(G(r,j))},
\end{aligned}\tag{162}
$$

where $E(s_1,s_2,s_3,s_4)=\sqrt{3/4e^{\tau}}<1, G(r,j)=\left\{\rho(fr,\widehat{S}r),\rho(gj,\breve{T}j),\rho(fr,\breve{T}j),\rho(gj,\widehat{S}r)\right\}$ and $\theta(t)=e^{\sqrt{t}}$. Since this inequality holds for all $r,j\in\chi$ with $f(r)\leq\breve{T}(j)$, so it is true for any $\Upsilon\in\Phi$,

$$\Upsilon(\rho(fr,\widehat{S}r),\rho(\widehat{S}r,\breve{T}j))<0.\tag{163}$$

Hence, we have

$$\theta(\rho(f(r),g(j)))\leq[\theta(A(r,j))]^{E(G(r,j))}.\tag{164}$$

Thus, $f,g,\widehat{S}$, and $\breve{T}$ are generalized Suzuki-type (θ,E)-rational contraction mappings. Therefore, all hypotheses of Theorem 2 are satisfied. Hence, $f,g,\widehat{S}$, and $\breve{T}$ have a common fixed point, that is, system (148) has at least one solution. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

Maximum Principle for the Space-Time Fractional Conformable Differential System Involving the Fractional Laplace Operator

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In this paper, the authors consider a IBVP for the time-space fractional PDE with the fractional conformable derivative and the fractional Laplace operator. A fractional conformable extremum principle is presented and proved. Based on the extremum principle, a maximum principle for the fractional conformable Laplace system is established. Furthermore, the maximum principle is applied to the linear space-time fractional Laplace conformable differential system to obtain a new comparison theorem. Besides that, the uniqueness and continuous dependence of the solution of the above system are also proved.

1. Introduction

Many fractional partial differential equations were used for modeling complex dynamic systems of engineering, physics, biology, and many other fields [1–4]. As a significant tool, the maximum principle plays an important role in the study of the complex dynamic systems without certain knowledge of the solutions [5–13]. In 2016, by using the maximum principle, Luchko and Yamamoto [14] obtained the uniqueness of both the strong and the weak solutions of the IBVP for a general time-fractional distributed order diffusion equation. In 2016, Jia and Li [15] applied the maximum principle to the classical solution and weak solution of a time-space fractional diffusion equation. Furthermore, they also deduced the maximum principle for a full fractional diffusion equation other than time-fractional and spatial-integer order diffusion equations. In 2019, Wang et al. [16] investigated the IBVP for Hadamard fractional differential equations with fractional Laplace operator $(-\Delta)^{\beta}$ by using the maximum principle.

There are diverse fractional derivatives, such as the Riemann–Liouville derivative, the Caputo fractional derivative, the left and right conformable derivatives, and other fractional derivatives [17–40]. In 2015, Abdeljawad [34] defined the left and right conformable derivatives. Depending on [34], Jarad et al. [35] introduced the fractional conformable derivatives and presented the fractional conformable derivative in the sense of Caputo. The extremum principle of the Caputo fractional conformable derivative is seldom regarded in the existing literature. In addition, the papers which mentioned the fractional conformable derivative do not include the fractional Laplace operator.

Motivated by the above works, in this context, the authors investigate the IBVP for a space-time Caputo fractional conformable diffusion system with the fractional Laplace operator. First, we provide a detailed proof of the Caputo fractional conformable extremum principle. Then, the new maximum principle is obtained by applying the extreme principle. As some applications of the maximum principle, a comparison principle for the space-time fractional Laplace conformable differential system is developed, and the properties of the solution of the system are given, such as the uniqueness and continuous dependence on the initial and boundary condition.

The article is organized as follows: in Section 2, the extremum principle for the Caputo fractional conformable derivative is established. In Section 3, the maximum principle of the space-time fractional Laplace conformable differential system is derived, which is used to obtain the comparison principle for the space-time fractional Laplace conformable differential system, and the properties of the solution of the above system are given in Section 4.

2. Problem Formulation and Extremum Principles

In this paper, we focus on a space-time Caputo fractional conformable system with the fractional Laplace operator:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha}u(x,t) + (-\Delta)^{\gamma}u(x,t) - a(x,t)u(x,t) = g(x,t), & (x,t)\in\Omega\times(a,b],\\ u(x,t)=0, & x\in\mathbb{R}^N\backslash\overline{\Omega},\ t\geq a, \end{cases} \tag{1}$$

where Ω represents an open and bounded domain in $\mathbb{R}^N\,(N\geq 1)$ in which boundary Γ is smooth and $a(x,t)\in\overline{\Omega}\times[a,b]$ is a bounded function. Here, ${}_a^{C\beta}D_t^{\alpha}$ is the left Caputo fractional conformable derivative. For a function $f\in C_{\alpha,a}^n$, the left Caputo fractional conformable derivative of order β is defined by

$${}_a^{C\beta}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t\left(\frac{(t-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{n-\beta-1}\frac{{}_a^nT^{\alpha}f(\tau)}{(\tau-a)^{1-\alpha}}\mathrm{d}\tau, \tag{2}$$

with $0<\beta<1$, $0<\alpha<1$, $n=[\beta]+1$, ${}_aT^{\alpha}f(t) = (t-a)^{1-\alpha}f'(t)$, ${}_a^nT^{\alpha} = \underbrace{{}_aT^{\alpha}\,{}_aT^{\alpha},\ldots,{}_aT^{\alpha}}_{n\,\text{times}}$, and $C_{\alpha,a}^n[a,b] = \{f\colon [a,b]\longrightarrow\mathbb{R}|_a^{n-1}T^{\alpha}f\in I_{\alpha}[a,b]\}$ (where $I_{\alpha}[a,b]$ is defined in Definition 1 of [34]). For detailed information of the Caputo fractional conformable derivative, see [35].

When $\phi\in C_{\text{loc}}^{1,1}(\mathbb{R}^N)\cap L^{\gamma}$, the fractional Laplace operator could be given by

$$(-\Delta)^{\gamma}\phi(x) = C_{N,\gamma}\int_{\mathbb{R}^N}\frac{\phi(x)-\phi(y)}{|x-y|^{N+2\gamma}}\mathrm{d}y, \tag{3}$$

with $C_{N,\gamma} = (\gamma 2^{2\gamma}\Gamma(N+2\gamma/2)/\pi^{N/2}\Gamma(1-\gamma))$, and

$$L_{\gamma} = \left\{\phi\colon\mathbb{R}^N\longrightarrow\mathbb{R}\Big|\int_{\mathbb{R}^N}\frac{|;\phi(x)|}{1+|x|^{N+\gamma}}\mathrm{d}x<\infty.\right\}. \tag{4}$$

Denote

$$H(\overline{\Omega}) = \left\{u(x,t)|u(x,t)\in C^{2,1}(\Omega\times(a,b)),u(x,t)\in C(\overline{\Omega}\times[a,b])\right\}. \tag{5}$$

Firstly, we can state two Caputo fractional conformable extremum principles.

Lemma 1. *If $f\in C_{\alpha,a}^1([a,b])$ reaches its maximum at a point $t_0\in[a,b]$, then*

$${}_a^{C\beta}D_{t_0}^{\alpha}f(t_0)\geq 0, \tag{6}$$

holds.

Proof. First, we introduce an auxiliary function

$$g(t) = f(t_0)-f(t)\geq 0,\quad t\in[a,b]. \tag{7}$$

Concurrently, $g(t)\in C_{\alpha,a}^1([a,b])$, $g(t_0)=0$, and ${}_a^{C\beta}D_t^{\alpha}g(t) = -{}_a^{C\beta}D_t^{\alpha}f(t)$.

By calculation, we notice that

$$\begin{aligned}{}_a^{C\beta}D_{\alpha}^{t_0}g(t_0) &= \frac{1}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}\cdot\frac{(\tau-a)^{1-\alpha}g'(\tau)}{(\tau-a)^{1-\alpha}}\mathrm{d}\tau\\ &= \frac{1}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g'(\tau)\mathrm{d}\tau\\ &= \frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g(\tau)\Big|_a^{t_0}\\ &\quad-\frac{\beta}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta-1}\cdot(\tau-a)^{\alpha-1}g(\tau)\mathrm{d}\tau.\end{aligned} \tag{8}$$

This is because

$$\begin{aligned}&\lim_{\tau\longrightarrow t_0}\frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}g(\tau)\\ &= \frac{\alpha^{\beta}}{\Gamma(1-\beta)}\lim_{\tau\longrightarrow t_0}\frac{g'(\tau)}{\beta((t_0-a)^{\alpha}-(\tau-a)^{\alpha})^{\beta-1}(-\alpha)(\tau-a)^{\alpha-1}}\\ &= 0.\end{aligned} \tag{9}$$

Therefore, formula (9) becomes

$$\begin{aligned}{}_a^{C\beta}D_{t_0}^{\alpha}g(t_0) &= -\frac{1}{\Gamma(1-\beta)}\left(\frac{(t_0-a)^{\alpha}}{\alpha}\right)^{-\beta}g(a)\\ &\quad-\frac{\beta}{\Gamma(1-\beta)}\int_a^{t_0}\left(\frac{(t_0-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta-1}\cdot(\tau-a)^{\alpha-1}g(\tau)\mathrm{d}\tau\\ &\leq 0.\end{aligned} \tag{10}$$

We can obtain ${}_{a}^{C\beta}D_{t_0}^{\alpha} f(t_0) \leq 0$.

The lemma is proved.

Using the same method, it is easy to obtain the following lemma. □

Lemma 2. *If $f \in C_{\alpha,a}^{1}([a,b])$ reaches its minimum at a point $t_0 \in [a,b]$, then*

$$ {}_{a}^{C\beta}D_{t_0}^{\alpha} f(t_0) \leq 0, \tag{11} $$

holds.

3. Maximum Principle

In this section, we focus on linear space-time Caputo fractional conformable Laplace system (1) with the initial-boundary condition:

$$ u(x,a) = \varphi(x), \quad x \in \Omega, \tag{12} $$

$$ u(x,t) = \mu(x,t), \quad (x,t) \in \Gamma \times [a,b]. \tag{13} $$

Theorem 1. *Let a function $u \in H(\overline{\Omega})$ satisfy linear space-time Caputo fractional conformable Laplace system (1), (12), and (13). Suppose $g(x,t) \leq 0$, $\forall (x,t) \in \Omega \times (a,b]$. Then, we have*

$$ u(x,t) \leq \max\left\{\max_{x\in\Omega}\varphi(x), \max_{(x,t)\in\Gamma\times[a,b]}\mu(x,t), 0\right\}, \quad \forall (x,t) \in \overline{\Omega}\times[a,b]. \tag{14} $$

Proof. We first suppose that inequality (14) is false; then, there exists a point $(x_0,t_0) \in \Omega \times (a,b]$ such that

$$ u(x_0,t_0) > \max\left\{\max_{x\in\Omega}\varphi(x), \max_{(x,t)\in\Gamma\times[a,b]}\mu(x,t), 0\right\} = M > 0. \tag{15} $$

Denote $\varepsilon = u(x_0,t_0) - M > 0$ and

$$ w(x,t) = u(x,t) + \frac{\varepsilon}{2}\frac{b-(t-a)}{b}, \quad (x,t) \in \overline{\Omega}\times[a,b]. \tag{16} $$

Besides, w implies

$$ \begin{aligned} & w(x,t) \leq u(x,t) + \frac{\varepsilon}{2}, \quad (x,t) \in \overline{\Omega}\times[a,b], \\ & w(x_0,t_0) \geq u(x_0,t_0) = \varepsilon + M \geq \varepsilon + u(x,t) \geq \varepsilon + w(x,t) - \frac{\varepsilon}{2} \\ & \geq \frac{\varepsilon}{2} + w(x,t),\ (x,t) \in (\Gamma\times[a,b]) \cup (\Omega\times\{a\}). \end{aligned} \tag{17} $$

The latter property implies that the maximum of w cannot be attained on $(\Gamma\times[a,b]) \cup (\Omega\times\{a\})$. Let $w(x_1,t_1) = \max_{(x,t)\in\overline{\Omega}\times[a,b]} w(x,t)$; then,

$$ w(x_1,t_1) \geq u(x_0,t_0) \geq \varepsilon + M > \varepsilon, \tag{18} $$

$$ (-\Delta)^{\gamma} w(x,t)|_{(x_1,t_1)} = C_{N,\gamma}\int_{\mathbb{R}^N}\frac{w(x_1,t_1)-w(x,t_1)}{|x_1 - x|^{N+2\gamma}}dx \geq 0. \tag{19} $$

By Lemma 1, we know

$$ {}_{a}^{C\beta}D_{t}^{\alpha} w(x,t)|_{(x_1,t_1)} \geq 0. \tag{20} $$

By calculation, we can show

$$ {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t-a)}{b}\right) = -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2b}\int_a^t\left(\frac{(t-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{-\beta}d\tau. \tag{21} $$

Assuming $u = (\tau - a/t - a)^{\alpha}$ and substituting into formula (21), we get

$$ \begin{aligned} {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t-a)}{b}\right) &= -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2b}\alpha^{\beta-1}\int_0^1 (t-a)^{1-\alpha\beta} \\ &\quad \cdot (1-u)^{-\beta}u^{1-\alpha}du \\ &= -\alpha^{\beta-1}(t-a)^{1-\alpha\beta}\frac{\varepsilon}{2b}\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta)}. \end{aligned} \tag{22} $$

Applying (19)–(22), it holds that

$$ \begin{aligned} & {}_{a}^{C\beta}D_{t}^{\alpha}u(x,t)|_{(x_1,t_1)} + (-\Delta)^{\gamma}u(x,t)|_{(x_1,t_1)} - a(x_1,t_1)u(x_1,t_1) - g(x_1,t_1) \\ & = {}_{a}^{C\beta}D_{t}^{\alpha}w(x,t)|_{(x_1,t_1)} - {}_{a}^{C\beta}D_{t}^{\alpha}\left(\frac{\varepsilon}{2}\frac{b-(t_1-a)}{b}\right) + (-\Delta)^{\gamma}w(x_1,t_1) \\ & \quad - a(x_1,t_1)\left(w(x_1,t_1) - \frac{\varepsilon}{2}\frac{b-(t_1-a)}{b}\right) - g(x_1,t_1) \\ & \geq \alpha^{\beta-1}(t_1-a)^{1-\alpha\beta}\frac{\varepsilon}{2b}\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta)} - a(x_1,t_1)\varepsilon\left(1 - \frac{b-(t_1-a)}{2b}\right) \\ & \geq 0. \end{aligned} \tag{23} $$

Equation (23) is in contradiction with (1).

The proof of the theorem is completed.

Similarly, the minimum principle can be obtained as follows. □

Theorem 2. *Let a function $u \in H(\overline{\Omega})$ satisfy linear space-time Caputo fractional conformable Laplace system (1), (12), and (13). Suppose $g(x,t) \geq 0$, $\forall (x,t) \in \Omega \times (a,b]$. Then, we have*

$$u(x,t) \geq \min\left\{\min_{x\in\Omega} \varphi(x), \min_{(x,t)\in\Gamma\times[a,b]} \mu(x,t), 0\right\}, \quad \forall (x,t) \in \overline{\Omega} \times [a,b]. \tag{24}$$

4. Some Applications of the Maximum Principle

Theorem 3. *Let $u(x,t) \in H(\overline{\Omega})$ be a solution of system (1) with initial-boundary values (12) and (13). Then,*

$$\|u\|_{C(\overline{\Omega}\times[a,b])} \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} + 2M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}, \tag{25}$$

where

$$M = \|g\|_{C(\overline{\Omega}\times[a,b])}. \tag{26}$$

Proof. We first present a function

$$w(x,t) = u(x,t) - M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(t-a)^{\alpha\beta}, \quad (x,t) \in \overline{\Omega} \times [a,b]. \tag{27}$$

If $u(x,t)$ is a solution of system (1), (12), and (13), then $w(x,t)$ is a solution of problem (1) with

$$\begin{aligned} g_1(x,t) &= g(x,t) - M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}\, {}_{a}^{C\beta}D_t^{\alpha}(t-a)^{\alpha\beta} \\ &= g(x,t) - M, \\ \mu_1(x,t) &= \mu(x,t) - M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(t-a)^{\alpha\beta}. \end{aligned} \tag{28}$$

Substitute $g_1(x,t)$ and $\mu_1(x,t)$ for $g(x,t)$ and $\mu(x,t)$, respectively. Owing to $g_1(x,t) \leq 0$, applying Theorem 1 (maximum principle), we have

$$w(x,t) \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\| + M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}\right\}. \tag{29}$$

Therefore,

$$u(x,t) \leq \max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} + 2M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}. \tag{30}$$

In a similar manner, we can get

$$u(x,t) \geq -\max\left\{\max_{x\in\Omega} \|\varphi(x)\|, \max_{(x,t)\in\Gamma\times[a,b]} \|\mu(x,t)\|\right\} - 2M \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}. \tag{31}$$

Combining (30) and (31), the theorem is proved. □

Theorem 4. *Let $u(x,t)$ satisfy IBVP (1), (12), and (13). $u(x,t)$ is continuous depending on the data given. That is, if*

$$\left\|g - g_1\right\|_{C(\overline{\Omega}\times[a,b])} \leq \varepsilon, \ \left\|\varphi(x) - \varphi_1(x)\right\|_{C(\overline{\Omega})} \leq \varepsilon_0, \left\|\mu(x,t) - \mu_1(x,t)\right\|_{C(\Gamma\times[a,b])} \leq \varepsilon_1, \tag{32}$$

then the estimation of the classical solution of $u(x,t)$ and $u_1(x,t)$,

$$\left\|u - u_1\right\|_{C(\overline{\Omega}\times[a,b])} \leq \max\{\varepsilon_0, \varepsilon_1\} + 2 \frac{\Gamma(2+\alpha\beta-\alpha-\beta)}{\beta\alpha^{\beta}\Gamma(1+\alpha\beta-\alpha)}(b-a)^{\alpha\beta}\varepsilon, \tag{33}$$

holds.

The demonstration process is similar to Theorem 3.

Theorem 5. *Let $u \in H(\overline{\Omega})$ be a solution of IBVP (1), (12), and (13). Assume $g(x,t) \leq 0$ and $a(x,t) \leq 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that*

$$u(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times [a,b], \tag{34}$$

if $\varphi(x) \leq 0$, and $\mu(x,t) \leq 0$.

Theorem 6. *Let $u \in H(\overline{\Omega})$ satisfy IBVP (1), (12), and (13). Assume $g(x,t) \geq 0$ and $a(x,t) \geq 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that*

$$u(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times [a,b], \tag{35}$$

if $\varphi(x) \geq 0$, and $\mu(x,t) \geq 0$.

The conclusion of Theorem 5 and Theorem 6 is obtained by Theorem 1.

Remark 1. Let $u \in H(\overline{\Omega})$ satisfy IBVP (1), (12), and (13). Assume $g(x,t) = a(x,t) = 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that

$$u(x,t) = 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \tag{36}$$

if $\varphi(x) = \mu(x,t) = 0$.

Theorem 7 (comparison theorem). Suppose $a(x,t) \geq 0$, $b(x,t) \geq 0$, and $b(x,t) > a(x,t)$, $\forall (x,t) \in \Omega \times (a,b]$. Assume $(u,v) \in H(\overline{\Omega}) \times H(\overline{\Omega})$ satisfies the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} u(x,t) + (-\Delta)^{\gamma} u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ {}_a^{C\beta}D_t^{\alpha} v(x,t) + (-\Delta)^{\gamma} v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, \quad v(x,t) = 0, \; x \in \mathbb{R}^N \backslash \overline{\Omega}, \; t \geq a, \\ u(x,a) \geq 0, \quad v(x,a) \geq 0, \; x \in \Omega, \\ u(x,t) \geq 0, \quad v(x,t) \geq 0, \; (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{37}$$

Then, it follows that

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{38}$$

Proof. Let $p(x,t) = u(x,t) + v(x,t), \forall (x,t) \in \overline{\Omega} \times [a,b]$. Then, by (37), we have

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} p(x,t) + (-\Delta)^{\gamma} p(x,t) - a(x,t)p(x,t) + b(x,t)p(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ p(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \; t \geq a, \\ p(x,a) \geq 0, \quad x \in \Omega, \\ p(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{39}$$

Thus, by (39) and Theorem 6, we obtain

$$p(x,t) \geq 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \text{i.e.} u(x,t) + v(x,t) \geq 0, \cdot (x,t) \in \overline{\Omega} \times [a,b]. \tag{40}$$

Using (37) and (40), we have that

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} u(x,t) + (-\Delta)^{\gamma} u(x,t) - (b(x,t) - a(x,t))u(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \; t \geq a, \\ u(x,a) \geq 0, \quad x \in \Omega, \\ u(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b], \end{cases} \tag{41}$$

$$\begin{cases} {}_a^{C\beta}D_t^{\alpha} v(x,t) + (-\Delta)^{\gamma} v(x,t) - (b(x,t) - a(x,t))v(x,t) \geq 0, & (x,t) \in \Omega \times (a,b], \\ v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}, \; t \geq a, \\ v(x,a) \geq 0, \quad x \in \Omega, \\ v(x,t) \geq 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{42}$$

Applying Theorem 6 to (41) and (42), we can get

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{43}$$

Thus, the conclusion holds.

Similarly, the following theorem holds. □

Theorem 8. *Suppose $a(x,t) \leq 0$, $b(x,t) \leq 0$, and $b(x,t) < a(x,t)$, $\forall (x,t) \in \Omega \times (a,b]$. Assume $(u,v) \in H(\overline{\Omega}) \times H(\overline{\Omega})$*

satisfies the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_{a}^{C\beta}D_t^{\alpha}u(x,t) + (-\Delta)^{\gamma}u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) \le 0, & (x,t) \in \Omega \times (a,b], \\ {}_{a}^{C\beta}D_t^{\alpha}v(x,t) + (-\Delta)^{\gamma}v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) \le 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \ge a, \\ u(x,a) \le 0, v(x,a) \le 0, \quad x \in \Omega, \\ u(x,t) \le 0,\ v(x,t) \le 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{44}$$

Then, it follows that

$$u(x,t) \le 0, v(x,t) \le 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{45}$$

Remark 2. Let $(u,v) \in H(\overline{\Omega}) \times H(\overline{\Omega})$ satisfy the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_{a}^{C\beta}D_t^{\alpha}u(x,t) + (-\Delta)^{\gamma}u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) = 0, & (x,t) \in \Omega \times (a,b], \\ {}_{a}^{C\beta}D_t^{\alpha}v(x,t) + (-\Delta)^{\gamma}v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) = 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \ge a, \\ u(x,a) = \varphi_1(x), v(x,a) = \varphi_2(x), \quad x \in \Omega, \\ u(x,t) = \mu_1(x,t),\ v(x,t) = \mu_2(x,t), \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{46}$$

Suppose $a(x,t) = b(x,t) = 0, \forall (x,t) \in \Omega \times (a,b]$. Then, it follows that

$$u(x,t) = 0, v(x,t) = 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \tag{47}$$

if $\varphi_1(x) = \varphi_2(x) = \mu_1(x,t) = \mu_2(x,t) = 0$.

Next, we focus on the following linear space-time fractional Laplace conformable differential system:

$$\begin{cases} {}_{a}^{C\beta}D_t^{\alpha}u(x,t) + (-\Delta)^{\gamma}u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) = g_1(x,t), & (x,t) \in \Omega \times (a,b], \\ {}_{a}^{C\beta}D_t^{\alpha}v(x,t) + (-\Delta)^{\gamma}v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) = g_2(x,t), & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \ge a, \\ u(x,a) = \varphi_1(x), v(x,a) = \varphi_2(x), \quad x \in \Omega, \\ u(x,t) = \mu_1(x,t),\ v(x,t) = \mu_2(x,t), \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{48}$$

Theorem 9. *Suppose $a(x,t) \le 0, b(x,t) \le 0, b(x,t) < a(x,t)$, $g_1(x,t) \le 0$, and $g_2(x,t) \le 0$, $\forall (x,t) \in \Omega \times (a,b]$; then, IBVP (48) has a unique solution on $H(\overline{\Omega}) \times H(\overline{\Omega})$.*

Proof. Let (u_1,v_1) and (u_2,v_2) be two solutions of IBVP (48). Denote

$$u(x,t) = u_1(x,t) - u_2(x,t), v(x,t) = v_1(x,t) - v_2(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [a,b], \tag{49}$$

satisfies the system

$$\begin{cases} {}_{a}^{C\beta}D_t^{\alpha}u(x,t) + (-\Delta)^{\gamma}u(x,t) - a(x,t)v(x,t) - b(x,t)u(x,t) = 0, & (x,t) \in \Omega \times (a,b], \\ {}_{a}^{C\beta}D_t^{\alpha}v(x,t) + (-\Delta)^{\gamma}v(x,t) - a(x,t)u(x,t) - b(x,t)v(x,t) = 0, & (x,t) \in \Omega \times (a,b], \\ u(x,t) = 0, v(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \ge a, \\ u(x,a) = 0, v(x,a) = 0, \quad x \in \Omega, \\ u(x,t) = 0, v(x,t) = 0, \quad (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{50}$$

Let $p(x,t) = u(x,t) + v(x,t),\ \forall (x,t) \in \overline{\Omega} \times [a,b]$. By (50), we have

$$\begin{cases} {}_{a}^{C\beta}D_t^{\alpha} p(x,t) + (-\Delta)^{\gamma} p(x,t) - (b(x,t) - a(x,t)) p(x,t) = 0, & (x,t) \in \Omega \times (a,b]. \\ p(x,t) = 0, & x \in \mathbb{R}^N \backslash \overline{\Omega},\ t \geq a, \\ p(x,a) = 0, & x \in \Omega, \\ p(x,t) = 0, & (x,t) \in \Gamma \times [a,b]. \end{cases} \tag{51}$$

Applying Theorem 8, we get

$$u(x,t) \leq 0, v(x,t) \leq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{52}$$

By the same way, using Theorem 8 to $-u(x,t)$ and $-v(x,t)$, we have

$$u(x,t) \geq 0, v(x,t) \geq 0, (x,t) \in \overline{\Omega} \times [a,b]. \tag{53}$$

Combining (52) and (53), we can get

$$u(x,t) = 0, v(x,t) = 0, \quad \forall (x,t) \in \overline{\Omega} \times [a,b]. \tag{54}$$

Thus, the conclusion holds. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Both authors contributed equally and approved the final manuscript.

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Research Article

Some New Tempered Fractional Pólya-Szegö and Chebyshev-Type Inequalities with Respect to Another Function

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In this present article, we establish certain new Pólya–Szegö-type tempered fractional integral inequalities by considering the generalized tempered fractional integral concerning another function Ψ in the kernel. We then prove certain new Chebyshev-type tempered fractional integral inequalities for the said operator with the help of newly established Pólya–Szegö-type tempered fractional integral inequalities. Also, some new particular cases in the sense of classical tempered fractional integrals are discussed. Additionally, examples of constructing bounded functions are considered. Furthermore, one can easily form new inequalities for Katugampola fractional integrals, generalized Riemann–Liouville fractional integral concerning another function Ψ in the kernel, and generalized fractional conformable integral by applying different conditions.

1. Introduction

The well-known Chebyshev functional [1] is defined by

$$\mathcal{T}(f_1, f_2) = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f_1(\theta) f_2(\theta)\mathrm{d}\theta - \frac{1}{x_2 - x_1}\left(\int_{x_1}^{x_2} f_1(\theta)\mathrm{d}\theta\right)\frac{1}{x_2 - x_1}\left(\int_{x_1}^{x_2} f_2(\theta)\mathrm{d}\theta\right), \tag{1}$$

where the functions f_1 and f_2 are integrable on $[x_1, x_2]$. If the functions f_1 and f_2 are synchronous on $[x_1, x_2]$, i.e.,

$$(f_1(\vartheta) - f_1(\zeta))(f_2(\vartheta) - f_2(\zeta)) \geq 0, \tag{2}$$

for any $\vartheta, \zeta \in [x_1, x_2]$, then $\mathcal{T}(f_1, f_2) \geq 0$. Functional (1) has gained more recognition due to its diverse applications in the fields of transform theory, numerical quadrature, probability, and statistical problems. Additionally, the researchers have established a large number of integral inequalities by utilizing functional (1). The interesting readers may consult [2–5]. In [6], Tassaddiq et al. recently established certain inequalities via fractional conformable integrals by considering functional (1).

In [7], Grüss introduced the following inequality:

$$\left|\mathcal{T}(f_1, f_2)\right| \le \frac{(M_1 - m_1)(N_1 - n_1)}{4}, \tag{3}$$

where the functions f_1 and f_2 are integrable on $[x_1, x_2]$ such that f_1 and f_2 satisfy the inequalities $m_1 \le f_1(\vartheta) \le M_1$ and $n_1 \le f_2(\zeta) \le N_1$, for all $\vartheta, \zeta \in [x_1, x_2]$ and for some constant $m_1, n_1, M_1, N_1 \in \mathbb{R}$.

In [8], Pólya–Szegö presented the following inequality:

$$\frac{\int_{x_1}^{x_2} f_1^2(\theta)\mathrm{d}\theta \int_{x_1}^{x_2} f_2^2(\theta)\mathrm{d}\theta}{\left(\int_{x_1}^{x_2} f_1(\theta) f_2(\theta)\mathrm{d}\theta\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 N_1}{m_1 n_1}} + \sqrt{\frac{m_1 n_1}{M_1 N_1}}\right)^2. \tag{4}$$

In [9], Dragomir and Diamond presented the following inequality with the help of Pólya–Szegö inequality:

$$\left|\mathcal{T}(f_1, f_2)\right| \le \frac{(M_1 - m_1)(N_1 - n_1)}{4(x_2 - x_1)^2 \sqrt{m_1 M_1 n_1 N_1}} \int_{x_1}^{x_2} f_1(\theta)\mathrm{d}\theta \cdot \int_{x_1}^{x_2} f_2(\theta)\mathrm{d}\theta, \tag{5}$$

where the functions f_1 and f_2 are positive and integrable on $[x_1, x_2]$ such that f_1 and f_2 satisfy the inequalities $m_1 \le f_1(\theta) \le M_1$ and $n_1 \le f_2(\zeta) \le N_1$, for all $\theta, \zeta \in [x_1, x_2]$ and for some constant $m_1, n_1, M_1, N_1 \in \mathbb{R}$.

In the last few decades, the researchers have considered that fractional integral inequalities are the most powerful tools for the development of both applied and pure mathematics. In [10], the authors presented some Grüss-type integral inequalities by considering fractional integrals. Some new integral inequalities in sense of Riemann–Liouville fractional integrals can be found in the work of Dahmani [11].

In [12], Sarikaya et al. gave the idea of generalized (k, s)-fractional integrals with applications. Set et al. [13] investigated some Grüss-type inequalities by considering generalized k-fractional integrals.

Very recently, the idea of fractional conformable and proportional fractional integral operators was proposed by Jarad et al. [14, 15]. Later on, Huang et al. [16] presented generalized Hermite–Hadamard-type inequalities by considering generalized fractional conformable integrals. In [17], Qi et al. established Chebyshev-type inequalities for generalized fractional conformable integrals.

In [18], Ntouyas et al. investigated some new Pólya–Szegö- and Chebyshev-type inequalities by considering Riemann–Liouville fractional integrals. The tempered fractional integral was first studied by Buschman [19], but Li et al. [20] and Meerschaert et al. [21] have described the associated tempered fractional calculus more explicitly. Fernandez and Ustaoğlu [22] investigated several analytic properties of the tempered fractional integral. In [23], Fahad et al. proposed the general form of the generalized tempered fractional integral concerning another function. In this paper, we investigate the said inequalities for the so-called tempered fractional integrals containing another function in the kernel.

The structure of the paper as follows.

In Section 2, some basic definitions are presented. Some new Pólya–Szegö-type for the so-called generalized tempered fractional integral in the sense of another function is presented in Section 3. In Section 4, we present some new generalized Chebyshev-type tempered fractional integral inequalities. In Section 5, certain new particular cases in terms of classical tempered fractional integrals are discussed. An example of constructing bounding functions is considered in Section 6. Finally, the concluding remarks are discussed in Section 7.

2. Preliminaries

In this section, we consider some well-known definitions and mathematical preliminaries.

Definition 1 (see [7]). Suppose that the functions $f_1, f_2\colon [x_1, x_1] \longrightarrow \mathbb{R}$ are positive with $\mathcal{A} \le f_1(\vartheta) \le \mathcal{B}$ and $\mathcal{C} \le f_2(\vartheta) \le \mathcal{D}$, for all $\vartheta \in [x_1, x_1]$, then the following inequality holds:

$$\left|\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f_1(\vartheta) f_2(\vartheta)\mathrm{d}\vartheta - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f_1(\vartheta)\mathrm{d}\vartheta \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f_2(\vartheta)\mathrm{d}\vartheta\right| \le \frac{1}{4}(\mathcal{B} - \mathcal{A})(\mathcal{D} - \mathcal{C}), \tag{6}$$

where the constants $\mathcal{B}, \mathcal{A}, \mathcal{C}, \mathcal{D} \in \mathbb{R}$ and 1/4 is the sharp of inequality (6).

Definition 2 (see [24, 25]). The function f_1 will be in the space $L_{p,r}]0, \infty[$ if

$$L_{p,r}[0, \infty[= \left\{ f_1\colon \|f_1\|_{L_{p,r}[0,\infty[} = \left(\int_r^s |f_1(\vartheta)|^p \vartheta^r \mathrm{d}\vartheta\right)^{1/p} < \infty, 1 \le p < \infty, r \ge 0 \right\}. \tag{7}$$

If we apply $r = 0$, then (7) gives

$$L_p[0,\infty[= \left\{ f_1 \colon \|f_1\|_{L_p[0,\infty[} = \left(\int_r^s |f_1(\vartheta)|^p \mathrm{d}\vartheta \right)^{1/p} < \infty, 1 \le p < \infty \right\}. \tag{8}$$

Definition 3 (see [26]). Suppose that the function $f_1 \in L_1[0,\infty[$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$ and having continuous derivative Ψ' on $[0,\infty[$ with $\Psi(0) = 0$. Then, the Lebesgue real-valued measurable function f_1 defined on $[0,\infty[$ is said to be in the space $X_\Psi^p(0,\infty)$, $(1 \le p < \infty)$ if

$$\|f_1\|_{X_\Psi^p} = \left(\int_r^s |f_1(\vartheta)|^p \Psi'(\vartheta) \mathrm{d}\vartheta \right)^{1/p} < \infty, 1 \le p < \infty. \tag{9}$$

When $p = \infty$, then

$$\|f_1\|_{X_\Psi^\infty} = \mathbf{ess} \sup_{0 \le \vartheta < \infty} [\Psi'(\vartheta) f_1(\varrho)]. \tag{10}$$

Note that, the space $X_\Psi^p(0,\infty)$ coincides with the space $L_p[0,\infty[$ if $\Psi(\vartheta) = \vartheta$ for $1 \le p < \infty$ and similarly with the space $L_{p,r}[1,\infty[$ if $\Psi(\vartheta) = \ln\vartheta$ for $1 \le p < \infty$.

Definition 4 (see [20–22]). The left-sided tempered fractional integral of order $\kappa > 0$ and $\tau \ge 0$ with $\mathfrak{R}(\kappa) > 0$ and $\mathfrak{R}(\tau) \ge 0$ is defined by

$$\left({}_{x_1}\mathscr{R}^{\kappa,\tau} f_1 \right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{\theta} e^{-\tau(\theta-\vartheta)} (\varrho - t)^{\kappa-1} f_1(\vartheta) \mathrm{d}\vartheta, x_1 < \theta. \tag{11}$$

Remark 1. By setting $\tau = 0$ in (11) yields the following Riemann–Liouville fractional integral, which is defined by

$$\left({}_{x_1}\mathscr{R}^{\kappa} f_1 \right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{\theta} (\theta - \vartheta)^{\kappa-1} f_1(\vartheta) \mathrm{d}\vartheta, \ x_1 < \theta. \tag{12}$$

The following results for (11) hold:

$${}_{x_1}\mathscr{R}^{\kappa,\tau} {}_{x_1}\mathscr{R}^{\lambda,\tau} f_1(\theta) = {}_{x_1}\mathscr{R}^{\kappa+\lambda,\tau} f_1(\theta). \tag{13}$$

We define the following one-sided tempered fractional integral.

Definition 5. The one-sided tempered fractional integral of order $\kappa > 0$, $\tau \ge 0$ is defined by

$$(\mathscr{R}_0^{\kappa,\tau} f_1)(\theta) = (\mathscr{R}^{\kappa,\tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_0^{\theta} e^{-\tau(\theta-\vartheta)} (\theta - \vartheta)^{\kappa-1} f_1(\vartheta) \mathrm{d}\vartheta. \tag{14}$$

Definition 6 (see [23]). Let the function f_1 be an integrable in the space $X_\Psi^p(0,\infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Then, the left-sided generalized tempered fractional integral of the function f_1 concerning another function Ψ in the kernel is defined by

$$\left({}_{x_1}^{\Psi}\mathscr{R}^{\kappa,\tau} f_1 \right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{\theta} e^{-\tau(\Psi(\theta)-\Psi(\theta))} (\Psi(\theta) - \Psi(\theta))^{\kappa-1} \Psi'(\theta) f_1(\theta) \mathrm{d}\vartheta, x_1 < \theta, \tag{15}$$

where $\tau \ge 0$, $\kappa, \in \mathbb{C}$ with $\mathfrak{R}(\kappa) > 0$, and $\Gamma(.)$ is the well-known gamma function.

Remark 2. The following results can be obtained:

(i) Applying Definition 6 for $\Psi(\theta) = \theta$, we get (11).

(ii) Applying Definition 6 for $\tau = 0$, then it will reduce to the left-sided generalized Riemann–Liouville fractional integral operator [27].

(iii) Applying Definition 6 for $\Psi(\theta) = \ln\theta$, then it will reduce to the following left-sided Hadamard tempered integral defined by [23]

$$\left({}_{x_1}^{\Psi}\mathscr{R}^{\kappa,\tau} f_1 \right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{\theta} \exp[-\tau(\ln\theta - \ln\vartheta)] (\ln\theta - \ln\vartheta)^{\kappa-1} \frac{f_1(\vartheta)}{\vartheta} \mathrm{d}\vartheta, x_1 < \theta. \tag{16}$$

(iv) Applying Definition 6 for $\Psi(\theta) = \theta^{\eta}/\eta$, $\eta > 0$ and $\tau = 0$, then it will reduce to the left-sided Katugampola [24] fractional integral.

(v) Applying Definition 6 for $\Psi(\theta) = \theta$ and $\tau = 0$, the left Riemann–Liouville fractional integral (12) will be obtained

(vi) Applying Definition 6 for $\Psi(\theta) = \theta^{\alpha+s}/\alpha + s$ and $\tau = 0$ (where $\alpha \in (0,1]$, $s \in \mathbb{R}$ and $\alpha + s \neq 0$), then it reduces to the left-sided generalized fractional conformable integral given by [28]

(vii) Applying Definition 6 for $\Psi(\theta) = (\theta - x_1)^{\alpha}/\alpha$, $\alpha > 0$, then it reduces to the fractional conformable integral defined by Jarad et al. [14].

In this article, we consider the following one-sided GTF-integral.

Definition 7. Let the function f_1 be integrable in the space $X_{\Psi}^{p}(0,\infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Then, the one-sided generalized tempered fractional integral of the function f_1 concerning another function Ψ in the kernel is defined by

$$\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau} f_1\right)(\theta) = \frac{1}{\Gamma(\kappa)} \int_0^{\theta} e^{-\tau(\Psi(\theta)-\Psi(\theta))} (\Psi(\theta) - \Psi(\theta))^{\kappa-1} \Psi'(\theta) f_1(\theta) \mathrm{d}\vartheta. \tag{17}$$

Definition 8. For $0 = x_0 < x_1 < \cdots < x_p < x_{p+1} = X$, we define the following subintegrals for the generalized tempered integral

$$\left({}^{\Psi}\mathscr{R}_{x_i,x_{i+1}}^{\kappa,\tau} f_1\right)(X) = \frac{1}{\Gamma(\kappa)} \int_{x_i}^{x_{i+1}} e^{-\tau(\Psi(X)-\Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) \mathrm{d}\vartheta. \tag{18}$$

Note that

$$\begin{aligned}
\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau} f_1\right)(X) = \sum_{i=0}^{p} {}^{\Psi}\mathscr{R}_{t_i,t_{i+1}}^{\kappa,\tau}(f_1)(X) &= \frac{1}{\Gamma(\kappa)} \int_0^{x_1} e^{-\tau(\Psi(X)-\Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) \mathrm{d}\vartheta \\
&+ \frac{1}{\Gamma(\kappa)} \int_{x_1}^{x_2} e^{-\tau(\Psi(X)-\Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) \mathrm{d}\vartheta \\
&+ \cdots + \frac{1}{\Gamma(\kappa)} \int_{x_p}^{X} e^{-\tau(\Psi(X)-\Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) \mathrm{d}\vartheta.
\end{aligned} \tag{19}$$

Remark 3. If we set $\Psi(X) = X$ and $\tau = 0$, then (18) will reduce to the subintegrals of Riemann–Liouville fractional integral defined by [18].

3. Pólya–Szegö-Type Tempered Fractional Integral Inequalities

In this section, we provide some new Pólya–Szegö-type tempered fractional integral inequalities for positive and integrable functions via tempered fractional integral (17) containing another function Ψ in the kernel.

