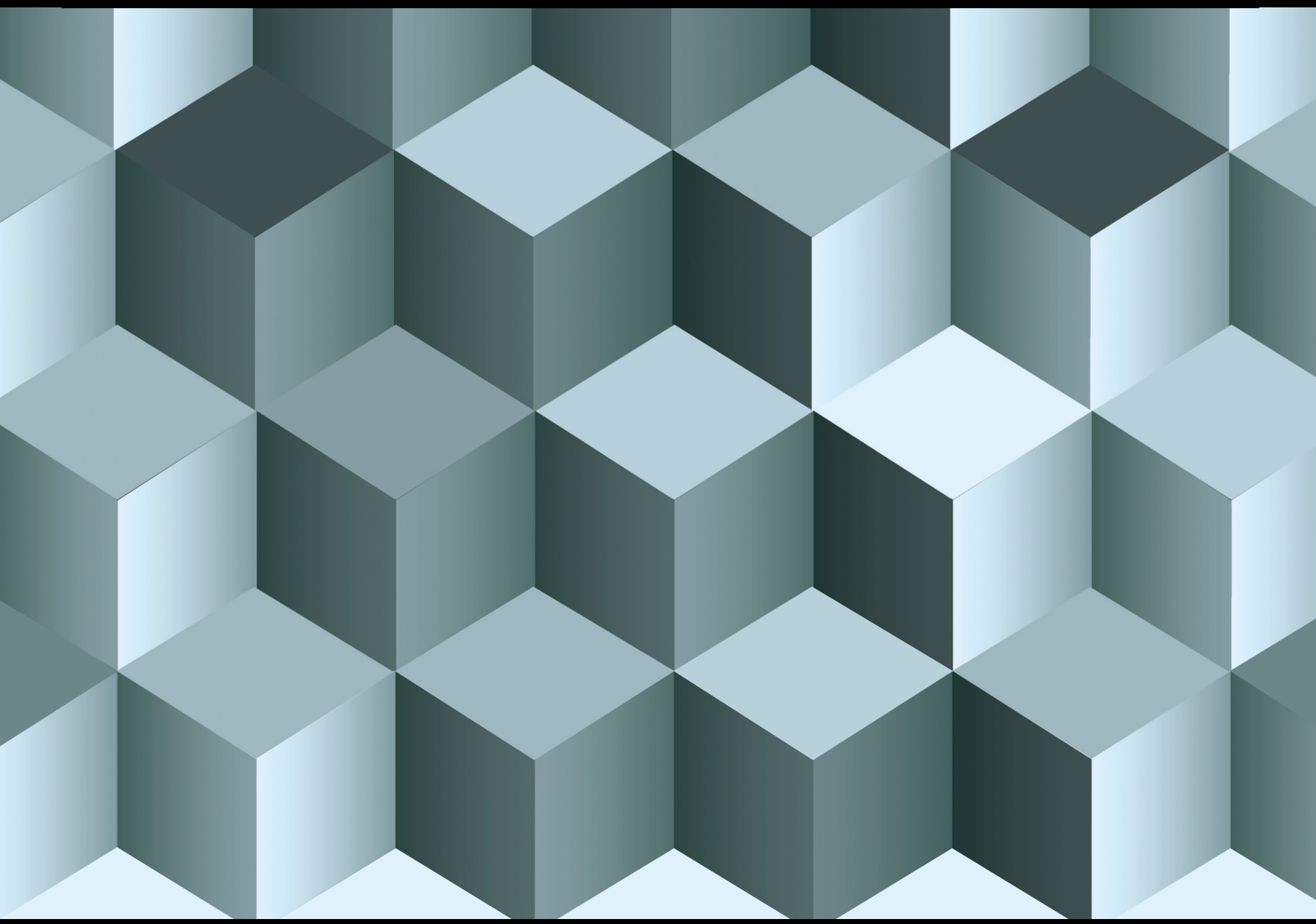
Journal of Function Spaces

Recent Advances in Function Spaces and its Applications in Fractional Differential Equations 2020

Lead Guest Editor: Xinguang Zhang
Guest Editors: Lishan Liu, Yong Hong Wu, Chuanjun Chen, and Liguang Wang



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Research Article

Higher-Order Compact Finite Difference for Certain PDEs in Arbitrary Dimensions

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In this paper, we first present the expression of a model of a fourth-order compact finite difference (CFD) scheme for the convection diffusion equation with variable convection coefficient. Then, we also obtain the fourth-order CFD schemes of the diffusion equation with variable diffusion coefficients. In addition, a fine description of the sixth-order CFD schemes is also developed for equations with constant coefficients, which is used to discuss certain partial differential equations (PDEs) with arbitrary dimensions. In this paper, various ways of numerical test calculations are prepared to evaluate performance of the fourth-order CFD and sixth-order CFD schemes, respectively, and the empirical results are proved to verify the effectiveness of the schemes in this paper.

1. Introduction

The standard strategy associated with generating higher order finite difference schemes to expand the stencil is proposed by Leonard [1]. The major disadvantage of these approaches is to widen the computational stencil with the increasing order of the approximation, leading to larger matrix bandwidths, which shall complicate the numerical analysis near the boundaries, while communication to implement on parallel computer architecture is raising demand. In consideration of the problems caused by noncompact finite difference methods, it is desirable to develop a class of schemes involving compact with high order.

In recent years, a great deal of efforts has been devoted to developing the designing schemes of CFD for solving various partial differential equations (see, e.g., [2–7]). A derivation of fourth-order and sixth-order compact difference schemes for the three dimensional Poisson equation is developed by Zhai and Feng [8] finite volume (FV) method, based on two different types of dual partitions. A new design of sixth-order compact finite-difference method with a nine-point stencil is developed by Nabavi et al. [9] to solve the Helmholtz equation in two-dimensional domain under the circumstance of Dirichlet and Neumann boundary. With the idea of the immersed interface method, third- and fourth-order compact finite difference schemes were proposed for solving the Helmholtz equations with discontinuous coefficient [10, 11]. It is worth noting that the advert of [12, 13] a new high-order finite difference discretization strategy, based on the Richardson extrapolation technique and an operator interpolation scheme, is explored to solve convection diffusion equations [14] which exploits an innovative adaptive scheme in terms of Adaptive Mesh Refinement (AMR) and Multigrid Algorithms to achieve a settlement of the fourth-order two-dimensional Poisson equation. In addition, a great number of studies reported by Bilbao and Hamilton [15] provides a two-step schemes (which operate over three time levels) of higher order accurate finite difference schemes applied for the wave equation in any number of spatial dimensions.

The main aim of the present work is to provide a general formulation and simple approach of higher order CFD

schemes for steady elliptic diffusion and convection-diffusion problems in any dimension. The primary concern of this paper focuses on the following convection-diffusion equation with Dirichlet boundary conditions

$$\begin{cases} -\Delta u + \boldsymbol{\alpha} \cdot \nabla u = f, \text{in } \Omega = [0,1]^d, \\ u|_{\partial\Omega} = g, \end{cases} \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is the velocity vector with dimension d, the source term f is known analytically throughout the domain Ω, and g is given on the boundary $\partial\Omega$, $\Delta u = \sum_{m=1}^{d} u_{x_m x_m}$ is the Laplacian operator, and $\nabla u = (u_{x_1}, \cdots, u_{x_d})$ is the divergence operator.

Obviously, it is not difficult to derive the fourth-order compact finite difference scheme, especially for the case when α is a constant vector. The scheme in response to simple procedure is introduced in Theorems 2 and 4 in the next section. The compact scheme of the sixth-order finite difference scheme for (1) with constant parameters will be completely described by Theorem 6 in Section 3.

The paper is organized as follows. In Section 2, the forth-order CFD schemes are derived for the convection-diffusion problems with variable convective coefficients in any dimension. Since then, two different fourth-order CFD schemes are linked to the consequent of steady elliptic diffusion equations with variable diffusive coefficients. The sixth-order CFD schemes are considered to be the solution of the convection-diffusion problems with constant convective coefficients in Section 3. The analysis consisted of two numerical examples is presented to verify the feasibility and high-order accuracy of the proposed methods in Section 4. This paper concludes with a discussion in Section 5.

2. Fourth-Order CFD Schemes for Convection-Diffusion Equations

2.1. Fourth-Order CFD Schemes for Convection-Diffusion Equations with Variable Convective Coefficients. Let $\pi_h = \{(x_{i_1}, \cdots, x_{i_d}) : x_{i_k} = i_k h, i_k = 0, \cdots, n+1, k = 1, \cdots, d\}$ be a uniform partition of Ω with the step size $h = 1/(n+1)$. Moreover, we introduce the central difference approximations $\delta_{x_m} u$ and $\delta_{x_m x_m} u$ as the first and second derivatives, respectively, by using the immediate neighbours. Denote $u_{i_1, \cdots, i_d}$ by the value of $u(x_1, \cdots, x_d)$ at the nodes $(x_{i_1}, \cdots, x_{i_d})$, and

$$\begin{aligned} \delta_{x_m} u\big|_{(x_{i_1}, \cdots, x_{i_d})} &= \frac{u_{i_1, \cdots, i_m+1, \cdots, i_d} - u_{i_1, \cdots, i_m-1, \cdots, i_d}}{2h}, \\ \delta_{x_m x_m} u\big|_{(x_{i_1}, \cdots, x_{i_d})} &= \frac{u_{i_1, \cdots, i_m+1, \cdots, i_d} - 2u_{i_1, \cdots, i_m, \cdots, i_d} + u_{i_1, \cdots, i_m-1, \cdots, i_d}}{h^2}, \end{aligned} \tag{2}$$

by the first order and second order central difference schemes, respectively, in the $x_m (m = 1, \cdots, d)$ direction.

By using the Taylor expansion, the problem (1) can be approximated by

$$\sum_{m=1}^{d} \left(-\delta_{x_m x_m} + \alpha_m \delta_{x_m}\right) u + \mathscr{R} u = f, \tag{3}$$

where

$$\mathscr{R} u = \frac{h^2}{6} \mathscr{R}^{(1)} u + O\left(h^4\right), \tag{4}$$

with

$$\mathscr{R}^{(1)} u = \sum_{m=1}^{d} \left(\frac{1}{2} u_{x_m x_m x_m x_m} - \alpha_m u_{x_m x_m x_m}\right). \tag{5}$$

Remark 1. By neglecting $\mathscr{R} u$ in (4), we obtain the simplest CFD scheme, with a second order accuracy, for (1), given by

$$\sum_{m=1}^{d} \left(-\delta_{x_m x_m} u + \alpha_m \delta_{x_m} u\right) = f, \tag{6}$$

which is a 5-point scheme in 2D and a 7-point one in 3D.

To obtain higher order CFD schemes, we will approximate the high order partial differential terms in (4) in a compact stencil.

Theorem 2. *For $\mathscr{R}^{(1)} u$ defined in (5), we have*

$$\begin{aligned} \mathscr{R}^{(1)} u = {} & \frac{1}{2} \mathscr{F}_1 - \frac{1}{2} \sum_{m=1}^{d} \sum_{n=1, n\neq m}^{d} \delta_{x_m x_m} \delta_{x_n x_n} u \\ & + \sum_{m=1}^{d} \alpha_m \sum_{n=1, n\neq m}^{d} \delta_{x_m} \delta_{x_n x_n} u - \sum_{m=1}^{d} a_{mm} \delta_{x_m x_m} u \\ & - \frac{1}{2} \sum_{m=1}^{d} \sum_{n=1, n\neq m}^{d} a_{mn} \delta_{x_m} \delta_{x_n} u - \sum_{m=1}^{d} a_m \delta_{x_m} u + O\left(h^2\right), \end{aligned} \tag{7}$$

where

$$\mathscr{F}_1 = \sum_{m=1}^{d} \left(-f_{x_m x_m} + \alpha_m f_{x_m}\right), \tag{8}$$

$$\begin{aligned} a_m &= \frac{1}{2} \left(-\Delta \alpha_m + \sum_{n=1}^{d} \alpha_n (\alpha_m)_{x_n}\right), \\ a_{mm} &= \frac{1}{2} \alpha_m^2 - (\alpha_m)_{x_m}, \\ a_{mn} &= \alpha_m \alpha_n - (\alpha_m)_{x_n} - (\alpha_n)_{x_m}, m \neq n, \end{aligned} \tag{9}$$

and $(\alpha_m)_{x_n}$ denote the derivation of α_m along the x_n − direction.

Proof. Differentiating both sides of (1) with respect to the m -th direction gives

$$-\sum_{n=1}^{d} u_{x_m x_n x_n} + \sum_{n=1}^{d} \alpha_n u_{x_m x_n x_n} + \sum_{n=1}^{d} (\alpha_n)_{x_m} u_{x_n} = f_{x_m}. \quad (10)$$

Differentiating both sides of (10) respect to the m-th direction again and then using (10) gives

$$\begin{aligned} &-\sum_{n=1}^{d} u_{x_m x_m x_n x_n} + \sum_{n=1}^{d} \alpha_n u_{x_m x_m x_n x_n} \\ &\quad + 2\sum_{n=1}^{d} (\alpha_n)_{x_m} u_{x_m x_n} + \sum_{n=1}^{d} (\alpha_n)_{x_m x_m} u_{x_n} = f_{x_m x_m}. \end{aligned} \quad (11)$$

Therefore, adding (11) with each direction, we get

$$\begin{aligned} \mathscr{R}^{(1)} u = {} & \frac{1}{2}\mathscr{F}_1 - \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1,n\neq m}^{d} u_{x_m x_m x_n x_n} \\ & - \sum_{m=1}^{d}\sum_{n=1,n\neq m}^{d} \alpha_m u_{x_m x_n x_n} - \sum_{m=1}^{d} a_{mm} u_{x_m x_m} \\ & - \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1,n\neq m}^{d} a_{mn} u_{x_m x_n} - \sum_{m=1}^{d} a_m u_{x_m}, \end{aligned} \quad (12)$$

where $\mathscr{F}_1$ appeared in (8). By using the central difference schemes

$$\begin{aligned} u_{x_m} &= \delta_{x_m} u + O(h^2), \\ u_{x_m x_m} &= \delta_{x_m x_m} u + O(h^2), \\ u_{x_m x_n} &= \delta_{x_m}\delta_{x_n} u + O(h^2), \\ u_{x_m x_n x_n} &= \delta_{x_m}\delta_{x_n x_n} u + O(h^2), \\ u_{x_m x_m x_n x_n} &= \delta_{x_m x_m}\delta_{x_n x_n} u + O(h^2), \end{aligned} \quad (13)$$

with $n \neq m$, we can get (7), which completes the proof of Theorem 2.

From equations (3), (4), (5), and Theorem 2, a fourth-order compact difference scheme for approximating (1) is obtained, which can be represented as follows

$$\begin{aligned} &\sum_{m=1}^{d}\sum_{\substack{n=1\\ n\neq m}}^{d} \left(c_{mn}\delta_{x_m}\delta_{x_n} + c_{mnn}\delta_{x_m}\delta_{x_n x_n} + c_{mmnn}\delta_{x_m x_m}\delta_{x_n x_n}\right) u \\ &\quad + \sum_{m=1}^{d} \left(c_m \delta_{x_m} + c_{mm}\delta_{x_m x_m}\right) u = -f + \frac{h^2}{12}\mathscr{F}_1, \end{aligned} \quad (14)$$

where

$$\begin{aligned} &c_m = -\alpha_m + \frac{h^2}{6} a_m, \quad c_{mm} = 1 + \frac{h^2}{6} a_{mm}, \\ &c_{mn} = \frac{h^2}{6} a_{mn}, \quad c_{mnn} = -\frac{h^2}{6}\alpha_m, \quad c_{mmnn} = \frac{h^2}{6}, \end{aligned} \quad (15)$$

and $a_m, a_{mm}, a_{mn}(m \neq n)$ are defined in (9).

Remark 3.

(i) If $d = 2$, the scheme (14) is no other than one suggested by Gupta et al. [16], and if $d = 3$, the scheme (14) is no other than the one suggested by Zhang [17].

(ii) If $\boldsymbol{\alpha}$ is a constant vector, it is easy to simplify the fourth-order CFD scheme (14) by reducing the terms (15) as

$$\begin{aligned} &c_m = -\alpha_m, \quad c_{mm} = 1 + \frac{h^2}{12}\alpha_m^2, \\ &c_{mn} = \frac{h^2}{6}\alpha_m\alpha_n, \quad c_{mnn} = -\frac{h^2}{6}\alpha_m, \quad c_{mmnn} = \frac{h^2}{6} \quad (m \neq n). \end{aligned} \quad (16)$$

2.2. Fourth-Order CFD Schemes for Diffusion Equations with Variable Diffusive Coefficients. Consider the d-dimensional diffusion equation

$$\begin{cases} -\nabla\cdot(\kappa\nabla u) = \tilde{f}, & \text{in } \Omega \subset \mathbb{R}^d, \\ u|_{\partial\Omega} = \varphi, \end{cases} \quad (17)$$

where the diffusive coefficient κ is a scalar function which is assumed to be positive. It can be rewritten as

$$-\sum_{n=1}^{d} \left(\kappa u_{x_n x_n} + \kappa_{x_n} u_{x_n}\right) = \tilde{f}. \quad (18)$$

Scheme I. By using the Taylor series expansion, the problem (17) can be approximated by

$$-\sum_{m=1}^{d} \left(\kappa\delta_{x_m x_m} + \kappa_{x_m}\delta_{x_m}\right) u + \tilde{\mathscr{R}} u = \tilde{f}. \quad (19)$$

$\tilde{\mathscr{R}} u$ is the same as (4) with different force terms, where

$$\tilde{\mathscr{R}} u = \frac{h^2}{6}\tilde{\mathscr{R}}^{(1)} u + O(h^4), \quad (20)$$

$$\tilde{\mathscr{R}}^{(1)} u = \sum_{m=1}^{d} \left(\frac{\kappa}{2} u_{x_m x_m x_m x_m} + \kappa_{x_m} u_{x_m x_m x_m}\right). \quad (21)$$

Theorem 4. *For $\tilde{\mathscr{R}}^{(1)}u$ defined in (21), we have*

$$\begin{aligned}\kappa\tilde{\mathscr{R}}^{(1)}u = &\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}\left(-\frac{\kappa^2}{2}\delta_{x_mx_m}\delta_{x_nx_n}-\kappa\delta_{x_m}\delta_{x_nx_n}\right.\\&\left.-\frac{1}{2}a_{mn}\delta_{x_m}\delta_{x_n}\right)u-\sum_{m=1}^{d}a_{mm}\delta_{x_mx_m}u\\&-\sum_{m=1}^{d}a_m\delta_{x_m}u+\frac{1}{2}\tilde{\mathscr{F}}_1+O\left(h^2\right),\end{aligned}\tag{22}$$

where

$$\tilde{\mathscr{F}}_1=\sum_{m=1}^{d}\left(-\kappa\tilde{f}_{x_mx_m}+\kappa_{x_m}\tilde{f}_{x_m}\right),\tag{23}$$

$$\begin{aligned}a_m&=\frac{1}{2}\sum_{n=1}^{d}\left(\kappa\kappa_{x_mx_nx_n}-\kappa_{x_n}\kappa_{x_mx_n}\right),\\a_{mm}&=\kappa\kappa_{x_mx_m}-\kappa_{x_m}^2+\frac{1}{2}\kappa\sum_{n=1}^{d}\kappa_{x_nx_n}-\frac{1}{2}\sum_{\substack{n=1\\n\neq m}}^{d}\kappa_{x_n}^2,\\a_{mn}&=2\kappa\kappa_{x_mx_n}-\kappa_{x_m}\kappa_{x_n},\ m\neq n.\end{aligned}\tag{24}$$

Proof. Differentiating both sides of (17) once and twice with respect to x_m, respectively, gives

$$\begin{aligned}&2\kappa_{x_m}u_{x_mx_m}+\sum_{n=1}^{d}\left(\kappa u_{x_nx_nx_m}+\kappa_{x_mx_n}u_{x_n}\right)\\&\quad+\sum_{\substack{n=1\\n\neq m}}^{d}\left(\kappa_{x_m}u_{x_nx_m}+\kappa_{x_n}u_{x_mx_n}\right)=-\tilde{f}_{x_m},\end{aligned}\tag{25}$$

$$\begin{aligned}&3\kappa_{x_m}u_{x_mx_mx_m}+\sum_{n=1}^{d}\left(\kappa u_{x_mx_mx_nx_n}+\kappa_{x_mx_mx_n}u_{x_n}\right)\\&\quad+\sum_{\substack{n=1\\n\neq m}}^{d}\left(2\kappa_{x_m}u_{x_mx_nx_n}+\kappa_{x_m}u_{x_mx_mx_n}\right.\\&\quad\left.+\kappa_{x_mx_m}u_{x_nx_n}+2\kappa_{x_mx_n}u_{x_mx_n}\right)=-\tilde{f}_{x_mx_m}.\end{aligned}\tag{26}$$

From (25), we can obtain the expression of $\kappa\kappa_{x_m}u_{x_mx_mx_m}$, and then, substituting (25) into (26) to eliminate the term concluding $u_{x_mx_mx_m}$ gives the expression of $\kappa^2u_{x_mx_mx_mx_m}$. By the definition of $\tilde{\mathscr{R}}^{(1)}u$, we have

$$\begin{aligned}\kappa\tilde{\mathscr{R}}^{(1)}u=&-\frac{\kappa^2}{2}\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}u_{x_mx_mx_nx_n}-\kappa\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}\kappa_{x_m}u_{x_mx_nx_n}\\&-\sum_{m=1}^{d}a_{mm}u_{x_mx_m}-\frac{1}{2}\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}a_{mn}u_{x_mx_n}\\&-\sum_{m=1}^{d}a_mu_{x_m}+\frac{1}{2}\tilde{\mathscr{F}}_1.\end{aligned}\tag{27}$$

By using the central difference schemes defined in (13), we can get (22),which completes the proof of Theorem 4.

From (19), (4), (21), and Theorem 4, a fourth-order compact difference scheme for approximating (17) is obtained by dropping off the higher order terms $O(h^4)$

$$\begin{aligned}&\sum_{m=1}^{d}c_m\delta_{x_m}u+\sum_{m=1}^{d}c_{mm}\delta_{x_mx_m}u+\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}c_{mn}\delta_{x_m}\delta_{x_n}u\\&+\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}c_{mnn}\delta_{x_m}\delta_{x_nx_n}u+\sum_{m=1}^{d}\sum_{\substack{n=1\\n\neq m}}^{d}c_{mmnn}\delta_{x_mx_m}\delta_{x_nx_n}u=\tilde{\mathscr{F}},\end{aligned}\tag{28}$$

where

$$\tilde{\mathscr{F}}=-\kappa\tilde{f}+\frac{h^2}{12}\tilde{\mathscr{F}}_1,\tag{29}$$

$$\begin{aligned}&c_m=\kappa\kappa_{x_m}+\frac{h^2}{6}a_m,\quad c_{mm}=\kappa^2+\frac{h^2}{6}a_{mm},\\&c_{mn}=\frac{h^2}{6}a_{mn},\quad c_{mnn}=\frac{h^2}{6}\kappa\kappa_{x_m},\quad c_{mmnn}=\frac{h^2}{12}\kappa^2,\quad m\neq n.\end{aligned}\tag{30}$$

$\tilde{\mathscr{F}}_1$ and $a_m, a_{mm}, a_{mn}(m\neq n)$ are defined by (23) and (24).

Scheme II. Now, we derive another fourth-order CFD scheme for approximating (17) based on the fourth-order CFD scheme for convection-diffusion equations discussed in Section 2.1. Since κ is positive, we can also rewrite (17) as the form

$$-\sum_{n=1}^{d}\left(u_{x_nx_n}+\frac{\kappa_{x_n}}{\kappa}u_{x_n}\right)=\frac{\tilde{f}}{\kappa}.\tag{31}$$

Let $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_d)$ with $\alpha_m=-\kappa_{x_m}/\kappa(m=1,\cdots,d)$, then (31) has the same form of (1). Substituting this $\boldsymbol{\alpha}$ into (14) and (15) and then using some direct calculations, a new fourth-order compact finite difference scheme is obtained as

$$\sum_{m=1}^{d}\sum_{\substack{n=1\\ n\neq m}}^{d}\left(c_{mn}\delta_{x_m}\delta_{x_n}+c_{mnn}\delta_{x_m}\delta_{x_nx_n}+c_{mmnn}\delta_{x_mx_m}\delta_{x_nx_n}\right)u + \sum_{m=1}^{d}\left(c_m\delta_{x_m}+c_{mm}\delta_{x_mx_m}\right)u=-\kappa\tilde{f}+\frac{h^2}{12}\tilde{\mathscr{F}}_1=\tilde{\mathscr{F}}, \tag{32}$$

where

$$\tilde{\mathscr{F}}=\sum_{m=1}^{d}\left(\kappa^2\tilde{f}_{x_mx_m}-\kappa\kappa_{x_m}\tilde{f}_{x_m}+\left(\kappa_{x_m}^2-\kappa\kappa_{x_mx_m}\right)\tilde{f}\right), \tag{33}$$

$$\begin{aligned}
c_m&=\kappa^2\kappa_{x_m}+\frac{h^2}{12}\left[\sum_{n=1}^{d}\left(\kappa^2\kappa_{x_mx_nx_n}+\kappa_{x_m}\kappa_{x_n}^2\right)-2\kappa\kappa_{x_m}\kappa_{x_mx_m}\right.\\
&\quad\left.-\kappa\sum_{n=1,n\neq m}^{d}\left(\kappa_{x_m}\kappa_{x_nx_n}+\kappa_{x_n}\kappa_{x_mx_n}\right)\right],\\
c_{mm}&=\kappa^3+\frac{h^2}{12}\kappa\left(2\kappa\kappa_{x_mx_m}-\kappa_{x_m}^2\right),\\
c_{mn}&=\frac{h^2}{6}\kappa\left(2\kappa\kappa_{x_mx_n}-\kappa_{x_m}\kappa_{x_n}\right),\\
c_{mnn}&=\frac{h^2}{6}\kappa^2\kappa_{x_m},\quad c_{mmnn}=\frac{h^2}{6}\kappa^3(m\neq n).
\end{aligned} \tag{34}$$

Remark 5.

(i) In practice, we should approximate the factors related to κ and $\tilde{f}$ with the accuracy $O(h^2)$ in (24), like κ_{x_m}, $\kappa_{x_mx_m}$, $\kappa_{x_mx_n}(m\neq n)$, $\kappa_{x_mx_mx_n}(m\neq n)$, and $\tilde{f}_{x_m}$, $\tilde{f}_{x_mx_m}$. All of these can be approximated by the corresponding central difference scheme to meet the requirement.

(ii) To guarantee the accuracy of the schemes (28) and (32), we also need to approximate κ_{x_m} and $\kappa_{x_mx_mx_m}$ with the fourth-order accuracy. For the purpose of this, the following approximation

$$\begin{aligned}
\kappa'(x)&=-\frac{1}{6h}\left[\kappa(x+h)-8\kappa\left(x+\frac{h}{2}\right)\right.\\
&\quad\left.+8\kappa\left(x-\frac{h}{2}\right)-\kappa(x-h)\right]+O\left(h^4\right),\\
\kappa'''(x)&=\frac{4}{h^3}\left[\kappa(x+h)-2\kappa\left(x+\frac{h}{2}\right)\right.\\
&\quad\left.+2\kappa\left(x-\frac{h}{2}\right)-\kappa(x-h)\right]+O\left(h^2\right),
\end{aligned} \tag{35}$$

will be introduced.

3. Sixth-Order CFD Schemes for Convection-Diffusion Equations

In the previous section, we have obtained the fourth-order CFD scheme for (1). Especially for the case that $\boldsymbol{\alpha}$ is a constant vector, this scheme seems very simple. Actually, after some further analysis, we can make it achieve higher order accuracy.

3.1. Sixth-Order CFD Schemes for Diffusion Equations with Variable Convection Coefficients. This study will offer a fresh insight into the following Theorem 6 to show the sixth-order CFD scheme for (1). By using the Taylor expansion, the problem (1) can be approximated by (3), where

$$\mathscr{R}u=\frac{h^2}{6}\mathscr{R}^{(1)}u+\frac{h^4}{120}\mathscr{R}^{(2)}u+O\left(h^6\right), \tag{36}$$

with

$$\mathscr{R}^{(1)}u=\sum_{n=1}^{d}\left(\frac{1}{2}u_{x_nx_nx_nx_n}-\alpha_n u_{x_nx_nx_nx_n}\right), \tag{37}$$

$$\mathscr{R}^{(2)}u=\sum_{n=1}^{d}\left(\frac{1}{3}u_{x_nx_nx_nx_nx_nx_n}-\alpha_n u_{x_nx_nx_nx_nx_n}\right). \tag{38}$$

Theorem 6. *Let $\mathscr{R}u$ be defined by (36), then we have*

$$\begin{aligned}
\mathscr{R}u&=\frac{h^2}{6}S_1u+\frac{h^4}{720}(S_2+S_3)u\\
&\quad+\frac{h^4}{180}(S_4+S_5-S_6+S_7+S_8)u\\
&\quad+\mathscr{F}^*+O\left(h^6\right),
\end{aligned} \tag{39}$$

where

$$\mathscr{F}^*=\frac{h^2}{12}\mathscr{F}_1+\frac{h^4}{720}(2\mathscr{F}_2-\mathscr{F}_3-4\mathscr{F}_4+4\mathscr{F}_5-4\mathscr{F}_6), \tag{40}$$

with $S_iu(i=1,\cdots,8)$ defined by (44), (59),(60), (65), (66), (71), (72), and (76) and $\mathscr{F}_i(i=1,\cdots,6)$ defined by (45), (53), (61), (67), (73), and (77) below.

This study will offer the compact sixth-order finite difference scheme for (1)

$$\sum_{m=1}^{d}\left(-\delta_{x_mx_m}+\alpha_m\delta_{x_m}\right)u+\mathscr{R}u=\mathscr{F}, \tag{41}$$

where $\mathscr{F}=f-\mathscr{F}^*$ and $\mathscr{R}U$ are defined in (39) by replacing the exact solution u by the approximating solution U and dropping off the remainder term $O(h^6)$.

To prove Theorem 6, we define $Q_i(i=1,2,\cdots,5)$ as follows:

$$\begin{aligned}
Q_1 &= \sum_{n=1}^{d} \alpha_n^2 u_{x_n x_n x_n x_n},\\
Q_2 &= 2\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} \alpha_n\alpha_k u_{x_n x_n x_n x_k},\\
Q_3 &= 2\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} \alpha_k u_{x_n x_n x_n x_n x_k},\\
Q_4 &= 2\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} \alpha_k u_{x_n x_n x_k x_k x_k},\\
Q_5 &= 2\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} u_{x_n x_n x_n x_k x_k},
\end{aligned} \tag{42}$$

and the following Lemmas 7-13 are needed.

Lemma 7. *Let $\mathscr{R}^{(1)}u$ be defined in (37) and $Q_k(k=1,\cdots,5)$ be defined in (42), then we have*

$$\begin{aligned}
\mathscr{R}^{(1)}u = S_1 u &+ \frac{h^2}{24}(Q_1 + 2Q_2 - Q_3 - 2Q_4 + Q_5)\\
&+ \frac{1}{2}\mathscr{F}_1 + O\left(h^4\right),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
S_1 u = \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} &\left(-\delta_{x_k x_k}\delta_{x_n x_n} + \alpha_k \delta_{x_k}\delta_{x_n x_n}\right.\\
&\left. + \alpha_n \delta_{x_k x_k}\delta_{x_n} - \alpha_n\alpha_k\delta_{x_k}\delta_{x_n}\right)u - \frac{1}{2}\sum_{n=1}^{d}\alpha_n^2\delta_{x_n x_n}u,
\end{aligned} \tag{44}$$

$$\mathscr{F}_1 = \sum_{n=1}^{d}\left(-f_{x_n x_n} + \alpha_n f_{x_n}\right). \tag{45}$$

Proof. Differentiating both sides of (1) once and twice with respect to $x_n(n=1,\cdots,d)$, respectively, gives

$$\begin{aligned}
-\sum_{k=1}^{d} u_{x_k x_k x_n} + \sum_{k=1}^{d}\alpha_k u_{x_k x_n} &= f_{x_n},\\
-\sum_{k=1}^{d} u_{x_k x_k x_n x_n} + \sum_{k=1}^{d}\alpha_k u_{x_k x_n x_n} &= f_{x_n x_n},
\end{aligned} \tag{46}$$

which leads to

$$u_{x_n x_n x_n} = -\sum_{\substack{k=1\\k\neq n}}^{d} u_{x_k x_k x_n} + \sum_{k=1}^{d}\alpha_k u_{x_k x_n} - f_{x_n}, \tag{47}$$

$$u_{x_n x_n x_n x_n} = -\sum_{\substack{k=1\\k\neq n}}^{d} u_{x_k x_k x_n x_n} + \sum_{k=1}^{d}\alpha_k u_{x_k x_n x_n} - f_{x_n x_n}. \tag{48}$$

Using (47) and (48) to eliminate the terms $u_{x_n x_n x_n}$ and $u_{x_n x_n x_n x_n}$ in (37) yields

$$\begin{aligned}
\mathscr{R}^{(1)}u = &-\frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} u_{x_k x_k x_n x_n} + \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left(\alpha_k u_{x_k x_n x_n} + \alpha_n u_{x_k x_k x_n}\right)\\
&- \frac{1}{2}\sum_{n=1}^{d}\sum_{k=1}^{d}\alpha_k\alpha_n u_{x_k x_n} + \frac{1}{2}\sum_{n=1}^{d}\left(-f_{x_n x_n} + \alpha_n f_{x_n}\right).
\end{aligned} \tag{49}$$

Taking account of the centered difference approximations

$$\begin{aligned}
u_{x_n x_n} &= \delta_{x_n x_n}u - \frac{h^2}{12}u_{x_n x_n x_n x_n} + O\left(h^4\right),\\
u_{x_k x_n} &= \delta_{x_k}\delta_{x_n}u - \frac{h^2}{6}\left(u_{x_k x_k x_k x_n} + u_{x_k x_n x_n x_n}\right) + O\left(h^4\right),\\
u_{x_k x_k x_n} &= \delta_{x_k x_k}\delta_{x_n}u - \frac{h^2}{12}\left(u_{x_k x_k x_k x_k x_n} + 2u_{x_k x_k x_n x_n x_n}\right) + O\left(h^4\right),\\
u_{x_k x_k x_n x_n} &= \delta_{x_k x_k}\delta_{x_n x_n}u - \frac{h^2}{12}\left(u_{x_k x_k x_n x_n x_n x_n} + u_{x_k x_k x_k x_k x_n x_n}\right)\\
&\quad + O\left(h^4\right),
\end{aligned} \tag{50}$$

with $n\neq k$, and noting the definition of S_1u given by (44), then we have

$$\begin{aligned}
\mathscr{R}^{(1)}u = &\frac{h^2}{24}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left[\left(u_{x_k x_k x_n x_n x_n x_n} + u_{x_k x_k x_k x_k x_n x_n}\right)\right.\\
&- \alpha_k\left(u_{x_n x_n x_n x_n x_k} + 2u_{x_n x_n x_k x_k x_k}\right)\\
&- \alpha_n\left(u_{x_k x_k x_k x_k x_n} + 2u_{x_k x_k x_n x_n x_n}\right)\\
&\left. + 2\alpha_n\alpha_k\left(u_{x_k x_k x_k x_n} + u_{x_k x_n x_n x_n}\right)\right]u\\
&+ \frac{h^2}{24}\sum_{n=1}^{d}\alpha_n^2 u_{x_n x_n x_n x_n} + S_1 u + \frac{1}{2}\mathscr{F}_1\\
= &\frac{h^2}{24}(Q_1 + 2Q_2 - Q_3 - 2Q_4 + Q_5)\\
&+ S_1 u + \frac{1}{2}\mathscr{F}_1 + O\left(h^4\right).
\end{aligned} \tag{51}$$

Therefore, the proof is completed when conditions (42) and (50) are satisfied.

Lemma 8. *Let $\mathscr{R}^{(2)}u$ be defined by (38) and $Q_k(k=1,\cdots,5)$ be defined by (42), then*

$$\mathscr{R}^{(2)}u = -\frac{1}{6}(4Q_1 + 2Q_2 - Q_3 - 2Q_4 + Q_5) + \frac{1}{3}\mathscr{F}_2, \quad (52)$$

where

$$\mathscr{F}_2 = \sum_{n=1}^{d}\left(-f_{x_nx_nx_nx_n} + 2\alpha_n f_{x_nx_nx_n}\right). \quad (53)$$

Proof. Differentiating both sides of (1) three times and four times with respect to $x_n(n=1,\cdots,d)$, respectively, gives rise to

$$u_{x_nx_nx_nx_nx_n} = -f_{x_nx_nx_n} - \sum_{\substack{k=1\\k\neq n}}^{d} u_{x_kx_kx_nx_nx_n} + \sum_{k=1}^{d}\alpha_k u_{x_kx_nx_nx_n}, \quad (54)$$

$$\begin{aligned} u_{x_nx_nx_nx_nx_nx_n} = -f_{x_nx_nx_nx_n} + \sum_{\substack{k=1\\k\neq n}}^{d}\big(&-u_{x_kx_kx_nx_nx_nx_n} \\ &+ \alpha_k u_{x_kx_nx_nx_nx_n}\big) + \alpha_n u_{x_nx_nx_nx_nx_n}. \end{aligned} \quad (55)$$

Using (54) and (55) to eliminate the terms $u_{x_nx_nx_n}$ and $u_{x_nx_nx_nx_n}$ in (38) yields

$$\begin{aligned} \mathscr{R}^{(2)}u &= \frac{1}{3}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\big(-u_{x_nx_nx_nx_nx_kx_k} + \alpha_k u_{x_nx_nx_nx_nx_k} \\ &\quad + 2\alpha_n u_{x_nx_nx_nx_kx_k} - 2\alpha_n\alpha_k u_{x_nx_nx_nx_k}\big) \\ &\quad - \frac{2}{3}\sum_{n=1}^{d}\alpha_n^2 u_{x_nx_nx_nx_n} + \frac{1}{3}\mathscr{F}_2 \\ &= -\frac{1}{6}(4Q_1 + 2Q_2 - Q_3 - 2Q_4 + Q_5) + \frac{1}{3}\mathscr{F}_2, \end{aligned} \quad (56)$$

which finishes the proof.

Lemma 9. *Let $\mathscr{R}u$ be defined by (36) and $Q_i(i=1,\cdots,5)$ be defined by (42), $S_1u, \mathscr{F}_1$, and $\mathscr{F}_2$ be defined by (44), (45), and (53) then*

$$\begin{aligned} \mathscr{R}u = \frac{h^4}{720}Q_1 + \frac{h^4}{180}(2Q_2 - Q_3 - 2Q_4 + Q_5) \\ + \frac{h^2}{6}S_1u + \frac{h^2}{6}\mathscr{F}_1 + \frac{h^4}{360}\mathscr{F}_2. \end{aligned} \quad (57)$$

Proof. The proof can be easily done by putting Lemma 7, Lemma 8., and the definition of $\mathscr{R}u$ together.

Now, we need to represent the terms $Q_i(i=1,\cdots,5)$ by using compact finite difference approximations.

Lemma 10. *Let Q_1 be defined by (42), then*

$$Q_1 = S_2u + S_3u - \mathscr{F}_3 + O(h^2), \quad (58)$$

where

$$S_2u = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_n^2\left[\left(\alpha_k\delta_{x_k}\delta_{x_nx_n} - \alpha_n\delta_{x_kx_k}\delta_{x_n}\right) - \delta_{x_kx_k}\delta_{x_nx_n}\right]u, \quad (59)$$

$$S_3u = \sum_{n=1}^{d}\sum_{k=1}^{d}\alpha_n^3\alpha_k\delta_{x_kx_n}u, \quad (60)$$

$$\mathscr{F}_3 = \sum_{n=1}^{d}\alpha_n^2\left(f_{x_nx_n} + \alpha_n f_{x_n}\right). \quad (61)$$

Proof. Substituting (47) into (48) yields

$$\begin{aligned} u_{x_nx_nx_nx_n} = \sum_{\substack{k=1\\k\neq n}}^{d}\left(-u_{x_kx_kx_nx_n} + \alpha_k u_{x_kx_nx_n} - \alpha_n u_{x_kx_kx_n}\right) \\ + \alpha_n\sum_{k=1}^{d}\alpha_k u_{x_kx_n} - \left(f_{x_nx_n} + \alpha_n f_{x_n}\right). \end{aligned} \quad (62)$$

Multiplying both sides of the above equation by α_n and then summing them over n, we can rewrite Q_1 defined in (42) as the form

$$\begin{aligned} Q_1 = \sum_{n=1}^{d}\alpha_n^2\sum_{\substack{k=1\\k\neq n}}^{d}\left[\left(\alpha_k u_{x_kx_nx_n} - \alpha_n u_{x_kx_kx_n}\right) - u_{x_kx_kx_nx_n}\right] \\ + \sum_{n=1}^{d}\alpha_n^2\sum_{k=1}^{d}\alpha_k\alpha_n u_{x_kx_n} - \sum_{n=1}^{d}\alpha_n^2\left(f_{x_nx_n} + \alpha_n f_{x_n}\right). \end{aligned} \quad (63)$$

Therefore, the expected result can be easily obtained by replacing the central difference approximations (13) in (63).

Lemma 11. *Let Q_2 be defined by (42),then we have*

$$Q_2 = S_4u + S_5u - \mathscr{F}_4 + O(h^2), \quad (64)$$

where

$$S_4u = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_n^2\alpha_k\delta_{x_nx_n}\delta_{x_k}u, \quad (65)$$

$$S_5u = \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\left(-\alpha_k\alpha_m\delta_{x_nx_n}\delta_{x_k}\delta_{x_m} + \alpha_n\alpha_k\alpha_m\delta_{x_n}\delta_{x_k}\delta_{x_m}\right)u, \tag{66}$$

$$\mathscr{F}_4 = \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_n\alpha_k f_{x_nx_k}. \tag{67}$$

Proof. Differentiating both sides of (1) with respect to x_mx_k, $k\neq m$, $(m,k=1,\cdots,d)$ yields

$$\begin{aligned}&-\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}u_{x_nx_nx_nx_k} - \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}u_{x_nx_nx_mx_k}\\&\quad+\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_nu_{x_nx_nx_k} + \frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\alpha_nu_{x_n}u_{x_k}u_{x_m}\\&=\frac{1}{2}\sum_{m=1}^{d}\sum_{\substack{k=1\\k\neq m k\neq m}}^{d}f_{x_mx_k}.\end{aligned} \tag{68}$$

Multiplying both sides of the above equation by $\alpha_m\alpha_k$ yields

$$\begin{aligned}Q_2 = &-\frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\alpha_m\alpha_ku_{x_nx_nx_mx_k}\\&+\frac{1}{2}\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\alpha_n\alpha_k\alpha_mu_{x_n}u_{x_k}u_{x_m}\\&+\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_n^2\alpha_ku_{x_nx_nx_k} - \frac{1}{2}\sum_{m=1}^{d}\sum_{\substack{k=1\\k\neq m}}^{d}\alpha_m\alpha_kf_{x_mx_k}.\end{aligned} \tag{69}$$

By using the centered difference approximations (13), we can obtain the results.

Lemma 12. *Let Q_2, Q_3, and Q_4 be defined in (42), then we have*

$$Q_2 - Q_3 - Q_4 = -S_6u + S_7u + \mathscr{F}_5 + O\left(h^2\right), \tag{70}$$

where

$$\begin{aligned}S_6u = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}&\left[\left(\alpha_k\delta_{x_k}\delta_{x_nx_n} + \alpha_n\delta_{x_n}\delta_{x_kx_k}\right)\delta_{x_mx_m}\right.\\&\left.-\alpha_m\left(\alpha_k\delta_{x_nx_n}\delta_{x_k} + \alpha_n\delta_{x_n}\delta_{x_kx_k}\right)\delta_{x_m}\right]u,\end{aligned} \tag{71}$$

$$S_7u = 2\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\alpha_n^2\delta_{x_nx_n}\delta_{x_kx_k}u, \tag{72}$$

$$\mathscr{F}_5 = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left(\alpha_kf_{x_nx_nx_k} + \alpha_nf_{x_nx_kx_k}\right). \tag{73}$$

Proof. By using the definitions of Q_2, Q_3, and Q_4 in (42), we have

$$\begin{aligned}Q_3 + Q_4 &= \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left[\alpha_k\left(u_{x_nx_n} + u_{x_kx_k}\right)_{x_nx_nx_k}\right.\\&\quad\left.+\alpha_n\left(u_{x_nx_n} + u_{x_kx_k}\right)_{x_nx_kx_k}\right]\\&= -\mathscr{F}_5 + Q_2 + \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left(\alpha_k^2 + \alpha_n^2\right)u_{x_nx_nx_kx_k}\\&\quad+\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\alpha_m\left(\alpha_ku_{x_mx_nx_nx_k} + \alpha_nu_{x_mx_nx_kx_k}\right)\\&\quad-\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\left(\alpha_ku_{x_mx_mx_nx_nx_k} + \alpha_nu_{x_mx_mx_nx_kx_k}\right).\end{aligned} \tag{74}$$

By using the centered difference approximations (13), we can obtain the results.

Lemma 13. *Let Q_4 and Q_5 be defined in (42), then we have*

$$Q_5 - Q_4 = S_8u - \mathscr{F}_6 + O\left(h^2\right), \tag{75}$$

where

$$\begin{aligned}S_8u = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}&\left(\alpha_m\delta_{x_m}\delta_{x_nx_n}\delta_{x_kx_k}\right.\\&\left.-\delta_{x_mx_m}\delta_{x_nx_n}\delta_{x_kx_k}\right)u,\end{aligned} \tag{76}$$

$$\mathscr{F}_6 = \sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}f_{x_nx_nx_kx_k}. \tag{77}$$

Proof. Differentiating both sides of (1) twice with respect to x_n and then twice with respect to $x_k, k\neq n$, and then summing them over n and k gives

$$\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{m=1}^{d}\left(-u_{x_mx_mx_nx_nx_kx_k}+\alpha_m u_{x_mx_nx_nx_kx_k}\right)=\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} f_{x_nx_nx_kx_k}, \tag{78}$$

which can be rewritten as

$$\begin{aligned}&\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\left(-u_{x_nx_nx_nx_nx_kx_k}+\alpha_n u_{x_nx_nx_nx_kx_k}\right.\\&\left.-u_{x_kx_kx_nx_nx_kx_k}+\alpha_k u_{x_kx_nx_nx_kx_k}\right)+\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\\&\cdot\left(-u_{x_mx_mx_nx_nx_kx_k}+\alpha_m u_{x_mx_nx_nx_kx_k}\right)\\&=\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} f_{x_nx_nx_kx_k}.\end{aligned} \tag{79}$$

By using the definition of Q_4 and Q_5 defined in (42), we have

$$\begin{aligned}Q_5-Q_4=&\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d}\sum_{\substack{m=1\\m\neq n,m\neq k}}^{d}\left(\alpha_m u_{x_mx_nx_nx_kx_k}\right.\\&\left.-u_{x_mx_mx_nx_nx_kx_k}\right)-\sum_{n=1}^{d}\sum_{\substack{k=1\\k\neq n}}^{d} f_{x_nx_nx_kx_k}.\end{aligned} \tag{80}$$

By using the centered difference approximations (13),we can obtain the results.

Proof of Theorem 6. The proof can be easily done by putting Lemmas 9–13 and the definition of $\mathscr{F}^*$ in (40) together.

3.2. Sixth-Order CFD Schemes for Diffusion Equations with Constant Convective Coefficients. It should be noted that the sixth-order CFD scheme given in (41) is valid for any dimension. For simplicity, we write down here the most commonly used schemes for (1), i.e., for 2 and 3 dimensions, in a concise way.

(i) If $d=2$, the terms including the summation symbol $\sum_{n=1}^{d}\sum_{k=1k\neq n}^{d}\sum_{m=1m\neq n,m\neq k}^{d}$ in S_i, $(i=1,\cdots,8)$, i.e., S_5, S_6, S_7, and S_8, vanish. Therefore, the sixth-order CFD scheme can be described as

$$\begin{aligned}&\sum_{n=1}^{2}\sum_{\substack{k=1\\k\neq n}}^{2}\left(c_{nk}\delta_{x_n}\delta_{x_k}+c_{nkk}\delta_{x_n}\delta_{x_kx_k}+c_{nnkk}\delta_{x_nx_n}\delta_{x_kx_k}\right)u\\&+\sum_{n=1}^{2}c_n\delta_{x_n}u+\sum_{n=1}^{2}c_{nn}\delta_{x_nx_n}u=f-\mathscr{F}^*,\end{aligned} \tag{81}$$

where

$$\begin{aligned}c_n&=\alpha_n,\\c_{nn}&=1+\frac{\alpha_n^2h^2}{12}\left(1-\frac{\alpha_n^2h^2}{60}\right),\\c_{nk}&=\frac{\alpha_n\alpha_kh^2}{12}\left(1-\frac{\left(\alpha_n^2+\alpha_k^2\right)h^2}{120}\right),\\c_{nkk}&=\frac{\alpha_nh^2}{6}\left(1-\frac{\left(\alpha_n^2-9\alpha_k^2\right)h^2}{120}\right),\\c_{nnkk}&=\frac{h^2}{12}+\frac{\left(\alpha_n^2+\alpha_k^2\right)h^4}{160},\end{aligned} \tag{82}$$

and $\mathscr{F}^*$ appeared in (40).

(ii) If $d=3$, all of the S_i, $(i=1,\cdots,8)$ exist, and the sixth-order CFD scheme can be described as

$$\begin{aligned}&\sum_{n=1}^{3}\sum_{\substack{k=1\\k\neq n}}^{3}\left(c_{nk}\delta_{x_n}\delta_{x_k}+c_{nkk}\delta_{x_n}\delta_{x_kx_k}+c_{nnkk}\delta_{x_n}\delta_{x_kx_k}\right)u\\&+\sum_{n=1}^{3}\left(a_n\delta_{x_n}+a_{nn}\delta_{x_nx_n}\right)u+\left(c_{123}\delta_{x_1}\delta_{x_2}\delta_{x_3}\right.\\&\left.+c_{112233}\delta_{x_1x_1}\delta_{x_2x_2}\delta_{x_3x_3}\right)u+\left(c_{1123}\delta_{x_1x_1}\delta_{x_2}\delta_{x_3}\right.\\&\left.+c_{1223}\delta_{x_1}\delta_{x_2x_2}\delta_{x_3}+c_{1233}\delta_{x_1}\delta_{x_2}\delta_{x_3x_3}\right)u\\&+\left(c_{12233}\delta_{x_1}\delta_{x_2x_2}\delta_{x_3x_3}+c_{11233}\delta_{x_1x_1}\delta_{x_2}\delta_{x_3x_3}\right.\\&\left.+c_{11223}\delta_{x_1x_1}\delta_{x_2x_2}\delta_{x_3}\right)u=f-\mathscr{F}^*,\end{aligned} \tag{83}$$

where the forms of c_n, c_{nn}, c_{nk}, c_{nkk}, c_{nnkk}, and $k\neq n$ are the same as that defined in (82), except in this case $n=1,2,3$ and

$$\begin{aligned}c_{123}&=\frac{\alpha_1\alpha_2\alpha_3h^4}{30},\\c_{112233}&=\frac{h^4}{30},\\c_{mnkk}&=\frac{\alpha_m\alpha_nh^4}{90},\\c_{mnnkk}&=\frac{\alpha_mh^4}{90},\end{aligned} \tag{84}$$

where $m,n,k=1,2,3$, but they are different from each other.

4. Numerical Results

The CFD schemes derived in Section 2.1 for 2D and 3D convection-diffusion equations are discussed in literature [16, 17], respectively, and the corresponding numerical results have been completed to established. Hence, two examples in this filed are shown to validate the accuracy of the CFD schemes proposed in Section 2.2 and Section 3.2.

Table 1: Numerical results for the fourth-order CFD method in 2D.

h	Scheme I				Scheme II			
	Case 1		Case 2		Case 1		Case 2	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
1/4	6.613*E*-03		1.279*E*-03		6.613*E*-03		1.270*E*-03	
1/8	3.995*E*-04	4.04	8.355*E*-05	3.93	3.994*E*-04	4.04	8.341*E*-05	3.92
1/16	2.557*E*-05	3.96	5.273*E*-06	3.98	2.556*E*-05	3.96	5.271*E*-06	3.98
1/32	1.594*E*-06	4.00	3.330*E*-07	3.98	1.594*E*-06	4.00	3.330*E*-07	3.98
1/64	9.958*E*-08	4.00	2.082*E*-08	3.99	9.958*E*-06	4.00	2.082*E*-08	3.99

Table 2: Numerical results for the fourth-order CFD method in 3D.

h	Scheme I				Scheme II			
	Case 1		Case 2		Case 1		Case 2	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
1/4	1.294*E*-02		4.226*E*-03		1.294*E*-02		4.211*E*-03	
1/8	7.523*E*-04	4.10	2.560*E*-04	4.04	7.524*E*-04	4.10	2.558*E*-04	4.04
1/16	4.740*E*-05	3.98	1.587*E*-05	4.01	4.740*E*-05	3.98	1.587*E*-05	4.01
1/32	2.954*E*-06	4.00	9.898*E*-07	4.00	2.955*E*-06	4.00	9.898*E*-07	4.00
1/64	1.849*E*-07	3.99	6.192*E*-08	3.99	1.850*E*-07	3.99	6.192*E*-08	3.99

Table 3: Numerical results for the sixth-order CFD method in 2D with $\boldsymbol{\alpha} = \mathrm{Re} \cdot (1, 1)$.

h	$\mathrm{Re} = 1$		$\mathrm{Re} = 10$		$\mathrm{Re} = 10^2$		$\mathrm{Re} = 10^3$	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
1/4	2.278*E*-04		5.292*E*-03		0.5348*E*-01		0.4837*E*-01	
1/8	3.708*E*-06	5.94	8.547*E*-05	5.95	0.2204*E*-01	1.28	0.1380*E*-01	1.81
1/16	5.732*E*-08	6.02	1.339*E*-06	5.99	0.1437*E*-02	3.94	0.3918*E*-02	1.82
1/32	9.020*E*-10	5.99	2.093*E*-08	6.00	0.1925*E*-04	6.22	0.1078*E*-02	1.86
1/64	1.409*E*-11	6.00	3.271*E*-10	6.00	0.3043*E*-06	5.98	0.2463*E*-03	2.13

Example 14. In this example, the diffusive coefficient is regarded as $\kappa(x_1,\cdots,x_d) = e^{\sum_{i=1}^{d} x_i}$, and then, the forcing function f and the Dirichlet boundary conditions are prescribed to satisfy the exact solution $u(x_1,\cdots,x_d) = \prod_{i=1}^{d} \sin \pi x_i$. To exhibit the numerical results, we denote two schemes as

(i) Scheme I: (28) with coefficients (30)

(ii) Scheme II: (32) with coefficients (34)

As the computation of the coefficients, we take the following two cases:

(i) Case 1: coefficients (30) and (34) computed analytically

(ii) Case 2: coefficients (30) and (34) computed by the approximation introduced in Remark 3.

The information relevant to maximum errors and convergence rates are listed in Tables 1 and 2 for 2D and 3D problems, severally.

From these tables, it offers a brief result that both two schemes are published in almost the same errors with a fourth-order convergence rate. Although the errors in essence were affected by the ways of computing the coefficients, the consequence of convergence rate still remains $O(h^4)$.

Example 15. An example to validate the accuracy of the CFD schemes is proposed in Section 3.2, considering the convection-diffusion equation, constant convection vector coefficients $\boldsymbol{\alpha}$ is regarded as $\mathrm{Re} \cdot \mathbf{e}$, where $\mathbf{e}$ is d-dimensional vector with all unit elements. The forcing function $f(x_1,\cdots, x_d)$ and the Dirichlet boundary conditions are prescribed to satisfy the exact solution $u(x_1,\cdots,x_d) = \prod_{i=1}^{d} \sin \pi x_i$. Re is a representative of the constant conducted to reflect the ratio of the convection to diffusion and simulate the Reynolds number.

For the different Re, the refinement of the mesh size h is undertaken to observe the convergence rate. The maximum errors and the convergence rate of 2D and 3D problems are listed as $1 \le \mathrm{Re} \le 10^3$ in Tables 3 and 4 and in Figure 1, clearly.

It has become apparent that Re is not very large leading to the rapid decrease of errors with the refinement of the mesh

Table 4: Numerical results for the sixth-order CFD method in 3D with $\boldsymbol{\alpha} = \mathrm{Re} \cdot (1, 1, 1)$.

h	Re = 1		Re = 10		Re = 10^2		Re = 10^3	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
1/4	1.195*E*-03		1.047*E*-02		4.286*E*-02		3.798*E*-02	
1/8	1.814*E*-05	6.042	1.589*E*-04	6.043	2.303*E*-02	0.90	1.482*E*-02	1.36
1/16	2.924*E*-07	5.955	2.542*E*-06	5.966	1.795*E*-03	3.68	5.782*E*-03	1.36
1/32	4.550*E*-09	6.006	3.985*E*-08	5.995	2.865*E*-05	5.97	1.976*E*-03	1.55
1/64	7.106*E*-11	6.001	6.251*E*-10	5.994	4.547*E*-07	5.98	4.488*E*-04	2.14

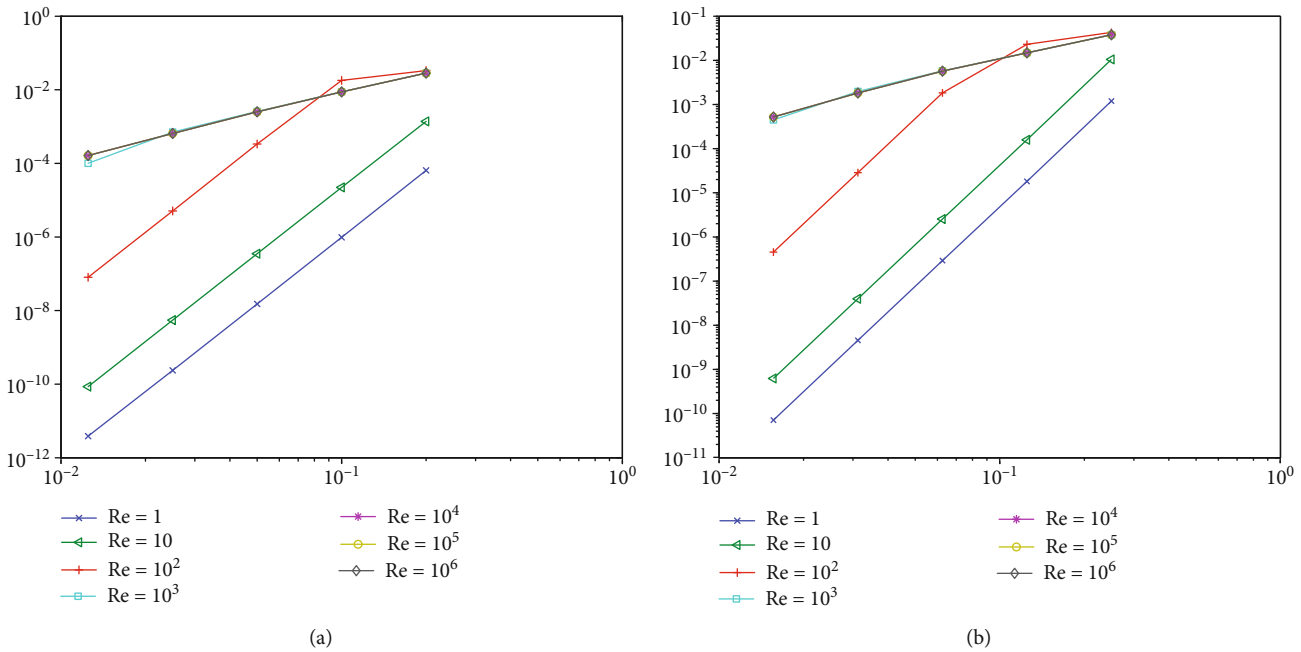

Figure 1: (a) Test of the sixth-order CFD method in 2D; (b) test of the sixth-order CFD method in 3D.

size h and a sixth-order convergence rate. But with the increase of Re, the computed accuracy is reversely affected. For the large Re, only a second-order convergence rate can be achieved. It is expected that the deterioration of computed accuracy is attributed to a reason of the convection-dominated equation (1), and when the Re increases to be infinite, it approximates a first-order hyperbolic equation.

5. Conclusions

Compact finite-difference (CFD) schemes seem to be simple and powerful ways contributing to a deeper understanding of high accuracy and low computational cost. Compared with the traditional explicit finite-difference schemes of the same-order, the effects of compact schemes are significantly more accurate with the benefit of smaller stencil sizes. In the experimented present work, the fourth-order CFD schemes generate fresh insight into a general formulation of convection-diffusion equations with variable convective coefficients and elliptic diffusion equations with variable diffusive coefficients in any dimension. When the convective coefficients are constants, the sixth-order CFD schemes provide an important opportunity to solve convection-diffusion problems. Numerical results of preliminary work show the efficiency of the proposed CFD schemes. A significant analysis and discussion on similar approaches may be extended to elliptic diffusion problems with positive diffusive tensor in any dimension and are under investigation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interests regarding the publication of this paper.

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Research Article

Normality of the p-Harmonic and Log-p-Harmonic Mappings

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In this paper, the concepts of p-harmonic mappings and log-p-harmonic mappings in the unit disk have been introduced and studied by many researchers. We proved the normality of the p-harmonic mappings and log-p-harmonic mappings, which extend the related results of harmonic mappings of earlier authors.

1. Introduction and Preliminaries

For real-valued harmonic functions defined in $\mathbb{D}$, Lappan [1] established that φ is normal if

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\frac{|grad\varphi(z)|}{1+\varphi^2(z)}<\infty, \tag{1}$$

where grad φ is the gradient vector of φ. In [2], the authors also proved geometric properties of real-valued harmonic normal functions. Namely, a real-valued harmonic function φ with the property

$$\iint_{\mathbb{D}}\left(\frac{|grad\varphi(z)|}{1+\varphi^2(z)}\right)^2 d\Omega<\infty, \tag{2}$$

is normal.

Recently, many authors considered the properties of the complex-valued harmonic mappings and harmonic quasiconformal mappings in [3–13]. We are motivated to establish the topic of normality for complex-valued p-harmonic mappings and log-p-harmonic mappings defined in the unit disk. An important concept related with normal harmonic functions is the Bloch function, which was studied by Colonna in [14]. It is a classical result of Lewy [15] that a harmonic mapping is locally univalent in a domain Ω if and only if its Jacobian does not vanish. In terms of the canonical decomposition, the Jacobian of harmonic mappings $f=h+\bar{g}$ is given by $J_f=|h'|^2-|g'|^2$, and thus, a locally univalent harmonic mapping in a simply connected domain Ω will be sense-preserving if $|h'|>|g'|$.

Following the above ideas, particularly the definition of Bloch harmonic function given by Colonna [14], we will prove that the polyharmonic mapping F and log-p-harmonic mapping f defined in the unit disk $\mathbb{D}$ are normal if they satisfy a Lipschitz type condition. Further, for the complex-valued polyharmonic mappings and log-p-harmonic mappings, we give out some additional conditions for which are normal. These conditions cannot be omitted. A 2p-times continuously differentiable complex-valued function $F(z)=u(z)+iv(z)$ in a domain $D\subseteq\mathbb{C}$ is polyharmonic mapping or p-harmonic if $F(z)$ satisfies the p-harmonic equation

$$\Delta^p F=\Delta\left(\Delta^{p-1}\right)F=0, \tag{3}$$

where the Laplacian operator

$$\Delta F=4F_{z\bar{z}}=\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}. \tag{4}$$

As we see in Proposition 1 in [16], we know that a mapping F is polyharmonic in a simply connected domain $D\subseteq\mathbb{C}$ if and only if F has the following representation

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z), \tag{5}$$

where each G_{p-k+1} is harmonic for $k \in \{1,\cdots,p\}$. When $p = 1$, the mapping F is called harmonic. When $p = 2$, the mapping F is called biharmonic. f is called log-p-harmonic mapping if $\log f$ is p-harmonic mapping. When $p = 1$, the mapping f is called log-harmonic. When $p = 2$, the mapping f is called log-biharmonic, which can be regarded as generalizations of holomorphic functions. So we say that f is called log-p-harmonic mapping in a simply connected domain $D \subseteq \mathbb{C}$ if and only if f has the form

$$f(z) = \prod_{k=1}^{p} \left[g_{p-k+1}(z)\right]^{|z|^{2(k-1)}}, \tag{6}$$

where each g_{p-k+1} is log-harmonic for $k \in \{1,\cdots,p\}$.

