## Fuzzy Nonlinear Programming with Applications in Decision Making

Euest Editars: Tin-Chih Taly Chen, Deng-Feng Li, T. Warren Liac, and Yi-Chi Wang

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## Journal of Applied Mathematics

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Guest Editors: Tin-Chih Toly Chen, Deng-Feng Li, T. Warren Liao, and Yi-Chi Wang

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## Contents

Fuzzy Nonlinear Programming with Applications in Decision Making, Tin-Chih Toly Chen, Deng-Feng Li, T. Warren Liao, and Yi-Chi Wang
Volume 2014, Article ID 509157, 2 pages

Multiple Attribute Decision Making Based on Hesitant Fuzzy Einstein Geometric Aggregation
Operators, Xiaoqiang Zhou and Qingguo Li
Volume 2014, Article ID 745617, 14 pages

Multiproject Resources Allocation Model under Fuzzy Random Environment and Its Application to Industrial Equipment Installation Engineering, Jun Gang, Jiuping Xu, and Yinfeng Xu Volume 2013, Article ID 818731, 19 pages

Optimality Condition and Wolfe Duality for Invex Interval-Valued Nonlinear Programming Problems, Jianke Zhang
Volume 2013, Article ID 641345, 11 pages

Genetic Algorithm Optimization for Determining Fuzzy Measures from Fuzzy Data, Chen Li, Gong Zeng-tai, and Duan Gang Volume 2013, Article ID 542153, 11 pages

A Biobjective Fuzzy Integer-Nonlinear Programming Approach for Creating an Intelligent Location-Aware Service, Yu-Cheng Lin and Toly Chen Volume 2013, Article ID 423415, 11 pages

Correlation Measures of Dual Hesitant Fuzzy Sets, Lei Wang, Mingfang Ni, and Lei Zhu Volume 2013, Article ID 593739, 12 pages

Stationary Points-I: One-Dimensional p-Fuzzy Dynamical Systems, Joao de Deus M. Silva, Jefferson Leite, Rodney C. Bassanezi, and Moiseis S. Cecconello
Volume 2013, Article ID 495864, 11 pages

Generalized Fuzzy Bonferroni Harmonic Mean Operators and Their Applications in Group Decision Making, Jin Han Park and Eun Jin Park
Volume 2013, Article ID 604029, 14 pages

Parametric Extension of the Most Preferred OWA Operator and Its Application in Search Engine's
Rank, Xiuzhi Sang and Xinwang Liu
Volume 2013, Article ID 273758, 10 pages

A Fuzzy Nonlinear Programming Approach for Optimizing the Performance of a Four-Objective Fluctuation Smoothing Rule in a Wafer Fabrication Factory, Horng-Ren Tsai and Toly Chen Volume 2013, Article ID 720607, 15 pages

## Editorial

# Fuzzy Nonlinear Programming with Applications in Decision Making 

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Fuzzy set theory provides an intuitive and computationally simple way to deal with uncertain and ambiguous properties. Decisions represented in terms of fuzzy sets also offer flexibility in their implementations. On the other hand, nonlinear programming relaxes the strict assumptions and constraints in linear programming and hence is more practicable to handle many decision making problems, which are better represented in the form of nonlinear programming models. Combining fuzzy set theory with nonlinear programming enables it to address the issue of uncertain parameters; the resulting fuzzy nonlinear programming problem involving fuzzy parameters can be viewed as an even more practicable approach than the conventional nonfuzzy one. In light of advanced computing systems, fuzzy nonlinear programming becomes one of the most promising approaches to solve practical application problems. The purpose of this special issue is to provide recent advances in developing fuzzy nonlinear programming methods and their applications to practicable and flexible decision making. The target audiences are researchers in fuzzy mathematics, operations research, information management, and system engineering, as well as practicing managers/engineers. After a strict review process, ten articles from researchers around the world were finally accepted. A brief summary of each is described below.
J. d. D. M. Silva et al. discussed the properties of pfuzzy dynamical systems that are variational systems of which dynamic behaviors are regulated by a Mamdani-type fuzzy system. They used a case study to illustrate a 1-dimensional p-fuzzy dynamical system and presented some theorems on
the conditions of existence and uniqueness of stationary points.
H.-R. Tsai and T. Chen proposed a fuzzy nonlinear programming (FNLP) approach for optimizing the scheduling performance of a four-factor fluctuation smoothing rule in a wafer fabrication factory. The proposed methodology considered the uncertainty in the remaining cycle time of a job and optimized a fuzzy four-factor fluctuation-smoothing rule to sequence the jobs in front of each machine. The fuzzy four-factor fluctuation-smoothing rule had five adjustable parameters, the optimization of which resulted in a FNLP problem.

Most preferred ordered weighted average (MP-OWA) operators are a new kind of neat OWA operators in the aggregation operator families. It considers the preferences of all alternatives across the criteria and provides unique aggregation characteristics in decision making. X. Sang and X. Liu established the parametric form of the MP-OWA operator to deal with uncertain preference information, including the most commonly used maximum, minimum, and average aggregation operators. A special form of the parametric MPOWA operator with the power function was also proposed in their study.

The Bonferroni mean (BM) operator is an important aggregation technique which reflects the correlation of aggregated arguments. J. H. Park and E. J. Park studied the desirable properties of the fuzzy Bonferroni harmonic mean (FBHM) operator and the fuzzy ordered Bonferroni harmonic mean (FOBHM) operator. To consider the correlation of any three
aggregated arguments, J. H. Park and E. J. Park developed the generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and the generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator. Based on these new operators, they proposed a new approach to multiple-attribute group decision making and illustrated the new approach with a practical example.
J. Gang et al. studied a multiproject resource allocation problem in a bilevel organization. To solve the problem, a bilevel multiproject resource allocation model with fuzzy or random variables was established, in which the opinions of decision makers from two levels were considered. On the upper level, the company manager aims to allocate the company's resources to multiple projects to minimize the total costs. On the lower level, each project manager attempts to schedule their resource-constrained projects, with minimization of the project duration as the main objective. To search for the optimal solution, a hybrid approach of adaptive particle swarm optimization, adaptive hybrid genetic algorithm, and fuzzy random simulation was proposed.
Y.-C. Lin and T. Chen created an intelligent locationaware service, in which a timely service was recommended to the user without changing the user's pace. To the user, there were two goals to achieve: one is to reach the service location just in time and the other is to get to the destination as soon as possible. To optimize the two objectives at the same time and to consider the uncertainty in the dynamic environment, a biobjective fuzzy integer-nonlinear programming problem was formulated and solved.
L. Wang et al. defined the correlation measures of dual hesitant fuzzy sets (DHFSs) and discussed their properties. A direct transfer algorithm for complex matrix synthesis was also proposed that helps to construct an equivalent correlation matrix for clustering DHFSs. L. Wang et al. also proved that the direct transfer algorithm is equivalent to the transfer closure algorithm, but its asymptotic time complexity and space complexity are superior to the latter.
J. Zhang extended the concepts of preinvex and invex to the interval-valued functions. Under the assumption of invexity, the Karush-Kuhn-Tucker optimality of sufficient and necessary conditions was also proved for interval-valued nonlinear programming problems. J. Zhang also proved the Wolfe duality theorem for invex interval-valued nonlinear programming problems, based on the concept of having no duality gaps in weak and strong senses.

Fuzzy measures and fuzzy integrals have been successfully used in many real applications. However, the determination of fuzzy measures is still a challenging task. L. Chen, Z.-T. Gong, and G. Duan proposed a more generalized type of fuzzy measures by means of genetic algorithm and the Choquet integral. In this paper, they firstly defined the $\sigma-\lambda$ rule then defined and characterized the Choquet integrals of interval-valued functions and fuzzy functions based on the $\sigma-\lambda$ rule.
X. Zhou and Q. Li defined an accuracy function of hesitant fuzzy elements (HFEs) and developed a new method to compare two HFEs. Then, based on Einstein operators, they gave some new operational laws on HFEs and some desirable properties of these operations. Several new hesitant fuzzy
aggregation operators, including the hesitant fuzzy Einstein weighted geometric $\left(\mathrm{HFEWG}_{\varepsilon}\right)$ operator and the hesitant fuzzy Einstein ordered weighted geometric ( $\mathrm{HFEOWG}_{\varepsilon}$ ) operator, were also developed as the extensions of the weighted geometric operator and the ordered weighted geometric (OWG) operator with hesitant fuzzy information, respectively.

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Tin-Chih Toly Chen<br>Deng-Feng Li<br>T. Warren Liao<br>Yi-Chi Wang

# Multiple Attribute Decision Making Based on Hesitant Fuzzy Einstein Geometric Aggregation Operators 

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#### Abstract

We first define an accuracy function of hesitant fuzzy elements (HFEs) and develop a new method to compare two HFEs. Then, based on Einstein operators, we give some new operational laws on HFEs and some desirable properties of these operations. We also develop several new hesitant fuzzy aggregation operators, including the hesitant fuzzy Einstein weighted geometric $\left(\mathrm{HFEWG}_{\varepsilon}\right)$ operator and the hesitant fuzzy Einstein ordered weighted geometric ( $\mathrm{HFEWG}_{\varepsilon}$ ) operator, which are the extensions of the weighted geometric operator and the ordered weighted geometric (OWG) operator with hesitant fuzzy information, respectively. Furthermore, we establish the connections between the proposed and the existing hesitant fuzzy aggregation operators and discuss various properties of the proposed operators. Finally, we apply the $\mathrm{HFEWG}_{\varepsilon}$ operator to solve the hesitant fuzzy decision making problems.


## 1. Introduction

Atanassov [1, 2] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a nonmembership function. It is more suitable to deal with fuzziness and uncertainty than the ordinary fuzzy set proposed by Zadeh [3] characterized by one membership function. Information aggregation is an important research topic in many applications such as fuzzy logic systems and multiattribute decision making as discussed by Chen and Hwang [4]. Research on aggregation operators with intuitionistic fuzzy information has received increasing attention as shown in the literature. Xu [5] developed some basic arithmetic aggregation operators based on intuitionistic fuzzy values (IFVs), such as the intuitionistic fuzzy weighted averaging operator and intuitionistic fuzzy ordered weighted averaging operator, while Xu and Yager [6] presented some basic geometric aggregation operators for aggregating IFVs, including the intuitionistic fuzzy weighted geometric operator and intuitionistic fuzzy ordered weighted geometric operator. Based on these basic aggregation operators proposed in [6] and [5],
many generalized intuitionistic fuzzy aggregation operators have been investigated [5-30]. Recently, Torra and Narukawa [31] and Torra [32] proposed the hesitant fuzzy set (HFS), which is another generalization form of fuzzy set. The characteristic of HFS is that it allows membership degree to have a set of possible values. Therefore, HFS is a very useful tool in the situations where there are some difficulties in determining the membership of an element to a set. Lately, research on aggregation methods and multiple attribute decision making theories under hesitant fuzzy environment is very active, and a lot of results have been obtained for hesitant fuzzy information [33-43]. For example, Xia et al. [38] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [37] gave several series of hesitant fuzzy aggregation operators with the help of quasiarithmetic means. Wei [35] explored several hesitant fuzzy prioritized aggregation operators and applied them to hesitant fuzzy decision making problems. Zhu et al. [43] investigated the geometric Bonferroni mean combining the Bonferroni mean and the geometric mean under hesitant fuzzy environment. Xia and Xu [36] presented some hesitant fuzzy operational
laws based on the relationship between the HFEs and the IFVs. They also proposed a series of aggregation operators, such as hesitant fuzzy weighted geometric (HFWG) operator and hesitant fuzzy ordered weighted geometric (HFOWG) operator. Furthermore, they applied the proposed aggregation operators to solve the multiple attribute decision making problems.

Note that all aggregation operators introduced previously are based on the algebraic product and algebraic sum of IFVs (or HFEs) to carry out the combination process. However, the algebraic operations include algebraic product and algebraic sum, which are not the unique operations that can be used to perform the intersection and union. There are many instances of various $t$-norms and $t$-conorms families which can be chosen to model the corresponding intersections and unions, among which Einstein product and Einstein sum are good alternatives for they typically give the same smooth approximation as algebraic product and algebraic sum, respectively. For intuitionistic fuzzy information, Wang and Liu [10, 11, 44] and Wei and Zhao [30] developed some new intuitionistic fuzzy aggregation operators with the help of Einstein operations. For hesitant fuzzy information, however, it seems that in the literature there is little investigation on aggregation techniques using the Einstein operations to aggregate hesitant fuzzy information. Therefore, it is necessary to develop some hesitant fuzzy information aggregation operators based on Einstein operations.

The remainder of this paper is structured as follows. In Section 2, we briefly review some basic concepts and operations related to IFS and HFS. we also define an accuracy function of HFEs to distinguish the two HFEs having the same score values, based on which we give the new comparison laws on HFEs. In Section 3, we present some new operations for HFEs and discuss some basic properties of the proposed operations. In Section 4, we develop some novel hesitant fuzzy geometric aggregation operators with the help of Einstein operations, such as the $\mathrm{HFEWG}_{\varepsilon}$ operator and the $\mathrm{HFEOWG}_{\varepsilon}$ operator, and we further study various properties of these operators. Section 5 gives an approach to solve the multiple attribute hesitant fuzzy decision making problems based on the $\mathrm{HFEOWG}_{\varepsilon}$ operator. Finally, Section 6 concludes the paper.

## 2. Preliminaries

In this section, we briefly introduce Einstein operations and some notions of IFS and HFS. Meantime, we define an accuracy function of HFEs and redefine the comparison laws between two HFEs.
2.1. Einstein Operations. Since the appearance of fuzzy set theory, the set theoretical operators have played an important role and received more and more attention. It is well known that the $t$-norms and $t$-conorms are the general concepts including all types of the specific operators, and they satisfy the requirements of the conjunction and disjunction operators, respectively. There are various $t$-norms and $t$-conorms families that can be used to perform the corresponding intersections and unions. Einstein sum $\oplus_{\varepsilon}$ and Einstein product
$\otimes_{\varepsilon}$ are examples of $t$-conorms and $t$-norms, respectively. They are called Einstein operations and defined as [45]

$$
\begin{array}{r}
x \otimes_{\varepsilon} y=\frac{x \cdot y}{1+(1-x) \cdot(1-y)}, \quad x \otimes_{\varepsilon} y=\frac{x+y}{1+x \cdot y}  \tag{1}\\
\forall x, y \in[0,1]
\end{array}
$$

2.2. Intuitionistic Fuzzy Set. Atanassov [1, 2] generalized the concept of fuzzy set [3] and defined the concept of intuitionistic fuzzy set (IFS) as follows.

Definition 1. Let $U$ be fixed an IFSA on $U$ is given by;

$$
\begin{equation*}
A=\left\{\left\langle x, \mu_{A}(x), v_{A}(x)\right\rangle \mid x \in U\right\} \tag{2}
\end{equation*}
$$

where $\mu_{A}: U \rightarrow[0,1]$ and $\nu_{A}: U \rightarrow[0,1]$, with the condition $0 \leq \mu_{A}(x)+\nu_{A}(x) \leq 1$ for all $x \in U$. Xu [5] called $\widetilde{a}=\left(\mu_{\widetilde{a}}, v_{\tilde{a}}\right)$ an IFV.

For IFVs, Wang and Liu [11] introduced some operations as follows.

Let $\lambda>0, \widetilde{a}_{1}=\left(\mu_{\tilde{a}_{1}}, v_{\tilde{a}_{1}}\right)$ and $\widetilde{a}_{2}=\left(\mu_{\tilde{a}_{2}}, v_{\tilde{a}_{2}}\right)$ be two IFVs; then
(1)
) $\widetilde{a}_{1} \otimes_{\varepsilon} \widetilde{a}_{2}=\left(\frac{\mu_{\widetilde{a}_{1}}+\mu_{\tilde{a}_{2}}}{1+\mu_{\tilde{a}_{1}} \mu_{\tilde{a}_{2}}}, \frac{v_{\widetilde{a}_{1}} \nu_{\tilde{a}_{2}}}{1+\left(1-v_{\tilde{a}_{1}}\right)\left(1-v_{\tilde{a}_{2}}\right)}\right)$
(2)

$$
\widetilde{a}_{1} \otimes_{\varepsilon} \widetilde{a}_{2}=\left(\frac{\mu_{\tilde{a}_{1}} \mu_{\widetilde{a}_{2}}}{1+\left(1-\mu_{\tilde{a}_{1}}\right)\left(1-\mu_{\tilde{a}_{2}}\right)}, \frac{v_{\widetilde{a}_{1}}+v_{\widetilde{a}_{2}}}{1+v_{\widetilde{a}_{1}} v_{\widetilde{a}_{2}}}\right)
$$

(3)

$$
\begin{equation*}
\tilde{a}_{1}^{\wedge_{\varepsilon} \lambda}=\left(\frac{2 v_{\tilde{a}_{1}}^{\lambda}}{\left(2-v_{\tilde{a}_{1}}\right)^{\lambda}+v_{\tilde{a}_{1}}^{\lambda}}, \frac{\left(1+\mu_{\tilde{a}_{1}}\right)^{\lambda}-\left(1-\mu_{\tilde{a}_{1}}\right)^{\lambda}}{\left(1+\mu_{\tilde{a}_{1}}\right)^{\lambda}+\left(1-\mu_{\tilde{a}_{1}}\right)^{\lambda}}\right) \tag{3}
\end{equation*}
$$

2.3. Hesitant Fuzzy Set. As another generalization of fuzzy set, HFS was first introduced by Torra and Narukawa [31, 32].

Definition 2. Let $X$ be a reference set; an HFS on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$.

To be easily understood, Xia and Xu use the following mathematical symbol to express the HFS:

$$
\begin{equation*}
H=\left\{\left.\frac{h_{H}(x)}{x} \right\rvert\, x \in X\right\} \tag{4}
\end{equation*}
$$

where $h_{H}(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $H$. For convenience, Xu and Xia [40] called $h_{H}(x)$ a hesitant fuzzy element (HFE).

Let $h$ be an HFE, $h^{-}=\min \{\gamma \mid \gamma \in h\}$, and $h^{+}=\max \{\gamma \mid$ $\gamma \in h\}$. Torra and Narukawa $[31,32]$ define the IFV $A_{\text {env }}(h)$ as the envelope of $h$, where $A_{\text {env }}(h)=\left(h^{-}, 1-h^{+}\right)$.

Let $\alpha>0, h_{1}$ and $h_{2}$ be two HFEs. Xia and Xu [36] defined some operations as follows:
(4)

(5)

$$
h_{1} \bigotimes h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1} \gamma_{2}\right\}
$$

(6) $\alpha h=\bigcup_{\gamma \in h}\left\{\gamma^{\alpha}\right\}$
(7) $h^{\alpha}=\bigcup_{\gamma \in h}\left\{1-(1-\gamma)^{\alpha}\right\}$.

In [36], Xia and Xu defined the score function of an HFE $h$ to compare the HFEs and gave the comparison laws.

Definition 3. Let $h$ be an HFE; $s(h)=(1 / n(h)) \sum_{\gamma \in h} \gamma$ is called the score function of $h$, where $n(h)$ is the number of values of $h$. For two HFEs $h_{1}$ and $h_{2}$, if $s\left(h_{1}\right)>s\left(h_{2}\right)$, then $h_{1}>h_{2}$; if $s\left(h_{1}\right)=s\left(h_{2}\right)$, then $h_{1}=h_{2}$.

From Definition 3, it can be seen that all HFEs are regarded as the same if their score values are equal. In hesitant fuzzy decision making process, however, we usually need to compare two HFEs for reordering or ranking. In the case where two HFEs have the same score values, they can not be distinguished by Definition 3. Therefore, it is necessary to develop a new method to overcome the difficulty.

For an IFV, Hong and Choi [46] showed that the relation between the score function and the accuracy function is similar to the relation between mean and variance in statistics. From Definition 3, we know that the score value of HFE $h$ is just the mean of the values in $h$. Motivated by the idea of Hong and Choi [46], we can define the accuracy function of HFE $h$ by using the variance of the values in $h$.

Definition 4. Let $h$ be an HFE; $k(h)=1-$ $\sqrt{(1 / n(h)) \sum_{\gamma \in h}(\gamma-s(h))^{2}}$ is called the accuracy function of $h$, where $n(h)$ is the number of values in $h$ and $s(h)$ is the score function of $h$.

It is well known that an efficient estimator is a measure of the variance of an estimate's sampling distribution in statistics: the smaller the variance, the better the performance of the estimator. Motivated by this idea, it is meaningful and appropriate to stipulate that the higher the accuracy degree of HFE, the better the HFE. Therefore, in the following, we develop a new method to compare two HFEs, which is based on the score function and the accuracy function, defined as follows.

Definition 5. Let $h_{1}$ and $h_{2}$ be two HFEs and let $s(\cdot)$ and $k(\cdot)$ be the score function and accuracy function of HFEs, respectively. Then
(1) if $s\left(h_{1}\right)<s\left(h_{2}\right)$, then $h_{1}$ is smaller than $h_{2}$, denoted by $h_{1}<h_{2}$;
(2) if $s\left(h_{1}\right)=s\left(h_{2}\right)$, then
(i) if $k\left(h_{1}\right)<k\left(h_{2}\right)$, then $h_{1}$ is smaller than $h_{2}$, denoted by $h_{1}<h_{2}$;
(ii) if $k\left(h_{1}\right)=k\left(h_{2}\right)$, then $h_{1}$ and $h_{2}$ represent the same information, denoted by $h_{1} \doteq h_{2}$. In particular, if $\gamma_{1}=\gamma_{2}$ for any $\gamma_{1} \in h_{1}$ and $\gamma_{2} \in h_{2}$, then $h_{1}$ is equal to $h_{2}$, denoted by $h_{1}=h_{2}$.

Example 6. Let $h_{1}=\{0.5\}, h_{2}=\{0.1,0.9\}, h_{3}=\{0.3,0.7\}$, $h_{4}=\{0.1,0.3,0.7,0.9\}, h_{5}=\{0.2,0.4,0.6,0.8\}$, and $h_{6}=$ $\{0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9\}$; then $s\left(h_{1}\right)=s\left(h_{2}\right)=$ $s\left(h_{3}\right)=s\left(h_{4}\right)=s\left(h_{5}\right)=s\left(h_{6}\right)=0.5, k\left(h_{1}\right)=1, k\left(h_{2}\right)=0.6$, $k\left(h_{3}\right)=0.8, k\left(h_{4}\right)=0.6838, k\left(h_{5}\right)=0.7764$, and $k\left(h_{6}\right)=$ 0.7418. By Definition 5, we have $h_{1} \succ h_{3}>h_{5}>h_{6}>h_{4}>h_{2}$.

## 3. Einstein Operations of Hesitant Fuzzy Sets

In this section, we will introduce the Einstein operations on HFEs and analyze some desirable properties of these operations. Motivated by the operational laws (1)-(3) on IFVs and based on the interconnection between HFEs and IFVs, we give some new operations of HFEs as follows.

Let $\alpha>0, h, h_{1}$, and $h_{2}$ be three HFEs; then
(8)

$$
\begin{align*}
& h_{1} \otimes_{\varepsilon} h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{\gamma_{1}+\gamma_{2}}{1+\gamma_{1} \gamma_{2}}\right\} \\
& h_{1} \otimes_{\varepsilon} h_{2}=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{\gamma_{1} \gamma_{2}}{1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)}\right\}  \tag{6}\\
& h^{\wedge} \alpha=\bigcup_{\gamma \in h}\left\{\frac{2 \gamma^{\alpha}}{(2-\gamma)^{\alpha}+\gamma^{\alpha}}\right\} \tag{10}
\end{align*}
$$

(9)

Proposition 7. Let $\alpha>0, \alpha_{1}>0, \alpha_{2}>0, h, h_{1}$ and $h_{2}$ be three HFEs; then
(1) $h_{1} \otimes_{\varepsilon} h_{2}=h_{2} \otimes_{\varepsilon} h_{1}$,
(2) $\left(h_{1} \otimes_{\varepsilon} h_{2}\right) \otimes_{\varepsilon} h_{3}=h_{1} \otimes_{\varepsilon}\left(h_{2} \otimes_{\varepsilon} h_{3}\right)$,
(3) $\left(h_{1} \otimes_{\varepsilon} h_{2}\right)^{\wedge_{\varepsilon} \alpha}=h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h_{2}^{\wedge_{\varepsilon} \alpha}$,
(4) $\left(h^{\wedge_{\varepsilon} \alpha_{1}}\right)^{\wedge_{\varepsilon} \alpha_{2}}=h^{\wedge_{\varepsilon}\left(\alpha_{1} \alpha_{2}\right)}$;
(5) $A_{\text {env }}\left(h^{\wedge_{\varepsilon} \alpha}\right)=\left(A_{\text {env }}(h)\right)^{\wedge_{\varepsilon} \alpha}$,
(6) $A_{\text {env }}\left(h_{1} \otimes_{\varepsilon} h_{2}\right)=A_{\text {env }}\left(h_{1}\right) \otimes_{\varepsilon} A_{e n v}\left(h_{2}\right)$.

Proof. (1) It is trivial.
(2) By the operational law (9), we have

$$
\begin{aligned}
& \left(h_{1} \otimes_{\varepsilon} h_{2}\right) \otimes_{\varepsilon} h_{3} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\{ \\
& \\
& \\
& \\
& \quad \times\left(\left(\gamma_{1} \gamma_{2} /\left(1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)\right) \gamma_{3}\right) \\
& \\
& \left.\left.\quad \times\left(1-\gamma_{3}\right)\right)^{-1}\right\}
\end{aligned}
$$

$$
\left.\begin{array}{rl}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\left(\gamma_{1} \gamma_{2} \gamma_{3}\right) \times(1\right. & +\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right) \\
& +\left(1-\gamma_{1}\right)\left(1-\gamma_{3}\right)+\left(1-\gamma_{2}\right) \\
& \left.\left.\times\left(1-\gamma_{3}\right)\right)^{-1}\right\} \\
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \gamma_{3} \in h_{3}}\left\{\left(\gamma_{1}\left(\gamma_{2} \gamma_{3} /\left(1+\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)\right)\right)\right)\right. \\
& \times\left(1+\left(1-\gamma_{1}\right)\right. \\
& \left.\left.\times\left(1-\left(\gamma_{2} \gamma_{3} /\left(1+\left(1-\gamma_{2}\right)\left(1-\gamma_{3}\right)\right)\right)\right)\right)^{-1}\right\}
\end{array}\right\}
$$

(3) Let $h=h_{1} \otimes_{\varepsilon} h_{2}$; then $h=h_{1} \otimes_{\varepsilon} h_{2}=$ $\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\gamma_{1} \gamma_{2} /\left(1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)\right\}$

$$
\begin{align*}
& \left(h_{1} \otimes_{\varepsilon} h_{2}\right)^{\wedge_{\varepsilon} \alpha} \\
& =h^{\wedge_{\varepsilon} \alpha}=\bigcup_{\gamma \in h}\left\{\frac{2 \gamma^{\alpha}}{(2-\gamma)^{\alpha}+\gamma^{\alpha}}\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\left(2\left(\gamma_{1} \gamma_{2} /\left(1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)\right)^{\alpha}\right)\right. \\
& \times\left(\left(2-\left(\gamma_{1} \gamma_{2} /\left(1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)\right)\right)^{\alpha}\right. \\
& \left.\left.+\left(\gamma_{1} \gamma_{2} /\left(1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)\right)\right)^{\alpha}\right)^{-1}\right\} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{2\left(\gamma_{1} \gamma_{2}\right)^{\alpha}}{\left(4-2 \gamma_{1}-2 \gamma_{2}+\gamma_{1} \gamma_{2}\right)^{\alpha}+\left(\gamma_{1} \gamma_{2}\right)^{\alpha}}\right\} \text {, } \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{2\left(\gamma_{1} \gamma_{2}\right)^{\alpha}}{\left(2-\gamma_{1}\right)^{\alpha}\left(2-\gamma_{2}\right)^{\alpha}+\left(\gamma_{1} \gamma_{2}\right)^{\alpha}}\right\} . \tag{8}
\end{align*}
$$

Since $h_{1}^{\wedge_{\varepsilon} \alpha}=\bigcup_{\gamma_{1} \in h}\left\{2 \gamma_{1}^{\alpha} /\left(\left(2-\gamma_{1}\right)^{\alpha}+\gamma_{1}^{\alpha}\right)\right\}$ and $h_{2}^{\wedge_{\varepsilon} \alpha}=$ $\bigcup_{\gamma_{2} \in h}\left\{2 \gamma_{2}^{\alpha} /\left(\left(2-\gamma_{2}\right)^{\alpha}+\gamma_{2}^{\alpha}\right)\right\}$, then

$$
\left.\begin{array}{rl}
h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h_{2}^{\wedge_{\varepsilon} \alpha} \\
= & \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\{ \\
& \cdot\left(\left(2 \gamma_{1}^{\alpha} /\left(\left(2-\gamma_{1}\right)^{\alpha}+\gamma_{1}^{\alpha}\right)\right)\right. \\
& \times\left(1+\left(1-\left(2 \gamma_{1}^{\alpha} /\left(\left(2-\gamma_{1}\right)^{\alpha}+\gamma_{1}^{\alpha}\right)\right)\right)\right. \\
& \left.\left.\quad \times\left(1-\left(2 \gamma_{2}^{\alpha} /\left(\left(2-\gamma_{2}^{\alpha}\right)^{\alpha}+\gamma_{2}^{\alpha}\right)\right)\right)\right)^{-1}\right\}
\end{array}\right\} \begin{aligned}
& \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{2\left(\gamma_{1} \gamma_{2}\right)^{\alpha}}{\left.\left(2-\gamma_{1}\right)^{\alpha}\left(2-\gamma_{2}\right)^{\alpha}+\left(\gamma_{1} \gamma_{2}\right)^{\alpha}\right\} .}\right.
\end{aligned}
$$

Thus $\left(h_{1} \otimes_{\varepsilon} h_{2}\right)^{\wedge_{\varepsilon} \alpha}=h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h_{2}^{\wedge_{\varepsilon} \alpha}$.
(4) Since $h^{\wedge_{\varepsilon} \alpha_{1}}=\bigcup_{\gamma \in h}\left\{2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right\}$, then

$$
\begin{align*}
& \left(h^{\wedge_{\varepsilon} \alpha_{1}}\right)^{\wedge_{\varepsilon} \alpha_{2}} \\
& =\bigcup_{\gamma \in h}\left\{\left(2\left(2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right)^{\alpha_{2}}\right)\right. \\
& \quad \times\left(\left(2-\left(2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right)\right)^{\alpha 2}\right. \\
& \left.\left.\quad \quad+\left(2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right)^{\alpha_{2}}\right)^{-1}\right\}  \tag{10}\\
& =\bigcup_{\gamma \in h}\left\{\frac{2 \gamma^{\left(\alpha_{1} \alpha_{2}\right)}}{\left.(2-\gamma)^{\left(\alpha_{1} \alpha_{2}\right)}+\gamma^{\left(\alpha_{1} \alpha_{2}\right)}\right\}}\right\} \\
& =h^{\wedge_{\varepsilon}\left(\alpha_{1} \alpha_{2}\right)} .
\end{align*}
$$

(5) By the definition of the envelope of an HFE and the operation laws (3) and (10), we have

$$
\begin{aligned}
& \left(A_{\text {env }}(h)\right)^{\wedge_{\varepsilon} \alpha} \\
& \quad=\left(h^{-}, 1-h^{+}\right)^{\wedge_{\varepsilon} \alpha} \\
& \quad=\left(\frac{2\left(h^{-}\right)^{\alpha}}{\left(2-h^{-}\right)^{\alpha}+\left(h^{-}\right)^{\alpha}}, \frac{\left[1+\left(1-h^{+}\right)\right]^{\alpha}-\left[1-\left(1-h^{+}\right)\right]^{\alpha}}{\left[1+\left(1-h^{+}\right)\right]^{\alpha}+\left[1-\left(1-h^{+}\right)\right]^{\alpha}}\right) \\
& \quad=\left(\frac{2\left(h^{-}\right)^{\alpha}}{\left(2-h^{-}\right)^{\alpha}+\left(h^{-}\right)^{\alpha}}, \frac{\left(2-h^{+}\right)^{\alpha}-\left(h^{+}\right)^{\alpha}}{\left(2-h^{+}\right)^{\alpha}+\left(h^{+}\right)^{\alpha}}\right) . \\
& A_{\text {env }}\left(h^{\wedge_{\varepsilon} \alpha}\right)
\end{aligned}
$$

$$
=A_{\mathrm{env}}\left(\bigcup_{\gamma \in h}\left\{\frac{2 \gamma^{\alpha}}{(2-\gamma)^{\alpha}+\gamma^{\alpha}}\right\}\right)
$$

$$
=\left(\frac{2\left(h^{-}\right)^{\alpha}}{\left(2-h^{-}\right)^{\alpha}+\left(h^{-}\right)^{\alpha}}, 1-\frac{2\left(h^{+}\right)^{\alpha}}{\left(2-h^{+}\right)^{\alpha}+\left(h^{+}\right)^{\alpha}}\right)
$$

$$
\begin{equation*}
=\left(\frac{2\left(h^{-}\right)^{\alpha}}{\left(2-h^{-}\right)^{\alpha}+\left(h^{-}\right)^{\alpha}}, \frac{\left(2-h^{+}\right)^{\alpha}-\left(h^{+}\right)^{\alpha}}{\left(2-h^{+}\right)^{\alpha}+\left(h^{+}\right)^{\alpha}}\right) . \tag{11}
\end{equation*}
$$

Thus, $A_{\text {env }}\left(h^{\wedge_{\varepsilon} \alpha}\right)=\left(A_{\text {env }}(h)\right)^{\wedge_{\varepsilon} \alpha}$.
(6) By the definition of the envelope of an HFE and the operation laws (2) and (9), we have

$$
\begin{aligned}
& A_{\text {env }}\left(h_{1}\right) \otimes_{\varepsilon} A_{\text {env }}\left(h_{2}\right) \\
& \quad=\left(h_{1}^{-}, 1-h_{1}^{+}\right) \otimes_{\varepsilon}\left(h_{2}^{-}, 1-h_{2}^{+}\right) \\
& \quad=\left(\frac{h_{1}^{-} h_{2}^{-}}{1+\left(1-h_{1}^{-}\right)\left(1-h_{2}^{-}\right)}, \frac{\left(1-h_{1}^{+}\right)+\left(1-h_{2}^{+}\right)}{1+\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)}\right)
\end{aligned}
$$

$$
\begin{align*}
& A_{\text {env }}\left(h_{1} \otimes_{\varepsilon} h_{2}\right) \\
&=A_{\text {env }}\left(\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{\gamma_{1} \gamma_{2}}{1+\left(1-\gamma_{1}\right)\left(1-\gamma_{2}\right)}\right\}\right) \\
&=\left(\frac{h_{1}^{-} h_{2}^{-}}{1+\left(1-h_{1}^{-}\right)\left(1-h_{2}^{-}\right)}, 1-\frac{h_{1}^{+} h_{2}^{+}}{1+\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)}\right) \\
&=\left(\frac{h_{1}^{-} h_{2}^{-}}{1+\left(1-h_{1}^{-}\right)\left(1-h_{2}^{-}\right)}, \frac{\left(1-h_{1}^{+}\right)+\left(1-h_{2}^{+}\right)}{1+\left(1-h_{1}^{+}\right)\left(1-h_{2}^{+}\right)}\right) . \tag{12}
\end{align*}
$$

Thus, $A_{\text {env }}\left(h_{1} \otimes_{\varepsilon} h_{2}\right)=A_{\text {env }}\left(h_{1}\right) \otimes_{\varepsilon} A_{\text {env }}\left(h_{2}\right)$.
Remark 8. Let $\alpha_{1}>0, \alpha_{2}>0$, and $h$ be an HFE. It is worth noting that $h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{2}} \doteq h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)}$ does not hold necessarily in general. To illustrate that, an example is given as follows.

Example 9. Let $h=(0.3,0.5), \alpha_{1}=\alpha_{2}=1$; then $h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{2}}=h \otimes_{\varepsilon} h=\bigcup_{\gamma_{i} \in h, \gamma_{j} \in h,(i, j=1,2)}\left\{\gamma_{i} \gamma_{j} /(1+(1-\right.$ $\left.\left.\left.\gamma_{i}\right)\left(1-\gamma_{j}\right)\right)\right\}=(0.0604,0.1111,0.2)$, and $h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)}=$ $h^{\wedge_{\varepsilon} 2}=\bigcup_{\gamma \in h}\left\{2 \gamma^{2} /\left((2-\gamma)^{2}+\gamma^{2}\right)\right\}=(0.0604,0.2)$. Clearly, $s\left(h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{2}}\right)=0.1238<0.1302=s\left(h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)}\right)$. Thus $h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{1}}<h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)}$.

However, if the number of the values in $h$ is only one, that is, $\mathrm{HFE} h$ is reduced to a fuzzy value, then the above result holds.

Proposition 10. Let $\alpha_{1}>0, \alpha_{2}>0$, and $h$ be an HFE, in which the number of the values is only one, that is, $h=\{\gamma\}$; then $h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{2}}=h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)}$.

Proof. Since $h^{\wedge_{\varepsilon} \alpha_{1}}=\bigcup_{\gamma \in h}\left\{2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right\}$ and $h^{\wedge_{\varepsilon} \alpha_{2}}=$ $\bigcup_{\gamma \in h}\left\{2 \gamma^{\alpha_{2}} /\left((2-\gamma)^{\alpha_{2}}+\gamma^{\alpha_{2}}\right)\right\}$, then

$$
\begin{aligned}
& h^{\wedge_{\varepsilon} \alpha_{1}} \otimes_{\varepsilon} h^{\wedge_{\varepsilon} \alpha_{1}} \\
& =\bigcup_{\gamma \in h}\left\{\left(\left(2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right)\right.\right. \\
& \left.\cdot\left(2 \gamma^{\alpha_{2}} /\left((2-\gamma)^{\alpha_{2}}+\gamma^{\alpha_{2}}\right)\right)\right) \\
& \quad \times\left(1+\left(1-\left(2 \gamma^{\alpha_{1}} /\left((2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right)\right)\right)\right. \\
& \left.\left.\quad \times\left(1-\left(2 \gamma^{\alpha_{2}} /\left((2-\gamma)^{\alpha_{2}}+\gamma^{\alpha_{2}}\right)\right)\right)\right)^{-1}\right\} \\
& =\bigcup_{\gamma \in h}\left\{\left(2 \gamma^{\alpha_{1}} \cdot 2 \gamma^{\alpha_{2}}\right) \times\left(\left[(2-\gamma)^{\alpha_{1}}+\gamma^{\alpha_{1}}\right] \cdot\left[(2-\gamma)^{\alpha_{2}}+\gamma^{\alpha_{2}}\right]\right.\right. \\
& \quad+\left[(2-\gamma)^{\alpha_{1}}-\gamma^{\alpha_{1}}\right] \\
& \left.\left.\cdot\left[(2-\gamma)^{\alpha_{2}}-\gamma^{\alpha_{2}}\right]\right)^{-1}\right\}
\end{aligned}
$$

$$
\begin{align*}
& =\bigcup_{\gamma \in h}\left\{\frac{2 \gamma^{\alpha_{1}+\alpha_{2}}}{(2-\gamma)^{\alpha_{1}+\alpha_{2}}+\gamma^{\alpha_{1}+\alpha_{2}}}\right\} \\
& =h^{\wedge_{\varepsilon}\left(\alpha_{1}+\alpha_{2}\right)} \tag{13}
\end{align*}
$$

Proposition 10 shows that it is consistent with the result (iii) in Theorem 2 in the literature [11].

## 4. Hesitant Fuzzy Einstein Geometric Aggregation Operators

The weighted geometric operator [47] and the ordered weighted geometric operator [48] are two of the most common and basic aggregation operators. Since their appearance, they have received more and more attention. In this section, we extend them to aggregate hesitant fuzzy information using Einstein operations.
4.1. Hesitant Fuzzy Einstein Geometric Weighted Aggregation Operator. Based on the operational laws (5) and (7) on HFEs, Xia and Xu [36] developed some hesitant fuzzy aggregation operators as listed below.

Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs; then.
(1) the hesitant fuzzy weighted geometric (HFWG) operator

$$
\begin{align*}
\operatorname{HFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right) & =\bigotimes_{j=1}^{n} h_{j}^{\omega_{j}} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right\}, \tag{14}
\end{align*}
$$

where $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ is the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ with $\omega_{j} \in[0,1]$ and $\sum_{j=1}^{n} \omega_{j}=1$.
(2) the hesitant fuzzy ordered weighted geometric (HFOWG) operator

$$
\begin{align*}
& \text { HFOWG }\left(h_{1}, h_{2}, \ldots, h_{n}\right) \\
& \quad=\bigotimes_{j=1}^{n} w_{j}^{h_{\sigma(j)}}  \tag{15}\\
& =\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \ldots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}\left\{\prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_{j}}\right\},
\end{align*}
$$

where $\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a permutation of $1,2, \ldots, n$, such that $h_{\sigma(j-1)}>h_{\sigma(j)}$ for all $j=2, \ldots, n$ and $w=$ $\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ is aggregation-associated vector with $w_{j} \in$ $[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$.

For convenience, let $H$ be the set of all HFEs. Based on the proposed Einstein operations on HFEs, we develop some new aggregation operators for HFEs and discuss their desirable properties.

Definition 11. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs. A hesitant fuzzy Einstein weighted geometric $\left(\mathrm{HFEWG}_{\varepsilon}\right)$ operator of dimension $n$ is a mapping $\operatorname{HFEWG}_{\varepsilon}: H^{n} \rightarrow H$ defined as follows:

$$
\begin{align*}
& \operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \\
& \quad=\bigotimes_{j=1}^{n} h_{j}^{\wedge_{\varepsilon} \omega_{j}}  \tag{16}\\
& \quad=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}\right\},
\end{align*}
$$

where $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ is the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ and $w_{j}>0, \sum_{j=1}^{n} w_{j}=1$. In particular, when $w_{j}=$ $1 / n, j=1,2, \ldots, n$, the $\operatorname{HFEWG}_{\varepsilon}$ operator is reduced to the hesitant fuzzy Einstein geometric $\left(\mathrm{HFEG}_{\varepsilon}\right)$ operator:

$$
\begin{align*}
& \operatorname{HFEG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{j}^{1 / n}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{1 / n}+\prod_{j=1}^{n} \gamma_{j}^{1 / n}}\right\} . \tag{17}
\end{align*}
$$

From Proposition 10, we easily get the following result.
Corollary 12. If all $h_{j}(j=1,2, \ldots, n)$ are equal and the number of values in $h_{j}$ is only one, that is, $h_{j}=h=\{\gamma\}$ for all $j=1,2, \ldots, n$, then

$$
\begin{equation*}
\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=h . \tag{18}
\end{equation*}
$$

Note that the $\mathrm{HFEWG}_{\varepsilon}$ operator is not idempotent in general; we give the following example to illustrate this case.

Example 13. Let $h_{1}=h_{2}=h_{3}=h=(0.3,0.7), w=$ $(0.4,0.25,0.35)^{T}$; then $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)=\{0.3,0.4137$, $0.3782,0.5126,0.4323,0.579,0.5342,0.7\}$. By Definition 3, we have $s\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.4812<0.5=s(h)$. Hence $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)<h$.

Lemma 14 (see $[18,49]$ ). Let $\gamma_{j}>0, w_{j}>0, j=1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
\prod_{j=1}^{n} \gamma_{j}^{w_{j}} \leq \sum_{j=1}^{n} w_{j} \gamma_{j} \tag{19}
\end{equation*}
$$

with equality if and only if $\gamma_{1}=\gamma_{2}=\cdots=\gamma_{n}$.
Theorem 15. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \succeq \operatorname{HFWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right), \tag{20}
\end{equation*}
$$

where the equality holds if only if all $h_{j}(j=1,2, \ldots, n)$ are equal and the number of values in $h_{j}$ is only one.

Proof. For any $\gamma_{j} \in h_{j}(j=1,2, \ldots, n)$, by Lemma 14, we have

$$
\begin{equation*}
\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{w_{j}}+\prod_{j=1}^{n} \gamma_{j}^{w_{j}} \leq \sum_{j=1}^{n} w_{j}\left(2-\gamma_{j}\right)+\sum_{j=1}^{n} w_{j} \gamma_{j}=2 . \tag{21}
\end{equation*}
$$

Then

$$
\begin{equation*}
\frac{2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}} \geq \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} . \tag{22}
\end{equation*}
$$

It follows that $s\left(\otimes_{\varepsilon j=1}^{n} h_{j}^{\wedge_{\varepsilon} \omega_{j}}\right) \geq s\left(\otimes_{\varepsilon j=1}^{n} h_{j}^{\omega_{j}}\right)$, which completes the proof of Theorem 15.

Theorem 15 tells us the result that the $\mathrm{HFEWG}_{\varepsilon}$ operator shows the decision maker's more optimistic attitude than the HFWA operator proposed by Xia and Xu [36] (i.e., (15)) in aggregation process. To illustrate that, we give an example adopted from Example 1 in [36] as follows.

Example 16. Let $h_{1}=(0.2,0.3,0.5), h_{2}=(0.4,0.6)$ be two HFEs, and let $w=(0.7,0.3)^{T}$ be the weight vector of $h_{j}(j=$ 1,2 ); then by Definition 11, we have

$$
\begin{align*}
\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}\right)= & \bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\frac{2 \prod_{j=1}^{2} \gamma_{j}^{\omega_{j}}}{\prod_{j=1}^{2}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{2} \gamma_{j}^{\omega_{j}}}\right\} \\
= & \{0.2482,0.2856,0.3276,0.3744, \\
& 0.4683,0.5288\} . \tag{23}
\end{align*}
$$

However, Xia and Xu [36] used the HFWG operator to aggregate the $h_{j}(j=1,2)$ and got

$$
\begin{align*}
& \operatorname{HFEG}\left(h_{1}, h_{2}\right) \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}}\left\{\prod_{j=1}^{2} \gamma_{j}^{w_{j}}\right\}  \tag{24}\\
& =\{0.2462,0.2781,0.3270,0.3693,0.4676,0.5281\} .
\end{align*}
$$

It is clear that $s\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}\right)\right)=0.3722>0.3694=$ $s\left(\operatorname{HFEG}\left(h_{1}, h_{2}\right)\right)$. Thus $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}\right) \succ \operatorname{HFEG}\left(h_{1}, h_{2}\right)$.

Based on Definition 11 and the proposed operational laws, we can obtain the following properties on $\mathrm{HFEWG}_{\varepsilon}$ operator.

Theorem 17. Let $\alpha>0, h_{j}(j=1,2, \ldots, n)$, be a collection of HFEs and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{i=1}^{n} w_{j}=1$. Then

$$
\begin{align*}
& H F E W G_{\varepsilon}\left(h_{1}^{\wedge_{\varepsilon} \alpha}, h_{2}^{\wedge_{\varepsilon} \alpha}, \ldots, h_{n}^{\wedge_{\varepsilon} \alpha}\right) \\
& \quad=\left(H F E W G_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right)^{\wedge_{\varepsilon} \alpha} . \tag{25}
\end{align*}
$$

Proof. Since $h_{j}^{\wedge_{\varepsilon} \alpha}=\bigcup_{\gamma \in h_{j}}\left\{2 \gamma_{j}^{\alpha} /\left(\left(2-\gamma_{j}\right)^{\alpha}+\gamma_{j}^{\alpha}\right)\right\}$ for all $j=$ $1,2, \ldots, n$, by the definition of $\mathrm{HFEWG}_{\varepsilon}$, we have

$$
\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\wedge_{\varepsilon} \alpha}, h_{2}^{\wedge_{\varepsilon} \alpha}, \ldots, h_{n}^{\wedge_{\varepsilon} \alpha}\right)
$$

$$
\begin{aligned}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\{ & \left(2 \prod_{j=1}^{n}\left(2 \gamma_{j}^{\alpha} /\left(\left(2-\gamma_{j}\right)^{\alpha}+\gamma_{j}^{\alpha}\right)\right)^{\omega_{j}}\right) \\
& \times\left(\prod_{j=1}^{n}\left(2-\left(2 \gamma_{j}^{\alpha} /\left(\left(2-\gamma_{j}\right)^{\alpha}+\gamma_{j}^{\alpha}\right)\right)\right)^{\omega_{j}}\right. \\
& \left.\left.+\prod_{j=1}^{n}\left(2 \gamma_{j}^{\alpha} /\left(\left(2-\gamma_{j}\right)^{\alpha}+\gamma_{j}^{\alpha}\right)\right)^{\omega_{j}}\right)^{-1}\right\}
\end{aligned}
$$

$$
\begin{equation*}
=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{j}^{\alpha \omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\alpha \omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\alpha \omega_{j}}}\right\} \tag{26}
\end{equation*}
$$

Since $\quad \operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)$
$\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right\}$, then

$$
\begin{aligned}
& \left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right)^{\wedge_{\varepsilon} \alpha} \\
& =\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\left(2 \left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}\right.\right.\right.\right. \\
& \\
& \left.\left.\left.\quad+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right)^{\alpha}\right)
\end{aligned}
$$

$$
\begin{align*}
& \times\left(\left(2-\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}\right.\right.\right.\right. \\
&\left.\left.\left.+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right)\right)^{\alpha} \\
&+\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}\right.\right. \\
&\left.\left.=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\begin{array}{l}
j=1 \\
\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}\right)^{\alpha}+\left(\prod_{j=1}^{n} \gamma_{j}^{\alpha \omega_{j}}\right)^{\alpha}
\end{array}\right)^{\alpha}\right)^{-1}\right\} \\
&=\bigcup_{\gamma_{1} \in h_{1}, \gamma_{2} \in h_{2}, \ldots, \gamma_{n} \in h_{n}}\left\{\begin{array}{l}
\left.2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)^{\alpha} \gamma_{j}^{\alpha \omega_{j}} \\
\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\alpha \omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\alpha \omega_{j}}
\end{array} .\right.
\end{align*}
$$

Theorem 18. Let $h$ be an HFE, $h_{j}(j=1,2, \ldots, n)$ a collection of HFEs, and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ the weight vector of $h_{j}$ $(j=1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{i=1}^{n} w_{j}=1$. Then

$$
\begin{align*}
& H F E W G_{\varepsilon}\left(h_{1} \otimes_{\varepsilon} h, h_{2} \otimes_{\varepsilon} h, \ldots, h_{n} \otimes_{\varepsilon} h\right)  \tag{28}\\
&=H F E W G_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} h
\end{align*}
$$

Proof. By the definition of $\mathrm{HFEWG}_{\varepsilon}$ and Einstein product operator of HFEs, we have
$\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} h$

$$
\begin{align*}
& =\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\frac{\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right) \cdot \gamma}{1+\left(1-\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right)\right)(1-\gamma)}\right\}  \tag{29}\\
& =\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\frac{2 \gamma \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}{(2-\gamma) \prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\gamma \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}\right\} .
\end{align*}
$$

Since $h_{j} \otimes_{\varepsilon} h=\bigcup_{\gamma_{j} \in h_{j}, \gamma \epsilon h}\left\{\gamma_{j} \gamma /\left(1+\left(1-\gamma_{j}\right)(1-\gamma)\right)\right\}$ for all $j=$ $1,2, \ldots, n$, by the definition of HFEWG $_{\varepsilon}$, we have
$\operatorname{HFEWG}_{\varepsilon}\left(h_{1} \otimes_{\varepsilon} h, h_{2} \otimes_{\varepsilon} h, \ldots, h_{n} \oplus_{\varepsilon} h\right)$

$$
+\prod_{j=1}^{n}\left(\gamma_{j} \gamma /\left(1+\left(1-\gamma_{j}\right)\right.\right.
$$

$$
\begin{aligned}
=\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\{ & \left(2 \prod_{j=1}^{n}\left(\gamma_{j} \gamma /\left(1+\left(1-\gamma_{j}\right)(1-\gamma)\right)\right)^{\omega_{j}}\right) \\
& \times\left(\prod _ { j = 1 } ^ { n } \left(2-\left(\gamma_{j} \gamma /\left(1+\left(1-\gamma_{j}\right)\right.\right.\right.\right.
\end{aligned}
$$

$$
\left.\left.\times(1-\gamma)))^{w_{j}}\right)^{-1}\right\}
$$

$$
=\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\frac{2 \prod_{j=1}^{n}\left(\gamma_{j} \gamma\right)^{\omega_{j}}}{\prod_{j=1}^{n}\left(\left(2-\gamma_{j}\right)(2-\gamma)\right)^{\omega_{j}}+\prod_{j=1}^{n}\left(\gamma_{j} \gamma\right)^{\omega_{j}}}\right\}
$$

$$
\begin{aligned}
& =\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\left(2 \prod_{j=1}^{n} \gamma^{\omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right. \\
& \times\left(\prod_{j=1}^{n}(2-\gamma)^{\omega_{j}} \cdot \prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma^{\omega_{j}}\right. \\
& \left.\left.\cdot \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)^{-1}\right\} \\
& =\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\left(2 \gamma^{\sum_{j=1}^{n} \omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right. \\
& \times\left((2-\gamma)^{\sum_{j=1}^{n} \omega_{j}} \cdot \prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}\right. \\
& \left.\left.+\gamma^{\sum_{j=1}^{n} \omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)^{-1}\right\} \\
& =\bigcup_{\gamma \in h, \gamma_{j} \in h_{j}, j=1, \ldots, n}\left\{\frac{2 \gamma \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}{(2-\gamma) \prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\gamma \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}\right\} .
\end{aligned}
$$

(30)

Based on Theorems 17 and 18, the following property can be obtained easily.

Theorem 19. Let $\alpha>0, h$ be an HFE, let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs, and let $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be the weight vector of $h_{j}(j=1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{i=1}^{n} w_{j}=1$. Then

$$
\begin{align*}
& \operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h, h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h, \ldots, h_{n}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h\right)  \tag{31}\\
&=\left(H F E W G_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} h\right)^{\wedge_{\varepsilon} \alpha} .
\end{align*}
$$

Theorem 20. Let $h_{j}$ and $h_{j}^{\prime}(j=1,2, \ldots, n)$ be two collections of HFEs and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{i=1}^{n} w_{j}=1$. Then
$\operatorname{HFEWG}_{\varepsilon}\left(h_{1} \otimes_{\varepsilon} h_{1}^{\prime}, h_{2} \otimes_{\varepsilon} h_{2}^{\prime}, \ldots, h_{n} \otimes_{\varepsilon} h_{n}^{\prime}\right)$

$$
\begin{equation*}
=\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} \operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right) \tag{32}
\end{equation*}
$$

Proof. By the definition of $\mathrm{HFEWG}_{\varepsilon}$ and Einstein product operator of HFEs, we have
$\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} \operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$

$$
\begin{align*}
& =\bigcup_{\gamma_{j} \in h_{j}, \gamma_{j}^{\prime} \in h_{j}^{\prime}, j=1, \ldots, n}\left\{\frac{\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right) \cdot\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}^{\prime}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right)}{1+\left(1-\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)\right)\right)\left(1-\left(2 \prod_{j=1}^{n} \gamma_{j}^{\prime \omega_{j}} /\left(\prod_{j=1}^{n}\left(2-\gamma_{j}^{\prime}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\prime \omega_{j}}\right)\right)\right)}\right\} \\
& =\bigcup_{\gamma_{j} \in h_{j}, \gamma_{j}^{\prime} \in h_{j}^{\prime}, j=1, \ldots, n}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\prime \omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}} \cdot \prod_{j=1}^{n}\left(2-\gamma_{j}^{\prime}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\prime \omega_{j}}}\right\} . \tag{33}
\end{align*}
$$

Since $h_{j} \otimes_{\mathcal{\varepsilon}} h_{j}^{\prime}=\bigcup_{\gamma_{j} \in h_{j}, \gamma_{j}^{\prime} \in h_{j}^{\prime}}\left\{\gamma_{j} \gamma_{j}^{\prime} /\left(1+\left(1-\gamma_{j}\right)\left(1-\gamma_{j}^{\prime}\right)\right)\right\}$ for all $j=1,2, \ldots, n$, by the definition of $\mathrm{HFEWG}_{\varepsilon}$, we have

$$
\begin{aligned}
& \operatorname{HFEWG}_{\varepsilon}\left(h_{1} \bigotimes_{\varepsilon} h_{1}^{\prime}, h_{2} \bigotimes_{\varepsilon} h_{2}^{\prime}, \ldots, h_{n} \bigotimes_{\varepsilon} h_{n}^{\prime}\right) \\
&=\bigcup_{\gamma_{j} \in h_{j}, \gamma_{j}^{\prime} \in h_{j}^{\prime}, j=1, \ldots, n}\left\{\left(2 \prod_{j=1}^{n}\left(\gamma_{j} \gamma_{j}^{\prime} /\left(1+\left(1-\gamma_{j}\right)\left(1-\gamma_{j}^{\prime}\right)\right)\right)^{\omega_{j}}\right)\right. \\
& \times\left(\prod _ { j = 1 } ^ { n } \left(2-\left(\gamma_{j} \gamma_{j}^{\prime} /\left(1+\left(1-\gamma_{j}\right)\right.\right.\right.\right. \\
&\left.\left.\left.\times\left(1-\gamma_{j}^{\prime}\right)\right)\right)\right)^{\omega_{j}} \\
&+\prod_{j=1}^{n}\left(\gamma_{j} \gamma_{j}^{\prime} /\left(1+\left(1-\gamma_{j}\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& \times\left(1-\gamma_{j}^{\prime}\right. \\
=\bigcup_{\gamma_{j} \in h_{j}, v_{j}^{\prime} \in h_{j}^{\prime}, j=1, \ldots, n} & \left\{\left(2 \prod_{j=1}^{n}\left(\gamma_{j} \gamma_{j}^{\prime}\right)^{\omega_{j}}\right)\right. \\
& \times\left(\prod_{j=1}^{n}\left[\left(2-\gamma_{j}\right)\left(2-\gamma_{j}^{\prime}\right)\right]^{\omega_{j}}\right. \\
& \left.\left.+\prod_{j=1}^{n}\left(\gamma_{j} \gamma_{j}^{\prime}\right)^{\omega_{j}}\right)^{-1}\right\} \\
=\bigcup_{\gamma_{j} \in h_{j}, \gamma_{j}^{\prime} \in h_{j}^{\prime}, j=1, \ldots, n} & \left\{\left(2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\prime \omega_{j}}\right)\right.
\end{aligned}
$$

$$
\begin{gather*}
\times\left(\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}} \cdot \prod_{j=1}^{n}\left(2-\gamma_{j}^{\prime}\right)^{\omega_{j}}\right. \\
\left.\left.\quad+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}} \cdot \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}\right)^{-1}\right\} \tag{34}
\end{gather*}
$$

Theorem 21. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs, $h_{\min }^{-}=\min _{j}\left\{h_{j}^{-} \mid h_{j}^{-}=\min \left\{\gamma_{j} \in h_{j}\right\}\right\}$, and $h_{\max }^{+}=\max _{j}\left\{h_{j}^{+} \mid\right.$ $\left.h_{j}^{+}=\max \left\{\gamma_{j} \in h_{j}\right\}\right\}$, and let $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be the weight vector of $h_{j}(j=1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{i=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
h_{\min }^{-} \preceq \operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h_{\max }^{+} \tag{35}
\end{equation*}
$$

where the equality holds if only if all $h_{j}(j=1,2, \ldots, n)$ are equal and the number of values in $h_{j}$ is only one.

Proof. Let $f(t)=(2-t) / t, t \in[0,1]$. Then $f^{\prime}(t)=-2 / t^{2}<0$. Hence $f(t)$ is a decreasing function. Since $h_{\text {min }}^{-} \leq h_{j}^{-} \leq \gamma_{j} \leq$ $h_{j}^{+} \leq h_{\max }^{+}$for any $\gamma_{j} \in h_{j}(j=1,2, \ldots, n)$, then $f\left(h_{\max }^{+}\right) \leq$ $f\left(\gamma_{j}\right) \leq f\left(h_{\min }^{-}\right)$; that is, $\left(2-h_{\max }^{+}\right) / h_{\max }^{+} \leq\left(2-\gamma_{j}\right) / \gamma_{j} \leq$ $\left(2-h_{\min }^{-}\right) / h_{\min }^{-}$. Then for any $\gamma_{j} \in h_{j}(j=1,2, \ldots, n)$, we have

$$
\begin{aligned}
& \prod_{j=1}^{n}\left(\frac{2-h_{\max }^{+}}{h_{\max }^{+}}\right)^{w_{j}} \\
& \leq \prod_{j=1}^{n}\left(\frac{2-\gamma_{j}}{\gamma_{j}}\right)^{w_{j}} \leq \prod_{j=1}^{n}\left(\frac{1-h_{\min }^{-}}{1+h_{\min }^{-}}\right)^{w_{j}} \\
& \Longleftrightarrow\left(\frac{2-h_{\max }^{+}}{h_{\max }^{+}}\right)^{\sum_{j=1}^{n} w_{j}} \leq \prod_{j=1}^{n}\left(\frac{2-\gamma_{j}}{\gamma_{j}}\right)^{w_{j}} \\
& \leq\left(\frac{1-h_{\min }^{-}}{1+h_{\min }^{-}}\right)^{\sum_{j=1}^{n} w_{j}} \Longleftrightarrow\left(\frac{2-h_{\max }^{+}}{h_{\max }^{+}}\right) \\
& \leq \prod_{j=1}^{n}\left(\frac{2-\gamma_{j}}{\gamma_{j}}\right)^{w_{j}} \leq\left(\frac{1-h_{\min }^{-}}{1+h_{\min }^{-}}\right) \Longleftrightarrow \frac{2}{h_{\max }^{+}} \\
& \quad \leq \prod_{j=1}^{n}\left(\frac{2-\gamma_{j}}{\gamma_{j}}\right)^{w_{j}}+1 \leq \frac{2}{h_{\min }^{-}} \Longleftrightarrow \frac{h_{\min }^{-}}{2} \\
& \quad \leq \frac{\prod_{j=1}^{n}\left(\left(2-\gamma_{j}\right) / \gamma_{j}\right)^{w_{j}}+1}{} \leq\left(\frac{h_{\max }^{+}}{2}\right) \Longleftrightarrow h_{\min }^{-} \\
& \quad \leq \frac{\prod_{j=1}^{n}\left(\left(2-\gamma_{j}\right) / \gamma_{j}\right)^{w_{j}}+1}{\prod_{\max }} \Longleftrightarrow h_{\min }^{+}
\end{aligned}
$$

$$
\begin{equation*}
\leq \frac{2 \prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{j}^{\omega_{j}}} \leq h_{\max }^{+} \tag{36}
\end{equation*}
$$

It follows that $h_{\text {min }}^{-} \leq s\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right) \leq h_{\max }^{+}$. Thus we have $h_{\min }^{-} \leq \operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h_{\max }^{+}$.

Remark 22. Let $h_{j}$ and $h_{j}^{\prime}(j=1,2, \ldots, n)$ be two collections of HFEs, and $h_{j}<h_{j}^{\prime}$ for all $j$; then $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \prec$ $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$ does not hold necessarily in general. To illustrate that, an example is given as follows.

Example 23. Let $h_{1}=(0.45,0.6), h_{2}=(0.6,0.7), h_{3}=$ $(0.5,0.6), h_{1}^{\prime}=(0.2,0.9), h_{2}^{\prime}=(0.45,0.95), h_{3}^{\prime}=(0.35,0.8)$, and $w=(0.5,0.3,0.2)^{T}$; then $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)=\{0.5024$, $0.5215,0.5286,0.5483,0.5791,0.6,0.6077,0.6291\} \quad$ and $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, h_{3}^{\prime}\right)=\{0.2778,0.3372,0.3835,0.4595$, $0.6088,0.7099,0.7833,0.8947\}$. By Definition 3, we have $s\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.5646$ and $s\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}\right.\right.$, $\left.\left.h_{3}^{\prime}\right)\right)=0.5568$. It follows that $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)>$ $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, h_{3}^{\prime}\right)$. Clearly, $h_{j} \prec h_{j}^{\prime}$ for $j=1,2,3$, but $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right) \succ \operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, h_{3}^{\prime}\right)$.
4.2. Hesitant Fuzzy Einstein Ordered Weighted Averaging Operator. Similar to the HFOWG operator introduced by Xia and Xu [36] (i.e., (15)), in what follows, we develop an (HFEOWG ${ }_{\varepsilon}$ ) operator, which is an extension of OWA operator proposed by Yager [50].

Definition 24. For a collection of the HFEs $h_{j}(j=$ $1,2, \ldots, n)$, a hesitant fuzzy Einstein ordered weighted averaging ( $\mathrm{HFEOWG}_{\varepsilon}$ ) operator is a mapping $\mathrm{HFEWG}_{\varepsilon}: H^{n} \rightarrow$ $H$ such that

$$
\begin{align*}
& \operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \\
&=\bigotimes_{j=1}^{n} h_{\sigma(j)}^{\wedge_{\varepsilon} w_{j}} \\
&=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1), \gamma_{\sigma(2)} \in h_{\sigma(2), \ldots, \gamma_{\sigma(n)} \in h_{\sigma(n)}}}}\left(2 \prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_{j}}\right)  \tag{37}\\
& \times\left(\prod_{j=1}^{n}\left(2-\gamma_{\sigma(j)}\right)^{w_{j}}\right. \\
&\left.\left.+\prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_{j}}\right)^{-1}\right\},
\end{align*}
$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2, \ldots, n)$, such that $h_{\sigma(j-1)} \succ h_{\sigma(j)}$ for all $j=2, \ldots, n$ and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ is aggregation-associated vector with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. In particular, if $w=$ $(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the HFEOWG $_{\varepsilon}$ operator is reduced to the $\mathrm{HFEA}_{\varepsilon}$ operator of dimension $n$ (i.e., (17)).

Note that the $\mathrm{HFEOWG}_{\varepsilon}$ weights can be obtained similar to the OWA weights. Several methods have been introduced to determine the OWA weights in [20, 21, 50-53].

Similar to the $\mathrm{HFEWG}_{\varepsilon}$ operator, the $\mathrm{HFEOWG}_{\varepsilon}$ operator has the following properties.

Theorem 25. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs and $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ the weight vector of $h_{j}(j=$ $1,2, \ldots, n)$ with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \succeq \operatorname{HFOWG}^{2}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \tag{38}
\end{equation*}
$$

where the equality holds if only if all $h_{j}(j=1,2, \ldots, n)$ are equal and the number of values in $h_{j}$ is only one.

From Theorem 25, we can conclude that the values obtained by the HFEOWG ${ }_{\varepsilon}$ operator are not less than the ones obtained by the HFOWA operator proposed by Xia and Xu [36]. To illustrate that, let us consider the following example.

Example 26. Let $h_{1}=(0.1,0.4,0.7), h_{2}=(0.3,0.5)$, and $h_{3}=(0.2,0.6)$ be three HFEs and suppose that $w=$ $(0.2,0.45,0.35)^{T}$ is the associated vector of the aggregation operator.

By Definitions 3 and 4, we calculate the score values and the accuracy values of $h_{1}, h_{2}$, and $h_{3}$ as follows, respectively:
$s\left(h_{1}\right)=s\left(h_{2}\right)=s\left(h_{3}\right)=0.5, k\left(h_{1}\right)=0.7551, k\left(h_{2}\right)=0.9$, $k\left(h_{3}\right)=0.8$.

According to Definition 5, we have $h_{2} \prec h_{3} \prec h_{1}$. Then $h_{\sigma(1)}=h_{2}, h_{\sigma(2)}=h_{3}, h_{\sigma(3)}=h_{1}$.

By the definition of HFEOWG ${ }_{\varepsilon}$, we have
$\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)$
$=\bigotimes_{j=1}^{3} h_{\sigma(j)}^{\wedge_{\varepsilon} w_{j}}$
$=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}}\left\{\frac{2 \prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_{j}}}{\prod_{j=1}^{3}\left(2-\gamma_{\sigma(j)}\right)^{w_{j}}+\prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_{j}}}\right\}$
$=\{0.1716,0.2787,0.3495,0.2939,0.4582,0.5598,0.1926$,
$0.3106,0.3877,0.3272,0.5047,0.6125\}$.

If we use the HFOWA operator, which was given by Xia and Xu [36] (i.e., (15)), to aggregate the HFEs $h_{j}(i=1,2,3)$, then we have
$\operatorname{HFOWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)$

$$
=\bigotimes_{j=1}^{3} h_{\sigma(j)}^{w_{j}}=\bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}}\left\{\prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_{j}}\right\}
$$

$$
\begin{align*}
= & \{0.1702,0.2764,0.3363,0.2790,0.4532,0.5513 \\
& 0.1885,0.3062,0.3724,0.3090,0.5020,0.6106\} . \tag{40}
\end{align*}
$$

Clearly, $s\left(\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right)\right)=0.3706>0.3629=$ $s\left(\operatorname{HFOWG}\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right)$. By Definition 3, we have $\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, h_{3}\right) \succ \operatorname{HFOWG}\left(h_{1}, h_{2}, h_{3}\right)$.

Theorem 27. Let $\alpha>0, h$ be an HFE, let $h_{j}$ and $h_{j}^{\prime}$ $(j=1,2, \ldots, n)$ be two collection of HFEs, and let $w=$ $\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be an aggregation-associated vector with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then
(1) $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\wedge_{\varepsilon} \alpha}, h_{2}^{\wedge_{\varepsilon} \alpha}, \ldots, h_{n}^{\wedge_{\varepsilon}^{\alpha}}\right)=$ $\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)\right)^{\wedge_{\varepsilon} \alpha}$,
(2) $\operatorname{HFEWG}_{\varepsilon}\left(h_{1} \otimes_{\varepsilon} h, h_{2} \otimes_{\varepsilon} h, \ldots, h_{n} \otimes_{\varepsilon} h\right)=\operatorname{HFEWG}_{\varepsilon}\left(h_{1}\right.$, $\left.h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} h$,
(3) $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h, h_{1}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h, \ldots, h_{n}^{\wedge_{\varepsilon} \alpha} \otimes_{\varepsilon} h\right)=$
$\left(\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} h\right)^{\wedge_{\varepsilon} \alpha}$,
(4) $\operatorname{HFEWG}_{\varepsilon}\left(h_{1} \otimes_{\varepsilon} h_{1}^{\prime}, h_{2} \otimes_{\varepsilon} h_{2}^{\prime}, \ldots, h_{n} \otimes_{\varepsilon} h_{n}^{\prime}\right)=$ $\operatorname{HFEWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \otimes_{\varepsilon} \operatorname{HFEWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$.

Theorem 28. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs and let $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be an aggregation-associated vector with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
h_{\min }^{-} \leq H F E O W G_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right) \leq h_{\max }^{+}, \tag{41}
\end{equation*}
$$

where $h_{\min }^{-}=\min _{j}\left\{h_{j}^{-} \mid h_{j}^{-}=\min \left\{\gamma_{j} \in h_{j}\right\}\right\}$ and $h_{\max }^{+}=$ $\max _{j}\left\{h_{j}^{+} \mid h_{j}^{+}=\max \left\{\gamma_{j} \in h_{j}\right\}\right\}$.

Besides the above properties, we can get the following desirable results on the $\mathrm{HFOWG}_{\varepsilon}$ operator.

Theorem 29. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs, and let $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be an aggregation-associated vector with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then

$$
\begin{equation*}
\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right) \tag{42}
\end{equation*}
$$

where $\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$ is any permutation of $\left(h_{1}, h_{2}, \ldots, h_{n}\right)$.
Proof. Let $\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\otimes_{\varepsilon j=1}^{n} h_{\sigma(j)}^{\wedge_{\varepsilon} w_{j}}$ and $\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)=\otimes_{\varepsilon j=1}^{n} h_{\sigma(j)}^{\prime}{ }^{\wedge_{\varepsilon} w_{j}}$. Since $\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$ is any permutation of $\left(h_{1}, h_{2}, \ldots, h_{n}\right)$, then we have $h_{\sigma(j)}=h_{\sigma(j)}^{\prime}(j=1,2, \ldots, n)$. Thus $\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=\operatorname{HFEOWG}_{\varepsilon}\left(h_{1}^{\prime}, h_{2}^{\prime}, \ldots, h_{n}^{\prime}\right)$.

Theorem 30. Let $h_{j}(j=1,2, \ldots, n)$ be a collection of HFEs, and let $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be an aggregation-associated vector with $w_{j} \in[0,1]$ and $\sum_{j=1}^{n} w_{j}=1$. Then
(1) if $w=(0,0, \ldots, 1)$, then $\operatorname{HFOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots, h_{n}\right)=$ $\min \left\{h_{1}, h_{2}, \ldots, h_{n}\right\}$;
(2) if $w=(1,0, \ldots, 0)$, then $\operatorname{HFOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots\right.$, $\left.h_{n}\right)=\max \left\{h_{1}, h_{2}, \ldots, h_{n}\right\} ;$
(3) if $w_{j}=1$ and $w_{i}=0(i \neq j)$, then $\operatorname{HFOWG}_{\varepsilon}\left(h_{1}, h_{2}, \ldots\right.$, $\left.h_{n}\right)=h_{\sigma(j)}$, where $h_{\sigma(j)}$ is the $j$ th largest of $h_{i}(i=$ $1,2, \ldots, n$ ).

## 5. An Application in Hesitant Fuzzy Decision Making

In this section, we apply the $\mathrm{HFEWG}_{\varepsilon}$ and $\mathrm{HFEOWG}_{\varepsilon}$ operators to multiple attribute decision making with hesitant fuzzy information.

For hesitant fuzzy multiple attribute decision making problems, let $Y=\left\{Y_{1}, Y_{2}, \ldots, Y_{m}\right\}$ be a discrete set of alternatives, let $A=\left\{A_{1}, A_{2}, \ldots, A_{n}\right\}$ be a collection of attributes, and let $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ be the weight vector of $A_{j}(j=1,2, \ldots, n)$ with $\omega_{j} \geq 0, j=1,2, \ldots, n$, and $\sum_{j=1}^{n} \omega_{j}=1$. If the decision makers provide several values for the alternative $Y_{i}(i=1,2, \ldots, m)$ under the attribute $A_{j}(j=$ $1,2, \ldots, n$ ) with anonymity, these values can be considered as an HFE $h_{i j}$. In the case where two decision makers provide the same value, the value emerges only once in $h_{i j}$. Suppose that the decision matrix $H=\left(h_{i j}\right)_{m \times n}$ is the hesitant fuzzy decision matrix, where $h_{i j}(i=1,2, \ldots, m, j=1,2, \ldots, n)$ are in the form of HFEs.

To get the best alternative, we can utilize the $\mathrm{HFEWG}_{\varepsilon}$ operator or the $\mathrm{HFEOWG}_{\varepsilon}$ operator; that is,

$$
\begin{align*}
h_{i} & =\operatorname{HFEWG}_{\varepsilon}\left(h_{i 1}, h_{i 2}, \ldots, h_{i n}\right) \\
& =\bigcup_{\gamma_{i 1} \in h_{i 1}, \gamma_{i 2} \in h_{i 2}, \ldots, \gamma_{i n} \in h_{i n}}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{i j}^{\omega_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{i j}\right)^{\omega_{j}}+\prod_{j=1}^{n} \gamma_{i j}^{\omega_{j}}}\right\} \tag{43}
\end{align*}
$$

or

$$
\begin{align*}
h_{i} & =\operatorname{HFEOWG}_{\varepsilon}\left(h_{i 1}, h_{i 2}, \ldots, h_{i n}\right) \\
& =\bigcup_{\gamma_{i \sigma(j)} \in h_{i \sigma(j), j=1,2, \ldots, n}}\left\{\frac{2 \prod_{j=1}^{n} \gamma_{i \sigma(j)}^{w_{j}}}{\prod_{j=1}^{n}\left(2-\gamma_{i \sigma(j)}\right)^{w_{j}}+\prod_{j=1}^{n} \gamma_{i \sigma(j)}^{w_{j}}}\right\} \tag{44}
\end{align*}
$$

to derive the overall value $h_{i}$ of the alternatives $Y_{i}(i=$ $1,2, \ldots, m)$, where $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ is the weight vector related to the HFEOWA $_{\varepsilon}$ operator, such that $w_{j} \geq 0, j=$ $1,2, \ldots, n$, and $\sum_{j=1}^{n} w_{j}=1$, which can be obtained by the normal distribution based method [20].

Then by Definition 3, we compute the scores $s\left(h_{i}\right)(i=$ $1,2, \ldots, m)$ of the overall values $h_{i}(i=1,2, \ldots, m)$ and use the scores $s\left(h_{i}\right)(i=1,2, \ldots, m)$ to rank the alternatives $Y=\left\{Y_{1}, Y_{2}, \ldots, Y_{m}\right\}$ and then select the best one (note that if there is no difference between the two scores $h_{i}$ and $h_{j}$, then we need to compute the accuracy degrees $k\left(h_{i}\right)$ and $k\left(h_{j}\right)$ of the overall values $h_{i}$ and $h_{j}$ by Definition 4, respectively, and then rank the alternatives $Y_{i}$ and $Y_{j}$ in accordance with Definition 5).

In the following, an example on multiple attribute decision making problem involving a customer buying a car,
which is adopted from Herrera and Martinez [54], is given to illustrate the proposed method using the $\mathrm{HFEOWG}_{\varepsilon}$ operator.

Example 31. Consider that a customer wants to buy a car, which will be chosen from five types $Y_{i}(i=1,2, \ldots, 5)$. In the process of choosing one of the cars, four factors are considered: $A_{1}$ is the consumption petrol, $A_{2}$ is the price, $A_{3}$ is the degree of comfort, and $A_{4}$ is the safety factor. Suppose that the characteristic information of the alternatives $Y_{i}(i=$ $1,2, \ldots, 5)$ can be represented by HFEs $h_{i j}(i=1,2, \ldots, 5 ; j=$ $1,2, \ldots, 4)$, and the hesitant fuzzy decision matrix is given in Table 1.