Lemma 1. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0) = 0$. Suppose that $\mathscr{U}_1, \mathscr{U}_2, \mathscr{V}_1$ and $\mathscr{V}_2$ are four positive and integrable functions on $[0,\infty)$ such that*

$$\begin{aligned}
(H_1)\, 0 < \mathscr{U}_1(\vartheta) \leq f_1(\vartheta) \leq \mathscr{U}_2(\vartheta), 0 < \mathscr{V}_1(\vartheta) \leq f_2(\vartheta) \\
\leq \mathscr{V}_2(\vartheta), \vartheta \in [0,\theta], \theta > 0.
\end{aligned} \tag{20}$$

Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2 f_1^2\right)(\theta)\, {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2 f_2^2\right)(\theta)}{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1\mathscr{V}_1 + \mathscr{U}_2\mathscr{V}_2) f_1 f_2\}(\theta)\right)^2} \leq \frac{1}{4}. \tag{21}$$

Proof. From the given hypothesis, we have

$$\left(\frac{\mathscr{U}_2(\vartheta)}{\mathscr{V}_1(\vartheta)}-\frac{f_1(\vartheta)}{f_2(\vartheta)}\right)\geq 0. \tag{22}$$

Similarly, we have

$$\left(\frac{f_1(\vartheta)}{f_2(\vartheta)}-\frac{\mathscr{U}_1(\vartheta)}{\mathscr{V}_2(\vartheta)}\right)\geq 0. \tag{23}$$

Taking product of (22) and (23), we get

$$\left(\frac{\mathscr{U}_2(\vartheta)}{\mathscr{V}_1(\vartheta)}-\frac{f_1(\vartheta)}{f_2(\vartheta)}\right)\left(\frac{f_1(\vartheta)}{f_2(\vartheta)}-\frac{\mathscr{U}_1(\vartheta)}{\mathscr{V}_2(\vartheta)}\right)\geq 0. \tag{24}$$

From (24), it can be written as

$$\left(\mathscr{U}_1(\vartheta)\mathscr{V}_1(\vartheta)+\mathscr{U}_2(\vartheta)\mathscr{V}_2(\vartheta)\right)f_1(\vartheta)f_2(\vartheta)\geq\mathscr{V}_1(\vartheta)\mathscr{V}_2(\vartheta)f_1^2(\vartheta)+\mathscr{U}_1(\vartheta)\mathscr{U}_2(\vartheta)f_2^2(\vartheta). \tag{25}$$

Now, taking product of (25) with $e^{-\tau(\Psi(\theta)-\Psi(\theta))}(\Psi(\theta)-\Psi(\theta))^{\kappa-1}\Psi'(\theta)/\Gamma(\kappa)$ and integrating the resultant identity with respect to ϑ over $(0,\theta)$, we have

$$\begin{aligned}&\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\left(\mathscr{U}_1(\vartheta)\mathscr{V}_1(\vartheta)+\mathscr{U}_2(\vartheta)\mathscr{V}_2(\vartheta)\right)f_1(\vartheta)f_2(\vartheta)\mathrm{d}\vartheta\\&\geq\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\mathscr{V}_1(\vartheta)\mathscr{V}_2(\vartheta)f_1^2(\vartheta)\mathrm{d}\vartheta\\&+\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\mathscr{U}_1(\vartheta)\mathscr{U}_2(\vartheta)f_2^2(\vartheta)\mathrm{d}\vartheta.\end{aligned} \tag{26}$$

With the aid of Definition 8, we can write

$${}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left[\left(\mathscr{U}_1\mathscr{V}_1+\mathscr{U}_2\mathscr{V}_2\right)f_1f_2\right](\theta)\geq{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2f_1^2\right)(\theta)+{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2f_2^2\right)(\theta). \tag{27}$$

By applying AM-GM inequality, i.e., $x_1+x_2\geq 2\sqrt{x_1x_2}, x_1,x_2\in\mathbb{R}^+$, we get

$${}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left[\left(\mathscr{U}_1\mathscr{V}_1+\mathscr{U}_2\mathscr{V}_2\right)f_1f_2\right](\theta)\geq 2\sqrt{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2f_1^2\right)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2f_2^2\right)(\theta)}. \tag{28}$$

It follows that

$$\begin{aligned}&{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2f_1^2\right)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2f_2^2\right)(\theta)\\&\leq\frac{1}{4}\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left[\left(\mathscr{U}_1\mathscr{V}_1+\mathscr{U}_2\mathscr{V}_2\right)f_1f_2\right](\theta)\right)^2,\end{aligned} \tag{29}$$

which gives the desired assertion (21). □

Corollary 1. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Suppose that $\mathscr{U}_1,\mathscr{U}_2,\mathscr{V}_1$, and $\mathscr{V}_2$ are four positive and integrable functions on $[0,\infty)$ such that*

$$\begin{aligned}&(H_2)0<m_1\leq f_1(\vartheta)\leq M_1<\infty, 0<n_1\leq f_2(\vartheta)\leq N_1<\infty,\\&\vartheta\in[0,\theta],\theta>0.\end{aligned} \tag{30}$$

Then, for $\kappa>0$, $\tau\geq 0$, and $\theta>0$, the following tempered fractional integral inequality holds:

$$\begin{aligned}&\frac{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2f_1^2\right)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2f_2^2\right)(\theta)}{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{f_1f_2\}(\theta)\right)^2}\\&\leq\frac{1}{4}\left(\sqrt{\frac{m_1n_1}{M_1N_1}}+\sqrt{\frac{M_1N_1}{m_1n_1}}\right)^2.\end{aligned} \tag{31}$$

Lemma 2. *Let all the conditions of Lemma 1 hold. Then, for* $\kappa, \lambda > 0$, $\tau \geq 0$, *and* $\theta > 0$, *the following tempered fractional integral inequality holds:*

$$\frac{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_2^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2\right)(\theta)}{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1 f_1\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1 f_2\right)(\theta) + {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_2 f_1\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_2 f_2\right)(\theta)\right)^2} \leq \frac{1}{4}. \tag{32}$$

Proof. From the hypothesis (H_1) defined by (20), we have

$$\left(\frac{\mathscr{U}_2(\vartheta)}{\mathscr{V}_1(\zeta)} - \frac{f_1(\vartheta)}{f_2(\zeta)}\right) \geq 0, \tag{33}$$

$$\left(\frac{f_1(\vartheta)}{f_2(\zeta)} - \frac{\mathscr{U}_1(\vartheta)}{\mathscr{V}_2(\zeta)}\right) \geq 0, \tag{34}$$

which follows that

$$\left(\frac{\mathscr{U}_1(\vartheta)}{\mathscr{V}_2(\zeta)} + \frac{\mathscr{U}_2(\vartheta)}{\mathscr{V}_1(\zeta)}\right)\frac{f_1(\vartheta)}{f_2(\zeta)} \geq \frac{f_1^2(\vartheta)}{f_2^2(\zeta)} + \frac{\mathscr{U}_1(\vartheta)\mathscr{U}_2(\vartheta)}{\mathscr{U}_1(\zeta)\mathscr{U}_2(\zeta)}. \tag{35}$$

Taking product on both sides of (18) by $\mathscr{V}_1(\zeta)\mathscr{V}_2(\zeta) f_2^2(\zeta)$, we obtain

$$\begin{aligned}&\mathscr{U}_1(\vartheta) f_1(\vartheta)\mathscr{V}_1(\zeta) f_2(\zeta) + \mathscr{U}_2(\vartheta) f_1(\vartheta)\mathscr{V}_2(\zeta) f_2(\zeta)\\ &\quad\geq \mathscr{V}_1(\zeta)\mathscr{V}_2(\zeta) f_1^2(\vartheta) + \mathscr{U}_1(\vartheta)\mathscr{U}_2(\vartheta) f_2^2(\zeta).\end{aligned} \tag{36}$$

Taking product of both sides of (36) with $e^{-\tau(\Psi(\theta)-\Psi(\theta))}(\Psi(\theta)-\Psi(\theta))^{\kappa-1}\Psi'(\theta)/\Gamma(\kappa)$ and integrating the resultant identity with respect to ϑ over $(0,\theta)$, we have

$$\begin{aligned}&\mathscr{V}_1(\zeta) f_2(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\mathscr{U}_1(\vartheta) f_1(\vartheta)\mathrm{d}\vartheta\\ &\quad+\mathscr{V}_2(\zeta) f_2(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\mathscr{U}_2(\vartheta) f_1(\vartheta)\mathrm{d}\vartheta\\ &\quad\geq \mathscr{V}_1(\zeta)\mathscr{V}_2(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta) f_1^2(\vartheta)\mathrm{d}\vartheta\\ &\quad+f_2^2(\zeta)\frac{1}{\Gamma(\kappa)}\int_0^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\mathscr{U}_1(\vartheta)\mathscr{U}_2(\vartheta)\mathrm{d}\vartheta,\end{aligned} \tag{37}$$

which by applying (8) becomes

$$\begin{aligned}&\mathscr{V}_1(\zeta) f_2(\zeta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1 f_1\right)(\theta) + \mathscr{V}_2(\zeta) f_2(\zeta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_2 f_1\right)(\theta)\\ &\quad\geq \mathscr{V}_1(\zeta)\mathscr{V}_2(\zeta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta) + f_2^2(\zeta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2\right)(\theta).\end{aligned} \tag{38}$$

Again, taking product of both sides of (38) with $e^{-\tau(\Psi(\theta)-\Psi(\zeta))}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\zeta)/\Gamma(\lambda)$ and integrating the resultant identity with respect to ζ over $(0,\theta)$ and then applying (17), we get

$$\begin{aligned}&{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_1 f_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_1 f_1\right)(\theta) + {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_2 f_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_2 f_1\right)(\theta)\\ &\quad\geq {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_1\mathscr{V}_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta) + {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_1\mathscr{U}_2\right)(\theta).\end{aligned} \tag{39}$$

By using AM-GM inequality, we get

$$\begin{aligned}&{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_1 f_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_1 f_1\right)(\theta) + {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_2 f_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_2 f_1\right)(\theta)\\ &\quad\geq 2\sqrt{{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{V}_1\mathscr{V}_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\mathscr{U}_1\mathscr{U}_2\right)(\theta)}.\end{aligned} \tag{40}$$

It follows that

$$\begin{aligned}&{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_1\mathscr{V}_2)(\theta)^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta)^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)\\&\leq\frac{1}{4}\left({}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_1f_2)(\theta)^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_1f_1)(\theta)+{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_2f_2)(\theta)^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_2f_1)(\theta)\right)^2,\end{aligned}\tag{41}$$

which completes the desired assertion (17). □

Corollary 2. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$ satisfying the hypothesis (H_2) defined by (16) and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Then, for $\kappa,\lambda>0$, $\tau\geq 0$, and $\theta>0$, the following tempered fractional integral inequality holds:*

$$\begin{aligned}&\frac{\gamma(\kappa,\tau\Psi(\theta))\gamma(\kappa,\tau\Psi(\theta))}{\tau^{\kappa+\lambda}\Gamma(\kappa)\Gamma(\lambda)}\frac{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta)^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_2^2\right)(\theta)}{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\right)^2}\\&\leq\frac{1}{4}\left(\sqrt{\frac{m_1n_1}{M_1N_1}}+\sqrt{\frac{M_1N_1}{m_1n_1}}\right)^2,\end{aligned}\tag{42}$$

where

$$\begin{aligned}{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}[1]&=\frac{1}{\Gamma(\kappa)}\int_0^{\theta}\exp[-\tau(\Psi(\theta)-\Psi(\vartheta))](\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)d\vartheta\\&=\frac{1}{\Gamma(\kappa)}\int_0^{\Psi(\theta)}e^{-\tau z}z^{\kappa-1}dz,\ (\Psi(0)=0)\\&=\frac{1}{\tau^{\kappa}\Gamma(\kappa)}\int_0^{\tau\Psi(\theta)}e^{-u}u^{\kappa-1}du\\&=\frac{\gamma(\kappa,\tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)},\end{aligned}\tag{43}$$

and $\gamma(\kappa,t)=\int_0^t e^{-u}u^{\kappa-1}du$ is the well-known incomplete gamma function (see [22]).

Lemma 3. *Suppose that all the conditions of Lemma 1 hold and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Then, for $\kappa,\lambda>0$, $\tau\geq 0$, and $\theta>0$, the following tempered fractional integral inequality holds:*

$${}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta)^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_2^2\right)(\theta)\leq{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\frac{\mathscr{U}_2f_1f_2}{\mathscr{V}_1}\right)(\theta)^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\frac{\mathscr{V}_2f_1f_2}{\mathscr{U}_1}\right)(\theta).\tag{44}$$

Proof. From the hypothesis (H_1) defined by (20), we have

$$\begin{aligned}&\frac{1}{\Gamma(\kappa)}\int_0^{\theta}e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)f_1^2(\vartheta)d\vartheta\\&\leq\frac{1}{\Gamma(\kappa)}\int_0^{\theta}e^{-\tau(\Psi(\theta)-\Psi(\vartheta))}(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)\frac{\mathscr{U}_2(\vartheta)}{\mathscr{V}_1(\vartheta)}f_1(\vartheta)f_2(\vartheta)d\vartheta,\end{aligned}\tag{45}$$

which in view of (17) yields

$$ {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta) \le {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(\frac{\mathscr{U}_2 f_1 f_2}{\mathscr{V}_1}\right)(\theta). \tag{46} $$

Similarly, one can obtain

$$ {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta) \le {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(\frac{\mathscr{V}_2 f_1 f_2}{\mathscr{U}_1}\right)(\theta). \tag{47} $$

Hence, the product of (46) and (47) yields the desired assertion (44). □

Corollary 3. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$ satisfying the hypothesis (H_2) defined by (30) and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Then, for $\kappa,\lambda>0$, $\tau\ge 0$, and $\theta>0$, the following tempered fractional integral inequality holds:*

$$ \frac{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_2^2\right)(\theta)}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1 f_2\right)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1 f_2\right)(\theta)} \le \frac{M_1 N_1}{m_1 n_1}. \tag{48} $$

4. Chebyshev-Type Tempered Fractional Integral Inequalities

In this section, certain Chebyshev-type inequalities via tempered fractional integral (20) are presented with the help of Pólya–Szegö integral inequality given by Lemma 1.

Theorem 1. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0,\infty[$, and its derivative Ψ' is continuous on $[0,\infty[$ with $\Psi(0)=0$. Suppose that $\mathscr{U}_1,\mathscr{U}_2,\mathscr{V}_1$, and $\mathscr{V}_2$ are four positive and integrable functions on $[0,\infty)$ satisfying the hypothesis (H_1) defined by (10). Then, for $\kappa,\lambda>0$, $\tau\ge 0$, and $\theta>0$, the following tempered fractional integral inequality holds:*

$$ \begin{aligned} &\left|\frac{\gamma(\kappa,\tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)}\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda,\tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)}\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\right| \\ &\quad \le \left|F_1(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) + F_2(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta)\right|^{1/2} \times \left|F_1(f_2,\mathscr{V}_1,\mathscr{V}_2)(\theta) + F_2(f_1,\mathscr{V}_1,\mathscr{V}_2)(\theta)\right|^{1/2}, \end{aligned} \tag{49} $$

where

$$ F_1(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) = \frac{\gamma(\lambda,\tau\Psi(\theta))}{4\tau^{\lambda}\Gamma(\lambda)}\,\frac{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1+\mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta), \tag{50} $$

$$ F_2(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) = \frac{\gamma(\kappa,\tau\Psi(\theta))}{4\tau^{\kappa}\Gamma(\kappa)}\,\frac{\left({}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\{(\mathscr{U}_1+\mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta). \tag{51} $$

Proof. By the given hypothesis, both the functions f_1 and f_2 are positive and integrable functions on $[0,\infty)$. Therefore, for $\vartheta,\zeta\in(0,\theta)$ with $\theta>0$, we define $\mathscr{A}(\vartheta,\zeta)$ by

$$ \begin{aligned} \mathscr{A}(\vartheta,\zeta) &= \left(f_1(\vartheta)-f_1(\zeta)\right)\left(f_2(\vartheta)-f_2(\zeta)\right) \\ &= f_1(\vartheta)f_2(\vartheta) + f_1(\zeta)f_2(\zeta) - f_1(\vartheta)f_2(\zeta) \\ &\quad - f_1(\zeta)f_2(\vartheta). \end{aligned} \tag{52} $$

Multiplying (52) by $(1/\Gamma(\kappa)\Gamma(\lambda))\exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]$ $\exp[-\tau(\Psi(\theta)-\Psi(\zeta))](\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}$ $\Psi'(\vartheta)\Psi'(\zeta)$ and double integrating the resultant identity with respect to ϑ and ζ over $(0,\theta)$ and then using (17), we obtain

$$ \begin{aligned} &\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^{\theta}\int_0^{\theta}\exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))] \\ &\quad \times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)\mathscr{A}(\vartheta,\zeta)d\vartheta d\zeta \\ &= \frac{\gamma(\kappa,\tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)}\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda,\tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)}\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) \\ &\quad - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta) - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta)\,{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta). \end{aligned} \tag{53} $$

By applying Cauchy–Schwartz inequality for double integrals, we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\right.\\
&\quad\left.\cdot(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)\mathscr{A}(\vartheta,\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\right|\\
&\leq\left[\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\right.\\
&\quad\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_1^2(\vartheta)\mathrm{d}\vartheta\mathrm{d}\zeta\\
&\quad+\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\\
&\quad\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_1^2(\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\\
&\quad-2\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\\
&\quad\left.\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_1(\vartheta)f_1(\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\right]^{1/2}\\
&\quad\times\left[\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\right.\\
&\quad\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_2^2(\vartheta)\mathrm{d}\vartheta\mathrm{d}\zeta\\
&\quad+\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\\
&\quad\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_2^2(\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\\
&\quad-2\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\\
&\quad\left.\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)f_2(\vartheta)f_2(\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\right]^{1/2}.
\end{aligned}\tag{54}$$

In view of (17) and (43), we get

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\kappa)\Gamma(\lambda)}\int_0^\theta\int_0^\theta \exp[-\tau(\Psi(\theta)-\Psi(\vartheta))]\exp[-\tau(\Psi(\theta)-\Psi(\zeta))]\times(\Psi(\theta)-\Psi(\vartheta))^{\kappa-1}\right.\\
&\quad\left.\cdot(\Psi(\theta)-\Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\vartheta)\mathscr{A}(\vartheta,\zeta)\mathrm{d}\vartheta\mathrm{d}\zeta\right|\\
&\leq\left[\frac{\gamma(\kappa,\tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)}{}^\Psi\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta)+\frac{\gamma(\lambda,\tau\Psi(\theta))}{\tau^\lambda\Gamma(\lambda)}{}^\Psi\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta)-2{}^\Psi\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta){}^\Psi\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta)\right]^{1/2}\\
&\quad\times\left[\frac{\gamma(\kappa,\tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)}{}^\Psi\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)+\frac{\gamma(\lambda,\tau\Psi(\theta))}{\tau^\lambda\Gamma(\lambda)}{}^\Psi\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)-2{}^\Psi\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta){}^\Psi\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta)\right]^{1/2}.
\end{aligned}\tag{55}$$

Applying Lemma 1 for $\mathscr{V}_1(\theta) = \mathscr{V}_2(\theta) = f_2(\theta) = 1$, we get

$$\frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta) \leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^{\lambda}\Gamma(\lambda)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)}. \tag{56}$$

It follows that

$$\begin{aligned}
&\frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta) \\
&\leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^{\lambda}\Gamma(\lambda)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} \\
&\quad - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta) \\
&= F_1(f_1, \mathscr{U}_1, \mathscr{U}_2)(\theta).
\end{aligned} \tag{57}$$

Similarly, one can get

$$\begin{aligned}
&\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_1^2\right)(\theta) - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta) \\
&\leq \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^{\kappa}\Gamma(\kappa)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} \\
&\quad - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta) \\
&= F_2(f_1, \mathscr{U}_1, \mathscr{U}_2)(\theta).
\end{aligned} \tag{58}$$

Again applying Lemma 1 for $\mathscr{U}_1(\theta) = \mathscr{U}_2(\theta) = f_1(\theta) = 1$, we get

$$\begin{aligned}
&\frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\left(f_2^2\right)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta) \\
&\leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^{\lambda}\Gamma(\lambda)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{V}_1 + \mathscr{V}_2)f_2\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(\mathscr{V}_1\mathscr{V}_2)(\theta)} \\
&\quad - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta) \\
&= F_1(f_2, \mathscr{U}_1, \mathscr{U}_2)(\theta),
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta) - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta){}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta) \\
&\leq \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^{\kappa}\Gamma(\kappa)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\lambda,\tau}\{(\mathscr{V}_1 + \mathscr{V}_2)f_2\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_1\mathscr{V}_2)(\theta)} \\
&\quad - {}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta){}^{\Psi}\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta) \\
&= F_2(f_2, \mathscr{U}_1, \mathscr{U}_2)(\theta).
\end{aligned} \tag{60}$$

Thus, by considering (53) to (60), we arrive at the desired assertion (49). This is the desired proof of Theorem 1. □

Theorem 2. *Suppose that all the conditions of Theorem 1 are satisfied. Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1f_2)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\right| \\
&\leq \left|F(f_1, \mathscr{U}_1, \mathscr{U}_2)(\theta)F(f_2, \mathscr{V}_1, \mathscr{V}_2)(\theta)\right|^{1/2},
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
F(f_1, \mathscr{U}_1, \mathscr{U}_2)(\theta) = {}&\frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^{\kappa}\Gamma(\kappa)} \frac{\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{{}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} \\
&-\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\right)^2.
\end{aligned} \tag{62}$$

Proof. Applying Theorem 1 for $\kappa = \lambda$, we get the desired result in (61). □

Remark 4. If we consider $\mathscr{U}_1 = m_1$, $\mathscr{U}_2 = M_1$, $\mathscr{V}_1 = n_1$, and $\mathscr{V}_2 = N_1$, then we have

$$F(f_1, m_1, M_1)(\theta) = \frac{(M_1 - m_1)^2}{4M_1m_1}\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\right)^2, \tag{63}$$

$$F(f_2, m_1, M_1)(\theta) = \frac{(N_1 - n_1)^2}{4N_1n_1}\left({}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\right)^2. \tag{64}$$

Corollary 4. *Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ and satisfying the hypothesis (H_2) given by (30). Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1f_2)(\theta) - {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\right| \\
&\leq \frac{(M_1 - m_1)(N_1 - n_1)}{4\sqrt{m_1M_1n_1N_1}} {}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathscr{R}_0^{\kappa,\tau}(f_2)(\theta).
\end{aligned} \tag{65}$$

5. Particular Cases

The following new Pólya–Szegö- and Chebyshev-type inequalities for classical tempered fractional integral (14) can be easily established.

Lemma 4. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$. Suppose that $\mathscr{U}_1, \mathscr{U}_2, \mathscr{V}_1$, and $\mathscr{V}_2$ are four positive and integrable functions on $[0,\infty)$ satisfying the hypothesis (H_1) defined by (20). Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:*

$$\frac{\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{V}_1\mathscr{V}_2 f_1^2\right)(\theta)\mathscr{R}_0^{\kappa,\tau}\left(\mathscr{U}_1\mathscr{U}_2 f_2^2\right)(\theta)}{\left(\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1\mathscr{V}_1 + \mathscr{U}_2\mathscr{V}_2)f_1 f_2\}(\theta)\right)^2} \leq \frac{1}{4}. \tag{66}$$

Proof. Applying Lemma 1 for $\Psi(\theta) = \theta$, we get Lemma 4. □

Lemma 5. *Let all the conditions of Lemma 4 are satisfied. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:*

$$\frac{\mathscr{R}_0^{\kappa,\tau}\left(f_1^2\right)(\theta)\mathscr{R}_0^{\lambda,\tau}\left(f_2^2\right)(\theta)\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_1\mathscr{V}_2)(\theta)}{\left(\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1 f_1)(\theta)\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_1 f_2)(\theta) + \mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_2 f_1)(\theta)\mathscr{R}_0^{\lambda,\tau}(\mathscr{V}_2 f_2)(\theta)\right)^2} \leq \frac{1}{4}. \tag{67}$$

Proof. Applying Lemma 2 for $\Psi(\theta) = \theta$, we get Lemma 5. □

Similarly, we can derive the particular case of Lemma 3. The following theorem represents the particular case of Theorem 1 in terms of classical tempered fractional integral.

Theorem 3. *Let the functions f_1 and f_2 be positive and integrable on $[0,\infty)$. Suppose that $\mathscr{U}_1, \mathscr{U}_2, \mathscr{V}_1$, and $\mathscr{V}_2$ are four positive and integrable functions on $[0,\infty)$ satisfying the hypothesis (H_1) defined by (20). Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:*

$$\begin{aligned}&\left|\frac{\gamma(\kappa,\tau\theta)}{\tau^{\kappa}\Gamma(\kappa)}\mathscr{R}_0^{\kappa,\tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda,\tau\theta)}{\tau^{\lambda}\Gamma(\lambda)}\mathscr{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) - \mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\mathscr{R}_0^{\lambda,\tau}(f_2)(\theta) - \mathscr{R}_0^{\kappa,\tau}(f_2)(\theta)\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta)\right| \\ &\leq \left|F_1(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) + F_2(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta)\right|^{1/2} \times \left|F_1(f_2,\mathscr{V}_1,\mathscr{V}_2)(\theta) + F_2(f_1,\mathscr{V}_1,\mathscr{V}_2)(\theta)\right|^{1/2},\end{aligned} \tag{68}$$

where

$$F_1(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) = \frac{\gamma(\lambda,\tau\theta)}{4\tau^{\lambda}\Gamma(\lambda)}\frac{\left(\mathscr{R}_0^{\kappa,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{\mathscr{R}_0^{\kappa,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} - \mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta), \tag{69}$$

$$F_2(f_1,\mathscr{U}_1,\mathscr{U}_2)(\theta) = \frac{\gamma(\kappa,\tau\theta)}{4\tau^{\kappa}\Gamma(\kappa)}\frac{\left(\mathscr{R}_0^{\lambda,\tau}\{(\mathscr{U}_1 + \mathscr{U}_2)f_1\}\right)^2}{\mathscr{R}_0^{\lambda,\tau}(\mathscr{U}_1\mathscr{U}_2)(\theta)} - \mathscr{R}_0^{\kappa,\tau}(f_1)(\theta)\mathscr{R}_0^{\lambda,\tau}(f_1)(\theta). \tag{70}$$

Proof. Applying Theorem 1 for $\Psi(\theta) = \theta$, then we get the desired Theorem 3. □

Similarly, we can derive particular result of Theorem 2.

6. Applications

In this section, we define a way for constructing four bounded functions and then utilize them to present certain estimates of Chebyshev-type tempered fractional integral inequalities of two unknown functions.

Let the unit function $\hbar(\theta)$ be defined by

$$\hbar(\theta) = \begin{cases} 1, \theta > 0, \\ 0, \theta \leq 0, \end{cases} \tag{71}$$

and let the Heaviside unit step function $\hbar_a(\theta)$ be defined by

$$\hbar_a(\theta) = \begin{cases} 1, \theta > a, \\ 0, \theta \leq a. \end{cases} \tag{72}$$

Suppose that the function $\mathscr{U}_1$ is piecewise continuous function on $[0, X]$ defined by

$$\begin{aligned}\mathscr{U}_1(x) &= m_{1_1}\left(\hbar_0(x) - \hbar_{x_1}(x)\right) + m_{1_2}\left(\hbar_{x_1}(x) - \hbar_{x_2}(x)\right) + \cdots + m_{1_{p+1}}\hbar_{x_p}(x)\\ &= m_{1_1}\hbar_0(x) + \left(m_{1_2} - m_{1_1}\right)\hbar_{x_1}(x) + \left(m_{1_3} - m_{1_2}\right)\hbar_{x_2}(x) + \cdots + \left(m_{1_{p+1}} - m_{1_p}\right)\hbar_{x_p}(x)\\ &= \sum_{i=0}^{p}\left(m_{1_{i+1}} - m_{1_i}\right)\hbar_{x_i}(x),\end{aligned} \tag{73}$$

where $m_{1_0} = 0$ and $0 = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = X$. Similarly, we define

$$\mathscr{U}_2(x) = \sum_{i=0}^{p}\left(M_{1_{i+1}} - M_{1_i}\right)\hbar_{x_i}(x), \tag{74}$$

$$\mathscr{V}_1(x) = \sum_{i=0}^{p}\left(n_{1_{i+1}} - n_{1_i}\right)\hbar_{x_i}(x), \tag{75}$$

$$\mathscr{V}_2(x) = \sum_{i=0}^{p}\left(N_{1_{i+1}} - N_{1_i}\right)\hbar_{x_i}(x), \tag{76}$$

where the constants $n_{1_0} = N_{1_0} = M_{1_0} = 0$. If there exists an integrable function f_1 on $[0, X]$ satisfying the hypothesis (H_1), then we have $m_{1_{i+1}} \leq f_1(x) \leq M_{1_{i+1}}$ for each $x \in (x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, p$.

Proposition 1. *Let the functions f_1 and f_2 be two positive and integrable on $[0, X]$. Assume that the functions $\mathscr{U}_1$, $\mathscr{U}_2$, $\mathscr{V}_1$, and $\mathscr{V}_2$ are defined by (73)–(76), respectively, and satisfying the hypothesis (H_1) defined by (30). Then, for $\kappa > 0$, the following inequality for tempered fractional integral holds:*

$$\begin{aligned}&\left(\sum_{i=0}^{p} n_{1_{i+1}} N_{1_{i+1}}\, {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1^2\right)(X)\right)^2 \left(\sum_{i=0}^{p} m_{1_{i+1}} M_{1_{i+1}}\, {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_2^2\right)(X)\right)^2\\ &\quad\leq \frac{1}{4}\sum_{i=0}^{p}\left(n_{1_{i+1}} N_{1_{i+1}} + m_{1_{i+1}} M_{1_{i+1}}\right) {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1 f_2\right)(X).\end{aligned} \tag{77}$$

Proof. By applying the Definition 8, we have

$${}^{\Psi}\mathscr{R}^{\kappa,\tau}_{0,X}\left(\mathscr{V}_1\mathscr{V}_2 f_1^2\right)(X) = \sum_{i=0}^{p} n_{1_{i+1}} N_{1_{i+1}}\, {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1^2\right)(X), \tag{78}$$

$${}^{\Psi}\mathscr{R}^{\kappa,\tau}_{0,X}\left(\mathscr{U}_1\mathscr{U}_2 f_2^2\right)(X) = \sum_{i=0}^{p} m_{1_{i+1}} M_{1_{i+1}}\, {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_2^2\right)(X), \tag{79}$$

$${}^{\Psi}\mathscr{R}^{\kappa,\tau}_{0,X}\left\{\left(\mathscr{U}_1\mathscr{V}_1 + \mathscr{U}_2\mathscr{V}_2\right) f_1 f_2\right\}(X) = \sum_{i=0}^{p}\left(m_{1_{i+1}} n_{1_{i+1}} + M_{1_{i+1}} N_{1_{i+1}}\right) {}^{\Psi}\mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1 f_2\right)(X). \tag{80}$$

Hence, by applying Lemma 1, we get the desired assertion (77). □

Proposition 2. *By setting $\Psi(\theta) = \theta$ in Proposition 1, then we arrive to the following result in terms of classical tempered fractional integral:*

$$\begin{aligned}&\left(\sum_{i=0}^{p} n_{1_{i+1}} N_{1_{i+1}} \mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1^2\right)(X)\right)^2 \left(\sum_{i=0}^{p} m_{1_{i+1}} M_{1_{i+1}} \mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_2^2\right)(X)\right)^2\\ &\quad\leq \frac{1}{4}\sum_{i=0}^{p}\left(n_{1_{i+1}} N_{1_{i+1}} + m_{1_{i+1}} M_{1_{i+1}}\right) \mathscr{R}^{\kappa,\tau}_{x_i,x_{i+1}}\left(f_1 f_2\right)(X).\end{aligned} \tag{81}$$

Remark 5. Throughout in the paper, if we apply $\Psi(\theta) = \theta$ and $\tau = 0$, then all the newly presented inequalities will be reduced to the work derived earlier by Ntouyas et al. [18].

7. Concluding Remarks

Certain new Pólya–Szegö- and Chebyshev-type inequalities by utilizing tempered fractional integral are presented in this paper. These inequalities generalized the existing inequalities. We can easily get the said Pólya–Szegö- and Chebyshev-type inequalities for Katugampola, generalized Riemann–Liouville, classical Riemann–Liouville, generalized conformable, and conformable fractional integrals by applying different conditions on function Ψ given in Remark 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally, and they have read and approved the final manuscript for publication.

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Research Article

Trapezium-Type Inequalities for k-Fractional Integral via New Exponential-Type Convexity and Their Applications

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In this paper, the authors investigated the concept of (s, m)-exponential-type convex functions and their algebraic properties. New generalizations of Hermite–Hadamard-type inequality for the (s, m)-exponential-type convex function ψ and for the products of two (s, m)-exponential-type convex functions ψ and ϕ are proved. Many refinements of the (H–H) inequality via (s, m)-exponential-type convex are obtained. Finally, several new bounds for special means and new error estimates for the trapezoidal and midpoint formula are provided as well. The ideas and techniques of this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

Theory of convexity also played a significant role in the development of theory of inequalities. Many famously known results in inequalities theory can be obtained using the convexity property of the functions. Hermite–Hadamard's double inequality is one of the most intensively studied results involving convex functions. This result provides us necessary and sufficient condition for a function to be convex. It is also known as classical equation of H–H inequality.

The Hermite–Hadamard inequality assert that if a function ψ: $J \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ is convex in J for $a_1, a_2 \in J$ and $a_1 < a_2$, then

$$\psi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \psi(\chi) d\chi \leq \frac{\psi(a_1) + \psi(a_2)}{2}. \quad (1)$$

Interested readers can refer to [1–29].

Definition 1 (see [30]). A function ψ: $[0, +\infty) \longrightarrow \mathfrak{R}$ is said to be s-convex in the second sense for a real number $s \in (0, 1]$ or ψ belongs to the class of K_s^2, if

$$\psi(\chi a_1 + (1 - \chi) a_2) \leq \chi^s \psi(a_1) + (1 - \chi)^s \psi(a_2), \quad (2)$$

holds for all $a_1, a_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

A s-convex function was introduced in Breckner's article in [30], and a number of properties and connections with s-convexity in the first sense are discussed in [10]. Usually, convexity means for s-convexity when $s = 1$. Dragomir et al. proved a variant of Hadamard's inequality in [5], which holds for s-convex functions in the second sense. In the last decade, many mathematicians added the rich literature in the field of mathematical inequalities involving fractional calculus (see [8, 9, 13, 15, 22, 25, 28, 31]).

Toader introduced the class of m-convex functions in [26].

Definition 2 (see [26]). A function $\psi\colon [0,a_2] \longrightarrow \mathfrak{R}$, $a_2>0$, is said to be m-convex, where $m\in(0,1]$, if

$$\psi(\chi\theta_1+m(1-\chi)\theta_2)\le\chi\psi(\theta_1)+m(1-\chi)\psi(\theta_2) \tag{3}$$

holds $\forall\,\theta_1,\theta_2\in[0,a_2]$ and $\chi\in[0,1]$. Otherwise, ψ is m-concave if $(-\psi)$ is m-convex.

In a recent paper, Eftekhari [6] defined the class of (s,m)-convex functions in the second sense as follows.

Definition 3. A function $\psi\colon[0,+\infty)\longrightarrow\mathfrak{R}$ is said to be (s,m)-convex for some fixed real numbers $s,m\in(0,1]$ if

$$\psi(\chi a_1+m(1-\chi)a_2)\le\chi^s\psi(a_1)+m(1-\chi)^s\psi(a_2) \tag{4}$$

holds $\forall\,a_1,a_2\in[0,+\infty)$ and $\chi\in[0,1]$.

Fractional integral inequalities are useful to find the uniqueness of solutions for certain fractional partial differential equations (see [19, 32]).

Let $\psi\in L[a_1,a_2]$. Then, Riemann–Liouville fractional integrals of order $\alpha>0$ with $a_1\ge0$ are defined as follows:

$$\begin{aligned}J^{\alpha}_{a_1^+}\psi(x)&=\frac{1}{\Gamma(\alpha)}\int_{a_1}^{x}(x-\chi)^{\alpha-1}\psi(\chi)\mathrm{d}\chi,\quad x>a_1,\\ J^{\alpha}_{a_2^-}\psi(x)&=\frac{1}{\Gamma(\alpha)}\int_{x}^{a_2}(\chi-x)^{\alpha-1}\psi(\chi)\mathrm{d}\chi,\quad x<a_2.\end{aligned} \tag{5}$$

For further details, one may see [14, 16, 18].

In [7, 17], there is given definition of k–fractional Riemann–Liouville integrals.

Let $\psi\in L[a_1,a_2]$. Then, k–fractional integrals of order $\alpha,k>0$ with $a_1\ge0$ are defined as follows:

$$\begin{aligned}{}^{k}J^{\alpha}_{a_1^+}\psi(x)&=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{x}(x-\chi)^{\alpha/k-1}\psi(\chi)\mathrm{d}\chi,\quad x>a_1,\\ {}^{k}J^{\alpha}_{a_2^-}\psi(x)&=\frac{1}{k\Gamma_k(\alpha)}\int_{x}^{a_2}(\chi-x)^{\alpha/k-1}\psi(\chi)\mathrm{d}\chi,\quad x<a_2,\end{aligned} \tag{6}$$

where $\Gamma_k(\alpha)$ is the k–gamma function defined as

$$\Gamma_k(\alpha)=\int_0^{+\infty}\chi^{\alpha-1}e^{-\chi^k/k}\mathrm{d}\chi. \tag{7}$$

We can notice that

$$\begin{aligned}&\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha),\\ &{}^{1}J^{0}_{a_1^+}\psi(x)={}^{1}J^{0}_{a_2^-}\psi(x)=\psi(x).\end{aligned} \tag{8}$$

By choosing $k=1$, the above k–fractional integrals yield Riemann–Liouville integrals.

Also that, the incomplete gamma function $\gamma(\vartheta,\theta)$ is defined for $\vartheta>0$ and $\theta\ge0$ by integral

$$\gamma(\vartheta,\theta)=\int_0^{\theta}e^{-\mu}\mu^{\vartheta-1}\mathrm{d}\mu. \tag{9}$$

The gamma function $\Gamma(\vartheta)$ is defined for $\vartheta>0$ by integral

$$\Gamma(\vartheta)=\int_0^{+\infty}e^{-\mu}\mu^{\vartheta-1}\mathrm{d}\mu. \tag{10}$$

Motivated by above results and literatures, we will give first in Section 2 the concept of (s,m)-exponential-type convex function, and we will study some of their algebraic properties. In Section 3, we will prove new generalizations of Hermite–Hadamard-type inequality for the (s,m)-exponential-type convex function ψ and for the products of two (s,m)-exponential-type convex functions ψ and ϕ. In Section 4, we will obtain some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power is (s,m)-exponential-type convex. In Section 5, some new bounds for special means and error estimates for the trapezoidal and midpoint formula will be provided. In Section 6, a briefly conclusion will be given as well.

2. Some Algebraic Properties of (s,m)-Exponential-Type Convex Functions

In this section, we will introduce a new definition, called (s,m)-exponential-type convex function, and we will study some basic algebraic properties of it.

Definition 4. A nonnegative function $\psi\colon J\longrightarrow\mathfrak{R}$ is said (s,m)-exponential-type convex for some fixed $s,m\in(0,1]$ if

$$\psi(\chi\theta_1+m(1-\chi)\theta_2)\le(e^{s\chi}-1)\psi(\theta_1)+m\left(e^{(1-\chi)s}-1\right)\psi(\theta_2) \tag{11}$$

holds for all $\theta_1,\theta_2\in J$ and $\chi\in[0,1]$.

Remark 1. For $m=s=1$, we get exponential- type convexity given by İşcan in [11].

Remark 2. (s,m)-exponential-type convex functions for some fixed $m\in(0,1]$ and $s\in[\ln2.5,1]$ have the range $[0,+\infty)$.

Proof. Let $\theta\in J$ be arbitrary for some fixed $m\in(0,1]$ and $s\in[\ln2.5,1]$. Using the Definition 4 for $\chi=1$, we have

$$\psi(\theta)\le(e^s-1)\psi(\theta)\Longrightarrow(e^s-2)\psi(\theta)\ge0\Longrightarrow\psi(\theta)\ge0. \tag{12}$$

□

Lemma 1. *For all $\chi\in[0,1]$ and for some fixed $m\in(0,1]$ and $s\in[ln2.5,1]$, the following inequalities $(e^{s\chi}-1)\ge\chi^s$ and $(e^{(1-\chi)s}-1)\ge(1-\chi)^s$ hold.*

Proof. The proof is evident. □

Proposition 1. *Every nonnegative (s,m)-convex function is (s,m)-exponential-type convex function for some fixed $m\in(0,1]$ and $s\in[ln2.5,1]$.*

Proof. By using Lemma 1, for some fixed $m \in (0,1]$ and $s \in [\ln 2.5, 1]$, we have

$$\begin{aligned}&\psi(\chi\theta_1 + m(1-\chi)\theta_2) \leq \chi^s\psi(\theta_1) + m(1-\chi)^s\psi(\theta_2)\\&\leq (e^{s\chi}-1)\psi(\theta_1) + m\left(e^{(1-\chi)s}-1\right)\psi(\theta_2).\end{aligned} \tag{13}$$

□

Theorem 1. *Let ψ, ϕ: $[a_1, a_2] \longrightarrow \mathfrak{R}$. If ψ and ϕ are (s,m)-exponential-type convex functions for some fixed $s, m \in (0,1]$, then the following holds:*

(1) $\psi + \phi$ is (s,m)-exponential-type convex function.

(2) For nonnegative real number c, $c\psi$ is (s,m)-exponential-type convex function.

Proof. By Definition 4, for some fixed $s, m \in (0,1]$, the proof is obvious. □

Theorem 2. *Let ψ: $[0, a_2] \longrightarrow J$ be m-convex function for $a_2 > 0$ and some fixed $m \in (0,1]$ and ϕ: $J \longrightarrow \mathfrak{R}$ is nondecreasing and (s,m)-exponential-type convex function for some fixed $s, m \in (0,1]$. Then, for the same fixed numbers $s, m \in (0,1]$, the function $\phi^\circ\psi$: $[0, a_2] \longrightarrow \mathfrak{R}$ is (s,m)-exponential-type convex.*

Proof. For all $\theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0,1]$ and for some fixed numbers $s, m \in (0,1]$, we have

$$\begin{aligned}&(\phi^\circ\psi)(\chi\theta_1 + m(1-\chi)\theta_2) = \phi(\psi(\chi\theta_1 + m(1-\chi)\theta_2))\\&\leq \phi(\chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2))\\&\leq (e^{s\chi}-1)(\phi^\circ\psi)(\theta_1) + m\left(e^{(1-\chi)s}-1\right)(\phi^\circ\psi)(\theta_2).\end{aligned} \tag{14}$$

□

Theorem 3. *Let ψ_i: $[a_1, a_2] \longrightarrow \mathfrak{R}$ be an arbitrary family of (s,m)-exponential-type convex functions for the same fixed $s, m \in (0,1]$ and let $\psi(\theta) = sup_i\psi_i(\theta)$. If $A = \{\theta \in [a_1, a_2]$: $\psi(\theta) < +\infty\} \neq \varnothing$, then A is an interval and ψ is (s,m)-exponential-type convex function on A.*

Proof. For all $\theta_1, \theta_2 \in A$ and $\chi \in [0,1]$ and for the same fixed numbers $s, m \in (0,1]$, we have

$$\begin{aligned}&\psi(\chi\theta_1 + m(1-\psi)\theta_2) = \sup_i\psi_i(\chi\theta_1 + m(1-\chi)\theta_2)\\&\leq \sup_i\left[(e^{s\chi}-1)\psi_i(\theta_1) + m\left(e^{(1-\chi)s}-1\right)\psi_i(\theta_2)\right]\\&\leq (e^{s\chi}-1)\sup_i\psi_i(\theta_1) + m\left(e^{(1-\chi)s}-1\right)\sup_i\psi_i(\theta_2)\\&= (e^{s\chi}-1)\psi(\theta_1) + m\left(e^{(1-\chi)s}-1\right)\psi(\theta_2) < +\infty.\end{aligned} \tag{15}$$

This means simultaneously that A is an interval, and ψ is (s,m)-exponential-type convex function on A. □

Theorem 4. *If the function ψ: $[a_1, a_2] \longrightarrow \mathfrak{R}$ is (s,m)-exponential-type convex for some fixed $s, m \in (0,1]$, then ψ is bounded on $[a_1, ma_2]$.*

Proof. Let $L = \max\{\psi(a_1), \psi(a_2/m)\}$ and $x \in [a_1, a_2]$ be an arbitrary point for some fixed $m \in (0,1]$. Then, there exists $\chi \in [0,1]$ such that $x = \chi a_1 + (1-\chi)a_2$. Thus, since $e^{s\chi} \leq e^s$ and $e^{(1-\chi)s} \leq e^s$ for some fixed $s \in (0,1]$, we have

$$\begin{aligned}\psi(x) &= \psi(\chi a_1 + (1-\chi)a_2) \leq (e^{s\chi}-1)\psi(a_1)\\&\quad + m\left(e^{(1-\chi)s}-1\right)\psi\left(\frac{a_2}{m}\right)\\&\leq (e^s-1)L + m(e^s-1)L = (m+1)L(e^s-1) = M.\end{aligned} \tag{16}$$

We have shown that ψ is bounded above from real number M. Interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 4 in [11]. □

3. New Generalizations of (H–H)-Type Inequality

In this section, we will establish some new generalizations of Hermite–Hadamard-type inequality for the (s,m)-exponential-type convex function ψ and for the products of two (s,m)-exponential-type convex functions ψ and ϕ.