For a continuously differentiable mapping f in D, we define

$$\begin{aligned} \Lambda_f(z) &= \max_{0\le\theta\le 2\pi} |f_z(z) + \exp(-2i\theta) f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|, \\ \lambda_f(z) &= \min_{0\le\theta\le 2\pi} |f_z(z) + \exp(-2i\theta) f_{\bar{z}}(z)| = \big||f_z(z)| - |f_{\bar{z}}(z)|\big|. \end{aligned} \tag{7}$$

Recently, many authors considered Landau-type theorems for harmonic mappings, biharmonic mappings, and p-harmonic mappings [16–23]. Li and Wang [24] introduced the log-p-harmonic mappings and derived two versions of Landau-type theorems. However, in virtue of being inspired by these results, we establish the normality of polyharmonic mappings and log-p-harmonic mappings.

2. Necessary Lemmas

In order to derive our main results, we need the following lemmas.

Lemma 1. [14]. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $|f(z)| < M$ for all $z \in \mathbb{D}$, then*

$$\Lambda_f(z) \le \frac{4M}{\pi\left(1 - |z|^2\right)}. \tag{8}$$

Lemma 2. [22]. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $|f(z)| < M$ for all $z \in \mathbb{D}$, then for $|z| = r < 1$, we have*

$$|f(z)| \le \frac{4Mr}{\pi(1-r)}. \tag{9}$$

Lemma 3. [25]. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$ and $\lambda_f(0) = 1$. If $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$, then*

$$|a_n| + |b_n| \le \frac{\Lambda^2 - 1}{n\Lambda}, \quad n = 2, 3, \cdots. \tag{10}$$

When $\Lambda > 1$, the above estimates are sharp for all $n = 2, 3, \cdots$, with the extremal functions $f_n(z)$ and $\overline{f_n(z)}$ as follows

$$f_n(z) = \Lambda^2 z - \left(\Lambda^3 - \Lambda\right)\int_0^z \frac{1}{\Lambda + z^{n-1}}\, dz. \tag{11}$$

When $\Lambda = 1$, then $f(z) = a_1 z + \overline{b_1 z}$ with $\|a_1| - |b_1\| = 1$.

Lemma 4. [19]. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$, then for each $z \in \mathbb{D}$,*

$$|f(z)| \le \Lambda z. \tag{12}$$

We recall that the chordal distance on the generalized complex plane $\widehat{\mathbb{C}}$, which is defined by

$$\begin{aligned} \chi(z_1, z_2) &= \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}; z_1, z_2 \in \mathbb{C}, \\ \chi(z,\infty) &= \frac{1}{\sqrt{1 + |z|^2}}. \end{aligned} \tag{13}$$

If P_{z_1}, P_{z_2} are the two points on the Riemann sphere, under stereographic projection, corresponding to z_1 and z_2, respectively, we have

$$|P_{z_1} - P_{z_2}| = \chi(z_1, z_2). \tag{14}$$

Therefore,

$$L(\Gamma) \ge \rho(z_1, z_2) \ge \chi(z_1, z_2), \tag{15}$$

where $\varrho(z_1, z_2)$ is the spherical distance of z_1 and z_2, Γ is any rectifiable curve in $\mathbb{C}$ with endpoints z_1, z_2, and

$$L(\Gamma) = \int_\Gamma \frac{|du|}{1 + |u|^2} \tag{16}$$

is the spherical length of Γ. On the basis of the paper, given $z_1, z_2 \in \mathbb{D}$, $\rho(z_1, z_2)$ denotes the hyperbolic distance between z_1, z_2. Therefore, if τ denotes the hyperbolic geodesic joining z_1 to z_2, then

$$\rho(z_1, z_2) = \int_\tau \frac{|dv|}{1 - |v|^2}. \tag{17}$$

More explicitly,

$$\rho(z_1, z_2) = \frac{1}{2}\log\frac{1+\lambda}{1-\lambda}, \tag{18}$$

where

$$\lambda = \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|. \tag{19}$$

With these notations, a polyharmonic mapping or log-p-harmonic mapping $f : \mathbb{D} \to \mathbb{C}$ is called a normal polyharmonic mapping or normal log-p-harmonic mapping, if

$$\sup_{z_1 \neq z_2} \frac{\chi(f(z_1), f(z_2))}{\rho(z_1, z_2)} < \infty. \tag{20}$$

The following lemma provides an alternative method for deciding when a polyharmonic mapping or log-p-harmonic mapping is normal.

Lemma 5. *Let $f(z)$ be a polyharmonic mapping or log-p-harmonic mapping in the unit disk $\mathbb{D}$, then f is normal if*

$$\|f\| := \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) \frac{|f_z| + |f_{\bar{z}}|}{1 + |f|^2} < \infty. \tag{21}$$

Proof. Suppose that $\|f\| < \infty$ and let $z_1, z_2 \in \mathbb{D}$. If $\tau : [0, 1] \to \mathbb{D}$ is the hyperbolic geodesic with endpoints z_1 and z_2,

$$\chi(f(z_1), f(z_2)) \leq \int_{f \circ \tau} \frac{|du|}{1 + |u|^2} = \int_0^1 \frac{|df(\tau(t))\tau'(t)|}{1 + |f(\tau(t))|^2} dt, \tag{22}$$

where df stands for the differential of f. From here and (21), we have

$$\begin{aligned} \chi(f(z_1), f(z_2)) &\leq \int_0^1 \frac{|df(\tau(t))\tau'(t)|}{1 + |f(\tau(t))|^2} dt \\ &\leq \int_0^1 \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\tau(t)|^2} \frac{|df(\tau(t))||\tau'(t)|}{1 + |f(\tau(t))|^2} dt \\ &= \int_0^1 \left\{ \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) \frac{|df(\tau(t))|}{1 + |f(\tau(t))|^2} \right\} \frac{|\tau'(t)|}{1 - |\tau(t)|^2} \\ &\leq \|f\| \cdot \int_0^1 \frac{|\tau'(t)|}{1 - |\tau(t)|^2} dt = \|f\| \cdot \rho(z_1, z_2). \end{aligned} \tag{23}$$

Hence, we obtain

$$\sup_{z_1 \neq z_2} \frac{\chi(f(z_1), f(z_2))}{\rho(z_1, z_2)} < \infty. \tag{24}$$

So it implies that f is normal.

3. Main Results and Their Proofs

In this section, we prove the normality of the polyharmonic mappings and log-p-harmonic mappings as follows.

Theorem 6. *Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a polyharmonic mapping in the unit disk $\mathbb{D}$ satisfying $F(0) = \lambda_F(0) - 1 = 0$. Suppose that for each $k \in \{1, \cdots, p\}$, we have*

$$\begin{aligned} &G_{p-k+1}(z) \text{ is harmonic in } \mathbb{D}, \text{ and} G_{p-k+1}(0) = 0; \\ &\left|G_{p-k+1}(z)\right| \leq M_{p-k+1}, \text{ and} \Lambda_{G_p}(z) \leq \Lambda_p, \text{ where } M_{p-k+1} \geq 0, \Lambda_p \geq 1. \end{aligned} \tag{25}$$

Then, F is normal polyharmonic mapping of the unit disk $\mathbb{D}$.

Proof. We may represent the harmonic functions $G_{p-k+1}(z)$ in series form as

$$\begin{aligned} G_{p-k+1}(z) &= \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^j, k \in \{1, 2, \cdots, p\}; \\ &\left| |a_{1,p}| - |b_{1,p}| \right| = \lambda_F(0) = \left| \left|\left(G_p\right)_z(0)\right| - \left|\left(G_p\right)_{\bar{z}}(0)\right| \right| \\ &= \lambda_{G_p}(0) = 1, |a_{1,p}| + |b_{1,p}| = \left|\left(G_p\right)_z(0)\right| + \left|\left(G_p\right)_{\bar{z}}(0)\right| \\ &= \Lambda_{G_p}(0) \leq \Lambda_p. \end{aligned} \tag{26}$$

Firstly, we calculate the boundedness of the derivative of F.

$$\begin{aligned} |F_z(z) + F_{\bar{z}}(z)| &\leq |F_z(z)| + |F_{\bar{z}}(z)| \\ &= \left|\left(G_p\right)_z(z) + \sum_{k=1}^{p-1} \left[|z|^{2k} \left(G_{p-k}\right)_z(z) + k G_{p-k}(z) \bar{z}^k z^{k-1} \right]\right| \\ &\quad + \left|\left(G_p\right)_{\bar{z}}(z) + \sum_{k=1}^{p-1} \left[|z|^{2k} \left(G_{p-k}\right)_{\bar{z}}(z) + k G_{p-k}(z) \bar{z}^{k-1} z^k \right]\right| \\ &\leq A_1 + A_2 + A_3 + A_4, \end{aligned} \tag{27}$$

where

$$\begin{aligned} A_1 &= \left| \left(G_p\right)_z(0) + \left(G_p\right)_{\bar{z}}(0) \right|, \\ A_2 &= \left| \sum_{k=1}^{p-1} |z|^{2k} \left[\left(G_{p-k}\right)_z(z) + \left(G_{p-k}\right)_{\bar{z}}(z) \right] \right|, \\ A_3 &= \left| \sum_{k=1}^{p-1} k G_{p-k}(z) \left(\bar{z}^k z^{k-1} + \bar{z}^{k-1} z^k \right) \right|, \\ A_4 &= \left| \left[\left(G_p\right)_z(z) - \left(G_p\right)_z(0) \right] + \left[\left(G_p\right)_{\bar{z}}(z) - \left(G_p\right)_{\bar{z}}(0) \right] \right|. \end{aligned} \tag{28}$$

By a simple calculation, we have

$$A_1 \leq \Lambda_{G_p}(0) \leq \Lambda_p. \tag{29}$$

Using Lemma 1, we have

$$A_2 \le \sum_{k=1}^{p-1} |z|^{2k}\left[\left|\left(G_{p-k}\right)_z(z)\right|+\left|\left(G_{p-k}\right)_{\bar{z}}(z)\right|\right] \le \sum_{k=1}^{p-1} r^{2k}\Lambda_{G_{p-k}}(z) \le \sum_{k=1}^{p-1} r^{2k}\frac{4M_{p-k}}{\pi(1-r^2)}. \tag{30}$$

By Lemma 2, we have

$$A_3 \le \sum_{k=1}^{p-1} |kG_{p-k}(z)|\left(|\bar{z}^k z^{k-1}|+|\bar{z}^{k-1}z^k|\right) \le \frac{8}{\pi(1-r)}\sum_{k=1}^{p-1} kM_{p-k}r^{2k}. \tag{31}$$

Using Lemma 3, we have

$$\begin{aligned} A_4 &\le \left|\left(G_p\right)_z(z)-\left(G_p\right)_z(0)\right|+\left|\left(G_p\right)_{\bar{z}}(z)-\left(G_p\right)_{\bar{z}}(0)\right| \\ &\le \sum_{n=2}^{\infty}\left(|a_{n,p}|+|b_{n,p}|\right)nr^{n-1} \\ &\le \sum_{n=2}^{\infty}\frac{\left(\Lambda_p^2-1\right)r^{n-1}}{\Lambda_p} = \frac{\Lambda_p^2-1}{\Lambda_p}\frac{r}{1-r}. \end{aligned} \tag{32}$$

By the above estimates, we obtain the following result

$$|F_z(z)+F_{\bar{z}}(z)| \le S_1(r), \tag{33}$$

where

$$S_1(r) = \frac{4}{\pi(1-r^2)}\sum_{k=1}^{p-1} M_{p-k}r^{2k} + \frac{8}{\pi(1-r)}\sum_{k=1}^{p-1} kM_{p-k}r^{2k} + \frac{\left(\Lambda_p^2-1\right)r}{\Lambda_p(1-r)} + \Lambda_p. \tag{34}$$

Now, differentiating $S_1(r)$, we have

$$S'_1(r) = \sum_{k=1}^{p-1}\frac{4M_{p-k}\left[2kr^{2k-1}\left(1-r^2\right)+2r^{2k+1}\right]}{\pi(1-r^2)^2} + \sum_{k=1}^{p-1}\frac{8kM_{p-k}\left[2kr^{2k+1}(1-r)+r^{2k}\right]}{\pi(1-r)^2} + \frac{\Lambda_p^2-1}{\Lambda_p(1-r)^2}. \tag{35}$$

In view of $\Lambda_p \ge 1$ and $r \in (0,1)$, after a simple calculation, it shows that $S'_1(r) > 0$. It is simple to verify that $S_1(r)$ is strictly increasing in $(0,1)$.

$$\lim_{r\to 0} S_1(r) = \Lambda_p,\ \lim_{r\to 1} S_1(r) = +\infty. \tag{36}$$

Obviously, $S_1(r)$ has only one pole $r = 1(|z| = r)$. In other words, $S_1(r)$ is bounded in the interval $(0,1)$.

Finally, we consider the boundedness of F for any $|z| = r_1$, then we have

$$\begin{aligned} |F(z)| &= \left|\sum_{n=1}^{\infty}\left(a_{n,p}z^n + \bar{b}_{n,p}\bar{z}^n\right) + \sum_{k=1}^{p-1}|z|^{2k}G_{p-k}(z)\right| \\ &\le |a_{1,p}z + \bar{b}_{1,p}\bar{z}| + \left|\sum_{n=2}^{\infty}\left(a_{n,p}z^n + \bar{b}_{n,p}\bar{z}^n\right)\right| \\ &\quad + \left|\sum_{k=1}^{p-1}|z|^{2k}G_{p-k}(z)\right| \le \Lambda_p r_1 + \frac{\Lambda_p^2-1}{\Lambda_p}\sum_{n=2}^{\infty}\frac{r_1^n}{n} \\ &\quad + \sum_{k=1}^{p-1} r_1^{2k}\frac{4M_{p-k}r_1}{\pi(1-r_1)} = \Lambda_p r_1 - \frac{\Lambda_p^2-1}{\Lambda_p}\left[r_1+\ln(1-r_1)\right] \\ &\quad + \frac{4r_1}{\pi(1-r_1)}\sum_{k=1}^{p-1} M_{p-k}r_1^{2k}. \end{aligned} \tag{37}$$

So $|F(z)| \le S_2(r_1)$, where

$$S_2(r_1) = \Lambda_p r_1 - \frac{\Lambda_p^2-1}{\Lambda_p}\left[r_1+\ln(1-r_1)\right] + \frac{4r_1}{\pi(1-r_1)}\sum_{k=1}^{p-1} M_{p-k}r_1^{2k}. \tag{38}$$

By the similar approach for differentiating $S_2(r_1)$, we have the following one

$$S'_2(r_1) = \sum_{k=1}^{p-1}\frac{4M_{p-k}\left[(2k+1)r_1^{2k}(1-r_1)+r_1^{2k+1}\right]}{\pi(1-r_1)^2} + \frac{\Lambda_p^2-1}{\Lambda_p^2(1-r_1)} + \frac{1}{\Lambda_p} \tag{39}$$

By elementary calculations, we get $S_2{}'(r_1) > 0$. It implies that $S_2(r_1)$ is increasing for $\Lambda_p \ge 1$ and $r_1 \in (0,1)$. It is simple to verify that $S_2(r_1)$ is bounded in $(0,1)$. Combined with Lemma 5 and Estimation (33), we conclude ultimately that F is normal polyharmonic mapping in the unit disk $\mathbb{D}$. The proof of this theorem is complete.

Theorem 7. *Let $F(z) = \sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$, be a polyharmonic mapping of $\mathbb{D}$ satisfying $F(0) = \lambda_F(0) - 1 = 0$. Suppose that for $k \in \{1,2,\cdots,p\}$, we have*

$$\begin{aligned} &G_{p-k+1}(z) \text{ is harmonic in } \mathbb{D}, \text{ and } G_{p-k+1}(0) = 0; \\ &\Lambda_{G_{p-k+1}}(z) \le \Lambda_{p-k+1} \text{ for all } z \in \mathbb{D}, \text{ where } \Lambda_{p-k+1} \ge 0, k \\ &\qquad = 2,3,\cdots,p, \text{ and } \Lambda_p \ge 1. \end{aligned} \tag{40}$$

Then, F is normal polyharmonic mapping in the unit disk $\mathbb{D}$.

Proof. We represent the harmonic functions $G_{p-k+1}(z)$ in series form as

$$G_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p-k+1} \bar{z}^j, \tag{41}$$

for each $k \in \{1, 2, \cdots, p\}$, and

$$|a_{1,p}| + |b_{1,p}| = \Lambda_F(0) = |(G_p)_z(0)| + |(G_p)_{\bar{z}}(0)| = \Lambda_{G_p}(0) \le \Lambda_p,$$

$$\left| |a_{1,p}| - |b_{1,p}| \right| = \lambda_F(0) = \left| |(G_p)_z(0)| - |(G_p)_{\bar{z}}(0)| \right| = \lambda_{G_p}(0) = 1 \tag{42}$$

To prove the normality of F, we first determine that the derivative of F is bounded in $\mathbb{D}$. Then, as in the proof of Theorem 6, we have

$$|F_z(z) + F_{\bar{z}}(z)| \le B_1 + B_2 + B_3 + B_4, \tag{43}$$

where

$$\begin{aligned} B_1 &\le |(G_p)_z(0)| + |(G_p)_{\bar{z}}(0)| = \Lambda_{G_p}(0) \le \Lambda_p, \\ B_2 &\le \sum_{k=1}^{p-1} |z|^{2k} \left[|(G_{p-k})_z(z)| + |(G_{p-k})_{\bar{z}}(z)| \right] \\ &\le \sum_{k=1}^{p-1} r^{2k} \Lambda_{G_{p-k}}(z) \le \sum_{k=1}^{p-1} \Lambda_{p-k} r^{2k}. \end{aligned} \tag{44}$$

By the condition (2) of Theorem 7 and Lemma 4, we have

$$B_3 \le \sum_{k=1}^{p-1} |k G_{p-k}(z)| \left(|\bar{z}^k z^{k-1}| + |\bar{z}^{k-1} z^k| \right) \le \sum_{k=1}^{p-1} 2k \Lambda_{p-k} r^{2k}. \tag{45}$$

Applying Lemma 3, we have

$$\begin{aligned} B_4 &\le |(G_p)_z(z) - (G_p)_z(0)| + |(G_p)_{\bar{z}}(z) - (G_p)_{\bar{z}}(0)| \\ &\le \sum_{n=2}^{\infty} n \left(|a_{n,p}| + |b_{n,p}| \right) r^{n-1} \le \frac{\Lambda_p^2 - 1}{\Lambda_p} \sum_{n=2}^{\infty} r^{n-1} \\ &= \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r}. \end{aligned} \tag{46}$$

By above estimates, we obtain that

$$\begin{aligned} |F_z(z) + F_{\bar{z}}(z)| &\le \sum_{k=1}^{p-1} \Lambda_{p-k} r^{2k} + \sum_{k=1}^{p-1} 2k \Lambda_{p-k} r^{2k} \\ &\quad + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p \\ &= \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p. \end{aligned} \tag{47}$$

Set

$$S_3(r) = \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p. \tag{48}$$

For differentiating $S_3(r)$, we have

$$S_3'(r) = \sum_{k=1}^{p-1} 2k(2k+1) \Lambda_{p-k} r^{2k-1} + \frac{\Lambda_p^2 - 1}{\Lambda_p (1-r)^2}. \tag{49}$$

By a simple calculation, we obtain $S_3'(r) > 0$. It shows that $S_3(r)$ is strictly increasing in $r \in (0, 1)$, and

$$\lim_{r \to 0} S_3(r) = \Lambda_p, \lim_{r \to 1} S_3(r) = +\infty. \tag{50}$$

This shows that $S_3(r)$ has only one pole $r = 1$, and it says that $S_3(r)$ is bounded in $(0, 1)$. So $|F_z(z) + F_{\bar{z}}(z)|$ is a finite value in the unit disk $\mathbb{D}$.

Now, we consider any z with $|z| = r_2$. Then, we have

$$\begin{aligned} |F(z)| &= \left| \sum_{n=1}^{\infty} \left(a_{n,p} z^n + \bar{b}_{n,p} \bar{z}^n \right) + \sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z) \right| \\ &\le |a_{1,p} z + \bar{b}_{1,p} \bar{z}| + \left| \sum_{n=2}^{\infty} \left(a_{n,p} z^n + \bar{b}_{n,p} \bar{z}^n \right) \right| \\ &\quad + \left| \sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z) \right| \le \Lambda_p r_2 + \frac{\Lambda_p^2 - 1}{\Lambda_p} \sum_{n=2}^{\infty} \frac{r_2^n}{n} \\ &\quad + \sum_{k=1}^{p-1} \Lambda_{p-k} r_2^{2k+1} = \sum_{k=1}^{p-1} \Lambda_{p-k} r_2^{2k+1} \\ &\quad - \frac{\Lambda_p^2 - 1}{\Lambda_p} [r_2 + \ln(1 - r_2)] + \Lambda_p r_2. := S_4(r_2) \end{aligned} \tag{51}$$

Differentiating $S_4(r_2)$, we have

$$S_4'(r_2) = \sum_{k=1}^{p-1} (2k+1) \Lambda_{p-k} r^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p (1-r_2)} + \frac{1}{\Lambda_p} \tag{52}$$

After elementary calculations, we have that $S_4'(r_2) > 0$. It implies that $S_4(r_2)$ is strictly increasing. It is simple to verify that $S_4(r_2)$ is finite for $r_2 \in (0, 1)$. It says that F is bounded in $\mathbb{D}$. Using these above estimates and Condition (21), we

conclude that F is normal polyharmonic mapping in $\mathbb{D}$. This proof of Theorem 7 is complete.

Finally, we establish the normality of log-p-harmonic mappings as follows.

Theorem 8. *Let $f(z) = \prod_{k=1}^{p} [g_{p-k+1}(z)]^{|z|^{2(k-1)}}$ be a log-p-harmonic mapping in the unit disk $\mathbb{D}$ satisfying $f(0) = g_p(0) = \lambda_f(0) = 1$. Suppose that for each $k \in \{1,\cdots,p\}$ we have*

$$\begin{aligned} &g_{p-k+1}(z) \text{ is } \log - \text{harmonic in } \mathbb{D}, \\ &\left|g_{p-k+1}(z)\right| \le M_{p-k+1}, \text{let } G_p = \log g_p,\ \Lambda_{G_p} \\ &\qquad \le \Lambda_p, \text{where } M_{p-k+1} \ge 1, \Lambda_p \ge 1. \end{aligned} \tag{53}$$

Then, f is a normal log-p-harmonic mapping in the unit disk $\mathbb{D}$.

Proof. Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$, for each $k \in \{1,\cdots,p\}$. We may represent the harmonic functions $G_{p-k+1}(z) = \log g_{p-k+1}$ in series form as

$$G_{p-k+1}(z) = \sum_{n=1}^{\infty} a_{j,p-k+1} z^j + \sum_{n=1}^{\infty} \bar{b}_{j,pk+1} \bar{z}^j. \tag{54}$$

Then, $F = \log f$ is a polyharmonic mapping in $\mathbb{D}$. We know that

$$\lambda_f(0) = \|f_z(0)| - |f_{\bar{z}}(0)\| = |f(0)|\,|F_z(0)| - |F_{\bar{z}}(0)\|, \tag{55}$$

and $f(0) = 1$, so it follows from $g_p(0) = \lambda_f(0) = 1$, we have $G_p(0) = \lambda_F(0) - 1 = 0$.

Obviously,

$$\begin{aligned} |G_{p-k+1}| &= |\log g_{p-k+1}| = |\log |g_{p-k+1}| + i \arg g_{p-k+1}| \\ &\le |\log |g_{p-k+1}\| + \pi, \end{aligned} \tag{56}$$

so we have

$$|G_{p-k+1}| \le \log M_{p-k+1} + \pi = \tilde{M}_{p-k+1}. \tag{57}$$

In order to prove the normality of f, it follows from Theorem 6 that we have

$$|F_z(z) + F_{\bar{z}}(z)| \le S_5(r), \tag{58}$$

where

$$\begin{aligned} S_5(r) = {} & \frac{4}{\pi(1-r^2)} \sum_{k=1}^{p-1} r^{2k} \tilde{M}_{p-k} + \frac{8}{\pi(1-r)} \sum_{k=1}^{p-1} k \tilde{M}_{p-k} r^{2k} \\ & + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p. \end{aligned} \tag{59}$$

So we obtain

$$\left|\frac{f_z(z)}{f(z)} + \frac{f_{\bar{z}}(z)}{f(z)}\right| \le S_5(r). \tag{60}$$

Differentiating $S_5(r)$, we have

$$\begin{aligned} S_5'(r) = {} & \sum_{k=1}^{p-1} \frac{4M_{p-k}\left[2kr^{2k-1}\left(1-r^2\right) + 2r^{2k+1}\right]}{\pi(1-r^2)^2} \\ & + \sum_{k=1}^{p-1} \frac{8k\tilde{M}_{p-k}\left[2kr^{2k-1}(1-r) + r^{2k}\right]}{\pi(1-r)^2} + \frac{\Lambda_p^2 - 1}{\Lambda_p (1-r)^2}. \end{aligned} \tag{61}$$

By elementary calculations, we get $S_5'(r) > 0$. It follows that $S_5(r)$ is strictly increasing in $(0,1)$, and

$$\lim_{r\to 0} S_5(r) = \Lambda_p, \quad \lim_{r\to 1} S_5(r) = +\infty. \tag{62}$$

It implies that $S_5(r)$ is bounded in $(0,1)$. Hence, there exists a finite value m_1 such that

$$\left|\frac{f_z(z)}{f(z)} + \frac{f_{\bar{z}}(z)}{f(z)}\right| \le m_1. \tag{63}$$

Finally, we consider any z with $|z| = r_3$; then, we have

$$\begin{aligned} |\log f(z)| = |F(z)| &= \Big|\sum_{n=1}^{\infty} \left(a_{n,p} z^n + \bar{b}_{n,p} \bar{z}^n\right) + \sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z)\Big| \\ &\le |a_{1,p} z + \bar{b}_{1,p} \bar{z}| + \Big|\sum_{n=2}^{\infty} \left(a_{n,p} z^n + \bar{b}_{n,p} \bar{z}^n\right)\Big| \\ &\quad + \Big|\sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z)\Big| \le \Lambda_p r_3 + \frac{\Lambda_p^2 - 1}{\Lambda_p} \sum_{n=2}^{\infty} \frac{r_3^n}{n} \\ &\quad + \sum_{k=1}^{p-1} \frac{4\tilde{M}_{p-k} r_3^{2k+1}}{\pi(1-r_3)} = \sum_{k=1}^{p-1} \frac{4\tilde{M}_{p-k} r_3^{2k+1}}{\pi(1-r_3)} \\ &\quad - \frac{\Lambda_p^2 - 1}{\Lambda_p} [r_3 + \ln(1-r_3)] + \Lambda_p r_3 = S_6(r_3). \end{aligned} \tag{64}$$

Differentiating $S_6(r_3)$, have the following result

$$\begin{aligned} S_6'(r_3) = {} & \sum_{k=1}^{p-1} \frac{4\tilde{M}_{p-k}(2k+1)(1-r_3) r_3^{2k} + r_3^{2k+1}}{\pi(1-r_3)^2} \\ & + \frac{\Lambda_p^2 - 1}{\Lambda_p (1-r_3)} + \frac{1}{\Lambda_p} \end{aligned} \tag{65}$$

It is not difficult to verify that $S_6'(r_3) > 0$, which means that $S_6(r_3)$ is strictly increasing. It is also easily seen that $S_6(r_3)$ is bounded in $(0,1)$. Hence, there is a finite value m_2 such that

$$|\log f(z)| \leq m_2. \tag{66}$$

Applying Lemma 5 and (63), (66), we obtain f is normal in $\mathbb{D}$. The proof of Theorem 8 is complete.

Data Availability

The data used to support the findings of this study are included with the article.

Conflicts of Interest

We declare that we have no competing interests.

Acknowledgments

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Research Article

A Novel Global Energy and Local Energy-Based Legendre Polynomial Approximation for Image Segmentation

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Active contour model (ACM) is a powerful segmentation method based on differential equation. This paper proposes a novel adaptive ACM to segment those intensity inhomogeneity images. Firstly, a novel signed pressure force function is presented with Legendre polynomials to control curve contraction. Legendre polynomials can approximate regional intensities corresponding to evolving curve. Secondly, global term of our model characterizes difference of Legendre coefficients, and local energy term characterizes fitting evolution curve of interested region. Final contour evolution will minimize the energy function. Thirdly, a correction term is employed to improve the performance of curve evolution according to the initial contour position, so wherever the initial contour being in the image, the object boundaries can be detected. Fourthly, our model combines the advantages of two classical models such as good topological changes and computational simplicity. The new model can classify regions with similar intensity values. Compared with traditional models, experimental results show effectiveness and efficiently of the new model.

1. Introduction

Image segmentation [1], image denoising [2, 3], and image reconstruction [4] are all basic tasks in image processing field. The goal of image segmentation is to partition an image set into several meaningful subsets according to different feature information. It is difficult to segment images with noise, low contrast, and intensity inhomogeneity. Many segmentation methods based on partial differential equation (PDE) model [1] have been proposed with the development of PDE [5–10] and stochastic theory [11, 12]. One of the famous segmentation methods is ACM, which can extract the desired object by an evolving curve based on a variational framework. The existing ACM methods classify with edge ACM [1] and region ACM [13].

The edge ACM utilizes image gradient to guide the contours toward the boundaries of desired objects, and segmentation result relies on the location of the initial contour. Region model applies intensity and texture information to guide curve evolution. Chan-Vase (CV) [13] model is a classical region-based model, which is derived from Mumford-Shah (MS) model [14].

Intensity inhomogeneity image segmentation is still a challenging problem. Li et al. propose a local binary fitting (LBF) [15] model with a kernel function to avoid reinitialization and perform better than original CV model. Wang et al. propose a local Gaussian distribution fitting (LGDF) [16] model with local intensity mean and variance information. Zhou et al. introduce a global and local intensity information (LGIF) [17] model, which can achieve high segmentation accuracy while with heavy computational complexity. Wang et al. propose a local CV (LCV) [18] model to deal with the computation problem. LCV model cannot perform well on intensity inhomogeneous images and sometimes leads to edge leakage because the average convolution operator is employed in local region. Li et al. propose a LIC model [19] based on K-means clustering method. Zhang et al. develop LIC model and propose a locally statistical ACM (LSACM)

[20]. LSACM collects pixels belonging to the same class and realizes soft segmentation. LIC model and LSACM model are all with high computational cost. Mukherjee and Acton [21] propose a L2S model to deal with intensity inhomogeneity images. L2S model is also with heavy computational complexity due to Legendre basis functions. Shi and Pan [22] present a LGBF to deal with intensity inhomogeneous image segmentation problem.

This paper proposes a novel Legendre polynomial approximation with adaptive global energy based on our previous model [23]. For presenting a qualitative expansion on previous work, this paper compares our model with representative models on noisy images, low-contrast images, real-world images, neuron images, and dendritic spine image, and the experiments also give a clear description of the effect of correction term on the initial contour position. The detailed process reasoning of Legendre coefficient of contour is given in our work. The main contributions of this paper are presented as follows. Firstly, a novel signed pressure force (SPF) function with Legendre polynomials is proposed. Regions are represented by a set of Legendre basis function. Secondly, our model has a global term and a local term. The local term is motivated by SBGFRLS model and GAC model and then combines the advantages of them. The global term is designed by utilizing the Legendre basis function and difference between foreground and background. Thirdly, a correction term is applied to control the direction of curve evolution. Experimental results show that our model is robust and efficient to segment intensity inhomogeneity images.

This rest of this paper is organized as follows. In Section 2, we review L2S and SBGFRLS models briefly. The new model and corresponding algorithm are presented in Section 3. In Section 4, the proposed model is validated by some experiments on synthetic and real images and compared with other ACM. Conclusion is drawn in Section 5.

2. The Related Works

2.1. The SBGFRLS Model. Zhang et al. propose the SBGFRLS method [24] in 2009, which incorporates the advantages of GAC model and CV model. New SPF function is constructed according to intensity information inside and outside the contour curve. Let Ω be a bounded open subset of R^2 and $I : [0, a] \times [0, b] \to R^+$ be a given image. The level set evolution equation is as follows:

$$\frac{\partial \phi}{\partial t} = \mathrm{SPF}(I(x)) \cdot \alpha |\nabla \phi|, x \in \Omega, \tag{1}$$

where α is the balloon force, which controls the evolution speed of the level set. The SPF function is defined as follows:

$$\mathrm{SPF}(I(x)) = \frac{I(x) - c_1 + c_2/2}{\max\left(|I(x) - c_1 + c_2/2|\right)}, \ x \in \Omega, \tag{2}$$

where c_1 and c_2 are mean grey values and can be computed as

$$\begin{aligned} c_1(\phi) &= \frac{\int_\Omega I(x) \cdot H(\phi) dx}{\int_\Omega H(\phi) dx}, \\ c_2(\phi) &= \frac{\int_\Omega I(x) \cdot (1\text{-}H(\phi)) dx}{\int_\Omega (1 - H(\phi)) dx}. \end{aligned} \tag{3}$$

The model utilizes region information to stop the curve evolution, which is more efficient and less sensitive to noise. SBGFRLS model and CV model cannot deal with images with intensity inhomogeneity because they all consider global image intensities merely.

2.2. The L2S Model. Mukherjee and Acton [21] propose a Legendre polynomial-based segmentation model. They replace c_1 and c_2 in the original CV model as two smooth functions ${c^m}_1(x)$ and ${c_2}^m$. Then, the energy function of the L2S is as follows:

$$\begin{aligned} E^{L2S}(\phi, A, B) = &\int_\Omega \left|f(x) - A^T P(x)\right|^2 H(\phi(x)) dx + \lambda_1 \|A\|_2^2 \\ &+ \int_\Omega \left|f(x) - B^T P(x)\right|^2 (1 - H(\phi(x))) dx \\ &+ \lambda_1 \|B\|_2^2 + \nu \int_\Omega \delta_\varepsilon(\phi) \frac{\nabla \phi}{|\nabla \phi|} dx. \end{aligned} \tag{4}$$

where$\lambda_1 \geq 0, \lambda_2 \geq 0$are fixed scalars. $A = (\alpha_0, \cdots, \alpha_N)^T$ and $B = (\beta_0, \cdots, \beta_N)^T$ are the coefficient vectors for the inside region and outside region, respectively. The last term in Eq. (7) is the regularization item, which introduces smoothness in the zero curve. Let perform$\partial E^{L2S}/\partial A = 0, \partial E^{L2S}/\partial B = 0$, so$\widehat{A}$ and $\widehat{B}$ (Appendix A gives a detailed calculation of coefficient vectors, and Appendix B gives the existence and computability of coefficient vectors) are, respectively, computed as follows:

$$\begin{aligned} &\widehat{A} = [K + \lambda_1 I]^{-1} P, \quad \widehat{B} = [L + \lambda_2 I]^{-1} Q, \\ &[K]_{i,j} = \left\langle \sqrt{H(\phi(x))} P_i(x), \sqrt{H(\phi(x))} P_j(x) \right\rangle, \\ &[L]_{i,j} = \left\langle \sqrt{1 - H(\phi(x))} P_i(x), \sqrt{1 - H(\phi(x))} P_j(x) \right\rangle. \end{aligned} \tag{5}$$

$\langle , \rangle$ denotes the inner product operator. The vectors P and Q are obtained as $P = \int_\Omega P(x) f(x) H(\phi(x)) dx$, $Q = \int_\Omega P(x) f(x)$ $(1\text{-}H(\phi(x)))dx$. By minimizing Eq. (7), the corresponding variational level set formulation is as follows:

$$\begin{aligned} \frac{\partial \phi}{\partial t} = &\left[-\left|f(x) - \widehat{A}^T P(x)\right|^2 + \left|f(x) - \widehat{B}^T P(x)\right|^2 \right] \delta_\varepsilon(\phi) \\ &+ \nu \delta_\varepsilon(\phi) \ \mathrm{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right). \end{aligned} \tag{6}$$

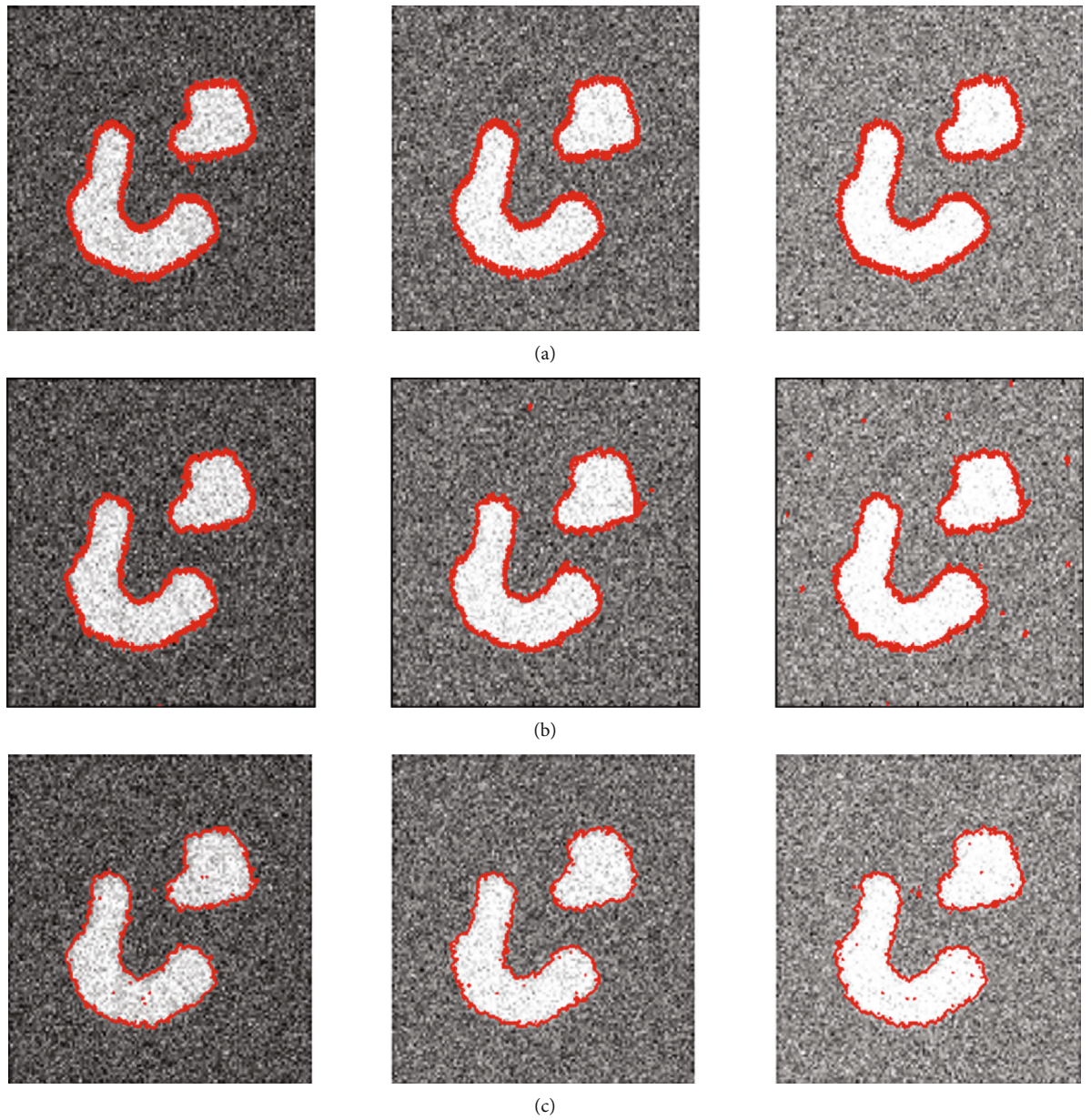


Figure 1: Segmental results for images with noise: (a) CV model segmentation results with noise image variance of 0.1, 0.2, and 0.3; (b) SBGFRLS model segmentation results with noise image variance of 0.1, 0.2, and 0.3; (c) Our model segmentation results with noise image variance of 0.1, 0.2, and 0.3.

The model approximates foreground and background by computing $\widehat{A}^T P(x)$ and $\widehat{B}^T P(x)$, respectively, which can perform well for images with inhomogeneous intensity.

3. A Novel Adaptive Segmentation Model

3.1. Model Construction. Motivated by the SBGFRLS model, we propose a novel ACM based on Legendre polynomial. The energy function of our model includes two parts, the local term and the global term:

$$E = E^L + E^G, \tag{7}$$

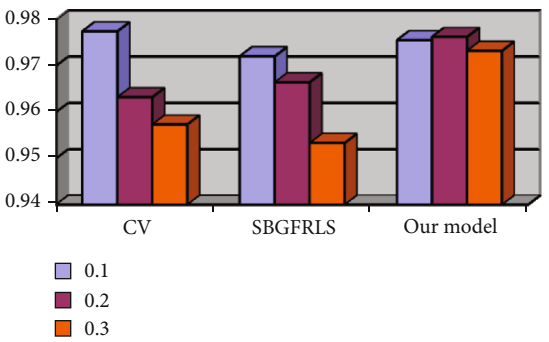
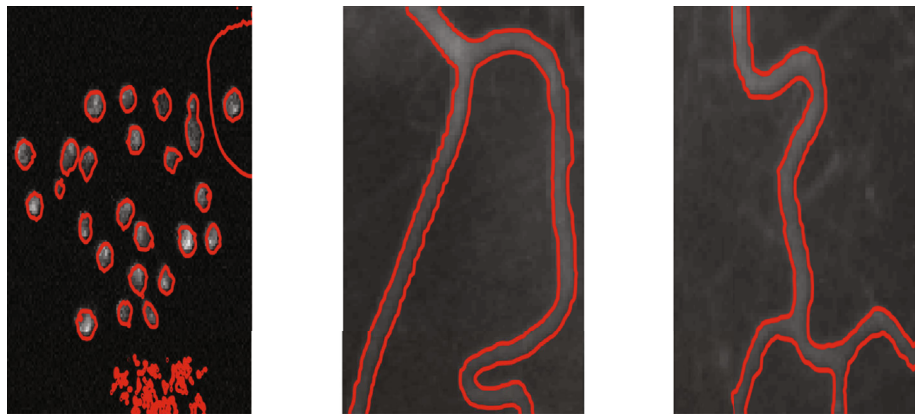
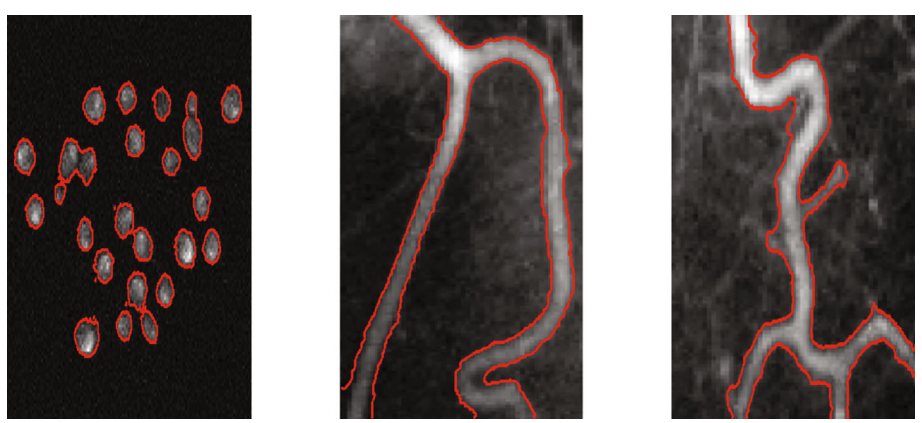


Figure 2: The corresponding Dice values of the segmental results in Figure 1.



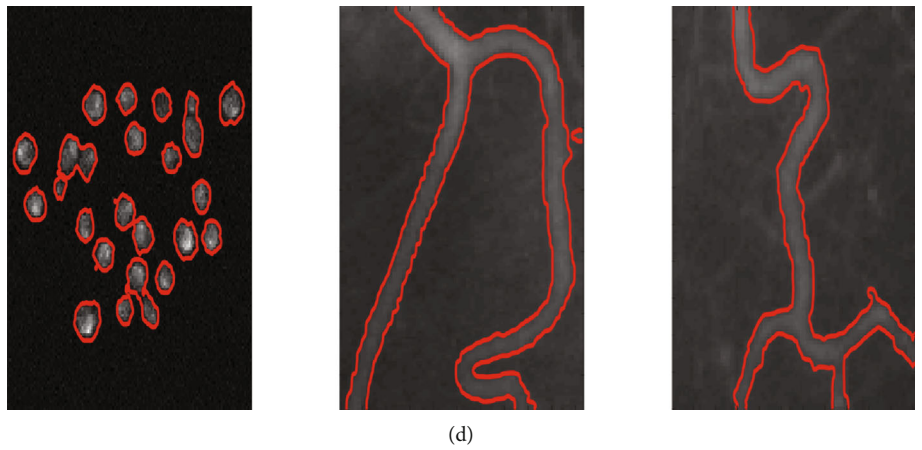
(a)

(b)



(c)

Figure 3: Continued.



(d)

(e)

Figure 3: The comparisons of LBF model, LIC model, L2S model, LSACM model, and our proposed model on segmenting images with intensity inhomogeneity: (a) results of LBF model; (b) results of LIC model; (c) results of L2S model; (d) result of LSACM model; (e) result of our proposed model.

where E^L and E^G denote the local term and the global term, respectively.

3.1.1. The Local Energy. The local energy term E^L is derived from the SBGFRLS model, which utilizes the global image mean intensity values inside and outside of the evolving curve C, respectively. In order to smooth the level set function to maintain the interface regular, div $(\nabla\phi/|\nabla\phi|)$ is curvature of evolving curve C is introduced. The level set formulation is as follows:

$$\frac{\partial\phi}{\partial t} = \mathrm{SPF}(I(x))\left(\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \alpha\right)|\nabla\phi|, \quad x \in \Omega. \tag{8}$$

The balloon force α can be regarded as a correction term, which is responsible for controlling the contour shrinking or expanding rate; then, the term div $(\nabla\phi/|\nabla\phi|) + \alpha$ can be negative value or positive value, so we can improve the performance of the SPF function to some extent. The SPF function will drive the contour to expand or shrink according to the location of the region of interest. However, the final obtained curve C can hardly extract local image feature for images with intensity inhomogeneity. The main reason is that the SPF model is constructed based on the intensity inside and outside the objects that are homogeneous. To solve these problem, our model introduces the Legendre polynomials to replace the scalars c_1 and c_2 in Eq. (5) by two smooth functions $c_1^m(x)$ and $c_2^m(x)$, which can represent by a linear combination of a few Legendre basis functions:

$$c^m{}_1(x) = \sum \alpha_k P_k(x), c_2{}^m(x) = \sum \beta_k P_k(x), \tag{9}$$

which can make these functions have the smoothness and flexibility, where $P_k$is a multidimensional Legendre

Table 1: The corresponding DICE values of the segmental results in Figure 3.

LBF	LIC	L2S	LSACM	Ours
0.6198	0.9061	0.8532	0.7893	0.9348
0.9654	0.8994	0.8823	0.9033	0.9311
0.9759	0.8669	0.8238	0.9035	0.9366

polynomial with degree k, and the 2-D polynomial is defined as follows:

$$p_k(x,y) = P_k(x)P_k(y), X = (x,y) \in \Omega \subset [-1,1]^2,$$
$$P_k(x) = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}^2 (x-1)^{k-i}(x+1)^i. \tag{10}$$

We can infer from the above equation that the highest degree of the 1D basis is m, and the highest degree of 2D basis is $(m+1)^2$. When $m=0$, two smooth functions will reduce to the scalars c_1 and c_2 in Eq. (5). Legendre polynomials' primary objective is to perform segmentation in the presence of an inhomogeneity intensity field. So the new SPF function is defined as follows:

$$\mathrm{SPF}^{\mathrm{new}}(I(x)) = \frac{I(x) - \widehat{A}^T P(x) + \widehat{B}^T P(x)/2}{\max\left(\left|I(x) - \widehat{A}^T P(x) + \widehat{B}^T P(x)/2\right|\right)}, \quad x \in \Omega, \tag{11}$$

where $\widehat{A}^T P(x)$ and $\widehat{B}^T P(x)$ are approximate to the gray values inside and outside of the evolving curve. $|\nabla\phi|$ can be replaced by $\delta(\phi)$ in (19) to increase the speed of curve evolution, and then the local term is as follows

$$E^L = \delta(\phi)\cdot \mathrm{SPF}^{\mathrm{new}}\left(I(x)\left(\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \alpha\right)\right) \quad x \in \Omega. \tag{12}$$

For each category of images, a correction term will decide the curve to evolve from inside to outside or from outside to inside, so that the initial contour being anywhere in the image can detect the object boundaries. At the same time, a new SPF can effectively drive curve to stop contours at weak edge, even images in the presence of inhomogeneity intensity.

3.1.2. The Global Energy. The main role of global energy affects the speed and accuracy of the evolution curve. Therefore, it is necessary to choose an appropriate global term. Most of parameters are selected manually, but we design an adaptive parameter strategy based on the difference between the foreground and background. The foreground and the background can be modeled by a set of Legendre basis functions in our model and can be represented in a lower dimensional subspace. When the gray value of background is less than the gray value of foreground, a novel adaptive global term can be written by utilizing Legendre basis functions as follows:

$$E^G = \frac{\mathrm{mean}\left(\widehat{A} - \widehat{B}\right)}{\min\left(\widehat{A} - \widehat{B}\right) - \max\left(\widehat{A} - \widehat{B}\right)}, \tag{13}$$

where mean(•), max(•), and mic(•) are the average function, the maximum function, and the minimum function, respectively, where $\widehat{A}$ and $\widehat{B}$ in Eq. (19) denote the coefficient vectors for the foreground and background. Each index satisfies the following relationship: $\max(\widehat{A} - \widehat{B}) > \mathrm{mean}(\widehat{A} - \widehat{B}) > \min(\widehat{A} - \widehat{B})$, and the data shows that the global term has values in the range $[-1, 1]$. An adaptive term can better meet the dynamic changes of the evolution curve than the constant term, which can improve the speed and accuracy of model segmentation. When the background gray value is greater than the foreground gray value, the global term can be set as zero.

Then, the final proposed model is as follows:

$$E = \delta(\phi)\cdot \mathrm{SPF}^{\mathrm{new}}(I(x))\left(\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \alpha\right) + \frac{\mathrm{mean}\left(\widehat{A} - \widehat{B}\right)}{\min\left(\widehat{A} - \widehat{B}\right) - \max\left(\widehat{A} - \widehat{B}\right)} \tag{14}$$

3.2. Algorithm Procedure

(a) Initialization:α, m, and σ

(b) Initialize the level set function ϕ as

$$\phi(x, t=0) = \begin{cases} -\rho & x \in \mathrm{outside}(C) \\ 0 & x \in C \\ \rho & x \in \mathrm{inside}(C) \end{cases} \qquad \rho > 0 \tag{15}$$

(c) Compute A, B, and $\mathrm{SPF}^{\mathrm{new}}(I(x))$by Eq. (11) and Eq. (17), respectively

(d) Evolve the level set function according to Eq. (20)

(e) Let $\phi = 1$ if $\phi > 0$; otherwise, $\phi = -1$

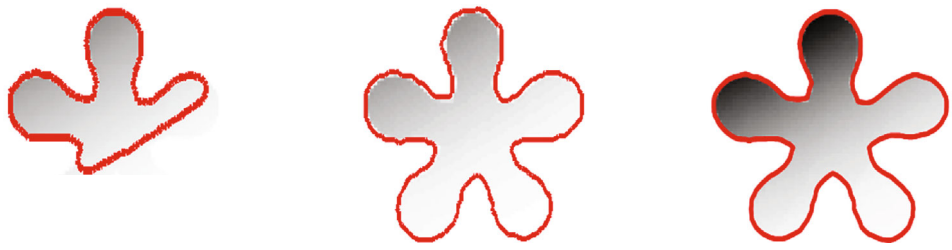
(f) Regularize the level set function ϕ with a Gaussian filter, i.e., $\phi = \phi * G_\sigma$

(g) Check whether the evolution of the level set has converged. If not, return to stage d

The step (e) serves as an optional segmentation procedure.

4. Experimental Results

In this section, the experimental results of our proposed model will be presented on a series of synthetic and real images. All experimental are implemented in Matlab R2014a on a 3.30-GHz PC. The initial contour can be chosen as rectangle, ellipse, and multiball manually according



(a) (b) (c)

Figure 4: Detected contour of regions of interest by LCV model, LSACM model, and our model: (a) results of LCV model; (b) results of LSACM model; (c) results of our model.

to image. The correction term α is important for curve evolution and should be set according to each category of images. We choose $m = 1$ and find that images with intensity inhomogeneity can be adequately modeled. When $m > 3$, it will require inversion of a larger matrix and need computation more expensive. To measure the quality of the extracted objects in segmentation, the Dice coefficients are used to evaluate the corresponding segmentation results of both our method and the other method. The Dice index $D \in [0, 1]$ represents the difference between the segmental result R_1 and the ground truth R_2, and larger value implies better segmentation results. The Dice is defined as

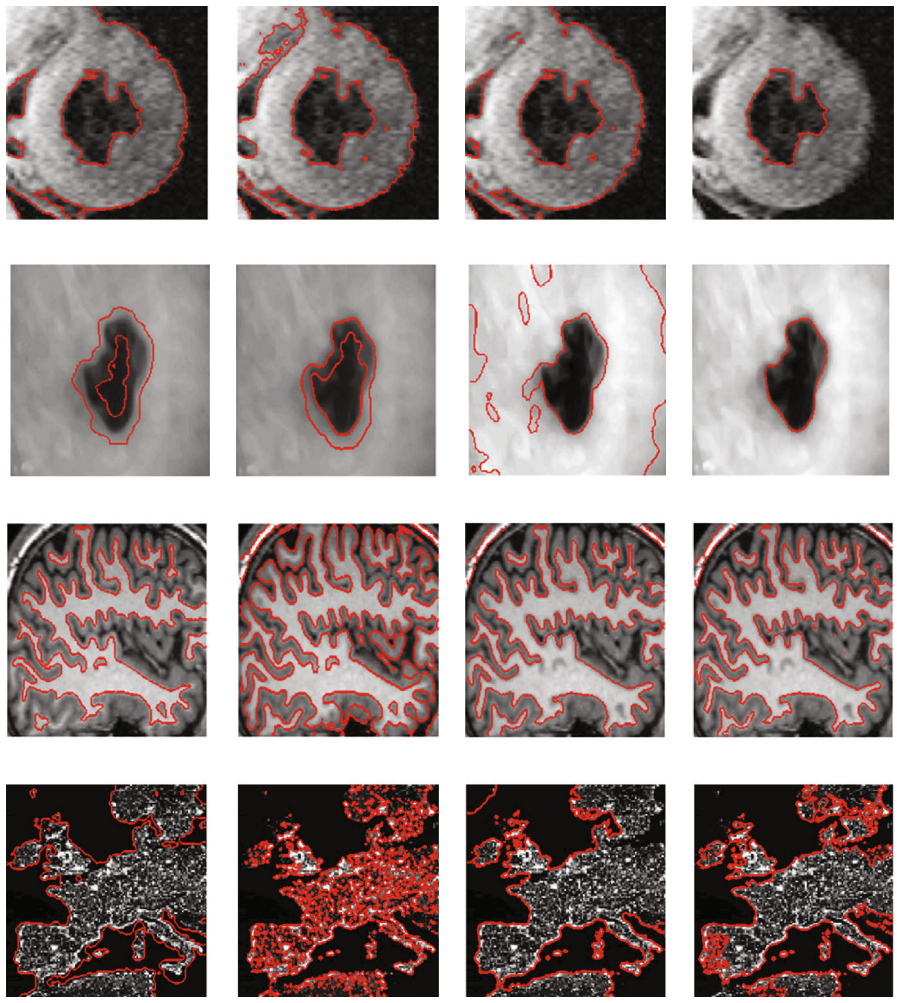
$$D(R_1, R_2) = \frac{2\text{Area}(R_1 \cap R_2)}{\text{Area}(R_1) + \text{Area}(R_2)}. \tag{16}$$

4.1. Results for Noisy Images. Figure1 shows the effectiveness of our model for noisy image segmentation. The image (two objects, [25]) in the first, second, and third row shows the corresponding segmentation results by CV model, SBGFRLS model, and our model. The first column, second column, and third column are images with Gaussian noise of standard deviations 0.1, 0.2, and 0.3, respectively. As shown in Figure2, the Dice value of our model is more stable with the variance increasing and our model can obtain better results than other models.

Table 2: Running time and the DICE value for images shown in Figure 4.

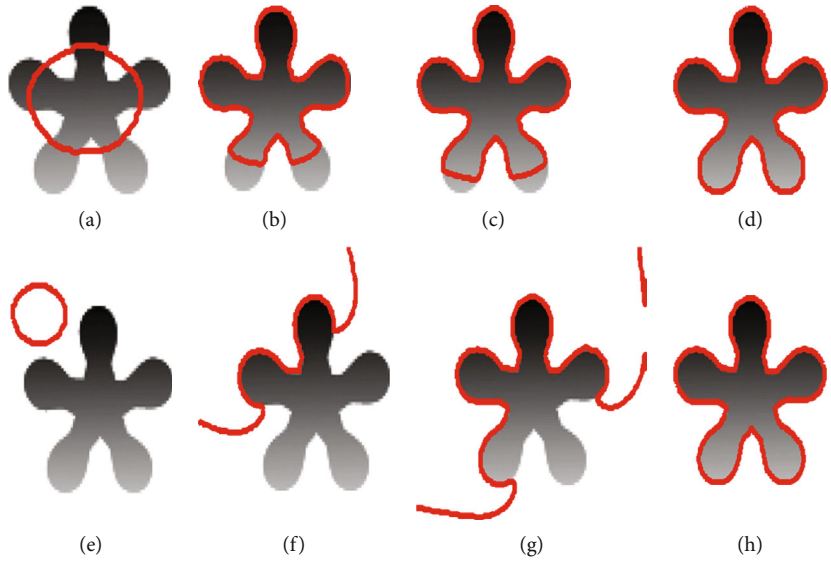
Image	CPU running time		
	LCV	LSACM	Ours
First	1.9375	37.2031	2.4375
Second	7.1719	35.8125	7.7031
Image	The DICE value		
First	0.4708	0.9745	0.9757
Second	0.9607	0.9966	0.9968

4.2. Robustness to Intensity Inhomogeneity. Figure 3 shows the segmentation results for yeast fluorescence micrograph image and two real blood vessel images with intensity inhomogeneity. The edges of these images are difficult to distinguish clearly, which makes the segmentation of such images challenging. For quantitatively evaluation, we give a comparison of the Dice value in Table 1; it can be seen from Figure 3 and Table 1 that our proposed model is proved to be more efficient in segmenting images with intensity inhomogeneity and more accurate in terms of segmentation accuracy than the other four methods.



(a) (b) (c) (d)

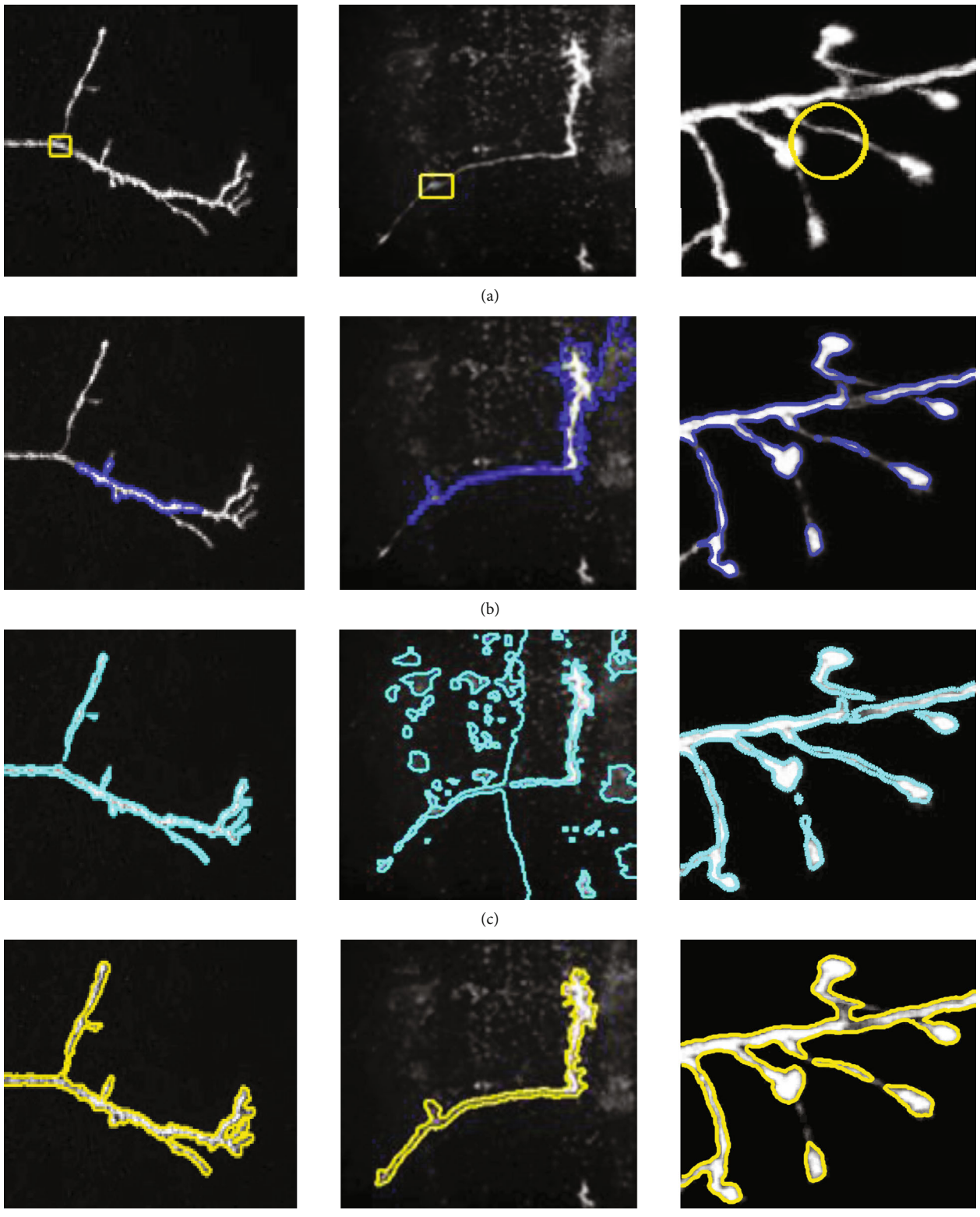
Figure 5: The comparisons of LSACM model, LBF model, LIC model, and our model on segmenting images with the intensity inhomogeneity: (a) results of LSACM model; (b) results of LBF model; (c) results of LIC model; (d) results of our model.



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 6: The process of segmentation using our model: (a) initial contour; (b) final contour, 20 iterations; (c) final contour, 40 iterations; (d) final contour, 100 iterations; (e) initial contour; (f) final contour, 60 iterations; (g) final contour, 120 iterations; (h) final contour, 240 iterations.



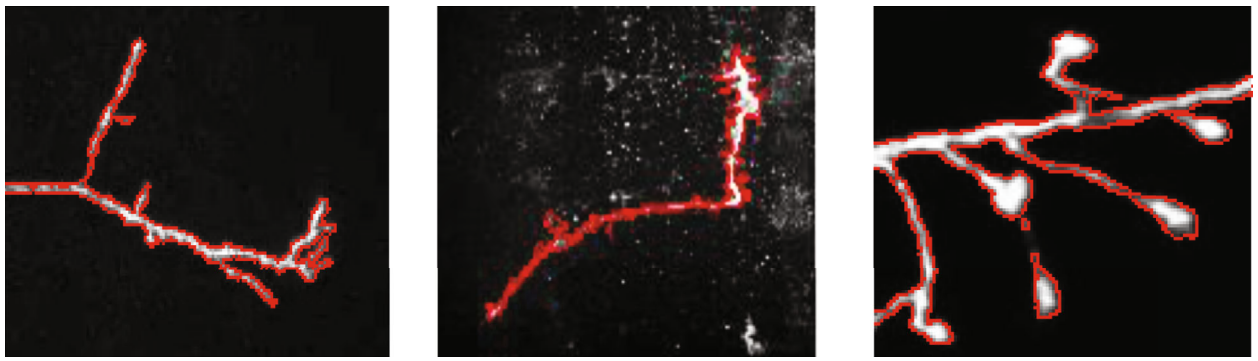
(a)

(b)

(c)

(d)

Figure 7: Continued.



(e)

Figure 7: Comparison results of our model with CV, LBF, and L2S models: (a) initial contours; (b) final segmentation results using the CV model; (c) final segmentation results using the LBF model; (d) final segmentation results using the L2S model; (e) final segmentation results using our model.