To use $\operatorname{HFEOWG}_{\varepsilon}$ operator, we first reorder the $h_{i j}(j=$ $1,2, \ldots, 4)$ for each alternative $Y_{i}(i=1,2, \ldots, 5)$. According to Definitions 3 and 4, we compute the score values and accuracy degrees of $s\left(h_{i j}\right)(i=1,2, \ldots, 5 ; j=1,2, \ldots, 4)$ as follows:

$$
\begin{array}{cl}
s\left(h_{11}\right)=0.45, \quad s\left(h_{12}\right)=0.75, & s\left(h_{13}\right)=0.3 \\
s\left(h_{14}\right)=0.3, \quad k\left(h_{13}\right)=0.9184, & k\left(h_{14}\right)=0.9 \\
s\left(h_{21}\right)=0.5, \quad s\left(h_{22}\right)=0.7, & s\left(h_{23}\right)=0.7 \\
s\left(h_{24}\right)=0.5, \quad k\left(h_{21}\right)=0.7551, & k\left(h_{24}\right)=0.8129, \\
k\left(h_{22}\right)=0.8367, \quad k\left(h_{23}\right)=0.9 \\
s\left(h_{31}\right)=0.85, \quad s\left(h_{32}\right)=0.4, & s\left(h_{33}\right)=0.35 \\
s\left(h_{34}\right)=0.4, \quad k\left(h_{32}\right)=0.8367, & k\left(h_{34}\right)=0.7764 \\
s\left(h_{41}\right)=0.6, \quad s\left(h_{42}\right)=0.6, & s\left(h_{43}\right)=0.3 \\
s\left(h_{44}\right)=0.4, \quad k\left(h_{41}\right)=0.772, & k\left(h_{42}\right)=0.8367 \\
s\left(h_{51}\right)=0.5, \quad s\left(h_{52}\right)=0.3, & s\left(h_{53}\right)=0.5 \\
s\left(h_{54}\right)=0.35, \quad k\left(h_{51}\right)=0.8367, & k\left(h_{53}\right)=0.8129 . \tag{45}
\end{array}
$$

Then by Definition 5, we have

$$
\begin{array}{llll}
h_{1 \sigma(1)}=h_{12}, & h_{1 \sigma(2)}=h_{11}, & h_{1 \sigma(3)}=h_{13}, & h_{1 \sigma(4)}=h_{14} ; \\
h_{2 \sigma(1)}=h_{23}, & h_{2 \sigma(2)}=h_{22}, & h_{2 \sigma(3)}=h_{24}, & h_{2 \sigma(4)}=h_{21} ; \\
h_{3 \sigma(1)}=h_{31}, & h_{3 \sigma(2)}=h_{32}, & h_{3 \sigma(3)}=h_{34}, & h_{3 \sigma(4)}=h_{33} ; \\
h_{4 \sigma(1)}=h_{42}, & h_{4 \sigma(2)}=h_{41}, & h_{4 \sigma(3)}=h_{44}, & h_{4 \sigma(4)}=h_{43} ; \\
h_{5 \sigma(1)}=h_{51}, & h_{5 \sigma(2)}=h_{53}, & h_{5 \sigma(3)}=h_{54}, & h_{5 \sigma(4)}=h_{52} . \tag{46}
\end{array}
$$

Suppose that $w=(0.1835,0.3165,0.3165,0.1835)^{T}$ is the weighted vector related to the HFEOWA ${ }_{\varepsilon}$ operator and it is derived by the normal distribution based method [20]. Then we utilize the HFEOWA operator to obtain the hesitant $^{\text {op }}$

Table 1: Hesitant fuzzy decision making matrix.

|  | $A_{1}$ | $A_{2}$ | $A_{3}$ | $A_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $Y_{1}$ | $\{0.4,0.5\}$ | $\{0.7,0.8\}$ | $\{0.2,0.3,0.4\}$ | $\{0.2,0.4\}$ |
| $Y_{2}$ | $\{0.2,0.5,0.8\}$ | $\{0.5,0.7,0.9\}$ | $\{0.6,0.8\}$ | $\{0.2,0.5,0.6,0.7\}$ |
| $Y_{3}$ | $\{0.8,0.9\}$ | $\{0.2,0.4,0.6\}$ | $\{0.2,0.3,0.4,0.5\}$ | $\{0.1,0.3,0.5,0.7\}$ |
| $Y_{4}$ | $\{0.3,0.4,0.6,0.8,0.9\}$ | $\{0.4,0.6,0.8\}$ | $\{0.1,0.2,0.4,0.5\}$ | $\{0.2,0.3,0.5,0.6\}$ |
| $Y_{5}$ | $\{0.3,0.5,0.7\}$ | $\{0.2,0.3,0.4\}$ | $\{0.2,0.5,0.6,0.7\}$ | $\{0.1,0.3,0.4,0.6\}$ |

fuzzy elements $h_{i}(i=1,2,3,4,5)$ for the alternatives $X_{i}$ ( $i=1,2,3,4,5$ ). Take alternative $X_{1}$ for an example; we have

$$
\begin{align*}
h_{1}= & \operatorname{HFEOWG}_{\varepsilon}\left(h_{11}, h_{12}, \ldots, h_{14}\right) \\
= & \bigcup_{\gamma_{1 \sigma(j)} \in h_{1 \sigma(j), j}, j, 2,3,4}\left\{\frac{2 \prod_{j=1}^{4} \gamma_{1 \sigma(j)}^{w_{j}}}{\prod_{j=1}^{4}\left(2-\gamma_{1 \sigma(j)}\right)^{w_{j}}+\prod_{j=1}^{4} \gamma_{1 \sigma(j)}^{w_{j}}}\right\} \\
= & \{0.3220,0.3642,0.3635,0.4099,0.3974,0.4470, \\
& 0.3473,0.3921,0.3914,0.4403,0.4272,0.4794, \\
& 0.3327,0.3760,0.3753,0.4228,0.4101,0.4607, \\
& 0.3587,0.4046,0.4039,0.4539,0.4405,0.4938\} \tag{47}
\end{align*}
$$

The results can be obtained similarly for the other alternatives; here we will not list them for vast amounts of data. By Definition 3, the score values $s\left(h_{i}\right)$ of $h_{i}(i=1,2,3,4,5)$ can be computed as follows:

$$
\begin{gather*}
s\left(h_{1}\right)=0.4048, \quad s\left(h_{2}\right)=0.5758, \quad s\left(h_{3}\right)=0.4311 \\
s\left(h_{4}\right)=0.4479, \quad s\left(h_{5}\right)=0.3620 \tag{48}
\end{gather*}
$$

According to the scores $s\left(h_{i}\right)$ of the overall hesitant fuzzy values $h_{i}(i=1,2,3,4,5)$, we can rank all the alternatives $X_{i}$ : $X_{2} \succ X_{4} \succ X_{3} \succ X_{1} \succ X_{5}$. Thus the optimal alternative is $X_{2}$.

If we use the HFWG operator introduced by Xia and Xu [36] to aggregate the hesitant fuzzy values, then

$$
\begin{gather*}
s\left(h_{1}\right)=0.3960, \quad s\left(h_{2}\right)=0.5630, \quad s\left(h_{3}\right)=0.4164 \\
s\left(h_{4}\right)=0.4344, \quad s\left(h_{5}\right)=0.3548 \tag{49}
\end{gather*}
$$

By Definition 5, we have $X_{2}>X_{4}>X_{3}>X_{1}>X_{5}$.
Note that the rankings are the same in such two cases, but the overall values of alternatives by the $\mathrm{HFEOWG}_{\varepsilon}$ operator are not smaller than the ones by the HFOWG operator. It shows that the attitude of the decision maker using the proposed $\mathrm{HFEOWG}_{\varepsilon}$ operator is more optimistic than the one using the HFOWG operator introduced by Xia and Xu [36] in aggregation process. Therefore, according to the decision makers' optimistic (or pessimistic) attitudes, the different hesitant fuzzy aggregation operators can be used to aggregate the hesitant fuzzy information in decision making process.

## 6. Conclusions

The purpose of multicriteria decision making is to select the optimal alternative from several alternatives or to get their ranking by aggregating the performances of each alternative under some attributes, which is the pervasive phenomenon in modern life. Hesitancy is the most common problem in decision making, for which hesitant fuzzy set can be considered as a suitable means allowing several possible degrees for an element to a set. Therefore, the hesitant fuzzy multiple attribute decision making problems have received more and more attention. In this paper, an accuracy function of HFEs has been defined for distinguishing between the two HFEs having the same score values, and a new order relation between two HFEs has been provided. Some Einstein operations on HFEs and their basic properties have been presented. With the help of the proposed operations, several new hesitant fuzzy aggregation operators including the $\mathrm{HFEWG}_{\varepsilon}$ operator and $\mathrm{HFEOWG}_{\varepsilon}$ operator have been developed, which are extensions of the weighted geometric operator and the OWG operator with hesitant fuzzy information, respectively. Moreover, some desirable properties of the proposed operators have been discussed and the relationships between the proposed operators and the existing hesitant fuzzy aggregation operators introduced by Xia and Xu [36] have been established. Finally, based on the $\mathrm{HFEOWG}_{\varepsilon}$ operator, an approach of hesitant fuzzy decision making has been given and a practical example has been presented to demonstrate its practicality and effectiveness.

## Conflict of Interests

The authors declared that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Multiproject Resources Allocation Model under Fuzzy Random Environment and Its Application to Industrial Equipment Installation Engineering 

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#### Abstract

This paper focuses on a multiproject resource allocation problem in a bilevel organization. To solve this problem, a bilevel multiproject resource allocation model under a fuzzy random environment is proposed. Two levels of decision makers are considered in the model. On the upper level, the company manager aims to allocate the company's resources to multiple projects to achieve the lowest cost, which include resource costs and a tardiness penalty. On the lower level, each project manager attempts to schedule their resource-constrained project, with minimization of project duration as the main objective. In contrast to prior studies, uncertainty in resource allocation has been explicitly considered. Specifically, our research uses fuzzy random variables to model uncertain activity durations and resource costs. To search for the optimal solution of the bilevel model, a hybrid algorithm made up of an adaptive particle swarm optimization, an adaptive hybrid genetic algorithm, and a fuzzy random simulation algorithm is also proposed. Finally, the efficiency of the proposed model and algorithm is evaluated through a practical case from an industrial equipment installation company. The results show that the proposed model is efficient in dealing with practical resource allocation problems in a bilevel organization.


## 1. Introduction

Because more and more construction companies must deal with multiple projects at the same time, both the theory and practice of multiproject resource allocation problems (MPRAP) are being paid increasing attention in the construction industry. In existing researches, resource allocation has often been considered only a constraint in multiple project scheduling problems and thus MPRAP has often been called a resource-constrained multiple project scheduling problem [1]. The importance and the wide-ranging applicability of multiproject resource allocation methods have been more widely accepted in recent years [2-6]. Fricke and Shenhar [2] investigated the differences associated with the resource allocation between multiproject management and single project management. Ben-Zvi and Lechler [5] tested several multiproject resource allocation strategies in realistic environments using a heuristic simulation tool. Xu
and Zhang [6] proposed a resource-constrained scheduling model with multiple projects and applied it to a large-scale water conservancy and hydropower construction project.

All this research has assisted in the improvement of multiple project resource allocation. However, it is still commonly assumed that a single manager oversees all projects. In today's industrial climate, managers face an increasingly complicated decision environment. As a result, a single manager has difficulties in dealing with across project resource allocation in addition to resource management within projects. In this case, a bilevel organizational structure is frequently used for project management [7]. This type of organization structure has largely been used in the construction industry and the software industry. In this bilevel organization, a central authority (company manager) determines the allocation across several projects. Once resources have been assigned to a project, a project manager then schedules the activities within a single project using the assigned resources. Thus,
the project resource allocation in a bilevel organization is a bilevel decision-making problem. Jennergren and Müller [8] originally proposed a bilevel resource allocation problem, in which they discussed a simple case made up of a headquarter and two divisions. Yang and Sum [9] discussed a bilevel resource allocation problem using a systematic analysis, in which they defined the resource allocation and project scheduling rules, the performances of which were then evaluated through experimentation. Yang and Sum [7] further extended this research as they examined the performance of due date, resource allocation, project release, and activity scheduling rules at the same time. This research had a positive impact on multiproject resource allocation as several resource allocation methods were compared. However, litter research has focused on an optimization decision-making problem. As a result, some managers are still confused about planning for optimal resource allocation in a bilevel and multiproject environment. Therefore, in this paper we will discuss a multiproject resource allocation problem using a bilevel optimization programming model.

In addition to the complexities of the bilevel structure, uncertainty is also frequently considered in resource allocation problems. Methods for dealing with uncertainty in decision-making mainly include random, fuzzy, and interval mathematical programming [10]. In resource allocation, the uncertainty traditionally has been assumed to be random. Golenko-Ginzburg and Gonik [11, 12] considered a resourceconstrained network project scheduling problem with a random activity duration dependent on the resource amounts assigned to that activity. Cohen et al. [13] addressed a resource allocation problem in stochastic, finite-capacity, multiproject systems using a cross entropy methodology. Bidot et al. [14] summarized these stochastic methods for dealing with uncertainty in practical project management. Though probability theory has been successfully applied to resource allocation problems, sometimes some uncertain parameters cannot be modeled using random theory because of several factors such as the lack of statistical data. In this case, probability theory can be replaced by fuzzy set theory as introduced by Zadeh [15]. Mjelde [16] first applied fuzzy set theory to a resource allocation problem. Following this research, many papers focus on resource allocation problems under a fuzzy environment [17-19] where the duration was often modelled as a fuzzy variable. In the research mentioned above, fuzziness and randomness were often considered separate aspects. But in reality, we may face a hybrid uncertain environment where fuzziness and randomness coexist in a decision-making process. In a project, some activities may be rarely performed for which the duration times can be described by fuzzy variables, while some other activities may have been processed many times before so duration times can be summarized using random variables. In this case, the fuzzy random variable, which was first proposed by Kwakernaak [20], can be a useful tool for the optimization of a resource assignment with mixed fuzziness and randomness uncertainty, because it is able to deal with the two types of uncertainty simultaneously.

Hence, this paper focuses on this type of bilevel multiproject resource allocation problem (BLMPRAP) under
a fuzzy random environment, in which we attempt to find an optimal allocation scheme using a bilevel programming model. In this model, the decision maker on the upper level is the company manager whose aim is to determine an optimal scheme for the allocation of company resource among multiple projects. The objective of the company manager is to minimize total cost which consists of resource cost and a tardiness penalty, while at the same time considering the lower-level decision-making. On the lower level, each project manager attempts to schedule their project in the most efficient way using the assigned resources. Contrary to the company manager, the project manager's objectives are focused on the project duration and finishing time of the project, rather than the cost. Therefore, the minimization of project duration is considered the objective on the lower level. In addition, the uncertainty associated with activity duration and resource costs is also explicitly considered in the model. Specifically, our research uses fuzzy random variables to model the activity duration and resource costs. Moreover, we also focus on a solution method for the proposed bilevel resource allocation model and two main heuristic methods are discussed in the algorithm section, and a solution method which integrates these two algorithms is proposed. Finally, a representative case is used to test the model and algorithm.

The remainder of this paper is organized as follows. In Section 2, two key problems are discussed: why the bilevel model is used for this problem and how to model the uncertain resource allocation environment using fuzzy random variables. Based on this analysis, the bilevel model for the considered multiproject resource allocation problem with fuzzy random variables in a hierarchical organization is detailed in Section 3. To solve the proposed model in Section 3, a solution algorithm based on the PSO and GA is introduced in Section 4. Then in Section 5, this proposed model and algorithm are applied to a practical case, which reflects the effectiveness of the proposed methodology in dealing with practical problems. Finally, concluding remarks and future research directions are outlined in Section 6.

## 2. Key Problem Statement

The problem considered in this paper is a bilevel resource allocation problem with multiple projects under a fuzzy random environment. In this section we explain why this problem should be solved using a bilevel programming model and outline the procedure for modeling uncertain activity duration and resource cost using fuzzy random variables.
2.1. Bilevel Resource Management Framework. In practice, more and more companies are concurrently managing multiple projects with limited resources. In order to service each project better, a hierarchical (bilevel) organizational structure which consists of a company level and a project level is being used by many companies such as construction companies, software companies, and some production companies. In these cases, the company managers need to deal with hierarchical decision-making.

To handle these decentralized optimal planning problems in a hierarchical (or multiple-level) organization which has
more than one decision maker, multilevel mathematical programming has been proposed [21]. Bilevel programming indicates that the hierarchical organization is composed of only two levels and is a sequence of two optimization problems in which the constraints region of one is determined by the solution of the second [22]. There are some common features in bilevel programming [23-25]. (1) There are interactive decision-making units within a hierarchical or bilevel structure in the organization. (2) The lower-level executes its decisions after, and in consideration of, the decisions of the upper level or leader. (3) Both the leader and the follower independently seek to maximize or minimize their own objectives, and often these objectives are in conflict. (4) The mutual influence between the leader and follower when making a decision is reflected in both the objective function and the constraints. Therefore, to assist in resolving the conflict between the two levels, bilevel programming is an appropriate method for dealing with the decision-making in a bilevel organization.

In this paper, resource allocation is considered across multiple projects in a bilevel organization which includes two levels of managers (i.e., company managers and project managers). On the upper level, the company managers are generally responsible for corporate planning and coordination between the multiple project groups with the aim of maximizing the company's income. In the construction industry, they generally control and manage key company resources, such as large-scale equipment and senior engineering staff. However, resources are generally limited and some are also very expensive. To save costs, company managers have to make detailed resource assignment plans over multiple projects. Cost is dependent on the practical project schedule as a tardiness penalty occurs if the project duration exceeds its contracted finishing time, so company managers must also consider the project managers' decisions when planning their resource allocation over multiple projects. On the lower level, project managers also maintain a reasonable level of resources called the "project resource." The manager of each project is responsible for resource allocation (including project resources and assigned company resources) over multiple project activities to ensure that the project is completed on time, so they also have to develop a resource-constrained project schedule after the company resources assignment plan has been completed.

Usually, there are different objectives between the company's projects and the project managers. The company managers desire a resources assignment plan that achieves a lower cost and a shorter duration. However, at the same time, the achievement of these objectives is dependent on not only the upper-level decision-making, but also the actions of the project managers. The project managers pay more attention to the finishing time or the cost of a single project, which may be in contradiction to the company's benefit. Company managers know that the project managers make decisions based on the assigned company resources, so to some degree they have some influence on the project managers' decisions through the different resource allocation schemes. Therefore, the considered multiproject resource allocation is a decisionmaking problem in a bilevel organization with a degree of
conflict in terms of benefits. It is appropriate to solve this problem using bilevel programming. In bilevel programming, the decision maker on the upper level is the company manager who seeks to allocate company resources to multiple projects at the lowest cost. On the lower level, each project manager attempts to schedule their project with the objective of project duration minimization under resource constraints. The bilevel resource assignment problem is illustrated in Figure 1.

### 2.2. Uncertain Activity Duration and Resource Cost. The fuzzy

 random environment has been studied and applied to many areas such as inventory problem [26], vehicle routing [27], logistics network design [28], and water resources allocation [29]. These studies show the necessity of considering fuzzy random environment in practical problems. With this background and evidence, there is a strong motivation for considering a fuzzy random environment for the BLMPRAP.In real conditions, uncertainty analysis is always an important consideration for managers in many areas of operations, such as the uncertainty that exists in activity durations, resource requirements, and operating costs. In this paper, our main consideration is project activity duration and unit resource cost uncertainty.

Activity durations are always uncertain because of a lack of knowledge and in previous studies they have often modeled these uncertainties as random or fuzzy variables. However, there are often circumstances where both fuzzy and random factors exist in a complex uncertain environment. For example, a company plans to install a boiler in a power plant construction project in October, but they do not have enough experience or historical data on this type of project. In this case, fuzzy variables are used to model the activity durations. At the same time, some known information associated with the activity duration, such as the effects of the weather, can be modeled as a random variable. For example, a shower may slow down the transportation speed of some necessary equipment or extreme temperatures may lead to lower work efficiency. From the local statistical information, in October, it is predicted with a probability of 0.6 to rain, with a probability of 0.3 to be fine, and a probability of 0.1 to be cloudy. Therefore, the weather can be modelled as a discrete random variable. In this situation, activity durations considering both fuzzy factors and random factors can be modelled as fuzzy random variables as shown in Figure 2. This means that more information is modelled into the variable, so more precise data can be obtained for solving practical problems through the use of fuzzy random variables rather than fuzzy variables or random variables, which results in a more precise solution to the model.

The situation is similar for resource costs. For example, as the gasoline price and the crane operators' wages are expected to rise, the cost of crane operations will also go up. However, it is very difficult to obtain a precise value because of the many uncertainties. In this case, an interval [ $a, b$ ] is used to model the changing cost. Further, based on the analysis of historical data, the cost is most possibly at around $\rho$, which is an expected value of a random variable which follows a normal distribution $N(\mu, \sigma)$. Then, the cost of the crane operations


Figure 1: Flow chart for the considered bilevel resource assignment problem.


Figure 2: Employing fuzzy random variables to model activity durations.


Figure 3: Employing a fuzzy random variable to express the unit cost for company resources.
can be described as a fuzzy random variable $\tilde{\bar{c}}=(a, \rho, b)$ with $\rho \sim N(\mu, \sigma)$ as Figure 3.

Considering the bilevel structure and uncertain environment simultaneously, the BLMPRAP under a fuzzy random environment can be stated as follows. A company has contracted n projects at the same time, though the company managers are unable to fully manage these projects, so for effective management they take charge of only some key resources and establish project groups to manage the projects. The problem the company managers face is how
to assign the company resources to each project group, while the project manager has to schedule their project with some resource constraints. To deal with this uncertainty, the activity durations and resource costs are modeled as fuzzy random variables. The decision-making framework for the proposed bilevel multiproject resource assignment problem is illustrated in Figure 4.

## 3. Modelling

To solve the multiproject resource allocation problem in a bilevel organization, a bilevel programming model under a fuzzy random environment is constructed. The mathematical description for this problem is given as follows.
3.1. Assumptions. To model the problem more efficiently, the following assumptions are adopted.
(1) The bilevel resource assignment problem consists of multiple resources and multiple projects. There are no new projects during the scheduled resource allocation periods.
(2) The problem has two levels of decision makers, that is, company managers on the upper level and project managers on the lower level. The managerial objective on the upper level is to minimize the total cost for all projects, and the objective on the lower level is to minimize the project duration.
(3) A single project consists of a number of activities each with several optional execution modes. Each mode is a combination of duration and resource requirements [30]. Activities cannot be interrupted, and every activity must be performed in only one mode.
(4) Each activity needs multiple types of resources. The unit cost for each company resource and the duration for each activity are modelled as fuzzy random variables.


Fuzzy random activity duration
Figure 4: Decision-making framework for the considered BLMPRAP.
(5) The company manager is responsible for resource allocation during multiple projects. Resources assigned to all projects do not exceed the limited quantities in any time period.

### 3.2. Notations

## Indices

$n$ : the project index, $n=1,2, \ldots, N$;
$i$ : the activity index, $i=1,2, \ldots, I_{n}$;
$j$ : the mode index, $j=1,2, \ldots, J_{i}$;
$k$ : the resource index, $k=1,2, \ldots, K$;
$t$ : the project time period index, $t=1,2, \ldots, T$;
$p$ : the resource assignment time period index, $p=$ $1,2, \ldots, P$.

## Parameters

$R_{k p}$ : the total quantity of resource $k$ in time period $p$;
$\widetilde{\bar{d}}_{n i j}$ : duration of activity $i$ executed in mode $j$ in project $n$;
$\operatorname{Pr}_{n}(i)$ : set of immediate predecessors of activity $i$ in project $n$;
$r_{n i j k}$ : amount of resource $k$ required to execute activity $i$ in mode $j$ in project $n$;
$T_{n}$ : the scheduled finishing time of project $n$; $T_{n}^{*}$ : the predetermined finishing time of project $n$; $t_{n} i^{\mathrm{EF}}$ : the early finishing time of activity $i$ in project $n$; $t_{n} i^{\mathrm{LF}}$ : the late finishing time of activity $i$ in project $n$; $D T_{n}$ : the overdue time of project $n$;
$t_{n i j}$ : the processing finishing time of activity $i$ in mode $j$ in project $n$;
$\tilde{\bar{c}}_{k p}$ : the unit cost of resource $k$ in period $p$; $c p_{n}$ : the unit overdue penalty cost of project $n$.

## Decision Variables

$R_{n k p}$ : the quantity of resource $k$ assigned to project $n$ in time period $p$;
$x_{n i j t}= \begin{cases}1, & \text { if activity } i \text { is executed in mode } j \text { and is } \\ \text { scheduled to be finished in time } t\end{cases}$
3.3. Multiproject Resource Allocation. The problem the company manager on the upper level faces is how to allocate the limited company resources over several projects in each period (generally the period is one week); in other words, they need to decide the quantity to be allocated to each project in each period for each type of resource. With this in mind, the decision variables for the upper level are $R_{n k p}$.

For resource allocation problems, minimization of the total cost or maximization of the total profit is often considered as the decision objective [5]. The cost is made up of the resource costs and the total tardiness penalty for the multiple projects. Resource costs occur when resources are allocated to a project group, so the resource cost can be stated as $\sum_{n=1}^{N} \widetilde{\bar{c}}_{k} R_{n k p}$ for each type of resource in every period. The project tardiness penalty occurs when a project finishing time exceeds its predetermined finishing time. $D T_{n}$ represents the overdue time of project $n$. It is a function on the finish time as in

$$
D T_{n}= \begin{cases}T_{n}-T_{n}^{*}, & \text { if } T_{n} \geq T_{n}^{*}  \tag{2}\\ 0, & \text { otherwise }\end{cases}
$$

Therefore, the total tardiness penalty can be stated as $\sum_{n=1}^{N} c p_{n} D T_{n}$, and the total cost can be described as

$$
\begin{equation*}
\sum_{n=1}^{N} c p_{n} D T_{n}+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \tilde{\bar{c}}_{k p} R_{n k p} \tag{3}
\end{equation*}
$$

In this equation, the unit resource cost $\widetilde{\bar{c}}_{k p}$ is uncertain because of the many changing influences such as gasoline prices and wages. In this paper, we consider a hybrid uncertain environment involving both fuzziness and randomness. To deal with this uncertainty, $\tilde{\bar{c}}_{k p}$ is modelled as a fuzzy random variable which means that the total cost is also a fuzzy random variable. Technically, it is not possible to derive a precise minimum total cost and an optimal solution. In a practical decision-making process, the decision makers usually choose a satisfactory solution with a certain deviation rather than an optimal solution. In these cases, chanceconstrained programming, which was first introduced by Charnes and Cooper [31], is often used. It is assumed that the goal of decision makers is to minimize the objective value on the condition of chance level $\alpha$, where $\alpha$ is the predetermined confidence level which is provided as an appropriate safety margin by the decision maker. Generally, the value of $\alpha$ is bigger than 0.5 [32].

In order to introduce the chance-constrained programming, the concept of a chance measure for the fuzzy random variables is first explained. Let $\xi$ be a fuzzy random variable defined on $(\Omega, \mathscr{A}, \operatorname{Pr})$, and $f: \mathbf{R} \rightarrow \mathbf{R}$ is a real-valued continuous function. Then a primitive chance of a fuzzy random event characterized by $f(\xi) \leq 0$ is a function from $(0,1]$ to $[0,1]$, defined as in the following [33] equation:

$$
\begin{align*}
& \operatorname{Ch}\{f(\xi) \leq 0\}(\beta) \\
& \quad=\sup _{\beta \in[0,1]}\{\omega \mid \operatorname{Pr}\{\omega \in \Omega \mid \operatorname{Pos}\{f(\xi) \leq 0\} \geq \alpha\} \geq \beta\}, \tag{4}
\end{align*}
$$

where $\operatorname{Pos}\{\cdot\}$ is the possibility of the fuzzy event and $\operatorname{Pr}\{\cdot\}$ is the probability of the random event. $\beta$ is referred to as the predetermined confidence levels associated with the probability measure of the random event $\operatorname{Pos}\{f(\xi) \leq 0\} \geq \alpha$. Generally, decision makers tend to take the same confidence levels between the parameters $\alpha$ and $\beta$.

From the definition of the chance measure, we can derive the following equation:

$$
\begin{align*}
& \operatorname{Ch}\{f(\xi) \leq 0\}(\beta)  \tag{5}\\
& \geq \alpha \Longleftrightarrow \operatorname{Pr}\{\omega \in \Omega \mid \operatorname{Pos}\{f(\xi) \leq 0\} \geq \alpha\} \geq \beta
\end{align*}
$$

Since it is not possible to derive a precise minimum objective, the decision makers descend to seek a minimum objective value $F_{1}$ on the condition of possibility level $\alpha$ at probability level $\beta$. Then, the fuzzy random objective can be transformed into the chance constraint $\operatorname{Ch}\left\{\sum_{n=1}^{N} c p_{n} D T_{n}+\right.$ $\left.\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \tilde{\bar{c}}_{k p} R_{n k p} \leq F_{1}\right\}(\beta) \geq \alpha$. That is, $\operatorname{Pr}\{\omega$ | $\left.\operatorname{Pos}\left\{\sum_{n=1}^{N} c p_{n} D T_{n}+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \tilde{\bar{c}}_{k p} R_{n k p} \leq F_{1}\right\} \geq \alpha\right\} \geq$ $\beta$. Finally, the uncertain model is transformed to a chanceconstrained model, and the following objective function and constraint are obtained:

$$
\begin{equation*}
\min F_{1} \tag{6}
\end{equation*}
$$

subject to (s.t.)

$$
\begin{align*}
& \operatorname{Pr}\left\{\omega \mid \operatorname{Pos}\left\{\sum_{n=1}^{N} c p_{n} D T_{n}+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \widetilde{\bar{c}}_{k p} R_{n k p} \leq F_{1}\right\}\right.  \tag{7}\\
& \quad \geq \alpha\} \geq \beta
\end{align*}
$$

Resource constraints must be met for all types of resource allocation problems. That is, for each type of resource, the total quantity allocated to every project cannot exceed the ownership quantity of the company in each period. This constraint is described as

$$
\begin{equation*}
\sum_{n=1}^{N} R_{n k p} \leq R_{k p}, \quad \forall k=1, \ldots, K ; p=1, \ldots, P \tag{8}
\end{equation*}
$$

Equations ((4)-(6)) make up the resource allocation model as in (9). In this model, the project finishing time is determined by solving the lower-level model. It can also be seen that the decisions on the lower level have an effect on the resource allocation on the upper level as follows:

$$
\begin{aligned}
& \text { min } F_{1} \\
& \text { s.t. } \quad \operatorname{Pr}\left\{\omega \mid \operatorname{Pos}\left\{\sum_{n=1}^{N} c p_{n} D T_{n}+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \overline{\bar{c}}_{k p} R_{n k p} \leq F_{1}\right\}\right. \\
& \geq \alpha\} \geq \beta \\
& \sum_{n=1}^{N} R_{n k p} \leq R_{k p}, \quad \forall k=1, \ldots, K ; p=1, \ldots, P \\
& D T_{n}= \begin{cases}T_{n}-T_{n}^{*}, & \text { if } T_{n} \geq T_{n}^{*} \\
0, & \text { otherwise }\end{cases}
\end{aligned}
$$

where $T_{n}$ is solved in the lower-level model.
3.4. Resource-Constrained Project Scheduling. When the resources are allocated to each project, the project manager has to consider how to make use of these resources to finish the project more quickly. Therefore, each project manager is faced with a resource-constrained project scheduling problem. Usually, the resources consist of company resources and project resources. In this paper, we only consider the company resources when scheduling the project.

For project scheduling, the minimization of project duration is often considered as the decision objective [1]. In this paper, the finishing time of the last activity is used to describe the project duration. This finishing time must be located in the range between the early finishing time and the late finishing time of the project after consideration of the entire range of possible execution modes. This can be stated as $\sum_{j=I}^{m_{I}} \sum_{t=t_{n} I^{\mathrm{EF}}}^{t_{n} I^{\mathrm{LF}}} t x_{n I j t}$. Here, $t_{n} I^{\mathrm{EF}}$ and $t_{n} I^{\mathrm{LF}}$ are the early finishing time and the late finishing time of activity $I$ in project $n$, respectively. Therefore, the objective function can be described as follows:

$$
\begin{equation*}
T_{n}=\sum_{j=I_{t=t_{n}} I^{\mathrm{EF}}}^{m_{n} I x_{n I j t} . . . . ~ t_{n} I^{\mathrm{F}}} \tag{10}
\end{equation*}
$$

In addition, some constraints must be met. First, each activity must be scheduled and its finish time must be in the range of its earliest finishing time and its latest possible finishing while ensuring that all activities are adequately arranged and there is only one execution mode for each activity. So we can get the following constraint:

$$
\begin{equation*}
\sum_{j=1}^{m_{n}{ }^{i}} \sum_{t=t_{n} i^{\mathrm{EF}}}^{t_{n}{ }^{\mathrm{LF}}} x_{n i j t}=1, \quad i=1,2, \ldots, I . \tag{11}
\end{equation*}
$$

In the scheduling problem, precedence is the basic term which ensures the rationality of the arrangement. $\sum_{j=1}^{m_{n} l} \sum_{t=t_{n}}^{t_{l} l^{\mathrm{LE}}} t x_{n l j t}$ is denoted as the actual finishing time of activity $l$ in project $n$. This must be between the earliest finishing time and the latest finishing time when the activity is scheduled in a certain executed mode. $\sum_{j=1}^{m_{n} i} \sum_{t=t_{n} i^{i^{\mathrm{EF}}}}^{t_{n}{ }^{\mathrm{LF}}}\left(t-\widetilde{d}_{i j}\right) x_{i j}$ is the starting time of the immediately following activity $i$ in project $n$. Generally, the beginning time of each activity must be posterior to the finishing time of its immediate predecessors. However, under a fuzzy random environment, the duration of activity $i$ is a fuzzy random variable. In this case, it is difficult to meet this constraint strictly. Decision makers always hope to meet the constraint using an expected value for the fuzzy random variable. From the definition of Puri and Ralescu [34], the expected value $E(\xi)$ of a fuzzy random variable $\xi$ can be calculated using the following equation:

$$
\begin{align*}
(E(\xi))_{\alpha} & =\int_{\Omega} \xi_{\alpha} d P \\
& =\left\{\int_{\Omega} f(\omega) d P(\omega): f \in L^{1} P, f(\omega) \text { a.s. }[P]\right\}, \tag{12}
\end{align*}
$$

where $\int_{\Omega} \xi_{\alpha} d P$ is the Aumann integral of $\xi_{\alpha}$ about $P$ and $L^{1} P$ denote all of the integrable function $f: \Omega \rightarrow R$ about the probability measure $P$.

The fuzzy expected value reflects the center value that the fuzzy random variable tends towards and describes the fuzzy random variable statistical properties. After going through the fuzzy expected operation above, all fuzzy random durations are transformed into fuzzy durations. Then the expected value operator of the fuzzy variables based on a fuzzy measure [33] can be used to transform the fuzzy duration into a crisp duration. This can be calculated using

$$
\begin{align*}
E^{\mathrm{Me}}(E(\tilde{\bar{d}}))= & \int_{0}^{+\infty} \operatorname{Me}\{E(\tilde{\bar{d}}) \geq r\} d r  \tag{13}\\
& -\int_{0}^{+\infty} \operatorname{Me}\{E(\tilde{\bar{d}}) \leq r\} d r
\end{align*}
$$

where Me is a type of fuzzy measure. Let $A$ be a fuzzy event; then $\operatorname{Me}\{A\}=\lambda \operatorname{Pos}\{A\}+(1-\lambda) \operatorname{Nec}\{A\}$. $\lambda$ are the optimistic and pessimistic indices, respectively, to determine the combined attitude of the decision maker.

From the fuzzy random expected value operator and the fuzzy expected value operator, the expected precedence constraints can be obtained as

$$
\begin{align*}
& \sum_{j=1}^{m_{n} l} \sum_{t=t_{n}} t_{n}^{\mathrm{EF}} l^{\mathrm{LF}} t x_{n l j t} \\
& \leq \sum_{j=1}^{m_{n} i} \sum_{t=t_{n} \mathrm{i}^{\mathrm{EF}}}^{t_{n} i^{\mathrm{LF}}}\left(t-\operatorname{Me}\left\{E\left(\tilde{\bar{d}}_{n i j}\right)\right\}\right) x_{n i j t}  \tag{14}\\
& \quad l \in P(i), i=1,2, \ldots, I_{n} .
\end{align*}
$$

In addition to the precedence constraints, resource constraints must be considered as well in this problem. In each period, the available resource quantity is $R_{n k p}$ which is allocated as part of the upper-level decisions. The resource constraint is described in (15). It ensures that the amount of resources $k$ used by all activities does not exceed its limited quantity $R_{n k p}$ in any period as follows:

$$
\begin{gather*}
\sum_{i=1}^{I_{n}} \sum_{j=1}^{m_{n i}} r_{n i j k} \sum_{s=t}^{t+\mathrm{Me}\left\{E\left(\tilde{\bar{d}}_{i j}\right)\right\}-1} x_{i j s} \leq R_{n k p}  \tag{15}\\
k \in K, t=1,2, \ldots, T
\end{gather*}
$$

The objective function and the constraints form the resource-constrained project scheduling model as in

$$
\begin{array}{ll}
\min & T_{n}=\sum_{j=I}^{m_{n} I} \sum_{t=t_{n} I^{\mathrm{EF}}}^{t_{n} I^{\mathrm{LF}}} t x_{n I j t} \\
\text { s.t. } & \sum_{j=1}^{m_{n}{ }^{i}} \sum_{t=t_{n}{ }^{\mathrm{EF}}}^{t_{n} i^{\mathrm{LF}}} x_{n i j t}=1, \quad i=1,2, \ldots, I
\end{array}
$$

$$
\begin{align*}
& \sum_{j=1}^{m_{n} l} \sum_{t=t_{n}{ }^{\mathrm{EF}}}^{t_{n} l^{\mathrm{LF}}} t x_{n l j t} \\
& \leq \sum_{j=1}^{m_{n} i} \sum_{t=t_{n}{ }^{\mathrm{EF}}}^{\mathrm{E}_{n} i^{\mathrm{LF}}}\left(t-\operatorname{Me}\left\{E\left(\tilde{\bar{d}}_{n i j}\right)\right\}\right) x_{n i j t}, \\
& \quad l \in P(i), i=1,2, \ldots, I_{n} \\
& \sum_{i=1}^{I_{n}} \sum_{j=1}^{m_{n i}} r_{n i j k}^{t+\operatorname{Me}\left\{E\left(\tilde{\bar{a}}_{i j}\right)\right\}-1} \sum_{s=t} x_{i j s} \leq R_{n k p}, \\
& \quad k \in K, t=1,2, \ldots, T \\
& x_{n i j t} \in\{0,1\}, \quad \forall n, i, j, t . \tag{16}
\end{align*}
$$

3.5. The Completed Bilevel Model. There are two levels of decision makers in the considered BLMPRAP. The decision maker on the upper level, the company manager, hopes to allocate the company resources to multiple projects at the lowest cost. The cost consists of resource costs and the tardiness penalty, so the upper-level decision maker is able to control the resource cost through appropriate allocation. The tardiness penalty is dependent on the finishing time of all projects, which in turn is determined by the specific project managers through their project schedule. In this situation, the company manager must consider the decision of the project managers. The company manager does know that the project managers must schedule their projects under the resource constraints. Therefore, the company manager can influence the decision-making of project managers on the lower-level model using different resource allocation schemes.

On the lower level, each project manager attempts to make a more efficient schedule under the resource constraints. The objective is often to minimize the finishing time of the project, although this may conflict with the company's objective. This is another reason why such a problem needs to be modeled as a bilevel programming model. In addition, uncertainty also impacts the decision. In this paper, the uncertain resource cost and activity duration are described using fuzzy random variables. On the upper level, possibility theory is used to deal with the uncertain resource cost. On the lower level, an expected value operation is used to cope with the uncertain activity duration. In sum, the complete bilevel programming model can be established based on the above discussion as in (17). In the model, the chance constrains are nonlinear when being transformed to crisp equations. As a result, the proposed model is a nonlinear bilevel optimization model under a fuzzy random environment as follows:

$$
\begin{array}{ll}
\min _{R_{n k p}} & F_{1} \\
\text { s.t. } & \operatorname{Pr}\left\{\omega \mid \operatorname{Pos}\left\{\sum_{n=1}^{N} c p_{n} D T_{n}\right.\right. \\
& \left.\left.+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \tilde{\bar{c}}_{k p} R_{n k p} \leq F_{1}\right\} \geq \alpha\right\} \geq \beta
\end{array}
$$

$$
\begin{aligned}
& \sum_{n=1}^{N} R_{n k p} \leq R_{k p}, \quad \forall k=1, \ldots, K ; p=1, \ldots, P \\
& D T_{n}= \begin{cases}T_{n}-T_{n}^{*}, & \text { if } T_{n} \geq T_{n}^{*} \\
0, & \text { otherwise }\end{cases}
\end{aligned}
$$

where $T_{n}$ is solved in the following model :

$$
\begin{align*}
& \min _{x_{n i t}} T_{n}=\sum_{j=I}^{m_{n} I} \sum_{t=t_{n} I^{\mathrm{EF}}}^{t_{n} I^{\mathrm{LF}}} t x_{n I j t}, \quad n=1, \ldots, N \\
& \text { s.t. } \quad \sum_{j=1}^{m_{n} i} \sum_{t=t_{n} i^{\mathrm{EF}}}^{t_{n}{ }^{\mathrm{LF}}} x_{n i j t}=1, \quad i=1,2, \ldots, I \\
& \sum_{j=1}^{m_{n}} \sum_{t=t_{n}}^{l} \sum_{t_{n}}^{l^{\mathrm{EF}}} t x_{n l j t} \\
& \leq \sum_{j=1}^{m_{n}{ }^{i}} \sum_{t=t_{n} \mathrm{i}^{\mathrm{EF}}}^{t_{n} i^{\mathrm{LF}}}\left(t-\mathrm{Me}\left\{E\left(\tilde{\bar{d}}_{n i j}\right)\right\}\right) x_{n i j t}, \\
& l \in P(i), i=1,2, \ldots, I_{n} \\
& \sum_{i=1}^{I_{n}} \sum_{j=1}^{m_{n i}} r_{n i j k}^{t+\operatorname{Me}\left\{\left(\tilde{\bar{d}}_{i j}\right)\right\}-1} \sum_{s=t} x_{i j s} \leq R_{n k p}, \\
& k \in K, t=1,2, \ldots, T \\
& x_{n i j t} \in\{0,1\}, \quad \forall n, i, j, t . \tag{17}
\end{align*}
$$

## 4. Fuzzy Random Simulation-Based aPSO-hGA

The proposed model is a bilevel programming model, which is considered as a strong NP-hard problem [35, 36]. It is often difficult to obtain an analytical optimal solution for such problems, and the most commonly used methods are to obtain a numerically optimal solution or a numerically efficient solution using an approximation or heuristic algorithm. For bilevel programming model, the particle swarm optimization algorithm (PSO) has been proposed in some research and has had good results [37, 38]. An important motivation for using PSO to solve bilevel programming is that PSO is usually quicker than other algorithms, since it often takes much more time to solve a bilevel model than a single level one. In our model, not only the bilevel structure of the model but also the considered multimode resourceconstrained project scheduling problems greatly increase the computing complexity and solution speed. In the bilevel model, the lower-level models are the constraints of the upper-level model. If the found solution is not optimal, then the final solution may not be feasible. This leads to a bilevel model which cannot be solved. Therefore, unlike the upper-level model, an algorithm with higher accuracy and stability needs to be chosen. In this paper, the multimode
resource-constrained project scheduling on the lower level is solved using a genetic algorithm (GA). In addition, to deal with the fuzzy random chance constraints, a fuzzy random simulation procedure is proposed. In this case, a hybrid algorithm of an adaptive PSO and a GA based on fuzzy random simulation (fuzzy random simulationbased aPSO-hGA) is proposed to solve the proposed nonlinear bilevel optimization model under a fuzzy random environment.
4.1. Framework for the Proposed Solution Algorithm. To solve the bilevel model, a particle swarm optimization is proposed to search for the solution to the upper level. At the beginning of the algorithm, some feasible solutions (particles: $R_{n k p}$ ) for the upper level decision variables which meet the constraints of upper level model are generated. Then the solutions are set into the lower-level model. A genetic algorithm is used to find the optimal solution $\left(x_{n i j t}\right)$ to the lower-level model. Both the solutions to the upper level and the lower level consist of the final feasible solutions ( $\left\{R_{n k p}, x_{n i j t}\right\}$ ) of the model, and they are evaluated and the correlative fitness values are calculated using a fuzzy random simulation procedure. Then the pbest, gbest, and lbest are recorded, and new solutions are generated through an update of the particles. This program goes on until the stop condition is met. In addition, in order to improve convergence speed and search efficiency, a float coding method and a parameter adaptation method are proposed for the PSO algorithm, respectively. The proposed solution approach is then a hybrid of the PSO and GA, and its overall procedure can be seen in Figure 5.
4.2. Solving the Resource Allocation Using an Improved aPSO. To solve the bilevel model, an improved adaptive PSO is introduced to cope with the upper-level programming. In contrast to classical PSO, to improve the convergence speed, a float coding method, which is capable of incorporating various constraints in its implementation [39] and has been used in PSO for solving bilevel programming problem [37], is proposed to generate the initial particles for the upperlevel variables. At the same time, a parameter adaptation regulation is applied to improve the search efficiency of the PSO. In addition, to deal with the uncertainty on the upper level, a fuzzy random simulation procedure is proposed to calculate the fitness value of each particle. The procedure for the improved adaptive PSO is as follows.

Step 1. Set the parameters for the adaptive PSO: swarm_size, iteration_max, $c_{p}, c_{g}, c_{l}$, inertia weight_max, and inertia weight_min.

Step 2. Initialize the velocity and the position of the upper-level model. Each particle is represented as $\theta=\left\{R_{111}, R_{112}, \ldots, R_{11 P}, R_{121}, R_{122}, \ldots, R_{1 K P}, \ldots, R_{N K P}\right\}$. In order to generate the random particle positions for the upper-level variables, float coding method is adopted. Thus, every particle represents a real dimensional position. At the same time, the constraints $\sum_{n=1}^{N} R_{n k p} \leq R_{k p}$, for all $k=1, \ldots, K, p=1, \ldots, P$, are also incorporated into the coding to ensure that the generated particles meet the
constraints on the upper level. That is, for any given $k$ and $p$, $R_{1 k p}$ is generated randomly within the range $\left[0, R_{k p}\right.$ ] while $R_{n k p}$ is generated within the range $\left[0, R_{k p}-\sum_{i=1}^{n-1} R_{i k p}\right]$.

Step 3. Solve the lower-level programming with the initialization result of the upper-level variables using the proposed adaptive hybrid genetic algorithm for the multimode resource-constrained project scheduling problem.

Step 4. Calculate each fitness value using a fuzzy random simulation procedure using the calculated results of the lower level: $x_{n i j t}$ and $T_{n}$. Here, the objective value of the upper-level model is considered the fitness value which is estimated using Procedure 1.

Step 5. Update the pbest, gbest, and lbest.
Step 5.1. Update pbest: for $s=1 \cdots S$, if $\operatorname{Fitness}\left(\theta_{s}\right)<$ Fitness $($ pbest $)$, pbest $=\theta_{s}$.

Step 5.2. Update gbest: for $s=1 \cdots S$, if Fitness $\left(\theta_{s}\right)<$ Fitness(gbest), gbest $=\theta_{s}$.

Step 5.3. Update pbest: for $s=1 \cdots S$, among all pbest from $K$ neighbors of the sth particle, set the personal best which obtains the least fitness value to be lbest.

Step 6. Update the inertia weight for iteration $\tau$ using the following equations:

$$
\begin{align*}
& \bar{\omega}=\frac{\sum_{s=1}^{S} \sum_{h=1}^{H}\left|\omega_{s h}\right|}{S \cdot H}, \\
& \omega^{*}= \begin{cases}\left(1-\frac{1.8 \tau}{T}\right) \omega^{\max }, & 0 \leq \tau \leq \frac{T}{2}, \\
\left(0.2-\frac{0.2 \tau}{T}\right) \omega^{\max }, & \frac{T}{2} \leq \tau \leq T,\end{cases}  \tag{18}\\
& \Delta w=\frac{\left(\omega^{*}-\bar{\omega}\right)}{\omega^{\max }}\left(w^{\max }-w^{\min }\right), \\
& w=w+\Delta w, \\
& w=w^{\max } \text { if } w>w^{\max }, \\
& w=w^{\min } \quad \text { if } w>w^{\min } .
\end{align*}
$$

Step 7. Update the velocity and the position of each particle using the following equations:

$$
\begin{aligned}
\omega_{n k p}(\tau+1)= & w(\tau) \omega_{n k p}(\tau)+c_{p} u_{r}\left(\psi_{n k p}-\theta_{n k p}(\tau)\right) \\
& +c_{g} u_{r}\left(\psi_{g h}-\theta_{n k p}(\tau)\right)+c_{l} u_{r}\left(\psi_{n k p}^{L}-\theta_{n k p}(\tau)\right) \\
\theta_{n k p}(\tau+1)= & \theta_{n k p}(\tau)+\omega_{n k p}(\tau+1)
\end{aligned}
$$

For any given $k$ and $p$
If $\theta_{n k p}(\tau+1)>R_{k p}-\sum_{i=1}^{n-1} R_{i k p}$, then set
$\theta_{n k p}(\tau+1)=R_{k p}-\sum_{i=1}^{n-1} R_{i k p}$ and $\omega_{n k p}(\tau+1)=0$


Figure 5: The overall procedure of the proposed solution algorithm.

Step 1: Let $n=1, F_{n}=-\infty, f\left(\widetilde{\bar{c}}_{k p}\right)=\sum_{n=1}^{N} c p_{n} D T_{n}+\sum_{p=1}^{P} \sum_{k=1}^{K} \sum_{n=1}^{N} \tilde{\bar{c}}_{k p} R_{n k p}$.
Step 2: Generate $\omega$ from $\Omega$ according to the probability measure $\operatorname{Pr}$ of the fuzzy random variables $\tilde{\bar{c}}_{\text {kp }}$.
Step 3: Generate a determined vector $f\left(\widetilde{\bar{c}}_{k p}(\omega)\right)$ uniformly from the $\alpha$-cut of fuzzy vector $f\left(\widetilde{\bar{c}}_{k p}(\omega)\right)$.
Step 4: If $f\left(\widetilde{\bar{c}}_{k p}(\omega)\right) \leq F_{n}$, then let $F_{n}=f\left(\overline{\bar{c}}_{k p}(\omega)\right)$. Return to step 3, and repeat $M$ times.
Step 5: If $n=N$, set $N^{\prime}=\beta N$ and return the $N^{\prime}$ th least element in $F_{1}, F_{2}, \ldots, F_{N}$ as the fitness value; else go to step 2 , and $n=n+1$.

Procedure 1: Calculate fitness value using fuzzy random simulation.

| Activity | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Priority | 3 | 2 | 6 | 8 | 12 | 13 | 1 | 16 | 7 | 14 | 4 | 10 | 5 | 9 | 11 | 15 |
| Mode | 1 | 1 | 2 | 2 | 1 | 1 | 3 | 2 | 3 | 1 | 2 | 3 | 2 | 1 | 2 | 2 |

Figure 6: An individual solution composed of priority-based and multistage-based chromosomes.

If $\theta_{n k p}(\tau+1)<0$, then set
$\theta_{n k p}(\tau+1)=0$ and $\omega_{n k p}(\tau+1)=0$.

Step 8. If the stopping criterion is met, that is, $\tau=T$, stop. Otherwise, $\tau=\tau+1$ and go to Step 2.
4.3. Solving the Project Scheduling Using a-hGA. In the considered problem, the multimode resource-constrained project scheduling problem (MRCPSP) is discussed on the lower level. For the MRCPSP, many types of heuristic algorithms such as simulated annealing [40], PSO [41], and genetic algorithm [42] have been applied in previous research. Zhang et al. [41] compared the performance of the several types of algorithms. The results show that the genetic algorithm has a higher percentage in finding the optimal schedule. Hence, the adaptive hybrid genetic algorithm which is proposed by Kim et al. [42] is introduced to solve the lowerlevel problem. Let $P(t)$ and $C(t)$ be the parents and offspring in the current generation $t$. The detailed procedure of the proposed genetic algorithm is as follows.

Step 1. Set the initial value and parameters for the genetic algorithm: population size, crossover rate $p_{c}$, mutation rate $p_{m}$, and maximum generation $g_{\text {max }}$.

Step 2. Generate the initial population $p(0)$ using an activity priority and multistage-based encoding routine. The individual solution is composed of two chromosomes where the first shows the feasible activity finish sequence and the second consists of activity mode assignments [42]. An example with 16 activities is illustrated in Figure 6.

Step 3. Evaluate $p(t)$ using the priority-based decoding routine. The objective function is used as the fitness function.

Step 4. Create $C(t)$ from $P(t)$ by order-based crossover operator for activity finish priority. The procedure is explained bellow and an example is illustrated in Figure 7.

Step 4.1. Select a set of positions from one parent in activity priority at random.

Step 4.2. Produce a child by copying the cites in these positions into the corresponding positions.

Step 4.3. Delete the cites which are already selected form the second parent. The resulting sequence of sites contains the sites the child needs.

Step 4.4. Place the cites into the unfixed positions of the child from left to right according to the order of the sequence to produce one offspring.

Step 5. Create $C(t)$ from $P(t)$ using the neighborhood search mutation routine for activity mode.

Step 5.1. Select a set of pivot genes randomly from the current chromosome.

Step 5.2. Pick up the genes, and search for the neighbors until the bound of the activity mode.

Step 5.3. Evaluate the neighbors; choose the best neighbor.

Step 5.4. If the best neighbor is better than the current, replace the current with the neighbor.

Step 6. Climb $C(t)$ using the iterative hill climbing routine method.

Step 6.1. Select the optimum chromosome in the current generation.


Figure 7: Order-based crossover for activity priority.

Step 6.2. Randomly generate as many new chromosomes as the population size in the neighborhood of the optimal one.

Step 6.3. Select a chromosome with the optimal value of fitness among the set of the neighborhood.

Step 6.4. Compare with the optimal one in the current generation and the optimal one in the neighborhood; choose the better and put it into the current generation to be the optimum chromosome instead of the original one.

Step 7. Apply the heuristic for adaptively regulating GA parameters. Select $P(t+1)$ from $P(t)$ and $C(t)$ using elitist selection routine. The regulation is as follows:

Here, the $p(t)$ can be $p_{c}(t)$ or $p_{M}(t) \cdot p^{\prime}(t)$ are the new parameters amended. And when it is $p_{c}(t), \alpha=0.05$; when it is $p_{M}(t), \alpha=0.015 . \overline{f_{\text {par }_{\text {size }}}(t)}$ and $\overline{f_{\text {off }}^{\text {size }}}$ (t) are the average fitness values of parents and offspring in the current generation $t$. par ${ }_{\text {size }}$ and off size are the parent size and offspring size satisfying constraints.

Step 8. Repeat the above stages 3 to 7 after $t+1 \rightarrow t$ until the stop condition is met, that is, $t \geq g_{\max }$.

## 5. Case Study

In this section, computational experiments that were carried out on a real application are presented. Through an illustrative example on the data set adopted from a case study, the proposed method is validated and the efficiency of the algorithm is tested. The data for resource allocation, project scheduling, and others involved in the case are from an industrial equipment installation company (company X ) and an electric power design institute in Sichuan province, China. The case is introduced to demonstrate the potential real world applications of the proposed methods.
5.1. Presentation of Case Problem. Company X is a stateowned large-scale comprehensive installation and construction company with total assets of 460 million RMB and more than 3000 workers, which always contracts for multiple projects at the same time. To manage these projects, many project groups are found. These project groups can purchase some materials and equipment by themselves. However, some other resources must be allocated from the company such as large-scale equipment and professional staff.

The company has contracted for an installation engineering project at the HP power plant construction project in Luzhou. This is made up of two projects: the installation projects of 1 and 2 power units. At the same time, the company has also contracted for another installation project: an equipment installation project for a sewage treatment construction engineering project in Luzhou. Hence, the company is managing three projects at the same time. For management convenience, each project is managed by a project group who takes charge of the project scheduling and resource allocation within their project. However, some important resources such as large-scale installation equipment are still controlled by the company manager. The problem the company manager faces is how to allocate these resources over the three projects so as to gain maximal company income. This is a good example of the proposed bilevel resource allocation problem.

In this case, each power plant construction engineering installation project consists of 12 activities, while the sewage treatment construction engineering equipment installation project has 11 activities. The flow charts are illustrated in Figures 8 and 9. Every activity has several optional modes, and every activity in a certain mode has a certain duration and some resource requirements. Each activity duration is modelled as a discrete triangular fuzzy random variable. The project managers traditionally use days as time units. The corresponding data is as follows in Table 1. However, each project has resource limitations including manpower, materials, and equipments, with some key resources being managed by company managers. In this paper, we consider four resources, including cranes (CR), concreting machinery (CM), welding outfits (WO), and electrical equipment (EE). The total quantities and unit costs for these equipments are shown in Table 2. In this case, all the resources are assigned to projects at the beginning of each week. Hence, we use weeks as the unit time for resource allocation. It is assumed that the other resources are sufficient for all three projects.


Figure 8: The activity precedence of installation project for thermal power plant construction engineering.


Figure 9: The activity precedence of installation project for sewage treatment construction engineering.
5.2. Computing Results. In order to run the program for the proposed PSO-GA algorithm, the parameters for the PSO algorithm were set as follows: swarm_size=40, iteration_max $=200$, inertia weight_max $=1$, weight_min $=0$, position acceleration constant $c_{p}=0.3, c_{g}=0.3$, and $c_{l}=$ 0.1 . For the GA, an order-based crossover is used as the crossover operator at a rate of 0.4. A neighborhood search mutation with a rate of 0.05 was used. The population size was set as 40 and the maximum cycle number was equal to 400 . In this case, the predetermined finishing times for the three projects were September 15 , October 31, and September 20, respectively. The unit overdue penalty costs are $50,000 \mathrm{RMB} /$ day, $30,000 \mathrm{RMB} /$ day, and $40,000 \mathrm{RMB} /$ day, respectively.

The computer running environment was an intercore 2 Duo 2.26 GHz clock pulse with 2048 MB memory. The program was written using MATLAB 2007. After 3.12 minutes on average, the optimal solutions for the bilevel programming were determined.

The partial assignment scheme for these resources is shown in Table 3 and Figure 10. The integrated project schedules are illustrated in Figure 12. From Table 3 and Figure 10, the following can be seen. (1) The resource should be assigned to projects dynamically since the demanded quantity changes over time. (2) It is not necessary to allocate resources during the projects for each period because the assigned quantity is the same in some continuous periods. (3) This also reflects that the allocation period length has an effect on the allocation results. From Figure 10, we can see
the following. (1) The finishing time and chosen modes for project 1 and project 2 are different although they have same resources requirements. (2) Activities which need the same resources have been staggered because the total resource quantity is insufficient to implement these activities at the same time. This indicates that the scheduling is impacted by the resource allocation over the three projects although the company manager does not control the project scheduling directly. Hence, the resource allocation on the upper level can impact the decision on the lower level. Moreover, the results show that existing company resources cannot ensure that all three projects can finish on time. In this case, the company managers have to allocate more resources to the first project, which has a higher tardiness penalty. These detailed results can assist the decision makers on both the upper and lower levels to make the appropriate resources allocation plans.
5.3. Model Analysis. In this section, the proposed model is analyzed through a comparison with other resource allocation methods and an analysis is given for three uncertain models.
5.3.1. Assignment Method Comparison. Traditionally, resource allocation planning over multiple projects is executed using a resource-constrained multiple project scheduling model (RCMPS). This model is dependent on an assumption that a single manager oversees all projects. That is, there is only one level manager who is responsible for the overall project resource allocation and for the resource allocation

TABLE 1: The activity duration and resource consumption for the installation projects.

| No. | Mode | Duration | Resource requirement |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | CR | CM | WO | EE |
|  |  |  | 16 | 10 | 30 | 6 |
| A | 1 | $\{(32,34,36), 0.5 ;(35,38,41), 0.3 ;(42,48,54), 0.2\}$ | 6 | 4 | 8 | 2 |
|  | 2 | $\{(28,30,32), 0.5 ;(30,33,35), 0.3 ;(35,40,45), 0.2\}$ | 8 | 5 | 12 | 2 |
| B | 1 | $\{(20,23,26), 0.4 ;(22,25,28), 0.3 ;(25,30,35), 0.3\}$ | 3 | 2 | 4 | 2 |
|  | 2 | $\{(15,18,21), 0.4 ;(18,21,24), 0.3 ;(20,23,26), 0.3\}$ | 4 | 2 | 6 | 2 |
|  | 3 | $\{(12,15,18), 0.4 ;(15,18,21), 0.3 ;(20,24,28), 0.3\}$ | 5 | 4 | 8 | 2 |
| C | 1 | $\{(20,23,26), 0.3 ;(22,25,28), 0.3 ;(25,30,35), 0.4\}$ | 2 | 3 | 0 | 0 |
|  | 2 | $\{(15,18,21), 0.3 ;(18,21,24), 0.3 ;(20,23,26), 0.4\}$ | 3 | 3 | 0 | 0 |
| D | 1 | $\{(24,26,28), 0.3 ;(26,28,30), 0.3 ;(30,32,34), 0.4\}$ | 2 | 0 | 4 | 0 |
|  | 2 | $\{(20,23,26), 0.3 ;(22,25,28), 0.3 ;(25,30,35), 0.4\}$ | 2 | 0 | 6 | 0 |
|  | 3 | $\{(18,20,22), 0.3 ;(20,22,24), 0.3 ;(22,24,26), 0.4\}$ | 3 | 0 | 8 | 0 |
| E | 1 | $\{(28,30,32), 0.3 ;(30,33,35), 0.3 ;(35,40,45), 0.4\}$ | 2 | 0 | 3 | 0 |
|  | 2 | $\{(24,26,28), 0.3 ;(28,31,34), 0.3 ;(30,34,38), 0.4\}$ | 3 | 0 | 4 | 0 |
|  | 3 | $\{(20,23,26), 0.3 ;(22,25,28), 0.3 ;(25,28,31), 0.4\}$ | 4 | 0 | 5 | 0 |
| F | 1 | $\{(15,17,19), 0.4 ;(17,19,21), 0.3 ;(20,22,24), 0.3\}$ | 2 | 4 | 0 | 0 |
|  | 2 | $\{(10,12,14), 0.4 ;(12,15,18), 0.3 ;(15,18,21), 0.3\}$ | 3 | 6 | 0 | 0 |
| G | 1 | $\{(15,17,19), 0.4 ;(17,19,21), 0.3 ;(20,22,24), 0.3\}$ | 2 | 0 | 8 | 1 |
|  | 2 | $\{(10,12,14), 0.4 ;(12,15,18), 0.4 ;(15,18,21), 0.2\}$ | 3 | 0 | 10 | 2 |
| H | 1 | $\{(3,5,7), 1.0\}$ | 0 | 0 | 0 | 4 |
|  | 2 | $\{(5,7,9), 1.0\}$ | 0 | 0 | 0 | 6 |
| I | 1 | $\{(32,34,36), 0.3 ;(35,38,41), 0.3 ;(42,48,54), 0.4\}$ | 2 | 2 | 4 | 0 |
|  | 2 | $\{(28,30,32), 0.3 ;(30,33,35), 0.3 ;(35,40,45), 0.4\}$ | 4 | 3 | 6 | 0 |
| J | 1 | $\{(20,23,26), 0.5 ;(22,25,28), 0.4 ;(25,30,35), 0.1\}$ | 2 | 0 | 0 | 2 |
|  | 2 | $\{(15,18,21), 0.5 ;(18,21,24), 0.4 ;(20,23,26), 0.1\}$ | 2 | 0 | 0 | 2 |
|  | 3 | $\{(12,15,18), 0.5 ;(15,18,21), 0.4 ;(20,24,28), 0.1\}$ | 3 | 0 | 0 | 3 |
| K | 1 | $\{(28,30,32), 0.3 ;(30,33,35), 0.3 ;(35,40,45), 0.4\}$ | 1 | 0 | 3 | 1 |
|  | 2 | $\{(22,25,27), 0.3 ;(25,28,31), 0.3 ;(30,33,36), 0.4\}$ | 2 | 0 | 5 | 2 |
| L | 1 | $\{(15,17,19), 0.5 ;(17,19,21), 0.4 ;(20,22,24), 0.1\}$ | 0 | 0 | 4 | 2 |
|  | 2 | $\{(13,15,17), 0.5 ;(15,17,19), 0.4 ;(17,20,23), 0.1\}$ | 0 | 0 | 6 | 3 |
|  | 3 | $\{(12,14,16), 0.5 ;(14,15,16), 0.4 ;(16,17,18), 0.1\}$ | 0 | 0 | 8 | 4 |

Table 2: The total quantity and unit cost of the company resources.

| Resource | CR | CM | WO | EE |
| :--- | :---: | :---: | :---: | :---: |
| Total quantity | 16 | 10 | 30 | 6 |
| Cost | $(780, \rho, 960)$, | $(1020, \rho, 1300)$, | $(500, \rho, 700)$, | $(920, \rho, 1240)$, |
| (unit: CN/day) | $\rho \sim N(850,50)$ | $\rho \sim N(1160,80)$ | $\rho \sim N(600,60)$ | $\rho \sim N(1100,75)$ |

in each specific project. However, a bilevel organization structure is frequently used to manage projects which have two levels of managers. In this case, a bilevel optimization assignment model (BLOAM) is proposed to allocate the company resources over multiple projects. In this model, the company managers are responsible for the multiproject resources allocation in each period, while the project managers are responsible for the resource allocation for each specific project. In practice, other several assignment methods can also be used in the bilevel multiproject environment. One of these is called the Simple Weight Allocation Method (SWAM). The SWAM gives a weigh to each project and then
assigns resources to the projects according to the weight at the beginning of each resource period. Another method is the First In System First Served (FISFS) method which gives priority to the project that has been waiting the longest when resource conflicts occur. In addition, the MINPDD method gives priority to the project that has the earliest project due date and the MINPSLK method gives priority to the project that has the smallest project slack. There is a common characteristic in these four methods, as all of them only set the allocation regulation before the project implementation rather than making a detailed allocation plan for each time period.

Table 3: An optimal resource assignment scheme after 20 experiments.

| Time | CR |  |  | CM |  |  | WO |  |  | EE |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $P_{1}$ | $P_{2}$ | $P_{3}$ | $P_{1}$ | $P_{2}$ | $P_{3}$ | $P_{1}$ | $P_{2}$ | $P_{3}$ | $P_{1}$ | $P_{2}$ | $P_{3}$ |
| 1-8 | 8 | 8 | 0 | 5 | 5 | 0 | 12 | 12 | 0 | 2 | 2 | 0 |
| 9-12 | 4 | 4 | 8 | 2 | 2 | 4 | 6 | 6 | 10 | 2 | 2 | 2 |
| 13-16 | 6 | 6 | 4 | 5 | 5 | 0 | 8 | 8 | 12 | 0 | 0 | 2 |
| 17-20 | 7 | 4 | 4 | 4 | 4 | 2 | 19 | 11 | 0 | 2 | 2 | 0 |
| 21 | 5 | 4 | 4 | 0 | 0 | 4 | 12 | 8 | 8 | 0 | 2 | 2 |
| 22 | 4 | 4 | 4 | 0 | 0 | 6 | 9 | 13 | 6 | 2 | 2 | 0 |
| 23-24 | 5 | 4 | 4 | 0 | 0 | 4 | 10 | 12 | 8 | 4 | 2 | 0 |
| 25-26 | 5 | 4 | 6 | 4 | 4 | 2 | 4 | 8 | 8 | 4 | 0 | 2 |



Figure 10: The allocation plan for Resource CR during 38 weeks.

To test these methods above, four performance measures were used: total cost, project finishing time, actual usage, and total resource transfers. Actual usage refers to the proportion of the used resources compared to the total assigned resources. In practice, in order to improve resource usage, the assigned resource is transferred to each project at the beginning of each time period. At the end, idle resources should be released back into company's resource pool if they are not required for a project in the next time period. This resource transfer is used to record the transferred resource quantity between the company resource pool and the projects.

The computation results from the five methods are shown in Table 4. It is seen that, in comparison with other allocation methods, the proposed bilevel optimization allocation method can save cost more than $11.45 \%$ ( $600,000 \mathrm{RMB}$ ). The finishing time is also acceptable since it is shorter than the other three methods. Moreover, it also has the best resource usage of the five methods while the resource transfer is at an intermediate level. On the contrary, the SWAM can be seen to be unacceptable because of the high cost and the low resource usage. The other three methods have comparable performances with each other. However, all of these methods show a higher cost and lower resource usage than the proposed bilevel optimization method. Hence, the proposed bilevel optimization allocation method is efficient
in reducing costs, shortening project duration, and improving resource usage. It also shows that it is necessary to decide on a resource allocation plan using the bilevel optimization method rather than only making allocation regulation before project implementation.
5.3.2. Uncertainty Analysis. Uncertainty is an important consideration in this study. In particular, fuzzy random variables which integrate fuzzy factor and random factor are used to model the uncertain activity durations and resource costs because of the lack of precise data. Besides the fuzzy random variables, we can also use fuzzy variable or random variables to deal with the uncertainty. If only fuzzy factors are considered, then some important random information such as the weather has to be ignored. The situation is similar when only considering the random factors. Taking the duration of activity $A$ as an example, because of the lack of precise data, some experts are invited to estimate the duration. From the experts, the activity duration will be in the interval $[32,54]$ with a most possible value of 40 . Then the duration can be modelled as fuzzy number $(32,40,54)$. This is the situation without considering random factors. In addition, the activity is estimated to be implemented in October. According to the weather data, the probability of sunshine, cloud, and rain are $0.5,0.3$, and 0.2 , respectively. In this case, a new estimated value for each type of weather can be obtained. Similar to the previous estimation, these values are also fuzzy numbers. Hence, the estimated duration which considers the weather can be obtained as follows:

$$
\tilde{\bar{d}}_{A}= \begin{cases}(32,34,36), & 0.5  \tag{21}\\ (35,38,41), & 0.3 \\ (42,48,54), & 0.2\end{cases}
$$

Compared with only fuzzy factors, the fuzzy random duration has more information which can lead to a more precise calculation. If we only consider the random factors, then the fuzzy number in the fuzzy random number has to be replaced using a crisp number. The crisp number is chosen in its interval randomly. This could lead to unstable, even false computation results.

A test was also made by solving the proposed bilevel optimization model using the three types of uncertain data. In the comparison, the fuzzy random model and fuzzy random found different solutions by adjusting the optimistic and pessimistic index $\lambda$, while the random model found different solutions by choosing different crisp numbers randomly. Table 5 shows the comparison of the three model types based on the upper level model objective. It can be seen that the proposed model with fuzzy random variables has a much better performance than the others, not only in the average value of the results, but also in the stability.
5.4. Algorithm Evaluation. In this paper, a hybrid algorithm made up of an adaptive PSO, a GA, and a fuzzy random simulation was proposed to solve a bilevel resource allocation problem. In order to test the efficiency of the algorithm, a comparison with other solution methods was conducted.

Table 4: An optimal resource assignment scheme after 20 experiments.

| Assign methods | Total cost <br> (unit: T RMB) | Project 1 | Finishing time |  | Project 2 |
| :--- | :---: | :---: | :---: | :---: | :---: |



Figure 11: The schedules of three projects on the lower level.

The most common solution strategy for bilevel model is to transform it into a single level using the Karush-Kuhn-Tucker (KKT) conditions. However, this is difficult when variables only take integer values in the inner models. This also means that it can not be solved using common commercial solvers. Hence, it is more appropriate to solve the problem using a heuristic algorithm. In this paper, an improved adaptive PSO was proposed to deal with the upper level model. First, a comparison of the improved aPSO and original PSO was carried out. The average convergence curves are shown in Figure 11. It is shown that the proposed improved aPSO is faster and has an improved solution accuracy compared to the original PSO.

In addition, in our problem, the lower-level model is also better to be solved using a heuristic algorithm. Traditionally, researchers have tended to use the same algorithm to solve both the upper level and lower level. However, for a multimode resource-constrained project scheduling problem, the genetic algorithm shows a significantly higher percentage of success in finding the optimal solution although it may be slower. If the optimal solution to the lower-level model cannot be found, then the final solution may not be feasible. Hence, an adaptive genetic algorithm was proposed to deal with the lower-level model in this paper. In order to test the efficiency of the proposed hybrid algorithm, other bilevel algorithms such as PSO-PSO, GA-GA, and GA-PSO were


Figure 12: The convergence curves of the three PSO algorithms.
Table 5: Comparisons among the three types of uncertainty.

| Type of uncertain | Best result | Worst result | Average result |
| :--- | :---: | :---: | :---: |
| Fuzzy random | 5240 | 5386 | 5292 |
| Fuzzy | 5259 | 5421 | 5306 |
| Random | 5204 | 5506 | 5318 |

Table 6: Performance of the proposed algorithms based on 50 experiments.

| Performance | PSO-GA | PSO-PSO | GA-PSO | GA-GA |
| :--- | :---: | :---: | :---: | :---: |
| Best result | $5,234,486$ | $5,382,321$ | $5,323,732$ | $5,224,494$ |
| Average result | $5,318,216$ | $5,547,291$ | $5,388,462$ | $5,267,782$ |
| Computing time | 3.23 | 2.15 | 5.29 | 14.84 |

also tested over 50 experiments. In the experiments, the PSO [41] and the GA [42] were used to solve the lower-level model. The proposed improved aPSO and the GA [43] were used to search for the solution to the upper-level model. The performance of these algorithms is shown in Table 6. The results indicate that the proposed bilevel hybrid algorithm based PSO for the upper level and GA for the lower level is able to find better solutions than either the PSO-PSO or GA-PSO, and it has a faster computation speed than that of the GA-GA. Hence, the proposed algorithm is shown to be efficient for solving the proposed bilevel multiple project resource allocation problem.

## 6. Conclusion

This paper presented a bilevel optimization model for company resource allocation among multiple projects in a hierarchical organization. There are two levels of decision makers in the model. The decision maker on the upper level is the company manager who hopes to allocate company resource to multiple projects at a lower cost. This cost consists of the resource costs and the tardiness penalty. On the lower level, each project manager attempts to schedule their project with the objective of minimization of project duration under
resource constraints and multiple modes. In addition, the uncertainty associated with activity duration and resource cost has been explicitly considered in the model. Specifically, our research used fuzzy random variables to model the activity duration and resource costs. Then a hybrid algorithm made up of an adaptive PSO and a GA based on fuzzy random simulation was also applied to search for the optimal solution to the bilevel model. In the algorithm, an adaptive PSO was introduced to cope with the upper level programming, while an adaptive hybrid genetic algorithm was embedded into the PSO to solve the lower-level model. Finally, the efficiency of the proposed model and algorithm was evaluated using a practical case and various computing attributes. In contrast to prior studies, the proposed model shows that it was able to deal with a multiproject resource allocation in a bilevel optimization such as in most of construction companies, software companies, and some production companies. The limitation of the proposed model is that it does not allow for new projects to be added during the scheduled resource allocation periods. This is an interest area for our future research. In addition, in future research we also expect to investigate additional methods for dealing with the uncertainty in resource management such as using interval mathematical programming, which has been successfully applied to environmental management [10, 44].

## Conflict of Interests

The authors declare that they have no conflict of interests.

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# Optimality Condition and Wolfe Duality for Invex Interval-Valued Nonlinear Programming Problems 

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#### Abstract

The concepts of preinvex and invex are extended to the interval-valued functions. Under the assumption of invexity, the Karush-Kuhn-Tucker optimality sufficient and necessary conditions for interval-valued nonlinear programming problems are derived. Based on the concepts of having no duality gap in weak and strong sense, the Wolfe duality theorems for the invex interval-valued nonlinear programming problems are proposed in this paper.


## 1. Introduction

In real world applications of mathematical programming, one cannot ignore the possibility that a small uncertainty in the data can make the usual optimal solution completely meaningless from a practical viewpoint. So the major difficulty we are faced with is how to seek a solution for these real world optimization problems. There are several optimization models to deal with these problems. If the coefficients of optimization problem are assumed as random variables with known distributions, the problem can be categorized as the stochastic optimization problem. Stochastic optimization is a widely used and a standard approach to deal with uncertainty; for the detail of this topic one can see the books written by Birge and Louveaux [1], Kall and Mayer [2], and Prékopa [3]. If the coefficients of optimization problem are assumed as fuzzy variables, the problem can be categorized as the fuzzy optimization problem. The book written by Delgado et al. [4] gives the main stream of this topic. However, there are several drawbacks of stochastic optimization and fuzzy optimization in real world applications. Firstly, the specifications of the distributions and membership functions in the stochastic optimization problems and fuzzy optimization problems are very subjective. Secondly, the approach of stochastic optimization (fuzzy optimization) requires the evaluation of the solution on the whole uncertainty set in order to determine its expected cost, which is computationally hard in general.

Finally, one cannot guarantee that the real cost matches the expected cost in stochastic optimization, since the expected cost is only an estimator of the possible solutions.

In recent years, some deterministic frameworks of optimization methods are studied to overcome the drawbacks of stochastic optimization and fuzzy optimization. One of these deterministic optimization methods is robust optimization, which is the worst case based method and does not need a probability distribution on the uncertainty set. The earliest date of studies on robust optimization can be back to 1973 ([5]); Soyster proposed the first robust model for linear optimization problems with uncertain data. However, the model is very conservative in the sense that they protect against the worst case scenario. The interest in robust formulations in the optimization community was revived in the 1990s. A number of important robust formulations and applications were introduced by Ben-Tal et al. [6], El Ghaoui et al. [7, 8] and Bertsimas and Sim [9], who provided a detailed analysis of the robust optimization framework in linear optimization and general convex programming. In robust optimization, the considered uncertainty set plays a crucial role, since it determines the level of protection of the solution. The solution of robust optimization models might be too conservative if all scenarios are considered. Another one of these deterministic optimization methods is intervalvalued optimization, which provides an alternative choice
for considering the uncertainty into the optimization problems. The coefficients in the interval-valued optimization are assumed as closed intervals. The bounds of uncertain data in interval-valued optimization are easier to be handled than specifying the distributions and membership functions in stochastic optimization and fuzzy optimization problems, respectively.

Duality theory has played a fundamental role in the area of constrained optimization and has been studied for over a century. The duality theory for interval linear programming problems with real-valued objective function was discussed by Rohn [10]. Wu [11-14] has studied the duality theory for interval-valued programming problems. In [11], Wu has proposed the Wolfe duality for interval-valued nonlinear programming problems. The Lagrangian duality for intervalvalued nonlinear programming problems was also studied by Wu in [13]. Although the Wolfe and Lagrangian duality theory obtained in [11-13] can be applied to the problems of interval-valued linear programming, the results obtained using this method will be complicated. Based on the concept of a scalar product of closed intervals, Wu [14] has proposed the new weak and strong duality theorems for interval-valued linear programming problems. Zhou and Wang [15] have established the optimality sufficient condition and a mixed dual model for interval-valued nonlinear programming problems. However, these results were mainly established for the interval-valued programming problems involving the optimization of convex objective functions over convex feasible regions. In real world applications, not all practical problems fulfill the requirements of convexity. Then, generalized convex functions [16-21] have been introduced in order to weaken as much as possible the convexity requirements for results related to optimality conditions and duality results.

In this paper, we study the Karush-Kuhn-Tucker optimality sufficient and necessary conditions for interval-valued optimization problems under the assumption of generalized convexity. We extend the concepts of preinvex and invex for real-valued functions to interval-valued functions. Under the assumption of invexity, the Karush-Kuhn-Tucker optimality sufficient and necessary conditions for interval-valued optimization problems are derived for the purpose of proving the strong duality theorems. By using the concept of having no duality gap in weak and strong sense, the strong duality theorems in weak and strong sense are then proposed. The results in this paper improve and extend the results of Wu in [11-14] for interval-valued nonlinear optimization problems.

In Section 2 we present some basic concepts and properties for closed intervals and interval-valued functions, respectively. In Section 3, The Wolfe's primal and dual pair problems are proposed for interval-valued optimization problems. In Section 4, We extend the concepts of preinvex and invex for real-valued functions to interval-valued functions. Under the assumption of invexity, the Karush-Kuhn-Tucker optimality sufficient and necessary conditions for interval-valued optimization problems are derived. In Section 5, we discuss the solvability for Wolfe's primal and dual problems under the assumption of invexity. In Section 6, the duality theorems in
weak and strong sense are established for the invex intervalvalued nonlinear optimization problems.