Theorem 5. *Let ψ: $[a_1, ma_2] \longrightarrow \mathfrak{R}$ be (s,m)-exponential-type convex function for some fixed $s, m \in (0,1]$ and $a_1 < ma_2$. If $\psi \in L_1([a_1, ma_2])$, then*

$$\begin{aligned}&\frac{1}{\left(e^{s/2}-1\right)}\psi\left(\frac{a_1 + ma_2}{2}\right) \leq \frac{1}{(ma_2 - a_1)}\\&\left\{\int_{a_1}^{ma_2}\psi(x)\mathrm{d}x + m\int_{a_1/m}^{a_2}\psi(x)\mathrm{d}x\right\}\\&\leq \left(\frac{e^s - s - 1}{s}\right)\left[\psi(a_1) + \psi(a_2) + m\left(\psi\left(\frac{a_1}{m^2}\right) + \psi(a_2)\right)\right].\end{aligned} \tag{17}$$

Proof. Let denote, respectively,

$$\theta_1 = \chi a_1 + m(1-\chi)a_2, \theta_2 = (1-\chi)\frac{a_1}{m} + \chi a_2, \quad \forall \chi \in [0,1]. \tag{18}$$

Using (s,m)-exponential-type convexity of ψ, we have

$$\begin{aligned}&\psi\left(\frac{a_1 + ma_2}{2}\right) = \psi\left(\frac{\theta_1 + m\theta_2}{2}\right)\\&= \psi\left(\frac{[\chi a_1 + m(1-\chi)a_2] + [(1-\chi)a_1 + m\chi a_2]}{2}\right)\\&\leq \left(e^{s/2}-1\right)[\psi(\chi a_1 + m(1-\chi)a_2) + \psi((1-\chi)a_1 + m\chi a_2)].\end{aligned} \tag{19}$$

Now, integrating on both sides in the last inequality with respect to χ over $[0,1]$, we get

$$\psi\left(\frac{a_1+ma_2}{2}\right)\le\left(e^{s/2}-1\right)\times\left[\int_0^1\psi(\chi a_1+m(1-\chi)a_2)\mathrm{d}\chi\right.$$
$$\left.+\int_0^1\psi\left((1-\chi)\frac{a_1}{m}+\chi a_2\right)\mathrm{d}\chi\right]$$
$$=\frac{\left(e^{s/2}-1\right)}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\mathrm{d}x+m\int_{a_1/m}^{a_2}\psi(x)\mathrm{d}x\right\},\tag{20}$$

which completes the left-side inequality. For the right-side inequality, using (s,m)-exponential-type convexity of ψ, we obtain

$$\frac{1}{(ma_2-a_1)}\left\{\int_{a_1}^{ma_2}\psi(x)\mathrm{d}x+m\int_{\frac{a_1}{m}}^{a_2}\psi(x)\mathrm{d}x\right\}$$
$$=\int_0^1\psi(\chi a_1+m(1-\chi)a_2)\mathrm{d}\chi+\int_0^1\psi\left((1-\chi)\frac{a_1}{m}+\chi a_2\right)\mathrm{d}\chi$$
$$\le\int_0^1\left[(e^{s\chi}-1)\psi(a_1)+m\left(e^{(1-\chi)s}-1\right)\psi(a_2)\right]\mathrm{d}\chi$$
$$+\int_0^1\left[(e^{s\chi}-1)\psi(a_2)+m\left(e^{(1-\chi)s}-1\right)\psi\left(\frac{a_1}{m^2}\right)\right]\mathrm{d}\chi$$
$$=\left(\frac{e^s-s-1}{s}\right)\left[\psi(a_1)+\psi(a_2)+m\left(\psi\left(\frac{a_1}{m^2}\right)+\psi(a_2)\right)\right],\tag{21}$$

which gives the right-side inequality. □

Corollary 1. *By taking $m=s=1$ in Theorem 5, we get (Theorem 5, [11]).*

Theorem 6. *Assume that ψ,ϕ: $[a_1,ma_2]\longrightarrow\mathfrak{R}$ are, respectively, (s_1,m) and (s_2,m)-exponential-type convex functions for the same fixed $m\in(0,1]$ and for some fixed $s_1,s_2\in(0,1]$, where $s_1<s_2$ and $a_1<ma_2$. If ψ,ϕ are synchronous functions and $\psi,\phi,\psi\phi\in L_1([a_1,ma_2])$, then*

$$\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\mathrm{d}\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)\mathrm{d}\theta$$
$$\le\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\phi(\theta)\mathrm{d}\theta\tag{22}$$
$$\le A(s_1,s_2)M_m(a_1,a_2)+B(s_1,s_2)N(a_1,a_2),$$

where

$$M_m(a_1,a_2)=\psi(a_1)\phi(a_1)+m^2\psi(a_2)\phi(a_2),N(a_1,a_2)$$
$$=\psi(a_1)\phi(a_2)+\psi(a_2)\phi(a_1),$$
$$A(s_1,s_2)=\frac{e^{s_1+s_2}+s_1+s_2-1}{s_1+s_2}-\frac{s_1(e^{s_2}-1)+s_2(e^{s_1}-1)}{s_1s_2},$$
$$B(s_1,s_2)=\frac{e^{s_2}-e^{s_1}}{s_2-s_1}-\frac{s_1(e^{s_2}-1)+s_2(e^{s_1}-1)}{s_1s_2}+1.\tag{23}$$

Proof. Let us denote $\theta=\chi a_1+m(1-\chi)a_2$ for all $\chi\in[0,1]$. Using the property of the (s_1,m) and (s_2,m)-exponential-type convex functions ψ and ϕ, respectively, we have

$$\psi(\chi a_1+m(1-\chi)a_2)\le(e^{s_1\chi}-1)\psi(a_1)+m\left(e^{(1-\chi)s_1}-1\right)\psi(a_2),$$
$$\phi(\chi a_1+m(1-\chi)a_2)\le(e^{s_2\chi}-1)g(a_1)+m\left(e^{(1-\chi)s_2}-1\right)\phi(a_2).\tag{24}$$

Multiplying above inequalities on both sides, we get

$$\psi(\chi a_1+m(1-\chi)a_2)\phi(\chi a_1+m(1-\chi)a_2)$$
$$\le\left[(e^{s_1\chi}-1)\psi(a_1)+m\left(e^{(1-\chi)s_1}-1\right)\psi(a_2)\right]$$
$$\times\left[(e^{s_2\chi}-1)g(a_1)+m\left(e^{(1-\chi)s_2}-1\right)\phi(a_2)\right]$$
$$=(e^{s_1\chi}-1)(e^{s_2\chi}-1)\psi(a_1)\phi(a_1)\tag{25}$$
$$+m\left[(e^{s_1\chi}-1)\left(e^{(1-\chi)s_2}-1\right)\psi(a_1)\phi a_2)\right.$$
$$\left.+(e^{s_2\chi}-1)\left(e^{(1-\chi)s_1}-1\right)\psi(a_2)\phi(a_1)\right]$$
$$+m^2\left(e^{(1-\chi)s_1}-1\right)\left(e^{(1-\chi)s_2}-1\right)\psi(a_2)\phi(a_2).$$

Applying Chebyshev integral inequality (see [23]), we obtain

$$\int_0^1\psi(\chi a_1+m(1-\chi)a_2)\phi(\chi a_1+m(1-\chi)a_2)\mathrm{d}\chi$$
$$\ge\int_0^1\psi(\chi a_1+m(1-\chi)a_2)\mathrm{d}\chi\cdot\int_0^1\phi(\chi a_1+m(1-\chi)a_2)\mathrm{d}\chi.\tag{26}$$

Changing the variable of integration, we get

$$\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\phi(\theta)\mathrm{d}\theta$$
$$\ge\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\mathrm{d}\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)\mathrm{d}\theta,\tag{27}$$

which completes the left-side inequality. For the right-side inequality, integrating on both sides of the inequality (25) with respect to χ over $[0,1]$, we have

$$\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\mathrm{d}\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)\mathrm{d}\theta$$
$$\leq\left[\int_0^1\left(e^{s_1\chi}-1\right)\left(e^{s_2\chi}-1\right)\mathrm{d}\chi\right]\psi(a_1)\phi(a_1)$$
$$+m\left[\left(\int_0^1\left(e^{s_1\chi}-1\right)\left(e^{(1-\chi)s_2}-1\right)\mathrm{d}\chi\right)\psi(a_1)\phi(a_2)\right.$$
$$\left.+\left(\int_0^1\left(e^{s_2\chi}-1\right)\left(e^{(1-\chi)s_1}-1\right)\mathrm{d}\chi\right)\psi(a_2)\phi(a_1)\right]$$
$$+m^2\left[\int_0^1\left(e^{(1-\chi)s_1}-1\right)\left(e^{(1-\chi)s_2}-1\right)\mathrm{d}\chi\right]\psi(a_2)\phi(a_2)$$
$$=A(s_1,s_2)M_m(a_1,a_2)+B(s_1,s_2)N(a_1,a_2),\tag{28}$$

which give the right-side inequality. □

4. Refinements of (H–H)-Type Inequality for k – Fractional Integral

In this section, we will obtain some refinements of the (H–H) inequality via (s,m)-exponential-type convex functions.

Lemma 2. *Suppose $0<w\leq 1$ and a mapping $\psi\colon [wa_1,a_2]\longrightarrow\mathfrak{R}$ is differentiable on (wa_1,a_2) with $0<a_1<a_2$. If $\psi'\in L_1[wa_1,a_2]$, then the following equality for k–fractional integral holds:*

$$\frac{\psi(wa_1)+\alpha/k\psi(a_2)}{\alpha/k+1}-\frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}}\,{}^kJ^{\alpha}_{a_2^-}\psi(wa_1)$$
$$=\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\int_0^1\left[\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right]\psi'(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi,\tag{29}$$

where $\alpha,k>0$ and $\Gamma(\cdot)$ is the Euler Gamma function.

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\int_0^1\left[\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right]\psi'(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi$$
$$=\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left\{\int_0^1\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}\psi'(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi\right.$$
$$\left.-\int_0^1\psi'(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi\right\}$$
$$=\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left[\left(\frac{\alpha}{k}+1\right)\left\{\frac{\chi^{\alpha/k}\psi(w(1-\chi)a_1+\chi a_2)}{a_2-wa_1}\Big|_0^1\right.\right.$$
$$\left.-\int_0^1\frac{\psi(w(1-\chi)a_1+\chi a_2)}{a_2-wa_1}\frac{\alpha}{k}\chi^{\alpha/k-1}\mathrm{d}\chi\right\}$$
$$\left.-\frac{\psi(w(1-\chi)a_1+\chi a_2)}{a_2-wa_1}\Big|_0^1\right]$$
$$=\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left[\left(\frac{\alpha}{k}+1\right)\times\left\{\frac{\psi(a_2)}{a_2-wa_1}-\frac{\alpha}{k(a_2-wa_1)}\right.\right.$$
$$\left.\left.\int_0^1\chi^{\alpha/k-1}\psi(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi\right\}\right]$$
$$=\frac{\psi(wa_1)+\alpha/k\psi(a_2)}{\alpha/k+1}-\frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}}\,{}^kJ^{\alpha}_{a_2^-}\psi(wa_1),\tag{30}$$

which completes the proof.

Lemma 3. *Suppose $0<w\leq 1$ and a mapping $\psi\colon [wa_1,a_2]\longrightarrow\mathfrak{R}$ is differentiable on (wa_1,a_2) with $0<a_1<a_2$. If $\psi'\in L_1[wa_1,a_2]$, then the following equality for k–fractional integral holds:*

$$\frac{\psi(wa_1)+\psi(a_2)}{w+1}-\frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}}$$
$$\left\{{}^kJ^{\alpha}_{a_1^+}\psi(a_2)+{}^kJ^{\alpha}_{a_2^-}\psi(wa_1)\right\}$$
$$=\left(\frac{a_2-wa_1}{w+1}\right)\int_0^1\left[\chi^{\alpha/k}-(1-\chi)^{\alpha/k}\right]\psi'(w(1-\chi)a_1+\chi a_2)\mathrm{d}\chi.\tag{31}$$

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2 - wa_1}{w+1}\right)\int_0^1\left[\chi^{\alpha/k} - (1-\chi)^{\alpha/k}\right]\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi$$
$$= \left(\frac{a_2 - wa_1}{w+1}\right)\left[\int_0^1 \chi^{\alpha/k}\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi - \int_0^1 (1-\chi)^{\alpha/k}\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi\right]$$
$$= \left(\frac{a_2 - wa_1}{w+1}\right)\left[I_1 - I_2\right], \tag{32}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \chi^{\alpha/k}\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi \\ &= \frac{\chi^{\alpha/k}\psi\left(w(1-\chi)a_1 + \chi a_2\right)}{a_2 - wa_1}\Big|_0^1 - \int_0^1 \frac{\psi\left(w(1-\chi)a_1 + \chi a_2\right)}{a_2 - wa_1}\frac{\alpha}{k}\chi^{\alpha/k-1}\mathrm{d}\chi \\ &= \frac{\psi(a_2)}{a_2 - wa_1} - \frac{\alpha}{k(a_2 - wa_1)}\int_0^1 \chi^{\alpha/k-1}\psi\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi \\ &= \frac{\psi(a_2)}{a_2 - wa_1} - \frac{\Gamma_k(\alpha+k)}{\left(a_2 - wa_1\right)^{\alpha/k+1}}\,{}^{k}J_{a_2^-}^{\alpha}\psi(wa_1), \end{aligned} \tag{33}$$

$$\begin{aligned} I_2 &= \int_0^1 (1-\chi)^{\alpha/k}\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi \\ &= \frac{(1-\chi)^{\alpha/k}\psi\left(w(1-\chi)a_1 + \chi a_2\right)}{a_2 - wa_1}\Big|_0^1 - \int_0^1 \frac{\psi\left(w(1-\chi)a_1 + \chi a_2\right)}{a_2 - wa_1}\frac{\alpha}{k}(1-\chi)^{\alpha/k-1}(-1)\mathrm{d}\chi \\ &= -\frac{\psi(wa_1)}{a_2 - wa_1} + \frac{\alpha}{k(a_2 - wa_1)}\int_0^1 (1-\chi)^{\alpha/k-1}\psi\left(w(1-\chi)a_1 + \chi a_2\right)\mathrm{d}\chi \\ &= -\frac{\psi(wa_1)}{a_2 - wa_1} + \frac{\Gamma_k(\alpha+k)}{\left(a_2 - wa_1\right)^{\alpha/k+1}}\,{}^{k}J_{a_1^+}^{\alpha}\psi(a_2). \end{aligned} \tag{34}$$

Combining equations (33) and (34) in (32) and multiplying by $a_2 - wa_1/w + 1$, we get (31), which completes the proof. □

Theorem 7. *Suppose $0 < w \le 1$ and a mapping $\psi\colon (0, a_2/mw] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2/mw)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s,m)-exponential-type convex on $(0, a_2/mw]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s, m \in (0,1]$, then the following inequality for k–fractional integral holds:*

$$\left|\frac{\psi(wa_1) + \alpha/k\psi(a_2)}{\alpha/k+1} - \frac{\Gamma_k(\alpha+k)}{\left(a_2 - wa_1\right)^{\alpha/k}}\,{}^{k}J_{a_2^-}^{\alpha}\psi(wa_1)\right| \le \left(\frac{a_2 - wa_1}{\alpha/k+1}\right)\left[U_1(\alpha,k,p) + U_2(\alpha,k,p)\right]^{1/p} \times\left[\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(wa_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right]^{1/q}, \tag{35}$$

where

$$\begin{aligned} U_1(\alpha,k,p) &= \int_0^{1/\sqrt{[\alpha]}(\alpha/k+1)^k}\left(1 - \left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}\right)^p\mathrm{d}\chi, \\ U_2(\alpha,k,p) &= \int_{1/\sqrt{[\alpha]}(\alpha/k+1)^k}^{1}\left(\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k} - 1\right)^p\mathrm{d}\chi. \end{aligned} \tag{36}$$

Proof. From Lemma 2, Hölder's inequality and (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned} &\left|\frac{\psi(wa_1) + \alpha/k\psi(a_2)}{\alpha/k+1} - \frac{\Gamma_k(\alpha+k)}{\left(a_2 - wa_1\right)^{\alpha/k}}\,{}^{k}J_{a_2^-}^{\alpha}\psi(wa_1)\right| \\ &\le \left(\frac{a_2 - wa_1}{\alpha/k+1}\right)\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k} - 1\right|\left|\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\right|\mathrm{d}\chi \\ &\le \left(\frac{a_2 - wa_1}{\alpha/k+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k} - 1\right|^p\mathrm{d}\chi\right)^{1/p}\left(\int_0^1\left|\psi'\left(w(1-\chi)a_1 + \chi a_2\right)\right|^q\mathrm{d}\chi\right)^{1/q} \\ &\le \left(\frac{a_2 - wa_1}{\alpha/k+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k} - 1\right|^p\mathrm{d}\chi\right)^{1/p} \times\left(\int_0^1\left|\left(e^{(1-\chi)s} - 1\right)\left|\psi'(wa_1)\right|^q + m\left(e^{s\chi} - 1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right|\mathrm{d}\chi\right)^{1/q} \\ &= \left(\frac{a_2 - wa_1}{\alpha/k+1}\right)\left[U_1(\alpha,k,p) + U_2(\alpha,k,p)\right]^{1/p} \times\left[\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(wa_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right]^{1/q}, \end{aligned} \tag{37}$$

which completes the proof. □

Theorem 8. *Suppose $0 < w \le 1$ and a mapping $\psi\colon (0, a_2/mw] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2/mw)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s,m)-exponential-type convex on $(0, a_2/mw]$ for $q \ge 1$, then for some fixed $s, m \in (0,1]$, then the following inequality for k–fractional integral holds:*

$$\begin{aligned}
&\left|\frac{\psi(wa_1)+\alpha/k\psi(a_2)}{\alpha/k+1}-\frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}}{}^{k}J^{\alpha}_{a_2^-}\psi(wa_1)\right|\\
&\leq\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left(\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}\right)^{1-1/q}\\
&\times\Bigg[\left|\psi'(wa_1)\right|^q\left\{-\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}-\frac{2}{s}e^{s\left(1-1/\sqrt{[\alpha]}\,(\alpha/k+1)^k\right)}-\left(\frac{\alpha}{k}+1\right)\frac{e^s}{s^{\alpha/k}}\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)\right.\\
&\left.+\left(\frac{\alpha}{k}+1\right)e^s\left(s^{-\alpha/k}\right)\gamma_{\left(1-1/\sqrt{[\alpha]}\,(\alpha/k+1)^k\right)}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)+\frac{1}{s}\right\}\\
&+m\left|\psi'\left(\frac{a_2}{m}\right)\right|\left\{\frac{2e^{s/\sqrt{[\alpha]}\,(\alpha/k+1)^k}}{s}-\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}-\left(\frac{\alpha}{k}+1\right)\left(\frac{1}{(-s)^{\alpha/k}}\right)\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)\right.\\
&\left.+\left(\frac{\alpha}{k}+1\right)\left(\frac{1}{(-s)^{\alpha/k}}\right)\gamma_{1-1/\sqrt{[\alpha]}\,(\alpha/k+1)^k}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)-\frac{e^s}{s}\right\}\Bigg]^{1/q}.
\end{aligned}\tag{38}$$

Proof. From Lemma 2, power mean inequality and (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(wa_1)+\alpha/k\psi(a_2)}{\alpha/k+1}-\frac{\Gamma_k(\alpha+k)}{(a_2-wa_1)^{\alpha/k}}{}^{k}J^{\alpha}_{a_2^-}\psi(wa_1)\right|\\
&\leq\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right|\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|\mathrm{d}\chi\\
&\leq\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right|\mathrm{d}\chi\right)^{1-1/q}\times\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right|\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|^q\mathrm{d}\chi\right)^{1/q}\\
&\leq\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right|\mathrm{d}\chi\right)^{1-1/q}\times\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\chi^{\alpha/k}-1\right|\left[\left(e^{(1-\chi)s}-1\right)\left|\psi'(wa_1)\right|^q+m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right]\mathrm{d}\chi\right)^{1/q}\\
&=\left(\frac{a_2-wa_1}{\alpha/k+1}\right)\left(\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}\right)^{1-1/q}\\
&\times\Bigg[\left|\psi'(wa_1)\right|^q\left\{-\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}-\frac{2}{s}e^{s\left(1-1/\sqrt{[\alpha]}\,(\alpha/k+1)^k\right)}-\left(\frac{\alpha}{k}+1\right)\frac{e^s}{s^{\alpha/k}}\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)\right.\\
&\left.+\left(\frac{\alpha}{k}+1\right)\frac{e^s}{s^{\alpha/k}}\gamma_{1-\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)+\frac{1}{s}\right\}\\
&+m\left|\psi'\left(\frac{a_2}{m}\right)\right|\left\{\frac{2e^{s/\sqrt{[\alpha]}\,(\alpha/k+1)^k}}{s}-\frac{2\alpha}{k(\alpha/k+1)^{k/\alpha+1}}-\left(\frac{\alpha}{k}+1\right)\left(\frac{1}{(-s)^{\alpha/k}}\right)\right.\\
&\left.\cdot\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)+\left(\frac{\alpha}{k}+1\right)\left(\frac{1}{(-s)^{\alpha/k}}\right)\gamma_{1-1/\sqrt{[\alpha]}\,(\alpha/k+1)^k}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt{[\alpha]}\,(\alpha/k+1)^k}\right)-\frac{e^s}{s}\right\}\Bigg]^{1/q},
\end{aligned}\tag{39}$$

which completes the proof. □

Theorem 9. *Suppose $0 < w \leq 1$ and a mapping $\psi\colon (0, a_2/m] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2/m)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s,m)-exponential-type convex on $(0, a_2/m]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s, m \in (0,1]$, then the following inequality for k–fractional integral holds:*

$$\left|\frac{\psi(wa_1)+\psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2 - wa_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{a_1^+}\psi(a_2) + {}^kJ^{\alpha}_{a_2^-}\psi(wa_1)\right\}\right|$$
$$\leq \frac{2(a_2 - wa_1)}{w+1}\left(\frac{k}{\alpha p + k}\right)^{1/p}\left[\left(\frac{e^s - s - 1}{s}\right)\left(|\psi'(wa_1)|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right]^{1/q}. \tag{40}$$

Proof. From Lemma 3, Hölder's inequality and (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{\psi(wa_1)+\psi(a_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(a_2 - wa_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{a_1^+}\psi(a_2) + {}^kJ^{\alpha}_{a_2^-}\psi(wa_1)\right\}\right|$$
$$\leq \left(\frac{a_2 - wa_1}{w+1}\right)\int_0^1 \left|\chi^{\alpha/k} - (1-\chi)^{\alpha/k}\right|\left|\psi'(w(1-\chi)a_1 + \chi a_2)\right| \mathrm{d}\chi$$
$$\leq \left(\frac{a_2 - wa_1}{w+1}\right)\left[\int_0^1 \chi^{\alpha/k}\left|\psi'(w(1-\chi)a_1 + \chi a_2)\right|\mathrm{d}\chi + \int_0^1 (1-\chi)^{\alpha/k}\left|\psi'(w(1-\chi)a_1 + \chi a_2)\right|\mathrm{d}\chi\right]$$
$$\leq \left(\frac{a_2 - wa_1}{w+1}\right)\left[\left(\int_0^1 \chi^{\alpha/kp} d\chi\right)^{1/p}\left(\int_0^1 \left|\psi'(w(1-\chi)a_1 + \chi a_2)\right|^q \mathrm{d}\chi\right)^{1/q} + \left(\int_0^1 (1-\chi)^{\alpha/kp}\mathrm{d}\chi\right)^{1/p}\left(\int_0^1 \left|\psi'(w(1-\chi)a_1 + \chi a_2)\right|^q \mathrm{d}\chi\right)^{1/q}\right]$$
$$\leq \left(\frac{a_2 - wa_1}{w+1}\right)\left[\left(\int_0^1 \chi^{\alpha/kp}\mathrm{d}\chi\right)^{1/p} \times \left(\int_0^1 \left[\left(e^{(1-\chi)s} - 1\right)|\psi'(wa_1)|^q + m\left(e^{s\chi} - 1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right]\mathrm{d}\chi\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 (1-\chi)^{\alpha/kp}\mathrm{d}\chi\right)^{1/p} \times \left(\int_0^1 \left[\left(e^{(1-\chi)s} - 1\right)|\psi'(wa_1)|^q + m\left(e^{s\chi} - 1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right]\mathrm{d}\chi\right)^{1/q}\right]$$
$$= \frac{2(a_2 - wa_1)}{w+1}\left(\frac{k}{\alpha p + k}\right)^{1/p}\left[\left(\frac{e^s - s - 1}{s}\right)\left(|\psi'(wa_1)|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right]^{1/q}, \tag{41}$$

which completes the proof. □

Theorem 10. *Suppose $0 < w \leq 1$ and a mapping $\psi\colon (0, a_2/m] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2/m)$ with $0 < a_1 < a_2$. If $|\psi'|^q$ is (s,m)-exponential-type convex on $(0, a_2/m]$ for $q \geq 1$, then for some fixed $s, m \in (0,1]$, then the following inequality for k–fractional integral holds:*

$$\left|\frac{\psi(wa_1)+\psi(a_2)}{w+1}-\frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{a_1^+}\psi(a_2)+{}^kJ^{\alpha}_{a_2^-}\psi(wa_1)\right\}\right|$$
$$\leq\left(\frac{a_2-wa_1}{w+1}\right)\left(\frac{k}{\alpha+k}\right)^{1-1/q}$$
$$\times\Bigg[\left\{|\psi'(wa_1)|^q\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,s\right)e^s s^{-\alpha/k-1}-\frac{1}{\alpha/k+1}\right)+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{\Gamma(\alpha/k+1,-s)-\Gamma(\alpha/k+1)}{(-s)^{\alpha/k}s}-\frac{1}{\alpha/k+1}\right)\right\}^{1/q}$$
$$+\Bigg\{|\psi'(wa_1)|^q\left((-1)^{\alpha/k-1}s^{-\alpha/k-1}\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,-s\right)\right)-\frac{1}{\alpha/k+1}\right)$$
$$+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{(\alpha/k+1)e^s ks^{-\alpha/k-1}(\Gamma(\alpha/k+1,s)-\Gamma(\alpha/k+1))}{k+\alpha}-\frac{1}{\alpha/k+1}\right)\Bigg\}^{1/q}\Bigg]. \tag{42}$$

Proof. From Lemma 3, power mean inequality and (s,m)-exponential-type convexity of $|\psi'|^q$, we have

$$\left|\frac{\psi(wa_1)+\psi(a_2)}{w+1}-\frac{\Gamma_k(\alpha+k)}{(w+1)(a_2-wa_1)^{\alpha/k}}\left\{{}^kJ^{\alpha}_{a_1^+}\psi(a_2)+{}^kJ^{\alpha}_{a_2^-}\psi(wa_1)\right\}\right|$$
$$\leq\left(\frac{a_2-wa_1}{w+1}\right)\int_0^1\left|\chi^{\alpha/k}-(1-\chi)^{\alpha/k}\right|\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|\mathrm{d}\chi$$
$$\leq\left(\frac{a_2-wa_1}{w+1}\right)\times\left[\int_0^1\chi^{\alpha/k}\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|\mathrm{d}\chi+\int_0^1(1-\chi)^{\alpha/k}\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|\mathrm{d}\chi\right]$$
$$\leq\left(\frac{a_2-wa_1}{w+1}\right)\Bigg[\left(\int_0^1\chi^{\alpha/k}\mathrm{d}\chi\right)^{1-1/q}\left(\int_0^1\chi^{\alpha/k}\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|^q\mathrm{d}\chi\right)^{1/q}+\left(\int_0^1(1-\chi)^{\alpha/k}\mathrm{d}\chi\right)^{1-1/q}$$
$$\left(\int_0^1(1-\chi)^{\alpha/k}\left|\psi'(w(1-\chi)a_1+\chi a_2)\right|^q\mathrm{d}\chi\right)^{1/q}\Bigg]$$
$$\leq\left(\frac{a_2-wa_1}{w+1}\right)\left(\int_0^1\chi^{\alpha/k}\mathrm{d}\chi\right)^{1-1/q}\times\Bigg[\left(\int_0^1\chi^{\alpha/k}\left[\left(e^{(1-\chi)s}-1\right)|\psi'(wa_1)|^q+m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right]\mathrm{d}\chi\right)^{1/q}$$
$$+\left(\int_0^1(1-\chi)^{\alpha/k}\left[\left(e^{(1-\chi)s}-1\right)|\psi'(wa_1)|^q+m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right]\mathrm{d}\chi\right)^{1/q}\Bigg]$$
$$=\left(\frac{a_2-wa_1}{w+1}\right)\left(\frac{k}{\alpha+k}\right)^{1-1/q}$$
$$\times\Bigg[\left\{|\psi'(wa_1)|^q\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,s\right)e^s s^{-\alpha/k-1}-\frac{1}{\alpha/k+1}\right)+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{\Gamma(\alpha/k+1,-s)-\Gamma(\alpha/k+1)}{(-s)^{\alpha/k}s}-\frac{1}{\alpha/k+1}\right)\right\}^{1/q}$$
$$+\Bigg\{|\psi'(wa_1)|^q\left((-1)^{\alpha/k-1}s^{-\alpha/k-1}\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,-s\right)\right)-\frac{1}{\alpha/k+1}\right)$$
$$+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{(\alpha/k+1)e^s ks^{-\alpha/k-1}(\Gamma(\alpha/k+1,s)-\Gamma(\alpha/k+1))}{k+\alpha}-\frac{1}{\alpha/k+1}\right)\Bigg\}^{1/q}\Bigg], \tag{43}$$

which completes the proof. ◻

5. Applications

Let consider the following two special means for different positive-real numbers $a_1 < a_2$.

(1) The arithmetic mean:

$$\mathscr{A}(a_1, a_2) = \frac{a_1 + a_2}{2}. \tag{44}$$

(2) The generalized log-mean:

$$\mathscr{L}_l(a_1, a_2) = \left[\frac{a_2^{l+1} - a_1^{l+1}}{(l+1)(a_2 - a_1)}\right]^{1/l}; \ l \in \mathfrak{R}\backslash\{-1, 0\}. \tag{45}$$

Dragomir et al. [5] have proved that, for $s \in (0,1)$, where $1 \le l \le 1/s$, the function $f(x) = x^{ls}$, $x > 0$ is s-convex function. Then, from Proposition 1, it is also s-exponential convex function for some fixed $s \in [\ln 2.5, 1)$.

Using Section 4, we have the following interesting results:

Proposition 2. *Let $0 < a_1 < a_2$, $0 < w \le 1$, and $q > 1$ such that $p^{-1} + q^{-1} = 1$. Then, for some fixed $s \in [ln2.5, 1)$, where $1 \le l \le 1/s$, we have*

$$\begin{aligned}\left|\mathscr{A}\left((wa_1)^{ls}, a_2^{ls}\right) - \mathscr{L}_{ls}^{ls}(wa_1, a_2)\right| &\le \frac{ls(a_2 - wa_1)}{\sqrt{[p]}2} \\ &\times \left(\frac{1}{p+1}\right)^{1/p}\left(\frac{e^s - s - 1}{s}\right)^{1/q}\mathscr{A}^{1/q}\left((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).\end{aligned} \tag{46}$$

Proof. Consider the s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 7 for $\alpha = 1 = k$, we obtain the required result. ◻

Proposition 3. *Let $0 < a_1 < a_2$, $0 < w \le 1$ and $q \ge 1$. Then, for some fixed $s \in [ln2.5, 1)$, where $1 \le l \le 1/s$, we get*

$$\begin{aligned}\left|\mathscr{A}\left((wa_1)^{ls}, a_2^{ls}\right) - \mathscr{L}_{ls}^{ls}(wa_1, a_2)\right| &\le \frac{ls(a_2 - wa_1)}{4^{(1-1/q)}} \\ &\times \left(\frac{2(s-2)e^s + 8e^{s/2} - s^2 - 2s - 4}{2s^2}\right)^{1/q} \\ &\mathscr{A}^{1/q}\left((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).\end{aligned} \tag{47}$$

Proof. Consider the s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 8 for $\alpha = 1 = k$, the result is obvious.

Proposition 4. *Let $0 < a_1 < a_2$, $0 < w \le 1$ and $q > 1$, such that $p^{-1} + q^{-1} = 1$. Then, for some fixed $s \in [ln2.5, 1)$, where $1 \le l \le 1/s$, we obtain*

$$\begin{aligned}&\left|\frac{2}{w+1}\left(\mathscr{A}\left((wa_1)^{ls}, a_2^{ls}\right) - \mathscr{L}_{ls}^{ls}(wa_1, a_2)\right)\right| \\ &\le \sqrt{[q]}2\frac{ls(a_2 - wa_1)}{w+1} \times \left(\frac{1}{p+1}\right)^{1/p}\left(\frac{e^s - s - 1}{s}\right)^{1/q} \\ &\mathscr{A}^{1/q}\left((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).\end{aligned} \tag{48}$$

Proof. Consider the s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 9 for $\alpha = 1 = k$, the result is evident. ◻

Proposition 5. *Let $0 < a_1 < a_2$, $0 < w \le 1$ and $q \ge 1$. Then, for some fixed $s \in [ln2.5, 1)$, where $1 \le l \le 1/s$, we have*

$$\begin{aligned}\left|\frac{2}{w+1}\left(\mathscr{A}\left((wa_1)^{ls}, a_2^{ls}\right) - \mathscr{L}_{ls}^{ls}(wa_1, a_2)\right)\right| &\le \frac{ls(a_2 - wa_1)}{2(w+1)} \\ &\times \left(\frac{2(s-2)e^s + 8e^{s/2} - s^2 - 2s - 4}{2s^2}\right)^{1/q} \\ &\mathscr{A}^{1/q}\left((wa_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).\end{aligned} \tag{49}$$

Proof. Consider the s-exponential convex function $\psi(x) = x^{ls}$, $x > 0$, and using Theorem 10 for $\alpha = 1 = k$, we obtain the required result.

At the end, let consider some applications of the integral inequalities obtained above, to find new bounds for the trapezoidal and midpoint formula.

For $a_2 > 0$, let $\mathscr{U}$: $0 = \chi_0 < \chi_1 <, \cdots, < \chi_{n-1} < \chi_n = a_2$ is a partition of $[0, a_2]$.

We denote

$$\begin{aligned}\mathscr{T}(\mathscr{U}, \psi) &= \sum_{i=0}^{n-1}\left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2}\right)h_i, \\ \int_0^{a_2}\psi(x)\mathrm{d}x &= \mathscr{T}(\mathscr{U}, \psi) + \mathscr{R}(\mathscr{U}, \psi),\end{aligned} \tag{50}$$

where $\mathscr{R}(\mathscr{U}, \psi)$ is the remainder term and $h_i = \chi_{i+1} - \chi_i$ for $i = 0, 1, 2, \ldots, n-1$.

Using above notations, we are in position to prove the following error estimations. ◻

Proposition 6. *Suppose a mapping ψ: $(0, a_2] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2)$ with $a_2 > 0$. If $|\psi'|^q$ is s-exponential-type convex on $(0, a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:*

$$|\mathscr{R}(\mathscr{U},\psi)| \leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{e^s - s - 1}{s}\right)^{1/q} \times \sum_{i=0}^{n-1} h_i^2 \left[\left(\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q\right)\right]^{1/q}. \tag{51}$$

Proof. Using Theorem 7 on subinterval $[\chi_i, \chi_{i+1}]$ of closed interval $[0, a_2]$, for all $i = 0, 1, 2, \ldots, n-1$ and $w = \alpha = k = m = 1$, we get

$$\left|\left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2}\right)h_i - \int_{\chi_i}^{\chi_{i+1}} \psi(x)\mathrm{d}x\right| \leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{1/p} \left(\frac{e^s - s - 1}{s}\right)^{1/q} \times h_i^2 \left[\left(\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q\right)\right]^{1/q}. \tag{52}$$

Summing inequality (52) over i from 0 to $n-1$ and using the property of modulus, we obtain the desired inequality (51). □

Proposition 7. *Suppose a mapping ψ: $(0, a_2] \longrightarrow \mathfrak{R}$ is differentiable on $(0, a_2)$ with $a_2 > 0$. If $|\psi'|^q$ is s-exponential-type convex on $(0, a_2]$ for $q \geq 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:*

$$|\mathscr{R}(\mathscr{U},\psi)| \leq \left(\frac{e^s - s - 1}{2s}\right)^{1/q} \sum_{i=0}^{n-1} h_i^2 \left[\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q\right]^{1/q}. \tag{53}$$

Proof. Apply the same technique as in Proposition 6 but using Theorem 8.

6. Conclusion

In this article, the authors showed new generalizations of trapezium-type inequality for the new class of functions, the so-called (s, m)-exponential-type convex function ψ and for the products of two (s, m)-exponential-type convex functions ψ and ϕ. We have obtained refinements of the (H–H) inequality for functions using (s, m)-exponential-type convex and founded new bounds for special means and for the error estimates for the trapezoidal and midpoint formula. We hope that current work will attract the attention of researchers working in mathematical analysis, fractional calculus, quantum calculus, postquantum calculus, and other related fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Fully Degenerate Daehee Numbers and Polynomials of the Second Kind

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In a study, Carlitz introduced the degenerate exponential function and applied that function to Bernoulli and Eulerian numbers and degenerate special functions have been studied by many researchers. In this paper, we define the fully degenerate Daehee polynomials of the second kind which are different from other degenerate Daehee polynomials and derive some new and interesting identities and properties of those polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\longrightarrow\infty} \frac{1}{p^N}\sum_{n=0}^{p^N-1} f(x), \quad \text{(see [1–3]).} \tag{1}$$

From (1), we have

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \quad \text{where } f_n(x) = f(x+n),\ f'(l) = \left.\frac{df(x)}{dx}\right|_{x=l}. \tag{2}$$

In particular, if $n = 1$, then

$$I_0(f_1) - I_0(f) = f'(0). \tag{3}$$

The *Stirling numbers of the first kind* are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0), \tag{4}$$

and the *Stirling numbers of the second kind* are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \tag{5}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 1)$ (see [4, 5]).

From (4) and (5), we can derive the following equations:

$$\left(e^t - 1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \tag{6}$$

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!}, \quad (n \geq 0), \quad \text{(see [4–6]).} \tag{7}$$

In addition,

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n+1}(n-1)!\frac{t^n}{n!}, \quad [4, 5]. \tag{8}$$

The *Bernoulli polynomials of order r* are defined by the following generating function:

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}, \quad (9)$$

(see [7–9]).

Carlitz's degenerate Bernoulli polynomials of order r is defined by the generating function to be

$$\sum_{n=0}^{\infty}\beta_n^{(r)}(x\,|\,\lambda)\frac{t^n}{n!}=\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}, \quad (10)$$

where $\lambda\in\mathbb{R}$ (see [10]). By (10), we know that

$$\begin{aligned}\lim_{\lambda\longrightarrow 0}\sum_{n=0}^{\infty}\beta_n^{(r)}(x\,|\,\lambda)\frac{t^n}{n!}&=\lim_{\lambda\longrightarrow 0}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}\\&=\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\end{aligned} \quad (11)$$

and thus, we obtain

$$\lim_{\lambda\longrightarrow 0}\beta_n^{(r)}(x\,|\,\lambda)=B_n^{(r)}(x). \quad (12)$$

In [11], the degenerate Bernoulli polynomials are defined as

$$\left(\frac{\log(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad (13)$$

which are different from Carlitz's degenerate numbers and polynomials.

By (13), we know that

$$\lim_{\lambda\longrightarrow 0}b_{n,\lambda}(x)=B_n^{(r)}(x). \quad (14)$$

Note that by (3),

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{x+x_1+\cdots+x_r/\lambda}d\mu_0(x_1)\ldots d\mu_0(x_r)\\&=\left(\frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\end{aligned} \quad (15)$$

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{x+x_1+\cdots+x_r/\lambda}d\mu_0\cdots d\mu_0(x_r)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\frac{x+x_1+\cdots+x_r}{\lambda}}{n}\lambda^n t^n d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}.\end{aligned} \quad (16)$$

By (15) and (16), we know that

$$b_{n,\lambda}^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}d\mu_0(x_1)\cdots d\mu_0(x_r). \quad (17)$$

The *higher-order Daehee polynomials* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}, \quad (18)$$

(see [12–15]).

In particular, if $r=1$, then $D_n^{(1)}(x)=D_n(x)$ is called the *Daehee polynomials*.

By replacing t as $\log(1+t)$ in (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{1}{n!}(\log(1+t))^n\\&=\left(\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{1}{n!}\right)\left(n!\sum_{l=0}^{\infty}S_1(l,n)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_n^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!},\end{aligned} \quad (19)$$

and so, we obtain

$$D_n^{(r)}(x)=\sum_{m=0}^{n}B_m^{(r)}(x)S_1(n,m). \quad (20)$$

Note that by (3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+x_1+\cdots+x_r}\mathrm{d}\mu_0(x_1)\cdots\mathrm{d}\mu_0(x_r) \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x+x_1+\cdots+x_r}{n}t^n\mathrm{d}\mu_0(x_1)\cdots\mathrm{d}\mu_0(x_r) \\ &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_n\mathrm{d}\mu_0(x_1)\cdots\mathrm{d}\mu_0(x_r)\right)\frac{t^n}{n!},\end{aligned} \tag{21}$$

and thus, we know that

$$D_n^{(r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_n\mathrm{d}\mu_0(x_1)\cdots\mathrm{d}\mu_0(x_r). \tag{22}$$

Carlitz introduced the degenerate Bernoulli polynomials in [10] and the degeneration of special functions have been studied (see [6, 16–29]).

In particular, the degenerate Stirling numbers of the second kind with a generating function are defined as

$$\frac{1}{m!}\left((1+\lambda t)^{1/\lambda}-1\right)^m = \sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!}, \tag{23}$$

where m is a given nonnegative integer in [6, 10, 22, 30].

After introducing Daehee numbers and polynomials [31], it plays an important role of developing various generalized polynomials, and interesting properties are obtained (see [8, 15, 21, 22, 28, 30–35]).

In this paper, we define the new degenerate Daehee polynomials and numbers which are called the degenerate Daehee polynomials of the second kind and investigate identities and properties of new polynomials.

2. Fully Degenerate Daehee Polynomials of the Second Kind

Let us assume that $\lambda \in \mathbb{R}$. By (3), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+y/\lambda}\mathrm{d}\mu_0(y) \\ &\quad= \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}(1+\lambda\log(1+t))^{x/\lambda}.\end{aligned} \tag{24}$$

By (24), we define the *degenerate Daehee polynomials of the second kind* by the generating function to be

$$\begin{aligned}&\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}(1+\lambda\log(1+t))^{x/\lambda} \\ &\quad= \sum_{n=0}^{\infty} D_n(x\mid\lambda)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

In the special case, $x=0$, and $D_n(\lambda) = D_n(0\mid\lambda)$ are called the *degenerate Daehee numbers of the second kind.*

Note that

$$\begin{aligned}\lim_{\lambda\longrightarrow 0}\sum_{n=0}^{\infty} D_n(x\mid\lambda)\frac{t^n}{n!} &= \lim_{\lambda\longrightarrow 0}\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1} \\ &\quad\cdot(1+\lambda\log(1+t))^{x/\lambda} \\ &= \frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!},\end{aligned} \tag{26}$$

and thus, we know that

$$\lim_{\lambda\longrightarrow 0} D_n(x\mid\lambda) = D_n(x), \quad (n\geq 0). \tag{27}$$

From (7) and (24), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+y/\lambda}\mathrm{d}\mu_0(y) \\ &\quad= \sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}\lambda^l(\log(1+t))^l\mathrm{d}\mu_0(y) \\ &\quad= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}\mathrm{d}\mu_0(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

By (25) and (28), we have

$$D_{n,\lambda} = \sum_{k=0}^{n} S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}\mathrm{d}\mu_0(y), \quad (n\geq 0). \tag{29}$$

Since

$$\binom{\frac{x+y}{\lambda}}{l} = \frac{((x+y)/\lambda)(((x+y)/\lambda)-1)(((x+y)/\lambda)-2)\ldots(((x+y)/\lambda)-l+1)}{l!}$$
$$= \frac{(x+y)(x+y-\lambda)(x+y-2\lambda)\ldots(x+y-(l-1)\lambda)}{\lambda^l l!}, \tag{30}$$

by (29) and (30), we have

$$D_n(\lambda) = \sum_{k=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}(x+y)_{l,\lambda}\mathrm{d}\mu_0(y), \quad (n\geq 0), \tag{31}$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

Thus, by (29) and (30), we have the following theorem which is Witt's type formula about degenerate Daehee polynomials of the second kind.