4.3. Results for Low-Contrast Images. Figure 4 and Table 2 show the effectiveness of our model for low-contrast images [26]. The first column, second column, and third column show the final contours by LCV model, LSACM model, and our model. Figure 4 shows the two low-contrast images with some weak boundaries, and our model obtains smoother curve and detects well the object's boundary, but the LCV model and LSACM model can obtain incomplete boundaries. As shown in Table 2, our model can effectively segment images with weak boundary and low contrast.

4.4. Results for Real-World Image. Figure 5 shows the ability of dealing with intensity inhomogeneous images (ultrasound medical image [24], skin image [25], brain MRI image [24], and Europe-night-light image [13]) in the real world, which captured from the digital camera. In order to further show the advantage of our model, we compare the segmentation results with the three models (LSACM, LBF, and LIC). Our model extracts the object contours accurately, whereas the other three methods produce oversegmentation and undersegmentation. For example, the result of the first image using LSACM, LBF, and LIC shows that the edge of image is oversegmented; the result of the second image using LSACM, LBF, and LIC shows that the edge of the cancer is oversegmented; the result of the third image using LSACM is oversegmented; and the result of the third image using LBF and LIC is undersegmented. Because the local term and global term are considered simultaneously, our model performs better than the other methods.

4.5. The Influence of Parameter α. Figure 6 shows the flexibility of the initial contour position in our model. Different initial contour position in image [24] is tested for the new model. The sign of parameter α is selected according to the image's characteristics, such as intensity and the position of initial contour. Since the gray value of the background is much larger than the object, the negative parameter α only needs 100 iterations in our experiment of first row, and the positive parameter α needs 480 iterations in our experiment of second row. The corresponding segmentation process of curve evolution illustrates the appropriate parameters that can speed up the evolution of curve.

4.6. Application on Neuron Images and Dendritic Spine Image. Figure 7 presents the results for dendritic spine image and neuron images which imaged by confocal microscope [21]. We validated the performance of our model on a set of neuron images by comparing with other methods. CV model and LBF model failed to obtain the object true boundary. The fourth and fifth rows in Figure 7 show that the L2S model and our model can extract similar results, but some fine details have been erased through L2S model. That suggests that our model based on local term and global term can be more effective than L2S model based on global term, which significantly improves the segmentation accuracy.

5. Conclusion

A novel adaptive segmentation model for images in the presence of low contrast, noise, weak edge, and intensity inhomogeneity is proposed in this paper. Regions are represented by a set of Legendre basis function, so Legendre polynomials are introduced to deal with intensity inhomogeneous image segmentation problem. The local and global information are all considered, and GAC model and SBGFRLS model are combined in our model. Our model has good topological changes and computational simplicity. The evolution direction can be chosen adaptively according to the parameter α, and the shape of initial contour can be selected on rectangular or elliptical manually. Experimental results show that our model is available and effective.

Appendix

A. The Process of Solving $\widehat{A}$ and $\widehat{B}$

In this appendix, we deduce the corresponding coefficients $\widehat{A}$ and $\widehat{B}$ in Eq. (11). Keeping ϕ fixed and minimizing the energy $E^{L2S}(\phi, A, B)$ with respect to the constant A and$B$. We perform$\partial E^{L2S}/\partial A = 0$, so the Eq. (7) can be expresses as

$$\frac{\partial\left(E^{L2S}\right)}{\partial A}=2\int_{\Omega}\frac{\partial\left(-A^{T}P(x)\right)}{\partial A}*\cdot\left(f(x)-A^{T}P(x)\right)H(\phi(x))\mathrm{d}x+2\lambda_{1}A=0 \tag{A.1}$$

then

$$2\int_{\Omega}\left(-P(x)f(x)H(\phi(x))+P(x)P(x)^{T}AH(\phi(x))\right)\mathrm{d}x+2\lambda_{1}A=0 \tag{A.2}$$

$$\int_{\Omega}P(x)P(x)^{T}AH(\phi(x))\mathrm{d}x+\lambda_{1}A=\int_{\Omega}(P(x)f(x)H(\phi(x)))\mathrm{d}x \tag{A.3}$$

$$\left(\int_{\Omega}P(x)P(x)^{T}H(\phi(x))\mathrm{d}x+\lambda_{1}I\right)A=\int_{\Omega}(P(x)f(x)H(\phi(x)))\mathrm{d}x \tag{A.4}$$

Let

$$\begin{aligned}P&=\int_{\Omega}P(x)f(x)H(\phi(x))dx,\\K&=\int_{\Omega}P(x)P(x)^{T}H(\phi(x))dx,\\&[K+\lambda_{1}I]\widehat{A}=P,\\\widehat{A}&=[K+\lambda_{1}I]^{-1}P.\end{aligned} \tag{A.5}$$

$[K]$ is $N\times N$ Gramian matrices, whose (i,j) th entry can be represented as

$$[K]_{i,j}=\left\langle\sqrt{H(\phi(x))}P_{i}(x),\ \sqrt{H(\phi(x))}P_{j}(x)\right\rangle. \tag{A.6}$$

Then, we perform $\partial E^{L2S}/\partial B=0$, so the Eq. (7) can be expresses as

$$\frac{\partial\left(E^{L2S}\right)}{\partial B}=2\int_{\Omega}\frac{\partial\left(-B^{T}P(x)\right)}{\partial A}*\left(f(x)-B^{T}P(x)\right)\cdot(1-H(\phi(x)))\mathrm{d}x+2\lambda_{1}B=0 \tag{A.7}$$

then

$$2\int_{\Omega}\left(-P(x)f(x)H(\phi(x))+P(x)P(x)^{T}B(1-H(\phi(x)))\right)dx+2\lambda_{1}B=0 \tag{A.8}$$

$$\int_{\Omega}P(x)P(x)^{T}B(1-H(\phi(x)))\Big)dx+\lambda_{1}B=\int_{\Omega}(P(x)f(x)(1-H(\phi(x))))dx \tag{A.9}$$

$$\left(\int_{\Omega}P(x)P(x)^{T}(1-H(\phi(x)))\right)dx+\lambda_{1}I)B=\int_{\Omega}(P(x)f(x)(1-H(\phi(x))))dx \tag{A.10}$$

Let

$$\begin{aligned}Q&=\int_{\Omega}P(x)f(x)(1-H(\phi(x)))dx,\\L&=\int_{\Omega}P(x)P(x)^{T}(1-H(\phi(x)))dx,\\&[L+\lambda_{1}I]\widehat{B}=Q,\\\widehat{B}&=[L+\lambda_{1}I]^{-1}Q.\end{aligned} \tag{A.11}$$

B. Coefficient Vectors are Invertible and Existing

However, computing the coefficient vectors needs a matrix inversion step. Here, we concretely show that the coefficient vectors $\widehat{A}$ and $\widehat{B}$ are invertible if the matrices $[K]$ and $[L]$ in (11) are invertible. $[L]$ is $N\times N$Gramian matrices, whose (i,j)th entry are obtained as the Eq. (11). Since $[K]$ and $[L]$ are a Gramian matrix, it is full rank if the polynomials $\sqrt{H(\phi(x))}P(x)$, $\sqrt{(1-H(\phi(x)))}P_{i}(x)(i=1,\cdots,N)$ are linearly independent. The regularized version of the Heaviside function is as $H(\phi)=1/2(1+(2/\pi)\arctan(\phi/\varepsilon))$, so it is easy to find that the functions $H(\phi(x))$ and $1-H(\phi(x))$ are bounded in (0, 1). Since the polynomials $P_{i}(x)$ are linearly independent themselves, it clearly shows that the polynomials $\sqrt{H(\phi(x))}P_{i}(x)$, $(i=1,\cdots,N)$ are linearly independent. So the coefficient vectors are existing.

Data Availability

All experimental images come from reference literatures, and we also point out the source one by one in the manuscript.

Conflicts of Interest

The authors declare there is no conflict of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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Research Article

Solutions for a Singular Hadamard-Type Fractional Differential Equation by the Spectral Construct Analysis

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In this paper, we consider the existence of positive solutions for a Hadamard-type fractional differential equation with singular nonlinearity. By using the spectral construct analysis for the corresponding linear operator and calculating the fixed point index of the nonlinear operator, the criteria of the existence of positive solutions for equation considered are established. The interesting point is that the nonlinear term possesses singularity at the time and space variables.

1. Introduction

In this paper, we focus on the existence of positive solutions for the following singular Hadamard-type fractional differential equation:

$$\begin{cases} \mathscr{D}_{\mathbf{t}}^{\alpha}\mathscr{D}_{\mathbf{t}}^{\beta}z(t) = f\left(t, z(t), -\mathscr{D}_{\mathbf{t}}^{\beta}z(t)\right), \ 1 < t < e, \\ z(1) = \sigma z(1) = \sigma z(e) = 0, \mathscr{D}_{\mathbf{t}}^{\beta}z(1) = \sigma\mathscr{D}_{\mathbf{t}}^{\beta}z(1) = \sigma\mathscr{D}_{\mathbf{t}}^{\beta}z(e) = 0, \end{cases} \tag{1}$$

where $2 < \alpha, \beta \le 3$; σ is a differential operator denoted by $t(d/dt)$, that is, $\sigma z(t) = t(d/dt)z(t)$; $\mathscr{D}_{\mathbf{t}}^{\alpha}, \mathscr{D}_{\mathbf{t}}^{\beta}$ are the Hadamard fractional derivative of order α, β; and $f \in (1, e) \times (0, +\infty) \times (0, +\infty)$, $[0, +\infty)$ is a continuous function with singularity at $t = 0, 1$ and $u = 0, v = 0$.

Singularity refers to a point or a domain where the given mathematical object is not defined or not "well-behaved." Near a singular point or zone, a minor change of the variable will lead to major changes of the property of the target object. Many physical phenomena in natural sciences and engineering often exhibit some singular behaviour. For example, Fisk [1] found that in certain materials the quantum fluctuations at absolute zero may push a system into a different phase or state, as result, the process loses its continuity, and then, the singular behaviour happens near the quantum critical points. In fluid mechanics, when a fluid is subjected to a severe impact to form a fracture, singular points or singular domains also follow the fracture. Normally, at singular points and domains, the extreme behaviour such as blow-up phenomena [2, 3], impulsive influence [4–9], and chaotic system [10–13], often leads to some difficulties for people in understanding and predicting the corresponding natural problems. Hence, the study of singularity for complex systems governed by differential equations [14–27] is important and interesting in deepening the understanding of the internal laws of dynamic system.

On the other hand, since the fractional differential operator is nonlocal, some often use it to describe viscoelastic behaviour and memory phenomena in various natural science fields such as the silicone gel with the property of weak frequency dependency [28, 29] and advection dispersion in anomalous diffusion [30–34]. In most cases, some are interested in the qualitative properties of solutions for the corresponding fractional equations; for the detail, see [35–65]. In

particular, in order to obtain the qualitative properties of solutions, many nonlinear analysis methods, such as fixed point theorems [66–71], iterative techniques [72–80],variational methods [81–98], and upper and lower solution methods [29, 44], have been developed and employed to study the qualitative properties and numerical results of solutions for various types of differential equations. For example, by using the fixed point index theory, Wang [69] established the existence and multiplicity of positive solutions for the following nonlocal singular fractional differential equation:

$$\begin{cases} D_{0_+}^{\alpha} u(t) + f(t, u(t) = 0, 0 < t < 1, n-1 < \alpha \le n, \\ u(0) = u'(0) == u^{(n-2)}(0) = 0, D_{0_+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0_+}^{\gamma} u(t) dV(t), \end{cases} \tag{2}$$

where $D_{0_+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative, $n \ge 3, 0 < \beta < 1, 0 \le \gamma < \alpha - 1, \eta \in (0, 1], f(t, x)$ may be singular at $t = 0, 1$, and $x = 0, a(t) \in L^1[0, 1] \cap C(0, 1)$ is nonnegative. In recent years, to enrich the theory of fractional calculus, the fractional derivative and integral were extended to many different forms such as Hadamard, Erdelyi-Kober, Hilfer derivatives, and integrals. In particular, it is more difficult to obtain the qualitative properties of solutions for Hadamard-type fractional differential equations since Hadamard derivatives possess a singular logarithmic kernel [99–106].

In this paper, we focus on the existence of positive solutions for the Hadamard-type fractional differential equation (1) with singularity in space variables. Our work has some new contributions. Firstly, the equation contains a Hadamard-type fractional derivative which has a singular logarithmic kernel. Secondly, the nonlinearity can have strong singularity in time and space variables. Thirdly, a new limit condition of integral type is introduced to overcome the difficulty of singularity. The rest of this paper is organized as follows. In Section 2, we firstly introduce the concept of Hadamard fractional integral and differential operators and then give the logarithmic kernel and Green function of the boundary value problem and their properties. Our main results are summarized in Section 3.

2. Preliminaries and Lemmas

Before the main results, we firstly recall the definition of the Hadamard-type fractional integrals and derivatives; for detail, see [107].

Let $\alpha \in \mathbb{C}, \mathbf{R}e(\alpha) > 0, n = [\mathbf{R}e(\alpha)]$ and (a, b) be a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$(I_a^{\alpha} x)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{x(s)}{s} ds, t \in (a, b), \tag{3}$$

and the α left Hadamard fractional derivative is defined by

$$(\mathscr{D}_{\mathbf{t}}^{\alpha}) x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \frac{x(s)}{s} ds, t \in (a, b). \tag{4}$$

In what follows, we consider the following linear auxiliary problem:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\beta} z(t) = x(t), 1 < t < e, \\ z(1) = \sigma z(1) = \sigma z(e) = 0. \end{cases} \tag{5}$$

It follows from [99] that problem (5) has a unique solution

$$z(t) = \int_1^e H(t, s) x(s) \frac{ds}{s}, \tag{6}$$

where

$$H(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} (\ln t)^{\beta-1} (1 - \ln s)^{\beta-2} - (\ln t - \ln s)^{\beta-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\beta-1} (1 - \ln s)^{\beta-2}, & 1 \le t \le s \le e, \end{cases} \tag{7}$$

is Green's function of equation (5). Now, let $x(t) = -\mathscr{D}_{\mathbf{t}}^{\beta} z(t)$, then the Hadamard-type fractional differential equation (1) reduces to the following convenient form:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = f\left(t, \int_1^e H(t, s) x(s) \frac{ds}{s}, x(t)\right), t \in (1, e), \\ x(1) = \sigma z(1) = \sigma x(e) = 0. \end{cases} \tag{8}$$

It follows from (6) that equation (8) is equivalent to the following integral equation:

$$x(t) = \int_1^e G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s)\right)\right] \frac{ds}{s}, \tag{9}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1} (1 - \ln s)^{\alpha-2} - (\ln t - \ln s)^{\alpha-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\alpha-1} (1 - \ln s)^{\alpha-2}, & 1 \le t \le s \le e. \end{cases} \tag{10}$$

As a result, in order to find the positive solutions of equation (1), it is sufficient to search the fixed point of the following operator:

$$Tx(t) = \int_1^e G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s)\right)\right] \frac{ds}{s}. \tag{11}$$

Lemma 1 (see [99]). *Let $\psi_i(t) = \ln t(1 - \ln t)^{i-2}, i = \alpha, \beta, t \in [1, e]$. Then, Green's functions H, G has the following properties:*

$$H, G \in C([1, e] \times [1, e], \mathbb{R}^+). \tag{12}$$

(i) For all $t, s \in [1, e]$, the following inequalities hold:

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} (\ln t)^{\beta-1} \psi_\beta(s) \le H(t, s) \le \frac{1}{\Gamma(\beta)} \psi_\beta(s), \\ &\frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \psi_\alpha(s) \le G(t, s) \le \frac{1}{\Gamma(\alpha)} \psi_\alpha(s) \end{aligned} \tag{13}$$

Let Q be a cone of Banach space E, for any $0 < r < R < \infty$, define

$$\begin{aligned} &\bar{Q}_r = \{x \in Q : \|x\| < r\}, \partial Q_r = \{x \in Q : \|x\| = r\}, \bar{Q}_R \setminus Q_r \\ &\quad = \{x \in Q : r \le \|x\| \le R\}. \end{aligned} \tag{14}$$

Now, we state the following lemmas which will be used in the rest of the paper.

Lemma 2 (see [108]). *Assume $T : \bar{Q}_r \to Q$ is a completely continuous operator.*

(i) If there exists $x_0 \in Q \setminus \{\theta\}$ such that

$$x - Tx \ne \mu x_0, x \in \partial Q_r, \mu \ge 0, \tag{15}$$

then, the fixed point index $i(T, Q_r, Q) = 0$

(ii) If

$$Tx \ne \mu x, x \in \partial Q_r, \mu \ge 1, \tag{16}$$

then, the fixed point index $i(T, Q_r, Q) = 1$

Lemma 3 (Krein-Rutmann, see [108]). *Let $L : E \to E$ be a continuous linear operator, P be a total cone, and $L(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a positive constant c such that $cL(\psi) \ge \psi$, then the spectral radius $\rho(L) \ne 0$ and has a positive eigenfunction corresponding to its first eigenvalue $\lambda = \rho(L)^{-1}$.*

Lemma 4 (Gelfand's formula, see [108]). *For a bounded linear operator L and the operator norm $\|\cdot\|$, the spectral radius of L^n satisfies*

$$\rho(L) = \lim_{n \to +\infty} \|L^n\|^{1/n}. \tag{17}$$

In this paper, we use the following assumption:

(B1) $f \in C((0, 1) \times (0,+\infty) \times (0,+\infty), [0,+\infty))$ and for any $0 < r < R < +\infty$,

$$\lim_{n \to +\infty} \sup_{u, v \in \bar{Q}_R Q_r} \int_{\Omega_n} \psi_\alpha(s) f(s, u(s), v(s)) ds = 0, \tag{18}$$

where $\Omega_n = [1, 1 + (1/n)] \cup [e - (1/n), e]$.

Now, let $E = C[1, e]$, then E is a Banach space equipped with the norm $\|x\| = \max_{t \in [1,e]} |x(t)|$. Let $P = \{x \in E : x(t) \ge 0, t \in [1, e]\}$ and

$$Q = \left\{x \in P : x(t) \ge (\ln t)^{\alpha-1} \|x\|, t \in [1, e]\right\}, \tag{19}$$

then, Q is a cone in the Banach space E and $\bar{Q}_R Q_r \subset Q \subset P$. Let us define a nonlinear operator $T : \bar{Q}_R \setminus Q_r \to P$ and a linear operator $L : E \to E$:

$$\begin{aligned} (Tx)(t) &= \int_1^e G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s)\right)\right] \frac{ds}{s}, t \in [1, e], \\ (Lx)(t) &= \int_1^e G(t, s) x(s) \frac{ds}{s}, t \in [1, e]. \end{aligned} \tag{20}$$

Thus, in order to solve equation (1), we only need to find the fixed point of operator equation $x = Tx$. To do this, we firstly establish some lemmas.

Lemma 5. *$L : Q \to Q$ is a completely continuous operator with the spectral radius $\rho(L) \ne 0$. Moreover, L has a positive eigenfunction ω^* corresponding to the first eigenvalue $\mu_1 = (\rho(L))^{-1}$.*

Proof. Firstly, it follows from Lemma 2 that, for any $x \in Q$, one has

$$\begin{aligned} \|Lx\| &= \max_{t \in [1,e]} \int_1^e G(t, s) x(s) \frac{ds}{s} \le \frac{1}{\Gamma(\alpha)} \int_1^e \psi_\alpha(s) x(s) \frac{ds}{s}, \\ Lx(t) &\ge \frac{1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \int_1^e \psi_\alpha(s) x(s) \frac{ds}{s}, \end{aligned} \tag{21}$$

which imply that $L : Q \to Q$. Since $H(t, s), t \in [1, e] \times [1, e]$ is uniform continuity, then the operator $L : Q \to Q$ is completely continuous.

On the other hand, by (10), we know that there exists a $\tau_0 \in (1, e)$ such that $G(\tau_0, \tau_0) > 0$. Thus, from the continuity of G, there exists a closed interval $[c, d] \subset (1, e)$ such that $\tau_0 \in (c, d)$ and $G(t, s) > 0$ for all $t, s \in [c, d]$. Now, we take $x \in Q$ such that $x(\tau_0) > 0$ and $x(t) = 0$ for all $t \in [c, d]$. Then, for all $t \in [c, d]$, one has

$$(Lx)(t) = \int_1^e G(t, s) x(s) \frac{ds}{s} \ge \int_c^d G(t, s) x(s) \frac{ds}{s} > 0. \tag{22}$$

Thus, there exists $\mu > 0$ such that $\mu(Lx)(t) \ge x(t)$ for $t \in [0, 1]$. From Krein-Rutmann's theorem, we know that

the spectral radius $\rho(L) \neq 0$ and T have a positive eigenfunction ω^* that satisfies $\mu_1 L\omega^* = \omega^*$, where $\mu_1 = (\rho(L))^{-1}$ is its first eigenvalue. The proof is completed.

Lemma 6. *Suppose that (B1) holds, then the operator $T : \bar{Q}_R \setminus Q_r \to Q$ is completely continuous.*

Proof. Firstly, for any $x \in \bar{Q}_R \setminus Q_r$, $t \in [1, e]$, it follows from Lemma 2 that

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s},\end{aligned} \tag{23}$$

that is

$$\|Tx\| \leq \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}. \tag{24}$$

Similarly, one also has

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\geq \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\geq (\ln t)^{\alpha-1}\|Tx\|,\end{aligned} \tag{25}$$

which implies $T(Q) \subset Q$ and then $T(\bar{Q}_R \setminus Q_r) \subset Q$.

On the other hand, from (B1), we know that there exists a natural number l such that

$$\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} < \frac{1}{2}. \tag{26}$$

Thus, for any $x \in \bar{Q}_R Q_r$ and $1 + (1/l) \leq t \leq e - 1/l$, we have

$$\left(\ln\left(1+\frac{1}{l}\right)\right)^{\alpha-1} r \leq x(t) \leq \|x\| = R, t \in [1, e], \tag{27}$$

$$\begin{aligned}&\frac{(\ln(1+(1/l)))^{\alpha-1}(\ln(1+(1/l)))^{\beta-1}r}{(\beta-1)\Gamma(\beta+1)} \leq \frac{(\ln t)^{\alpha-1}(\ln t)^{\beta-1}}{\Gamma(\beta)} \\ &\cdot \int_1^e \psi_\beta(\tau)x(\tau)\frac{d\tau}{\tau}\|x\| \leq \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} \\ &\leq \frac{1}{\Gamma(\beta)}\int_1^e \psi_\beta(\tau)x(\tau)\frac{d\tau}{\tau} \leq \frac{R}{(\beta-1)\Gamma(\beta+1)}, t \in [1, e].\end{aligned} \tag{28}$$

Take

$$M_1 = \max_{(t,x,y)\in\Omega_l^*}\{f(t,x,y)\}, \tag{29}$$

where

$$\begin{aligned}\Omega_l^* = &\left[1+\frac{1}{l}, e-\frac{1}{l}\right] \times \left[\frac{(\ln(1+(1/l)))^{\alpha-1}(\ln(1+(1/l)))^{\beta-1}R}{(\beta-1)\Gamma(\beta+1)},\right. \\ &\left.\frac{r}{(\beta-1)\Gamma(\beta+1)}\right] \times \left[\left(\ln\left(1+\frac{1}{l}\right)\right)^{\alpha-1} r, R\right].\end{aligned} \tag{30}$$

So, it follows from (26), (27) and (28) that

$$\begin{aligned}&\sup_{x\in\bar{Q}_R\setminus Q_r}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\leq \sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\quad + \sup_{x\in\bar{Q}_R\setminus Q_r}\int_{1+1/l}^{e-1/l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\leq \frac{1}{2} + M_1\int_1^e \psi_\alpha(s)\frac{ds}{s} < +\infty,\end{aligned} \tag{31}$$

which implies T is uniformly bounded for any bounded set.

Secondly, we shall prove that $T : \bar{Q}_R \setminus Q_r \to Q$ is continuous. To do this, let $x_n, x_0 \in \bar{Q}_R Q_r$ and $\|x_n - x_0\| \to 0$ $(n \to \infty)$. For any $\varepsilon > 0$, it follows from (B1) that there exists a natural number $m > 0$ such that

$$\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_m}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} < \frac{\varepsilon}{4}. \tag{32}$$

On the other hand, it follows form the fact that $f(t,u,v)$ is uniformly continuous on$\Omega_m^* = [1 + (1/m), e - 1/m] \times [((\ln(1 + (1/m)))^{\alpha-1}(\ln(1 + (1/m)))^{\beta-1}R/(\beta-1)\Gamma(\beta+1)), (r/(\beta-1)\Gamma(\beta+1))] \times [(\ln(1 + (1/m)))^{\alpha-1}r, R]$, (21) that

$$\begin{aligned}\lim_{n\to+\infty}\Bigg|&f\left(s, \int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau}, x_n(s)\right) \\ &- f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\Bigg| = 0,\end{aligned} \tag{33}$$

holds uniformly on $s \in [1 + (1/m), e - (1/m)]$. Thus, the

Lebesgue control convergence theorem ensures

$$\int_{1+(1/m)}^{e-(1/m)} \psi_\alpha(s)\left|f\left(s,\int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau},x_n(s)\right)\right. \\ \left.-f\left(s,\int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau},x_0(s)\right)\right|\frac{ds}{s}\to 0,\ \text{as } n\to\infty. \tag{34}$$

In other words, for the above $\varepsilon>0$, there exists a positive integer N such that $n>N$, we have

$$\int_{1+1/m}^{e-1/m} \psi_\alpha(s)\left|f\left(s,\int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau},x_n(s)\right)\right. \\ \left.-f\left(s,\int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau},x_0(s)\right)\right|\frac{ds}{s}<\frac{\varepsilon}{2}. \tag{35}$$

Thus, by (32) and (35), for any $n>N$, we have

$$\begin{aligned}\|Tx_n-Tx_0\|&\le 2\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_m}\psi_\alpha(s)\left[f\left(s,\int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau},x_0(s)\right)\right]\frac{ds}{s}\\&+\int_{1+(1/m)}^{e-(1/m)}\psi_\alpha(s)\left|f\left(s,\int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau},x_n(s)\right)\right.\\&\left.-f\left(s,\int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau},x_0(s)\right)\right|\frac{ds}{s}<2\times\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned} \tag{36}$$

Therefore, $T:\bar{Q}_RQ_r\to Q$ is continuous.

In the end, we shall prove that T is equicontinuous. Firstly, it follows from (B1) that for any $\varepsilon>0$, there exists a positive integer k such that

$$\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_k}\psi_\alpha(s)\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}<\frac{\varepsilon}{4}. \tag{37}$$

Take

$$M_2=\max_{(t,x,y)\in\Omega_k^*}\{f(t,x,y)\}, \tag{38}$$

where

$$\Omega_k^*=\left[1+\frac{1}{k},e-\frac{1}{k}\right]\times\left[\frac{(\ln(1+1/k))^{\alpha-1}(\ln(1+(1/k)))^{\beta-1}r}{(\beta-1)\Gamma(\beta+1)},\right. \\ \left.\frac{R}{(\beta-1)\Gamma(\beta+1)}\right]\times\left[\left(\ln\left(1+\frac{1}{k}\right)\right)^{\alpha-1}r,R\right]. \tag{39}$$

Notice that $G(t,s)$ is uniformly continuous on $[1,e]\times[1,e]$, so for the above $\varepsilon>0$ and fixed $s\in[1+(1/k),e-(1/k)]$, there exists $\delta>0$, when $|t_1-t_2|<\delta$, $t_1,t_2\in[1,e]$, we have

$$|G(t_1,s)-G(t_2,s)|\le[2M_2(e-1)]^{-1}\varepsilon. \tag{40}$$

It follows from the above argument that, for $|t_1-t_2|<\delta$, $t_1,t_2\in[1,e]$, one has

$$\begin{aligned}|Tx(t)-Tx(t')|&\le 2\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_k}\psi_\alpha(s)\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}\\&+\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{1+(1/k)}^{e-(1/k)}|G(t,s)-G(t',s)|\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}\\&<2\times\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned} \tag{41}$$

which implies that T is an equicontinuous operator. According to the Arzela-Ascoli Theorem, $T:\bar{Q}_R\setminus Q_r\to Q$ is completely continuous. The proof is completed.

3. Main Results

We state the main results of this paper as follows.

Theorem 7. *Let μ_1 be the first eigenvalue of L, assume (B1) holds and*

$$\limsup_{\substack{u+v\to+\infty\\ v\to+\infty}}\frac{f(t,u,v)}{v}<\mu_1<\liminf_{\substack{u\to0^+\\ v\to0^+}}\frac{f(t,u,v)}{u+v}, \tag{42}$$

uniformly holds on $t\in[1,e]$. Then, equation (1) has at least one positive solution.

Proof. Firstly, by Lemma 6, we know that $T:\bar{Q}_R\setminus Q_r\to Q$ is completely continuous. Thus, it follows from the extension theorem of a completely continuous operator (see Theorem 8.3 on page 56 of [108]) that, for any $R>0$, there exists a completely continuous operator $T^*:\bar{Q}_R\to Q$. Thus, without loss of the generality, we shall still write it as T.

Next, it follows from (42) that there exists $r>0$ such that

$$f(t,u,v)\ge\mu_1(u+v),\ |v|\le\frac{r}{(\beta-1)\Gamma(\beta+1)},\ |x|\le r,\ t\in[1,e]. \tag{43}$$

Since, for any $x\in\partial Q_r$, we have

$$\left|\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}\right|\le\frac{r}{(\beta-1)\Gamma(\beta+1)},\ |x(s)|\le r, \tag{44}$$

thus, for $x\in\partial Q_r$, from (43) and (44), one gets

$$\begin{aligned}(Tx)(t)&=\int_1^e G(t,s)\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}\\&\ge\mu_1\int_1^e G(t,s)\left|\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}+x(s)\right|\frac{ds}{s}\\&\ge\mu_1(Lx)(t),\ t\in[1,e].\end{aligned} \tag{45}$$

Let ω^* be the positive eigenfunction corresponding to μ_1,

i.e., $\omega^* = \mu_1 L\omega^*$. In what follows, we shall use the contradiction method to show

$$x - Tx \neq \mu\omega^*, x \in \partial Q_r, \mu \geq 0. \tag{46}$$

Firstly, we can suppose that T has no fixed points on $x \in \partial Q_r$ (otherwise, the proof is finished). In this case, let us suppose there exist $x_0 \in \partial Q_r$ and $\mu_0 \geq 0$ such that $x_0 - Tx_0 = \mu_0\omega^*$. This implies that $\mu_0 > 0$ and then

$$x_0 = Tx_0 + \mu_0\omega^* \geq \mu_0\omega^*. \tag{47}$$

Let $\bar{\mu} = \sup\{\mu \mid x_0 \geq \mu\omega^*\}$, then $\bar{\mu} \geq \mu_0$, $x_0 \geq \bar{\mu}\omega^*$, $\mu_1 Lx_0 \geq \mu_1\bar{\mu}L\omega^* = \bar{\mu}\omega^*$. Thus, it follows from (45) that

$$x_0 = Tx_0 + \mu_0\omega^* \geq \mu_1 Lx_0 + \mu_0\omega^* \geq \bar{\mu}\omega^* + \mu_0\omega^* = (\bar{\mu} + \mu_0)\omega^*, \tag{48}$$

which contradicts with the definition of $\bar{\mu}$. So (46) holds and from Lemma 2, we have

$$i(T, Q_r, Q) = 0. \tag{49}$$

On the other hand, it follows from (42) that there exists $R_1 > r$ and $0 < \kappa < 1$ such that

$$f(t, u, v) \leq \kappa\mu_1|v|, \text{for } |u + v| \geq R_1,\ |v| \geq R_1. \tag{50}$$

Let $\tilde{L}x = \kappa\mu_1 Lx$. Obviously, $\tilde{L} : E \to E$ is also a bounded linear operator and $\tilde{L}(Q) \subset Q$. Since μ_1 is the first eigenvalue of operator L and $0 < \kappa < 1$, we have

$$r^{-1}\left(\tilde{L}\right) = (\kappa\mu_1 r(L))^{-1} = \kappa^{-1} > 1. \tag{51}$$

From Gelfand's formula, we have

$$\kappa = \lim_{n\to+\infty}\left\|\tilde{L}^n\right\|^{1/n}. \tag{52}$$

Now, choose $\varepsilon_0 = 1/2(1 - \kappa)$, it follows from (52) that there exists an enough large integer $N > 0$ such that when $n \geq N$, one has

$$\|\tilde{L}^n\| \leq [\kappa + \varepsilon_0]^n. \tag{53}$$

Let $\tilde{L}^0 = I$ be the identity operator and define

$$\|x\|^* = \sum_{i=1}^{N}[\kappa + \varepsilon_0]^{N-i}\|\tilde{L}^{i-1}x\|, x \in E, \tag{54}$$

then $\|\cdot\|^*$ is still the norm of E. Let

$$M = \sup_{x\in\partial Q_{R_1}}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}, \tag{55}$$

then from (31), we have $M < +\infty$. Take

$$R_2 > \max\left\{R_1, \frac{2}{\varepsilon_0}M^*\right\}, \tag{56}$$

where $M^* = \|M\|^*$. Notice that $\|x\|^* > [\kappa + \varepsilon_0]^{N-1}\|x\|$, so we can choose a large enough $R > R_2$ such that $\|x\| \geq R$ implies $\|x\|^* > R_2$.

Next, we shall show

$$Tx \neq \mu x, x \in \partial Q_R, \mu \geq 1. \tag{57}$$

Otherwise, there exist $x_1 \in \partial Q_R$ and $\mu^* \geq 1$ such that $Tx_1 = \mu^* x_1$. Let $\tilde{x}(t) = \min\{x_1(t), R_1\}$ and

$$D(x_1) = \{t \in [1, e]: x_1(t) > R_1\}. \tag{58}$$

Since $x_1 \in C[1, e]$, $x_1(t) \leq \|x_1\| = R$, there exists $1 < t_0 \leq e$ such that $x_1(t_0) = R$. So for any $t \in [1, e]$, we get $\tilde{x}(t) = \min\{x_1(t), R_1\} \leq \min\{R, R_1\} = R_1$ and $\tilde{x}(t_0) = \min\{x_1(t_0), R_1\} = \min\{R, R_1\} = R_1$, which implies $\|\tilde{x}(t)\| = R_1$, i.e., $\tilde{x} \in \partial Q_{R_1}$. Thus by Lemma 2, we have

$$\begin{aligned}
\mu^* x_1 = (Tx_1)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x_1(\tau)\frac{d\tau}{\tau}, x_1(s)\right)\right]\frac{ds}{s} \\
&\leq \int_{D(x_1)} G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x_1(\tau)\frac{d\tau}{\tau}, x_1(s)\right)\right]\frac{ds}{s} \\
&\quad + \int_{[1,e]\setminus D(x_1)} \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau}, \tilde{x}(s)\right)\right]\frac{ds}{s} \\
&\leq \kappa\mu_1\int_1^e G(t,s)x_1(s)\frac{ds}{s} + \int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau}, \tilde{x}(s)\right)\right]\frac{ds}{s} \\
&\leq \left(\tilde{L}x_1\right)(t) + M, t \in [1, e].
\end{aligned} \tag{59}$$

So it follows from $\tilde{L}(Q) \subset Q$ and (59), we have

$$0 \leq \left(\tilde{L}^j(Tx_1)\right)(t) \leq \left(\tilde{L}^j\left(\tilde{L}x_1 + M\right)\right)(t), j = 0, 1, 2, \cdots, N - 1. \tag{60}$$

Since Q is a normal cone with normality constant 1 and (60), one has

$$\left\|\tilde{L}^j(Tx_1)\right\| \leq \left\|\tilde{L}^j\left(\tilde{L}x_1 + M\right)\right\|, j = 0, 1, 2, \cdots, N - 1, \tag{61}$$

which leads to

$$\begin{aligned}
\|Tx_1\|^* &= \sum_{i=1}^{N}[\kappa + \varepsilon_0]^{N-i}\left\|\tilde{L}^{i-1}(Tx_1)\right\| \\
&\leq \sum_{i=1}^{N}[\kappa + \varepsilon_0]^{N-i}\left\|\tilde{L}^{i-1}\left(\tilde{L}x_1 + M\right)\right\| = \left\|\tilde{L}x_1 + M\right\|^*.
\end{aligned} \tag{62}$$

According to the selection of R_2, we have $M^* < (\varepsilon_0/2)R_2$. Thus, it follows from the fact $\|x\| \geq R$ implies $\|x\|^* > R_2$ and

(53), (54), and (62) that

$$\begin{aligned}\mu^*\|x_1\|^* &= \|Tx_1\|^* \le \|\tilde{L}x_1\|^* + M^* = \sum_{i=1}^{N}[\kappa+\varepsilon_0]^{N-i}\|\tilde{L}^i x_1\| + M^* \\ &= [\kappa+\varepsilon_0]\sum_{i=1}^{N-1}[\kappa+\varepsilon_0]^{N-i-1}\|\tilde{L}^i x_1\| + \|\tilde{L}^N x_1\| + M^* \\ &\le [\kappa+\varepsilon_0]\sum_{i=1}^{N-1}[\kappa+\varepsilon_0]^{N-i-1}\|\tilde{L}^i x_1\| + [\kappa+\varepsilon_0]^N\|x_1\| + M^* \\ &= [\kappa+\varepsilon_0]\sum_{i=1}^{N}[\kappa+\varepsilon_0]^{N-i}\|\tilde{L}^{i-1}x_1\| + M^* \le [\kappa+\varepsilon_0]\|x_1\|^* \\ &\quad + \frac{\varepsilon_0}{2}R_2 \le [\kappa+\varepsilon_0]\|x_1\|^* + \frac{\varepsilon_0}{2}\|x_1\|^* = \left[\frac{1}{4}\kappa+\frac{3}{4}\right]\|x_1\|^*.\end{aligned} \tag{63}$$

Notice that $\mu^* \ge 1$, we have $(1/4)\kappa + 3/4 \ge 1$, and then $\kappa \ge 1$, which is a contradiction with $0 < \kappa < 1$. So (57) is indeed valid and it follows from Lemma 2 that

$$i(T, Q_R, Q) = 1. \tag{64}$$

Thus, (49) and (64) lead to

$$i\left(T, Q_R \setminus \bar{Q}_r, Q\right) = i(T, Q_R, Q) - i(T, Q_r, Q) = 1, \tag{65}$$

which implies T has at least one fixed point on $Q_R \setminus \bar{Q}_r$. Consequently, equation (1) has at least one positive solution.

Theorem 8. *Let μ_1 be the first eigenvalue of L, suppose (B1) holds, and*

$$\limsup_{\substack{u\to 0^+\\ v\to 0^+}} \frac{f(t,u,v)}{v} < \mu_1, \quad \liminf_{u+v\to+\infty} \frac{f(t,u,v)}{u+v} > \tilde{\mu}_1 \tag{66}$$

uniformly on $t \in [1, e]$, where $\tilde{\mu}_1$ is any eigenvalues of L. Then, equation (1) has at least one positive solution.

To prove Theorem 8, we need some preliminaries and lemma. For any enough small $0 < \varepsilon < 1$, define

$$(L_\varepsilon x)(t) = \int_{1+\varepsilon}^{e-\varepsilon} G(t,s)x(s)\frac{ds}{s}, \ t \in [1, e]. \tag{67}$$

By Lemma 5, we know $L_\varepsilon : Q \to Q$ is still a completely continuous linear operator with the spectral radius $\rho(L_\varepsilon) \ne 0$, and L_ε has a positive eigenfunction ω_ε corresponding to its first eigenvalue $\mu_\varepsilon = (\rho(L_\varepsilon))^{-1}$.

Lemma 9. *There exists an eigenvalue $\tilde{\mu}_1$ of L such that*

$$\lim_{\varepsilon\to 0^+} \mu_\varepsilon = \tilde{\mu}_1. \tag{68}$$

Proof. Let $\varepsilon_m \to 0\,(m \to \infty)$ satisfying $\varepsilon_1 \ge \varepsilon_2 \ge \cdots \ge \varepsilon_m \ge \cdots$. Thus, for $\omega \in Q$ and $k > m$, one has

$$\left(L_{\varepsilon_m}\omega\right)(t) \le \left(L_{\varepsilon_k}\omega\right)(t) \le (L\omega)(t), \quad t \in [1, e], \tag{69}$$

and then

$$\left(L^n_{\varepsilon_m}\omega\right)(t) \le \left(L^n_{\varepsilon_k}\omega\right)(t) \le (L^n\omega)(t), \quad t \in [1, e], n = 2, 3, \cdots, \tag{70}$$

where $L^n_{\varepsilon_m} = L(L^{n-1}_{\varepsilon_m})$, $n = 2, 3, \cdots$. It follows from the fact that Q is a normal cone with normality constant 1 that

$$\|L^n_{\varepsilon_m}\| \le \|L^n_{\varepsilon_k}\| \le \|L^n\|, n = 1, 2, \cdots. \tag{71}$$

By Gelfand's formula, we have $\mu_{\varepsilon_m} \ge \mu_{\varepsilon_k} \ge \mu_1$, where μ_1 is the first eigenvalue of L, which implies $\{\mu_{\varepsilon_m}\}$ is monotonous with lower boundedness μ_1. Let

$$\lim_{m\to+\infty} \mu_{\varepsilon_m} = \tilde{\mu}_1, \tag{72}$$

we assert $\tilde{\mu}_1$ is an eigenvalue of L.

In fact, let ω_{ε_m} be positive eigenfunctions of L_{ε_m} corresponding to μ_{ε_m} with $\|\omega_{\varepsilon_m}\| = 1$, $m = 1, 2, \cdots$. So we have

$$\omega_{\varepsilon_m}(t) = \mu_{\varepsilon_m}L_{\varepsilon_m}\omega_{\varepsilon_m}(t) = \mu_{\varepsilon_m}\int_{1+\varepsilon_m}^{e-\varepsilon_m} G(t,s)\omega_{\varepsilon_m}(s)\frac{ds}{s}, \quad t \in [1, e]. \tag{73}$$

Since

$$\begin{aligned}\left\|L_{\varepsilon_m}\omega_{\varepsilon_m}\right\| &= \max_{1\le t\le e}\int_{1+\varepsilon_m}^{e-\varepsilon_m} G(t,s)\omega_{\varepsilon_m}(s)\frac{ds}{s} \le \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)\frac{ds}{s} \\ &< +\infty, (m = 1, 2, \cdots),\end{aligned} \tag{74}$$

$\{L_{\varepsilon_m}\omega_{\varepsilon_m}\} \subset Q$ is uniformly bounded.

Moreover, for any m and $t_1, t_2 \in [1, e]$, one has

$$|L_{\varepsilon_m}\omega_{\varepsilon_m}(t_1) - L_{\varepsilon_m}\omega_{\varepsilon_m}(t_2)| \le \int_{\varepsilon_m}^{1-\varepsilon_m} |G(t_1,s) - G(t_2,s)|\omega_{\varepsilon_m}(s)\frac{ds}{s}. \tag{75}$$

Notice that $G(t, s)$ is uniformly continuous on $[1, e] \times [1, e]$; then, (75) implies $\{L_{\varepsilon_m}\omega_{\varepsilon_m}\} \subset Q$ is equicontinuous. Thus, from the Arzela-Ascoli theorem and $\lim_{m\to+\infty}\mu_{\varepsilon_m} = \tilde{\mu}_1$, we know $\omega_{\varepsilon_m} \to \omega_0$ as $m \to +\infty$, which yields $\|\omega_0\| = 1$. Take the limit for two sides of (73), we obtain

$$\omega_0(t) = \tilde{\mu}_1\int_1^e G(t,s)\omega_0(s)\frac{ds}{s}, \quad t \in [1, e], \tag{76}$$

that is, $\omega_0 = \tilde{\mu}_1 L\omega_0$, and $\tilde{\mu}_1$ is an eigenvalue of L.

Proof of Theorem 8. Firstly, it follows from (66) that for any $t \in [1, e]$, there exists $r > 0$ such that

$$f(t,u,v) \le \mu_1 v, |u| \le \frac{r}{(\beta-1)\Gamma(\beta+1)}, |v| \le r. \tag{77}$$

Thus, for any $x \in \partial Q_r$, since

$$\left|\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}\right| \le \frac{r}{(\beta-1)\Gamma(\beta+1)}, |x(s)| \le \|x\| = r, \tag{78}$$

we have

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\le \mu_1 \int_1^e G(t,s)x(s)\frac{ds}{s} = \mu_1 (Lx)(t), t \in [1, e].\end{aligned} \tag{79}$$

In the following, we prove that

$$Tx \ne \mu x, \text{ for any } x \in \partial Q_r, \mu \ge 1. \tag{80}$$

Firstly, we may suppose that T has no fixed point on ∂Q_r (otherwise, the proof is completed). If (80) is not true, i.e., there exist $x_0 \in \partial Q_r$ and $\mu_0 \ge 1$ satisfying $Tx_0 = \mu_0 x_0$. Thus $\mu_0 > 1$, it follows from (79) that

$$\mu_0 x_0 = Tx_0 \le \mu_1 Lx_0. \tag{81}$$

From induction, we have

$$\mu_0^n x_0 \le \mu_1^n L^n x_0, n = 1, 2, \cdots, \tag{82}$$

which implies that

$$\|L^n\| \ge \frac{\|L^n x_0\|}{\|x_0\|} \ge \frac{\mu_0^n \|x_0\|}{\mu_1^n \|x_0\|} = \frac{\mu_0^n}{\mu_1^n}. \tag{83}$$

Thus, by Gelfand's formula, we have

$$r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \ge \frac{\mu_0}{\mu_1} > \frac{1}{\mu_1}, \tag{84}$$

which contradicts with $r(L) = \mu_1^{-1}$. Therefore, (80) is valid and from Lemma 2, we get

$$i(T, Q_r, Q) = 1. \tag{85}$$

On the other hand, for any fixed small enough $0 < \varepsilon < 1$, take

$$\rho_\varepsilon = \frac{\Gamma(\alpha-1)(\ln(1+\varepsilon))^{\beta-1}}{(\beta-1)\Gamma(\alpha+\beta-2)} + (\ln(1+\varepsilon))^{\alpha-1}. \tag{86}$$

By (66) and $\lim_{\varepsilon\to 0^+} \mu_\varepsilon = \tilde{\mu}_1$, we can find a sufficiently small $\varepsilon > 0$ and $R > r$ such that

$$f(t,u,v) \ge \mu_\varepsilon(u+v), u+v \ge \rho_\varepsilon R, \ t \in [1, e], \tag{87}$$

where μ_ε is the first eigenvalue of L_ε.

Let ω_ε be the positive eigenfunction of L_ε corresponding to μ_ε, which implies $\omega_\varepsilon = \mu_\varepsilon L_\varepsilon \omega_\varepsilon$. For any $x \in \partial Q_R$, $s \in [1+\varepsilon, e-\varepsilon]$, we have

$$\begin{aligned}&\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} + x(s) \ge \frac{1}{\Gamma(\beta)}(\ln s)^{\beta-1}\int_1^e \psi_\beta(\tau)(\ln\tau)^{\alpha-1}\frac{d\tau}{\tau}\|x\| \\ &\quad + (\ln s)^{\alpha-1}\|x\| \ge \left(\frac{\Gamma(\alpha-1)(\ln(1+\varepsilon))^{\beta-1}}{(\beta-1)\Gamma(\alpha+\beta-2)} + (\ln(1+\varepsilon))^{\alpha-1}\right)R = \rho_\varepsilon R.\end{aligned} \tag{88}$$

Thus, it follows from (87) and (88) that

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\ge \int_{1+\varepsilon}^{e-\varepsilon} G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\ge \mu_\varepsilon \int_{1+\varepsilon}^{e-\varepsilon} G(t,s)\left(\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} + x(s)\right)\frac{ds}{s} \\ &\ge \mu_\varepsilon \int_{1+\varepsilon}^{e-\varepsilon} G(t,s)x(s)\frac{ds}{s} = \mu_\varepsilon (L_\varepsilon x)(t), t \in [1, e].\end{aligned} \tag{89}$$

Similar to the proof of Theorem 7, we obtain

$$x - Tx \ne \mu\omega_\varepsilon, x \in \partial Q_R, \mu \ge 0. \tag{90}$$

According to Lemma 2, we have

$$i(T, Q_R, Q) = 0. \tag{91}$$

Combining (80) and (91), one has

$$i\left(T, Q_R \setminus \bar{Q}_r, Q\right) = i(T, Q_R, Q) - i(T, Q_r, Q) = -1. \tag{92}$$

Hence, T has at least one fixed point on $Q_R \setminus \bar{Q}_r$, and then equation (1) has at least a positive solution. The proof is completed.

4. Conclusion

Singular behaviour is a class of important natural phenomena in many physical science, mathematics, engineering, and bioscience. So, the study for singularity is an interesting and challenging problem. In this paper, we consider the existence of positive solutions for a Hadamard-type fractional differential equation with singular nonlinearity by introducing a new limit-type growth condition. The main advantage of the assumption is that it provides an effective method for handling the singularity at space variables. This assumption is valid and reasonable and easier to get the solution of the target equation.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out by the collaboration of all authors. All authors read and approved the final manuscript.

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Research Article

The Existence of Positive Solution of a Nonlinear Problem on Unit Circle

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We consider the following nonlinear problem $u''(x) + \lambda u(x) + f(x,u) = 0, \quad x \in S^1$, (P) where S^1 is a unit circle, $\lambda \in \mathbb{R}$ is a parameter, and $f(x,u)$ is a nonlinearity. By using the variational method, we obtain the existence of a positive solution to (P) for all $\lambda \in \mathbb{R}$. This phenomenon is different from the corresponding boundary value problem.

1. Introduction

We consider the following nonlinear problem

$$u''(x) + \lambda u(x) + f(x,u) = 0, \quad x \in S^1, \tag{1}$$

where S^1 is a unit circle, λ is a parameter and the nonlinear term f satisfies.

(f_1) $f(x,t)$ is a continuous function on $S^1 \times \mathbb{R}$, and $f(x,t)$ is odd on t.

The second-order differential equation (1) is a nonlinear elliptic equation model arising from studying some physics processes or geometric problems. For example, (1) is the problem of oscillations [1, 2] of a spherical thick shell made of an elastic material when the nonlinearity f satisfies a superlinear condition as $t \longrightarrow +\infty$, namely,

(f_2) $\lim_{t\to+\infty} f(x,t)/t = +\infty$ uniformly for $x \in S^1$.

For $\lambda = 1$, $f(x,t) = g(x)|t|^{p-1}t$ with $p \in \mathbb{R}$ and $g(x)$: $S^1 \longrightarrow \mathbb{R}$, (1) is also used to model the planar Minkowski problem [3–5].

As is well known, the strong maximum principle is a powerful tool for studying positive solutions of a nonlinear elliptic equation, see for example [6, 7]. In [8], the existence of positive solutions to the nonlinear elliptic equation can also be obtained by a standard argument, based on transforming the partial differential equation to the equivalent ordinary differential equation. The iterative method can also be used in the existence of a solution to nonlinear elliptic equation (see [9]). If the parameter $\lambda > 0$ is small, the related Green function of the operator $d/dx^2 + \lambda$ on S^1 is positive. When the Green function is positive, the upper and lower solution method and Schauder's fixed point theorem are also suitable to obtain the positive solution to some nonlinear elliptic equations, see for example [10–12]. However, the Green function of operator $d/dx^2 + \lambda$ on S^1 may be sign-changing for some $\lambda > 0$.

In this paper, we aim at studying the existence of a positive solution of nonlinear equation (1) for all $\lambda \in \mathbb{R}$ and the effect of asymptotic properties of nonlinearity $f(x,t)$ when $t \longrightarrow 0$, to the positivity of solution of (1). Our contribution includes two aspects. Firstly, we establish the following Theorem 1 to show that a nonnegative solution is strictly positive by the analysis based on the Taylor expansion theorem.

Theorem 1. *Assume $\lambda \in \mathbb{R}$, f satisfies (f_1) and $f(x,t)/t$ is bounded as $t \longrightarrow 0$. Let $v \in C^2(S^1)$ be a nonnegative solution of (1). If $v \equiv 0$, then v is strictly positive.*

Indeed, we can also apply the Green function or the strong maximum principle to study the positive of the solution. However, the direct application of Green function for

studying the strictly positive of solution often needs more assumptions on parameter λ and the information of sign of the non-homogeneous term. By using Theorem 1, we could estimate the strictly positive of solution to (1) for all $\lambda \in \mathbb{R}$ as the following applications.

Secondly, as an application, we give a strictly positive solution of (1) for all $\lambda \in \mathbb{R}$ by using Theorem 1 and the variational method. To study the existence of positive solution to (1) under a general nonlinearity, we also need the following general assumptions.

(f_3) $\lim_{t\to 0} f(x,t)/t = 0$ uniformly for $x \in S^1$;

(f_4) there exists $\alpha > 2$ such that, for every $x \in S^1$ and $t \in \mathbb{R}$

$$\alpha F(x,t) \le t f(x,t); \tag{2}$$

(f_5) there exists $T > 0$ such that

$$\inf_{x\in S^1, |t|>T} F(x,t) > 0; \tag{3}$$

(f_6) for every $x \in S^1$, $f(x,t)/t$ is an increasing function of t on $[0, +\infty)$.

Theorem 2. *Assume (f_1) - (f_6) hold. For $\lambda < 0$, (1) has a positive solution.*

We see that the existence of a positive solution in Theorem 2 depends on the assumption that $\lambda < 0$. It is natural to ask whether there exist nontrivial solutions of (1) for some $\lambda \ge 0$. To shed some light to this problem, we give the existence of positive solutions to (1) with some special nonlinearity as follows.

Theorem 3. *Assume $f(x,u) = g(x)|u(x)|^{p-1}u(x)$ with $p > 1$. Let $\lambda \ge 0$. If $g \in C(S^1)$ is nonnegative with $\int_{S^1} |g(x)|^{2/(1-p)} dx < +\infty$, then (1) has a positive solution.*

From Theorems 2 and 3, we obtain that (1) may own a positive solution for all $\lambda \in \mathbb{R}$, which is different from the boundary value problem [13–16]. To illustrate that the conclusions above for (1) is a new phenomenon that different from the existence of a positive solution to boundary value problem, we consider the following problem

$$\begin{aligned} &u''(x) + \lambda u(x) + g(x)|u(x)|^{p-1}u(x) = 0 \quad x \in (0, 2\pi), \\ &u(0) = u(2\pi) = 0. \end{aligned} \tag{4}$$

Let $u \in C^2([0, 2\pi])$ be a solution of (4), we have

$$\left(\lambda - \frac{1}{4}\right)\int_0^{2\pi} \sin\left(\frac{x}{2}\right)u(x)dx + \int_0^{2\pi} g(x)|u(x)|^{p-1} \sin\left(\frac{x}{2}\right)dx = 0. \tag{5}$$

If $\lambda > 1/4$, it follows that there is no positive solution to the boundary value problem (4) with $g(x) \ge 0$.

Since the proof of Theorem 2 depends on (f_1) and $\lambda < 0$, it is impossible to deduce Theorem 3 by a similar method for Theorem 2. In the next section, we apply a reverse Hölder's inequality and constrained variational method to prove Theorem 3.

2. Proof of Theorems

We first give our new method to estimate the positivity of the solution of (1) as follows. By assumption, we obtain

$$-v''(x) - \lambda v(x) = f(x, v(x)), x \in [0, 2\pi]. \tag{6}$$

We will prove that v is strictly positive by the method of contradiction. Assume $v(\theta_0) = 0$ for some $\theta_0 \in [0, 2\pi]$. We choose $\theta_0 = \inf\{x \in [0, 2\pi]: v(x) = 0\}$. Since $v \equiv 0$, we have two points θ_r and θ_l in $[0, 2\pi]$ such that $\theta_0 \le \theta_r < \theta_l$ and

$$v(x) = 0 \text{ for } x \in [\theta_0, \theta_r]; \text{ and } v(x) > 0 \text{ for } x \in (\theta_r, \theta_l). \tag{7}$$

Since 0 is the minimum value of v on $[0, 2\pi]$, we have

$$v'(x) = 0 \text{ for } x \in [\theta_0, \theta_r]. \tag{8}$$

By (f_1), we see that $f(x, 0) = 0$. It follows from (6) and (7) that

$$v''(x) = 0 \text{ for } x \in [\theta_0, \theta_r]. \tag{9}$$

By Taylor's theorem, we give a formula of $v(x)$ for all $x \in (\theta_r, \theta_l)$ as follows,

$$v(x) = \frac{v''(\xi_x)}{2}(x - \theta_r)^2 \quad \text{for some} \quad \xi_x \in (\theta_r, x). \tag{10}$$

We claim that there exists a strictly monotone decreasing sequence $\{\theta_n\} \subset (\theta_r, \theta_l)$ such that

$$\theta_n \xrightarrow{n} \theta_r \quad \text{and} \quad |v''(\theta_n)| > 0. \tag{11}$$

Otherwise, there exists a small $\delta > 0$ such that $|v''(x)| = 0$ for all $x \in (\theta_r, \theta_r + \delta)$. By applying (10), we derive that

$$v(x) = 0 \text{ for } x \in (\theta_r, \theta_r + \delta), \tag{12}$$

which contradicts to the second part of (7). By (9), we see that $v''(\theta_r) = 0$. Let θ_1 be given by (11), then we have

$$\rho_1 = \sup\left\{x \in [\theta_r, \theta_1]: |v''(x)| = \sup_{\rho\in[\theta_r,\theta_1]} |v''(\rho)|\right\} > \theta_r. \tag{13}$$

For $k \in \mathbb{N}$, we can define a sequence of ρ_k by

$$\rho_{k+1} = \sup\left\{x \in (\theta_r, \rho_k - \delta_k):\ |v''(x)| = \sup_{\rho\in(\theta_r,\rho_k-\delta_k)} |v''(\rho)|\right\}, \tag{14}$$

where $\delta_k = (\rho_k - \theta_r)/2$. From (11), we see that $\theta_n \xrightarrow{n} \theta_r$. Fix $k \in \mathbb{N}$, if $i \in \mathbb{N}$ is large enough, then $\theta_i \in (\theta_r, \rho_k)$; and it follows from (14) that $|v''(\rho_k)| \geq |v''(\theta_i)| > 0$. Following this way, we obtain a strictly monotone decreasing sequence $\{\rho_n\} \subset (\theta_r, \theta_l)$ such that $\rho_n \xrightarrow{n} \theta_r$ and

$$|v''(\rho_n)| > 0, \quad |v''(x)| \leq |v''(\rho_n)| \text{ for } x \in (\theta_r, \rho_n). \tag{15}$$

For each $n \in \mathbb{N}$, via applying (10) with $x = \rho_n$, there exists $\xi_n \in (\theta_r, \rho_n)$ such that

$$v(\rho_n) = \frac{v(\xi_n)}{2}(\rho_n - \theta_r)^2 \text{ and } \left|v''(\xi_n)\right| \leq \left|v''(\rho_n)\right|. \tag{16}$$

Let $x = \rho_n$ in (6), we have

$$-v''(\rho_n) - \lambda v(\rho_n) = f(\rho_n, v(\rho_n)). \tag{17}$$

Substituting $v(\rho_n)$ in (17) by using the first part of (16), and then multiplying (17) by $1/v''(\rho_n)$, we deduce that

$$-1 - \lambda\frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2 = \frac{f(\rho_n, v(\rho_n))}{v(\rho_n)}\frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2. \tag{18}$$

By using the second part of (16) and the assumption that $f(\theta, t)/t$ is bounded, we derive that

$$\frac{f(\rho_n, v(\rho_n))}{v(\rho_n)} = O(1) \text{ and } \frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2 = o(1) \text{ as } n \longrightarrow +\infty. \tag{19}$$

From (18), we get a contradiction that $1 = o(1)$ as $n \longrightarrow +\infty$.

To prove Theorem 2, we firstly obtain a nonnegative solution of (1) via a minimum of I on manifold N; then, we prove that it is strictly positive by using Theorem 1. The process is standard for studying the existence of nonnegative solutions to nonlinear elliptic equations by the method involving the Nehari manifold. For completeness, we give the main steps. The following deformation lemma is needed.

Lemma 1. *(see, for example, Lemma 2.3 in [17]). Let X be a Banach space, $I \in C^1(X, \mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon, \delta > 0$ such that*

$$\inf_{\phi=0}\frac{I'(u)\phi}{\|\phi\|} \geq 8\varepsilon/\delta \text{ for all } u \in \{u : \|u - v\| \leq 2\delta, c - 2 \leq \varepsilon I(u) \leq c + 2\varepsilon\}. \tag{20}$$

We denote by I^d the set $\{u \in X : I(u) \leq d\}$ for any $d \in \mathbb{R}$. Then, there exists $\eta \in C([0, 1] \times X, X)$ such that

(i) $\eta(t, u) = u$, if $t = 0$ or if $u \in \{u \in X : \inf_{w\in S}\|u - w\| \leq 2\delta, c - 2\varepsilon \leq I(u) \leq c + 2\varepsilon\}$,

(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$

(iii) $\eta(t, \cdot)$ is an homeomorphism of X for any $t \in [0, 1]$

(iv) $I(\eta(\cdot, u))$ is non increasing for any $u \in X$

Let S be parameterized by angle x, $H := H^1(S)$ be the Sobolev space equipped with the usual normal

$$\|u\| = \left\{\int_S \left(u'\right)^2 + u^2 dx\right\}^{1/2}. \tag{21}$$

Define $F(x, u) = \int_0^u f(x, t)dt$. Under the assumptions $(f_1) - (f_3)$, we have the following C^1 variational function of (1).

$$I(u) = \frac{1}{2}\int_{S^1} u'^2 dx - \frac{\lambda}{2}\int_{S^1} u^2 dx - \int_{S^1} F(x, u)dx. \tag{22}$$

The related Nehari manifold is defined by

$$N := \left\{u \in H^1(S^1) \setminus \{0\}: \int_{S^1} u'^2 dx - \lambda\int_{S^1} u^2 dx - \int_{S^1} f(x, u)udx = 0\right\}. \tag{23}$$

Similar to Lemma 4.1 in [17], we have the following lemma.

Lemma 2. *Assume $(f_1) - (f_6)$ hold. Let $u \in H$ be nontrivial and $h(t) = I(tu)$ for $t > 0$. If $\lambda < 0$, there exists $t_u > 0$ such that $t_u u \in N$, $h(t_u) = \max_{t>0} h(t)$. The map $u \longrightarrow t_u$ is continuous, and $u \longrightarrow t_u u$ defines a homeomorphism of the unit sphere of H with N.*

Proof. If $\lambda < 0$, we see that $1/2\int_{S^1} u'^2 dx - \lambda/2\int_{S^1} u^2 dx > 0$ for the nontrivial $u \in H$. For any $u \in N$, let $g(t) = 1/t\int_{S^1} f(x, tu)dx$. Then g is an increasing function of t by (f_6). By the definition of $h(t) = I(tu)$, we see that $h(0) = 0$, and $t_u u \in N$ if and only if $\int_{S^1} u'^2 dx - \lambda\int_{S^1} u^2 dx = 1/t_u\int_{S^1} f(x, t_u u)udx$, which is equivalent to $h'(t_u) = 0$. Under the assumptions $(f_3) - (f_5)$,

we see that $h(t) > 0$ for small $t > 0$ and $h(t) < 0$ for large t. Therefore, there exists a unique t_u such that $h'(t_u)=0$ and $t_u u \in N$. By (f_4), there exists a constant $C_0 > 0$ such that

$$C_0(|u|^\alpha - 1) \le uf(x,u), \tag{24}$$

where $\alpha > 2$ is given by (f_4).

Assume that there exists a nontrivial sequence $u_n \xrightarrow{n} u$ in $H^1(S^1)$ with $u=0$, we have a sequence $\{t_{u_n}\}$ that $t_{u_n} u_n \in N$. This and (f_4) imply that

$$\begin{aligned} Ct_{u_n}\|u_n\|^2 &\ge t_{u_n}\left(\int_{S^1} u_n'^2 dx - \lambda\int_{S^1} u_n^2 dx\right) \\ &= \int_{S^1} f\left(x, t_{u_n}u_n\right)u_n dx \\ &\ge \int_{S^1} C_0\left(t_{u_n}^\alpha |u_n|^\alpha - 1\right)dx. \end{aligned} \tag{25}$$

It follows that $\{t_{u_n}\}$ is bounded. And a subsequence of $\{t_{u_n}\}$ converges to a number t_0. By using (25) and the uniqueness of map $t_u = t(u)$, we see that $t_0 = t_u$. That is, t_{u_n} converge to t_u. The inverse map of $u \longrightarrow t_u u$ is the retraction $u \longrightarrow u/\|u\|$.