## 2. Preliminaries

Let us denote by $\mathscr{F}$ the class of all closed intervals in $R$ if $A=\left[a^{L}, a^{U}\right] \in \mathscr{I}$ denotes a closed interval, where $a^{L}$ and $a^{U}$ mean the lower and upper bounds of $A$, respectively. Let $A=\left[a^{L}, a^{U}\right]$ and $B=\left[b^{L}, b^{U}\right]$ be in $\mathscr{F}$; we have
(i) $A+B=\{a+b: a \in A$ and $b \in B\}=\left[a^{L}+b^{L}, a^{U}+b^{U}\right]$;
(ii) $-A=\{-a: a \in A\}=\left[-a^{U},-a^{L}\right]$;
(iii) $A \times B=\{a b: a \in A$ and $b \in B\}=\left[\min _{a b}, \max _{a b}\right]$, where $\min _{a b}=\min \left\{a^{L} b^{L}, a^{L} b^{U}, a^{U} b^{L}, a^{U} b^{U}\right\}$ and $\max _{a b}=\max \left\{a^{L} b^{L}, a^{L} b^{U}, a^{U} b^{L}, a^{U} b^{U}\right\}$.

Then, we can see that

$$
\begin{align*}
A-B & =A+(-B)=\left[a^{L}-b^{U}, a^{U}-b^{L}\right],  \tag{1}\\
k A & =\{k a: a \in A\} \\
& = \begin{cases}{\left[k a^{L}, k a^{U}\right]} & \text { if } k \geq 0, \\
|k|\left[-a^{U},-a^{L}\right] & \text { if } k<0,\end{cases} \tag{2}
\end{align*}
$$

where $k$ is a real number. The real number $a \in R$ can be regarded as a closed interval $A_{a}=[a, a]$. Let $B \in \mathscr{F}$ be a closed interval; we write that $a+B$ will mean $A_{a}+B$. For more details on the topic of interval analysis, one can refer to [22].

We say that $A$ and $B$ are comparable if and only if $A \preceq B$ or $A \succeq B$. We write that $A \leq_{L U} B$ if and only if $a^{L} \leq b^{L}$ and $a^{U} \leq b^{U}$ and that $A<_{L U} B$ if and only if $A \leq_{L U} B$ and $A \neq B$; that is, the following (a1) or (a2) or (a3) is satisfied:
(al) $a^{L}<b^{L}$ and $a^{U} \leq b^{U}$;
(a2) $a^{L} \leq b^{L}$ and $a^{U}<b^{U}$;
(a3) $a^{L}<b^{L}$ and $a^{U}<b^{U}$.
Therefore if $A$ and $B$ are not comparable, then the following (b1) or (b2) or (b3) or (b4) or (b5) or (b6) is satisfied:
(b1) $a^{L} \leq b^{L}$ and $a^{U}>b^{U}$; (b2) $a^{L}<b^{L}$ and $a^{U} \geq b^{U}$;
(b3) $a^{L}<b^{L}$ and $a^{U}>b^{U}$; (b4) $a^{L} \geq b^{L}$ and $a^{U}<b^{U}$;
(b5) $a^{L}>b^{L}$ and $a^{U} \leq b^{U}$; (b6) $a^{L}>b^{L}$ and $a^{U}<b^{U}$.
In other words, if $A$ and $B$ are not comparable, then $A \neq B$ and $A \supseteq B$ or $A \subseteq B$.

The function $f: R^{n} \rightarrow \mathscr{F}$ defined on the Euclidean space $R^{n}$ is called interval-valued function if $f(\mathbf{x})=f\left(x_{1}, \ldots, x_{n}\right)$ is a closed interval in $R$ for each $\mathbf{x} \in R^{n} . f$ can be also written as $f(\mathbf{x})=\left[f^{L}(\mathbf{x}), f^{U}(\mathbf{x})\right]$, where $f^{L}$ and $f^{U}$ are real-valued functions defined on $R^{n}$ and satisfy $f^{L}(\mathbf{x}) \leq f^{U}(\mathbf{x})$ for every $\mathbf{x} \in R^{n} . \mathrm{Wu}([23])$ has shown the concepts of limit, continuity, and two kinds of differentiation of interval-valued function.

Definition 1 (see [23]). Let $X$ be an open set in $R$. An intervalvalued function $f: X \rightarrow \mathscr{I}$ with $f(x)=\left[f^{L}(x), f^{U}(x)\right]$ is
called weakly differentiable at $x_{0}$ if the real-valued functions $f^{L}$ and $f^{U}$ are differentiable at $x_{0}$ (in the usual sense).

Let $A, B \in \mathscr{F}$; if there exists a $C \in \mathscr{J}$ such that $A=B+C$, then $C$ is called the Hukuhara difference. One also writes $C=$ $A \ominus B$, when we say that the Hukuhara difference $C$ exists, which means that $a^{L}-b^{L} \leq a^{U}-b^{U}$ and $C=\left[a^{L}-b^{L}, a^{U}-b^{U}\right]$.

Definition 2 (see [23]). Let $X$ be an open set in $R$. An intervalvalued function $f: X \rightarrow \mathscr{J}$ is called $H$-differentiable at $x_{0}$ if there exists a closed interval $A\left(x_{0}\right) \in \mathscr{I}$ such that the limits

$$
\begin{align*}
& \lim _{h \rightarrow 0_{+}} \frac{f\left(x_{0}+h\right) \ominus f\left(x_{0}\right)}{h},  \tag{3}\\
& \lim _{h \rightarrow 0_{+}} \frac{f\left(x_{0}\right) \ominus f\left(x_{0}-h\right)}{h}
\end{align*}
$$

both exist and are equal to $A\left(x_{0}\right)$. In this case, $A\left(x_{0}\right)$ is called the $H$-derivative of $f$ at $x_{0}$.

Let $f$ be an interval-valued function defined on $R^{n}$. One says that $f$ is continuous at $\mathbf{c} \in R^{n}$ if

$$
\begin{equation*}
\lim _{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x})=f(\mathbf{c}) \tag{4}
\end{equation*}
$$

Definition 3 (see [23]). Let $f$ be an interval-valued function defined on $X \subseteq R^{n}$ and let $\mathbf{x}_{0}=\left(x_{1}^{0}, \ldots, x_{n}^{0}\right) \in X$ be fixed.
(i) One say that $f$ is weakly continuously differentiable at $\mathbf{x}_{0}$ if the real-valued functions $f^{L}$ and $f^{U}$ are continuously differentiable at $\mathbf{x}_{0}$ (i.e., all the partial derivatives of $f^{L}$ and $f^{U}$ exist on some neighborhoods of $\mathbf{x}_{0}$ and are continuous at $\mathbf{x}_{0}$ ).
(ii) One says that $f$ is continuously $H$-differentiable at $\mathbf{x}_{0}$ if all of the partial H-derivatives $\left(\left(\partial f / \partial x_{1}\right)_{H}, \ldots,\left(\partial f / \partial x_{n}\right)_{H}\right)$ exist on some neighborhoods of $\mathbf{x}_{0}$ and are continuous at $\mathbf{x}_{0}$ (in the sense of interval-valued function).

Proposition 4 (see [23]). Let $f$ be an interval-valued function defined on $X \subseteq R^{n}$. If $f$ is $H$-differentiable at $\mathbf{x}_{0} \in X$, then $f$ is weakly differentiable at $\mathbf{x}_{0}$; if $f$ is continuously H-differentiable at $\mathbf{x}_{0} \in X$, then $f$ is weakly continuously differentiable at $\mathbf{x}_{0}$.

## 3. The Wolfe's Primal and Dual Problems

In this section, we introduce the Wolfe's primal and dual pair problems for conventional nonlinear programming problem following Wu in [12]. We consider the interval-valued optimization problem as follows:

$$
\begin{align*}
\text { (IVP) } & F(\mathbf{x}) \\
\text { subject to } & \\
& g_{i}(\mathbf{x}) \leq 0, \quad i=1, \ldots, m \\
& h_{i}(\mathbf{x}) \leq 0, \quad i=1, \ldots, m \\
& \mathbf{x} \geq \mathbf{0}, \tag{5}
\end{align*}
$$

where $F: R^{n} \rightarrow \mathscr{J}$ is an interval-valued function and $g_{i}:$ $R^{n} \rightarrow R$ and $h_{i}: R^{n} \rightarrow R, i=1, \ldots, m$, are real-valued functions.

We denote by

$$
\begin{equation*}
X=\left\{\mathbf{x} \in R^{n}: \mathbf{x} \geq 0, g_{i}(\mathbf{x}) \leq 0, h_{i}(\mathbf{x}) \leq 0, i=1, \ldots, m\right\} \tag{6}
\end{equation*}
$$

the feasible set of primal problem (IVP). We also denote by

$$
\begin{equation*}
\operatorname{Obj}_{P}(F, X)=\{F(\mathbf{x}): \mathbf{x} \in X\} \tag{7}
\end{equation*}
$$

the set of all objective values of primal problem (IVP).
Definition 5 (see [12]). Let $\mathbf{x}^{*}$ be a feasible solution of primal problem (IVP). One says that $\mathbf{x}^{*}$ is a nondominated solution of problem (IVP) if there exists no $\widetilde{\mathbf{x}} \in X$ such that $F(\widetilde{\mathbf{x}}) \prec F\left(\mathbf{x}^{*}\right)$. In this case, $F\left(\mathbf{x}^{*}\right)$ is called the nondominated objective value of $F$.

We denote by $\operatorname{Min}(F, X)=\left\{F\left(\mathbf{x}^{*}\right): \quad \mathbf{x}^{*}\right.$ is a non-dominated solution of (IVP) $\}$ the set of all nondominated objective values of problem (IVP).

If we assume that the interval-valued function $F$ and the real-valued functions $g_{i}$ and $h_{i}, i=1, \ldots, m$ are differentiable on $R_{+}^{n}$, the dual problem of (IVP) is formulated as follows:
(DIVP)

$$
\begin{array}{r}
\max \quad F(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} \cdot g_{i}(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} \cdot h_{i}(\mathbf{x}) \\
\text { subject to } \nabla F^{L}(\mathbf{x})+\nabla F^{U}(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}(\mathbf{x}) \\
+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}(\mathbf{x})=\mathbf{0}  \tag{8}\\
\boldsymbol{\mu}=\left(\mu_{1}, \ldots, \mu_{m}\right) \geq \mathbf{0}, \quad i=1, \ldots, m \\
\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \geq \mathbf{0}, \\
i=1, \ldots, m \\
\mathbf{x} \geq \mathbf{0} .
\end{array}
$$

We denote by $Y$ the feasible set of dual problem (DIVP) consisting of elements $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in R_{+}^{n} \times R_{+}^{m} \times R_{+}^{m}$. We write

$$
\begin{equation*}
H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})=F(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} \cdot g_{i}(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} \cdot h_{i}(\mathbf{x}) \tag{9}
\end{equation*}
$$

and denote by

$$
\begin{equation*}
\operatorname{Obj}_{D}(H, Y)=\{H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}):(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in Y\} \tag{10}
\end{equation*}
$$

the set of all objective values of primal problem (DIVP).
Definition 6 (see [12]). Let ( $\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}$ ) be a feasible solution of primal problem (DIVP). One says that $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ is a nondominated solution of problem (DIVP) if there exists no $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ such that $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right) \prec H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$. In this case, $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ is called the nondominated objective value of problem (DIVP).

We denote by $\operatorname{Max}(H, Y)=\left\{H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right):\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)\right.$ is a nondominated solution of (DIVP) $\}$ the set of all nondominated objective values of problem (DIVP).

## 4. The KKT Optimality Conditions for Interval-Valued Optimization Problems

In this section, we extend the concepts of preinvex and invex for real-valued functions to interval-valued functions. Under the assumption of invexity, we propose the KKT optimality sufficient and necessary conditions for intervalvalued optimization problems.

### 4.1. Preinvexity and Invexity of the Interval-Valued Functions.

 The concept of convexity plays an important role in the optimization theory. In recent years, the concept of convexity has been generalized in several directions using novel and innovative techniques. An important generalization of convex functions is the introduction of preinvex function, which was introduced by Weir and Mond ([19]) and by Weir and Jeyakumar ([20]). Yang et al. ([21]) has established the characterization of prequasi-invex functions under the condition of lower semicontinuity, upper semicontinuity, and semistrict prequasi-invexity, respectively.Definition 7 (see $[19,20]$ ). A set $K \subseteq R^{n}$ is said to be invex if there exists a vector function $\eta: R^{n} \times R^{n} \rightarrow R^{n}$ such that

$$
\begin{equation*}
\mathbf{x}, \mathbf{y} \in K, \lambda \in[0,1] \Longrightarrow \mathbf{y}+\lambda \eta(\mathbf{x}, \mathbf{y}) \in K \tag{11}
\end{equation*}
$$

Definition 8 (see [19, 20]). Let $K \subseteq R^{n}$ be an invex set with respect to $\eta: R^{n} \times R^{n} \rightarrow R^{n}$. Let $f: K \rightarrow R$. One says that $f$ is preinvex if

$$
\begin{array}{r}
f(\mathbf{y}+\lambda \eta(\mathbf{x}, \mathbf{y})) \leq \lambda f(\mathbf{x})+(1-\lambda) f(\mathbf{y}), \\
\forall \mathbf{x}, \mathbf{y} \in K, \lambda \in[0,1] . \tag{12}
\end{array}
$$

Hanson has also introduced the concept of invex function in [17].

Definition 9 (see [17]). Let $K \subseteq R^{n}$ be an invex set with respect to $\eta: R^{n} \times R^{n} \rightarrow R^{n}$. Let $f: K \rightarrow R$. One says that $f$ is invex if

$$
\begin{equation*}
f(\mathbf{x})-f(\mathbf{y}) \geq \eta^{T}(\mathbf{x}, \mathbf{y}) \nabla f(\mathbf{y}) . \tag{13}
\end{equation*}
$$

Pini ([18]) has shown that, if $f$ is defined on an invex set $K \subseteq R^{n}$ and if it is preinvex and differentiable, then $f$ is also invex with respect to $\eta$, but the converse is not true in general. Wu has extended the concept of convexity to the intervalvalued functions in [11-14].

Now, we extend the concepts of preinvexity and invexity to the interval-valued functions.

Definition 10. Let $K \subseteq R^{n}$ be an invex set with respect to $\eta$ : $K \times K \rightarrow R^{n}$, and let $f=\left[f^{L}(\mathbf{x}), f^{U}(\mathbf{x})\right]$ be an intervalvalued function defined on $K$. One says that $f$ is $L U$-preinvex at $\mathbf{x}^{*}$ with respect to $\eta$ if

$$
\begin{equation*}
f\left(\mathbf{x}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq_{L U} \lambda f\left(\mathbf{x}^{*}\right)+(1-\lambda) f(\mathbf{x}) \tag{14}
\end{equation*}
$$

for each $\lambda \in(0,1)$ and each $\mathbf{x} \in K$.

Definition 11. Let $K \subseteq R^{n}$ be an invex set with respect to $\eta$ : $K \times K \rightarrow R^{n}$, and let $f=\left[f^{L}(\mathbf{x}), f^{U}(\mathbf{x})\right]$ be an intervalvalued function defined on $K$. One says that $f$ is invex at $\mathbf{x}^{*}$ if the real-valued functions $f^{L}$ and $f^{U}$ are invex at $\mathbf{x}^{*}$.

It is obvious that the particular case of H -differentiable $L U$-convex interval-valued function is obtained by choosing $\eta(\mathbf{x}, \mathbf{y})=\mathbf{x}-\mathbf{y}$ in H -differentiable invex interval-valued function, but H-differentiable invex interval-valued function may not be H -differentiable $L U$-convex interval-valued function.

Example 12. Consider that $f: R \rightarrow R, f(x)=\left[1-e^{-x^{2}}\right.$, $1-0.2 e^{-x^{2}}$; this interval-valued function is invex since $f^{L}$ and $f^{U}$ have a unique global minimizer at $x^{*}=0$, where $\left(f^{L}\right)^{\prime}=\left(f^{U}\right)^{\prime}=0$ and is therefore invex. However, $f$ is not $L U$-convex at $x^{*}$ and therefore not $L U$-preinvex. As $x^{*}=0$ and $f^{L}\left(x^{*}\right)=0$, then for $\lambda \in(0,1)$. Consider the following:

$$
\begin{gather*}
\lambda f^{L}\left(x^{*}\right)+(1-\lambda) f^{L}(y)=(1-\lambda) f^{L}(y) \\
f^{L}\left(\lambda x^{*}+(1-\lambda) y\right)=f^{L}((1-\lambda) y) \tag{15}
\end{gather*}
$$

Taking $y=5, \lambda=0.5$, we get $(1-\lambda) f^{L}(y) \approx 0.5<f^{L}((1-$ $\lambda) y) \approx 0.998$. Then, $(1-\lambda) f^{L}(y)<f^{L}((1-\lambda) y), \forall y \in R$, so the real-valued function $f^{L}$ is not convex at $x^{*}=0$ and the interval-valued function $f$ is not $L U$-convex at $x^{*}=0$.

Proposition 13. Let $K \subseteq R^{n}$ be an invex set with respect to $\eta: K \times K \rightarrow R^{n}$, and let $f=\left[f^{L}(\mathbf{x}), f^{U}(\mathbf{x})\right]$ be an intervalvalued function defined on $K$. The interval-valued function $f$ is $L U$-preinvex at $\mathbf{x}^{*}$ with respect to $\eta$ if and only if the realvalued functions $f^{L}$ and $f^{U}$ are preinvex at $\mathbf{x}^{*}$ with respect to the same $\eta$.

Proof. By Definition 10, we have

$$
\begin{align*}
& f^{L}\left(\mathbf{x}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq\left[\lambda f\left(\mathbf{x}^{*}\right)+(1-\lambda) f(\mathbf{x})\right]^{L} \\
& f^{U}\left(\mathbf{x}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq\left[\lambda f\left(\mathbf{x}^{*}\right)+(1-\lambda) f(\mathbf{x})\right]^{U} \tag{16}
\end{align*}
$$

Since $\lambda>0$ and $1-\lambda>0$, then

$$
\begin{align*}
& f^{L}\left(\mathbf{x}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq \lambda f^{L}\left(\mathbf{x}^{*}\right)+(1-\lambda) f^{L}(\mathbf{x})  \tag{17}\\
& f^{U}\left(\mathbf{x}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq \lambda f^{U}\left(\mathbf{x}^{*}\right)+(1-\lambda) f^{U}(\mathbf{x})
\end{align*}
$$

The proof is complete.
Proposition 14. Let $K \subseteq R^{n}$ be an invex set with respect to $\eta: K \times K \rightarrow R^{n}$, and let $f=\left[f^{L}(\mathbf{x}), f^{U}(\mathbf{x})\right]$ be an intervalvalued function defined on $K$. If the interval-valued function $f$ is LU-preinvex with respect to $\eta$ and $H$-differentiable at $\mathbf{x}^{*}$, then the interval-valued functions $f$ is invex at $\mathbf{x}^{*}$ with respect to the same $\eta$.

Proof. From Definition 10 and Proposition 13, we have

$$
\begin{align*}
& f^{L}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq \lambda f^{L}(\mathbf{x})+(1-\lambda) f^{L}\left(\mathbf{x}^{*}\right) \\
& f^{U}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right) \leq \lambda f^{U}(\mathbf{x})+(1-\lambda) f^{U}\left(\mathbf{x}^{*}\right) \tag{18}
\end{align*}
$$

We can rewrite the two above inequalities as

$$
\begin{align*}
& f^{L}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right)-f^{L}\left(\mathbf{x}^{*}\right) \leq \lambda\left[f^{L}(\mathbf{x})-f^{L}\left(\mathbf{x}^{*}\right)\right] \\
& f^{U}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right)-f^{U}\left(\mathbf{x}^{*}\right) \leq \lambda\left[f^{U}(\mathbf{x})-f^{U}\left(\mathbf{x}^{*}\right)\right] \tag{19}
\end{align*}
$$

Since $\lambda>0,1-\lambda>0$, and the interval-valued function $f$ is H -differentiable at $\mathbf{x}^{*}$, then the real-valued functions $f^{L}$ and $f^{U}$ are differentiable at $\mathbf{x}^{*}$ by Definition 3. Divide by $\lambda$ to obtain

$$
\begin{align*}
& \frac{1}{\lambda}\left[f^{L}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right)-f^{L}\left(\mathbf{x}^{*}\right)\right] \leq f^{L}(\mathbf{x})-f^{L}\left(\mathbf{x}^{*}\right) \\
& \frac{1}{\lambda}\left[f^{U}\left(\mathbf{x}^{*}+\lambda \eta\left(\mathbf{x}^{*}, \mathbf{x}\right)\right)-f^{U}\left(\mathbf{x}^{*}\right)\right] \leq f^{U}(\mathbf{x})-f^{U}\left(\mathbf{x}^{*}\right) \tag{20}
\end{align*}
$$

Taking the limit as $\lambda \rightarrow 0_{+}$, we get

$$
\begin{align*}
\eta^{T}\left(\mathbf{x}^{*}, \mathbf{x}\right) \cdot \nabla f^{L}\left(\mathbf{x}^{*}\right) & \leq f^{L}(\mathbf{x})-f^{L}\left(\mathbf{x}^{*}\right),  \tag{21}\\
\eta^{T}\left(\mathbf{x}^{*}, \mathbf{x}\right) \cdot \nabla f^{U}\left(\mathbf{x}^{*}\right) & \leq f^{U}(\mathbf{x})-f^{U}\left(\mathbf{x}^{*}\right)
\end{align*}
$$

From the two above inequalities, we can see that $f^{L}$ and $f^{U}$ are invex at $\mathbf{x}^{*}$ with respect to the same $\eta$. By Definition 11 , it can be shown that the interval-valued function $f$ is invex at $\mathbf{x}^{*}$ with respect to the same $\eta$.
4.2. The KKT Optimality Conditions for Invex Interval-Valued Optimization Problems. Now we consider the following two real-valued optimization problems:

$$
\begin{array}{r}
\left(\mathrm{P}_{\mathrm{LU}}\right) \quad \min \quad f(\mathbf{x})=F^{L}(\mathbf{x})+F^{U}(\mathbf{x}) \\
\text { subject to } \quad g_{i}(\mathbf{x}) \leq 0, \quad i=1, \ldots, m, \\
h_{i}(\mathbf{x}) \leq 0, \quad i=1, \ldots, m \\
\mathbf{x} \geq \mathbf{0}, \\
\left(\mathrm{D}_{\mathrm{LU}}\right) \quad \max \quad f(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})=H^{L}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})+H^{U}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \\
\text { subject to } \nabla F^{L}(\mathbf{x})+\nabla F^{U}(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}(\mathbf{x}) \\
\\
+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}(\mathbf{x})=\mathbf{0}, \\
\boldsymbol{\mu}=\left(\mu_{1}, \ldots, \mu_{m}\right) \geq \mathbf{0}, \quad i=1, \ldots, m \\
\boldsymbol{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \geq \mathbf{0}, \quad i=1, \ldots, m,
\end{array}
$$

Wu ([12]) has proposed the following result.
Proposition 15 (see [12]). (1) If $\mathbf{x}^{*}$ is an optimal solution of problem $\left(P_{L U}\right)$, then $\mathbf{x}^{*}$ is a nondominated solution of problem (IVP);
(2) If $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ is an optimal solution of problem $\left(D_{L U}\right)$, then $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ is a nondominated solution of problem (DIVP).

Now, we show that the KKT conditions are necessary and sufficient for optimality under the assumptions of invexity and modified Slater condition is satisfied.

Let us rename the constraint functions $h_{i}$ for $i=1, \ldots, m$ as $g_{m+i}=h_{i}$ for $i=1, \ldots, m$. Let $J\left(\mathbf{x}^{*}\right)$ denote the set of active constraints at $\mathbf{x}^{*}$, which is defined by

$$
\begin{equation*}
J\left(\mathbf{x}^{*}\right)=\left\{i: g_{i}\left(\mathbf{x}^{*}\right)=0 \text { for } i=1, \ldots, 2 m\right\} \tag{23}
\end{equation*}
$$

Theorem 16 (KKT necessary conditions for $\mathrm{P}_{\mathrm{LU}}$ ). Suppose that $\mathbf{x}^{*}$ is an optimal solution of the problem of $P_{L U}$ and there exists a point $\widehat{\mathbf{x}}$ such that $g_{i}(\widehat{\mathbf{x}})<0$ and that $g_{i}\left(x^{*}\right)=0$ for all $i \in J\left(x^{*}\right)$. Suppose, also, that $f(\mathbf{x})$ and $g_{i}$ are differentiable for $i=1, \ldots, 2 m$ at $\mathbf{x}^{*}$ and $f(\mathbf{x})$ and $g_{i}$ are invex with respect to the same vector function $\eta\left(\mathbf{x}, \mathbf{x}^{*}\right)$. Then there exist $0 \leq \mu_{i}$, $\lambda_{i} \in R$ for $i=1, \ldots, m$ such that

$$
\begin{gather*}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0}, \\
\mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m . \tag{24}
\end{gather*}
$$

Proof. Since $J\left(\mathbf{x}^{*}\right)$ denote the set of active constraints at $\mathbf{x}^{*}$. Then,

$$
\begin{equation*}
g_{i}\left(\mathbf{x}^{*}\right)=0, \quad \forall i \in J\left(\mathbf{x}^{*}\right) . \tag{25}
\end{equation*}
$$

If we can show that

$$
\begin{equation*}
\mathbf{y}^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right) \leq 0 \quad\left(\forall i \in J\left(\mathbf{x}^{*}\right)\right) \Longrightarrow \mathbf{y}^{T} \nabla f\left(\mathbf{x}^{*}\right) \geq 0 \tag{26}
\end{equation*}
$$

the result will follow as in $[16,24]$ by applying Farkas' Lemma, where $f(\mathbf{x})=F^{L}(\mathbf{x})+F^{U}(\mathbf{x})$ is a real-valued function.

Assume that (26) does not hold; then there exists $\mathbf{y}=$ $\left(y_{1}, \ldots, y_{n}\right) \in R^{n}$ such that

$$
\begin{equation*}
\mathbf{y}^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right) \leq 0\left(\forall i \in J\left(\mathbf{x}^{*}\right)\right) \Longrightarrow \mathbf{y}^{T} \nabla f\left(\mathbf{x}^{*}\right)<0 \tag{27}
\end{equation*}
$$

Since the Slater-type condition holds, then

$$
\begin{equation*}
g_{i}(\widehat{\mathbf{x}})-g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{28}
\end{equation*}
$$

By the invexity of $g_{i}$, we have

$$
\begin{equation*}
\left[\eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) . \tag{29}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{30}
\end{equation*}
$$

for all $\rho>0$. Therefore, for some positive $\sigma>0$ are small enough such that

$$
\begin{equation*}
g_{i}\left(\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]\right) \leq g_{i}\left(\mathbf{x}^{*}\right)=0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{31}
\end{equation*}
$$

which can shown that $\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]$ is a feasible solution of $\mathrm{P}_{\mathrm{LU}}$. Since $\mathbf{x}^{*}$ is an optimal solution of the problem of $\mathrm{P}_{\mathrm{LU}}$, we have

$$
\begin{equation*}
f\left(\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]\right) \geq f\left(\mathbf{x}^{*}\right) \tag{32}
\end{equation*}
$$

then

$$
\begin{equation*}
\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla f\left(\mathbf{x}^{*}\right) \geq 0 \tag{33}
\end{equation*}
$$

for all $\rho>0$. When $\rho \rightarrow 0^{+}$, we have

$$
\begin{equation*}
\mathbf{y}^{T} \nabla f\left(\mathbf{x}^{*}\right) \geq 0 \tag{34}
\end{equation*}
$$

which contradicts to (27). Then, (26) is satisfied. By applying Farkas' Lemma and setting $y_{i}=0$ for $i \notin J\left(\mathbf{x}^{*}\right)$, it can be shown that there exists $0 \leq y_{i}^{*} \in R,\left(i \in J\left(\mathbf{x}^{*}\right)\right)$ such that

$$
\begin{equation*}
\nabla f\left(\mathbf{x}^{*}\right)+\sum_{i \in J\left(\mathbf{x}^{*}\right)} y_{i}^{*} \nabla g_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0} \tag{35}
\end{equation*}
$$

From (35), $f(\mathbf{x})=F^{L}(\mathbf{x})+F^{U}(\mathbf{x})$ and $y_{i}^{*}=\mu_{i}, i=1, \ldots, m$; $y_{i}^{*}=\lambda_{i}, i=m+1, \ldots, 2 m ; g_{i}\left(\mathbf{x}^{*}\right)=h_{i}\left(\mathbf{x}^{*}\right)$ if $i=m+1, \ldots, 2 m$. Then, we get

$$
\begin{gather*}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0} \\
\mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m \tag{36}
\end{gather*}
$$

The result follows.
Theorem 17 (KKT necessary conditions for (IVP)). Suppose that $\mathbf{x}^{*}$ is a nondominated solution of primal problem (IVP) and there exists a point $\widehat{\mathbf{x}}$ such that $g_{i}(\widehat{\mathbf{x}})<0$ and that $g_{i}\left(x^{*}\right)=$ 0 for all $i \in J\left(x^{*}\right)$. Suppose, also, that $F(\mathbf{x})$ is $H$-differentiable and $g_{i}$ are differentiable for $i=1, \ldots, 2 m$ at $\mathbf{x}^{*}$ and $F(\mathbf{x})$ and $g_{i}$ are invex with respect to the same vector function $\eta\left(\mathbf{x}, \mathbf{x}^{*}\right)$. Then there exist $0 \leq \mu_{i}, \lambda_{i} \in R$ for $i=1, \ldots, m$ such that

$$
\begin{gather*}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0}, \\
\mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m . \tag{37}
\end{gather*}
$$

Proof. Since $J\left(\mathbf{x}^{*}\right)$ denote the set of active constraints at $\mathbf{x}^{*}$. Then,

$$
\begin{equation*}
g_{i}\left(\mathbf{x}^{*}\right)=0, \quad \forall i \in J\left(\mathbf{x}^{*}\right) \tag{38}
\end{equation*}
$$

Suppose that there exists $\mathbf{y} \in R^{n}$ such that

$$
\begin{align*}
\mathbf{y}^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right) \leq 0 & \left(\forall i \in J\left(\mathbf{x}^{*}\right)\right),  \tag{39}\\
\mathbf{y}^{T} \nabla F^{L}\left(\mathbf{x}^{*}\right)<0, & \mathbf{y}^{T} \nabla F^{U}\left(\mathbf{x}^{*}\right)<0 .
\end{align*}
$$

Since the Slater-type condition holds, then

$$
\begin{equation*}
g_{i}(\widehat{\mathbf{x}})-g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{40}
\end{equation*}
$$

by the invexity of $g_{i}$, we have

$$
\begin{equation*}
\left[\eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{41}
\end{equation*}
$$

then

$$
\begin{equation*}
\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla g_{i}\left(\mathbf{x}^{*}\right)<0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{42}
\end{equation*}
$$

for all $\rho>0$. Therefore, for some positive $\sigma>0$ are small enough such that

$$
\begin{equation*}
g_{i}\left(\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]\right) \leq g_{i}\left(\mathbf{x}^{*}\right)=0, \quad i \in J\left(\mathbf{x}^{*}\right) \tag{43}
\end{equation*}
$$

which can show that $\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]$ is a feasible solution of primal problem (IVP). Since $\mathbf{x}^{*}$ is a nondominated solution of primal problem (IVP), there exists no feasible solution $\mathbf{x}$ such that $F(\mathbf{x}) \prec F\left(\mathbf{x}^{*}\right)$, which means that there exists no feasible solution $x$ such that the following are satisfied.
(1) $F^{L}(\mathbf{x})<F^{L}\left(\mathbf{x}^{*}\right)$, and $F^{U}(\mathbf{x}) \leq F^{U}\left(\mathbf{x}^{*}\right)$;
(2) $F^{L}(\mathbf{x}) \leq F^{L}\left(\mathbf{x}^{*}\right)$, and $F^{U}(\mathbf{x})<F^{U}\left(\mathbf{x}^{*}\right)$;
(3) $F^{L}(\mathbf{x})<F^{L}\left(\mathbf{x}^{*}\right)$, and $F^{U}(\mathbf{x})<F^{U}\left(\mathbf{x}^{*}\right)$.

That is to say, we have the following results for the feasible solution $\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]$ of primal problem (IVP):

$$
\begin{align*}
& F^{L}\left(\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]\right) \geq F^{L}\left(\mathbf{x}^{*}\right) \\
& \quad \text { or } F^{U}\left(\mathbf{x}^{*}+\sigma\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]\right) \geq F^{U}\left(\mathbf{x}^{*}\right) \tag{44}
\end{align*}
$$

then

$$
\begin{align*}
& {\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla F^{L}\left(\mathbf{x}^{*}\right) \geq 0} \\
& \quad \text { or }\left[\mathbf{y}+\rho \eta\left(\widehat{\mathbf{x}}, \mathbf{x}^{*}\right)\right]^{T} \nabla F^{U}\left(\mathbf{x}^{*}\right) \geq 0 \tag{45}
\end{align*}
$$

for all $\rho>0$. When $\rho \rightarrow 0^{+}$, we have

$$
\begin{equation*}
\mathbf{y}^{T} \nabla F^{L}\left(\mathbf{x}^{*}\right) \geq 0 \quad \text { or } \quad \mathbf{y}^{T} \nabla F^{U}\left(\mathbf{x}^{*}\right) \geq 0 \tag{46}
\end{equation*}
$$

which contradicts to (39). Therefore, we conclude that the system of inequalities presented in (39) has no solution. According to Farkas' lemma [24] and setting $y_{i}=0$ for $i \notin J\left(\mathbf{x}^{*}\right)$, it can be shown that there exists $0 \leq y_{i}^{*} \in R,(i \in$ $\left.J\left(\mathbf{x}^{*}\right)\right)$ such that

$$
\begin{equation*}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i \in J\left(\mathbf{x}^{*}\right)} y_{i}^{*} \nabla g_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0} \tag{47}
\end{equation*}
$$

From (47), $y_{i}^{*}=\mu_{i}, i=1, \ldots, m ; y_{i}^{*}=\lambda_{i}, i=m+$ $1, \ldots, 2 m ; g_{i}\left(\mathbf{x}^{*}\right)=h_{i}\left(\mathbf{x}^{*}\right)$ if $i=m+1, \ldots, 2 m$. Then, we get

$$
\begin{gather*}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0}, \\
\mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m . \tag{48}
\end{gather*}
$$

The result follows.

We can also show that the KKT sufficient condition holds under the assumption of invexity.

Theorem 18 (KKT sufficient conditions). Suppose that the interval-valued function $F$ is $H$-differentiable and $g_{i}$ is differentiable for $i=1, \ldots, 2 m$ at $\mathbf{x}^{*}$ and $F, h_{i}$, and $g_{i}$ are invex with respect to the same vector function $\eta\left(\mathbf{x}, \mathbf{x}^{*}\right)$. If there exist Lagrange multipliers $0 \leq \mu_{i}, \lambda_{i} \in R$ for $i=1, \ldots, m$ such that

$$
\begin{gather*}
\nabla F\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=[\mathbf{0}, \mathbf{0}]  \tag{49}\\
\mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m \tag{50}
\end{gather*}
$$

where $[\mathbf{0}, \mathbf{0}]=([0,0], \ldots,[0,0])$ with $n$ components, then $\mathbf{x}^{*}$ is a nondominated solution of primal problem (IVP).

Proof. Suppose the contrary that $\mathbf{x}^{*}$ is not a nondominated solution of (IVP). Then, there exists a feasible solution $\widetilde{\mathbf{x}} \in$ $X$ such that $F(\widetilde{\mathbf{x}}) \prec F\left(\mathbf{x}^{*}\right)$. From Definition 11 and the assumptions, it can be shown that $F^{L}, F^{U}, g_{i}$, and $h_{i}$ are invex at $\mathbf{x}^{*}$ with respect to the same vector function $\eta\left(\mathbf{x}, \mathbf{x}^{*}\right)$ for all $i=1, \ldots, m$.

From the feasibility of $\widetilde{\mathbf{x}} \in X$, we get

$$
\begin{equation*}
g_{i}(\widetilde{\mathbf{x}}) \leq 0, \quad h_{i}(\widetilde{\mathbf{x}}) \leq 0 \quad i=1, \ldots, m \tag{51}
\end{equation*}
$$

From (49), we have

$$
\begin{align*}
& \nabla F^{L}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0}  \tag{52}\\
& \nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0} .
\end{align*}
$$

Since $h_{i}$ and $g_{i}$ are invex at $\mathbf{x}^{*}$ with respect to the same $\eta$, $\mu_{i} \geq 0$ and $\lambda_{i} \geq 0$ for all $i=1, \ldots, m$. Then

$$
\begin{align*}
& -\sum_{i=1}^{m} \mu_{i} \nabla g_{i}^{T}\left(\mathbf{x}^{*}\right) \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) \geq \sum_{i=1}^{m} \mu_{i}\left[g_{i}\left(\mathbf{x}^{*}\right)-g_{i}(\widetilde{\mathbf{x}})\right]  \tag{53}\\
& -\sum_{i=1}^{m} \lambda_{i} \nabla h_{i}^{T}\left(\mathbf{x}^{*}\right) \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) \geq \sum_{i=1}^{m} \lambda_{i}\left[h_{i}\left(\mathbf{x}^{*}\right)-h_{i}(\widetilde{\mathbf{x}})\right]
\end{align*}
$$

From (50) and (51), we have

$$
\begin{align*}
& \sum_{i=1}^{m} \mu_{i}\left[g_{i}\left(\mathbf{x}^{*}\right)-g_{i}(\widetilde{\mathbf{x}})\right] \geq \sum_{i=1}^{m} \mu_{i} g_{i}\left(\mathbf{x}^{*}\right)=0 \\
& \sum_{i=1}^{m} \lambda_{i}\left[h_{i}\left(\mathbf{x}^{*}\right)-h_{i}(\widetilde{\mathbf{x}})\right] \geq \sum_{i=1}^{m} \lambda_{i} h_{i}\left(\mathbf{x}^{*}\right)=0 \tag{54}
\end{align*}
$$

From (52), we get

$$
\begin{align*}
{\left[\nabla F^{L}\left(\mathbf{x}^{*}\right)\right]^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right)=} & -\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) \\
& -\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right)  \tag{55}\\
{\left[\nabla F^{U}\left(\mathbf{x}^{*}\right)\right]^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right)=} & -\sum_{i=1}^{m} \mu_{i} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right)^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) \\
& -\sum_{i=1}^{m} \lambda_{i} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) \tag{56}
\end{align*}
$$

Since the interval-valued function $F$ is invex at $\mathbf{x}^{*}$ with respect to $\eta$, then $F^{L}$ and $F^{U}$ are invex at $\mathbf{x}^{*}$ with respect to the same $\eta$. We have

$$
\begin{align*}
& F^{L}(\widetilde{\mathbf{x}}) \geq F^{L}\left(\mathbf{x}^{*}\right)+\left[\nabla F^{L}\left(\mathbf{x}^{*}\right)\right]^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right),  \tag{57}\\
& F^{U}(\widetilde{\mathbf{x}}) \geq F^{U}\left(\mathbf{x}^{*}\right)+\left[\nabla F^{U}\left(\mathbf{x}^{*}\right)\right]^{T} \eta\left(\widetilde{\mathbf{x}}, \mathbf{x}^{*}\right) . \tag{58}
\end{align*}
$$

By (53)-(55), (57), we obtain

$$
\begin{equation*}
F^{L}(\widetilde{\mathbf{x}}) \geq F^{L}\left(\mathbf{x}^{*}\right) \tag{59}
\end{equation*}
$$

Similarly, from (53)-(54), (56), and (58), we have

$$
\begin{equation*}
F^{U}(\widetilde{\mathbf{x}}) \geq F^{U}\left(\mathbf{x}^{*}\right) \tag{60}
\end{equation*}
$$

which contradicts that $F(\widetilde{\mathbf{x}}) \prec F\left(\mathbf{x}^{*}\right)$. The result follows.

## 5. Solvability

In this section, we discuss the solvability for Wolfe's primal and dual problems.

Lemma 19. Let $F, g_{i}, h_{i}$, and $i=1, \ldots, m$, be continuously differentiable on $R_{+}^{n}$. Suppose that $\widehat{\mathbf{x}}$ is a feasible solution of primal problem (IVP) and $(\mathbf{x}, \mu, \lambda)$ is a feasible solution of dual problem (DIVP). If $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, are invex at $\mathbf{x}$ with respect to the same vector function $\eta(\widehat{\mathbf{x}}, \mathbf{x})$, then the following statements hold true.
(i) If $F^{U}(\mathbf{x}) \geq F^{U}(\widehat{\mathbf{x}})$, then $F^{L}(\widehat{\mathbf{x}}) \geq H^{L}(\mathbf{x}, \mu, \lambda)$.
(ii) If $F^{U}(\mathbf{x})>F^{U}(\widehat{\mathbf{x}})$, then $F^{L}(\widehat{\mathbf{x}})>H^{L}(\mathbf{x}, \mu, \lambda)$.
(iii) If $F^{L}(\mathbf{x}) \geq F^{L}(\widehat{\mathbf{x}})$, then $F^{U}(\widehat{\mathbf{x}}) \geq H^{U}(\mathbf{x}, \mu, \lambda)$.
(iv) If $F^{L}(\mathbf{x})>F^{L}(\widehat{\mathbf{x}})$, then $F^{U}(\widehat{\mathbf{x}})>H^{U}(\mathbf{x}, \mu, \lambda)$.

Proof. From Definitions 3 and 11, it can be shown that $F^{L}$ and $F^{U}$ are continuously differentiable on $R_{+}^{n}$ and invex at $\mathbf{x}$ with respect to the same $\eta(\widehat{\mathbf{x}}, \mathbf{x})$.

Since $\widehat{\mathbf{x}}$ is a feasible solution of primal problem (IVP), then

$$
\begin{equation*}
g_{i}(\widehat{\mathbf{x}}) \leq 0, \quad h_{i}(\widehat{\mathbf{x}}) \leq 0, \tag{61}
\end{equation*}
$$

for all $i=1, \ldots, m$. Then we have

$$
F^{L}(\widehat{\mathbf{x}}) \geq F^{L}(\mathbf{x})+\left[\nabla F^{L}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})
$$

(by the invexity of $F^{L}$ )

$$
\begin{aligned}
= & F^{L}(\mathbf{x})-\left[\nabla F^{U}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})-\sum_{i=1}^{m} \mu_{i} \nabla g_{i}(\mathbf{x})^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x}) \\
& -\sum_{i=1}^{m} \lambda_{i} \nabla h_{i}(\mathbf{x})^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})
\end{aligned}
$$

(since $(\mathbf{x}, \mu, \lambda)$ is a feasible solution of dual problem (DIVP))

$$
\begin{aligned}
\geq & F^{L}(\mathbf{x})+F^{U}(\mathbf{x})-F^{U}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \mu_{i}\left[g_{i}(\mathbf{x})-g_{i}(\widehat{\mathbf{x}})\right] \\
& +\sum_{i=1}^{m} \lambda_{i}\left[h_{i}(\mathbf{x})-h_{i}(\widehat{\mathbf{x}})\right]
\end{aligned}
$$

(by the invexity of $F^{U}, g_{i}, h_{i}, \mu_{i}, \lambda_{i} \geq 0$ )

$$
\geq F^{L}(\mathbf{x})+F^{U}(\mathbf{x})-F^{U}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \mu_{i} g_{i}(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x})
$$

(by (61), $\mu_{i}, \lambda_{i} \geq 0$ )
$\geq F^{L}(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} g_{i}(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x})$,
if $F^{U}(\mathbf{x})-F^{U}(\widehat{\mathbf{x}}) \geq 0$
$=H^{L}(\mathbf{x}, \mu, \lambda)$.

Then statement (i) holds true. If $F^{U}(\mathbf{x})>F^{U}(\widehat{\mathbf{x}})$, then

$$
\begin{equation*}
F^{L}(\widehat{\mathbf{x}})>F^{L}(\mathbf{x})+\sum_{i=1}^{m} \mu_{i} g_{i}(\mathbf{x})+\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x})=H^{L}(\mathbf{x}, \mu, \lambda) \tag{63}
\end{equation*}
$$

it can be shown that statement (ii) holds. On the other hand, considering the real-valued function $F^{U}$, we can also obtain statements (iii) and (iv) by using the similar arguments.

Lemma 20. Let $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, be continuously differentiable on $R_{+}^{n}$. Suppose that $\widehat{\mathbf{x}}$ is a feasible solution of primal problem (IVP) and ( $\mathbf{x}, \mu, \lambda$ ) is a feasible solution of dual problem (DIVP). If $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, are invex at $\mathbf{x}$ with respect to the same vector function $\eta(\widehat{\mathbf{x}}, \mathbf{x})$, then the following statements hold true.
(i) If $F^{L}(\mathbf{x}) \leq F^{L}(\widehat{\mathbf{x}})$, then $F^{L}(\widehat{\mathbf{x}}) \geq H^{L}(\mathbf{x}, \mu, \lambda)$.
(ii) If $F^{L}(\mathbf{x})<F^{L}(\widehat{\mathbf{x}})$, then $F^{L}(\widehat{\mathbf{x}})>H^{L}(\mathbf{x}, \mu, \lambda)$.
(iii) If $F^{U}(\mathbf{x}) \leq F^{U}(\widehat{\mathbf{x}})$, then $F^{U}(\widehat{\mathbf{x}}) \geq H^{U}(\mathbf{x}, \mu, \lambda)$.
(iv) If $F^{U}(\mathbf{x})<F^{U}(\widehat{\mathbf{x}})$, then $F^{U}(\widehat{\mathbf{x}})>H^{U}(\mathbf{x}, \mu, \lambda)$.

Proof. From Definitions 3 and 11, it can be shown that $F^{L}$ and $F^{U}$ are continuously differentiable on $R_{+}^{n}$ and invex at $\mathbf{x}$ with respect to the same $\eta(\widehat{\mathbf{x}}, \mathbf{x})$. Consider the following:

$$
\begin{aligned}
F^{L}(\widehat{\mathbf{x}}) & -H^{L}(\mathbf{x}, \mu, \lambda) \\
& =F^{L}(\widehat{\mathbf{x}})-F^{L}(\mathbf{x})-\sum_{i=1}^{m} \mu_{i} g_{i}(\mathbf{x})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x}) \\
& \geq\left[\nabla F^{L}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})-\sum_{i=1}^{m} \mu_{i} \mathcal{g}_{i}(\mathbf{x})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x})
\end{aligned}
$$

(by the invexity of $F^{L}$ )

$$
\begin{aligned}
\geq & {\left[\nabla F^{L}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x}) } \\
& +\left[-\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \mu_{i} g_{i}(\mathbf{x})\right] \\
& +\left[-\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\mathbf{x})\right] \\
\geq & {\left[\nabla F^{L}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x}) } \\
& +\left[-\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \mu_{i} \nabla g_{i}(\mathbf{x})^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})\right] \\
& +\left[-\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}})+\sum_{i=1}^{m} \lambda_{i} \nabla h_{i}(\mathbf{x})^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})\right]
\end{aligned}
$$

(by the invexity of $g_{i}, h_{i}$ and $\mu_{i}, \lambda_{i} \geq 0$ )

$$
\begin{aligned}
= & \left\{\left[\nabla F^{L}(\mathbf{x})\right]^{T}+\sum_{i=1}^{m} \mu_{i} \nabla g_{i}(\mathbf{x})^{T}+\sum_{i=1}^{m} \lambda_{i} \nabla h_{i}(\mathbf{x})^{T}\right\} \eta(\widehat{\mathbf{x}}, \mathbf{x}) \\
& -\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}}) \\
= & -\left[\nabla F^{U}(\mathbf{x})\right]^{T} \eta(\widehat{\mathbf{x}}, \mathbf{x})-\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}})
\end{aligned}
$$

(since $(\mathbf{x}, \mu, \lambda)$ is a feasible solution of dual problem (DIVP))

$$
\begin{align*}
& \geq F^{U}(\mathbf{x})-F^{U}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \mu_{i} g_{i}(\widehat{\mathbf{x}})-\sum_{i=1}^{m} \lambda_{i} h_{i}(\widehat{\mathbf{x}}) \\
& =F^{U}(\mathbf{x})-H^{U}(\widehat{\mathbf{x}}, \mu, \lambda) \\
& \left.\geq 0, \quad \text { if } F^{L}(\mathbf{x}) \leq F^{L}(\widehat{\mathbf{x}}) \quad \text { (using Lemma } 19(\mathrm{iii})\right) . \tag{64}
\end{align*}
$$

Then statement (i) holds true. If $F^{L}(\mathbf{x})<F^{L}(\widehat{\mathbf{x}})$, then statement (ii) holds true by using Lemma 19(iv). On the other hand, we can also obtain statements (iii) and (iv) by using the similar arguments and Lemma 19(i) and (ii), respectively.

Proposition 21. Let $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, be continuously differentiable on $R_{+}^{n}$. Suppose that $\widehat{\mathbf{x}}$ is a feasible solution of primal problem (IVP) and ( $\mathbf{x}, \mu, \lambda$ ) is a feasible solution of dual problem (DIVP). If $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, are invex at $\mathbf{x}$ with respect to the same vector function $\eta(\widehat{\mathbf{x}}, \mathbf{x})$, then the following statements hold true.
(i) If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are comparable, then $F(\widehat{\mathbf{x}}) \succeq$ $H(\mathbf{x}, \mu, \lambda)$.
(ii) If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are not comparable, then $F^{L}(\widehat{\mathbf{x}})>$ $H^{L}(\mathbf{x}, \mu, \lambda)$ or $F^{U}(\widehat{\mathbf{x}})>H^{U}(\mathbf{x}, \mu, \lambda)$.

Proof. If $F(\mathbf{x}) \succeq F(\widehat{\mathbf{x}})$, then $F(\widehat{\mathbf{x}}) \succeq H(\mathbf{x}, \mu, \lambda)$ using Lemma 19(i) and (iii). On the other hand, if $F(\mathbf{x}) \preceq F(\widehat{\mathbf{x}})$, then $F(\widehat{\mathbf{x}}) \succeq H(\mathbf{x}, \mu, \lambda)$ using Lemma 20(i) and (iii).

If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are not comparable, then we have
(1) $F^{L}(\mathbf{x}) \leq F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x})>F^{U}(\widehat{\mathbf{x}})$;
(2) $F^{L}(\mathbf{x})<F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x}) \geq F^{U}(\widehat{\mathbf{x}})$;
(3) $F^{L}(\mathbf{x})<F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x})>F^{U}(\widehat{\mathbf{x}})$;
(4) $F^{L}(\mathbf{x}) \geq F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x})<F^{U}(\widehat{\mathbf{x}})$;
(5) $F^{L}(\mathbf{x})>F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x}) \leq F^{U}(\widehat{\mathbf{x}})$;
(6) $F^{L}(\mathbf{x})>F^{L}(\widehat{\mathbf{x}})$ and $F^{U}(\mathbf{x})<F^{U}(\widehat{\mathbf{x}})$.

By using Lemma 19 (ii) and (iv), and Lemma 20 (ii) and (iv), it can be shown that $F^{L}(\widehat{\mathbf{x}})>H^{L}(\mathbf{x}, \mu, \lambda)$ or $F^{U}(\widehat{\mathbf{x}})>$ $H^{U}(\mathbf{x}, \mu, \lambda)$.

Theorem 22 (solvability). Let $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, be invex with respect to the same vector function $\eta$ and continuously differentiable on $R_{+}^{n}$. Suppose that $\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right)$ is a feasible solution of dual problem (DIVP) and $H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \in$ $\operatorname{Obj}_{P}(F, X)$. Then ( $\mathbf{x}^{*}, \mu^{*}, \lambda^{*}$ ) solves dual problem (DIVP); that is, $H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \in \operatorname{Max}(H, Y)$.

Proof. Suppose that $\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right)$ is not a nondominated solution of dual problem (DIVP). Then there exists a feasible solution ( $\mathbf{x}, \mu, \lambda$ ) of dual problem (DIVP) such that $H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \prec H(\mathbf{x}, \mu, \lambda)$. According to the assumption of $H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \in \operatorname{Obj}_{P}(F, X)$, there exists a feasible solution $\widehat{\mathbf{x}}$ of primal problem (IVP) such that

$$
\begin{equation*}
F(\widehat{\mathbf{x}})=H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \prec H(\mathbf{x}, \mu, \lambda) . \tag{65}
\end{equation*}
$$

It means that the following (a1) or (a2) or (a3) is satisfied:
(al) $F^{L}(\widehat{\mathbf{x}})<H^{L}(\mathbf{x}, \mu, \lambda)$ and $F^{U}(\widehat{\mathbf{x}}) \leq H^{U}(\mathbf{x}, \mu, \lambda)$;
(a2) $F^{L}(\widehat{\mathbf{x}}) \leq H^{L}(\mathbf{x}, \mu, \lambda)$ and $F^{U}(\widehat{\mathbf{x}})<H^{U}(\mathbf{x}, \mu, \lambda)$;
(a3) $F^{L}(\widehat{\mathbf{x}})<H^{L}(\mathbf{x}, \mu, \lambda)$ and $F^{U}(\widehat{\mathbf{x}})<H^{U}(\mathbf{x}, \mu, \lambda)$.
If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are comparable. Then, from Proposition 21(i), we get $F(\widehat{\mathbf{x}}) \succeq H(\mathbf{x}, \mu, \lambda)$, which contradicts (65). If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are not comparable, we have $F^{L}(\widehat{\mathbf{x}})>H^{L}(\mathbf{x}, \mu, \lambda)$ or $F^{U}(\widehat{\mathbf{x}})>H^{U}(\mathbf{x}, \mu, \lambda)$ by using Proposition 21(ii), which contradicts one of (al)-(a3). We complete the proof.

Theorem 23 (solvability). Let $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, be invex with respect to the same vector function $\eta$ and continuously differentiable on $R_{+}^{n}$. Suppose that $\mathbf{x}^{*}$ is a feasible solution of primal problem (IVP) and $F\left(\mathbf{x}^{*}\right) \in \operatorname{Obj}_{D}(H, Y)$. Then $\mathbf{x}^{*}$ solves primal problem (IVP); that is, $F\left(\mathbf{x}^{*}\right) \in$ $\operatorname{Min}(F, X)$.

Proof. Suppose that $\mathbf{x}^{*}$ is not a nondominated solution of primal problem (IVP). Then there exists a feasible solution $\mathbf{x}$ of primal problem (IVP) such that $F(\mathbf{x}) \prec F\left(\mathbf{x}^{*}\right)$. According to the assumption of $F\left(\mathbf{x}^{*}\right) \in \mathrm{Obj}_{D}(H, Y)$, there exists a feasible solution $(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ of dual problem (DIVP) such that

$$
\begin{equation*}
H(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})=F\left(\mathbf{x}^{*}\right) \succ F(\mathbf{x}) \tag{66}
\end{equation*}
$$

It means that the following (c1) or (c2) or (c3) is satisfied:
(c1) $F^{L}(\mathbf{x})<H^{L}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ and $F^{U}(\mathbf{x}) \leq H^{U}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$;
(c2) $F^{L}(\mathbf{x}) \leq H^{L}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ and $F^{U}(\mathbf{x})<H^{U}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$;
(c3) $F^{L}(\mathbf{x})<H^{L}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ and $F^{U}(\mathbf{x})<H^{U}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$.
If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are comparable, then, from Proposition 21(i), we get $F(\mathbf{x}) \succeq H(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$, which contradicts (66). If $F(\mathbf{x})$ and $F(\widehat{\mathbf{x}})$ are not comparable, we have $F^{L}(\mathbf{x})>H^{L}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ or $F^{U}(\mathbf{x})>H^{U}(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ by using Proposition 21(ii), which contradicts one of (cl)-(c3). We complete the proof.

Theorem 24 (solvability). Let $F, g_{i}$, and $h_{i}, i=1, \ldots, m$, be invex with respect to the same vector function $\eta$ and continuously differentiable on $R_{+}^{n}$. Suppose that $\mathbf{x}^{*}$ is a feasible solution of primal problem (IVP) and ( $\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda}$ ) is a feasible solution of dual problem (DIVP). If $F\left(\mathbf{x}^{*}\right)=H(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$, then $\mathbf{x}^{*}$ solves primal problem (IVP) and $(\widehat{\mathbf{x}}, \widehat{\mu}, \widehat{\lambda})$ solves dual problem (DIVP).

Proof. The proof follows Theorems 22 and 23.

## 6. Duality Theorems

In this section, we present the weak and strong duality theorems under the assumption of invexity. Our results generalize the results of duality theorems by Wu in [11, 12].

Under the assumption convexity, $\mathrm{Wu}([11,12])$ has introduced two kinds of concepts of no duality gap and studied strong duality theorems.

Definition 25 (see [11, 12]). Two kinds of concepts of no duality gap are presented below.
(i) We say that the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense if and only if $\operatorname{Min}(F, X) \bigcap \operatorname{Max}(H, Y) \neq \varnothing$.
(ii) We say that the primal problem (IVP) and dual problem (DIVP) have no duality gap in strong sense if and only if there exist $F\left(\mathbf{x}^{*}\right) \in \operatorname{Min}(F, X)$ and $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right) \in \operatorname{Max}(H, Y)$ such that $F\left(\mathbf{x}^{*}\right)=$ $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$.
$\mathrm{Wu}([11,12])$ has shown that the primal problem (IVP) and dual problem (DIVP) have no duality gap in strong sense which implies that the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense.

Now, we establish strong duality theorems in weak and strong sense under the assumption of invexity, respectively.

Theorem 26 (strong duality theorem in weak sense). Let $F$, $g_{i}$, and $h_{i}, i=1, \ldots, m$, be invex with respect to the same vector function $\eta$ and continuously differentiable on $R_{+}^{n}$. If one of following conditions is satisfied:
(i) there exists a feasible solution $\mathbf{x}^{*}$ of primal problem $(I V P)$ such that $F\left(\mathbf{x}^{*}\right) \in O b j_{D}(H, Y)$,
(ii) there exists a feasible solution $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ of dual problem (DIVP) such that $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right) \in \operatorname{Obj}_{P}(F, X)$,

Then the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense.

Proof. Since the condition (i) is satisfied, from Theorem 23, it can be shown that $F\left(\mathbf{x}^{*}\right) \in \operatorname{Min}(F, X)$. According to the assumption of $F\left(\mathbf{x}^{*}\right) \in \operatorname{Obj}_{D}(H, Y)$, there exists a feasible solution ( $\widehat{\mathbf{x}}, \widehat{\boldsymbol{\mu}}, \widehat{\lambda}$ ) of dual problem (DIVP) such that $F\left(\mathbf{x}^{*}\right)=H(\widehat{\mathbf{x}}, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\lambda}})$. Using the similar arguments in the proof of Theorem 22 by looking at (65), we have $H(\widehat{\mathbf{x}}, \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\lambda}}) \in$ $\operatorname{Max}(H, Y)$. Suppose that condition (ii) is satisfied; from Theorem 22, we have $H\left(\mathbf{x}^{*}, \mu^{*}, \lambda^{*}\right) \in \operatorname{Max}(H, Y)$. Since $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right) \in \operatorname{Obj}_{P}(F, X)$, there exists a feasible solution $\widehat{\mathbf{x}}$ of primal problem (IVP) such that $F(\widehat{\mathbf{x}})=H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$. Using the similar arguments in the proof of Theorem 23 by looking at (66), we have $F(\widehat{\mathbf{x}}) \in \operatorname{Min}(F, X)$. Then, the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense.

Theorem 27 (strong duality theorem in strong sense). Let $F, g_{i}, h_{i}, i=1, \ldots, m$, be invex with respect to the same vector function $\eta$ and continuously differentiable on $R_{+}^{n}$. Suppose that $\mathbf{x}^{*}$ is a solution of the problem $P_{L U}$ (also is a nondominated solution of primal problem (IVP) by Proposition 15). If there exists a point $\widehat{\mathbf{x}}$ such that $g_{i}(\widehat{\mathbf{x}})<0$ and that $g_{i}\left(x^{*}\right)=0$ for all $i \in J\left(x^{*}\right), i=1, \ldots, 2 m$, and $g_{i}=h_{i}$ if $i=m+1, \ldots, 2 m$. Then the primal problem (IVP) and dual problem (DIVP) have no duality gap in strong sense; that is to say, there exist $\mathbf{0} \leq \mu^{*}, \lambda^{*} \in R^{m}$ such that $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \lambda^{*}\right)$ solves dual problem (DIVP) and $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)=F\left(\mathbf{x}^{*}\right)$.

Proof. According to the assumptions and Theorem 16, there exist $\mathbf{0} \leq \mu^{*}, \lambda^{*} \in R^{m}$ such that

$$
\begin{gathered}
\nabla F^{L}\left(\mathbf{x}^{*}\right)+\nabla F^{U}\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i}^{*} \cdot \nabla g_{i}\left(\mathbf{x}^{*}\right) \\
+\sum_{i=1}^{m} \lambda_{i}^{*} \cdot \nabla h_{i}\left(\mathbf{x}^{*}\right)=\mathbf{0}, \\
\mu_{i}^{*} g_{i}\left(\mathbf{x}^{*}\right)=0=\lambda_{i}^{*} h_{i}\left(\mathbf{x}^{*}\right) \quad \forall i=1, \ldots, m .
\end{gathered}
$$

It can be shown that $\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)$ is a feasible solution of dual problem (DIVP) and $H\left(\mathbf{x}^{*}, \boldsymbol{\mu}^{*}, \boldsymbol{\lambda}^{*}\right)=F\left(\mathbf{x}^{*}\right)+\sum_{i=1}^{m} \mu_{i}^{*} \cdot g_{i}\left(\mathbf{x}^{*}\right)+$ $\sum_{i=1}^{m} \lambda_{i}^{*} \cdot h_{i}\left(\mathbf{x}^{*}\right)=F\left(\mathbf{x}^{*}\right)$. Using Theorem 24, we complete the proof.

## 7. Conclusion

The Karush-Kuhn-Tucker optimality conditions and duality for interval-valued nonlinear optimization problems under the assumption of invexity are represented in this paper. Our results generalize the results of Wu in [11, 12]. Intervalvalued optimization provides a deterministic framework for studying mathematical programming problems in the face of data uncertainty. The result of Karush-Kuhn-Tucker optimality conditions can be also used to obtain the nondominated solution of interval-valued optimization problems. In the future research, we may extend to consider the Karush-KuhnTucker optimality conditions and duality for multiobjective interval-valued nonlinear optimization problems under the assumption of generalized convexity.

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## Research Article

# Genetic Algorithm Optimization for Determining Fuzzy Measures from Fuzzy Data 

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#### Abstract

Fuzzy measures and fuzzy integrals have been successfully used in many real applications. How to determine fuzzy measures is a very difficult problem in these applications. Though there have existed some methodologies for solving this problem, such as genetic algorithms, gradient descent algorithms, neural networks, and particle swarm algorithm, it is hard to say which one is more appropriate and more feasible. Each method has its advantages. Most of the existed works can only deal with the data consisting of classic numbers which may arise limitations in practical applications. It is not reasonable to assume that all data are real data before we elicit them from practical data. Sometimes, fuzzy data may exist, such as in pharmacological, financial and sociological applications. Thus, we make an attempt to determine a more generalized type of general fuzzy measures from fuzzy data by means of genetic algorithms and Choquet integrals. In this paper, we make the first effort to define the $\sigma-\lambda$ rules. Furthermore we define and characterize the Choquet integrals of interval-valued functions and fuzzy-number-valued functions based on $\sigma$ - $\lambda$ rules. In addition, we design a special genetic algorithm to determine a type of general fuzzy measures from fuzzy data.


## 1. Introduction

Fuzzy measures [1-4] and fuzzy integrals [5-9] have been applied successfully in multiattributes decision-making [10, 11], classification [12, 13], information fusion [14-18], nonlinear multiregression [19], feature selection [20, 21] and image processing. The reason of success is from the highly nonadditive and non-linear characteristics of fuzzy measures and fuzzy integrals. Fuzzy measure is the generalization of classical measure by using nonadditivity instead of additivity, which makes fuzzy measure be able to describe the importance of each individual information source (attribute or classifier) as well as the interaction [13], among them.

The Choquet integral [22-26] with respect to fuzzy measure is often used in information fusion and data mining as a nonlinear aggregation tool. The nonadditivity of fuzzy measures can effectively describe the interaction among the contributions from each attribute toward some target. Some works have shown successful applications of the Choquet
integral in nonlinear multiregressions, classifications, and decisionmakings [19, 25, 27-30], where the values of fuzzy measure are usually regarded as unknown parameter to be elicited from training data sets.

Most of existed works can only deal with the data consisting of classic numbers which may arise limitations in practical applications. It is not reasonable to assume that all data are real data before we elicit them from practical data. Sometimes, fuzzy data may exist, such as in pharmacological, financial, and sociological applications. Genetic algorithm (GA) is a stochastic search method for optimization problems based on the mechanics of natural selection and natural genetics. GA has demonstrated considerable success in providing good solutions to many complex optimization problems and received more and more attentions during the past three decades. The advantage of GA just make it able to obtain the global optimal solution fairly. In addition, compared with the traditional methods, GA has the ability to avoid getting stuck at a local optimal solution, since GA search from
a single point. Thus, we make an attempt to determine a more generalized type of general fuzzy measures from fuzzy data by means of genetic algorithms and Choquet integrals.

The rest of this study is organized as follows. In Section 2, the basic definitions of fuzzy measures based on $\sigma-\lambda$ rules are reviewed. Section 3 briefly introduces the basic concepts on the Choquet integral of real-valued functions based on rules and gives the operational schemes of its on discrete sets. In Section 4, we formulate the problems to be solved. Section 5 uses genetic algorithm optimization to determine fuzzy measures from real-valued data. Section 6, introduces the Choquet integral of interval-valued functions based on rules. Consequently, we use genetic algorithm optimization to determine fuzzy measures from interval-valued data. Section 7 discusses the Choquet integral of fuzzy numbervalued functions based on rules, and then uses genetic algorithm optimization to determine fuzzy measures from fuzzy number-valued data. Finally, conclusions are made in Section 8.

## 2. Fuzzy Measure Based on $\sigma-\lambda$ Rules

Definition 1. Let $X$ be a nonempty set and $\mathscr{A}$ a $\sigma$-algebra on the $X$. A set function $\mu$ is called a fuzzy measure based on $\sigma-\lambda$ rules if

$$
\mu\left(\bigcup_{i=1}^{\infty} A_{i}\right)= \begin{cases}\frac{1}{\lambda}\left\{\prod_{i=1}^{\infty}\left[1+\lambda \mu\left(A_{i}\right)\right]-1\right\}, & \lambda \neq 0  \tag{1}\\ \sum_{i=1}^{\infty} \mu\left(A_{i}\right), & \lambda=0\end{cases}
$$

where $\lambda \in(-(1 / \sup \mu), \infty) \bigcup\{0\},\left\{A_{i}\right\} \subset \mathscr{A}$, and $A_{i} \cap A_{j}=\emptyset$ for all $i, j=1,2, \ldots$ and $i \neq j$.

Particularly, if $\lambda=0$, then $\sigma-\lambda$ rule is $\sigma$-additivity.
Definition 2. Let $\mathscr{A}$ be a $\sigma$-algebra on the $X$. $\mu$ is called Sugeno measure based on $\sigma-\lambda$ rules if $\mu$ satisfies $\sigma-\lambda$ rules and $\mu(X)=$ 1. Briefly we denoted $g_{\lambda}$.

Remark 3. In Definition 1, if $n=2$, then

$$
\mu(A \cup B)= \begin{cases}\mu(A)+\mu(B)+\lambda \mu(A) \mu(B), & \lambda \neq 0  \tag{2}\\ \mu(A)+\mu(B), & \lambda=0\end{cases}
$$

Remark 4. In Definition 2, $g_{\lambda}$ is a classical probability measure if $\lambda=0$, and it can be represented by a wide range of nonadditive measure as long as we select proper parameters, many scholars think that it is a very important kind of nonadditive measure [31-33].

Example 5. Three workers, $x_{1}, x_{2}$, and $x_{3}$, are engaged in producing the same kind of products; the efficiencies of every people are given as follows: $\mu\left(x_{1}\right)=5, \mu\left(x_{2}\right)=6$, and $\mu\left(x_{3}\right)=8$. Then we can get the joint efficiencies by use of $\sigma-\lambda$ rules as shown in Table 1.

Remark 6. In Example 5, $x_{i}$ can be viewed as a attribute, $i=1,2,3$, we can calculate the contribution of their joint attributes by use of $\sigma-\lambda$ rules if we only know the contribution of individual attribute $g_{\lambda}\left(x_{i}\right), i=1,2,3$.

Table 1: The values of set function $\mu$ in Example 5.

| Set | Value of $\mu$ |
| :--- | :---: |
| $E_{1}=\left\{x_{1}\right\}$ | 5 |
| $E_{2}=\left\{x_{2}\right\}$ | 6 |
| $E_{3}=\left\{x_{1}, x_{2}\right\}$ | $11+30 \lambda$ |
| $E_{4}=\left\{x_{3}\right\}$ | 8 |
| $E_{5}=\left\{x_{1}, x_{3}\right\}$ | $13+40 \lambda$ |
| $E_{6}=\left\{x_{2}, x_{3}\right\}$ | $14+48 \lambda$ |
| $E_{7}=\left\{x_{1}, x_{2}, x_{3}\right\}$ | $240 \lambda^{2}+118 \lambda+19$ |

Theorem 7. Let $g_{\lambda}$ be a Sugeno measure based on $\sigma-\lambda$ rules. If $\lambda \geq 0$, then $g_{\lambda}$ has the monotonicity.

Proof. Let $E, F \in \mathscr{F}$ and $E \subset F$. Since $F=E \cup(F-E)$, this implies that

$$
\begin{gather*}
g_{\lambda}(F)=g_{\lambda}(E \cup(F-E))=g_{\lambda}(E)  \tag{3}\\
+g_{\lambda}(F-E)+\lambda g_{\lambda}(E) g_{\lambda}(F-E) \\
g_{\lambda}(F) \geq g_{\lambda}(E) \tag{4}
\end{gather*}
$$

for $\lambda \geq 0$ and $g_{\lambda} \geq 0$.
Due to the limitation of the classical measure, Sugeno, the Japanese scholar, presents set functions called fuzzy measures which use the monotonicity instead of the additivity. In practical applications, we often use regular fuzzy measure [32] on finite sets.

Definition 8 (see [28]). Let $X$ be a finite set and $2^{X}$ be the power set of $X$. Set function $\mu: 2^{X} \rightarrow[0,1]$ is called a regular fuzzy measure defined on let $2^{X}$ if the following conditions are satisfied:
(1) $\mu(\emptyset)=0, \mu(X)=1$;
(2) if $E \in 2^{X}, G \in 2^{X}$, and $E \subset G$, then $\mu(E) \leq \mu(G)$.

Definition 9 (see [28]). Let $X$ be a finite set and $2^{X}$ be the power set, of $X$. A fuzzy measure $\mu: 2^{X} \rightarrow[0,1]$ is called a regular $\lambda$-fuzzy measure defined on let $2^{X}$ if the following conditions are satisfied:
(1) $\mu(\emptyset)=0, \mu(X)=1$;
(2) if $\mu(A \cup B)=\mu(A)+\mu(B)+\lambda \mu(A) \mu(B)$, where $A \subset$ $X, B \subset X, A \cap B=\emptyset$ and $\lambda \in(-1, \infty)$.

Theorem 10. Let $g_{\lambda}$ be a Sugeno measure based on $\sigma-\lambda$ rules. Then $g_{\lambda}$ is a regular $\lambda$-fuzzy measure defined on $\mathscr{A}$.

Proof. We could prove that $g_{\lambda}(\emptyset)=0$. Otherwise for every $\lambda \geq 0, A \in \mathscr{A}$, we have

$$
\begin{align*}
g_{\lambda}(A) & =g_{\lambda}(A \cup \emptyset) \\
& =g_{\lambda}(A)+g_{\lambda}(\emptyset)+\lambda g_{\lambda}(A) g_{\lambda}(\emptyset),  \tag{5}\\
g_{\lambda}(\emptyset) & =-\lambda g_{\lambda}(A) g_{\lambda}(\emptyset) .
\end{align*}
$$

Since $g_{\lambda}(\emptyset) \neq 0$, we have $\lambda g_{\lambda}(\emptyset)=-1$; it is a contradiction. Furthermore, we obtain $g_{\lambda}(X)=1$, and $g_{\lambda}$ has the monotonicity by Definition 2 and Theorem 7.

Denoting finite set $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, the value $g_{\lambda i}=$ $g_{\lambda}\left(x_{i}\right)$ for all $i=1,2, \ldots, n$ is called measure density.

Theorem 11. The parameter $\lambda$ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the following equation:

$$
\begin{equation*}
\prod_{i=1}^{n}\left(1+\lambda g_{\lambda i}\right)=1+\lambda \tag{6}
\end{equation*}
$$

Proof. We can prove the above theorem by Theorem 10 and [32].

If we know the values of Sugeno measure based on $\sigma-\lambda$ on singleton sets, we can use Theorem 11 to obtain the values of $\lambda$ and then use Definition 1 to obtain the values on the other sets. It implies that a Sugeno measure based on $\sigma-\lambda$ can be determined by measure densities.

Theorem 12 (see [28]). If one knows the measure density $g_{\lambda i}$ on finite set $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, then there is only one solution $\lambda$ obtained from $\prod_{i=1}^{n}\left(1+\lambda g_{\lambda i}\right)=1+\lambda$.

## 3. The Choquet Integrals of Real-Valued Function Based on $\sigma-\lambda$ Rules

Definition 13 (see [34]). A regular fuzzy number, denoted by $\widetilde{a}$, is a fuzzy subset of $\mathbb{R}$ with membership function $m: \mathbb{R} \rightarrow$ $[0,1]$ satisfying the following conditions.
(RFN1) there exists at least one number $a_{0} \in \mathbb{R}$ such that $a_{0}=1$;
(RFN2) $m(t)$ is nondecreasing on $\left(-\infty, a_{0}\right.$ ] and nonincreasing on $\left[a_{0},+\infty\right)$;
(RFN3) $m(t)$ is upper semicontinuous; that is, $\lim _{t \rightarrow t_{0}^{+}}=m\left(t_{0}\right)$ if $t_{0}<a_{0}$ and $\lim _{t \rightarrow t_{0}^{-}}=m\left(t_{0}\right)$ if $t_{0}>a_{0}$;
(RFN4) $\int_{-\infty}^{\infty} m(t) d t<\infty$.
The set of all regular fuzzy numbers is denoted by $\mathcal{N}$.
Let $f: X \rightarrow(-\infty, \infty)$ be a measurable function with respect to $\mathscr{A}$; that is, $f$ satisfies the condition that

$$
\begin{equation*}
\{x \mid f(x) \geq \alpha\} \in \mathscr{A} \tag{7}
\end{equation*}
$$

for any $\alpha \in \mathbb{R}$.
From now on, we suppose that all functions defined on $X$ appearing as an integrand of the Choquet integral in this paper are measurable.

Definition 14 (see [22]). Let ( $X, \mathscr{A}$ ) be a measurable space, and let $g_{\lambda}$ be a Sugeno measure based on $\sigma-\lambda$ rules on $\mathscr{A}$.

The Choquet integral of a real-valued function $f: X \rightarrow$ $(-\infty,+\infty)$ is defined as

$$
\text { (c) } \begin{align*}
\int_{X} f d g_{\lambda}= & \int_{-\infty}^{0}\left[g_{\lambda}\left(F_{\alpha}\right)-g_{\lambda}(X)\right] d \alpha \\
& +\int_{0}^{\infty} g_{\lambda}\left(F_{\alpha}\right) d \alpha \tag{8}
\end{align*}
$$

where

$$
\begin{equation*}
F_{\alpha}=\{x \mid f(x) \geq \alpha\} \tag{9}
\end{equation*}
$$

for $\alpha \in(-\infty,+\infty)$, if both of Riemann integrals exist and at least one of them has finite value.

Let $A \in \mathscr{A}$, then Choquet integral of a nonnegative realvalued function $f: X \rightarrow(0,+\infty)$ is defined as

$$
\begin{equation*}
\text { (c) } \int_{A} f d g_{\lambda}=\int_{0}^{\infty} g_{\lambda}\left(A \cap F_{\alpha}\right) d \alpha . \tag{10}
\end{equation*}
$$

Without loss of the generality, Yang et al. [34] have proposed a new scheme to calculate the value of a Choquet integral with a real-valued integrand.

When $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, for any function $f: X \rightarrow$ $(-\infty,+\infty)$, both $\left[g_{\lambda}\left(F_{\alpha}\right)-g_{\lambda}(X)\right]$ and $g_{\lambda}\left(F_{\alpha}\right)$ are functions of $\alpha$ with bounded variance; therefore, their Riemann integrals with respect to $\alpha$ exist and are finite. So, Choquet integral (c) $\int f d g_{\lambda}$ is well defined. To calculate the value of the Choquet integral of a given real-valued function $f$, usually the values of $f$, that is, $f\left(x_{1}\right), f\left(x_{2}\right), \ldots, f\left(x_{n}\right)$, should be sorted in a nondecreasing order, so that $f\left(x_{1}^{\prime}\right) \leq f\left(x_{2}^{\prime}\right) \leq$ $\cdots \leq f\left(x_{n}^{\prime}\right)$, where $\left\{x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{n}^{\prime}\right\}$ is a certain permutation of $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. Then, the value of the Choquet integral is obtained by

$$
\begin{align*}
\text { (c) } \int f d g_{\lambda}=\sum_{i=1}^{n}[ & \left.f\left(x_{i}^{\prime}\right)-f\left(x_{i-1}^{\prime}\right)\right]  \tag{11}\\
& \cdot g_{\lambda}\left(\left\{x_{i}^{\prime}, x_{i+1}^{\prime}, \ldots, x_{n}^{\prime}\right\}\right)
\end{align*}
$$

where $f\left(x_{0}^{\prime}\right)=0$.
Example 15. Let $X=\left\{x_{1}, x_{2}, x_{3}\right\}, g_{\lambda}\left(x_{1}\right)=0.1, g_{\lambda}\left(x_{2}\right)=$ $0.2, g_{\lambda}\left(x_{3}\right)=0.3, f\left(x_{1}\right)=0.3, f\left(x_{2}\right)=0.7$, and $f\left(x_{3}\right)=$ 0.5. We can obtain $\prod_{i=1}^{3}\left(1+\lambda g_{\lambda_{i}}\right)=1+\lambda$ by Theorem 11 . Furthermore we get $\lambda=3.1$.