Theorem 1. *For each $n\geq 0$, we have*

$$D_n(\lambda) = \sum_{k=0}^{n} S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}\mathrm{d}\mu_0(y)$$
$$= \sum_{k=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}(x+y)_{l,\lambda}\mathrm{d}\mu_0(y). \tag{32}$$

By replacing t as e^t-1 in (25), we obtain the following:

$$\sum_{n=0}^{\infty} D_n(x\mid\lambda)\frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty} D_n(x|\lambda)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_m(x\mid\lambda)S_2(m,n)\right)\frac{t^n}{n!}. \tag{33}$$

On the other hand,

$$\sum_{n=0}^{\infty} D_n(x\mid\lambda)\frac{(e^t-1)^n}{n!} = \frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda}$$
$$= \sum_{n=0}^{\infty} B_n(x\mid\lambda)\frac{t^n}{n!}, \tag{34}$$

where $B_n(x\mid\lambda)$ is the degenerate Bernoulli polynomials of the second kind of order $r\in\mathbb{Z}$ which are defined by the generating function to be

$$\left(\frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n^{(r)}(x\mid\lambda)\frac{t^n}{n!} \tag{35}$$

(see [20, 25]).

In particular, if $r=1$, $B_n^{(1)}(x\mid\lambda) = B_n(x\mid\lambda)$ is called the degenerate Bernoulli polynomials of the second kind.

For positive integer d with $d\equiv 1\,(\mathrm{mod}\,2)$, if we put $f(x) = (1+\lambda\log(1+t))^{x/\lambda}$, then, by (2), we obtain

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+d/\lambda}\mathrm{d}\mu_0(y)$$
$$-\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x/\lambda}\mathrm{d}\mu_0(y) \tag{36}$$
$$= \sum_{l=0}^{d-1}\frac{1}{\lambda}(1+\lambda\log(1+t))^{x/\lambda}\log(1+\lambda\log(1+t)).$$

By (36), we have

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x/\lambda}\mathrm{d}\mu_0(x)$$
$$= \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{d/\lambda}-1}\sum_{l=0}^{d-1}(1+\lambda\log(1+t))^{l/\lambda}$$
$$= d\sum_{l=0}^{d-1}\frac{(1/\lambda)\log(1+(\lambda/d)d\log(1+t))}{(1+(\lambda/d)d\log(1+t))^{d/\lambda}-1}$$
$$\cdot\left(1+\frac{\lambda}{d}d\log(1+t)\right)^{(d/\lambda)(l/d)}. \tag{37}$$

Note that by (6),

$$\frac{(1/\lambda)\log(1+(\lambda/d)d\log(1+t))}{(1+(\lambda/d)d\log(1+t))^{(d/\lambda)}-1}(1+(\lambda/d)d\log(1+t))^{(d/\lambda)(l/d)}$$
$$= \sum_{n=0}^{\infty} B_n\left(\frac{l}{d}\,\middle|\,\frac{\lambda}{d}\right)\frac{(d\log(1+t))^n}{n!}$$
$$= \sum_{n=0}^{\infty} B_n\left(\frac{l}{d}\,\middle|\,\frac{\lambda}{d}\right)\frac{d^n}{n!}n!\sum_{m=n}^{\infty} S_1(m,n)\frac{x^m}{m!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m\left(\frac{l}{d}\,\middle|\,\frac{\lambda}{d}\right)d^m S_1(n,m)\right)\frac{t^n}{n!}. \tag{38}$$

By (24), (37), and (38), we have

$$\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x/\lambda}\mathrm{d}\mu_0(x)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{d-1} B_m\left(\frac{l}{d}\,\middle|\,\frac{\lambda}{d}\right)d^{m+1}S_1(n,m)\right)\frac{t^n}{n!}. \tag{39}$$

Hence, by (33), (34), and (39), we obtain the following theorem which shows the relationship between degenerate Daehee polynomials of the second kind and degenerate Bernoulli polynomials of the second kind.

Theorem 2. *For nonnegative integer n and $d \in \mathbb{N}$ with $d \equiv 1\,(mod\,2)$, we have*

$$\begin{aligned} B_n(x \mid \lambda)(x) &= \sum_{m=0}^{n} D_{m,\lambda}(x) S_2(n,m), \\ D_{n,\lambda} &= \sum_{m=0}^{n}\sum_{l=0}^{d-1} B_m\left(\frac{l}{d}\,\middle|\,\frac{\lambda}{d}\right) d^{m+1} S_1(n,m). \end{aligned} \tag{40}$$

By (25), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} D_n(x \mid \lambda)\frac{t^n}{n!} &= \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right) \\ &\quad\cdot (1+\lambda\log(1+t))^{x/\lambda} \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}\lambda^n(\log(1+t))^n\right) \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}\lambda^n n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!}\right) \\ &= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}(x)_{l,\lambda} S_1(n,l)\frac{x^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(x)_{l,\lambda} S_1(k,l) D_{n-k}(\lambda)\right)\frac{t^n}{n!}. \end{aligned} \tag{41}$$

By comparing the coefficients on both sides of (41), we obtain the following theorem.

Theorem 3. *For nonnegative integer n, we have*

$$D_n(x \mid \lambda) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(x)_{l,\lambda} S_1(k,l) D_{n-k}(\lambda). \tag{42}$$

Note that if we put $f(x) = (1+\lambda\log(1+t))^{x/\lambda}$, then

$$\begin{aligned} f'(0) &= \frac{1}{\lambda}\log(1+\lambda\log(1+t)) \\ &= \frac{1}{\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\lambda^n}{n}(\log(1+t))^n \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\lambda^{n-1}}{n}n!\sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n}(-1)^{m+1}\lambda^{m-1}(m-1)! S_1(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{43}$$

and thus, by (3), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} f(x_1)\,d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) \\ &= \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}(1+\lambda\log(1+t))^{1/\lambda} \\ &\quad - \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1} \\ &= \sum_{n=0}^{\infty}\left(D_n(1 \mid \lambda) - D_n(\lambda)\right)\frac{t^n}{n!}. \end{aligned} \tag{44}$$

Moreover,

$$\begin{aligned} &((1/\lambda)\log(1+\lambda\log(1+t)))\sum_{l=0}^{n-1}(1+\lambda\log(1+t))^{l/\lambda} \\ &= \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}}(1+\lambda\log(1+t))^n \\ &\quad - \frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}} \\ &= \sum_{r=0}^{\infty}\{D_r(n \mid \lambda) - D_r(\lambda)\}\frac{t^r}{r!}, \end{aligned} \tag{45}$$

and, by (43), we obtain

$$
\begin{aligned}
&(1/\lambda)\log(1+\lambda\log(1+t))\sum_{l=0}^{n-1}(1+\lambda\log(1+t))^{l/\lambda}\\
&=\left(\sum_{p=0}^{\infty}\sum_{m=1}^{p}(-\lambda)^{m-1}(m-1)!S-1(p,m)\frac{t^p}{p!}\right)\\
&\cdot\left(\sum_{l=0}^{n-1}\sum_{s=0}^{\infty}\sum_{q=0}^{s}(l)_{q,\lambda}S_1(s,q)\frac{t^s}{s!}\right)\\
&=\sum_{r=0}^{\infty}\left(\sum_{l=0}^{n-1}\sum_{p=0}^{r}\sum_{m=1}^{p}\sum_{q=0}^{r-p}\binom{r}{p}(l)_{q,\lambda}(-\lambda)^{m-1}\right.\\
&\left.\cdot(m-1)!S_1(p,m)S_1(r-p,q)\right)\frac{t^r}{r!}.
\end{aligned}\tag{46}
$$

From (43), (44), and (46), we obtain the following theorem which represents a recurrence relations between degenerate Daehee polynomials of the second kind and degenerate Daehee numbers of the second kind.

Theorem 4. *For each nonnegative integer r, we have*

$$D_r(1\mid\lambda)-D_r(\lambda)=\sum_{m=1}^{r}(-\lambda)^{m-1}(m-1)!S_1(r,m).\tag{47}$$

Moreover, for each positive integer $n\geq 2$,

$$
\begin{aligned}
D_r(n\mid\lambda)-D_r(\lambda)=&\sum_{l=0}^{n-1}\sum_{p=0}^{r}\sum_{m=1}^{p}\sum_{q=0}^{r-p}\binom{r}{p}(l)_{q,\lambda}(-\lambda)^{m-1}\\
&\cdot(m-1)!S_1(p,m)S_1(r-p,q).
\end{aligned}\tag{48}
$$

3. Higher-Order Degenerate Daehee Polynomials of the Second Kind

In this section, we consider the *higher-order degenerate Daehee polynomials of the second kind* given by the generating function as follows: for the given positive real number r,

$$
\begin{aligned}
&\left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^r(1+\lambda\log(1+t))^{x/\lambda}\\
&=\sum_{n=0}^{\infty}D_n^{(r)}(x\mid\lambda)\frac{t^n}{n!}.
\end{aligned}\tag{49}
$$

In particular, if $x=0$, $D_n^{(r)}(0\mid\lambda)=D_n^{(r)}(\lambda)$ are called the *higher-order degenerate Daehee numbers of the second kind*.

Note that

$$
\begin{aligned}
\lim_{\lambda\longrightarrow 0}\sum_{n=0}^{\infty}D_n^{(r)}(x\mid\lambda)\frac{t^n}{n!}&=\lim_{\lambda\longrightarrow 0}\left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^r\\
&\cdot(1+\lambda\log(1+t))^{x/\lambda}\\
&=\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x\\
&=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}\tag{50}
$$

From (35), we note that

$$
\begin{aligned}
&\left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^r(1+\lambda\log(1+t))^{x/\lambda}\\
&=\sum_{n=0}^{\infty}B_n^{(r)}(x\mid\lambda)\frac{(\log(1+t))^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{B_n^{(r)}(x\mid\lambda)}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m^{(r)}(x\mid\lambda)S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{51}
$$

In addition, by replacing t by e^t-1 in (49), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}D_n^{(r)}(x\mid\lambda)\frac{(e^t-1)^n}{n!}&=\sum_{n=0}^{\infty}\frac{D_n^{(r)}(x\mid\lambda)}{n!}n!\sum_{m=n}^{\infty}S_2(n,m)\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}D_m^{(r)}(x\mid\lambda)S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned}\tag{52}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}D_n^{(r)}(x\mid\lambda)\frac{(e^t-1)^n}{n!}&=\left(\frac{(1/\lambda)\log(1+\lambda t)}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}\\
&=\sum_{n=0}^{\infty}B_n^{(r)}(x\mid\lambda)\frac{t^n}{n!}.
\end{aligned}\tag{53}
$$

Hence, by (51)–(53), we obtain the following theorem.

Theorem 5. *For $n\geq 0$, we have*

$$D_n^{(r)}(x\mid\lambda)=\sum_{m=0}^{n}B_m^{(r)}(x\mid\lambda)S_1(n,m).\tag{54}$$

Moreover,

$$B_n^{(r)}(x \mid \lambda) = \sum_{m=0}^{n} D_m^{(r)}(x \mid \lambda) S_2(n,m). \tag{55}$$

In particular, if $r = 1$, then we know that Theorem 5 is a generalization of Theorem 1 and Theorem 2.

Note that for each $k, r \in \mathbb{N}$,

$$\begin{aligned}
&\left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{r}(1+\lambda\log(1+t))^{x/\lambda} \\
&= \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{k} \\
&\quad\cdot \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{r-k}(1+\lambda\log(1+t))^{x/\lambda} \\
&= \left(\sum_{n=0}^{\infty} D_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} D_n^{(r-k)}(x \mid \lambda)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} D_m^{(k)}(\lambda) D_{n-m}^{(r-k)}(x \mid \lambda)\right)\frac{t^n}{n!}.
\end{aligned} \tag{56}$$

It is well known that for each $k \in \mathbb{Z}$,

$$\left(\frac{t}{\log(1+t)}\right)^{k}(1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-k+1)}(x)\frac{t^n}{n!}. \tag{57}$$

By (23), (7), and (57), we have

$$\begin{aligned}
&\left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{r}(1+\lambda\log(1+t))^{x/\lambda} \\
&= \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{-r}(1+\lambda\log(1+t))^{x/\lambda} \\
&= \sum_{n=0}^{\infty} B_n^{(n+r+1)}(x+1)\frac{1}{n!}\left((1+\lambda\log(1+t))^{1/\lambda}-1\right)^{n} \\
&= \left(\sum_{n=0}^{\infty} B_n^{(n+r+1)}(x+1)\right)\left(\sum_{m=n}^{\infty} S_{2,\lambda}(m,n)\frac{(\log(1+t))^{m}}{m!}\right) \\
&= \sum_{p=0}^{\infty}\sum_{n=0}^{p} B_n^{(n+r+1)}(x+1) S_{2,\lambda}(p,n)\frac{1}{p!}(\log(1+t))^{p} \\
&= \sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\sum_{n=0}^{p} B_n^{(n+r+1)}(x+1) S_{2,\lambda}(q-l,n) S_1(q,p)\right)\frac{t^q}{q!}.
\end{aligned} \tag{58}$$

By (56) and (58), we obtain the following theorem.

Theorem 6. *For each $n, q \geq 0$, we have*

$$\begin{aligned}
D_n^{(r)}(x \mid \lambda) &= \sum_{m=0}^{n}\binom{n}{m} D_m^{(k)}(\lambda) D_{n-m}^{(r-k)}(x \mid \lambda), \\
D_q^{(r)}(x \mid \lambda) &= \sum_{p=0}^{q}\sum_{n=0}^{p} B_n^{(n+r+1)}(x+1) S_{2,\lambda}(q-l,n) S_1(q,p).
\end{aligned} \tag{59}$$

Note that by (3),

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+x_1+\cdots+x_r/\lambda} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \left(\frac{(1/\lambda)\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{1/\lambda}-1}\right)^{r}(1+\lambda\log(1+t))^{x/\lambda} \\
&= \sum_{n=0}^{\infty} D_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}.
\end{aligned} \tag{60}$$

By (17) and (60), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+x_1+\cdots+x_r/\lambda} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\frac{x+x_1+\cdots+x_r}{\lambda}}{n}\lambda^n \\
&\quad\cdot (\log(1+t))^n d\mu(x_1)\cdots d\mu_0(x_r) \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}\frac{1}{n!}n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(+x_1+\cdots+x_r)_{m,\lambda} S_1(n,m)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_1(n,m)\beta_m^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{61}$$

Thus, by (60) and (61), we obtain the following theorem which shows that higher-order degenerate Daehee polynomials of the second kind are represented by linear combination of the higher-order Carlitz's type degenerate Bernoulli polynomials.

Theorem 7. *For each nonnegative integer n and each integer r,*

$$D_n^{(r)}(x \mid \lambda) = \sum_{m=0}^{n} S_1(n,m)\beta_m^{(r)}(x). \tag{62}$$

By (7) and (23), we have

$$
\begin{aligned}
&(1+\lambda\log(1+t))^{x+x_1+\cdots+x_r/\lambda}\\
&=\left((1+\lambda\log(1+t))^{1/\lambda}-1+1\right)^{x+x_1+\cdots+x_r}\\
&=\sum_{m=0}^{\infty}\binom{x+x_1+\cdots+x_r}{m}\left((1+\lambda\log(1+t))^{1/\lambda}-1\right)^{m}\\
&=\sum_{m=0}^{\infty}(x+x_1+\cdots+x_r)_m\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{(\log(1+t))^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(x+x_1+\cdots+x_r)_m S_{2,\lambda}(n,m)\frac{(\log(1+t))^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(x+x_1+\cdots+x_r)_m S_{2,\lambda}(n,m)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(x+x_1+\cdots+x_r)_m S_{2,\lambda}(k,m)S_1(n,k)\right)\frac{t^n}{n!},
\end{aligned}
\tag{63}
$$

and so by (22), (60), and (63), we obtain

$$
\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{x+x_1+\cdots+x_r/\lambda}d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_m d\mu_0(x_1)\cdots d\mu_0(x_r)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)D_m^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{64}
$$

By (60) and (64), we obtain the following theorem.

Theorem 8. *For each nonnegative integer m,*

$$
D_n^{(r)}(x\mid\lambda)=\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)D_m^{(r)}(x). \tag{65}
$$

Theorem 8 shows that higher-order degenerate Daehee polynomials are related closely to Daehee polynomials of order r.

4. Conclusion

In the past two decades, the degenerations of special functions and their applications have been studied as a new area of mathematics. In this paper, we considered the degenerate Daehee numbers and polynomials by using p-adic invariant integral on $\mathbb{Z}_p$ which are different from Kim's degenerate Daehee polynomials. We derive some new and interesting properties of those polynomials.

Next, from the definition of the higher-order degenerate Daehee numbers and Daehee polynomials of the second kind, we found the relationship between the degenerate Bernoulli polynomials, the first and second Stirling numbers, the Bernoulli polynomials, degenerate Stirling numbers of the second kind, and those numbers and polynomials.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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Research Article

Bounds for the Remainder in Simpson's Inequality via n-Polynomial Convex Functions of Higher Order Using Katugampola Fractional Integrals

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The goal of this paper is to derive some new variants of Simpson's inequality using the class of n-polynomial convex functions of higher order. To obtain the main results of the paper, we first derive a new generalized fractional integral identity utilizing the concepts of Katugampola fractional integrals. This new fractional integral identity will serve as an auxiliary result in the development of the main results of this paper.

1. Introduction and Preliminaries

The following inequality known in the literature as Simpson's inequality [1].

$$\left|\int_a^b \Lambda(t)\mathrm{d}t - \frac{b-a}{6}\left[\Lambda(a) + 4\Lambda\left(\frac{a+b}{2}\right) + \Lambda(b)\right]\right| \leq \frac{(b-a)^4}{2880}\left\|\Lambda^{(4)}\right\|_\infty, \tag{1}$$

where Λ: $[a,b] \longrightarrow \mathbb{R}$ is a four times continuously differentiable function on (a,b) and

$$\left\|\Lambda^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|\Lambda^{(4)}(t)\right| < \infty. \tag{2}$$

Simpson's inequality plays a significant role in analysis [2–4]. Over the years, it has been extended and generalized in different directions using novel and innovative approaches. The survey by Dragomir et al. [5] is very informative regarding the developments of Simpson's inequality and its applications.

In recent years, the fractional calculus [6–10] is often known as noninteger calculus which has become a powerful tool in mathematics because it provides a good tool to describe physical memory. Fractional calculus has wide applications in real life through its help in solving different physical problems [11–20]. The classic definition of Riemann–Liouville fractional integrals is one of the most basic concepts in fractional calculus which is defined as:

Definition 1. Let $\Lambda \in L[a,b]$. Then Riemann–Liouville integrals $J^\alpha_{a^+}\Lambda$ and $J^\alpha_{b^-}\Lambda$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\begin{aligned} J^\alpha_{a^+}\Lambda(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\Lambda(t)\mathrm{d}t, \quad x > a,\\ J^\alpha_{b^-}\Lambda(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\Lambda(t)\mathrm{d}t, \quad x < b, \end{aligned} \tag{3}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}\mathrm{d}x, \tag{4}$$

is a well known gamma function.

In past few decades, several successful attempts have been made in generalizing the classical concepts of fractional calculus. Erdelyi–Kober operator is a significant generalization of fractional integrals introduced and was studied by

Arthur Erdelyi and Hermann Kober. But there is a drawback that one cannot get the Hadamard version of the derivatives and integrals from Erdelyi–Kober operators. Katugampola [21] gave a well-defined concept of fractional integrals as:

Definition 2. Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then the left and right sides of Katugampola fractional integrals of order $\alpha > 0$ of $\Lambda \in X_c^{\varrho}(a, b)$ are defined by

$$\begin{aligned} {}^{\varrho}I_{a^+}^{\alpha}\Lambda(x) &= \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\varrho-1}}{(x^{\varrho} - t^{\varrho})^{1-\alpha}} \Lambda(t)\mathrm{d}t, \\ {}^{\varrho}I_{a^+}^{\alpha}\Lambda(x) &= \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\varrho-1}}{(t^{\varrho} - x^{\varrho})^{1-\alpha}} \Lambda(t)\mathrm{d}t, \end{aligned} \tag{5}$$

if the integral exists.

If we take $\rho = 1$, then we can recapture Riemann–Liouville fractional integrals from the Katugampola fractional integrals. It worth to mention here that Erdelyi–Kober operators and Katugampola fractional integrals are not equivalent to each other.

Sarikaya et al. [22] are the first authors to utilize the concepts of Riemann–Liouville fractional integrals in obtaining the fractional analogues of Hermite–Hadamard's inequality. This idea inspired several inequalities expert, and resultantly huge number of articles have been written on the fractional analogues of classical inequalities. For example, Hu et al. [23] obtained some new fractional analogues of integral inequalities using Katugampola fractional integrals. Nie et al. [24] obtained k-fractional analogues of Simpson's inequality. Peng et al. [25] also obtained some new fractional analogues of Simpson's inequality. Set [26] obtained fractional analogues of Ostrowski's inequality. Wu et al. [27] obtained fractional analogues of inequalities using k-th order differentiable functions. Kermausuor [28] obtained new Simpson type inequalities involving Katugampola fractional integrals essentially using the class of Breckner type s-convex function.

Recently, Toplu et al. [29] introduced the notion of n-polynomial convex functions as follows.

Definition 3. Let $n \in \mathbb{N}$. Then a nonnegative function $\Lambda: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a n-polynomial convex function if the inequality

$$\Lambda(tx + (1-t)y \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\Lambda(x) + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]\Lambda(y), \tag{6}$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Many researchers have also derived several new Hermite–Hadamard's like inequalities [30–36] using the concept of n-polynomial convex functions. We would like to point out here that the class of n-polynomial convex functions generalize the class of convex functions if we take $n = 1$, then we have the class of 1-polynomial convex functions which is just the classical convex functions. Also we can get other type of convexities: for example, for $n = 2$, we have 2-polynomial convexity. Another point of pondering here is that every n-polynomial convex function is an h-convex function with the function $h(t) = 1/n\sum_{s=1}^{n}[1 - (1-t)^s]$. So more generally, every nonnegative convex function is also an n-polynomial convex function.

The idea behind the study of this article is to extend the notion of n-polynomial convex functions with the introduction of higher order n-polynomial convex functions. We derive a new fractional integral identity using the concepts of Katugampola fractional integrals. This new identity will serve as an auxiliary result in the development of some new fractional analogues of Simpson's inequalities using the concept of n-polynomial convex functions of higher order.

Before we move to our main results, we would like to introduce the notion of n-polynomial convex functions of higher order.

Definition 4. Let $n \in \mathbb{N}$. Then a nonnegative function $\Lambda: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a higher order n-polynomial convex function if the inequality

$$\Lambda(tx + (1-t)y \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\Lambda(x) + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]\Lambda(y) - \mu(t^{\sigma}(1-t) + t(1-t)^{\sigma})\|y-x\|^{\sigma}, \tag{7}$$

holds for every $x, y \in I$, $\sigma > 0$, and $t \in [0, 1]$.

Remark 1. Note that if $\sigma = 0$ in (7), then the class of n-polynomial convex functions of higher order reduces to the class of n-polynomial convex functions. If $\sigma = 2$, then we have a new class of strongly n-polynomial convex functions. If $n = 1$, then we have the class of higher-order convex functions [37]. And along with $n = 1$, if we have $\sigma = 2$, then the class of n-polynomial convex functions of higher order reduces to the class of strongly convex functions [38]. From this, it is evident that the class of n-polynomial convex functions of higher order is quite unifying one as it relates several other unrelated classes of convexity [39–45].

2. Main Results

In this section, we discuss our main results.

2.1. A Key Lemma. We now derive the main auxiliary result of the paper.

Lemma 1. *Let $\alpha, \varrho > 0$, and let $\Lambda\colon [a^{\varrho}, b^{\varrho}] \longrightarrow \mathbb{R}$ be a differentiable function on $(a^{\varrho}, b^{\varrho})$, with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^{\varrho}, b^{\varrho}])$, then*

$$\begin{aligned}
&\frac{1}{(i+1)(i+2)}\left[\Lambda(a^{\varrho}) + (i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right) + \Lambda(b^{\varrho})\right] \\
&\quad - \frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{(b^{\varrho}-a^{\varrho})^{\alpha}}\left[{}^{\varrho}I^{\alpha}_{a^{+}}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + {}^{\varrho}I^{\alpha}_{b^{-}}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right] \\
&\quad = \frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[\int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)t^{\varrho-1}\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t - \int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)t^{\varrho-1}\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t\right].
\end{aligned} \tag{8}$$

Proof. Consider

$$\begin{aligned}
I &= \frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[\int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)t^{\varrho-1}\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t - \int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}t^{\varrho-1}\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t\right)\right] \\
I &= \frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[I_1 - I_2\right].
\end{aligned} \tag{9}$$

Integrating by parts I_1, we have

$$\begin{aligned}
I_1 &= \int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)t^{\varrho-1}\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t \\
&= -\frac{i+1}{\varrho(b^{\varrho}-a^{\varrho})}\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)\Lambda\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\Bigg|_0^1 - \frac{\alpha}{b^{\varrho}-a^{\varrho}}\int_0^1 t^{\alpha\varrho-1}\Lambda\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t, \\
&= \frac{\Lambda(a^{\varrho})}{\varrho(b^{\varrho}-a^{\varrho})(i+2)} + \frac{i+1}{\varrho(i+2)(b^{\varrho}-a^{\varrho})}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) - \frac{(i+1)^{\alpha}\varrho^{\alpha-1}\Gamma(\alpha+1)^{\varrho}}{(b^{\varrho}-a^{\varrho})^{\alpha+1}}I^{\alpha}_{a^{+}}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right).
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
I_2 &= \int_0^1\left(\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right)t^{\varrho-1}\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\mathrm{d}t \\
&= -\frac{\Lambda(b^{\varrho})}{\varrho(b^{\varrho}-a^{\varrho})(i+2)} - \frac{i+1}{\varrho(i+2)(b^{\varrho}-a^{\varrho})}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right) + \frac{(i+1)^{\alpha}\varrho^{\alpha-1}\Gamma(\alpha+1)^{\varrho}}{(b^{\varrho}-a^{\varrho})^{\alpha+1}}I^{\alpha}_{b^{-}}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right).
\end{aligned} \tag{11}$$

By substituting the values of I_1 and I_2 in I, we get the required result.

This completes the proof. □

2.2. Results and Discussions. We now derive the main results using Lemma 1.

Theorem 1. *Let $\alpha, \varrho > 0$ and let $\Lambda\colon [a^{\varrho}, b^{\varrho}] \longrightarrow \mathbb{R}$ be a differentiable function on $(a^{\varrho}, b^{\varrho})$, with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^{\varrho}, b^{\varrho}])$. If $|\Lambda'|$ is higher-order n-polynomial convex function of order $\sigma > 0$, then*

$$\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda(a^{\varrho}) + \Lambda(b^{\varrho}) + (i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right)\right] - \frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{(b^{\varrho}-a^{\varrho})^{\alpha}}\left[{}^{\varrho}I^{\alpha}_{a^+}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right)\right.\right. \\
&\quad \left.\left. + {}^{\varrho}I^{\alpha}_{b^-}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right]\right| \\
&\leq \frac{(b^{\varrho}-a^{\varrho})}{(i+1)(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(2s-i+1)+i^{s+1}-1}{(i+1)^{s}(s+1)}\left(\left|\Lambda'(a^{\varrho})\right| + \left|\Lambda'(b^{\varrho})\right|\right)\right) - \frac{4\mu}{(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right].
\end{aligned} \tag{12}$$

Proof. By using Lemma 1 and because $|\Lambda'|$ is higher-order n-polynomial convex function of order $\sigma > 0$, we have

$$\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda(a^{\varrho}) + \Lambda(b^{\varrho}) + (i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right)\right] - \frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{(b^{\varrho}-a^{\varrho})^{\alpha}}\left[{}^{\varrho}I^{\alpha}_{a^+}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right)\right.\right. \\
&\quad \left.\left. + {}^{\varrho}I^{\alpha}_{b^-}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right]\right| \\
&\leq \frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[\int_0^1\left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1}\left|\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t + \int_0^1\left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1}\left|\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t\right] \\
&= \frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[I_1 + I_2\right].
\end{aligned} \tag{13}$$

Note that

$$\begin{aligned}
I_1 &= \int_0^1\left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1}\left|\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t \\
&= \frac{1}{\varrho}\int_0^1\left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right|\left|\Lambda'\left(\frac{i+x}{i+1}a^{\varrho} + \frac{1-x}{i+1}b^{\varrho}\right)\right| \mathrm{d}x \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1\left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \mathrm{d}x\right)\left(\int_0^1\left|\Lambda'\left(\frac{i+x}{i+1}a^{\varrho} + \frac{1-x}{i+1}b^{\varrho}\right)\right| \mathrm{d}x\right)\right] \\
&\leq \frac{1}{\varrho}\left(\int_0^1\left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \mathrm{d}x\right)\left(\int_0^1\left(\frac{1}{n}\sum_{s=1}^{n}\left(1 - \left(1 - \left(\frac{i+x}{i+1}\right)\right)^{s}\right)\left|\Lambda'(a^{\varrho})\right|\right.\right. \\
&\quad \left.\left. + \frac{1}{n}\sum_{s=1}^{n}\left(1 - \left(\frac{i+x}{i+1}\right)^{s}\right)\left|\Lambda'(b^{\varrho})\right| - \mu\left(x^{\sigma}(1-x) + x(1-x)^{\sigma}\right)\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\mathrm{d}x\right)\right) \\
&= \frac{1}{\varrho(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(1+s)-1}{(i+1)^{s}(1+s)}\left|\Lambda'(a^{\varrho})\right| + \frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(s-i)+i^{s+1}}{(i+1)^{s}(1+s)}\left|\Lambda'(b^{\varrho})\right| - \frac{2\mu}{(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right)^{1/q}\right].
\end{aligned} \tag{14}$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1} \left|\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t \\
&= \frac{1}{\varrho}\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \left|\Lambda'\left(\frac{1-x}{i+1}a^{\varrho} + \frac{i+x}{i+1}b^{\varrho}\right)\right| \mathrm{d}x \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \mathrm{d}x\right)\left(\int_0^1 \left|\Lambda'\left(\frac{1-x}{i+1}a^{\varrho} + \frac{i+x}{i+1}b^{\varrho}\right)\right| \mathrm{d}x\right)\right] \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \mathrm{d}x\right)\left(\int_0^1 \left(\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(1-\left(\frac{1-x}{i+1}\right)\right)^{s}\right)\left|\Lambda'\left(a^{\varrho}\right)\right|\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{1-x}{i+1}\right)^{s}\right)\left|\Lambda'\left(b^{\varrho}\right)\right| - \mu\left(x^{\sigma}(1-x) + x(1-x)^{\sigma}\right)\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\mathrm{d}x\right)\right)\right] \\
&= \frac{1}{\varrho(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(s-i)+i^{s+1}}{(i+1)^{s}(1+s)}\left|\Lambda'\left(a^{\varrho}\right)\right| + \frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(1+s)-1}{(i+1)^{s}(1+s)}\left|\Lambda'\left(b^{\varrho}\right)\right| - \frac{2\mu}{(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right)\right],
\end{aligned} \tag{15}
$$

where we have used the fact that $|1/i+2-x^{\alpha}/i+1| \leq 1/i+2$ for all $x \in [0,1]$. Using I_1 and I_2 in (13), we get the required result. □

Theorem 2. *Let $\alpha, \varrho > 0$, and let $\Lambda\colon [a^{\varrho}, b^{\varrho}] \longrightarrow \mathbb{R}$ be a differentiable function on $(a^{\varrho}, b^{\varrho})$, with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^{\varrho}, b^{\varrho}])$. If $|\Lambda'|^{q}$ is higher-order n-polynomial convex function of order $\sigma > 0$, where $r^{-1} + q^{-1} = 1$ and $q > 1$, then*

$$
\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda\left(a^{\varrho}\right) + \Lambda\left(b^{\varrho}\right) + (i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right)\right]\right. \\
&\quad \left. - \frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{\left(b^{\varrho}-a^{\varrho}\right)^{\alpha}}\left[{}^{\varrho}I_{a^{+}}^{\alpha}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + {}^{\varrho}I_{b^{-}}^{\alpha}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right]\right| \\
&\leq \frac{\left(b^{\varrho}-a^{\varrho}\right)}{(i+1)(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(1+s)-1}{(i+1)^{s}(1+s)}\left|\Lambda'\left(a^{\varrho}\right)\right|^{q} + \frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(s-i)+i^{s+1}}{(i+1)^{s}(1+s)}\left|\Lambda'\left(b^{\varrho}\right)\right|^{q} - \frac{2\mu}{(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right)^{1/q}\right. \\
&\quad \left. + \left(\frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(s-i)+i^{s+1}}{(i+1)^{s}(1+s)}\left|\Lambda'\left(a^{\varrho}\right)\right|^{q} + \frac{1}{n}\sum_{s=1}^{n}\frac{(i+1)^{s}(1+s)-1}{(i+1)^{s}(1+s)}\left|\Lambda'\left(b^{\varrho}\right)\right|^{q} - \frac{2\mu}{(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right)^{1/q}\right].
\end{aligned} \tag{16}
$$

Proof. Using Lemma 1, Holder's inequality and because $|\Lambda\prime|^{q}$ is a higher-order n-polynomial convex function of order $\sigma > 0$, we have

$$
\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda\left(a^{\varrho}\right) + \Lambda\left(b^{\varrho}\right) + (i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + \Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right)\right]\right. \\
&\quad \left. - \frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{\left(b^{\varrho}-a^{\varrho}\right)^{\alpha}}\left[{}^{\varrho}I_{a^{+}}^{\alpha}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right) + {}^{\varrho}I_{b^{-}}^{\alpha}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right]\right| \\
&\leq \frac{\varrho\left(b^{\varrho}-a^{\varrho}\right)}{i+1}\left[\int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1}\left|\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho} + \frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t + \int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1}\left|\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\right| \mathrm{d}t\right] \\
&= \frac{\varrho\left(b^{\varrho}-a^{\varrho}\right)}{i+1}\left[I_3 + I_4\right].
\end{aligned} \tag{17}
$$

Note that

$$\begin{aligned}
I_3 &= \int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1} \left|\Lambda'\left(\frac{i+t^\varrho}{i+1}a^\varrho + \frac{1-t^\varrho}{i+1}b^\varrho\right)\right| \mathrm{d}t \\
&= \frac{1}{\varrho}\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right| \left|\Lambda'\left(\frac{i+x}{i+1}a^\varrho + \frac{1-x}{i+1}b^\varrho\right)\right| \mathrm{d}x \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right|^r \mathrm{d}x\right)^{1/r} \left(\int_0^1 \left|\Lambda'\left(\frac{i+x}{i+1}a^\varrho + \frac{1-x}{i+1}b^\varrho\right)\right|^q \mathrm{d}x\right)^{1/q}\right] \\
&\leq \frac{1}{\varrho}\left[\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right|^r \mathrm{d}x\right)^{1/r} \left(\left(\int_0^1 \left(\frac{1}{n}\sum_{s=1}^n \left(1-\left(1-\left(\frac{i+x}{i+1}\right)\right)^s\right)|\Lambda'(a^\varrho)|^q + \frac{1}{n}\sum_{s=1}^n \left(1-\left(\frac{i+x}{i+1}\right)^s\right)|\Lambda'(b^\varrho)|^q\right.\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.\left. - \mu\left(x^\sigma(1-x) + x(1-x)^\sigma\right)\|b^\varrho - a^\varrho\|^\sigma \mathrm{d}x\right)^{1/q}\right)\right)\right] \\
&= \frac{1}{\varrho(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}|\Lambda'(a^\varrho)|^q + \frac{1}{n}\sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}|\Lambda'(b^\varrho)|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)}\|b^\varrho - a^\varrho\|^\sigma\right)^{1/q}\right].
\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}
I_4 &= \int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1} \left|\Lambda'\left(\frac{1-t^\varrho}{i+1}a^\varrho + \frac{i+t^\varrho}{i+1}b^\varrho\right)\right| \mathrm{d}t \\
&= \frac{1}{\varrho}\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right| \left|\Lambda'\left(\frac{1-x}{i+1}a^\varrho + \frac{i+x}{i+1}b^\varrho\right)\right| \mathrm{d}x \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right|^r \mathrm{d}x\right)^{1/r} \left(\int_0^1 \left|\Lambda'\left(\frac{1-x}{i+1}a^\varrho + \frac{i+x}{i+1}b^\varrho\right)\right|^q \mathrm{d}x\right)^{1/q}\right] \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^\alpha}{i+1}\right|^r \mathrm{d}x\right)^{1/r} \left(\int_0^1 \left(\frac{1}{n}\sum_{s=1}^n \left(1-\left(1-\left(\frac{1-x}{i+1}\right)\right)^s\right)|\Lambda'(a^\varrho)|^q + \frac{1}{n}\sum_{s=1}^n \left(1-\left(\frac{1-x}{i+1}\right)^s\right)|\Lambda'(b^\varrho)|^q - \mu\left(x^\sigma(1-x) + x(1-x)^\sigma\right)\|b^\varrho - a^\varrho\|^\sigma \mathrm{d}x\right)\right)^{1/q}\right] \\
&= \frac{1}{\varrho(i+2)}\left[\left(\frac{1}{n}\sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}|\Lambda'(a^\varrho)|^q + \frac{1}{n}\sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}|\Lambda'(b^\varrho)|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)}\|b^\varrho - a^\varrho\|^\sigma\right)^{1/q}\right],
\end{aligned} \tag{19}$$

where $|1/i+2 - x^\alpha/i+1| \leq 1/i+2$ for all $x \in [0,1]$. Using I_3 and I_4 in (17), we get the required result. □

Theorem 3. *Let $\alpha, \varrho > 0$, and let $\Lambda\colon [a^\varrho, b^\varrho] \longrightarrow \mathbb{R}$ be a differentiable function on (a^ϱ, b^ϱ), with $0 \leq \alpha < b$ such that $\Lambda' \in L_1([a^\varrho, b^\varrho])$. If $|\Lambda\prime|^q$ is a higher-order n-polynomial convex function of order $\sigma > 0$ where $q > 1$, then*

$$\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda(a^\varrho) + \Lambda(b^\varrho) + (i+1)\left(\Lambda\left(\frac{ia^\varrho + b^\varrho}{i+1}\right) + \Lambda\left(\frac{a^\varrho + ib^\varrho}{i+1}\right)\right)\right]\right. \\
&\qquad \left. - \frac{(i+1)^{\alpha-1}\varrho^\alpha\Gamma(\alpha+1)}{(b^\varrho - a^\varrho)^\alpha}\left[{}^\varrho I^\alpha_{a^+}\Lambda\left(\frac{ia^\varrho + b^\varrho}{i+1}\right) + {}^\varrho I^\alpha_{b^-}\Lambda\left(\frac{a^\varrho + ib^\varrho}{i+1}\right)\right]\right| \\
&\leq \frac{(b^\varrho - a^\varrho)}{(i+1)(i+2)^{1/r}}\left[\begin{aligned}
&\left(\frac{1}{i(i+2)}\sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}|\Lambda'(a^\varrho)|^q + \frac{1}{n(i+2)}\sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}|\Lambda'(b^\varrho)|^q - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)}\|b^\varrho - a^\varrho\|^\sigma\right)^{1/q} \\
&+\left(\frac{1}{n(i+2)}\sum_{s=1}^n \frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}|\Lambda'(a^\varrho)|^q + \frac{1}{n(i+2)}\sum_{s=1}^n \frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}|\Lambda'(b^\varrho)|^q - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)}\|b^\varrho - a^\varrho\|^\sigma\right)^{1/q}
\end{aligned}\right].
\end{aligned} \tag{20}$$

Proof. Using Lemma 1, Power mean inequality, and because $|\Lambda\prime|^q$ is a higher-order n-polynomial convex function of order $\sigma > 0$, we have

$$\begin{aligned}
&\left|\frac{1}{(i+1)(i+2)}\left[\Lambda(a^{\varrho})+\Lambda(b^{\varrho})+(i+1)\left(\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right)+\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right)\right]\right.\\
&\left.\quad-\frac{(i+1)^{\alpha-1}\varrho^{\alpha}\Gamma(\alpha+1)}{(b^{\varrho}-a^{\varrho})^{\alpha}}\left[{}^{\varrho}I^{\alpha}_{a^+}\Lambda\left(\frac{ia^{\varrho}+b^{\varrho}}{i+1}\right)+{}^{\varrho}I^{\alpha}_{b^-}\Lambda\left(\frac{a^{\varrho}+ib^{\varrho}}{i+1}\right)\right]\right|\\
&\quad\leq\frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[\int_0^1\left|\frac{1}{i+2}-\frac{t^{\alpha\varrho}}{i+1}\right|t^{\varrho-1}\left|\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho}+\frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\right|\mathrm{d}t+\int_0^1\left|\frac{1}{i+2}-\frac{t^{\alpha\varrho}}{i+1}\right|t^{\varrho-1}\left|\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho}+\frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\right|\mathrm{d}t\right]\\
&\quad=\frac{\varrho(b^{\varrho}-a^{\varrho})}{i+1}\left[I_5+I_6\right].
\end{aligned}\tag{21}$$

Note that

$$\begin{aligned}
I_5&=\int_0^1\left|\frac{1}{i+2}-\frac{t^{\alpha\varrho}}{i+1}\right|t^{\varrho-1}\left|\Lambda'\left(\frac{i+t^{\varrho}}{i+1}a^{\varrho}+\frac{1-t^{\varrho}}{i+1}b^{\varrho}\right)\right|\mathrm{d}t\\
&=\frac{1}{\varrho}\int_0^1\left|\frac{1}{i+2}-\frac{x^{\alpha}}{i+1}\right|\left|\Lambda'\left(\frac{i+x}{i+1}a^{\varrho}+\frac{1-x}{i+1}b^{\varrho}\right)\right|\mathrm{d}x\\
&\leq\frac{1}{\varrho}\left[\left(\int_0^1\left|\frac{1}{i+2}-\frac{x^{\alpha}}{i+1}\right|\mathrm{d}x\right)^{1/r}\left(\int_0^1\left|\frac{1}{i+2}-\frac{x^{\alpha}}{i+1}\right|\left|\Lambda'\left(\frac{i+x}{i+1}a^{\varrho}+\frac{1-x}{i+1}b^{\varrho}\right)\right|^q\mathrm{d}x\right)^{1/q}\right]\\
&\leq\frac{1}{\varrho}\left[\left(\int_0^1\left|\frac{1}{i+2}-\frac{x^{\alpha}}{i+1}\right|\mathrm{d}x\right)^{1/r}\left(\int_0^1\left|\frac{1}{i+2}-\frac{x^{\alpha}}{i+1}\right|\left(\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(1-\left(\frac{i+x}{i+1}\right)\right)^s\right)\left|\Lambda'(a^{\varrho})\right|^q+\frac{1}{n}\sum_{s=1}^{n}\left(1-\left(\frac{i+x}{i+1}\right)^s\right)\left|\Lambda'(b^{\varrho})\right|^q\right.\right.\right.\\
&\qquad\left.\left.\left.-\mu\left(x^{\sigma}(1-x)+x(1-x)^{\sigma}\right)\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\mathrm{d}x\right)^{1/q}\right]\\
&=\frac{1}{\varrho(i+2)^{1/r}}\left[\left(\frac{1}{n(i+2)}\sum_{s=1}^{n}\frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}\left|\Lambda'(a^{\varrho})\right|^q+\frac{1}{n(i+2)}\sum_{s=1}^{n}\frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}\left|\Lambda'(b^{\varrho})\right|^q\right.\right.\\
&\qquad\left.\left.-\frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)}\left\|b^{\varrho}-a^{\varrho}\right\|^{\sigma}\right)^{1/q}\right].
\end{aligned}\tag{22}$$

Similarly,

$$
\begin{aligned}
I_6 &= \int_0^1 \left|\frac{1}{i+2} - \frac{t^{\alpha\varrho}}{i+1}\right| t^{\varrho-1} \left|\Lambda'\left(\frac{1-t^{\varrho}}{i+1}a^{\varrho} + \frac{i+t^{\varrho}}{i+1}b^{\varrho}\right)\right| dt \\
&= \frac{1}{\varrho}\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \left|\Lambda'\left(\frac{1-x}{i+1}a^{\varrho} + \frac{i+x}{i+1}b^{\varrho}\right)\right| dx \\
&\leq \frac{1}{\varrho}\left[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| dx\right)^{1/r} \left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \left|\Lambda'\left(\frac{1-x}{i+1}a^{\varrho} + \frac{i+x}{i+1}b^{\varrho}\right)\right|^q dx\right)^{1/q}\right] \\
&\leq \frac{1}{\varrho}\Bigg[\left(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| dx\right)^{1/r} \Bigg(\int_0^1 \left|\frac{1}{i+2} - \frac{x^{\alpha}}{i+1}\right| \Bigg(\frac{1}{n}\sum_{s=1}^{n}\left(1 - \left(1 - \left(\frac{1-x}{i+1}\right)^s\right)\right)\left|\Lambda'\left(a^{\varrho}\right)\right|^q + \frac{1}{n}\sum_{s=1}^{n}\left(1 - \left(\frac{1-x}{i+1}\right)\right)^s \left|\Lambda'\left(b^{\varrho}\right)\right|^q \\
&\quad - \mu\left(x^{\sigma}(1-x) + x(1-x)^{\sigma}\right)\left\|b^{\varrho} - a^{\varrho}\right\|^{\sigma} dx\Bigg)\Bigg)^{1/q}\Bigg] \\
&= \frac{1}{\varrho(i+2)^{1/r}}\Bigg[\Bigg(\frac{1}{i(i+2)}\sum_{s=1}^{n}\frac{(i+1)^s(s-i)+i^{s+1}}{(i+1)^s(1+s)}\left|\Lambda'\left(a^{\varrho}\right)\right|^q + \frac{1}{n(i+2)}\sum_{s=1}^{n}\frac{(i+1)^s(1+s)-1}{(i+1)^s(1+s)}\left|\Lambda'\left(b^{\varrho}\right)\right|^q \\
&\quad - \frac{2\mu}{(i+2)(\sigma+1)(\sigma+2)}\left\|b^{\varrho} - a^{\varrho}\right\|^{\sigma}\Bigg)^{1/q}\Bigg],
\end{aligned} \tag{23}
$$

where we have used the fact that $|1/i+2 - x^{\alpha}/i+1| \leq 1/i+2$ for all $x \in [0,1]$. Using I_5 and I_6 in (21), we get the required result. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Composition Formulae for the k-Fractional Calculus Operators Associated with k-Wright Function

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In this article, the k-fractional-order integral and derivative operators including the k-hypergeometric function in the kernel are used for the k-Wright function; the results are presented for the k-Wright function. Also, some of special cases related to fractional calculus operators and k-Wright function are considered.