By using the properties of I in Lemma 2. We obtain the following Lemma 3.

Lemma 3. *Assume $(f_1)-(f_6)$ hold and $\lambda < 0$. Let $c_1 = \inf_{u\in N} I(u)$, then $c_1 > 0$. If $v \in N$ and $I(v) = c_1$, then v is a critical point of I on H.*

Proof. Let $c_2 \coloneqq \inf_{u\in H, u=0} \max_{t\ge 0} I(tu)$ and $c \coloneqq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, where $\Gamma \coloneqq \{\gamma \in C([0,1], H)\colon I(\gamma(1)) < 0, \gamma(0) = 0\}$. We prove this Lemma by the following two steps.

Step 1. $c_1 = c_2 = c > 0$.

By Lemma 2, we see that $I(t_u u) = \max_{t\ge 0} I(tu)$ if and only if $t_u u \in N$. This and the definitions of c_1, c_2 imply that $c_1 = c_2$. By $(f_1)-(f_2)$, we see that $I(tu) < 0$ for $u=0$ and $t > 0$ large. Hence, $c \le c_2$. From Lemma 2, we see that the Nehari manifold N separates H into two parts. By assumptions $(f_1)-(f_3)$, we deduce that $\inf_{u\in N}\|u\|>0$ and there exists $r > 0$ such that

$$\min_{\|u\|\le r} I(u) = 0, \qquad \inf_{\|u\|=r} I(u) > 0. \tag{26}$$

It follows that the pass $\gamma \in \Gamma$ has to cross N and $c \ge \inf_{\|u\|=r} I(u) > 0$. These facts imply that $c_1 \le c$.

Step 2. v is a critical point of I.

Assume that $I'(v)=0$. Then there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$\inf_{\phi=0} \frac{I'(u)\phi}{\|\phi\|} \ge \varepsilon \quad \text{for all} \quad \|u - v\| \le 3\delta. \tag{27}$$

For $\varepsilon \coloneqq \{c/2, \varepsilon\delta/8\}$ and $S \coloneqq B(v,\delta)$, Lemma 1 yields a deformation η such that

(i) $\eta(1,u) = u$ if $u \in \{u \in H : c - \varepsilon \le I(u) \le c + 2\varepsilon\}$,

(ii) $\eta(1, I^{c+\varepsilon} \cap B(v,\delta)) \subset I^{c-\varepsilon}$

(iii) $I(\eta(1,u)) \le I(u)$, $u \in H$

We deduce that $\max_{t>0} I(\eta(1, tv)) \le c - \varepsilon < c = c_2$, which contradicts the definition of c_2.

By Lemma 3, we see that $c_1 = \inf_{u\in N} I(u) > 0$. There exists a sequence of $\{u_n\} \subset N$ such that $I(u_n) \longrightarrow c_1$ as $n \longrightarrow +\infty$, that is,

$$\frac{1}{2}\int_{S^1} u'^2 dx - \frac{\lambda}{2}\int_{S^1} u^2 dx - \int_{S^1} F(x,u)dx = c_1 + o(1), \text{ as } n \longrightarrow +\infty, \tag{28}$$

$$\int_{S^1} u'^2 dx - \lambda\int_{S^1} u^2 dx - \int_{S^1} f(x,u)u dx = 0, \tag{29}$$

Let $\alpha > 2$ be given by (f_4), it follows from (28) and (29) that

$$\frac{\alpha-2}{2}\left(\int_{S^1} u'^2 dx - \lambda\int_{S^1} u^2 dx\right) \le 2c_1 \quad \text{for large} \quad n. \tag{30}$$

Since $\lambda < 0$, we have that $\{\|u_n\|\}$ is bounded. Then, there exists a weak limitation v of the sequence $\{u_n\}$ in H, i.e., up to a subsequence, $u_n \rightharpoonup v$ weakly in H and $u_n \longrightarrow v$ uniformly on S^1 as $n \longrightarrow +\infty$. This together with the assumptions $(f_1)-(f_3)$ imply that

$$\lim_{n\to+\infty}\int_{S^1} f(x,u_n)u_n dx = \int_{S^1} f(x,v)v dx. \tag{31}$$

It follows from (29) and (31) that $\lim_{n\to+\infty}\|u_n - v\| = 0$, which implies that $v \in N$ and $c_1 = \inf_{u\in N} I(u) = I(v)$. By Lemma 3, we see that $I'(v) = 0$. Let $v^+ = \max\{0, v\}$ and $v^- = -\max\{0, -v\}$. Then, we deduce that $I(v^+)v^+ = I(v)v^+ = I(v^-)v^- = I(v)v^- = 0$. If v is a sign-changing function, then v^- and v^+ are nontrivial. It follows that $v^-, v^+ \in N$. We thus deduce the following contradiction

$$c_1 = I(v) = I(v^+) + I(v^-) \ge 2\inf_{u\in N} I(u) = 2c_1 > 0. \tag{32}$$

So, v is not a sign-changing function. By the second part of (f_1), we see that $I(v) = I(-v)$. Without loss of generality, we assume that v is nonnegative. A standard regularity

shows that v is C^2 on S^1. Since the nonlinear term f satisfies assumptions (f_1) and (f_3), we can apply Theorem 1 to get that v is strictly positive.

Let us consider the constrained optimal problem

$$c = \inf_{F(u)=1} I(u), \tag{33}$$

where

$$I(u) = \int_S \left(u'\right)^2 - \lambda u^2 dx, \tag{34}$$

$$F(u) = \int_S g(x)|u|^{p+1} dx. \tag{35}$$

For $\lambda \geq 0$, it follows from reverse Hölder inequality and (34) that

$$\begin{aligned} \inf_{F(u)=1} I(u) &= \inf_{F(u)=0} \frac{I(u)}{\{F(u)\}^{2/p+1}} \\ &\geq \inf_{F(u)=0} \frac{-\lambda \int_{S^1} |u|^2 dx}{\left\{\int_{S^1} g(x)^{2/(1-p)} d\theta\right\}^{(1-p)/2} \int_{S^1} |u|^2 dx} \\ &= -\lambda \left\{\int_{S^1} g(x)^{2/(1-p)} d\theta\right\}^{(p-1)/2} > -\infty, \end{aligned} \tag{36}$$

which implies that $I(u)$ is bounded in $H^1(S)$ from below under the constrained condition that $F(u) = 1$. By the definition of I, F in (34) and (35), there exists a nonnegative minimizing sequence $\{u_n\} \subset H^1(S)$ such that $F(u_n) = 1$ and

$$I(u_n) \longrightarrow \inf_{F(u)=1} I(u) \text{as} n \longrightarrow +\infty. \tag{37}$$

It follows from $2/(p-1) > (p-1)/(p+1)$ and the Hölder inequality that

$$\int_{S^1} g(x)^{-(p-1)/(p+1)} dx \leq C \left| \int_{S^1} g(x)^{2/(1-p)} dx \right|^{\frac{(p-1)^2}{2(p+1)}} < +\infty. \tag{38}$$

Since $F(u_n) = 1$, by Hölder inequality we see that

$$\begin{aligned} \int_{S^1} u_n^2 dx &\leq \left(\int_{S^1} g(x)^{-(p-1)/(p+1)} dx\right)^{(p+1)/(p+2)} \left(\int_{S^1} g(x) u_n^{p+1} dx\right)^{2/(p+1)} \\ &= \left(\int_{S^1} g(x)^{-(p-1)/(p+1)} dx\right)^{(p+1)/(p+2)} < +\infty. \end{aligned} \tag{39}$$

This together with $I(u_n) \overset{(n\to+\infty)}{\longrightarrow} \inf_{F(u)=1} I(u)$ imply that $\{u_n\}$is bounded in $H^1(S)$. So, there is a subsequence such that

$$u_n \xrightarrow{n} v \quad \text{weakly} \quad \text{in} \quad H^1\left(S^1\right). \tag{40}$$

By the compactness of Sobolev imbedding, up to a subsequence, we have

$$u_n \xrightarrow{n} v \quad \text{strongly} \quad \text{in} \quad C^\alpha\left(S^1\right) \quad \text{for} \quad \alpha \in (0, 1/2). \tag{41}$$

It follows that $F(|v|) = 1$, and therefore v=0.

$$\int_0^{2\pi} g(\theta)|v|^{p+1} d\theta = \lim_{n\to+\infty} \int_0^{2\pi} g(\theta)|u_n|^{p+1} d\theta = 1. \tag{42}$$

By the Fatou lemma, we get

$$\inf_{F(u)=1} I(u) \leq I(|v|) \leq \lim_{n\to+\infty} I(u_n) = \inf_{F(u)=1} I(u), \tag{43}$$

So, $|v|$ is a minimizer of I under constrained condition that $F(u) = 1$. There exists $\omega \in \mathbb{R}$ such that

$$I'(|v|) = \omega F'(|v|) \quad \text{in} \quad H^{-1}(S). \tag{44}$$

It follows from the regularity theorem that $|v| \in C^2(S)$ is a classical nonnegative solution of

$$-u'' - \lambda u = \frac{\omega(p+1)}{2} g(x)|u|^{p-1} u, x \in S^1. \tag{45}$$

Let $w = (\omega(p+1)/2)^{1/(p-1)} |v|$, we see that w is nonnegative solution of (1) with $f(x,u) = g(x)|u|^{p-1}u$. By applying Theorem 1, we see that w is positive.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Three-Order Multipoint Boundary Value Problems for p-Laplacian Operator on Time Scales

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In this paper, the existence of positive solutions for the nonlinear four-point singular BVP for there-order with p-Laplacian operator on time scales will be studied. By using the fixed-point theory, the existence of positive solutions for nonlinear singular boundary value problem with p-Laplacian operator on time scales is obtained.

1. Introduction

In recent years, the nonlinear boundary value problems have been extensively studied. Recently, for the existence of positive solutions of multipoint boundary value problems, some authors have obtained the existence results. The differential equations offer wonderful tools for describing various natural phenomena arising from natural sciences and engineering, many numerical and analytical results, for example [1–20]. However, the multipoint boundary value problems treated in the above mentioned references do not discuss the problems with singularities and the there-order p-Laplacian operator. For the singular case of multipoint boundary value problems for higher-order p-Laplacian operator, with the author's acknowledge, no one has studied the existence of positive solutions in this case.

In this paper, we study the following equation with p-Laplacian on time scale:

$$\left(\phi_p\left(u^{\Delta\Delta}\right)\right)^{\nabla} + g(t)f\left(u(t), u^{\Delta}(t)\right) = 0, \quad 0 < t < T, \tag{1}$$

with the following boundary value conditions:

$$\begin{cases} u(0) = 0, \\ u^{\Delta}(0) - M_0\left(u^{\Delta\Delta}(\xi)\right) = 0, \quad u^{\Delta}(T) + M_1\left(u^{\Delta\Delta}(\eta)\right) = 0, \end{cases} \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $(1/p) + (1/q) = 1$. $\xi, \eta \in (0, T)$ is prescribed and $\xi < \eta$, $g : (0, T) \to [0,\infty)$, M_0, M_1 are both nondecreasing continuous odd functions defined on $(-\infty, +\infty)$.

A time scale $\mathbf{T}$ is a nonempty subset and closed subset of $\mathbf{R}$. By an internal $(0, T)$, we always mean the intersection of the real internal $(0, T)$ with the given time scale, that is $(0, T) \bigcap \mathbf{T}$. The operators σ and ρ from $\mathbf{T}$ to $\mathbf{T}$ which defined by [21],

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \rho(t) = \sup\{\tau \in \mathbf{T} \mid \tau < t\} \in \mathbf{T}. \tag{3}$$

are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbf{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

In this paper, by constructing an integral equation which is equivalent to the problem (1), (2), we research the existence of positive solutions for nonlinear singular boundary value problem (1), (2) when g and f satisfy some suitable conditions.

2. Preliminaries and Lemmas

For convenience, in the rest of this article, $\mathbf{T}$ is a closed subset of $\mathbf{R}$ with $0 \in \mathbf{T}_k$, $T \in \mathbf{T}^k$.

Letting

$$B = \{u \in C_{ld}[0, T]: u(0) = 0\}. \tag{4}$$

Then, B is a Banach space with the norm $\|u\| = \max_{t\in[0,T]} |u^{\Delta}(t)|$. Suppose

$$K = \{u \in B: \ u^{\Delta}(t) \geq 0, \ u^{\Delta}(t) \text{ is concave function on } t \in [0, T]\}. \tag{5}$$

Obviously, K is a cone in B and $0 \leq u^{\Delta}(t) \leq \|u\|$ on $[0, T]$. Set $K_r = \{u \in K : \|u\| \leq r\}$.

In the rest of the paper, we make the following assumptions:

(H_1) $f \in C([0,+\infty)^2, [0,+\infty))$;

(H_2) $g \in C_{ld}((0, T), [0,+\infty))$ and there exists $t_0 \in (0, T)$ which satisfy

$$g(t_0) > 0, 0 < \int_0^T g(t)\nabla t < \infty; \tag{6}$$

(H_3) M_0, $M_1 \in ((-\infty,+\infty), R)$ are both increasing, continuous, odd functions, and at least one of them satisfies the condition that there exists one $b > 0$ which satisfy

$$0 < M_i(v) \leq bv, \forall v \geq 0, i = 0 \text{ or } 1. \tag{7}$$

By direct account, From paper [22], we can easy to obtain the following results.

Lemma 1. *Suppose condition (H_2) holds. Then, there exists a constant $L \in (0, (T/2))$ satisfies*

$$0 < \int_L^{T-L} g(t)\nabla t < \infty. \tag{8}$$

Furthermore, the function

$$A(t) = \int_L^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s + \int_t^{T-L} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\Delta s, \quad t \in [L, T-L], \tag{9}$$

is positive ld-continuous function on $[L, T-L]$, therefore, $A(t)$ has minimum on $[L, T-L]$. Hence, we suppose that there exists constant $B' > 0$ which satisfy $A \geq B'$ on $t \in [L, T-L]$.

Lemma 2. *Suppose that conditions (H_1), (H_2), (H_3) hold, $u(t) \in K \cap C^2(0, T)$ is a solution of boundary value problems (1), (2) if and only if $u(t) \in B$ is a solution of the following integral equation*

$$u(t) = \int_0^t w(s)\Delta s, \tag{10}$$

where

$$w(t) = \begin{cases} M_0\phi_q\left(\int_\xi^\delta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_0^t \phi_q\left(\int_s^\delta g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s, 0 \leq t \leq \delta, \\ M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_t^T \phi_q\left(\int_\delta^s g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s, \delta \leq t \leq T. \end{cases} \tag{11}$$

Here, δ is a unique solution of the equation $g_1(t) = g_2(t)$, where

$$g_1(t) = M_0\phi_q\left(\int_\xi^\delta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_0^t \phi_q\left(\int_s^\delta g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s,$$

$$g_2(t) = M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_t^T \phi_q\left(\int_\delta^s g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s. \tag{12}$$

Equation $g_1(t) = g_2(t)$ has a unique solution in $(0, T)$. Because $g_1(t)$ is strictly monotone increasing on $[0, T)$, and

$g_1(0) = 0$, $g_2(t)$ is strictly monotone decreasing on $(0, T]$, and $g_2(T) = 0$.

Proof. Necessity. By the equation of the boundary condition and (H_3), we have

$$u^{\Delta\Delta}(\xi) \geq 0, u^{\Delta\Delta}(\eta) \leq 0. \tag{13}$$

Then, there exist a constant $\delta \in [\xi, \eta] \subset (0, T)$ which satisfy $u^{\Delta\Delta}(\delta) = 0$. Firstly, by integrating the equation of the problems (1) on (δ, t), we have

$$\phi_p\left(u^{\Delta\Delta}(t)\right) = \phi_p\left(u^{\Delta\Delta}(\delta)\right) - \int_\delta^t g(s) f\left(u(s), u^\Delta(s)\right)\Delta s, \tag{14}$$

then

$$u^{\Delta\Delta}(t) = -\phi_q\left(\int_\delta^t g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right), \tag{15}$$

$$u^\Delta(t) = u^\Delta(\delta) - \int_\delta^t \phi_q\left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r)\right)\Delta r\right)\nabla s. \tag{16}$$

By $u^{\Delta\Delta}(\delta) = 0$ and condition (2), let $t = \eta$ on (15), we have

$$u^{\Delta\Delta}(1) = -M_1\left(u^{\Delta\Delta}(\eta)\right) = M_1\phi_q\left(\int_\delta^\eta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right). \tag{17}$$

Then, we have

$$\begin{aligned} u^\Delta(\delta) &= M_1\phi_q\left(\int_\delta^\eta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right) \\ &\quad + \int_\delta^T \phi_q\left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r)\right)\Delta r\right)\nabla s. \end{aligned} \tag{18}$$

Then

$$\begin{aligned} u^{\Delta\Delta}(t) &= M_1\phi_q\left(\int_\delta^\eta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right) \\ &\quad + \int_t^T \phi_q\left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r)\right)\Delta r\right)\nabla s. \end{aligned} \tag{19}$$

Therefore, by integrating the above equation (19) on $(0, t)$, we can east to have

$$\begin{aligned} u(t) &= \int_0^t M_1\phi_q\left(\int_\delta^\eta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right)\nabla t \\ &\quad + \int_0^t \left(\int_{s_1}^T \phi_q\left(\int_\delta^s g(r) f\left(u(r), u^\Delta(r)\right)\Delta r\right)\nabla s\right)\nabla s_1. \end{aligned} \tag{20}$$

Similarly, for $t \in (0, \delta)$, by integrating the equation of problems (1) on $(0, \delta)$, we have

$$\begin{aligned} u(t) &= \int_0^t M_0\phi_q\left(\int_\xi^\delta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right)\nabla t \\ &\quad + \int_0^t \left(\int_{s_1}^T \phi_q\left(\int_s^\delta g(r) f\left(u(r), u^\Delta(r)\right)\Delta r\right)\nabla s\right)\nabla s_1. \end{aligned} \tag{21}$$

Therefore, for any $t \in [0, T]$, $u(t)$ can be expressed as equation $u(t) = \int_0^t w(s)\Delta s$, where $w(t)$ is expressed as Lemma 3.

Sufficiency. Suppose $u(t) = \int_0^t w(s)\Delta s$. Then we have

$$u^{\Delta\Delta}(t) = \begin{cases} \phi_q\left(\int_t^\delta g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right) \geq 0, & 0 \leq t < \delta, \\ -\phi_q\left(\int_\delta^t g(s) f\left(u(s), u^\Delta(s)\right)\Delta s\right) \leq 0, & \delta < t \leq T, \end{cases} \tag{22}$$

So, $\left(\phi_p(u^{\Delta\Delta})\right)^\nabla + g(t) f(u(t), u^\Delta(t)) = 0$, $0 < t < T$, $t \neq \delta$. These imply that the equation (1) holds. Furthermore, we can easily obtain the boundary value equations of (2). This completes the proof of Lemma 3.

Now, we define an operator $T : K \to C^2[0, T]$ given by

$$(Tu)(t) = \int_0^t w(s)\Delta s, \tag{23}$$

where $w(t)$ is given by (15). And we can easily obtain the following Lemma.

Lemma 3. *Let $u \in K$ and L in Lemma 1. Then*

$$u^\Delta(t) \geq L\|u\|, t \in [L, T - L]. \tag{24}$$

Lemma 4. *Suppose that conditions (H_1), (H_2) hold, then for $L \in (0, T/2)$ in Lemma 1, we have*

$$u(t) \leq \frac{1}{L} u^\Delta(t), t \in [L, T - L]. \tag{25}$$

Proof. If $u(t)$ is the solution of problem (1), (2), then $u^\Delta(t)$ is a concave function, and $u(t) \geq 0$, $u^\Delta(t) \geq 0, t \in [0, T]$.

Thus for $t \in [L, T - L]$, we have $u^\Delta(t) \geq L\|u^\Delta\|$. Then by $u(t) = \int_0^t u^\Delta(s)\Delta s \leq \|u^\Delta\|$, we have

$$u(t) \leq \frac{1}{L} u^\Delta(t), t \in [L, 1 - L]. \tag{26}$$

The proof is complete.

Remark. : Obviously, we can obtain the following results,

$$w(0) - M_0 w'(\xi) = 0, \quad w(1) + M_1 w'(\eta) = 0. \tag{27}$$

Furthermore, by Arzela-Ascoli Theorem, it is easy to obtain the following Lemma.

Lemma 5. *$T : K \to K$ is completely continuous.*

For convenience, we set

$$R^* = 2\left(B'\right)^{-1}, R_* = \left[(b+1)\phi_q\left(\int_0^T g(r)\Delta r\right)\right]^{-1}. \tag{28}$$

where B' and L are given as Lemma 1.

3. The Existence of Single and Many Positive Solution

In this section, we present the following five main results.

Theorem 6. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_1) For $Lr \le u_2 \le r$, $0 \le u_1 \le (1/L)u_2$, we have $f(u_1, u_2) \ge (mr)^{p-1}$;

(A_2) For $0 \le u_2 \le R, 0 \le u_1 \le (1/L)u_2$, we have $f(u_1, u_2) \le (MR)^{p-1}$,where $m \in (R^*, \infty)$, $M \in (0, R_*)$. Then, the boundary value problem (1), (2) has at last one solution u such that $\|u\|$ lies between r and R.

The proof of Theorem 6. From Condition (H_3), for $\forall v \ge 0$, we can suppose that $0 < M_0(v) \le bv$, and $r < R$. By Lemma 3, for any $u \in K$, we can obtain that

$$u^{\Delta}(t) \ge L\|u\|, t \in [L, T-L]. \tag{29}$$

We define the following two open subset Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \Omega_2 = \{u \in K : \|u\| < R\}. \tag{30}$$

For any $u \in \partial\Omega_1$, by (29), we have

$$r = \|u\| \ge u^{\Delta}(t) \ge L\|u\| = Lr, t \in [L, T-L]. \tag{31}$$

For $t \in [L, T-L]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.

(i) If $\delta \in [L, T-L]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned}
2(Tu)^{\Delta}(\delta) &\ge \int_0^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\quad + \int_{\delta}^{T} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge \int_L^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\quad + \int_{\delta}^{T-L} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge mrA(\delta) \ge mrB' > 2r = 2\|u\|.
\end{aligned} \tag{32}$$

Then, with the case of $\delta \in [L, T-L]$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

(ii) If $\delta \in (T-L, T]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned}
(Tu)^{\Delta}(\delta) &\ge M_0\phi_q\left(\int_{\xi}^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\
&\quad + \int_0^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge \int_L^{T-L} \phi_q\left(\int_s^{T-L} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge mrA(1-L) \ge mrB' > 2r > r = \|u\|.
\end{aligned} \tag{33}$$

Then, with the case of $\delta \in (T-L, T]$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

(iii) If $\delta \in (0, L)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned}
(Tu)^{\Delta}(\delta) &\ge M_1\phi_q\left(\int_{\delta}^{\eta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\
&\quad + \int_{\delta}^{1} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge \int_L^{T-L} \phi_q\left(\int_L^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\ge mrA(L) \ge mrB' > 2r > r = \|u\|.
\end{aligned} \tag{34}$$

Then, with the case of $\delta \in (0, L)$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

Therefore, for any case of $\delta \in (0, T-L]$, we all easy to obtain that

$$\|Tu\| > \|u\|, \forall u \in \partial\Omega_1. \tag{35}$$

Then, by fixed point theorem of cone expansion-compression type in [23, 24], we have

$$i(T, \Omega_1, K) = 0. \tag{36}$$

Secondly, for $u \in \partial\Omega_2$, using $u^{\Delta}(t) \le \|u\| = R$, from (A_2), we can easily know that

$$\begin{aligned}(Tu)^{\Delta}(\delta) &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\\ &\quad + \int_0^T \phi_q\left(\int_s^{\delta} g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s\\ &\le bMR\phi_q\left(\int_0^T g(r)\Delta r\right) + MR\phi_q\left(\int_0^T g(r)\Delta r\right)\\ &= (b+1)MR\phi_q\left(\int_0^T g(r)\Delta r\right) \le R = \|u\|.\end{aligned} \tag{37}$$

Thus, we have

$$\|Tu\| < \|u\|, \forall u \in \partial\Omega_2. \tag{38}$$

Then, by fixed point theorem of cone expansion-compression type in [23, 24], we have

$$i(T, \Omega_2, K) = 1. \tag{39}$$

Therefore, by (36), (39), $r < R$ we have

$$i\left(T, \Omega_2 \setminus \bar{\Omega}_1, K\right) = 1. \tag{40}$$

Then, operator T has at last one fixed point $u \in (\Omega_2 \setminus \bar{\Omega}_1)$, and $r \le \|u\| \le R$. This completes the proof of Theorem 6.

Theorem 7. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_3) $f^0 = \lim_{u_2\to 0}\max_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi \in [0, (R_*/4)^{p-1})$;

(A_4) $f_\infty = \lim_{u_2\to\infty}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \psi \in ((2R^*/L)^{p-1}, \infty)$.

Then, the boundary value problem (1), (2) has at last one solution u.

The proof of Theorem 7.

First, by $0 \le \lim_{u_2\to 0}\max_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi < (R_*/4)^{p-1}$, letting $\varepsilon = (R_*/4)^{p-1} - \varphi > 0$, we know that there exists an appropriately small positive number ρ which satisfy as $0 \le u_2 \le \rho$, $u_2 \ne 0$, and we have

$$f(u_1,u_2) \le (\varphi+\varepsilon)(u_2)^{p-1} \le (R_*/4)^{p-1}\rho^{p-1} = ((R_*/4)\rho)^{p-1}. \tag{41}$$

Then, letting $R = \rho$, $0 < M = R_*/4 < R_*$, thus by the above equation, we can have

$$f(u_1,u_2) \le (MR)^{p-1}, 0 \le u_2 \le R, 0 \le u_1 \le \frac{1}{L}u_2. \tag{42}$$

So condition (A_2) in Theorem 6 holds.

Next, by $(2R^*/L)^{p-1} < \lim_{u_2\to\infty}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \psi < \infty$, letting $\varepsilon = \psi - (2R^*/L)^{p-1}$, we know that there exists an adequately big positive number $r \ne R$ which satisfy as $u_2 \ge Lr$, $0 \le u_1 \le (1/L)u_2$, and we have

$$f(u_1,u_2) \ge (\psi-\varepsilon)(u_2)^{p-1} \ge \left(\frac{2R^*}{L}\right)^{p-1}(Lr)^{p-1} = (2R^*r)^{p-1}, \tag{43}$$

Letting $m = 2R^* > R^*$, thus by the above equation, we have that (A_1) in Theorem 6 holds. Therefore, by Theorem 6, we can easily obtain the results of Theorem 7 holds. The proof of Theorem 3.2 is complete.

Corollary 8. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_5) $f^\infty = \lim_{u_2\to\infty}\max_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \lambda \in [0, (R_*/4)^{p-1})$;

(A_6) $f_0 = \lim_{u_2\to 0}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi \in ((2R^*/L)^{p-1}, \infty)$.

Then, the boundary value problem (1), (2) has at last one solution u.

The proof of Corollary 8. Similar to the proof of Theorem 7, we can obtain Corollary 8.

Theorem 9. *Suppose that conditions (H_1), (H_2), (H_3), and (A_2) in Theorem 6 hold. Assume that f also satisfy.*

(A_7) $f_0 = \lim_{u_2\to 0}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$;

(A_8) $f_\infty = \lim_{u_2\to\infty}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$.

Then, the boundary value problem (1), (2) have at least two solutions u_1, u_2 which satisfy

$$0 < \|u_1\| < R < \|u_2\|. \tag{44}$$

The proof of Theorem 9.

Firstly, by $\lim_{u_2\to 0}\min_{0\le u_1\le (1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$, for any $M > (2/B'L)$, there exists a constant $\rho_* \in (0, R)$ which satisfy

$$f(u_1,u_2) \ge (Mu_2)^{p-1}, 0 < u_2 \le \rho_*, 0 \le u_1 \le \frac{1}{L}u_2. \tag{45}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, similar to the previous proof of Theorem 6, for any $u \in \partial\Omega_{\rho_*}$, from the above

discussion and Lemma 2, we can have from three perspectives

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\rho_*}. \tag{46}$$

Then, by fixed point theorem of cone expansion-compression type, we can have

$$i\left(T, \Omega_{\rho_*}, K\right) = 0. \tag{47}$$

Secondly, for any $\bar{M} > (2/B'L)$, by $\lim_{u_2\to\infty} \min_{0\leq u_1\leq(1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = +\infty$, there exists a constant $\rho_0 > 0$ which satisfy

$$f(u_1, u_2) \geq \left(\bar{M}u_2\right)^{p-1}, u_2 > \rho_0, 0 \leq u_1 \leq \frac{1}{L}u_2. \tag{48}$$

Therefore, we choose a constant $\rho^* > \max\{R, \rho_0/L\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2, we can easily obtain

$$u^{\Delta}(t) \geq L\|u\| = L\rho^* > \rho_0, t \in [L, T-L]. \tag{49}$$

Then, by the above discussion and also similar to the previous proof of Theorem 6, we can also have from three perspectives

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\rho^*}. \tag{50}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 0. \tag{51}$$

Finally, imitating the latter proof of Theorem 6, for any $u \in \partial\Omega_R$, by (A_2), setting $\Omega_R = \{u \in K : \|u\| < R\}$, we can also easy to have

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega_R. \tag{52}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i(T, \Omega_R, K) = 1. \tag{53}$$

Therefore, by (47), (51), (53), $\rho_* < R < \rho^*$ we have

$$i\left(T, \Omega_R \setminus \bar{\Omega}_{\rho_*}, K\right) = 1, i\left(T, \Omega_{\rho^*} \setminus \bar{\Omega}_R, K\right) = -1. \tag{54}$$

Then, T have fixed point $u_1 \in \Omega_R \setminus \bar{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho^*} \setminus \bar{\Omega}_R$.

Obviously, u_1, u_2 are all positive solutions of problem (1), (2) and $0 < \|u_1\| < R < \|u_2\|$. The proof of Theorem 9 is complete.

Theorem 10. *Suppose that conditions (H_1), (H_2), (H_3) and (A_1) in Theorem 6 hold. Assume that f also satisfy.*

(A_9) $f^0 = \lim_{u_2\to 0} \max_{0\leq u_1\leq(1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$;

(A_{10}) $f^{\infty} = \lim_{u_2\to\infty} \max_{0\leq u_1\leq(1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$.

Then, the boundary value problem (1), (2) have at least two solutions u_1, u_2 which satisfy $0 < \|u_1\| < r < \|u_2\|$.

The proof of Theorem 10.

Firstly, by $\lim_{u_2\to\infty} \max_{0\leq u_1\leq(1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$, for $\eta_1 \in (0, R_*)$, there exists a constant $\rho_* \in (0, r)$ which satisfy

$$f(u_1, u_2) \leq (\eta_1 u_2)^{p-1}, 0 < u_2 \leq \rho_*, 0 \leq u_1 \leq \frac{1}{L}u_2. \tag{55}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (23), we have

$$\begin{aligned}
\|Tu\| = (Tu)^{(n-2)}(\delta) &\leq M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^{\delta} g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\
&\leq M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\
&\quad + \phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\
&\leq (b+1)\eta_1\rho_*\phi_q\left(\int_0^T g(r)dr\right) \leq \rho_* = \|u\|.
\end{aligned} \tag{56}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T, \Omega_{\rho_*}, K\right) = 1. \tag{57}$$

Secondly, letting $f^*(x) = \max_{0\leq u_{n-1}\leq x, 0\leq u_1\leq(1/L)u_2} f(u_1, u_2)$, we can easy to know that $f^*(x)$ is monotone increasing with respect to $x \geq 0$.

Therefore by $\lim_{u_2\to\infty} \max_{0\leq u_1\leq(1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$, we can easy to have $\lim_{x\to\infty} (f^*(x)/x^{p-1}) = 0$.

Therefore, for any $\eta_2 \in (0, R_*)$, there exists a constant $\rho^* > r$ which satisfy

$$f^*(x) \leq (\eta_2 x)^{p-1}, x \geq \rho^*. \tag{58}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for any $u \in \partial\Omega_{\rho^*}$, by (4.8), we have

$$\begin{aligned}\|Tu\| = (Tu)^{\Delta}(\delta) &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\\ &+ \int_0^T \phi_q\left(\int_s^{\delta} g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s\\ &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\\ &+ \phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\\ &\le (b+1)\phi_q\left(\int_0^T g(r)f^*(\rho^*)dr\right)\\ &\le (b+1)\eta_2\rho^*\phi_q\left(\int_0^T g(r)dr\right) \le \rho^* = \|u\|.\end{aligned} \tag{59}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 1. \tag{60}$$

Finally, imitating the previous proof of Theorem 6, for any $u \in \partial\Omega_r$, by (A_1), setting $\Omega_r = \{u \in K : \|u\| < r\}$, For any $u \in \partial\Omega_r$, we can also easy to have

$$\|Tu\| \ge \|u\|, \forall u \in \partial\Omega_r. \tag{61}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i(T, \Omega_r, K) = 0. \tag{62}$$

Therefore, by (57), (60), (62), $\rho_* < r < \rho^*$, we have

$$i\left(T, \Omega_r \setminus \bar{\Omega}_{\rho_*}, K\right) = -1, i\left(T, \Omega_{\rho^*} \setminus \bar{\Omega}_r, K\right) = 1. \tag{63}$$

Then, T have fixed point $u_1 \in \Omega_r \setminus \bar{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho_*} \setminus \bar{\Omega}_r$.

Obviously, u_1, u_2 are all positive solutions of problem (1),(2) and $0 < \|u_1\| < r < \|u_2\|$. The proof of Theorem 10 is complete.

4. Application

Example. Consider the following three-order BVP with p-Laplacian

$$\begin{cases}\left(\phi_p\left(u''\right)\right)' + \dfrac{1}{64\pi^4}t^{-\frac{1}{2}}(1-t)\left[\ln u^5 + e^{\left(u'\right)^2}\right] = 0, 0 < t < 1,\\ u(0) = 0,\\ u'(0) - u''(0.25) = 0, u'(1) + 5u''(0.3) = 0,\end{cases} \tag{64}$$

where $p = 4, \xi = 0.25, \eta = 0.3$, $B' = 1/4$,

$$g(t) = \frac{1}{64\pi^4}t^{-\frac{1}{2}}(1-t), f(u_1, u_2) = \ln u^5 + e^{\left(u'\right)^2}. \tag{65}$$

Then obviously, $q = 4/3, \int_0^1 g(t)dt = 1/64\pi^3, f_{\infty} = +\infty$, $f_0 = +\infty$,

$$M_0(v) = v < 2v = bv, M_1(v) = 5v, \forall v \ge 0, \tag{66}$$

so conditions (H_1), (H_2), (H_3), (A_7), (A_8) hold.

Next, $\phi_q(\int_0^1 g(t)dt) = 1/4\pi$, $R_* = 4\pi/3$, we choose $R = 3$, $M = 2$ and for $B' = 1/4$, because of the monotone increasing of $f(u_1, u_2)$ on $[0,\infty) \times [0,\infty)$, then

$$f(u_1, u_2) \le f(12, 3) = 286, \quad 0 \le u_2 \le 3, \quad 0 \le u_1 \le 4u_2. \tag{67}$$

Therefore, using $0 < M < R_*$, we have $(MR)^{p-1} = 328$, we know

$$f(u_1, u_2) \le (MR)^{p-1}, \quad 0 \le u_2 \le 3, \quad 0 \le u_1 \le 4u_2, \tag{68}$$

so conditions (A_2) holds. Then, by Theorem 9, the Example has at least two positive solutions v_1, v_2 and $0 < \|v_1\| < 3 < \|v_2\|$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

A Novel Speckle Noise Removal Algorithm Based on ADMM and Energy Minimization Method

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Speckle noise removal in medical ultrasound images is a challenging task. In this paper, a new model is proposed to removal speckle noise, alternating direction method of multipliers algorithm is employed to solve the new energy minimization model. The convexity, existence, and uniqueness of the new energy minimization model's solution are proved. Series of experiments are designed in this paper. Numerical results show that the new algorithm can reduce the step effect effectively obtain good results in visual effect and quantitative measures by comparing with some traditional models.

1. Introduction

With the rapid development of science and technology, computer network and image equipment are widely used. There are more and more application fields of digital image, including pattern recognition, medical imaging, video processing, remote sensing, and other fields. At the same time, people have higher and higher requirements for the quality of digital image, and the digital image processing technology has attracted more and more attention of researchers.

Image denoising technology is mainly to input a degraded image with noise into the computer for processing, to eliminate the noise in the image, make it closer to the original image, and facilitate the subsequent processing. In the early image denoising work, it can be divided into filter based on convolution methods [1, 2], wavelet based on image denoising methods [3, 4], depth based on learning method, and partial differential equation (PDE) based methods [5–8].

Medical ultrasound image reflects the difference of acoustic parameters in media and can obtain information different from optical, X-ray and Y-ray. Ultrasound has a good ability to distinguish human soft tissues, which is helpful to identify micropathological changes in biological tissues. But in the process of medical ultrasound image transmission, speckle noise will be produced, which will lead to a significant decline in the quality of ultrasound image and cover up the damage of some important tissues. How to eliminate speckle noise in ultrasound image while retaining its important features is a challenge.

In Ref. [9], Loupas et al. proved by experiments that the noise in medical ultrasound image is no longer the multiplicative noise in the form of Rayleigh distribution, and also proved in the paper that the form of medical ultrasound degraded image can be written:

$$f = u + \sqrt{u}n \ , \tag{1}$$

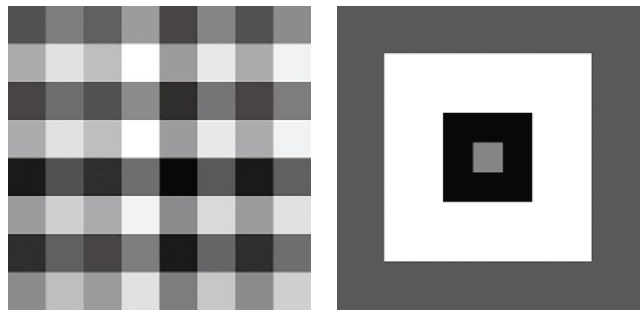
where f is a noisy image and n represents the Gaussian random noise with zeros mean and standard deviation σ.

In the paper [2], Kristian et al. derive the corresponding data fidelity term according to the distribution characteristics of Gaussian noise and the degradation model of medical ultrasonic noise image:

$$\int_{\Omega} \frac{(f-u)^2}{u} dx \ , \tag{2}$$

Initialize $k = 0$, $u_0 = f$, $\beta_0^1 = 0$, $\beta_0^2 = 0$
Given parameters α, μ and κ.
Repeat
  z_{k+1} is updated by the positive solution of the cubic equation (19);
  d_{k+1} is updated by the equation (20);
  u_{k+1} is updated by the equation (23);
  $\beta_{k+1}^1 = \beta_k^1 - \kappa\mu(z_{k+1} - u_{k+1})$;
  $\beta_{k+1}^2 = \beta_k^2 - \kappa\mu(d_{k+1} - \nabla u_{k+1})$;
Until a stopping condition is satisfied.
Final Input: u

Algorithm 1: ADMM algorithm for the new model.



(a) (b)

Figure 1: Original images. (a) syn1(256×256). (b) syn2(512×512).

With the passage of time, mathematical research is constantly improving, so stochastic theory [10–14] and PDE [15–20] have been fully developed, and Fractional theory [21, 22], wavelet [23], and statistical information [24] are all employed to deal with intensity inhomogeneity or noise. In this paper, the method of partial differential equation is used to solve speckle noise. In the numerical algorithm, although the use of image denoising is effective, it will not only bring some ladder effect but also low iterative efficiency. Therefore, the optimization algorithm has received great attention in recent years. Many efficient optimization algorithms have been proposed, such as Split Bregman type iterative [25], the Douglas-Rachford method [26], primal-dual algorithm [27], and Alternating Direction Method of Multipliers algorithm (ADMM) [26, 28]. In this paper, we will use the ADMM algorithm to solve the new energy minimization model.

The rest of this paper is as follows. In Section 2, we introduce the new energy minimization model and the ADMM algorithm. In Section 3, we adopt the ADMM algorithm to solve the new energy minimization model. Section 4 shows some numerical results and compares them with other existing models. The conclusion is drawn in Section 5.

2. Background

2.1. The TV Model. In 1992, the total variation (TV) model was proposed, which is the classical total variational model. The forms of TV model are generally as follows:

$$\min \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_\Omega |u - u_0|^2 dx, \tag{3}$$

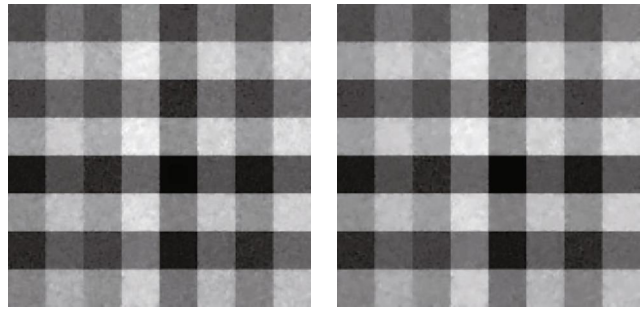
where $u : \Omega \to R$ is a clean image and u_0 is a noisy image affected by Gaussian random noise with mean zero and standard deviation σ. $\lambda > 0$ is regularization parameter which can balance fidelity term and regularized term in TV model. ∇ represents the gradient operator, which is defined as follows:

$$\nabla u = \left(\partial_x^+ u, \partial_y^+ u\right), \tag{4}$$

where ∂_x^+, ∂_y^+ is the forward discrete derivation operators, which are defined as:

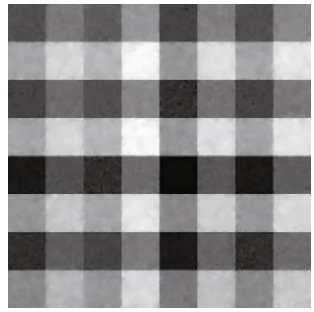
$$\partial_x^+ u = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } 1 \le j \le N-1 \\ 0 & \text{if } j = N \end{cases}, \quad \partial_y^+ u = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } 1 \le i \le M-1 \\ 0 & \text{if } i = M \end{cases}, \tag{5}$$

2.2. The New Energy Minimization Model. The new energy minimization model is an adaptive total variation model,



(a)

(b)



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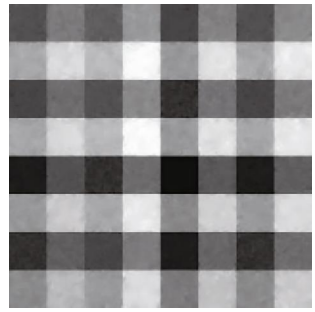

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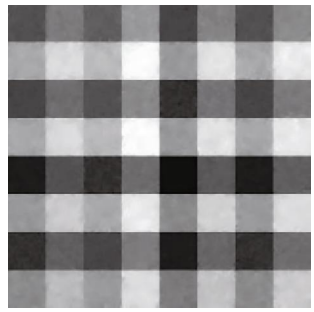
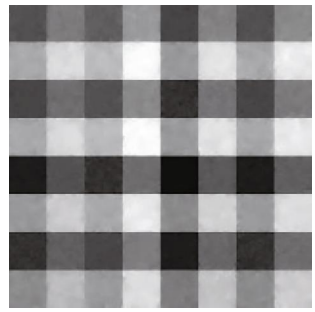
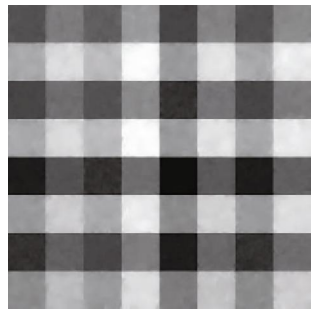


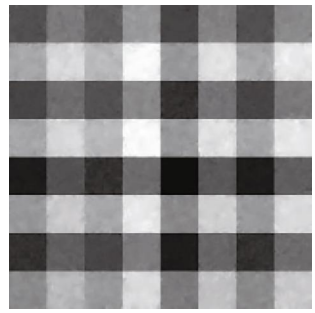
(e)



(f)

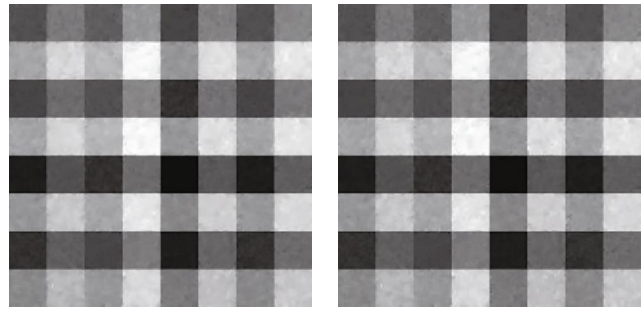


(g)



(h)

Figure 2: Continued.



(i) (j)

Figure 2: Numerical experiments with different parameters δ with noise standard deviation $\sigma = 2$. (a) Restored image by $\delta = 1$. (b) Restored image by $\delta = 5$. (c) Restored image by $\delta = 10$. (d) Restored image by $\delta = 20$. (e) Restored image by $\delta = 30$. (f) Restored image by $\delta = 40$. (g) Restored image by $\delta = 50$. (h) Restored image by $\delta = 60$. (i) Restored image by $\delta = 100$. (j)Restored image by $\delta = 200$.

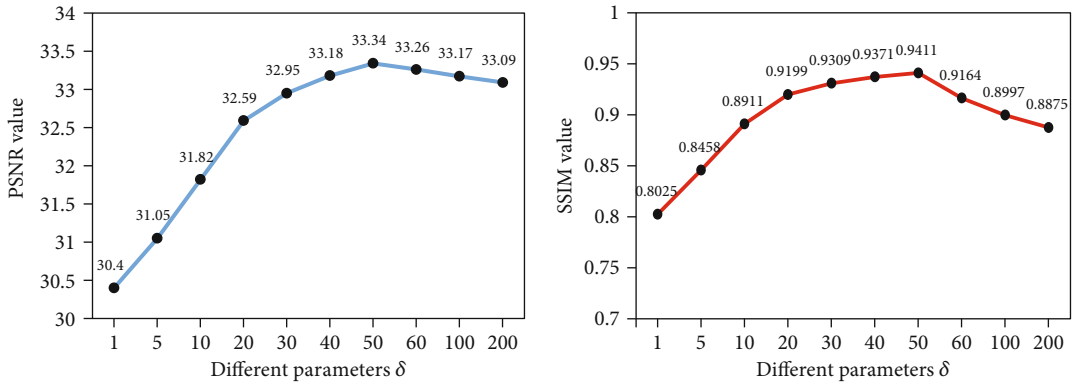
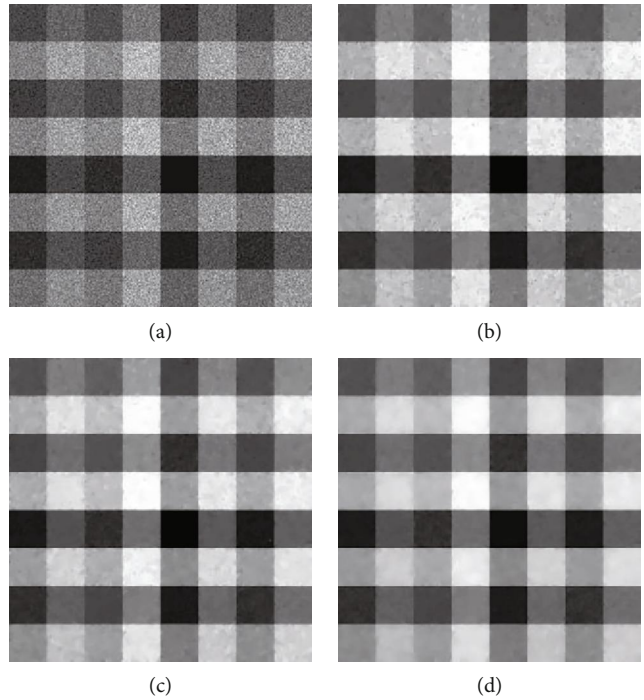


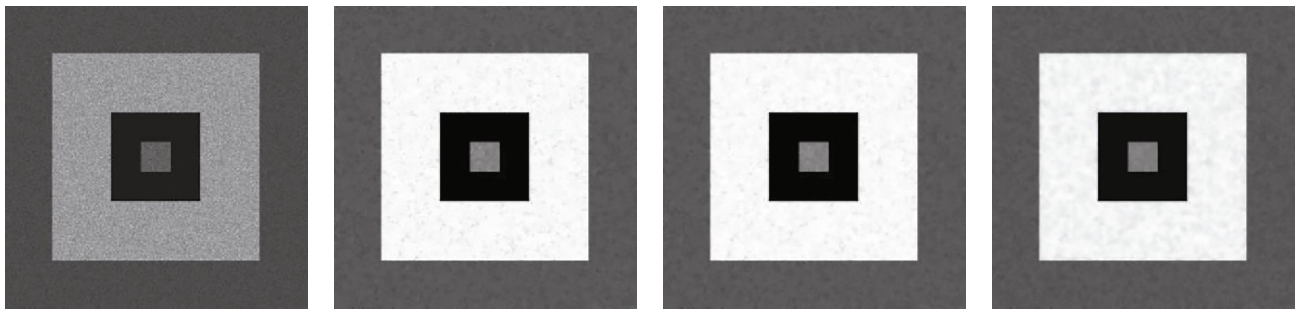
Figure 3: The PSNR and SSIM values of different δ.



(a) (b)

(c) (d)

Figure 4: Numerical result of the "syn1" image with noise standard deviation $\sigma = 2$. (a) Noisy image. (b) Restored image by the JIN's model. (c) Restored image by finite difference for the new model; (d) Restored image by ADMM for the new model.



(a) (b) (c) (d)

Figure 5: Numerical result of the "syn2" image with noise standard deviation $\sigma = 3$. (a) Noisy image. (b) Restored image by the JIN's model. (c) Restored image by finite difference for the new model. (d) Restored image by ADMM for the new model.

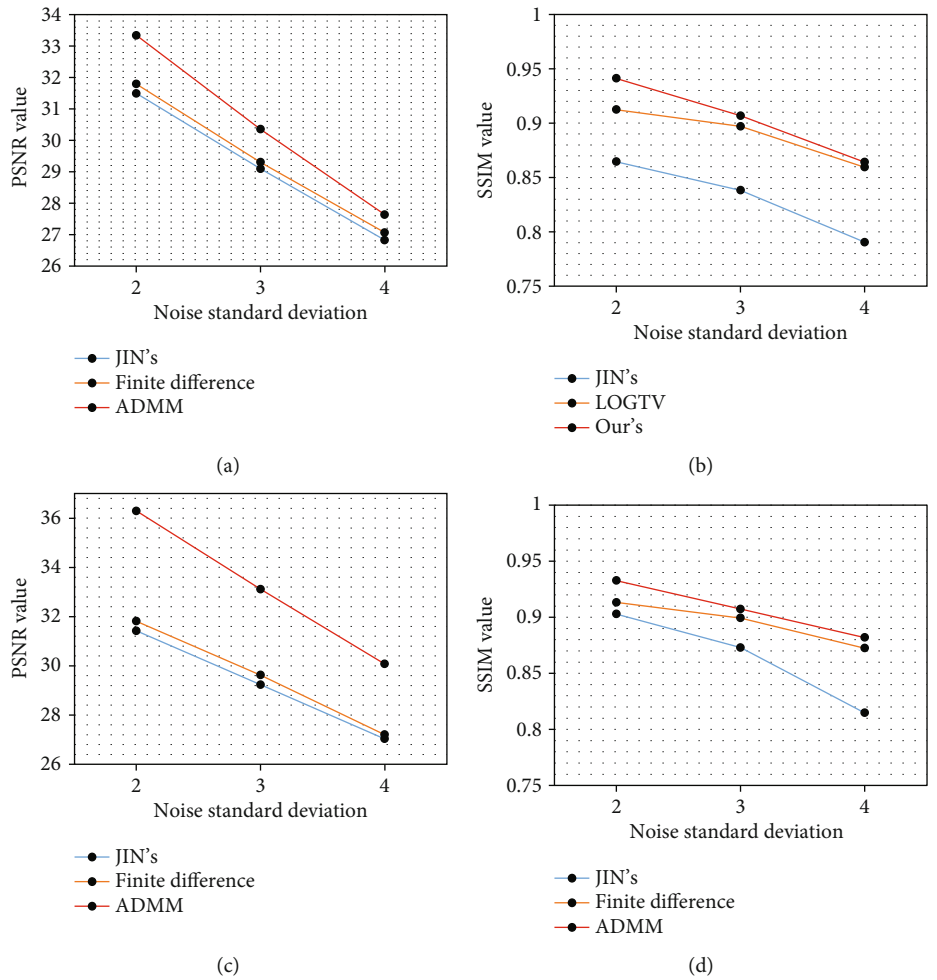


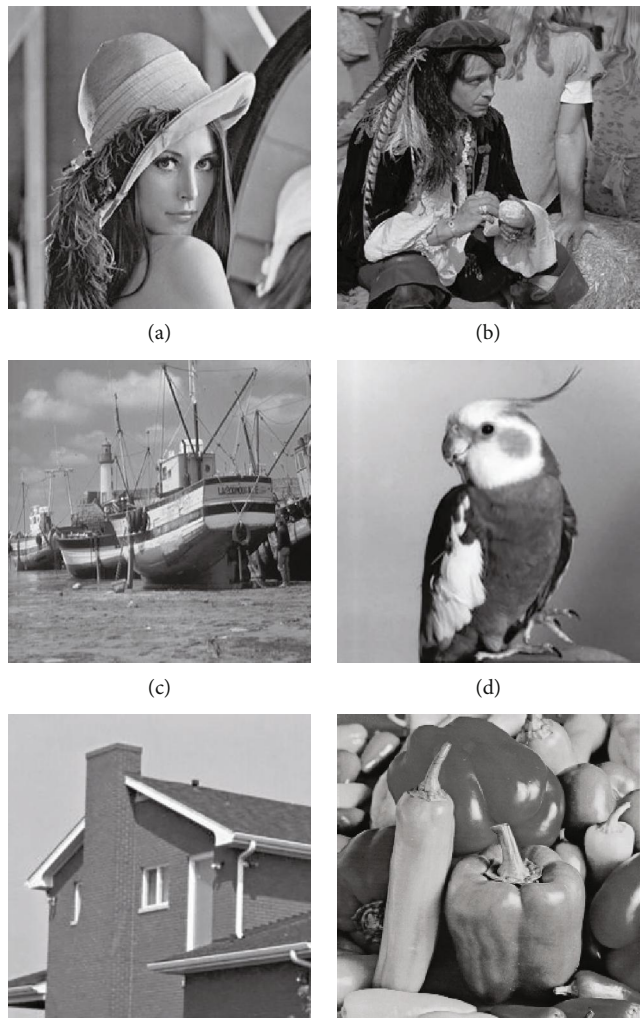
Figure 6: Folded-line contrast map. (a, b) PSNR and SSIM values of different models with different noise standard deviations ("syn1" image). (c, d) PSNR and SSIM values of different models with different noise standard deviations ("syn2" image).

which is defined by a new regularization term and the variance of speckle noise, as follows:

$$\min_{u}\left[\int_{\Omega}\varphi(|\nabla u|)dx+\frac{\alpha}{2}\int_{\Omega}\frac{(f-u)^2}{u}dx\right], \tag{6}$$

Where $\phi(s) = s\ln(\delta + s)$, $\int_{\Omega}((f-u)^2/u)\mathrm{d}x$ is the fidelity term, and α represents the regularization parameter which can balance fidelity term and regularization term.

Firstly, the energy model (6) is convex, which guarantees the existence of the minimal solution of the model (6).



(a) (b) (c) (d) (e) (f)

Figure 7: Original images: (a) lena (512 × 512); (b) pirate (512 × 512); (c) boat (512 × 512); (d) bird (256 × 256); (e) house (256 × 256); (f) peppers (256 × 256).

Theorem 1. *The energy functional $E(u)$ is convex. That is to say, for any u_1, $u_2 \in \Omega$ and $t \in [0, 1]$, we have:*

$$E(tu_1 + (1-t)u_2) \le tE(u_1) + (1-t)E(u_2) \quad , \tag{7}$$

where $E(u) = \int_\Omega \varphi(|\nabla u|)dx + (\alpha/2)\int_\Omega ((f-u)^2/u)dx$.

Proof. The proof of Theorem 1 is given in the appendix A.

Secondly, the uniqueness of the minimum solution of the model (6) can also be proved.

Theorem 2. *If u_1 and u_2 are two minimize solutions of model (6), then we have $u_1 = u_2$.*

Proof. The proof of Theorem 2 is given in the appendix B.

2.3. Alternating Direction Method of Multipliers. Alternating Direction Method of Multipliers is a computational framework for solving optimization problems, which is suitable for solving distributed convex optimization problems. ADMM decomposes a large global problem into several smaller, easy-to-solve local problems, and then solves them alternately. Its essence is the further development of the augmented Lagrange algorithm.

For linear constrained minimization problem,

$$\min_{x,y} \quad f(x) + g(y) \quad such \quad that \quad Ax + By = b, \tag{8}$$

where $x \in R^n$, $y \in R^m$, $A \in R^{k\times n}$, $B \in R^{k\times m}$, $b \in R^k$, and function $f : R^n \longrightarrow R$ and function $g : R^m \longrightarrow R$ are convex and lower semicontinuous functions.

By introducing a Lagrangian multiplier vector or dual variable $\beta \in R^k$, then the augmented Lagrange function of problem (8) is

$$L_\mu(x, y, \beta; \mu) = f(x) + g(y) - \beta^T(Ax + By - b) + \frac{\mu}{2}\|Ax + By - b\|_2^2 \tag{9}$$

where $\mu > 0$ is a penalty parameter. By the algorithm of Alternating Direction Method of Multipliers (ADMM), the solution (x^{k+1}, y^{k+1}) is achieved with the following iteration:

$$\begin{cases} x^{k+1} = \arg\min_x L_\mu\left(x, y^k, \beta^k; \mu\right) \\ y^{k+1} = \arg\min_y L_\mu\left(x^{k+1}, y, \beta^k; \mu\right) \\ \beta^{k+1} = \beta^k - \kappa\mu\left(Ax^{k+1} + By^{k+1} - b\right) \end{cases}, \tag{10}$$

where $\sigma \in (0, (1+\sqrt{5}/2))$ is a positive parameter. The convergence of the ADMM algorithm was proved in [29].

3. The ADMM Algorithm for the Energy Minimization Model

In this subsection, we will describe the ADMM algorithm for solving new energy minimization model. Firstly, the model (6) can be transformed into the following discretized version:

$$\min_u \left[\langle \phi(|\nabla u|),\ 1\rangle + \frac{\alpha}{2}\left\langle \frac{(f-u)^2}{u},\ 1\right\rangle\right], \tag{11}$$

To solve the problem with the ADMM algorithm, we introduce two new variables: z, $d = (d_1,\ d_2)$. Apply these two variables, the unconstrained problem (11) is written as the following constrained problem:

$$\begin{aligned} &\min_u \left[\langle \phi(|d|),\ 1\rangle + \frac{\alpha}{2}\left\langle \frac{(f-z)^2}{z},\ 1\right\rangle\right], \\ &\text{subject } that\ z = u,\ \ d = \nabla u \end{aligned} \tag{12}$$

Next, we make

$$\begin{aligned} A &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},\ x = \begin{bmatrix} z \\ d \end{bmatrix} \\ B &= \begin{bmatrix} -I \\ -\nabla \end{bmatrix},\ y = [u],\ \beta = \begin{bmatrix} \beta^1 \\ \beta^2 \end{bmatrix} \end{aligned}, \tag{13}$$

Now, we letting

$$f(x) = \langle \phi(|d|),\ 1\rangle + \frac{\alpha}{2}\left\langle \frac{(f-z)^2}{z},\ 1\right\rangle \text{ and } g(y) = 0, \tag{14}$$

So according to constraint problem (8), the constrained problem (12) can be solved by the ADMM algorithm. The augmented Lagrange function of problems (9) is

$$\begin{aligned} L_\mu\left(z, d, u, \beta^1, \beta^2; \mu\right) = \langle \phi(|d|),\ 1\rangle &+ \frac{\alpha}{2}\left\langle \frac{(f-z)^2}{z},\ 1\right\rangle \\ &- \left(\beta^1\right)^T(z-u)\ - \left(\beta^2\right)^T(d-\nabla u) \\ &+ \frac{\mu}{2}\|z-u\|_2^2 + \frac{\mu}{2}\|d-\nabla u\|_2^2, \end{aligned} \tag{15}$$

According to the structure of the ADMM algorithm, the problem (15) can be rewritten as the following form:

$$\begin{cases} (z_{k+1}, d_{k+1}) = \arg\min_{z\geq 0,\ d} L_\mu\left(z, d, u_k, \beta_k^1, \beta_k^2; \mu\right) \\ u_{k+1} = \arg\min_d L_\mu\left(z_{k+1}, d_{k+1}, u, \beta_k^1, \beta_k^2; \mu\right) \\ \beta_{k+1}^1 = \beta_k^1 - \kappa\mu(z_{k+1} - u_{k+1}) \\ \beta_{k+1}^2 = \beta_k^2 - \kappa\mu(d_{k+1} - \nabla u_{k+1}) \end{cases}, \tag{16}$$

Since the two new variables z and d in problem (16) are not related to each other, so we can divide two simple minimization subproblem as follows,

$$\begin{cases} z_{k+1} = \min_{z\geq 0}\left[\frac{\alpha}{2}\left\langle \frac{(f-z)^2}{z},\ 1\right\rangle - \left(\beta_k^1\right)^T z + \frac{\mu}{2}\|z-u_k\|_2^2\right], \\ d_{k+1} = \min_d\left[\langle \phi_i(|d|),\ 1\rangle - \left(\beta_k^2\right)^T d + \frac{\mu}{2}\|d-\nabla u_k\|_2^2\right] \end{cases} \tag{17}$$

First, for the $z-subproblem$, we can obtain the corresponding Euler-Lagrange equation:

$$\frac{\alpha}{2}\left(1 - \frac{f^2}{z^2}\right) - \beta_k^1 + \mu(z-u_k) = 0, \tag{18}$$

Simplifying the Euler-Lagrange equation above, we can obtain three complicated solutions by the roots formula of the cubic equation:

$$2\mu z^3 + \left(\alpha - 2\beta_k^1 - 2\mu u_k\right)z^2 - \alpha f^2 = 0, \tag{19}$$

For equation (19), there is only one positive real number solution (the proof is given in appendix C).

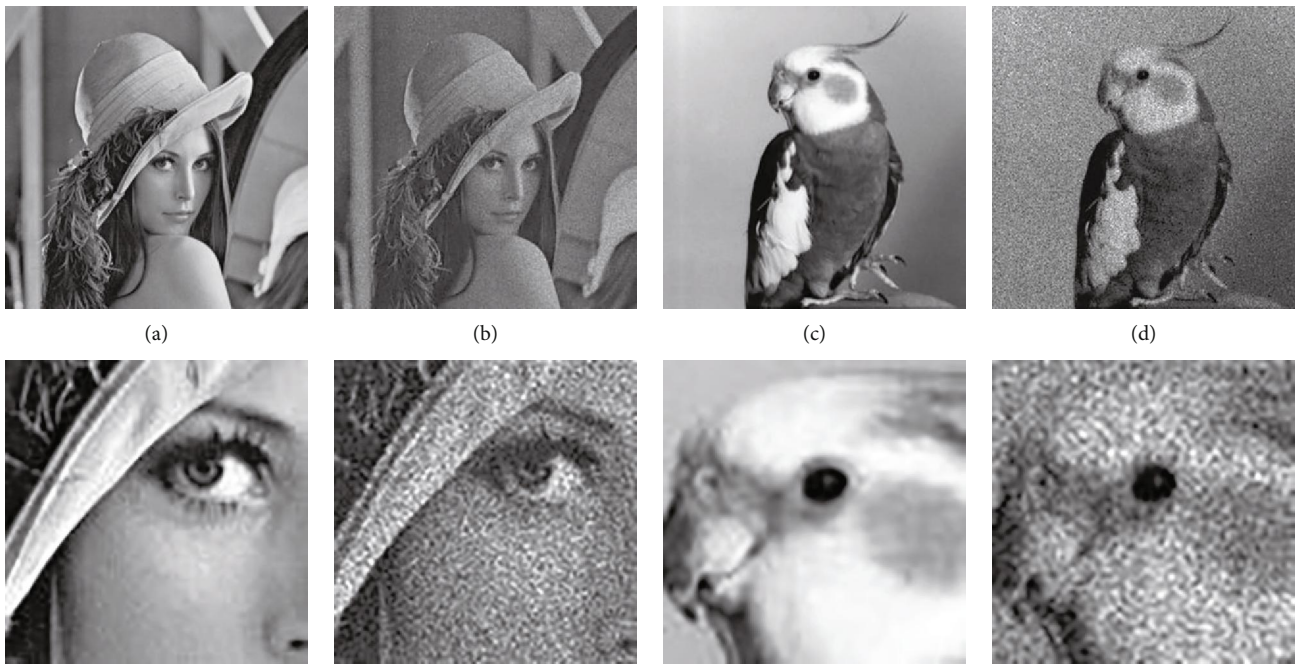


FIGURE 8: Numerical result of the "lena" and "bird" images with noise standard deviation $\sigma = 2$. (a, c) Original images. (b, d) Noisy images. (e–h) About the detailed image of (a–d).