By Definition 1, we can get the following results:

$$
\begin{align*}
g_{\lambda}\left(\left\{x_{1}, x_{2}\right\}\right) & =g_{\lambda}\left(x_{1}\right)+g_{\lambda}\left(x_{2}\right)+\lambda g_{\lambda}\left(x_{1}\right) \cdot g_{\lambda}\left(x_{2}\right) \\
& =0.1+0.2+3.1 \times 0.1 \times 0.2  \tag{12}\\
& =0.362 .
\end{align*}
$$

Similarly, $g_{\lambda}\left(\left\{x_{1}, x_{3}\right\}\right)=0.493, g_{\lambda}\left(\left\{x_{2}, x_{3}\right\}\right)=0.593, g_{\lambda}\left(\left\{x_{1}\right.\right.$, $\left.\left.x_{2}, x_{3}\right\}\right)=1$.

Table 2: Data of Example 16.

| $i$ | $f_{i 1}$ | $f_{i 2}$ | $f_{i 3}$ | $f_{i 4}$ | $f_{i 5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 0.6 | 0.8 | 0.8 | 1 |
| 2 | 0.8 | 0.5 | 0.7 | 1 | 0.7 |
| 3 | 0.8 | 0.8 | 0.5 | 0.8 | 0.9 |
| 4 | 0.5 | 0.5 | 1 | 0.7 | 0.6 |
| 5 | 0.2 | 0.9 | 0.7 | 0.6 | 0.3 |
| 6 | 0.7 | 0.3 | 0.9 | 0.6 | 0.8 |
| 7 | 1 | 0.6 | 1 | 0.5 | 0.8 |
| 8 | 0.5 | 0.8 | 0.4 | 0.7 | 0.5 |
| 9 | 0.4 | 0.7 | 0.6 | 0.9 | 0.2 |
| 10 | 0.7 | 0.5 | 0.6 | 0.8 | 0.8 |

By Definition 14, we can get the Choquet integrals of $f(x)$ with respect to $g_{\lambda}$ as follows:

$$
\text { (c) } \begin{align*}
\int f d g_{\lambda}= & (0.3-0) \cdot 1+(0.5-0.3) \\
& \cdot g_{\lambda}\left(\left\{x_{3}, x_{2}\right\}\right)+(0.7-0.5)  \tag{13}\\
& \cdot g_{\lambda}\left(x_{2}\right)=0.4586 .
\end{align*}
$$

Example 16. Let $X=\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right\}, g_{\lambda}\left(x_{1}\right)=0.1$, $g_{\lambda}\left(x_{2}\right)=0.2, g_{\lambda}\left(x_{3}\right)=0.3, g_{\lambda}\left(x_{4}\right)=0.15$, and $g_{\lambda}\left(x_{5}\right)=$ 0.175 (Table 2).

By Theorem 11, we obtain that $\prod_{i=1}^{5}\left(1+\lambda g_{\lambda i}\right)=1+\lambda$, and with the aid of Mathematica software, we calculate that $\lambda=0.218$; furthermore, we get Choquet integrals which are shown in Table 3.

## 4. Questions Description: Determine Fuzzy Measures

In this section, we formulate our problems to be solved.
If we regard fuzzy integrals as multi-input single-output systems, we can obtain the Data through handling these systems. Suppose that we have several information sources $x_{1}, x_{2}, \ldots, x_{n}, n \geq 2$ and a given object $y$. Let $X=$ $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$; we have the data with sample size $m$ as shown in Table 4, where $f_{i j}$ is the $i$ th value of source $x_{j}$ and $E_{i}$ is the $i$ th value of object.

We hope to find a Sugeno measure $g_{\lambda}$ on measurable space $\left(X, 2^{X}\right)$, such that $E_{i}=(c) \int f_{i} d g_{\lambda}, i=1,2, \ldots, m$, where function $f_{i}$ is defined by $f_{i}\left(x_{j}\right)=f_{i j}, j=1,2, \ldots, n$ for $i=1,2, \ldots, m$.

If such a Sugeno measure $g_{\lambda}$ does not exist, we hope to find the optimally approximate solution. This is just the inverse problem of synthetic evaluation. We can also use the least square method to transform the above problem to a constrained optimization problem. An optimization problem is described as follows:

$$
\begin{equation*}
\min (V)=\sqrt{\frac{1}{m} \sum_{i=1}^{m}\left(E_{i}-(c) \int f_{i} d g_{\lambda}\right)^{2}} \tag{14}
\end{equation*}
$$

A result of $\min (V)=0$ also means that a precise solution is found.

Here, we discuss this problem in three aspects. The first one is the values of $f_{i j}$, and $E_{i}(i=1,2, \ldots, m, j=1,2, \ldots, n)$ are a real-valued data. The second one is the values of $f_{i j}$, and $E_{i}(i=1,2, \ldots, m, j=1,2, \ldots, n)$ are an interval-valued data. The last one is $f_{i j}$, and $E_{i}(i=1,2, \ldots, m, j=1,2, \ldots, n)$ are a fuzzy-valued data.

## 5. Using Genetic Algorithm to Determine Fuzzy Measures from Real-Valued Data

In this section, we use genetic algorithm to determine fuzzy measures from real-valued data.
5.1. Genetic Algorithm (GA). Genetic algorithm (GA) is a stochastic search method for optimization problems based on the mechanics of natural selection and natural genetics (i.e., survival of the fittest). GA has demonstrated considerable success in providing good solutions to many complex optimization problems and received more and more attentions during the past three decades. When the objective functions to be optimized in the optimization problems are multimodal or the search spaces are particularly irregular, algorithms need to be highly robust in order to avoid getting stuck at a local optimal solution. The advantage of GA just makes it able to obtain the global optimal solution fairly. In addition, GA does not require the specific mathematical analysis of optimization problems, which makes GA easily coded by users who are not necessarily good at mathematics and algorithms.
5.1.1. The Decimal Coding. Chromosome $V$ is denoted by $V=$ $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$, where gene $a_{i} \in[0,1]$ for all $i=1,2, \ldots, n$, and $a_{1}=g_{\lambda 1}, a_{2}=g_{\lambda 2}, \ldots$, and $a_{n-1}=g_{\lambda n-1}, a_{n}=\lambda$.
5.1.2. The Decoding. Find the formula

$$
\begin{equation*}
g_{\lambda n}=\frac{1}{\lambda}\left(\frac{1+\lambda}{\prod_{i=1}^{n-1}\left(1+\lambda g_{\lambda i}\right)}-1\right) \tag{15}
\end{equation*}
$$

by $g_{\lambda i}(i=1,2, \ldots, n-1), \lambda$, and Definition 1. Furthermore, the values of $g_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ for all $k=1,2, \ldots, n$ can be obtained by Definition 1 . The encoding can guarantee to get a feasible solution. That is, the solution satisfies Definition 1, and it will not undermine the feasibility of the solution no matter what kind of genetic operation (crossover or mutation) be used to chromosome.
5.1.3. The Arithmetic Crossover. Use the crossover probability $P_{c}$ to choose two parent chromosomes $V^{1}=\left(a_{1}^{1}, a_{2}^{1}, \ldots, a_{n}^{1}\right)$

Table 3: Results of Example 16.

| $(c) \int f_{1} d g_{\lambda}$ | (c) $\int f_{2} d g_{\lambda}$ | (c) $\int f_{3} d g_{\lambda}$ | (c) $\int f_{4} d g_{\lambda}$ | (c) $\int f_{5} d g_{\lambda}$ |
| :--- | :---: | :---: | :---: | :---: |
| 0.8 | 0.7 | 0.71 | 0.79 | 0.58 |
| $(c) \int f_{6} d g_{\lambda}$ | (c) $\int f_{7} d g_{\lambda}$ | $(c) \int f_{8} d g_{\lambda}$ | (c) $\int f_{9} d g_{\lambda}$ | (c) $\int f_{10} d g_{\lambda}$ |
| 0.67 | 0.79 | 0.56 | 0.56 | 0.65 |

Table 4: Data.

| $x_{1}$ | $x_{2}$ | $\cdots$ | $x_{n}$ | $y$ |
| :--- | :---: | :---: | :---: | :---: |
| $f_{11}$ | $f_{12}$ | $\ldots$ | $f_{1 n}$ | $E_{1}$ |
| $f_{21}$ | $f_{22}$ | $\ldots$ | $f_{2 n}$ | $E_{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $f_{m 1}$ | $f_{m 2}$ | $\cdots$ | $f_{m n}$ | $E_{m}$ |

and $V^{2}=\left(a_{1}^{2}, a_{2}^{2}, \ldots, a_{n}^{2}\right)$ and use the arithmetic crossover to get two offspring chromosomes $V^{3}$ and $V^{4}$ :

$$
\begin{align*}
V^{3}=\alpha V^{1}+(1-\alpha) V^{2}= & \left(\alpha a_{1}^{1}+(1-\alpha) a_{1}^{2}, \alpha a_{2}^{1}\right. \\
& \left.+(1-\alpha) a_{2}^{2}, \ldots, \alpha a_{n}^{1}+(1-\alpha) a_{n}^{2}\right), \\
V^{4}=(1-\alpha) V^{1}+\alpha V^{2}= & (1-\alpha) a_{1}^{1}+\alpha a_{1}^{2},(1-\alpha) a_{2}^{1} \\
& \left.+\alpha a_{2}^{2}, \ldots,(1-\alpha) a_{n}^{1}+\alpha a_{n}^{2}\right), \tag{16}
\end{align*}
$$

where $\alpha \in[0,1]$.
5.1.4. The Nonuniform Mutation. Select parent chromosome $V=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ according to the mutation probability $P_{m}$ and take the mutation to $a_{k}$ for the random generation of $k$ in $[1, n]$. Let

$$
\begin{equation*}
a_{k}^{\prime}=a_{k}+\phi\left(t, a_{k}^{U}-a_{k}\right) \tag{17}
\end{equation*}
$$

or

$$
\begin{equation*}
a_{k}^{\prime}=a_{k}-\phi\left(t, a_{k}-a_{k}^{L}\right) \tag{18}
\end{equation*}
$$

where $a_{k}^{U}$ and $a_{k}^{L}$ are the upper and lower bound of $a_{k}$, respectively, and $t$ is the number of generations. Function $\phi(t, b)$ is defined as follows:

$$
\begin{equation*}
\phi(t, b)=b \cdot r \cdot\left(1-\frac{t}{T}\right)^{b} \tag{19}
\end{equation*}
$$

where $r$ is a random number of $[0,1], T$ is the largest number of generations, and $b$ is the parameter. Obviously, $\lim _{n \rightarrow T} \phi(t, b)=0$.
5.1.5. The Evaluation Function. Evaluation function is defined by

$$
\begin{equation*}
\min (V)=\sqrt{\frac{1}{n} \sum_{i=1}^{n}\left(E_{i}-(c) \int f_{i} d g_{\lambda}\right)^{2}} \tag{20}
\end{equation*}
$$

where (c) $\int f_{i} d g_{\lambda}$ is defined by Definition 14 . Use the objective function as the evaluation function of a single chromosome.

The genetic algorithm procedure is summarized as follows.

Step 1. Initialize pop size chromosomes randomly.
Step 2. Update the chromosomes by crossover and mutation operations.

Step 3. Calculate the evaluation function for all chromosomes.

Step 4. Select the chromosomes by spinning the roulette wheel.

Step 5. Repeat the Step 2 to Step 3 for a given number of cycles.

Step 6. Report the best chromosome as the optimal solution.

### 5.2. Examples and Results

Example 17. A railway administration chooses 15 passengers randomly to evaluate the passenger train plan in the administration (Table 5). Customer bases its overall scores on transfer times, in-train congestion, travel time, and ticket price, and also they have a score for each of four aspects. Let $X=$ $\left\{x_{1}, x_{2}, x_{3}, x_{4},\right\}$ be the attributes of transfer times, congestion, travel time, and ticket price, respectively. We want to know which attribute is the most important for passengers. Here, $P_{c}=0.85, P_{m}=0.3$, and the population size is 20 .

We can obtain that the transfer times are the most important for passengers from Table 6 and the convergence rate of Example 17 as shown in Figure 1.

## 6. Using Genetic Algorithm to Determine Fuzzy Measures from Interval-Valued Data

The intervals are derived from many practical application problems, when instead of knowing the precise values $x$ of some quantity $X$ we know only the intervals $[\underline{x}, \bar{x}]$, in which $X$, ranges. Since the comparison of two values or quantities is the basic and most frequently used step in optimization, interval-valued function plays an important role in interval computation development.
6.1. The Choquet Integrals of Interval-Valued Function Based on $\sigma-\lambda$ Rules. With the definitions of the preceding subsections and from Wu et al. [35], we assume that $R^{+}=[0,+\infty)$,

Table 5: Data of Example 17 (the customer evaluation).

| Customer | Transfer times | Congestion | Travel time | Ticket price | Evaluation $E_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 0 | 0.3 | 1 | 0.2 |
| 2 | 0.3 | 0.8 | 0.5 | 0.6 |  |
| 3 | 0.7 | 0.9 | 1 | 0.7 | 0.8 |
| 4 | 0.1 | 0 | 0.4 | 0.3 | 0.15 |
| 5 | 0.5 | 0.4 | 0.5 | 0.5 | 0.7 |
| 6 | 1 | 0.3 | 0.8 | 0.8 | 0.5 |
| 7 | 0.3 | 1 | 0 | 1 | 0.7 |
| 8 | 0.9 | 0.6 | 0.5 | 0.7 | 0.5 |
| 9 | 0.6 | 0.8 | 0.2 | 0.4 |  |
| 10 | 0.4 | 1 | 0.5 | 0.6 |  |
| 11 | 0.8 | 0.4 | 0.7 | 0 | 0.75 |
| 12 | 1 | 0.8 | 1 | 0.3 | 0.5 |
| 13 | 0.2 | 0.6 | 0.2 | 0.5 | 0.2 |
| 14 | 0 | 0.3 | 0.8 | 0.9 | 0.3 |
| 15 | 0.4 | 0.2 |  | 0 | 0.3 |

Table 6: Results of Example 17.

| Set | $\lambda$ | $g_{\lambda}$ | Error | Number of <br> generation |
| :--- | :---: | :---: | :---: | :---: |
| $\left\{x_{1}\right\}$ |  | 0.2969 |  |  |
| $\left\{x_{2}\right\}$ | 0.965 | 0.1842 | 0.0154 | 100 |
| $\left\{x_{3}\right\}$ |  | 0.1347 |  |  |
| $\left\{x_{4}\right\}$ |  | 0.1529 |  |  |



Figure 1: The convergence rate of Example 17.
$I\left(R^{+}\right)=\left\{\bar{r}:\left[r^{-}, r^{+}\right] \subset R^{+}\right\}$is the set of interval numbers, and $\bar{F}(X)$ is the set of all interval numbers $\bar{f}$.

Interval numbers satisfy the following basic operations:
(1) $\bar{r} * \bar{p}=\left[r^{-} * p^{-}, r^{+} * p^{+}\right](*+\mathrm{V} \wedge)$;
(2) $k \cdot \bar{r}=\left[k r^{-}, k r^{+}\right],\left(k \in \mathbb{R}^{+}\right)$;
(3) $\bar{r} \leq \bar{p} \Leftrightarrow r^{-} \leq p^{-}, r^{+} \leq p^{+}$;
(4) $d(\bar{r}, \bar{p})=\max \left\{\left|r^{-}-p^{-}\right|,\left|r^{+}-p^{+}\right|\right\}$;
(5) if $d\left(\bar{r}_{n}, \bar{r}\right) \rightarrow 0$, then $\bar{r}_{n} \rightarrow \bar{r}$.

Definition 18 (see [34]). An interval-valued function $\bar{f}$ : $X \rightarrow I\left(R^{+}\right)$is measurable if both $f^{-}(x)$ and $f^{+}(x)$ are measurable function of $x$, where $\bar{f}(x)=\left[f^{-}(x), f^{+}(x)\right]$, $f^{-}(x)$ is the left end point of interval $\bar{f}(x)$, and $f^{+}(x)$ is the right end point of interval $\bar{f}(x)$.
Definition 19. Let $\left(X, \mathscr{A}, g_{\lambda}\right)$ be a nonadditive measure space based on $\sigma$ - $\lambda$ rules $\bar{f}: X \rightarrow I\left(R^{+}\right)$a measurable function in $X$ and $E \in \mathscr{A}$. Then the Choquet integral of $\bar{f}$ with respect to $g_{\lambda}$ is defined by

$$
\begin{equation*}
\text { (c) } \int_{E} \bar{f} d g_{\lambda}=:\left\{(c) \int_{E} \bar{f} d g_{\lambda} \mid g \in S_{\bar{f}(x)}\right\} \tag{21}
\end{equation*}
$$

if $(c) \int_{E} \bar{f} d g_{\lambda}$ is a closed interval on $I\left(R^{+}\right)$, where $S_{\bar{f}(x)}=\{g \mid$ $\left.g: X \rightarrow R^{+}\right\}$is a measurable selection on $\bar{f}(x)$.

Theorem 20. Let $\bar{f}: X \rightarrow I\left(R^{+}\right)$be a measurable intervalvalued function on $X$, and let $g_{\lambda}$ be a Sugeno measure based on $\sigma-\lambda$ rules on $\mathscr{A}$. The Choquet integral of $\bar{f}$ with respect to $g_{\lambda}$ is

$$
\begin{equation*}
\text { (c) } \int \bar{f} d g_{\lambda}=\left[(c) \int f^{-} d g_{\lambda},(c) \int f^{+} d g_{\lambda}\right], \tag{22}
\end{equation*}
$$

where $f^{-}(x)$ is the left end point of interval $\bar{f}(x)$ and $f^{+}(x)$ is the right end point of interval $\bar{f}(x)$, for every $x \in X$.

Proof. We can prove the above theorem by Theorem 10 and [34].

Using the continuity and the monotonicity of the Choquet integral with the nonnegativity and the monotonicity of the fuzzy measures, we may obtain the following theorem.
Theorem 21. Let $\bar{f}: X \rightarrow I\left(R^{+}\right)$be a measurable intervalvalued function on $X$, let $g_{\lambda}$ be a sugeno measure based on
$\sigma-\lambda$ rules on $\mathscr{A}$, and $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. Then, the value of Choquet integral of $\bar{f}$ with respect to $g_{\lambda}$ is obtained by
(c) $\int \bar{f} d g_{\lambda}$

$$
\begin{align*}
= & {\left[\sum_{i=1}^{n}\left[f^{-}\left(x_{i}^{\prime}\right)-f^{-}\left(x_{i-1}^{\prime}\right)\right]\right.} \\
& \cdot g_{\lambda}\left(\left\{x_{i}^{\prime}, x_{i+1}^{\prime}, \ldots, x_{n}^{\prime}\right\}\right)  \tag{23}\\
& \sum_{i=1}^{n}\left[f^{+}\left(x_{i}^{\prime \prime}\right)-f^{+}\left(x_{i-1}^{\prime \prime}\right)\right] \\
& \left.\cdot g_{\lambda}\left(\left\{x_{i}^{\prime \prime}, x_{i+1}^{\prime \prime}, \ldots, x_{n}^{\prime \prime}\right\}\right)\right]
\end{align*}
$$

where $f^{-}\left(x_{0}^{\prime}\right)=0, f^{+}\left(x_{0}^{\prime \prime}\right)=0$, the values of $f^{-}$and $f^{+}$, that is, $f^{-}\left(x_{1}\right), f^{-}\left(x_{2}\right), \ldots, f^{-}\left(x_{n}\right)$, and $f^{+}\left(x_{1}\right), f^{+}\left(x_{2}\right), \ldots, f^{+}\left(x_{n}\right)$ should be sorted in a nondecreasing order, so that $f^{-}\left(x_{1}^{\prime}\right) \leq$ $f^{-}\left(x_{2}^{\prime}\right) \leq \cdots \leq f^{-}\left(x_{n}^{\prime}\right)$ and $f^{+}\left(x_{1}^{\prime \prime}\right) \leq f^{+}\left(x_{2}^{\prime \prime}\right) \leq \cdots \leq f^{+}\left(x_{n}^{\prime \prime}\right)$, respectively, and $\left\{x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{n}^{\prime}\right\}$ and $\left\{x_{1}^{\prime \prime}, x_{2}^{\prime \prime}, \ldots, x_{n}^{\prime \prime}\right\}$ are a certain permutation of $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, respectively.
6.2. Using Genetic Algorithm to Determine Fuzzy Measures from Interval-Valued Data. In this subsection, we use genetic algorithm to determine fuzzy measures from interval-valued data.
6.2.1. Genetic Algorithm (GA). In this subsection, the all algorithms are the same to the Section 5.1 but the evaluation function. Here, the valuation function is defined by

$$
\begin{align*}
& \operatorname{Eval}(V)=\left(\frac { 1 } { m } \sum _ { i = 1 } ^ { m } \operatorname { m i n } \left\{\left((c) \int f_{i}^{-} d g_{\lambda}-E_{i}^{-}\right)^{2}\right.\right. \\
&\left.\left.\left((c) \int f_{i}^{+} d g_{\lambda}-E_{i}^{+}\right)^{2}\right\}\right)^{1 / 2} \tag{24}
\end{align*}
$$

where (c) $\int f_{i}^{-} d g_{\lambda}$ and (c) $\int f_{i}^{+} d g_{\lambda}$ are defined by Definition 14 and Theorem 20.

### 6.2.2. Examples and Results

Example 22. If the evaluation information in Example 17 is represented by the interval-valued fuzzy numbers as shown in Table 7, then we redetermine fuzzy measures from intervalvalued data by using genetic optimization.

We can obtain that the transfer times are the most important for passengers from Table 8 and the convergence rate of Example 22 as shown in Figure 2.

## 7. Using Genetic Algorithm to Determine Fuzzy Measures from Fuzzy-Valued Data

In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. suppose, for


Figure 2: The convergence rate of Example 22.
example, service quality is an intangible asset of enterprises that is related to customers judgments about the overall quality of a firm. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications), and experimental science. The concept takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty.

Definition 23 (see [33, 36, 37]). Let ( $X, \widetilde{\mathscr{F}}$ ) be a fuzzy measurable space, and let $\mathscr{F}^{*}(\mathbb{R})$ be the class of all fuzzy subsets of $\mathbb{R}$. A fuzzy-valued function $\tilde{f}: X \rightarrow \mathscr{F}^{*}(\mathbb{R})$ is called a measurable function if for every $\lambda \in(0,1]$, its $\alpha$-cut $f_{\alpha}(x)$ is measurable, where

$$
\begin{equation*}
f_{\alpha}(x)=(\tilde{f}(x))_{\alpha}=\{r \mid \tilde{f}(x)(r) \geq \alpha\} . \tag{25}
\end{equation*}
$$

Remark 24. A measurable fuzzy-valued function is a especially measurable fuzzy set-value function.

Let $\widetilde{f}: X \rightarrow \widetilde{\mathscr{F}}, \forall \lambda \in(0,1]$, and $(\widetilde{f}(x))_{\alpha}=$ $\left[(\tilde{f}(x))_{\alpha}^{-},(\tilde{f}(x))_{\alpha}^{+}\right]$. We will simply denote that

$$
\begin{equation*}
f_{\alpha}^{-}(x)=(\tilde{f}(x))_{\alpha}^{-}, f_{\alpha}^{+}(x)=(\tilde{f}(x))_{\alpha}^{+} \tag{26}
\end{equation*}
$$

Obviously, $f_{\alpha}^{-}(x)$ and $f_{\alpha}^{+}(x)$ are real functions.
From now on, we suppose that all functions defined on $X$ appearing as an integrand of the Choquet integral in this paper are measurable.

According to Theorem 10, $g_{\lambda}$ is a signed fuzzy measure on $\mathscr{A}$. Therefore, we may give the following definition referring to [34].

Fuzzy-valued function $\tilde{f}: X \rightarrow \widetilde{\mathscr{F}}$ is said to be a $C$-integrally bounded, if there exists a Choquet integrable function $h: X \rightarrow R^{+}$such that $|\dot{x}| \leq h(t)$ for $\dot{x} \in[\tilde{f}(t)]_{0}$.

Definition 25. Let $\tilde{f}: X \rightarrow \widetilde{\mathscr{F}}$ be a measurable fuzzy-valued function on $X$, and let $g_{\lambda}$ be a Sugeno measure based on $\sigma$ $\lambda$ rules. Assume that $\tilde{f}$ is $C$-integrally bounded. $\tilde{f}$ is called Choquet integrable with respect to $g_{\lambda}$ if

$$
\begin{equation*}
\left\{\left[(c) \int \tilde{f} \mathrm{~d} \mu\right]_{\lambda}=:\left\{(c) \int g \mathrm{~d} \mu \mid g \in S_{f_{\lambda}}\right\}, 0 \leq \lambda \leq 1\right\} \tag{27}
\end{equation*}
$$

Table 7: Data of Example 22 (the customer evaluation).

| Customer | Transfer times | Congestion | Travel time | Ticket price | Evaluation $E_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | $[0.00,0.00]$ | $[0.00,0.00]$ | $[0.25,0.35]$ | $[1.00,1.00]$ | $[0.15,0.25]$ |
| 2 | $[0.25,0.35]$ | $[0.15,0.25]$ | $[0.45,0.55]$ | $[0.55,0.65]$ | $[0.45,0.55]$ |
| 3 | $[0.65,0.75]$ | $[0.85,0.95]$ | $[1.00,1.00]$ | $[0.65,0.75]$ | $[0.75,0.85]$ |
| 4 | $[0.05,0.15]$ | $[0.00,0.00]$ | $[0.35,0.45]$ | $[0.25,0.35]$ | $[0.10,0.20]$ |
| 5 | $[0.45,0.55]$ | $[0.35,0.45]$ | $[0.45,0.55]$ | $[0.55,0.65]$ | $[0.45,0.55]$ |
| 6 | $[1.00,1.00]$ | $[0.25,0.35]$ | $[0.75,0.85]$ | $[0.75,0.85]$ | $[0.65,0.75]$ |
| 7 | $[0.25,0.35]$ | $[1.00,1.00]$ | $[0.00,0.00]$ | $[1.00,1.00]$ | $[0.45,0.55]$ |
| 8 | $[0.85,0.95]$ | $[0.55,0.65]$ | $[0.45,0.55]$ | $[0.65,0.75]$ | $[0.65,0.75]$ |
| 9 | $[0.55,0.65]$ | $[0.75,0.85]$ | $[0.15,0.25]$ | $[0.35,0.45]$ | $[0.45,0.55]$ |
| 10 | $[0.35,0.45]$ | $[1.00,1.00]$ | $[0.45,0.55]$ | $[0.00,0.00]$ | $[0.35,0.45]$ |
| 11 | $[0.75,0.85]$ | $[0.35,0.45]$ | $[1.00,1.00]$ | $[0.25,0.35]$ | $[0.55,0.65]$ |
| 12 | $[1.00,1.00]$ | $[0.75,0.85]$ | $[0.65,0.75]$ | $[0.45,0.55]$ | $[0.70,0.80]$ |
| 13 | $[0.15,0.25]$ | $[0.55,0.65]$ | $[1.00,1.00]$ | $[0.65,0.75]$ | $[0.45,0.55]$ |
| 14 | $[0.00,0.00]$ | $[0.25,0.35]$ | $[0.15,0.25]$ | $[0.85,0.95]$ | $[0.15,0.25]$ |
| 15 | $[0.35,0.45]$ | $[0.15,0.25]$ | $[0.75,0.85]$ | $[0.00,0.00]$ | $[0.25,0.35]$ |

Table 8: Results of Example 22.

| Set | $\lambda$ | $g_{\lambda}$ | Error | Number of <br> generation |
| :--- | :---: | :---: | :---: | :---: |
| $\left\{x_{1}\right\}$ |  | 0.3186 |  |  |
| $\left\{x_{2}\right\}$ | 0.937 | 0.1819 | 0.010 | 100 |
| $\left\{x_{3}\right\}$ |  | 0.1407 |  |  |
| $\left\{x_{4}\right\}$ |  | 0.1344 |  |  |

confirms a unique fuzzy number $\widetilde{a} \in \widetilde{\mathscr{F}}$, which is denoted by (c) $\int \tilde{f} \mathrm{~d} \mu=\widetilde{a}$, where $S_{f_{\lambda}}=\left\{g: X \rightarrow R, g \in f_{\lambda}\right.$ is a measurable selection of $\left.f_{\lambda}\right\}$.

The exact membership function of the Choquet integral with respect to Sugeno fuzzy measure $g_{\lambda}$ for fuzzy-valued integrand is rather difficult to be found. In a simpler but common case where $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ is finite, according to Definition 25, the calculation of the Choquet integral with a fuzzy-valued function comes down to that of the Choquet integral with an interval-valued function. Here, let us look at examples to show how to calculate the fuzzy-valued Choquet integral with respect to a Sugeno measure $g_{\lambda}$.

Example 26. Let $\tilde{f}\left(x_{1}\right)=[0,1,1], \widetilde{f}\left(x_{2}\right)=[0.5,0.5,1.5]$, and $\tilde{f}\left(x_{3}\right)=[0,1,2]$. Here, a triangular fuzzy number is denoted by $\left[a_{l}, a_{0}, a_{r}\right]$, where $a_{l}, a_{0}, a_{r} \in \mathbb{R}$ and $a_{l} \leq a_{0} \leq a_{r}$. Set function $g_{\lambda}$ is a sugeno measure based on $\sigma-\lambda$ rules. Function $\tilde{f}$ is triangular fuzzy valued. The membership function of $\widetilde{f}\left(x_{1}\right), \widetilde{f}\left(x_{2}\right)$, and $\widetilde{f}\left(x_{3}\right)$ are

$$
\begin{gather*}
m_{1}(t)=m_{\tilde{f}\left(x_{1}\right)}(t)= \begin{cases}t, & t \in[0,1], \\
0, & t \notin[0,1],\end{cases}  \tag{28}\\
m_{2}(t)=m_{\tilde{f}\left(x_{2}\right)}(t)= \begin{cases}1.5-t, & t \in[0.5,1.5], \\
0, & t \notin[0.5,1.5],\end{cases} \tag{29}
\end{gather*}
$$

$$
m_{3}(t)=m_{\tilde{f}\left(x_{3}\right)}(t)= \begin{cases}t, & t \in[0,1]  \tag{30}\\ 2-t, & t \in(1,2] \\ 0, & t \notin[0,2]\end{cases}
$$

respectively. They are shown in Figure 1. The $\alpha$-cut of $\tilde{f}$ is represented by interval

$$
\begin{aligned}
\tilde{f}_{\alpha}\left(x_{1}\right) & =M_{\alpha}^{\tilde{f}\left(x_{1}\right)}=\left\{t \mid m_{\tilde{f}\left(x_{1}\right)}(t) \geq \alpha\right\}=[\alpha, 1] ; \\
\tilde{f}_{\alpha}\left(x_{2}\right) & =M_{\alpha}^{\tilde{f}\left(x_{2}\right)}=\left\{t \mid m_{\tilde{f}\left(x_{2}\right)}(t) \geq \alpha\right\} \\
& =[0.5,1.5-\alpha] ; \\
\tilde{f}_{\alpha}\left(x_{3}\right) & =M_{\alpha}^{\tilde{f}\left(x_{3}\right)}=\left\{t \mid m_{\tilde{f}\left(x_{3}\right)}(t) \geq \alpha\right\} \\
& =[\alpha, 2-\alpha], \quad \text { where } \alpha \in[0,1] .
\end{aligned}
$$

It is easy to get $\left[\widetilde{f}_{\alpha}\left(x_{1}\right)\right]_{l}=\alpha,\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{l}=0.5,\left[\widetilde{f}_{\alpha}\left(x_{3}\right)\right]_{l}=$ $\alpha,\left[\tilde{f}_{\alpha}\left(x_{1}\right)\right]_{r}=1,\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{r}=1.5-\alpha$, and $\left[\tilde{f}_{\alpha}\left(x_{3}\right)\right]_{r}=2-\alpha$.

We conclude that
(1) When $0 \leq \alpha \leq 0.5$, we get $\left.\left[\tilde{f}_{\alpha}\left(x_{1}\right)\right]_{l}=\tilde{f}_{\alpha}\left(x_{3}\right)\right]_{l} \leq$ $\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{l}$. Using Definition 14, we can let $x_{1}^{\prime}=x_{1}, x_{2}^{\prime}=$ $x_{3}, x_{3}^{\prime}=x_{2}$, and then

$$
\begin{align*}
& {\left[(c) \int \tilde{f}_{\alpha} d g_{\lambda}\right]_{l}=\left\{\left[f\left(x_{1}^{\prime}\right)\right]_{l}-\left[f\left(x_{0}^{\prime}\right)\right]_{l}\right\}} \\
& \\
& \quad \cdot g_{\lambda}\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right)+\left\{\left[f\left(x_{2}^{\prime}\right)\right]_{l}-\left[f\left(x_{1}^{\prime}\right)\right]_{l}\right\} \\
& g_{\lambda}\left(x_{2}^{\prime}, x_{3}^{\prime}\right)+\left\{\left[f\left(x_{3}^{\prime}\right)\right]_{l}-\left[f\left(x_{2}^{\prime}\right)\right]_{l}\right\} \cdot g_{\lambda}\left(x_{3}^{\prime}\right) \\
& \quad=[\alpha-0] \cdot 1+[\alpha-\alpha] \cdot 0.5931+[0.5-\alpha] \cdot 0.3  \tag{32}\\
& \quad=0.7 \alpha+0.15
\end{align*}
$$

where $\left[f\left(x_{0}^{\prime}\right)\right]_{l}=0,\left[f\left(x_{0}^{\prime}\right)\right]_{r}=0$.

Table 9: Data of Example 27 (The customer evaluation).

| Customer | Transfer times | Congestion | Travel time | Ticket price | Evaluation $E_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | $[-0.1,0.00,0.10]$ | $[-0.1,0.00,0.10]$ | $[0.20,0.30,0.40]$ | $[0.90,1.00,1.10]$ | $[0.10,0.20,0.30]$ |
| 2 | $[0.20,0.30,0.40]$ | $[0.70,0.80,0.90]$ | $[0.40,0.50,0.60]$ | $[0.50,0.60,0.70]$ | $[0.40,0.50,0.60]$ |
| 3 | $[0.60,0.70,0.80]$ | $[0.80,0.90,1.00]$ | $[0.90,1.00,1.10]$ | $[0.60,0.70,0.80]$ | $[0.70,0.80,0.90]$ |
| 4 | $[0.00,0.10,0.20]$ | $[-0.1,0.00,0.10]$ | $[0.30,0.40,0.50]$ | $[0.20,0.30,0.40]$ | $[0.05,0.15,0.25]$ |
| 5 | $[0.40,0.50,0.60]$ | $[0.30,0.40,0.50]$ | $[0.40,0.50,0.60]$ | $[0.50,0.60,0.70]$ | $[0.40,0.50,0.60]$ |

Table 10: The triangular fuzzy number changed into the interval number.

| Customer | Transfer times | Congestion | Travel time |
| :--- | :---: | :---: | :---: |
| 1 | $[0.1 \alpha-0.1,-0.1 \alpha+0.1]$ | $[0.1 \alpha-0.1,-0.1 \alpha+0.1]$ | $[0.1 \alpha+0.2,-0.1 \alpha+0.4]$ |
| 2 | $[0.1 \alpha+0.2,-0.1 \alpha+0.4]$ | $[0.1 \alpha+0.7,-0.1 \alpha+0.9]$ | $[0.1 \alpha+0.4,-0.1 \alpha+0.6]$ |
| 3 | $[0.1 \alpha+0.6,-0.1 \alpha+0.8]$ | $[0.1 \alpha+0.8,-0.1 \alpha+1.0]$ | $[0.1 \alpha+0.9,-0.1 \alpha+1.1]$ |
| 4 | $[0.1 \alpha,-0.1 \alpha+0.2]$ | $[0.1 \alpha-0.1,-0.1 \alpha+0.1]$ | $[0.1 \alpha+0.3,-0.1 \alpha+0.5]$ |
| 5 | $[0.1 \alpha+0.4,-0.1 \alpha+0.6]$ | $[0.1 \alpha+0.3,-0.1 \alpha+0.5]$ |  |
| Customer | Ticket price | Evaluation $E_{i}$ |  |
| 1 | $[0.1 \alpha+0.9,-0.1 \alpha+1.1]$ | $[0.1 \alpha+0.1,-0.1 \alpha+0.3]$ |  |
| 2 | $[0.1 \alpha+0.5,-0.1 \alpha+0.7]$ | $[0.1 \alpha+0.7,-0.1 \alpha+0.6]$ |  |
| 3 | $[0.1 \alpha+0.6,-0.1 \alpha+0.8]$ | $[0.1 \alpha+0.05,-0.1 \alpha+0.9]$ |  |
| 4 | $[0.1 \alpha+0.2,-0.1 \alpha+0.4]$ | $[0.1 \alpha+0.4,-0.1 \alpha+0.6]$ |  |
| 5 | $[0.1 \alpha+0.5,-0.1 \alpha+0.7]$ |  |  |

Table 11: Results of Example 27.

| Set | $\lambda$ | $g_{\lambda}$ | Error | Number of <br> generation |
| :--- | :---: | :---: | :---: | :---: |
| $\left\{x_{1}\right\}$ |  | 0.4292 |  |  |
| $\left\{x_{2}\right\}$ | 0.05 | 0.2428 | 0.0049 | 100 |
| $\left\{x_{3}\right\}$ |  | 0.1586 |  |  |
| $\left\{x_{4}\right\}$ |  | 0.1524 |  |  |

When $0 \leq \alpha \leq 0.5$, we get $\left[\tilde{f}_{\alpha}\left(x_{1}\right)\right]_{r} \leq\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{r} \leq$ $\left[\tilde{f}_{\alpha}\left(x_{3}\right)\right]_{r}$. Using Definition 14 we can let $x_{1}^{\prime}=x_{1}, x_{2}^{\prime}=$ $x_{2}$, and $x_{3}^{\prime}=x_{3}$, and then

$$
\begin{align*}
& {\left[(c) \int \tilde{f}_{\alpha} d g_{\lambda}\right]_{r}=}\left\{\left[f\left(x_{1}^{\prime}\right)\right]_{r}-\left[f\left(x_{0}^{\prime}\right)\right]_{r}\right\} \\
& \cdot g_{\lambda}\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right) \\
&+\left\{\left[f\left(x_{2}^{\prime}\right)\right]_{r}-\left[f\left(x_{1}^{\prime}\right)\right]_{r}\right\}, \\
& g_{\lambda}\left(x_{2}^{\prime}, x_{3}^{\prime}\right)+\left\{\left[f\left(x_{3}^{\prime}\right)\right]_{r}-\left[f\left(x_{2}^{\prime}\right)\right]_{r}\right\} \cdot g_{\lambda}\left(x_{3}^{\prime}\right)  \tag{33}\\
&= {[1-0] \cdot 1+[1.5-\alpha-1] } \\
& \cdot 0.593+[2-\alpha-1.5+\alpha] \cdot 0.3 \\
&= 1.4465-0.593 \alpha .
\end{align*}
$$

That is, $(c) \int \tilde{f}_{\alpha} d g_{\lambda}=[0.7 \alpha+0.15,1.45-0.592]$.
(2) When $0.5<\alpha \leq 1$, we get $\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{l}<\left[\tilde{f}_{\alpha}\left(x_{3}\right)\right]_{l}=$ $\left[\tilde{f}_{\alpha}\left(x_{1}\right)\right]_{l}$. Using Definition 14, we can let $x_{1}^{\prime}=x_{2}, x_{2}^{\prime}=$ $x_{1}$, and $x_{3}^{\prime}=x_{3}$, and then

$$
\begin{align*}
& {\left[(c) \int \tilde{f}_{\alpha} d g_{\lambda}\right]_{l}=}\left\{\left[f\left(x_{1}^{\prime}\right)\right]_{l}-\left[f\left(x_{0}^{\prime}\right)\right]_{l}\right\} \\
& \cdot g_{\lambda}\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right) \\
&+\left\{\left[f\left(x_{2}^{\prime}\right)\right]_{l}-\left[f\left(x_{1}^{\prime}\right)\right]_{l}\right\},  \tag{34}\\
& g_{\lambda}\left(x_{2}^{\prime}, x_{3}^{\prime}\right)+\left\{\left[f\left(x_{3}^{\prime}\right)\right]_{l}-\left[f\left(x_{2}^{\prime}\right)\right]_{l}\right\} \cdot g_{\lambda}\left(x_{3}^{\prime}\right) \\
&= 0.5 \cdot 1+[\alpha-\alpha] \cdot 0.3+[\alpha-0.5] \cdot 0.493 \\
&= 0.493 \alpha+0.2535
\end{align*}
$$

where $\left[f\left(x_{0}^{\prime}\right)\right]_{l}=0,\left[f\left(x_{0}^{\prime}\right)\right]_{r}=0$.
When $0.5<\alpha \leq 1$, we get $\left[\tilde{f}_{\alpha}\left(x_{2}\right)\right]_{r} \leq\left[\tilde{f}_{\alpha}\left(x_{1}\right)\right]_{r} \leq$ $\left[\tilde{f}_{\alpha}\left(x_{3}\right)\right]_{r}$. Using Definition 14, we can let $x_{1}^{\prime}=x_{2}, x_{2}^{\prime}=$ $x_{1}$, and $x_{3}^{\prime}=x_{3}$, and then

$$
\begin{aligned}
{\left[(c) \int \tilde{f}_{\alpha} d g_{\lambda}\right]_{r}=} & \left\{\left[f\left(x_{1}^{\prime}\right)\right]_{r}-\left[f\left(x_{0}^{\prime}\right)\right]_{r}\right\} \\
& \cdot g_{\lambda}\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right) \\
& +\left\{\left[f\left(x_{2}^{\prime}\right)\right]_{r}-\left[f\left(x_{1}^{\prime}\right)\right]_{r}\right\}
\end{aligned}
$$



Figure 3: The convergence rate of Example 27.

$$
\begin{align*}
& g_{\lambda}\left(x_{2}^{\prime}, x_{3}^{\prime}\right)+\left\{\left[f\left(x_{3}^{\prime}\right)\right]_{r}-\left[f\left(x_{2}^{\prime}\right)\right]_{r}\right\} \cdot g_{\lambda}\left(x_{3}^{\prime}\right) \\
&= {[1.5-\alpha] \cdot 1+[1-(1.5-\alpha)] } \\
& \cdot 0.4931+[(2-\alpha)-1] \cdot 0.3 \\
&= 1.55-0.8 \alpha . \tag{35}
\end{align*}
$$

That is, (c) $\int \tilde{f}_{\alpha} d g_{\lambda}=[0.493 \alpha+0.2535,1.55-0.8 \alpha]$.
Example 27. If the evaluation information in Example 17 is represented by the triangular fuzzy number as shown in Table 9, then we redetermine fuzzy measures from fuzzyvalued data by using genetic algorithm.

Fist, we use the method as shown in Example 26 to change the triangular fuzzy number of Table 9 into interval number as shown in Table 10; that is, the optimization of triangular fuzzy number comes down to that of the interval number.

We can obtain that the transfer times are the most important for passengers from Table 11 and the convergence rate of Example 27 as shown in Figure 3, which represents the convergence rate of Example 27, where the solid line represents the convergence rate of the target value about the best optimal solution, and dotted line represents the convergence rate of average value of the target value about all the solutions.

## 8. Conclusions and Remarks

In this paper, we have considered the genetic algorithm optimization for determining fuzzy measures from fuzzy data. We have gotten joint measures by use of single measure with the aid of $\sigma-\lambda$ rules. Then we have formulated our problems to be solved; that is, how to determine the fuzzy measures from fixed data. Furthermore, we have introduced the Choquet integral of interval-valued functions and then given
the genetic algorithm optimization to determine fuzzy measures from interval-valued data. Finally we have discussed the Choquet integral of fuzzy number-valued functions. Consequently, we have given the genetic algorithm optimization to determine fuzzy measures from fuzzy number-valued data.

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## Research Article

# A Biobjective Fuzzy Integer-Nonlinear Programming Approach for Creating an Intelligent Location-Aware Service 

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#### Abstract

An intelligent location-aware service is created in the present study, in which a timely service is provided to a user without changing the user's pace. To the user, there are two goals to achieve-one is to reach the service location just in time; the other is to get to the destination as soon as possible. To consider the two objectives at the same time and to allow for the uncertainty in the dynamic environment, a biobjective fuzzy integer-nonlinear programming problem is formulated and solved. To illustrate the applicability of the proposed methodology, an experiment has been performed. According to the experimental results, the user's waiting time was reduced by $61 \%$ using the proposed methodology.


## 1. Introduction

Ambient intelligence (AmI), coined by European Commission in 2001, features an environment in which the environment supports the people inhabiting it in an unobtrusive/ transparent, interconnected, adaptable, dynamic, embedded, and intelligent way [1, 2]. An AmI system has the following characteristics.
(1) Interfacing with human senses rather than focusing on computer-based input and output devices.
(2) Sensors/detectors are embedded in everyday objects that can communicate with each other.
(3) Environment is sensitive to a user's needs and even can anticipate the user's needs or behavior.

In fact, a few basic AmI technologies and systems have already been applied in our everyday lives, such as thermostats, movement sensors that control lighting, and movement sensors linked to a security alarm for detecting intruders. AmI related topics include context-aware computing, ubiquitous computing, pervasive computing, everywhere computing, human/artificial intelligence, machine learning, agent-based software, and robotics.

Basically, AmI technologies can be divided into the following categories: ubiquitous computing, context awareness, intelligence, and natural user-system interaction. In the literature, several researchers have established the procedure for developing AmI systems. For example, in Cook et al. [2], the procedure is composed of three steps.
(1) Sensing features of the users and their environment.
(2) Reasoning about the accumulated data.
(3) Selecting actions to take that will benefit the users in the environment.

Garzotto and Valoriani [3] detailed the steps of developing an AmI system including requirements, specifications, mockups, functional prototypes, and beta-systems. After summarizing the viewpoints of these studies, a procedure of eight steps can be proposed as motion decomposition, motion analysis, scenario generation, law, privilege, and data security consideration, human-system interface design, data/message transmission, data analysis, and performance evaluation.

In short, the existing AmI methods have the following problems.
(1) There is a lack of a systematic procedure for developing AmI applications.
(2) Cost-effect analyses of AmI applications have seldom been done. That may be because most successful AmI applications are based on massive government support.
(3) The usefulness of some AmI technologies is being questioned.
(4) No AmI application is successful in every respect.

The objective of this study is to create an intelligent location-aware service (LAS), which can be regarded as an innovative application of AmI, ubiquitous computing, and mobile commerce. A context-aware service (CAS) system provides relevant information and/or services to a user through the interpretation of the user's context [4]. A LAS is a special CAS that utilizes the location of a user to adapt the service accordingly [4].

Krevl and Ciglarič [5] proposed a framework for developing location-based distributed applications. The framework comprises of three layers-the client layer, the application/server layer, and the database layer. Allamanis et al. [6] observed that LASs can foster the creation of online social networks. Savage et al. [7] designed a restaurant recommendation LAS system, which processed the user's location, preferences, feelings, and transportation mode with decision trees to make a recommendation. In addition, a discrete hidden Markov model was also built to reduce the misclassification in the decision tree. The LAS system developed by Zhang et al. [8] considered location features, social features, and implicit patterns and then recommended online social groups to users that may be interested in.

However, it is not easy for a LAS to be successful, because of the uncertainties arising from changes in a user's speed, positioning inaccuracy, unstable network connections, human-assisted service preparation, and others. To deal with such uncertainties, fuzzy and probabilistic (stochastic) methods are two main streams; however, fuzzy methods are easier to implement than probabilistic methods and do not rely on strict assumptions [9-12]. In the literature, fuzzy methods have been applied to various LASs. For example, Kotsakis and Ketselidis [13] indicated that the uncertainties of a dynamic environment increase the difficulty to interpret the context. In such a circumstance, linguistic terms may be a better choice to describe user's subjective feelings. Mateo et al. [14] established a set of fuzzy inference rules to help a user select the most suitable nearby restaurant. In Anagnostopoulos and Hadjiefthymiades [15], a user's context is represented through fuzzy variables assuming fuzzy values, which then become inputs to the fuzzy inference rules. Chen et al. [16] developed a LAS system that can inform users of the best station to park their cars in during the peak period. Simple fuzzy inference rules were established to make a recommendation.

In short, the existing LAS systems have the following problems.
(1) There is considerable uncertainty in detecting a user's position. For example, the detection error may be up to 20 meters using the global positioning system on
the cell phone [17]; that is large enough to mislead the recommendation process. Such an uncertainty has not been properly considered.
(2) There is a gap between theory and practice of LAS. More emphasis is put on the correctness of a service than on the timeliness of providing the service. Some LAS systems attempt to recommend the nearest service location to the user; however, timeliness is a different concept that forces the LAS system to go along with a user's pace.
An intelligent LAS is created in this study, in which a user sends a request to the system server through his/her mobile phone and reaches the recommended service location exactly as the requested service becomes available; that is just in time (JIT). JIT is traditionally a production strategy to reduce waste; however, it has seldom been applied to AmI, ubiquitous computing, or LAS. In this study, the general methodology or technology to address this problem is fuzzy mathematical programming, which is usually used to optimize a specific target under an uncertain environment [18-20].

Compared with the existing methods, the proposed methodology has the following innovative characteristics.
(1) The location and speed of a user are highly uncertain, and therefore they are given in fuzzy numbers.
(2) Since fuzzy numbers are used to represent the location and speed of a user, the time to go through a path, denoted as the "path duration," is also a fuzzy number.
(3) In addition, the service preparation time is also given in fuzzy numbers to consider its uncertainty. Since most of the services are subject to human intervention, such a treatment should be reasonable. As a result, the available time of each service location is also a fuzzy number.
(4) Two goals are to be achieved. The first one is to find a timely manner to reach the service location. The other is to reach the destination as soon as possible. That leads to a biobjective decision-making problem. To assist decision-making, a biobjective fuzzy integer nonlinear programming (2o-FINLP) model is formulated and solved.

The differences between the proposed methodology and some existing methods were summarized in Table 1.

The remaining of this paper is organized as follows. First, the architecture and operation procedure of the intelligent LAS are detailed in Section 2. To reach the service location just in time and to reach the destination as soon as possible, a 2o-FINLP problem is formulated and solved in Section 3. To illustrate the applicability of the proposed methodology, an experiment has been performed. Finally, Section 4 concludes this paper and lists some topics for future investigation.

## 2. The Intelligent LAS

The first step of developing an AmI system is to propose the scenario for usability evaluation. The scenario of

Table 1: Differences between the proposed methodology and some existing methods.

| Method | General methodology <br> or technology | Emphasis | Number of objectives |
| :--- | :--- | :--- | :--- |
| Kotsakis and Ketselidis [13] | Linguistic modeling | Correctness of <br> recommendation <br> Correctness of <br> recommendation | Single |
| Mateo et al. [14] | Fuzzy inference rules | Correctness of <br> recommendation | Single |
| Anagnostopoulos and <br> Hadjiefthymiades [15] | Linguistic modeling <br> and Fuzzy inference <br> rules | Fuzzy inference rules | Correctness of <br> recommendation <br> Correctness and timeliness <br> of recommendation |



Figure 1: The scenario.
the intelligent LAS is shown in Figure 1. While most of the scenarios are presented in short films, Figure 1 can still clearly express the concepts of the new system.

The system architecture of the intelligent LAS is illustrated in Figure 2. There are four main parts in the system: users, the communication and request system, the system server, and service locations. The interactions among them refer to the data flow diagram (DFD) in Figure 3 and are briefly described here. At first, the communication and request system establishes a ubiquitous environment for the users to access the system server and then the service locations. Then, the system server, as a third-party service provider, selectively bridges the gap between a user and the service locations. That is, only the JIT service location will be contacted by the system server on behalf of the user. The system server, finally, can get a commission from the service locations as a reward.

## 3. Determining the Fuzzy JIT Service Location and Path

The variables and parameters used in the proposed methodology are defined as follows.
(1) $i$ : there are $n$ nodes in the traffic network indicated with node $i=1 \sim n$.
(2) $k$ : there are $m$ service locations in the traffic network indicated with service location $k=1 \sim m$.
(3) $l_{j i}$ or $\widetilde{l}_{j i}$ : the length of the path connecting nodes $i$ and $j . i, j=1 \sim n ; i \neq j . l_{j i}=\infty$ if there is no connection between the two nodes. The start point and destination are nodes 1 and $n$, respectively. In addition, no back path is allowed; namely, $l_{j i}=\infty$ if $i>j$.
(4) $d_{i}$ or $\tilde{d}_{i}$ : the distance from the start point to node $i$. Obviously, $d_{1}=0$, and $d_{n}=\max _{i} d_{i}$. In addition, $d_{i}$ is determined by the distances of nodes connected to node $i$ as follows:

$$
\begin{gather*}
d_{i}=d_{j}+l_{j i} \\
j<i . \tag{1}
\end{gather*}
$$

(5) $d_{(k)}$ or $\widetilde{d}_{(k)}$ : the distance from the start point to service location $k . k=1 \sim m$.
(6) $d_{i, j}$ or $\tilde{d}_{i, j}$ : the distance from node $i$ to node $j$. Obviously, $d_{1, j}=d_{j}$.


Figure 2: The system architecture.
(7) $d_{i,(k)}$ or $\widetilde{d}_{i,(k)}$ : the distance from node $i$ to service location $k$.
(8) $p$ or $\tilde{p}$ : the service preparation time. Most of the services are prepared manually. For this reason, the preparation time is subject to the skills and mental and physical conditions of the staff and, therefore, may be highly uncertain. For these reasons, in this study, the service preparation time is given in triangular fuzzy numbers.
(9) $r$ : the positioning accuracy.
(10) $s$ or $\widetilde{s}$ : the user's speed.
(11) $t_{c}$ : the current time.
(12) $v_{k}$ or $\widetilde{v}_{k}$ : the available time of service location $k$.
3.1. Preliminary. Triangular fuzzy numbers are used in this study to represent uncertain variables. For this reason, some arithmetic operations on triangular fuzzy numbers are introduced [21].

Theorem 1 (arithmetic operations on triangular fuzzy numbers). $\widetilde{A}=\left(A_{1}, A_{2}, A_{3}\right)$ and $\widetilde{B}=\left(B_{1}, B_{2}, B_{3}\right)$ are two triangular fuzzy numbers:
(1) (Fuzzy addition)

$$
\widetilde{A}(+) \widetilde{B}=\left(A_{1}+B_{1}, A_{2}+B_{2}, A_{3}+B_{3}\right),
$$

(2) (Fuzzy subtraction)

$$
\widetilde{A}(-) \widetilde{B}=\left(A_{1}-B_{3}, A_{2}-B_{2}, A_{3}-B_{1}\right)
$$

(3) (Scalar multiplication)

$$
k \widetilde{A}=\left(k A_{1}, k A_{2}, k A_{3}\right) \text { if } k \geq 0
$$

(4) (Fuzzy multiplication)

$$
\widetilde{A}(\times) \widetilde{B} \cong\left(A_{1} B_{1}, A_{2} B_{2}, A_{3} B_{3}\right) \text { if } A_{1}, B_{1} \geq 0,
$$

(5) (Fuzzy division)

$$
\begin{equation*}
\widetilde{A}(/) \widetilde{B} \cong\left(\frac{A_{1}}{B_{3}}, \frac{A_{2}}{B_{2}}, \frac{A_{3}}{B_{1}}\right) \text { if } A_{1} \geq 0 ; B_{1}>0 . \tag{3}
\end{equation*}
$$



Figure 3: The DFD of the intelligent LAS.

Subsequently, we reviewed the ways of converting fuzzy objective functions and constraints. For a constraint $\widetilde{A} \leq \widetilde{B}$ in which both $\widetilde{A}$ and $\widetilde{B}$ are triangular fuzzy numbers, Klir and Yuan [22] proposed the following way to convert it:

$$
\begin{align*}
& A_{1} \leq B_{1} \\
& A_{2} \leq B_{2}  \tag{4}\\
& A_{3} \leq B_{3}
\end{align*}
$$

Another common way to convert the constraint is to defuzzify the related terms before comparison, say, using the center of gravity (COG) function [23]:

$$
\begin{equation*}
\frac{A_{1}+A_{2}+A_{3}}{3} \leq \frac{B_{1}+B_{2}+B_{3}}{3} \tag{5}
\end{equation*}
$$

The third way is to assess the possibility [24] that the constraint is satisfied by

$$
\begin{equation*}
\operatorname{pos}(\widetilde{A} \leq \widetilde{B})=\sup _{A \leq B} \min \left(\mu_{\widetilde{A}}(A), \mu_{\widetilde{B}}(B)\right) . \tag{6}
\end{equation*}
$$

Theorem 2 (possibility comparison of two triangular fuzzy numbers). To ensure that the possibility is $100 \%$, the following requirement has to be met:

$$
\begin{equation*}
A_{2} \leq B_{2} \tag{7}
\end{equation*}
$$

## The required proof is straightforward.

Based on these concepts and theorems, in the following section, we describe how to calculate the fuzzy speed of a user and the fuzzy travel time of a path.
3.2. Fuzzy Speed and Travel Time. According to the detection results of the GPS system, the distance that the user went
through within time $t$ is $q$. Then, considering the positioning uncertainty $r$, the fuzzy path length is ( $q-2 r, q, q+2 r$ ). The fuzzy speed is

$$
\begin{equation*}
\tilde{s}=\left(\frac{q-2 r}{t}, \frac{q}{t}, \frac{q+2 r}{t}\right) . \tag{8}
\end{equation*}
$$

The fuzzy travel time of a path with length $l_{j i}$ is

$$
\begin{align*}
\tilde{l}_{j i} & =l_{j i}(/)\left(\frac{q-2 r}{t}, \frac{q}{t}, \frac{q+2 r}{t}\right) \\
& =\left(\frac{l_{i j} t}{q+2 r}, \frac{l_{i j} t}{q}, \frac{l_{i j} t}{q-2 r}\right), \tag{9}
\end{align*}
$$

according to (3).
3.3. o-FINLP Model. To find the fuzzy JIT service location and path, the waiting time is to be minimized:

$$
\begin{equation*}
\operatorname{Min} \widetilde{Z}_{1}=\min _{k}\left(\widetilde{p}(-) \widetilde{d}_{(k)}\right) \tag{10}
\end{equation*}
$$

In addition, the preparation of the required service cannot start before the available time of the service location. To consider that, the objective function is modified:

$$
\begin{equation*}
\operatorname{Min} \widetilde{Z}_{1}=\min _{k}\left(\widetilde{p}(+) \max \left(0, \widetilde{v}_{k}-t_{c}\right)(-) \tilde{d}_{(k)}\right) \tag{11}
\end{equation*}
$$

On the other hand, the user also needs to reach the destination as soon as possible:

$$
\begin{equation*}
\operatorname{Min} \widetilde{Z}_{2}=\tilde{d}_{n} \tag{12}
\end{equation*}
$$

As a result, the following 20 -FINLP problem is to be solved:

$$
\begin{gather*}
\operatorname{Min} \widetilde{Z}_{1}=\min _{k}\left(\widetilde{p}(+) \max \left(0, \widetilde{v}_{k}-t_{c}\right)(-) \tilde{d}_{(k)}\right)  \tag{13}\\
\operatorname{Min} \widetilde{Z}_{2}=\widetilde{d}_{n}
\end{gather*}
$$

s.t.

$$
\begin{gather*}
\tilde{p}(+) \max \left(0, \tilde{v}_{k}-t_{c}\right) \geq \tilde{d}_{(k)},  \tag{14}\\
\tilde{d}_{i}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(\tilde{d}_{j}(+) \tilde{l}_{j i}\right), \quad i=1 \sim n,  \tag{15}\\
\sum_{j<i, l_{j i} \neq \infty} x_{j i}=1, \quad i=1 \sim n  \tag{16}\\
x_{j i} \in\{0,1\}, \quad i=1 \sim n ; j<i ; l_{j i} \neq \infty .
\end{gather*}
$$

Constraint (14) is to guarantee that the user will be in time. The binary variable $x_{j i}$ is used to indicate whether the path connecting nodes $i$ and $j$ has been selected or not.
3.4. The Equivalent Crisp Problem. To solve the FINLP problem, it has to be converted into a crisp problem. First, according to (2)~(3),

$$
\begin{align*}
& \widetilde{Z}_{1}=\left(Z_{11}, Z_{12}, Z_{13}\right) \\
& =\min _{k}\left(p_{1}+\max \left(0, v_{k 1}-t_{c}\right)-d_{(k) 3},\right.  \tag{17}\\
& p_{2}+\max \left(0, v_{k 2}-t_{c}\right)-d_{(k) 2}, \\
& \left.p_{3}+\max \left(0, v_{k 3}-t_{c}\right)-d_{(k) 1}\right) .
\end{align*}
$$

In a similar way,

$$
\begin{equation*}
\widetilde{Z}_{2}=\left(Z_{21}, Z_{22}, Z_{23}\right)=\left(d_{n 1}, d_{n 2}, d_{n 3}\right) . \tag{18}
\end{equation*}
$$

In the literature, Loukil et al. [25] mentioned five ways used to deal with multiobjective optimization problems: the simultaneous (or Pareto) approach, the utility (or compromise) approach, the goal programming (or satisfying) approach, the hierarchical approach, and the interactive approach [26]. In this study, both the hierarchical approach and the utility approach are applied to handle the 2o-FINLP problem. First, $\widetilde{Z}_{1}$ is optimized before $\widetilde{Z}_{2}$, according to the hierarchical approach. Such a treatment is reasonable, because the user will first encounter that decision problem to determine the JIT service location, that is, service location $k^{*}$. After determining the JIT service location, the path duration to the JIT service location becomes fixed, and the second objective reduces to the determination of the remaining path from the JIT service location to the destination.

Subsequently, to apply the utility approach to deal with the two objectives, we need to define the utility of a fuzzy objective (or variable). Here, we are of the opinion that the defuzzification result of a fuzzy variable embodies its
utility. So, in the proposed methodology, the COG function is applied to defuzzify a fuzzy variable and derive its utility:

$$
\begin{align*}
& U\left(\widetilde{Z}_{1}\right)=\frac{Z_{1}+Z_{2}+Z_{3}}{3} \\
& =\min _{k}\left(\frac { 1 } { 3 } \left(p_{1}+\max \left(0, v_{k 1}-t_{c}\right)-d_{(k) 3}\right.\right. \\
&  \tag{19}\\
& \quad+p_{2}+\max \left(0, v_{k 2}-t_{c}\right)-d_{(k) 2} \\
& \left.\left.\quad+p_{3}+\max \left(0, v_{k 3}-t_{c}\right)-d_{(k) 1}\right)\right), \\
& U\left(\widetilde{Z}_{2}\right)=\frac{Z_{21}+Z_{22}+Z_{23}}{3}=\frac{d_{n 1}+d_{n 2}+d_{n 3}}{3} .
\end{align*}
$$

The first constraint, that is, constraint (14), is converted in the same way:

$$
\begin{align*}
& \frac{p_{1}+p_{2}+p_{3}}{3}+\max \left(0, \frac{v_{k 1}+v_{k 2}+v_{k 3}}{3}-t_{c}\right) \\
& \quad \geq \frac{d_{(k) 1}+d_{(k) 2}+d_{(k) 3}}{3} \tag{20}
\end{align*}
$$

In addition, the possibility that constraint (14) is satisfied can be measured with

$$
\begin{align*}
& \operatorname{pos}\left(\widetilde{p}(+) \max \left(0, \widetilde{v}_{k}-t_{c}\right) \geq \tilde{d}_{(k)}\right) \\
& \quad=\sup _{p+\max \left(0, v_{k}-t_{c}\right) \geq d_{(k)}} \min \left(\mu_{\tilde{p}}(p), \mu_{\tilde{v}_{k}}\left(v_{k}\right), \mu_{\tilde{d}_{(k)}}\left(d_{(k)}\right)\right) . \tag{21}
\end{align*}
$$

To ensure that the possibility is $100 \%$, the following constraint has to be satisfied:

$$
\begin{equation*}
p_{2}+\max \left(0, v_{k 2}-t_{c}\right) \geq d_{(k) 2} \tag{22}
\end{equation*}
$$

Further, (15) is equal to

$$
\begin{align*}
\tilde{d}_{i}=\left(d_{i 1}, d_{i 2}, d_{i 3}\right)=( & \sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 1}+l_{j i 1}\right), \\
& \sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 2}+l_{j i 2}\right)  \tag{23}\\
& \left.\times \sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 3}+l_{j i 3}\right)\right) .
\end{align*}
$$

So

$$
\begin{align*}
& d_{i 1}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 1}+l_{j i 1}\right), \\
& d_{i 2}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 2}+l_{j i 2}\right),  \tag{24}\\
& d_{i 3}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 3}+l_{j i 3}\right) .
\end{align*}
$$

Finally, the 2o-FINLP model can be transformed into a twostep crisp model.

Step 1.

$$
\begin{aligned}
\operatorname{Min} U\left(\widetilde{Z}_{1}\right)=\min _{k}\left(\frac{1}{3}\right. & \left(p_{1}+\max \left(0, v_{k 1}-t_{c}\right)-d_{(k) 3}\right. \\
& +p_{2}+\max \left(0, v_{k 2}-t_{c}\right)-d_{(k) 2} \\
& \left.\left.+p_{3}+\max \left(0, v_{k 3}-t_{c}\right)-d_{(k) 1}\right)\right)
\end{aligned}
$$

s.t.

$$
\begin{align*}
& p_{2}+\max \left(0, v_{k 2}-t_{c}\right) \geq d_{(k) 2}, \\
& \frac{p_{1}+p_{2}+p_{3}}{3}+\max \left(0, \frac{v_{k 1}+v_{k 2}+v_{k 3}}{3}-t_{c}\right)  \tag{26}\\
& \geq \frac{d_{(k) 1}+d_{(k) 2}+d_{(k) 3}}{3}, \\
& d_{i 1}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 1}+l_{j i 1}\right), \quad i=1 \sim(m), \\
& d_{i 2}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 2}+l_{j i 2}\right), \quad i=1 \sim(m),  \tag{27}\\
& d_{i 3}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 3}+l_{j i 3}\right), \quad i=1 \sim(m), \\
& \sum_{j<i, l_{j i} \neq \infty} x_{j i}=1, \quad i=1 \sim(m),  \tag{28}\\
& x_{j i} \in\{0,1\}, \quad i=1 \sim(m) ; j<i ; l_{j i} \neq \infty . \tag{29}
\end{align*}
$$

Step 1 model has at most 1 objective function, $3(m)+1$ real variables, $(1 / 2)(m)((m)-1)$ binary variables, and $4(m)+3$ constraints. The objective function (25) is to minimize the defuzzified value of the minimum waiting time among the $m$ service locations. Constraints (26) are to guarantee that the user will not be late. Equations (27) are used to calculate the fuzzy distance from the start point to each node. Among the paths to a node, only one of them will be chosen, as required by (28). Binary variables are defined in (29).

Step 2.

$$
\begin{equation*}
\operatorname{Min} U\left(\widetilde{Z}_{2}\right)=\frac{d_{n 1}+d_{n 2}+d_{n 3}}{3} \tag{30}
\end{equation*}
$$

s.t.

$$
\begin{align*}
& d_{i 1}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 1}+l_{j i 1}\right), \quad i=\left(k^{*}\right) \sim n, \\
& d_{i 2}=\sum_{j<i, l_{j i} \neq \infty} x_{j i}\left(d_{j 2}+l_{j i 2}\right), \quad i=\left(k^{*}\right) \sim n,  \tag{31}\\
& d_{i 3}=\sum_{j<i, l_{l i} \neq \infty} x_{j i}\left(d_{j 3}+l_{j i 3}\right), \quad i=\left(k^{*}\right) \sim n, \\
& \sum_{j<i, l_{j i} \neq \infty} x_{j i}=1, \quad i=\left(k^{*}\right) \sim n,  \tag{32}\\
& x_{j i} \in\{0,1\}, \quad i=\left(k^{*}\right) \sim n ; j<i ; l_{j i} \neq \infty . \tag{33}
\end{align*}
$$

Table 2: The available times of the three service locations.

| $k$ | Node \# | $\widetilde{\nu}_{k}$ |
| :--- | :---: | :---: |
| 1 | 7 | $(8,10,11)$ |
| 2 | 8 | 0 |
| 3 | 9 | $(7,9,11)$ |

Table 3: The fuzzy JIT paths to the service locations.

| $k$ | Node $\#$ | $U *\left(\widetilde{Z}_{1}\right)$ | Fuzzy JIT path |
| :---: | :---: | :---: | :---: |
| 1 | 7 | 6.3 | $1 \rightarrow 4 \rightarrow 7$ |
| 2 | 8 | No feasible solution | No feasible solution |
| 3 | 9 | 2.3 | $1 \rightarrow 4 \rightarrow 7 \rightarrow 9$ |

Step 2 model has at most 1 objective function, $3(n-(1))+1$ real variables, $(1 / 2)(n-(1))(n-(1)-1)$ binary variables, and $3(m)+1$ constraints. The objective function (30) is to minimize the defuzzified value of the arrival time at the destination. Equations (31) are used to calculate the fuzzy distance from the start point to each node. Among the paths to a node, only one of them will be chosen, as required by (32). Binary variables are defined in (33).

To determine the fuzzy JIT service location and path, the following procedure is proposed.
(1) For service location $k$, calculate its JIT path using the Step 1 model. Assume the optimal objective function value of Step 1 model is $U *\left(\widetilde{Z}_{1}\right)(k)$.
(2) Find the $k^{*}$ that minimizes $U *\left(\widetilde{Z}_{1}\right)\left(k^{*}\right)$. Node $\left(k^{*}\right)$ is the JIT service location.
(3) Determine the remaining path from service location $k^{*}$ to the destination using Step 2 model.
3.5. An Experiment. The experimental region is located in the Seatwen District of Taichung City, Taiwan. The area of the experimental region is about 3 square kilometers (see Figure 4). In this figure, there are three possible service locations at different places-nodes 7, 8, and 9 (indicated with red pins). These service locations represent fast food restaurants with drive-in facilities. The abstracted road map is shown in Figure 5. The length of each path is expressed with a triangular fuzzy number. Assume $\widetilde{p}=(6,8,13) ; t_{c}$ $=0$. The available times of the three service locations were summarized in Table 2.

As an example, the Step 1 model for finding the fuzzy JIT path to service location 1 (node 7) is illustrated in Box 1. The optimal objective function value $U *\left(\widetilde{Z}_{1}\right)(1)$ is 6.3 . Namely, the user only needs to wait 6.3 minutes after he/she reaches service location 1 (node 7). The fuzzy JIT path to this service location is $1 \rightarrow 4 \rightarrow 7$. In the same way, the fuzzy JIT paths to other service locations have been obtained and were summarized in Table 3. Obviously, the fuzzy JIT service location in this experiment is service location 3 (node 9). The corresponding path is $1 \rightarrow 4 \rightarrow 7 \rightarrow 9$.

Subsequently, to determine the remaining path from the fuzzy JIT service location to the destination, Step 2 model

Table 4: Comparison of the results.

| Model | JIT service location | JIT path | Remaining path | Waiting time |
| :--- | :---: | :---: | :---: | :---: |
| Fuzzy | 3 (node 9) | $1 \rightarrow 4 \rightarrow 7 \rightarrow 9$ | $9 \rightarrow 13 \rightarrow 14$ | 2.3 |
| Crisp | 1 (node 7) | $1 \rightarrow 4 \rightarrow 7$ | $7 \rightarrow 10 \rightarrow 13 \rightarrow 14$ | 6 |



Figure 4: The experimental region.
is to be solved. For that, the Lingo code is shown in Box 2. The optimal remaining path is $9 \rightarrow 13 \rightarrow 14$, and $U *\left(\widetilde{Z}_{2}\right)=$ 23.7.
3.6. Global Optimality of the Solution. To solve the 2o-FINLP problem, the existing optimization packages generally use branch-and-bound algorithms to approximate the optimal solution, which cannot guarantee the global optimality of the solution. Chen and Wu [27] mentioned a forced reoptimization procedure to tackle this problem, which is considered useful to the present study. The concept is to add another constraint to the defuzzified models. For example, for Step 1 model,

$$
\begin{equation*}
U\left(\widetilde{Z}_{1}\right) \leq U\left(\widetilde{Z}_{1}\right)(\zeta)+\Delta \tag{34}
\end{equation*}
$$

where $U\left(\widetilde{Z}_{1}\right)(\zeta)$ is the optimal objective function value of Step 1 model after the $\zeta$-th round of optimization. $\Delta$ is the required minimum improvement. The process stops if no improvement can be gained after another round of reoptimization. Similarly, for Step 2 model,

$$
\begin{equation*}
U\left(\widetilde{Z}_{2}\right) \leq U\left(\widetilde{Z}_{2}\right)(\zeta)+\Delta \tag{35}
\end{equation*}
$$

In the previous experiment, we set $\Delta=1$. After a round of the forced re-optimization, the solutions remain unchanged, which means that the solution is already global optimal.
3.7. Comparison with the Crisp Case. For a comparison, we also examined the crisp case, in which only the core values (i.e., values with membership equal to 1 ) of fuzzy variables
and parameters are considered and their uncertainties are ignored. After optimization, the JIT service location is service location 1 (node 7) with a waiting time of 6 . The JIT path and the remaining path are $1 \rightarrow 4 \rightarrow 7$ and $7 \rightarrow 10 \rightarrow 13 \rightarrow$ 14 , respectively. Obviously, the results are quite different from the above analysis considering the uncertainty of variables (see Table 4). If the waiting time after reaching the service location is compared, then the fuzzy model achieves a better performance.

## 4. Conclusions

This study created an intelligent LAS for a dynamic environment, in which a user requires a timely service when going to his/her destination. Obviously, there are two objectives to the user-one is to choose the JIT service location; the other is to reach the destination as soon as possible. In order to optimize these two objectives simultaneously, a $20-$ FINLP problem was to be solved. Another feature of the proposed methodology is the incorporation of fuzzy variables to consider the uncertainty of dynamic factors. To solve the 2o-FINLP problem, first the two objectives are processed in a hierarchical manner. Subsequently, the utility of either fuzzy objective function value is defined; that serves as the basis for comparing feasible solutions.

The proposed system is a generic framework and can be applied to various service industries, such as the fast food industry. In Taiwan, for example, the scale of the fast food industry has reached NT\$ 22 billion until 2011. In this industry, the drive-in facility is universally used to provide real-time services to consumers. However, consumers must still wait in the driveway for the ordering, billing, and preparation of meals. To eliminate the problem of waiting, some meals can be prepared in advance; however, that leads to food preservation problems. The proposed JIT LAS service is expected to solve both of these problems.

The contribution of this study is twofold.
(1) It is the first attempt to apply the concept of JIT to LAS.
(2) Most existing fuzzy LASs use fuzzy reference rules that have not been optimized. In contrast, this study optimized the selection of service location using fuzzy mathematical programming.
To illustrate the applicability of the proposed methodology, an experiment has been performed. According to the experimental results,
(1) the user's waiting time was indeed reduced by the proposed methodology;
(2) after considering the uncertainties of various parameters, the recommendation results indeed became


Figure 5: The abstracted road map.

```
\(\min =56-d 71-d 72-d 73\);
\(18 \geq d 72\);
\(56 \geq d 71+d 72+d 73\);
\(d 11=0\);
\(d 12=0\);
\(d 13=0\);
\(d 31=d 11+1\);
\(d 32=d 12+2 ;\)
\(d 33=d 13+5 ;\)
\(d 41=d 11+3 ;\)
\(d 42=d 12+5 ;\)
\(d 43=d 13+8 ;\)
\(d 71=x 37 * d 31+2 * x 37+x 47 * d 41+6 * x 47\);
\(d 72=x 37 * d 32+4 * x 37+x 47 * d 42+7 * x 47\);
\(\mathrm{d} 73=x 37 * d 33+4 * x 37+x 47 * d 43+8 * x 47\);
\(x 37+x 47=1\);
@bin(x37); @bin(x47);
```

Box 1: Step 1 model for finding the fuzzy JIT path to service location 1 (node 7).

```
\(\min =d 141+d 142+d 143 ;\)
\(d 71=9\);
\(d 72=12 ;\)
\(d 73=16 ;\)
\(d 101=d 71+3 ;\)
\(d 102=d 72+6 ;\)
\(d 103=d 73+9\);
\(d 91=11\);
\(d 92=16 ;\)
\(d 93=20\);
\(d 121=d 91+1 ;\)
\(d 122=d 92+1 ;\)
\(d 123=d 93+5 ;\)
\(d 131=d 91+4 ;\)
\(d 132=d 92+6 ;\)
\(d 133=d 93+7 ;\)
\(d 141=x 1214 * d 121+5 * x 1214+x 1314 * d 131+1 * x 1314 ;\)
\(d 142=x 1214 * d 122+8 * x 1214+x 1314 * d 132+2 * x 1314 ;\)
\(d 143=x 1214 * d 123+9 * x 1214+x 1314 * d 133+4 * x 1314\);
\(x 1214+x 1314=1\);
@bin(x1214); @bin(x1314);
```

Box 2: Step 2 model for determining the remaining path.
different. Nevertheless, such a treatment reduced the risk of unsuitable recommendation.

However, there are some limitations that need to be acknowledged and addressed regarding the proposed methodology.
(1) Although the existing optimization packages can be easily applied to solve the problem of finding the JIT service location, the acquisition cost of the optimization package undoubtedly will increase the budget for establishing the system, which is unfavorable to the promotion of the proposed methodology.
(2) In addition, if any existing optimization package is used as the problem solver, an additional interface will need to be developed to convert the data received by the system server to the model file of the optimization package, which may not be easy.
(3) In theory, the JIT path problem is equivalent to a restricted longest path problem and may not be solved effectively when the problem scale is large. Fortunately, in real life, the JIT path problem occurs mostly in small areas.
(4) In addition, in practical applications, a lot of users may send their requests to the system server simultaneously, which causes a great burden on the system server.

To address these difficulties, an algorithm or heuristic can be developed in future studies to replace the optimization package. There are many algorithms to find the shortest or longest path in a network that can serve as a basis for developing the algorithm/heuristic. Such an algorithm/heuristic can be easily programmed and integrated with the other modules on the system server. However, one drawback of an algorithm/heuristic is that the solution obtained is not always globally optimal. In addition, the concept of JIT can be applied to other fields of LAS or AmI, in order to provide more support to users in a dynamic environment.

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# Research Article 

# Correlation Measures of Dual Hesitant Fuzzy Sets 

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#### Abstract

The dual hesitant fuzzy sets (DHFSs) were proposed by Zhu et al. (2012), which encompass fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and fuzzy multisets as special cases. Correlation measures analysis is an important research topic. In this paper, we define the correlation measures for dual hesitant fuzzy information and then discuss their properties in detail. One numerical example is provided to illustrate these correlation measures. Then we present a direct transfer algorithm with respect to the problem of complex operation of matrix synthesis when reconstructing an equivalent correlation matrix for clustering DHFSs. Furthermore, we prove that the direct transfer algorithm is equivalent to transfer closure algorithm, but its asymptotic time complexity and space complexity are superior to the latter. Another real world example, that is, diamond evaluation and classification, is employed to show the effectiveness of the association coefficient and the algorithm for clustering DHFSs.