1. Introduction and Preliminaries

Fractional calculus was introduced in 1695, but in the last two decades researchers have been able to use it properly on the account of availability of computational resources. In many areas of application of fractional calculus, the researchers found significant applications in science and engineering. In the literature, many applications of fractional calculus are available in astrophysics, biosignal processing, fluid dynamics, nonlinear control theory, stochastic dynamical system, and so on. Also, a number of researchers [1–10] have studied in-depth level of properties, applications, and various directions of extensions of Gauss hypergeometric function of fractional integration.

Recently, in a series of research publications on generalized classical fractional calculus operators, research by Mubeen and Habibullah [11] has been published on the integral part of the Riemann-Liouville version and its applications; an alternative definition for the k-Riemann–Liouville fractional derivative was introduced by Dorrego [12]. The left- and right-hand operators of Saigo k-fractional integration and differentiation associated with the k-Gauss hypergeometric function defined by Gupta and Parihar [13] (see also [14]) are as follows:

$$\left(I_{0+,k}^{\tau,\delta,\gamma} f\right)(x) = \frac{x^{(-\tau-\delta)/k}}{k\Gamma_k(\tau)} \int_0^x (x-t)^{(\tau/k)-1} {}_2F_{1,k}\left((\tau+\delta,k), (-\gamma,k); (\tau,k); \left(1-\frac{t}{x}\right)\right) f(t)\mathrm{d}t; \quad \Re(\tau)>0, k>0, \tag{1}$$

$$\left(I_{-,k}^{\tau,\delta,\gamma} f\right)(x) = \frac{1}{k\Gamma_k(\tau)} \int_x^\infty (t-x)^{(\tau/k)-1} t^{(-\tau-\delta)/k} {}_2F_{1,k}\left((\tau+\delta,k), (-\gamma,k); (\tau,k); \left(1-\frac{x}{t}\right)\right) f(t)\mathrm{d}t; \quad \Re(\tau)>0, k>0, \tag{2}$$

where ${}_2F_{1,k}((\tau,k), (\delta,k); (\gamma,k); x)$ is the k-Gauss hypergeometric function defined by [11] for $x \in \mathbb{C}, |x|<1$, $\Re(\gamma) > \Re(\delta) > 0$:

$${}_2F_{1,k}((\tau,k), (\delta,k); (\gamma,k); x) = \sum_{n=0}^{\infty} \frac{(\tau)_{n,k} (\delta)_{n,k} x^n}{(\gamma)_{n,k} n!}. \tag{3}$$

The corresponding fractional differential operators have their respective forms as

$$\left(D_{0+,k}^{\tau,\delta,\gamma} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I_{0+,k}^{-\tau+n,-\delta-n,\tau+\gamma-n} f\right)(x); \quad \Re(\tau) > 0, k > 0; \quad n = [\Re(\tau) + 1]$$

$$\left(D_{0+,k}^{\tau,\delta,\gamma} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{x^{((\tau+\delta)/k)}}{k\Gamma_k(-\tau+n)} \int_0^x (x-t)^{-(\tau/k)+n-1} \times {}_2F_{1,k}\left((-\tau-\delta,k), (-\gamma-\tau+n,k); (-\tau+n,k); \left(1-\frac{t}{x}\right)\right) f(t)\mathrm{d}t, \tag{4}$$

$$\left(D_{-,k}^{\tau,\delta,\gamma} f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I_{-,k}^{-\tau+n,-\delta-n,\tau+\gamma} f\right)(x); \quad \Re(\tau) > 0, k > 0; \quad n = [\Re(\tau) + 1]$$

$$\left(D_{-,k}^{\tau,\delta,\gamma} f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{k\Gamma_k(-\tau+n)} \int_x^\infty (t-x)^{-((\tau+n)/k)-1} t^{(\tau+\delta)/k} \times {}_2F_{1,k}\left((-\tau-\delta,k), (-\gamma-\tau,k); (-\tau+n,k); \left(1-\frac{x}{t}\right)\right) f(t)\mathrm{d}t, \tag{5}$$

where $x > 0, \tau \in \mathbb{C}, \Re(\tau) > 0, k > 0$, and $[\Re(\tau)]$ is the integer part of $\Re(\tau)$.

Remark 1. If we set $k = 1$ in equations (1), (2), (4), and (5), operators reduce to Saigo's fractional integral and derivative operators stated in [5], respectively.

Now, we consider the following basic results for our study.

Lemma 1 (see [13], pp. 497, Eq. 4.2). *Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}, \Re(\zeta) > \max[0, \Re(\delta-\gamma)]$; then,*

$$\left(I_{0+,k}^{\tau,\delta,\gamma} t^{(\zeta/k)-1}\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta)\Gamma_k(\zeta-\delta+\gamma)}{\Gamma_k(\zeta-\delta)\Gamma_k(\zeta+\tau+\gamma)} x^{((\zeta-\delta)/k)-1}. \tag{6}$$

Lemma 2 (see [13], pp. 497, Eq. 4.3). *Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}$, $\Re(\tau) > 0$, $k \in \Re^+(0,\infty)$, and $\Re(\zeta) > \max[\Re(-\delta), \Re(-\gamma)]$; then,*

$$\left(I_{-,k}^{\tau,\delta,\gamma} t^{-(\zeta/k)}\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta+\delta)\Gamma_k(\zeta+\gamma)}{\Gamma_k(\zeta)\Gamma_k(\zeta+\tau+\delta+\gamma)} x^{(-\zeta-\delta)/k}. \tag{7}$$

Lemma 3 (see [13], pp. 500, Eq. 6.2). *Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}$, $n = [\Re(\tau)] + 1$, $k \in \Re^+(0,\infty)$ such that $\Re(\zeta) > \max[0, \Re(-\tau-\delta-\gamma)]$; then,*

$$\left(D_{0+,k}^{\tau,\delta,\gamma} t^{(\zeta/k)-1}\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta)\Gamma_k(\zeta+\delta+\gamma+\tau)}{\Gamma_k(\zeta+\gamma)\Gamma_k(\zeta+\delta+n-nk)} x^{((\zeta+\delta+n)/k)-n-1}. \tag{8}$$

Lemma 4 (see [13], pp. 500, Eq. 6.3). *Let $\tau, \delta, \gamma, \zeta \in \mathbb{C}$, and $n = [\Re(\tau)] + 1$, $k \in \Re^+$, and $\Re(\zeta) > \max[\Re(-\tau-\gamma), \Re(\delta-nk+n)]$; then,*

$$\left(D_{-,k}^{\tau,\delta,\gamma} t^{-(\zeta/k)}\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\zeta-\delta-n+nk)\Gamma_k(\zeta+\tau+\gamma)}{\Gamma_k(\zeta)\Gamma_k(\zeta-\delta+\gamma)} x^{((-\zeta+\delta+n)/k)-n}. \tag{9}$$

Recently, Gehlot and Prajapati [15] studied the concept of the generalized k-Wright function, which is presented in the following definition, and its connection with other special functions. It is the generalization of Mittag-Leffler function and many other special functions (see also, [16–21]). These special functions have found many important applications in solving problems of physics, biology, engineering, and applied sciences.

The k-Wright function is defined for $k \in \mathbb{R}^+; x, a_i, b_j \in \mathbb{C}$, $\varepsilon_i, \varsigma_j \in \mathbb{R}\,(\varepsilon_i, \varsigma_j \neq 0; i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, and $(a_i + \varepsilon_i n)$, $(b_j + \varsigma_j n) \in \mathbb{C}\backslash kz^-$ as

$$ {}_p\Psi_q^k(x) = {}_p\Psi_q^k\left[\begin{matrix} (a_1,\varepsilon_1), \ldots, (a_p,\varepsilon_p); \\ (b_1,\varsigma_1), \ldots, (b_q,\varsigma_q); \end{matrix}\ x\right] = {}_p\Psi_q^k\left[\begin{matrix} (a_i,\varepsilon_i)_{1,p}; \\ (b_j,\varsigma_j)_{1,q}; \end{matrix}\ x\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i+\varepsilon_i n)}{\prod_{j=1}^q \Gamma_k\left(b_j+\varsigma_j n\right)\,(x)^n/n!}, \tag{10}$$

with the convergence conditions described as

$$\sum_{j=1}^{q} \frac{\varsigma_j}{k} - \sum_{i=1}^{p} \frac{\varepsilon_i}{k} > -1. \tag{11}$$

Remark 2. When we put $k = 1$ in (10), the k-Wright function reduces to Wright function which is stated in [22].

The following relation of the k-Wright function in terms of the generalized k-Mittag-Leffler function, k-Bessel function, k-hypergeometric function, and Mittag-Leffler family function is defined as follows by giving the appropriate values of the parameters:

(1) For $p = 1, q = 2$, the generalized k-Mittag-Leffler function from Gehlot [17] is

$$ {}_1\Psi_2^k(x) = {}_1\Psi_2^k \left[\begin{matrix} (\vartheta, \omega k); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n\omega,k}}{\Gamma_k(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{k,\varepsilon,\varsigma}^{\vartheta,\omega}(x), \tag{12} $$

Here, Díaz and Pariguan [23] introduced the k-Pochhammer symbol and k-gamma function as follows:

$$ (\mathfrak{I})_{n,k} = \begin{cases} \dfrac{\Gamma_k(\mathfrak{I} + nk)}{\Gamma_k(\mathfrak{I})}, & k \in \mathfrak{R}, \ \mathfrak{I} \in \mathbb{C}/\{0\}; \\ \mathfrak{I}(\mathfrak{I} + k) \ldots (\mathfrak{I} + (n-1)k), & n \in \mathbb{N}, \ \mathfrak{I} \in \mathbb{C}, \end{cases} \tag{13} $$

and the relation with classical Euler's gamma function is as follows:

$$ \Gamma_k(\mathfrak{I}) = k^{(\mathfrak{I}/k)-1} \Gamma\left(\frac{\mathfrak{I}}{k}\right), \tag{14} $$

where $\mathfrak{I} \in \mathbb{C}$, $k \in \mathfrak{R}$, and $n \in \mathbb{N}$. For more information on the k-Pochhammer symbol, k-special functions, and fractional Fourier transforms, refer to Romero and Cerutti's [24] articles.

(2) For $p = 1, q = 2$, the generalized k-Mittag-Leffler function from Dorrego and Cerutti [16] is

$$ {}_1\Psi_2^k(x) = {}_1\Psi_2^k \left[\begin{matrix} (\vartheta, k); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n,k}}{\Gamma_k(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{k,\varepsilon,\varsigma}^{\vartheta}(x). \tag{15} $$

(3) For $p = 1, q = 3$, the k-Bessel function of the first kind from Cerutti [25] is

$$ {}_1\Psi_3^k(x) = {}_1\Psi_3^k \left[\begin{matrix} (\vartheta, k); \\ (\varsigma + 1, \varepsilon), (\vartheta, 0)(k, k); \end{matrix} \; \frac{-xk}{2} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n (\vartheta)_{n,k}}{\Gamma_k(\varepsilon n + \varsigma + 1)} \frac{(x/2)^n}{(n!)^2} = J_{k,\varsigma}^{\vartheta,\varepsilon}(x). \tag{16} $$

(4) For $p = 3, q = 3$, the k-hypergeometric function with three parameters from Mubeen et al. [26] is

$$ {}_3\Psi_3^k(x) = {}_3\Psi_3^k \left[\begin{matrix} (\vartheta, k), (\varphi, k), (\varsigma, 0); \\ (\varsigma, k), (\vartheta, 0)(\varphi, 0); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n,k} (\varphi)_{n,k} x^n}{(\varsigma)_{n,k} n!} = {}_2F_{1,k}((\vartheta, k), (\varphi, k); (\varsigma, k); x). \tag{17} $$

(5) For $k = 1, p = 1, q = 2$, the generalized Mittag-Leffler function from Shukla and Prajapati [20] is

$$ {}_1\Psi_2^1(x) = {}_1\Psi_2^1 \left[\begin{matrix} (\vartheta, \omega); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{(\vartheta)_{n\omega}}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon,\varsigma}^{\vartheta,\omega}(x). \tag{18} $$

(6) For $k = 1, p = 1, q = 2$, the generalized Mittag-Leffler function from Prabhakar [19] is

$$ {}_1\Psi_2^1(x) = {}_1\Psi_2^1 \left[\begin{matrix} (\vartheta, 1); \\ (\varsigma, \varepsilon), (\vartheta, 0); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon,\varsigma}^{\vartheta}(x). \tag{19} $$

(7) For $k = 1$, $p = 1$, and $q = 2$, the Mittag-Leffler function from Wiman [21] is

$$ {}_1\Psi_1^1(x) = {}_1\Psi_1^1 \left[\begin{matrix} (1, 1); \\ (\varsigma, \varepsilon); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\varepsilon n + \varsigma)} \frac{x^n}{n!} = E_{\varepsilon,\varsigma}(x). \tag{20} $$

(8) For $k = 1$, $p = 1$, and $q = 2$, the Mittag-Leffler function from Mittag-Leffler [18] is

$$ {}_1\Psi_1^1(x) = {}_1\Psi_1^1 \left[\begin{matrix} (1, 1); \\ (1, \varepsilon); \end{matrix} \; x \right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\varepsilon n + 1)} \frac{x^n}{n!} = E_{\varepsilon}(x). \tag{21} $$

2. Saigo k-Fractional Integration in terms of k-Wright Function

In this section, we present the composition formulas of k-fractional integrals (1) and (2), involving the k-Wright function.

Theorem 1. *Let* $\tau, \delta, \gamma, \in \mathbb{C}; k \in \mathbb{R}^+$, $c, \varepsilon_i, \varsigma_j \in \mathbb{R}\,(\varepsilon_i, \ \varsigma_j \neq 0;\ i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, *and* $\nu > 0$ *such that* $\mathfrak{R}(\tau) > 0$, $\mathfrak{R}(\zeta) > \max[0, \mathfrak{R}(\delta - \gamma)]$, *and* $\mathfrak{R}(\zeta + \gamma - \delta) > 0$. *If condition (11) is satisfied and* $I_{0+,k}^{\tau,\delta,\gamma}$ *be the left-sided integral operator of the generalized k-fractional integration associated* *with k-Wright function, then the following equation holds true:*

$$\begin{aligned}&\left(I_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}{}_{p}\Psi_{q}^{k}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\(b_j,\varsigma_j)_{1,q};\end{matrix}ct^{\nu/k}\right]\right\}\right)(x)\\&=x^{(\zeta-\delta/k)-1}{}_{p+2}\Psi_{q+2}^{k}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),(\zeta,\nu),(\zeta+\gamma-\delta,\nu);\\(b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),(\zeta-\delta,\nu),(\zeta+\tau+\gamma,\nu);\end{matrix}kcx^{\nu/k}\right].\end{aligned}\tag{22}$$

Proof. We indicate the R.H.S. of equation (22) by I_1, and invoking equation (10), we obtain

$$\begin{aligned}I_1&=I_{0+,k}^{\tau,\delta,\gamma}\left(t^{(\zeta/k)-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\varepsilon_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\varsigma_j n)}\frac{(ct^{(\nu/k)})^n}{n!}\right)(x)\\&=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\varepsilon_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\varsigma_j n)}\frac{c^n}{n!}I_{0+,k}^{\tau,\delta,\gamma}\left(t^{((\zeta+\nu n)/k)}\right)(x).\end{aligned}\tag{23}$$

Now applying equation (6), we get

$$I_1=x^{((\zeta-\delta)/k)-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\varepsilon_i n)\Gamma_k(\zeta+\nu n)\Gamma_k(\zeta+\gamma-\delta+\nu n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\varsigma_j n)\Gamma_k(\zeta-\delta+\nu n)\Gamma_k(\zeta+\tau+\gamma+\nu n)}\frac{(kcx^{\nu/k})^n}{n!}.\tag{24}$$

Now, interpreting definition (10) on the aforementioned equation, we arrive at the desired result (22). □

Theorem 2. *Let $\tau,\delta,\gamma,\in\mathbb{C}$; $k\in\mathbb{R}^+$, $c,\varepsilon_i,\varsigma_j\in\mathbb{R}\,(\varepsilon_i,\varsigma_j\neq 0$; $i=1,2,\ldots,p$; $j=1,2,\ldots,q)$, and $\nu>0$ such that $\Re(\tau)>0$ and $\Re(\zeta+\tau)>\max[-\Re(\delta),-\Re(\gamma)]$, with $\Re(\delta)\neq\Re(\gamma)$. If condition (11) is satisfied and $I_{-,k}^{\tau,\delta,\gamma}$ be the right-sided integral operator of the generalized k-fractional integration associated with k-Wright function, then the following equation holds true:*

$$\begin{aligned}&\left(I_{-,k}^{\tau,\delta,\gamma}\left\{t^{(-\tau-\zeta)/k}{}_{p}\Psi_{q}^{k}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\(b_j,\varsigma_j)_{1,q};\end{matrix}ct^{(-\nu/k)}\right]\right\}\right)(x)\\&=x^{(-\tau-\zeta-\delta)/k}{}_{p+2}\Psi_{q+2}^{k}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p)(\tau+\zeta+\delta,\nu),(\tau+\zeta+\gamma,\nu);\\(b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),(\tau+\zeta,\nu),(2\tau+\zeta+\delta+\gamma,\nu);\end{matrix}kcx^{(-\nu/k)}\right].\end{aligned}\tag{25}$$

Proof. The finding is similar to that of Theorem 1. So, we omit the details. □

Corollary 1. *By assuming $k=1$ in (22) and (25), the result becomes*

$$\begin{aligned}&\left(I_{0+}^{\tau,\delta,\gamma}\left\{t^{\zeta-1}{}_{p}\Psi_{q}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\(b_j,\varsigma_j)_{1,q};\end{matrix}ct^{\nu}\right]\right\}\right)(x)\\&=x^{\zeta-\delta-1}{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),(\zeta,\nu),(\zeta+\gamma-\delta,\nu);\\(b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),(\zeta-\delta,\nu),(\zeta+\tau+\gamma,\nu);\end{matrix}cx^{\nu}\right],\end{aligned}\tag{26}$$

and

$$\left(I_{-}^{\tau,\delta,\gamma}\left\{t^{-\tau-\zeta}{}_{p}\Psi_{q}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\ (b_j,\varsigma_j)_{1,q};\end{matrix} ct^{-\nu}\right]\right\}\right)(x)$$
$$= x^{-\tau-\zeta-\delta}{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p)(\tau+\zeta+\delta,\nu),\ (\tau+\zeta+\gamma,\nu);\\ (b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),\ (\tau+\zeta,\nu),\ (2\tau+\zeta+\delta+\gamma,\nu);\end{matrix} cx^{-\nu}\right]. \quad (27)$$

3. Saigo k-Fractional Differentiation in terms of k-Wright Function

In this section, we present the composition formulas of k-fractional derivatives (4) and (5), involving the k-Wright function.

Theorem 3. *Let $\tau,\delta,\gamma\in\mathbb{C}$; $k\in\mathbb{R}^+$, $c,\varepsilon_i,\varsigma_j\in\mathbb{R}\,(\varepsilon_i,\varsigma_j\neq 0$; $i=1,2,\ldots,p$; $j=1,2,\ldots,q)$, and $\nu>0$ such that $\Re(\tau)>0$, $\Re(\zeta)>\max[0,\Re(-\tau-\delta-\gamma)]$, $\Re(\zeta+\gamma+\delta)>0$. If condition (11) is satisfied and $D_{0+,k}^{\tau,\delta,\gamma}$ be the left-sided differential operator of the generalized k-fractional differentiation associated with k-Wright function, then the following equation holds true:*

$$\left(D_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}{}_{p}\Psi_{q}^{k}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\ (b_j,\varsigma_j)_{1,q};\end{matrix} ct^{\nu/k}\right]\right\}\right)(x)$$
$$= x^{((\zeta+\delta)/k)-1}{}_{p+2}\Psi_{q+2}^{k}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),\ (\zeta,\nu),\ (\zeta+\delta+\gamma+\tau,\nu);\\ (b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),\ (\zeta+\gamma,\nu),\ (\zeta+\delta,1-k+\nu);\end{matrix} cx^{((\nu+1)/k)-1}\right]. \quad (28)$$

Proof. For simplicity, let I_2 denote the left side of (28). Using definition (10), we obtain

$$I_2=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\varepsilon_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\varsigma_j n)}\frac{c^n}{n!}D_{0+,k}^{\tau,\delta,\gamma}\left(t^{((\zeta+\nu n)/k)-1}\right)(x). \quad (29)$$

Now, applying equation (8), we obtain

$$I_2=x^{((\zeta+\delta)/k)-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma_k(a_i+\varepsilon_i n)\Gamma_k(\zeta+\nu n)\Gamma_k(\zeta+\delta+\gamma+\tau+\nu n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\varsigma_j n)\Gamma_k(\zeta+\gamma+\nu n)\Gamma_k(\zeta+\delta+n-nk+\nu n)n!}\left(cx^{((\nu+1)/k)-1}\right)^n. \quad (30)$$

In accordance with (10), the required result is (28). This completes the proof of Theorem 3. □

Theorem 4. *Let $\tau,\delta,\gamma,\in\mathbb{C}$; $k\in\mathbb{R}^+$, $c,\varepsilon_i,\varsigma_j\in\mathbb{R}\,(\varepsilon_i,\varsigma_j\neq 0$; $i=1,2,\ldots,p$; $j=1,2,\ldots,q)$, and $\nu>0$ such that $\Re(\tau)>0$, $\Re(\zeta)>\max[\Re(\tau+\delta)+n-\Re(\gamma)]$, $\Re(\tau+\delta-\gamma)+n\neq 0$, where $n=[\Re(\tau)+1]$. If condition (11) is satisfied and $D_{-,k}^{\tau,\delta,\gamma}$ be the right-sided differential operator of the generalized k-fractional differentiation associated with k-Wright function, then thefollowing equation holds true:*

$$\left(D_{-,k}^{\tau,\delta,\gamma}\left\{t^{(\tau-\zeta)/k}{}_{p}\Psi_{q}^{k}\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\ (b_j,\varsigma_j)_{1,q};\end{matrix} ct^{(-\nu/k)}\right]\right\}\right)(x)$$
$$= x^{(\tau-\zeta+\delta)/k}{}_{p+2}\Psi_{q+2}^{k}\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),\ (\zeta-\tau-\delta,\nu+k-1),\ (\zeta+\gamma,\nu);\\ (b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),\ (\zeta-\tau,\nu),\ (\zeta-\tau-\delta+\gamma,\nu);\end{matrix} cx^{((-\nu+1)/k)-1}\right]. \quad (31)$$

Proof. The proof is parallel to that of Theorem 3. Therefore, we omit the details. □

Corollary 2. *By letting $k = 1$ in (28) and (31), the equation becomes*

$$\left(D_{0+}^{\tau,\delta,\gamma}\left\{t^{\zeta-1}\,{}_p\Psi_q^k\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\(b_j,\varsigma_j)_{1,q};\end{matrix}\; ct^{\nu}\right]\right\}\right)(x) = x^{\zeta+\delta-1}\,{}_{p+2}\Psi_{q+2}^k\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),(\zeta,\nu),(\zeta+\delta+\gamma+\tau,\nu);\\(b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),(\zeta+\gamma,\nu),(\zeta+\delta,\nu);\end{matrix}\; cx^{\nu}\right], \tag{32}$$

$$\left(D_{-}^{\tau,\delta,\gamma}\left\{t^{\tau-\zeta}\,{}_p\Psi_q^k\left[\begin{matrix}(a_i,\varepsilon_i)_{1,p};\\(b_j,\varsigma_j)_{1,q};\end{matrix}\; ct^{-\nu}\right]\right\}\right)(x) = x^{\tau-\zeta+\delta}\,{}_{p+2}\Psi_{q+2}^k\left[\begin{matrix}(a_1,\varepsilon_1),\ldots,(a_p,\varepsilon_p),(\zeta-\tau-\delta,\nu),(\zeta+\gamma,\nu);\\(b_1,\varsigma_1),\ldots,(b_q,\varsigma_q),(\zeta-\tau,\nu),(\zeta-\tau-\delta+\gamma,\nu);\end{matrix}\; cx^{-\nu}\right]. \tag{33}$$

4. Special Cases and Concluding Remarks

Being very general, the results given in (22), (25), (28), and (31) can yield a wide number of special cases by assigning some appropriate values to the parameters involved. Now, as shown in the following, we are explaining a few corollaries.

Corollary 3. *If we put $p = 1$ and $q = 2$ in Theorems 1 and 2, then we get the following interesting results on the right known as k-Mittag-Leffler function:*

$$\left(I_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}E_{k,\varepsilon,\varsigma}^{\vartheta,\omega}\left[ct^{\nu/k}\right]\right\}\right)(x) = x^{((\zeta-\delta)/k)-1}\,{}_3\Psi_4^k\left[\begin{matrix}(\vartheta,\omega k),(\zeta,\nu),(\zeta+\gamma-\delta,\nu);\\(\varsigma,\varepsilon),(\vartheta,0),(\zeta-\delta,\nu),(\zeta+\tau+\gamma,\nu);\end{matrix}\; kcx^{\nu/k}\right], \tag{34}$$

and

$$\left(I_{-,k}^{\tau,\delta,\gamma}\left\{t^{(-\tau-\zeta)/k}E_{k,\varepsilon,\varsigma}^{\vartheta,\omega}\left[ct^{(-\nu/k)}\right]\right\}\right)(x) = x^{(-\tau-\zeta-\delta)/k}\,{}_3\Psi_4^k\left[\begin{matrix}(\vartheta,\omega k),(\tau+\zeta+\delta,\nu),(\tau+\zeta+\gamma,\nu);\\(\varsigma,\varepsilon),(\vartheta,0),(\tau+\zeta,\nu),(2\tau+\zeta+\delta+\gamma,\nu);\end{matrix}\; kcx^{(-\nu/k)}\right]. \tag{35}$$

Corollary 4. *If we put $p = 1$ and $q = 3$ in Theorems 1 and 2, then we get the following interesting results on the right known as k-Bessel function of the first kind:*

$$\left(I_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}J_{k,\varsigma}^{\vartheta,\varepsilon}\left[ct^{\nu/k}\right]\right\}\right)(x) = x^{((\zeta-\delta)/k)-1}\,{}_3\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\zeta,\nu),(\zeta+\gamma-\delta,\nu);\\(\varsigma+1,\varepsilon),(\vartheta,0),(\zeta-\delta,\nu),(\zeta+\tau+\gamma,\nu),(k,k);\end{matrix}\; \frac{-ck^2x^{\nu/k}}{2}\right], \tag{36}$$

and

$$\left(I_{-,k}^{\tau,\delta,\gamma}\left\{t^{(-\tau-\zeta)/k}J_{k,\varsigma}^{\vartheta,\varepsilon}\left[ct^{(-\nu/k)}\right]\right\}\right)(x) = x^{(-\tau-\zeta-\delta)/k}\,{}_3\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\tau+\zeta+\delta,\nu),(\tau+\zeta+\gamma,\nu);\\(\varsigma+1,\varepsilon),(\vartheta,0),(\tau+\zeta,\nu),(2\tau+\zeta+\delta+\gamma,\nu),(k,k);\end{matrix}\; \frac{-ck^2x^{(-\nu/k)}}{2}\right]. \tag{37}$$

Corollary 5. *If we put $p = 3$ and $q = 3$ in Theorems 1 and 2, then we get the following results on the right known as k-hypergeomrtric function:*

$$\begin{aligned}&\left(I_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}{}_2F_{1,k}\left((\vartheta,k),(\varphi,k);(\varsigma,k);ct^{\nu/k}\right)\right\}\right)(x)\\&\quad=x^{((\zeta-\delta)/k)-1}{}_5\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\varphi,k),(\varsigma,0),(\zeta,\nu),(\zeta+\gamma-\delta,\nu);\\(\varsigma,k),(\varphi,0),(\vartheta,0),(\zeta-\delta,\nu),(\zeta+\tau+\gamma,\nu);\end{matrix}\;kcx^{\nu/k}\right],\end{aligned}\tag{38}$$

$$\begin{aligned}&\left(I_{-,k}^{\tau,\delta,\gamma}\left\{t^{(-\tau-\zeta)/k}{}_2F_{1,k}\left((\vartheta,k),(\varphi,k);(\varsigma,k);ct^{(-\nu/k)}\right)\right\}\right)(x)\\&\quad=x^{(-\tau-\zeta-\delta)/k}{}_5\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\varphi,k),(\varsigma,0),(\tau+\zeta+\delta,\nu),(\tau+\zeta+\gamma,\nu);\\(\varsigma,k),(\varphi,0),(\vartheta,0),(\tau+\zeta,\nu),(2\tau+\zeta+\delta+\gamma,\nu);\end{matrix}\;kcx^{(-\nu/k)}\right].\end{aligned}\tag{39}$$

Corollary 6. *If we put $p = 1$ and $q = 2$ in Theorems 3 and 4, then we get the following results on the right known as k-Mittag-Leffler function:*

$$\begin{aligned}&\left(D_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}E_{k,\varepsilon,\varsigma}^{\vartheta,\omega}\left[ct^{\nu/k}\right]\right\}\right)(x)\\&\quad=x^{((\zeta+\delta)/k)-1}{}_3\Psi_4^k\left[\begin{matrix}(\vartheta,\omega k),(\zeta,\nu),(\zeta+\delta+\gamma+\tau,\nu);\\(\varsigma,\varepsilon),(\vartheta,0),(\zeta+\gamma,\nu),(\zeta+\delta,1-k+\nu);\end{matrix}\;cx^{((\nu+1)/k)-1}\right],\end{aligned}\tag{40}$$

$$\begin{aligned}&\left(D_{-,k}^{\tau,\delta,\gamma}\left\{t^{(\tau-\zeta)/k}E_{k,\varepsilon,\varsigma}^{\vartheta,\omega}\left[ct^{(-\nu/k)}\right]\right\}\right)(x)\\&\quad=x^{(\tau-\zeta+\delta)/k}{}_3\Psi_4^k\left[\begin{matrix}(\vartheta,\omega k),(\zeta-\tau-\delta,\nu+k-1),(\zeta+\gamma,\nu);\\(\varsigma,\varepsilon),(\vartheta,0),(\zeta-\tau,\nu),(\zeta-\tau-\delta+\gamma,\nu);\end{matrix}\;cx^{((-\nu+1)/k)-1}\right].\end{aligned}\tag{41}$$

Corollary 7. *If we put $p = 1$ and $q = 3$ in Theorems 3 and 4, then we get the following results on the right known as k-Bessel function of the first kind:*

$$\begin{aligned}&\left(D_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}J_{k,\varsigma}^{\vartheta,\varepsilon}\left[ct^{\nu/k}\right]\right\}\right)(x)\\&\quad=x^{((\zeta+\delta)/k)-1}{}_3\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\zeta,\nu),(\zeta+\delta+\gamma+\tau,\nu);\\(\varsigma+1,\varepsilon),(\vartheta,0),(\zeta+\gamma,\nu),(\zeta+\delta,1-k+\nu),(k,k);\end{matrix}\;\frac{-ckx^{((\nu+1)/k)-1}}{2}\right],\end{aligned}\tag{42}$$

$$\begin{aligned}&\left(D_{-,k}^{\tau,\delta,\gamma}\left\{t^{(\tau-\zeta)/k}J_{k,\varsigma}^{\vartheta,\varepsilon}\left[ct^{(-\nu/k)}\right]\right\}\right)(x)\\&\quad=x^{(\tau-\zeta+\delta)/k}{}_3\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\zeta-\tau-\delta,\nu+k-1),(\zeta+\gamma,\nu);\\(\varsigma+1,\varepsilon),(\vartheta,0),(\zeta-\tau,\nu),(\zeta-\tau-\delta+\gamma,\nu),(k,k);\end{matrix}\;\frac{-ckx^{((-\nu+1)/k)-1}}{2}\right].\end{aligned}\tag{43}$$

Corollary 8. *If we put $p = 3$ and $q = 3$ in Theorems 3 and 4, then we get the following results on the right known as k-hypergeomrtric function:*

$$\begin{aligned}&\left(D_{0+,k}^{\tau,\delta,\gamma}\left\{t^{(\zeta/k)-1}{}_2F_{1,k}\left((\vartheta,k),(\varphi,k);(\varsigma,k);ct^{\nu/k}\right)\right\}\right)(x)\\&= x^{((\zeta+\delta)/k)-1}{}_5\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\varphi,k),(\varsigma,0),(\zeta,\nu),(\zeta+\delta+\gamma+\tau,\nu);\\(\varsigma,k),(\vartheta,0),(\varphi,0),(\zeta+\gamma,\nu),(\zeta+\delta,1-k+\nu);\end{matrix}\, cx^{((\nu+1)/k)-1}\right].\end{aligned} \tag{44}$$

$$\begin{aligned}&\left(D_{-,k}^{\tau,\delta,\gamma}\left\{t^{((\tau-\zeta)/k)}{}_2F_{1,k}\left((\vartheta,k),(\varphi,k);(\varsigma,k);ct^{(-\nu/k)}\right)\right\}\right)(x)\\&= x^{(\tau-\zeta+\delta)/k}{}_5\Psi_5^k\left[\begin{matrix}(\vartheta,k),(\varphi,k),(\varsigma,0)(\zeta-\tau-\delta,\nu+k-1),(\zeta+\gamma,\nu);\\(\varsigma,k),(\vartheta,0),(\varphi,0),(\zeta-\tau,\nu),(\zeta-\tau-\delta+\gamma,\nu);\end{matrix}\, cx^{((-\nu+1)/k)-1}\right].\end{aligned} \tag{45}$$

The advantage of the generalized k-fractional calculus operators, which are also called by many authors as the general operator, is that they generalize Saigo's fractional calculus operators and classical Riemann–Liouville (R-L) operators. For $k \longrightarrow 1$, operators (1), (2), (4), and (5) reduce to Saigo's [5] fractional integral and differentiation operators. If we take $\delta = -\tau$, (1), (2), (4), and (5) reduce the operators to k-Riemann–Liouville as follows:

$$\begin{aligned}\left(I_{0+,k}^{\tau,-\tau,\gamma}f\right)(x) &= \left(I_{0+,k}^{\tau}f\right)(x),\\ \left(I_{-,k}^{\tau,-\tau,\gamma}f\right)(x) &= \left(I_{-,k}^{\tau}f\right)(x),\\ \left(D_{0+,k}^{\tau,-\tau,\gamma}f\right)(x) &= \left(D_{0+,k}^{\tau}f\right)(x),\\ \left(D_{-,k}^{\tau,-\tau,\gamma}f\right)(x) &= \left(D_{-,k}^{\tau}f\right)(x).\end{aligned} \tag{46}$$

Due to the most general character of the k-Wright function, numerous other interesting special cases from (22), (25), (28), and (31) can be given in the form of k-Struve function, k-Wright-type function, and many more, but due to lack of space, they are not represented here.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

On Coupled Systems for Hilfer Fractional Differential Equations with Nonlocal Integral Boundary Conditions

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In this paper, we study a coupled system involving Hilfer fractional derivatives with nonlocal integral boundary conditions. Existence and uniqueness results are obtained by applying Leray-Schauder alternative, Krasnoselskii's fixed point theorem, and Banach's contraction mapping principle. Examples illustrating our results are also presented.

1. Introduction

The theory of fractional differential equations has been widely used in pure mathematics and applications in the fields of physics, biology, and engineering. There are many interesting results for qualitative analysis and applications. We refer the interested reader, in fractional calculus, to the classical reference texts such as [1–7]. In the literature, there exist several different definitions of fractional integrals and derivatives, and the most popular of them are Riemann–Liouville, Caputo, and other less-known such as Hadamard fractional derivative and the Erdeyl–Kober fractional derivative. A generalization of derivatives of both Riemann–Liouville and Caputo was given by Hilfer in [8] as

$$ {}^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}u(t), \tag{1}$$

where $n-1<\alpha<n$, $0\le\beta\le 1$, $t>a\ge 0$, and $D^n = (\mathrm{d}^n/\mathrm{d}t^n)$.

He named it as generalized fractional derivative of order α and a type β. Many authors call it the Hilfer fractional derivative. We notice that when $\beta=0$, the Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative:

$$ {}^{H}D^{\alpha,0}u(t) = D^{n}I^{n-\alpha}u(t). \tag{2}$$

When $\beta=1$, the Hilfer fractional derivative corresponds to the Caputo fractional derivative:

$$ {}^{H}D^{\alpha,1}u(t) = I^{n-\alpha}D^{n}u(t). \tag{3}$$

Such derivative interpolates between the Riemann–Liouville and Caputo derivative. Some properties and applications of the Hilfer derivative are given in [9, 10] and the references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see, for example [11–16] and references therein. Nonlocal boundary value problems for Hilfer fractional derivative were studied in [17].

To the best of our knowledge, there is no work carried out on systems of boundary value problems with Hilfer fractional derivative in the literature. This paper come to fill this gap. Thus, the objective of the present work is to introduce a new class of coupled systems of Hilfer-type fractional differential equations with nonlocal integral boundary conditions and develop the existence and uniqueness of solutions. In precise terms, we consider the following coupled system:

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t, x(t), y(t)), & t \in [a,b], \\ {}^{H}D^{\alpha_1,\beta_1}y(t) = g(t, x(t), y(t)), & t \in [a,b], \\ x(a) = 0, x(b) = \sum_{i=1}^{m} \theta_i I^{\varphi_i} y(\xi_i), \\ y(a) = 0, y(b) = \sum_{j=1}^{n} \zeta_j I^{\psi_j} x(z_j), \end{cases} \tag{4}$$

where ${}^{H}D^{\alpha,\beta}$ and ${}^{H}D^{\alpha_1,\beta_1}$ are the Hilfer fractional derivatives of orders α and α_1, $1 < \alpha, \alpha_1 < 2$, and parameters β and β_1, respectively, $0 \le \beta, \beta_1 \le 1$, and I^{φ_i}, I^{ψ_j} are the Riemann–Liouville fractional integrals of order $\varphi_i > 0$ and $\psi_j > 0$, respectively, the points ξ_i, $z_j \in [a,b]$, $a \ge 0$, f, g: $[a,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and θ_i, $\zeta_j \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are given real constants.

The paper is organized as follows. We present our main results in Section 3, by applying Leray–Schauder alternative, Krasnoselskii's fixed point theorem, and Banach's contraction mapping principle, while Section 2 contains some preliminary concepts related to our problem. Examples are constructed to illustrate the main results.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [2, 5].

Definition 1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function u: $[a, \infty) \longrightarrow \mathbb{R}$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) \mathrm{d}s, \tag{5}$$

provided the right-hand side exists on (a, ∞).

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function is defined by

$${}^{\mathrm{RL}}D^{\alpha}u(t) \coloneqq D^n I^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) \mathrm{d}s, \tag{6}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is pointwise defined on (a, ∞).

Definition 3. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function is defined by

$${}^{\mathrm{C}}D^{\alpha}u(t) \coloneqq I^{n-\alpha} D^n u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n \cdot u(s) \mathrm{d}s, \quad n-1 < \alpha < n, \tag{7}$$

provided the right-hand side is pointwise defined on (a, ∞).

Lemma 1 (see [18]). *If $g \in C^n[a,b]$, $n-1 < \alpha < n$, and $0 \le \beta \le 1$, then*

(1) $I^{\alpha H}D^{\alpha,\beta}g(t) = g(t) - \sum_{k=1}^{n} ((t-a)^{\gamma-k}/\Gamma(\gamma-k+1)) (d/dt)^{n-k} I^{(1-\beta)(n-\alpha)} g(a)$

(2) ${}^{H}D^{\alpha,\beta} I^{\alpha} g(t) = g(t)$

The following lemma deals with a linear variant of problem (4).