Second, for the $d-subproblem$, we will apply the shrink operator to obtain the solution d. The form of the solution d is as follows:

$$\begin{aligned} d_{k+1} &= \min_{d}\left[\langle \phi_i(|d|),\ 1\rangle - \left(\beta_k^2\right)^T d + \frac{\mu}{2}\|d-\nabla u_k\|_2^2\right] \\ &= shrink\left(\frac{\beta_k^2}{\mu}+\nabla u_k, \frac{\phi_i'(|d|)}{\mu}\right), \end{aligned} \tag{20}$$

where the shrink operator $shrink(a,\theta)$ is defined [30] as:

$$shrink(a,\theta) = \max\left(\|a\|_1 - \theta, 0\right)\frac{a}{\|a\|_1}, \tag{21}$$

Last, the $u-subproblem$ is a least square problem as follow:

$$u_{k+1} = \min_{u}\|Bu+b\|_2^2\ , \tag{22}$$

where $b = (z_{k+1} - \beta_k^1/\mu,\ d_{k+1} - \beta_k^2/\mu)$. So the solution u_{k+1} of the least square problem (22) equivalent to the solution of the $B^TBu = -B^Tb$. That is to say, the solution is as follows:

$$\left(I + \nabla^T\nabla\right)\ u = z_{k+1} - \frac{\beta_k^1}{\mu} + \nabla^T\left(d_{k+1} - \frac{\beta_k^2}{\mu}\right)\ , \tag{23}$$

To sum up, the ADMM algorithm can be summarized in Algorithm 1.

4. Experimental Results

In this section, we show five sets of experimental results. Test images include synthetic, natural, and real medical ultrasound images. In addition, the denoising effect of the model is compared with that of the existing models, such as TV model [7], ATV model [5], JIN's model [8], and finite difference for the new model.

For the algorithm 1, the stopping condition is that the solution of two adjacent iterations satisfies:

$$\frac{\|u_k - u_{k+1}\|_2}{\|u_k\|_2} < 10^{-3}\ \text{ and }\ k > M\ , \tag{24}$$

where M is the maximum iteration numbers; $M = 100$ for algorithm 1. u_k represents the results of the kth iterations. For the JIN's model and finite difference for the new model, we calculated the noise deviation reduction (NDR) at each iteration as a convergence condition;

$$NDR = mean\frac{(f-u_k)}{\sqrt{u_k}}\ , \tag{25}$$

And the stopping condition (NDR) for finite difference for the new model is as follow

$$\left|\sqrt{NDR} - \sigma\right| < 10^{-1}\ , \tag{26}$$

FIGURE 9: Numerical result of the "lena" image with noise standard deviation $\sigma = 2$. (a) Restored image by the TV model. (b) Restored image by the JIN's mode. (c) Restored image by finite difference for the new model. (d) Restored image by ADMM for the new model. (e–h) About the detailed image of (a–d), respectively.

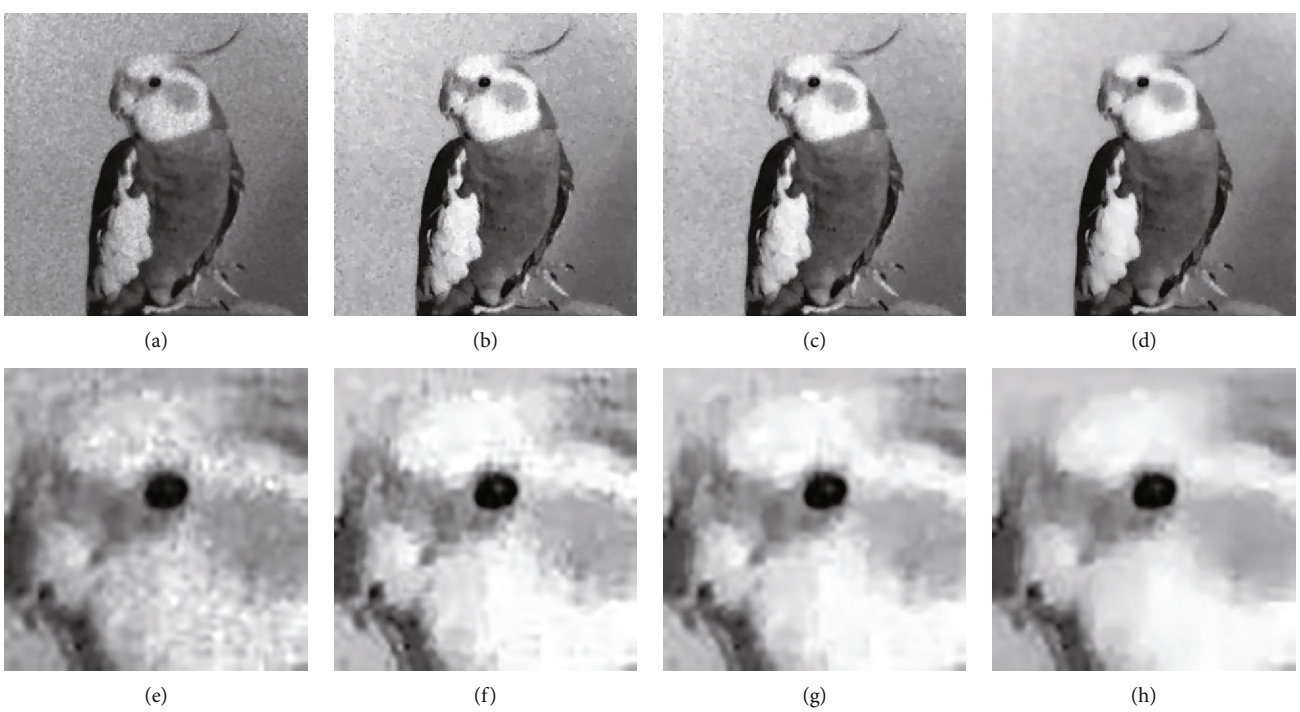
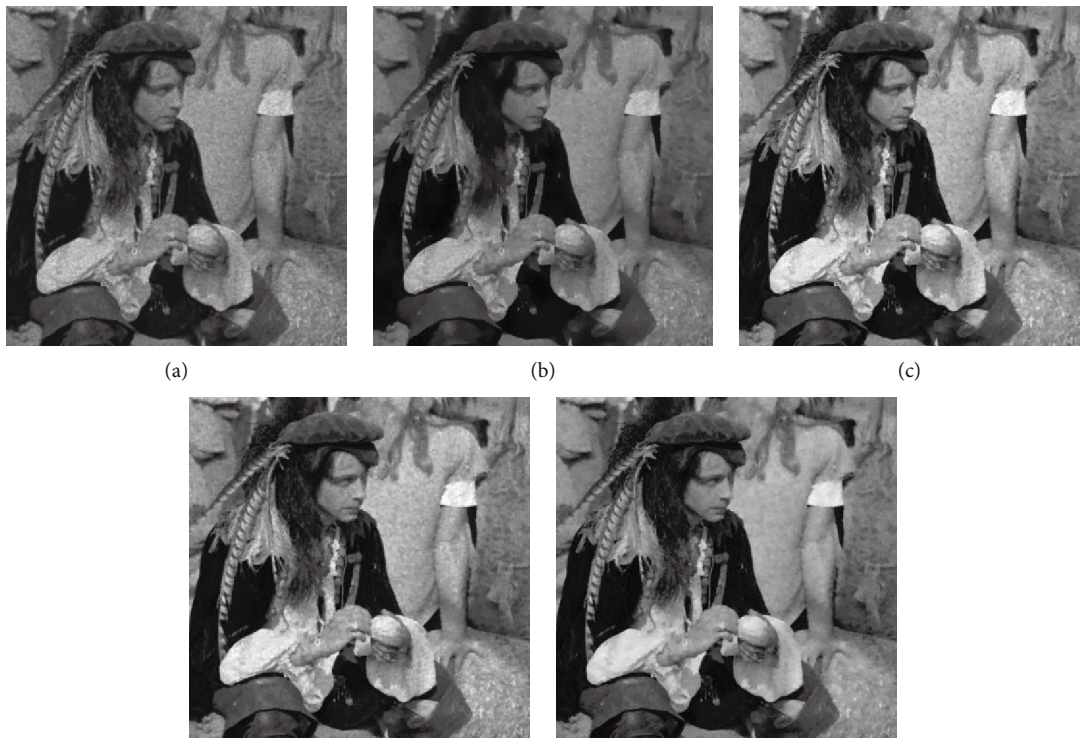


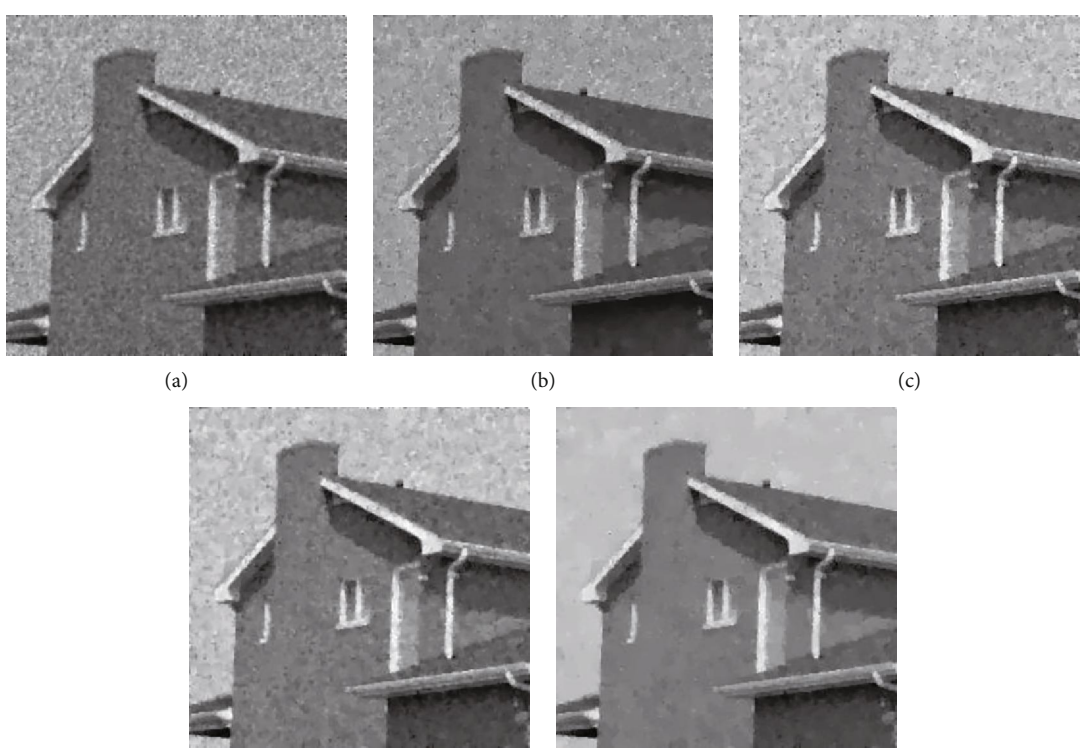
FIGURE 10: Numerical result of the "bird" image with noise standard deviation $\sigma = 2$. (a) Restored image by the TV model. (b) Restored image by the JIN's mode. (c) Restored image by finite difference for the new model. (d) Restored image by ADMM for the new model. (e–h) About the detailed image of (a–d), respectively.



(a) (b) (c)

(d) (e)

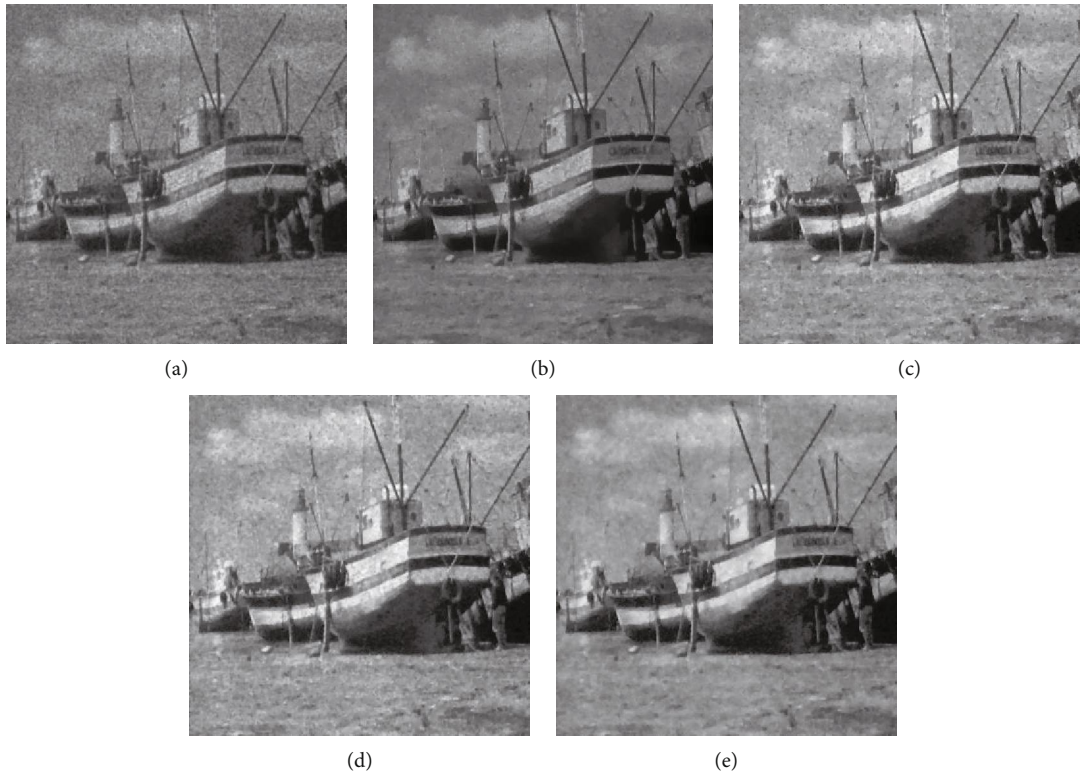
Figure 11: Numerical result of the "pirate" image with noise standard deviation $\sigma = 3$. (a) Restored image by the TV model; (b) Restored image by the ATV model; (c) Restored image by the JIN's model; (d) Restored image by finite difference for the new model; (e) Restored image by ADMM for the new model.



(a) (b) (c)

(d) (e)

Figure 12: Numerical result of the "house" image with noise standard deviation $\sigma = 3$. (a) Restored image by the TV model. (b) Restored image by the ATV model. (c) Restored image by the JIN's model. (d) Restored image by finite difference for the new model. (e) Restored image by ADMM for the new model.



(a) (b) (c) (d) (e)

Figure 13: Numerical result of the "boat" image with noise standard deviation $\sigma = 4$. (a) Restored image by the TV model. (b) Restored image by the ATV model. (c) Restored image by the JIN's model. (d) Restored image by finite difference for the new model. (e) Restored image by ADMM for the new model.

The effect of image denoising can be evaluated from two aspects. The first is the subjective aspect: it is judged by the subjective consciousness of peoples. The second is the objective aspect: it evaluates the image denoising situation through scientific indicators. In this paper, we evaluate the effect of image denoising by calculating the peak signal-to-noise ratio (PSNR) and structural similarity, which are defined as follows:

$$PSNR(u, \bar{u}) = 10 \log_{10} \frac{255^2 mn}{\|u - \bar{u}\|_2^2}, \quad (27)$$

$$SSIM(u, \bar{u}) = \frac{(2\mu_u \mu_{\bar{u}} + c_1)(2\sigma_{u\bar{u}} + c_2)}{(\mu_u{}^2 + \mu_{\bar{u}}{}^2 + c_1)(\sigma_u{}^2 + \sigma_{\bar{u}}{}^2 + c_2)}, \quad (28)$$

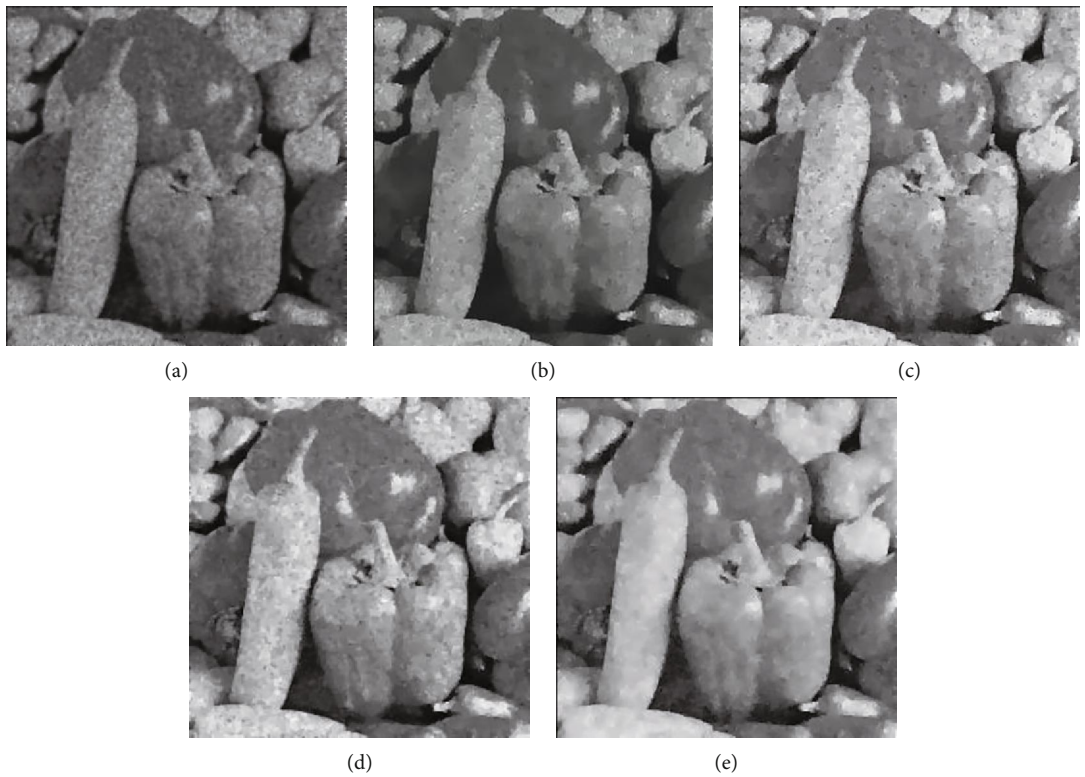
where $u \in R^{m \times n}$ is the clean image and $\bar{u} \in R^{m \times n}$ is the restored image. μ_u and $\mu_{\bar{u}}$ are the mean intensity of u and $\bar{u}$, respectively. σ_u and $\sigma_{\bar{u}}$ are the standard deviation of u and $\bar{u}$, respectively. $\sigma_{u\bar{u}}$ is the covariance of u and $\bar{u}$, and c_1 and c_2 are some constants for stability.

Best denoising performance are given in bold.

4.1. The Effect of Different Parameters of the Model on Denoising. In the first experiment, we find the optimal parameters δ value in the algorithm 1 for original image in Figure 1. We choose "syn1" and the noise level is $\sigma = 2$. Figure 2 shows that numerical experiments with different parameters δ in algorithm 1. Figure 3 shows that the different PSNR and SSIM values when different δ values are used in algorithm 1. Here, we can see that the optimal values of PSNR and SSIM values are at $\delta = 50$. Therefore, in the following experiment, we choose $\delta = 50$ in algorithm 1.

4.2. Denoising Effect of the ADMM Algorithm. In the second experiment, we mainly evaluate the denoising effect of the ADMM algorithm by testing image "syn1" and "syn2". Figures 4(a) and 5(a) show the noise images, and the noise deviation are $\sigma = 2$ and $\sigma = 3$, respectively. Figures 4(b) and 5(b) show the restored image by the JIN's model. Figures 4(c) and 5(c) show the restored image by finite difference for the new model. Figures 4(d) and 5(d) show the restored image by ADMM for the new model. In addition, Figure 6 shows that the different PSNR and SSIM values when used different models and algorithm. In Figure 6, we can find that our new algorithm has better denoising effect than JIN model and finite difference for the new model.

4.3. Comparison with TV Model, ATV Model, JIN's Model, and Finite Difference for the New Model. In the third experiment, we compare the denoising effects of the TV model, ATV model, JIN's model, and finite difference for the new model with ADMM for the new model by



(a) (b) (c) (d) (e)

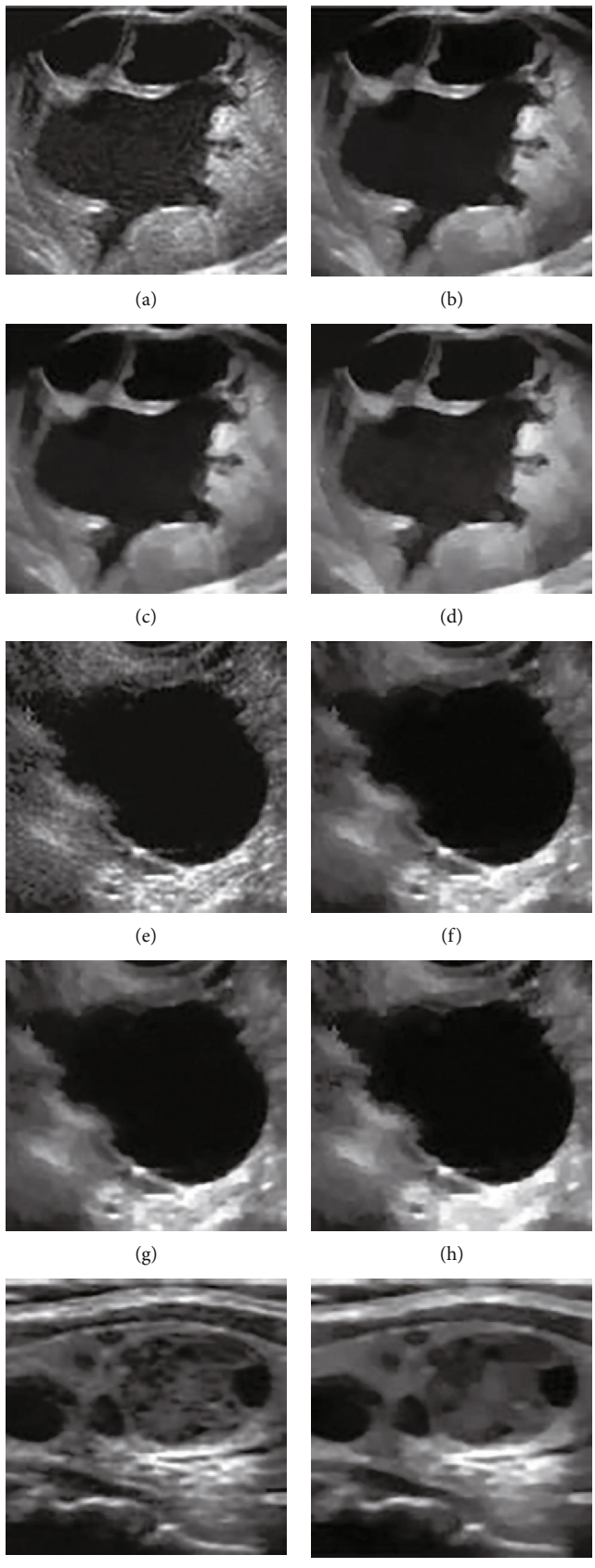
Figure 14: Numerical result of the "peppers" image with noise standard deviation $\sigma = 4$. (a) Restored image by the TV model. (b) Restored image by the ATV model. (c) Restored image by the JIN's model. (d) Restored image by finite difference for the new model. (e) Restored image by ADMM for the new model.

Table 1: The PSNR of the restored images by the different model.

Image	σ	TV (PSNR/SSIM)	ATV (PSNR/SSIM)	JIN's (PSNR/SSIM)	Finite difference for the new model (PSNR/SSIM)	ADMM for the new model (PSNR/SSIM)
Lena	2	29.49/0.8306	29.94/0.8731	29.98/0.8665	30.65/0.8831	**30.83/0.9042**
Bird	2	28.99/0.6907	30.28/0.7849	30.36/0.7774	31.07/0.8173	**31.74/0.8747**
Pirate	2	28.33/0.8975	27.22/0.8530	28.51/0.9042	28.74/0.9060	**28.81/0.9153**
House	2	27.45/0.6185	29.06/0.8101	28.77/0.6876	29.74/0.7447	**30.35/0.8157**
Boat	2	27.26/0.8240	27.90/0.8574	28.35/0.8622	28.64/0.8661	**28.81/0.8808**
Peppers	2	27.60/0.7183	28.66/0.8377	29.46/0.8195	29.53/0.8274	**29.64/0.8430**
Lena	3	27.73/0.7609	27.98/0.8149	28.49/0.8298	28.75/0.8266	**28.85/0.8717**
Bird	3	27.36/0.6451	27.92/0.6930	28.48/0.7307	29.13/0.7701	**29.73/0.8456**
Pirate	3	26.35/0.8392	26.38/0.8332	26.81/0.8578	26.98/0.8583	**27.05/0.8720**
House	3	26.08/0.5752	27.33/0.6611	27.24/0.6551	27.82/0.6896	**28.25/0.7846**
Boat	3	25.79/0.7481	26.47/0.7986	26.68/0.8029	26.82/0.7989	**27.03/0.8218**
Peppers	3	25.93/0.6758	26.70/0.7409	27.13/0.7446	27.37/0.7516	**27.57/0.8004**
Lena	4	26.17/0.6925	26.94/0.8129	26.82/0.7625	27.19/0.7641	**27.25/0.8194**
Bird	4	25.66/0.5679	26.49/0.6584	27.25/0.7263	27.51/0.7282	**27.63/0.7994**
Pirate	4	24.03/0.7652	24.33/0.7205	24.75/0.7787	25.77/0.8159	**25.79/0.8306**
House	4	24.63/0.5130	25.30/0.5760	26.18/0.6399	26.43/0.6437	**26.51/0.7385**
Boat	4	24.47/0.6807	24.95/0.7338	25.24/0.7370	25.51/0.7388	**25.61/0.7745**
Peppers	4	24.49/0.6153	25.59/0.7153	25.63/0.6800	25.83/0.7011	**26.19/0.7600**

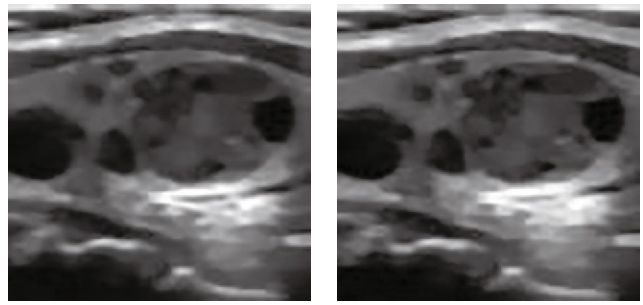
subjective and objective evaluation criteria. The test original images are shown in Figure 7 ("lena," "pirate," "boat," "bird," "house," and "peppers").

Firstly, in the subjective evaluation, we mainly focus on the details of the restored images. The first line of Figure 8 shows the original images ("lena" and "bird") and the noise



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)

Figure 15: Continued.



(k) (l)

Figure 15: Numerical result of the real ultrasound image. (a, e, i) Real ultrasound images. (b, f, j) Restored image by the JIN's model. (c, g, k) Restored image by finite difference for the new model. (d, h, l) Restored image by ADMM for the new model.

images with noise deviation is $\sigma = 2$. The second line is the corresponding detail images in the first line. Figures 9 and 10 show the restoration results of the noisy images ("lena" and "bird"). Figures 9(a), 9(d), 10(a), and 10(d) were restored images of the TV model, JIN's model, finite difference for the new model, and ADMM for the new model, respectively. Figures 9(e), 9(h), 10(e), and 10(h) are corresponding details images of Figures 9(a), 9(d), 10(a), and 10(d), respectively. By observing at the detailed images in Figures 9 and 10, we can see that the denoising effects of the four models are different. Figures 9(e), 9(f), 10(e), and 10(f)show that the denoising effect of TV model JIN's model are worse in subjective vision, and Figures 9(g), 9(h), 10(g), and 10(h) show that the denoising effect of finite difference for the new model and ADMM for the new model are well in subjective vision. In addition, we can clearly see the staircase effect in the details images obtained by the TV model and JIN's model. At the same time, by comparing with finite difference for the new model, the staircase effect of the new model is greatly reduced after it is repaired by the ADMM algorithm. This means that using the ADMM algorithm to solve the new energy minimization model has a better effect than other model algorithms in removing speckle noise.

Secondly, in the objective evaluation, we compare the denoising effects of different models with PSNR and SSIM values. Figures 11, 12, 13, and 14(a) shows the restoration results for images through TV model; Figures 11, 12, 13, and 14(b) shows the restoration results for images through ATV model; Figures 11, 12, 13, and 14(c) show the restoration results for images through JIN's model; Figures 11, 12, 13 and 14(d) show the restoration results for images through finite difference for the new model; Figures 11, 12, 13, and 14(e) show the restoration results for images through ADMM for the new model. The noise versions of "boat" and "house" and "pirate" and "bird" are obtained by degradation model (1) with standard deviation 3 and 4, respectively. Table 1 shows the PSNR and SSIM values for different test images by using the TV model, ATV model, JIN's model, finite difference for the new model, and ADMM for the new model. From these values, we observe that the denoising effect of the ADMM for the new model is better than other models and finite difference for the new model.

4.4. Denoising of Real Ultrasound Images. The last experiment is mainly to test the denoising effect of the new algorithm for solving the real ultrasound image and compare it with other denoising models. However, the real ultrasound image does not have the original image, so it cannot be evaluated using the original PSNR and SSIM. In 2005, Buades et al. [31] proposed an evaluation method without original image. The method evaluates the image denoising effect based on the difference and the estimated noise images between the real ultrasound image and the restored image. The difference image and the estimated noise image are defined as:

$$f - \bar{u} \text{ and } \frac{f - \bar{u}}{\sqrt{\bar{u}}}, \tag{29}$$

where $f \in R^{m\times n}$ is the real ultrasound image and $\bar{u} \in R^{m\times n}$ is the restored image. The image denoising effect can be evaluated based on the difference image and the similarity of the estimated noise image, that is, based on the residual image (the difference between the difference image and estimated noise image). If the texture of the residual image is small, the denoising effect is well.

Figure 15 shows that the experimental results of real ultrasound images by applying JIN's model, finite difference for the new model, and ADMM for the new model. Figure 16 shows the residual image after applying the JIN model, finite difference for the new model, and ADMM for the new model to restore the real ultrasound image. Visually speaking, the new model has fewer textures of residual images under the new algorithm, which shows that our method is better in denoising effect.

5. Conclusion

In this paper, we introduce the ADMM algorithm to optimize the denoising effect of the model for the new speckle noise recovery model based on adaptive variational method. The new algorithm is a computational framework for solving optimization problems, which is suitable for solving distributed convex optimization problems. The numerical experiments results show the effectiveness of

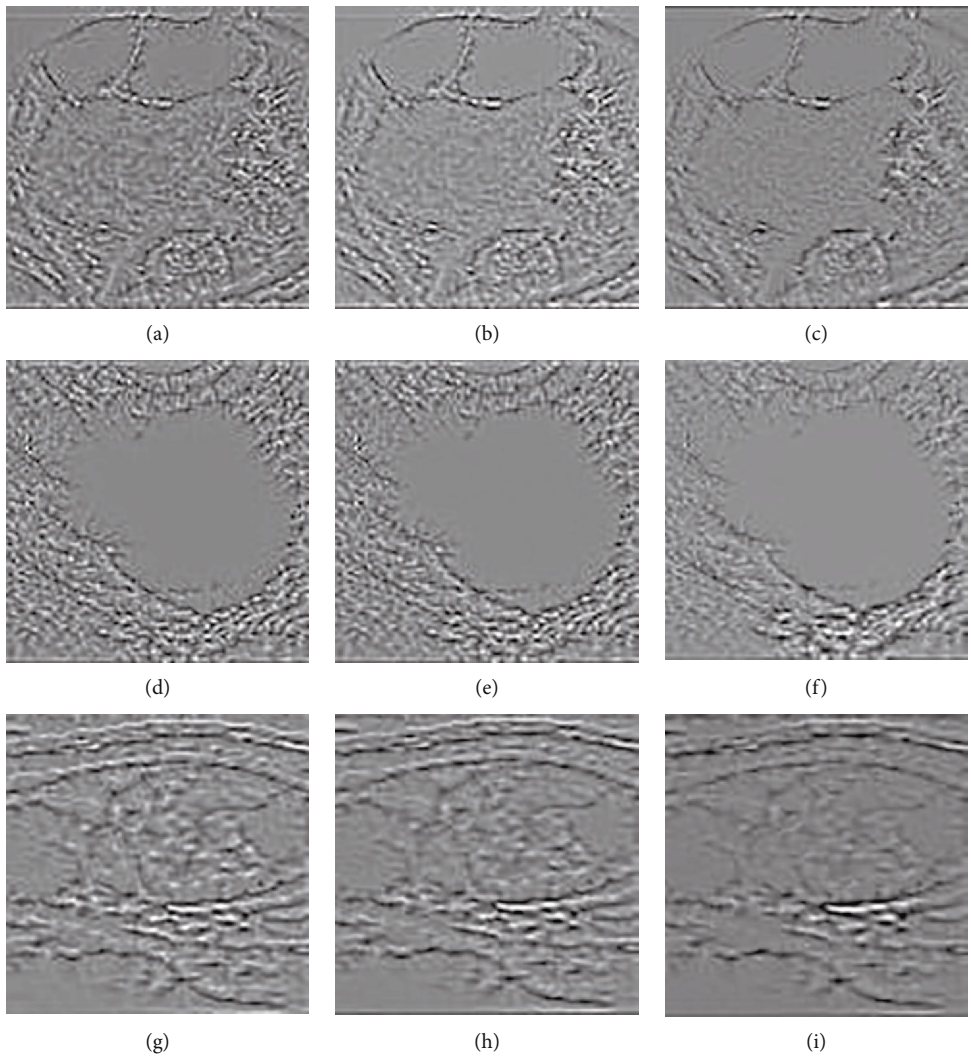


Figure 16: Residual image. (a, d, g) Residual image by the JIN's model. (b, e, h) Residual image by finite difference for the new model. (c, f, i) Residual image by ADMM for the new model.

the new method. In addition, by comparing with some existing methods, the experiment results show the high efficiency of the new method in image restoration. In recent years, nonconvex model denoising methods have received more and more attention, so it is possible to study nonconvex models in the next work.

Appendix

A. The Proof of Theorem 3.1.1

Proof. Firstly, the function $\phi(s)$ is convex, according to Definition 2.1.2, for any $u_1,\ u_2 \in \Omega$, and $t \in [0, 1]$, we have

$$\int_\Omega \varphi(|\nabla(tu_1 + (1-t)u_2)|)\mathrm{d}x \le t\int_\Omega \varphi(|\nabla u_1|)\mathrm{d}x + (1-t)\int_\Omega \varphi(|\nabla u_2|)\mathrm{d}x \tag{A.1}$$

meanwhile, we have

$$\begin{aligned}
&\int_\Omega \frac{(f-(tu_1+(1-t)u_2))^2}{tu_1+(1-t)u_2}\mathrm{d}x \\
&= \int_\Omega \left(\frac{f^2}{tu_1+(1-t)u_2} - 2f + tu_1 + (1-t)u_2\right)\mathrm{d}x \\
&\le \int_\Omega \left(\frac{f^2(tu_2+(1-t)u_1)(tu_1+(1-t)u_2)}{(tu_1+(1-t)u_2)u_1u_2} - 2f + tu_1 + (1-t)u_2\right)\mathrm{d}x \\
&= \int_\Omega \left(\frac{f^2(tu_2+(1-t)u_1)}{u_1u_2} - 2f + tu_1 + (1-t)u_2\right)\mathrm{d}x \\
&= \int_\Omega \left(\frac{tf^2}{u_1} - 2tf + tu_1 + \frac{(1-t)f^2}{u_2} - 2(1-t)f + (1-t)u_2\right)\mathrm{d}x \\
&= \int_\Omega t\frac{(f-u_1)^2}{u_1} + (1-t)\frac{(f-u_2)^2}{u_2}\mathrm{d}x,
\end{aligned} \tag{A.2}$$

Therefore,

$$E(tu_1 + (1-t)u_2) \le tE(u_1) + (1-t)E(u_2) \quad . \tag{A.3}$$

This proof is established.

B. The Proof of Theorem 3.1.2

Proof. According to Theorem 3.1.1, we have

$$\begin{aligned} E\left(\frac{u_1+u_2}{2}\right) &= \int_\Omega \varphi\left(\left|\nabla\left(\frac{1}{2}u_1 + \frac{1}{2}u_2\right)\right|\right)\mathrm{d}x + \alpha\int_\Omega \frac{(f-(u_1+u_2/2))^2}{u_1+u_2/2}\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega \varphi(|\nabla u_1|)\mathrm{d}x + \frac{1}{2}\int_\Omega \varphi(|\nabla u_2|)\mathrm{d}x + \alpha\int_\Omega \\ &\quad \cdot\left(\frac{f^2\left(1/4(u_1+u_2)^2 - 1/4(u_1-u_2)^2\right)}{(u_1+u_2/2)u_1u_2} - 2f + \frac{1}{2}u_1 + \frac{1}{2}u_2\right)\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega \varphi(|\nabla u_1|)\mathrm{d}x + \frac{1}{2}\int_\Omega \varphi(|\nabla u_2|)\mathrm{d}x \\ &\quad + \alpha\int_\Omega\left(\frac{(f-u_1)^2}{2u_1} + \frac{(f-u_2)^2}{2u_2} - \frac{f^2(u_1-u_2)^2}{2(u_1+u_2)u_1u_2}\right)\mathrm{d}x \\ &= \frac{1}{2}E(u_1) + \frac{1}{2}E(u_2) - \alpha\int_\Omega \frac{f^2(u_1-u_2)^2}{2(u_1+u_2)u_1u_2}\mathrm{d}x \\ &= E(u_1) - \alpha\int_\Omega \frac{f^2(u_1-u_2)^2}{2(u_1+u_2)u_1u_2}\mathrm{d}x \end{aligned} \tag{B.1}$$

If $u_1 \neq u_2$, then the above assumption give a contradiction that u_1 is not a minimize solution.

C. The Proof of Cubic Equation (24) Only One Positive Real Number Solution

Proof. For the cubic equation, it has three solutions. Now, we assume (z^1, z^2, z^3) are solution of cubic equation (24). According to Vieta theorem, we have

$$\begin{aligned} z^1 + z^2 + z^3 &= -\frac{\alpha - 2\beta_k^1 - 2\mu u_k}{2\mu} \\ z^1z^2 + z^1z^3 + z^2z^3 &= 0 \\ z^1z^2z^3 &= \frac{\alpha f^2}{2\mu} > 0 \end{aligned} \quad , \tag{C.1}$$

It is easy to know that the cubic equation has at least one real solution and $z^i \neq 0$ for all $i = 1,\ 2,\ 3$. Now, let us analyze the three solutions, without loss of generality, we assume $z^1 + z^2 = \xi$, $z^1z^1 = \rho$, and z^3 are a real solution.

The first case: assume z^3 is positive. So we can know $z^1z^2 = \rho > 0$ and $z^1 + z^2 = \xi < 0$. Meanwhile, (z^1, z^2) are solution of equation $z^2 - \xi z + \rho = 0$; thus, $z^1 = \xi + \sqrt{\xi^2 - 4\rho}/2$ and $z^2 = \xi - \sqrt{\xi^2 - 4\rho}/2$.

If $\xi^2 - 4\rho < 0$, then z^1 and z^2 are complex numbers. If $\xi^2 - 4\rho \ge 0$, then $\xi \pm \sqrt{\xi^2 - 4\rho}/2 < 0$. Thus, z^1 and z^2 are negative.

The second case: assume z^3 is negative. So we can know $z^1z^2 = \rho < 0$ and $z^1 + z^2 = \xi < 0$. Meanwhile, (z^1, z^2) are solution of equation $z^2 - \xi z + \rho = 0$; thus, $z^1 = \xi + \sqrt{\xi^2 - 4\rho}/2$ and $z^2 = \xi - \sqrt{\xi^2 - 4\rho}/2$.

Because of $\xi^2 - 4\rho > \xi^2 > 0$, therefore, z^1 and z^2 are real number, z^1 is positive, and z^2 is negative.

This proof is established.

Data Availability

The experimental data are obtained by MATLAB R2017a, 2.93 GHz cup, 4 G ram, and windows 7.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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Research Article

Investigations for a Type of Variable Coefficient Fractional Subdiffusion Equation with Multidelay

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A difference scheme is constructed for a type of variable coefficient time fractional subdiffusion equation with multidelay. Stability and convergence results of the scheme are obtained, and theoretical results are proved by two numerical tests.

1. Introduction

This paper will consider the following variable coefficient time fractional subdiffusion equation with arbitrary multidelay

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} - (D(x)u_x)_x &= f\left(u(x,t), u(x,t-s_1), u\left(x,t-s_q\right), x, t\right),\\ &(x,t)\in(0,1)\times(0,T],\\ u(x,t) &= \phi(x,t), x\in[0,1], t\in[-s,0],\\ u(0,t) &= \gamma(t), u(1,t) = \beta(t), t\in(0,T],\end{aligned} \tag{1}$$

where $0 < c_0 \le D(x) \le c_1$ is the variable diffusion coefficient, $s_j > 0, j = 1, 2, \cdots, q$ are delay terms, and $s = \max_{1\le i\le q}\{s_i\}$; time fractional partial derivative $(\partial^\alpha u/\partial t^\alpha)(0 < \alpha < 1)$ is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial u(x,\xi)}{\partial \xi}d\xi, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

For simplicity, we consider the following equation with arbitrary two different delays:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} - (D(x)u_x)_x &= f(u(x,t), u(x,t-s_1), u(x,t-s_2), x, t),\\ &(x,t)\in(0,1)\times(0,T],\end{aligned} \tag{3}$$

$$u(x,t) = \phi(x,t), x\in[0,1], \quad t\in[-s,0], \tag{4}$$

$$u(0,t) = \gamma(t), u(1,t) = \beta(t), t\in(0,T], \tag{5}$$

where $s_2 > s_1 > 0$ are two delay terms and satisfy $s_1 = (m_1 + \delta_1)\tau$, $s_2 = (m_2 + \delta_2)\tau$, $m_2 > m_1 > 0$ are integers, and $\delta_1, \delta_2 \in [0,1]$, $s = (m_2 + 1)\tau$.

Applications of delay differential equations (DDEs) can be found in many fields, such as electrotechnics, population dynamics, biology, and economics [1–5]. However, one can obtain the analytical solutions of DDEs in few cases. The researchers turn to numerical methods for solving DDEs in general cases [6–9]. Most studies about DDEs consider one delay; in fact, more than one delays should be considered in some systems. Tian and Kuang considered H-method and GP-stability for DDEs with several delay terms [10]; the applications of such DDEs can be found in [11–14].

Fractional delay differential equations (FDDEs) are widely used in automatic control, population dynamics,

finance, etc. [15–17]. Numerical solutions for FDDEs can be found in [18–25], where [18–20] considered numerical solutions for FDDEs, and [21–25] considered numerical solutions for fractional delay partial differential equations (FDPDEs). However, the researchers seldom considered the numerical methods for FDDEs with multidelays; some studies considered such FDDEs without theoretical analysis.

In this paper, there is an effective difference scheme for solving systems (3)–(5) with multidelay. Some stability and convergence results of the scheme are obtained by mathematical proof, and the theoretical results are proved by two numerical tests.

We organize the paper as the follows. Section 2 constructs a numerical scheme to solve (3)–(5). Section 3 provides the stability and convergence results by proof. Section 4 presents two numerical tests. Section 5 gives a brief discussion.

2. The Construction of the Second Finite Difference Method

The following assumptions are assumed to be true in this paper.

(H1) Assume $D(x)$ to be sufficiently smooth function which fulfills $0 < c_0 \le D(x) \le c_1$

(H2) Suppose that $f(u(x,t), u(x,t-s_1), u(x,t-s_2), x, t)$ is sufficiently smooth and satisfies [26, 27]

$$\begin{aligned}&|f(\mu+\varepsilon_1, \nu+\varepsilon_2, \iota+\varepsilon_3, x, t) - f(\mu, \nu, \iota, x, t)|\\ &\quad \le c_2|\varepsilon_1| + c_3|\varepsilon_2| + c_4|\varepsilon_3|,\end{aligned} \tag{6}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrary real numbers and $c_2 > 0, c_3 > 0$, and $c_4 > 0$ are three constants.

Taking $M > 0$ and $N > 0$ to be two integers, we have space and time stepsize $h = 1/M$ and $\tau = T/N$, respectively, and discrete points $x_i = ih$ and $t_k = k\tau$. Assume $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_k | -(m_2+1) \le k \le N\}$, and $U_i^k = u(x_i, t_k)$. Define the following grid function space on $\Omega_{h\tau}$:

$$\mathscr{W} = \left\{ v_i^k \mid 0 \le i \le M, -(m_2+1) \le k \le N \right\} [rgb]1.00,0.00,0.00, \tag{7}$$

We have

$$\begin{aligned}&\delta_x v_{i+1/2}^k = \frac{v_{i+1}^k - v_i^k}{h}, \delta_x^2 v_i^k = \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \delta_x (D\delta_x v)_i^k\\ &\quad = \left(D_{i+1/2}\delta_x v_{i+1/2}^k - D_{i-1/2}\delta_x v_{i-1/2}^k \right)/h,\\ &\delta_t^\alpha v^k = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[d_0 v^k - \sum_{j=1}^{k-1} \left(d_{k-j-1} - d_{k-j} \right) v^j - d_{k-1} v^0 \right],\end{aligned} \tag{8}$$

where $d_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j \ge 0$.

We introduce the following two Lemmas,

Lemma 1 [28]. *Suppose $0 < \alpha < 1$, $y \in C^2[0, t_k]$; it holds that*

$$\begin{aligned}&\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{y'(s)ds}{(t_k - s)^\alpha} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \right.\\ &\quad \left. \cdot \left[d_0 y(t_k) - \sum_{j=1}^{k-1} \left(d_{k-j-1} - d_{k-j} \right) y(t_j) - d_{k-1} y(t_0) \right] \right|\\ &\quad \le \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - \left(1 + 2^{-\alpha}\right) \right] \max_{0 \le t \le t_k} |y''(t)| \tau^{2-\alpha}.\end{aligned} \tag{9}$$

Lemma 2 [29]. *Assume $0 < \alpha < 1$; then, it holds that*

(i) d_j decrease monotonically with j increases, and $0 < d_j \le 1$

(ii) $d_0 = 1$, $\sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) = d_0 - d_{k-1}$

Considering (3) at the point (x_i, t_k), we have

$$\begin{aligned}&\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) - \frac{\partial}{\partial x} \left(D(x) \frac{\partial u(x,t)}{\partial x} \right) \Big|_{(x_i, t_k)}\\ &\quad = f(u(x_i, t_k), u(x_i, t_k - s_1), u(x_i, t_k - s_2), x_i, t_k), 0 \le i\\ &\quad \le M, 0 \le k \le N.\end{aligned} \tag{10}$$

From Lemma 1, we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = \delta_t^\alpha U_i^k + r_i^k, \tag{11}$$

where

$$r_i^k = \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - \left(1 + 2^{-\alpha}\right) \right] \max_{0 \le t \le t_k} \left| \frac{\partial^2 u(x_i, t_k)}{\partial t^2} \right| \tau^{2-\alpha}. \tag{12}$$

From Taylor's expansion, we have

$$\begin{aligned}D_{i+1/2}\delta_x U_{i+1/2}^k &= D_{i+1/2} \frac{\partial u}{\partial x}(x_{i+1/2}, t_k)\\ &\quad + \frac{h^2}{24} D_{i+1/2} \frac{\partial^3 u}{\partial x^3}(x_{i+1/2}, t_k) + O\left(h^3\right),\end{aligned} \tag{13}$$

$$\begin{aligned}D_{i-1/2}\delta_x U_{i-1/2}^k &= D_{i-1/2} \frac{\partial u}{\partial x}(x_{i-1/2}, t_k)\\ &\quad + \frac{h^2}{24} D_{i-1/2} \frac{\partial^3 u}{\partial x^3}(x_{i-1/2}, t_k) + O\left(h^3\right).\end{aligned} \tag{14}$$

Subtracting (13) and (14) and dividing the result by h, we obtain

$$\begin{aligned}\delta_x(D\delta_x U)_i^k &= \frac{1}{h}\left(D_{i+1/2}\delta_x U_{i+1/2}^k - D_{i-1/2}\delta_x U_{i-1/2}^k\right)\\ &= \frac{1}{h}\left(D_{i+1/2}\frac{\partial u}{\partial x}(x_{i+1/2},t_k) - D_{i-1/2}\frac{\partial u}{\partial x}(x_{i-1/2},t_k)\right)\\ &\quad + O(h^2) = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,t)}{\partial x}\right)\Big|_{(x_i,t_k)} + O(h^2),\end{aligned} \tag{15}$$

$$\begin{aligned}&f(u(x_i,t_k),u(x_i,t_k-s_1),u(x_i,t_k-s_2),x_i,t_k)\\ &\quad = f\left(2U_i^{k-1}-U_i^{k-2},U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right)\\ &\quad + \tau^2\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)f_\mu\left(\rho_k,U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right).\end{aligned} \tag{16}$$

where $U_i^{k-m_1-\delta_1} = (1-\delta_1)U_i^{k-m_1} + \delta_1 U_i^{k-m_1+1}$, $U_i^{k-m_2-\delta_2} = (1-\delta_2)U_i^{k-m_2} + \delta_2 U_i^{k-m_2+1}$, $\eta_i^k \in (t_k - s_2, t_k)$, and ρ_k is a real number between $u(x_i,t_k)$ and $2U_i^{k-1} - U_i^{k-2}$.

Substituting (11), (15), and (16) into (10), we obtain

$$\begin{aligned}&\delta_t^\alpha U_i^k - \delta_x(D\delta_x U)_i^k\\ &\quad = f\left(2U_i^{k-1}-U_i^{k-2},U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right) + R_i^k,\end{aligned} \tag{17}$$

where

$$\begin{aligned}R_i^k &= r_i^k + \tau^2\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)f_\mu\left(\rho_k,U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right)\\ &\quad + O(h^2).\end{aligned} \tag{18}$$

From Assumptions (H1) and (H2), we can easily obtain

$$\left|R_i^k\right| \le C_R\left(\tau^{2-\alpha}+h^2\right), \quad 0\le i\le M, 1\le k\le N. \tag{19}$$

We have the discretization of (4) and (5):

$$U_i^k = \phi(x_i,t_k), \quad 0\le i\le M, -(m_2+1)\le k\le 0, \tag{20}$$

$$U_0^k = \gamma(t_k), \quad U_M^k = \beta(t_k), \quad 1\le k\le N. \tag{21}$$

Replacing U_i^k by u_i^k in (17), (20), and (21), and omitting R_i^k, we have the following numerical scheme:

$$\begin{aligned}&\delta_t^\alpha u_i^k - \delta_x(D\delta_x u)_i^k\\ &\quad = f\left(2u_i^{k-1}-u_i^{k-2},u_i^{k-m_1-\delta_1},u_i^{k-m_2-\delta_2},x_i,t_k\right),\end{aligned} \tag{22}$$

$$u_i^k = \phi(x_i,t_k), \quad 0\le i\le M, -(m_2+1)\le k\le 0, \tag{23}$$

$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k), \quad 1\le k\le N. \tag{24}$$

3. The Solvability, Convergence, and Stability of the Difference Scheme

Assume the following to be defined on Ω_h

$$\mathscr{V}_{h,0} = \{u \mid u = (u_0,u_1,\cdots,u_M), u_0 = u_M = 0\}. \tag{25}$$

If $u, v \in \mathscr{V}_{h,0}$, the following inner products and corresponding norms are introduced

$$\begin{aligned}(u,v) &= h\sum_{i=1}^{M-1} u_i v_i,\\ \|u\| &= \sqrt{(u,u)},\\ \|u\|_\infty &= \max_{1\le i\le M-1}|u_i|,\\ \langle\delta_x u,\delta_x v\rangle &= h\sum_{i=0}^{M-1}(\delta_x u_{i+1/2})(\delta_x v_{i+1/2}),\\ |\delta_x u|_1 &= \sqrt{\langle\delta_x u,\delta_x u\rangle},\\ \langle\delta_x u,\delta_x v\rangle_D &= h\sum_{i=0}^{M-1} D(x_{i+1/2})(\delta_x u_{i+1/2})(\delta_x v_{i+1/2}),\\ |\delta_x u|_{1D} &= \sqrt{\langle\delta_x u,\delta_x u\rangle_D}.\end{aligned} \tag{26}$$

We can obtain Lemma 3 from Assumption (H1):

Lemma 3. *For $\forall u \in \mathscr{V}_{h,0}$, we have $\sqrt{c_0}|\delta_x u|_1 \le |\delta_x u|_{1D} \le \sqrt{c_1}|\delta_x u|_1$.*

Lemma 4 [30]. *For $\forall u \in \mathscr{V}_{h,0}$, one can obtain*

$$\|u\|_\infty \le |\delta_x u|_1/2, \|u\| \le |\delta_x u|_1/\sqrt{6}. \tag{27}$$

We also introduce Lemma 5 to be utilized in the following proof.

Lemma 5 [30]. *Assuming the sequence $\{F^k \ge 0 \mid k \ge 0\}$ satisfies*

$$F^{k+1} \le A + B\tau\sum_{i=1}^{k} F^l, \quad k = 0,1,\cdots, \tag{28}$$

then

$$F^{k+1} \le A\exp(Bk\tau), \quad k = 0,1,2,\cdots, \tag{29}$$

where $A \ge 0$ and $B \ge 0$ are two constants.

Theorem 6. *Under the assumptions of (H1) and (H2), the solution of the difference schemes (22)–(24) is unique.*

Proof. Difference schemes (22)–(24) can be reformed as follows:

When $k = 1$, we have

$$-\frac{D_{i-1/2}}{h^2}u_{i-1}^1 + \left(\frac{1}{\lambda} + \frac{D_{i-1/2} + D_{i+1/2}}{h^2}\right)u_i^1 - \frac{D_{i+1/2}}{h^2}u_{i+1}^1 = \frac{1}{\lambda}u_i^0 + f_i^1, \tag{30}$$

When $k \geq 2$, we have

$$\begin{aligned} &-\frac{D_{i-1/2}}{h^2}u_{i-1}^k + \left(\frac{1}{\lambda} + \frac{D_{i-1/2} + D_{i+1/2}}{h^2}\right)u_i^k - \frac{D_{i+1/2}}{h^2}u_{i+1}^k \\ &\quad = \frac{1}{\lambda}\left[\sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)u_i^j + d_{k-1}u_i^0\right] + f_i^k, \end{aligned} \tag{31}$$

where $\lambda = \tau^\alpha\Gamma(2-\alpha)$, $f_i^k = f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k)$, $k = 1, 2, \cdots$. From (30) and (31), we can see that the schemes (22)–(24) are a linear tridiagonal system with strictly diagonally dominant coefficient matrix. Thus, schemes (22)–(31) have a unique solution.

Denote $e_i^k = U_i^k - u_i^k$, $0 \leq i \leq M$, $-(m_2+1) \leq k \leq N$, subtracting (22)–(24) from (17), (20), and (21), respectively, the following error equations can be obtained:

$$\delta_t^\alpha e_i^k - \delta_x(D\delta_x e)_i^k = p_i^k + R_i^k, \tag{32}$$

$$e_i^k = 0, \quad 0 \leq i \leq M, -(m_2+1) \leq k \leq 0, \tag{33}$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 1 \leq k \leq N. \tag{34}$$

where $p_i^k = f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-m_1-\delta_1}, U_i^{k-m_2-\delta_2}, x_i, t_k) - f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k)$.

Theorem 7. *Denote $u(x,t)$ to be the solution of systems (3)–(5), and (22)–(24) has numerical solution $\{u_i^k \mid 0 \leq i \leq M, -(m_2+1) \leq k \leq N\}$. Then, we have*

$$\|e^k\|_\infty \leq C\left(\tau^{2-\alpha} + h^2\right), 1 \leq k \leq N, \tag{35}$$

where C is a positive constant independent of h and τ.

Proof. Multiplying (32) with $h\delta_t^\alpha e_i^k$, one can have

$$\begin{aligned} &h\sum_{i=1}^{M-1}\left(\delta_t^\alpha e_i^k\right)\left(\delta_t^\alpha e_i^k\right) - h\sum_{i=1}^{M-1}\left(\delta_x(D\delta_x e)_i^k\right)\left(\delta_t^\alpha e_i^k\right) \\ &\quad = h\sum_{i=1}^{M-1}p_i^k\left(\delta_t^\alpha e_i^k\right) + h\sum_{i=1}^{M-1}\left(R_i^k\right)\left(\delta_t^\alpha e_i^k\right), \\ &\qquad 1 \leq i \leq M-1, 0 \leq k \leq N-1. \end{aligned} \tag{36}$$

Then each term of (36) will be estimated.

$$h\sum_{i=1}^{M-1}\left(\delta_t^\alpha e_i^k\right)\left(\delta_t^\alpha e_i^k\right) = \left\|\delta_t^\alpha e_i^k\right\|^2. \tag{37}$$

By the discrete Green formula, we have

$$\begin{aligned} &-h\sum_{i=1}^{M-1}\left(\delta_x(D\delta_x e)_i^k\right)\left(\delta_t^\alpha e_i^k\right) \\ &= h\sum_{i=0}^{M-1}D_{i+1/2}\left(\delta_x e_{i+1/2}^k\right)\left(\delta_t^\alpha\delta_x e_{i+1/2}^k\right) \\ &= \frac{1}{\lambda}h\sum_{i=0}^{M-1}D_{i+1/2}\left(\delta_x e_{i+1/2}^k\right)\left[\delta_x e_{i+1/2}^k\right. \\ &\quad \left. - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)\delta_x e_{i+1/2}^j - d_{k-1}\delta_x e_{i+1/2}^0\right] \\ &= \frac{1}{\lambda}\left\{\left|\delta_x e^k\right|_{1D}^2 - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)\left\langle\delta_x e^k, \delta_x e^j\right\rangle_D\right. \\ &\quad \left. - d_{k-1}\left\langle\delta_x e^k, \delta_x e^0\right\rangle_D\right\} \\ &\geq \frac{1}{\lambda}\left\{\left|\delta_x e^k\right|_{1D}^2 - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)\frac{\left|\delta_x e^k\right|_{1D}^2 + \left|\delta_x e^j\right|_{1D}^2}{2}\right. \\ &\quad \left. - d_{k-1}\frac{\left|\delta_x e^k\right|_{1D}^2 + \left|\delta_x e^0\right|_{1D}^2}{2}\right\} \\ &= \frac{1}{\lambda}\left\{\frac{\left|\delta_x e^k\right|_{1D}^2}{2} - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)\frac{\left|\delta_x e^j\right|_{1D}^2}{2}\right\}. \end{aligned} \tag{38}$$

From the Cauchy-Schwarz inequality, we have

$$\begin{aligned} h\sum_{i=1}^{M-1}\left(R_i^k\right)\left(\delta_t^\alpha e_i^k\right) &\leq h\sum_{i=1}^{M-1}\left(\frac{\left(R_i^k\right)^2}{2\varepsilon} + \frac{\varepsilon}{2}\left(\delta_t^\alpha e_i^k\right)^2\right) \\ &= \frac{1}{2\varepsilon}\|R_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2. \end{aligned} \tag{39}$$

From the Cauchy-Schwarz inequality and Assumption (H2), we have

$$\begin{aligned} h\sum_{i=1}^{M-1}p_i^k\left(\delta_t^\alpha e_i^k\right) &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{2\varepsilon}h\sum_{i=1}^{M-1}\left(p_i^k\right)^2 \\ &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{2\varepsilon}h\sum_{i=1}^{M-1}\left[c_3\left|2e_i^{k-1} - e_i^{k-2}\right|\right. \\ &\quad \left. + c_4\left|e_i^{k-m_1-\delta_1}\right| + c_5\left|e_i^{k-m_2-\delta_2}\right|\right]^2 \\ &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{3c_3^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(2e_i^{k-1} - e_i^{k-2}\right)^2 \\ &\quad + \frac{3c_4^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(e_i^{k-m_1-\delta_1}\right)^2 \\ &\quad + \frac{3c_5^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(e_i^{k-m_2-\delta_2}\right)^2 \end{aligned}$$

$$\le \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{24c_3^2}{2\varepsilon}\left(\|e^{k-1}\|^2+\|e^{k-2}\|^2\right) + \frac{3c_4^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_1}\|^2,\|e^{k-m_1+1}\|^2\right\}\right) + \frac{3c_5^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_2}\|^2,\|e^{k-m_2+1}\|^2\right\}\right). \tag{40}$$

Inserting (37)–(40) into (36), we obtain

$$\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{\lambda}\left\{\frac{\left|\delta_x e^k\right|_{1D}^2}{2} - \sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\frac{\left|\delta_x e^j\right|_{1D}^2}{2}\right\} \le \frac{1}{2\varepsilon}\|R_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{24c_3^2}{2\varepsilon}\left(\|e^{k-1}\|^2+\|e^{k-2}\|^2\right) + \frac{3c_4^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_1}\|^2,\|e^{k-m_1+1}\|^2\right\}\right) + \frac{3c_5^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_2}\|^2,\|e^{k-m_2+1}\|^2\right\}\right). \tag{41}$$

Multiplying (41) by 2λ, and letting $\varepsilon = 1$, we have

$$\left|\delta_x e^k\right|_{1D}^2 \le \sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_{1D}^2 + \lambda\|R_i^k\|^2 + 24\lambda c_3^2\left(\|e^{k-1}\|^2+\|e^{k-2}\|^2\right) + 3\lambda c_4^2\left(\max\left\{\|e^{k-m_1}\|^2,\|e^{k-m_1+1}\|^2\right\}\right) + 3\lambda c_5^2\left(\max\left\{\|e^{k-m_2}\|^2,\|e^{k-m_2+1}\|^2\right\}\right). \tag{42}$$

From Lemmas 3 and 4, and noticing (19), we obtain

$$\left|\delta_x e^k\right|_1^2 \le \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_1^2 + \frac{\lambda}{c_0}C_R^2\left(\tau^{2-\alpha}+h^2\right)^2 + \frac{4\lambda c_3^2}{c_0}\left(\left|\delta_x e^{k-1}\right|_1^2 + \left|\delta_x e^{k-2}\right|_1^2\right) + \frac{\lambda c_4^2}{2c_0}\left(\max\left\{\left|\delta_x e^{k-m_1}\right|_1^2, \left|\delta_x e^{k-m_1+1}\right|_1^2\right\}\right) + \frac{\lambda c_5^2}{2c_0}\left(\max\left\{\left|\delta_x e^{k-m_2}\right|_1^2, \left|\delta_x e^{k-m_2+1}\right|_1^2\right\}\right), \tag{43}$$

denoting

$$C_k = \frac{1}{c_0}\Gamma(2-\alpha)\max\left\{C_R^2, 4c_3^2, c_4^2/2, c_5^2/2\right\} > 0. \tag{44}$$

Noticing $\lambda = \tau^\alpha\Gamma(2-\alpha)$, one can obtain $\tau^\alpha < 1$ from $0 < \tau < 1$. By (43), one can have

$$\left|\delta_x e^k\right|_1^2 \le \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_1^2 + C_k\left[\left(\tau^{2-\alpha}+h^2\right)^2 + \left|\delta_x e^{k-1}\right|_1^2 + \left|\delta_x e^{k-2}\right|_1^2 + \max\left\{\left|\delta_x e^{k-m_1}\right|_1^2, \left|\delta_x e^{k-m_1+1}\right|_1^2\right\} + \max\left\{\left|\delta_x e^{k-m_2}\right|_1^2, \left|\delta_x e^{k-m_2+1}\right|_1^2\right\}\right]. \tag{45}$$

From Lemmas 2 and 5, we have

$$\begin{aligned}\left|\delta_x e^k\right|_1^2 &\le C_k\exp\left(4C_k + \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\right)\left(\tau^{2-\alpha}+h^2\right)^2 \\ &= C_k\exp\left(4C_k + \frac{c_1}{c_0}\left(1-d_{k-1}\right)\right)\left(\tau^{2-\alpha}+h^2\right)^2 \\ &\le C_1\left(\tau^{2-\alpha}+h^2\right)^2,\end{aligned} \tag{46}$$

where $C_1 = C_k\exp\left(4C_k + (c_1/c_0)\right)$. From Lemma 4, we have

$$\|e^k\|_\infty \le \sqrt{C_1\left(\tau^{2-\alpha}+h^2\right)^2} = C\left(\tau^{2-\alpha}+h^2\right). \tag{47}$$

We complete the proof.