## 1. Introduction

Correlation indicates how well two variables move together in a linear fashion. In other words, correlation reflects a linear relationship between two variables. It is an important measure in data analysis, in particular in decision making, medical diagnosis, pattern recognition, and other real world problems [1-7]. Zadeh [8] introduced the concept of fuzzy sets (FSs) whose basic component is only a membership function with the nonmembership function being one minus the membership function. In fuzzy environments, Hung and $\mathrm{Wu}[9]$ used the concept of "expected value" to define the correlation coefficient of fuzzy numbers, which lies in $[-1,1]$. Hong $[10]$ considered the computational aspect of the $T_{\omega}$-based extension principle when the principle is applied to a correlation coefficient of $L-R$ fuzzy numbers and gave the exact solution of a fuzzy correlation coefficient without programming or the aid of computer resources. Atanassov [11, 12] gave a generalized form of fuzzy set, called intuitionistic fuzzy set (IFS), which is characterized by a membership function and a non-membership function. In intuitionistic fuzzy environments, Gerstenkorn and Mańko [13] defined a function measuring the correlation of IFSs and introduced a coefficient of such a correlation. Bustince and Burillo [14] introduced the concepts of correlation and correlation coefficient of interval-valued intuitionistic fuzzy
sets (IVIFSs) [12]. Hung [15] and Mitchell [16] derived the correlation coefficient of intuitionistic fuzzy sets from a statistical viewpoint by interpreting an intuitionistic fuzzy set as an ensemble of ordinary fuzzy sets. Hung and Wu [17] proposed a method to calculate the correlation coefficient of intuitionistic fuzzy sets by means of "centroid." Xu [18] gave a detailed survey on association analysis of intuitionistic fuzzy sets and pointed out that most existing methods deriving association coefficients cannot guarantee that the association coefficient of any two intuitionistic fuzzy sets equals one if and only if these two intuitionistic fuzzy sets are the same. Szmidt and Kacprzyk [5] discussed a concept of correlation for data represented as intuitionistic fuzzy set adopting the concepts from statistics and proposed a formula for measuring the correlation coefficient (lying in $[-1,1]$ ) of intuitionistic fuzzy sets. Robinson and Amirtharaj [19] defined the correlation coefficient of interval vague sets lying in the interval $[0,1]$ and proposed a new method for computing the correlation coefficient of interval vague sets lying in the interval $[-1,1]$ using a-cuts over the vague degrees through statistical confidence intervals which is presented by an example. Instead of using point-based membership as in fuzzy sets, interval-based membership is used in a vague set. In [20], Robinson and Amirtharaj presented a detailed comparison between vague sets and intuitionistic fuzzy sets and defined the correlation coefficient of vague sets through
simple examples. Hesitant fuzzy sets (HFSs) were originally introduced by Torra [21, 22]. In hesitant fuzzy environments, Chen et al. [23] derived some correlation coefficient formulas for HFSs and applied them to two real world examples by using clustering analysis under hesitant fuzzy environments. Xu and Xia [24] defined the correlation measures for hesitant fuzzy information and then discussed their properties in detail.

Recently, Dubois and Prade introduced the definition of dual hesitant fuzzy set. Dual hesitant fuzzy set can reflect human's hesitance more objectively than the other classical extensions of fuzzy set (intuitionistic fuzzy set, type-2 fuzzy set (T-2FS) [25], hesitant fuzzy set, etc.). The motivation to propose the DHFSs is that when people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. They further indicated that DHFSs can better deal with the situations that permit both the membership and the nonmembership of an element to a given set having a few different values, which can arise in a group decision making problem. For example, in the organization, some decision makers discuss the membership degree 0.6 and the nonmembership 0.3 of an alternative $A$ that satisfies a criterion $x$. Some possibly assign $(0.8,0.2)$, while the others assign ( $0.7,0.2$ ). No consistency is reached among these decision makers. Accordingly, the difficulty of establishing a common membership degree and a non-membership degree is not because we have a margin of error (intuitionistic fuzzy set) or some possibility distribution values (type-2 fuzzy set), but because we have a set of possible values (hesitant fuzzy set). For such a case, the satisfactory degrees can be represented by a dual hesitant fuzzy element $\{(0.6,0.8,0.7),(0.3,0.2)\}$, which is obviously different from intuitionistic fuzzy number $(0.8,0.2)$ or $(0.7,0.2)$ and hesitant fuzzy number $\{0.6,0.8,0.7\}$. The aforementioned measures, however, cannot be used to deal with the correlation measures of dual hesitant fuzzy information. Thus, it is very necessary to develop some theories for dual hesitant fuzzy sets. However, little has been done about this issue. In this paper, we mainly discuss the correlation measures of dual hesitant fuzzy information. To do this, the remainder of the paper is organized as follows. Section 2 presents some basic concepts related to DHFSs, HFSs, and IFSs. In Section 3, we propose some correlation measures of dual hesitant fuzzy elements, obtain several important conclusions, and given an example to illustrate the correlation measures. In Section 4, we propose a direct transfer clustering algorithm based on DHFSs and then use a numerical example to illustrate our algorithm. Finally, Section 5 concludes the paper with some remarks and presents future challenges.

## 2. Preliminaries

### 2.1. DHFSs, HFSs, and IFSs

Definition 1 (see [26]). Let $X$ be a fixed set then a dual hesitant fuzzy set (DHFS) $D$ on $X$ is described as;

$$
\begin{equation*}
D=\{\langle x, h(x), g(x)\rangle \mid x \in X\} \tag{1}
\end{equation*}
$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set $D$, respectively, with the conditions

$$
\begin{equation*}
0 \leq \gamma, \quad \eta \leq 1, \quad 0 \leq \gamma^{+}+\eta^{+} \leq 1, \tag{2}
\end{equation*}
$$

where $\gamma \in h(x), \eta \in g(x), \gamma^{+} \in h^{+}(x)=\cup_{\gamma \in h(x)} \max \{\gamma\}$, and $\eta^{+} \in g^{+}(x)=\cup_{\eta \in g(x)} \max \{\eta\}$ for all $x \in X$. For convenience, the pair $d_{E}(x)=\left(h_{E}(x), g_{E}(x)\right)$ is called a dual hesitant fuzzy element (DHFE), denoted by $d=(h, g)$, with the conditions $\gamma \in h(x), \eta \in g(x), \gamma^{+} \in h^{+}(x)=U_{\gamma \in h(x)} \max \{\gamma\}$, and $\eta^{+} \in$ $g^{+}(x)=\cup_{\eta \in g(x)} \max \{\eta\}, 0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^{+}+\eta^{+} \leq 1$.

Definition 2 (see [21, 22]). Let $X$ be a fixed set; a hesitant fuzzy set (HFS) $A$ on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$
\begin{equation*}
A=\left\{\left\langle x, h_{A}(x)\right\rangle \mid x \in X\right\}, \tag{3}
\end{equation*}
$$

where $h_{A}(x)$ is a set of values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $A$. For convenience, we call $h_{A}(x)$ a hesitant fuzzy element (HFE). We use $\left\langle x, h_{A}\right\rangle$ for all $x \in X$ to represent HFSs.

Definition 3 (see [11, 12]). Let $X$ be a fixed set, an intuitionistic fuzzy set (IFS) $A$ on $X$ is an object having the form

$$
\begin{equation*}
A=\left\{\left\langle x, \mu_{A}(x), \nu_{A}(x)\right\rangle \mid x \in X\right\} \tag{4}
\end{equation*}
$$

which is characterized by a membership function $\mu_{A}$ and a non-membership function $\nu_{A}$, where $\mu_{A}: X \rightarrow[0,1]$ and $v_{A}: X \rightarrow[0,1]$, with the condition $0 \leq \mu_{A}(x)+\nu_{A}(x) \leq 1$, for all $x \in X$. We use $\left\langle x, \mu_{A}, v_{A}\right\rangle$ for all $x \in X$ to represent IFSs considered in the rest of the paper without explicitly mentioning it. Furthermore, $\pi_{A}(x)=1-\mu_{A}(x)-\nu_{A}(x)$ is called a hesitancy degree or an intuitionistic index of $x$ in $A$. In the special case $\pi(x)=0$, that is, $\mu_{A}(x)+\nu_{A}(x)=1$, the IFS $A$ reduces to an FS.
2.2. Correlation Coefficients of HFSs and IFSs. Many approaches $[4,13,17,20,21]$ have been introduced to compute the correlation coefficients of IFSs. Let $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be a discrete universe of discourse, for any two $A$ and $B$ on $X$.

The correlation of the IFSs $A$ and $B$ is defined as [13]

$$
\begin{equation*}
C_{\mathrm{IFS}_{1}}(A, B)=\sum_{i=1}^{n}\left(u_{A}\left(x_{i}\right) u_{B}\left(x_{i}\right)+v_{A}\left(x_{i}\right) u_{B}\left(x_{i}\right)\right), \tag{5}
\end{equation*}
$$

Then, the correlation coefficient of the IFSs $A$ and $B$ is defined as

$$
\begin{align*}
\rho_{\mathrm{IFS}_{1}}(A, B)= & \frac{C_{\mathrm{IFS}_{1}}(A, B)}{\left(C_{\mathrm{IFS}_{1}}(A, A) \cdot C_{\mathrm{IFS}_{1}}(B, B)\right)^{1 / 2}} \\
= & \left(\sum_{i=1}^{n}\left(u_{A}\left(x_{i}\right) u_{B}\left(x_{i}\right)+v_{A}\left(x_{i}\right) u_{B}\left(x_{i}\right)\right)\right) \\
& \times\left(\left(\left(\sum_{i=1}^{n}\left(u_{A}^{2}\left(x_{i}\right)+v_{A}^{2}\left(x_{i}\right)\right)\right)\right.\right. \\
& \left.\left.\cdot\left(\sum_{i=1}^{n}\left(u_{B}^{2}\left(x_{i}\right)+v_{B}^{2}\left(x_{i}\right)\right)\right)\right)^{1 / 2}\right)^{-1} . \tag{6}
\end{align*}
$$

In [23], Chen et al. defined the correlation and correlation coefficient for HFSs as follows, respectively:

$$
\begin{align*}
& C_{\mathrm{HFS}_{1}}(A, B)=\sum_{i=1}^{n}\left(\frac{1}{l_{i}} \sum_{j=1}^{l_{i}} h_{A_{\sigma(j)}}\left(x_{i}\right) h_{B_{\sigma(j)}}\left(x_{i}\right)\right) \\
& \rho_{\mathrm{IFS}_{1}}(A, B)= \frac{C_{\mathrm{HFS}_{1}}(A, B)}{\left(C_{\mathrm{HFS}_{1}}(A, A) \cdot C_{\mathrm{HFS}_{1}}(B, B)\right)^{1 / 2}} \\
&=\left(\sum_{i=1}^{n}\left(\frac{1}{l_{i}} \sum_{j=1}^{l_{i}} h_{A_{\sigma(j)}}\left(x_{i}\right) h_{B_{\sigma(j)}}\left(x_{i}\right)\right)\right) \\
& \times\left(\left(\left(\sum_{i=1}^{n}\left(\frac{1}{l_{i}} \sum_{j=1}^{l_{i}} h_{A_{\sigma(j)}}^{2}\left(x_{i}\right)\right)\right)\right.\right. \\
&\left.\left.\cdot\left(\sum_{i=1}^{n}\left(\frac{1}{l_{i}} \sum_{j=1}^{l_{i}} h_{B_{\sigma(j)}^{2}}^{2}\left(x_{i}\right)\right)\right)\right)^{1 / 2}\right)^{-1}, \tag{7}
\end{align*}
$$

where $l_{i}=\max \left\{l\left(h_{A}\left(x_{i}\right)\right), l\left(h_{B}\left(x_{i}\right)\right)\right\}$ for each $x_{i}$ in $X$, and $l\left(h_{A}\left(x_{i}\right)\right)$ and $l\left(h_{B}\left(x_{i}\right)\right)$ represent the number of values in $h_{A}\left(x_{i}\right)$ and $h_{B}\left(x_{i}\right)$, respectively. We will talk about $l_{i}$ in detail in the next section.

## 3. Correlation Measures of DHFEs

In this section, we first introduce the concept of correlation and correlation coefficient for DHFSs and then propose several correlation coefficient formulas and discuss their properties.

We arrange the elements in $d_{E}(x)=\left(h_{E}(x), g_{E}(x)\right)$ in decreasing order and let $\gamma_{E}^{\sigma(i)}(x)$ be the $i$ th largest value in $h_{E}(x)$ and $\eta_{E}^{\sigma(j)}(x)$ the $j$ th largest value in $g_{E}(x)$. Let $l_{h}\left(d_{E}\left(x_{i}\right)\right)$ the number of values in $h_{E}\left(x_{i}\right)$ and $l_{g}\left(d_{E}\left(x_{i}\right)\right)$ be the number of values in $g_{E}\left(x_{i}\right)$. For convenience, $l\left(d\left(x_{i}\right)\right)=$ $\left(l_{h}\left(d\left(x_{i}\right)\right), l_{g}\left(d\left(x_{i}\right)\right)\right)$. In most cases, for two DHFSs $A$ and $B, l\left(d_{A}\left(x_{i}\right)\right) \neq l\left(d_{B}\left(x_{i}\right)\right)$; that is, $l_{h}\left(d_{A}\left(x_{i}\right)\right) \neq l_{h}\left(d_{B}\left(x_{i}\right)\right)$,
$l_{g}\left(d_{A}\left(x_{i}\right)\right) \neq l_{g}\left(d_{B}\left(x_{i}\right)\right)$. To operate correctly, we should extend the shorter one until both of them have the same length when we compare them. In [24, 27], Xu and Xia extended the shorter one by adding different values in hesitant fuzzy environments. Similarly, Torra [21] also applied this ideal to derive some correlation coefficient formulas for HFSs. In fact, we can extend the shorter one by adding any value in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. The same situation can also be found in many existing references [13, 14].

We define several correlation coefficients for DHFEs.
Definition 4. For two DHFSs $A$ and $B$ on $X$, the correlation of $A$ and $B$, denoted as $C_{\mathrm{DHFS}_{1}}(A, B)$, is defined by

$$
\begin{align*}
& C_{\mathrm{DHFS}_{1}}(A, B)=\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right. \\
&\left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right) . \tag{8}
\end{align*}
$$

Definition 5. For two DHFSs $A$ and $B$ on $X$, the correlation coefficient of $A$ and $B$, denoted as $\rho_{\mathrm{DHFS}_{1}}(A, B)$, is defined by:

$$
\begin{align*}
& \rho_{\mathrm{DHFS}_{1}}(A, B) \\
& =\frac{C_{\mathrm{DHFS}_{1}}(A, B)}{\left(C_{\mathrm{DHFS}_{1}}(A, A) \cdot C_{\mathrm{DHFS}_{1}}(B, B)\right)^{1 / 2}} \\
& =\left(\sum _ { i = 1 } ^ { n } \left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right.\right. \\
& \left.\left.\quad+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right)\right) \\
& \\
& \times\left(\left(\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right.\right. \\
& \cdot \sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)\right.  \tag{9}\\
& \left.\left.\left.\quad+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right)^{1 / 2}\right)^{-1} .
\end{align*}
$$

Definition 6. For two DHFSs $A$ and $B$ on $X$, the correlation coefficient of $A$ and $B$, denoted as $\rho_{\mathrm{DHFS}_{2}}(A, B)$, is defined by
$\rho_{\mathrm{DHFS}_{2}}(A, B)$

$$
\begin{align*}
&= \frac{C_{\mathrm{DHFS}_{1}}(A, B)}{\max \left\{C_{\mathrm{DHFS}_{1}}(A, A), C_{\mathrm{DHFS}_{1}}(B, B)\right\}} \\
&=\left(\sum _ { i = 1 } ^ { n } \left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right.\right. \\
&\left.\left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right)\right) \\
& \times\left(\operatorname { m a x } \left\{\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right),\right.\right. \\
& \sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)\right. \\
&\left.\left.\left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right\}\right)^{-1} . \tag{10}
\end{align*}
$$

Theorem 7. For two DHFSs $A$ and $B$, the correlation coefficient of $A$ and $B$, denoted as $\rho_{D H F S_{i}}(A, B)(i=1,2)$, should satisfy the following properties:
(1) $0 \leq \rho_{D H F S_{i}}(A, B) \leq 1$;
(2) $A=B \Rightarrow \rho_{D H F S_{i}}(A, B)=1$;
(3) $\rho_{D H F S_{i}}(A, B)=\rho_{D H F S_{i}}(B, A) ; i=1,2$.

Proof. (1) The inequality $0 \leq \rho_{\mathrm{DHFS}_{1}}(A, B)$ and $0 \leq$ $\rho_{\mathrm{DHFS}_{2}}(A, B)$ is obvious. Below let us prove that $\rho_{\mathrm{DHFS}_{1}}(A, B) \leq 1, \rho_{\mathrm{DHFS}_{2}}(A, B) \leq 1$ :

$$
\begin{aligned}
& C_{\mathrm{DHFS}_{1}}(A, B) \\
& =\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right. \\
& \left.\quad+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right) \\
& =\sum_{i=1}^{n}\left(\sum_{j=1}^{l_{h(i)}} \frac{\gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)}{l_{h(i)}}\right. \\
& \left.\quad+\sum_{k=1}^{l_{g(i)}} \frac{\eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)}{l_{g(i)}}\right)
\end{aligned}
$$

$$
\begin{align*}
& =\left(\sum_{j=1}^{l_{h(1)}} \frac{\gamma_{A \sigma_{(j)}}\left(x_{1}\right)}{\sqrt{l_{h(1)}}} \cdot \frac{\gamma_{B \sigma_{j(j)}}\left(x_{1}\right)}{\sqrt{l_{h(1)}}}\right. \\
& +\sum_{j=1}^{l_{h(2)}} \frac{\gamma_{A \sigma_{(j)}}\left(x_{2}\right)}{\sqrt{l_{h(2)}}} \cdot \frac{\gamma_{B \sigma_{(j)}}\left(x_{2}\right)}{\sqrt{l_{h(2)}}} \\
& \left.+\cdots+\sum_{j=1}^{l_{h(n)}} \frac{\gamma_{A \sigma_{(j)}}\left(x_{n}\right)}{\sqrt{l_{h(n)}}} \cdot \frac{\gamma_{B \sigma_{(j)}}\left(x_{n}\right)}{\sqrt{l_{h(n)}}}\right) \\
& +\left(\sum_{k=1}^{l_{g(1)}} \frac{\eta_{A \sigma_{k \mid}}\left(x_{1}\right)}{\sqrt{l_{g(1)}}} \cdot \frac{\eta_{B \sigma_{k( }}\left(x_{1}\right)}{\sqrt{l_{g(1)}}}\right. \\
& +\sum_{k=1}^{l_{g(2)}} \frac{\eta_{A \sigma_{(k)}}\left(x_{2}\right)}{\sqrt{g_{g(2)}}} \cdot \frac{\eta_{B \sigma_{k(k}}\left(x_{2}\right)}{\sqrt{l_{g(2)}}} \\
& \left.+\cdots+\sum_{k=1}^{l_{g(n)}} \frac{\eta_{A \sigma_{k \mid}}\left(x_{n}\right)}{\sqrt{l_{g(n)}}} \cdot \frac{\eta_{B \sigma_{(k)}}\left(x_{n}\right)}{\sqrt{l_{g(n)}}}\right) . \tag{11}
\end{align*}
$$

Using the Cauchy-Schwarz inequality

$$
\begin{align*}
& \left(x_{1} y_{1}+x_{2} y_{2}+\cdots+x_{n} y_{n}\right)^{2}  \tag{12}\\
& \quad \leq\left(x_{1}^{2}+x_{2}^{2}+\cdots+x_{n}^{2}\right) \cdot\left(y_{1}^{2}+y_{2}^{2}+\cdots+y_{n}^{2}\right)
\end{align*}
$$

where $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in R^{n},\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in R^{n}$, we obtain

$$
\left.\begin{array}{l}
C_{\mathrm{DHFS}_{1}}(A, B)^{2} \\
\leq\left(\sum_{j=1}^{l_{h(1)}} \frac{\gamma_{A \sigma_{(j)}}^{2}\left(x_{1}\right)}{l_{h(1)}}+\sum_{j=1}^{l_{h(2)}} \frac{\gamma_{A \sigma_{(j)}}^{2}\left(x_{2}\right)}{l_{h(2)}}\right. \\
+\cdots+\sum_{j=1}^{l_{h(n)}} \frac{\gamma_{A \sigma_{(j)}}^{2}\left(x_{n}\right)}{l_{h(n)}}+\sum_{k=1}^{l_{g(1)}} \frac{\eta_{A \sigma_{(k)}}^{2}\left(x_{1}\right)}{l_{g(1)}} \\
\quad+\sum_{k=1}^{l_{g(2)}} \frac{\eta_{A \sigma_{(k)}}^{2}\left(x_{2}\right)}{l_{g(2)}}+\cdots+\sum_{k=1}^{\sum_{j=1}} \frac{l_{g(n)}}{l_{A \sigma_{(k)}}^{2}\left(x_{n}\right)} \\
l_{g(n)}
\end{array}\right)
$$

$$
\begin{align*}
& =\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right) \\
& \cdot \sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right) \\
& =C_{\mathrm{DHFS}_{1}}(A, A) \cdot C_{\mathrm{DHFS}_{1}}(B, B) . \tag{13}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
C_{\mathrm{DHFS}_{1}}(A, B) \leq\left(C_{\mathrm{DHFS}_{1}}(A, A)\right)^{1 / 2} \cdot\left(C_{\mathrm{DHFS}_{1}}(B, B)\right)^{1 / 2} . \tag{14}
\end{equation*}
$$

So, $0 \leq \rho_{\mathrm{DHFS}_{1}}(A, B) \leq 1$.
In fact, we have

$$
\begin{align*}
& \left(x_{1}^{2}+x_{2}^{2}+\cdots+x_{n}^{2}\right) \cdot\left(y_{1}^{2}+y_{2}^{2}+\cdots+y_{n}^{2}\right) \\
& \leq\left(\max \left(\left(x_{1}^{2}+x_{2}^{2}+\cdots+x_{n}^{2}\right),\left(y_{1}^{2}+y_{2}^{2}+\cdots+y_{n}^{2}\right)\right)\right)^{2} \\
& \sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right) \\
& \quad \cdot \sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right) \\
& \leq\left(\operatorname { m a x } \left\{\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right.\right. \\
& \left.\left.\sum_{i=1}^{n}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right\}\right)^{2} \tag{15}
\end{align*}
$$

Then

$$
\begin{align*}
& \left(C_{\mathrm{DHFS}_{1}}(A, A) \cdot C_{\mathrm{DHFS}_{1}}(B, B)\right)^{1 / 2}  \tag{16}\\
& \quad \leq \max \left\{C_{\mathrm{DHFS}_{1}}(A, A), C_{\mathrm{DHFS}_{1}}(B, B)\right\} .
\end{align*}
$$

We also obtain $0 \leq \rho_{\mathrm{DHFS}_{2}}(A, B) \leq 1$.
(2) and (3) are straightforward.

Moreover, from the proof of Theorem 7, we have Theorem 8 easily.

Theorem 8. For two DHFSs $A$ and $B$ on $X$, then $\rho_{{D H F S_{1}}}(A, B) \geq \rho_{D H F S_{2}}(A, B)$.

However, from Theorem 7, we notice that all the above correlation coefficients cannot guarantee that the correlation coefficient of any two DHFSs equals one if and only if these two DHFSs are the same. Thus, how to derive the correlation coefficients of the DHFSs satisfying this desirable property is an interesting research topic. To solve this issue, in what follows, we develop a new method to calculate the correlation coefficient of the DHFSs $A$ and $B$.

Definition 9. For two DHFSs $A$ and $B$ on $X$, the correlation coefficient of $A$ and $B$, denoted as $\rho_{\mathrm{DHFS}_{3}}(A, B)$, is defined by
$\rho_{\mathrm{DHFS}_{3}}(A, B)$

$$
\begin{equation*}
=\left(\frac{1}{2 n} \sum_{i=1}^{n}\left(\frac{\Delta \gamma_{\min }^{\lambda}+\Delta \gamma_{\max }^{\lambda}}{\Delta \gamma_{i}^{\lambda}+\Delta \gamma_{\max }^{\lambda}}+\frac{\Delta \eta_{\min }^{\lambda}+\Delta \eta_{\max }^{\lambda}}{\Delta \eta_{i}^{\lambda}+\Delta \eta_{\max }^{\lambda}}\right)\right)^{1 / \lambda} \tag{17}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta \gamma_{i}^{\lambda}=\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}}\left|\gamma_{A \sigma_{(j)}}\left(x_{i}\right)-\gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right|^{\lambda}, \\
& \Delta \gamma_{\min }^{\lambda}=\min _{i}\left\{\Delta \gamma_{i}^{\lambda}\right\}, \quad \Delta \gamma_{\max }^{\lambda}=\max _{i}\left\{\Delta \gamma_{i}^{\lambda}\right\}, \\
& \Delta \eta_{i}^{\lambda}=\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}}\left|\eta_{A \sigma_{(k)}}\left(x_{i}\right)-\eta_{B \sigma_{(k)}}\left(x_{i}\right)\right|^{\lambda},  \tag{18}\\
& \Delta \gamma_{\text {min }}^{\lambda}=\min _{i}\left\{\Delta \eta_{i}^{\lambda}\right\}, \quad \Delta \gamma_{\max }^{\lambda}=\max _{i}\left\{\Delta \eta_{i}^{\lambda}\right\}, \\
& \lambda>0 .
\end{align*}
$$

Equation (17) is motivated by the generalized idea provided by Xu [18]. Obviously, the greater the value of $\rho_{\mathrm{DHFS}_{3}}(A, B)$, the closer $A$ to $B$. By Definition 9, we have Theorem 10.

Theorem 10. The correlation coefficient $\rho_{D_{H F S_{3}}}(A, B)$ satisfies the following properties:
(1) $0 \leq \rho_{D H F S_{3}}(A, B) \leq 1$;
(2) $A=B \Leftrightarrow \rho_{D H F S_{3}}(A, B)=1$;
(3) $\rho_{D H F S_{3}}(A, B)=\rho_{D H F S_{3}}(B, A)$.

Proof. (1) The inequality $0 \leq \rho_{\mathrm{DHFS}_{3}}(A, B)$ is obvious. Below let us prove that $\rho_{\mathrm{DHFS}_{3}}(A, B) \leq 1$ :

$$
\begin{aligned}
& \frac{\Delta \gamma_{\min }^{\lambda}+\Delta \gamma_{\max }^{\lambda}}{\Delta \gamma_{i}^{\lambda}+\Delta \gamma_{\max }^{\lambda}}+\frac{\Delta \eta_{\min }^{\lambda}+\Delta \eta_{\max }^{\lambda}}{\Delta \eta_{i}^{\lambda}+\Delta \eta_{\max }^{\lambda}} \text { for } i=1,2, \ldots, n \\
& \quad=\frac{\left(\Delta \gamma_{\min }^{\lambda} / \Delta \gamma_{\max }^{\lambda}\right)+1}{\left(\Delta \gamma_{i}^{\lambda} / \Delta \gamma_{\max }^{\lambda}\right)+1}+\frac{\left(\Delta \eta_{\min }^{\lambda} / \Delta \eta_{\max }^{\lambda}\right)+1}{\left(\Delta \eta_{i}^{\lambda} / \Delta \eta_{\max }^{\lambda}\right)+1} \\
& \quad \leq 2 .
\end{aligned}
$$

We obtain
$\rho_{\mathrm{DHFS}_{3}}(A, B)$

$$
\begin{align*}
& =\frac{1}{2 n} \sum_{i=1}^{n}\left(\frac{\Delta \gamma_{\min }^{\lambda}+\Delta \gamma_{\max }^{\lambda}}{\Delta \gamma_{i}^{\lambda}+\Delta \gamma_{\max }^{\lambda}}+\frac{\Delta \eta_{\min }^{\lambda}+\Delta \eta_{\max }^{\lambda}}{\Delta \eta_{i}^{\lambda}+\Delta \eta_{\max }^{\lambda}}\right)  \tag{20}\\
& \leq \frac{1}{2 n} \cdot 2 n \\
& =1
\end{align*}
$$

(2) and (3) are obvious.

Usually, in practical applications, the weight of each element $x_{i} \in X$ should be taken into account, and, so, we present the following weighted correlation coefficient. Assume that the weight of the element $x_{i} \in X$ is $w_{i}(i=$ $1,2, \ldots, n)$ with $w_{i} \in[0,1]$ and $\sum_{i=1}^{n} w_{i}=1$; then we extend the correlation coefficient formulas given:

$$
\begin{align*}
& \rho_{\text {DHFS }_{-w 1}}(A, B) \\
& =\left(\sum _ { i = 1 } ^ { n } w _ { i } \left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right.\right. \\
& \left.\left.\quad+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right)\right) \\
& \times\left(\left(\sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right.\right. \\
& \cdot \sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)\right. \\
& \left.\left.\left.\quad+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right)^{1 / 2}\right)^{-1}, \tag{21}
\end{align*}
$$

$$
\begin{align*}
& \rho_{\text {DHFS }_{-w 2}}(A, B) \\
& =\left(\sum _ { i = 1 } ^ { n } w _ { i } \left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{A \sigma_{(j)}}\left(x_{i}\right) \gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right.\right. \\
& \left.\left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}\left(x_{i}\right) \eta_{B \sigma_{(k)}}\left(x_{i}\right)\right)\right) \\
& \times\left(\operatorname { m a x } \left\{w _ { i } \sum _ { i = 1 } ^ { n } \left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{\text {hi }}} \gamma_{A \sigma_{(j)}}^{2}\left(x_{i}\right)\right.\right.\right.  \tag{22}\\
& \left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{A \sigma_{(k)}}^{2}\left(x_{i}\right)\right), \\
& \sum_{i=1}^{n} w_{i}\left(\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}} \gamma_{B \sigma_{(j)}}^{2}\left(x_{i}\right)\right. \\
& \left.\left.\left.+\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}} \eta_{B \sigma_{(k)}}^{2}\left(x_{i}\right)\right)\right\}\right)^{-1}, \\
& \rho_{\text {DHFS }_{\text {w }}}(A, B) \\
& =\left(\frac{1}{2} \sum_{i=1}^{n} w_{i}\left(\frac{\Delta \gamma_{\text {min }}^{\lambda}+\Delta \gamma_{\text {max }}^{\lambda}}{\Delta \gamma_{i}^{\lambda}+\Delta \gamma_{\text {max }}^{\lambda}}+\frac{\Delta \eta_{\text {min }}^{\lambda}+\Delta \eta_{\text {max }}^{\lambda}}{\Delta \eta_{i}^{\lambda}+\Delta \eta_{\text {max }}^{\lambda}}\right)\right)^{1 / \lambda}, \tag{23}
\end{align*}
$$

where

$$
\begin{align*}
& \Delta \gamma_{i}^{\lambda}=\frac{1}{l_{h(i)}} \sum_{j=1}^{l_{h(i)}}\left|\gamma_{A \sigma_{(j)}}\left(x_{i}\right)-\gamma_{B \sigma_{(j)}}\left(x_{i}\right)\right|^{\lambda}, \\
& \Delta \gamma_{\text {min }}^{\lambda}=\min _{i}\left\{\Delta \gamma_{i}^{\lambda}\right\}, \quad \Delta \gamma_{\text {max }}^{\lambda}=\max _{i}\left\{\Delta \gamma_{i}^{\lambda}\right\}, \\
& \Delta \eta_{i}^{\lambda}=\frac{1}{l_{g(i)}} \sum_{k=1}^{l_{g(i)}}\left|\eta_{A \sigma_{(k)}}\left(x_{i}\right)-\eta_{B \sigma_{(k)}}\left(x_{i}\right)\right|^{\lambda},  \tag{24}\\
& \Delta \gamma_{\text {min }}^{\lambda}=\min _{i}\left\{\Delta \eta_{i}^{\lambda}\right\}, \quad \Delta \gamma_{\text {max }}^{\lambda}=\max _{i}\left\{\Delta \eta_{i}^{\lambda}\right\}, \\
& \lambda>0 .
\end{align*}
$$

Note that all these formulas satisfy the properties in Theorem 7.

In what follows, we use a medical diagnosis problem in $[28,29]$ to illustrate the developed correlation coefficient formulas. Actually, this is also a pattern recognition problem.

Example 11. To make a proper diagnosis $Q=\left\{Q_{1}\right.$ (viral fever), $Q_{2}$ (malaria), $Q_{3}$ (typhoid), $Q_{4}$ (stomach problem), and $Q_{5}$ (chest problem)\} for a patient with the given values of the symptoms, $S=\left\{S_{1}\right.$ (temperature), $S_{2}$ (headache), $S_{3}$ (cough), $S_{4}$ (stomach pain), and $S_{5}$ (chest pain)\}, Xu [18] considered all possible diagnoses and symptoms as HFEs. Utilizing DHFSs can take much more information into account; the more values we obtain from patients, the greater epistemic certainty we have. So, in this paper, we use DHFEs to deal with such cases; each symptom is described by a DHFE, which is described by two sets $\left(\gamma_{i j}\right)$ and $\left(\eta_{i j}\right)$. $\left(\gamma_{i j}\right)$ indicates the degree that symptoms characteristic $S_{i}$ satisfies the considered diagnoses $G_{j}$ and $\left(\eta_{i j}\right)$ indicates the degree that the symptoms characteristic $S_{i}$ does not satisfy the considered diagnoses $G_{j}$. The data are given in Table 1 . The set of patients is $P=\{\mathrm{Al}$, Bob, Joe, Ted $\}$. The symptoms which can be also described by DHFEs are given in Table 2. We need to seek a diagnosis for each patient.

We utilize the correlation coefficient $\rho_{\text {DHFS } 1}$ to derive a diagnosis for each patient. All the results for the considered patients are listed in Table 3. From the arguments in Table 3, we can find that Ted suffers from viral fever, Al and Joe from malaria, and Bob from stomach problem.

If we utilize the correlation coefficient formulas $\rho_{\mathrm{DHFS}}^{2}$ and $\rho_{\text {DHFS3 }}$ to derive a diagnosis, then the results are listed in Tables 4 and 5, respectively.

From Tables 3-5 we know that the results obtained by different correlation coefficient formulas are different. That is because these correlation coefficient formulas are based on different linear relationships.

## 4. Clustering Method Based on Direct Transfer Algorithm for HFSs

Based on clustering algorithms for IFSs [30, 31], and HFSs [23], and the correlation coefficient formulas developed previously for DHFSs, in what follows, we propose a direct

Table 1: Symptoms characteristic of the considered diagnoses.

|  | Temperature | Headache | Cough | Stomach pain | Chest pain |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Viral fever | $\{(0.6,0.4,0.3)$, | $\{(0.7,0.5,0.3,0.2)$, | $\{(0.5,0.3)$, | $\{(0.5,0.4,0.3,0.2,0.1)$, | $\{(0.5,0.4,0.2,0.1)$, |
|  | $(0.2,0.0)\}$ | $(0.3,0.1)\}$ | $(0.5,0.4,0.2)\}$ | $(0.5,0.3)\}$ | $(0.5,0.4,0.3)\}$ |
| Malaria | $\{(0.9,0.8,0.7)$, | $\{(0.5,0.3,0.2,0.1)$, | $\{(0.2,0.1)$, | $\{(0.6,0.5,0.3,0.2,0.1)$, | $\{(0.4,0.3,0.2,0.1)$, |
|  | $(0.1,0.0)\}$ | $(0.4,0.3)\}$ | $(0.7,0.6,0.5)\}$ | $(0.3,0.2)\}$ | $(0.6,0.5,0.4)\}$ |
| Typhoid | $\{(0.6,0.3,0.1)$, | $\{(0.9,0.8,0.7,0.6)$, | $\{(0.5,0.3)$, | $\{(0.5,0.4,0.3,0.2,0.1)$, | $\{(0.6,0.4,0.3,0.2)$, |
|  | $(0.3,0.2)\}$ | $(0.1,0.0)\}$ | $(0.5,0.4,0.3)\}$ | $(0.5,0.4)\}$ | $(0.4,0.3,0.2)\}$ |
| Stomach problem | $\{(0.5,0.4,0.2)$, | $\{(0.4,0.3,0.2,0.1)$, | $\{(0.4,0.3)$, | $\{(0.9,0.8,0.7,0.6,0.5)$, | $\{(0.5,0.4,0.2,0.1)$, |
|  | $(0.5,0.3)\}$ | $(0.4,0.3)\}$ | $(0.6,0.5,0.4)\}$ | $(0.1,0.0)\}$ | $(0.5,0.4,0.3)\}$ |
| Chest problem | $\{(0.3,0.2,0.1)$, | $\{(0.5,0.3,0.2,0.1)$, | $\{(0.3,0.2)$, | $\{(0.7,0.6,0.5,0.3,0.2)$, | $\{(0.8,0.7,0.6,0.5)$, |
|  | $(0.7,0.6)\}$ | $(0.5,0.3)\}$ | $(0.6,0.4,0.3)\}$ | $(0.2,0.1)\}$ | $(0.2,0.1,0.0)\}$ |

Table 2: Symptoms characteristic of the considered patients.

|  | Temperature | Headache | Cough | Stomach pain | Chest pain |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Al | $\{(0.9,0.7,0,5),(0.1,0.0)\}$ | $\begin{gathered} \{(0.4,0.3,0.2,0.1), \\ (0.5,0.4)\} \end{gathered}$ | $\{(0.4,0.3),(0.5,0.4,0.2)\}$ | $\begin{gathered} \{(0.6,0.5,0.4,0.2, \\ 0.1),(0.3 .0 .2)\} \end{gathered}$ | $\{(0.4,0.3,0.2,0.1),(0.5,0.4,0.3)\}$ |
| Bob | $\{(0.5,0.4,0.2),(0.5,0.3)\}$ | $\begin{gathered} \{(0.5,0.4,0.3,0.1) \\ (0.4,0.3)\} \end{gathered}$ | $\{(0.2,0.1),(0.7,0.6,0.5)\}$ | $\begin{gathered} \{(0.9,0.8,0.6,0.5 \\ 0.4),(0.1,0.0)\} \end{gathered}$ | $\{(0.5,0.4,0.3,0.2),(0.5,0.4,0.3)\}$ |
| Joe | $\{(0.9,0.7,0.6),(0.1,0.0)\}$ | $\begin{gathered} \{(0.7,0.4,0.3,0.1), \\ (0.2,0.1)\} \end{gathered}$ | $\{(0.3,0.2),(0.5,0.4,0.3)\}$ | $\begin{gathered} \{(0.6,0.4,0.3,0.2 \\ 0.1),(0.4,0.3)\} \end{gathered}$ | $\{(0.6,0.3,0.2,0.1),(0.4,0.3,0.2)\}$ |
| Ted | $\{(0.8,0.7,0.5),(0.2,0.1)\}$ | $\begin{gathered} \{(0.6,0.5,0.4,0.2), \\ (0.4,0.3)\} \\ \hline \end{gathered}$ | $\{(0.5,0.3),(0.5,0.4,0.3)\}$ | $\begin{gathered} \{(0.6,0.4,0.3,0.2 \\ 0.1),(0.4,0.3)\} \\ \hline \end{gathered}$ | $\{(0.5,0.4,0.2,0.1),(0.5,0.4,0.3)\}$ |

Table 3: Values of $\rho_{\text {DhFS }_{1}}$ for each patient to the considered set of possible diagnoses.

|  | Viral fever | Malaria | Typhoid | Stomach <br> problem | Chest <br> problem |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Al | 0.9257 | 0.9620 | 0.7957 | 0.8680 | 0.7110 |
| Bob | 0.8380 | 0.8791 | 0.8041 | 0.9922 | 0.9035 |
| Joe | 0.9427 | 0.9521 | 0.8757 | 0.9329 | 0.7026 |
| Ted | 0.9718 | 0.9472 | 0.8890 | 0.8790 | 0.7644 |

transfer algorithm to clustering analysis with respect to the problem of complex operation of matrix synthesis when reconstructing analogical relation to equivalence relation clustering under hesitant fuzzy environments. Before doing this, some concepts are introduced firstly.

Definition 12. Let $A_{j}(j=1,2, \ldots, m)$ be $m$ DHFs; then $C=$ $\left(\rho_{i j}\right)_{m \times m}$ is called an association matrix, where $\rho_{i j}=\rho\left(A_{i}, A_{j}\right)$ is the association coefficient of $A_{i}$ and $A_{j}$, which has the following properties:
(1) $0 \leq \rho_{i j} \leq 1$, for all $i, j=1,2, \ldots, m$;
(2) $\rho_{i j}=1$ if and only if $A_{i}=A_{j}$;
(3) $\rho_{i j}=\rho_{j i}$, for all $i, j=1,2, \ldots, m$.

Definition 13 (see $[23,30]$ ). Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an association matrix; if $C^{2}=C \circ C=\left(\bar{\rho}_{i j}\right)_{m \times m}$, then $C^{2}$ is called a composition matrix of $C$, where $\bar{\rho}_{i j}=\max \left\{\min \left\{\rho_{i k}, \rho_{k j}\right\}\right\}$, for all $i, j=1,2, \ldots, m$.

Based on Definition 13, we have the following theorem.

Theorem 14 (see $[23,30]$ ). Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an association matrix; then the composition matrix $C^{2}$ is also an association matrix.

Theorem 15 (see $[23,30])$. Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an association matrix; then, for any nonnegative integer $k$, the composition matrix $C^{2^{k+1}}$ derived from $C^{2^{k+1}}=C^{2^{k}} \circ C^{2^{k}}$ is also an association matrix.

Definition 16 (see [23, 30]). Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an association matrix, if $C^{2} \subseteq C$, that is,

$$
\begin{equation*}
\max _{k}\left\{\min \left\{\rho_{i k}, \rho_{k j}\right\}\right\} \leq \rho_{i j}, \quad \forall i, j=1,2, \ldots, m \tag{25}
\end{equation*}
$$

then $C$ is called an equivalent association matrix.
By the transitivity principle of equivalent matrix, we can easily prove the following theorem.

Theorem 17 (see [23, 30, 32]). Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an association matrix; then, after the finite times of compositions: $C \rightarrow C^{2} \rightarrow C^{4} \rightarrow \cdots \rightarrow C^{2 k} \rightarrow \cdots$, there must exist a positive integer $k$ such that $C^{2^{k}}=C^{2^{k+1}}$, and $C^{2^{k}}$ is also an equivalent association matrix.

Definition 18 (see [23, 30, 31]). Let $C=\left(\rho_{i j}\right)_{m \times m}$ be an equivalent correlation matrix. Then we call $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ the $\lambda$-cutting matrix of $C$, where

$$
{ }_{\lambda} \rho_{i j}=\left\{\begin{array}{ll}
0 & \text { if } \rho_{i j}<\lambda,  \tag{26}\\
1 & \text { if } \rho_{i j} \geq \lambda,
\end{array} \quad i, j=1,2, \ldots, m\right.
$$

and $\lambda$ is the confidence level with $\lambda \in[0,1]$.

TABLE 4: Values of $\rho_{\mathrm{DHFS}_{2}}$ for each patient to the considered set of possible diagnoses.

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest problem |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Al | 0.8718 | 0.8888 | 0.7571 | 0.8286 | 0.7591 |
| Bob | 0.7586 | 0.8451 | 0.7960 | 0.9852 | 0.8889 |
| Joe | 0.9075 | 0.8607 | 0.8152 | 0.8712 | 0.6500 |
| Ted | 0.8764 | 0.9140 | 0.8835 | 0.8678 | 0.7550 |

TABLE 5: Values of $\rho_{\mathrm{DHFS}_{3}}$ for each patient to the considered set of possible diagnoses.

|  | Viral fever | Malaria | Typhoid | Stomach problem | Chest problem |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Al | 0.8085 | 0.7739 | 0.7900 | 0.7515 | 0.8026 |
| Bob | 0.7480 | 0.7714 | 0.6824 | 0.7547 | 0.8006 |
| Joe | 0.7925 | 0.7887 | 0.7548 | 0.6878 | 0.8100 |
| Ted | 0.8063 | 0.7244 | 0.7516 | 0.7386 | 0.8230 |

Next, a traditional transfer closure algorithm is given as follows.

Step 1. Let $\left\{A_{1}, A_{2}, \ldots, A_{m}\right\}$ be a set of DHFSs in $X=$ $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. We can calculate the correlation coefficients of the DHFSs and then construct a correlation matrix $C=$ $\left(\rho_{i j}\right)_{m \times m}$, where $\rho_{i j}=\rho\left(A_{i}, A_{j}\right)$.

Step 2. Check whether $C=\left(\rho_{i j}\right)_{m \times m}$ is an equivalent correlation matrix; that is, check whether it satisfies $C^{2} \subseteq$ $C$, where

$$
\begin{array}{r}
C^{2}=C \circ C=\left(\bar{\rho}_{i j}\right)_{m \times m}, \quad \bar{\rho}_{i j}=\max _{k}\left\{\min \left\{\rho_{i k}, \rho_{k j}\right\}\right\},  \tag{27}\\
i, j=1,2, \ldots, m .
\end{array}
$$

If it does not hold, we construct the equivalent correlation matrix $C^{2^{k}}: C \rightarrow C^{2} \rightarrow C^{4} \rightarrow \cdots \rightarrow C^{2 k} \rightarrow \cdots$, until $C^{2^{k}}=C^{2^{k+1}}$.

Step 3. For a confidence level $\lambda$, we construct a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ through Definition 18 in order to classify the DHFSs $A_{j}(j=1,2, \ldots, m)$. If all elements of the $i$ th line (column) are the same as the corresponding elements of the $j$ th line (column) in $C_{\lambda}$, then the DHFSs $A_{i}$ and $A_{j}$ are of the same type. By means of this principle, we can classify all these $m A_{j}(j=1,2, \ldots, m)$.

By analyzing the aforementioned transfer closure algorithm, this algorithm has one drawback such as complex operation of matrix synthesis when reconstructing the equivalent correlation matrix. In this paper, we have the following theorem of the correlation coefficients in dual hesitant fuzzy environment.

Theorem 19. For all $x, y \in A$, for the confidence level $\lambda$, if $\exists x_{1}, x_{2}, x_{3}, \ldots, x_{p}$, when $\rho\left(x, x_{1}\right) \geq \lambda, \rho\left(x_{1}, x_{2}\right) \geq \lambda$, $\rho\left(x_{2}, x_{3}\right) \geq \lambda, \ldots, \rho\left(x_{p}, y\right) \geq \lambda$, then $x, x_{1}, x_{2}, \ldots, x_{p}$ and $y$ are of the same type.

Proof. we are motivated by the generalized idea based on the transitivity principle of ordinary equivalent relation $R$ : for all $x, y \in A$ (here, $A$ is an ordinary set, not a fuzzy set), $\exists x_{1}, x_{2}, x_{3} \ldots, x_{p}$, when $\left(x, x_{1}\right) \in R,\left(x_{1}, x_{2}\right) \in$ $R, \ldots,\left(x_{k}, x_{k+1}\right) \in R, \ldots,\left(x_{p}, y\right) \in R$, we can have $(x, y) \in R$.

And from Definition 18, we can see that the $\lambda$-cutting matrix of $C$ is an ordinary correlation matrix, which completes the proof of Theorem 19.

From the above theoretical analysis, we propose a direct transfer algorithm for clustering DHFSs as follows.

Step 1. Let $A=\left\{A_{1}, A_{2}, \ldots, A_{m}\right\}$ be a set of DHFSs in $X=$ $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. We can calculate the correlation coefficients of the DHFSs and then construct a correlation matrix $C=$ $\left(\rho_{i j}\right)_{m \times m}$, where $\rho_{i j}=\rho\left(A_{i}, A_{j}\right)$.

Step 2. By setting the threshold to the confidence level $\lambda$, we can construct a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$. If $\rho_{i j}=1$, this means that the DHFSs $A_{i}$ and $A_{j}$ are of the same type. By means of this principle, we can classify all these $m A_{j}(j=$ $1,2, \ldots, m)$.

We can see that the transfer closure algorithm must construct the equivalent correlation matrix $C \rightarrow C^{2} \rightarrow$ $C^{4} \rightarrow \cdots \rightarrow C^{2 k} \rightarrow \cdots$, until $C^{2^{k}}=C^{2^{k+1}}$ and then construct a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ through Definition 18 in order to classify the DHFSs $A_{j}(j=$ $1,2, \ldots, m)$. Simply, the transfer algorithm only constructs a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ by setting the threshold to the confidence level $\lambda$ and then classifies the DHFSs $A_{j}(j=$ $1,2, \ldots, m)$ directly. In what follows, we will talk about the relationship between the transfer closure algorithm and the direct transfer algorithm.

Theorem 20. The clustering results are the same by the transfer closure algorithm and the direct transfer algorithm, at the same confidence level.

Proof. (1) For a confidence level $\lambda$, for all $A_{i}, A_{j} \in A$, $\exists x_{1}, x_{2}, x_{3} \ldots, x_{p}$, if $\rho\left(A_{i}, x_{1}\right) \geq \lambda, \rho\left(x_{1}, x_{2}\right) \geq \lambda, \rho\left(x_{2}, x_{3}\right) \geq$ $\lambda, \ldots, \rho\left(x_{p}, A_{j}\right) \geq \lambda$, then $A_{i}$ and $A_{j}$ are of the same type by the direct transfer algorithm.

Assume that we construct the equivalent correlation matrix $C^{2^{k}}$ when we employ the transfer closure algorithm. We must prove that $\rho_{\mathrm{C}^{2}}\left(A_{i}, x_{j}\right) \geq \lambda$. Consider

$$
\begin{align*}
& \rho_{\mathrm{C}^{2}}\left(A_{i}, A_{j}\right) \\
& \quad=\max _{A_{q} \in A}\left\{\min \left\{\rho_{\mathrm{C}}\left(A_{i}, A_{q}\right), \rho_{\mathrm{C}}\left(A_{q}, A_{j}\right)\right\}\right\}, \\
& \quad i, j, q=1,2, \ldots, m \\
& =\max _{A_{q} \in A_{,}, A_{i} \neq A_{j}}\left\{\max \left\{\min \left\{\rho_{\mathrm{C}}\left(A_{i}, A_{q}\right), \rho_{\mathrm{C}}\left(A_{q}, A_{j}\right)\right\}\right\},\right. \\
& \left.\quad \rho_{\mathrm{C}}\left(A_{i}, A_{j}\right)\right\} \\
& \geq \rho_{\mathrm{C}}\left(A_{i}, A_{j}\right) . \tag{28}
\end{align*}
$$

So $C^{2} \supseteq C$, and, for the same reason, we have $C^{2^{k}} \supseteq$ $C^{2^{k-1}} \supseteq C^{2^{k-2}}, \ldots, \supseteq C^{2} \supseteq C$. Consider

$$
\begin{aligned}
& \rho_{\mathrm{C}^{k^{k}}}\left(A_{i}, A_{j}\right) \\
& =\max _{A_{q} \in A}\left\{\min \left\{\rho_{\mathrm{C}^{2^{k}}}\left(A_{i}, A_{q}\right), \rho_{\mathrm{C}^{k}}\left(A_{q}, A_{j}\right)\right\}\right\}, \\
& \quad i, j, q=1,2, \ldots, m
\end{aligned}
$$

$$
=\max _{A_{q} \in A, A_{q} \neq A_{x_{p}}}\left\{\operatorname { m a x } \left\{\min \left\{\rho_{\mathrm{C}^{2^{k}}}\left(A_{i}, A_{p}\right), \rho_{\mathrm{C}^{k}}\left(A_{p}, A_{j}\right)\right\},\right.\right.
$$

$$
\min \left\{\rho_{\mathrm{C}^{2}}\left(A_{i}, A_{x_{p}}\right)\right.
$$

$$
\left.\left.\left.\rho_{\mathrm{C}^{2^{k}}}\left(A_{x_{p}}, A_{j}\right)\right\}\right\}\right\}
$$

$$
\geq \min \left\{\rho_{\mathrm{C}^{k}}\left(A_{i}, A_{x_{p}}\right), \rho_{\mathrm{C}^{k^{k}}}\left(A_{x_{p}}, A_{j}\right)\right\}
$$

$$
\geq \min \left\{\rho_{C^{2}}\left(A_{i}, A_{x_{p-1}}\right)\right.
$$

$$
\left.\rho_{\mathrm{C}^{k}}\left(A_{x_{p-1}}, A_{x_{p}}\right), \rho_{\mathrm{C}^{2^{k}}}\left(A_{x_{p}}, A_{j}\right)\right\}
$$

$$
\geq \min \left\{\rho_{\mathrm{C}^{2}}\left(A_{i}, A_{x_{1}}\right), \rho_{\mathrm{C}^{2}}\left(A_{x_{1}}, A_{x_{2}}\right)\right.
$$

$$
, \ldots, \rho_{C^{2}}\left(A_{i}, A_{x_{p-1}}\right), \rho_{C^{2}}\left(A_{x_{p-1}}, A_{x_{p}}\right)
$$

$$
\left.\rho_{\mathrm{C}^{k}}\left(A_{x_{p}}, A_{j}\right)\right\}
$$

$$
\geq \min \left\{\rho_{C}\left(A_{i}, A_{x_{1}}\right), \rho_{C}\left(A_{x_{1}}, A_{x_{2}}\right)\right.
$$

$$
, \ldots, \rho_{C}\left(A_{i}, A_{x_{p-1}}\right)
$$

$$
\left.\rho_{C}\left(A_{x_{p-1}}, A_{x_{p}}\right), \rho_{C}\left(A_{x_{p}}, A_{j}\right)\right\}
$$

$\geq \lambda$.

For a confidence level $\lambda$, when we get that $A_{i}$ and $A_{j}$ are of the same type using the direct transfer algorithm, we can also have the same clustering results by the transfer closure algorithm.
(2) For a confidence level $\lambda$, for all $A_{i}, A_{j} \in A, \exists$ the equivalent correlation matrix $C^{2^{k}}, \rho_{C^{2^{k}}}\left(A_{i}, x_{j}\right) \geq \lambda$, then, $A_{i}$ and $A_{j}$ are of the same type by the transfer closure algorithm. Let

$$
\begin{align*}
& \rho_{\mathrm{C}^{2^{k}}}\left(A_{i}, A_{j}\right) \\
& \quad=\max _{A_{q} \in A}\left\{\min \left\{\rho_{\mathrm{C}^{k-1}}\left(A_{i}, A_{q}\right), \rho_{\mathrm{C}^{k-1}}\left(A_{q}, A_{j}\right)\right\}\right\} . \tag{30}
\end{align*}
$$

Then $\exists x_{1}, \rho_{C^{2^{k}}}\left(A_{i}, A_{j}\right)=\min \left\{\rho_{\mathrm{C}^{2^{k-1}}}\left(A_{i}, A_{x_{1}}\right), \rho_{\mathrm{C}^{k-1}}\left(A_{x_{1}}\right.\right.$, $\left.\left.A_{j}\right)\right\} \geq \lambda, \rho_{C^{2^{k-1}}}\left(A_{i}, A_{x_{1}}\right) \geq \lambda, \rho_{C^{k-1}}\left(A_{x_{1}}, A_{j}\right) \geq \lambda$.

So $A_{i}$ and $A_{j}$ are the same type in $C^{2^{k-1}}$ by the direct transfer algorithm.

For the same reason, $\exists x_{2}, \rho_{\mathrm{C}^{k-2}}\left(A_{i}, A_{x_{2}}\right) \geq \lambda, \rho_{\mathrm{C}^{k-2}}\left(A_{x_{2}}\right.$, $\left.A_{x_{1}}\right) \geq \lambda . \exists x_{3}, \rho_{C^{2^{k-2}}}\left(A_{x_{1}}, A_{x_{3}}\right) \geq \lambda, \rho_{C^{2^{k-2}}}\left(A_{x_{3}}, A_{j}\right) \geq \lambda$.
$A_{i}$ and $A_{j}$ are the same type in $C^{2^{k-2}}$ by the direct transfer algorithm.

So, $\exists x_{1}, x_{2}, x_{3}, \ldots, x_{2^{k}}, \rho_{C}\left(A_{i}, A_{x_{1}}\right) \geq \lambda, \rho_{C}\left(A_{x_{1}}, A_{x_{2}}\right) \geq$ $\lambda, \rho_{C}\left(A_{x_{2}}, A_{x_{3}}\right) \geq \lambda, \ldots, \rho_{\mathrm{C}}\left(A_{x^{2 k}}, A_{j}\right) \geq \lambda$.
$A_{i}$ and $A_{j}$ are the same type in $C$ by the direct transfer algorithm.

For a confidence level $\lambda$, when we get $A_{i}$ and $A_{j}$ are of the same type using the transfer closure algorithm, we can also have the same clustering results by the direct transfer algorithm, which completes the proof.

We assume $A=\left\{A_{1}, A_{2}, \ldots, A_{m}\right\}$ to be a set of DHFSs, and we construct the equivalent correlation matrix $C^{2^{k}}: C \rightarrow$ $C^{2} \rightarrow C^{4} \rightarrow \cdots \rightarrow C^{2 k} \rightarrow \cdots$, until $C^{2^{k}}=C^{2^{k+1}}$ and then construct a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ for the transfer closure algorithm. Consequently, the running time of the transfer closure algorithm is $T_{\text {tca }}=O\left(\mathrm{~km}^{3}+\mathrm{km}^{2}\right)$; by the same arguments, the direct transfer algorithm requires $T_{\mathrm{dta}}=O\left(m^{2}\right)$ time on the same example. And we have established $S_{\text {tca }}=O\left(\mathrm{~m}^{2}\right)$ space bound at least for the step of constructing the equivalent correlation matrix based on the transfer closure algorithm, while, for the transfer algorithm, it constructs a $\lambda$-cutting matrix $C_{\lambda}=\left({ }_{\lambda} \rho_{i j}\right)_{m \times m}$ by setting the threshold to the confidence level $\lambda$ and needs $S_{\text {tca }}=O(m)$ space bound. We can see that the computational complexity of both two algorithms ranges depends on the number of $m$, and the direct transfer algorithm exhibits better behavior.

Below, we conduct experiments in order to demonstrate the effectiveness of the proposed clustering algorithm for DHFSs.

Example 21. Every diamond is a miracle of time and place and chance. Like snowflakes, no two are exactly alike. Every consumer shopping for diamonds is faced with endless diamond combinations. In addition to different diamond combinations, prices are also influenced by market supply and demand conditions, fashion trends, and so forth. While

Table 6: Diamond data set.

|  | "D" color | "FL" clarity | "3 excellent" cut | "1 carat" weight |
| :--- | :---: | :---: | :---: | :---: |
| $A_{1}$ | $\{(0.5,0.4,0.3) ;(0.4,0.2)\}$ | $\{(0.6,0.5) ;(0.3,0.2,0.1)\}$ | $\{(0.6,0.4,0.3) ;(0.4,0.2,0.1)\}$ | $\{(0.6) ;(0.4)\}$ |
| $A_{2}$ | $\{(0.8,0.7,0.6) ;(0.2,0.1)\}$ | $\{(0.7,0.6) ;(0.3,0.2,0.1)\}$ | $\{(0.7,0.6,0.5) ;(0.3,0.2,0.1)\}$ | $\{(0.7) ;(0.2)\}$ |
| $A_{3}$ | $\{(0.9,0.8,0.7) ;(0.1,0.0)\}$ | $\{(0.8,0.7) ;(0.2,0.1,0.0)\}$ | $\{(0.8,0.7,0.6) ;(0.2,0.1,0.0)\}$ | $\{(0.9) ;(0.1)\}$ |
| $A_{4}$ | $\{(0.4,0.3,0.1) ;(0.6,0.5)\}$ | $\{(0.6,0.5) ;(0.4,0.2,0.1)\}$ | $\{(0.6,0.5,0.4) ;(0.3,0.2,0.1)\}$ | $\{(0.3) ;(0.6)\}$ |
| $A_{5}$ | $\{(0.6,0.5,0.4) ;(0.3,0.2)\}$ | $\{(0.3,0.2) ;(0.6,0.5,0.4)\}$ | $\{(0.3,0.2,0.1) ;(0.6,0.5,0.4)\}$ | $\{(0.1) ;(0.8)\}$ |
| $A_{6}$ | $\{(0.6,0.5,0.4) ;(0.4,0.2)\}$ | $\{(0.7,0.6) ;(0.3,0.2,0.1)\}$ | $\{(0.2,0.1,0.0) ;(0.7,0.2,0.1)\}$ | $\{(0.8) ;(0.1)\}$ |
| $A_{7}$ | $\{(0.8,0.6,0.5) ;(0.2,0.1)\}$ | $\{(0.6,0.5) ;(0.3,0.2,0.1)\}$ | $\{(0.4,0.3,0.2) ;(0.5,0.4,0.3)\}$ | $\{(0.5) ;(0.4)\}$ |
| $A_{8}$ | $\{(0.7,0.6,0.5) ;(0.2,0.0)\}$ | $\{(0.4,0.3) ;(0.6,0.5,0.4)\}$ | $\{(0.6,0.5,0.4) ;(0.4,0.3,0.2)\}$ | $\{(0.8) ;(0.2)\}$ |
| $A_{9}$ | $\{(0.4,0.3,0.2) ;(0.6,0.5)\}$ | $\{(0.4,0.3) ;(0.6,0.5,0.4)\}$ | $\{(0.2,0.1,0.0) ;(0.8,0.6,0.5)\}$ | $\{(0.2) ;(0.6)\}$ |
| $A_{10}$ | $\{(0.2,0.1,0.0) ;(0.7,0.6)\}$ | $\{(0.8,0.6) ;(0.2,0.1,0.0)\}$ | $\{(0.6,0.5,0.3) ;(0.4,0.2,0.1)\}$ | $\{(0.7) ;(0.3)\}$ |

consumers' tastes and budgets change, most seek to find a fair price for the diamond they choose. Until the middle of the twentieth century, there was no agreed upon standard by which diamonds could be judged. No matter how beautiful a diamond may look you simply cannot see its true quality. GIA created the first and now globally accepted standard for describing diamonds: color, clarity, cut, and carat weight. Concerning color, the less color in the stone there is, the more desirable and valuable it is. Grades run from " $D$ " to "X." Clarity measures the amount, size, and placement of internal "inclusions", and external "blemishes." Grades run from "Flawless" to "included." Cut does not refer to a diamond's shape but to the proportion and arrangement of its facets and the quality of workmanship. Grades range from "excellent" to "poor." Carat refers to a diamond's weight. Generally speaking, the higher the carat weight, the more

$$
C=\left(\begin{array}{llllllllll}
1.0000 & 0.9495 & 0.9010 & 0.9227 & 0.7571 & 0.8270 & 0.9542 & 0.9123 & 0.7778 & 0.9241  \tag{31}\\
0.9495 & 1.0000 & 0.9853 & 0.8053 & 0.6436 & 0.8948 & 0.9457 & 0.9404 & 0.6226 & 0.8260 \\
0.9010 & 0.9853 & 1.0000 & 0.7164 & 0.5146 & 0.8697 & 0.8904 & 0.9076 & 0.4867 & 0.7811 \\
0.9227 & 0.8053 & 0.7164 & 1.0000 & 0.8174 & 0.7353 & 0.8490 & 0.7463 & 0.8484 & 0.9025 \\
0.7571 & 0.6436 & 0.5146 & 0.8174 & 1.0000 & 0.6003 & 0.8240 & 0.6997 & 0.9316 & 0.5822 \\
0.8270 & 0.8948 & 0.8697 & 0.7353 & 0.6003 & 1.0000 & 0.9048 & 0.8750 & 0.6948 & 0.8540 \\
0.9542 & 0.9457 & 0.8904 & 0.8490 & 0.8240 & 0.9048 & 1.0000 & 0.9105 & 0.7993 & 0.8018 \\
0.9123 & 0.9404 & 0.9076 & 0.7463 & 0.6997 & 0.8750 & 0.9105 & 1.0000 & 0.6826 & 0.7612 \\
0.7778 & 0.6226 & 0.4867 & 0.8484 & 0.9316 & 0.6948 & 0.7993 & 0.6826 & 1.0000 & 0.7206 \\
0.9241 & 0.8260 & 0.7811 & 0.9025 & 0.5822 & 0.8540 & 0.8018 & 0.7612 & 0.7206 & 1.0000
\end{array}\right)
$$

Step 2. We give a detailed sensitivity analysis with respect to the confidence level, and, by (26), we get all the possible classifications of the ten diamonds; see Table 7 and Figure 1.

From the above numerical analysis, under the group setting, the experts' evaluation information usually does not reach an agreement for the objects that need to be classified. Example 21 clearly shows that the clustering algorithm based on DHFSs provides a proper way to resolve this issue.

In the following, a comparison is made among the method proposed in this paper, Chen et al.'s method [23], and Zhao et al.'s method [31] in Table 8.
expensive the stone. Two diamonds of equal carat weight, however, can have very different quality and price when the other three Cs are considered. We choose a "perfect" diamond whose 4C is "D" color, "FL" clarity, " 3 excellent" cut, and "lcarat" weight. For the convenience of analysis, the weight vector of these attributes is $w=(0.25,0.25,0.25,0.25)$. Here, there are ten diamonds. In order to better make the assessment, several evaluation organizations are requested. The normalized evaluation diamond data, represented by DHFSs, are displayed in Table 6.

Now we utilize the direct transfer algorithm to cluster the ten diamonds, which involves the following steps.

Step 1. Utilize (21) to calculate the association coefficients, and then construct an association matrix:

Table 7: The clustering result of 10 diamonds.

| Class | Confidence level | Dual hesitant fuzzy clustering algorithm |
| :--- | :---: | :---: |
| 10 | $0.9853<\lambda \leq 1$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ |
| 9 | $0.9542<\lambda \leq 0.9853$ | $\left\{A_{2}, A_{3}\right\},\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ |
| 8 | $0.9495<\lambda \leq 0.9542$ | $\left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ |
| 7 | $0.9404<\lambda \leq 0.9495$ | $\left\{A_{1}, A_{2}, A_{3}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ |
| 6 | $0.9316<\lambda \leq 0.9404$ | $\left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ |
| 5 | $0.9241<\lambda \leq 0.9316$ | $\left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\},\left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{10}\right\}$ |
| 4 | $0.9227<\lambda \leq 0.9241$ | $\left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}, A_{10}\right\},\left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\}$ |
| 3 | $0.9123<\lambda \leq 0.9227$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{7}, A_{8}, A_{10}\right\},\left\{A_{5}, A_{9}\right\},\left\{A_{6}\right\}$ |
| 2 | $0.8484<\lambda \leq 0.9123$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{6}, A_{7}, A_{8}, A_{10}\right\},\left\{A_{5}, A_{9}\right\}$ |
| 1 | $0<\lambda \leq 0.8484$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |

Table 8: Comparisons of the derived results.

| Classes | The results derived by the direct transfer algorithm method | The results derived by Chen et al.'s transfer algorithm method | The results derived by Zhao et al.s Boole method |
| :---: | :---: | :---: | :---: |
| 10 | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$, $\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ | $\left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}$, $\left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\}$ | $\begin{aligned} & \left\{A_{1}\right\},\left\{A_{2}\right\},\left\{A_{3}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ & \left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{aligned}$ |
| 9 | $\begin{aligned} & \left\{A_{2}, A_{3}\right\},\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ & \left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{aligned}$ | $\begin{aligned} & \left\{A_{2}, A_{3}\right\},\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\} \\ & \left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{aligned}$ | $\begin{aligned} & \left\{A_{2}, A_{3}\right\},\left\{A_{1}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ & \left\{A_{6}\right\},\left\{A_{7}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{aligned}$ |
| 8 | $\begin{gathered} \left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{2}, A_{3}\right\},\left\{A_{1}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 7 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \quad\left\{A_{6}\right\},\left\{A_{8}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 6 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\}, \\ \left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\}, \\ \left\{A_{4}\right\},\left\{A_{5}\right\},\left\{A_{6}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\},\left\{A_{4}\right\},\left\{A_{5}\right\}, \\ \left\{A_{6}\right\},\left\{A_{9}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 5 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{10}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\},\left\{A_{10}\right\} \end{gathered}$ |
| 4 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}, A_{10}\right\} \\ \quad\left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{4}\right\},\left\{A_{6}\right\} \end{gathered}$ |
| 3 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{7}, A_{8}, A_{10}\right\} \\ \left\{A_{5}, A_{9}\right\},\left\{A_{6}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{7}, A_{8}, A_{10}\right\} \\ \left\{A_{5}, A_{9}\right\},\left\{A_{6}\right\} \end{gathered}$ |
| 2 | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{6}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{6}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\} \end{gathered}$ | $\begin{gathered} \left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{6}, A_{7}, A_{8}, A_{10}\right\}, \\ \left\{A_{5}, A_{9}\right\} \end{gathered}$ |
| 1 | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ | $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}\right\}$ |



Figure 1: The clustering result of 10 diamonds.
the clustering results have much to do with the threshold; the smaller the confidence level is, the more detailed the clustering will be.

## 5. Conclusions

Dual hesitant fuzzy set, as an extension of fuzzy set, can describe the situation that people have hesitancy when they make a decision more objectively than other extensions of fuzzy set (interval-valued fuzzy set, intuitionistic fuzzy set, type-2 fuzzy set, and fuzzy multiset). In this paper, the correlation coefficients for DHFSs have been studied. Their properties have been discussed, and the differences and correlations among them have been investigated in detail. We have made the clustering analysis under dual hesitant fuzzy environments with one typical real world example. To further extend the application range of the present clustering algorithm, in particular for the case that needs to assign weights for different experts, it will be necessary to generalize the original definition of DHFSs.

Given that DHFSs are a suitable technique of denoting uncertain information that is widely encountered in daily life and the latent applications of our algorithm in the field of data
mining, information retrieval and pattern recognition, and so forth, may be the directions for future research.

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## Research Article

# Stationary Points-I: One-Dimensional p-Fuzzy Dynamical Systems 

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p-Fuzzy dynamical systems are variational systems whose dynamic is obtained by means of a Mamdani type fuzzy rule-based system. In this paper, we will show the 1-dimensional p-fuzzy dynamical systems and will present theorems that establish conditions of existence and uniqueness of stationary points. Besides the obtained analytical results, we will present examples that illustrate and confirm the obtained mathematical results.

## 1. Introduction

Variational equations or deterministic dynamical systems (difference and differential equations) constitute a powerful tool for modeling when the state variables depend on variations throughout time. The efficiency of a deterministic model depends on knowledge of the relationships between variables and their variations. In addition, in many situations such relations are only partially known; therefore, the modeling with deterministic variational systems, or even with stochastic ones, may not be adequate. In addition, fuzzy systems derived from deterministic models, which have subjectivity regarding some parameters, are not appropriate when we have only incomplete information of the phenomenon being analyzed. Thus, the use of a rule-based system can be adopted as an alternative for modeling partially known phenomena or those carried with imprecision.

Fuzzy rule-based systems have been used with success in some areas as control, decision taking, recognition systems, and so forth. This success is due to its simplicity and interrelation with humans way of reasoning. Fuzzy rulebased systems are conceptually simple [1]. Such systems are basically threefold: an input (fuzzifier), an inference mechanism composed of a base of fuzzy rules together with
an inference method, and, finally, an output (defuzzifier) stage (see Figure 1).

There are two main types of fuzzy rule-based systems, the Mamdani fuzzy systems and the Takagi-Sugeno fuzzy systems [2]. The main characteristic of the Mamdani type systems is that both the antecedent and consequent are expressed by linguistic terms, while in the Takagi-Sugeno type systems only the antecedent is expressed by linguistic terms and the consequent is expressed by functions.

The Takagi-Sugeno fuzzy systems are more restrictive than the Mamdani fuzzy systems because they require an $a$ priori function in the output. However, due to the existence of theoretical methods for Takagi-Sugeno fuzzy systems stability analysis [3-8], the latter has become more used. On the other hand, Mamdani type systems are used as a "black box" and are still criticized because they lack a study on its stability [9, 10].

Fuzzy variational equations have been used in different methods. Some attempts to contemplate subjectivity original from aleatory processes such as Hukuhara's derivative, differential inclusions, and Zadeh's extension [11] have been already proposed. In these methods, the adopted process for studying the variational systems is always derived from deterministic classical systems.


Figure 1: Architecture of a fuzzy rule-based system.

In this paper, we will present the p-fuzzy systems whose dynamics is not based in formal concepts of variations originated from derivative or explicit differences or differential inclusions. In the p-fuzzy dynamical systems the dynamic (iterative process) is obtained by means of a Mamdani's fuzzy rule-based system. The main advantage of this method with respect to the other ones is the simplicity of the involved mathematics, just because the interactive method is supplied by the Mamdani controller.

Formally, a p-fuzzy system in $\mathbb{R}^{n}$ is a discrete dynamic system:

$$
x_{k+1}=F\left(x_{k}\right)
$$

$$
\begin{equation*}
x_{o} \text { given and } x_{k} \in \mathbb{R}^{n}, \tag{1}
\end{equation*}
$$

where the $F$ function is given by $F\left(x_{k}\right)=x_{k}+\Delta\left(x_{k}\right)$ and $\Delta\left(x_{k}\right) \in \mathbb{R}^{n}$ is obtained by means of a fuzzy rule-based system; that is, $\Delta\left(x_{i}\right)$ is the defuzzification value of the rulebased system. The architecture of a p-fuzzy system can be visualized in Figure 2.

The name p-fuzzy dynamical systems or purely fuzzy was chosen to differentiate it from other fuzzy systems given by variational equations.

In this paper, we will focus on the one-dimensional pfuzzy systems, which are always associated with a Mamdani fuzzy system, where the defuzzification method is the centroid. We have chosen this method because it is widely used and more general to deal with weight mean of linguistic variables [12, pages 242, 243].

Analogous to the inhibited variational models in which one has stationary solutions, our objective is to present results that establish the necessary and sufficient conditions for the existence of a stationary point.


Figure 2: Architecture of a p-fuzzy system.


Figure 3: Family of successive fuzzy subsets.

## 2. Preliminaries

In this section, we introduce the main concepts for the development of the work presented in this paper.

### 2.1. Definitions

Definition 1 (support). Let $A$ be a fuzzy subset of $X$; the support of $A$, denoted $\operatorname{supp}(A)$, is the crisp subset of $X$ whose elements all have nonzero membership grades in $A$; that is,

$$
\begin{equation*}
\operatorname{supp}(A)=\{x \in X ; A(x)>0\} \tag{2}
\end{equation*}
$$

Definition 2 ( $\alpha$-cut). An $\alpha$-level set of a fuzzy subset $A$ of $X$ is a crisp set denoted by $[A]^{\alpha}$ and defined by
(i) $[A]^{0}=\overline{\operatorname{supp}(A)},(\alpha=0)$,
(ii) $[A]^{\alpha}=\{x \in X ; A(x) \geq \alpha\}$, if $\alpha \in(0,1]$.

Definition 3 (fuzzy number). A fuzzy number $A$ is a fuzzy set $A \subset \mathbb{R}$ satisfying the following conditions:
(i) $[A]^{\alpha} \neq \varnothing, \forall \alpha \in[0,1]$,
(ii) $[A]^{\alpha}$ is a closed interval, $\forall \alpha \in[0,1]$,
(iii) the $\operatorname{supp}(A)$ is bounded.

Definition 4. Consider a one-dimensional p-fuzzy system given by (1).

Consider that $x^{*}$ is a stationary point of (1) if

$$
\begin{equation*}
F\left(x^{*}\right)=x^{*}+\Delta\left(x^{*}\right)=x^{*} \Longleftrightarrow \Delta\left(x^{*}\right)=0 . \tag{3}
\end{equation*}
$$

Definition 5. Let $\left\{A_{i}\right\}_{1 \leq i \leq k}$ be any finite family of normal fuzzy subsets associated with the fuzzy variable $x$. Assume that $\left\{A_{i}\right\}_{1 \leq i \leq k}$ is a family of successive fuzzy subsets (Figure 3) if,
(i) $\operatorname{supp}\left(A_{i}\right) \cap \operatorname{supp}\left(A_{i+1}\right) \neq \varnothing$, for each $1 \leq i<k$;


Figure 4: Mamdani's inference process for $A^{*}$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$.
(ii) $\bigcap_{j=i, i+2} \operatorname{supp}\left(A_{j}\right)$ has at maximum only one element for each $1 \leq i<k-1$; that is, $\operatorname{supp}\left(A_{i}\right) \cap$ $\operatorname{supp}\left(A_{i+2}\right) \neq \phi$, if and only if, $\max \left\{x \in \operatorname{supp}\left(A_{i}\right)\right\}=$ $\min \left\{x \in \operatorname{supp}\left(A_{i+2}\right)\right\} ;$
(iii) $\bigcup_{i=1, k} \operatorname{supp}\left(A_{i}\right)=U$, where $U$ is the domain of the fuzzy variable $x$;
(iv) given $z_{1} \in \operatorname{supp}\left(A_{i}\right)$ and $z_{2} \in \operatorname{supp}\left(A_{i+1}\right)$, if $A_{i}\left(z_{1}\right)=$ 1 and $A_{i+1}\left(z_{2}\right)=1$, then necessarily $z_{1}<z_{2}$ for each $1 \leq i<k$.

Definition 6. Consider a family of successive fuzzy subsets $\left\{A_{i}\right\}_{1 \leq i \leq k}$ that describe the antecedent of a fuzzy system associated with the p-fuzzy system (1). We say that $A^{*}$ is an equilibrium viable set of (1) if $A^{*}$ contains stationary points of (1).

If for some $1 \leq i<k$ there are $z_{1}, z_{2} \in\left[A_{i} \cup A_{i+1}\right]^{0}$ such that $\Delta\left(z_{1}\right) \cdot \Delta\left(z_{2}\right)<0$, then $A^{*}$ is given by $A^{*}=\left[A_{i} \cap A_{i+1}\right]^{0}$. If for all $z_{1}, z_{2} \in\left[A_{i} \cup A_{i+1}\right]^{0}, 1 \leq i<k, \Delta\left(z_{1}\right) \cdot \Delta\left(z_{2}\right)>0$, then $A^{*}=\left[A_{k}\right]^{0}$.

A p-fuzzy system depends on the fuzzy system associated with it, that is, it depends on the rule-base, on the inference method and on the defuzzification method used. In Definition 6, a sufficient condition for $\Delta\left(z_{1}\right) \cdot \Delta\left(z_{2}\right)<0$ is that the p-fuzzy system be associated with a fuzzy system whose rule-base in $A^{*}=\left[A_{i} \cup A_{i+1}\right]^{0}$ is of the type
$R_{1}$ : if $x$ is $A_{i}$ then $\Delta$ is $B$,
$R_{2}$ : if $x$ is $A_{i+1}$ then $\Delta$ is $C$,
where $\operatorname{supp}(B) \subset \mathbb{R}^{-}$and $\operatorname{supp}(C) \subset \mathbb{R}^{+}$or vice versa. When $\operatorname{supp}(B) \subset \mathbb{R}^{-}$and $\operatorname{supp}(C) \subset \mathbb{R}^{+}$we have that the equilibrium viable set $A^{*}$ is of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$. If, on the other hand, we have that $\operatorname{supp}(B) \subset \mathbb{R}^{+}$and $\operatorname{supp}(C) \subset \mathbb{R}^{-}$we say that $A^{*}$ is of the type $\left(A_{i}, A_{i+1}\right) \rightarrow$ ( $B, C$ ).