Lemma 2. *Let φ_i, $\psi_j > 0$, ξ_i, $z_j \in [a,b]$, $a \ge 0$, θ_i, $\zeta_j \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $1 < \alpha, \alpha_1 < 2$, $0 \le \beta, \beta_1 \le 1$, $\gamma = \alpha + 2\beta - \alpha\beta$, $\gamma_1 = \alpha_1 + 2\beta_1 - \alpha_1\beta_1$, $h, h_1 \in C([a,b], \mathbb{R})$, and*

$$\Lambda = \frac{(b-a)^{\gamma+\gamma_1-2}}{\Gamma(\gamma)\Gamma(\gamma_1)} - \left(\sum_{i=1}^{m} \theta_i \frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1 + \varphi_i)} \right) \cdot \left(\sum_{j=1}^{n} \zeta_j \frac{(z_j - a)^{\gamma+\psi_j-1}}{\Gamma(\gamma + \psi_j)} \right) \neq 0. \tag{8}$$

Then, the system

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = h(t), & t \in [a,b], \\ {}^{H}D^{\alpha_1,\beta_1}y(t) = h_1(t), & t \in [a,b], \\ x(a) = 0, \\ x(b) = \sum_{i=1}^{m} \theta_i I^{\varphi_i} y(\xi_i), \\ y(a) = 0, \\ y(b) = \sum_{j=1}^{n} \zeta_j I^{\psi_j} x(z_j), \end{cases} \tag{9}$$

is equivalent to the following integral equations:

$$x(t) = I^{\alpha}h(t) + \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}h_1(\xi_i) - I^{\alpha}h(b)\right) + \left(\sum_{i=1}^{m}\theta_i\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}h(z_j) - I^{\alpha_1}h_1(b)\right)\Bigg] \tag{10}$$

$$y(t) = I^{\alpha_1}h_1(t) + \frac{(t-a)^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\Bigg[\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}h(z_j) - I^{\alpha_1}h_1(b)\right) + \left(\sum_{j=1}^{n}\zeta_j\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}h_1(\xi_i) - I^{\alpha}h(b)\right)\Bigg]. \tag{11}$$

Proof. Operating fractional integral I^{α} on both sides of the first equation in (9) and using Lemma 1, we obtain

$$x(t) - \sum_{k=1}^{2}\frac{(t-a)^{\gamma-k}}{\Gamma(\gamma-k+1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2-k} I^{(1-\beta)(2-\alpha)}x(a) = I^{\alpha}h(t). \tag{12}$$

Then, we have, since $(1-\beta)(2-\alpha) = 2-\gamma$,

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\left(\frac{d}{dt}\right)I^{2-\gamma}x(t)|_{t=a} + \frac{(t-a)^{\gamma-2}}{\Gamma(\gamma-1)}I^{2-\gamma}x(t)|_{t=a} + I^{\alpha}h(t) \\ &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}{}^{H}D^{\gamma-1,\beta}x(t)|_{t=a} + \frac{(t-a)^{\gamma-2}}{\Gamma(\gamma-1)}I^{2-\gamma}x(t)|_{t=a} + I^{\alpha}h(t). \end{aligned} \tag{13}$$

Then,

$$x(t) = \frac{c_1}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{c_2}{\Gamma(\gamma-1)}(t-a)^{\gamma-2} + I^{\alpha}h(t), \tag{14}$$

where

$$\begin{aligned} c_1 &= {}^{H}D^{\gamma-1,\beta}x(t)|_{t=a}, \\ c_2 &= I^{2-\gamma}x(t)|_{t=a}. \end{aligned} \tag{15}$$

By a similar way, we obtain

$$y(t) = \frac{d_1}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1} + \frac{d_2}{\Gamma(\gamma_1-1)}(t-a)^{\gamma_1-2} + I^{\alpha_1}h_1(t). \tag{16}$$

By setting

$$\begin{aligned} d_1 &= {}^{H}D^{\gamma_1-1,\beta}y(t)|_{t=a}, \\ d_2 &= I^{2-\gamma}y(t)|_{t=a}, \end{aligned} \tag{17}$$

from the boundary conditions $x(a) = 0$ and $y(a) = 0$, we obtain $c_2 = 0$ and $d_2 = 0$. Then, we obtain

$$\begin{aligned} x(t) &= \frac{c_1}{\Gamma(\gamma)}(t-a)^{\gamma-1} + I^{\alpha}h(t), \\ y(t) &= \frac{d_1}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1} + I^{\alpha_1}h_1(t), \end{aligned} \tag{18}$$

$$\begin{aligned} \sum_{i=1}^{m}\theta_i I^{\varphi_i}y(\xi_i) &= d_1\sum_{i=1}^{m}\frac{\theta_i(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)} + \sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}h_1(\xi_i), \\ \sum_{j=1}^{n}\zeta_i I^{\psi_i}x(z_j) &= c_1\sum_{j=1}^{n}\frac{\zeta_j(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)} + \sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}h(z_j). \end{aligned} \tag{19}$$

From $x(b) = \sum_{i=1}^{m}\theta_i I^{\varphi_i}y(\xi_i)$ and $y(b) = \sum_{j=1}^{n}\zeta_j I^{\psi_j} x(z_j)$, we have

$$\begin{aligned} c_1 &= \frac{1}{\Lambda}\Bigg[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}h_1(\xi_i) - I^{\alpha}h(b)\right) + \left(\sum_{i=1}^{m}\theta_i\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}h(z_j) - I^{\alpha_1}h_1(b)\right)\Bigg], \\ d_1 &= \frac{1}{\Lambda}\Bigg[\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}h(z_j) - I^{\alpha_1}h_1(b)\right) + \left(\sum_{j=1}^{n}\zeta_j\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}h_1(\xi_i) - I^{\alpha}h(b)\right)\Bigg]. \end{aligned} \tag{20}$$

Substituting the values of c_1 and d_1 in (18), we obtain solutions (10) and (11). The converse follows by direct computation. This completes the proof.

3. Main Results

Let $\mathscr{C} = C([a,b],\mathbb{R})$, $a \geq 0$, denote the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$. The space $X = \{x\colon x(t) \in C^2([a,b],\mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|,\ t \in [a,b]\}$ is a Banach space. Let $Y = \{y\colon |y(t)| \in C^2([a,b],\mathbb{R})\}$ with the norm $\|y\| = \sup\{|y(t)|,\ t \in [a,b]\}$. It is obvious that the product space $(X \times Y, \|(x, y)\|)$ is Banach space with the norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 2, we define two operators $\mathcal{K}\colon X \times Y \longrightarrow X \times Y$ by

$$\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix}, \tag{21}$$

where

$$\begin{aligned}
\mathcal{K}_1(x, y)(t) &= I^{\alpha} f_{x,y}(t) + \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i} g_{x,y}(\xi_i) - I^{\alpha} f_{x,y}(b)\right)\right.\\
&\quad\left.+\left(\sum_{i=1}^{m}\theta_i\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j} f_{x,y}(z_j) - I^{\alpha_1} g_{x,y}(b)\right)\right],\\
\mathcal{K}_2(x, y)(t) &= I^{\alpha_1} g_{x,y}(t) + \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j} f_{x,y}(z_j) - I^{\alpha_1} g_{x,y}(b)\right)\right.\\
&\quad\left.+\left(\sum_{j=1}^{n}\zeta_j\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i} g_{x,y}(\xi_i) - I^{\alpha} f_{x,y}(b)\right)\right],
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
f_{x,y}(t) &= f(t, x(t), y(t)),\\
g_{x,y}(t) &= g(t, x(t), y(t)), \quad t \in [a, b].
\end{aligned} \tag{23}$$

For computational convenience, we set

$$\begin{aligned}
M_1 = {}&\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1+\alpha-1}}{\Gamma(\gamma_1)\Gamma(\alpha+1)}\right.\\
&\left.+\left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}|\zeta_j|\frac{(z_j-a)^{\alpha+\psi_j}}{\Gamma(\alpha+\psi_j+1)}\right)\right],
\end{aligned} \tag{24}$$

$$\begin{aligned}
M_2 = {}&\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\alpha_1+\varphi_i}}{\Gamma(\alpha_1+\varphi_i+1)}\right)\right.\\
&\left.+\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}\left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\right],
\end{aligned} \tag{25}$$

$$\begin{aligned}
M_3 = {}&\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma_1)}\left[\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\left(\sum_{j=1}^{n}|\zeta_j|\frac{(z_j-a)^{\alpha+\psi_j}}{\Gamma(\alpha+\psi_j+1)}\right)\right.\\
&\left.+\frac{(b-a)^{\alpha}}{\Gamma(\alpha_1+1)}\left(\sum_{i=1}^{m}|\zeta_j|\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\right],
\end{aligned} \tag{26}$$

$$\begin{aligned}
M_4 = {}&\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{(b-a)^{\gamma_1-1}}{|\Lambda|\Gamma(\gamma_1)}\left[\frac{(b-a)^{\gamma+\alpha_1-1}}{\Gamma(\gamma)\Gamma(\alpha_1+1)}\right.\\
&\left.+\left(\sum_{j=1}^{n}|\zeta_j|\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\alpha_1+\varphi_i}}{\Gamma(\alpha_1+\varphi_i+1)}\right)\right].
\end{aligned} \tag{27}$$

Banach's contraction mapping principle is applied in the first result to prove existence and uniqueness of solutions of system (4).

Theorem 1. *Suppose that $f, g\colon [a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions. In addition, we assume that*

(H_1) There exist constants $\ell_i, n_i, i = 1, 2$, such that, for all $t \in [a, b]$ and $x_i, y_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le l_1|x_1 - x_2| + l_2|y_1 - y_2| \tag{28}$$

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le n_1|x_1 - x_2| + n_2|y_1 - y_2|. \tag{29}$$

Then, system (4) has a unique solution on $[a, b]$, if

$$(M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4)(n_1 + n_2) < 1. \tag{30}$$

Proof. *Define $sup_{t\in[a,b]} f(t, 0, 0) = N_1 < \infty$ and $sup_{t\in[a,b]} g(t, 0, 0) = N_2 < \infty$ such that*

$$r > \frac{(M_1 + M_3)N_1 + (M_2 + M_4)N_2}{1 - [(M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4)(n_1 + n_2)]}. \tag{31}$$

Now, we will show that the set $\mathcal{K}B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y\colon \|(x, y)\| \le r\}$. For any $(x, y) \in B_r$, $t \in [a, b]$, we find that

$$\begin{aligned}
|f(t, x(t), y(t))| &= |f(t, x(t), y(t)) - f(t, 0, 0) + f(t, 0, 0)|\\
&\le |f(t, x(t), y(t)) - f(t, 0, 0)| + |f(t, 0, 0)|\\
&\le \ell_1\|x\| + \ell_2\|y\| + N_1,\\
|g(t, x(t), y(t))| &\le n_1\|x\| + n_2\|y\| + N_2.
\end{aligned} \tag{32}$$

For $(x, y) \in B_r$, we have

$$
\begin{aligned}
&\left|\mathscr{K}_1(x, y)(t)\right| \\
&\leq I^{\alpha}\left|f_{x,y}\right|(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left|g_{x,y}\right|(\xi_i) + I^{\alpha}\left|f_{x,y}\right|(b)\right)\right. \\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i| \frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\left|\zeta_j\right| I^{\alpha+\psi_j}\left|f_{x,y}\right|(z_j) + I^{\alpha_1}\left|g_{x,y}\right|(b)\right)\right] \\
&\leq I^{\alpha}\left(\left|f_{x,y} - f_{0,0}\right| + \left|f_{0,0}\right|\right)(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right. \\
&\quad \times\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left(\left|g_{x,y} - g_{0,0}\right| + \left|g_{0,0}\right|\right)(\xi_i) + I^{\alpha}\left(\left|f_{x,y} - f_{0,0}\right|\right) + \left|f_{0,0}\right|(b)\right) \\
&\quad + \left(\sum_{i=1}^{m}|\theta_i| \frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\left|\zeta_j\right| I^{\alpha+\psi_j}\left(\left|f_{x,y} - f_{0,0}\right| + \left|f_{0,0}\right|\right)(z_j)\right. \\
&\quad \left.\left. + I^{\alpha_1}\left(\left|g_{x,y} - g_{0,0}\right|\right) + \left|g_{0,0}\right|(b)\right)\right] \qquad (33) \\
&\leq I^{\alpha}\left(\ell_1\|x\| + \ell_2\|y\| + N_1\right)(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\right. \\
&\quad \times\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left(n_1\|x\| + n_2\|y\| + N_2\right)(\xi_i) + I^{\alpha}\left(\ell_1\|x\| + \ell_2\|y\| + N_1\right)(b)\right) \\
&\quad + \left(\sum_{i=1}^{m}|\theta_i| \frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\left|\zeta_j\right| I^{\alpha+\psi_j}\left(\ell_1\|x\| + \ell_2\|y\| + N_1\right)(z_j)\right. \\
&\quad \left.\left. + I^{\alpha_1}\left(n_1\|x\| + n_2\|y\| + N_2\right)(b)\right)\right] \\
&\leq M_1\left(\ell_1\|x\| + \ell_2\|y\| + N_1\right) + M_2\left(n_1\|x\| + n_2\|y\| + N_2\right) \\
&= \left(M_1\ell_1 + M_2 n_1\right)\|x\| + \left(M_1\ell_2 + M_2 n_2\right)\|y\| + M_1 N_1 + M_2 N_2 \\
&\leq \left[M_1\left(\ell_1 + \ell_2\right) + M_2\left(n_1 + n_2\right)\right]r + M_1 N_1 + M_2 N_2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left\|\mathscr{K}_1(x, y)\right\| \leq{}& \left[M_1\left(\ell_1 + \ell_2\right) + M_2\left(n_1 + n_2\right)\right]r + M_1 N_1 \\
&+ M_2 N_2.
\end{aligned} \tag{34}
$$

Similarly, we have

$$
\begin{aligned}
\left|\mathscr{K}_2(x, y)(t)\right| \leq{}& M_3\left(\ell_1\|x\| + \ell_2\|y\| + N_1\right) + M_4 \\
&\cdot\left(n_1\|x\| + n_2\|y\| + N_2\right),
\end{aligned} \tag{35}
$$

and hence

$$
\left\|\mathscr{K}_2(x, y)\right\| \leq \left[M_3\left(\ell_1 + \ell_2\right) + M_4\left(n_1 + n_2\right)\right]r + M_3 N_1 + M_4 N_2. \tag{36}
$$

Consequently, it follows that

$$
\begin{aligned}
\|\mathscr{K}(x, y)\| \leq{}& \left[M_1\left(\ell_1 + \ell_2\right) + M_2\left(n_1 + n_2\right)\right]r + M_1 N_1 + M_2 N_2 \\
&+ \left[M_3\left(\ell_1 + \ell_2\right) + M_4\left(n_1 + n_2\right)\right]r + M_3 N_1 + M_4 N_2 \leq r,
\end{aligned} \tag{37}
$$

which implies $\mathscr{K}B_r \subset B_r$. Next, we will show that the operator $\mathscr{K}$ is a contraction mapping. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, we obtain

$$
\begin{aligned}
&\left|\mathcal{K}_1(x_1, y_1)(t) - \mathcal{K}_1(x_2, y_2)(t)\right| \\
&\quad \le I^{\alpha}\left|f_{x_1,y_1} - f_{x_2,y_2}\right|(b) \\
&\qquad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left|g_{x_1,y_1} - g_{x_2,y_2}\right|(\xi_i) + I^{\alpha}\left|f_{x_1,y_1} - f_{x_2,y_2}\right|(b)\right)\right. \\
&\qquad + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}\left|f_{x_1,y_1} - f_{x_2,y_2}\right|(z_j)\right. \\
&\qquad \left.\left. + I^{\alpha_1}\left|g_{x_1,y_1} - g_{x_2,y_2}\right|(b)\right)\right] \\
&\quad \le \left(\ell_1\|x_1 - x_2\| + \ell_2\|y_1 - y_2\|\right) I^{\alpha}(1)(b) \\
&\qquad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\left(n_1\|x_1 - x_2\| + n_2\|y_1 - y_2\|\right)\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}(1)(\xi_i)\right.\right. \\
&\qquad + \left(\ell_1\|x_1 - x_2\| + \ell_2\|y_1 - y_2\|\right) I^{\alpha}(1)(b) \\
&\qquad + \left(\ell_1\|x_1 - x\|_2 + \ell_2\|y_1 - y_2\|\right)\left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i - a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}(1)(z_j)\right) \\
&\qquad \left.\left. + \left(n_1\|x_1 - x\|_2 + n_2\|y_1 - y_2\|\right) I^{\alpha_1}(1)(b)\right)\right] \\
&\quad \le M_1\left(\ell_1\|x_1 - x\|_2 + \ell_2\|y_1 - y_2\|\right) + M_2\left(n_1\|x_1 - x\|_2 + n_2\|y_1 - y_2\|\right) \\
&\quad = (M_1\ell_1 + M_2 n_1)\|x_1 - x\|_2 + (M_1\ell_2 + M_2 n_2)\|y_1 - y_2\|.
\end{aligned} \tag{38}
$$

Therefore, we obtain the following inequality:

$$
\begin{aligned}
\left\|\mathcal{K}_1(x_1, y_1) - \mathcal{K}_1(x_2, y_2)\right\| &\le \left[M_1(\ell_1 + \ell_2) + M_2(n_1 + n_2)\right] \\
&\quad \cdot \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).
\end{aligned} \tag{39}
$$

In addition, we also obtain

$$
\begin{aligned}
\left\|\mathcal{K}_2(x_1, y_1) - \mathcal{K}_2(x_2, y_2)\right\| &\le \left[M_3(\ell_1 + \ell_2) + M_4(n_1 + n_2)\right] \\
&\quad \cdot \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).
\end{aligned} \tag{40}
$$

From (39) and (40), it yields

$$
\begin{aligned}
\left\|\mathcal{K}(x_1, y_1) - \mathcal{K}(x_2, y_2)\right\| &\le \left[(M_1 + M_3)(\ell_1 + \ell_2)\right. \\
&\quad \left. + (M_2 + M_4)(n_1 + n_2)\right] \\
&\quad \times \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).
\end{aligned} \tag{41}
$$

As $(M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4)(n_1 + n_2) < 1$, therefore, $\mathcal{K}$ is a contraction operator. By Banach's fixed point theorem, the operator $\mathcal{K}$ has a unique fixed point, which is the unique solution of (4) on $[a, b]$. The proof is completed.

Now, we prove our second existence result via Leray–Schauder alternative.

Lemma 3 (Leray–Schauder alternative, see [19]). *Let $F\colon E \longrightarrow E$ be a completely continuous operator. Let*

$$
\xi(F) = \{x \in E\colon x = \lambda F(x) \text{ for some}, \quad 0 < \lambda < 1\}. \tag{42}
$$

Then, either the set $\xi(F)$ is unbounded, or F has at least one fixed point.

Theorem 2. *Assume that there exist real constants $u_i, v_i \ge 0$ for $i = 1, 2$ and $u_0, v_0 > 0$ such that, for any $x_i \in \mathbb{R}$ $(i = 1, 2)$, we have*

$$
\begin{aligned}
\left|f(t, x_1, x_2)\right| &\le u_0 + u_1|x_1| + u_2|x_2|, \\
\left|g(t, x_1, x_2)\right| &\le v_0 + v_1|x_1| + v_2|x_2|.
\end{aligned} \tag{43}
$$

If $(M_1 + M_3)u_1 + (M_2 + M_4)v_1 < 1$ and $(M_1 + M_3)u_2 + (M_2 + M_4)v_2 < 1$, where $M_1, M_2, M_3,$ and M_4 are given in (24)–(27), then (4) has at least one solution on $[a, b]$.

Proof. By continuity of the functions f and g on $[a, b] \times \mathbb{R} \times \mathbb{R}$, the operator $\mathcal{K}$ is continuous. We will show that the operator $\mathcal{K}\colon X \times Y \longrightarrow X \times Y$ is completely continuous. Let $\Phi \subset X \times Y$ be bounded. Then, there exist positive constants L_1 and L_2 such that

$$
|f(t, x, y)| \le L_1, \ |g(t, x, y)| \le L_2, \quad \forall (x, y) \in \Phi. \tag{44}
$$

Then, for any $(x, y) \in \Phi$, we have

$$\begin{aligned}
&\left|\mathscr{K}_1(x,y)(t)\right| \le I^{\alpha}\left|f_{x,y}\right|(b)\\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left|g_{x,y}\right|(\xi_i) + I^{\alpha}\left|f_{x,y}\right|(b)\right)\right.\\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}\left|f_{x,y}\right|(z_j) + I^{\alpha_1}\left|g_{x,y}\right|(b)\right)\right]\\
&\le L_1 I^{\alpha}(1)(b)\\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(L_2\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}(1)(\xi_i) + L_1 I^{\alpha}(1)(b)\right)\right.\\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(L_1\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}(1)(z_j) + L_2 I^{\alpha_1}(1)(b)\right)\right]\\
&\le L_1 M_1 + L_2 M_2,
\end{aligned} \tag{45}$$

which yields

$$\left\|\mathscr{K}_1(x,y)\right\| \le L_1 M_1 + L_2 M_2. \tag{46}$$

Similarly, we obtain that

$$\left\|\mathscr{K}_2(x,y)\right\| \le L_1 M_3 + L_2 M_4. \tag{47}$$

Hence, from the above inequalities, we obtain that the set $\mathscr{K}\Phi$ is uniformly bounded. Next, we are going to prove that the set $\mathscr{K}\Phi$ is equicontinuous. For any $(x,y)\in\Phi$ and $\tau_1,\tau_2\in[a,b]$ such that $\tau_1<\tau_2$, we have

$$\begin{aligned}
&\left|\mathscr{K}_1(x,y)(\tau_2) - \mathscr{K}_1(x,y)(\tau_1)\right|\\
&\le I^{\alpha}\left|f_{x,y}\right|(\tau_2) - I^{\alpha}\left|f_{x,y}\right|(\tau_1)\\
&\quad + \frac{\left[(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}\right]}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}\left|g_{x,y}\right|(\xi_i) + I^{\alpha}\left|f_{x,y}\right|(b)\right)\right.\\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}\left|f_{x,y}\right|(z_j) + I^{\alpha_1}\left|g_{x,y}\right|(b)\right)\right]\\
&\le L_1\left|\int_a^{\tau_1}\frac{(\tau_2-s)^{\alpha-1}-(\tau_1-a)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|\\
&\quad + \frac{\left[(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}\right]}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(L_2\sum_{i=1}^{m}|\theta_i| I^{\alpha_1+\varphi_i}(1)(\xi_i) + L_1 I^{\alpha}(1)(b)\right)\right.\\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(L_1\sum_{j=1}^{n}|\zeta_j| I^{\alpha+\psi_j}(1)(z_j) + L_2 I^{\alpha_1}(1)(b)\right)\right]\\
&\le \frac{L_1}{\Gamma(\alpha+1)}\left[2(\tau_2-\tau_1)^{\alpha} + (\tau_2-a)^{\alpha} - (\tau_1-a)^{\alpha}\right]\\
&\quad + \frac{\left[(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}\right]}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(L_2\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\alpha_1+\varphi_i}}{\Gamma(\alpha_1+\varphi_i+1)} + L_1\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\right.\\
&\quad \left. + \left(\sum_{i=1}^{m}|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(L_1\sum_{j=1}^{n}|\zeta_j|\frac{(z_j-a)^{\alpha+\psi_j}}{\alpha+\psi_j+1} + L_2\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha+1)}\right)\right].
\end{aligned} \tag{48}$$

Therefore, we obtain

$$\left|\mathscr{K}_1(x,y)(\tau_2)-\mathscr{K}_1(x,y)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{49}$$

Analogously, we can obtain the following inequality:

$$\left|\mathscr{K}_2(x,y)(\tau_2)-\mathscr{K}_2(x,y)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{50}$$

Hence, the set $\mathscr{K}\Phi$ is equicontinuous. By applying the Arzelá–Ascoli theorem, the set $\mathscr{K}\Phi$ is relative compact which implies that the operator $\mathscr{K}$ is completely continuous. Lastly, we shall show that the set $\xi=\{(x,y)\in X\times Y\colon (x,y)=\lambda\mathscr{K}(x,y),\quad 0\le\lambda\le 1\}$ is bounded. Let any $(x,y)\in\xi$, then $(x,y)=\lambda\mathscr{K}(x,y)$. For any $t\in[a,b]$, we have

$$\begin{aligned} x(t)&=\lambda\mathscr{K}_1(x,y)(t),\\ y(t)&=\lambda\mathscr{K}_2(x,y)(t). \end{aligned} \tag{51}$$

Then, we obtain

$$\begin{aligned} \|x\|&\le(u_0+u_1\|x\|+u_2\|y\|)M_1+(v_0+v_1\|x\|+v_2\|y\|)M_2,\\ \|y\|&\le(u_0+u_1\|x\|+u_2\|y\|)M_3+(v_0+v_1\|x\|+v_2\|y\|)M_4, \end{aligned} \tag{52}$$

which imply that

$$\begin{aligned} \|x\|+\|y\|\le{}&(M_1+M_3)u_0+(M_2+M_4)v_0\\ &+[(M_1+M_3)u_1+(M_2+M_4)v_1]\|x\|\\ &+[(M_1+M_3)u_2+(M_2+M_4)v_2]\|y\|. \end{aligned} \tag{53}$$

Thus, we obtain

$$\|(x,y)\|\le\frac{(M_1+M_3)u_0+(M_2+M_4)v_0}{M^*}, \tag{54}$$

where $M^*=\min\{1-(M_1+M_3)u_1-(M_2+M_4)v_1,\ 1-(M_1+M_3)u_2-(M_2+M_4)v_2\}$, which shows that the set ξ is bounded. Therefore, by applying Lemma 3, the operator $\mathscr{K}$ has at least one fixed point. Therefore, we deduce that problem (4) has at least one solution on $[a,b]$. The proof is complete.

The last existence theorem is based on Krasnoselskii's fixed point theorem.

Lemma 4 (Krasnoselskii's fixed point theorem, see [20]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A and B be operators such that (i) $Ax+By\in M$, where $x,y\in M$, (ii) A is compact and continuous, and (iii) B is a contraction mapping. Then, there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 3. *Assume that $f,g\colon[a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions satisfying assumption (H_1) in Theorem 1. In addition, we suppose and there exist two positive constants P,Q such that, for all $t\in[a,b]$ and $x_i,y_i\in\mathbb{R}, i=1,2$,*

$$\begin{aligned} |f(t,x_1,x_2)|&\le P,\\ |g(t,x_1,x_2)|&\le Q. \end{aligned} \tag{55}$$

If

$$\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(\ell_1+\ell_2)+\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha+1)}(n_1+n_2)<1, \tag{56}$$

then problem (4) has at least one solution on $[a,b]$.

Proof. To apply Lemma 4, we decompose the operator $\mathscr{K}$ into four operators $\mathscr{K}_{1,1},\mathscr{K}_{1,2},\mathscr{K}_{2,1}$, and $\mathscr{K}_{2,2}$ as

$$\begin{aligned} \mathscr{K}_{1,1}(x,y)(t)={}&\frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}g_{x,y}(\xi_i)-I^{\alpha}f_{x,y}(b)\right)\right.\\ &\left.+\left(\sum_{i=1}^{m}\theta_i\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}f_{x,y}(z_j)-I^{\alpha_1}g_{x,y}(b)\right)\right],\\ \mathscr{K}_{1,2}(x,y)(t)={}&I^{\alpha}f_{x,y}(t),\\ \mathscr{K}_{2,1}(x,y)(t)={}&\frac{(t-a)^{\gamma_1-1}}{\Lambda\Gamma(\gamma_1)}\left[\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\left(\sum_{j=1}^{n}\zeta_j I^{\alpha+\psi_j}f_{x,y}(z_j)-I^{\alpha_1}g_{x,y}(b)\right)\right.\\ &\left.+\left(\sum_{j=1}^{n}\zeta_j\frac{(z_j-a)^{\gamma+\psi_j-1}}{\Gamma(\gamma+\psi_j)}\right)\left(\sum_{i=1}^{m}\theta_i I^{\alpha_1+\varphi_i}g_{x,y}(\xi_i)-I^{\alpha}f_{x,y}(b)\right)\right],\\ \mathscr{K}_{2,2}(x,y)(t)={}&I^{\alpha_1}g_{x,y}(t). \end{aligned} \tag{57}$$

Note that $\mathcal{K}_1(x,y)(t) = \mathcal{K}_{1,1}(x,y)(t) + \mathcal{K}_{1,2}(x,y)(t)$ and $\mathcal{K}_2(x,y)(t) = \mathcal{K}_{2,1}(x,y)(t) + \mathcal{K}_{2,2}(x,y)(t)$. Also, observe that the ball B_δ is a closed, bounded, and convex subset of the Banach space $\mathcal{C}$. Let $B_\delta = \{(x,y) \in X \times Y \colon \|(x,y)\| \le \delta\}$ be a ball, where a constant $\delta \ge \max\{M_1P + M_2Q, M_3P + M_4Q\}$. Now, we will show that $\mathcal{K}B_\delta \subset B_\delta$ for satisfying condition (i) of Lemma 4. Setting $x = (x_1, x_2)$ and $y = (y_1, y_2) \in B_\delta$ and using condition (55), then we have, as in Theorem 2 that

$$\left|\mathcal{K}_{1,1}(x_1,x_2)(t) + \mathcal{K}_{1,2}(y_1,y_2)(t)\right| \le M_1P + M_2Q \le \delta. \tag{58}$$

Similarly, we can find that

$$\left|\mathcal{K}_{2,1}(x_1,x_2)(t) + \mathcal{K}_{2,2}(y_1,y_2)(t)\right| \le M_3P + M_4Q \le \delta. \tag{59}$$

That yields $\mathcal{K}_1x + \mathcal{K}_2y \in B_\delta$. To show that the operator $(\mathcal{K}_{1,2}, \mathcal{K}_{2,2})$ is a contraction mapping satisfying condition (iii) of Lemma 4, for $(x_1,y_1), (x_2,y_2) \in B_\delta$, we have

$$\begin{aligned}\left|\mathcal{K}_{1,2}(x_1,y_1)\right|(t) - \mathcal{K}_{1,2}(x_2,y_2)(t)\big| &\le I^\alpha\left|\left|f_{x_1,y_1} - f_{x_2,y_2}\right|(t)\right.\\ &\le \left(\ell_1\|x_1 - x_2\| + \ell_2\|y_1 - y_2\|\right)I^\alpha(1)(b)\\ &\le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}(\ell_1 + \ell_2)\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right)\end{aligned} \tag{60}$$

$$\begin{aligned}\left|\mathcal{K}_{2,2}(x_1,y_1)(t) - \mathcal{K}_{2,2}(x_2,y_2)(t)\right| &\le I^{\alpha_1}\left|g_{x_1,y_1} - g_{x_2,y_2}\right|(t)\\ &\le \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}(n_1 + n_2)\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).\end{aligned} \tag{61}$$

It follows from (60) and (61) that

$$\begin{aligned}&\left\|\left(\mathcal{K}_{1,2}, \mathcal{K}_{2,2}\right)(x_1,y_1) - \left(\mathcal{K}_{1,2}, \mathcal{K}_{2,2}\right)(x_2,y_2)\right\|\\ &\quad\le \left[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}(\ell_1 + \ell_2) + \frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}(n_1 + n_2)\right]\\ &\qquad\cdot\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right),\end{aligned} \tag{62}$$

which is a contraction by inequality in (56). Therefore, condition (iii) of Lemma 4 is satisfied. Next, we will show that the operator $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1})$ satisfies condition (ii) of Lemma 4. By applying the continuity of the functions f, g on $[a,b] \times \mathbb{R} \times \mathbb{R}$, we can conclude that the operator $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1})$ is continuous. For each $(x,y) \in B_\delta$, one has

$$\begin{aligned}&\left|\mathcal{K}_{1,1}(x,y)(t)\right|\\ &\le \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(Q\sum_{i=1}^m|\theta_i|I^{\alpha_1+\varphi_i}(1)(\xi_i) + PI^\alpha(1)(b)\right)\right.\\ &\quad\left.+\left(\sum_{i=1}^m|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(P\sum_{j=1}^n|\zeta_j|I^{\alpha+\psi_j}(1)(z_j) + QI^{\alpha_1}(1)(b)\right)\right]\\ &= \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(Q\sum_{i=1}^m|\theta_i|\frac{(\xi_i-a)^{\alpha_1+\varphi_i}}{\Gamma(\alpha_1+\varphi_i+1)} + P\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)\right.\\ &\quad\left.+\left(\sum_{i=1}^m|\theta_i|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(P\sum_{j=1}^n|\zeta_j|\frac{(z_j-a)^{\alpha+\psi_j}}{\Gamma(\alpha+\psi_j+1)} + Q\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\right]\\ &\coloneqq P^*,\end{aligned} \tag{63}$$

and similarly

$$\left|\mathscr{K}_{2,1}(x,y)(t)\right| \leq Q^*. \tag{64}$$

Then, we obtain the following fact:

$$\left\|\left(\mathscr{K}_{1,1},\mathscr{K}_{2,1}\right)(x,y)\right\| \leq P^* + Q^*, \tag{65}$$

which implies that the set $(\mathscr{K}_{1,1},\mathscr{K}_{2,1})B_\delta$ is uniformly bounded. In the next step, we will show that the set $(\mathscr{K}_{1,1},\mathscr{K}_{2,1})B_\delta$ is equicontinuous. For $\tau_1,\tau_2 \in [a,b]$ such that $\tau_1 < \tau_2$ and for any $(x,y) \in B_\delta$, we can prove that

$$\begin{aligned}
&\left|\mathscr{K}_{1,1}(x,y)(\tau_2) - \mathscr{K}_{1,1}(x,y)(\tau_1)\right| \\
&\leq \frac{\left[(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}\right]}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\tau_1)}\left(Q\sum_{i=1}^{m}\left|\theta_i\right|I^{\alpha_1+\varphi_i}(1)(\xi_i) + PI(1)(b)\right)\right. \\
&\quad\left. + \left(\sum_{i=1}^{m}\left|\theta_i\right|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(P\sum_{j=1}^{n}\left|\zeta_j\right|I^{\alpha+\psi_j}(1)(z_j) + QI^{\alpha_1}(1)(b)\right)\right] \\
&\leq \frac{\left[(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}\right]}{|\Lambda|\Gamma(\gamma)}\left[\frac{(b-a)^{\gamma_1-1}}{\Gamma(\gamma_1)}\left(Q\sum_{i=1}^{m}\left|\theta_i\right|\frac{(\xi_i-a)^{\alpha_1+\varphi_i}}{\Gamma(\alpha_1+\varphi_i+1)} + P\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)\right. \\
&\quad\left. + \left(\sum_{i=1}^{m}\left|\theta_i\right|\frac{(\xi_i-a)^{\gamma_1+\varphi_i-1}}{\Gamma(\gamma_1+\varphi_i)}\right)\left(P\sum_{j=1}^{n}\left|\zeta_j\right|\frac{(z_j-a)^{\alpha+\psi_j}}{\Gamma(\alpha+\psi_j+1)} + Q\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\right].
\end{aligned} \tag{66}$$

Therefore, we have $|\mathscr{K}_{1,1}(x,y)(\tau_2) - \mathscr{K}_{1,1}(x,y)(\tau_1)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Similarly, we can show that $|\mathscr{K}_{2,1}(x,y)(\tau_2) - \mathscr{K}_{2,1}(x,y)(\tau_1)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$.

Thus, $|(\mathscr{K}_{1,1},\mathscr{K}_{2,1})(x,y)(\tau_2) - (\mathscr{K}_{1,1},\mathscr{K}_{2,1})(x,y)(\tau_1)|$ tends to zero as $\tau_1 \longrightarrow \tau_2$. Therefore, the set $(\mathscr{K}_{1,1},\mathscr{K}_{2,1})B_\delta$ is equicontinuous. By applying the Arzelá–Ascoli theorem, the operator $(\mathscr{K}_{1,1},\mathscr{K}_{2,1})$ is compact on B_δ. By application of Lemma 4, we have that problem (4) has at least one solution on $[a,b]$. This completes the proof.

Example 1. Consider the coupled system of Hilfer fractional differential equations with nonlocal integral boundary conditions of the form

$$\begin{cases}
{}^{H}D^{(3/2),(1/2)}x(t) = f(t,x(t),y(t)), \quad t \in \left[\dfrac{1}{3},\dfrac{10}{3}\right], \\
{}^{H}D^{(5/4),(2/3)}y(t) = g(t,x(t),y(t)), \quad t \in \left[\dfrac{1}{3},\dfrac{10}{3}\right], \\
x\left(\dfrac{1}{3}\right) = 0, \quad x\left(\dfrac{10}{3}\right) = \dfrac{1}{2}I^{1/3}y\left(\dfrac{2}{3}\right) + \dfrac{2}{3}I^{1/2}y(1) + \dfrac{3}{4}I^{3/5}y\left(\dfrac{4}{3}\right) + \dfrac{4}{5}I^{2/3}y\left(\dfrac{5}{3}\right), \\
y\left(\dfrac{1}{3}\right) = 0, \quad y\left(\dfrac{10}{3}\right) = \dfrac{3}{2}I^{3/2}x(2) + \dfrac{4}{3}I^{5/3}x\left(\dfrac{7}{3}\right) + \dfrac{5}{4}I^{7/4}y\left(\dfrac{8}{3}\right) + \dfrac{6}{5}I^{9/5}x(3).
\end{cases} \tag{67}$$

Here, $\alpha = 3/2$, $\alpha_1 = 5/4$, $\beta = 1/2$, $\beta_1 = 2/3$, $a = 1/3$, $b = 10/3$, $m = n = 4$, $\theta_i = (i/(i+1))$, $\zeta_i = ((2+i)/(1+i))$, $\varphi_i = (i/(i+2))$, and $\psi_i = ((2i+1)/(i+1))$, $i = 1,2,3,4$, $\xi_j = (j/3)$, $j = 2,3,4,5$, and $z_r = (r/3)$, $r = 6,7,8,9$. Then, we can compute constants as $\gamma = 7/4$, $\gamma_1 = 41/21$, $\Lambda = 6.371398411$, $M_1 = 12.56809051$, $M_2 = 3.253588460$, $M_3 = 48.72536839$, and $M_4 = 22.05071608$.

(i) Let the nonlinear functions f and g be defined by

$$f(t,x,y)=\frac{9e^{-t}}{9t^2+3239}\left(\frac{x^2+2|x|}{1+|x|}\right)+\frac{3}{3t+539}\sin|y|+\frac{2}{3}, \tag{68}$$

$$g(t,x,y)=\frac{9}{10(3t+11)^2}\tan^{-1}|x|+\left(\frac{3\cos^2 t}{479+3t}\right)\frac{|y|}{1+|y|}+\frac{1}{2}. \tag{69}$$

It is easy to check that f and g satisfy (H_1) in Theorem 1 as $|f(t,x_1,y_1)-f(t,x_2,y_2)|\le(1/180)|x_1-x_2|+(1/180)|y_1-y_2|$ and $|g(t,x_1,y_1)-g(t,x_2,y_2)|\le(1/160)|x_1-x_2|+(1/160)|y_1-y_2|$. Setting $\ell_1=\ell_2=(1/180)$ and $n_1=n_2=(1/160)$, we have

$$(M_1+M_3)(\ell_1+\ell_2)+(M_2+M_4)(n_1+n_2)=0.9973422390<1. \tag{70}$$

Therefore, by applying Theorem 1, we deduce that problem (67) with (68)–(69) has a unique solution (x,y) on $[1/3,10/3]$.

(ii) Given the functions f and g by

$$f(t,x,y)=\frac{3e^{-|y|}}{5(3t+50)}\left(\frac{|x|^9}{1+x^8}\right)+\frac{9}{(3t+29)^2}\left(\frac{y^4}{1+|y|^3}\right)+\frac{3}{4}, \tag{71}$$

$$g(t,x,y)=\frac{3e^{-y^2}\sin|x|}{(19+x^2)(14+3t)}+\frac{3y\cos^2(xy)}{4(50+3t)}+\frac{4}{7}, \tag{72}$$

we can check that

$$\begin{aligned}|f(t,x,y)|&\le\frac{3}{4}+\frac{1}{85}|x|+\frac{1}{100}|y|,\\ |g(t,x,y)|&\le\frac{4}{7}+\frac{1}{95}|x|+\frac{1}{68}|y|.\end{aligned} \tag{73}$$

which satisfy conditions in Theorem 2 by $u_0=3/4$, $v_0=4/7$, $u_1=1/85$, $v_1=1/95$, $u_2=1/100$, and $v_2=1/68$. Hence, we find that $(M_1+M_3)u_1+(M_2+M_4)v_1=0.9874606169<1$ and $(M_1+M_3)u_2+(M_2+M_4)v_2=0.9850567146<1$. The conclusion of Theorem 2 implies that problem (67) with (71)-(72) has at least one solution (x,y) on $[1/3,10/3]$.

(iii) Define the functions f and g by

$$f(t,x,y)=\frac{3e^{-t^2}}{3t+29}\left(\frac{|x|}{1+|x|}\right)+\frac{9}{9t^2+134}\sin|y|+\frac{2}{3}, \tag{74}$$

$$g(t,x,y)=\frac{9}{5(3t+5)^2}\tan^{-1}|x|+\left(\frac{9}{(3t+14)^2}\right)\left(\frac{|y|}{1+|y|}\right)+\frac{1}{2}. \tag{75}$$

Observe that condition (H_1) in Theorem 1 is satisfied for nonlinear functions f and g with Lipschitz constants $\ell_1=1/10$, $\ell_2=1/15$, $n_1=1/20$, and $n_2=1/25$. Next, we can find that $(M_1+M_3)(\ell_1+\ell_2)+(M_2+M_4)(n_1+n_2)=12.49296389>1$. Then, Theorem 1 cannot be used to obtain the existence criteria for the investigated problem. However, we calculate that $|f(t,x,y)|\le 5/6$, $|g(t,x,y)|\le(108+5\pi)/200$, and

$$\begin{aligned}&\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}(\ell_1+\ell_2)+\frac{(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}(n_1+n_2)\\ &=0.9650966816<1.\end{aligned} \tag{76}$$

Hence, all assumptions of Theorem 3 hold. Therefore, by Theorem 3, problem (67) with (74)-(75) has at least one solution (x,y) on $[1/3,10/3]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Pade Method for Construction of Numerical Algorithms for Fractional Initial Value Problem

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In this paper, we propose an efficient method for constructing numerical algorithms for solving the fractional initial value problem by using the Pade approximation of fractional derivative operators. We regard the Grunwald–Letnikov fractional derivative as a kind of Taylor series and get the approximation equation of the Taylor series by Pade approximation. Based on the approximation equation, we construct the corresponding numerical algorithms for the fractional initial value problem. Finally, we use some examples to illustrate the applicability and efficiency of the proposed technique.

1. Introduction

In the past decades, fractional differential equations were successfully applied to many problems in engineering, physics, chemistry, biology, economics, control theory, biophysics, and so on [1–5]. It is significant to obtain the exact or numerical solutions of fractional nonlinear equations. Since most fractional differential equations do not have exact analytic solutions, numerical techniques and seminumerical methods are used extensively, such as the homotopy analysis and homotopy perturbation method [6–8], variational iteration method [9–11], orthogonal polynomial method [12, 13], fractional Adams method [14], and some other methods [15].

Among all the numerical methods, the direct numerical method [16–18] is the basic one. Due to the special property of the fractional derivative, most of these methods have their inbuilt deficiencies mostly due to the calculation of Adomian polynomials, the Lagrange multiplier, divergent results, and some other huge computational works.

In this paper, we propose an efficient method for constructing numerical algorithms for solving the fractional initial value problem by using the Pade approximation of fractional derivative operators. The advantage of the proposed technique is that efficient numerical methods can be constructed without calculation of long historical terms of the fractional derivative.