To discuss the stability of the difference schemes (22)–(24), we consider the following problem:

$$\frac{\partial^\alpha v}{\partial t^\alpha} - \left(D(x)v_x\right)_x = f\left(v(x,t), v(x,t-s_1), v(x,t-s_2), x, t\right), \quad (x,t)\in(0,1)\times(0,T], \tag{48}$$

$$v(x,t) = \phi(x,t) + \psi(x,t), \quad x\in[0,1], \quad t\in[-s,0], \tag{49}$$

$$v(0,t) = \gamma(t), \quad u(1,t) = \beta(t), \quad t\in(0,T], \tag{50}$$

where $\psi(x,t)$ is the perturbation caused by $\phi(x,t)$. The following difference scheme solving for (48)–(50) can be obtained:

$$\delta_t^\alpha u_i^k - \delta_x\left(D\delta_x u\right)_i^k = f\left(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k\right), \tag{51}$$

$$u_i^k = \phi(x_i,t_k) + \psi_i^k, \quad 0\le i\le M, -(m_2+1)\le k\le 0, \tag{52}$$

$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k), \quad 1\le k\le N. \tag{53}$$

Table 1: Simulation results of Example 10, where $h = 1/2000$ and $\alpha = 0.4,0.6,0.8$.

τ	$\alpha = 0.4$ $e(h,\tau)$	Rate_τ	$\alpha = 0.6$ $e(h,\tau)$	Rate_τ	$\alpha = 0.8$ $e(h,\tau)$	Rat_{e_τ}
1/50	4.244e-005	*	1.752e-004	*	6.464e-004	*
1/100	1.609e-005	1.399	6.903e-005	1.344	2.851e-004	1.181
1/150	8.950e-006	1.446	3.963e-005	1.369	1.758e-004	1.192
1/200	6.401e-006	1.165	2.724e-005	1.303	1.253e-004	1.178

Table 2: Simulation results of Example 10, where $\tau = 1/2000$ and $\alpha = 0.4,0.6,0.8$.

h	$\alpha = 0.4$ $e(h,\tau)$	Rate_h	$\alpha = 0.6$ $e(h,\tau)$	Rate_h	$\alpha = 0.8$ $e(h,\tau)$	Rate_h
1/10	6.213e-002	*	6.072e-002	*	5.889e-002	*
1/20	1.534e-002	2.018	1.500e-002	2.017	1.455e-002	2.017
1/40	3.842e-003	1.997	3.758e-003	1.997	3.654e-003	1.994
1/80	9.599e-004	2.001	9.396e-004	2.000	9.185e-004	1.992

Denote

$$\omega_i^k = v_i^k - u_i^k, \quad 0 \le i \le M, -(m_2+1) \le k \le N. \tag{54}$$

where $\bar{C}$ is a bounded constant independent of h and τ.

Definition 8. Assume u_i^k satisfy (22)–(24) and v_i^k satisfy (51)–(53), a numerical scheme for (3)–(5) is stable if we have

$$\left\|\omega^k\right\|_\infty \le \bar{C} \max_{-(m_2+1)\le j\le 0} \left|\psi^j\right|_1, \tag{55}$$

Similar to the proof of Theorem 7, the following stability result can be obtained.

Theorem 9. *Under the same condition as Theorem 7, numerical schemes (22)–(24) satisfy* $\|\omega^k\|_\infty \le \widehat{C} \max_{-(m_2+1)\le j\le 0} |\psi^j|_1$.

4. Numerical Test

Now, we present the following numerical tests to testify the schemes (22)–(24). Introducing the following notations,

$$\begin{aligned} e(h,\tau) &= \max_{1\le k\le N} \left\|U^k - u^k\right\|_\infty, \\ \text{Rate}_\tau &= \frac{\log\left(e(h,\tau_1)/e(h,\tau_2)\right)}{\log\left(\tau_1/\tau_2\right)}, \\ \text{Rate}_h &= \frac{\log\left(e(h_1,\tau)/e(h_2,\tau)\right)}{\log\left(h_1/h_2\right)}. \end{aligned} \tag{56}$$

when considering for Rate_τ, h should be fixed and small enough. While considering for Rate_h, τ should be fixed and small enough.

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \dfrac{\partial u}{\partial x}\left(\left(x^2+1\right)\dfrac{\partial u}{\partial x}\right) = \dfrac{u(x,t-0.1)}{1+u^2(x,t-0.05)} + G(x,t), \\ (x,t) \in (0,1)\times(0,T], \\ u(x,t) = t^{2+\alpha}\cos(2\pi x), \quad x \in [0,1], \quad t \in [-0.1,0], \\ u(0,t) = t^{2+\alpha}, \quad u(1,t) = t^{2+\alpha}, \quad t \in (0,1]. \end{cases} \tag{57}$$

Example 10. Let $D(x) = x^2 + 1$.

Equation (57) has exact solution $u(x,t) = t^{2+\alpha}\cos(2\pi x)$, where

$$\begin{aligned} G(x,t) = &\left[\frac{\Gamma(3+\alpha)}{\Gamma(3)}t^2 + 4\pi^2\left(x^2+1\right)t^{2+\alpha}\right]\cos(2\pi x) \\ &+ 4\pi x t^{2+\alpha}\sin(2\pi x) \\ &- \frac{(t-0.1)^{2+\alpha}\cos(2\pi x)}{1+(t-0.05)^{4+2\alpha}\cos^2(2\pi x)}. \end{aligned} \tag{58}$$

Table 1 presents the maximum errors in the temporal directions, where $\alpha = 0.4,0.6,0.8$, and $h = 1/2000$. From the results, we can draw a conclusion that the convergence order in the temporal directions is $2-\alpha$. Table 2 presents the maximum errors in the spatial directions, where $\alpha = 0.4,0.6,0.8$, and $\tau = 1/2000$. From the results, we can draw a conclusion that the convergence order in the spatial directions is 2.

Figure 1 gives the error planes of Example 10, where $\tau = h = 1/100$, and $\alpha = 0.1,0.3,0.7,0.9$. Figure 1 tells us that bigger α can bring bigger error.

Example 11. Let $D(x) = x^2 + 1$

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \dfrac{\partial u}{\partial x}\left(x^2\dfrac{\partial u}{\partial x}\right) = -u(x,t)^2 + u(x,t-0.1) + u(x,t-0.2) + G(x,t), (x,t) \in (0,1)\times(0,T], \\ u(x,t) = e^x x^2(1-x)^2 t^{2+\alpha}, \quad x \in [0,1], \quad t \in [-0.2,0], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,1]. \end{cases} \tag{59}$$

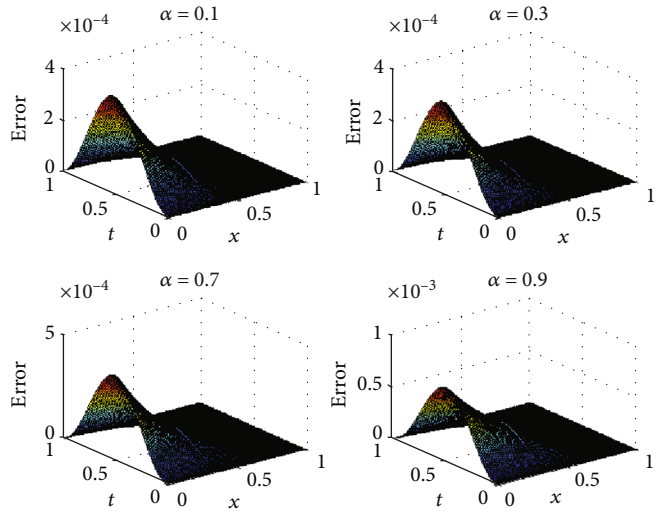

Figure 1: Error planes of Example 10, where $\tau = h = 1/100$.

Table 3: Simulation results of Example 11, where $h = 1/2000$ and $\alpha = 0.3,0.5,0.8$.

τ	$\alpha = 0.3$ $e(h,\tau)$	Rate_τ	$\alpha = 0.5$ $e(h,\tau)$	Rate_τ	$\alpha = 0.8$ $e(h,\tau)$	Rate_τ
1/10	9.029e-005	*	2.277e-004	*	9.049e-004	*
1/20	2.712e-005	1.735	7.924e-005	1.523	3.941e-004	1.199
1/40	8.183e-006	1.729	2.763e-005	1.520	1.713e-004	1.202
1/80	2.530e-006	1.693	9.707e-006	1.509	7.451e-005	1.201

Table 4: Simulation results of Example 11, where $\tau = 1/2000$ and $\alpha = 0.3,0.5,0.8$.

h	$\alpha = 0.3$ $e(h,\tau)$	Rate_h	$\alpha = 0.5$ $e(h,\tau)$	Rate_h	$\alpha = 0.8$ $e(h,\tau)$	Rate_h
1/8	8.064e-003	*	7.917e-003	*	7.624e-003	*
1/16	2.100e-003	1.941	2.057e-003	1.945	1.971e-003	1.952
1/32	5.272e-004	1.994	5.163e-004	1.994	4.957e-004	1.991
1/64	1.319e-004	1.998	1.293e-004	1.998	1.250e-004	1.987

Equation (59) has exact solution $u(x,t) = e^x x^2 (1-x)^2 t^{2+\alpha}$, where

$$\begin{aligned} G(x,t) = {} & e^{2x} x^4 (1-x)^4 t^{4+2\alpha} - e^x x^2 (1-x)^2 (t-0.1)^{2+\alpha} \\ & - e^x x^2 (1-x)^2 (t-0.2)^{2+\alpha} \\ & + \frac{\Gamma(3+\alpha)}{2} t^2 e^x x^2 (1-x)^2 \\ & - e^x t^{2+\alpha} \left(6x^2 - 18x^3 + 5x^4 + 8x^5 + x^6\right). \end{aligned} \tag{60}$$

Tables 3 and 4 show the computational results for a different α of Example 11. When $h = 1/2000$, the maximum errors and convergence orders are provided in Table 3. It can be shown that the $2-\alpha$ order accuracy in temporal direction is verified, while when $\tau = 1/2000$, from Table 4, we can see that the 2 order accuracy in spatial direction is verified.

5. Conclusion

This paper provides a finite difference scheme for solving a type of variable coefficient time fractional subdiffusion equation with multidelay. The unconditional stability and the global convergence of the scheme in the maximum norm are proved. Numerical experiments were provided to support the theoretical results and testify the efficiency of the difference scheme.

Data Availability

The author declares that the readers can access the data supporting the conclusions of the study online directly.

Conflicts of Interest

The author declares that there is no conflicts of interests regarding the publication of this article.

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Research Article

Optimal Expected Utility of Dividend Payments with Proportional Reinsurance under VaR Constraints and Stochastic Interest Rate

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In this paper, we consider the problem of maximizing the expected discounted utility of dividend payments for an insurance company taking into account the time value of ruin. We assume the preference of the insurer is of the CRRA form. The discounting factor is modeled as a geometric Brownian motion. We introduce the VaR control levels for the insurer to control its loss in reinsurance strategies. By solving the corresponding Hamilton-Jacobi-Bellman equation, we obtain the value function and the corresponding optimal strategy. Finally, we provide some numerical examples to illustrate the results and analyze the VaR control levels on the optimal strategy.

1. Introduction

In recent years, dividend optimization problems for insurance company have attracted extensive attention. The problem of optimal dividend was proposed by De Finetti in 1957. He suggested that a company would seek to find a strategy in order to maximize the accumulated value of expected discounted dividends up to the ruin time. Since then, many results on optimal dividend problems have been obtained. Some literatures on the optimal dividend include Avanzi et al. [1], Avanzi et al. [2], Avram et al. [3], Bayraktar et al. [4], Dai et al. [5], Dai et al. [6], Hubalek and Schachermayer [7], Li and Wu [8], Loeffen [9], Ng [10], Thonhauser and Albrecher [11], Yao et al. [12], and Yin and Wen [13].

On the other hand, reinsurance can effectively reduce the risk, so reinsurance is an important way for an insurer to control its risk exposure. Many interesting results have been obtained under reinsurance strategy in recent years. See, for example, Azcue and Muler [14], who studied the optimal reinsurance in the framework of Cramér-Lundberg model. Zhou and Yuen [15] considered reinsurance for a diffusion model with capital injection under variance premium principle. For some related discussions, among others, we refer the reader to Li and Shen [16], Liang and Palmowski [17], Wen and Yin [18], Wu [19], and Yao et al. [20].

In practice, Value-at-Risk (VaR) is often used to measure and control the risk of an insurance company. Recently, Chen et al. [21] investigated the optimal reinsurance strategies and the minimum probability of ruin with VaR constraints. Zhang et al. [22] discussed the optimal reinsurance strategies and the survival probability under dynamic VaR constraints. Bi and Cai [23] studied the equilibrium strategies under the mean-variance criterion with VaR constraints. Other works about optimal reinsurance problems with VaR constraints can be found in Cai and Tan [24], Cai et al. [25], Cheung et al. [26], and Yiu [27].

The interest rate is the main factor of an insurance company. And, it is affected by many factors. So, we must pay attention to the current situation of interest rate and its changing trend. It is more reasonable to assume that the interest rate is a function of time. Eisenberg [28] solved the optimal dividends under a stochastic interest rate for the case of the Brownian motion. In this paper, we are going to study the problem of maximizing the expected discounted utility of dividend, taking into account both reinsurance under VaR constraints and a stochastic interest rate. The insurer's

surplus process is approximated by a Brownian motion with drift. By solving the corresponding Hamilton-Jacobi-Bellman equation, we obtain the value function and the corresponding optimal strategies with and without VaR constraints.

This paper is organized as follows. In Section 2, we provide a general formulation of the optimal reinsurance-dividend problem. Then, we investigate the problem of the expected discounted utility of dividend maximization under noncheap reinsurance assumption. We give the explicit expressions for the expected discounted utility of dividend payments and the optimal strategies in Section 3. In Section 4, we consider the optimization problem with VaR constraints and solve the optimization problem using the results derived in Section 3. We also present some numerical examples to illustrate the results in Section 5.

2. Model Settings and Problem Formulations

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

2.1. Reserve Process of an Insurer. The reserve process $\{R_t\}_{t\geq 0}$ of an insurer is modeled by

$$dR_t = (1+\theta)adt - dY_t,\ R_0 = x \tag{1}$$

for the aggregate cumulative amount of claims counted up to time t:

$$dY_t = adt - bdW_t, \quad Y_0 = 0, \tag{2}$$

where a and b are positive constants, $x \geq 0$ is the initial surplus, $(1+\theta)a$ is the premium rate with the safety loading $\theta > 0$, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion.

In the following, the insurer is allowed to pay dividends. The cumulated dividends are described by a process $C = (C_t)_{t\geq 0}$, which is an adapted and nondecreasing process. C_t represents the total dividends up to time t. In this paper, we assume that $(C_t)_{t\geq 0}$ is absolutely continuous with respect to the Lebesgue measure. We suppose that the process C admits almost surely a density process denoted by $c_t \geq 0$ modeling the intensity of the dividend payments in continuous time. To manage the risk, the insurer could choose a reinsurance policy q_t at time t. When $q_t \in [0,1]$, it means the insurer purchases proportional reinsurance. In this case, the insurer only pays its $(1-q_t)Y_t$, while the reinsurer pays the rest $q_t Y_t$. A strategy π is said to be admissible, if the ruin cannot be caused by the dividend payment, and (q_t, c_t) is $\mathcal{F}_t$-progressively measurable and satisfies $0 \leq q_t \leq 1$, $c_t \geq 0$ for all $t \geq 0$. We denote the set of all admissible strategies by Π. The set of admissible strategies consists of nonnegative, nondecreasing, adapted, cádlág process. Let X_t^π denote the insurers total wealth at time t under the strategy $\pi_t = (q_t, c_t)$. Then, the dynamic of X_t^π is given by

$$\begin{aligned} dX_t^\pi &= ((1+\theta)a - (1-q_t)a)dt + b(1-q_t)dW_t \\ &\quad - (1+\eta)q_t a dt - c_t d = (\theta - \eta q_t)a dt \\ &\quad + b(1-q_t)dW_t - c_t dt,\ t \geq 0, \end{aligned} \tag{3}$$

where $\eta(>\theta)$ is the safety loading of the reinsurer and the condition of $\eta > \theta$ is required for avoiding the insurers arbitrage.

As a risk measure, we consider that dividends are discounted by the geometric Brownian motion

$$e^{\{-r-mt-\delta B_t\}}, \quad \text{if} \quad m > \frac{\delta^2}{2}, \tag{4}$$

where $r > 0$, $m > 0$, and $\delta \geq 0$. Let further $\{B_t\}_{t\geq 0}$ be a standard Brownian motion independent of $\{W_t\}_{t\geq 0}$. We suppose that the underlying filtration $\{\mathcal{F}_t\}$ is generated by $\{W_t, B_t\}$. The time of ruin for this process is defined by $\tau^\pi = \inf\{t \geq 0 : X_t^\pi \leq 0\}$. We define the return function corresponding to π to be

$$V^\pi(r,x) = \mathbb{E}_x\left[\int_0^{\tau^\pi} e^{-r-ms-\delta B_s} u(c_s)ds + \int_0^{\tau^\pi} e^{-r-ms-\delta B_s}\lambda ds\right], \tag{5}$$

where u is some fixed utility function, $\lambda > 0$, $\mathbb{E}_x$ means expectation with respect to $\mathbb{P}_x(\cdot) = P(\cdot | X_0 = x)$. Usually, it is assumed that $u : \mathbb{R}_{t\geq 0} \to \mathbb{R}_{t\geq 0}$ is differentiable and nonnegative. $e^{-\beta s}\lambda$ can be interpreted as the present value of an amount which the insurer earns as long as the company is alive. In this way, the lifetime of the portfolio becomes part of the value function and is weighted according to the choice of λ. We seek to find an optimal strategy π^* so that

$$V(r,x) = \max_{\pi\in\Pi} V^\pi(r,x) = V^{\pi^*}(r,x). \tag{6}$$

In the present paper, we consider the Constant Relative Risk Aversion (CRRA) utility function:

$$u(c) = \frac{c^p}{p}, \quad p \in (0.1). \tag{7}$$

2.2. Value-at-Risk Constraints for the Proportional Reinsurance and Dividend Strategy. Under the reinsurance-dividend strategy $\pi_t = (q_t, c_t)$, the insurer's wealth process $\{X_t^\pi, t \geq 0\}$ is a risk process. The insurer may or has to use the risk measure of VaR to control its wealth for avoiding huge loss. For $h > 0$ is small enough and for s in the interval $[t, t+h)$, we approximate π_s by π_t, $s \in [t, t+h)$. This is a reasonable approximation since the insurance strategy can only be adjusted at discrete time, and the decision is

made based on the surplus at time t. Thus, the loss of the insurer in time interval $[t, t+h]$ can be expressed as $\Delta X_{t,h}^{\pi} := X_t^{\pi} - X_{t+h}^{\pi}$. Then, the SDE (3) admits a solution

$$X_{t+h}^{\pi} = X_t^{\pi} + \int_t^{t+h} (\theta - \eta q_s) a ds + \int_t^{t+h} b(1-q_s) dW_s - \int_t^{t+h} c_s d = X_t^{\pi} + (\theta - \eta q_t) ah - c_t h + b(1-q_t) \int_t^{t+h} dW_s. \tag{8}$$

Thus,

$$\Delta X_{t,h}^{\pi} = -(\theta - \eta q_t) ah + c_t h - b(1-q_t) \int_t^{t+h} dW_s. \tag{9}$$

For a given risk level $k \in (0,1)$ and a given horizon $h > 0$, we denote the conditional VaR of $\Delta X_{t,h}^{\pi}$ conditioning on $\mathscr{F}_t$ by $VaR_t^{k,h,\pi}$, namely,

$$\mathrm{VaR}_t^{k,h,\pi} := \inf \left\{ L \in \mathbb{R}; \quad \mathbb{P}\left(\Delta X_{t,h}^{\pi} \geq L \mid \mathscr{F}_t\right) \leq k \right\}. \tag{10}$$

Proposition 1. *Given risk level $k \in (0,1)$ and time length $h > 0$. We have*

$$VaR_t^{k,h,\pi} = c_t h - (\theta - \eta q_t) ah - \Phi^{-1}(k) b(1-q_t) \sqrt{h}, \tag{11}$$

where $\Phi^{-1}(\cdot) = \inf\{x \in \mathbb{R} : \Phi(x) \geq k\}$ is the inverse function of the cumulative standard normal distribution function $\Phi(x) = \int_{-\infty}^{x} (1/\sqrt{2\pi}) e^{-z^2/2} dz$.

Proof. We have

$$\begin{aligned} &\mathbb{P}\left(\Delta X_{t,h}^{\pi} \geq L \mid \mathscr{F}_t\right) \\ &= \mathbb{P}\left(-(\theta - \eta q_t) ah + c_t h - b(1-q_t) \int_t^{t+h} dW_s \geq L \mid \mathscr{F}_t \right) \\ &= \mathbb{P}\left(b(1-q_t) \int_t^{t+h} dW_s \leq c_t h - (\theta - \eta q_t) ah - L \mid \mathscr{F}_t \right) \\ &= \mathbb{P}\left(\frac{1}{\sqrt{h}} \int_t^{t+h} dW_s \leq \frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b (1-q_t)} \mid \mathscr{F}_t \right) \\ &= \Phi\left(\frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b (1-q_t)} \right), \end{aligned} \tag{12}$$

where the last equality follows from the fact that $1/\sqrt{h} \int_t^{t+h} d W_s$ conditional on the filtration $\mathscr{F}_t$ is a standard normal random variable. Then, we have

$$\begin{aligned} \mathbb{P}\left(\Delta X_{t,h}^{\pi} \geq L \mid \mathscr{F}_t\right) \leq k &\Leftrightarrow \Phi\left(\frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b (1-q_t)} \right) \leq k \\ &\Leftrightarrow L \geq c_t h - (\theta - \eta q_t) ah \\ &\quad - \Phi^{-1}(k) b (1-q_t) \sqrt{h}. \end{aligned} \tag{13}$$

Thus, we obtain (11).

In this paper, we will derive the optimal strategy under the constraint

$$\mathrm{VaR}_t^{k,h,\pi} \leq \bar{\mathrm{VaR}}, \quad t \geq 0. \tag{14}$$

3. Solution to the Optimization Problem without VaR Constraints

In this section, we will solve the optimal reinsurance-dividend problem without VaR constraints. The HJB equation corresponding to the problem (6) is given by

$$\max_{0 \leq q \leq 1, c \geq 0} \left\{ [(\theta - \eta q) a - c] V_x + \frac{1}{2} b^2 (1-q)^2 V_{xx} + m V_r + \frac{\delta^2}{2} V_{rr} + e^{-r}(u(c) + \lambda) \right\} = 0, \tag{15}$$

with the boundary condition

$$V(0,0) = 0. \tag{16}$$

We conjecture that the value function can be written in the form $V(r,x) = e^{-r} F(x)$. In this case, the HJB equation becomes

$$\max_{0 \leq q \leq 1, c \geq 0} \left\{ [(\theta - \eta q) a - c] F'(x) + \frac{1}{2} b^2 (1-q)^2 F''(x) + \left(\frac{\delta^2}{2} - m \right) F(x) + u(c) + \lambda \right\} = 0. \tag{17}$$

Note that supremum in (17) without constraints is realized for

$$\begin{aligned} q(x) &= 1 + \frac{\eta a}{b^2} \frac{F'(x)}{F''(x)}, \\ c(x) &= \left(F'(x) \right)^{-\frac{1}{1-p}}. \end{aligned} \tag{18}$$

Denote $\alpha = 1 + (2b^2(m - \delta^2/2)/\eta^2 a^2)$. Let

$$B = -\frac{2b^2}{\eta^2 a^2} \frac{1-p}{1-\alpha+\alpha p} > 0, \tag{19}$$

$$D = \frac{2b^2}{\alpha \eta^2 a} (\eta - \theta) > 0. \tag{20}$$

We will solve our optimization problem under the assumption:

Assumption 2. For noncheap reinsurance, assume that $\eta > \theta$, $\alpha(1-p) > 1$ and $(2-\alpha(1-p))\eta - 2\theta < 0$.

For the optimal value function $V(r,x)$, we can derive the following property.

Lemma 3. *The optimal value function $V(r,x)$ satisfies $V_x \geq 0$, $V_r \leq 0$, and $V_{xx} < 0$.*

Proof. It is easy to see that $V(r,x)$ is increasing in x and decreasing in r. We prove the property of $V_{xx} < 0$. For $0 < y < x$, denote $\tau_y^\pi = \inf\{t : X_t^\pi = y\}$. From the dynamic programming principle, we claim that $V(r,x)$ satisfies the following equality:

$$V(r,x) = \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{21}$$

We first show the $\leq$ part. Let $\varepsilon > 0$. There exists $\tilde{\pi} \in \Pi$ such that $V^{\tilde{\pi}}(r,x) > V(r,x) - \varepsilon$. We choose the strategy $\bar{\pi} = \pi \mathbf{1}_{s\in[0,\tau_y)} + \tilde{\pi}\mathbf{1}_{s\in[\tau_y,\tau)}$ where π is an admissible strategy. Then,

$$\begin{aligned}
V(r,x) &\geq V^{\bar{\pi}}(r,x) \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) d + \int_{\tau_y^\pi}^{\tau^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds \right] \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + \int_0^{\tau^\pi - \tau_y^\pi} e^{-r-m\left(s+\tau_y^\pi\right) - \delta B_{s+\tau_y^\pi}} \right. \\
&\quad \left. \cdot \left(u\left(c_{s+\tau_y^\pi}\right) + \lambda \right) ds \right] \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V^{\bar{\pi}}(r,y) \right] \\
&\geq \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V(r,y) \right] - \varepsilon,
\end{aligned} \tag{22}$$

and therefore, we have

$$V(r,x) \geq \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{23}$$

For the other direction, since

$$\begin{aligned}
V^\pi(r,x) &= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V^\pi(r,y) \right] \\
&\leq \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V(r,y) \right],
\end{aligned} \tag{24}$$

we have

$$V(r,x) \leq \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{25}$$

Thus, (21) is proved.

For $h > 0$, let Π^h be the set of strategies π such that

$$\int_0^\zeta (\theta - \eta q_s) a ds + \int_0^\zeta b(1-q_s) dW_s - \int_0^\zeta c_s ds = -h, \tag{26}$$

on the set $\zeta < \infty$, where ζ is a stopping time defined by

$$\zeta = \inf \left\{ t \geq 0 : \int_0^t (\theta - \eta q_s) a ds + \int_0^t b(1-q_s) dW_s - \int_0^t c_s ds = -h \right\}. \tag{27}$$

By putting $h = x - y$ from (21), we obtain that

$$V(r,x) = \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + e^{-m\zeta} V(r,y) \right]. \tag{28}$$

Then,

$$\begin{aligned}
V(r,x) - V(r,x-h) &= \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds + \left(e^{-m\zeta} - 1 \right) V(r,x-h) \right] \\
&= \max_{\pi\in\Pi} \left\{ \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s} (u(c_s) + \lambda) ds \right] \right. \\
&\quad \left. + \mathbb{E}_x \left[e^{-m\zeta} - 1 \right] V(r,x-h) \right\}.
\end{aligned} \tag{29}$$

Since $\mathbb{E}_x[e^{-m\zeta}] < 1$ and $V(r,x)$ is a nondecreasing function of x, the right-hand side of (29) is a decreasing function of x. Thus, $V(r,x) - V(r,x-h)$ is decreasing in x. Hence, V_x is also decreasing and $V_{xx} < 0$. This finishes the proof.

Proposition 4. *If Assumption 2 is true, then on $(0, x^*]$, the function v solving (15) is given by*

$$v(r,x) = \frac{e^{-r}}{m - \delta^2/2} \left\{ \left[\left(\frac{1}{\alpha} - 1 \right) (\eta - \theta) a + \frac{(1-p)^2(1-\alpha)}{p(1-\alpha+\alpha p)} e^{\frac{\xi}{1-p}} \right] e^{-\xi} + \lambda \right\}, \tag{30}$$

where B and D are given in (19) and (20) and $\xi = g^{-1}(x)$, where g^{-1} is the inverse of function g:

$$g(\xi) = (1-p) B e^{\frac{\xi}{1-p}} + D\xi + Q_1, \tag{31}$$

for $Q_1 = -(1-p)Be^{\xi_0/1-p} - D\xi_0$ *and* ξ_0 *satisfies*

$$e^{-\xi_0}\frac{1-\alpha}{\alpha}(\eta-\theta)a + \frac{(1-p)^2(1-\alpha)}{p(1-\alpha-\alpha p)}e^{\frac{p\xi_0}{1-p}} + \lambda = 0. \tag{32}$$

Above

$$x^* = g(\xi^*), \tag{33}$$

where

$$\xi^* = (1-p)\ln\left(\frac{b^2-\eta aD}{\eta aB}\right). \tag{34}$$

The maximizers $q^*(x)$ and $c^*(x)$ are given by

$$\begin{aligned} q^*(x) &= 1 - \frac{\eta a}{b^2}g'(\xi), \\ c^*(x) &= (e^r v_x)^{-\frac{1}{1-p}}. \end{aligned} \tag{35}$$

Proof. From the expression of $q(x) = 1 + (\eta a/b^2)(F'(x)/F''(x))$ and $c(x) = (F'(x))^{-1/1-p}$, we have that the HJB equation of (17) becomes

$$\begin{aligned} &(\theta-\eta)aF'(x) - \frac{\eta^2a^2}{2b^2}\frac{\left(F'(x)\right)^2}{F''(x)} + \left(\frac{\delta^2}{2}-m\right)F(x) \\ &\quad + \frac{1-p}{p}\left(F'(x)\right)^{-\frac{p}{1-p}} + \lambda = 0. \end{aligned} \tag{36}$$

Letting $x = g(\xi)$ and $F'(g(\xi)) = e^{-\xi}$, then we have $F''(g(\xi)) = -e^{-\xi}/g'(\xi)$. Plugging it into (36) and taking derivatives with ξ produce

$$(\theta-\eta)ae^{-\xi} + \frac{\eta^2a^2}{2b^2}g'(\xi)e^{-\xi} + \left(\frac{\delta^2}{2}-m\right)F(g(\xi)) + \frac{1-p}{p}e^{\frac{p}{1-p}\xi} + \lambda = 0. \tag{37}$$

We take derivatives with ξ, and we get

$$\frac{\eta^2a^2}{2b^2}g''(\xi) - \left(\frac{\eta^2a^2}{2b^2} + m - \frac{\delta^2}{2}\right)g'(\xi) - (\theta-\eta)a + e^{\frac{1}{1-p}\xi} = 0. \tag{38}$$

Denote $h(\xi) = g'(\xi)$, we have

$$h(\xi) = g'(\xi) = Ae^{\alpha\xi} + Be^{\frac{\xi}{1-p}} + D, \tag{39}$$

where $A = h(0) - (2b^2/\alpha\eta^2 a)(\eta-\theta) + (2b^2/\eta^2a^2)(1-p/1-\alpha+\alpha p)$. In view of $F''(x) < 0$, the value of A should be nonnegative. Then, $h(\xi) > 0$. If A is strictly positive, from (37) and (38), we get $F(x) \sim K_0 x^{\alpha-1/\alpha}$, where K_0 is some fixed constant. Note that

$$\begin{aligned} F(x) &\le \mathbb{E}\left\{\int_0^\infty \left[e^{r-ms-\delta B_s}u(x+(\theta+1)as+bB_s)+\lambda\right]ds\right\} \\ &\le Kx^p + \frac{\lambda e^{-r}}{m-\delta^2/2}, \end{aligned} \tag{40}$$

where $K > 0$ is some fixed constant. Therefore, by Assumption 2, we must choose A to be equal zero. Recall that $h(\xi) = g'(\xi)$ and (37), we obtain the representation of the function

$$\begin{aligned} F(x) &= \frac{\lambda}{m-\delta^2/2} + \frac{e^{-\xi}}{m-\delta^2/2}\left[-(\eta-\theta)a + \frac{\eta^2a^2}{2b^2}g'(\xi) + \frac{1-p}{p}e^{\frac{\xi}{1-p}}\right] \\ &= \frac{\lambda}{m-\delta^2/2} + \frac{e^{-\xi}}{m-\delta^2/2}\left[-(\eta-\theta)a + \frac{\eta^2a^2}{2b^2}Be^{\frac{\xi}{1-p}} + \frac{\eta^2a^2}{2b^2}D + \frac{1-p}{p}e^{\frac{\xi}{1-p}}\right]. \end{aligned} \tag{41}$$

Thus, we have $v(r,x)$ given in (30). Since $F(0) = 0$ and $g'(\xi_0) = Be^{\xi_0/1-p} + D$, we get (31) and (32).

Since

$$\begin{cases} q(x) = 1 - \dfrac{\eta a}{b^2}g'(\xi), \\ g'(\xi) > 0, \end{cases} \tag{42}$$

we get $q(x) < 1$. Since $g''(\xi) = (B/1-p)e^{\xi/1-p} > 0$, to satisfy the required condition $q(x) > 0$, we need to show that the inequality $(\eta a/b^2)g'(\xi) < 1$ is true. It is easy to verify that ξ^* given in (34) is the unique solution of the equation $(\eta a/b^2)g'(\xi) = 1$. In view of Assumption 2 and $(\eta a/b^2)g'(\xi) = 2(\eta-\theta)/\alpha\eta(1-p)$, we have $(\eta a/b^2)g'(\xi_0) < 1$, so we have $0 < g'(\xi) < b^2/\eta a$ for $\xi_0 < \xi < \xi^*$, i.e., $0 < q(x) < 1$ on $(0, x^*]$.

It follows from the facts $F'(x) = e^{-\xi} > 0$ and $F''(x) = -e^{-\xi}/g'(\xi) < 0$ that $F(x)$ is increasing and strictly concave. Substituting the values of F' and F'' into the left of (17), we have

$$\begin{aligned} &\max_{0\le q\le 1, c\ge 0}\left\{(\theta-\eta q)aF'(x) + \frac{1}{2}b^2(1-q)^2F''(x) \right. \\ &\qquad \left. + \left(\frac{\delta^2}{2}-m\right)F(x) - cF'(x) + u(c) + \lambda\right\} \\ &\quad = \max_{0\le q\le 1, c\ge 0}\left\{(\theta-\eta q)ae^{-\xi} + \frac{1}{2}b^2(1-q)^2\frac{e^{-\xi}}{g'(\xi)} \right. \\ &\qquad \left. + \left(\frac{\delta^2}{2}-m\right)F(x) - ce^{-\xi} + u(c) + \lambda\right\}. \end{aligned} \tag{43}$$

Since $u(\cdot)$ is an increasing and concave function and $0 < g'(\xi) < b^2/\eta a$ for $\xi_0 < \xi < \xi^*$, therefore, the maximizers

q^* and c^* are given by $q^*(x) = 1 - (\eta a/b^2)g'(\xi)$ and $c(x) = e^{\xi/1-p}$ with $x = g(\xi)$. Hence, Equation (43) can be rewritten as

$$(\theta - \eta)ae^{-\xi} + \frac{\eta^2 a^2}{2b^2} g'(\xi)e^{-\xi} + \left(\frac{\delta^2}{2} - m\right)F(x) + \frac{1-p}{p}e^{\frac{p}{1-p}\xi} + \lambda = 0. \tag{44}$$

It follows from (30) and (31) that (44) is true. This finishes the proof.

Since $q^*(x) \le 0$ on the interval (x^*, ∞), it is reasonable to let $q^*(x) = 0$; then, the HJB equation is equivalent to the following equation:

$$\theta a v_x + \frac{1}{2}b^2 v_{xx} + m v_r + \frac{\delta^2}{2} v_{rr} - c v_x + e^{-r}u(c) + \lambda = 0. \tag{45}$$

We also conjecture that the value function can be written in the form $V(x, r) = e^{-r}F(x)$. In this case, the HJB equation of (45) becomes

$$\theta a F'(x) + \frac{1}{2}b^2 F''(x) - \left(m - \frac{\delta^2}{2}\right)F(x) + \frac{1-p}{p}\left(F'(x)\right)^{\frac{-p}{1-p}} + \lambda = 0. \tag{46}$$

with the boundary conditions

$$F'(x^*) = e^{-\xi^*}, \tag{47}$$

$$\frac{F'(x^*)}{F''(x^*)} = -\frac{b^2}{\eta a}. \tag{48}$$

If we set $F(x) = G(x) + (\lambda/m - \delta^2/2)$, then the HJB equation of (46) becomes

$$\theta a G'(x) + \frac{1}{2}b^2 G''(x) - \left(m - \frac{\delta^2}{2}\right)G(x) + \frac{1-p}{p}\left(G'(x)\right)^{\frac{-p}{1-p}} = 0, \tag{49}$$

with the boundary conditions

$$G'(x^*) = e^{-\xi^*} \text{ and } \frac{G'(x^*)}{G''(x^*)} = -\frac{b^2}{\eta a}. \tag{50}$$

Similar to the proof of Proposition 5 in Liang and Palmowski [17], we have the following results whose proof is omitted.Summarizing Propositions 4 and 5 gives the following result.

Proposition 5. *Assume that Assumption 2 holds. Suppose also that F on (x^*, ∞) solves (46)–(48). Then $F'(x) > 0$, $F''(x) < 0$ on (x^*, ∞), and the the maximum in*

$$\max_{0 \le q \le 1}\left\{(\theta - \eta q)aF'(x) + \frac{1}{2}b^2(1-q)^2F''(x)\right\} = 0 \tag{51}$$

is attained when $q = 0$ in this case.

Theorem 6. *Assume that Assumption 2 holds. Then the value function $V(r, x)$ is given by*

$$V(r,x) = \begin{cases} \dfrac{e^{-r}}{m - \delta^2/2}\left\{\left[\left(\dfrac{1}{\alpha} - 1\right)(\eta - \theta)a + \dfrac{(1-p)^2(1-\alpha)}{p(1-\alpha+\alpha p)}e^{\frac{\xi}{1-p}}\right]e^{-\xi} + \lambda\right\}, & \text{if } 0 < x \le x^*, \\ e^{-r}F(x), & \text{if } x > x^*, \end{cases} \tag{52}$$

with B and D given in (19) and (20) and $x = g(\xi)$, where g is defined in (31). Moreover, x^* is given in (33), and $F(x)$ solves (46). The corresponding optimal dividend strategy is

$$c^*(x) = (e^r v_x)^{\frac{-1}{1-p}}, \tag{53}$$

and the optimal reinsurance proportion is

$$q^*(x) = \begin{cases} 1 - \dfrac{\eta a}{b^2}g'(\xi), & \text{if } 0 < x \le x^*, \\ 0, & \text{if } x > x^*. \end{cases} \tag{54}$$

Proof. If $0 < x \le x^*$, denote $V(r, x) = e^{-r}F(x)$, where

$$F(x) = \frac{\lambda}{m - \delta^2/2} + \frac{e^{-\xi}}{m - \delta^2/2}\left[-(\eta - \theta)a + \frac{\eta^2 a^2}{2b^2}Be^{\frac{\xi}{1-p}} + \frac{\eta^2 a^2}{2b^2}D + \frac{1-p}{p}e^{\frac{\xi}{1-p}}\right]. \tag{55}$$

Since $F(x)$ is a continuously differentiable solution of Equation (17), it is easy to obtain that $e^{-r}F(x)$ is a solution of (15). If $x > x^*$, denote $V(r, x) = e^{-r}F(x)$, where $F(x)$ is a continuously differentiable solution of Equation (36); it is easy to obtain that $e^{-r}F(x)$ is a solution of (15).

Table 1: Parameters of the insurance market and the geometric Brownian motion.

a	b	p	η	θ	r	m	δ	λ
0.03	0.5	0.01	0.8	0.4	1	2	1	0.5

From now on, we prove the optimality of strategy

$$\pi^* = (q_t^*, c_t^*) = \left(q\left(X_t^{\pi^*}\right), c\left(X_t^{\pi^*}\right)\right). \tag{56}$$

Let π be an admissible strategy with dividend and reinsurance strategy. Select $0 < \xi_1 < x < \xi_2$. Define

$$\tau_x = \inf\{t \geq 0 : X_t^\pi = x\}. \tag{57}$$

It follows from the Itô's formula that we have

$$\begin{aligned} e^{-r-m\left(t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}\right)-\delta B_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}} F\left(X^\pi_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}\right) &= \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}\left\{F'(X_s^\pi)(\theta-\eta q_s)a + \frac{1}{2}b^2(1-q_s)^2F''(X_s^\pi) - c_sF'(X_s^\pi)\right\}ds \\ &\quad - \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}\left(m-\frac{\delta^2}{2}\right)F(X_s^\pi)d + \delta\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F(X_s^\pi)dB_s \\ &\quad + \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F'(X_s^\pi)b(-q_s)dW_s \leq e^{-r}F(x) + \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F'(X_s^\pi)b(1-q_s)dW_s \\ &\quad + \delta\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F(X_s^\pi)dB_s - \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds. \end{aligned} \tag{58}$$

Taking expectations on the both sides of the above inequality yields

$$\begin{aligned} &\mathbb{E}_x\left[e^{-r-m\left(t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}\right)-\delta B_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}}F\left(X^\pi_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}\right)\right] \\ &\quad \leq e^{-r}F(x) - \mathbb{E}_x\left[\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds\right]. \end{aligned} \tag{59}$$

Note that the $\pi = \pi^*$ equality holds. We claim that $\{e^{-r-mt-\delta B_t}F(X_t^\pi)\}_{t\geq 0}$ is uniformly integrable. Then, letting $\xi_1 \downarrow 0$, $\xi_2 \to \infty$, and by applying the dominated convergence theorem and the monotone convergence theorem, we have

$$\mathbb{E}_x\left[\int_0^\tau e^{-r-ms-\delta B_s}(u(c_s^*)+\lambda)ds\right] = e^{-r}F(x). \tag{60}$$

Thus,

$$e^{-r}F(x)\begin{cases} \geq V^\pi(r,x), & \text{if } \pi \text{ arbitrary}, \\ = V^{\pi^*}(r,x), & \text{if } \pi = \pi^*. \end{cases} \tag{61}$$

And, we can conclude that $V(r,x) \geq V^{\pi^*}(r,x) = e^{-r}F(x) \geq \sup_\pi V^\pi(r,x) = V(r,x)$. By the concavity of F, we have

$$F(x) \leq F'(0)x. \tag{62}$$

Thus, for each $T \geq 0$,

$$\begin{aligned} \mathbb{E}_x\left[e^{-r-mT-\delta B_T}F\left(X_T^{\pi^*}\right)\right] &\leq \mathbb{E}_x\left[e^{-r-mT-\delta B_T}\sup_{0\leq t\leq T}F\left(X_t^{\pi^*}\right)\right] \\ &\leq \mathbb{E}_x\left[e^{-r-mT-\delta B_T}F'(0)\sup_{0\leq t\leq T}F\left(X_t^{\pi^*}\right)\right] \\ &\leq e^{-r-\left(m-\frac{\delta^2}{2}\right)T}F'(0)\mathbb{E}_x\left[\sup_{0\leq t\leq T}X_t^{\pi^*}\right] \\ &\leq F'(0)\mathbb{E}_x\left[\sup_{0\leq t\leq T}X_t^{\pi^*}\right]. \end{aligned} \tag{63}$$

We introduce now a new stochastic process Y_t having the following dynamics:

$$dY_t = \theta a dt + b(1-q_t^*)dW_t, \quad Y_0 = x. \tag{64}$$

Then, one can observe that $\{Y_t\}_{t\geq 0}$ is a submartingale and $X_t^{\pi^*} \leq Y_t$ for any $t \geq 0$. We denote $Y_T^* = \sup_{0\leq t\leq T}Y_t$. Then, by the Cauchy-Swartz inequality and Doob's maximal inequality for submartingale, we have

$$\begin{aligned} \mathbb{E}_x\left[\sup_{0\leq t\leq T}X_t^{\pi^*}\right] &\leq \mathbb{E}_x(Y_T^*) \leq \sqrt{\mathbb{E}_x(Y_T^*)^2} \leq 2\sqrt{\mathbb{E}_x(Y_T)^2} \\ &\leq K_1T + K_2\sqrt{T} < \infty, \end{aligned} \tag{65}$$

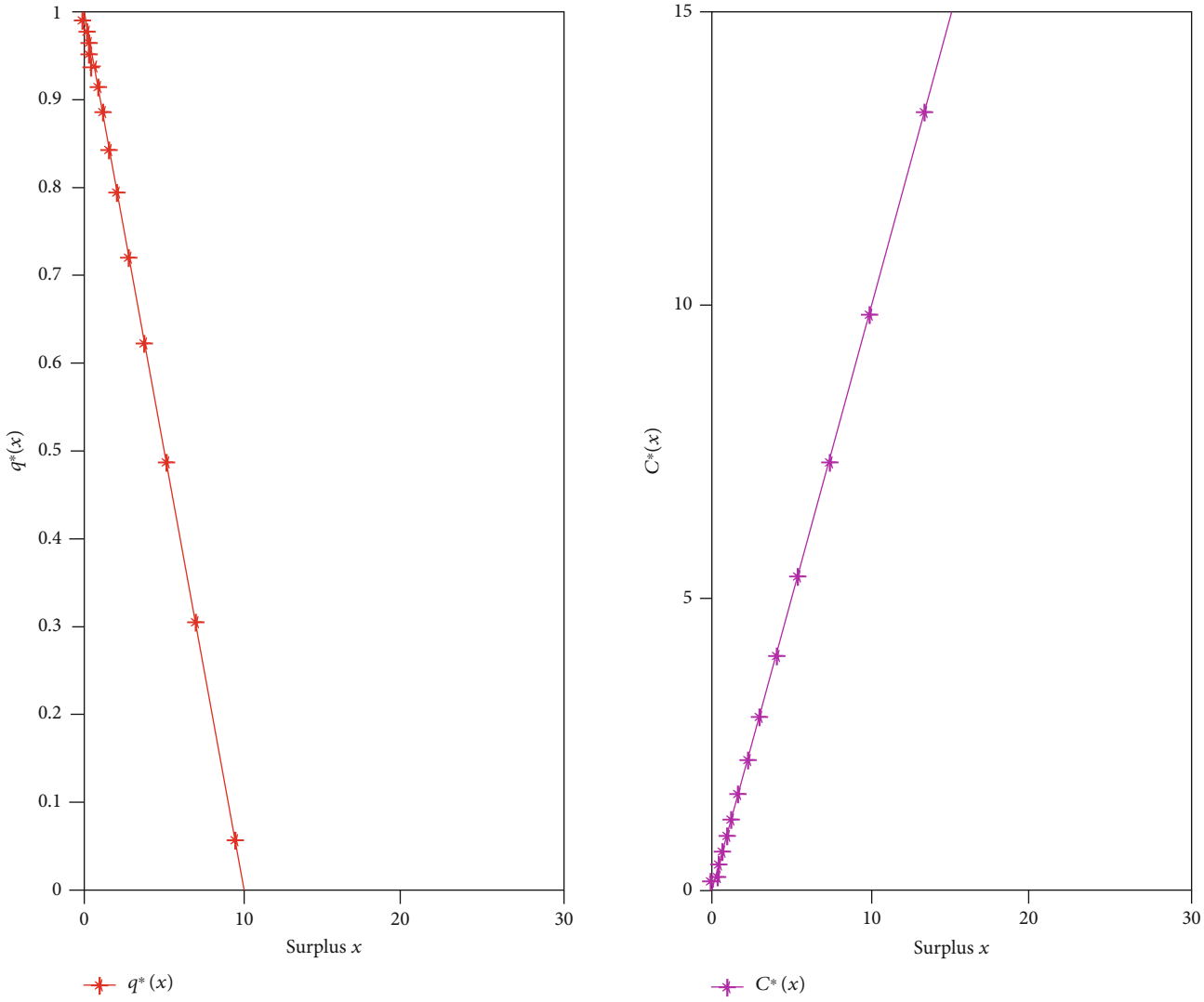

FIGURE 1: $q^*(x)$ and $c^*(x)$ change with x without VaR constraint.

where K_1 and K_2 are some fixed constants. Therefore, in view of (64) and (65), we can conclude that $e^{-r-mt-\delta B_t} F(X_t^*)$ is uniformly integrable. This completes our proof.

Remark 7. Letting $m = \beta$ and $r = \delta = 0$ and $\lambda \to 0$ in Theorem 6, the result reduces to the result of Theorem 7 in Liang and Palmowski [17].

4. Solution to the Optimization Problem under VaR Constraints

In this section, we will use the results in Section 3 to solve the optimal problem with VaR constraints (14). To do so, we make the following assumption.

Assumption 8. We assume that $\bar{\text{VaR}} \geq -\Phi^{-1}(k)b\sqrt{h} - \theta ah$ and $\Phi^{-1}(k) < 0$ or equivalently $k < 1/2$.

Under Assumption 8, the VaR constraint $\text{VaR}_t^{k,h,\pi} \leq \bar{\text{VaR}}$ is equivalent to

$$c(x)h + \left(\Phi^{-1}(k)b\sqrt{h} + \eta ah\right)q(x) - \Phi^{-1}(k)b\sqrt{h} - \theta ah - \bar{VaR} \leq 0. \tag{66}$$

Note that $-\Phi^{-1}(k)b\sqrt{h} - \theta ah - \bar{\text{VaR}} \leq 0$, so there exists at least one strategy $(q(x), c(x)) \equiv (0, 0)$ that satisfies (66). Then, the control space defined in Assumption 8 is not empty. The strategy under VaR constraints should satisfy

$$\begin{cases} 0 \leq q(x) \leq 1, \\ c(x) \geq 0, \\ c(x)h + \left(\Phi^{-1}(k)b\sqrt{h} + \eta ah\right)q(x) - \Phi^{-1}(k)b\sqrt{h} - \theta ah - \bar{\text{VaR}} \leq 0. \end{cases} \tag{67}$$

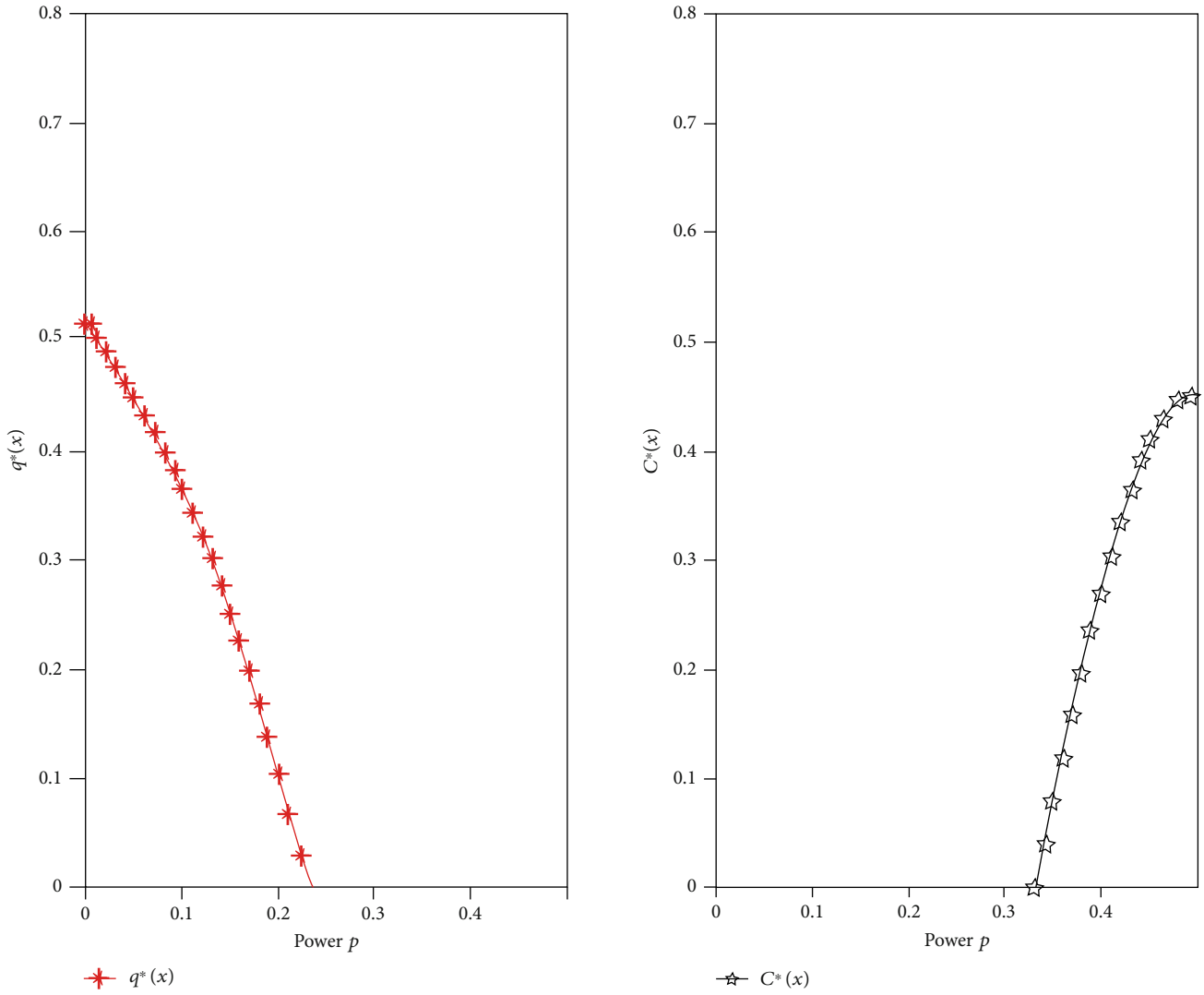

FIGURE 2: $q^*(x)$ and $c^*(x)$ change with p without VaR constraint.

TABLE 2: VaR control levels and levels with time interval $h = 1/250$.

	Case 1	Case 2	Case 3
$\bar{VaR}$	0.08	0.1	0.1
k	0.01	0.01	0.03

If the strategy $(q(x), c(x))$ defined in Theorem 6 satisfies the last inequality in (67), then the strategy $(q^*(x), c^*(x))$ is also a solution with the constraint, namely $(q^*(x), c^*(x)) = (q^*_{VaR}(x), c^*_{VaR}(x))$. Otherwise, if the strategy $(q(x), c(x))$ defined in the first and second equations in (67) is outside the control space at the initial point x, then the optimal strategy is just the boundary of the control space. If the strategy $(q(x), c(x))$ defined in the first and second equations in (67) is inside the control space at the initial point x, but it leaves the control space at sometime, we define the first exit point $\bar{x}$ of the control space as

$$\bar{x} = \inf\{x > 0 : c(x)h + Eq(x) - F > 0, \text{or } q(x) < 0, \text{or } c(x) < 0\}, \tag{68}$$

where $E = \Phi^{-1}(k)b\sqrt{h} + \eta ah$ and $F = \bar{VaR} + \Phi^{-1}(k)b\sqrt{h} + \theta ah$. Then, under the VaR constraints, the optimal strategy is $\pi^*_{\text{VaR}} = (q^*_{\text{VaR}}(x), c^*_{\text{VaR}}(x))$ with

$$q^*_{\text{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0, \bar{x} \wedge x^*], \\ q^*(\bar{x} \wedge x^*), & \text{if } x > \bar{x} \wedge x^*, \end{cases} \tag{69}$$

$$c^*_{\text{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0, \bar{x}], \\ c^*(\bar{x}), & \text{if } x > \bar{x}. \end{cases} \tag{70}$$

5. Numerical Examples

In this section, we illustrate the results obtained in Sections 3 and 4 by numerical examples. We set the model parameters

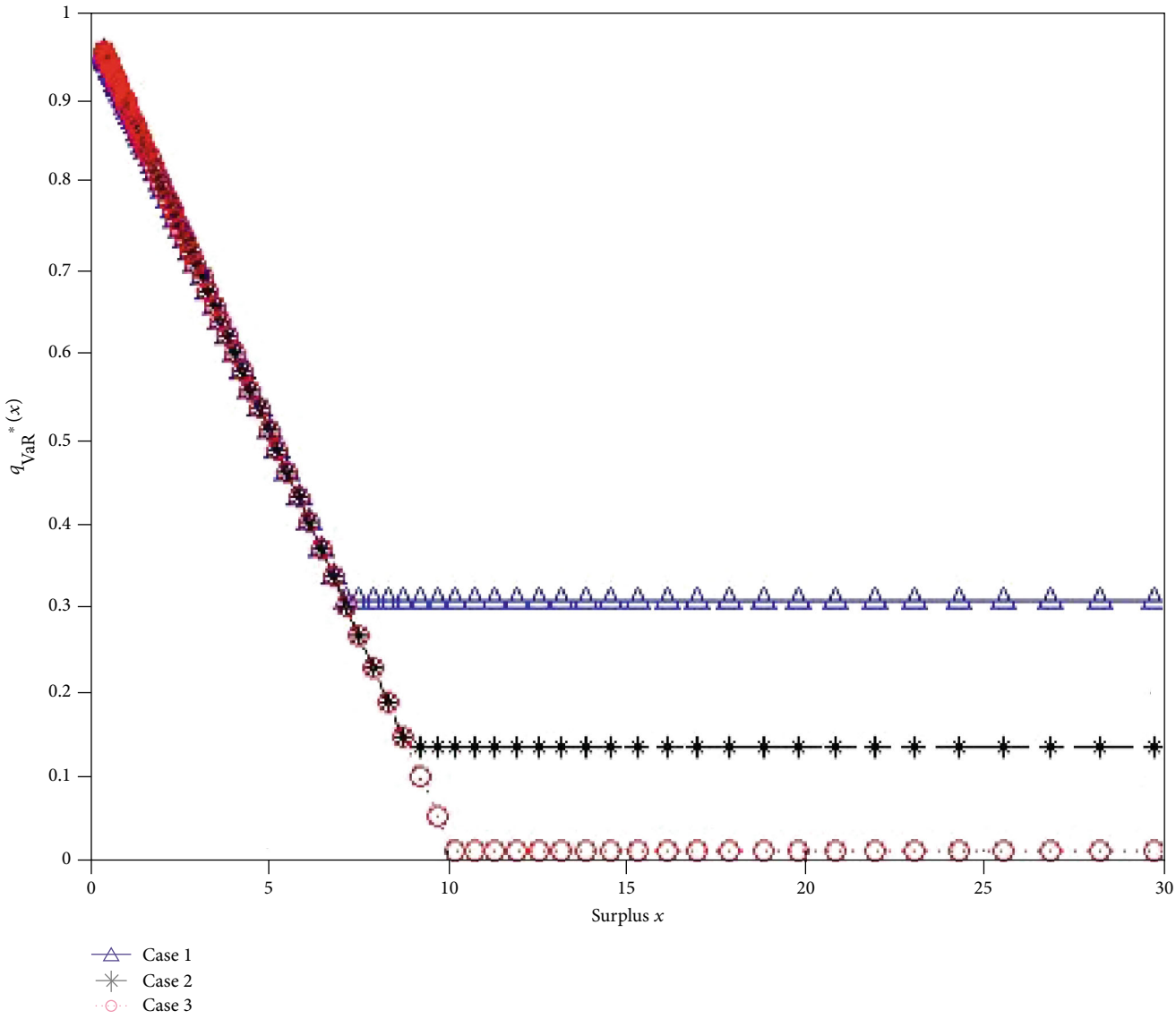

FIGURE 3: $q^*(x)$ change with x under the VaR constraints.

of the insurers reserve process and the financial market in Table 1.

First, we plot the optimal dividend and optimal reinsurance proportion for surplus x ranging from 0 to 30 without VaR constraint. The strategies are shown in Figure 1. From Figure 1, we find the optimal threshold x^* equals to 10.3337 according to Theorem 6, which implies that the insurer will take all the claims without buying any reinsurance when the surplus is larger than 10.3337. From this graph, we can also observe that the optimal dividend rate $c^*(x)$ is nearly linear increasing with the surplus x, while the optimal reinsurance proportion is decreasing with x. When its surplus is sufficiently small, the insurer will divert most of its risk incurred by claims to the reinsurer, which seems intuitively reasonable. It is a reasonable result because when the insurer has a bigger initial wealth x, the insurer would like to retain a bigger proportion $(1 - q(x))$ of its insurance risks and increase the dividend rate.

Second, we plot the optimal dividend and optimal reinsurance proportion as the functions of $p \in [0, 1]$ for fixed $x = 5$ without VaR constraint. The strategies are shown in Figure 2. From Figure 2, we find that the optimal dividend rate increases with increasing p, while the optimal reinsurance proportion decreases. This phenomenon can be explained by noting that the parameter $1 - p$ represents the risk aversion of the insurer, so the insurer will retain more proportion of the insurance risks and give more dividend payment.

Third, we use the model parameters of the insurers reserve process market as in Table 2 and consider the VaR constraints for three different cases as in Table 2, where the time interval h is equal to 1/250, and the VaR control levels of $\bar{\text{VaR}}$ are set so that (66) holds. Cases 1 of Table 2: for the VaR control level $\bar{\text{VaR}}$ and risk level k given in Case 1 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain

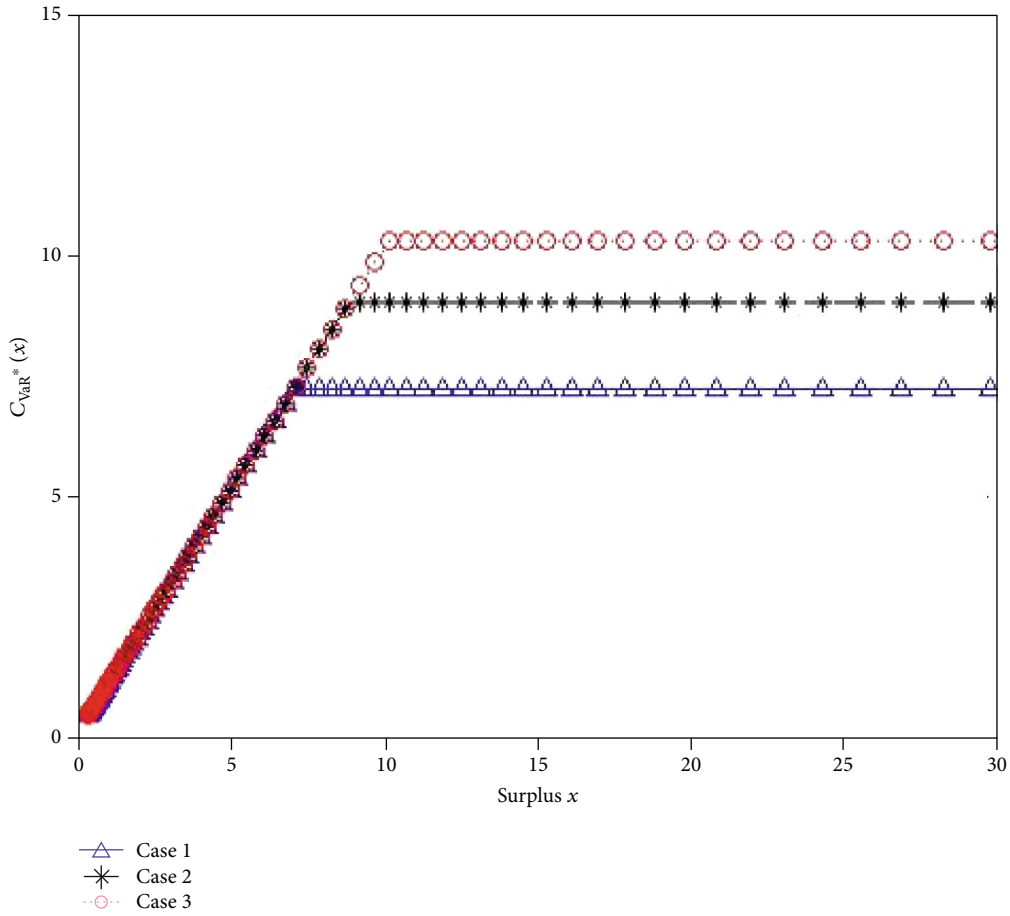

FIGURE 4: $c^*(x)$ change with x under the VaR constraints.

the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q^*_{\mathrm{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0,7.1249], \\ 0.3085, & \text{if } x > 7.1249, \end{cases} \quad c^*_{\mathrm{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0,7.1249], \\ 7.2076, & \text{if } x > 7.1249. \end{cases} \tag{71}$$

Cases 2 of Table 2: for the VaR control level $\bar{\mathrm{Va}}\mathrm{R}$ and risk level k given in Case 2 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q^*_{\mathrm{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0,8.9208], \\ 0.1353, & \text{if } x > 8.9208, \end{cases} \quad c^*_{\mathrm{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0,8.9208], \\ 9.0151, & \text{if } x > 8.9208. \end{cases} \tag{72}$$

Cases 3 of Table 2: for the VaR control level $\bar{\mathrm{Va}}\mathrm{R}$ and risk level k given in Case 3 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q^*_{\mathrm{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0,10.2729], \\ 0.0141, & \text{if } x > 10.2729, \end{cases} \quad c^*_{\mathrm{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0,10.2729], \\ 10.2791, & \text{if } x > 10.2729. \end{cases} \tag{73}$$

Under the VaR constraints, both the reinsurance strategy $1 - q^*_{\mathrm{VaR}}(x)$ and the $c^*_{\mathrm{VaR}}(x)$ investment strategy are increasing while the initial capital x increases, which are presented in Figures 3 and 4. It is a reasonable result because when the insurer has a bigger initial wealth x, the insurer would like to retain a bigger proportion of its insurance risks and a larger dividend rate. The upper bounds represent the effect of the VaR constraints on the strategies, which implies that to limit the loss of the insurer at the VaR control level $\overline{\mathrm{VaR}}$, the insurer has to limit both the amounts of dividends and the retained insurance risks within certain scope. And, we can see that the upper bounds of the reinsurance strategy $1 - q^*_{\mathrm{VaR}}(x)$ and the rate of dividend in case 2 are larger than those in case 1, which means that the insurer can retain more insurance risks and pay more dividend payment in Case 2 than in Case 1. This finding is consistent with the fact that the insurer has a tougher VaR control level of 0.08 in case 1 than that of the VaR control level 0.1 in case 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

Authors' Contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Research Article

Existence Theorems for Fractional Semilinear Integrodifferential Equations with Noninstantaneous Impulses and Delay

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In this paper, we consider a class of fractional semilinear integrodifferential equations with noninstantaneous impulses and delay. By the semigroup theory and fixed point theorems, we establish various theorems for the existence of mild solutions for the problem. An example involving partial differential equations with noninstantaneous impulses is given to show the application of our main results.