To understand the dynamics of the p-fuzzy system we need to understand how the rule-based system works more specifically, given $x \in A^{*}$, how is $\Delta(x)$ obtained? In the following section we will describe this process.
2.2. Output Defuzzification of the Fuzzy System. Let $A^{*}=$ $\left[A_{i} \cap A_{i+1}\right]^{0}=\left[c_{1}, c_{2}\right]$ be an equilibrium viable set of the p-fuzzy system. To facilitate the notation, we will indicate
by $r$ the membership function of $A_{i}$, by $s$ the membership function of $A_{i+1}$,

$$
\begin{equation*}
z_{1}=\min _{x \in \operatorname{supp}\left(A_{i}\right)}\{r(x)=1\}, \quad z_{2}=\max _{x \in \operatorname{supp}\left(A_{i+1}\right)}\{s(x)=1\}, \tag{4}
\end{equation*}
$$

and by $f$ and $g$ the membership functions of $C$ and $D$ (Figure 4), respectively. Assume that the p-fuzzy system, in the equilibrium viable set $A^{*}$, is of the type $\left(A_{i}, A_{i+1}\right) \rightarrow$ (C, B).

For each $x \in A^{*}, \Delta(x)$ is the $\mathbf{R}$ region centroid abscise, with $\mathbf{R}$ limited by the membership function of the fuzzy output, $\widehat{\Delta} x$, (see Figure 4). Thus,

$$
\begin{align*}
\Delta(x)= & \left(\int_{b}^{f^{-1}(m)} t f(t) d t+\int_{f^{-1}(m)}^{0} m t d t+\int_{0}^{g^{-1}(n)} n t d t\right. \\
& \left.+\int_{g^{-1}(n)}^{a} t g(t) d t\right) \\
\times & \left(\int_{0}^{n} g^{-1}(t) d t-\int_{0}^{m} f^{-1}(t) d t\right)^{-1} \tag{5}
\end{align*}
$$

where $(n, m)=(r(x), s(x))$. Equation (5) can be rewritten as

$$
\begin{equation*}
\Delta(x)=\frac{h_{1}(n)+h_{2}(m)}{A(m, n)} \tag{6}
\end{equation*}
$$

where

$$
\begin{gather*}
h_{1}(n)=\int_{0}^{g^{-1}(n)} n t d t+\int_{g^{-1}(n)}^{a} t g(t) d t  \tag{7}\\
h_{2}(m)=\int_{b}^{f^{-1}(m)} t f(t) d t+\int_{f^{-1}(m)}^{0} m t d t  \tag{8}\\
A(m, n)=\int_{0}^{n} g^{-1}(t) d t-\int_{0}^{m} f^{-1}(t) d t \tag{9}
\end{gather*}
$$

2.3. Preliminary Results. In this section, we will present some important technical results for the main demonstrations of the theorems that establish sufficient conditions for uniqueness and existence of the stationary point of the p-fuzzy systems. Here, the presented results are technical enough and are important only for demonstrating the theorems in
the next section. Therefore, the reader will be able, without loss of continuity, to skip them if desired.

The following results are referred to the equilibrium viable set of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$ (Figure 4). From now on, it will be assumed that the functions $r, s, f$, and $g$ (which represent the membership functions $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$ and $\mu_{C}$, resp.) are continuous.

Lemma 7. The function $h_{1}(7)$ is increasing, and its range is given $b y, \operatorname{Im}\left(h_{1}\right)=\left[0, \int_{0}^{a} \operatorname{tg}(t) d t\right]$.

Proof. Since $g$ is continuous in $[0, a]\left(g^{-1}\right.$ is limited in $\left.[0, a]\right)$, then the function $h_{1}$ is differentiable, and

$$
\begin{equation*}
h_{1}^{\prime}(n)=\left(\int_{0}^{g^{-1}(n)} n t d t\right)^{\prime}+\left(\int_{g^{-1}(n)}^{a} t g(t) d t\right)^{\prime} \tag{10}
\end{equation*}
$$

Using the derivative properties,

$$
\begin{align*}
h_{1}^{\prime}(n)= & \int_{0}^{g^{-1}(n)} t d t+n\left(\int_{0}^{g^{-1}(n)} t d t\right)^{\prime}  \tag{11}\\
& -\left(\int_{a}^{g^{-1}(n)} \operatorname{tg}(t) d t\right)^{\prime}
\end{align*}
$$

and using the chain rule and the fundamental theorem of calculus, we obtain

$$
\begin{align*}
h_{1}^{\prime}(n)= & \int_{0}^{g^{-1}(n)} t d t+n g^{-1}(n)\left(g^{-1}\right)^{\prime}(n)  \tag{12}\\
& -n g^{-1}(n)\left(g^{-1}\right)^{\prime}(n)
\end{align*}
$$

Hence,

$$
\begin{equation*}
h_{1}^{\prime}(n)=\int_{0}^{g^{-1}(n)} t d t=\frac{\left(g^{-1}(n)\right)^{2}}{2}>0 \tag{13}
\end{equation*}
$$

Therefore, $h_{1}$ is increasing.
Since,

$$
\begin{gather*}
h_{1}(0)=\int_{0}^{g^{-1}(0)} 0 t d t+\int_{g^{-1}(0)}^{a} \operatorname{tg}(t) d t=\int_{a}^{a} \operatorname{tg}(t) d t=0 \\
h_{1}(1)=\int_{0}^{g^{-1}(1)} t d t+\int_{g^{-1}(1)}^{a} \operatorname{tg}(t) d t=\int_{0}^{a} t g(t) d t \tag{14}
\end{gather*}
$$

then $\operatorname{Im}\left(h_{1}\right)=\left[0, \int_{0}^{a} t g(t) d t\right]$.
Lemma 8. The function $h_{2}$ is decreasing, and its range is given by $\operatorname{Im}\left(h_{2}\right)=\left[\int_{b}^{0} t f(t) d t, 0\right]$.

Proof. Analogous to the demonstration of Lemma 7.
Lemma 9. Let $\phi: I=\left[d_{1}, d_{2}\right] \rightarrow \mathbb{R}$ be a function of the class $C^{2}$. If $\phi^{\prime \prime}(z)>0, \forall z \in\left(d_{1}, d_{2}\right)$ and $\phi\left(d_{1}\right)<0$, then $\phi$ has at maximum one root in $I$.


Figure 5: System p-fuzzy output with $g(t)>f(-t)$.

Proof of Lemma 9. Assume there are $z_{1}, z_{2} \in I\left(z_{1}<z_{2}\right)$ such that $\phi\left(z_{1}\right)=\phi\left(z_{2}\right)=0$. Since $\phi^{\prime \prime}(z)>0$, we have that $\phi$ is not constant. Hence, by Rolle's Theorem, $\exists c \in\left(z_{1}, z_{2}\right)$ such that $\phi^{\prime}(c)=0$. Hence $c$ is a minimum point, because $\phi^{\prime \prime}(c)>0$. But, $\phi\left(d_{1}\right)<0 \Rightarrow \phi(c)>0$. Since $\phi$ is continuous $\exists z_{o} \in$ $\left(z_{1}, z_{2}\right)$ such that $\phi(c)>\phi\left(z_{o}\right)>0$, it is nonsense! Therefore $\phi$ has at maximum one root.

Lemma 10. If $g(t)>f(-t), \forall t \in[0,-b]$, then $g^{-1}(k)>$ $-f^{-1}(k) \forall k \in[0,1]$.

Proof. The proof is simple (see Figure 5).
Lemma 11. If $g(t)>f(-t), \forall t \in[0,-b]$ then for $m, n \in[0,1]$ with $m \leq n$ one has that

$$
\begin{equation*}
\Delta(x)=\frac{h_{1}(n)+h_{2}(m)}{A(m, n)}>0 . \tag{15}
\end{equation*}
$$

Proof. The proof is simple.
Lemma 12. If $g(t)<f(-t), \forall t \in[0, a]$, and $m, n \in[0,1] m \geq$ $n$, then $\Delta(x)<0$.

## Proof. It is analogous to the proof of Lemma 11.

## 3. Stationary Point

In this section, we will enunciate and prove a theorem that guarantees the existence of at least one stationary point for each equilibrium viable set of the p-fuzzy system. For this, we will use again Figure 4 to motivate the presented results in this section.

Theorem 13 (existence). Let $S$ be a $p$-fuzzy system and $A^{*}$ an equilibrium viable set of $S$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$. Then, there is at least one stationary point of $S$ in $A^{*}$. That is, $\exists x^{*} \in A^{*}$ such that $\Delta\left(x^{*}\right)=0$.

Proof. Given $x \in A^{*}$, from Definition 4, then $x$ is a stationary point if and only if

$$
\begin{equation*}
\Delta(x)=0 \Longleftrightarrow h_{1}(n)+h_{2}(m)=0 . \tag{16}
\end{equation*}
$$

If $\mu_{A_{i}}\left(c_{1}\right)=0$, then

$$
\begin{equation*}
\Delta\left(c_{1}\right)=h_{1}\left(\mu_{A_{i}}\left(c_{1}\right)\right)+h_{2}\left(\mu_{A_{i+1}}\left(c_{1}\right)\right)=h_{1}(0)+h_{2}(0)=0 \tag{17}
\end{equation*}
$$

and, therefore $c_{1}$ is a stationary point. If $\mu_{A_{i+1}}\left(c_{2}\right)=0$ one has that $\Delta\left(c_{2}\right)=0$; hence $c_{2}$ is a stationary point. Now, assume that $\mu_{A_{i}}\left(c_{1}\right)>0$ and $\mu_{A_{i+1}}\left(c_{2}\right)>0$. Since $\mu_{A_{i+1}}\left(c_{1}\right)=0$, then from Lemmas 7 and 8 one has that $h_{1}\left(\mu_{A_{i}}\left(c_{1}\right)\right)>0$ and $h_{2}\left(\mu_{A_{i+1}}\left(c_{1}\right)\right)=0$. Therefore

$$
\begin{equation*}
\Delta\left(c_{1}\right)=h_{1}\left(\mu_{A_{i}}\left(c_{1}\right)\right)+h_{2}\left(\mu_{A_{i+1}}\left(c_{1}\right)\right)=h_{1}\left(\mu_{A_{i}}\left(c_{1}\right)\right)>0 \tag{18}
\end{equation*}
$$

Thus, from Lemma 7,

$$
\begin{equation*}
\Delta\left(c_{2}\right)=h_{1}\left(\mu_{A_{i}}\left(c_{2}\right)\right)+h_{2}\left(\mu_{A_{i+1}}\left(c_{2}\right)\right)=h_{2}\left(\mu_{A_{i+1}}\left(c_{2}\right)\right)<0 \tag{19}
\end{equation*}
$$

Since $\Delta$ is continuous, by Bolzano's Intermediate Value Theorem, $\exists x^{*} \in\left[c_{1}, c_{2}\right]$ such that $\Delta\left(x^{*}\right)=0$; therefore, $x^{*}$ is a stationary point.

Remark 14. If in Theorem 13 we consider $A^{*}$ as an equilibrium viable set of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(B, C)$, the result is analogous; that is, there exists a stationary point $x^{*} \in A^{*}$.
3.1. Local Stationary Points-Symmetrical Output. If $A^{*}$ is an equilibrium viable set, where the membership functions of the consequents, $B$ and $C$, are symmetrical functions, then the stationary point in $A^{*}$ is unique. Except when $\mu_{A_{i}}\left(c_{1}\right)=0$ or $\mu_{A_{i+1}}\left(\mathcal{c}_{2}\right)=0$ possibly occurs. Then we have the following proposition.

Proposition 15. Let $S$ be a p-fuzzy system and $A^{*}$ an equilibrium viable set of $S$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$. If the membership functions of $B$ and $C$, respectively, $\mu_{B}$ and $\mu_{C}$, are monotonous and symmetric, that is, $\mu_{C}(t)=\mu_{B}(-t)$, then there exists an equilibrium point in $A^{*}$ :

$$
\begin{equation*}
x^{*}=\max _{x \in A^{*}}\left[\min \left(\mu_{A_{i}}(x), \mu_{A_{i+1}}(x)\right)\right] . \tag{20}
\end{equation*}
$$

Proof. Since $\mu_{B}(t)=\mu_{C}(-t)$ then $\mu_{B}(-a)=\mu_{C}(a)=0=$ $\mu_{B}(b) \Rightarrow b=-a$, because $\mu_{B}$ U' is monotonous. Yet, we have that $\mu_{B}(t)=\mu_{C}(-t) \Rightarrow \mu_{B}^{-1}\left(\mu_{C}(t)\right)=-t=-\mu_{B}^{-1}\left(\mu_{B}(t)\right) \Rightarrow$ $\mu_{B}^{-1}(y)=-\mu_{C}^{-1}(y)$.

Then, $\Delta\left(z_{o}\right)=0$, if and only if, $h_{1}(n)=-h_{2}(m)$. Since $b=-a$, from (8), then we obtain

$$
\begin{equation*}
h_{2}(m)=\int_{-a}^{\mu_{B}^{-1}(m)} t \mu_{B}(t) d t+\int_{\mu_{B}^{-1}(m)}^{0} m t d t . \tag{21}
\end{equation*}
$$

If we perform a change in the variable $u=-t$, we have

$$
\begin{gather*}
h_{2}(m)=\int_{\mathrm{a}}^{-\mu_{\mathrm{B}}^{-1}(m)} u \mu_{B}(-u) d u+\int_{-\mu_{B}^{-1}(m)}^{0} m u d u \\
\Longrightarrow h_{2}(m)=\int_{a}^{\mu_{C}^{-1}(m)} u \mu_{C}(u) d u+\int_{\mu_{C}^{-1}(m)}^{0} m u d u=-h_{1}(m) . \tag{22}
\end{gather*}
$$

That is, $h_{2}=-h_{1}$. Hence, $h_{1}(n)=-h_{2}(m) \Leftrightarrow h_{1}(n)=$ $h_{1}(m) \Leftrightarrow m=n$ (because $h_{1}$ is increasing: Lemma 7); that proves the proposition.

Remark 16. If $\mu_{A_{i}}\left(c_{1}\right) \neq 0$, then $x^{*}=\max _{x \in A^{*}}\left[\min \left(\mu_{A_{i}}(x)\right.\right.$, $\left.\left.\mu_{A_{i+1}}(x)\right)\right]$ is the only stationary point in $A^{*}$. Besides that, if the system $S$ is of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(B, C)$, then the result of Proposition 15 is the same.

## 4. Uniqueness of the Stationary Point

In this section we will enunciate and prove theorems that establish condition for uniqueness of the stationary point of a one-dimensional p-fuzzy system. Initially we will consider a simpler case, when $A^{*} \subset\left[z_{1}, z_{2}\right]$ (Figure 4), where

$$
\begin{equation*}
z_{1}=\min _{x \in \operatorname{supp}\left(A_{i}\right)}\{r(x)=1\}, \quad z_{2}=\max _{x \in \operatorname{supp}\left(A_{i+1}\right)}\{s(x)=1\} . \tag{23}
\end{equation*}
$$

Theorem 17. Let $S$ be a p-fuzzy system and $A^{*}$ an equilibrium viable set of $S$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow(C, B)$. If the functions $\mu_{A_{i}}$ and $\mu_{A_{i+1}}$ are piecewise monotonous and $A^{*} \subset\left[z_{1}, z_{2}\right]$ then there exists only one stationary point in $A^{*}$.

Proof. Given $x \in A^{*}$ one has

$$
\begin{equation*}
\Delta(x)=h_{1}(n)+h_{2}(m)=h_{1}\left(\mu_{A_{i}}(x)\right)+h_{2}\left(\mu_{A_{i+1}}(x)\right) . \tag{24}
\end{equation*}
$$

Using Lemmas 7 and 8 and the chain rule we find that the derivative of $\Delta$ is

$$
\begin{align*}
\Delta^{\prime}(x)= & \frac{\left[\mu_{C}^{-1}\left(\mu_{A_{i}}(x)\right)\right]^{2}}{2} \mu_{A_{i}}^{\prime}(x)  \tag{25}\\
& -\frac{\left[\mu_{B}^{-1}\left(\mu_{A_{i+1}}(x)\right)\right]^{2}}{2} \mu_{A_{i+1}}^{\prime}(x)
\end{align*}
$$

Since in $A^{*}=\left[c_{1}, c_{2}\right] \mu_{A_{i}}$ is not increasing and $\mu_{A_{i+1}}$ is not decreasing, then $\mu_{A_{i}}^{\prime}(x) \leq 0$ and $\mu_{A_{i+1}}^{\prime}(x) \geq 0$, and, besides that, if $\mu_{A_{i}}^{\prime}(x)=0$, we have that $\mu_{A_{i+1}}^{\prime}(x) \neq 0$ and if $\mu_{A_{i+1}}^{\prime}(x)=$ 0 we obtain $\mu_{A_{i}}^{\prime}(x) \neq 0$. Then, from (25), $\Delta^{\prime}(x)<0$. This shows that $\Delta$ is decreasing. From Theorem 13, there exists a stationary point in $A^{*}$ then this point is unique.

Now, consider the more general case, when $A^{*} \not \subset\left[z_{1}, z_{2}\right]$, and divide it into two theorems. Initially, consider the case that the membership functions $\mu_{C}$ and $\mu_{B}$ are such that $\mu_{C}(t)>\mu_{B}(-t)$; next, consider the case where $\mu_{C}(t)<$ $\mu_{B}(-t)$.


Figure 6: Functions $h_{1}$ and $h_{2}$.


Figure 7: Functions $\xi, \delta_{1}$ and $\delta_{2}$.

### 4.1. Case 1: $\mu_{C}(t)>\mu_{B}(-t)$

Theorem 18 (uniqueness). Let $S$ be a $p-f u z z y$ system and $A^{*}$ an equilibrium viable set of $S$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow$ (C, B). If the functions $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$ and $\mu_{C}$ are continuously differentiable, $\mu_{A_{i}}$ and $\mu_{A_{i+1}}$ are piecewise monotonous $\mu_{B}$ and $\mu_{C}$ are strictly monotonous, such that
(i) $\mu_{C}(t)>\mu_{B}(-t), \forall t \in(0,-b)$,
(ii) $\mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)<\left(p^{3} / q^{3}\right), \forall p \in \operatorname{supp}(B), q \in$ $\operatorname{supp}(C)$ and $\mu_{B}(p)>\mu_{C}(q)$,
(iii) $\left[\mu_{A_{i+1}}^{\prime}(x) / \mu_{A_{i}}^{\prime}(x)\right] \leq 0, \forall x \in\left(z_{o}, c_{2}\right), \mu_{A_{i}}(x) \neq \mu_{A_{i+1}}(x)$.

Then, $S$ has only one stationary point, $x^{*}$ in $A^{*}$, and $x^{*} \in$ $\left(z_{0}, c_{2}\right]$.

Proof. For the sake of simplicity notation, we make $r=\mu_{A_{i}}$, $s=\mu_{A_{i+1}}, f=\mu_{B}$ and $g=\mu_{C}$.

Initially, we see that given $x \in\left(z_{o}, c_{2}\right.$ ] (Figure 4), $x$ determines only one $(n, m) \in[0,1]^{2}$ such that $n=r(x)$ and $m=s(x)$. By monotonicity of $r$, we have that for each $n \in\left[0, r\left(z_{o}\right)\right)$ there exists only one $m \in[0,1]$ such that $n=r(x)$ and $m=s(x)$. That is, each $(n, m)$, in this situation, determines only one $x \in\left(z_{0}, c_{2}\right]$.

By Theorem 13, there exists a stationary point $x^{*} \in$ $\left[c_{1}, c_{2}\right]=\left[c_{1}, z_{o}\right] \cup\left(z_{o}, c_{2}\right]$. Given $x \in\left[c_{1}, z_{o}\right] \Rightarrow m=s(x) \leq$ $n=r(x)$. Then, by Lemma $11 x^{*} \notin\left[c_{1}, z_{0}\right] \Rightarrow x^{*} \in\left(z_{0}, \mathcal{c}_{2}\right]$. That is equivalent to the existence of only one $\left(n^{*}, m^{*}\right)$, with $n^{*} \in\left[0, r\left(z_{o}\right)\right)$ such that $H\left(n^{*}, m^{*}\right)=0$.

Since for each $n \in\left[0, r\left(z_{o}\right)\right)$ there exists only one $m \in$ $[0,1]$ such that $n=r(x)$ and $m=s(x)$, then we may define a function $\delta_{2}:\left[0, r\left(z_{o}\right)\right) \rightarrow[0,1]$ such that $m=\delta_{2}(n)$ (Figure 7). We observe that $\delta_{2}$ is continuous, because $r$ and $s$ are continuous. Using the chain rule we get the derivative of $\delta_{2}$ to be

$$
\begin{equation*}
\delta_{2}^{\prime}(n)=\frac{s^{\prime}(x)}{r^{\prime}(x)} \stackrel{(\text { iii }}{\Rightarrow} \delta_{2}^{\prime \prime}(n) \leq 0, \quad \forall n \in D_{\delta_{2}} . \tag{26}
\end{equation*}
$$

By Lemmas 7 and 8, $h_{1}$ and $-h_{2}$ are increasing, and, by condition (i), it follows that (Figure 6)

$$
\begin{equation*}
\int_{0}^{a} t g(t) d t>-\int_{b}^{0} t f(t) d t \Longleftrightarrow h_{1}(1)>-h_{2}(1) . \tag{27}
\end{equation*}
$$

Then, given $n \in\left[0, h_{1}^{-1}\left(-h_{2}(1)\right)\right]$, there exists only one $m \in$ $[0,1]$ such that

$$
\begin{equation*}
h_{1}(n)=-h_{2}(m) \Longleftrightarrow h_{1}(n)+h_{2}(m)=0 . \tag{28}
\end{equation*}
$$

Therefore, we may define an injective function $\xi, m=\xi(n)$ (Figure 7) so that the inverse range of 0 by $H$ is given by

$$
\begin{equation*}
H^{-1}(0)=\{(n, m) ; m=\xi(n)\} \tag{29}
\end{equation*}
$$

where $H:\left[0, h_{1}^{-1}\left(-h_{2}(1)\right)\right] \times[0,1] \rightarrow \mathbb{R}$ is given by $H(n, m)=h_{1}(n)+h_{2}(m)$.

Since $\partial H / \partial n=h_{1}^{\prime}(n)=\left(g^{-1}(n)\right)^{2} / 2>0$ (Lemma 7) and $\partial H / \partial m=h_{2}^{\prime}(m)=-\left(f^{-1}(m)\right)^{2} / 2<0$ (Lemma 8) then, by the Implicity Function Theorem, $\xi$ is $k$ times differentiable, and, besides that,

$$
\begin{gather*}
\xi^{\prime}(n)=-\frac{d h_{1} / n}{d h_{2} / m}=\left[\frac{g^{-1}(n)}{f^{-1}(m)}\right]^{2}>0  \tag{30}\\
\forall n \in\left(0, h_{1}^{-1}\left(-h_{2}(1)\right)\right), m \in(0,1), m=\xi(n)
\end{gather*}
$$

Thus, $\xi$ is a strictly increasing function, and since $H(0,0)=0$ and $H\left(h_{1}^{-1}\left(-h_{2}(1)\right), 1\right)=0$, then $D_{\xi}=\left[0, h_{1}^{-1}\left(h_{2}(1)\right)\right]$ and $\operatorname{Im}_{\xi}=[0,1]$.

Given $m, n \in(0,1)$, there is only one $p \in(b, 0)$ such that $p=f^{-1}(m)$, and there exists only one $q \in(0, a)$ such that $q=$ $g^{-1}(n)$, since by assumption $f$ and $g$ are strictly monotonous.

From Lemma 11 we have that $m \leq n \Rightarrow H(m, n)>0$. Hence, $H(m, n)=0 \Rightarrow m>n$. Therefore, we are interested in the pairs $(m, n)$ such that $m>n$. Thus, we have

$$
\begin{equation*}
m>n \Longleftrightarrow f(p)>g(q) . \tag{31}
\end{equation*}
$$



Figure 8: p-Fuzzy system: existence of more than one stationary point.

Since $f$ and $g$ are monotonous, by Lagrange's Medium Value Theorem, we obtain

$$
\begin{gather*}
p=f^{-1}(m) \Longleftrightarrow\left(f^{-1}\right)^{\prime}(m)=\frac{1}{f^{\prime}(p)}  \tag{32}\\
q=g^{-1}(n) \Longleftrightarrow\left(g^{-1}\right)^{\prime}(n)=\frac{1}{g^{\prime}(q)} \tag{33}
\end{gather*}
$$

Then,

$$
\begin{align*}
m & >n \stackrel{(31)}{\Longleftrightarrow} f(p)>g(q) \stackrel{(i i)}{\Longleftrightarrow} \frac{g^{\prime}(q)}{f^{\prime}(p)} \\
& <\frac{p^{3}}{q^{3}} \stackrel{(32) e(33)}{\Longleftrightarrow} \frac{\left(f^{-1}\right)^{\prime}(m)}{\left(g^{-1}\right)^{\prime}(n)}<\frac{\left[f^{-1}(m)\right]^{3}}{\left[g^{-1}(n)\right]^{3}} . \tag{34}
\end{align*}
$$

Therefore,

$$
\begin{align*}
m> & n \Longrightarrow\left(f^{-1}\right)^{\prime}(m)\left[g^{-1}(n)\right]^{3} \\
& -\left(g^{-1}\right)^{\prime}(n)\left[f^{-1}(m)\right]^{3}>0 \tag{35}
\end{align*}
$$

By differentiation of (30), we obtain

$$
\begin{align*}
\xi^{\prime \prime}(n)= & \frac{-2 g^{-1}(n)}{\left[f^{-1}(m)\right]^{5}} \\
& \times\left\{\left(f^{-1}\right)^{\prime}(m)\left[g^{-1}(n)\right]^{3}-\left(g^{-1}\right)^{\prime}(n)\left[f^{-1}(m)\right]^{3}\right\} \tag{36}
\end{align*}
$$

and since $-2 g^{-1}(n) /\left[f^{-1}(m)\right]^{5}>0, \forall m, n \in(0,1)$, from (35), we have

$$
\begin{equation*}
\xi^{\prime \prime}(n)>0, \quad \forall n \in \stackrel{o}{D_{\xi}} . \tag{37}
\end{equation*}
$$

Now we take $I=D_{\xi} \cap D_{\delta_{2}}=D_{\xi} \cap\left[0, r\left(z_{o}\right)\right)$, and we define the function $\phi: I \rightarrow[0,1]$ such that

$$
\begin{equation*}
\phi(n)=\xi(n)-\delta_{2}(n) \tag{38}
\end{equation*}
$$

Then, from (26) and (37) we have $\phi^{\prime \prime}(n)>0, \forall n \in \stackrel{o}{I}$. Since $\xi(0)=0$ and by condition (iii) of Theorem 18, it follows that $\delta_{2}(0)>0$; then we have $\phi(0)<0$. Consequently, from Lemma 9, we have that there is only one $n^{*} \in I$ such that

$$
\begin{equation*}
\phi\left(n^{*}\right)=0 \stackrel{(38)}{\Longleftrightarrow} \xi\left(n^{*}\right)=\delta_{2}\left(n^{*}\right) \tag{39}
\end{equation*}
$$

Since $\xi=H^{-1}(0)$, then we obtain

$$
\begin{equation*}
0=H\left(n^{*}, \xi\left(n^{*}\right)\right) \stackrel{(39)}{=} H\left(n^{*}, \delta_{2}\left(n^{*}\right)\right) \tag{40}
\end{equation*}
$$

So, there exists only one $x^{*} \in\left(z_{0}, c_{2}\right], n^{*}=r\left(x^{*}\right)$ and $m^{*}=\delta_{2}\left(n^{*}\right)=s\left(x^{*}\right)$ such that

$$
\begin{equation*}
\Delta\left(x^{*}\right)=\frac{H\left(n^{*}, m^{*}\right)}{A\left(n^{*}, m^{*}\right)}=0 \tag{41}
\end{equation*}
$$

This finally proves the theorem.

### 4.2. Case 2: $\mu_{C}(t)<\mu_{B}(-t)$

Theorem 19 (uniqueness). Let $S$ be a $p$-fuzzy system and $A^{*}$ an equilibrium viable set of $S$ of the type $\left(A_{i}, A_{i+1}\right) \rightarrow$ ( $C, B$ ). If the functions $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$, and $\mu_{C}$ are continuously differentiable, $\mu_{A_{i}}$ and $\mu_{A_{i+1}}$ are piecewise monotonous, $\mu_{B}$ and $\mu_{C}$ are strictly monotonous, such that
(i) $\mu_{C}(t)<\mu_{B}(-t), \forall t \in(0, a)$,
(ii) $\mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)>p^{3} / q^{3}, \forall p \in \operatorname{supp}(B), q \in \operatorname{supp}(C)$ and $\mu_{B}(p)<\mu_{C}(q)$,
(iii) $\left[\mu_{A_{i}}^{\prime}(x) / \mu_{A_{i+1}}^{\prime}(x)\right] \leq 0, \forall x \in\left(c_{1}, z_{o}\right), \mu_{A_{i}}(x) \neq \mu_{A_{i+1}}(x)$.

Then, $S$ has only one stationary point, $x^{*}$ in $A^{*}$, and $x^{*} \in$ $\left[c_{1}, z_{o}\right.$ ).

Proof. It is analogous to the proof of Theorem 18.
4.3. Some Comments about Uniqueness Theorems. When we do not have $\mu_{C}(t)>\mu_{B}(-t)$ or $\mu_{C}(t)<\mu_{B}(-t)$, it is impossible to establish general conditions for uniqueness of stationary points. For example, consider the p-fuzzy system in Figure 8. This system has an equilibrium viable set, $A^{*}=\left[A_{1} \cap A_{2}\right]^{0}$.


Figure 9: Function $\Delta$ with $m_{1}=\varepsilon=0.1$.

The sets that describe the input variable have membership functions $\mu_{A_{1}}$ and $\mu_{A_{2}}$ :

$$
\begin{align*}
& \mu_{A_{1}}(x)= \begin{cases}\frac{1}{40} x, & \text { if } 0<x \leq 40 \\
\frac{-1}{50} x+\frac{9}{5}, & \text { if } 40<x \leq 90 \\
0, & \text { otherwise, }\end{cases} \\
& \mu_{A_{2}}(x)= \begin{cases}\frac{1}{55} x-\frac{1}{11}, & \text { if } 5<x \leq 60 \\
\frac{-1}{40} x+\frac{5}{2}, & \text { if } 60<x \leq 100 \\
0, & \text { otherwise, }\end{cases} \tag{42}
\end{align*}
$$

and the fuzzy sets that describe the output variable have membership functions: $\mu_{\mathrm{C}}(t)=(-1 / 2) t+1$ and

$$
\begin{align*}
& \mu_{B}(t) \\
& \quad= \begin{cases}\frac{1}{3} t+1_{\varepsilon}, & \text { if }-3<t \leq 3\left(m_{1}-1\right) \\
\frac{3\left(m_{1}-1\right)}{} t+m_{1}+\varepsilon, & \text { if } 3\left(m_{1}-1\right)<t \\
\frac{1}{1} t+1, & \leq \frac{3 \varepsilon\left(\varepsilon+m_{1}-1\right)\left(m_{1}-1\right)}{3\left(m_{1}-1\right)+\varepsilon^{2}} \\
0, & \text { if } \frac{3 \varepsilon\left(\varepsilon+m_{1}-1\right)\left(m_{1}-1\right)}{3\left(m_{1}-1\right)+\varepsilon^{2}} \\
0, t \leq 0 \\
\text { otherwise. }\end{cases} \tag{43}
\end{align*}
$$

For example, if we take in (43) $m_{1}=0.1$ and $\varepsilon=0.1$, the p-fuzzy system obtained has three stationary points in $A^{*}$, which can be visualized in Figure 9, which depicts the graphic of function $\Delta$.

Remark 20. Note that the function $\mu_{B}$ is not differentiable in all points into $\operatorname{supp}(B)$, which was a requirement made in the previous cases. However, it can be clearly constructed a function $\mu_{B}$ derivable in all points of $\operatorname{supp}(B)$. For example,


Figure 10: Function $\Delta$ with $\varepsilon=0.4, m_{1}=0.1$.
if we substitute the second sentence of $\mu_{B}$ by an adequate fourth degree polynomial, obviously $\mu_{B}$ will be derivable into $\operatorname{supp}(B)$.

Remark 21. If we take, for example, $\varepsilon=0.3$ and $m_{1}=$ 0.3 , we have that the obtained p -fuzzy system has only one stationary point (Figure 10). This shows that Theorems 18 and 19 establish only sufficient conditions for uniqueness of the stationary point.
4.4. Uniqueness for Triangular and Trapezoidal Membership Functions. In this section, we will list some important consequences of Theorems 18 and 19. We will also show that for triangular and trapezoidal membership functions, the stationary point is only one in $A^{*}$. However, before doing so, let us take a look at the following lemmas.

Lemma 22. If $\mu_{B}(p)>\mu_{C}(q)$ then $q>-p$, where $p=\mu_{B}^{-1}(m)$ and $q=\mu_{C}^{-1}(n)$.

Proof. In fact, we have that $\mu_{B}(p)>\mu_{C}(q) \Rightarrow m>n$ hence using Lemma 10 and the fact that $-\mu_{B}^{-1}$ isincreasing, once $\mu_{B}$ is increasing, then we have that

$$
\begin{equation*}
q=\mu_{C}^{-1}(n)>-\mu_{B}^{-1}(n)>-\mu_{B}^{-1}(m)=-p \tag{44}
\end{equation*}
$$

Lemma 23. If $\mu_{B}(p)<\mu_{C}(q)$, then $q<-p$, where $p=\mu_{B}^{-1}(m)$ and $q=\mu_{C}^{-1}(n)$.

Proof. Analogous to previous proof.
Corollary 24. Let $S$ be a $p$-fuzzy system and $A^{*}$ an equilibrium viable set of S. If $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$ and $\mu_{C}$ are triangular fuzzy numbers, then $S$ has only one stationary point in $A^{*}$.

Proof. We will prove the case where $S$ is $\left(\mu_{A_{i}}, \mu_{A_{i+1}}\right) \rightarrow$ $\left(\mu_{C}, \mu_{B}\right)$. If $S$ is $\left(\mu_{A_{i}}, \mu_{A_{i+1}}\right) \rightarrow\left(\mu_{B}, \mu_{C}\right)$, then the proof is analogous.


Figure 11: Population ( $x$ ).

If $a=b$, then $\mu_{A_{i}}$ and $\mu_{A_{i+1}}$ are symmetrical and by Proposition 15 the stationary point is only one:

$$
\begin{equation*}
x^{*}=\max _{x \in A^{*}}\left[\min \left(\mu_{A_{i}}(x), \mu_{A_{i+1}}(x)\right)\right] \tag{45}
\end{equation*}
$$

Assume that $a>b$; then $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$, and $\mu_{C}$ satisfy Theorem 18. In fact, (i) and (iii) are trivial. Since $\mu_{B}(t)=$ $-(1 / b) t+1$ and $\mu_{C}(t)=-(1 / a) t+1$, then $\mu_{B}(p)>\mu_{C}(q) \Rightarrow$ $b / a<p / q \Rightarrow \mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)<p / q$. From Lemma 22, we have that $q>-p \Rightarrow p / q<p^{3} / q^{3}$ and, therefore, we obtain $\mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)<p^{3} / q^{3}$, which satisfies (ii).

Now, assume that $a<b$; then $\mu_{A_{i}}, \mu_{A_{i+1}}, \mu_{B}$, and $\mu_{C}$ satisfy the Theorem 19. In fact, (i) and (iii) are trivial and $\mu_{B}(p)<\mu_{C}(q) \Rightarrow b / a>p / q \Rightarrow \mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)>p / q$. From Lemma 23 we get $q<-p \Rightarrow p / q>p^{3} / q^{3}$, and, therefore, $\mu_{C}^{\prime}(q) / \mu_{B}^{\prime}(p)>p^{3} / q^{3}$, which satisfies (ii). This concludes the proof.

Corollary 25. Let $S$ be a $p$-fuzzy system and $A^{*}$ an equilibrium viable set of $S$. If $\mu_{A_{i}}$ and $\mu_{A_{i+1}}$ are trapezoidal fuzzy numbers and $\mu_{B}$ and $\mu_{C}$ are triangular fuzzy numbers, then $S$ has only one stationary point in $A^{*}$.

Proof. It is analogous to the proof of Corollary 24.

## 5. Examples

In this section we will present some computational experiments that confirm the mathematical theory presented in the previous sections. The experiments had been carried out with Matlab software. For the experiments we will consider inhibited one-dimensional p-fuzzy systems. These systems can be used to model situations where the state variable is increasing (resp., decreasing) with a carrying capacity (resp. lower bound). These situations, in population dynamics, are described by inhibited models such as Gompertz's model, Verhulst's model, von Bertallanffy's models, and Asymptotic Exponential model.

The inhibited one-dimensional p-fuzzy systems are composed of the variables "Population" (Figure 11) and "Variation" (Figure 12). The rule-base of these systems is
(1) if Population is low $\left(A_{1}\right)$ then Variation is low positive (C);


Figure 12: Variation $(\Delta): \mu_{\mathrm{C}}(t)=\mu_{B}(-t)$.
(2) if Population is medium low $\left(A_{2}\right)$ then Variation is medium positive $(D)$;
(3) if Population is medium $\left(A_{3}\right)$ then Variation is high positive ( $E$ );
(4) if Population is medium high $\left(A_{4}\right)$ then Variation is medium positive $(D)$;
(5) if Population is high $\left(A_{5}\right)$ then Variation is low positive (C);
(6) if Population is the highest $\left(A_{6}\right)$ then Variation is low negative ( $B$ ).
5.1. Example 1. In this system the membership functions of $B$ and $C$ are $\mu_{B}(t)=t+1$ and $\mu_{C}(t)=1-t$. These functions are symmetric (Figure 12). Observing the rules we can identify an equilibrium viable set, $A^{*}=[200,280]$, where $A^{*}=A_{5} \cap A_{6}$, in which membership functions are given by size:

$$
\begin{align*}
& \mu_{A_{5}}(x)= \begin{cases}\frac{1}{50}(x-150), & \text { if } 150<x \leq 200 \\
1, & \text { if } 200<x \leq 220 \\
\frac{-1}{60}(x-280), & \text { if } 220<x \leq 270 \\
0, & \text { otherwise }\end{cases}  \tag{46}\\
& \mu_{A_{6}}(x)= \begin{cases}\frac{1}{70}(x-200), & \text { if } 200<x \leq 270 \\
1, & \text { if } 270<x \leq 300 \\
0, & \text { otherwise }\end{cases}
\end{align*}
$$

A simple calculation shows that $\mu_{A_{5}} \cap \mu_{A_{6}}=243.07$, which is the stationary point of the system, as shown in Proposition 15. This is the same result as that obtained from numerical experiments in Figure 13, where it is possible to observe the solution of the p-fuzzy system with initial condition $x_{0}=50$ converging to the stationary point $x^{*}=$ 243.07 (dotted line curve).
5.2. Example 2 (Application). Losses caused by fungi attacks reach up to the amazing and worrisome figure of $20 \%$ of the total harvested fruits. Among the recognized pathogens


Figure 13: Equilibrium of the p-fuzzy systems.


Figure 14: Population (A).
of these degenerative processes are the Colletotrichum $s p$. (bitter putrefaction); Alternaria $s p$ and Fusarium sp (carpelar putrefaction); Alternaria alternata (black putrefaction) and Penicillium $s p$. (blue mould). In general, putrefaction occurs always after harvesting, when the apples are kept in boxes awaiting the refrigeration process.

Let us consider a situation where an apple box filled with approximately 3000 fruits exists and there is a rotten apple in the center of the box that will contaminate the other apples. The dispersion of the disease occurs through the contact of the rotten apple with a healthy one. If we want to model the dispersion of the disease, we can only use the intuition because the available information we have is that after $n$ days all the apples will be contaminated. We do not have in hand a "force of infection of the disease" parameter, and we know little about the possible contacts between the rotten apple and a healthy one. Thus, any mathematical model or simulation of the phenomenon only will produce an approximate result. On the other hand, if we simply use the intuition, we can formulate some relative rules to the dispersion process such as the following:

> "If the rotten apple population is "low" then the variation (incidence) of rotten apples is "small"."

Let us consider then the p-fuzzy system with the following linguistic variables:


Figure 15: Variation ( $\Delta$ ).


FIgURE 16: Solutions: p-fuzzy system and adjust model.
(i) A: "rotten apple population" (Figure 14); L: low; Ml: medium low; M: medium; medium high; H : high,
(ii) $\Delta$ "Variation" (incidence of the disease, Figure 15); S: small; M: medium; G: great,
where the rule-base is:
(1) if Population ( $A$ ) is low (L), then Variation is small (S);
(2) if Population $(A)$ is medium low $(\mathrm{L})$, then Variation is medium (M);
(3) if Population ( $A$ ) is medium (M), then Variation is great (G);
(4) if Population $(A)$ is medium high (M), then Variation is medium ( M );
(5) if Population (A) is medium (G), then Variation is small (S).

This p-fuzzy system can be used to adjust the parameters of a deterministic differential equation model of the type

$$
\begin{equation*}
\frac{d A}{d t}=r A\left(1-\frac{A}{K}\right) \tag{47}
\end{equation*}
$$

for example, by the method of least squares. In Figure 16 it is possible to observe the solution of the p-fuzzy system (dotted line curve) and the adjust model (continuous line curve).

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# Research Article 

# Generalized Fuzzy Bonferroni Harmonic Mean Operators and Their Applications in Group Decision Making 

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#### Abstract

The Bonferroni mean (BM) operator is an important aggregation technique which reflects the correlations of aggregated arguments. Based on the BM and harmonic mean operators, H. Sun and M. Sun (2012) developed the fuzzy Bonferroni harmonic mean (FBHM) and fuzzy ordered Bonferroni harmonic mean (FOBHM) operators. In this paper, we study desirable properties of these operators and extend them, by considering the correlations of any three aggregated arguments instead of any two, to develop generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator. In particular, all these operators can be reduced to aggregate interval or real numbers. Then based on the GFWBHM and GFOWBHM operators, we present an approach to multiple attribute group decision making and illustrate it with a practical example.


## 1. Introduction

Multiple attribute group decision making (MAGDM) is the common phenomenon in modern life, which is to select the optimal alternative(s) from several alternatives or to get their ranking by aggregating the performances of each alternative under several attributes, in which the aggregation techniques play an important role. Considering the relationships among the aggregated arguments, we can classify the aggregation techniques into two categories: the ones which consider the aggregated arguments dependently and the others which consider the aggregated arguments independently. For the first category, the well-known ordered weighted averaging (OWA) operator [1,2] is the representative, on the basis of which, a lot of generalizations have been developed, such as the ordered weighted geometric (OWG) operator [3-5], the ordered weighted harmonic mean (OWHM) operator [6], the continuous ordered weighted averaging (C-OWA) operator [7], the continuous ordered weighted geometric (C-OWG) operator [8]. The second category can reduce to two subcategories: the first subcategory focuses on changing the weight vector of the aggregation operators, such as the Choquet integral-based aggregation operators [9], in which
the correlations of the aggregated arguments are measured subjectively by the decision makers, and the representatives of another subcategory are the power averaging (PA) operator [10] and the power geometric (PG) operator [11], both of which allow the aggregated arguments to support each other in aggregation process, on the basis of which the weighted vector is determined. The second subcategory focuses on the aggregated arguments such as the Bonferroni mean (BM) operator [12]. Yager [13] provided an interpretation of BM operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. Beliakov et al. [14] presented a composed aggregation technique called the generalized Bonferroni mean (GBM) operator, which models the average of the conjunctive expressions and the average of remaining. In fact, they extended the BM operator by considering the correlations of any three aggregated arguments instead of any two. However, both BM operator and the GBM operator ignore some aggregation information and the weight vector of the aggregated arguments. To overcome this drawback, Xia et al. [15] developed the generalized weighted Bonferroni mean (GWBM) operator as the weighted version of the GBM operator. Based on the GBM operator and geometric
mean operator, they also developed the generalized Bonferoni geometric mean (GWBGM) operator. The fundamental characteristic of the GWBM operator is that it focuses on the group opinions, while the GWBGM operator gives more importance to the individual opinions. Because of the usefulness of the aggregation techniques, which reflect the correlations of arguments, most of them have been extended to fuzzy, intuitionistic fuzzy, or hesitant fuzzy environment [16-20].

Harmonic mean is the reciprocal of arithmetic mean of reciprocal, which is a conservative average to be used to provide for aggregation lying between the max and min operators, and is widely used as a tool to aggregate central tendency data [21]. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. However, in many situations, the input arguments take the form of triangular fuzzy numbers because of time pressure, lack of knowledge, and people's limited expertise related with problem domain. Therefore, "how to aggregate fuzzy data by using the harmonic mean?" is an interesting research topic and is worth paying attention to. So Xu [21] developed the fuzzy harmonic mean operators such as fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean (FOWHM) operator and fuzzy hybrid harmonic mean (FHHM) operator, and applied them to MAGDM. Wei [22] developed fuzzy induced ordered weighted harmonic mean (FIOWHM) operator and then, based on the FWHM and FIOWHM operators, presented the approach to MAGDM. H. Sun and M. Sun [23] further applied the BM operator to fuzzy environment, introduced the fuzzy Bonferroni harmonic mean (FBHM) operator and the fuzzy ordered Bonferroni harmonic mean (FOBHM) operator, and applied the FOBHM operator to multiple attribute decision making. In this paper, we will develop some new harmonic aggregation operators, including the generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator, and apply them to MAGDM.

In order to do this, the remainder of this paper is arranged in following sections. Section 2 first reviews some aggregation operators, including the BM, GBM, and GWBM operators. Then, some basic concepts related to triangular fuzzy numbers and some operational laws of triangular fuzzy numbers are introduced. The desirable properties of the FBHM and FOBHM operators are discussed. We extend them, by considering the correlations of any three aggregated arguments instead of any two, to develop generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator. In particular, all these operators can be reduced to aggregate interval or real numbers. Section 3 presents an approach to MAGDM based on the GFWBHM and GFOWBHM operators. Section 4 illustrates the presented approach with a practical example, verifies and shows the advantages of the presented approach, and makes a comparative study to the existing approaches. Section 5 ends the paper with some concluding remarks.

## 2. Generalized Fuzzy Bonferroni Harmonic Mean Operators

The Bonferroni mean operator was initially proposed by Bonferroni [12] and was also investigated intensively by Yager [13].

Definition 1. Let $p, q \geq 0$ and let $a_{i}(i=1,2, \ldots, n)$ be a collection of nonnegative numbers. If

$$
\begin{equation*}
\mathrm{BM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\left(\frac{1}{n(n-1)} \sum_{\substack{i, j=1 \\ i \neq j}}^{n} a_{i}^{p} a_{j}^{q}\right)^{1 /(p+q)}, \tag{1}
\end{equation*}
$$

then $\mathrm{BM}^{p, q}$ is called the Bonferroni mean (BM) operator.
Beliakov et al. [14] further extended the BM operator by considering the correlations of any three aggregated arguments instead of any two.

Definition 2. Let $p, q, r \geq 0$ and let $a_{i}(i=1,2, \ldots, n)$ be a collection of nonnegative numbers. If

$$
\begin{align*}
& \operatorname{GBM}^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
& \quad=\left(\frac{1}{n(n-1)(n-2)} \sum_{\substack{i, j, k=1 \\
i \neq j \neq k}}^{n} a_{i}^{p} a_{j}^{q} a_{k}^{r}\right)^{1 /(p+q+r)}, \tag{2}
\end{align*}
$$

then $\mathrm{GBM}^{p, q, r}$ is called the generalized Bonferroni mean (GBM) operator.

In particular, if $r=0$, then the GBM operator reduces to the BM operator. However, it is noted that both BM operator and the GBM operator do not consider the situation that $i=j$ or $j=k$ or $i=k$, and the weight vector of the aggregated arguments is not also considered. To overcome this drawback, Xia et al. [15] defined the weighted version of the GBM operator.

Definition 3. Let $p, q, r \geq 0$ and let $a_{i}(i=1,2, \ldots, n)$ be a collection of nonnegative numbers with the weight vector $\underset{\sum^{n}}{w}=\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ such that $w_{i}>0, i=1,2, \ldots, n$ and $\sum_{i=1}^{n} w_{i}=1$. If

$$
\begin{align*}
& \operatorname{GWBM}^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
& \quad=\left(\sum_{i, j, k=1}^{n} w_{i} w_{j} w_{k} a_{i}^{p} a_{j}^{q} a_{k}^{r}\right)^{1 /(p+q+r)}, \tag{3}
\end{align*}
$$

then $\mathrm{GWBM}^{p, q, r}$ is called the generalized weighted Bonferroni mean (GWBM) operator.

Some special cases can be obtained as the change of the parameters as follows.
(1) If $r=0$, then the GWBM operator reduces to the following:

$$
\begin{align*}
& \mathrm{GWBM}^{p, q, 0}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
&=\left(\sum_{i, j, k=1}^{n} w_{i} w_{j} w_{k} a_{i}^{p} a_{j}^{q}\right)^{1 /(p+q)} \\
&=\left(\sum_{i=1}^{n} w_{i} w_{j} a_{i}^{p} a_{j}^{q} \sum_{k=1}^{n} w_{k}\right)^{1 /(p+q)}  \tag{4}\\
&=\left(\sum_{i=1}^{n} w_{i} w_{j} a_{i}^{p} a_{j}^{q}\right)^{1 /(p+q)}
\end{align*}
$$

which is the weighted Bonferroni mean (WBM) operator.
(2) If $q=0$ and $r=0$, then the GWBM operator reduces to the following:

$$
\begin{align*}
& \operatorname{GWBM}^{p, 0,0}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
&=\left(\sum_{i, j, k=1}^{n} w_{i} w_{j} w_{k} a_{i}^{p}\right)^{1 / p} \\
&=\left(\sum_{i=1}^{n} w_{i} a_{i}^{p} \sum_{j=1}^{n} w_{j} \sum_{k=1}^{n} w_{k}\right)^{1 / p}  \tag{5}\\
&=\left(\sum_{i=1}^{n} w_{i} a_{i}^{p}\right)^{1 / p}
\end{align*}
$$

which is the generalized weighted averaging operator. Furthermore, in this case, let us look at the GWBM operator for some special cases of $p$.
(1) If $p=1$, the GWBM operator reduces to the weighted averaging (WA) operator.
(2) If $p \rightarrow 0$, then the GWBM operator reduces to the weighted geometric (WG) operator.
(3) If $p \rightarrow+\infty$, then the GWBM operator reduces to the max operator.
The previous aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [24], which is a widely used tool to deal with uncertainty and fuzziness, described as follows.

Definition 4 (see [24]). A triangular fuzzy number $\hat{a}$ can be defined by a triplet $\left[a^{L}, a^{M}, a^{U}\right]$. The membership function $\mu_{\hat{a}}(x)$ is defined as

$$
\mu_{\widehat{a}}(x)= \begin{cases}0, & x<a^{L} \\ \frac{x-a^{L}}{a^{M}-a^{L}}, & a^{L} \leq x \leq a^{M} \\ \frac{x-a^{U}}{a^{M}-a^{U}}, & a^{M} \leq x \leq a^{U} \\ 0, & x>a^{U}\end{cases}
$$

where $a^{U} \geq a^{M} \geq a^{L} \geq 0, a^{L}$, and $a^{U}$ stand for the lower and upper values of $\hat{a}$, respectively, and $a^{M}$ stands for the modal value [24]. In particular, if any two of $a^{L}, a^{M}$, and $a^{U}$ are equal, then $\widehat{a}$ reduces to an interval number; if all $a^{L}, a^{M}$, and $a^{U}$ are equal, then $\hat{a}$ reduces to a real number. For convenience, we let $\Omega$ be the set of all triangular fuzzy numbers.

Let $\widehat{a}=\left[a^{L}, a^{M}, a^{U}\right]$ and $\widehat{b}=\left[b^{L}, b^{M}, b^{U}\right]$ be two triangular fuzzy numbers, then some operational laws defined as follows [24]:
(1) $\widehat{a}+\widehat{b}=\left[a^{L}, a^{M}, a^{U}\right]+\left[b^{L}, b^{M}, b^{U}\right]=\left[a^{L}+b^{L}, a^{M}+\right.$ $\left.b^{M}, a^{U}+b^{U}\right]$
(2) $\lambda \hat{a}=\lambda\left[a^{L}, a^{M}, a^{U}\right]=\left[\lambda a^{L}, \lambda a^{M}, \lambda a^{U}\right]$;
(3) $\hat{a} \times \hat{b}=\left[a^{L}, a^{M}, a^{U}\right] \times\left[b^{L}, b^{M}, b^{U}\right]=\left[a^{L} b^{L}, a^{M} b^{M}\right.$, $\left.a^{U} b^{U}\right]$;
(4) $1 / \widehat{a}=1 /\left[a^{L}, a^{M}, a^{U}\right]=\left[1 / a^{U}, 1 / a^{M}, 1 / a^{L}\right]$.

In order to compare two triangular fuzzy numbers, Xu [21] provided the following definition.

Definition 5. Let $\widehat{a}=\left[a^{L}, a^{M}, a^{U}\right]$ and let $\widehat{b}=\left[b^{L}, b^{M}, b^{U}\right]$ be two triangular fuzzy numbers; then the degree of possibility of $\widehat{a} \geq \widehat{b}$ is defined as follows:

$$
\begin{align*}
& p(\widehat{a} \geq \widehat{b}) \\
&= \delta \max \left\{1-\max \left(\frac{b^{M}-a^{L}}{a^{M}-a^{L}+b^{M}-b^{L}}, 0\right), 0\right\} \\
&+(1-\delta) \max \left\{1-\max \left(\frac{b^{U}-a^{M}}{a^{U}-a^{M}+b^{U}-b^{M}}, 0\right), 0\right\}, \tag{0,1}
\end{align*}
$$

which satisfies the following properties:

$$
\begin{gather*}
0 \leq p(\hat{a} \geq \widehat{b}) \leq 1, \quad p(\widehat{a} \geq \widehat{a})=0.5, \\
p(\widehat{a} \geq \widehat{b})+p(\widehat{b} \geq \widehat{a})=1 . \tag{8}
\end{gather*}
$$

Here, $\delta$ reflects the decision maker's risk-bearing attitude. If $\delta>0.5$, then the decision maker is risk lover; if $\delta=0.5$, then the decision maker is neutral to risk; if $\delta<0.5$, then the decision maker is risk avertor.

In the following, we will give a simple procedure for ranking of the triangular fuzzy numbers $\widehat{a}_{i}(i=1,2, \ldots, n)$. First, by using (7), we compare each $\widehat{a}_{i}$ with all the $\widehat{a}_{j}(j=$ $1,2, \ldots, n)$; for simplicity, let $p_{i j}=p\left(\widehat{a}_{i} \geq \widehat{a}_{j}\right)$, and then we develop a possibility matrix $[25,26]$ as

$$
P=\left(\begin{array}{cccc}
p_{11} & p_{12} & \cdots & p_{1 n}  \tag{9}\\
p_{21} & p_{22} & \cdots & p_{2 n} \\
& & \vdots & \\
p_{n 1} & p_{n 2} & \cdots & p_{n n}
\end{array}\right)
$$

where $p_{i j} \geq 0, p_{i j}+p_{j i}=1, p_{i i}=1 / 2, i, j=1,2, \ldots, n$.

Summing all elements in each line of matrix $P$, we have $p_{i}=\sum_{j=1}^{n} p_{i j}, i=1,2, \ldots, n$. Then, in accordance with the values of $p_{i}(i=1,2, \ldots, n)$, we rank the $\widehat{a}_{i}(i=1,2, \ldots, n)$ in descending order.

To aggregate the triangular fuzzy correlated information, based on the BM and weighted harmonic mean operators, H. Sun and M. Sun [23] developed the fuzzy Bonferroni harmonic mean operator. Because this operator considers the weight vector of the aggregated arguments, we redefine this operator as fuzzy weighted Bonferroni harmonic mean operator.

Definition 6 (see [23]). Let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ be a collection of triangular fuzzy numbers, let $w=\left(w_{1}\right.$, $\left.w_{2}, \ldots, w_{n}\right)^{T}$ be the weight vector of $\widehat{a}_{i}(i=1,2, \ldots, n)$, where $w_{i}$ indicates the importance degree of $\widehat{a}_{i}$, satisfying $w_{i}>0$, $i=1,2, \ldots, n$ and $\sum_{i=1}^{n} w_{i}=1$. If

$$
\begin{align*}
& \text { FWBHM }^{p, q}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
& \qquad \begin{aligned}
= & \frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q}\right)\right)^{1 /(p+q)}} \\
= & {\left[\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\right)\right)^{1 /(p+q)}},\right.} \\
& \frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{i}^{M}\right)^{p}\left(a_{j}^{M}\right)^{q}\right)\right)^{1 /(p+q)}} \\
& \left.\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\right)\right)^{1 /(p+q)}}\right]
\end{aligned}
\end{align*}
$$

where $p, q \geq 0$, then $\mathrm{FWBHM}^{p, q}$ is called the fuzzy weighted Bonferroni harmonic mean (FWBHM) operator.

In particular, if $w=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the FWBHM operator reduces to the following:

$$
\begin{equation*}
\operatorname{FBHM}^{p, q}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right)=\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q}\right)\right)^{1 /(p+q)}} \tag{11}
\end{equation*}
$$

which we call the fuzzy Bonferroni harmonic mean (FBHM) operator.

In addition, a special case can obtained as the change of parameter. If $q=0$, then the FWBHM operator reduces to the following:

$$
\begin{aligned}
& \text { FWBHM }^{p, 0}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
& \quad=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \hat{a}_{i}^{p}\right)\right)^{1 / p}}=\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} / \hat{a}_{i}^{p}\right) \sum_{j=1}^{n} w_{j}\right)^{1 / p}} \\
& =\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} / \widehat{a}_{i}^{p}\right)\right)^{1 / p}}
\end{aligned}
$$

$$
\begin{align*}
= & {\left[\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{i}^{L}\right)^{p}\right)\right)^{1 / p}}, \frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{i}^{M}\right)^{p}\right)\right)^{1 / p}},\right.} \\
& \left.\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{i}^{U}\right)^{p}\right)\right)^{1 / p}}\right] \tag{12}
\end{align*}
$$

which we call the fuzzy weighted generalized harmonic mean (FWGHM) operator.

On the basis of the operational laws of triangular fuzzy numbers, the FWBHM operator has the following properties.

Theorem 7. Let $p, q \geq 0$, and let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=$ $1,2, \ldots, n)$ be a collection of triangular fuzzy numbers, and the following are valid.
(1) Idempotency. If all $\widehat{a}_{i}(i=1,2, \ldots, n)$ are equal, that is, $\widehat{a}_{i}=$ $\widehat{a}$, for all $i$, then

$$
\begin{equation*}
\operatorname{FWBHM}^{p, q}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right)=\widehat{a} \tag{13}
\end{equation*}
$$

(2) Boundedness. $\hat{a}^{-} \leq \operatorname{FWBHM}^{p, q}\left(\hat{a}_{1}, \hat{a}_{2}, \ldots, \hat{a}_{n}\right) \leq \widehat{a}^{+}$, where $\hat{a}^{-}=\left[\min _{i}\left\{a_{i}^{L}\right\}, \min _{i}\left\{a_{i}^{M}\right\}, \min _{i}\left\{a_{i}^{U}\right\}\right]$ and $\hat{a}^{+}=$ $\left[\max _{i}\left\{a_{i}^{L}\right\}, \max _{i}\left\{a_{i}^{M}\right\}, \max _{i}\left\{a_{i}^{U}\right\}\right]$.
(3) Commutativity. Let $\hat{a}_{i}^{\prime}=\left[a_{i}^{\prime L}, a_{i}^{\prime M}, a_{i}^{\prime U}\right](i=1,2, \ldots, n) b e$ a collection of triangular fuzzy numbers, and then

$$
\begin{align*}
& \text { FWBHM }^{p, q}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
& \quad=\operatorname{FWBHM}^{p, q}\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right), \tag{14}
\end{align*}
$$

where $\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right)$ is any permutation of $\left(\hat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right)$.
Proof. Since (2) can be proven easily, we prove (1) and (3) as follows.
(1) Since $\widehat{a}_{i}=\widehat{a}$, we have

$$
\begin{align*}
& \text { FWBHM }^{p, q}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widehat{a}^{p} \widehat{a}^{q}\right)\right)^{1 /(p+q)}} \\
&=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widehat{a}^{p+q}\right)\right)^{1 /(p+q)}}  \tag{15}\\
&=\frac{\widehat{a}}{\left(\sum_{i=1}^{n} w_{i} \sum_{j=1}^{n} w_{j}\right)^{1 /(p+q)}} \\
&=\widehat{a} .
\end{align*}
$$

(3) Since $\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right)$ is any permutation of $\left(\hat{a}_{1}, \hat{a}_{2}\right.$, $\left.\ldots, \widehat{a}_{n}\right)$, then

$$
\begin{align*}
& \text { FWBHM }^{p, q}\left(\hat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \hat{a}_{i}^{p} a_{j}^{q}\right)\right)^{1 /(p+q)}} \\
&=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(\hat{a}_{i}^{\prime}\right)^{p}\left(\hat{a}_{j}^{\prime}\right)^{q}\right)\right)^{1 /(p+q)}}  \tag{16}\\
&=\operatorname{FWBHM}^{p, q}\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right) .
\end{align*}
$$

In particular, if the triangular fuzzy numbers $\widehat{a}_{i}=\left[a_{i}^{L}\right.$, $\left.a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ reduce to the interval numbers $\widetilde{a}_{i}=\left[a_{i}^{L}, a_{i}^{U}\right](i=1,2, \ldots, n)$, then the FWBHM operator (10) reduces to the uncertain weighted Bonferroni harmonic mean (UWBHM) operator as follows:

$$
\begin{align*}
& \text { UWBHM }^{p, q}\left(\widetilde{a}_{1}, \widetilde{a}_{2}, \ldots, \widetilde{a}_{n}\right) \\
& \qquad \begin{aligned}
& \frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widetilde{a}_{i}^{p} \widetilde{a}_{j}^{q}\right)\right)^{1 /(p+q)}} \\
& =\left[\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\right)\right)^{1 /(p+q)}},\right. \\
& \left.\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\right)\right)^{1 /(p+q)}}\right] .
\end{aligned}
\end{align*}
$$

If $w=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the UWBHM operator reduces to the uncertain Bonferroni harmonic mean (UBHM) operator as follows:

$$
\begin{align*}
& \mathrm{UBHM}^{p, q}\left(\widetilde{a}_{1}, \tilde{a}_{2}, \ldots, \widetilde{a}_{n}\right) \\
&=\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 / \tilde{a}_{i}^{p} \widetilde{a}_{j}^{q}\right)\right)^{1 /(p+q)}} \\
&= {\left[\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\right)\right)^{1 /(p+q)}},\right.}  \tag{18}\\
&\left.\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\right)\right)^{1 /(p+q)}}\right] .
\end{align*}
$$

If $a_{i}^{L}=a_{i}^{U}=a_{i}$, for all $i$, then the UWBHM operator (17) reduces to the weighted Bonferroni harmonic mean (WBHM) operator as follows:

$$
\begin{equation*}
\operatorname{WBHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / a_{i}^{p} a_{j}^{q}\right)\right)^{1 /(p+q)}} \tag{19}
\end{equation*}
$$

In this case, if $w=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the WBHM operator reduces to the Bonferroni harmonic mean (BHM) operator:

$$
\begin{equation*}
\operatorname{BHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 / a_{i}^{p} a_{j}^{q}\right)\right)^{1 /(p+q)}} . \tag{20}
\end{equation*}
$$

Example 8. Given a collection of triangular fuzzy numbers: $\widehat{a}_{1}=[2,3,4], \widehat{a}_{2}=[1,2,4], \widehat{a}_{3}=[2,4,6], \widehat{a}_{4}=[1,3,5]$, let $w=$ ( $0.3,0.1,0.2,0.4)^{T}$ be the weight vector of $\widehat{a}_{i}(i=1,2,3,4)$; then, by FWBHM operator (10) (let $p=q=2$ ), we have

$$
\begin{align*}
& \text { FWBHM }^{2,2}\left(\widehat{a}_{1}, \widehat{a}_{2}, \widehat{a}_{3}, \widehat{a}_{4}\right) \\
& =\left[\frac{1}{\left(\sum_{i, j=1}^{4}\left(w_{i} w_{j} /\left(a_{i}^{L}\right)^{2}\left(a_{j}^{L}\right)^{2}\right)\right)^{1 / 4}},\right. \\
& \frac{1}{\left(\sum_{i, j=1}^{4}\left(w_{i} w_{j} /\left(a_{i}^{M}\right)^{2}\left(a_{j}^{M}\right)^{2}\right)\right)^{1 / 4}},  \tag{21}\\
& \left.\frac{1}{\left(\sum_{i, j=1}^{4}\left(w_{i} w_{j} /\left(a_{i}^{U}\right)^{2}\left(a_{j}^{U}\right)^{2}\right)\right)^{1 / 4}}\right] \\
& =
\end{align*}
$$

Based on the OWA and FWBHM operators and Definition 5, we define fuzzy ordered weighted Bonferroni harmonic mean (FOWBHM) operator as follows.

Definition 9. Let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ be a collection of triangular fuzzy numbers. For $p, q \geq 0$, a fuzzy ordered weighted Bonferroni harmonic mean (FOWBHM) operator of dimension $n$ is a mapping FOWBHM ${ }^{p, q}: \Omega^{n} \rightarrow$ $\Omega$, that has an associated vector $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ such that $\omega_{i} \geq 0$ and $\sum_{i=1}^{n} \omega_{i}=1$. Furthermore,

$$
\operatorname{FOWBHM}^{p, q}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right)
$$

$$
=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \hat{a}_{\sigma(i)}^{p} \hat{a}_{\sigma(j)}^{q}\right)\right)^{1 /(p+q)}}
$$

$$
\begin{equation*}
=\left[\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{\sigma(i)}^{L}\right)^{p}\left(a_{\sigma(j)}^{L}\right)^{q}\right)\right)^{1 /(p+q)}}\right. \tag{22}
\end{equation*}
$$

$$
\left.\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{\sigma(i)}^{U}\right)^{p}\left(a_{\sigma(j)}^{U}\right)^{q}\right)\right)^{1 /(p+q)}}\right]
$$

where $\widehat{a}_{\sigma(i)}=\left[a_{\sigma(i)}^{L}, a_{\sigma(i)}^{M}, a_{\sigma(i)}^{U}\right](i=1,2, \ldots, n)$, and $(\sigma(1)$, $\sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2, \ldots, n)$ such that $\widehat{a}_{\sigma(i-1)} \geq \widehat{a}_{\sigma(i)}$ for all $i$.

However, if there is a tie between $\widehat{a}_{i}$ and $\widehat{a}_{j}$, then we replace each of $\widehat{a}_{i}$ and $\widehat{a}_{j}$ by their average $\left(\widehat{a}_{i}+\widehat{a}_{j}\right) / 2$ in process of aggregation. If $k$ items are tied, then we replace these by $k$ replicas of their average. The weighting vector $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ can be determined by using some weights determining methods like the normal distribution based method; see [27-29] for more details.

If $\omega=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the FOWBHM operator reduces to the FBHM operator; in addition, if $q=0$, then the FOWBHM operator reduces to the following:

FOWBHM $^{p, 0}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right)$

$$
\begin{align*}
& =\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widehat{a}_{\sigma(i)}^{p}\right)\right)^{1 / p}}=\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} / \widehat{a}_{\sigma(i)}^{p}\right) \sum_{j=1}^{n} w_{j}\right)^{1 / p}} \\
& =\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} / \widehat{a}_{\sigma(i)}^{p}\right)\right)^{1 / p}} \\
& =\left[\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{\sigma(i)}^{L}\right)^{p}\right)\right)^{1 / p}}, \frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{\sigma(i)}^{M}\right)^{p}\right)\right)^{1 / p}},\right. \\
& \left.\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} /\left(a_{\sigma(i)}^{U}\right)^{p}\right)\right)^{1 / p}}\right], \tag{23}
\end{align*}
$$

which we call the fuzzy ordered weighted generalized harmonic mean (FOWGHM) operator.

In particular, if the triangular fuzzy numbers $\widehat{a}_{i}=$ $\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ reduce to the interval numbers $\widetilde{a}_{i}=\left[a_{i}^{L}, a_{i}^{U}\right](i=1,2, \ldots, n)$, then the FOWBHM operator reduces to the uncertain ordered weighted Bonferroni harmonic mean (UOWBHM) operator as follows:
$\operatorname{UOWBHM}^{p, q}\left(\widetilde{a}_{i}, \widetilde{a}_{2}, \ldots, \widetilde{a}_{n}\right)$

$$
\begin{align*}
& =\frac{1}{\left(\sum_{i, j=1}^{n}\left(\omega_{i} \omega_{j} / \widetilde{a}_{\sigma(i)}^{p} \widetilde{a}_{\sigma(j)}^{q}\right)\right)^{1 /(p+q)}} \\
& =\left[\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{\sigma(i)}^{L}\right)^{p}\left(a_{\sigma(j)}^{L}\right)^{q}\right)\right)^{1 /(p+q)}},\right. \tag{24}
\end{align*}
$$

$$
\left.\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} /\left(a_{\sigma(i)}^{U}\right)^{p}\left(a_{\sigma(j)}^{U}\right)^{q}\right)\right)^{1 /(p+q)}}\right]
$$

where $\widetilde{a}_{\sigma(i)}=\left[a_{\sigma(i)}^{L}, a_{\sigma(i)}^{U}\right]$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2, \ldots, n)$ such that $\widetilde{a}_{\sigma(i-1)} \geq \widetilde{a}_{\sigma(i)}$ for all $i$.

If there is a tie between $\widetilde{a}_{i}$ and $\widetilde{a}_{j}$, then we replace each of $\widetilde{a}_{i}$ and $\widetilde{a}_{j}$ by their average $\left(\widetilde{a}_{i}+\widetilde{a}_{j}\right) / 2$ in process of aggregation. If $k$ items are tied, then we replace these by $k$ replicas of their average.

If $a_{i}^{L}=a_{i}^{U}=a_{i}$, for all $i$, then the UOWBHM operator reduces to the ordered weighted Bonferroni harmonic mean (OWBHM) operator as follows:

$$
\begin{equation*}
\operatorname{OWBHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\frac{1}{\left(\sum_{i, j=1}^{n}\left(\omega_{i} \omega_{j} / b_{i}^{p} b_{j}^{q}\right)\right)^{1 /(p+q)}}, \tag{25}
\end{equation*}
$$

where $b_{i}$ is the $i$ th largest of $a_{i}(i=1,2, \ldots, n)$. The OWBHM operator (25) has some special cases.
(1) If $\omega=(1,0, \ldots, 0)^{T}$, then

$$
\begin{equation*}
\operatorname{OWBHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\max \left\{a_{i}\right\}=b_{1} . \tag{26}
\end{equation*}
$$

(2) If $\omega=(0,0, \ldots, 1)^{T}$, then

$$
\begin{equation*}
\text { OWBHM }^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\min \left\{a_{i}\right\}=b_{n} . \tag{27}
\end{equation*}
$$

(3) If $\omega=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then

$$
\operatorname{OWBHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right)
$$

$$
=\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 / b_{i}^{p} b_{j}^{q}\right)\right)^{1 /(p+q)}}
$$

$$
\begin{equation*}
=\frac{1}{\left(\left(1 / n^{2}\right) \sum_{i, j=1}^{n}\left(1 / a_{i}^{p} a_{j}^{q}\right)\right)^{1 /(p+q)}} \tag{28}
\end{equation*}
$$

$$
=\operatorname{BHM}^{p, q}\left(a_{1}, a_{2}, \ldots, a_{n}\right) .
$$

Example 10. Let $\widehat{a}_{1}=[3,4,6], \widehat{a}_{2}=[1,2,4], \widehat{a}_{3}=[2,4,5]$, $\widehat{a}_{4}=[3,5,6]$, and $\widehat{a}_{5}=[2,5,7]$ be a collection of triangular fuzzy numbers. To rank these triangular fuzzy numbers, we first compare each triangular fuzzy number $\widehat{a}_{i}$ with all triangular fuzzy numbers $\widehat{a}_{j}(j=1,2,3,4,5)$ by using (7) (without loss of generality, set $\delta=0.5)$; let $p_{i j}=p\left(\widehat{a}_{i} \geq\right.$ $\left.\widehat{a}_{j}\right)(i, j=1,2,3,4,5)$, then we utilize these possibility degrees to construct the following matrix $P=\left(p_{i j}\right)_{5 \times 5}$ :

$$
P=\left(\begin{array}{ccccc}
0.500 & 1 & 0.667 & 0.333 & 0.375  \tag{29}\\
0 & 0.500 & 0 & 0 & 0 \\
0.333 & 1 & 0.500 & 0.125 . & 0.200 \\
0.667 & 1 & 0.875 & 0.500 & 0.467 \\
0.625 & 1 & 0.800 & 0.533 & 0.500
\end{array}\right) .
$$

Summing all elements in each line of matrix $P$, we have

$$
\begin{gather*}
p_{1}=2.875, \quad p_{2}=0.500, \quad p_{3}=2.158  \tag{30}\\
p_{4}=3.509, \quad p_{5}=3.458
\end{gather*}
$$

and then we rank the triangular fuzzy numbers $\widehat{a}_{i}(i=$ $1,2,3,4,5)$ in descending order in accordance with the values of $p_{i}(i=1,2,3,4,5)$ as follows:

$$
\begin{gather*}
\hat{a}_{\sigma(1)}=\widehat{a}_{4}, \quad \hat{a}_{\sigma(2)}=\widehat{a}_{5}, \quad \widehat{a}_{\sigma(3)}=\hat{a}_{1},  \tag{31}\\
\widehat{a}_{\sigma(4)}=\widehat{a}_{3}, \quad \widehat{a}_{\sigma(5)}=\widehat{a}_{2} .
\end{gather*}
$$

Suppose that the weighting vector of the FOWBHM operator is $\omega=(0.1117,0.2365,0.3036,0.2365,0.1117)^{T}$ (derived by the normal distribution based method [27]), and then by (22) (let $p=q=2$ ), we get

Both FWBHM and FOWBHM operators, however, can only deal with the situation in which there are correlations between any two aggregated arguments, but not the situation in which there exist connections among any three aggregated arguments. To solve this issue, motivated by Definition 3, we define the following.

Definition 11. Let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ be a collection of triangular fuzzy numbers and let $w=$ $\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}$ be the weight vector of $\widehat{a}_{i}(i=1,2, \ldots, n)$, where $w_{i}$ indicates the importance degree of $\widehat{a}_{i}$, satisfying $w_{i}>0, i=1,2, \ldots, n$ and $\sum_{i=1}^{n} w_{i}=1$. For $p, q, r \geq 0$, if

$$
\operatorname{GFWBHM}^{p, q, r}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right)
$$

$$
\begin{aligned}
& =\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q} \widehat{a}_{k}^{r}\right)\right)^{1 /(p+q+r)}} \\
& =\left[\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\left(a_{k}^{L}\right)^{r}\right)\right)^{1 /(p+q+r)}},\right.
\end{aligned}
$$

$$
\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{M}\right)^{p}\left(a_{j}^{M}\right)^{q}\left(a_{k}^{M}\right)^{r}\right)\right)^{1 /(p+q+r)}}
$$

$$
\begin{equation*}
\left.\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\left(a_{k}^{U}\right)^{r}\right)\right)^{1 /(p+q+r)}}\right] \tag{33}
\end{equation*}
$$

then GFWBHM ${ }^{p, q, r}$ is called generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator.

$$
\begin{align*}
& \text { FOWBHM }^{2,2}\left(\widehat{a}_{1}, \hat{a}_{2}, \widehat{a}_{3}, \widehat{a}_{4}, \widehat{a}_{5}\right) \\
& =\left[\frac{1}{\left(\sum_{i, j=1}^{5}\left(\omega_{i} \omega_{j} /\left(a_{\sigma(i)}^{L}\right)^{2}\left(a_{\sigma(j)}^{L}\right)^{2}\right)\right)^{1 / 4}},\right. \\
& \frac{1}{\left(\sum_{i, j=1}^{5}\left(\omega_{i} \omega_{j} /\left(a_{\sigma(i)}^{M}\right)^{2}\left(a_{\sigma(j)}^{M}\right)^{2}\right)\right)^{1 / 4}},  \tag{32}\\
& \left.\frac{1}{\left(\sum_{i, j=1}^{5}\left(\omega_{i} \omega_{j} /\left(a_{\sigma(i)}^{U}\right)^{2}\left(a_{\sigma(j)}^{U}\right)^{2}\right)\right)^{1 / 4}}\right] . \\
& =[1.901,3.632,5.509] \text {. }
\end{align*}
$$

In particular, if $w=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the GFWBHM operator reduces to the following:

$$
\begin{align*}
& \operatorname{GFBHM}^{p, q, r}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q} \widehat{a}_{k}^{r}\right)\right)^{1 /(p+q+r)}}, \tag{34}
\end{align*}
$$

which we call the generalized fuzzy Bonferroni harmonic mean (GFBHM) operator.

In addition, some special cases can be obtained as the change of parameters.
(1) If $r=0$, then the GFWBHM operator reduces to

$$
\begin{align*}
& \text { GFWBHM }^{p, q, 0}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q}\right)\right)^{1 /(p+q)}} \\
&=\frac{1}{\left(\left(\sum_{k=1}^{n} w_{k}\right) \sum_{i, j=1}^{n}\left(w_{i} w_{j} / \widehat{a}_{i}^{p} \widehat{a}_{j}^{q}\right)\right)^{1 /(p+q)}}  \tag{35}\\
&=\frac{1}{\left(\sum_{i, j=1}^{n}\left(w_{i} w_{j} / \hat{a}_{i}^{p} \widehat{a}_{j}^{q}\right)\right)^{1 /(p+q)}},
\end{align*}
$$

which is the FWBHM operator.
(2) If $q=0$ and $r=0$, then the GFWBHM operator reduces to

$$
\begin{align*}
& \text { GFWBHM }^{p, 0,0}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} / \widehat{a}_{i}^{p}\right)\right)^{1 / p}} \\
&=\frac{1}{\left(\left(\sum_{j=1}^{n} w_{j}\right)\left(\sum_{k=1}^{n} w_{k}\right) \sum_{i=1}^{n}\left(w_{i} / \widehat{a}_{i}^{p}\right)\right)^{1 / p}}  \tag{36}\\
&=\frac{1}{\left(\sum_{i=1}^{n}\left(w_{i} / \widehat{a}_{i}^{p}\right)\right)^{1 / p}},
\end{align*}
$$

which is FWGHM operator. In this case, if $p=1$, then FWGHM operator reduces to FWHM operator.

Similar to the FWBHM operator, the GFWBHM operator has the following properties.