Consider the following homogeneous fractional initial problem:

$$\begin{cases} {}_0D^{(\alpha)}y = f(y,t), & t \in [0,T], \\ y^{(k)}(0) = 0, & k = 0,1,\ldots,m-1, \end{cases} \tag{1}$$

where ${}_0D^{(\alpha)}$ is usually the Caputo derivative and $m = [\alpha]$. Given the condition $y^{(k)}(0) = 0, k = 0,1,\ldots,m-1$, the Caputo derivative is equivalent to the Grunwald–Letnikov derivative and the Riemann–Liouville derivative, e.g.,

$${}_0^C D^{(\alpha)} = {}_0^{\mathrm{RL}} D^{(\alpha)} = {}_0^{\mathrm{GL}} D^{(\alpha)}. \tag{2}$$

For the following linear nonhomogeneous fractional initial problem,

$$\begin{cases} {}_0D^{(\alpha)}y = f(y,t), & t \in [0,T], \\ y^{(k)}(0) = y_k, & k = 0,1,\ldots,m-1, \end{cases} \tag{3}$$

the following transform can be employed to make it be a homogeneous one with regard to $z(t)$:

$$y(t) = \sum_{k=0}^{m-1} y^{(k)}(0)t^k + z(t). \tag{4}$$

Based on the G algorithm [19], we have the following numerical scheme for problem (1):

$$h^{-\alpha}\sum_{j=0}^{N} c_{nj}^{(\alpha)} y_j = f(t_n, y_n), \quad n = 0, 1, \ldots, \left[\frac{t}{h}\right], \tag{5}$$

where $c_{nj}^{(\alpha)}$ and N are calculated according to different algorithms. Algorithm 5 is a technique which is easy to manipulate. However, this method involves huge computational work when $T \gg 0$ because in general, these methods need to compute many terms to get the approximation to the fractional derivative. To some extent, the short memory principle [20] can be used to tackle the problem, but low accuracy will be the cost. So, it is significant to find an efficient approximant to the fractional derivative, which is of low computation cost on the one hand and highly accurate on the other hand. In this paper, we put forward a reliable method for the construction of numerical algorithms for solving the fractional differential initial value problem by using Pade approximation to fractional derivative operators. The rest of the paper is organized as follows. In Section 2, we briefly list some basics of Pade approximation. In Section 3, we get the approximant of the fractional derivative operators. In Section 4, we propose a method for constructing algorithms for solving the fractional initial value problem by using Pade approximation. In Section 5, we use some examples to illustrate the applicability and efficiency of the proposed technique. Section 6 is the conclusion.

2. Some Basics for Pade Approximation

In numerical mathematics, Pade approximation [21] is believed to be the best approximation of a function by rational functions of a given order. Under this technique, the approximant's power series agrees with the power series of the function it is approximating. The Pade approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge. For these reasons, Pade approximants are often used in many fields of computations.

Given a function $f(x)$ and two integers $m > 0$ and $n > 1$, the Pade approximant of order $[m, n]$ is the rational function

$$R(x) = \frac{\sum_{j=0}^{m} a_j x^j}{1 + \sum_{k=0}^{n} b_k x^k}, \tag{6}$$

which agrees with $f(x)$ to the highest possible order, which amounts to

$$\begin{aligned} &f(0) = R(0), f'(0) = R'(0), \\ &f''(0) = R''(0), \ldots, f^{m+n}(0) = R^{m+n}(0). \end{aligned} \tag{7}$$

Equivalently, if $R(x)$ is expanded in a Maclaurin series, its first $m+n$ terms would cancel the first $m+n$ terms of $f(x)$, and as such $f(x) - R(x) = O(x^{m+n+1})$. The Pade approximant is unique for given m and n, that is, the coefficients $a_0, a_1, \ldots, a_m, b_1, \ldots, b_n$ can be uniquely determined. It is for reasons of uniqueness that the zeroth-order term at the denominator of $R(x)$ was chosen to be 1; otherwise, the numerator and denominator of $R(x)$ would have been unique only up to multiplication by a constant. The Pade approximant defined above is also denoted as $[m, n]_{f(x)}$.

3. Pade Approximant for the Fractional Derivative Operator

There are many definitions for the fractional derivative. The following equation is called the reverse Grunwald–Letnikov derivative:

$$D^{\alpha} f(x) = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(x - mh), \tag{8}$$

where

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha + 1)}{\Gamma(m+1)\Gamma(\alpha - m + 1)}. \tag{9}$$

For any given $\alpha > 0$, if we denote

$$\begin{aligned} E^{h} f(x) &= f(x + h), \\ E^{-h} f(x) &= f(x - h), \end{aligned} \tag{10}$$

we have

$$D^{\alpha} f(x) \approx h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh). \tag{11}$$

Therefore, we can denote

$$D^{\alpha} f(x) = h^{-\alpha}\left(1 - E^{-h}\right)^{\alpha} f(x). \tag{12}$$

So, we have

$$h^{\alpha} D^{\alpha} f(x) = \left(1 - E^{-h}\right)^{\alpha} f(x). \tag{13}$$

Let

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}, \tag{14}$$

be the $[n, n]$ Pade approximant to $(1 - x)^{\alpha}$; we have

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n} = (1 - x)^{\alpha} + O\left(x^{2n+1}\right). \tag{15}$$

Therefore,

$$\begin{aligned} &a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ &\quad = \left(1 + b_1 x + b_2 x^2 + \cdots + b_n x^n\right)(1 - x)^{\alpha} + O\left(h^{2n+1}\right). \end{aligned} \tag{16}$$

Replacing x in (16) with the operator E^{-h}, we have

$$a_0 + a_1E^{-h} + a_2E^{-2h} + \cdots + a_nE^{-nh} = \left(1 + b_1E^{-h} + b_2E^{-2h} + \cdots + b_nE^{-nh}\right)\left(1 - E^{-h}\right)^{\alpha} + O\left(E^{-h}\right)^{2n+1}, \tag{17}$$

where 1 can be understood as the identical operator. So, we get the following approximation:

$$h^{-\alpha}\left(a_0 + a_1E^{-h} + a_2E^{-2h} + \cdots + a_nE^{-nh}\right) \approx \left(1 + b_1E^{-h} + b_2E^{-2h} + \cdots + b_nE^{-nh}\right)\left(1 - E^{-h}\right)^{\alpha} h^{-\alpha}. \tag{18}$$

Noticing (13), we get

$$\left(1 - E^{-h}\right)^{\alpha} = h^{\alpha}D^{\alpha}. \tag{19}$$

So, we get

$$h^{-\alpha}\left(a_0 + a_1E^{-h} + a_2E^{-2h} + \cdots + a_nE^{-nh}\right) \approx \left(1 + b_1E^{-h} + b_2E^{-2h} + \cdots + b_nE^{-nh}\right)D^{\alpha}. \tag{20}$$

Then, we get the following approximation to fractional derivative operators:

$$\left(a_0 + a_1E^{-h} + a_2E^{-2h} + \cdots + a_nE^{-nh}\right) = h^{\alpha}\left(1 + b_1E^{-h} + b_2E^{-2} + \cdots + b_nE^{-nh}\right)D^{\alpha}. \tag{21}$$

Consider fractional initial value problem (1):

$$\begin{cases} D^{\alpha}y = f(t, y), & t \in [0, T], \\ y^{(k)}(0) = 0, & k = 0, 1, \ldots, m-1. \end{cases} \tag{22}$$

Denote

$$\begin{aligned} y(0) &= 0, \\ y(h) &= y_1, \ldots, y(nh) = y_n, \\ f(x_n, y_n) &= f_n, \end{aligned} \tag{23}$$

where h is the discretization step size.

From (21), we have

$$\left(a_0 + a_1E^{-h} + a_2E^{-2h} + \cdots + a_nE^{-nh}\right)y_n = h^{\alpha}\left(1 + b_1E^{-h} + b_2E^{-2h} + \cdots + b_nE^{-nh}\right)D^{\alpha}y_n. \tag{24}$$

Due to the linearity and commutativity of the operators E^{-h} and D^{α}, we have

$$\left(a_0y_n + a_1y_{n-1} + a_2y_{n-2} + \cdots + a_ny_0\right) = h^{\alpha}D^{\alpha}\left(y_n + b_1y_{n-1} + b_2y_{n-2} + \cdots + b_ny_0\right). \tag{25}$$

Noticing

$$D^{\alpha}y_n = f_n, \tag{26}$$

we get

$$\left(a_0y_n + a_1y_{n-1} + a_2y_{n-2} + \cdots + a_ny_0\right) = h^{\alpha}\left(f_n + b_1f_{n-1} + b_2f_{n-2} + \cdots + b_nf_0\right). \tag{27}$$

Then, we get the following numerical algorithm:

$$a_0y_n = -a_1y_{n-1} - a_2y_{n-2} - \ldots - a_ny_0 + h^{\alpha}\left(f_n + b_1f_{n-1} + b_2f_{n-2} + \cdots + b_nf_0\right). \tag{28}$$

This is a multiple step algorithm for problem (1), where a_is and b_is can be determined by the Pade approximation of $(1-x)^{\alpha}$ for any given α. From the construction process, we can see the local truncation error of this algorithm is $2n+1$.

4. Some Implicit Multistep Algorithms for the Fractional Initial Value Problem

Let $\alpha = 0.5$; the $[2,2]$ Pade approximant to $(1-x)^{0.5}$ is

$$\frac{16 - 20x + 5x^2}{16 - 12x + x^2}, \tag{29}$$

and the $[3,3]$ Pade approximant to $(1-x)^{0.5}$ is

$$\frac{64 - 112x + 56x^2 - 7x^3}{64 - 80x + 24x^2 - x^3}. \tag{30}$$

Then, from (28), we get the following two multistep algorithms:

$$16y_n = 20y_{n-1} - 5y_{n-2} + h^{0.5}\left(16f_n - 12f_{n-1} + f_{n-2}\right), \tag{31}$$

in which

$$\begin{aligned} y_n &= \frac{5}{4}y_{n-1} - \frac{5}{16}y_{n-2} + h^{05}\left(f_n - \frac{3}{4}f_{n-1} + \frac{1}{16}f_{n-2}\right) \\ &\quad \text{(algorithm 1)}, \\ 64y_n &= 112y_{n-1} - 56y_{n-2} + 7y_{n-3} \\ &\quad + h^{0.5}\left(64f_n - 80f_{n-1} + 24f_{n-2} - f_{n-3}\right), \end{aligned} \tag{32}$$

in which

$$y_n = \frac{7}{4}y_{n-1} - \frac{7}{8}y_{n-2} + \frac{7}{64}y_{n-3} + h^{0.5}\left(f_n - \frac{5}{4}f_{n-1} + \frac{3}{8}f_{n-2} - \frac{1}{64}f_{n-3}\right) \quad \text{(algorithm 2)}. \tag{33}$$

In the same way, from the following $[4,4]$ Pade approximant to $(1-x)^{0.5}$,

$$\frac{1 - (9/4) + (27/16)x^2 - (15/32)x^3 + (9/256)x^4}{1 - (7/4) + (15/16)x^2 - (5/32)x^3 + (1/256)x^4}, \tag{34}$$

we get the following (algorithm (3)):

$$y_n = \frac{9}{4}y_{n-1} - \frac{27}{16}y_{n-2} + \frac{15}{32}y_{n-3} - \frac{9}{256}y_{n+4} + h^{0.5}\left(f_n - \frac{7}{4}f_{n-1} + \frac{15}{16}f_{n-2} - \frac{5}{32}f_{n-3} + \frac{1}{256}f_{n-4}\right) \quad \text{(algorithm 3)}. \tag{35}$$

In the same way, the [2, 2] Pade approximant to $(1-x)^{0.8}$ is

$$\frac{1-(7/5)x+(21/50)x^2}{1-(3/5)x+(1/50)x^2}. \tag{36}$$

So, we get the following numerical algorithm:

$$y_n=\frac{7}{5}y_{n-1}-\frac{21}{50}y_{n-2}+h^{08}\left(f_n-\frac{3}{5}f_{n-1}+\frac{1}{50}f_{n-2}\right) \quad \text{(algorithm 4)}. \tag{37}$$

We can also get the following algorithm by using the [3, 3] Pade approximant to $(1-x)^{0.8}$:

$$\begin{aligned} y_n&=\frac{3211}{1680}y_{n-1}-\frac{6479}{6000}y_{n-2}+\frac{3287}{20000}y_{n-3}\\ &+h^{08}\left(f_n-\frac{1867}{1680}f_{n-1}+\frac{3791}{14000}f_{n-2}-\frac{1943}{420000}f_{n-3}\right)\\ &\text{(algorithm 5)}, \end{aligned} \tag{38}$$

and many other more complicated numerical schemes can be constructed by using the Pade approximant for any given α.

5. Numerical Tests

5.1. Numerical Test 1.

$$\begin{cases} D^{0.5}y=(\sqrt{2}/2)\left(y+\sqrt{1-y^2}\right) \\ y(0)=0 \end{cases} \tag{39}$$

The exact solution to this problem is $y=\sin(x)$. We use algorithms (1) and (2) to solve this problem. Because both algorithms (1) and (2) are implicit, we use the iterative method

$$y_n=g(y_n), \tag{40}$$

to get the value of y_n, where

$$g(y_n)=\frac{5}{4}y_{n-1}-\frac{5}{16}y_{n-2}+h^{0.5}\left(f_n-\frac{3}{4}f_{n-1}+\frac{1}{16}f_{n-2}\right), \tag{41}$$

for algorithm 1 and

$$\begin{aligned} g(y_n)&=\frac{7}{4}y_{n-1}-\frac{7}{8}y_{n-2}+\frac{7}{64}y_{n-3}\\ &+h^{0.5}\left(f_n-\frac{5}{4}f_{n-1}+\frac{3}{8}f_{n-2}-\frac{1}{64}f_{n-3}\right), \end{aligned} \tag{42}$$

for algorithm 2, respectively. The computational errors are listed in Table 1 ($h=(\text{pi}/2)/10$). We also compare algorithms (1) and (2) with the exact solution in Figure 1.

We also make the comparison with the corresponding G algorithm. The results are listed in Table 2. The comparison shows that algorithms (1) and (2) are more efficient than the corresponding Grunwald–Letnikov-based G algorithm.

5.2. Numerical Test 2.

$$\begin{cases} D^{0.8}y=y\cos\left(\dfrac{2\pi}{5}\right)+\sqrt{1-y^2}\,\sin\left(\dfrac{2\pi}{5}\right), \\ y(0)=0. \end{cases} \tag{43}$$

The exact solution to this problem is $y=\sin x$. We use the following algorithm (4)

$$y_n=\frac{7}{5}y_{n-1}-\frac{21}{50}y_{n-2}+h^{0.8}\left(f_n-\frac{3}{5}f_{n-1}+\frac{1}{50}f_{n-2}\right), \tag{44}$$

and algorithm (5)

$$\begin{aligned} y_n&=\frac{3211}{1680}y_{n-1}-\frac{6479}{6000}y_{n-2}+\frac{3287}{20000}y_{n-3}\\ &+h^{0.8}\left(f_n-\frac{1867}{1680}f_{n-1}+\frac{3791}{14000}f_{n-2}-\frac{1943}{420000}f_{n-3}\right), \end{aligned} \tag{45}$$

to get the numerical solution. Because both algorithms (4) and (5) are implicit schemes, we use the iterative method $y_n=g(y_n)$ to get the value of y_n, where

$$g(y_n)=\frac{7}{5}y_{n-1}-\frac{21}{50}y_{n-2}+h^{08}\left(f_n-\frac{3}{5}f_{n-1}+\frac{1}{50}f_{n-2}\right), \tag{46}$$

in algorithm (4) and

$$\begin{aligned} g(y_n)&=\frac{3211}{1680}y_{n-1}-\frac{6479}{6000}y_{n-2}+\frac{3287}{20000}y_{n-3}\\ &+h^{08}\left(f_n-\frac{1867}{1680}f_{n-1}+\frac{3791}{14000}f_{n-2}-\frac{1943}{420000}f_{n-3}\right), \end{aligned} \tag{47}$$

in algorithm (5). The errors of the numerical results are listed in Table 3 ($h=(\text{pi}/2)/10$).

The comparison with the corresponding G algorithm is listed in Table 4. The comparison shows that algorithms (4) and (5) are more efficient than the corresponding G algorithm.

5.3. Numerical Test 3.

$$\begin{cases} D^{0.8}y=x^{-0.8}\dfrac{\Gamma(5)}{\Gamma(4.2)}y, \\ y(0)=0. \end{cases} \tag{48}$$

The exact solution is $y=x^4$; we use algorithm 5 to solve this problem. The comparison between the exact and the numerical solution can be seen in Figure 2. One can see that the numerical solution is very accurate when $T\gg 1$, and the result is obtained in less than 50 steps. This is much more efficient than the G algorithm.

Table 1: Computational errors for algorithms (1) and (2).

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
Algorithm (1)	0.000	0.000	0.0155	0.0318	0.0455	0.0575	0.0716	0.0992	0.0783	0.0069
Algorithm (2)	0.000	0.000	0.000	0.0011	0.0028	0.0043	0.0049	0.0042	0.0211	0.0263

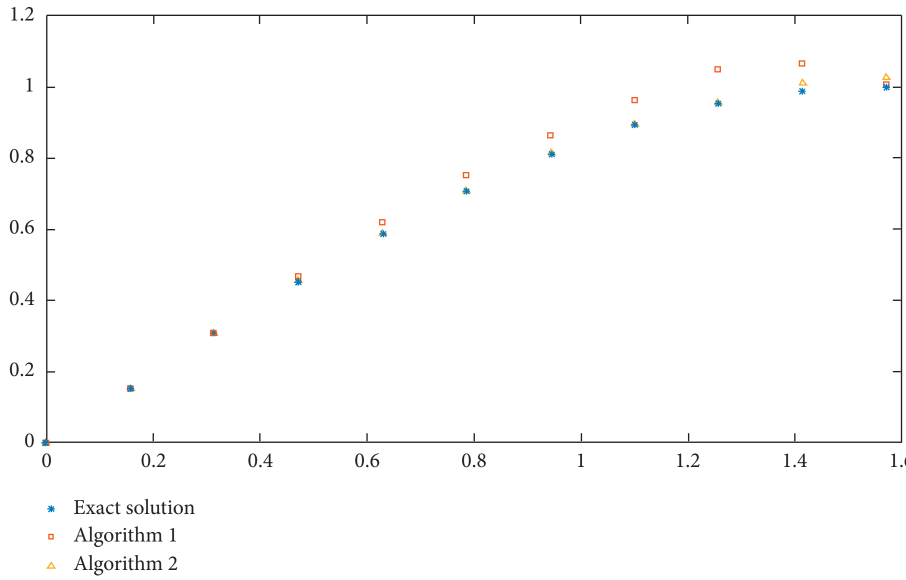

Figure 1: Comparison between algorithms (1) and (2) with the exact solution.

Table 2: Computational error comparison with the G algorithm.

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
Algorithm (1)	0.000	0.000	0.0155	0.0318	0.0455	0.0575	0.0716	0.0992	0.0783	0.0069
Algorithm (2)	0.000	0.000	0.000	0.0011	0.0028	0.0043	0.0049	0.0042	0.0211	0.0263
G-algm	0.1232	0.1782	0.1642	0.2589	0.2529	0.2762	0.3856	0.3842	0.5485	0.5478

Table 3: Computational errors for algorithms (3) and (4).

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
Algorithm (3)	0.000	0.000	0.0111	0.0123	0.0245	0.0267	0.0343	0.0355	0.0455	0.0058
Algorithm (4)	0.000	0.000	0.0000	0.0009	0.0019	0.0038	0.0037	0.0042	0.0111	0.0165

Table 4: Comparison with the G algorithm.

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
Algorithm (1)	0.000	0.000	0.0155	0.0318	0.0455	0.0575	0.0716	0.0992	0.0783	0.0069
Algorithm (2)	0.000	0.000	0.000	0.0011	0.0028	0.0043	0.0049	0.0042	0.0211	0.0263
G-algm	0.1255	0.1351	0.1385	0.1395	0.2452	0.2522	0.3836	0.3911	0.4263	0.5485

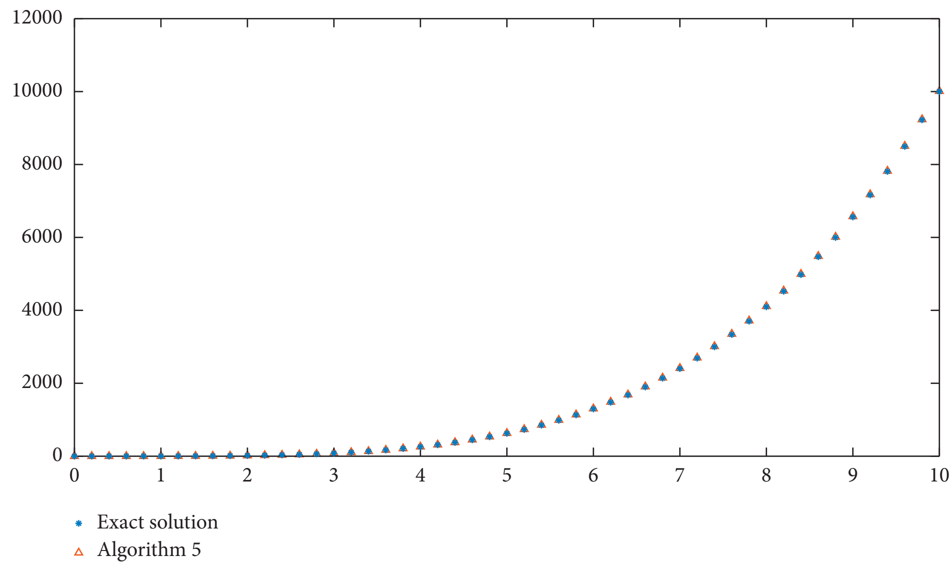

FIGURE 2: Comparison between algorithm (5) and the exact solution.

6. Conclusion

We can construct numerical algorithms for solving the fractional initial value problem based on the Pade approximation of fractional derivative operators. Numerical tests show that this method is more efficient than the corresponding G algorithms. Generally speaking, we can calculate more terms to achieve more accuracy when Grunwald–Letnikov-based method is employed, but that would lead to more computational work, while the algorithms derived from Pade approximation are proved to be more efficient.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Derivation of Bounds of an Integral Operator via Exponentially Convex Functions

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In this paper, bounds of fractional and conformable integral operators are established in a compact form. By using exponentially convex functions, certain bounds of these operators are derived and further used to prove their boundedness and continuity. A modulus inequality is established for a differentiable function whose derivative in absolute value is exponentially convex. Upper and lower bounds of these operators are obtained in the form of a Hadamard inequality. Some particular cases of main results are also studied.

1. Introduction

We start with the definition of convex function.

Definition 1 (see [1]). A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le t f(x) + (1-t) f(y), \tag{1}$$

holds for all $x, y \in [a,b]$ and $t \in [0,1]$. If inequality (1) is reversed, then the function f will be concave on $[a,b]$.

Convex functions are very useful in many areas of mathematics and other subjects due to their fascinating properties and characterizations. Their geometric and analytic interpretations provide straightforward proofs of many mathematical inequalities including Hadamard, Jensen, Hölder, and Minkowski [1–3]. Theoretically, convex functions have been generalized and extended as h-convex, m-convex, s-convex, (α,m)-convex, $(h-m)$-convex, (s,m)-convex, etc. Awan et al. [4] defined the function named exponentially convex function as follows:

Definition 2. A function $f\colon K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, where K is an interval, is said to be an exponentially convex function if

$$f(ta + (1-t)b) \le t\frac{f(a)}{e^{\alpha a}} + (1-t)\frac{f(b)}{e^{\alpha b}}, \tag{2}$$

holds for all $a, b \in K$, $t \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (2) is reversed, then f is called exponentially concave.

If $\alpha = 0$, then (2) gives inequality (1). For some recent citations and utilization of exponentially convex functions, one can see [5–14] and references therein. Our goal in this paper is to prove generalized integral inequalities for exponentially convex functions by using integral operators given in Definition 7. In the following, we give definitions of Riemann–Liouville fractional integrals:

Definition 3. Let $f \in L_1[a,b]$. Then, the left-sided and right-sided Riemann–Liouville fractional integral operators of order $\mu \in \mathbb{C}\,(\mathscr{R}(\mu) > 0)$ are defined by

$${}^{\mu}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t, \quad x > a, \tag{3}$$

$${}^{\mu}I_{b\_} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)\mathrm{d}t, \quad x < b. \tag{4}$$

A k-fractional analogue is given as follows:

Definition 4 (see [15]). Let $f \in L_1[a,b]$. Then, for $k>0$, the k-fractional integral operators of f of order $\mu \in \mathbb{C}, \mathscr{R}(\mu)>0$ are defined by

$$^{\mu}I^k_{a^+}f(x) = \frac{1}{k\Gamma_\kappa(\mu)}\int_a^x (x-t)^{(\mu/k)-1}f(t)\mathrm{d}t, \quad x>a, \tag{5}$$

$$^{\mu}I^k_{b\_}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{(\mu/k)-1}f(t)\mathrm{d}t, \quad x<b. \tag{6}$$

A more general definition of the Riemann–Liouville fractional integral operators is given as follows:

Definition 5 (see [16]). Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}(\mathscr{R}(\mu)>0)$ are defined by

$$^{\mu}_{g}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1}g'(t)f(t)\mathrm{d}t, \quad x>a, \tag{7}$$

$$^{\mu}_{g}I_{b\_}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1}g'(t)f(t)\mathrm{d}t, \quad x<b, \tag{8}$$

where $\Gamma(.)$ is the gamma function.

Definition 6. Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided k-fractional integral operators, $k>0$, of a function f with respect to another function g on $[a,b]$ of order $\mu, k \in \mathbb{C}, \mathscr{R}(\mu)>0$ are defined by

$$^{\mu}_{g}I^k_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{(\mu/k)-1}g'(t)f(t)\mathrm{d}t, \quad x>a, \tag{9}$$

$$^{\mu}_{g}I^k_{b\_}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{(\mu/k)-1}g'(t)f(t)\mathrm{d}t, \quad x<b, \tag{10}$$

where $\Gamma_k(.)$ is the k-gamma function.

A compact form of integral operators defined above is given as follows:

Definition 7 (see [17]). Let $f, g\colon [a,b] \longrightarrow \mathbb{R}, 0<a<b$ be the functions such that f be positive and $f \in L_1[a,b]$ and g be differentiable and strictly increasing. Also, let (ϕ/x) be an increasing function on $[a,\infty)$. Then, for $x \in [a,b]$, the left- and right-sided integral operators are defined by

$$\left(F^{\phi,g}_{a^+}f\right)(x) = \int_a^x K_g(x,t;\phi)f(t)\mathrm{d}(g(t)), \quad x>a, \tag{11}$$

$$\left(F^{\phi,g}_{b\_}f\right)(x) = \int_x^b K_g(t,x;\phi)f(t)\mathrm{d}(g(t)), \quad x<b, \tag{12}$$

where $K_g(x,y;\phi) = (\phi(g(x)-g(y))/(g(x)-g(y)))$.

Integral operators defined in (11) and (12) produce several fractional and conformable integral operators defined in [16, 18–25].

Remark 1. Integral operators given in (11) and (12) produce fractional and conformable integral operators as follows:

(i) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, then (11) and (12) integral operators coincide with (9) and (10) fractional integral operators.

(ii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu>0$, then (11) and (12) integral operators coincide with (7) and (8) fractional integral operators.

(iii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and g as identity function (11) and (12), integral operators coincide with (5 and 6) fractional integral operators.

(iv) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu>0$, and along with g as identity function (11) and (12), integral operators coincide with (3 and 4) fractional integral operators.

(v) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, $k=1$, and $g(x) = (x^{\rho}/\rho)$, $\rho>0$, then (11) and (12) produce Katugampola fractional integral operators defined by Chen and Katugampola in [18].

(vi) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, $k=1$, and $g(x) = (x^{\tau+s}/(\tau+s))$, $s>0$, then (11) and (12) produce generalized conformable integral operators defined by Khan and Khan in [22].

(vii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = ((x-a)^s/s)$, $s>0$, in (11) and $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = -((b-x)^s/s)$, $s>0$, in (12), respectively, then conformable (k,s)-fractional integrals are achieved as defined by Habib et al. in [20].

(viii) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = (x^{1+s}/(1+s))$, then (11) and (12) produce conformable fractional integrals defined by Sarikaya et al. in [24].

(ix) If we consider $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = ((x-a)^s/s)$, $s>0$, in (11) and $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = -((b-x)^s/s)$, $s>0$, in (12) with $k=1$, respectively, then conformable fractional integrals are achieved as defined by Jarad et al. in [21].

(x) If we consider $\phi(t) = t^{\mu/k}\mathscr{F}^{\sigma,k}_{\rho,\lambda}(w(t)^{\rho})$, then (11) and (12) produce generalized k-fractional integral operators defined by Tunc et al. in [25].

(xi) If we consider $\phi(t) = (\exp(-At)/\mu)$ and $A = ((1-\mu)/\mu)$, $\mu>0$, then following generalized fractional integral operators with exponential kernel are obtained [19]:

$$ {}^{\mu}_{g}E_{a^+} f(x) = \frac{1}{\mu}\int_a^x \exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right) f(t)\mathrm{d}t, \quad x>a, $$

$$ {}^{\mu}_{g}E_{b_-} f(x) = \frac{1}{\mu}\int_x^b \exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right) f(t)\mathrm{d}t, \quad x<b. \tag{13} $$

(xii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$ and $g(t) = \ln t$, then Hadamard fractional integral operators will be obtained [16, 23].

(xiii) If we consider $\phi(t) = (t^{\mu}/\Gamma(\mu))$ and $g(t) = -t^{-1}$, then Harmonic fractional integral operators given in [16] will be obtained and given as follows:

$$ \begin{aligned} {}^{\mu}R_{a^+} f(x) &= \frac{t^{\mu}}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\frac{f(t)}{t^{\mu+1}}\mathrm{d}t, \quad x>a, \\ {}^{\mu}R_{b_-} f(x) &= \frac{t^{\mu}}{\Gamma(\mu)}\int_a^x (t-x)^{\mu-1}\frac{f(t)}{t^{\mu+1}}\mathrm{d}t, \quad x<b. \end{aligned} \tag{14} $$

(xiv) If we consider $\phi(t) = t^{\mu}\ln t$, then left- and right-sided logarithmic fractional integrals are obtained in [19] and given as follows:

$$ \begin{aligned} {}^{\mu}_{g}\mathscr{L}_{a^+} f(x) &= \int_a^x (g(x)-g(t))^{\mu-1}\ln(g(x) \\ &\quad - g(t))g'(t)\mathrm{d}t, \quad x>a, \\ {}^{\mu}_{g}\mathscr{L}_{b_-} f(x) &= \int_a^x (g(t)-g(x))^{\mu-1}\ln(g(x) \\ &\quad - g(t))g'(t)\mathrm{d}t, \quad x<b. \end{aligned} \tag{15} $$

In the upcoming section, we will derive bounds of sum of the left- and right-sided integral operators defined in (11) and (12) for exponentially convex functions. These bounds lead to produce results associated to several kinds of well-known operators for exponentially convex functions, some of the results are presented in particular cases. Further in Section 3, bounds are presented in the form of a Hadamard inequality; several Hadamard type inequalities are obtained.

2. Bounds of Integral Operators and Their Consequences

Theorem 1. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a positive and exponentially convex function and $g\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on $[a,b]$. Then, for $x \in [a,b]$ the following inequality for integral operators (11) and (12) holds*

$$ \begin{aligned} &\left(F^{\phi,g}_{a^+} f\right)(x) + \left(F^{\phi,g}_{b^-} f\right)(x) \\ &\le \phi(g(x)-g(a))\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) + \phi(g(b) \\ &\quad - g(x))\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right). \end{aligned} \tag{16} $$

Proof. For the kernel of integral operator (11), we have

$$ K_g(x,t;\phi)g'(t) \le K_g(x,a;\phi)g'(t), \quad x \in (a,b] \text{ and } t \in [a,x). \tag{17} $$

An exponentially convex function satisfies the following inequality:

$$ f(t) \le \left(\frac{x-t}{x-a}\right)\frac{f(a)}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)\frac{f(x)}{e^{\alpha x}}. \tag{18} $$

Inequalities (17) and (18) lead to the following integral inequality:

$$ \begin{aligned} &\int_a^x K_g(x,t;\phi)g'(t)f(t)\mathrm{d}t \\ &\le K_g(x,a;\phi)\left(\frac{f(a)}{e^{\alpha a}}\int_a^x\left(\frac{x-t}{x-a}\right)g'(t)\mathrm{d}t \right. \\ &\quad \left. + \frac{f(x)}{e^{\alpha x}}\int_a^x\left(\frac{t-a}{x-a}\right)g'(t)\mathrm{d}t\right), \end{aligned} \tag{19} $$

while (19) gives

$$ \left(F^{\phi,g}_{a^+} f\right)(x) \le \phi(g(x)-g(a))\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right). \tag{20} $$

Again, for the kernel of integral operator (12), we have

$$ K_g(t,x;\phi)g'(t) \le K_g(b,x;\phi)g'(t), \quad t \in (x,b] \text{ and } x \in [a,b). \tag{21} $$

An exponentially convex function satisfies the following inequality:

$$ f(t) \le \left(\frac{t-x}{b-x}\right)\frac{f(b)}{e^{\alpha b}} + \left(\frac{b-t}{b-x}\right)\frac{f(x)}{e^{\alpha x}}. \tag{22} $$

Inequalities (21) and (22) lead to the following integral inequality:

$$ \begin{aligned} &\int_x^b K_g(t,x;\phi)g'(t)f(t)\mathrm{d}t \\ &\le K_g(b,x;\phi)\left(\frac{f(b)}{e^{\alpha b}}\int_x^b\left(\frac{t-x}{b-x}\right)g'(t)\mathrm{d}t \right. \\ &\quad \left. + \frac{f(x)}{e^{\alpha x}}\int_x^b\left(\frac{b-t}{b-x}\right)g'(t)\mathrm{d}t\right), \end{aligned} \tag{23} $$

while (23) gives

$$ \left(F^{\phi,g}_{b^-} f\right)(x) \le \phi(g(b)-g(x))\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right). \tag{24} $$

By adding (20) and (24), (16) can be achieved.

The following remark connected the abovementioned theorem with already known results. □

Remark 2

(1) For $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu>0$, and $\alpha = 0$ in (16), Corollary 1 in [26] can be achieved.

(2) For $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu>0$, $g(x) = x$, and $\alpha = 0$ in (16), Corollary 1 in [27] can be achieved.

(3) For $\alpha = 0$, in (16), Theorem 1 in [28] can be achieved.

Next results indicate upper bounds of several known fractional and conformable integral operators.

Proposition 1. *Let $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$. Then, (11) and (12) produce the fractional integral operators (7) and (8) as follows:*

$$\begin{aligned}\left(F_{a^+}^{(t^{\mu}/\Gamma(\mu)),g} f\right)(x) &\coloneqq {}_g^{\mu}I_{a^+} f(x),\\ \left(F_{b^-}^{(t^{\mu}/\Gamma(\mu)),g} f\right)(x) &\coloneqq {}_g^{\mu}I_{b^-} f(x).\end{aligned} \tag{25}$$

Further they satisfy the following bound for $\mu \geq 1$:

$$\begin{aligned}&\left({}_g^{\mu}I_{a^+} f\right)(x) + \left({}_g^{\mu}I_{b^-} f\right)(x)\\ &\quad\leq \frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right)\\ &\qquad + \frac{(g(b)-g(x))^{\mu}}{\Gamma(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{26}$$

Proposition 2. *Let $g(x) = I(x) = x$. Then, (11) and (12) produce integral operators defined in [29] as follows:*

$$\begin{aligned}\left(F_{a^+}^{\phi,I} f\right)(x) &\coloneqq \left({}_{a^+}I_{\phi} f\right)(x) = \int_a^x \frac{\phi(x-t)}{(x-t)} f(t)\mathrm{d}t,\\ \left(F_{b^-}^{\phi,I} f\right)(x) &\coloneqq \left({}_{b^-}I_{\phi} f\right)(x) = \int_x^b \frac{\phi(t-x)}{(t-x)} f(t)\mathrm{d}t.\end{aligned} \tag{27}$$

Further they satisfy the following bound:

$$\begin{aligned}&\left({}_{a^+}I_{\phi} f\right)(x) + \left({}_{b^-}I_{\phi} f\right)(x)\\ &\quad\leq \phi(x-a)\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) + \phi(b-x)\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{28}$$

Corollary 1. *If we take $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$, then (11) and (12) produce the fractional integral operators (9) and (10) as follows:*

$$\begin{aligned}\left(F_{a^+}^{(t^{\mu/k}/k\Gamma_k(\mu)),g} f\right)(x) &\coloneqq {}_g^{\mu}I_{a^+}^k f(x),\\ \left(F_{b^-}^{(t^{\mu/k}/k\Gamma_k(\mu)),g} f\right)(x) &\coloneqq {}_g^{\mu}I_{b^-}^k f(x).\end{aligned} \tag{29}$$

From (16), the following bound holds for $\mu \geq k$:

$$\begin{aligned}\left({}_g^{\mu}I_{a^+}^k f\right)(x) + \left({}_g^{\mu}I_{b^-}^k f\right)(x) &\leq \frac{(g(x)-g(a))^{\mu/k}}{k\Gamma_k(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right)\\ &\quad + \frac{(g(b)-g(x))^{\mu/k}}{k\Gamma_k(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{30}$$

Corollary 2. *If we take $\phi(t) = (t_{\mu}/\Gamma(\mu))$, $\mu > 0$, and $g(x) = I(x) = x$, then (11) and (12) produce left- and right-sided Riemann–Liouville fractional integral operators (3) and (4) as follows:*

$$\begin{aligned}\left(F_{a^+}^{(t^{\mu}/\Gamma(\mu)),I} f\right)(x) &\coloneqq {}^{\mu}I_{a^+} f(x),\\ \left(F_{b^-}^{(t^{\mu}/\Gamma(\mu)),I} f\right)(x) &\coloneqq {}^{\mu}I_{b^-} f(x).\end{aligned} \tag{31}$$

From (16), the following bound holds for $\mu \geq 1$:

$$\begin{aligned}&\left({}^{\mu}I_{a^+} f\right)(x) + \left({}^{\mu}I_{b^-} f\right)(x)\\ &\quad\leq \frac{(x-a)^{\mu}}{\Gamma(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) + \frac{(b-x)^{\mu}}{\Gamma(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{32}$$

Corollary 3. *If we take $\phi(t) = (t^{\mu/k}/k\Gamma_k(\mu))$ and $g(x) = I(x) = x$, then (11) and (12) produce the fractional integral operators (5) and (6) as follows:*

$$\begin{aligned}\left(F_{a^+}^{(t^{\mu/k}/k\Gamma_k(\mu)),I} f\right)(x) &\coloneqq {}^{\mu}I_{a^+}^k f(x),\\ \left(F_{b^-}^{(t^{\mu/k}/k\Gamma_k(\mu)),I} f\right)(x) &\coloneqq {}^{\mu}I_{b^-}^k f(x).\end{aligned} \tag{33}$$

From (16), the following bound holds for $\mu \geq k$:

$$\begin{aligned}&\left({}^{\mu}I_{a^+}^k f\right)(x) + \left({}^{\mu}I_{b^-}^k f\right)(x)\\ &\quad\leq \frac{(x-a)^{\mu/k}}{k\Gamma_k(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) + \frac{(b-x)^{\mu/k}}{k\Gamma_k(\mu)}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{34}$$

Corollary 4. *If we take $\phi(t) = (t^{\mu}/\Gamma(\mu))$, $\mu > 0$ and $g(x) = (x^{\rho}/\rho)$, $\rho > 0$, then (11) and (12) produce the fractional integral operators defined in [18] as follows:*

$$\begin{aligned}\left(F_{a^+}^{(t^{\mu}/\Gamma(\mu)),g} f\right)(x) &= \left({}^{\rho}I_{a^+}^{\mu} f\right)(x)\\ &= \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_a^x \left(x^{\rho}-t^{\rho}\right)^{\mu-1} t^{\rho-1} f(t)\mathrm{d}t,\\ \left(F_{b^-}^{(t^{\mu}/\Gamma(\mu)),g} f\right)(x) &= \left({}^{\rho}I_{b^-}^{\mu} f\right)(x)\\ &= \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_x^b \left(t^{\rho}-x^{\rho}\right)^{\mu-1} t^{\rho-1} f(t)\mathrm{d}t.\end{aligned} \tag{35}$$

From (16), they satisfy the following bound:

$$\begin{aligned}&\left({}^{\rho}I_{a^+}^{\mu} f\right)(x) + \left({}^{\rho}I_{b^-}^{\mu} f\right)(x)\\ &\quad\leq \frac{(x^{\rho}-a^{\rho})^{\mu}}{\Gamma(\mu)(\rho^{\mu})}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) + \frac{(b^{\rho}-x^{\rho})^{\mu}}{\Gamma(\mu)(\rho^{\mu})}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).\end{aligned} \tag{36}$$

Corollary 5. *If we take $\phi(t) = (t^\mu/\Gamma(\mu))$, $\mu > 0$, and $g(x) = (x^{n+1}/(n+1))$, $n > 0$, then (11) and (12) produce the fractional integral operators defined as follows:*

$$\begin{aligned}
\left(F_{a^+}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &= \left({}^n I_{a^+}^\mu f\right)(x) \\
&= \frac{(n+1)^{1-\mu}}{\Gamma(\mu)} \int_a^x \left(x^{n+1} - t^{n+1}\right)^{\mu-1} t^n f(t)\mathrm{d}t, \\
\left(F_{b^-}^{(t^\mu/\Gamma(\mu)),g} f\right)(x) &= \left({}^n I_{b^-}^\mu f\right)(x) \\
&= \frac{(n+1)^{1-\mu}}{\Gamma(\mu)} \int_x^b \left(t^{n+1} - x^{n+1}\right)^{\mu-1} t^n f(t)\mathrm{d}t.
\end{aligned} \tag{37}$$

From (16), they satisfy the following bound:

$$\begin{aligned}
&\left({}^n I_{a^+}^\mu f\right)(x) + \left({}^n I_{b^-}^\mu f\right)(x) \\
&\leq \frac{(x^{n+1} - a^{n+1})^\mu}{\Gamma(\mu)(n+1)^\mu}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(a)}{e^{\alpha a}}\right) \\
&+ \frac{(x^{n+1} - a^{n+1})^\mu}{\Gamma(\mu)(n+1)^\mu}\left(\frac{f(x)}{e^{\alpha x}} + \frac{f(b)}{e^{\alpha b}}\right).
\end{aligned} \tag{38}$$

Next, we prove boundedness and continuity of integral operators.

Theorem 2. *Let the assumptions of Theorem 1 are satisfied. If $f \in L_\infty[a,b]$, then integral operators defined in (11) and (12) are continuous.*

Proof. From (20), we have

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq \phi(g(x) - g(a))\|f\|_\infty\left(\frac{1}{e^{\alpha x}} + \frac{1}{e^{\alpha a}}\right). \tag{39}$$

It is given that (ϕ/x) is increasing on $[a,b]$:

$$\begin{aligned}
&\frac{\phi(g(x) - g(a))}{g(x) - g(a)} \leq \frac{\phi(g(b) - g(a))}{g(b) - g(a)}, \\
&\phi(g(x) - g(a)) \leq \frac{\phi(g(b) - g(a))g(x) - g(a)}{g(b) - g(a)}.
\end{aligned} \tag{40}$$

Further g is increasing; therefore, we have

$$\phi(g(x) - g(a)) \leq \phi(g(b) - g(a)). \tag{41}$$

Therefore, (39) gives

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq \phi(g(b) - g(a))\|f\|_\infty\left(e^{-\alpha x} + e^{-\alpha a}\right). \tag{42}$$

If $\alpha > 0$, then $e^{-\alpha x}$ is decreasing on $[a,b]$ and we get

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq 2e^{-\alpha a}\phi(g(b) - g(a))\|f\|_\infty. \tag{43}$$

If $\alpha < 0$, then $e^{-\alpha x}$ is increasing on $[a,b]$ and we get from (42)

$$\left(F_{a^+}^{\phi,g} f\right)(x) \leq 2e^{-\alpha b}\phi(g(b) - g(a))\|f\|_\infty. \tag{44}$$

Hence, $(F_{a^+}^{\phi,g} f)(x)$ is bounded and it is linear, and therefore, $(F_{a^+}^{\phi,g} f)(x)$ is continuous.

Similarly, continuity of $(F_{b^-}^{\phi,g} f)(x)$ can be proved.