1. Introduction

Over the last couple of decades, fractional differential equations have been applied successfully to model many phenomena in physics, engineering, chemistry, financial, biology, etc. Consequently, the subject of fractional differential equations has attracted more and more attention worldwide and for more details, see for example [1–19] and the references therein.

Meanwhile, differential equations with impulsive effects have been used widely as mathematical models for the study of many phenomena in physical, biology, optimal control model of economics, etc. Much attention has been paid to the existence of solutions for the differential equations with impulses in abstract space. For details, see [7, 20–28].

In [26], the author initially studied the differential equations with noninstantaneous impulsive effects as follows:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \cdots, N, \\ u(t) = g_k(t, u(t)), & t \in (t_k, s_k],\ k = 1, 2, \cdots, N, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup of bounded operators $\{T(t)\}_{t\geq 0}$ defined on a Banach space E.

In [20], the author studied the following integer order integrodifferential equations with instantaneous impulses in a Banach space E:

$$\begin{cases} u'(t) = Au(t) + f\left(t, u(\tau_1(t)), u(\tau_2(t)), \cdots, u(\tau_n(t)), \int_0^t h(t, s, u(\tau_{n+1}(s)))ds\right), \\ t \in [0, b], \quad t \neq t_k,\ k = 1, 2, \cdots, m, \\ u(0) + g(u) = u_0, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \end{cases} \tag{2}$$

where for any $t \geq 0$, the linear operator A is the infinitesimal generator of a compact, analytic semigroup, and the nonlinear term is Lipschitz continuous. The existence of mild solutions has been proved.

In this paper, we investigate the following fractional semilinear integrodifferential equations with noninstantaneous impulses and delay:

$$\begin{cases} {}^{c}D_t^{\beta}u(t) = A(t)u(t) + f\left(t, u(\tau_1(t)), u(\tau_2(t)),\cdots,u(\tau_n(t)), \int_0^t h(t,s,u(\tau_{n+1}(s)))ds\right), \\ t \in (s_k, t_{k+1}], \quad k = 0, 1, \cdots, N, \\ u(t) = l_k(t, u(t)), \quad t \in (t_k, s_k],\ k = 1, 2, \cdots, N, \\ u(0) + g(u) = u_0, \end{cases} \tag{3}$$

where $\beta \in (0, 1]$, ${}^{c}D_t^{\beta}$ is the Caputo's fractional derivative of order β, $A(t)$ is a closed and linear operator with domain $D(A)$ defined on a Banach space E, and the fixed points s_i and t_i satisfying $0 = s_0 < t_1 \leq s_1 < t_2 \leq \cdots < t_N \leq s_N < t_{N+1} = T_0$ are prefixed numbers. $f, l_k, h, g\ (k = 1, 2, \cdots, N)$ and $\tau_i\ (i = 1, 2, \cdots, n+1)$ are to be specified later.

Inspired by the results mentioned above, by the semigroup theory and fixed point theorems, we consider the existence of mild solutions for the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay (3). In [7], the authors discussed the existence of solutions for the fractional ordinary differential equation with a generalized impulsive term. In [20–23], the authors discussed the integer or fractional differential equations with instantaneous impulses and the linear operator A is independent of t. In [26, 29, 30], the authors discussed the integer-order differential equations with noninstantaneous impulses and the linear operator A is independent of t. In [31–33], the authors discussed the fractional differential equations with noninstantaneous impulses and the linear operator A is also independent of t. In this paper, we consider the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay, and the linear operator $A(t)$ is assumed to be dependent on t. Therefore, the mentioned results above are special cases of the problem investigated in this paper. Our results improve and generalize the results in References [7, 20–23, 26, 29–33].

The rest of this paper is organized as follows. In Section 2, we present the basic notation and preliminary results. In Section 3, we prove the existence of mild solutions. In Section 4, we give an illustrative example, followed by the conclusion of this paper in Section 5.

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space, $J = [0, T_0]$, $C(J, E) = \{u : J \longrightarrow E \text{ is continuous}\}$, $\mathrm{PC}(J, E) = \{u : J \longrightarrow E : u \in C((s_k, t_{k+1}], E)$ and there exist $u(t_k^-)$ and $u(t_k^+)$ with $u(t_k^-) = u(t_k), k = 1, 2, \cdots, N\}$ with the PC-norm $\|u\|_{\mathrm{PC}} = \sup\{\|u(t)\| : t \in J\}$.

Definition 1 [34]. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \tag{4}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2 [34]. The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \longrightarrow R$ is given by

$${}^{c}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{5}$$

where $\Gamma(\cdot)$ denotes the Gamma function, α is a fractional number, $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on R^+.

Definition 3 [35, 36]. Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$, we call $A(t)$ the generator of an β-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $U_\beta : R_+^2 \longrightarrow B(E)$ such that $\{\lambda^\beta : \mathrm{Re}\ \lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\beta I - A(s)\right)^{-1}u = \int_0^\infty e^{-\lambda(t-s)}U_\beta(t,s)udt, \quad \mathrm{Re}\ (\lambda) > \omega, u \in E. \tag{6}$$

In this case, $U_\beta(t, s)$ is called the β-resolvent family generated by $A(t)$.

Definition 4. A function $u \in PC(J, E)$ is called a mild solution of the problem (3), if $u(0) + g(u) = u_0$, and

$$u(t) = \begin{cases} U_\beta(t,0)(u_0 - g(u)) + \int_0^t U_\beta(t,s)f\left(s, u(\tau_1(s)), u(\tau_2(s)),\cdots,u(\tau_n(s)), \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)ds,\ t \in [0, t_1], \\ l_k(t, u(t)), \quad t \in (t_k, s_k],\ k = 1, 2, \cdots, N, \\ U_\beta(t, s_k)l_k(s_k, u(s_k)) + \int_{s_k}^t U_\beta(t,s)f\left(s, u(\tau_1(s)), u(\tau_2(s)),\cdots,u(\tau_n(s)), \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)ds,\ t \in (s_k, t_{k+1}],\ k = 1, 2, \cdots, N. \end{cases} \tag{7}$$

Lemma 5. *(Sadovskii fixed point theorem). Let B be a convex, closed, and bounded subset of a Banach space E and $Q : B \longrightarrow B$ be a condensing map. Then, Q has one fixed point in B.*

3. Main Results

For convenience in presentation, we give here the basic assumptions to be used later throughout the paper.

(H_1) The function $f : J \times E^{n+1} \longrightarrow E$ is continuous, and for all $r > 0$, there exist nonnegative Lebesgue integrable functions $L_i \in L^1(J, R^+)$ $(i = 1, 2, \cdots, n+1)$ such that for any $t \in J, x_i, y_i \in T_r = \{x \in E : \|x\| \leq r\}$ $(i = 1, 2, \cdots, n+1)$, we have

$$\|f(t, x_1, x_2, \cdots, x_{n+1}) - f(t, y_1, y_2, \cdots, y_{n+1})\| \leq \sum_{i=1}^{n+1} L_i(t)\|x_i - y_i\|. \tag{8}$$

(H_2) The function $h : J \times J \times E \longrightarrow E$ is continuous, and for all $r > 0$, there exists a nonnegative Lebesgue integrable function $L_h(t, s) \in L^1(J \times J, R^+)$ such that for any $t \in J, x, y \in T_r$, we have

$$\|h(t, s, x) - h(t, s, y)\| \leq L_h(t, s)\|x - y\|. \tag{9}$$

(H_3) $\tau_i : J \longrightarrow J$ are continuous such that $0 \leq \tau_i(t) \leq t$ $(i = 1, 2, \cdots, n+1)$ for all $t \in J$.

(H_4) The functions $l_k : J \times E \longrightarrow E$ and $g : \mathrm{PC}(J, E) \longrightarrow E$ are continuous, and for all $r > 0$, there exist nonnegative constants L_{l_k}, L_g such that for any $t \in J, x, y \in T_r, u, v \in \mathrm{PC}(J, E)$, we have

$$\|l_k(t, x) - l_k(t, y)\| \leq L_{l_k}\|x - y\|, k = 1, 2, \cdots, N,$$
$$\|g(u) - g(v)\| \leq L_g\|u - v\|_{\mathrm{PC}}. \tag{10}$$

(H_4') There is a function $\phi_k : J \longrightarrow R^+$ $(k = 1, 2, 3, \cdots, N)$ such that $\|l_k(t, x)\| \leq \phi_k(t)$ for every $t \in [t_k, s_k]$ and for every $x \in T_r$. Let $M_k := \sup_{t \in [t_k, s_k]} \phi_k(t) < \infty$, $k = 1, 2, 3, \cdots, N$.

(H_5) The function $f : J \times E^{n+1} \longrightarrow E$ satisfies the Carathéodory condition, i.e., $f(\cdot, x_1, x_2, \cdots, x_{n+1})$ is measurable for all $x_i \in E$, and $f(t, \cdot, \cdot, \cdots, \cdot)$ is continuous for a.e. $t \in J$, and for all $r > 0$, there exists $m_{f_i} \in L^1(J, R^+)$ $(i = 1, 2, \cdots, n+1)$ such that $\|f(t, x_1, x_2, \cdots, x_{n+1})\| \leq \sum_{i=1}^{n+1} m_{f_i}(t)\|x_i\|$ for $t \in J$, $x_i \in T_r$.

(H_6) The function $h : J \times J \times E \to E$ is continuous, and for all $r > 0$, there exists $m_h \in L^1(J \times J, R^+)$ such that $\|h(t, s, x)\| \leq m_h(t, s)\|x\|$ for $s, t \in J, x \in T_r$.

Theorem 6. *Assume $\rho = \max\{\Delta, M(\Delta + N_1 + N_2)\} < 1$ and conditions (H_1)–(H_4) hold, where $M = \max_{0 \leq s < t \leq T_0} \|U_\beta(t, s)\| < +\infty$, $\Delta = \max\{L_g, L_{l_k}, k = 1, 2, \cdots, N\}$,*

$$N_1 = \max\left\{\int_0^{t_1} \sum_{i=1}^{n} L_i(s)ds, \int_{s_k}^{t_{k+1}} \sum_{i=1}^{n} L_i(s)ds, k = 1, 2, \cdots, N\right\},$$
$$N_2 = \max\left\{\int_0^{t_1} L_{n+1}(s)\int_0^{t_1} L_h(s, \sigma)d\sigma ds, \int_{s_k}^{t_{k+1}} L_{n+1}(s)\int_0^{t_{k+1}} L_h(s, \sigma)d\sigma ds, k = 1, 2, \cdots, N\right\}. \tag{11}$$

Then, there exists a unique mild solution $u^*(t)$ of the problem (3) on $\mathrm{PC}(J, E)$ and $\|u^*\|_{PC} \leq r$, where

$$r \geq \max\left\{\frac{\|l_k(t, 0)\|}{1 - L_{l_k}}, \frac{M\left(\|g(0)\|_{\mathrm{PC}} + \int_0^{t_1}\|f(s, 0, 0, \cdots, 0)\|ds + \int_0^{t_1}\int_0^{s}\|h(s, \sigma, 0)\|d\sigma ds\right)}{1 - M\left(L_g + \int_0^{t_1}\sum_{i=1}^{n} L_i(s)ds + \int_0^{t_1} L_{n+1}(s)\int_0^{s} L_h(s, \sigma)d\sigma ds\right)}, \right.$$
$$\left. \frac{M\left(\|l_k(s_k, 0)\| + \int_{s_k}^{t_{k+1}}\|f(s, 0, 0, \cdots, 0)\|ds + \int_{s_k}^{t_{k+1}}\int_0^{s}\|h(s, \sigma, 0)\|d\sigma ds\right)}{1 - M\left(L_{l_k} + \int_{s_k}^{t_{k+1}}\sum_{i=1}^{n} L_i(s)ds + \int_{s_k}^{t_{k+1}} L_{n+1}(s)\int_0^{s} L_h(s, \sigma)d\sigma ds\right)}, \quad k = 1, 2, 3, \cdots, N\right\}. \tag{12}$$

Proof. Define an operator $Q : \mathrm{PC}(J.E) \longrightarrow \mathrm{PC}(J.E)$ by

$$(Qu)(t) = \begin{cases} U_\beta(t, 0)(u_0 - g(u)) + \int_0^t U_\beta(t, s) f\left(s, u(\tau_1(s)), \cdots, u(\tau_n(s)), \int_0^s h(s, \sigma, u(\tau_{n+1}(\sigma)))d\sigma\right)ds, & t \in [0, t_1], \\ l_k(t, u(t)), \quad t \in (t_k, s_k], \ k = 1, 2, \cdots, N, \\ U_\beta(t, s_k) l_k(s_k, u(s_k)) + \int_{s_k}^t U_\beta(t, s) f\left(s, u(\tau_1(s)), \cdots, u(\tau_n(s)), \int_0^s h(s, \sigma, u(\tau_{n+1}(\sigma)))d\sigma\right)ds, & t \in (s_k, t_{k+1}], \ k = 1, 2, \cdots, N. \end{cases} \tag{13}$$

Let $B_r = \{u \in \mathrm{PC}(J,E): \|u\|_{\mathrm{PC}} \le r\}$. We will prove $Q(B_r) \subset B_r$. Let $u \in B_r$, for $t \in [0, t_1]$, we have

$$\begin{aligned}\|(Qu)(t)\| &\le M\left(L_g\|u\|_{\mathrm{PC}} + \|g(0)\|_{\mathrm{PC}}\right) + M\int_0^t \sum_{i=1}^n L_i(s)\|u(\tau_i \\ &\cdot (s))\| + \|f(s,0,0,\cdots,0)\|ds + M\int_0^t L_{n+1}(s) \\ &\cdot \left(\int_0^s L_h(s,\sigma)\|u(\tau_{n+1}(\sigma))\|d\sigma\right. \\ &+ \left.\int_0^s \|h(s,\sigma,0)\|d\sigma\right)ds \le M\left(\|g(0)\|_{\mathrm{PC}}\right. \\ &+ \int_0^{t_1} \|f(s,0,0,\cdots,0)\|ds + \int_0^{t_1}\int_0^s \|h(s,\sigma,0)\| \\ &\cdot \|d\sigma ds) + M\left(L_g + \int_0^{t_1}\sum_{i=1}^n L_i(s)ds\right. \\ &+ \left.\int_0^{t_1} L_{n+1}(s)\int_0^s L_h(s,\sigma)d\sigma ds\right)r \le r.\end{aligned} \tag{14}$$

For all $t \in (s_k, t_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$\begin{aligned}\|(Qu)(t)\| &\le M\left(L_{l_k}r + \|l_k(s_k,0)\|\right) + M\int_{s_k}^{t_{k+1}} r\sum_{i=1}^n L_i(s) \\ &+ \|f(s,0,0,\cdots,0)\|ds + M\int_{s_k}^{t_{k+1}} rL_{n+1}(s) \\ &\cdot \left(\int_0^s L_h(s,\sigma)d\sigma + \int_0^s \|h(s,\sigma,0)\|d\sigma\right)ds \\ &\le M\left(\|l_k(s_k,0)\| + \int_{s_k}^{t_{k+1}} \|f(s,0,0,\cdots,0)\|ds\right. \\ &+ \left.\int_{s_k}^{t_{k+1}}\int_0^s \|h(s,\sigma,0)\|d\sigma ds\right) + M\left(L_{l_k}\right. \\ &+ \left.\int_{s_k}^{t_{k+1}}\sum_{i=1}^n L_i(s)ds + \int_{s_k}^{t_{k+1}} L_{n+1}(s)\right. \\ &\cdot \left.\int_0^s L_h(s,\sigma)d\sigma ds\right)r \le r.\end{aligned} \tag{15}$$

For all $t \in (t_k, s_k]$ $(k = 1, 2, \cdots, N)$, we have

$$\|(Qu)(t)\| \le L_{l_k}r + \|l_k(t,0)\| \le r. \tag{16}$$

From the above inequalities, we have that $Qu \in B_r$. Next, we prove that Q is a contraction map on B_r. For all $u, v \in \mathrm{PC}(J,E)$ and $t \in [0, t_1]$, we have

$$\begin{aligned}\|(Qu)(t) - (Qv)(t)\| &\le \|U_\beta(t,0)(g(u) - g(v))\| \\ &+ \int_0^t (t-s)^{\alpha-1}\|U_\beta(t,s)\| \\ &\times \|f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots, \\ &u(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma) \\ &- f(s,v(\tau_1(s)),v(\tau_2(s)),\cdots,v(\tau_n(s)), \\ &\int_0^s h(s,\sigma,v(\tau_{n+1}(\sigma)))d\sigma\|ds \le ML_g\|u \\ &- v\|_{\mathrm{PC}} + M\int_0^t \sum_{i=1}^n L_i(s)\|u(\tau_i(s)) \\ &- v(\tau_i(s))\|ds + M\int_0^t L_{n+1}(s) \\ &\cdot \left\|\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right. \\ &- \left.\int_0^s h(s,\sigma,v(\tau_{n+1}(\sigma)))d\sigma\right\|ds \\ &\le ML_g\|u - v\|_{\mathrm{PC}} + M\int_0^t \sum_{i=1}^n L_i(s)ds\|u \\ &- v\|_{\mathrm{PC}} + M\int_0^t L_{n+1}(s)\int_0^s L_h(s,\sigma)\|(u \\ &\cdot (\tau_{n+1}(\sigma)) - v(\tau_{n+1}(\sigma)))\|d\sigma ds \\ &\le ML_g\|u - v\|_{\mathrm{PC}} + M\int_0^t \sum_{i=1}^n L_i(s)ds\|u \\ &- v\|_{\mathrm{PC}} + M\int_0^t L_{n+1}(s)\int_0^s L_h(s,\sigma) \\ &\cdot d\sigma ds\|u - v\|_{\mathrm{PC}} \le M\left(L_g\right. \\ &+ \int_0^{t_1}\sum_{i=1}^n L_i(s)ds + \int_0^{t_1} L_{n+1}(s)\int_0^{t_1} L_h \\ &\cdot \left.(s,\sigma)d\sigma ds\right)\|u - v\|_{\mathrm{PC}} \le M\left(L_g + N_1\right. \\ &+ \left.N_2\right)\|u - v\|_{\mathrm{PC}}.\end{aligned} \tag{17}$$

For all $t \in (s_k, t_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$\begin{aligned}\|(Qu)(t) - (Qv)(t)\| &\le \|U_\beta(t,s_k)(l_k(s_k,u(s_k)) - l_k(s_k,v(s_k)))\| \\ &+ M\int_{s_k}^t \|f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots, \\ &u(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma) \\ &- f(s,v(\tau_1(s)),v(\tau_2(s)),\cdots,v(\tau_n(s)), \\ &\int_0^s h(s,\sigma,v(\tau_{n+1}(\sigma)))d\sigma\|ds \\ &\le ML_{l_k}\|u - v\|_{\mathrm{PC}} + M\int_{s_k}^t \sum_{i=1}^n L_i(s) \\ &\cdot \|u(\tau_i(s)) - v(\tau_i(s))\|ds\end{aligned}$$

$$
\begin{aligned}
&+M\int_{s_k}^{t} L_{n+1}(s)\left\|\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right.\\
&\left.-\int_0^s h(s,\sigma,v(\tau_{n+1}(\sigma)))d\sigma\right\|ds \le ML_{l_k}\|u-v\|_{\mathrm{PC}}\\
&+M\int_{s_k}^{t}\sum_{i=1}^{n} L_i(s)ds\|u-v\|_{\mathrm{PC}}+M\int_{s_k}^{t} L_{n+1}(s)\int_0^s L_h(s,\sigma)\\
&\cdot\|(u(\tau_{n+1}(\sigma))-v(\tau_{n+1}(\sigma)))\|d\sigma ds \le ML_{l_k}\|u-v\|_{\mathrm{PC}}\\
&+M\int_{s_k}^{t_{k+1}}\sum_{i=1}^{n} L_i(s)ds\|u-v\|_{\mathrm{PC}}+M\int_{s_k}^{t_{k+1}} L_{n+1}(s)\int_0^{t_{k+1}} L_h(s,\sigma)\\
&\cdot d\sigma ds\|u-v\|_{PC} \le M\left(L_{l_k}+N_1+N_2\right)\|u-v\|_{\mathrm{PC}}.
\end{aligned}
\tag{18}
$$

For all $t\in(t_k,s_k](k=1,2,\cdots,N)$, we have

$$
\|(Qu)(t)-(Qv)(t)\|=\|l_k(t,u(t))-l_k(t,v(t))\|\le L_{l_k}\|u-v\|_{\mathrm{PC}}. \tag{19}
$$

From the above results, for all $u,v\in\mathrm{PC}(J,E)$, we have

$$
\|Qu-Qv\|_{\mathrm{PC}}\le\rho\|u-v\|_{\mathrm{PC}}. \tag{20}
$$

Therefore, $Q:B_r\to B_r$ is a contraction map and there exists a unique mild solution u^* of the problem (3) in B_r and $\|u^*\|_{PC}\le r$.

Theorem 7. *Assume that the conditions (H_3), (H_4), (H_4'), (H_5), and (H_6) hold, $U_\beta(t,s)$ is compact for $t,s>0$ and*

$$
\begin{aligned}
\varrho:=\max\Bigg\{&L_{l_k},ML_{l_k},M\left(L_g+\int_0^{t_1}\sum_{i=1}^{n} m_{f_i}(s)ds\right.\\
&\left.+\int_0^{t_1} m_{f_{n+1}}(s)\int_0^{t_1} m_h(s,\sigma)d\sigma ds\right),\\
&M\left(M_k+\int_{s_k}^{t_{k+1}}\sum_{i=1}^{n} m_{f_i}(s)ds+\int_{s_k}^{t_{k+1}} m_{f_{n+1}}(s)\int_0^{t_{k+1}} m_h(s,\sigma)d\sigma ds\right),\\
&k=1,2,\cdots,N\Bigg\}<1.
\end{aligned}
\tag{21}
$$

Then, the problem (3) has a mild solution $u^*\in\mathrm{PC}(J,E)$ and $\|u^*\|_{\mathrm{PC}}\le r$, where

$$
r\ge\max\left\{M_k,\frac{M(\|u_0\|+\|g(0)\|)}{1-M\left(L_g+\int_0^{t_1}\sum_{i=1}^{n} m_{f_i}(s)ds+\int_0^{t_1} m_{f_{n+1}}(s)\int_0^{t_1} m_h(s,\sigma)d\sigma ds\right)},\frac{MM_k}{1-M\left(\int_{s_k}^{t_{k+1}}\sum_{i=1}^{n} m_{f_i}(s)ds+\int_{s_k}^{t_{k+1}} m_{f_{n+1}}(s)\int_0^{t_{k+1}} m_h(s,\sigma)d\sigma ds\right)},\quad k=1,2,3,\cdots,N\right\}. \tag{22}
$$

Proof. We introduce the decomposition $Q=Q_1+Q_2$, where $Q_j:\mathrm{PC}(J,E)\to\mathrm{PC}(J,E)$ $(j=1,2)$ are defined by

$$
(Q_1u)(t)=\begin{cases}U_\beta(t,0)(u_0-g(u)), & t\in[0,t_1],\\ l_k(t,u(t)), & t\in(t_k,s_k],\ k=1,2,\cdots,N,\\ U_\beta(t,s_k)l_k(s_k,u(s_k)), & t\in(s_k,t_{k+1}],\ k=1,2,\cdots,N,\end{cases}
$$

$$
(Q_2u)(t)=\begin{cases}\displaystyle\int_0^t U_\beta(t,s)f\left(s,u(\tau_1(s)),\cdots,u(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)ds, & t\in[0,t_1],\\ 0, & t\in(t_k,s_k],\ k=1,2,\cdots,N,\\ \displaystyle\int_{s_k}^t U_\beta(t,s)f\left(s,u(\tau_1(s)),\cdots,u(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)ds, & t\in(s_k,t_{k+1}],\ k=1,2,\cdots,N.\end{cases}
\tag{23}
$$

Let $B_r=\{u\in\mathrm{PC}(J,E):\|u\|_{\mathrm{PC}}\le r\}$, we then prove that the operator $Q=Q_1+Q_2$ is a condensing map on B_r. It is easy to see that B_r is a closed, bounded, and convex subset of $\mathrm{PC}(J,E)$.

Step 1. We prove $Q(B_r) \subset B_r$. Let $u \in B_r$. For $t \in [0, t_1]$, we have

$$\begin{aligned}\|(Qu)(t)\| &\le \|U_\beta(t,0)(u_0 - g(u))\| + \int_0^t \|U_\beta(t,s)\| \\ &\quad \times \|f(s, u(\tau_1(s)), u(\tau_2(s)), \cdots, u(\tau_n(s)), \\ &\quad \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))\|ds \le M(\|u_0\| + \|g(0)\| \\ &\quad + L_g\|u\|_{PC}) + M\int_0^t \sum_{i=1}^n m_{f_i}(s)\|u(\tau_i(s))\|ds \\ &\quad + M\int_0^t m_{f_{n+1}}(s)\left\|\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right\|ds \\ &\le M\left(\|u_0\| + \|g(0)\| + L_g r\right) + M\int_0^t \sum_{i=1}^n m_{f_i}(s)ds\|u\|_{PC} \\ &\quad + M\int_0^t m_{f_{n+1}}(s)\int_0^s m_h(s,\sigma)d\sigma ds\|u\|_{PC} \\ &\le M\left(\|u_0\| + \|g(0)\| + L_g r\right) + M\left(\int_0^{t_1} \sum_{i=1}^n m_{f_i}(s)ds\right. \\ &\quad \left. + \int_0^{t_1} m_{f_{n+1}}(s)\int_0^{t_1} m_h(s,\sigma)d\sigma ds\right) r \le r.\end{aligned} \tag{24}$$

For all $t \in (t_k, s_k]$ $(k = 1, 2, \cdots, N)$, we have

$$\|(Qu)(t)\| = \|(Q_1u)(t)\| = \|I_k(t, u(t))\| \le M_k \le r. \tag{25}$$

For all $t \in (s_k, t_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$\begin{aligned}\|(Qu)(t)\| &\le \|U_\beta(t,s_k)I_k(s_k,u(s_k))\| \\ &\quad + \int_{s_k}^t \|U_\beta(t,s)f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots,u(\tau_n(s)), \\ &\quad \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\|ds \\ &\le MM_k + M\int_{s_k}^t \sum_{i=1}^n m_{f_i}(s)\|u(\tau_i(s))\|ds \\ &\quad + M\int_{s_k}^t m_{f_{n+1}}(s)\left\|\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right\|ds \\ &\le MM_k + M\int_{s_k}^t \sum_{i=1}^n m_{f_i}(s)ds\|u\|_{PC} \\ &\quad + M\int_{s_i}^t m_{f_{n+1}}(s)\int_0^s m_h(s,\sigma)\|u(\tau_{n+1}(\sigma))\|d\sigma ds \\ &\le MM_k + M\int_{s_k}^{t_{k+1}} \sum_{i=1}^n m_{f_i}(s)ds\|u\|_{PC} \\ &\quad + M\int_{s_k}^{t_{k+1}} m_{f_{n+1}}(s)\int_0^{t_{k+1}} m_h(s,\sigma)d\sigma ds\|u\|_{PC} \\ &\le MM_k + M\left(\int_{s_k}^{t_{k+1}} \sum_{i=1}^n m_{f_i}(s)ds\right. \\ &\quad \left. + \int_{s_k}^{t_{k+1}} m_{f_{n+1}}(s)\int_0^{t_{k+1}} m_h(s,\sigma)d\sigma ds\right) r \le r.\end{aligned} \tag{26}$$

Hence, $Q(B_r) \subset B_r$.

Step 2. We prove that Q_1 is a contraction on B_r. For any u, $v \in B_r$ and $t \in [0, t_1]$, we have

$$\begin{aligned}\|(Q_1u)(t) - (Q_1v)(t)\| &\le \|U_\beta(t,0)(g(u) - g(v))\| \\ &\le ML_g\|u - v\|_{PC}.\end{aligned} \tag{27}$$

For all $t \in (t_k, s_k]$ $(k = 1, 2, \cdots, N)$, we have

$$\|(Q_1u)(t) - (Q_1v)(t)\| \le L_{I_k}\|u - v\|_{PC}. \tag{28}$$

For all $t \in (s_k, t_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$\begin{aligned}\|(Q_1u)(t) - (Q_1v)(t)\| &\le \|U_\beta(t,s_k)(I_k(s_k,u(s_k)) \\ &\quad - I_k(s_k,v(s_k)))\| \le ML_{I_k}\|u - v\|_{PC}.\end{aligned} \tag{29}$$

Thus, since $\varrho < 1$, we get that Q_1 is a contraction on B_r.

Step 3. We prove that Q_2 is completely continuous on B_r. Firstly, we prove that the operator Q_2 is continuous. Let $\{u_m\}_0^\infty \subset B_r$, and $u_m \to u \in B_r$. For any $t \in [0, t_1]$, by (H_3), we have

$$\begin{aligned}\|u_m(\tau_i(t)) - u(\tau_i(t))\| &\le \|u_m - u\|_{PC} \to 0 \ (m \to \infty), \\ &\quad i = 1, 2, \cdots, n+1.\end{aligned} \tag{30}$$

By (H_5),

$$\begin{aligned}&f\left(t, u_m(\tau_1(t)), u_m(\tau_2(t)), \cdots, u_m(\tau_n(t)), \int_0^s h(s,\sigma,u_m(\tau_{n+1}(\sigma)))d\sigma\right) \\ &\quad \to f\left(t, u(\tau_1(t)), u(\tau_2(t)), \cdots, u(\tau_n(t)), \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right), \\ &\quad (m \to \infty), \quad \text{a.e. } t \in [0, t_1], \\ &\left\|f\left(t, u_m(\tau_1(t)), \cdots, u_m(\tau_n(t)), \int_0^s h(s,\sigma,u_m(\tau_{n+1}(\sigma)))d\sigma\right)\right. \\ &\quad \left. - f\left(t, u(\tau_1(t)), u(\tau_2(t)), \cdots, u(\tau_n(t)), \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)\right\| \\ &\quad \le 2r\left(\sum_{i=1}^n m_{f_i}(t) + m_{f_{n+1}}(t)\int_0^t m_h(t,s)ds\right), \quad \text{a.e. } t \in [0, t_1].\end{aligned} \tag{31}$$

By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\|(Q_2u_m)(t)-(Q_2u)(t)\| \le M\int_0^t \|f(t,u_m(\tau_1(t)),\cdots,u_m(\tau_n(t)),\\ \int_0^s h(s,\sigma,u_m(\tau_{n+1}(\sigma)))d\sigma)\\ -f(t,u(\tau_1(t)),u(\tau_2(t)),\cdots,\\ u(\tau_n(t)),\int_0^s h(s,\sigma,u(\tau_{n+1}\\ \cdot(\sigma)))d\sigma)\|ds\to 0,\end{aligned} \tag{32}$$

as $m\to\infty$. Therefore, $\|Q_2u_m - Q_2u\|_{PC}\to 0$, as $m\to\infty$. For $t\in(s_k,t_{k+1}]\quad(k=1,2,\cdots,N)$, the proof is similar to that for $t\in[0,t_1]$. Hence, Q_2 is continuous. By the proof of Step 1, for any $t\in J$, it is easy to see that Q_2 maps a bounded set into a bounded set in B_r.

Secondly, we prove that $Q_2(B_r)$ is equicontinuous.

Case 1. For $[0,t_1]$, $0\le e_1<e_2\le t_1$, $u\in B_r$,

$$\begin{aligned}\|(Q_2u)(e_2)-(Q_2u)(e_1)\| \le \left\|\int_0^{e_1}\left(U_\beta(e_2,s)-U_\beta(e_1,s)\right)\right.\\ \times f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots,\\ u(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma)\\ \left.\cdot ds\right\| + \left\|\int_{e_1}^{e_2} U_\beta(e_2,s)\times f(s,u(\tau_1\right.\\ \left.\cdot(s)),u(\tau_2(s)),\cdots,u(\tau_n(s)),\right.\\ \left.\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)ds\|\\ \coloneqq I_1+I_2.\end{aligned} \tag{33}$$

By (H_5), we have

$$\begin{aligned}I_1 &\le \left(\int_0^{t_1}\sum_{i=1}^n m_{f_i}(s)ds+\int_0^{t_1} m_{f_{n+1}}(s)\int_0^{t_1} m_h(s,\sigma)d\sigma ds\right)\\ &\quad\cdot r\sup_{s\in[0,t_1]}\|U_\beta(e_2,s)-U_\beta(e_1,s)\|,\end{aligned}$$

$$I_2 \le M\left(\int_{e_1}^{e_2}\sum_{i=1}^n m_{f_i}(s)ds+\int_{e_1}^{e_2} m_{f_{n+1}}(s)\int_0^s m_h(s,\sigma)d\sigma ds\right)r. \tag{34}$$

Further, the compactness of $U_\beta(t,s)$ for $t,s>0$ implies that the $U_\beta(t,s)$ for $t,s>0$ is continuous in the sense of uniformly operator topology. Thus, $I_1\to 0$ as $e_2\to e_1$. It is easy to see that $I_2\to 0$ as $e_2-e_1\to 0$ since the functions $m_{f_i}\in L^1(J,R^+)$ $(i=1,2,\cdots,n+1)$ and $m_h(t,s)\in L^1(J\times J,R^+)$. Therefore, $\|(Q_2u)(e_2)-(Q_2u)(e_1)\|\to 0$ as $e_2-e_1\to 0$.

Case 2. For $(s_k,t_{k+1}]$ $(k=1,2,\cdots,N)$, $s_k\le e_1<e_2\le t_{k+1}$, $u\in B_r$,

$$\begin{aligned}\|(Q_2u)(e_2)-(Q_2u)(e_1)\| \le \left\|\int_{s_k}^{e_1}\left(U_\beta(e_2,s)-U_\beta(e_1,s)\right)\right.\\ \times f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots,u\\ \left.\cdot(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))\right.\\ \left.\cdot d\sigma)ds\right\| + \left\|\int_{e_1}^{e_2} U_\beta(e_2,s)\right.\\ \times f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots,u\\ \left.\cdot(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))\right.\\ \left.\cdot d\sigma)ds\right\| \coloneqq I_3+I_4.\end{aligned} \tag{35}$$

Similar to the proof for Case 1, we get $\|(Q_2u)(e_2)-(Q_2u)(e_1)\|\to 0$ as $e_2-e_1\to 0$. Therefore, $Q_2(B_r)$ is equicontinuous.

Finally, we prove that $Q_2(B_r)$ is precompact. For every fixed t $(0<t\le t_1)$ and $0<\varepsilon<t$, let $u\in B_r$ and define

$$\begin{aligned}(Q_{2,\varepsilon}u)(t) = \int_0^{t-\varepsilon} U_\beta(t,s)f(s,u(\tau_1(s)),u(\tau_2(s)),\cdots,u(\tau_n(s)),\\ \int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma)ds.\end{aligned} \tag{36}$$

Due to the compactness of $U(t,s)$ for $t,s>0$, the set $Y_\varepsilon(t)=\{(Q_{2,\varepsilon}u)(t)\colon u\in B_r\}$ is relatively compact in E for any ε $(0<\varepsilon<t)$. For any $u\in B_r$, it follows from

$$\begin{aligned}\|(Q_2u)(t)-(Q_{2,\varepsilon}u)(t)\| \le \left\|\int_{t-\varepsilon}^t U_\beta(t,s)f(s,u(\tau_1(s)),\cdots,u\right.\\ \left.\cdot(\tau_n(s)),\int_0^s h(s,\sigma,u(\tau_{n+1}(\sigma)))d\sigma\right)\\ \left.\cdot ds\right\| \le M\left(\int_{t-\varepsilon}^t\sum_{i=1}^n m_{f_i}(s)ds\right.\\ \left.+\|m_h\|\int_{t-\varepsilon}^t m_{f_{n+1}}(s)\int_0^{t_1} m_h(s,\sigma)d\sigma ds\right)r,\end{aligned} \tag{37}$$

that $Y(t)=\{(Q_2u)(t)\colon u\in B_r\}$ is totally bounded. Thus, $Y(t)$ is relatively compact in E. Similarly, we can prove that $Q_2(B_r)(t)\subset E$, $t\in(s_k,t_{k+1}]$ $(k=1,2,\cdots,N)$ is precompact. Using the Arzelà-Ascoli theorem, we have that $Q_2:\mathrm{PC}[J,E]\to\mathrm{PC}[J,E]$ is completely continuous. Hence, the operator $Q=Q_1+Q_2$ is a condensing operator on B_r. By Lemma 5, Q has a fixed point u^* in B_r, which is a mild solution of the problem (3).

Remark 8. In this paper, we consider the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay, and the linear operator $A(t)$ is assumed to be dependent on t. In conditions (H_1) and (H_2), $L_i \in L^1(J, R^+)$ $(i = 1, 2, \cdots, n+1)$ and $L_h(t,s) \in L^1(J \times J, R^+)$ are Lebesgue integrable functions, not constants. The mentioned results above are special cases of the problem investigated in this paper.

4. An Application

In order to show the application of the main results of this paper, we consider the following problem:

$$\begin{cases} {}^{c}D_t^{\alpha} u(x,t) = t\dfrac{\partial^2}{\partial x^2}u(x,t) + \dfrac{t}{1+t^2}u(x,\sin t) + \dfrac{1}{1+t^2}\left(\displaystyle\int_0^t u(x,\sin s)ds\right)^{1/2}, \\ (t,x) \in \bigcup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi], \\ u(x,t) = G_i(t, u(x,t)), \quad x \in [0,\pi],\ t \in (t_i, s_i], \\ u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le 1, \\ u(x,0) + \dfrac{1}{3}u\cos u = u_0(x), \quad 0 \le x \le \pi, \end{cases} \tag{38}$$

where $0 = s_0 < t_1 \le s_1 < \cdots < t_N \le s_N < t_{N+1} = 1$, $0 < \alpha \le 1$, $u_0(x) \in E = L^2[0,\pi]$ and the operator $A : D(A) \subset E \to E$ is defined by $A(t)(z) = t(\partial^2 u/\partial x^2)$ with

$$D(A) = \left\{ u \in E : u'' \in E, u(0) = u(\pi) = 0 \right\}. \tag{39}$$

It is well known that $A(t)$ generates a β-resolvent family on E.

Let

$$\begin{aligned} u(t) &= u(\cdot, t), \\ f(t,x,y) &= \frac{t}{1+t^2}x\sin t + \frac{1}{1+t^2}y^{1/2}, \\ l_i(t,u) &= G_i(\mathrm{t}, u(\cdot,t)), \\ g(u) &= \frac{1}{3}u\cos u. \end{aligned} \tag{40}$$

Then, (38) takes the abstract form (3). Assume that all the conditions of Theorems 6 and 7 are satisfied, then the problem (38) has a mild solution on $[0,1]$.

5. Conclusions

This paper investigates the existence of mild solutions for a class of fractional semilinear integrodifferential equations with noninstantaneous impulses and delay by the fixed point theorem and the semigroup theory. In [7, 20–23, 26, 29–33], the linear operator A is independent of t, while in this paper, the linear operator $A(t)$ is allowed to be dependent on t. Therefore, the differential equations in [7, 20–23, 26, 29–33] are special cases of the problem investigated in this paper. Hence, our results improve and generalize the results of References [7, 20–23, 26, 29–33].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Dynamic Inequalities in Quotients with General Kernels and Measures

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By utilizing the precepts related to the punctuation of time scales $\mathbb{T}$, we present some nouveau forms in quotients for Hardy's and related inequalities on time scales. In particular, some recent results for the Pólya-Knopp, Hardy-Hilbert, and Hardy-Littlewood-Pólya-type inequalities are presented.

1. Introduction

In [1], Hardy stated and proved the following integral inequality:

$$\int_0^\infty \left(\theta^{-1}\int_0^\theta f(\eta)d\eta\right)^\xi d\theta \le \left(\frac{\xi}{\xi-1}\right)^\xi \int_0^\infty f^\xi(\theta)d\theta, \quad \text{for } \xi > 1, \tag{1}$$

where f is a positive function and the constant $(\xi/(\xi-1))^\xi$ is the best ever. We rewrite (1) with the function $f^{\xi^{-1}}$ rather than f, and by assuming limit $\xi \to \infty$, we acquired the limiting instance of the inequality of Hardy known as the inequality of Pólya-Knopp (see [2]), that is,

$$\int_0^\infty \exp\left(\theta^{-1}\int_0^\theta \ln f(\eta)d\eta\right)d\theta \le e\int_0^\infty f(\theta)d\theta. \tag{2}$$

Lately, Kaijser et al. in [3] pointed out that both (1) and (2) are just special states of the much more general inequality of Hardy-Knopp for positive function f:

$$\int_0^\infty \Phi\left(\theta^{-1}\int_0^\theta f(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty \Phi(f(\theta))\frac{d\theta}{\theta}, \tag{3}$$

where $\Phi \in C((0,\infty),\mathbb{R})$ is a convex function. A popularization of inequality (3) with two weight functions is proved in [4]. Particularly, it was proved that if $0 < b \le \infty$, $\lambda : (0, b) \to \mathbb{R}$ is a nonnegative function and Φ is a convex on (a, c), then the inequality

$$\int_0^b \lambda(\theta)\Phi\left(\frac{1}{\theta}\int_0^\theta f(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^b \nu(\theta)\Phi(f(\theta))\frac{d\theta}{\theta}, \tag{4}$$

is available for all integrable functions $f : (0, b) \to \mathbb{R}$, such that $f(\theta) \in (a, c)$ and υ is defined by

$$v(\eta) := \eta \int_{\eta}^{b} \frac{u(\theta)}{\theta^2} d\theta, \quad \text{for } \eta \in (0, b). \tag{5}$$

Using the inequality of Jensen for convex functions and the theorem of Fubini, Kaijser et al. [5] established an inviting popularization of (1). Particularly, they proved that if $\lambda : (0, b) \to \mathbb{R}$ and $k : (0, b) \times (0, b) \to \mathbb{R}$ are nonnegative functions such that $0 < K(\theta) = \int_0^{\theta} k(\theta, \eta) d\eta < \infty$, for $\theta \in (0, b)$ and

$$v(\eta) := \eta \int_{\eta}^{b} \lambda(\theta) \frac{k(\theta, \eta)}{K(\theta)} \frac{d\theta}{\theta} < \infty, \quad \text{for } \eta \in (0, b), \tag{6}$$

then

$$\int_0^{\infty} \lambda(\theta) \Phi(A_k f(\theta)) \frac{d\theta}{\theta} \le \int_0^{\infty} v(\theta) \Phi(f(\theta)) \frac{d\theta}{\theta}, \tag{7}$$

where $\Phi \in C(I, \mathbb{R})$ is a convex function and

$$A_k f(\theta) := \frac{1}{K(\theta)} \int_0^{\theta} k(\theta, \eta) f(\eta) d\eta, \quad \theta \in (0, b). \tag{8}$$

For further popularization of (7), Krulić et al. [6] proved that if $(\Omega_1, \sum_1, \mu_1)$ and $(\Omega_2, \sum_2, \mu_2)$ are two measure spaces with positive σ-finite measures $\lambda : \Omega_1 \to \mathbb{R}$ and $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ which are nonnegative measurable functions such that $0 < K(\theta) = \int_{\Omega_2} k(\theta, \eta) d\mu_2(\eta) < \infty$, $\theta \in \Omega_1$, $0 < \xi \le \xi^* < \infty$, and v is defined by

$$v(\eta) := \left(\int_{\Omega_1} \lambda(\theta) \left(\frac{k(\theta, \eta)}{K(\theta)} \right)^{\xi^*/\xi} d\mu_1(\theta) \right)^{\xi/\xi^*} < \infty, \quad \eta \in \Omega_2, \tag{9}$$

then the inequality

$$\left(\int_{\Omega_1} \lambda(\theta) \Phi^{\xi^*/\xi}(A_k f(\theta)) d\mu_1(\theta) \right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta) \Phi(f(\eta)) d\mu_2(\eta) \right)^{1/\xi}, \tag{10}$$

is available for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(\Omega_2) \subseteq I$, where $\Phi \in C(I, \mathbb{R})$ is a convex function and A_k is defined by

$$A_k f(\theta) := \frac{1}{K(\theta)} \int_{\Omega_2} k(\theta, \eta) f(\eta) d\mu_2(\eta), \theta \in \Omega_1. \tag{11}$$

In [7], Iqbal et al. checked some new weighted Hardy-type inequalities on $(\Omega_1, \sum_1, \mu_1)$ and $(\Omega_2, \sum_2, \mu_2)$ measure spaces with positive σ-finite measures by replacing $k(\theta, \eta)$ by $k(\theta, \eta) f_2(\theta)$ and f by f_1/f_2, where $f_i : \Omega_2 \to \mathbb{R} (i = 1, 2)$ are measurable functions in (10) as follows:

$$\left(\int_{\Omega_1} \lambda(\theta) \left[\Phi\left(\frac{g_1(\theta)}{g_2(\theta)} \right) \right]^{\xi^*/\xi} d\mu_1(\theta) \right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta) \Phi\left(\frac{f_1(\eta)}{f_2(\eta)} \right) d\mu_2(\eta) \right)^{1/\xi}, \tag{12}$$

where $0 < \xi \le \xi^* < \infty$, $\Phi \in C(I, \mathbb{R})$ is a convex function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a nonnegative measurable function, and v, $g_i(\theta) (i = 1, 2)$ are defined by

$$v(\eta) := f_2(\eta) \left(\int_{\Omega_1} \lambda(\theta) \left(\frac{k(\theta, \eta)}{g_2(\theta)} \right)^{\xi^*/\xi} d\mu_1(\theta) \right)^{\xi/\xi^*} < \infty, \quad g_2(\theta) \ne 0, \ \eta \in \Omega_2,$$
$$g_i(\theta) := \int_{\Omega_2} k(\theta, \eta) f_i(\eta) d\mu_2(\eta) \quad (i = 1, 2). \tag{13}$$

In [8], the author founded the time scale version of (1). Particularly, he proved that if $\xi > 1$, $f \in C_{rd}([a_1, \infty]_{\mathbb{T}}, \mathbb{R})$ is a nonnegative function and $\int_{a_1}^{\infty} f^{\xi}(t) \Delta t$ exists, then

$$\int_{a_1}^{\infty} \left(\frac{1}{\sigma(t) - a_1} \int_{a_1}^{\sigma(t)} f(\theta) \Delta\theta \right)^{\xi} \Delta\eta \le \left(\frac{\xi}{\xi - 1} \right)^{\xi} \int_{a_1}^{\infty} f^{\xi}(\eta) \Delta\eta. \tag{14}$$

In [9], the authors proved the time scale version of (4), which is given by

$$\int_{a_1}^{b_1} \lambda(\theta) \Phi\left(\frac{1}{\sigma(\theta) - a_1} \int_{a_1}^{\sigma(\theta)} f(\eta) \Delta\eta \right) \frac{\Delta\theta}{\theta - a_1} \le \int_{a_1}^{b_1} v(\theta) \Phi(f(\theta)) \frac{\Delta\theta}{\theta - a_1}, \tag{15}$$

where $\Phi \in C((c_1, d_1), \mathbb{R})$ is a convex function, $\lambda \in C_{rd}([a_1, b_1]_{\mathbb{T}}, \mathbb{R})$ is a nonnegative function, and v is defined by

$$v(t) := (\eta - a_1) \int_t^{b_1} \frac{\lambda(\eta)}{(\theta - a_1)(\sigma(\theta) - a_1)} \Delta\theta, \quad \text{for } \eta \in (a_1, b_1)_{\mathbb{T}}. \tag{16}$$

In [10], the authors outstretched a number of Hardy-type inequalities with certain kernels on time scale. Namely, they proved that if $(\Omega_1, \sum_1, \mu_1)$ and $(\Omega_2, \sum_2, \mu_2)$ are two time scale measure spaces, $\zeta : \Omega_1 \to \mathbb{R}_+$ and $k : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$, which are nonnegative measurable functions such that

$$K(\theta) := \int_{\Omega_2} k(\theta, \eta) \Delta\eta < \infty, \quad \theta \in \Omega_1, \tag{17}$$

and υ is defined by

$$\upsilon(\eta) \coloneqq \int_{\Omega_1} \frac{k(\theta,\eta)\zeta(\theta)}{K(\theta)}\Delta\theta < \infty, \quad \eta \in \Omega_2, \tag{18}$$

then the inequality

$$\int_{\Omega_1} \zeta(\theta)\Phi\left(\frac{1}{K(\theta)}\int_{\Omega_2} k(\theta,\eta)f(\eta)\Delta\eta\right)\Delta\theta \le \int_{\Omega_2} \upsilon(\eta)\Phi(f(\eta))\Delta\eta, \tag{19}$$

is available for all $\Delta\mu_2$-integrable $f : \Omega_2 \to \mathbb{R}$ such that $f(\Omega_2) \subset I$ and $\Phi \in \mathrm{C}(I, \mathbb{R})$ is a convex function. For development of dynamic inequalities on time scale calculus, we refer the reader to articles [11–18].

The article is regimented as follows. In Section 2, we recall the precepts related to the punctuation of time scales. In Section 3, we prove our results and give some remarks. Particularly, we prove a general dynamic weighted Hardy-type inequality with a nonnegative kernel. In Section 4, we critique a few particular states of the obtained inequalities, related to power and exponential functions and to the most simplest shapes of kernels.

2. Preliminaries

In this section, we will premise some fundamental precepts and effects on time scales which will be beneficial for deducing our major results. The following definitions and theorems are referred from [19, 20].

A nonempty arbitrary locked subplot of the real numbers $\mathbb{R}$ is called a time scale which is denoted by $\mathbb{T}$. For $\tau \in \mathbb{T}$, if $\inf\ \Phi = \sup\ \mathbb{T}$ and $\sup\ \Phi = \inf\ \mathbb{T}$, then the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined as

$$\begin{aligned} \sigma(\tau) &\coloneqq \inf\{\theta \in \mathbb{T} : \theta > \tau\}, \\ \rho(\tau) &\coloneqq \sup\{\theta \in \mathbb{T} : \theta < \tau\}, \end{aligned} \tag{20}$$

respectively. The Δ-derivative of $\chi : \mathbb{T} \to \mathbb{R}$ at $\tau \in \mathbb{T}^k = \mathbb{T}/(\rho(\sup\ \mathbb{T}), \sup\ \mathbb{T}]$ is the number that enjoys the property that for all $\varepsilon > 0$, there exists a neighborhood $U \subset \mathbb{T}$ of $\tau \in \mathbb{T}^k$ such that

$$\left|[\chi(\sigma(\tau)) - \chi(\theta)] - \chi^{\Delta}(\tau)[\sigma(\tau) - \theta]\right| \le \varepsilon|\sigma(\tau) - \theta|, \quad \text{for all } \theta \in U. \tag{21}$$

Furthermore, χ is called a delta differentiable on $\mathbb{T}^k$ if it is delta differentiable at every $\tau \in \mathbb{T}^k$. Similarly, for $\tau \in \mathbb{T}_k$, we define the ∇-derivative of $\chi : \mathbb{T} \to \mathbb{R}$ at $\tau \in \mathbb{T}_k = \mathbb{T}/(\inf\ \mathbb{T}, \sigma(\inf\ \mathbb{T})]$ as the number that enjoys the property that for all $\varepsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $\tau \in \mathbb{T}_k$ such that

$$\left|\chi(\rho(\tau)) - \chi(\theta) - \chi^{\nabla}(\tau)(\rho(\tau) - \theta)\right| \le \varepsilon|\rho(\tau) - \theta|, \quad \text{for all } \theta \in V. \tag{22}$$

Moreover, χ is called a nabla differentiable on $\mathbb{T}_k$ if it is nabla differentiable at every $\tau \in \mathbb{T}_k$. For $\theta, t \in \mathbb{T}^k$, the delta integral of χ^{Δ} is defined as

$$\int_{\theta}^{t} \chi^{\Delta}(\tau)\Delta\tau \coloneqq \chi(t) - \chi(\theta). \tag{23}$$

Similarly, for $\theta, t \in \mathbb{T}_k$, the nabla integral of χ^{∇} is defined as

$$\int_{\theta}^{t} \chi^{\nabla}(\tau)\nabla\tau \coloneqq \chi(t) - \chi(\theta). \tag{24}$$

Now, let $\chi(\tau)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $\tau \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the diamond-$\alpha$ dynamic derivative $\chi^{\Diamond_\alpha}(\tau)$ is defined by

$$\chi^{\Diamond_\alpha}(\tau) \coloneqq \alpha\chi^{\Delta}(\tau) + (1-\alpha)\chi^{\nabla}(\tau), \quad 0 \le \alpha \le 1. \tag{25}$$

The diamond-α derivative debases to the standard Δ -derivative for $\alpha = 1$ or the standard ∇-derivative for $\alpha = 0$.

Next, we recall the inequality of Minkowski and the inequality of Jensen on time scales which are utilized in the proof of the major results.

Theorem 1. *Suppose $(\Omega, \mathscr{M}, \mu_\Delta)$ and $(\Lambda, \mathscr{L}, \lambda_\Delta)$ are two finite-dimensional time scale measure spaces, and let u, v, and g be positive functions on Ω, Λ, and $\Omega \times \Lambda$, respectively. If $\xi \ge 1$, then the inequality*

$$\begin{aligned} &\left[\int_{\Omega}\left(\int_{\Lambda} g(\theta,\eta)\upsilon(\eta)d\upsilon_\Delta(\eta)\right)^{\xi} u(\theta)d\mu_\Delta(\theta)\right]^{1/\xi} \\ &\quad \le \int_{\Lambda}\left(\int_{\Omega} g^{\xi}(\theta,\eta)u(\theta)d\mu_\Delta(\theta)\right)\upsilon(\eta)d\upsilon_\Delta(\eta), \end{aligned} \tag{26}$$

is available for all integrals in (26). If $0 < \xi < 1$ and

$$\begin{aligned} \int_{\Omega}\left(\int_{\Lambda} g\upsilon d\upsilon_\Delta\right)^{\xi} ud\mu_\Delta > 0, \\ \int_{\Lambda} g\upsilon d\upsilon_\Delta > 0, \end{aligned} \tag{27}$$

are available, then (26) is reversed. For $\xi < 0$, in addition with (27), if

$$\int_{\Omega} g^{\xi} ud\mu_\Delta > 0, \tag{28}$$

is available, then again (26) is reversed.

Theorem 2. *Let $e, l \in \mathbb{T}$ and $u, v \in \mathbb{R}$. Suppose that $g_1 \in C_{rd}([e,l]_{\mathbb{T}}, (u,v)_{\mathbb{T}})$ and $h_1 \in C_{rd}([e,l]_{\mathbb{T}}, \mathbb{R})$ are nonnegative with $\int_e^l h_1(s)\Delta s > 0$. If $\Phi \in C((u,v), \mathbb{R})$ is convex, then*

$$\Phi\left(\frac{\int_e^l h_1(s)g_1(s)\Delta s}{\int_e^l h_1(s)\Delta s}\right) \le \frac{\int_e^l \Phi(h_1(s))g_1(s)\Delta s}{\int_e^l h_1(s)\Delta s}. \tag{29}$$

3. Inequalities with General Kernels

In this section, we state and prove our major results. Before presenting the results, we labeled the following hypotheses.

(H1) $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time scale measure spaces

(H2) $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a positive measurable kernel and

$$K(\theta) := \int_{\Omega_2} k(\theta, \eta)\Delta\mu_2(\eta) < \infty, \quad \theta \in \Omega_1. \tag{30}$$

(H3) $u : \Omega_1 \to \mathbb{R}$ is a $\Delta\mu_2$-integrable and

$$v(\eta) := \left(\int_{\Omega_1} u(\theta)\left(\frac{k(\theta,\eta)}{K(\theta)}\right)^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{\xi/\xi^*} < \infty, \tag{31}$$

where $0 < \xi \le \xi^* < \infty$.

In what follows, we will prove the foundation theorem that will be the decisive step in establishing our major result.

Theorem 3. *Assume (H1)–(H3).*

If $\Phi \in C(I, \mathbb{R})$ is a positive convex, then the following inequality

$$\left(\int_{\Omega_1} u(\theta)(\Phi(A_k(f(\theta)))^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)\Phi(f(\eta))\Delta\mu_2(\eta)\right)^{1/\xi}, \tag{32}$$

is available for all nonnegative $\Delta\mu_2$-integrable function $f : \Omega_2 \to \mathbb{R}$ such that $Imf \subseteq I$, where $A_k f : \Omega_1 \to \mathbb{R}$ is defined by

$$A_k f(\theta) := \frac{1}{K(\theta)}\int_{\Omega_2} k(\theta, \eta)f(\eta)\Delta\mu_2(\eta), \quad \theta \in \Omega_1. \tag{33}$$

Proof. We begin with an evident identity

$$\int_{\Omega_1} u(\theta)(\Phi(A_k(f(\theta)))^{\xi^*/\xi}\Delta\mu_1(\theta) = \int_{\Omega_1} u(\theta)\left(\Phi\left(\frac{1}{K(\theta)}\int_{\Omega_2} k(\theta,\eta)f(\eta)\Delta\mu_2(\eta)\right)\right)^{\xi^*/\xi}\Delta\mu_1(\theta). \tag{34}$$

Utilizing the inequality of Jensen (29) and the theorem of Minkowski (2) on (34), we find that

$$\begin{aligned}
&\int_{\Omega_1} u(\theta)\left(\Phi(A_k(f(\theta)))\right)^{\xi^*/\xi}\Delta\mu_1(\theta) \\
&\le \int_{\Omega_1} u(\theta)\left(\frac{1}{K(\theta)}\int_{\Omega_2} k(\theta,\eta)\Phi(f(\eta))\Delta\mu_2(\eta)\right)^{\xi^*/\xi}\Delta\mu_1(\theta) \\
&= \left(\left(\int_{\Omega_1} u(\theta)\left(\frac{1}{K(\theta)}\int_{\Omega_2} k(\theta,\eta)\Phi(f(\eta))\Delta\mu_2(\eta)\right)^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{\xi/\xi^*}\right)^{\xi^*/\xi} \\
&\le \left(\int_{\Omega_2}\Phi(f(\eta))\left(\int_{\Omega_1} u(\theta)\left(\frac{k(\theta,\eta)}{K(\theta)}\right)^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{\xi/\xi^*}\Delta\mu_2(\eta)\right)^{\xi^*/\xi}.
\end{aligned} \tag{35}$$

Taking into computation definition (31) of v, it follows that

$$\int_{\Omega_1} u(\theta)\left(\Phi(A_k(f(\theta)))^{\xi^*/\xi}\Delta\mu_1(\theta) \le \left(\int_{\Omega_2} v(\eta)\Phi(f(\eta))\Delta\mu_2(\eta)\right)^{\xi^*/\xi}. \tag{36}$$

Finally, elevating (36) to the ξ^*th power, we acquired (33).

Remark 4. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$, and $\xi = \xi^* = 1$, inequality (32) reduces to inequality (19) premised in Section 1. Choosing $\mathbb{T} = \mathbb{R}$ in Theorem 3, then inequality (32) reduces to inequality (10).

In what follows, we labeled few particular convex functions starting with power functions.

Corollary 5. *Suppose the supposition of Theorem 3 be gratified only with $\Phi(x) = x^\xi$ for $\xi > 1$. Then, inequality (32) yields the following result:*

$$\left(\int_{\Omega_1} u(\theta)[A_k(f(\theta)]^{\xi^*}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)[f(\eta)]^{\xi}\Delta\mu_2(\eta)\right)^{1/\xi}. \tag{37}$$

Remark 6. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$, and $\xi = \xi^*$, Corollary 5 coincides with Corollary 3.3 in [10].

Now, considering Theorem 3 with $\Phi(x) = e^x$ and $f = \ln(g^\xi)$, we get the following inviting result.

Corollary 7. *Suppose the supposition of Theorem 3 is gratified only with $\Phi(x) = e^x$. If $0 < \xi \le \xi^* < \infty$, then inequality (32) yields the following result:*

$$\left(\int_{\Omega_1} u(\theta)\left[e^{\xi/K(\theta)\int_{\Omega_2} k(\theta,\eta)\,\ln\,(g(\eta))\Delta\mu_2(\eta)}\right]^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)[g(\eta)]^{\xi}\Delta\mu_2(\eta)\right)^{1/\xi}. \tag{38}$$

Remark 8. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$, and $\xi = \xi^*$, Corollary 7 coincides with Corollary 3.4 in [10]. Also, in the special case $\xi = 1$, Remark 8 coincides with Corollary 3.5 in [10].

Now, by using a special substitution, we obtain our central result; that is, if we replace $k(\theta,\eta)$ by $k(\theta,\eta)f_2(\eta)$ and f by f_1/f_2, where $f_j : \Omega_2 \to \mathbb{R}(j = 1, 2)$ are measurable functions, we obtain these results.

Theorem 9. *Assume (H1) and (H2) and v be defined on Ω_2 by*

$$v(\eta) := f_2(\eta)\left(\int_{\Omega_1} u(\theta)\left(\frac{k(\theta,\eta)}{g_2(\theta)}\right)^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{\xi/\xi^*} < \infty, \quad g_2(\theta) \neq 0, \tag{39}$$

where $0 < \xi \le \xi^* < \infty$. If $\Phi \in C(I, \mathbb{R})$ is a positive convex, then the following inequality

$$\left(\int_{\Omega_1} u(\theta)\left[\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\right]^{\xi^*/\xi}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\mu_2(\eta)\right)^{1/\xi}, \tag{40}$$

is available for all measurable functions $f_j : \Omega_2 \to \mathbb{R}(j = 1, 2)$ and

$$g_j(\theta) := \int_{\Omega_2} k(\theta,\eta)f_j(\eta)\Delta\mu_2(\eta) \quad (j = 1, 2). \tag{41}$$

Remark 10. Choosing $\mathbb{T} = \mathbb{R}$ in Theorem 9, then inequality (40) reduces to inequality (12) established by Iqbal et al. (Theorem 2.1, [7]).

As a special case of Theorem 9 for $\xi = \xi^*$, we get the next corollary. Also, we note that the function Φ need not to be positive.