Theorem 12. Let $p, q, r \geq 0$, and let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=$ $1,2, \ldots, n$ ) be a collection of triangular fuzzy numbers, and the following are valid.
(1) Idempotency. If all $\widehat{a}_{i}(i=1,2, \ldots, n)$ are equal, that is, $\widehat{a}_{i}=$ $\hat{a}$, for all $i$, then

$$
\begin{equation*}
\operatorname{GFWBHM}^{p, q, r}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right)=\widehat{a} . \tag{37}
\end{equation*}
$$

(2) Boundedness. $\hat{a}^{-} \leq$GFWBHM $^{p, q, r}\left(\widehat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right) \leq$ $\hat{a}^{+}$, where $\widehat{a}^{-}=\left[\min _{i}\left\{a_{i}^{L}\right\}, \min _{i}\left\{a_{i}^{M}\right\}, \min _{i}\left\{a_{i}^{U}\right\}\right]$ and $\widehat{a}^{+}=$ $\left[\max _{i}\left\{a_{i}^{L}\right\}, \max _{i}\left\{a_{i}^{M}\right\}, \max _{i}\left\{a_{i}^{U}\right\}\right]$.

$$
\begin{equation*}
\widehat{a}^{+}=\left[\max _{i}\left\{a_{i}^{L}\right\}, \max _{i}\left\{a_{i}^{M}\right\}, \max _{i}\left\{a_{i}^{U}\right\}\right] . \tag{38}
\end{equation*}
$$

(3) Commutativity. Let $\vec{a}_{i}^{\prime}=\left[a_{i}^{\prime L}, a_{i}^{\prime M}, a_{i}^{\prime U}\right](i=1,2, \ldots, n) b e$ a collection of triangular fuzzy numbers, and then

$$
\begin{align*}
& \text { GFWBHM }^{p, q, r}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
&=\text { GFWBHM }^{p, q, r}\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right), \tag{39}
\end{align*}
$$

where $\left(\hat{a}_{1}^{\prime}, \hat{a}_{2}^{\prime}, \ldots, \hat{a}_{n}^{\prime}\right)$ is any permutation of $\left(\hat{a}_{1}, \hat{a}_{2}, \ldots, \widehat{a}_{n}\right)$.
In particular, if the triangular fuzzy numbers $\widehat{a}_{i}=$ $\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ reduce to the interval numbers $\widetilde{a}_{i}=\left[a_{i}^{L}, a_{i}^{U}\right](i=1,2, \ldots, n)$, then the GFWBHM operator (24) reduces to the generalized uncertain weighted Bonferroni harmonic mean (GUWBHM) operator as follows:

$$
\begin{align*}
& \operatorname{GUWBHM}^{p, q, r}\left(\widetilde{a}_{1}, \widetilde{a}_{2}, \ldots, \widetilde{a}_{n}\right) \\
& =\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} / \widetilde{a}_{i}^{p} \widetilde{a}_{j}^{q} \widetilde{a}_{k}^{r}\right)\right)^{1 /(p+q+r)}} \\
& =\left[\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\left(a_{k}^{L}\right)^{r}\right)\right)^{1 /(p+q+r)}},\right. \\
& \left.\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\left(a_{k}^{U}\right)^{r}\right)\right)^{1 /(p+q+r)}}\right] . \tag{40}
\end{align*}
$$

If $w=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the GUWBHM operator reduces to the generalized uncertain Bonferroni harmonic mean (GUBHM):

$$
\begin{align*}
& \operatorname{GUBHM}^{p, q, r}\left(\widetilde{a}_{1}, \tilde{a}_{2}, \ldots, \widetilde{a}_{n}\right) \\
& \quad=\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 / \widetilde{a}_{i}^{p} \widetilde{a}_{j}^{q} \widetilde{a}_{k}^{r}\right)\right)^{1 /(p+q+r)}} \\
& =\left[\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 /\left(a_{i}^{L}\right)^{p}\left(a_{j}^{L}\right)^{q}\left(a_{k}^{L}\right)^{r}\right)\right)^{1 /(p+q+r)}},\right. \\
& \left.\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 /\left(a_{i}^{U}\right)^{p}\left(a_{j}^{U}\right)^{q}\left(a_{k}^{U}\right)^{r}\right)\right)^{1 /(p+q+r)}}\right] . \tag{41}
\end{align*}
$$

Furthermore, if $a_{i}^{L}=a_{i}^{U}=a_{i}$, for all $i$, then the GUWBHM operator reduces to the generalized weighted Bonferroni harmonic mean (GWBHM) operator:

$$
\begin{align*}
& \text { GWBHM }^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
& \quad=\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(w_{i} w_{j} w_{k} / a_{i}^{p} a_{j}^{q} a_{k}^{r}\right)\right)^{1 /(p+q+r)}} . \tag{42}
\end{align*}
$$

In this case, if $p=1$ and $q=r=0$, the GWBHM operator reduces to the weighted harmonic mean (WHM) operator.

Example 13. Consider the four triangular fuzzy numbers $\widehat{a}_{i}$ and their weight vector $w$ given in Example 8. Then by the GFWBHM operator (33) (without of generalization, let $p=$ $q=r=3$ ), we have

$$
\begin{align*}
& \text { GFWBHM }^{3,3,3}\left(\widehat{a}_{1}, \widehat{a}_{2}, \widehat{a}_{3}, \widehat{a}_{4}\right) \\
& =\left[\frac{1}{\left(\sum_{i, j, k=1}^{4}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{L}\right)^{3}\left(a_{j}^{L}\right)^{3}\left(a_{k}^{L}\right)^{3}\right)\right)^{1 / 9}}\right. \\
& \frac{1}{\left(\sum_{i, j, k=1}^{4}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{M}\right)^{3}\left(a_{j}^{M}\right)^{3}\left(a_{k}^{M}\right)^{3}\right)\right)^{1 / 9}}, \\
& \left.\frac{1}{\left(\sum_{i, j, k=1}^{4}\left(w_{i} w_{j} w_{k} /\left(a_{i}^{U}\right)^{3}\left(a_{j}^{U}\right)^{3}\left(a_{k}^{U}\right)^{3}\right)\right)^{1 / 9}}\right] \\
& =[1.21,2.89,4.59] . \tag{43}
\end{align*}
$$

Definition 14. Let $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ be a collection of triangular fuzzy numbers. For $p, q, r \geq 0$, a generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator of dimension $n$ is a mapping GFOWBHM ${ }^{p, q, r}: \Omega^{n} \rightarrow \Omega$, that has an associated vector $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)^{T}$ such that $\omega_{i} \geq 0$ and $\sum_{i=1}^{n} \omega_{i}=1$. Furthermore,

$$
\begin{align*}
& \text { GFOWBHM }^{p, q_{2} r}\left(\widehat{a}_{1}, \widehat{a}_{2}, \ldots, \widehat{a}_{n}\right) \\
& =\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} / \hat{a}_{\sigma(i)}^{p} \hat{a}_{\sigma(j)}^{q} \hat{a}_{\sigma(k)}^{r}\right)\right)^{1 /(p+q+r)}} \\
& =\left[\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} /\left(a_{\sigma(i)}^{L}\right)^{p}\left(a_{\sigma(j)}^{L}\right)^{q}\left(a_{\sigma(k)}^{L}\right)^{r}\right)\right)^{1 /(p+q+r)}},\right. \\
& \frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} /\left(a_{\sigma(i)}^{M}\right)^{p}\left(a_{\sigma(j)}^{M}\right)^{q}\left(a_{\sigma(k)}^{M}\right)^{r}\right)\right)^{1 /(p+q+r)}}, \\
& \left.\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} /\left(a_{\sigma(i)}^{U}\right)^{p}\left(a_{\sigma(j)}^{U}\right)^{q}\left(a_{\sigma(k)}^{U}\right)^{r}\right)\right)^{1 /(p+q+r)}}\right], \tag{44}
\end{align*}
$$

where $\widehat{a}_{\sigma(i)}=\left[a_{\sigma(i)}^{L}, a_{\sigma(i)}^{M}, a_{\sigma(i)}^{U}\right](i=1,2, \ldots, n)$, and $(\sigma(1)$, $\sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2, \ldots, n)$ such that $\widehat{a}_{\sigma(i-1)} \geq \widehat{a}_{\sigma(i)}$ for all $i$.

However, if there is a tie between $\widehat{a}_{i}$ and $\widehat{a}_{j}$, then we replace each of $\widehat{a}_{i}$ and $\widehat{a}_{j}$ by their average $\left(\widehat{a}_{i}+\widehat{a}_{j}\right) / 2$ in process of aggregation. If $k$ items are tied, then we replace these by $k$ replicas of their average.

If $\omega=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then the GFOWBHM operator reduces to the GFBHM operator. Moreover, some special cases can be obtained as the change of parameters. If $r=0$, then the GFOWBHM operator reduces to FOWBHM operator; if $r=0$ and $q=0$, then GFOWBHM operator reduces to FOWGHM operator. In particular, if the triangular fuzzy numbers $\widehat{a}_{i}=\left[a_{i}^{L}, a_{i}^{M}, a_{i}^{U}\right](i=1,2, \ldots, n)$ reduce to the interval numbers $\widetilde{a}_{i}=\left[a_{i}^{L}, a_{i}^{U}\right](i=1,2, \ldots, n)$, then the GFOWBHM operator reduces to the generalized uncertain ordered weighted Bonferroni harmonic mean (GUOWBHM) operator:
GUOWBHM $^{p, q, r}\left(\widetilde{a}_{i}, \widetilde{a}_{2}, \ldots, \widetilde{a}_{n}\right)$

$$
\begin{align*}
& =\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} / \widetilde{a}_{\sigma(i)}^{p} \widetilde{a}_{\sigma(j)}^{q} \widetilde{a}_{\sigma(k)}^{r}\right)\right)^{1 /(p+q+r)}} \\
& =\left[\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} /\left(a_{\sigma(i)}^{L}\right)^{p}\left(a_{\sigma(j)}^{L}\right)^{q}\left(a_{\sigma(k)}^{L}\right)^{r}\right)\right)^{1 /(p+q+r)}},\right. \\
& \left.\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} /\left(a_{\sigma(i)}^{U}\right)^{p}\left(a_{\sigma(j)}^{U}\right)^{q}\left(a_{\sigma(k)}^{U}\right)^{r}\right)\right)^{1 /(p+q+r)}}\right], \tag{45}
\end{align*}
$$

where $\widetilde{a}_{\sigma(i)}=\left[a_{\sigma(i)}^{L}, a_{\sigma(i)}^{U}\right]$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1,2, \ldots, n)$ such that $\widetilde{a}_{\sigma(i-1)} \geq \widetilde{a}_{\sigma(i)}$ for all $i$.

If $a_{i}^{L}=a_{i}^{U}=a_{i}$, for all $i=1,2, \ldots, n$, then the GUOWBHM operator reduces to the generalized ordered weighted Bonferroni harmonic mean (GOWBHM) operator:

$$
\begin{align*}
& \text { GOWBHM }^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
&=\frac{1}{\left(\sum_{i, j, k=1}^{n}\left(\omega_{i} \omega_{j} \omega_{k} / b_{i}^{p} b_{j}^{q} b_{k}^{r}\right)\right)^{1 /(p+q+r)}}, \tag{46}
\end{align*}
$$

where $b_{i}$ is the $i$ th largest of $a_{i}(i=1,2, \ldots, n)$. In this case, if $p=1$ and $q=r=0$, then the GOWBHM operator reduces to the ordered weighted harmonic mean (OWHM) operator.

The GOWBHM operator (46) has some special cases.
(1) If $\omega=(1,0, \ldots, 0)^{T}$, then

$$
\begin{equation*}
\text { GOWBHM }^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\max \left\{a_{i}\right\}=b_{1} \tag{47}
\end{equation*}
$$

(2) If $\omega=(0,0, \ldots, 1)^{T}$, then

$$
\begin{equation*}
\text { GOWBHM }^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\min \left\{a_{i}\right\}=b_{n} \tag{48}
\end{equation*}
$$

(3) If $\omega=(1 / n, 1 / n, \ldots, 1 / n)^{T}$, then

$$
\begin{align*}
& \operatorname{GOWBHM}^{p, q, r}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \\
&=\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 / b_{i}^{p} b_{j}^{q} b_{k}^{r}\right)\right)^{1 /(p+q+r)}}  \tag{49}\\
&=\frac{1}{\left(\left(1 / n^{3}\right) \sum_{i, j, k=1}^{n}\left(1 / a_{i}^{p} a_{j}^{q} a_{k}^{r}\right)\right)^{1 /(p+q+r)}},
\end{align*}
$$

In the following section, we will apply the developed operators to multiple attribute group decision making.

## 3. An Approach to Multiple Attribute Group Decision Making with Triangular Fuzzy Information

For a group decision making with triangular fuzzy information, let $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be a discrete set of $n$ alternatives, let $G=\left\{G_{1}, G_{2}, \ldots, G_{m}\right\}$ be the set of $m$ attributes, whose weight vector is $w=\left(w_{1}, w_{2}, \ldots, w_{m}\right)^{T}$ with $w_{i} \geq 0$ and $\sum_{i=1}^{m} w_{i}=1$, and let $D=\left\{d_{1}, d_{2}, \ldots, d_{s}\right\}$ be the set of decision makers, whose weight vector is $v=\left(v_{1}, v_{2}, \ldots, v_{s}\right)^{T}$, where $v_{k} \geq 0$ and $\sum_{k=1}^{s} v_{k}=1$. Suppose that $A^{(k)}=\left(\widehat{a}_{i j}^{(k)}\right)_{m \times n}$ is the decision matrix, where $\widehat{a}_{i j}^{(k)}=\left[a_{i j}^{L(k)}, a_{i j}^{M(k)}, a_{i j}^{U(k)}\right]$ is an attribute value, which takes the form of triangular fuzzy number, of the alternative $x_{j} \in X$ with respect to the attribute $G_{i} \in G$.

In the following, we apply the GFWBHM and GFOWBHM operators to group decision making with triangular fuzzy information.

Step 1. Normalize each attribute value $\widehat{a}_{i j}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)}=\left(\widehat{r}_{i j}^{(k)}\right)_{m \times n}$ $\left(\hat{r}_{i j}^{(k)}=\left[r_{i j}^{L(k)}, r_{i j}^{M(k)}, r_{i j}^{U(k)}\right]\right)$ using the following formulas:
$\widehat{r}_{i j}^{(k)}=\frac{\widehat{a}_{i j}^{(k)}}{\sum_{j=1}^{n} \widehat{a}_{i j}^{(k)}}$

$$
=\left[\frac{a_{i j}^{L(k)}}{\sum_{j=1}^{n} a_{i j}^{U(k)}}, \frac{a_{i j}^{M(k)}}{\sum_{j=1}^{n} a_{i j}^{M(k)}}, \frac{a_{i j}^{U(k)}}{\sum_{j=1}^{n} a_{i j}^{L(k)}}\right]
$$

for benefit attribute $G_{i}, i=1,2, \ldots, m$,

$$
j=1,2, \ldots, n, k=1,2, \ldots, s
$$

$$
\widehat{r}_{i j}^{(k)}=\frac{1 / \widehat{a}_{i j}^{(k)}}{\sum_{j=1}^{n}\left(1 / \widehat{a}_{i j}^{(k)}\right)}
$$

$$
=\left[\frac{1 / a_{i j}^{U(k)}}{\sum_{j=1}^{n}\left(1 / a_{i j}^{L(k)}\right)}, \frac{1 / a_{i j}^{M(k)}}{\sum_{j=1}^{n}\left(1 / a_{i j}^{M(k)}\right)}, \frac{1 / a_{i j}^{L(k)}}{\sum_{j=1}^{n}\left(1 / a_{i j}^{U(k)}\right)}\right]
$$

for cost attribute $G_{i}, i=1,2, \ldots, m$,

$$
\begin{equation*}
j=1,2, \ldots, n, k=1,2, \ldots, s \tag{51}
\end{equation*}
$$

Step 2. Utilize the GFWBHM operator (33) as follows:

$$
\begin{align*}
& \widehat{r}_{j}^{(k)}=\text { GFWBHM }^{p, q, r}\left(\widehat{r}_{1 j}^{(k)}, \widehat{r}_{2 j}^{(k)}, \ldots, \widehat{r}_{m j}^{(k)}\right) \\
&= \frac{1}{\left(\sum_{i, h, l=1}^{m}\left(w_{i} w_{h} w_{l} /\left(\hat{r}_{i j}^{(k)}\right)^{p}\left(\hat{r}_{h j}^{(k)}\right)^{q}\left(\widehat{r}_{l j}^{(k)}\right)^{r}\right)\right)^{1 /(p+q+r)}} \\
&= {\left[\left(\sum_{i, h, l=1}^{m}\left(\frac{w_{i} w_{h} w_{l}}{\left(r_{i j}^{L(k)}\right)^{p}\left(r_{h j}^{L(k)}\right)^{q}\left(r_{l j}^{L(k)}\right)^{r}}\right)\right)^{-1 /(p+q+r)}\right.} \\
&\left(\sum_{i, h, l=1}^{m}\left(\frac{w_{i} w_{h} w_{l}}{\left(r_{i j}^{M(k)}\right)^{p}\left(r_{h j}^{M(k)}\right)^{q}\left(r_{l j}^{M(k)}\right)^{r}}\right)\right)^{-1 /(p+q+r)} \\
&\left.\left(\sum_{i, h, l=1}^{m}\left(\frac{w_{i} w_{h} w_{l}}{\left(r_{i j}^{U(k)}\right)^{p}\left(r_{h j}^{U(k)}\right)^{q}\left(r_{l j}^{U(k)}\right)^{r}}\right)\right)^{-1 /(p+q+r)}\right] \tag{52}
\end{align*}
$$

to aggregate all the elements in the $j$ th column of $R^{(k)}$ and get the overall attribute value $\widehat{r}_{j}^{(k)}$ of the alternative $x_{j}$ corresponding to the decision maker $d_{k}$.

Step 3. Utilize the GFOWBHM operator (44):
$\widehat{r}_{j}=\operatorname{GFOWBHM}^{p, q, r}\left(\widehat{r}_{j}^{(1)}, \widehat{r}_{j}^{(2)}, \ldots, \widehat{r}_{j}^{(s)}\right)$
$=\frac{1}{\left(\sum_{k, h, l=1}^{s}\left(\omega_{k} \omega_{h} \omega_{l} /\left(\dot{\hat{r}}_{j}^{\sigma(k)}\right)^{p}\left(\dot{\hat{r}}_{j}^{\sigma(h)}\right)^{q}\left(\dot{\hat{r}}_{j}^{\sigma(l)}\right)^{r}\right)\right)^{1 /(p+q+r)}}$
$=\left[\left(\sum_{k, h, l=1}^{s}\left(\frac{\omega_{k} \omega_{h} \omega_{l}}{\left(\dot{r}_{j}^{L(\sigma(k))}\right)^{p}\left(\dot{r}_{j}^{L(\sigma(h))}\right)^{q}\left(\dot{r}_{j}^{L(\sigma(l))}\right)^{r}}\right)\right)^{-1 /(p+q+r)}\right.$,

$$
\begin{align*}
& \left(\sum_{k, h, l=1}^{s}\left(\frac{\omega_{k} \omega_{h} \omega_{l}}{\left(\dot{r}_{j}^{M(\sigma(k))}\right)^{p}\left(\dot{r}_{j}^{M(\sigma(h))}\right)^{q}\left(\dot{r}_{j}^{M(\sigma(l))}\right)^{r}}\right)\right)^{-1 /(p+q+r)} \\
& \left.\left(\sum_{k, h, l=1}^{s}\left(\frac{\omega_{k} \omega_{h} \omega_{l}}{\left(\dot{r}_{j}^{U(\sigma(k))}\right)^{p}\left(\dot{r}_{j}^{U(\sigma(h))}\right)^{q}\left(\dot{r}_{j}^{U(\sigma(l))}\right)^{r}}\right)\right)^{-1 /(p+q+r)}\right] \tag{53}
\end{align*}
$$

to aggregate the overall attribute values $\widehat{r}_{j}^{(k)}(k=1,2, \ldots, s)$ corresponding to the decision maker $d_{k}(k=1,2, \ldots, s)$ and get the collective overall attribute value $\widehat{r}_{j}$, where $\dot{\hat{r}}_{j}^{(\sigma(k))}=$ $\left[\dot{r}_{j}^{L(\sigma(k))}, \dot{r}_{j}^{M(\sigma(k))}, \dot{r}_{j}^{U(\sigma(k))}\right]$ is the $k$ th largest of the weighted data and $\dot{\hat{r}}_{j}^{(k)}\left(\dot{\hat{r}}_{j}^{(k)}=s v_{k} \widehat{r}_{j}^{(k)}, k=1,2, \ldots, s\right), \omega=\left(\omega_{1}, \omega_{2}\right.$, $\left.\ldots, \omega_{s}\right)^{T}$ is the weighting vector of the GFOWBHM operator, with $\omega_{k} \geq 0$ and $\sum_{k=1}^{s} \omega_{k}=1$.

Step 4. Compare each $\widehat{r}_{j}$ with all $\widehat{r}_{i}(i=1,2, \ldots, n)$ by using (7), and let $p_{i j}=p\left(\widehat{r}_{i} \geq \widehat{r}_{j}\right)$, and then construct the possibility matrix $P=\left(p_{i j}\right)_{n \times n}$, where $p_{i j} \geq 0, p_{i j}+p_{j i}=1, p_{i i}=0.5$, $i, j=1,2, \ldots, n$. Summing all elements in each line of matrix $P$, we have $p_{i}=\sum_{j=1}^{n} p_{i j}, i=1,2, \ldots, n$, and then reorder $\widehat{r}_{j}(j=1,2, \ldots, n)$ in descending order in accordance with the values of $p_{j}(j=1,2, \ldots, n)$.

Step 5. Rank all alternatives $x_{j}(j=1,2, \ldots, n)$ by the ranking of $\widehat{r}_{j}(j=1,2, \ldots, n)$, and then select the most desirable one.

Step 6. End.

## 4. Example Illustrations

In this section, we use a multiple attribute group decision making problem of determining what kind of airconditioning systems should be installed in a library (adopted from $[21,30]$ ) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives $x_{j}(j=1,2,3,4,5)$ are to be evaluated using triangular fuzzy numbers by the three decision makers $d_{k}(k=1,2,3)$ (whose weight vector is $\left.v=(0.4,0.3,0.3)^{T}\right)$ under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (i.e., $G_{1}$ : owning cost $\left(\$ / \mathrm{ft}^{2}\right)$, $G_{2}$ : operating cost $\left(\$ / \mathrm{ft}^{2}\right), G_{3}$ : performance $\left({ }^{*}\right), G_{4}$ : noise level ( Db ), $G_{5}$ : maintainability ( ${ }^{*}$ ), $G_{6}$ : reliability (\%), $G_{7}$ : flexibility ( ${ }^{*}$ ), $G_{8}$ : safety $\left({ }^{*}\right)$, where ${ }^{*}$ unit is from 0 to 1 scale, three attributes $G_{1}, G_{2}$, and $G_{4}$ are cost attributes, and the other five attributes are benefit attributes, and suppose that the weight vector of the attributes $G_{i}(i=1,2, \ldots, 8)$ is $\left.w=(0.05,0.08,0.14,0.12,0.18,0.21,0.05,0.17)^{T}\right)$ emerged from three impacts is Tables 1,2 , and 3.

In the following, we utilize the decision procedure to select the best air-conditioning system.

Table 1: Triangular fuzzy number decision matrix $A^{(1)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[3.5,4.0,4.7]$ | $[1.7,2.0,2.3]$ | $[3.5,3.8,4.2]$ | $[3.5,3.8,4.5]$ | $[3.3,3.8,4.0]$ |
| $G_{2}$ | $[5.5,6.0,6.5]$ | $[4.8,5.1,5.5]$ | $[4.5,5.2,5.5]$ | $[4.5,4.7,5.0]$ | $[5.5,5.7,6.0]$ |
| $G_{3}$ | $[0.7,0.8,0.9]$ | $[0.5,0.56,0.6]$ | $[0.5,0.6,0.7]$ | $[0.7,0.85,0.9]$ | $[0.6,0.7,0.8]$ |
| $G_{4}$ | $[35,40,45]$ | $[70,73,75]$ | $[65,68,70]$ | $[40,42,45]$ | $[50,55,60]$ |
| $G_{5}$ | $[0.4,0.45,0.5]$ | $[0.4,0.44,0.6]$ | $[0.7,0.76,0.8]$ | $[0.9,0.97,1.0]$ | $[0.5,0.54,0.6]$ |
| $G_{6}$ | $[95,98,100]$ | $[70,73,75]$ | $[80,83,90]$ | $[90,93,95]$ | $[85,90,95]$ |
| $G_{7}$ | $[0.3,0.35,0.5]$ | $[0.7,0.75,0.8]$ | $[0.8,0.9,1.0]$ | $[0.6,0.75,0.8]$ | $[0.4,0.5,0.6]$ |
| $G_{8}$ | $[0.7,0.74,0.8]$ | $[0.5,0.53,0.6]$ | $[0.6,0.68,0.7]$ | $[0.7,0.8,0.9]$ | $[0.8,0.85,0.9]$ |

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[4.0,4.3,4.5]$ | $[2.1,2.2,2.4]$ | $[5.0,5.1,5.2]$ | $[4.3,4.4,4.5]$ | $[3.0,3.3,3.5]$ |
| $G_{2}$ | $[6.0,6.3,6.5]$ | $[5.0,5.1,5.2]$ | $[4.5,4.7,5.0]$ | $[5.0,5.1,5.3]$ | $[7.0,7.5,8.0]$ |
| $G_{3}$ | $[0.7,0.8,0.9]$ | $[0.4,0.5,0.6]$ | $[0.5,0.55,0.6]$ | $[0.7,0.75,0.8]$ | $[0.7,0.8,0.9]$ |
| $G_{4}$ | $[37,38,39]$ | $[70,73,75]$ | $[65,66,67]$ | $[40,42,45]$ | $[50,52,55]$ |
| $G_{5}$ | $[0.4,0.5,0.6]$ | $[0.5,0.55,0.6]$ | $[0.8,0.85,0.9]$ | $[0.8,0.95,1.0]$ | $[0.4,0.44,0.5]$ |
| $G_{6}$ | $[92,93,95]$ | $[70,75,80]$ | $[83,84,85]$ | $[90,91,92]$ | $[90,93,95]$ |
| $G_{7}$ | $[0.4,0.45,0.5]$ | $[0.8,0.85,0.9]$ | $[0.7,0.73,0.8]$ | $[0.7,0.85,0.9]$ | $[0.4,0.45,0.5]$ |
| $G_{8}$ | $[0.6,0.7,0.8]$ | $[0.6,0.65,0.7]$ | $[0.5,0.6,0.7]$ | $[0.7,0.76,0.8]$ | $[0.7,0.8,0.9]$ |

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[4.3,4.4,4.6]$ | $[2.2,2.4,2.5]$ | $[4.5,4.8,5.0]$ | $[4.7,4.9,5.0]$ | $[3.1,3.2,3.4]$ |
| $G_{2}$ | $[6.4,6.7,7.0]$ | $[5.0,5.2,5.5]$ | $[4.7,4.8,4.9]$ | $[5.5,5.7,6.0]$ | $[6.0,6.5,7.0]$ |
| $G_{3}$ | $[0.8,0.85,0.9]$ | $[0.5,0.6,0.7]$ | $[0.6,0.7,0.8]$ | $[0.7,0.8,0.9]$ | $[0.7,0.75,0.8]$ |
| $G_{4}$ | $[36,38,40]$ | $[72,73,75]$ | $[67,68,70]$ | $[45,48,50]$ | $[55,57,60]$ |
| $G_{5}$ | $[0.4,0.46,0.5]$ | $[0.4,0.45,0.6]$ | $[0.8,0.95,1.0]$ | $[0.8,0.85,0.9]$ | $[0.5,0.55,0.6]$ |
| $G_{6}$ | $[93,94,95]$ | $[77,78,80]$ | $[85,87,90]$ | $[90,94,95]$ | $[90,96,100]$ |
| $G_{7}$ | $[0.4,0.5,0.6]$ | $[0.8,0.9,1.0]$ | $[0.8,0.86,0.9]$ | $[0.6,0.7,0.8]$ | $[0.5,0.57,0.6]$ |
| $G_{8}$ | $[0.7,0.78,0.8]$ | $[0.5,0.55,0.6]$ | $[0.6,0.68,0.7]$ | $[0.8,0.85,0.9]$ | $[0.8,0.85,0.9]$ |

Step 1. By using (51), we normalize each attribute value $\widehat{a}_{i j}^{(k)}$ in the matrices $A^{(k)}(k=1,2,3)$ into the corresponding element in the matrices $R^{(k)}=\left(\widehat{r}_{i j}\right)_{8 \times 5}(k=1,2,3)$ (Tables 4, 5, and 6).

Step 2. Utilize the GFWBHM operator (52) (let $p=q=r=3$ ) to aggregate all elements in the $j$ th column $R^{(k)}$ and get the overall attribute value $\widehat{r}_{j}^{(k)}$ :

$$
\begin{aligned}
& \widehat{r}_{1}^{(1)}=[0.1390,0.1753,0.2187], \\
& \widehat{r}_{2}^{(1)}=[0.1347,0.1586,0.1927], \\
& \widehat{r}_{3}^{(1)}=[0.1581,0.1852,0.2178], \\
& \widehat{r}_{4}^{(1)}=[0.1900,0.2289,0.2651], \\
& \widehat{r}_{5}^{(1)}=[0.1565,0.1911,0.2311], \\
& \widehat{r}_{1}^{(2)}=[0.1480,0.1851,0.2248],
\end{aligned}
$$

$$
\begin{align*}
& \widehat{r}_{2}^{(2)}=[0.1434,0.1706,0.1992], \\
& \widehat{r}_{3}^{(2)}=[0.1561,0.1792,0.2057], \\
& \widehat{r}_{4}^{(2)}=[0.1927,0.2228,0.2477], \\
& \widehat{r}_{5}^{(2)}=[0.1499,0.1761,0.2098], \\
& \widehat{r}_{1}^{(3)}=[0.1459,0.1811,0.2104], \\
& \widehat{r}_{2}^{(3)}=[0.1370,0.1607,0.1938], \\
& \widehat{r}_{3}^{(3)}=[0.1679,0.1921,0.2173], \\
& \widehat{r}_{4}^{(3)}=[0.1883,0.2138,0.2395], \\
& \widehat{r}_{5}^{(3)}=[0.1678,0.1922,0.2215] . \tag{54}
\end{align*}
$$

Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[0.12,0.16,0.21]$ | $[0.25,0.32,0.43]$ | $[0.14,0.17,0.21]$ | $[0.13,0.17,0.21]$ | $[0.14,0.17,0.22]$ |
| $G_{2}$ | $[0.15,0.18,0.21]$ | $[0.18,0.21,0.24]$ | $[0.18,0.20,0.25]$ | $[0.20,0.23,0.25]$ | $[0.16,0.19,0.21]$ |
| $G_{3}$ | $[0.18,0.23,0.30]$ | $[0.13,0.16,0.20]$ | $[0.13,0.17,0.23]$ | $[0.18,0.24,0.30]$ | $[0.15,0.20,0.27]$ |
| $G_{4}$ | $[0.22,0.26,0.32]$ | $[0.13,0.14,0.16]$ | $[0.14,0.15,0.17]$ | $[0.22,0.25,0.28]$ | $[0.16,0.19,0.23]$ |
| $G_{5}$ | $[0.11,0.14,0.17]$ | $[0.11,0.14,0.21]$ | $[0.20,0.24,0.28]$ | $[0.26,0.31,0.34]$ | $[0.14,0.17,0.21]$ |
| $G_{6}$ | $[0.21,0.22,0.24]$ | $[0.15,0.17,0.18]$ | $[0.18,0.19,0.21]$ | $[0.20,0.21,0.23]$ | $[0.19,0.21,0.23]$ |
| $G_{7}$ | $[0.08,0.11,0.18]$ | $[0.19,0.23,0.29]$ | $[0.22,0.28,0.36]$ | $[0.16,0.23,0.29]$ | $[0.11,0.15,0.21]$ |
| $G_{8}$ | $[0.18,0.21,0.24]$ | $[0.13,0.15,0.18]$ | $[0.15,0.19,0.21]$ | $[0.18,0.22,0.27]$ | $[0.21,0.24,0.27]$ |

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[0.15,0.16,0.19]$ | $[0.28,0.32,0.36]$ | $[0.13,0.14,0.15]$ | $[0.15,0.16,0.17]$ | $[0.19,0.21,0.25]$ |
| $G_{2}$ | $[0.17,0.18,0.19]$ | $[0.21,0.22,0.23]$ | $[0.21,0.24,0.26]$ | $[0.20,0.22,0.23]$ | $[0.13,0.15,0.17]$ |
| $G_{3}$ | $[0.18,0.24,0.30]$ | $[0.11,0.15,0.20]$ | $[0.13,0.16,0.20]$ | $[0.18,0.22,0.27]$ | $[0.18,0.24,0.30]$ |
| $G_{4}$ | $[0.25,0.27,0.29]$ | $[0.13,0.14,0.15]$ | $[0.15,0.15,0.16]$ | $[0.22,0.24,0.27]$ | $[0.18,0.20,0.21]$ |
| $G_{5}$ | $[0.11,0.15,0.21]$ | $[0.14,0.17,0.21]$ | $[0.22,0.26,0.31]$ | $[0.22,0.29,0.34]$ | $[0.11,0.13,0.17]$ |
| $G_{6}$ | $[0.21,0.21,0.22]$ | $[0.16,0.17,0.19]$ | $[0.19,0.19,0.20]$ | $[0.20,0.21,0.22]$ | $[0.20,0.21,0.22]$ |
| $G_{7}$ | $[0.11,0.14,0.17]$ | $[0.22,0.26,0.30]$ | $[0.19,0.22,0.27]$ | $[0.19,0.26,0.30]$ | $[0.19,0.14,0.17]$ |
| $G_{8}$ | $[0.15,0.20,0.26]$ | $[0.15,0.19,0.23]$ | $[0.13,0.17,0.23]$ | $[0.18,0.22,0.26]$ | $[0.18,0.23,0.29]$ |

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$.

|  | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $G_{1}$ | $[0.15,0.17,0.18]$ | $[0.28,0.30,0.35]$ | $[0.14,0.15,0.17]$ | $[0.14,0.15,0.16]$ | $[0.20,0.23,0.25]$ |
| $G_{2}$ | $[0.16,0.17,0.19]$ | $[0.20,0.22,0.24]$ | $[0.22,0.24,0.25]$ | $[0.18,0.20,0.22]$ | $[0.16,0.17,0.20]$ |
| $G_{3}$ | $[0.20,0.23,0.27]$ | $[0.12,0.16,0.21]$ | $[0.15,0.19,0.24]$ | $[0.17,0.22,0.27]$ | $[0.17,0.20,0.24]$ |
| $G_{4}$ | $[0.26,0.28,0.31]$ | $[0.14,0.15,0.16]$ | $[0.15,0.16,0.17]$ | $[0.21,0.22,0.25]$ | $[0.17,0.19,0.20]$ |
| $G_{5}$ | $[0.11,0.14,0.17]$ | $[0.11,0.14,0.21]$ | $[0.22,0.29,0.34]$ | $[0.22,0.26,0.31]$ | $[0.14,0.17,0.21]$ |
| $G_{6}$ | $[0.20,0.21,0.22]$ | $[0.17,0.17,0.18]$ | $[0.18,0.19,0.21]$ | $[0.20,0.21,0.22]$ | $[0.20,0.21,0.23]$ |
| $G_{7}$ | $[0.10,0.14,0.19]$ | $[0.21,0.25,0.32]$ | $[0.21,0.24,0.29]$ | $[0.15,0.20,0.26]$ | $[0.13,0.16,0.19]$ |
| $G_{8}$ | $[0.18,0.21,0.24]$ | $[0.13,0.15,0.18]$ | $[0.15,0.18,0.21]$ | $[0.21,0.23,0.26]$ | $[0.21,0.23,0.26]$ |

Step 3. Utilize the GFOWBHM operator (53) (suppose that its weight vector is $\omega=(0.243,0.514,0.243)^{T}$ determined by using the normal distribution based method [27] (let $\delta=0.5$ and $p=q=r=3$ ) to aggregate the overall attribute value $\widehat{r}_{j}^{(k)}(k=1,2,3)$, corresponding to the decision maker $d_{k}(k=$ $1,2,3$ ), and get the collective overall attribute value $\widehat{r}_{j}$ :

$$
\begin{align*}
& \widehat{r}_{1}=[0.1459,0.1816,0.2195], \\
& \widehat{r}_{2}=[0.1395,0.1649,0.1962], \\
& \widehat{r}_{3}=[0.1592,0.1837,0.2111],  \tag{55}\\
& \widehat{r}_{4}=[0.1909,0.2218,0.2493], \\
& \widehat{r}_{5}=[0.1552,0.1830,0.2171] .
\end{align*}
$$

Step 4. Compare each $\widehat{r}_{j}$ with all $\widehat{r}_{i}(i=1,2,3,4,5)$ by using (7) (without loss of generality, set $\delta=0.5$ ), and let $p_{i j}=p\left(\widehat{r}_{i} \geq\right.$ $\widehat{r}_{j}$ ), and then construct a possibility matrix:

$$
P=\left(\begin{array}{ccccc}
0.5 & 0.7387 & 0.4598 & 0 & 0.4610  \tag{56}\\
0.2613 & 0.5 & 0.1638 & 0 & 0.1921 \\
0.5402 & 0.8362 & 0.5 & 0 & 0.5010 \\
1 & 1 & 1 & 0.5 & 1 \\
0.5390 & 0.8079 & 0.4990 & 0 & 0.5
\end{array}\right)
$$

Summing all elements in each line of matrix $P$, we have

$$
\begin{gather*}
p_{1}=2.1595, \quad p_{2}=1.1171, \quad p_{3}=2.3774 \\
p_{4}=4.5, \quad p_{5}=2.3459 \tag{57}
\end{gather*}
$$

and then reorder $\widehat{r}_{j}(j=1,2,3,4,5)$ in descending order in accordance with the values of $p_{j}(j=1,2,3,4,5)$ :

$$
\begin{equation*}
\widehat{r}_{4}>\widehat{r}_{3}>\widehat{r}_{5}>\widehat{r}_{1}>\widehat{r}_{2} \tag{58}
\end{equation*}
$$

Step 5. Rank all the alternatives $x_{j}(j=1,2,3,4,5)$ by the ranking of $\widehat{r}_{j}(j=1,2,3,4,5)$ :

$$
\begin{equation*}
x_{4}>x_{3}>x_{5}>x_{1}>x_{2} \tag{59}
\end{equation*}
$$

and thus the most desirable alternative is $x_{4}$.
From the previous analysis, the results obtained by the proposed approach are very similar to the ones obtained Xu's approach [21], but our approach is more flexible than that of Xu [21] because it can provide the decision makers more choices as parameters are assigned different values.

Table 7: Comparison of the proposed approach with other approaches.

|  | Xu's approach [21] | Wei's approach [22] | Sun and Sun's approach [23] | Proposed approach |
| :---: | :---: | :---: | :---: | :---: |
| Problem type | MAGDM | MAGDM | MADM | MAGDM |
| Application area | Air-conditioning system selection | Investment of money | EPR system selection | Air-conditioning system selection |
| Decision information | Triangular fuzzy decision matrix | Triangular fuzzy decision matrix | Triangular fuzzy decision matrix | Triangular fuzzy decision matrix |
| Solution method |  |  |  |  |
| Aggregation stage | FWHM operator | FIOWHM operator |  | GFWBHM operator |
| Exploitation stage | FHHM operator | FWHM operator | FOBHM operator | GFOWBHM operator |
| Ranking stage | Complementary matrix | Complementary matrix | Possibility matrix | Possibility matrix |
| Final decision | Ranking of alternatives | Ranking of alternatives | Ranking of alternatives | Ranking of alternatives |

## 5. Comparison of the Proposed Approach with Other Approaches

In this section, we compare the proposed approach with other approaches. The approaches to be compared here are the approaches proposed by Xu [21], Wei [22], and H . Sun and M. Sun [23], respectively.

Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things and not on how they are themselves. The differences in four approaches are the following.
(1) The H. Sun and M. Sun's approach is only suitable for solving multiple attribute decision making (MADM), while the proposed approach and Xu's and Wei's approaches are suitable for solving MAGDM because the approaches provide the aggregation stage in aggregation process.
(2) The Xu's and Wei's approaches have simple computation process than the proposed approach and H. Sun and M. Sun's approach, while the proposed approach and H . Sun and M. Sun's approach are more flexible than Xu's and Wei's approaches because these can provide the decision makers more choices as parameters are assigned different values.
(3) The Wei's approach uses the weights of decision makers as the order inducing variables in aggregation stage, while other approaches use the weights of decision makers to determine the order positions of the overall attribute values in exploitation stage.

Others of relative comparison with Xu's, Wei's, and H. Sun and $M$. Sun's approaches are shown in Table 7.

## 6. Conclusions

In this paper, we have extended the GWBM operator to the triangular fuzzy environment and developed the fuzzy harmonic aggregation operators including the FWBHM and GFWBHM operators. Based on the these operators and Yager's OWA operator, we have developed the FOWBHM
operator and the GFOWBHM operator, respectively, and discussed their properties and special cases. It has been pointed out that if all the input fuzzy data reduce to the interval or numerical data, then the GFWBHM operator reduces to the GUWBHM operator and GWBHM operator, respectively; the GFOWBHM operator reduces to the GUOWBHM operator and GOWBHM operator, respectively. In these situations, the WHM (resp., OWHM) operator is the special case of the GWBHM (resp. GOWBHM) operator. Based on the GFWBHM and GFOWBHM operators, we have developed an approach to multiple attribute group decision making with triangular fuzzy information and have also applied the proposed approach to the problem of determining what kind of air-conditioning systems should be installed in the library. Furthermore, the comparison of the proposed approach with other existing approaches is presented. The merit of the proposed approach is that it is more flexible than the classical ones because it can provide the decision makers more choices as parameters are assigned different values. Apparently, the proposed aggregation techniques and decision making method can also extended to the intervalvalued triangular fuzzy environment.

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# Parametric Extension of the Most Preferred OWA Operator and Its Application in Search Engine's Rank 

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#### Abstract

Most preferred ordered weighted average (MP-OWA) operator is a new kind of neat (dynamic weight) OWA operator in the aggregation operator families. It considers the preferences of all alternatives across the criteria and provides unique aggregation characteristics in decision making. In this paper, we propose the parametric form of the MP-OWA operator to deal with the uncertainty preference information, which includes MP-OWA operator as its special case, and it also includes the most commonly used maximum, minimum, and average aggregation operators. A special form of parametric MP-OWA operator with power function is proposed. Some properties of the parametric MP-OWA operator are provided and the advantages of them in decision making problems are summarized. The new proposed parametric MP-OWA operator can grasp the subtle preference information of the decision maker according to the application context through multiple aggregation results. They are applied to rank search engines considering the relevance of the retrieved queries. An search engine ranking example illustrates the application of parametric MP-OWA operator in various decision situations.


## 1. Introduction

The ordered weighted averaging (OWA) operator, which was introduced by Yager [1], provides for aggregation lying between maximum and minimum operators and has received more and more attention since its appearance [2,3]. The OWA operator has been used in a wide range of applications, such as neural networks $[4,5]$, database systems [6, 7], fuzzy logic controllers [8, 9], decision making [10-12], expert systems [13-15], database query management and data mining [16, 17], and lossless image compression $[18,19]$.

Until now, according to the weight assignment methods, the existing OWA operators can be classified into two categories: one is the static OWA operators having weights depending on the serial position, and the other is dynamic or neat OWA operators having weights depending on the aggregated elements. The static OWA operators include the maximum entropy operator [20], minimum variance operator [21], the maximum Rényi entropy operator [22], least square deviation operator and chi-square operator [23], exponential OWA operator [24], linguistic ordered weighted averaging operator [5, 25], and intuitionistic fuzzy ordered weighted distance operator [26-28].

For neat OWA operator with dynamic weights, Yager [29, 30] proposed the families of neat OWA operator called basic defuzzification distribution (BADD) OWA operator and parameterized and-like and or-like OWA operators. Marimin et al. [31,32] used neat OWA operator to aggregate the linguistic labels for expressing fuzzy preferences in group decision making problem. Peláez and Doña [33, 34] introduced majority additive OWA (MA-OWA) operator and quantified MA-OWA (QMA-OWA) operator to model the majority idea in group decision making problem. Liu and Lou [35] extended BADD OWA operator to additive neat OWA (ANOWA) operator for decision making problem. Wu et al. [36] introduced an argument-dependent approach based on maximizing the entropy.

Recently, Emrouznejad [3] proposed a new kind of neat OWA operator called most preferred OWA (MP-OWA) operator, which considers the preferences of alternatives across all the criteria. It has an interesting characteristic that the aggregation combines static OWA operator with dynamic OWA operator together. That is, because the weights correlate with internal aggregated elements in the way of neat OWA operator, and the aggregated elements must be ordered decreasingly when aggregating, which is the same as that
of static-weight OWA operator, consequently, the MP-OWA operator not only has the advantage of neat OWA operator in which the weighting vector is relevant with the aggregated elements values rather than the positions, but also utilizes the most preferred information, which is connected with the maximum frequency of all scales for each criteria. Some extension researches about MP-OWA operator and the application can be found in the literature [ $3,4,11,37,38$ ].

In this paper, we propose parametric MP-OWA operator families, which combine the characteristics of MP-OWA operator with ordinary neat OWA operator together. We also propose the family of parametric MP-OWA operator with power function; it is quite useful as it includes the current MP-OWA operator as a special case and also includes multiple situations because of the aggregation results ranging between the minimum and the maximum. Meanwhile, some properties of the parametric MP-OWA operator and the MP-OWA operator family with power function are provided and analyzed, which can be used as the basis to apply our new parametric MP-OWA operator in practice. Moreover, we discuss the advantages of our new parametric MPOWA operator, which not only helps decision makers realize viewing the decision making problem completely through considering the preference relation and the parameter $(r)$, but also offers another kind of method for decision making problems based on preference information. We apply the proposed method to decision making problem concentrated on ranking search engines and get different rankings through changing the values of parameter $(r)$, which can help decision makers recognize the best search engines indirectly as well. It is necessary to stress that the proposed method can develop an amazingly wide range of decision making problems with preference relations such as information aggregation and group decision making.

This paper is organized as follows: Section 2 reviews some basic concepts of neat OWA operator and MP-OWA operator. Section 3 gives a general form of parametric MP-OWA operator and develops a particular member of MP-OWA operator with power function; some properties and advantages are also discussed. Section 4 gives an example of ranking search engines using the proposed approach. Section 5 summarizes the main results and draws conclusions.

## 2. Preliminaries

2.1. Neat OWA Operator. Yager [29] proposed neat OWA operator, which means the weighting vector not only depends on position indexes of the aggregated elements, but also the aggregated values.

Assume $x_{1}, x_{2}, \ldots, x_{n}$ is a collection of numbers; the aggregation of neat OWA operator is indicated as follows:

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} y_{i} \tag{1}
\end{equation*}
$$

where $y_{i}$ is the $i$ th largest value of $x_{i}$ and $w_{i}$ is the weights to be a function of the ordered aggregated elements $y_{i}$, which is denoted as follows:

$$
\begin{equation*}
w_{i}=f_{i}\left(y_{1}, y_{2}, \ldots, y_{n}\right) \tag{2}
\end{equation*}
$$

The weights are required to satisfy two conditions:
(1) $w_{i} \in[0,1]$ for each $i$,
(2) $\sum_{i=1}^{n} w_{i}=1$.

In this case, (1) can be rewritten as follows:

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} f_{i}\left(y_{1}, y_{2}, \ldots, y_{n}\right) y_{i} \tag{3}
\end{equation*}
$$

If $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ are inputs, $Y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ are inputs ordered, and $Z=\left(z_{1}, z_{2}, \ldots, z_{n}\right)=\operatorname{Perm}\left(x_{1}, x_{2}, \ldots\right.$, $\left.x_{n}\right)$ is any permutation of the inputs, then the OWA operator is neat if

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} y_{i} \tag{4}
\end{equation*}
$$

is the same for the assignment $Z=Y$.
Later, Yager and Filev [30] introduced the first family of neat OWA operator namely BADD OWA operator. The weighting vector is another collection of elements $v_{i}(i=$ $1,2, \ldots, n)$, such that

$$
\begin{equation*}
v_{i}=\frac{x_{i}^{\alpha}}{\sum_{j=1}^{n} x_{j}^{\alpha}}, \tag{5}
\end{equation*}
$$

where $\alpha \in[0,+\infty)$. It can be easily seen that BADD OWA operator has properties as follows:
(1) $v_{i} \in[0,1]$ for each $i$,
(2) $\sum_{i=1}^{n} v_{i}=1$.

From (5), the weighting vector of BADD OWA operator is expressed as follows:

$$
\begin{align*}
W(\alpha) & =\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T} \\
& =\left(\frac{x_{1}^{\alpha}}{\sum_{j=1}^{n} x_{j}^{\alpha}}, \frac{x_{2}^{\alpha}}{\sum_{j=1}^{n} x_{j}^{\alpha}}, \ldots, \frac{x_{n}^{\alpha}}{\sum_{j=1}^{n} x_{j}^{\alpha}}\right)^{T} . \tag{6}
\end{align*}
$$

Accordingly, the aggregation expression is denoted as follows:

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} x_{i}=\frac{\sum_{i=1}^{n} x_{i}^{\alpha+1}}{\sum_{j=1}^{n} x_{j}^{\alpha}} \tag{7}
\end{equation*}
$$

Liu [39] proposed a generalization BADD OWA operator with weighted functional average, which is also called additive neat OWA (ANOWA) operator, where

$$
\begin{align*}
W_{f} & =\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T} \\
& =\left(\frac{f\left(x_{1}\right)}{\sum_{j=1}^{n} f\left(x_{j}\right)}, \frac{f\left(x_{2}\right)}{\sum_{j=1}^{n} f\left(x_{j}\right)}, \ldots, \frac{f\left(x_{n}\right)}{\sum_{j=1}^{n} f\left(x_{j}\right)}\right)^{T},  \tag{8}\\
& F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} x_{i}=\frac{\sum_{i=1}^{n} f\left(x_{i}\right) x_{i}}{\sum_{j=1}^{n} f\left(x_{j}\right)}, \tag{9}
\end{align*}
$$

where $f\left(x_{i}\right)$ can be any form of a continuous function. When $f\left(x_{i}\right)$ takes the form of power function, that is $f\left(x_{i}\right)=x_{i}^{\alpha}$, it turns into BADD OWA operator.
2.2. The MP-OWA Operator. The MP-OWA operator, which was proposed by Emrouznejad [3], is based on the most popular criteria for all alternatives and considers the preferences of alternatives across all criteria. Suppose $Z=$ $\left\{Z_{1}, Z_{2}, \ldots, Z_{m} ; m \geqslant 2\right\}$ is a set of alternatives to be ranked, $C=\left\{C_{1}, C_{2}, \ldots, C_{n} ; n \geqslant 2\right\}$ is a group of criteria to rate the alternatives, $S=\left\{S_{1}, S_{2}, \ldots, S_{r}\right\}$ satisfying $S_{1}<S_{2}<\cdots<S_{r}$ is a given scale set, and $S_{j i} \in S$ is the scale value of alternative $A_{j}$ for criteria $C_{i}$. Then, the matrix of preference rating given to alternatives for each criteria is shown in Table 1.

Meanwhile, the frequency $N_{k i}(k \in[1, r], i \in[1, n])$ of scale $S_{k}$ given to criteria $C_{i}$ is summarized in Table 2.

The frequency of the most popular scale for each criteria can be written as follows:

$$
\begin{align*}
V= & \left(v_{1}, v_{2}, \ldots, v_{n}\right)^{T} \\
= & \left(\max \left\{N_{k 1}: \forall k\right\}, \max \left\{N_{k 2}: \forall k\right\}, \ldots,\right.  \tag{10}\\
& \left.\quad \max \left\{N_{k n}: \forall k\right\}\right)^{T} .
\end{align*}
$$

Accordingly, the weighting vector of MP-OWA operator can be expressed as follows:

$$
\begin{align*}
W & =\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T} \\
& =\left(\frac{v_{1}}{\sum_{k=1}^{n} v_{k}}, \frac{v_{2}}{\sum_{k=1}^{n} v_{k}}, \ldots, \frac{v_{n}}{\sum_{k=1}^{n} v_{k}}\right)^{T} ; \tag{11}
\end{align*}
$$

of course, $\sum_{i=1}^{n} w_{i}=1$.
The aggregation is expressed as follows:

$$
\begin{equation*}
F_{W}\left(Z_{k}\right)=F_{W}\left(z_{k 1}, z_{k 2}, \ldots, z_{k n}\right)=\sum_{i=1}^{n} w_{i} y_{k i} \tag{12}
\end{equation*}
$$

where $y_{k i}$ is the $i$ th largest value of $z_{k i}$.
From (11), it is clear that the weight is independent of the ordering of set $V$; the more frequency of $S_{k}$ given to criteria $C_{j}$ is, the bigger the corresponding weight is. That is, the MPOWA operator overemphasizes the opinions of the majority and ignores those of the minority.

## 3. Parametric MP-OWA Operator

In this section, we firstly propose the general form of parametric MP-OWA operator, and some propositions are proposed. Then, we develop a particular family of parametric MP-OWA operator with power function, and some properties are also discussed.
3.1. The General Form of Parametric MP-OWA Operator. Similar to the extensions of OWA operator to the parametric form of BADD operator and ANOWA operator [30, 39], we will extend the MP-OWA operator to a parametric format, that can represent the preference information more flexibly, and MP operator becomes a special case of it.

TABLE 1: Matrix of preference rating of $n$ criteria with $m$ alternatives.

|  |  | Criteria |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $C_{1}$ | $\ldots$ | $C_{i}$ | $\ldots$ | $C_{n}$ |
| Alternatives | $Z_{1}$ | $S_{11}$ | $\cdots$ | $S_{1 i}$ | $\cdots$ | $S_{1 n}$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
|  | $Z_{j}$ | $S_{j 1}$ | $\cdots$ | $S_{j i}$ | $\cdots$ | $S_{j n}$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
|  | $Z_{m}$ | $S_{m 1}$ | $\cdots$ | $S_{m i}$ | $\cdots$ | $S_{m n}$ |

Table 2: Matrix of frequency that scale gives to criteria.

|  |  |  | Criteria |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $C_{1}$ | $\ldots$ | $C_{i}$ | $\cdots$ | $C_{n}$ |
| Scales | $S_{1}$ | $N_{11}$ | $\cdots$ | $N_{1 i}$ | $\cdots$ | $N_{1 n}$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
|  | $S_{k}$ | $N_{k 1}$ | $\cdots$ | $N_{k i}$ | $\cdots$ | $N_{k n}$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
|  | $S_{r}$ | $N_{r 1}$ | $\cdots$ | $N_{r i}$ | $\cdots$ | $N_{r n}$ |

Definition 1. For aggregated matrix $Z=\left(Z_{1}, Z_{2}, \ldots, Z_{m}\right)$, $Z_{j}=\left(z_{j 1}, z_{j 2}, \ldots, z_{j n}\right)(j \in[1, m]) . v_{i}=\max \left\{N_{k i}, k \in\right.$ $[1, r]\}(i \in[1, n])$, and $f\left(v_{i}\right) \geqslant 0$, where $N_{k i}$ is the frequency of each scale for criteria. The vector of the maximum frequency function can be written as follows:

$$
\begin{equation*}
V_{f}=\left(f\left(v_{1}\right), f\left(v_{2}\right), \ldots, f\left(v_{n}\right)\right)^{T} \tag{13}
\end{equation*}
$$

The weighting vector is defined as follows:

$$
\begin{align*}
W_{f} & =\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T} \\
& =\left(\frac{f\left(v_{1}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}, \frac{f\left(v_{2}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}, \ldots, \frac{f\left(v_{n}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}\right)^{T} . \tag{14}
\end{align*}
$$

Here, $f\left(v_{i}\right)$ can be substituted for many specific functions.
It is obvious that $w_{i}$ satisfies the normalization properties of $w_{i} \geqslant 0$ and $\sum_{i=1}^{n} w_{i}=1$.

The parametric MP-OWA operator aggregation is

$$
\begin{equation*}
F_{f}(Z)=\sum_{j=1}^{m} \sum_{i=1}^{n} w_{i} y_{j i}=\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{f\left(v_{i}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)} y_{j i} \tag{15}
\end{equation*}
$$

where $y_{j i}$ is the $i$ th largest value of $z_{j i}$.
In (14), if $f\left(v_{i}\right)=v_{i}(i \in[1, n])$, (15) is the same as (12); that is, MP-OWA operator becomes a special case of the parametric MP-OWA operator.

Next, we will give some properties of our new proposed parametric MP-OWA operator.

Definition 2. Assume $F_{f}$ is a parametric MP-OWA operator with a weighting vector $W_{f}$; the degree of orness $\left(W_{f}\right)$ is defined as follows:

$$
\begin{equation*}
\operatorname{orness}(W)=\sum_{i=1}^{n} \frac{n-i}{n-1} \frac{f\left(v_{i}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)} . \tag{16}
\end{equation*}
$$

Next, some propositions of the parametric MP-OWA operator are described as:

Proposition 3. Assume $F_{f}$ is the aggregation result with parametric MP-OWA operator and $f\left(v_{i}\right)$ is the $i$ th value of the set $V$.
(1) Boundary. If $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is the aggregated elements, then
$\min _{1 \leqslant i \leqslant n}\left\{f\left(x_{i}\right)\right\} \leqslant F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \leqslant \max _{1 \leqslant i \leqslant n}\left\{f\left(x_{i}\right)\right\}$.
(2) Commutativity. If $x_{i}$ and $x_{i}^{(k)}$ are the ith largest values of the aggregated sets $X$ and $X^{K}$, respectively, then

$$
\begin{equation*}
F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=F_{f}\left(x_{1}^{(k)}, x_{2}^{(k)}, \ldots, x_{n}^{(k)}\right) \tag{18}
\end{equation*}
$$

where $\left(x_{1}^{(k)}, x_{2}^{(k)}, \ldots, x_{n}^{(k)}\right)$ is any permutation of the arguments $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$.
(3) Monotonicity. If $x_{i}$ and $y_{i}$ are the ith largest values of the aggregated sets $X$ and $Y$, respectively, and $y_{i} \leqslant x_{i}$, then

$$
\begin{equation*}
F\left(y_{1}, y_{2}, \ldots, y_{n}\right) \leqslant F\left(x_{1}, x_{2}, \ldots, x_{n}\right), \tag{19}
\end{equation*}
$$

where the vector $V$ is the same as both aggregated vectors.
(4) Idempotency. If $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is the aggregated elements, and $x_{i}=x(i=1,2, \ldots, n)$, then

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=x \tag{20}
\end{equation*}
$$

Apparently, if we set $f\left(v_{i}\right)=v_{i}(i \in[1, n])$, parametric MP-OWA operator turns into MP-OWA operator, and the conclusions of Proposition 3 are also correct.

Proposition 4. Assume $F$ is the MP-OWA operator aggregation result and $v_{i}$ is the ith value of set $V$.
(1) Boundary. If $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is the aggregated elements, then

$$
\begin{equation*}
\min _{1 \leqslant i \leqslant n}\left\{x_{i}\right\} \leqslant F\left(x_{1}, x_{2}, \ldots, x_{n}\right) \leqslant \max _{1 \leqslant i \leqslant n}\left\{x_{i}\right\} . \tag{21}
\end{equation*}
$$

(2) Commutativity. If $x_{i}$ and $x_{i}^{(k)}$ are the ith largest values of the aggregated sets $X$ and $X^{K}$, respectively, then

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=F\left(x_{1}^{(k)}, x_{2}^{(k)}, \ldots, x_{n}^{(k)}\right) \tag{22}
\end{equation*}
$$

where $\left(x_{1}^{(k)}, x_{2}^{(k)}, \ldots, x_{n}^{(k)}\right)$ is any permutation of the arguments $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$.
(3) Monotonicity. If $x_{i}$ and $y_{i}$ are the ith largest values of the aggregated sets $X$ and $Y$, respectively, and $y_{i} \leqslant x_{i}$, for each $i(i=1,2, \ldots, n)$, then

$$
\begin{equation*}
F\left(y_{1}, y_{2}, \ldots, y_{n}\right) \leqslant F\left(x_{1}, x_{2}, \ldots, x_{n}\right) \tag{23}
\end{equation*}
$$

where $V$ is the same vector as both aggregated values of $y_{i}$ and $x_{i}$.
(4) Idempotency. If $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ is the aggregated elements, and $x_{i}=x(i=1,2, \ldots, n)$, then

$$
\begin{equation*}
F\left(x_{1}, x_{2}, \ldots, x_{n}\right)=x \tag{24}
\end{equation*}
$$

3.2. Parametric MP-OWA Operator with Power Function. Similar to ANOWA operator (8), which takes the form of power function and becomes BADD OWA operator, we study the family of parametric MP-OWA operator with power function, and the function $f\left(v_{i}\right)$ is given in the following form as:

$$
\begin{equation*}
f\left(v_{i}\right)=v_{i}^{r} \tag{25}
\end{equation*}
$$

where $r$ is a real number.
From (14), the weighting vector of parametric MP-OWA operator can be rewritten as follows:

$$
\begin{align*}
W_{f} & =\left(\frac{f\left(v_{1}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}, \frac{f\left(v_{2}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}, \ldots, \frac{f\left(v_{n}\right)}{\sum_{j=1}^{n} f\left(v_{j}\right)}\right)^{T}  \tag{26}\\
& =\left(\frac{v_{1}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \frac{v_{2}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \ldots, \frac{v_{n}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}\right)^{T} .
\end{align*}
$$

Accordingly, from (15), the aggregation can be expressed as follows:

$$
\begin{equation*}
F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} y_{i}=\sum_{i=1}^{n} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}} y_{i}, \tag{27}
\end{equation*}
$$

where $y_{i}$ is the $i$ th largest value of $x_{i}$.
Regarding (16), the orness equation can also be described as follows:

$$
\begin{equation*}
\operatorname{orness}\left(W_{f}\right)=\sum_{i=1}^{n} \frac{n-i}{n-1} w_{i}=\sum_{i=1}^{n} \frac{n-i}{n-1} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \tag{28}
\end{equation*}
$$

when $r=1$ in (25), parametric MP-OWA operator becomes ordinary MP-OWA operator.

Remark 5. Generally speaking, in the parametric MP-OWA operator (15), $f\left(v_{i}\right)$ can take any function forms, such as power function, exponential function, or other function forms. Here, we only take the form of power function. The reasons for this decision are as follows: (1) Power function for parametric MP-OWA operator with $f\left(v_{i}\right)=v_{i}^{\alpha}$ can deduce the ordinary MP-OWA operator very naturally with $\alpha=1$. But parametric MP-OWA operator with other forms cannot do it. (2) The parameter in power function and other functions does not have any common, which makes parametric MP-OWA operator different from both in expressions and final aggregation results, so that, they do not need to be put together and compared with each other. (3) Because we have extended the MP-OWA operator to the parametric format, we can compare the results on various parameter values. But the comparisons of both different function formats and different parameter values of each format will be complicated; neither much facts, nor much help to problem understanding can be observed.

From (13), the maximum frequency vector $V_{f}$ with power function can also be denoted as follows:

$$
\begin{align*}
V_{f} & =\left(f\left(v_{1}\right), f\left(v_{2}\right), \ldots, f\left(v_{n}\right)\right)^{T}  \tag{29}\\
& =\left(v_{1}^{r}, v_{2}^{r}, \ldots, v_{n}^{r}\right)^{T} .
\end{align*}
$$

For parametric MP-OWA operator, $f\left(v_{i}\right)=v_{i}^{r}$ is a monotonic function with argument $v_{i}$. If parameter $r>0, f\left(v_{i}\right)$ increases with $v_{i}$. With the increasing of $v_{i}$, larger (smaller) aggregated elements will be given more (less) emphasis. If $r<0, f\left(v_{i}\right)$ decreases with $v_{i}$. With the increasing of $v_{i}$, larger (smaller) aggregated elements will be given less (more) emphasis.

Therefore, if the decision maker wants to put more emphasis on large aggregated elements and less emphasis on small aggregated elements, he or she can choose $r>0$; if he or she wants to put more emphasis on small aggregated elements and less emphasis on large elements, $r<0$ can be selected.

Some properties of parametric MP-OWA operator with power function $f\left(v_{i}\right)=v_{i}^{r}$ are discussed in the following.

Theorem 6. Assume $F_{f}$ is the parametric MP-OWA operator, $f\left(v_{i}\right)=v_{i}^{r}$ is the ith value of set $V$ and $x_{i}$ is the $i$ th value of set $X$.
(1) For $r \rightarrow-\infty$, the orness is $(n-k) /(n-1)$, and $F_{f}(X)=((n-k) /(n-1)) x_{k}$, where $k$ is the index of the $\min _{1 \leqslant i \leqslant n}\left\{v_{i}\right\}$.
(2) For $r=0$, the orness is $1 / 2$, and $F_{f}(X)=\operatorname{avg}\left\{x_{i}\right\}$.
(3) For $r \rightarrow+\infty$, the orness is $(n-l) /(n-1)$, and $F_{f}(X)=((n-l) /(n-1)) x_{l}$, where $l$ is the index of the $\max _{1 \leqslant i \leqslant n}\left\{v_{i}\right\}$.

Proof. See Appendix A.
Remark 7. By using different values of parameter $r$ for parametric MP-OWA operator, people can get different weighting vectors for decision making. For example, if the decision makers have no subjective preference for aggregated elements, they can select $r=0$ or MP-OWA operator. If they want to underweight large aggregated elements and overweight small aggregated elements, parameter $r<0$ is the right choice; when the parameter $r$ decreases to a certain negative number, the weights according to large aggregated elements reach zero; that is, the decision makers would neglect the influence of large aggregated elements and stress the small elements to the ultimate aggregation results. On the contrary, they can choose $r>0$.

Theorem 8. Assume $F_{f}$ is the parametric MP-OWA operator, $f\left(v_{i}\right)$ is the ith values of the set $V$, and $f\left(v_{i}\right)=v_{i}^{r}(i \in[1, n])$. If $r_{1}>r_{2}$, then orness $\left(W_{r_{1}}\right)>\operatorname{orness}\left(W_{r_{2}}\right)$.

Proof. See Appendix B.
3.3. Advantages of the Parametric MP-OWA Operator in Decision Making. Compared with the MP-OWA operator, the advantages of the parametric MP-OWA operator are summarized as follows:
(1) It extends the MP-OWA operator to a parametric form, which brings about more flexibility in practice. The parametric MP-OWA operator can generate multiple weighting vectors through changing the values of the parameter $r$; people may select appropriate weighting vector to reflect their preferences,
which provide more flexibility for decision making. However, the MP-OWA operator obtains merely one weighting vector, which does not reflect any attitude of the decision makers to the aggregated elements, and people could not change the ultimate aggregation result any more.
(2) It provides a power function as a specific form to compute the weighting vector. Decision makers can choose different values of parameter $r$ according to their interest and actual application context.
(3) It offers another kind of method for problems concentrated on ranking search engines. Parametric MPOWA operator is based on the use of multiple decision making process, where a group of queries retrieved from selected search engines are used to look for an optimal ranking of the search engines. It can also identify which are the best search engines at the same time.
(4) It is necessary to stress that the proposed method can develop an amazingly wide range of decision making problems with preference relations, such as information aggregation and group decision making.

## 4. The Application of Parametric MP-OWA Operator in Ranking Internet Search Engine Results

4.1. Background. Emrouznejad [3] used OWA operator to measure the performance of search engines by factors such as average click rate, total relevancy of the returned results, and the variance of the clicked results. In their study, a number of students were asked to execute the sample search queries. They classified the results into three categories: relevant, undecided, and irrelevant documents, whose values are 2,1 , and 0 , respectively. Each participant was asked to evaluate the result items and the results are shown denoted as matrix $Z$ in Table 3.

The frequencies of all scales for each query are shown in Table 4.
4.2. Computing Process. To further understand what the influence of parametric MP-OWA operator on the results of decision making will be, the weighting vectors, aggregation results, and ranking lists are computed and compared with the MP-OWA operator.

From (10), it is obvious that the maximum frequency of each query in Table 4 is

$$
\begin{equation*}
V=(9,7,5,8,6,5,4,7,6,4,6,6)^{T} \tag{30}
\end{equation*}
$$

Next, we will use $r=-4,-3,-2,-1,0,1,2,3,4$ of power function for parametric MP-OWA operator to rank the search engines, and the ranks are compared with those of MP-OWA operator. Take $r=2$, for example; the computing process is as follows.

From (25), we get

$$
\begin{equation*}
f\left(v_{i}\right)=v_{i}^{2}, \quad i \in[1,12] \tag{31}
\end{equation*}
$$

Table 3: Matrix of judgment for sample queries.

| Queries/search engines | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| LookSmart | 2 | 1 | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 1 |
| Lycos | 2 | 1 | 0 | 2 | 1 | 1 | 2 | 2 | 0 | 1 | 1 | 2 |
| Altavista | 2 | 2 | 1 | 2 | 1 | 0 | 2 | 1 | 2 | 2 | 1 | 0 |
| Msn | 2 | 1 | 2 | 0 | 0 | 2 | 1 | 2 | 2 | 1 | 1 | 2 |
| Yahoo | 1 | 2 | 2 | 2 | 1 | 1 | 0 | 0 | 2 | 2 | 1 | 1 |
| Teoma | 2 | 2 | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 2 | 1 | 0 |
| WiseNut | 2 | 1 | 2 | 2 | 1 | 0 | 1 | 2 | 2 | 0 | 0 | 0 |
| MetaCrawler | 1 | 2 | 0 | 2 | 2 | 2 | 0 | 2 | 0 | 1 | 2 | 2 |
| ProFusion | 2 | 2 | 2 | 0 | 1 | 1 | 2 | 2 | 2 | 0 | 1 | 2 |
| WebFusion-Max | 2 | 2 | 1 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 2 | 2 |
| WebFusion-OWA | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |

Table 4: The frequencies of all scales for each query.

| Queries/scales | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 4 | 2 | 3 | 2 | 4 | 1 | 3 | 3 | 1 | 3 |
| 1 | 2 | 4 | 2 | 1 | 6 | 4 | 3 | 3 | 2 | 4 | 4 | 2 |
| 2 | 9 | 7 | 5 | 8 | 2 | 5 | 4 | 7 | 6 | 4 | 6 | 6 |

From (29), the maximum frequency vector with power function is

$$
\begin{align*}
V_{f(r=2)} & =\left(v_{1}^{2}, v_{2}^{2}, \ldots, v_{12}^{2}\right)^{T}  \tag{32}\\
& =(81,49,25,64,36,25,16,49,36,16,36,36)^{T} .
\end{align*}
$$

Take the result $V_{f(r=2)}$ into (26); obtain the corresponding weighting vector $W_{f}$ as follows:

$$
\begin{align*}
W_{f(r=2)}= & \left(\frac{f\left(v_{1}\right)}{\sum_{j=1}^{12} f\left(v_{j}\right)}, \frac{f\left(v_{2}\right)}{\sum_{j=1}^{12} f\left(v_{j}\right)}, \ldots, \frac{f\left(v_{12}\right)}{\sum_{j=1}^{12} f\left(v_{j}\right)}\right)^{T} \\
= & (0.17,0.10,0.05,0.14,0.08,0.05,0.03,0.10, \\
& 0.08,0.03,0.08,0.08)^{T} . \tag{33}
\end{align*}
$$

Take the result $W_{f(r=2)}$ into (33) and the matrix $Z$ of Table 3 into (27); the aggregation result is

$$
\begin{align*}
F_{f(r=2)}= & Z W_{f} \\
= & (1.17,1.39,1.44,1.44,1.39,1.41,  \tag{34}\\
& 1.28,1.44,1.48,1.55,1.74)^{T} .
\end{align*}
$$

It is noticed that matrix $Z$ of Table 3 must be ordered decreasingly in each row before information aggregation.

With the same method, we get other aggregation results with parameter $r=-4,-3,-2,-1,0,1,3,4$, which are displayed in Table 5, the last column of which is calculated with the MP-OWA operator by Emrouznejad [3].

Correspondingly, the aggregation results of parametric MP-OWA operator with parameter $r=-4,-3,-2,-1,0,1$, 2, 3, 4 and MP-OWA operator are listed in Table 6. And the ranks given to each search engine using parametric MP-OWA operator with power function and MP-OWA operator are shown in Table 7.

### 4.3. Comparisons and Some Discussions

(1) From Table 5, it is seen that if $r>0$, the larger (smaller) of the values $r$ are, the larger (smaller) of the values $f\left(v_{i}\right)$ are, and the weights of search engines become larger (smaller) correspondingly. That is, more (less) emphasis would be put on larger (smaller) aggregated elements. For example, no matter $r=$ $4,3,2$, or $1, v_{1}=9$ has the largest weights, whereas $v_{4}=v_{7}=4$ has the smallest weights.
(2) When $r=4, w_{10}=0.01$; that is, there is almost no emphasis put on the smallest aggregated element. As the monotonicity of function $f\left(v_{i}\right)$ with $v_{i}$, if $r$ continues to increase, there will appear more zero weights, and the aggregation results may lose more information.
(3) If $r \leqslant 0$, the larger (smaller) of the values $v_{i}$ are, the smaller (larger) of the values $f\left(v_{i}\right)$ are, and the weights of search engines become smaller (larger) correspondingly. That is, more (less) emphasis would be put on smaller (bigger) aggregated elements. For example, no matter what $r=0,-1,-2,-3$, or -4 , $v_{1}=9$ has the smallest weights, whereas $v_{4}=v_{7}=4$ has the largest weights.
(4) When $r=-1, w_{1}=0$; that is, there is no emphasis put on the largest aggregated element. As the monotonicity of function $f\left(v_{i}\right)$ with $v_{i}$, if $r$ continues to

TABLE 5: Weights given to search engines with different values of parameter $r$.

| Methods/weights |  | Parametric Mp-OWA operator parameter $r$ |  |  |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $V_{0}$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | MP-OWA operator |
| $w_{1}$ | 9 | 0.01 | 0.02 | 0.03 | 0.05 | 0.08 | $\mathbf{0 . 1 2}$ | 0.17 | 0.23 | 0.29 | $\mathbf{0 . 1 2}$ |
| $w_{2}$ | 7 | 0.03 | 0.04 | 0.05 | 0.07 | 0.08 | $\mathbf{0 . 1 0}$ | 0.10 | 0.11 | 0.11 | $\mathbf{0 . 1 0}$ |
| $w_{3}$ | 5 | 0.10 | 0.11 | 0.10 | 0.10 | 0.08 | $\mathbf{0 . 0 7}$ | 0.05 | 0.04 | 0.03 | $\mathbf{0 . 0 7}$ |
| $w_{4}$ | 8 | 0.02 | 0.03 | 0.04 | 0.06 | 0.08 | $\mathbf{0 . 1 1}$ | 0.14 | 0.16 | 0.18 | $\mathbf{0 . 1 1}$ |
| $w_{5}$ | 6 | 0.05 | 0.06 | 0.07 | 0.08 | 0.08 | $\mathbf{0 . 0 8}$ | 0.08 | 0.07 | 0.06 | $\mathbf{0 . 0 8}$ |
| $w_{6}$ | 5 | 0.10 | 0.11 | 0.10 | 0.10 | 0.08 | $\mathbf{0 . 0 7}$ | 0.05 | 0.04 | 0.03 | $\mathbf{0 . 0 7}$ |
| $w_{7}$ | 4 | 0.25 | 0.21 | 0.16 | 0.12 | 0.08 | $\mathbf{0 . 0 5}$ | 0.03 | 0.02 | 0.01 | $\mathbf{0 . 0 5}$ |
| $w_{8}$ | 7 | 0.03 | 0.04 | 0.05 | 0.07 | 0.08 | $\mathbf{0 . 1 0}$ | 0.10 | 0.11 | 0.11 | $\mathbf{0 . 1 0}$ |
| $w_{9}$ | 6 | 0.05 | 0.06 | 0.07 | 0.08 | 0.08 | $\mathbf{0 . 0 8}$ | 0.08 | 0.07 | 0.06 | $\mathbf{0 . 0 8}$ |
| $w_{10}$ | 4 | 0.25 | 0.21 | 0.16 | 0.12 | 0.08 | $\mathbf{0 . 0 5}$ | 0.03 | 0.02 | 0.01 | $\mathbf{0 . 0 5}$ |
| $w_{11}$ | 6 | 0.05 | 0.06 | 0.07 | 0.08 | 0.08 | $\mathbf{0 . 0 8}$ | 0.08 | 0.07 | 0.06 | $\mathbf{0 . 0 8}$ |
| $w_{12}$ | 6 | 0.05 | 0.06 | 0.07 | 0.08 | 0.08 | $\mathbf{0 . 0 8}$ | 0.08 | 0.07 | 0.06 | $\mathbf{0 . 0 8}$ |

Table 6: Aggregation results given to search engines with different values of parameter $r$.

| Methods/search engines | Parametric Mp-OWA operator with parameter $r$ |  |  |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | MP-OWA operator |
| LookSmart | 0.77 | 0.82 | 0.87 | 0.93 | 1 | $\mathbf{1 . 0 8}$ | 1.17 | 1.27 | 1.38 | $\mathbf{1 . 0 8}$ |
| Lycos | 1.1 | 1.13 | 1.16 | 1.2 | 1.25 | $\mathbf{1 . 3 2}$ | 1.39 | 1.47 | 1.55 | $\mathbf{1 . 3 2}$ |
| Altavista | 1.21 | 1.24 | 1.26 | 1.29 | 1.33 | $\mathbf{1 . 3 8}$ | 1.44 | 1.51 | 1.58 | $\mathbf{1 . 3 8}$ |
| MSN | 1.21 | 1.24 | 1.26 | 1.29 | 1.33 | $\mathbf{1 . 3 8}$ | 1.44 | 1.51 | 1.58 | $\mathbf{1 . 3 8}$ |
| Yahoo | 1.1 | 1.13 | 1.16 | 1.2 | 1.25 | $\mathbf{1 . 3 2}$ | 1.39 | 1.47 | 1.55 | $\mathbf{1 . 3 2}$ |
| Teoma | 0.95 | 1.03 | 1.1 | 1.17 | 1.25 | $\mathbf{1 . 3 3}$ | 1.41 | 1.49 | 1.57 | $\mathbf{1 . 3 3}$ |
| WiseNut | 0.80 | 0.86 | 0.92 | 1 | 1.08 | $\mathbf{1 . 1 8}$ | 1.28 | 1.38 | 1.48 | $\mathbf{1 . 1 8}$ |
| MetaCrawler | 1.21 | 1.24 | 1.26 | 1.29 | 1.33 | $\mathbf{1 . 3 8}$ | 1.44 | 1.51 | 1.58 | $\mathbf{1 . 3 8}$ |
| ProFusion | 1.46 | 1.45 | 1.42 | 1.41 | 1.42 | $\mathbf{1 . 4 4}$ | 1.48 | 1.53 | 1.59 | $\mathbf{1 . 4 4}$ |
| WebFusion-Max | 1.51 | 1.51 | 1.5 | 1.49 | 1.5 | $\mathbf{1 . 5 2}$ | 1.55 | 1.6 | 1.65 | $\mathbf{1 . 5 2}$ |
| WebFusion-OWA | 1.59 | 1.61 | 1.62 | 1.64 | 1.67 | $\mathbf{1 . 7 0}$ | 1.74 | 1.78 | 1.82 | $\mathbf{1 . 7 0}$ |

decrease, there will appear more zero weights, and the aggregation would lose more information.
(5) When $r=1$, the weights and the aggregation results are the same as those of MP-OWA operator, which are labeled in bold in Tables 5 and 6.
It is shown that MP-OWA operator is a special case of parametric MP-OWA operator with function function on condition of $r=1$.
(6) Here, we only list a few values of parametric MP-OWA operator with power function, but we have included all the ranking with this method.
Because when $r>0$, although the weight and the aggregation results of each search engines both change steadily, the rank remains the same; when $r \leqslant$ 0 , the rank shows the similar regularity as well. In other words, with different values of parameter $r$, we get two kinds of aggregation results; the conditions are $r>0$ and $r \leqslant 0$.
(7) It can also be seen that the ranks of most search engines on each method keep the same, especially the WebFusion-OWA, WebFusion-Max, ProFusion,
and LookSmart. It is noticeably that no matter what methods we use, search engines WebFusion-OWA, WebFusion-Max, ProFusion, and LookSmart remains in the first, second, and third place, respectively. From the result, we also deduce the best search engines indirectly.