For a differentiable function f, as $|f'|$ is exponentially convex, the following result holds: □

Theorem 3. *Let $f: I \longrightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is exponentially convex and $g: I \longrightarrow \mathbb{R}$ is a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on I, then for $a, b \in I$, $a < b$, the following inequalities for integral operators holds:*

$$\left|F_{a^+}^{\phi,g}(f * g)(x)\right| \leq \phi(g(x) - g(a))\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right), \tag{45}$$

$$\left|F_{b^-}^{\phi,g}(f * g)(x)\right| \leq \phi(g(b) - g(x))\left(\frac{|f'(b)|}{e^{\alpha b}} + \frac{|f'(x)|}{e^{\alpha x}}\right), \tag{46}$$

where

$$\begin{aligned}
F_{a^+}^{\phi,g}(f * g)(x) &= \int_a^x K_g(x,t;\phi)g'(t)f'(t)\mathrm{d}t, \\
F_{b^-}^{\phi,g}(f * g)(x) &= \int_x^b K_g(t,x;\phi)g'(t)f'(t)\mathrm{d}t.
\end{aligned} \tag{47}$$

Proof. An exponentially convex function $|f'|$ satisfies the following inequality:

$$|f'(t)| \leq \left(\frac{x-t}{x-a}\right)\frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)\frac{|f'(x)|}{e^{\alpha x}}. \tag{48}$$

From which, we can write

$$f'(t) \leq \left(\frac{x-t}{x-a}\right)\frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)\frac{|f'(x)|}{e^{\alpha x}}. \tag{49}$$

Inequalities (17) and (49) lead to the following integral inequality:

$$\begin{aligned}
&\int_a^x K_g(x,t;\phi)g'(t)f'(t)\mathrm{d}t \\
&\leq K_g(x,a;\phi)\left(\frac{|f'(a)|}{e^{\alpha a}}\int_a^x\left(\frac{x-t}{x-a}\right)g'(t)\mathrm{d}t\right. \\
&\left.+\left(\frac{|f'(x)|}{e^{\alpha x}}\right)\int_a^x\left(\frac{t-a}{x-a}\right)g'(t)\mathrm{d}t\right),
\end{aligned} \tag{50}$$

while (50) gives

$$F_{a^+}^{\phi,g}(f * g)(x) \leq \phi(g(x) - g(a))\left[\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right)\right]. \tag{51}$$

From (48), we can write

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)\frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)\frac{|f'(x)|}{e^{\alpha x}}\right). \tag{52}$$

Adopting the same method as we did for (49), the following integral inequality holds:

$$F_{a^+}^{\phi,g}(f*g)(x) \geq -\phi(g(x)-g(a))\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{|f'(x)|}{e^{\alpha x}}\right). \tag{53}$$

From (51) and (53), (45) can be achieved.

An exponentially convex function $|f'|$ satisfies the following inequality:

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)\frac{|f'(b)|}{e^{\alpha b}}+\left(\frac{b-t}{b-x}\right)\frac{|f'(x)|}{e^{\alpha x}}. \tag{54}$$

From which, we can write

$$f'(t) \leq \left(\frac{t-x}{b-x}\right)\frac{|f'(b)|}{e^{\alpha b}}+\left(\frac{b-t}{b-x}\right)\frac{|f'(x)|}{e^{\alpha x}}. \tag{55}$$

Inequalities (21) and (55) lead the following integral inequality:

$$\begin{aligned}&\int_x^b K_g(t,x;\phi)g'(t)f'(t)\mathrm{d}t\\&\leq K_g(b,x;\phi)\left(\frac{|f'(b)|}{e^{\alpha b}}\int_x^b\left(\frac{x-t}{b-x}\right)g'(t)\mathrm{d}t\right.\\&\left.+\frac{|f'(x)|}{e^{\alpha x}}\int_x^b\left(\frac{b-t}{b-x}\right)g'(t)\mathrm{d}t\right),\end{aligned} \tag{56}$$

while (56) gives

$$F_{b^-}^{\phi,g}(f*g)(x) \leq \phi(g(b)-g(x))\left(\frac{|f'(b)|}{e^{\alpha b}}+\frac{|f'(x)|}{e^{\alpha x}}\right). \tag{57}$$

From (54), we can write

$$f'(t) \geq -\left(\left(\frac{t-x}{b-x}\right)\frac{|f'(b)|}{e^{\alpha b}}+\left(\frac{b-t}{b-x}\right)\frac{|f'(x)|}{e^{\alpha x}}\right). \tag{58}$$

Adopting the same method as we did for (55), the following inequality holds:

$$F_{b^-}^{\phi,g}(f*g)(x) \geq -\phi(g(b)-g(x))\left(\frac{|f'(b)|}{e^{\alpha b}}+\frac{|f'(x)|}{e^{\alpha x}}\right). \tag{59}$$

From (57) and (59), (46) can be achieved. □

3. Hadamard Type Inequalities for Exponentially Convex Function

In this section, we prove the Hadamard type inequality for an exponentially convex function. In order to prove this inequality result, we need the following lemma.

Lemma 1 (see [30]). *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be an exponentially convex function. If f is exponentially symmetric, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{e^{\alpha x}}, \quad x \in [a,b]. \tag{60}$$

Theorem 4. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be positive, exponentially convex, and symmetric about $((a+b)/2)$ and $g\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also, let (ϕ/x) be an increasing function on $[a,b]$. Then, for $a,b \in I$, $a<b$, the following estimations of Hadamard type are valid.*

$$\begin{aligned}&h(\alpha)f\left(\frac{a+b}{2}\right)\left(\left(F_{b^-}^{\phi,g}(1)\right)(a)+\left(F_{a^+}^{\phi,g}(1)\right)(b)\right)\\&\leq\left(\left(F_{b^-}^{\phi,g}f\right)(a)+\left(F_{a^+}^{\phi,g}f\right)(b)\right)\\&\leq 2\phi(g(b)-g(a))\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right),\end{aligned} \tag{61}$$

where $h(\alpha)=e^{\alpha b}$ for $\alpha<0$ and $h(\alpha)=e^{\alpha a}$ for $\alpha \geq 0$.

Proof. For the kernel of integral operator (11), we have

$$K_g(x,a;\phi)g'(x) \leq K_g(b,a;\phi)g'(x), \quad x \in (a,b]. \tag{62}$$

An exponentially convex function satisfies the following inequality:

$$f(x) \leq \left(\frac{x-a}{b-a}\right)\frac{f(b)}{e^{\alpha b}}+\left(\frac{b-x}{b-a}\right)\frac{f(a)}{e^{\alpha a}}. \tag{63}$$

Inequalities (62) and (63) lead the following integral inequality:

$$\begin{aligned}&\int_a^b K_g(x,a;\phi)g'(x)f(x)\mathrm{d}x\\&\leq K_g(b,a;\phi)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b\left(\frac{x-a}{b-a}\right)g'(x)\mathrm{d}x\right.\\&\left.+\frac{f(a)}{e^{\alpha a}}\int_a^b\left(\frac{b-x}{b-a}\right)g'(x)\mathrm{d}x\right),\end{aligned} \tag{64}$$

while (64) gives

$$\left(F_{b^-}^{\phi,g}f\right)(a) \leq \phi(g(b)-g(a))\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right). \tag{65}$$

On the contrary, for the kernel of integral operator (12), we have

$$K_g(b,x;\phi)g'(x) \leq K_g(b,a;\phi)g'(x). \tag{66}$$

Inequalities (63) and (66) lead the following integral inequality:

$$\begin{aligned}&\int_a^b K_g(b,x;\phi)g'(x)f(x)\mathrm{d}x\\&\le K_g(b,a;\phi)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b\left(\frac{x-a}{b-a}\right)g'(x)\mathrm{d}x\right.\\&\left.+\frac{f(a)}{e^{\alpha a}}\int_a^b\left(\frac{b-x}{b-a}\right)g'(x)\mathrm{d}x\right),\end{aligned}\tag{67}$$

while the abovementioned inequality gives

$$\left(F_{a^+}^{\phi,g}f\right)(b)\le\phi\left(g(b)-g(a)\right)\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\tag{68}$$

From (65) and (68), the following inequality can be obtained:

$$\left(F_{a^+}^{\phi,g}f\right)(b)+\left(F_{b^-}^{\phi,g}f\right)(a)\le 2\phi\left(g(b)-g(a)\right)\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\tag{69}$$

Now, using Lemma 1 and multiplying (60) with $K_g(x,a;\phi)g\prime(x)$, then integrating over $[a,b]$, we have

$$\begin{aligned}&\int_a^b K_g(x,a;\phi)f\left(\frac{a+b}{2}\right)g'(x)dx\\&\le\int_a^b\frac{1}{e^{\alpha x}}K_g(x,a;\phi)g'(x)f(x)\mathrm{d}x.\end{aligned}\tag{70}$$

From which, we have

$$f\left(\frac{a+b}{2}\right)\left(F_{b^-}^{\phi,g}(1)\right)(a)\le\frac{1}{h(\alpha)}\left(F_{b^-}^{\phi,g}f\right)(a).\tag{71}$$

Again using Lemma 1 and multiplying (60) with $K_g(b,x;\phi)g\prime(x)$, then integrating over $[a,b]$, we have

$$\begin{aligned}&\int_a^b K_g(b,x;\phi)f\left(\frac{a+b}{2}\right)g'(x)\mathrm{d}x\\&\le\int_a^b\frac{1}{e^{\alpha x}}K_g(b,x;\phi)g'(x)f(x)\mathrm{d}x.\end{aligned}\tag{72}$$

From which, we have

$$f\left(\frac{a+b}{2}\right)\left(F_{a^+}^{\phi,g}(1)\right)(b)\le\frac{1}{h(\alpha)}\left(F_{a^+}^{\phi,g}f\right)(b).\tag{73}$$

From (71) and (73), the following inequality can be achieved:

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left(F_{b^-}^{\phi,g}(1)\right)(a)+\left(F_{a^+}^{\phi,g}(1)\right)(b)\\&\le\frac{1}{h(\alpha)}\left(\left(F_{b^-}^{\phi,g}f\right)(a)+\left(F_{a^+}^{\phi,g}f\right)(b)\right).\end{aligned}\tag{74}$$

From (69) and (74), (61) can be achieved. □

Remark 3. For $\alpha=0$, in (61), Theorem 3 in [28] can be achieved.

Corollary 6. *If we put $\phi(t)=(t^{\mu/k}/k\Gamma_k(\mu))$, then the inequality (61) produces the following Hadamard type inequality:*

$$\begin{aligned}&h(\alpha)f\left(\frac{a+b}{2}\right)\left({}_g^{\mu}I_{b_-}^k(1)(a)+{}_g^{\mu}I_{a^+}^k(1)(b)\right)\\&\le\left({}_g^{\mu}I_{b_-}^k f(a)+{}_g^{\mu}I_{a^+}^k f(b)\right)\\&\le\frac{2\left(g(b)-g(a)\right)^{\mu/k}}{(k\Gamma_k(\mu))}\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\end{aligned}\tag{75}$$

Corollary 7. *If we put $\phi(t)=(t^{\mu}/\Gamma(\mu))$, then the inequality (61) produces the following Hadamard type inequality:*

$$\begin{aligned}&h(\alpha)f\left(\frac{a+b}{2}\right)\left({}_g^{\mu}I_{b_-}(1)(a)+{}_g^{\mu}I_{a^+}(1)(b)\right)\\&\le\left({}_g^{\mu}I_{b_-}f(a)+{}_g^{\mu}I_{a^+}f(b)\right)\\&\le\frac{2\left(g(b)-g(a)\right)^{\mu}}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\end{aligned}\tag{76}$$

Corollary 8. *If we put $\phi(t)=(t^{\mu/k}/k\Gamma_k(\mu))$ and g as identity function, then the inequality (61) produces the following Hadamard type inequality:*

$$\begin{aligned}&h(\alpha)f\left(\frac{a+b}{2}\right)\left({}^{\mu}I_{b_-}^k(1)(a)+{}^{\mu}I_{a^+}^k(1)(b)\right)\\&\le\left({}^{\mu}I_{b_-}^k f(a)+{}^{\mu}I_{a^+}^k f(b)\right)\\&\le\frac{2(b-a)^{\mu/k}}{k\Gamma_k(\mu)}\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\end{aligned}\tag{77}$$

Corollary 9. *If we put $\phi(t)=(t^{\mu}/\Gamma(\mu))$ and g as identity function, then the inequality (61) produces the following Hadamard type inequality:*

$$\begin{aligned}&h(\alpha)f\left(\frac{a+b}{2}\right)\left({}^{\mu}I_{b_-}(1)(a)+{}^{\mu}I_{a^+}(1)(b)\right)\\&\le\left({}^{\mu}I_{b_-}f(a)+{}^{\mu}I_{a^+}f(b)\right)\\&\le\frac{2(b-a)}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}}+\frac{f(a)}{e^{\alpha a}}\right).\end{aligned}\tag{78}$$

4. Concluding Remarks

We have studied an integral operator for exponentially convex functions; this operator has direct consequences to several fractional and conformable integral operators. We have obtained bounds of the integral operator in different forms. In Theorem 1, upper bounds of this operator are studied for an exponentially convex function and several special cases have been presented in the form of propositions and corollaries. The boundedness is studied in Theorem 2. In Theorem 3, we have obtained results for differentiable function f such that $|f'|$ is exponentially convex. A version of the Hadamard inequality is proved in Theorem 4 which leads to its several variants for fractional and conformable integral operators.

Data Availability

No data were used to support this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Extended Bessel-Maitland Function and Integral Operators Associated with Fractional Calculus

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The aim of this paper is to introduce a presumably and remarkably altered integral operator involving the extended generalized Bessel-Maitland function. Particular properties are considered for the extended generalized Bessel-Maitland function connected with fractional integral and differential operators. The integral operator connected with operators of the fractional calculus is also observed. We point out important links to known findings from some individual cases with our key outcomes.

1. Introduction and Preliminaries

The Bessel-Maitland function $J_{\vartheta}^{\varsigma}(.)$ is a generalization of Bessel function introduced by Ed. Maitland Wright [1] through a series representation as follows:

$$J_{\vartheta}^{\varsigma}(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma(\varsigma m + \vartheta + 1)m!}, \quad (z, \vartheta \in \mathbb{C}, \varsigma > 0). \tag{1}$$

In fact, Watson's book [2] finds the application of the Bessel-Maitland function in the diverse field of engineering, chemical and biological sciences, and mathematical physics.

Further, Pathak [3] defined generalization of the Bessel-Maitland function $J_{\vartheta,q}^{\varsigma,\delta}(.)$ in the form as follows:

$$J_{\vartheta,q}^{\varsigma,\delta}(z) = \sum_{m=0}^{\infty} \frac{(\delta)_{qm}(-z)^m}{\Gamma(\varsigma m + \vartheta + 1)m!}, \tag{2}$$

where $z \in \mathbb{C}\backslash(-\infty, 0]$; $\varsigma, \vartheta, \delta \in \mathbb{C}$, $\Re(\varsigma) \geq 0$, $\Re(\vartheta) \geq -1$, $\Re(\delta) \geq 0$; $q \in (0,1) \cup \mathbb{N}$, and $(\delta)_{qm}$ is known as generalized Pochhammer symbol which is defined as

$$(\delta)_0 = 1, \, (\delta)_{qm} = \frac{\Gamma(\delta + qm)}{\Gamma(\delta)}. \tag{3}$$

Since the implementation of the Bessel-Maitland function in 1983, a number of extensions and generalizations have been introduced and examined with different applications (see information in [4–9]).

By motivation of these investigations and applications of the Bessel-Maitland function, Suthar et al. [7] defined the generalized Bessel-Maitland function (2) in the following manner:

$$J_{\vartheta,q}^{\varsigma,\delta;c}(z;p) = \sum_{m=0}^{\infty} \frac{\mathscr{B}_p(\delta + qm, c - \delta)(c)_{qm}(-z)^m}{B(\delta, c-\delta)\Gamma(\varsigma m + \vartheta + 1)m!},$$
$$(p > 0, q \in \mathbb{N}, \Re(c) > \Re(\delta) > 0), \tag{4}$$

which is known as extended generalized Bessel-Maitland function; here, $\mathscr{B}_p(s,t)$ is the extended beta function (see [10]).

$$\mathscr{B}_p(s,t) = \int_0^1 z^{s-1}(1-z)^{t-1} e^{-p/z(1-z)} \mathrm{d}z,$$
$$(\Re(p) > 0, \Re(s) > 0, \Re(t) > 0). \tag{5}$$

For $p = 0$, (5) reduces to beta function (see, e.g., [11], Section 1.1).

Remark 1

(i) The particular case of equation (4), when $p = 0$, reduces to (2) and when $p = q = 0$, reduces to (1).

(ii) When $q = 1, \vartheta = \vartheta - 1$ and $z = -z$ in (4) reduce to the extended Mittag–Leffler function defined by Ozarslan and Yilmaz ([12], equation (4)).

Definition 1. The space of Lebesgue measurable of real or complex valued function $\mathscr{L}(a,b)$ for our study of the significance of fractional calculus is defined as follows:

$$\mathscr{L}(a,b)=\left(f:\ \|f\|_1=\int_a^b f(x)\mathrm{d}x<\infty\right). \tag{6}$$

Definition 2. The Riemann–Liouville (R-L) fractional integral operators $\mathfrak{I}_{a+}^{\ell}$ and $\mathfrak{I}_{b-}^{\ell}$ are defined respectively as (see, e.g., [13]) follows:

$$\left(\mathfrak{I}_{a+}^{\ell}f\right)(x)=\frac{1}{\Gamma(\ell)}\int_a^x (x-\lambda)^{\ell-1}f(\lambda)\mathrm{d}\lambda,\quad (x>a), \tag{7}$$

$$\left(\mathfrak{I}_{b-}^{\ell}f\right)(x)=\frac{1}{\Gamma(\ell)}\int_x^b (\lambda-x)^{\ell-1}f(\lambda)\mathrm{d}\lambda,\quad (x<b), \tag{8}$$

where $f(x)\in\mathscr{L}(a,b),\ell\in\mathbb{C}$, and $\mathfrak{R}(\ell)>0$.

Definition 3. For $f(x)\in\mathscr{L}(a,b);\ell\in\mathbb{C},\mathfrak{R}(\ell)>0$ and $n=[\mathfrak{R}(\ell)]+1$, the Riemann–Liouville fractional differential operators $\mathscr{D}_{a+}^{\ell}$ are defined by (see, e.g., [13])

$$\left(\mathscr{D}_{a+}^{\ell}f\right)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\mathfrak{I}_{a+}^{n-\ell}f\right)(x). \tag{9}$$

Also, $\mathscr{D}_{a+}^{\ell,\nu}$ of order $0<\ell<1$ and class $0<\nu<1$ with reference to x which is the generalized form of (9) (see [13–15]) is defined as follows:

$$\left(\mathscr{L}_{a+}^{\ell,\nu}f\right)=\left(\mathfrak{I}_{a+}^{\nu(1-\ell)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathfrak{I}_{a+}^{(1-\nu)(1-\ell)}f\right)\right)(x). \tag{10}$$

On setting $\nu=0$ in (10), it reduces $\mathscr{L}_{a+}^{\ell}$ specified in (9) to the fractional differential operator.

We found the following baseline findings for our study.

Lemma 1 (Mathai and Haubold [16]). *If $\ell,u,\in\mathbb{C},\mathfrak{R}(\ell)>0$, $\mathfrak{R}(u)>0$, then*

$$\left(\mathfrak{I}_{a+}^{\ell}(\lambda-a)^{u-1}\right)(x)=\frac{\Gamma(u)}{\Gamma(\ell+\mu)}(x-a)^{\ell+u-1}. \tag{11}$$

Lemma 2 (Srivastava and Manocha [17]). *If a function $f(z)$ is analytic and has a power series representation $f(z)=\sum_{m=0}^{\infty}a_m z^m$ in the disc $|z|<\mathfrak{R}$, then*

$${}_0\mathscr{L}_z^{\ell}\left\{z^{u-1}f(z)\right\}=\frac{\Gamma(u)}{\Gamma(\ell+u)}\sum_{m=0}^{\infty}\frac{a_m(u)_m}{(\ell+u)_m}z^m. \tag{12}$$

Lemma 3 (Srivastava and Tomovski [18]). *Let $x>a,0<\ell<1,0\le\nu\le1,\mathfrak{R}(\vartheta)>0$ and $\mathfrak{R}(\ell)>0$. Then, the subsequent result holds true for $\mathscr{L}_{a+}^{\ell,\nu}f$ as follows:*

$$\mathscr{L}_{a+}^{\ell,\nu}\left[(\lambda-a)^{\vartheta-1}\right](x)=\frac{\Gamma(\vartheta)}{\Gamma(\vartheta-\ell)}(x-a)^{\vartheta-\ell-1}. \tag{13}$$

We also provided the subsequent established facts and rules in this article.

Fubini's theorem (Dirichlet formula) (Samko et al. [15])

$$\int_a^b \mathrm{d}z\int_a^z f(z,t)\mathrm{d}t=\int_a^b \mathrm{d}t\int_t^b f(z,t)\mathrm{d}z. \tag{14}$$

We define the following integral operator in terms of extended generalized Bessel function for $\delta,\omega\in\mathbb{C},\mathfrak{R}(\varsigma)>0$ and $\mathfrak{R}(\vartheta)>0$ for our further analysis of fractional calculus, then the integral operator

$$\left(\mathfrak{I}_{a+;\vartheta,q}^{\omega;\varsigma,\delta;c}f\right)(x)=\int_a^x (x-\lambda)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(x-\lambda)^{\varsigma};p\right)f(\lambda)\mathrm{d}\lambda, \tag{15}$$

where $x>$ a.

If we put $p=0$ to the operator, then (15) reduces

$$\left(\mathfrak{I}_{a+;\vartheta,q}^{\omega;\varsigma,\delta}f\right)(x)=\int_a^x (x-\lambda)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta}\left(\omega(x-\lambda)^{\varsigma}\right)f(\lambda)\mathrm{d}\lambda. \tag{16}$$

If $\omega=0$, then (16) reduces the integral operator to the R-L fractional integral operator described in (7).

2. Integral Operators with Extended Generalized Bessel-Maitland Function in the Kernel

In this part, we consider the composition of the fractional integral and derivative of Riemann–Liouville and the fractional derivative of Hilfer with the extended generalized Bessel-Maitland function defined by (4).

Theorem 1. *Suppose $\varsigma,\vartheta,\delta,c\in\mathbb{C}$, $\mathfrak{R}(\varsigma)>0,\mathfrak{R}(\vartheta)>-1$, $\mathfrak{R}(c)>\mathfrak{R}(\delta)>0$, $(p>0)$, and $q,n\in\mathbb{N}$, then the following result holds true:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n z^{\vartheta}\left(J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma};p\right)\right)=z^{\vartheta-n}J_{\vartheta-n,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma};p\right). \tag{17}$$

Proof. Using (4), we see that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n z^{\vartheta}\left(J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma};p\right)\right)&=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n z^{\vartheta}\sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+qm,c-\delta)(c)_{qm}(-\omega z^{\varsigma})^m}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\\&=\sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+qm,c-\delta)(c)_{qm}(-\omega)^m}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n z^{\varsigma m+\vartheta}.\end{aligned} \tag{18}$$

Using the identity,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} x^{m} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, \quad m \geq n, \tag{19}$$

and after simplifying, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n} z^{\vartheta}\left(J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma}; p\right)\right) = z^{\vartheta-n} \sum_{m=0}^{\infty} \frac{\mathscr{B}_{p}(\delta+qm, c-\delta)(c)_{qm}\left(-\omega z^{\varsigma}\right)^{m}}{B(\delta, c-\delta)\Gamma(\varsigma m+\vartheta-n+1)m!}. \tag{20}$$

Finally, it can be expressed by using (4) again, and we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n} z^{\vartheta}\left(J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma}; p\right)\right) = z^{\vartheta-n} J_{\vartheta-n,q}^{\varsigma,\delta;c}\left(\omega z^{\varsigma}; p\right). \tag{21}$$

□

Corollary 1. *Suppose* $\varsigma, \vartheta, \delta \in \mathbb{C}$, $\mathfrak{R}(\varsigma) > 0, \mathfrak{R}(\vartheta) > -1$, $\mathfrak{R}(\delta) > 0, p = 0, q \in (0,1) \cup \mathbb{N}$, *and* $n \in \mathbb{N}$, *then the following result holds true:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n} z^{\vartheta}\left(J_{\vartheta,q}^{\varsigma,\delta}\left(\omega z^{\varsigma}\right)\right) = z^{\vartheta-n} J_{\vartheta-n,q}^{\varsigma,\delta}\left(\omega z^{\varsigma}\right). \tag{22}$$

Theorem 2. *If* $x > a\,(a \in \mathfrak{R}_{+} = (0,\infty))$, $\delta, \ell, \vartheta, \omega \in \mathbb{C}$, $\mathfrak{R}(\vartheta) > -1, \mathfrak{R}(\ell) > 0, p > 0, q \in \mathbb{N}$, *then*

$$\begin{aligned}&\mathfrak{I}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)\right](x)\\ &\quad = (x-a)^{\vartheta+\ell} J_{\vartheta+\ell,q}^{\varsigma,\delta;c}\left(\omega(x-a)^{\varsigma}; p\right),\end{aligned} \tag{23}$$

$$\begin{aligned}&\mathscr{L}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)\right](x)\\ &\quad = (x-a)^{\vartheta-\ell} J_{\vartheta-\ell,q}^{\varsigma,\delta;c}\left(\omega(x-a)^{\varsigma}; p\right),\end{aligned} \tag{24}$$

$$\begin{aligned}&\mathscr{L}_{a+}^{\ell,v}\left[(\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)\right](x)\\ &\quad = (x-a)^{\vartheta-\ell} J_{\vartheta-\ell,q}^{\varsigma,\delta;c}\left(\omega(x-a)^{\varsigma}; p\right),\end{aligned} \tag{25}$$

Proof

(i) Using (4) and (7), we have

$$\begin{aligned}&\mathfrak{I}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)\right](x)\\ &= \frac{1}{\Gamma(\ell)} \int_{a}^{x} (\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)(x-\lambda)^{\ell-1}\mathrm{d}\lambda\\ &= \sum_{m=0}^{\infty} \frac{\mathscr{B}_{p}(\delta+mq, c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta, c-\delta)\Gamma(\varsigma m+\vartheta+1)m!} \frac{1}{\Gamma(\ell)}\\ &\cdot \int_{a}^{x} (\lambda-a)^{\vartheta+\varsigma m}(x-\lambda)^{\ell-1}\mathrm{d}\lambda\\ &= \sum_{m=0}^{\infty} \frac{\mathscr{B}_{p}(\delta+mq, c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta, c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\\ &\cdot \left(\mathfrak{I}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta+\varsigma m}\right]\right)(x).\end{aligned} \tag{26}$$

By use of (11), we have

$$\begin{aligned}&\mathfrak{I}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta} J_{\vartheta,q}^{\varsigma,\delta;c}\left(\omega(\lambda-a)^{\varsigma}; p\right)\right](x)\\ &= \sum_{m=0}^{\infty} \frac{\mathscr{B}_{p}(\delta+mq, c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta, c-\delta)\Gamma(\varsigma m+\vartheta+1)m!} \frac{\Gamma(\varsigma m+\vartheta+1)}{\Gamma(\varsigma m+\vartheta+\ell+1)}(x-a)^{\varsigma m+\vartheta+\ell}\\ &= (x-a)^{\vartheta+\ell} \sum_{m=0}^{\infty} \frac{\mathscr{B}_{p}(\delta+mq, c-\delta)}{B(\delta, c-\delta)} \frac{(c)_{mq}}{\Gamma(\varsigma m+\vartheta+\ell+1)} \frac{\left(-\omega(x-a)^{\varsigma}\right)^{m}}{m!}\\ &= (x-a)^{\vartheta+\ell} J_{\vartheta+\ell,q}^{\varsigma,\delta;c}\left(\omega(x-a)^{\varsigma}; p\right).\end{aligned} \tag{27}$$

This occupies in the (23) statement.

(ii) On using (9), we have

$$\mathscr{L}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right](x)$$
$$=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\mathfrak{J}_{a+}^{n-\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right]\right)(x), \tag{28}$$

and using (23), this takes the following form:

$$\mathscr{L}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right](x)$$
$$=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left[(x-a)^{\vartheta-\ell+n}J_{\vartheta-\ell+n,q}^{\varsigma,\delta;c}(\omega(x-a)^{\varsigma};p)\right]. \tag{29}$$

Applying (17), we get

$$\mathscr{L}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta-\ell,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right](x)$$
$$=(x-a)^{\vartheta-\ell}J_{\varsigma,\vartheta-\ell}^{\gamma,q;c}(\omega(x-a)^{\varsigma};p). \tag{30}$$

This completes the desired proof (24).

(iii) By using (4), we obtain

$$\mathscr{L}_{a+}^{\ell,\nu}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right](x)$$
$$=\left(\mathscr{L}_{a+}^{\ell,\nu}\left[\sum_{m=0}^{\infty}\frac{\mathscr{B}_{p}(\delta+mq,c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}(\lambda-a)^{\varsigma m+\vartheta}\right]\right)(x)$$
$$=\sum_{m=0}^{\infty}\frac{\mathscr{B}_{p}(\delta+mq,c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\left(\mathscr{L}_{a+}^{\ell,\nu}\left[(\lambda-a)^{\varsigma m+\vartheta}\right]\right)(x). \tag{31}$$

By applying (13), we get

$$\mathscr{L}_{a+}^{\ell,\nu}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p)\right](x)$$
$$=\sum_{m=0}^{\infty}\frac{\mathscr{B}_{p}(\delta+mq,c-\delta)(c)_{mq}(-\omega)^{m}}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\frac{\Gamma(\varsigma m+\vartheta+1)}{\Gamma(\varsigma m+\vartheta-\ell+1)}(x-a)^{\varsigma m+\vartheta-\ell}$$
$$=(x-a)^{\vartheta-\ell}\sum_{m=0}^{\infty}\frac{\mathscr{B}_{p}(\delta+mq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{mq}}{\Gamma(\varsigma m+\vartheta-\ell+1)}\frac{(-\omega(x-a)^{\varsigma})^{m}}{m!}$$
$$=(x-a)^{\vartheta-\ell}J_{\vartheta-\ell,q}^{\varsigma,\delta;c}(\omega(\lambda-a)^{\varsigma};p), \tag{32}$$

which brings in the necessary proof. □

Corollary 2. *If* $x>a\,(a\in\mathfrak{R}_{+}=(0,\infty)),\delta,\ell,\vartheta,\omega\in\mathbb{C}$, $\mathfrak{R}(\vartheta)>-1$, $\mathfrak{R}(\ell)>0$, $p=0,q\in(0,1)\cup\mathbb{N}$, *then Theorem 2 reduces respectively to*

$$\mathfrak{J}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta}(\omega(\lambda-a)^{\varsigma})\right](x)=(x-a)^{\vartheta+\ell}J_{\vartheta+\ell,q}^{\varsigma,\delta}(\omega(x-a)^{\varsigma}),$$
$$\mathscr{L}_{a+}^{\ell}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta}(\omega(\lambda-a)^{\varsigma})\right](x)=(x-a)^{\vartheta-\ell}J_{\vartheta-\ell,q}^{\varsigma,\delta}(\omega(x-a)^{\varsigma}),$$
$$\mathscr{L}_{a+}^{\ell,\nu}\left[(\lambda-a)^{\vartheta}J_{\vartheta,q}^{\varsigma,\delta}(\omega(\lambda-a)^{\varsigma})\right](x)=(x-a)^{\vartheta-\ell}J_{\vartheta-\ell,q}^{\varsigma,\delta}(\omega(x-a)^{\varsigma}). \tag{33}$$

3. Some Properties of the Operator $(\mathfrak{J}_{a+;\vartheta,q}^{\omega;\varsigma,\delta;c}f)(x)$

In this section, we derive several continuity properties of the generalized fractional integral operator.

Theorem 3. *If* $\delta,\omega\in\mathbb{C},\mathfrak{R}(\varsigma)>0,\mathfrak{R}(\vartheta)>-1,p>0,q\in\mathbb{N}$, *and* $\mathfrak{R}(\mu)>0$, *then*

$$\left(\mathfrak{J}_{a+;\vartheta,q}^{\omega;\varsigma,\delta;c}\left[(\lambda-a)^{\mu-1}\right]\right)(x)=(x-a)^{\mu+\vartheta}\Gamma(\mu)J_{\vartheta+\mu,q}^{\varsigma,\delta;c}(\omega(x-a)^{\varsigma};p). \tag{34}$$

Proof. From (4) and (15), we obtain

$$\begin{aligned}\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[(\lambda-a)^{\mu-1}\right]\right)(x) &= \int_a^x (x-\lambda)^{\vartheta}(\lambda-a)^{\mu-1}J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)\mathrm{d}\lambda \\ &= \sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)(c)_{mq}(-\omega)^m}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\left(\int_a^x(\lambda-a)^{\mu-1}(x-\lambda)^{\vartheta+\varsigma m}\mathrm{d}\lambda\right) \\ &= \sum_{m=0}^{\infty}\frac{B_p(\delta+mq,c-\delta)(c)_{mq}(-\omega)^m}{B(\delta,c-\delta)m!}\frac{1}{\Gamma(\varsigma m+\vartheta+1)}\times\int_a^x(\lambda-a)^{\mu-1}(x-\lambda)^{\vartheta+\varsigma m}\mathrm{d}\lambda \\ &= \sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)(c)_{mq}(-\omega)^m}{B(\delta,c-\delta)m!}\mathfrak{I}^{\varsigma m+\vartheta+1}_{a+}\left[(\lambda-a)^{\mu-1}\right](x) \\ &= \sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)(c)_{mq}}{B(\delta,c-\delta)}\frac{(-\omega)^m}{m!}\frac{\Gamma(\mu)}{\Gamma(\varsigma m+\vartheta+\mu+1)}(x-a)^{\varsigma m+\vartheta+\mu} \\ &= (x-a)^{\vartheta+\mu}\Gamma(\mu)J^{\varsigma,\delta;c}_{\vartheta+\mu,q}\left(\omega(x-a)^{\varsigma};p\right).\end{aligned} \tag{35}$$

This completes the desired proof. □

Corollary 3. *If* $\delta,\omega\in\mathbb{C},\mathfrak{R}(\varsigma)>0,\mathfrak{R}(\vartheta)>-1,\mathfrak{R}(\ell)>0$, $q\in(0,1)\cup\mathbb{N}$, *and* $\mathfrak{R}(\mu)>0$, *then*

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta}_{a+;\vartheta,q}\left[(\lambda-a)^{\mu-1}\right]\right)(x)=(x-a)^{\mu+\vartheta}\Gamma(\mu)J^{\varsigma,\delta}_{\vartheta+\mu,q}\left(\omega(x-a)^{\varsigma}\right). \tag{36}$$

Theorem 4. *If* $\delta,\varsigma,\vartheta,\omega,c\in\mathbb{C},\mathfrak{R}(\varsigma)>0,\mathfrak{R}(\vartheta)>-1$, $\mathfrak{R}(c)>0$, $p>0$, *and* $q\in\mathbb{N}$, *then*

$$\left\|\mathfrak{I}^{\omega;\varsigma,\delta,c}_{a+;\vartheta,q}\varphi\right\|_1\le\mathscr{S}\|\varphi\|_1, \tag{37}$$

where

$$\mathscr{S}=(b-a)^{\mathfrak{R}(\vartheta)+1}\sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)\left|(c)_{mq}\right|}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)(\mathfrak{R}(\vartheta)+\mathfrak{R}(\varsigma)m+1)}\frac{|-\omega(b-a)^{\varsigma}|^m}{m!}. \tag{38}$$

Proof. From (4), (6), and (15), we have

$$\begin{aligned}\left\|\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\varphi\right\|_1 &= \int_a^b\left|\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\varphi\right)(x)\right|\mathrm{d}x \\ &= \int_a^b\left|\int_a^x(x-\lambda)^{\vartheta}J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)\varphi(\lambda)\mathrm{d}\lambda\right|\mathrm{d}x.\end{aligned} \tag{39}$$

By exchanging the integration order and using Dirichlet formula (14), we have

$$\begin{aligned}\left\|\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\varphi\right\|_1 &\le \sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)\left|(c)_{mq}\right||(-\omega)^m|}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!} \\ &\quad\times\int_a^b\left[\int_{\lambda}^b(x-\lambda)^{\mathfrak{R}(\vartheta)+\mathfrak{R}(\varsigma)m}\mathrm{d}x\right]|\varphi(\lambda)|\mathrm{d}\lambda.\end{aligned} \tag{40}$$

Setting $u=x-\lambda$, we obtain

$$\left\|\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\varphi\right\|_1\le\sum_{m=0}^{\infty}\frac{\mathscr{B}_p(\delta+mq,c-\delta)\left|(c)_{mq}\right||(-\omega)^m|}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)m!}\int_a^b\left[\frac{u^{\mathfrak{R}(\vartheta)+\mathfrak{R}(\varsigma)m+1}}{\mathfrak{R}(\vartheta)+\mathfrak{R}(\varsigma)m+1}\right]_0^{b-a}\times|\varphi(\lambda)|\mathrm{d}\lambda. \tag{41}$$

This can also be written as

$$\left\|\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\varphi\right\|_1\le\left\{(b-a)^{\mathfrak{R}(\vartheta)+1}\sum_{m=0}^{\infty}\frac{B_p(\delta+mq,c-\delta)\left|(c)_{mq}\right||(-\omega(b-a)^{\varsigma})|^m}{B(\delta,c-\delta)\Gamma(\varsigma m+\vartheta+1)(\mathfrak{R}(\vartheta)+\mathfrak{R}(\varsigma)m+1)m!}\right\}.\int_a^b|\varphi(\lambda)|\mathrm{d}\lambda=\mathscr{S}\|\varphi\|_1. \tag{42}$$

This completes the desired proof. □

Corollary 4. *If* $\delta, \varsigma, \vartheta, \omega, \in \mathbb{C}, \Re(\varsigma) > 0, \Re(\vartheta) > -1,$ *and* $q \in (0,1) \cup \mathbb{N}$, *then*

$$\left\| \mathfrak{I}^{\omega;\varsigma,\delta}_{a+;\vartheta,q} \varphi \right\|_1 \leq \mathfrak{H} \|\varphi\|_1, \tag{43}$$

where

$$\mathfrak{H} = (b-a)^{\Re(\vartheta)+1} \sum_{m=0}^{\infty} \frac{\left|(\delta)_{mq}\right|}{\Gamma(\varsigma m + \vartheta + 1)(\Re(\vartheta) + \Re(\varsigma) m + 1)} \frac{|-\omega(b-a)^{\varsigma}|^m}{m!}. \tag{44}$$

Theorem 5. *If* $\ell, \delta, \varsigma, \vartheta, \omega \in \mathbb{C}, \Re(\varsigma) > 0, \Re(\vartheta) > -1, \Re(\delta) > 0, \Re(\ell) > 0, p > 0, q \in \mathbb{N},$ *and* $x > a$, *for any function* $f \in \mathscr{L}(\varsigma, \vartheta)$, *then the result holds true:*

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta+\ell,q} f\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+} f\right]\right)(x). \tag{45}$$

Proof. From (7) and (15), we have

$$\begin{aligned}
\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) &= \frac{1}{\Gamma(\ell)} \int_a^x (x-\lambda)^{\ell-1} \left[\mathfrak{I}^{\omega;\varsigma,\delta,c}_{a+;\vartheta,q} f\right](\lambda) \mathrm{d}\lambda \\
&= \frac{1}{\Gamma(\ell)} \int_a^x (x-\lambda)^{\ell-1} \left[\int_a^{\lambda} (\lambda-u)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(\lambda-u)^{\varsigma}; p\right) f(u) \mathrm{d}u\right] \mathrm{d}\lambda.
\end{aligned} \tag{46}$$

By interchanging the order of integration and using (14), we have

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) = \int_a^x \left[\frac{1}{\Gamma(\ell)} \int_u^x (x-\lambda)^{\ell-1} (\lambda-u)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(\lambda-u)^{\varsigma}; p\right) \mathrm{d}\lambda\right] \times f(u) \mathrm{d}u. \tag{47}$$

Setting $\lambda - u = \rho$, we obtain

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) = \int_a^x \left[\frac{1}{\Gamma(\ell)} \int_0^{x-u} (x-u-\rho)^{\ell-1} (\rho)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega\rho^{\varsigma}; p\right) \mathrm{d}\rho\right] f(u) \mathrm{d}u. \tag{48}$$

By applying (7) and (23), we get

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) = \int_a^x \left[(x-u)^{\ell+\vartheta} J^{\varsigma,\delta;c}_{\vartheta+\ell,q}\left(\omega(x-u)^{\varsigma}; p\right)\right] f(u) \mathrm{d}u, \tag{49}$$

thus, using (15), we get

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q} f\right]\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta+\ell,q} f\right)(x). \tag{50}$$

To demonstrate the second part, we start from the right side of (45), and using (7) and (15), we have

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x) = \int_a^x (x-\lambda)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)\left[\mathfrak{I}^{\ell}_{a+}f\right](\lambda)\mathrm{d}\lambda$$
$$= \int_a^x \left[(x-\lambda)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)\left(\frac{1}{\Gamma(\ell)}\int_a^{\lambda}(\lambda-u)^{\ell-1}f(u)\mathrm{d}u\right)\right]\mathrm{d}\lambda \tag{51}$$
$$= \int_a^x \left[\frac{1}{\Gamma(\ell)}\int_a^{\lambda}(\lambda-u)^{\ell-1}(x-\lambda)^{\vartheta} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)f(u)\mathrm{d}u\right]\mathrm{d}\lambda.$$

By interchanging the order of integration and using (14), we obtain

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x) = \int_a^x \frac{1}{\Gamma(\ell)}\left[\int_u^x (x-\lambda)^{\vartheta}(\lambda-u)^{\ell-1} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega(x-\lambda)^{\varsigma};p\right)\mathrm{d}\lambda\right]\times f(u)\mathrm{d}u. \tag{52}$$

Setting $x-\lambda=\rho$, we have

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x) = \int_a^x \frac{1}{\Gamma(\ell)}\left[\int_0^{x-u}\rho^{\vartheta}(x-u-\rho)^{\ell-1} J^{\varsigma,\delta;c}_{\vartheta,q}\left(\omega\rho^{\varsigma};p\right)\mathrm{d}\rho\right]f(u)\mathrm{d}u. \tag{53}$$

Again, by using (7) and applying (23), we get

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x) = \int_a^x (x-u)^{\vartheta+\ell} J^{\varsigma,\delta;c}_{\vartheta+\ell,q}\left(\omega(x-u)^{\varsigma};p\right)f(u)\mathrm{d}u. \tag{54}$$

Finally, using (15), we obtain

$$\left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta;c}_{a+;\vartheta+\ell,q}f\right)(x). \tag{55}$$

Thus, (50) and (55) complete the desired proof of (45). □

Corollary 5. *If $\ell,\delta,\varsigma,\vartheta,\omega\in\mathbb{C}$, $\Re(\varsigma)>0$, $\Re(\vartheta)>-1$, $\Re(\delta)>0$, $\Re(\ell)>0$, $q\in(0,1)\cup\mathbb{N}$, and $x>a$, for any function $f\in\mathscr{L}(\varsigma,\vartheta)$, then Theorem 5 takes the form:*

$$\left(\mathfrak{I}^{\ell}_{a+}\left[\mathfrak{I}^{\omega;\varsigma,\delta}_{a+;\vartheta,q}f\right]\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta}_{a+;\vartheta+\ell,q}f\right)(x) = \left(\mathfrak{I}^{\omega;\varsigma,\delta}_{a+;\vartheta,q}\left[\mathfrak{I}^{\ell}_{a+}f\right]\right)(x). \tag{56}$$

4. Concluding Remark and Discussion

The newly defined integral operators involving the extended generalized Bessel-Maitland function is investigated here. Various special cases of the paper's related results may be analyzed by taking appropriate values of the relevant parameters. For example, as given in remarks (i) and (ii), we obtain the undeniable result due to Gauhar et al. [19, 20]. For a number of other special cases, we refer to [21] and leave the findings to interested readers.

Data Availability

No data used to support the study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to the present investigation. All authors read and approved the final manuscript.

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