## 5. Conclusions

We have presented a new kind of neat OWA operator based on MP-OWA operator in aggregation families when considering the decision maker's preference for all alternatives across the criteria. It is very useful for decision makers, since it not only considers the preference of alternatives across all the criteria, but also provides multiple aggregation results according to their preferences and application context to choose. We have discussed several properties and have studied particular cases such as minimum, average, and maximum aggregation results.

We have analyzed the applicability of parametric MPOWA operator that gets more comprehensive results than MP-OWA operator. We have concentrated on an application in ranking search engines based on multiple decision making

Table 7: Comparison of ranks given to search engines with different values of parameter $r$.

| Methods/search engines | Parametric Mp-OWA operator with parameter $r$ |  |  |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | MP-OWA operator |
| WebFusion-OWA | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| WebFusion-Max | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| ProFusion | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 |
| Altavista | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| Msn | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| MetaCrawler | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 5 |
| Teoma | 9 | 9 | 9 | 9 | 9 | 7 | 7 | 7 | 7 | 6 |
| Lycos | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 8 | 8 | 7 |
| Yahoo | 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 | 9 | 8 |
| WiseNut | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 9 |
| LookSmart | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 10 |

process, where a group of queries are used to look for an optimal search engines list. And the decision makers can realize viewing the decision making problem completely through considering the preference relation and the corresponding parameter $r$. It is observed that no matter what values of the parameter $r$ are, the ranking of some search engines keeps the same. It also implies which the best search engines are. Finally, it should be noted that the proposed method can also be applied to a wide range of decision making problems with preference relations, such as information aggregation and group decision making.

In our future research, we expect to further propose parametric MP-OWA operator through employing other type of preference information such as linguistic variables and type-2 fuzzy set. We will develop different type of applications not only in decision theory but also in other fields such as engineering and economics.

## Appendices

## A. Proof of Theorem 6

Proof. (1) For $r \rightarrow-\infty$, we get

$$
\lim _{r \rightarrow-\infty} \frac{v_{i}^{r}}{v_{j}^{r}}=\lim _{r \rightarrow-\infty}\left(\frac{v_{i}}{v_{j}}\right)^{r}= \begin{cases}0, & \text { if } v_{i}>v_{j}  \tag{A.1}\\ 1, & \text { if } v_{i}=v_{j}, \\ +\infty, & \text { if } v_{i}<v_{j}\end{cases}
$$

From (A.1), it is also right that

$$
\lim _{r \rightarrow-\infty}\left(\frac{v_{i}}{v_{j}}\right)^{-r}= \begin{cases}+\infty, & \text { if } v_{i}>v_{j},  \tag{A.2}\\ 1, & \text { if } v_{i}=v_{j}, \\ 0, & \text { if } v_{i}<v_{j}\end{cases}
$$

Accordingly, from (A.2), when $n$ is a large integer, it is obvious that

$$
\lim _{r \rightarrow-\infty} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}=\lim _{r \rightarrow-\infty} \frac{1}{\sum_{j=1}^{n}\left(v_{i} / v_{j}\right)^{-r}}= \begin{cases}0, & \text { if } v_{i}>v_{j}  \tag{A.3}\\ 0, & \text { if } v_{i}=v_{j} \\ 1, & \text { if } v_{i}<v_{j}\end{cases}
$$

When $r \rightarrow-\infty$, combining (A.3) with (28), we obtain

$$
\begin{align*}
\operatorname{orness}(W) & =\lim _{r \rightarrow-\infty} \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}} \\
& =\lim _{r \rightarrow-\infty} \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{1}{\sum_{j=1}^{n}\left(v_{i} / v_{j}\right)^{-r}}  \tag{A.4}\\
& =\frac{n-j}{n-1},
\end{align*}
$$

where $j$ is the index of minimum $v_{i}$.
Accordingly, combine (A.3) with (27), and the aggregation result is

$$
\begin{equation*}
F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\min \left\{x_{i}\right\}=x_{k}, \tag{A.5}
\end{equation*}
$$

where $k$ is the index of minimum $x_{i}$.
(2) For $r=0$, from (28), we get

$$
\begin{align*}
\operatorname{orness}(W) & =\sum_{i=1}^{n} \frac{n-i}{n-1} \frac{1}{\sum_{j=1}^{n}\left(v_{i} / v_{j}\right)^{-r}} \\
& =\sum_{i=1}^{n} \frac{n-i}{n-1} \frac{1}{n}  \tag{A.6}\\
& =\frac{1}{2},
\end{align*}
$$

such that from (27), the aggregation result is

$$
\begin{equation*}
F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i=1}^{n} \frac{1}{n} x_{i}=\operatorname{avg}\left\{x_{i}\right\} . \tag{A.7}
\end{equation*}
$$

(3) For $r \rightarrow+\infty$, we obtain
$\lim _{r \rightarrow+\infty} \frac{v_{i}^{r}}{v_{j}^{r}}=\lim _{r \rightarrow+\infty} \frac{1}{\left(v_{i} / v_{j}\right)^{r}}= \begin{cases}+\infty, & \text { if } v_{i}>v_{j}, \\ 1, & \text { if } v_{i}=v_{j}, \\ 0, & \text { if } v_{i}<v_{j} .\end{cases}$

From (A.8), it is also right that

$$
\lim _{r \rightarrow+\infty}\left(\frac{v_{i}}{v_{j}}\right)^{-r}= \begin{cases}0, & \text { if } v_{i}>v_{j}  \tag{A.9}\\ 1, & \text { if } v_{i}=v_{j} \\ +\infty, & \text { if } v_{i}<v_{j}\end{cases}
$$

From the conclusion of (A.9), when $n$ is a large integer, it is obvious that

$$
\lim _{r \rightarrow+\infty} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}=\lim _{r \rightarrow+\infty} \frac{1}{\sum_{j=1}^{n}\left(v_{i} / v_{j}\right)^{-r}}= \begin{cases}1, & \text { if } v_{i}>v_{j}  \tag{A.10}\\ 0, & \text { if } v_{i}=v_{j} \\ 0, & \text { if } v_{i}<v_{j}\end{cases}
$$

Combining (28) with (A.10), the orness level is

$$
\begin{align*}
\operatorname{orness}(W) & =\lim _{r \rightarrow+\infty} \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{v_{i}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \\
& =\lim _{r \rightarrow+\infty} \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{1}{\sum_{j=1}^{n}\left(v_{i} / v_{j}\right)^{-r}},  \tag{A.11}\\
& =\frac{n-j}{n-1},
\end{align*}
$$

where $j$ is the index of maximum $v_{i}$.
Accordingly, combine (27) with (A.10), and the aggregation result is

$$
\begin{equation*}
F_{f}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\max \left\{x_{i}\right\}=x_{l} . \tag{A.12}
\end{equation*}
$$

The proof of Theorem 6 is completed.

## B. Proof of Theorem 8

Proof. In (29), let vector $V$ satisfy $v_{1} \geqslant v_{2} \geqslant \cdots \geqslant v_{n}$, which can be written as follows:

$$
\begin{align*}
W_{\lambda} & =\left(w_{1}, w_{2}, \ldots, w_{n}\right)^{T}, \\
& =\left(\frac{v_{1}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \frac{v_{2}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}, \ldots, \frac{v_{n}^{r}}{\sum_{j=1}^{n} v_{j}^{r}}\right)^{T} . \tag{B.1}
\end{align*}
$$

From (B.1), the derivative of function $w_{i}$ with variable $r$ is as follows:

$$
\begin{equation*}
\frac{\partial w_{i}}{\partial r}=\frac{v_{i}^{r} \ln v_{i} \sum_{i=1}^{n} v_{i}^{r}-v_{i}^{r} \sum_{i=1}^{n} v_{i}^{r} \ln v_{i}}{\left(\sum_{i=1}^{n} v_{i}^{r}\right)^{2}} \tag{B.2}
\end{equation*}
$$

Accordingly, from (28) and (B.2), the derivative of orness $\alpha$ with variable $r$ is formed as follows:

$$
\begin{align*}
\frac{\partial \alpha}{\partial r}= & \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{\partial w_{i}}{\partial r} \\
= & \sum_{i=1}^{n} \frac{n-i}{n-1} \frac{v_{i}^{r} \ln v_{i} \sum_{i=1}^{n} v_{i}^{r}-v_{i}^{r} \sum_{i=1}^{n} v_{i}^{r} \ln v_{i}}{\left(\sum_{i=1}^{n} v_{i}^{r}\right)^{2}} \\
= & \frac{1}{n-1} \frac{1}{\left(\sum_{i=1}^{n} v_{i}^{r}\right)^{2}} \\
& \quad \times\left[(n-1)\left(0+v_{1}^{r} v_{2}^{r} \ln \frac{v_{1}}{v_{2}}+\cdots+v_{1}^{r} v_{n}^{r} \ln \frac{v_{1}}{v_{n}}\right)\right. \\
& \quad+(n-2)\left(v_{2}^{r} v_{1}^{r} \ln \frac{v_{2}}{v_{1}}+0+\cdots+v_{2}^{r} v_{n}^{r} \ln \frac{v_{2}}{v_{n}}\right) \\
& \quad+(n-3)\left(v_{3}^{r} v_{1}^{r} \ln \frac{v_{3}}{v_{1}}+\cdots\right. \\
= & \frac{1}{n-1} \frac{1}{\left.\left.\left(\sum_{i=1}^{n} v_{i}^{r}\right)^{r} v_{2}^{r} \ln \frac{v_{3}}{v_{2}}+\cdots+0\right)+\cdots+0\right]} \sum_{i=1}^{n} \sum_{i<j}^{n}\left(v_{i}^{r} v_{j}^{r}\right) \frac{v_{i}}{v_{j}} .
\end{align*}
$$

Since $v_{1} \geqslant v_{2} \geqslant \cdots \geqslant v_{n}$, it is concluded that $\partial \alpha / \partial r>0$. Namely, orness $\alpha$ increases monotonically with parameter $r$.

So when $r_{1}>r_{2}$, orness $\left(W_{r_{1}}\right)>\operatorname{orness}\left(W_{r_{2}}\right)$.
The proof of Theorem 8 is completed.

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# A Fuzzy Nonlinear Programming Approach for Optimizing the Performance of a Four-Objective Fluctuation Smoothing Rule in a Wafer Fabrication Factory 

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#### Abstract

In theory, a scheduling problem can be formulated as a mathematical programming problem. In practice, dispatching rules are considered to be a more practical method of scheduling. However, the combination of mathematical programming and fuzzy dispatching rule has rarely been discussed in the literature. In this study, a fuzzy nonlinear programming (FNLP) approach is proposed for optimizing the scheduling performance of a four-factor fluctuation smoothing rule in a wafer fabrication factory. The proposed methodology considers the uncertainty in the remaining cycle time of a job and optimizes a fuzzy four-factor fluctuation-smoothing rule to sequence the jobs in front of each machine. The fuzzy four-factor fluctuation-smoothing rule has five adjustable parameters, the optimization of which results in an FNLP problem. The FNLP problem can be converted into an equivalent nonlinear programming (NLP) problem to be solved. The performance of the proposed methodology has been evaluated with a series of production simulation experiments; these experiments provide sufficient evidence to support the advantages of the proposed method over some existing scheduling methods.


## 1. Introduction

In complex manufacturing systems, such as wafer fabrication factories, job scheduling is subject to many sources of uncertainty or randomness [1]. Such uncertainty or randomness is partly due to manual operations, including the loading and unloading of jobs, the setup or repair of machines, and visual inspections. The other two sources of uncertainty, the unexpected releases of emergency orders and machine breakdowns, are beyond the control of a wafer fabrication factory. The literature provides probabilistic (stochastic) and fuzzy methods that can consider uncertainty or randomness. However, it is difficult to identify the probability distribution of each parameter, which means that a probabilistic (stochastic) method is not easy to use. In addition, fuzzy methods are advantageous because subjective factors can be considered, such as human interpretations of the scheduling performance
and the tradeoffs of different scheduling objectives. For example, a job 3 months late and a job 3 days late are both late. However, the first job is difficult to accept, while the second is still acceptable. In other words, there are different degrees of acceptance, even if both jobs have been delayed. Second, to one scheduler, one objective may be much more important than the other, but to another scheduler, the two objectives may be equally important. The concepts of "acceptability" and "relative importance" can both be suitably modeled by a fuzzy method. For example, Murata et al. [2] used trapezoidal fuzzy numbers (TrFNs) to represent the satisfaction levels of due dates. In the literature, due dates, processing times, and precedence relations have been fuzzified [2-4].

Many existing fuzzy scheduling methods take the form of fuzzy inference rules, such as "if the release time is early and the number of operations is large, then the job priority is high" [ 5,6$]$. A fuzzy scheduling system usually uses a number of
fuzzy inference rules and can be divided into two types: Mamdani [7] and Takagi-Sugeno-Kang (TSK). For example, Xiong et al. [5] scheduled a flexible manufacturing system (FMS) using two fuzzy dispatching rules of the TSK type. Murata et al. [2] used six fuzzy rules to move jobs between different priority classes. Lee et al. [8] established two fuzzy inference rules to select a combination of some existing dispatching rules for scheduling a flexible manufacturing system. Tan and Tang [9] applied Taguchi's design of experiment (DOE) techniques to improve the design of some fuzzy dispatching rules for a test facility. In Benincasa et al. [6], up to 27 fuzzy inference rules (each with three inputs and one output) were established to schedule automated guided vehicles. Dong and Liu [3] built an adaptive neurofuzzy inference system (ANFIS) to schedule a job shop. For any two jobs, the inputs to the ANFIS were the differences between the two jobs, and the output from the ANFIS determined the sequence of the two jobs. If the output was greater than zero, then the first job was to be processed before the second job. Murata et al. [2] considered a job shop with 10 machines and jobs with different priorities. A fuzzy linear programming (FLP) problem was solved to optimize the total reward. However, the fuzziness came from the satisfaction level of the due date rather than from the parameters. A good review of the literature on fuzzy scheduling methods can be found in Dubois et al. [10].

On the other hand, a scheduling problem can be formulated as a mathematical programming problem. For example, Biggs and Laughton [11] optimized a nonlinear programming (NLP) model for electric power scheduling. A recursive quadratic programming approach was proposed to solve the NLP problem. In Pedro [12], the $F_{m}|p r m u| C_{\max }$ problem was formulated as a mixed integer programming (MIP) model. The optimal solution of the mathematical programming problem gives the optimal schedule for the manufacturing system. However, sometimes the mathematical programming problem is not easy to solve, and some soft computing methods can be applied to search for the optimal solution of the mathematical programming problem [13-15]. For example, Chiang and Fu [16] minimized the number of tardy jobs for a job shop in which machines have sequence-dependent setup times. An NLP problem with a linear objective function and some quadratic constraints was solved by the application of a genetic algorithm (GA). Ishibuchi and Murata [13] applied a similar approach for multiobjective flow shop scheduling. In the NLP model of Connors et al. [17], the inventory level was estimated with a nonlinear equation and then the holding costs were minimized. Chen and Yao applied a deterministic fluid network [18] in order to find an optimal solution. Both Murata et al. [2] and Ishibuchi et al. [19] built FLP models to solve scheduling problems and maximized the satisfaction levels of the due dates. Murata et al. applied GA to solve the FLP problem. For the same purpose, Ishibuchi et al. [19] applied a hybrid GA with neighborhood search. Xia and Wu [15] combined particle swarm optimization (PSO) and simulated annealing (SA) for flexible job shop scheduling.

A summary of existing fuzzy scheduling methods is shown in Table 1. The existing approaches have the following problems.

Table 1: Some fuzzy scheduling methods.

| References | Fuzziness | Model | Soft computing <br> method |
| :--- | :---: | :---: | :---: |
| Xiong et al. [5] | Fuzzy rule | MIP | TSK |
| Lee et al. [8] | Fuzzy rule | MIP | single-rule-based |
| Benincasa et al. [6] | Fuzzy rule <br> Fuzzified <br> objective | MIP | Mamdani |
| Ishibuchi et al. [19] | Hybrid GA |  |  |
| Tan and Tang [9] | Fuzzy rule <br> Fuzzified <br> objective | MIP | Mamdani |
| Murata et al. [2] | FLP <br> Dong and Liu [3] | Fuzzy rule | MIP |

(1) Dubois et al. [10] distinguished two categories of fuzzy scheduling methods: methods that represent preference profiles and methods that model uncertainty distributions. However, in most studies that use fuzzy methods, the fuzziness comes from the fuzzification of the scheduling objective (which belongs to the first category) or from the fuzzy rules that are subjectively chosen by the scheduler rather than from fuzzy parameters (which belong to the second category). In other words, processing time, due date, and precedence relations are all free from uncertainty in these studies.
(2) Although an NLP problem is not easy to solve, the soft computing method applied to solve the NLP problem may also pose a considerable challenge.
(3) The combination of NLP with fuzziness results in a FNLP approach that considers the uncertainty of parameters and does not need to make simplifying assumptions, and therefore has the potential to solve realistic scheduling problems effectively. However, very few studies applied FNLP methods. Most of them are used for scheduling small manufacturing systems [20].

To tackle these problems, an FNLP approach is proposed in this study to optimize the performance of a job dispatching rule in a wafer fabrication factory. In other words, this study is not going to optimize the performance of a schedule for a wafer fabrication factory, which is known to be an NP-hard problem but to optimize the performance of a dispatching rule in a wafer fabrication factory. The use of some special types of fuzzy numbers establishes the corresponding subcategories for these FNLP models, such as the type-2 FNLP (with type-2 fuzzy numbers), the intervalvalued FNLP (with interval fuzzy numbers), the intuitionistic FNLP (with intuitionistic fuzzy numbers), and the intervalvalued intuitionistic FNLP [21], which has been of interest to researchers in recent years. Fares and Kaminska [22] solved two multiple-objective FNLP problems to find the optimal sets of circuit parameter values for a bipolar emitter follower circuit and an unbuffered two-stage complementary metaloxide semiconductor (CMOS) op-amp. Chen and Wang [23] defined the yield competitiveness of a semiconductor
product, which is uncertain and can be enhanced by allocating more capacity to the product; this functions in a nonlinear way. They therefore constructed an FNLP model to optimize the effects of capacity reallocation on the yield competitiveness of a semiconductor product, which was then converted to an NLP problem to be solved.

In the proposed methodology, the fuzziness comes from the uncertainty of the remaining cycle time, that is, the time still needed to complete a job; this time is highly uncertain [24]. In the proposed methodology, the remaining cycle time of a job is estimated with a triangular fuzzy number (TFN). There are various types of fuzzy numbers with different shapes. Among them, a TFN is easily implemented and has been used for numerous applications (e.g., [23-25]). Subsequently, the remaining cycle time estimate is fed into a four-factor fluctuation smoothing rule [25] to sequence the jobs in front of a machine. The four-factor fluctuation smoothing rule is fuzzified in this way, and the slack of a job is also expressed by a TFN. The fuzzy fluctuationsmoothing dispatching rule has five adjustable parameters, the optimization of which constitutes an FNLP problem. To convert the FNLP problem into a more tractable form, $\alpha$-cut operations are also applied.

The unique features of the proposed methodology include the following.
(1) Combining fuzziness and NLP: scheduling decisions represented in terms of fuzzy sets are flexible in their implementations. NLP relaxes the strict assumptions and constraints of linear programming (LP) and is highly practicable.
(2) Considering the uncertainty in the remaining cycle time: dispatching rules that consider dynamic information, such as the remaining cycle time, are more effective for highly complex manufacturing systems [26]. To this end, an effective fuzzy back propagation network (FBPN) approach is applied.
(3) Establishing a fuzzy dispatching rule directly from the existing rules: a fuzzy dispatching rule is deduced by fuzzifying the four-objective fluctuation-smoothing dispatching rule [25] and diversifying the slack. This rule accepts the fuzzy remaining cycle time as an input and uses a fuzzy value to represent the slack of each job.
(4) Diversifying the slack by solving an FNLP model: the emergence of ties may lead to incorrect scheduling results. In the proposed methodology, to reduce the number of ties, the slacks of jobs are diversified by maximizing the standard deviation [27], which leads to a FNLP problem. The FNLP problem is not easy to solve; therefore, this study applies $\alpha$-cut operations [28].
(5) Optimizing four objectives simultaneously: the proposed fuzzy rule fuses four dispatching rules in a nonlinear way. In contrast, most existing methods optimize the weighted sum of multiple objectives (e.g., $[13,15]$ ).


Figure 1: The flowchart of the proposed methodology.

The rest of this paper is organized as follows. Section 2 is divided into four parts: $\alpha$-cut operations, effective FBPN, fuzzified dispatching, and FNLP. First, the concepts of $\alpha$ cuts and $\alpha$-cut operations are introduced. The next part explains the effective FBPN approach that estimates the remaining cycle time of a job with a fuzzy number. The next part shows that the four-objective fluctuation smoothing rule is fuzzified so that it can accept the remaining cycle time estimate as an input. The final part of Section 2 explains the role of the FNLP. To obtain the best values of the parameters in the fuzzy four-objective fluctuation smoothing rule, and to diversify the slack, an FNLP model is built. To solve the FNLP problem, $\alpha$-cut operations are applied. Section 3 details how a series of production simulation experiments are carried out to assess the advantages and disadvantages of the proposed methodology. Finally, the conclusions of this study are made in Section 4.

## 2. Methodology

The flow chart of Figure 1 illustrates the steps of the proposed methodology.

Subsequently, the variables and parameters that will be used in the proposed methodology are defined as follows.
(1) $\mathrm{CT}_{j}$ : the cycle time of job $j$.
(2) $\widetilde{\mathrm{CTE}_{j}}$ : the estimated cycle time of job $j ; \widetilde{\mathrm{CTE}_{j}}=$ $\left(\mathrm{CTE}_{j 1}, \mathrm{CTE}_{j 2}, \mathrm{CTE}_{j 3}\right)$.
(3) $\mathrm{DD}_{j}$ : the due date of job $j$.
(4) $R_{j}$ : the release time of job $j ; j=1 \sim n$.
(5) $\mathrm{RCT}_{j u}$ : the remaining cycle time of job $j$ from step $u$.
(6) $\widetilde{\mathrm{RCTE}}_{j u}$ : the estimated remaining cycle time of job $j$ from step $u ; \widetilde{\operatorname{RCTE}}_{j u}=\left(\operatorname{RCTE}_{j u 1}, \operatorname{RCTE}_{j u 2}\right.$, RCTE $_{j u 3}$ ).
(7) $\mathrm{RPT}_{j u}$ : the remaining processing time of job $j$ from step $u$.
(8) $\mathrm{SCT}_{j u}$ : the step cycle time of job $j$ until step $u$.
(9) $\mathrm{SK}_{j u}$ or $\widetilde{\mathrm{SK}}_{j u}$ : the slack of job $j$ at step $u$.
(10) $t$ : the current time.
(11) $\mathrm{TPT}_{j}$ : the total processing time of job $j$.
(12) $\lambda$ : mean release rate.
(13) $x_{j p}$ : the inputs to the three-layer BPN of job $j, p=$ $1 \sim P$.
(14) $h_{l}$ : the output from hidden-layer node $l, l=1 \sim L$.
(15) $w_{l}^{o}$ : the connection weight between hidden-layer node $l$ and the output node.
(16) $w_{p l}^{h}$ : the connection weight between input node $p$ and hidden-layer node $l, P=1 \sim P ; l=1 \sim L$.
(17) $\theta_{l}^{h}$ : the threshold on hidden-layer node $l$.
(18) $\widetilde{\theta}^{o}$ : the threshold on the output node; $\widetilde{\theta}^{o}=\left(\theta_{1}^{o}, \theta_{2}^{o}, \theta_{3}^{o}\right)$.

All fuzzy parameters in the proposed methodology are given in TFNs
2.1. $\alpha$ Cuts and $\alpha$-Cut Operations. The $\alpha$-cut operations are applied to solve the FNLP problem. For this reason, the concepts of $\alpha$ cuts and $\alpha$-cut operations are introduced as follows.

Definition 1 ( $\alpha$ cuts). $\widetilde{A}$ is a fuzzy number. The $\alpha$ cut of $\widetilde{A}$ is an interval number given by

$$
\begin{equation*}
A(\alpha)=\left\{x \mid x \in R, \mu_{\widetilde{A}}(x) \geq \alpha\right\}=\left[A_{L}(\alpha), A_{R}(\alpha)\right] \tag{1}
\end{equation*}
$$

Definition 2 (arithmetic of fuzzy numbers based on $\alpha$-cut operations). Given two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, and their
$\alpha$ cuts $A(\alpha)$ and $B(\alpha)$, the arithmetic operations of $\widetilde{A}$ and $\widetilde{B}$ based on their $\alpha$ cuts are as follows:

$$
\begin{align*}
& A(\alpha)(+) B(\alpha) \\
& =\left[A_{L}(\alpha), A_{R}(\alpha)\right](+)\left[B_{L}(\alpha), B_{R}(\alpha)\right]  \tag{2}\\
& =\left[A_{L}(\alpha)+B_{L}(\alpha), A_{R}(\alpha)+B_{R}(\alpha)\right], \\
& A(\alpha)(-) B(\alpha) \\
& =\left[A_{L}(\alpha), A_{R}(\alpha)\right](-)\left[B_{L}(\alpha), B_{R}(\alpha)\right]  \tag{3}\\
& =\left[A_{L}(\alpha)-B_{R}(\alpha), A_{R}(\alpha)-B_{L}(\alpha)\right], \\
& A(\alpha)(\times) B(\alpha) \\
& =\left[A_{L}(\alpha), A_{R}(\alpha)\right](\times)\left[B_{L}(\alpha), B_{R}(\alpha)\right] \\
& =\left[\operatorname { m i n } \left(A_{L}(\alpha) B_{L}(\alpha), A_{L}(\alpha) B_{R}(\alpha),\right.\right. \\
& \left.A_{R}(\alpha) B_{L}(\alpha), A_{R}(\alpha) B_{R}(\alpha)\right),  \tag{4}\\
& \max \left(A_{L}(\alpha) B_{L}(\alpha), A_{L}(\alpha) B_{R}(\alpha),\right. \\
& \left.\left.A_{R}(\alpha) B_{L}(\alpha), A_{R}(\alpha) B_{R}(\alpha)\right)\right], \\
& k A(\alpha)=\left[k A_{L}(\alpha), k A_{R}(\alpha)\right], \quad k \geq 0,  \tag{5}\\
& A^{2}(\alpha)=\left\{\begin{array}{cl}
{\left[0, \max \left(A_{L}^{2}(\alpha), A_{R}^{2}(\alpha)\right)\right],} & \text { If } A_{L} \leq 0 \leq A_{R}, \\
{\left[\min \left(A_{L}^{2}(\alpha), A_{R}^{2}(\alpha)\right),\right.} & \\
\left.\max \left(A_{L}^{2}(\alpha), A_{R}^{2}(\alpha)\right)\right], & \text { otherwise. }
\end{array}\right.  \tag{6}\\
& A(\alpha)(/) B(\alpha) \\
& =\left[A_{L}(\alpha), A_{R}(\alpha)\right](/)\left[B_{L}(\alpha), B_{R}(\alpha)\right] \\
& =\left[\frac{A_{L}(\alpha)}{B_{R}(\alpha)}, \frac{A_{R}(\alpha)}{B_{L}(\alpha)}\right]  \tag{7}\\
& A_{L}(\alpha) \geq 0, B_{L}(\alpha)>0,
\end{align*}
$$

where $(+),(-),(\times)$, and (/) denote fuzzy addition, subtraction, multiplication, and division, respectively. Equation (5) is equivalent to

$$
\begin{align*}
A^{2}(\alpha)=[ & \max \left(A_{L}(\alpha) A_{R}(\alpha), 0\right) \\
& \cdot \min \left(\frac{A_{L}(\alpha)}{A_{R}(\alpha)}, \frac{A_{R}(\alpha)}{A_{L}(\alpha)}\right),  \tag{8}\\
& \left.\max \left(A_{L}^{2}(\alpha), A_{R}^{2}(\alpha)\right)\right]
\end{align*}
$$

if $A_{L}(\alpha), A_{R}(\alpha) \neq 0$. The $\min ()$ and $\max ()$ function can be replaced by

$$
\begin{align*}
& x=\min (a, b) \Longleftrightarrow x \leq a ; x \leq b ;(x-a)(x-b)=0 \\
& x=\max (a, b) \Longleftrightarrow x \geq a ; x \geq b ;(x-a)(x-b)=0 \tag{9}
\end{align*}
$$

Theorem 3 (average of fuzzy numbers based on $\alpha$ cuts). Given $n$ fuzzy numbers $\widetilde{A}_{j}=\left[A_{j L}(\alpha), A_{j R}(\alpha)\right], j=1 \sim n$, the average of these fuzzy numbers can be derived as

$$
\begin{equation*}
\bar{A}(\alpha)=\left[\bar{A}_{L}(\alpha), \bar{A}_{R}(\alpha)\right]=\left[\frac{\sum_{j=1}^{n} A_{j L}(\alpha)}{n}, \frac{\sum_{j=1}^{n} A_{j R}(\alpha)}{n}\right] . \tag{10}
\end{equation*}
$$

Proof. Theorem 3 can be directly derived from (2) and (5).
2.2. The Effective FBPN Approach for Estimating the Remaining Cycle Time. Before any job is scheduled, the remaining cycle time of each job needs to be estimated. In this work, the effective FBPN approach is applied and the remaining cycle time is estimated with a fuzzy value.

In the effective FBPN approach, jobs are classified into $K$ categories using fuzzy c-means (FCM). First, in order to facilitate the subsequent calculations and problem solving, all raw data are normalized [29]. Then, we place the (normalized) attributes of job $j$ in vector $\mathbf{x}_{j}=\left[x_{j p}\right]$.

FCM classifies jobs by minimizing the following objective function:

$$
\begin{equation*}
\operatorname{Min} \sum_{k=1}^{K} \sum_{j=1}^{n} \mu_{j(k)}^{m} e_{j(k)}^{2}, \tag{11}
\end{equation*}
$$

where $K$ is the required number of categories; $n$ is the number of jobs; $\mu_{j(k)}$ indicates that job $j$ belongs to category $k ; e_{j(k)}$ measures the distance from job $j$ to the centroid of category $k ; m \in[1, \infty)$ is a parameter to adjust the fuzziness and is usually set to 2 . The procedure of FCM is as follows
(1) Produce a preliminary clustering result: the performance of FCM is sensitive to the initial conditions.
(2) (Iterations) calculate the centroid of each category as

$$
\begin{gather*}
\bar{x}_{(k)}=\left\{\bar{x}_{(k) p}\right\} ; \quad p=1 \sim P, \\
\bar{x}_{(k) p}=\frac{\sum_{j=1}^{n} \mu_{j(k)}^{m} x_{j p}}{\sum_{j=1}^{n} \mu_{j(k)}^{m}}, \\
\mu_{j(k)}=\frac{1}{\sum_{q=1}^{K}\left(e_{j(k)} / e_{j(q)}\right)^{2 /(m-1)}},  \tag{12}\\
e_{j(k)}=\sqrt{\sum_{\text {all } p}\left(x_{j p}-\bar{x}_{(k) p}\right)^{2}},
\end{gather*}
$$

where $\bar{x}_{(k)}$ is the centroid of category $k . \mu_{j(k)}^{(t)}$ is the membership function that indicates job $i$ belongs to category $k$ after the $t$ th iteration.
(3) Remeasure the distance from each job to the centroid of each category, and then recalculate the corresponding membership.
(4) Stop if the following condition is met. Otherwise, return to step (2):

$$
\begin{equation*}
\max _{k} \max _{j}\left|\mu_{j(k)}^{(t)}-\mu_{j(k)}^{(t-1)}\right|<d \tag{13}
\end{equation*}
$$

where $d$ is a real number representing the threshold for the convergence of membership.
Finally, the separate distance test ( $S$ test) proposed by Xie and Beni [30] can be applied to determine the optimal number of categories $K$ :

Min

## S

$$
\begin{align*}
& \text { subject to } \quad J_{m}=\sum_{k=1}^{K} \sum_{j=1}^{n} \mu_{j(k)}^{m} e_{j(k)}^{2} \\
& e_{\min }^{2}=\min _{k 1 \neq k 2}\left(\sum_{\text {allp }}\left(\bar{x}_{(k 1) p}-\bar{x}_{(k 2) p}\right)^{2}\right)  \tag{14}\\
& \\
& S=\frac{J_{m}}{n \times e_{\min }^{2}} \\
& \\
& K \in Z^{+} .
\end{align*}
$$

The $K$ value that minimizes $S$ determines the optimal number of categories.

After clustering, a three-layer FBPN is used to estimate the cycle times of jobs for each category. The configuration of the three-layer FBPN is as follows. First, the inputs are the $P$ parameters associated with the $j$ th job. Subsequently, there is only a single hidden layer; the hidden layer has twice as many neurons as the input layer. In addition, Chen and Wang [31] and Chen and Lin [32] have described how an NLP model can be constructed to adjust the connection weights and thresholds in an FBPN; this problem is not easy to solve. In the proposed methodology, only the threshold on the output node ( $\widetilde{\theta}^{o}$ ) will be adjusted. This way is much simpler and can also achieve good results. In other words, only $\widetilde{\theta}^{o}$ is fuzzy, while the other parameters are crisp. In this way, the fuzzy remaining cycle time estimate is generated with minimal effort. This makes the FBPN approach an effective one. The output from the three-layer FBPN is the (normalized) estimated remaining cycle time $\left(N\left(\widetilde{\mathrm{RCTE}}_{j u}\right)\right)$ of the training examples, where $N()$ is the normalization function.

The procedure for determining the parameter values is now described. First, to determine the value of each parameter and $\theta_{2}^{o}$, the FBPN is treated as a crisp network. Some algorithms are applicable for this purpose, such as gradient descent algorithms, conjugate gradient algorithms, the Levenberg-Marquardt algorithm, and others. In this study, the Levenberg-Marquardt algorithm is applied. The Levenberg-Marquardt algorithm was designed for training with second-order speed without having to compute the Hessian matrix. It uses approximation and updates the network parameters in a Newton-like way [33].

Subsequently, $\theta_{3}^{o}$ is to be determined, so that the actual value will be less than the upper bound of the network output.


Figure 2: The estimation results by the effective FBPN approach.

Assume that the adjustment made to the threshold on the output node is denoted as $\Delta \theta^{\circ}=\theta_{3}^{o}-\theta_{2}^{o}$. The optimal value of $\Delta \theta^{\circ}$ should be set as follows:

$$
\begin{equation*}
\Delta \theta^{o *}=\min _{j}\left(\ln \left(\frac{1}{N\left(\mathrm{RCT}_{j u}\right)}-1\right)-\ln \left(\frac{1}{o_{j 2}}-1\right)\right) . \tag{15}
\end{equation*}
$$

In a similar way, $\theta_{1}^{o}$ can be determined so that each actual value will be greater than the appropriate lower bound. The optimal value of $\Delta \theta^{\circ}$ can be obtained as:

$$
\begin{equation*}
\Delta \theta^{* *}=\max _{j}\left(\ln \left(\frac{1}{N\left(\mathrm{RCT}_{j u}\right)}-1\right)-\ln \left(\frac{1}{o_{j 2}}-1\right)\right) . \tag{16}
\end{equation*}
$$

This FBPN approach can generate a very precise interval of the remaining cycle time for each job, thereby reducing the risk of misscheduling. An instance has been analyzed in Figure 2 to evaluate the performance of this method. To provide a comparison, a statistical analysis method is also applied to this instance, in which the relationship between the remaining cycle time and job attributes is fitted with a multiple regression equation. The results are shown in Figure 3. Compared with the effective FBPN approach, the statistical analysis method is not only inaccurate but also not precise enough. The remaining cycle time estimated by the statistical analysis method is therefore prone to errors, which may result in incorrect scheduling.
2.3. The Fuzzy Four-Objective Dispatching Rule. Lu et al. [26] proposed two fluctuation smoothing rules-the fluctuation smoothing policy for mean cycle time (FSMCT) and the fluctuation smoothing policy for variation of cycle time (FSVCT). FSMCT is aimed at minimizing the mean cycle time, while FSVCT is aimed at minimizing the variance of cycle time:


Figure 3: The estimation results by the statistical analysis approach.
(FSMCT)

$$
\begin{equation*}
\mathrm{SK}_{j u}\left(\mathrm{FSMCT}^{2}\right)=\frac{j}{\lambda}-\mathrm{RCTE}_{j u} \tag{17}
\end{equation*}
$$

(FSVCT)

$$
\begin{equation*}
\mathrm{SK}_{j u}(\mathrm{FSVCT})=R_{j}-\mathrm{RCTE}_{j u} . \tag{18}
\end{equation*}
$$

Jobs with the smallest slack values are given the highest priorities.

If the remaining cycle time is estimated with a TFN, then we have two fuzzy fluctuation smoothing rules as
(fuzzy FSMCT)

$$
\begin{equation*}
\widetilde{\mathrm{SK}}_{j u}(\mathrm{FSMCT})=\frac{j}{\lambda}-\widetilde{\mathrm{RCTE}}_{j u} \tag{19}
\end{equation*}
$$

(fuzzy FSVCT)

$$
\begin{equation*}
\widetilde{\mathrm{SK}}_{j u}(\mathrm{FSVCT})=R_{j}-\widetilde{\mathrm{RCTE}}_{j u} \tag{20}
\end{equation*}
$$

To determine the sequence of jobs, the fuzzy slacks must be compared. To this end, various methods have been proposed in the literature, such as a method based on the probability measure [34], a coefficient of variance (CV) index [35], a method that uses the area between the centroid point and the original point [36], and a method based on the fuzzy mean and standard deviation [37]. For a comparison of these methods, refer to Zhu and Xu [37]. In this study, the method based on the fuzzy mean and standard deviation is applied because it is relatively simple and can yield reasonable comparison results. To put this in context, the following theorem is introduced.

Table 2: An example $(\lambda=1.18)$.

| Number | $R_{j}$ | $j$ | $\widetilde{\mathrm{RCTE}}_{j u}$ | $\widetilde{\mathrm{SK}}_{j u}(\mathrm{FFSMCT})$ | $\widetilde{\mathrm{SK}}_{j u}(\mathrm{FFSVCT})$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 102 | 159 | $(1200,1399,1458)$ | $(-1324,-1265,-1066)$ | $(-1357,-1297,-1099)$ |
| 2 | 756 | 37 | $(976,1127,1176)$ | $(-1145,-1096,-945)$ | $(-421,-371,-221)$ |
| 3 | 826 | 37 | $(1086,1223,1299)$ | $(-1269,-1192,-1055)$ | $(-474,-397,-261)$ |
| 4 | 652 | 86 | $(1618,1822,1976)$ | $(-1904,-1750,-1546)$ | $(-1325,-1170,-967)$ |
| 5 | 208 | 55 | $(455,530,557)$ | $(-511,-484,-410)$ | $(-350,-322,-248)$ |
| 6 | 783 | 84 | $(1742,2040,2158)$ | $(-2088,-1969,-1671)$ | $(-1376,-1257,-960)$ |
| 7 | 800 | 96 | $(2039,2366,2549)$ | $(-2468,-2285,-1959)$ | $(-1750,-1566,-1240)$ |
| 8 | 478 | 52 | $(848,942,992)$ | $(-949,-898,-805)$ | $(-515,-464,-371)$ |
| 9 | 469 | 65 | $(992,1116,1176)$ | $(-1122,-1061,-938)$ | $(-708,-647,-524)$ |
| 10 | 699 | 32 | $(853,995,1031)$ | $(-1005,-968,-827)$ | $(-333,-296,-155)$ |
| 11 | 836 | 85 | $(1830,2151,2311)$ | $(-2240,-2079,-1759)$ | $(-1476,-1315,-995)$ |
| 12 | 497 | 45 | $(794,883,918)$ | $(-881,-845,-757)$ | $(-422,-386,-298)$ |
| 13 | 596 | 101 | $(1700,2047,2170)$ | $(-2086,-1962,-1615)$ | $(-1575,-1451,-1105)$ |
| 14 | 798 | 34 | $(975,1146,1256)$ | $(-1228,-1118,-948)$ | $(-459,-348,-178)$ |
| 15 | 197 | 79 | $(659,743,800)$ | $(-734,-677,-593)$ | $(-604,-546,-463)$ |
| 16 | 804 | 85 | $(1819,2092,2318)$ | $(-2247,-2020,-1748)$ | $(-1515,-1288,-1016)$ |
| 17 | 163 | 78 | $(560,647,708)$ | $(-643,-581,-495)$ | $(-546,-484,-398)$ |
| 18 | 457 | 44 | $(685,810,839)$ | $(-803,-773,-649)$ | $(-383,-353,-229)$ |
| 19 | 523 | 100 | $(1547,1851,2042)$ | $(-1958,-1767,-1463)$ | $(-1520,-1328,-1025)$ |

Theorem 4. The fuzzy mean and standard deviation of $a$ triangular fuzzy number $\widetilde{A}=\left(x_{0}-a, x_{0}, x_{0}+b\right)$ can be derived as

$$
\begin{gather*}
\mu_{\widetilde{A}}=x_{0}+\frac{b-a}{3} \\
\sigma_{\widetilde{A}}=\sqrt{\frac{a^{2}+a b+b^{2}}{18}} . \tag{21}
\end{gather*}
$$

Proof. Refer to Zhu and Xu [37]. It is, in fact, the center-ofgravity (COG) method.

The following definition details a method based on the fuzzy mean and standard deviation.

Definition 5. For any two fuzzy numbers $\widetilde{A}$ and $\widetilde{B} \in F(R)$, the sequence of $\widetilde{A}$ and $\widetilde{B}$ can be determined according to their fuzzy means and standard deviations as follows.
(1) $\mu_{\widetilde{A}}>\mu_{\widetilde{B}}$ if and only if $\widetilde{A}>\widetilde{B}$.
(2) $\mu_{\widetilde{A}}<\mu_{\widetilde{B}}$ if and only if $\widetilde{A} \prec \widetilde{B}$.
(3) If $\mu_{\widetilde{A}}=\mu_{\widetilde{B}}$, then
(i) $\sigma_{\widetilde{A}}>\sigma_{\widetilde{B}}$ if and only if $\widetilde{A} \prec \widetilde{B}$.
(ii) $\sigma_{\widetilde{A}}<\sigma_{\widetilde{B}}$ if and only if $\widetilde{A}>\widetilde{B}$.
(iii) $\sigma_{\widetilde{A}}=\sigma_{\widetilde{B}}$ if and only if $\widetilde{A}=\widetilde{B}$.

Consider the example in Table 2. The sequencing results by the two fuzzified rules are

$$
\begin{aligned}
& \text { Fuzzy FSMCT: } 7 \rightarrow 11 \rightarrow 16 \rightarrow 6 \rightarrow 13 \rightarrow 4 \rightarrow \\
& 19 \rightarrow 1 \rightarrow 3 \rightarrow 14 \rightarrow 2 \rightarrow 9 \rightarrow 10 \rightarrow 8 \rightarrow 12 \rightarrow \\
& 18 \rightarrow 15 \rightarrow 17 \rightarrow 5 .
\end{aligned}
$$

Fuzzy FSVCT: $7 \rightarrow 13 \rightarrow 19 \rightarrow 16 \rightarrow 11 \rightarrow 1 \rightarrow$ $6 \rightarrow 4 \rightarrow 9 \rightarrow 15 \rightarrow 17 \rightarrow 8 \rightarrow 3 \rightarrow 12 \rightarrow 2 \rightarrow$ $14 \rightarrow 18 \rightarrow 5 \rightarrow 10$.

Chen [25] combined four traditional dispatching rulesEDD, critical ratio (CR), the fluctuation smoothing policy for mean cycle time (FSMCT)-and the fluctuation smoothing policy for variation of cycle time (FSVCT), and proposed the four-objective dispatching rule. In the four-objective dispatching rule, the slack of job $j$ at processing step $u$ is defined as

$$
\begin{align*}
\mathrm{SK}_{j u}= & \left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u}-\min _{j} \mathrm{RCTE}_{j u}}{\max _{j} \mathrm{RCTE}_{j u}-\min _{j} \mathrm{RCTE}_{j u}}\right)^{\eta}  \tag{22}\\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{9},
\end{align*}
$$

where $\alpha, \beta, \gamma$, and $\eta$ and are positive real numbers that satisfy the following constraints:

If $\alpha=1$ then $\beta, \gamma, \vartheta=0 ; \eta=-1$, and vice versa,
If $\beta=1$ then $\alpha=0 ; \gamma, \eta, \vartheta=-1$, and vice versa,
If $\eta=1$ then $\alpha, \beta=0 ; \gamma, \mathcal{\vartheta}=1$, and vice versa.

Jobs with the smallest slack values will be given the highest priorities. There are many possible models that can form the combinations of $\alpha, \beta, \gamma, \eta$, and $\vartheta$. For example,
(Linear model) $\quad \alpha=1-2 \beta-\gamma ; \quad \gamma=\mathcal{\vartheta}=\eta+\alpha$,
(Nonlinear model) $\quad \alpha=(1-2 \beta-\gamma)^{u}, \quad u \in Z^{+}$;

$$
\begin{equation*}
\gamma=\vartheta=(\eta+\alpha)^{v}, \quad v=1,3,5, \ldots, \tag{24}
\end{equation*}
$$

(Logarithmic model 1) $\quad \alpha=\frac{\ln (2-2 \beta-\gamma)}{\ln 2}$;

$$
\gamma=\vartheta=\frac{\ln (1.5 \eta+\alpha+2.5)}{\ln 2}-1 .
$$

The values of $\alpha$ and $\beta$ are within $\left[\begin{array}{ll}0 & 1\end{array}\right]$.
If the remaining cycle time is estimated with a triangular fuzzy number, then (22) becomes

$$
\begin{align*}
\widetilde{\mathrm{SK}}_{j u}= & \left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\widetilde{\operatorname{RCTE}}_{j u}-\min _{j} \widetilde{\mathrm{RCTE}}_{j u}}{\max _{j} \widetilde{\mathrm{RCTE}}_{j u}-\min _{j} \widetilde{\mathrm{RCTE}}_{j u}}\right)^{\eta}  \tag{25}\\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{\vartheta}
\end{align*}
$$

Job $j$ is processed before job $k$ if $\widetilde{\mathrm{SK}}_{j u}<\widetilde{\mathrm{SK}}_{k u}$. In Wang et al. [27], to diversify the slack, the standard deviation of the slack was maximized:

$$
\begin{equation*}
\sigma_{\mathrm{SK}_{j u}}=\sqrt{\frac{\sum_{j=1}^{n}\left(\mathrm{SK}_{j u}-\overline{\mathrm{SK}}_{u}\right)^{2}}{n-1}} . \tag{26}
\end{equation*}
$$

When the job slack is a fuzzy value,

$$
\begin{equation*}
\sigma_{\widetilde{\mathrm{KK}}_{j u}}=\sqrt{\frac{\sum_{j=1}^{n}\left(\widetilde{\mathrm{SK}}_{j u}-\overline{\widetilde{\mathrm{SK}}}_{u}\right)^{2}}{n-1}} . \tag{27}
\end{equation*}
$$

Maximizing $\sigma_{\widetilde{\mathrm{SK}}_{j u}}$ is equivalent to maximizing $\sum_{j=1}^{n}\left(\widetilde{\mathrm{SK}}_{j u}-\overline{\widetilde{\mathrm{SK}}}_{u}\right)^{2}$. Finally, the following FNLP problem is to be solved:

$$
\begin{align*}
& \operatorname{Max} \quad \widetilde{Z}_{1}=\sum_{j=1}^{n}\left(\widetilde{S K}_{j u}-\overline{\widetilde{S K}}_{u}\right)^{2} \\
& \text { subject to } \quad \widetilde{\mathrm{SK}}_{j u}=\left(\frac{j-1}{n-1}\right)^{\alpha} \\
& \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\widetilde{\mathrm{RCTE}}_{j u}-\min _{j} \widetilde{\mathrm{RCTE}}_{j u}}{\max _{j} \widetilde{\mathrm{RCTE}}_{j u}-\min _{j} \widetilde{\mathrm{RCTE}}_{j u}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{9} \\
& j=1 \sim n \\
& \alpha=1-2 \beta-\gamma ; \quad \gamma=\mathcal{\vartheta}=\eta+\alpha \\
& \text { (or) } \alpha=(1-2 \beta-\gamma)^{u}, \quad u \in Z^{+} \text {; } \\
& \gamma=\mathfrak{\vartheta}=(\eta+\alpha)^{v}, \quad v=1,3,5, \ldots \\
& \text { (or) } \alpha=\frac{\ln (2-2 \beta-\gamma)}{\ln 2} \text {; } \\
& \gamma=\vartheta=\frac{\ln (1.5 \eta+\alpha+2.5)}{\ln 2}-1 \\
& 0 \leq \alpha, \quad \beta \leq 1 . \tag{28}
\end{align*}
$$

The proposed FNLP problem is intractable and may need to be converted into an equivalent NLP problem to be solved. First, (27) can be decomposed to

$$
\widetilde{\mathrm{SK}}_{j u}=\left(\mathrm{SK}_{j u 1}, \mathrm{SK}_{j u 2}, \mathrm{SK}_{j u 3}\right),
$$

$$
\left.\begin{array}{rl}
\mathrm{SK}_{j u 1}= & \left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta}, \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u 1}-\min _{j} \mathrm{RCTE}_{j u 1}}{\max _{j} \mathrm{RCTE}_{j u 3}-\min _{j} \mathrm{RCTE}_{j u 1}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{9}, \\
\mathrm{SK}_{j u 2}=\left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u 2}-\min _{j} \mathrm{RCTE}_{j u 2}}{\max _{j} \mathrm{RCTE}_{j u 2}-\min _{j} \mathrm{RCTE}_{j u 2}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{\vartheta} \cdot \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{9}, \\
\mathrm{SK}_{j u 3}=\left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\gamma} \\
\max _{j} R_{j}-\min _{j} R_{j}
\end{array}\right)
$$

The $\alpha$-cut of $\widetilde{S K}_{j u}$ is

$$
\begin{align*}
\mathrm{SK}_{j u}(\alpha)= & {\left[\mathrm{SK}_{j u L}(\alpha), \mathrm{SK}_{j u R}(\alpha)\right] } \\
= & {\left[\mathrm{SK}_{j u 1}+\alpha\left(\mathrm{SK}_{j u 2}-\mathrm{SK}_{j u 1}\right)\right.}  \tag{30}\\
& \left.\mathrm{SK}_{j u 3}+\alpha\left(\mathrm{SK}_{j u 2}-\mathrm{SK}_{j u 3}\right)\right] .
\end{align*}
$$

Subsequently, the objective function is equal to

$$
\begin{align*}
\widetilde{Z}_{1} & =\sum_{j=1}^{n}\left(\widetilde{\mathrm{SK}}_{j u}-\overline{\widetilde{\mathrm{SK}}}_{u}\right)^{2} \\
& =\sum_{j=1}^{n}\left(\widetilde{\mathrm{SK}}_{j u}-\frac{\sum_{k=1}^{n} \widetilde{\mathrm{SK}}_{k u}}{n}\right)^{2}  \tag{31}\\
& =\frac{1}{n^{2}} \sum_{j=1}^{n}\left((n-1) \widetilde{\mathrm{SK}}_{j u}-\sum_{k=1}^{j-1} \widetilde{\mathrm{SK}}_{k u}-\sum_{k=j+1}^{n} \widetilde{\mathrm{SK}}_{k u}\right)^{2} .
\end{align*}
$$

The $\alpha$-cut of $\widetilde{Z}_{1}$ is

$$
\begin{align*}
& {\left[Z_{1 L}(\alpha), Z_{1 R}(\alpha)\right]} \\
& \qquad \begin{array}{l}
=\frac{1}{n^{2}} \sum_{j=1}^{n}\left[(n-1) \mathrm{SK}_{j u L}(\alpha)-\sum_{k=1}^{j-1} \mathrm{SK}_{k u R}(\alpha)\right. \\
\\
\quad-\sum_{k=j+1}^{n} \mathrm{SK}_{k u R}(\alpha),(n-1) \mathrm{SK}_{j u R}(\alpha) \\
\\
\left.\quad-\sum_{k=1}^{j-1} \mathrm{SK}_{k u L}(\alpha)-\sum_{k=j+1}^{n} \mathrm{SK}_{k u L}(\alpha)\right]^{2}
\end{array}
\end{align*}
$$

where

$$
\begin{align*}
\xi(\alpha)= & (n-1) \mathrm{SK}_{j u L}(\alpha)-\sum_{k=1}^{j-1} \mathrm{SK}_{k u R}(\alpha) \\
& -\sum_{k=j+1}^{n} \mathrm{SK}_{k u R}(\alpha) \\
\zeta(\alpha)= & (n-1) \mathrm{SK}_{j u R}(\alpha)-\sum_{k=1}^{j-1} \mathrm{SK}_{k u L}(\alpha)  \tag{35}\\
& -\sum_{k=j+1}^{n} \mathrm{SK}_{k u L}(\alpha)
\end{align*}
$$

Equation (33) is equivalent to

$$
\begin{gather*}
Z_{1 L}(\alpha)=\frac{1}{n^{2}} \sum_{j=1}^{n} \Xi(\alpha) \cdot \Theta(\alpha) \\
\Xi(\alpha) \geq=\xi(\alpha) \zeta(\alpha) \\
\Xi(\alpha) \geq=0 \\
\Xi(\alpha)(\Xi(\alpha)-(\alpha) \zeta(\alpha))=0 \\
\Theta(\alpha) \leq \frac{\xi(\alpha)}{\zeta(\alpha)}  \tag{36}\\
\Theta(\alpha) \leq \frac{\zeta(\alpha)}{\xi(\alpha)}, \\
\left(\Theta(\alpha)-\frac{\zeta(\alpha)}{\xi(\alpha)}\right)\left(\Theta(\alpha)-\frac{\xi(\alpha)}{\zeta(\alpha)}\right)=0
\end{gather*}
$$

Similarly, (34) can be replaced by

$$
\begin{gather*}
Z_{1 R}(\alpha)=\frac{1}{n^{2}} \sum_{j=1}^{n} \Upsilon(\alpha), \\
\Upsilon(\alpha) \geq=\xi^{2}(\alpha),  \tag{37}\\
\Upsilon(\alpha) \geq=\zeta^{2}(\alpha), \\
\left(\Upsilon(\alpha)-\xi^{2}(\alpha)\right)\left(\Upsilon(\alpha)-\zeta^{2}(\alpha)\right)=0 .
\end{gather*}
$$

In order to facilitate the solving of the problem, the usual practice is to defuzzify the fuzzy objective function $\widetilde{Z}_{1}$, using the center-of-gravity defuzzification [28]:

$$
\begin{align*}
D\left(\widetilde{Z}_{1}\right) & =\frac{\sum_{\alpha=0}^{1} \alpha Z_{1 L}(\alpha)+\sum_{\alpha=0}^{1} \alpha Z_{1 R}(\alpha)}{\sum_{\alpha=0}^{1} \alpha+\sum_{\alpha=0}^{1} \alpha}  \tag{38}\\
& =\frac{\sum_{\alpha=0}^{1} \alpha Z_{1 L}(\alpha)+\sum_{\alpha=0}^{1} \alpha Z_{1 R}(\alpha)}{11}
\end{align*}
$$

where $D()$ is the defuzzification function. Finally, the following NLP model is optimized instead of the original FNLP problem:
$\operatorname{Max} \quad Z_{2}=\frac{\sum_{\alpha=0}^{1} \alpha Z_{1 L}(\alpha)+\sum_{\alpha=0}^{1} \alpha Z_{1 R}(\alpha)}{11}$
subject to $Z_{1 L}(\alpha)=\frac{1}{n^{2}} \sum_{j=1}^{n} \Xi(\alpha) \cdot \Theta(\alpha)$

$$
\begin{aligned}
& \Xi(\alpha) \geq=\xi(\alpha) \zeta(\alpha) \\
& \Xi(\alpha) \geq=0 \\
& \Xi(\alpha)(\Xi(\alpha)-(\alpha) \zeta(\alpha))=0 \\
& \Theta(\alpha) \leq \frac{\xi(\alpha)}{\zeta(\alpha)} \\
& \Theta(\alpha) \leq \frac{\zeta(\alpha)}{\xi(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
& \left(\Theta(\alpha)-\frac{\zeta(\alpha)}{\xi(\alpha)}\right)\left(\Theta(\alpha)-\frac{\xi(\alpha)}{\zeta(\alpha)}\right)=0 \\
& Z_{1 R}(\alpha)=\frac{1}{n^{2}} \sum_{j=1}^{n} \Upsilon(\alpha) \\
& \Upsilon(\alpha) \geq=\xi^{2}(\alpha) \\
& \Upsilon(\alpha) \geq=\zeta^{2}(\alpha) \\
& \left(\Upsilon(\alpha)-\xi^{2}(\alpha)\right)\left(\Upsilon(\alpha)-\zeta^{2}(\alpha)\right)=0 \\
& \xi(\alpha)=(n-1) \mathrm{SK}_{j u L}(\alpha) \\
& -\sum_{k=1}^{j-1} \mathrm{SK}_{k u R}(\alpha)-\sum_{k=j+1}^{n} \mathrm{SK}_{k u R}(\alpha) \\
& \zeta(\alpha)=(n-1) \operatorname{SK}_{j u R}(\alpha) \\
& -\sum_{k=1}^{j-1} \mathrm{SK}_{k u L}(\alpha)-\sum_{k=j+1}^{n} \mathrm{SK}_{k u L}(\alpha) \\
& \mathrm{SK}_{j u L}(\alpha)=\mathrm{SK}_{j u 1}+\alpha\left(\mathrm{SK}_{j u 2}-\mathrm{SK}_{j u 1}\right) \\
& \mathrm{SK}_{j u R}(\alpha)=\mathrm{SK}_{j u 3}+\alpha\left(\mathrm{SK}_{j u 2}-\mathrm{SK}_{j u 3}\right) \\
& \mathrm{SK}_{j u 1}=\left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u 1}-\min _{j} \mathrm{RCTE}_{j u 1}}{\max _{j} \mathrm{RCTE}_{j u 3}-\min _{j} \mathrm{RCTE}_{j u 1}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{\vartheta} \\
& \mathrm{SK}_{j u 2}=\left(\frac{j-1}{n-1}\right)^{\alpha} \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta} \\
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u 2}-\min _{j} \mathrm{RCTE}_{j u 2}}{\max _{j} \mathrm{RCTE}_{j u 2}-\min _{j} \mathrm{RCTE}_{j u 2}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{9} \\
& \mathrm{SK}_{j u 3}=\left(\frac{j-1}{n-1}\right)^{\alpha} \\
& \cdot\left(\frac{\mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}{\max _{j} \mathrm{RPT}_{j u}-\min _{j} \mathrm{RPT}_{j u}}\right)^{\beta}
\end{aligned}
$$

$$
\begin{align*}
& \cdot\left(\frac{R_{j}-\min _{j} R_{j}}{\max _{j} R_{j}-\min _{j} R_{j}}\right)^{\gamma} \\
& \cdot\left(\frac{\mathrm{RCTE}_{j u 3}-\min _{j} \mathrm{RCTE}_{j u 1}}{\max _{j} \mathrm{RCTE}_{j u 3}-\min _{j} \mathrm{RCTE}_{j u 1}}\right)^{\eta} \\
& \cdot\left(\frac{\mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}{\max _{j} \mathrm{SCT}_{j u}-\min _{j} \mathrm{SCT}_{j u}}\right)^{\vartheta} \\
& j=1 \sim n \\
& (\text { Linear model }) \alpha=1-2 \beta-\gamma ; \\
& \gamma=\vartheta=\eta+\alpha \\
& 0 \leq \alpha, \quad \beta \leq 1 \tag{39}
\end{align*}
$$

Consider the example in Table 3. Both the proposed methodology and Wang et al.'s method are applied to this example. In the proposed methodology, the optimal objective function value $Z_{2}^{*}$ is 187272 when the parameters $\alpha, \beta, \gamma, \eta$, and $\vartheta$ are equal to $0.980,0.560,-1.100,-1.100$, and -2.082 , respectively. The slacks of the jobs are shown in Figure 4. Please note that in this figure the $x$-axis is converted to logarithmic values for clarity. In contrast to this, Wang et al.s method cannot consider the uncertainty in the remaining cycle time, and therefore only the center value of the remaining cycle time is considered. The slacks obtained by using Wang et al.s method are shown in Figure 5, in which $\sigma_{\mathrm{SK}_{j u}}^{*}=93$. The optimal values of the parameters $\alpha, \beta, \gamma, \eta$, and $\vartheta$ are equal to $0.938,0.587,-1.112,-1.112$, and -2.050 , respectively. Obviously, one has the following.
(1) After considering the uncertainty of the remaining cycle time, the best values of the five parameters changed, and the slacks of jobs became different. This might result in different sequencing results.
(2) There were 14 ties in Wang et al.'s method. Conversely, after considering the uncertainty in the remaining cycle time, the proposed methodology successfully diversified the slacks of the jobs and reduced the number of ties to 11 . In this regard, the advantage of the proposed methodology over Wang et al.'s approach is $21 \%$.
(3) In the method of Wang et al., if there is a long tail in the remaining cycle time of a job on the right-hand side, then the slack of the job will be underestimated. Conversely, the slack will be overestimated if there is a long tail on the left-hand side.

## 3. Simulation Experiment

A real wafer fabrication factory mainly used for the production of dynamic random access memory (DRAM) was simulated. The wafer fabrication factory is located in Taiwan's Taichung Science Park and has a monthly capacity of about 25,000 wafers. However, the following assumptions were made to generate data that are less noisy than real-world data.

Table 3: An example.

| Number | $R_{j}$ | $j$ | $\widetilde{\mathrm{RCTE}}_{j u}$ | $\mathrm{SCT}_{j u}$ | $\mathrm{RPT}_{j u}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 102 | 159 | $(1200,1399,1458)$ | 881 | 560 |
| 2 | 756 | 37 | $(976,1127,1176)$ | 227 | 451 |
| 3 | 826 | 37 | $(1086,1223,1299)$ | 157 | 489 |
| 4 | 652 | 86 | $(1618,1822,1976)$ | 331 | 729 |
| 5 | 208 | 55 | $(455,530,557)$ | 775 | 212 |
| 6 | 783 | 84 | $(1742,2040,2158)$ | 200 | 816 |
| 7 | 800 | 96 | $(2039,2366,2549)$ | 183 | 946 |
| 8 | 478 | 52 | $(848,942,992)$ | 505 | 377 |
| 9 | 469 | 65 | $(992,1116,1176)$ | 514 | 446 |
| 10 | 699 | 32 | $(853,995,1031)$ | 284 | 398 |
| 11 | 836 | 85 | $(1830,2151,2311)$ | 147 | 860 |
| 12 | 497 | 45 | $(794,883,918)$ | 486 | 353 |
| 13 | 596 | 101 | $(1700,2047,2170)$ | 387 | 819 |
| 14 | 798 | 34 | $(975,1146,1256)$ | 185 | 458 |
| 15 | 197 | 79 | $(659,743,800)$ | 786 | 297 |
| 16 | 804 | 85 | $(1819,2092,2318)$ | 179 | 837 |
| 17 | 163 | 78 | $(560,647,708)$ | 820 | 259 |
| 18 | 457 | 44 | $(685,810,839)$ | 526 | 324 |
| 19 | 523 | 100 | $(1547,1851,2042)$ | 460 | 740 |



Figure 4: The fuzzy slacks obtained by the proposed methodology.
(1) The distributions of the times between machine breakdowns are exponential.
(2) The distribution of the time required to repair a machine is uniform.
(3) The percentages of jobs with different priorities released into the wafer fabrication factory are controlled.
(4) A job has equal chances to be processed on each alternative machine or head that is available at a step.
(5) A job cannot proceed to the next step until the processing of every wafer in the job has been finished.
(6) No preemption is allowed.

In the simulated wafer fabrication factory, there are more than 10 types of memory products and more than 500 workstations for performing single-wafer or batch operations using $58 \mathrm{~nm} \sim 110 \mathrm{~nm}$ technologies. Jobs released into the fabrication factory are assigned three types of priorities, that


Figure 5: The fuzzy slacks obtained by using Wang et al.'s method.
is, "normal," "hot," and "super hot." Usually, a job will only be "super hot" if it is part of an emergency order; "super hot" jobs will be processed first. The large scale and the reentrant process flows of this wafer fabrication factory exacerbate the difficulties of job dispatching. Currently, the longest average cycle time exceeds three months with a variation of more than 300 hours. The managers of this wafer fabrication factory are therefore seeking better dispatching rules to replace FIFO and EDD, in order to shorten the average cycle times and ensure on-time delivery to customers.

One hundred replications of the simulation were successively run. The simulation horizon of each replication was twenty-four months. The warm-up period was the first four months. The time required for each simulation replication was about 45 minutes using a PC with Intel Dual E2200 2.2 GHz CPUs and 1.99 G RAM.

To make comparisons with some existing approaches, eight methods were tested. FIFO, EDD, shortest remaining processing time (SRPT), CR, FSVCT, FSMCT, the nonlinear fluctuation smoothing rule (NFS), and the four-objective slack-diversifying rule (4o-SDR) [25] were applied to schedule the simulated wafer fabrication factory. The data of 1000 jobs were collected and separated by product types and priorities.

For FIFO, jobs were sequenced on each machine first by their priorities, then by their arrival times at the machine. For EDD, jobs were also sequenced first by their priorities, then by their due dates. The performance of EDD depends on how jobs' due dates are determined. In the experiment, the due date of each job was determined as follows:

$$
\begin{equation*}
\mathrm{DD}_{j}=R_{j}+(\Psi-1.5 * \text { priority }) * \mathrm{TPT}_{j}, \tag{40}
\end{equation*}
$$

where $\Psi$ indicates the cycle time multiplier.
FSVCT and FSMCT consisted of two stages. First, jobs were scheduled based on FIFO, in which the remaining cycle times of all jobs were recorded and averaged at each step. Then, FSVCT/FSMCT policy was applied to schedule the jobs based on the average remaining cycle times obtained earlier. In other words, jobs were sequenced on each machine first by their priorities, and then by their slack values, which were determined by (17) and (18). With SRPT, the remaining processing time of each job was calculated. Then, jobs were sequenced first by their priorities, then by their remaining
processing times. With CR , jobs were sequenced first by their priorities, then by their critical ratios. NFS is a nonlinear fusion of FSMCT and FSVCT. In the simulation experiment, a weight of 0.8 was given to FSMCT. With $40-\mathrm{SDR}$, the remaining cycle time of a job was estimated using the fuzzy cmeans and back propagation network (FCM-BPN) approach [31]; it was a crisp value. The five adjustable parameters were set to $(\alpha, \beta, \gamma, \eta, \vartheta)=(0.6,0.2,0,-0.6,0)$ after initial scenarios had been examined.

In the proposed methodology, the remaining cycle time of a job was estimated using the effective FBPN approach; it was a fuzzy value. After the fuzzy remaining cycle time estimate had been fed into the fuzzy four-objective fluctuation smoothing rule, an FNLP problem was solved to determine the values of the five parameters in the rule, so as to optimize the scheduling performance.

The average cycle time, cycle time standard deviation, the number of tardy jobs, and the maximum lateness of all cases were calculated to assess the scheduling performance. The results are summarized in Tables 4, 5, 6, and 7.

According to the experimental results, the following points can be made.
(1) In various respects, the proposed methodology was obviously superior to the existing dispatching rules. For example, the fuzzy four-objective fluctuation smoothing rule was the best at reducing the average cycle time for all cases. Its advantage over the current rule FIFO was $26 \%$ on average. The average cycle time is one of the most important scheduling goals of a wafer fabrication factory; such experimental results are very valuable. Of the traditional scheduling rules, SRPT performed well for reducing the average cycle times but posed the risk of high cycle time variation.
(2) On-time delivery is another important scheduling objective. The maximum lateness is often used to assess this. The proposed methodology can effectively reduce the maximum lateness and so can enhance on-time delivery. This tends to improve the firm's customer relationships. The performance levels of the two traditional methods in this field, EDD and CR, were not as good as expected.
(3) Through reduced cycle time variability, we can more accurately estimate the cycle time and promise our customer a more reliable due date. The fuzzy fourobjective fluctuation smoothing rule has very good performance in this respect, with an average advantage of $28 \%$ over the existing scheduling rules.
(4) The number of tardy jobs is another indicator by which one can assess on-time delivery. The proposed methodology outperformed the existing methods in most cases.
In order to confirm the advantages of the proposed methodology over the existing methods, a Wilcoxon signedrank test [38] was used to test the following hypotheses.
$H_{a 0}$ : When shortening the average cycle time, the scheduling performance of the proposed methodology is the same as that of the existing approach being compared.

TABLE 4: The performance levels of various approaches for average cycle time.

| Average cycle <br> time (hrs) | $A$ <br> (normal) | $A$ <br> (hot) | $A$ <br> (super hot) | $B$ <br> (normal) | $B$ <br> (hot) |
| :--- | :---: | :---: | :---: | :---: | :---: |
| FIFO | 1254 | 400 | 317 | 1278 | 426 |
| EDD | 1094 | 345 | 305 | 1433 | 438 |
| SRPT | 948 | 350 | 308 | 1737 | 457 |
| CR | 1148 | 355 | 300 | 1497 | 440 |
| FSMCT | 1313 | 347 | 293 | 1851 | 470 |
| FSVCT | 1014 | 382 | 315 | 1672 | 475 |
| NFS | 1456 | 407 | 321 | 1452 | 421 |
| 4o-SDR | 1183 | 347 | 271 | 1160 | 339 |
| The proposed | 932 | 274 | 265 | 810 | 269 |
| methodology |  |  |  |  |  |

Table 5: The performance levels of various approaches for maximum lateness.

| The <br> maximum <br> lateness (hrs) | $A$ <br> (normal) | $A$ <br> (hot) | $A$ <br> (super hot) | $B$ <br> (normal) | $B$ <br> (hot) |
| :--- | :---: | :---: | :---: | :---: | :---: |
| FIFO | 401 | -122 | 164 | 221 | 172 |
| EDD | 295 | -181 | 144 | 336 | 185 |
| SRPT | 584 | -142 | 174 | 718 | 194 |
| CR | 302 | -159 | 138 | 423 | 192 |
| FSMCT | 875 | -165 | 125 | 856 | 171 |
| FSVCT | 706 | -112 | 174 | 686 | 260 |
| NFS | 627 | 10 | 161 | 331 | 151 |
| 4o-SDR | 360 | -152 | 118 | 21 | 94 |
| The proposed | 287 | -145 | 112 | 25 | 106 |
| methodology |  |  |  |  |  |

TABLE 6: The performance levels of various approaches for cycle time standard deviation.

| Cycle time <br> standard <br> deviation (hrs) | $A$ <br> (normal) | $A$ <br> (hot) | $A$ <br> (super hot) | $B$ <br> (normal) | $B$ <br> (hot) |
| :--- | :---: | :---: | :---: | :---: | :---: |
| FIFO | 55 | 24 | 25 | 87 | 51 |
| EDD | 129 | 25 | 22 | 50 | 63 |
| SRPT | 248 | 31 | 22 | 106 | 53 |
| CR | 69 | 29 | 18 | 58 | 53 |
| FSMCT | 419 | 33 | 16 | 129 | 104 |
| FSVCT | 280 | 37 | 27 | 201 | 77 |
| NFS | 64 | 40 | 19 | 37 | 26 |
| 4o-SDR | 71 | 41 | 22 | 30 | 29 |
| The proposed | 68 | 20 | 23 | 27 | 34 |
| methodology |  |  |  |  |  |

$H_{a 1}$ : When shortening the average cycle time, the scheduling performance of the proposed methodology is better than that of the existing approach being compared.

Table 7: The performance levels of various approaches for number of tardy jobs.

| Number of <br> tardy jobs | $A$ <br> (normal) | $A$ <br> (hot) | $A$ <br> (super hot) | $B$ <br> (normal) | $B$ <br> (hot) |
| :--- | :---: | :---: | :---: | :---: | :---: |
| FIFO | 79 | 0 | 12 | 16 | 5 |
| EDD | 71 | 0 | 12 | 19 | 5 |
| SRPT | 37 | 0 | 12 | 19 | 5 |
| CR | 79 | 0 | 12 | 19 | 5 |
| FSMCT | 58 | 0 | 12 | 19 | 5 |
| FSVCT | 56 | 0 | 12 | 18 | 5 |
| NFS | 58 | 0 | 12 | 19 | 5 |
| 4o-SDR | 79 | 0 | 12 | 19 | 5 |
| The proposed | 37 | 0 | 12 | 19 | 5 |
| methodology |  |  |  |  |  |

$H_{b 0}$ : When reducing the maximum lateness, the scheduling performance of the proposed methodology is the same as that of the existing approach being compared.
$H_{b 1}$ : When reducing the maximum lateness, the scheduling performance of the proposed methodology is better than that of the existing approach being compared.
$H_{c 0}$ : When reducing the cycle time standard deviation, the scheduling performance of the proposed methodology is the same as that of the existing approach being compared.
$H_{c 1}$ : When reducing the cycle time standard deviation, the scheduling performance of the proposed methodology is better than that of the existing approach being compared.
$H_{d 0}$ : When reducing the number of tardy jobs, the scheduling performance of the proposed methodology is the same as that of the existing approach being compared.
$H_{d 1}$ : When reducing the number of tardy jobs, the scheduling performance of the proposed methodology is better than that of the existing approach being compared.

The results are summarized in Table 8. The null hypothesis $H_{a 0}$ was rejected at $\alpha=0.025$, which showed that the fuzzy slack-diversifying fluctuation-smoothing rule was superior to seven existing approaches at reducing the average cycle time. With regard to maximum lateness, the advantage of the fuzzy slack-diversifying fluctuation-smoothing rule over FIFO, SRPT, and FSVCT was significant. Similar results could be observed with cycle time standard deviation. However, the advantage of the fuzzy slack-diversifying fluctuationsmoothing rule was not statistically significant for the number of tardy jobs.

## 4. Conclusions and Directions for Future Research

Multiobjective scheduling is an important task in a wafer fabrication factory. It is also a difficult task owing to the uncertainty and complexity of the wafer fabrication system.

Table 8: Results of the Wilcoxon sign-rank test.

|  | $H_{a 0}$ <br> (the average <br> cycle time) | $H_{b 0}$ <br> (the <br> maximum <br> lateness) | $H_{c 0}$ <br> (cycle time <br> standard <br> deviation) | $H_{d 0}$ <br> (the number <br> of tardy jobs) |
| :--- | :--- | :---: | :---: | :---: |
| FIFO | $2.02^{* *}$ | $2.02^{* *}$ | 1.21 | 0.54 |
| EDD | $2.02^{* *}$ | 1.21 | $1.75^{*}$ | 1.21 |
| SRPT | $2.02^{* *}$ | $2.02^{* *}$ | $1.75^{*}$ | 0.67 |
| CR | $2.02^{* *}$ | $1.75^{*}$ | 1.48 | 1.21 |
| FSMCT | $2.02^{* *}$ | 1.48 | $1.75^{*}$ | 1.21 |
| FSVCT | $2.02^{* *}$ | $2.02^{* *}$ | $2.02^{* *}$ | 0.54 |
| NFS | $2.02^{* *}$ | $2.02^{* *}$ | 0.54 | 1.21 |
| $4 \mathrm{o}-$ SDR | $2.02^{* *}$ | -0.08 | 0.34 | 0.76 |
| ${ }^{*} P<0.05$. |  |  |  |  |
| ${ }^{* * *} P<0.025$. |  |  |  |  |
| ${ }^{* * *} P<0.01$. |  |  |  |  |

This study demonstrates that an FNLP approach can consider such uncertainties and optimize the performance of multiobjective scheduling in a wafer fabrication factory.

The proposed methodology starts from the estimation of the remaining cycle time for each job. To this end, an effective FBPN approach has been proposed. The estimated remaining cycle time from the FBPN is a fuzzy number. After the fuzzy remaining cycle time estimate is fed to the fourfactor fluctuation smoothing rule, the rule is fuzzified, and the slack of each job is expressed by a fuzzy number. To reduce the number of ties, the slacks of jobs need to be diversified, which results in an FNLP problem. Since the FNLP problem is not easy to solve, some $\alpha$-cut operations are applied to convert it into an equivalent NLP problem.

After a simulation study, the following points are concluded
(1) The simulation experiment results showed that the proposed method indeed enhanced the scheduling performance of the wafer fabrication factory in four respects-average cycle time, maximum lateness, cycle time standard deviation, and number of tardy jobs.
(2) Consideration of the uncertainty in the remaining cycle time has a considerable degree of influence on scheduling performance. The proposed FBPN approach not only enhances the accuracy of estimating the remaining cycle time but can also generate a precise range for the remaining cycle time. This effectively avoids incorrect scheduling.
(3) The experimental results in this study also confirmed that the scheduling performance of a complex production system can be significantly improved by optimizing the existing dispatching rule with an FNLP model.

However, the proposed methodology only optimizes the performance of a dispatching rule in a wafer fabrication factory; this does not optimize every aspect of scheduling for a wafer fabrication factory. Although slack diversification
optimizes the four-objective fluctuation smoothing rule, there are other approaches that can achieve the same effect. Further, some soft computing techniques can be applied to solve the FNLP problem. All of these issues constitute directions for future research.

## Abbreviations

| 4o-SDR: | Four-objective slackdiversifying |
| :--- | :--- |
| ANFIS: | Adaptive neurofuzzy inference system |
| COG: | Center of gravity |
| CR: | Critical ratio |
| CV: | Coefficient of variance |
| DOE: | Design of experiment |
| EDD: | Earliest due date |
| FBPN: | Fuzzy back propagation network |
| FCM: | Fuzzy c-means |
| FLP: | Fuzzy linear programming |
| FMS: | Flexible manufacturing system |
| FNLP: | Fuzzy nonlinear programming |
| FSMCT: | The fluctuation smoothing policy for mean |
|  | cycle time |
| FSVCT: | The fluctuation smoothing policy for cycle |
|  | time variation |
| GA: | Genetic algorithm |
| LP: | Linear programming |
| MIP: | Mixed integer programming |
| NFS: | Nonlinear fluctuation smoothing |
| NLP: | Nonlinear programming |
| PSO: | Particle swarm optimization |
| SA: | Simulated annealing |
| SRPT: | Shortest remaining processing time |
| TFN: | Triangular fuzzy number |
| TrFN: | Trapezoidal fuzzy number |
| TSK: | Takagi-Sugeno-Kang. |

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