Corollary 11. *Assume (H1) and (H2) and v be defined on Ω_2 by*

$$v(\eta) := f_2(\eta)\int_{\Omega_1} u(\theta)\frac{k(\theta,\eta)}{g_2(\theta)}\Delta\mu_1(\theta) < \infty, \quad g_2(\theta) \neq 0. \tag{42}$$

If $\Phi \in C(I, \mathbb{R})$ is a positive convex, then the following inequality

$$\int_{\Omega_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\Delta\mu_1(\theta) \le \int_{\Omega_2} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\mu_2(\eta), \tag{43}$$

is available for all measurable functions $f_j : \Omega_2 \to \mathbb{R}(j = 1, 2)$ and g_j is defined by (41).

Corollary 12. *Suppose the supposition of Theorem 9 is gratified only with $\Phi(x) = x^{\xi}$ for $\xi > 1$. Then, inequality (40) yields the following result:*

$$\left(\int_{\Omega_1} u(\theta)\left(\frac{g_1(\theta)}{g_2(\theta)}\right)^{\xi^*}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)\left(\frac{f_1(\eta)}{f_2(\eta)}\right)^{\xi}\Delta\mu_2(\eta)\right)^{1/\xi}. \tag{44}$$

Corollary 13. *Suppose the supposition of Theorem 9 is gratified only with $\Phi(x) = e^x$. If $0 < \xi \le \xi^* < \infty$, then inequality (40) yields the following result:*

$$\left(\int_{\Omega_1} u(\theta)e^{\xi^* g_1(\theta)/\xi g_2(\theta)}\Delta\mu_1(\theta)\right)^{1/\xi^*} \le \left(\int_{\Omega_2} v(\eta)e^{f_1(\eta)/f_2(\eta)}\Delta\mu_2(\eta)\right)^{1/\xi}. \tag{45}$$

Remark 14. Choosing $\mathbb{T} = \mathbb{R}$ in Corollary 11, then (43) takes the form

$$\int_{\Omega_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)d\mu_1(\theta) \le \int_{\Omega_2} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\mu_2(\eta), \tag{46}$$

where

$$v(\eta) := f_2(\eta)\int_{\Omega_1} u(\theta)\frac{k(\theta,\eta)}{g_2(\theta)}d\mu_1(\theta) < \infty, \quad g_2(\theta) \neq 0, \tag{47}$$

which is the same result due to Iqbal et al. (Corollary 2.2, [7]).

Corollary 15. *For the Lebesgue diamond-α scale measures $\Delta\mu_1(\theta) = \Diamond_\alpha\theta$, $\Delta\mu_2(\eta) = \Diamond_\alpha\eta$, $\Omega_1 = [a_1, b_1]_{\mathbb{T}}$, and $\Omega_2 = [c_1, d_1]_{\mathbb{T}}$, inequality (40) takes the form*

$$\left(\int_{a_1}^{b_1} u(\theta)\left[\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\right]^{\xi^*/\xi}\Diamond_\alpha\theta\right)_{1/\xi^*} \le \left(\int_{c_1}^{d_1} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Diamond_\alpha\eta\right)^{1/\xi}, \tag{48}$$

where $0 < \xi \le \xi^* < \infty$, $\Phi \in C(I, \mathbb{R})$ *is a positive convex, and*

$$v(\eta) = f_2(\eta)\left(\int_{a_1}^{b_1} u(\theta)\left(\frac{k(\theta,\eta)}{g_2(\theta)}\right)^{\xi^*/\xi} \Diamond_\alpha \theta\right)^{\xi/\xi^*} < \infty, \quad g_2(\theta) \neq 0,$$
$$g_j(\theta) := \int_{c_1}^{d_1} k(\theta,\eta) f_j(\eta) \Diamond_\alpha \eta \quad (j = 1, 2). \tag{49}$$

Remark 16. If we set $\xi = \xi^* = 1$, then inequality (48) takes the form

$$\int_{a_1}^{b_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\Diamond_\alpha \theta \le \int_{c_1}^{d_1} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Diamond_\alpha \eta, \tag{50}$$

which coincided with inequality (7) in [21].

Remark 17. If we set $\Phi(\theta) = -\ln|\theta|$, then inequality (50) takes the form

$$\int_{a_1}^{b_1} u(\theta) \ln\left|\frac{g_1(\theta)}{g_2(\theta)}\right| \Diamond_\alpha \theta \ge \int_{c_1}^{d_1} v(\eta) \ln\left|\frac{f_1(\eta)}{f_2(\eta)}\right| \Diamond_\alpha \eta, \tag{51}$$

which coincided with Corollary 4 in [21].

Remark 18. In Remark 16, if we set $f_2(\eta) = 1$, $g_1(\theta) = g(\theta)$, $f_1(\eta) = f(\eta)$, and $\Phi \in C([0,\infty), \mathbb{R})$ be a convex function, then

$$g_2(\theta) := \int_{c_1}^{d_1} k(\theta,\eta) \Diamond_\alpha \eta = K(\theta), \quad \forall \theta \in [a_1, b_1]_{\mathbb{T}}. \tag{52}$$

We assume that $K(\theta) > 0$ and define

$$v(\eta) := \int_{a_1}^{b_1} \frac{u(\theta)k(\theta,\eta)}{K(\theta)} \Diamond_\alpha \theta, \quad \forall \theta \in [c_1, d_1]_{\mathbb{T}}. \tag{53}$$

Then, inequality (50) takes the form

$$\int_{a_1}^{b_1} u(\theta)\Phi\left(\frac{g(\theta)}{K(\theta)}\right)\Diamond_\alpha \theta \le \int_{c_1}^{d_1} v(\eta)\Phi(f(\eta))\Diamond_\alpha \eta, \tag{54}$$

which coincided with inequality (2.7) in [22] (Theorem 2.2).

Remark 19. Apply Remark 8 for $\Phi(\theta) = -\ln|\theta|$, which is convex with domain $(0, \infty)$. Then, inequality (54) takes the form

$$\int_{a_1}^{b_1} u(\theta) \ln\left(\frac{|g(\theta)|}{K(\theta)}\right)\Diamond_\alpha \theta \ge \int_{c_1}^{d_1} v(\eta) \ln(|f(\eta)|)\Diamond_\alpha \eta, \tag{55}$$

which coincided with inequality (2.10) in [22] (Corollary 2.3).

Remark 20. Apply Remark 18 for $\Phi(\theta) = e^\theta, \theta \ge 0$. Then, inequality (54) takes the form

$$\int_{a_1}^{b_1} u(\theta) e^{g(\theta)/K(\theta)} \Diamond_\alpha \theta \le \int_{c_1}^{d_1} v(\eta) e^{f(\eta)} \Diamond_\alpha \eta, \tag{56}$$

which coincided with inequality (2.11) in [22] (Corollary 2.4).

Remark 21. If we set $\alpha = 1$, then the delta version form of inequality (50) takes the form

$$\int_{a_1}^{b_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\Delta\theta \le \int_{c_1}^{d_1} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta, \tag{57}$$

which coincided with inequality (8) in [21].

Remark 22. If we set $\alpha = 0$, then the nabla version form of inequality (50) takes the form

$$\int_{a_1}^{b_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\nabla\theta \le \int_{c_1}^{d_1} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\nabla\eta, \tag{58}$$

which coincided with inequality (9) in [21].

Remark 23. In particular, if $\mathbb{T} = \mathbb{R}$ and $[a_1, b_1]_{\mathbb{T}} = [c_1, d_1]_{\mathbb{T}}$, then inequality (50) takes the form

$$\int_{a_1}^{b_1} u(\theta)\Phi\left(\frac{g_1(\theta)}{g_2(\theta)}\right)d\theta \le \int_{c_1}^{d_1} v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\eta, \tag{59}$$

where

$$v(\eta) := f_2(\eta)\int_{a_1}^{b_1} u(\theta)\left(\frac{k(\theta,\eta)}{g_2(\theta)}\right)d\theta, \quad \forall \eta \in [a_1, b_1]_{\mathbb{T}}, g_2(\theta) \neq 0,$$
$$g_j(\theta) := \int_{a_1}^{b_1} k(\theta,\eta) f_j(\eta) d\eta, \quad \forall \theta \in [a_1, b_1]_{\mathbb{T}}, j = 1, 2. \tag{60}$$

This result is given in [23, 24].

4. Inequalities with Special Kernels

The next theorem states the general result for Hardy's inequality in quotient.

Theorem 24. *Suppose* $0 \le \xi \le \xi^* < \infty$ *and* u *be a weight function. Define* v *on* $[e, l)_{\mathbb{T}}$ *by*

$$v(\eta) := f_2(\eta)\left(\int_\eta^l \left(\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta\right)^{-\xi^*/\xi} u(\theta)\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{61}$$

If $\Phi \in C(I, \mathbb{R})$ *is a positive convex, then the following inequality*

$$\left(\int_e^l u(\theta)\left(\Phi\left(\frac{\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta}{\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}, \tag{62}$$

is available for all measurable functions $f_j : [e,l)_{\mathbb{T}} \to \mathbb{R}$ $(j = 1, 2)$.

Proof. Rewrite inequality (40) with $\Omega_1 = \Omega_2 = [e,l)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$. Define the kernel $k : [e,l)_{\mathbb{T}} \times [e,l)_{\mathbb{T}} \to \mathbb{R}$ by

$$k(\theta,\eta) := \begin{cases} 1, & \text{if } 0 \le e \le \eta \le \sigma(\theta) \le l, \\ 0, & \text{if } \eta > \sigma(\theta). \end{cases} \tag{63}$$

Then, g_j in (41) takes the form

$$g_j(\theta) := \int_e^{\sigma(\theta)} f_j(\eta)\Delta\eta. \tag{64}$$

Substituting $g_j(\theta)(j = 1, 2)$ in (40), we acquired (47).

Corollary 25. *If we apply Theorem 24 for* $\Phi(\theta) = \theta^{\xi}$, $\xi \ge 1$, *then (62) takes the form*

$$\left(\int_e^l u(\theta)\left(\frac{\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta}{\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta}\right)^{\xi^*}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l v(\eta)\left(\frac{f_1(\eta)}{f_2(\eta)}\right)^{\xi}\Delta\eta\right)^{1/\xi}, \tag{65}$$

where

$$v(\eta) := f_2(\eta)\left(\int_\eta^l\left(\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta\right)^{-\xi^*/\xi} u(\theta)\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{66}$$

Remark 26. If we put in Corollary 25 $f_2(\eta) = 1$, $\xi = \xi^*$, and $l = \infty$, then we acquired Hardy's inequality on time scale

$$\int_e^\infty \frac{u(\theta)}{\sigma(\theta) - e}\left(\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta\right)^{\xi}\Delta\theta \le \int_e^\infty v(\eta) f_1^{\xi}(\eta)\Delta\eta, \tag{67}$$

where

$$v(\eta) := \int_\eta^\infty \frac{u(\theta)}{\sigma(\theta) - e}\Delta\theta. \tag{68}$$

Remark 27. If we put in Theorem 24 $\mathbb{T} = \mathbb{R}$, $e = 0$, and $l = \infty$, we see that $\sigma(\theta) = \theta$ and then (62) takes the form

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\theta f_1(\eta)d\eta}{\int_0^\theta f_2(\eta)d\eta}\right)\right)^{\xi^*/\xi} d\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\eta\right)^{1/\xi}, \tag{69}$$

where

$$v(\eta) := f_2(\eta)\left(\int_\eta^\infty\left(\int_0^\theta f_2(\eta)d\eta\right)^{-\xi^*/\xi} u(\theta)d\theta\right)^{\xi/\xi^*} < \infty, \tag{70}$$

which is the same result due to Iqbal et al. (Theorem 2.4, [7]).

Example 1. If we put $\Phi(\theta) = \theta^{\xi}$, $\xi \ge 1$, and a particular weight function

$$u(\theta) := \frac{1}{\theta^2}\left(\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta\right)^{\xi^*/\xi}, \quad \theta[\in e, l)_{\mathbb{T}}, \tag{71}$$

in (62), we obtain

$$v(\eta) := f_2(\eta)\left(\int_\eta^l \frac{1}{\theta^2}\Delta\theta\right)^{\xi/\xi^*}, \tag{72}$$

and inequality (62) becomes

$$\left(\int_e^l\left(\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta\right)^{\xi^*}\left(\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta\right)^{\xi^*(1/\xi-1)}\frac{\Delta\theta}{\theta^2}\right)^{1/\xi^*} \le \left(\int_e^l\left(\int_\eta^l \frac{1}{\theta^2}\Delta\theta\right)^{\xi/\xi^*} f_1^{\xi}(\eta) f_2^{1-\xi}(\eta)\Delta\eta\right)^{1/\xi}. \tag{73}$$

For $\xi^* = \xi$, inequality (73) reduces to

$$\int_e^l\left(\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta\right)^{\xi}\left(\int_e^{\sigma(\theta)} f_2(\eta)\Delta\eta\right)^{1-\xi}\frac{\Delta\theta}{\theta^2} \le \int_e^l\left(\int_\eta^l \frac{1}{\theta^2}\Delta\theta\right) f_1^{\xi}(\eta) f_2^{1-\xi}(\eta)\Delta\eta. \tag{74}$$

Remark 28. When $\mathbb{T} = \mathbb{R}$, $e = 0$, $l = \infty$, and $f_2(\eta) = 1$, we see that $\sigma(\theta) = \theta$, and then (74) takes the form

$$\int_0^\infty\left(\frac{1}{\theta}\int_0^\theta f_1(\eta)d\eta\right)^{\xi}\frac{d\theta}{\theta} \le \int_0^\infty f_1^{\xi}(\eta)\frac{d\eta}{\eta}. \tag{75}$$

Now, for the convex function $\Phi \in C(I, \mathbb{R})$ defined by $\Phi(\theta) = \exp(\theta)$, we can give the general form of Pólya-Knopp's inequality in quotient.

Theorem 29. *Suppose $0 \le \xi \le \xi^* < \infty$ and u be a weight function defined on $[e,l)_{\mathbb{T}}$. Define v on $[e,l)_{\mathbb{T}}$ by*

$$v(\eta) := \ln f_2(\eta)\left(\int_\eta^l \left(\int_e^{\sigma(\theta)} \ln f_2(\eta)\Delta\eta\right)^{-\xi^*/\xi} u(\theta)\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{76}$$

Then, the following inequality:

$$\left(\int_e^l u(\theta)\left(\exp\left(\frac{\int_e^{\sigma(\theta)} \ln f_1(\eta)\Delta\eta}{\int_e^{\sigma(\theta)} \ln f_2(\eta)\Delta\eta}\right)\right)^{\xi/\xi^*}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l v(\eta)\exp\left(\frac{\ln f_1(\eta)}{\ln f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}, \tag{77}$$

is available for all positive measurable functions $f_j : [e,l)_{\mathbb{T}} \to \mathbb{R}(j = 1, 2)$.

Proof. We rewrite inequality (40) with $\Omega_1 = \Omega_2 = [e,l)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$, and $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(\theta) = \exp(\theta)$; we obtain

$$\left(\int_e^l u(\theta)\left(\exp\left(\frac{g_1(\theta)}{g_2(\theta)}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l v(\eta)\exp\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{78}$$

Define $k(\theta,\eta)$ such as Theorem 24. Substituting $g_j(\theta)$ $(j = 1, 2)$ defined by (64) in (78), we acquired

$$\left(\int_e^l u(\theta)\left(\exp\left(\frac{\int_e^{\sigma(\theta)} f_1(\eta)\Delta\eta}{\int_e^{\sigma(\theta)} f_2(\eta)\mathrm{i}\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l v(\eta)\exp\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{79}$$

Replacing f_j by $\ln f_j (j = 1, 2)$ in (79), we acquired (77).

Remark 30. If we put in Theorem 29 $\mathbb{T} = \mathbb{R}$, $e = 0$, and $l = \infty$, we see that $\sigma(\theta) = \theta$ and then (77) takes the form

$$\left(\int_0^\infty u(\theta)\left(\exp\left(\frac{\int_0^\theta \ln f_1(\eta)d\eta}{\int_0^\theta \ln f_2(\eta)d\eta}\right)\right)^{\xi^*/\xi}d\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\exp\left(\frac{\ln f_1(\eta)}{\ln f_2(\eta)}\right)d\eta\right)^{1/\xi}, \tag{80}$$

where

$$v(\eta) := \ln f_2(\eta)\left(\int_\eta^\infty \left(\int_0^\theta \ln f_2(\eta)d\eta\right)^{-\xi^*/\xi} u(\theta)d\theta\right)^{\xi/\xi^*} < \infty, \tag{81}$$

which is the same result due to Iqbal et al. (Corollary 2.6, [7]).

Remark 31. If we put $f_2(\eta) = \exp(.)$, $\xi = \xi^*$, and $l = \infty$ in Theorem 29, we have

$$\int_e^\infty u(\theta)\exp\left(\frac{\int_e^{\sigma(\theta)} \ln f_1(\eta)\Delta\eta}{\sigma(\theta) - e}\right)\Delta\theta \le \int_e^\infty v(\eta)f_1(\eta)\Delta\eta, \tag{82}$$

where

$$v(\eta) := \int_\eta^\infty \frac{u(\theta)}{\sigma(\theta) - e}\Delta\theta < \infty. \tag{83}$$

Remark 32. In a private case of Remark 31, when $\mathbb{T} = \mathbb{R}$ and $e = 0$, we see that $\sigma(\theta) = \theta$, and inequality (82) reduces to

$$\int_0^\infty u(\theta)\exp\left(\frac{\int_0^\theta \ln f_1(\eta)d\eta}{\theta}\right)d\theta \le \int_0^\infty v(\eta)f_1(\eta)d\eta, \tag{84}$$

where

$$v(\eta) := \int_\eta^\infty \frac{u(\theta)}{\theta}d\theta < \infty. \tag{85}$$

Example 2. Particularly, for the weight function

$$u(\theta) := \frac{1}{\theta^2}\left(\int_e^{\sigma(\theta)} \ln f_2(\eta)\Delta\eta\right)^{\xi^*/\xi}, \quad \theta \in [e,l)_{\mathbb{T}}, \tag{86}$$

in Theorem 29, we obtain

$$v(\eta) := \left(\int_\eta^l \frac{1}{\theta^2}\Delta\theta\right)^{\xi/\xi^*} \ln f_2(\eta), \tag{87}$$

and inequality (77) becomes

$$\left(\int_e^l \frac{1}{\theta^2}\left(\int_e^{\sigma(\theta)} \ln f_2(\eta)\Delta\eta\right)^{\xi^*/\xi}\left(\exp\left(\frac{\int_e^{\sigma(\theta)} \ln f_1(\eta)\Delta\eta}{\int_e^{\sigma(\theta)} \ln f_2(\eta)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_e^l\left(\int_\eta^l \frac{1}{\theta^2}\Delta\theta\right)^{\xi/\xi^*} \ln f_2(\eta)\exp\left(\frac{\ln f_1(\eta)}{\ln f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{88}$$

Remark 33. If we put $\mathbb{T} = \mathbb{R}$, $\xi = \xi^*$, $f_2(\eta) = \exp(.)$, $e = 0$, and $l = \infty$ in (88), we see that $\sigma(\theta) = \theta$, and inequality (88) reduces to

$$\int_0^\infty \exp\left(\frac{1}{\theta}\int_0^\theta \ln f_1(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty f_1(\eta)\frac{d\eta}{\eta}. \tag{89}$$

The next general result is for Hardy-Hilbert's inequality.

Theorem 34. *Suppose $0 \le \xi \le \xi^* < \infty$ and u be a weight function defined on $[0,\infty)_{\mathbb{T}}$. Define v on $[0,\infty)_{\mathbb{T}}$ by*

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{u(\theta)}{(\theta+\eta)^{s\xi^*/\xi}}\left(\int_0^\infty \frac{f_2(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{-\xi^*/\xi}\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{90}$$

If $\Phi \in C(I,\mathbb{R})$ is a positive convex, then the following inequality

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty (f_1(\eta)/(\theta+\eta)^s)\Delta\eta}{\int_0^\infty (f_2(\eta)/(\theta+\eta)^s)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}, \tag{91}$$

is available for all measurable functions $f_j : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ $(j = 1, 2)$.

Proof. We rewrite (40) with $\Omega_1 = \Omega_2 = [0,\infty)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$. Define the kernel $k : [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ by

$$k(\theta,\eta) := \frac{(\eta/\theta)^{s-2/\xi}}{(\theta+\eta)^s}, \quad \xi > 1. \tag{92}$$

Then, g_j defined in (64) becomes

$$g_j(\theta) := \int_0^\infty \left(\frac{\eta}{\theta}\right)^{s-2/\xi}(\theta+\eta)^{-s}f_j(\eta)\Delta\eta := \theta^{2-s/\xi}\int_0^\infty \eta^{s-2/\xi}\frac{f_j(\eta)}{(\theta+\eta)^s}\Delta\eta. \tag{93}$$

Substituting $g_j(\theta)(j = 1, 2)$ in (40), we acquired

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty \eta^{s-2/\xi}(f_1(\eta)/(\theta+\eta)^s)\Delta\eta}{\int_0^\infty \eta^{s-2/\xi}(f_2(\eta)/(\theta+\eta)^s)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{94}$$

Writing $f_j(\eta)$ instead of $\eta^{(s-2)/\xi}f_j(\eta)$ in (94), we obtain (91).

Remark 35. If we put in Theorem 34 $\mathbb{T} = \mathbb{R}$, then (91) takes the form

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty (f_1(\eta)/(\theta+\eta)^s)d\eta}{\int_0^\infty (f_2(\eta)/(\theta+\eta)^s)d\eta}\right)\right)^{\xi^*/\xi}d\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\eta\right)^{1/\xi}, \tag{95}$$

where

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{u(\theta)}{(\theta+\eta)^{s\xi^*/\xi}}\left(\int_0^\infty \frac{f_2(\eta)}{(\theta+\eta)^s}d\eta\right)^{-\xi^*/\xi}d\theta\right)^{\xi/\xi^*} < \infty, \tag{96}$$

which is the same result due to Iqbal et al. (Theorem 2.8, [7]).

Example 3. For $0 < \alpha < (s\xi^*)/\xi$ and taking the particular weight function

$$u(\theta) := \theta^{\alpha-1}\left(\int_0^\infty \frac{f_2(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{\xi^*/\xi}, \quad \theta \in [0,\infty)_{\mathbb{T}}, \tag{97}$$

we obtain,

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{\theta^{\alpha-1}}{(\theta+\eta)^{s\xi^*/\xi}}\Delta\theta\right)^{\xi/\xi^*}. \tag{98}$$

Let the function $\Phi : [0,\infty) \to \mathbb{R}$ be defined by $\Phi(\theta) = \theta^\xi$; then, inequality (91) becomes

$$\left(\int_0^\infty \theta^{\alpha-1}\left(\int_0^\infty \frac{f_1(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{\xi^*}\left(\int_0^\infty \frac{f_2(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{\xi^*(1/\xi-1)}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty\left(\int_0^\infty \frac{\theta^{\alpha-1}}{(\theta+\eta)^{s\xi^*/\xi}}\Delta\theta\right)^{\xi^*/\xi}(f_1(\eta))^\xi(f_2(\eta))^{1-\xi}\Delta\eta\right)^{1/\xi}. \tag{99}$$

Remark 36. When $\mathbb{T} = \mathbb{R}$ in Example 3, we see that

$$v(\eta) := \eta^{\alpha\xi/\xi^*-s}f_2(\eta)\left(B\left(\alpha, \frac{s\xi^*}{\xi}-\alpha\right)\right)^{\xi/\xi^*}, \tag{100}$$

and then (99) takes the form

$$\left(\int_0^\infty \theta^{\alpha-1}\left(\int_0^\infty \frac{f_1(\eta)}{(\theta+\eta)^s}d\eta\right)^{\xi^*}\left(\int_0^\infty \frac{f_2(\eta)}{(\theta+\eta)^s}d\eta\right)^{\xi^*(1/\xi-1)}d\theta\right)^{1/\xi^*} \le \left(B\left(\alpha, \frac{s\xi^*}{\xi}-\alpha\right)\right)^{1/\xi^*}\left(\int_0^\infty \eta^{\alpha\xi/\xi^*-s}(f_1(\eta))^\xi(f_2(\eta))^{1-\xi}d\eta\right)^{1/\xi}, \tag{101}$$

where B is the beta function, which is the same result due to Iqbal et al. (Example 2.9, [7]).

In the forthcoming theorem, we give the Hardy-Littlewood-Pólya inequality in quotient.

Theorem 37. *Suppose $0 \le \xi \le \xi^* < \infty$ and u be a weight function defined on $[0,\infty)_{\mathbb{T}}$. Define v on $[0,\infty)_{\mathbb{T}}$ by*

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{u(\theta)}{(\max\{\theta,\eta\})^{s\xi^*/\xi}}\left(\int_0^\infty \frac{f_2(\eta)}{(\max\{\theta,\eta\})^s}\Delta\eta\right)^{-\xi^*/\xi}\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{102}$$

If $\Phi \in C(I,\mathbb{R})$ is a positive convex, then the following inequality

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty (f_1(\eta)/(\max\{\theta,\eta\})^s)\Delta\eta}{\int_0^\infty (f_2(\eta)/(\max\{\theta,\eta\})^s)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}, \tag{103}$$

is available for all measurable functions $f_j : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ $(j = 1, 2)$.

Proof. Rewrite inequality (40) with $\Omega_1 = \Omega_2 = [0,\infty)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$. Define the kernel $\Phi : [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ by

$$k(\theta,\eta) := \left(\frac{\eta}{\theta}\right)^{s-2/\xi}(\max\{\theta,\eta\})^{-s}. \tag{104}$$

Then, g_j defined in (64) becomes

$$g_j(\theta) := \int_0^\infty \frac{(\eta/\theta)^{s-2/\xi}}{(\max\{\theta,\eta\})^s} f_j(\eta)\Delta\eta := \theta^{2-s/\xi}\int_0^\infty \eta^{s-2/\xi}\frac{f_j(\eta)}{(\max\{\theta,\eta\})^s}\Delta\eta. \tag{105}$$

Substituting $g_j(\theta)(j = 1, 2)$ in (40), we acquired

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty \eta^{s-2/\xi}((f_1(\eta)/\max\{\theta,\eta\})^s)\Delta\eta}{\int_0^\infty \eta^{s-2/\xi}(f_2(\eta)/(\max\{\theta,\eta\})^s)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{106}$$

Writing $f_j(\eta)$ instead of $\eta^{(s-2)/\xi}f_j(\eta)$ in (106), we obtain (103).

Remark 38. When $\mathbb{T} = \mathbb{R}$ in Theorem 37, then (103) takes the form

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty ((f_1(\eta)/\max\{\theta,\eta\})^s)d\eta}{\int_0^\infty (f_2(\eta)/(\max\{\theta,\eta\})^s)d\eta}\right)\right)^{\xi^*/\xi}d\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\eta\right)^{1/\xi}, \tag{107}$$

where

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{u(\theta)}{(\max\{\theta,\eta\})^{s\xi^*/\xi}}\left(\int_0^\infty \frac{f_2(\eta)}{(\max\{\theta,\eta\})^s}d\eta\right)^{-\xi^*/\xi}d\theta\right)^{\xi/\xi^*} < \infty, \tag{108}$$

which is the same result due to Iqbal et al. (Theorem 2.10, [7]).

Example 4. For $0 < \alpha < (s\xi^*)/\xi$ and taking the particular weight function

$$u(\theta) := \theta^{\alpha-1}\left(\int_0^\infty \frac{f_2(\eta)}{(\max\{\theta,\eta\})^s}\Delta\eta\right)^{\xi/\xi^*}, \quad \theta \in [0,\infty)_{\mathbb{T}}, \tag{109}$$

we obtain

$$v(\eta) := f_2(\eta)\left(\int_0^\infty \frac{\theta^{\alpha-1}}{(\max\{\theta,\eta\})^{s\xi^*/\xi}}\Delta\theta\right)^{\xi^*/\xi}. \tag{110}$$

Let the function $\Phi : [0,\infty) \to \mathbb{R}$ be defined by $\Phi(\theta) = \theta^\xi$, then inequality (103) becomes

$$\left(\int_0^\infty \theta^{\alpha-1}\left(\int_0^\infty \frac{f_1(\eta)}{(\max\{\theta,\eta\})^s}\Delta\eta\right)^{\xi^*}\left(\int_0^\infty \frac{f_2(\eta)}{(\max\{\theta,\eta\})^s}\Delta\eta\right)^{\xi^*(1/\xi-1)}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty\left(\int_0^\infty \frac{\theta^{\alpha-1}}{(\max\{\theta,\eta\})^{s\xi^*/\xi}}\Delta\theta\right)^{\xi/\xi^*}(f_1(\eta))^\xi(f_2(\eta))^{1-\xi}\Delta\eta\right)^{1/\xi}. \tag{111}$$

Remark 39. When $\mathbb{T} = \mathbb{R}$ in Example 4, we see that

$$v(\eta) := \eta^{\alpha\xi/\xi^*-s}f_2(\eta)\left(\frac{s\xi^*}{\alpha(s\xi^*-\alpha\xi)}\right)^{\xi/\xi^*}, \tag{112}$$

and then inequality (111) becomes

$$\left(\int_0^\infty \theta^{\alpha-1}\left(\int_0^\infty \frac{f_1(\eta)}{(\max\{\theta,\eta\})^s}d\eta\right)^{\xi^*}\left(\int_0^\infty \frac{f_2(\eta)}{(\max\{\theta,\eta\})^s}d\eta\right)^{\xi^*(1/\xi-1)}d\theta\right)^{1/\xi^*} \le \left(\frac{s\xi^*}{\alpha(s\xi^*-\alpha\xi)}\right)^{1/\xi^*}\left(\int_0^\infty \eta^{\alpha\xi/\xi^*-s}f_1^{\xi}(\eta)f_2^{1-\xi}(\eta)d\eta\right)^{1/\xi}, \tag{113}$$

which is the same result due to Iqbal et al. (Example 2.11, [7]).

Now, we give the result for the Hardy-Hilbert-type inequality in quotient.

Theorem 40. *Let $0 \le \xi \le \xi^* < \infty$ and u be a weight function defined on $[0,\infty)_{\mathbb{T}}$. Define v on $[0,\infty)_{\mathbb{T}}$ by*

$$v(\eta) := f_2(\eta)\left(\int_0^\infty u(\theta)\left(\frac{\ln\eta-\ln\theta}{\eta-\theta}\right)^{\xi^*/\xi}\left(\int_0^\infty \frac{\ln\eta-\ln\theta}{\eta-\theta}f_2(\eta)\Delta\eta\right)^{-\xi^*/\xi}\Delta\theta\right)^{\xi/\xi^*} < \infty. \tag{114}$$

If $\Phi \in C(I,\mathbb{R})$ is a positive convex, then the following inequality

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)f_1(\eta)\Delta\eta}{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)f_2(\eta)\Delta\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}, \tag{115}$$

is available for all measurable functions $f_j : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ $(j = 1, 2)$.

Proof. Rewrite inequality (40) with $\Omega_1 = \Omega_2 = [0,\infty)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$. For $\alpha \in (0,1)$, we define the kernel $k : [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ by

$$k(\theta,\eta) := \frac{\ln(\eta/\theta)}{\eta-\theta}\left(\frac{\theta}{\eta}\right)^{\alpha}. \tag{116}$$

Then, g_j defined in (64) takes the form

$$g_j(\theta) := \int_0^\infty \frac{\ln\eta-\ln\theta}{\eta-\theta}\left(\frac{\theta}{\eta}\right)^{\alpha}f_j(\eta)\Delta\eta := \theta^{\alpha}\int_0^\infty \frac{\ln\eta-\ln\theta}{\eta-\theta}\eta^{-\alpha}f_j(\eta)\Delta\eta. \tag{117}$$

Substituting $g_j(\theta)(j = 1, 2)$ in (40), we acquired

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)\eta^{-\alpha}f_1(\eta)\Delta\eta}{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)\eta^{-\alpha}f_2(\eta)\mathrm{i}\eta}\right)\right)^{\xi^*/\xi}\Delta\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)\Delta\eta\right)^{1/\xi}. \tag{118}$$

Writing $f_j(\eta)$ instead of $\eta^{-\alpha}f_j(\eta)$ in (118), we obtain (115).

Remark 41. If we put $\mathbb{T} = \mathbb{R}$ in Theorem 34, we acquired the Hardy-Hilbert-type inequality in [7] (Theorem 2.12):

$$\left(\int_0^\infty u(\theta)\left(\Phi\left(\frac{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)f_1(\eta)d\eta}{\int_0^\infty((\ln\eta-\ln\theta)/\eta-\theta)f_2(\eta)d\eta}\right)\right)^{\xi^*/\xi}d\theta\right)^{1/\xi^*} \le \left(\int_0^\infty v(\eta)\Phi\left(\frac{f_1(\eta)}{f_2(\eta)}\right)d\eta\right)^{1/\xi}, \tag{119}$$

where

$$v(\eta) := f_2(\eta)\left(\int_0^\infty u(\theta)\left(\frac{\ln\eta-\ln\theta}{\eta-\theta}\right)^{\xi^*/\xi}\left(\int_0^\infty \frac{\ln\eta-\ln\theta}{\eta-\theta}f_2(\eta)d\eta\right)^{-\xi^*/\xi}d\theta\right)^{\xi/\xi^*} < \infty. \tag{120}$$

Example 5. For $\alpha \in (0,1)$ and for

$$u(\theta) := \theta^{-\alpha}\left(\frac{\ln(\eta/\theta)}{\eta-\theta}f_2(\eta)\Delta\eta\right)^{\xi^*/\xi}, \quad \theta \in [0,\infty)_{\mathbb{T}}, \tag{121}$$

in (114), we obtain

$$v(\eta) := \eta^{(1-\alpha)\xi/\xi^*-1}f_2(\eta)C, \tag{122}$$

where

$$C := \left(\int_0^\infty z^{-\alpha}\left(\frac{\ln z}{z-1}\right)^{\xi^*/\xi}\Delta z\right)^{\xi/\xi^*}. \tag{123}$$

Let $\xi \ge 1$ and the function $\Phi : [0,\infty) \to \mathbb{R}$ be defined by $\Phi(\theta) = \theta^{\xi}$. Then, inequality (115) becomes

$$\left(\int_0^\infty \theta^{-\alpha}\left(\int_0^\infty \frac{\ln(\eta/\theta)}{\eta-\theta}f_1(\eta)\Delta\eta\right)^{\xi^*}\left(\int_0^\infty \frac{\ln(\eta/\theta)}{\eta-\theta}f_2(\eta)\Delta\eta\right)^{\xi^*(1/\xi-1)}\Delta\theta\right)^{1/\xi^*} \le \left(C\int_0^\infty \eta^{(1-\alpha)\xi/\xi^*-1}f_1^{\xi}(\eta)f_2^{1-\xi}(\eta)\Delta\eta\right)^{1/\xi}. \tag{124}$$

Remark 42. When $\mathbb{T} = \mathbb{R}$ in Example 5, we see that

$$u(\theta) := \theta^{-\alpha}\left(\frac{\ln(\eta/\theta)}{\eta-\theta}f_2(\eta)d\eta\right)^{\xi^*/\xi}, \quad \theta \in [0,\infty), \qquad v(\eta) := \eta^{(1-\alpha)\xi/\xi^*-1}f_2(\eta)C, \tag{125}$$

where

$$C := \left(\int_0^\infty z^{-\alpha}\left(\frac{\ln z}{z-1}\right)^{\xi^*/\xi}dz\right)^{\xi/\xi^*}, \tag{126}$$

and then inequality (124) becomes

$$\left(\int_0^\infty \theta^{-\alpha}\left(\int_0^\infty \frac{\ln(\eta/\theta)}{\eta-\theta} f_1(\eta)d\eta\right)^{\xi^*}\left(\int_0^\infty \frac{\ln(\eta/\theta)}{\eta-\theta} f_2(\eta)d\eta\right)^{\xi^*\left(\xi^{-1}-1\right)} d\theta\right)^{1/\xi^*} \leq \left(C\int_0^\infty \eta^{(1-\alpha)\xi/\xi^*-1} f_1^{\xi}(\eta) f_2^{1-\xi}(\eta)d\eta\right)^{1/\xi}, \tag{127}$$

which is the same result due to Iqbal et al. (Example 2.13, [7]).

5. Conclusions

In this paper, with the help of special kernels, we obtained some nouveau forms for Hardy's and related inequalities on time scales in quotients and also we presented some recent results for the Pólya-Knopp-, Hardy-Hilbert-, Hardy-Littlewood-Pólya-type inequalities. Furthermore, we got some continuous inequalities as special cases of the obtained dynamic inequalities.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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Research Article

Improvement on Conformable Fractional Derivative and Its Applications in Fractional Differential Equations

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In this paper, we made improvement on the conformable fractional derivative. Compared to the original one, the improved conformable fractional derivative can be a better replacement of the classical Riemann-Liouville and Caputo fractional derivative in terms of physical meaning. We also gave the definition of the corresponding fractional integral and illustrated the applications of the improved conformable derivative to fractional differential equations by some examples.

1. Introduction

The fractional order derivative has always been an interesting research topic in the theory of functional space for many years [1–11]. Various types of fractional derivatives were introduced, among which the following Riemann-Liouville and Caputo are the most widely used ones.

(1) *Riemann-Liouville Definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$$ {}_a^{RL}D^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx. \tag{1} $$

(2) *Caputo Definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$$ {}_a^{C}D^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx. \tag{2} $$

Both Riemann-Liouville definition and Caputo definition are defined via fractional integrals. Therefore, these two fractional derivatives inherit some nonlocal behaviors including historical memory and future dependence. All definitions including (1) and (2) above satisfy the property that the fractional derivative is linear. This is the only property inherited from the 1st derivative. However, the existing fractional derivatives do not satisfy the following properties which the integral derivatives have.

(1) Most of the fractional derivatives except Caputo-type derivatives do not satisfy $D^{\alpha}(1) = 0$, if α is not a natural number

(2) All fractional derivatives do not obey the familiar product rule for two functions:

$$ D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f) \tag{3} $$

(3) All fractional derivatives do not obey the familiar quotient rule for two functions:

$$ D^{\alpha}(fg) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2} \tag{4} $$

(4) All fractional derivatives do not obey the chain rule:

$$ D^{\alpha}f(g(t)) = f^{\alpha}(g(t))D^{\alpha}(g) \tag{5} $$

(5) Fractional derivatives do not have corresponding Rolle's theorem

(6) Fractional derivatives do not have a corresponding mean value theorem

(7) In general, all fractional derivatives do not obey

$$D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f \tag{6}$$

(8) The Caputo definition assumes that the function f is differentiable

To overcome some of these difficulties, Khalil et al. [12] proposed a new interesting factional derivative definition called conformable derivative that extends the familiar limit definition of the derivative of a function given by the following.

Definition 1. Given a function $f : [0,+\infty) \longrightarrow R$, then the conformable fractional derivative of f of order α is defined by

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{7}$$

for $t > 0$ and $\alpha \in (0, 1]$. If f is α differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0+} T_{\alpha}(f)(t)$ exists, then define $T_{\alpha}(f)(0) = \lim_{t \to 0+} T_{\alpha}(f)(t)$. It is easy to see that if f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} f'(t)$. One can find functions which are α-differentiable at a point but not differentiable at this point.

As a result of the above definition, the authors in [12] showed that the conformable derivative obeys the product rule and quotient rule and has results similar to Rolle's theorem and the mean value theorem in classical calculus.

The conformable fractional derivative has two advantages over the classical fractional derivatives. First, the conformable fractional derivative definition is natural and it satisfies most of the properties which the classical integral derivative has such as linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions, Rolle's theorem, and mean value theorem. Second, the conformable derivative bring us a lot of convenience when it is applied for modelling many physical problems, because the differential equations with conformable fractional derivative are easier to solve numerically than those associated with the Riemann-Liouville or Caputo fractional derivative. In fact, many researchers have already applied conformable fractional derivative to many fields and a lot of corresponding techniques were developed [13–20].

However, there are still shortcomings or disadvantages for the conformable derivative. If we look at the Riemann-Liouville and Caputo fractional derivative definition, we have

$$\begin{aligned}
\lim_{\alpha \to 0} {}_{a}^{RL}D^{\alpha}(f)(t) &= \lim_{\alpha \to 0} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{a}^{t} \frac{f(x)}{(t-x)^{\alpha}}\, dx = f(t), \\
\lim_{\alpha \to 0} {}_{a}^{C}D^{\alpha}(f)(t) &= \lim_{\alpha \to 0} \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \frac{f'(x)}{(t-x)^{\alpha}}\, dx = f(t) - f(a),
\end{aligned} \tag{8}$$

for $\alpha \in (0, 1]$, and

$$\begin{aligned}
\lim_{\alpha \to n-1} {}_{a}^{RL}D^{\alpha}(f)(t) &= \lim_{\alpha \to n-1} \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx = f^{n-1}(t), \\
\lim_{\alpha \to n-1} {}_{a}^{C}D^{\alpha}(f)(t) &= \lim_{\alpha \to n-1} \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx = f^{n-1}(t) - f^{n-1}(a),
\end{aligned} \tag{9}$$

for $\alpha \in (n-1, n]$, $n = 2, 3, \cdots$, whereas

$$\lim_{\alpha \to n-1} T_{\alpha}(f)(t) = \lim_{\alpha \to n-1} t^{(n-\alpha)} f^{(n)}(t) = t f^{(n)}(t), \tag{10}$$

for $\alpha \in (n-1, n]$, $n = 1, 2, \cdots$.

That means the physical meaning of the classical Riemann-Liouville and Caputo fractional derivative is quite different from that of the conformable derivative, especially when $\alpha \in (n-1, n]$ and close to $n-1$, $n = 1, 2, \cdots$. We graph the conformable derivatives of $y = \sin x$ and $y = e^{x}$ in Figures 1 and 2 and make comparison with the Riemann-Liouville and Caputo fractional derivatives. We can see if the Riemann-Liouville and Caputo fractional derivatives are replaced by conformable derivative, a large error will occur. To overcome this difficulty, we propose a kind of modified conformable fractional derivative in Section 2. This modified conformable fractional derivative is a local operator on the one hand and approximates the Riemann-Liouville and Caputo fractional derivative better on the other hand. So it is a better choice to replace the classical Riemann-Liouville and Caputo fractional derivative with the improved conformable fractional derivative, especially when $\alpha \in (n-1, n]$ and close to $n-1$, $n = 1, 2, \cdots$.

2. Improvement on Conformable Fractional Derivative

First, we give the definition for the improved Caputo-type conformable fractional derivative ${}_{a}^{C}\tilde{T}_{\alpha}(t)$ and the improved Riemann-Liouville-type conformable fractional derivative ${}_{a}^{RL}\tilde{T}_{\alpha}(t)$ for $0 \le \alpha \le 1$.

Definition 2. Given a function $f : R \longrightarrow R$, the improved Caputo-type conformable fractional derivative of f of order α is defined by

$${}_{a}^{C}\tilde{T}_{\alpha}(t) = \lim_{\varepsilon \to 0} \left[(1-\alpha)(f(t) - f(a)) + \alpha \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon} \right], \tag{11}$$

where $-\infty < a < t < +\infty$, a is a given number.

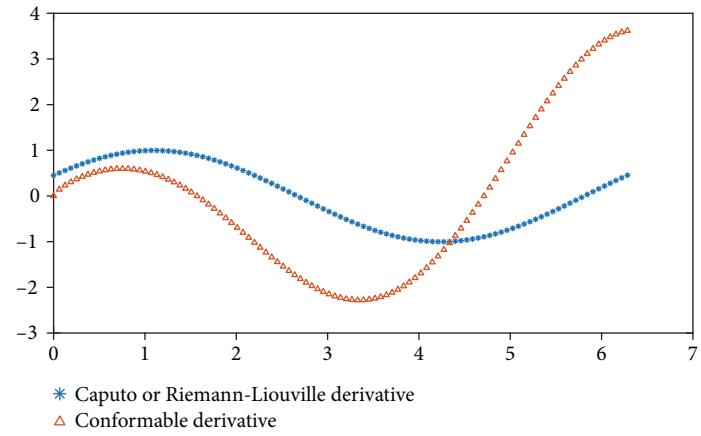

FIGURE 1: 0.3th derivative of $y = \sin x$.

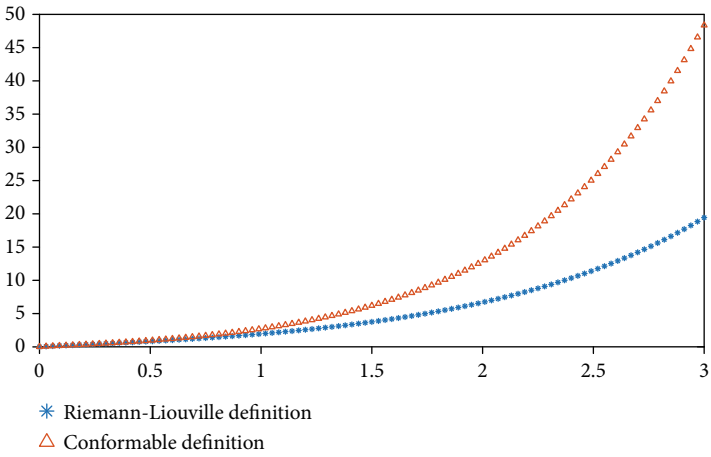

FIGURE 2: 0.2th derivative of $y = e^x$.

The improved Riemann-Liouville-type conformable fractional derivative of f of order α is defined by

$$ {}_a^{RL}\tilde{T}_\alpha(t) = \lim_{\varepsilon\to 0}\left[(1-\alpha)f(t) + \alpha\frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon}\right], \tag{12} $$

where $-\infty < a < t < +\infty$, a is a given number.

It is easy to see

$$ \begin{aligned} \lim_{\alpha\to 0_a}{}^{C}\tilde{T}_\alpha(t) &= (f(t)-f(a)) = \lim_{\alpha\to 0_a}{}^{C}D^\alpha(t), \\ \lim_{\alpha\to 0_a}{}^{RL}\tilde{T}_\alpha(t) &= f(t) = \lim_{\alpha\to 0_a}{}^{RL}D^\alpha(t). \end{aligned} \tag{13} $$

If $\alpha = 1$, both ${}_a^{C}\tilde{T}_a(t)$ and ${}_a^{RL}\tilde{T}_\alpha(t)$ coincide with $f'(t)$. In Definition 2, we introduce a to let ${}_a^{RL}\tilde{T}_a(t)$ and ${}_a^{RL}\tilde{T}_\alpha(t)$ have a kind of historical memory as the Caputo and Riemann-Liouville fractional derivative have.

For $n < \alpha \le n+1$, $n = 1, 2, \cdots$, we give the following.

Definition 3. Given a function $f : R \to R$, the improved Caputo-type conformable fractional derivative of f of order α is defined by

$$ \begin{aligned} {}_a^{C}\tilde{T}_\alpha(t) = \lim_{\varepsilon\to 0}\Bigg[&(n+1-\alpha)\left(f^{(n)}(t)-f^{(n)}(a)\right) \\ &+ (\alpha-n)\frac{f^{(n)}\left(t+\varepsilon(t-a)^{n+1-\alpha}\right)-f^{(n)}(t)}{\varepsilon}\Bigg], \end{aligned} \tag{14} $$

where $-\infty < a < t < +\infty$.

The improved Riemann-Liouville-type conformable fractional derivative is defined by

$$ \begin{aligned} {}_a^{RL}\tilde{T}_a(t) = \lim_{\varepsilon\to 0}\Bigg[&(n+1-\alpha)f^{(n)}(x) \\ &+ (\alpha-n)\frac{f^{(n)}\left(t+\varepsilon(t-a)^{n+1-\alpha}\right)-f^{(n)}(t)}{\varepsilon}\Bigg], \end{aligned} \tag{15} $$

where $-\infty < a < t < +\infty$.

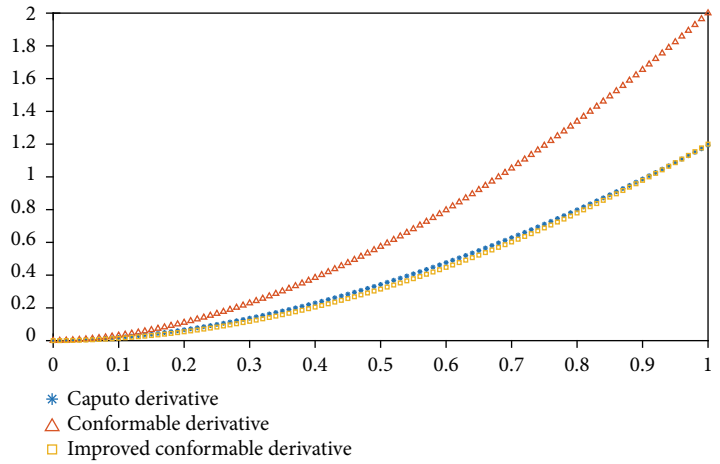

FIGURE 3: 0.2th fractional derivative of $y = x^2$.

It is easy to see

$$\lim_{\alpha \to n_a} {}^{C}\tilde{T}_{\alpha}(t) = \left(f^{(n)}(t) - f^{(n)}(a)\right) = \lim_{\alpha \to n_a} {}^{C}D^{\alpha}(t),$$
$$\lim_{\alpha \to n_a} {}^{RL}\tilde{T}_{\alpha}(t) = f^{(n)}(t) = \lim_{\alpha \to n_a} {}^{RL}D^{\alpha}(t). \tag{16}$$

If $\alpha = n + 1$, both ${}_{a}^{C}\tilde{T}_{a}(t)$ and ${}_{a}^{RL}\tilde{T}_{\alpha}(t)$ coincide with $f^{(n+1)}(t)$.

Remark 4. As a result of Definitions 2 and 3, we can easily show that if f is 1-differentiable at t, then

$${}_{a}^{C}\tilde{T}_{a}(f)(t) = (1-\alpha)(f(t) - f(a)) + \alpha(t-a)^{1-\alpha} f'(t),$$
$${}_{a}^{RL}\tilde{T}_{a}(f)(t) = (1-\alpha) f(t) + \alpha(t-a)^{1-\alpha} f'(t), \tag{17}$$

for $\alpha \in (0, 1]$. If f is $(n+1)$-differentiable at t, then

$${}_{a}^{C}\tilde{T}_{a}(f)(t) = (n+1-\alpha)\left(f^{(n)}(t) - f^{(n)}(a)\right) + (\alpha - n)(t-a)^{n+1-\alpha} f^{(n+1)}(t), \tag{18}$$

for $\alpha \in (n, n+1]$, $n = 1, 2, \cdots$.

In fact, compared with the conformable fractional derivative, the improved conformable fractional derivative can be a better replacement of the Caputo and Riemann-Liouville fractional derivative. One can see in Figures 3–9 that the improved conformable fractional derivative approximates the Caputo and Riemann-Liouville fractional derivative in a better way for most elementary functions especially when α is larger than and close to n and even when α is close to $n + 0.5$, $n = 0, 1, 2, \cdots$.

Based on the results in [12], one can prove the following theorem.

Theorem 5. *We can easily show that the improved conformable fractional derivative satisfies the following properties if* $0 \le \alpha \le 1$.

(1) ${}_{a}^{C}\tilde{T}_{\alpha}(mf + ng) = m\,{}_{a}^{C}\tilde{T}_{\alpha}(f) + n\,{}_{a}^{C}\tilde{T}_{\alpha}(g)$

(2) ${}_{a}^{RL}\tilde{T}_{\alpha}(mf + ng) = m\,{}_{a}^{RL}\tilde{T}_{\alpha}(f) + n\,{}_{a}^{RL}\tilde{T}_{\alpha}(g)$

(3) ${}_{a}^{RL}\tilde{T}_{\alpha}(fg) = (1-\alpha)\,{}_{a}^{RL}\tilde{T}_{\alpha}(f)g + f\,{}_{a}^{RL}\tilde{T}_{\alpha}(g) - (1-\alpha)fg$

(4) ${}_{a}^{RL}\tilde{T}_{\alpha}(f(g(x))) = (1-\alpha) f(g(x)) + \alpha f'(g(x)) T_{\alpha}(g(x))$

Theorem 6. *We get the following improved conformable fractional derivatives of certain functions for* $0 \le \alpha \le 1$.

(1) ${}_{0}^{C}\tilde{T}_{\alpha}(t^p) = {}_{0}^{RL}\tilde{T}_{\alpha}(t^p) = (1-\alpha)(t^p) + \alpha p t^{p-\alpha}$

(2) ${}_{0}^{C}\tilde{T}_{\alpha}(\lambda) = 0$, *for any constant* λ

(3) ${}_{0}^{RL}\tilde{T}_{\alpha}(\lambda) = (1-\alpha)$, *for any constant* λ

(4) ${}_{0}^{C}\tilde{T}_{\alpha}(e^t) = (1-\alpha)(e^t - 1) + \alpha t^{1-\alpha} e^t$

(5) ${}_{0}^{RL}\tilde{T}_{\alpha}(e^t) = (1-\alpha)(e^t) + \alpha t^{1-\alpha} e^t$

(6) ${}_{0}^{C}\tilde{T}_{\alpha}(\sin t) = {}_{0}^{RL}\tilde{T}_{\alpha}(\sin t) = (1-\alpha)\sin t + \alpha t^{1-\alpha}\cos t$

(7) ${}_{0}^{C}\tilde{T}_{\alpha}(\cos t) = (1-\alpha)(\cos t - 1) - \alpha t^{1-\alpha}\sin t$

(8) ${}_{0}^{RL}\tilde{T}_{\alpha}(\cos t) = (1-\alpha)(\cos t) - \alpha t^{1-\alpha}\sin t$

(9) ${}_{0}^{RL}\tilde{T}_{\alpha}(e^{(1/a)t^a})$

Proof. It is very easy to verify properties (1)–(8) if we take into account the conclusions in [12]. We only prove property (9) here. From [12], we have

$$T_{\alpha}\left(e^{(1/a)t^a}\right) = e^{(1/a)t^a}. \tag{19}$$

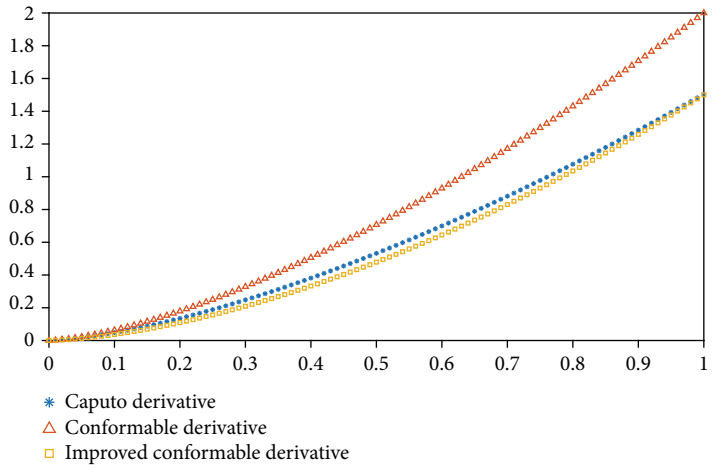

Figure 4: 0.5th fractional derivative of $y = x^2$.

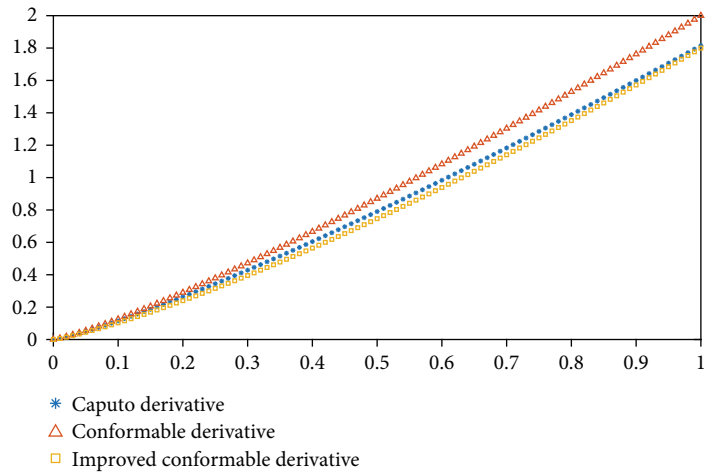

Figure 5: 0.8th fractional derivative of $y = x^2$.

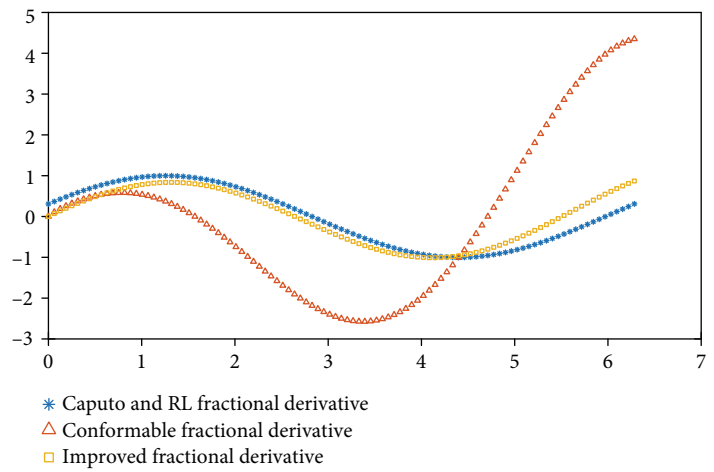

Figure 6: 0.2th fractional derivative of $y = \sin x$.

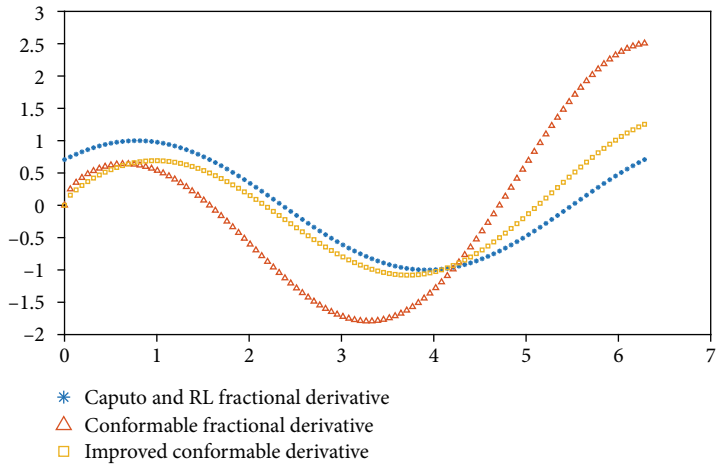

Figure 7: 0.5th fractional derivative of $y = \sin x$.

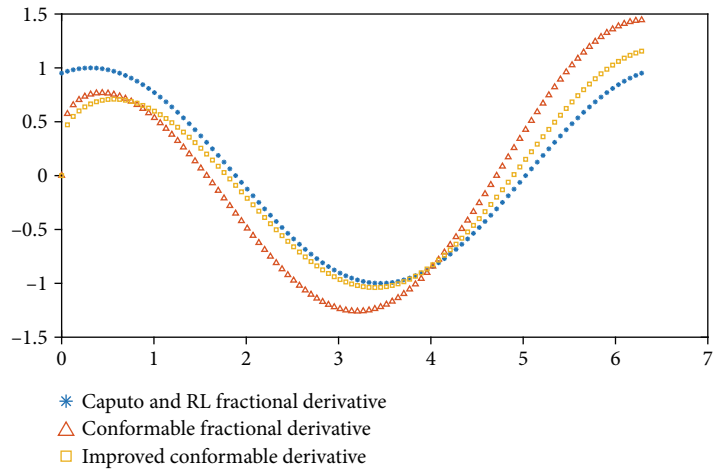

Figure 8: 0.8th fractional derivative of $y = \sin x$.

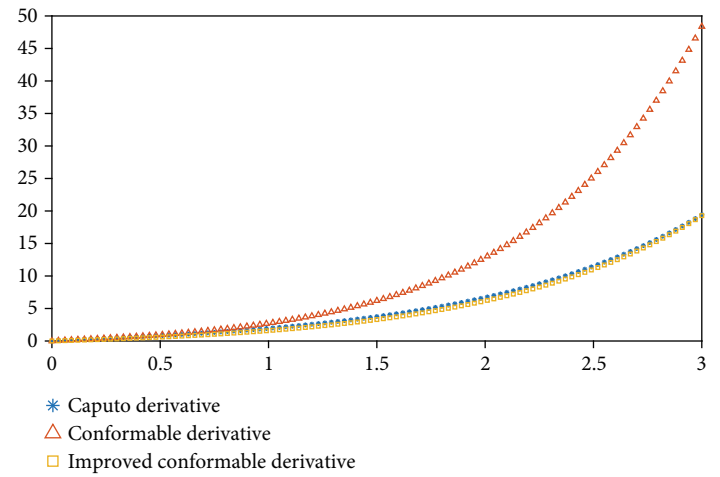

Figure 9: 0.2th fractional derivative of $y = e^x$.

Therefore,

$$
\begin{aligned}
{}_0^{RL}\tilde{T}_\alpha\left(e^{(1/a)t^a}\right) &= (1-\alpha)e^{(1/a)t^a} + \alpha T_\alpha\left(e^{(1/a)t^a}\right) \\
&= (1-\alpha)e^{(1/a)t^a} + \alpha e^{(1/a)t^a} = e^{(1/a)t^a}. \qquad (20)
\end{aligned}
$$

Now, we give the following definition for α fractional integral.

Definition 7. For $\alpha \in (0, 1]$ and continuous function f, let

$$
I_\alpha(f) = \frac{1}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt. \qquad (21)
$$

When $\alpha = 1, I_1(f) = \int_0^x f(t)dt$ and coincides with the usual Riemann integral.

Theorem 8. *Suppose $y(0) = 0$, we have*

(1) $I_\alpha \cdot {}_0^{RL}\tilde{T}_\alpha(y(t)) = {}_0^{RL}\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t)$

(2) $I_\alpha \cdot {}_0^{C}\tilde{T}_\alpha(y(t)) = {}_0^{C}\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t)$

Proof. We only prove (1); (2) can be proved in the same way.

$$
\begin{aligned}
I_\alpha \cdot {}_0^{RL}\tilde{T}_\alpha(f(x)) &= \frac{1-\alpha}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt \\
&+ \int_0^x f'(t) e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt = \frac{1-\alpha}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt \\
&+ \int_0^x e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} df(t) = f(t) e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)}\Big|_0^x \\
&- \frac{1-\alpha}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt \\
&+ \frac{1-\alpha}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt = f(x).
\end{aligned} \qquad (22)
$$

Let $(1-\alpha)y(x) + \alpha t^{1-\alpha} y'(x) = f(x)$; by using a variation of the constant method, we can get

$$
y(x) = \frac{1}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt. \qquad (23)
$$

Let

$$
I_\alpha(f) = \frac{1}{\alpha}\int_0^x \frac{f(t)}{t^{1-\alpha}} e^{\left(1-\alpha/\alpha^2\right)\left(t^\alpha - x^\alpha\right)} dt, \qquad (24)
$$

we have ${}_0^{RL}\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t)$.

3. Applications of Improved Conformable Fractional Derivative

We solve fractional differential equations by using the improved conformable fractional derivatives.

Example 1. Consider the following fractional differential equation:

$$
y^{(0.5)} = \frac{2}{\Gamma(2.5)} x^{3/2}, \quad y(0) = 0. \qquad (25)
$$

If $y^{(0.5)}$ is understood as the Riemann-Liouville or Caputo fractional derivative, the solution to this problem is

$$
y = x^2. \qquad (26)
$$

Now, we take $y^{(0.5)}$ as the improved conformable fractional derivative; the problem reduces to

$$
\begin{cases}
\dfrac{1}{2}y + \dfrac{1}{2}\sqrt{x}y' = \dfrac{2}{\Gamma(2.5)} x^{3/2}, \\
y(0) = 0.
\end{cases} \qquad (27)
$$

We can solve this problem through the variation constant method and get the following solution:

$$
y = \frac{4}{\Gamma(2.5)}\left(x^{3/2} - \frac{3}{2}x + \frac{3}{2}\sqrt{x} - \frac{3}{4}\right) + \frac{3}{\Gamma(2.5)} e^{-2\sqrt{x}}. \qquad (28)
$$

If $y^{(0.5)}$ is understood as the conformable fractional derivative, the problem reduces to

$$
\begin{cases}
\sqrt{x}y' = \dfrac{2}{\Gamma(2.5)} x^{3/2}, \\
y(0) = 0,
\end{cases} \qquad (29)
$$

and the solution to this problem is

$$
y = \frac{1}{\Gamma(2.5)} x^2. \qquad (30)
$$

We compare the three solutions in Figure 10. We can see that the improved conformable fractional derivative solution is much better than the solution obtained by using the conformable fractional derivative. That means if we prefer a replacement of the Riemann-Liouville or Caputo fractional derivative, the improved conformable fractional derivative is a better choice.

Example 2. In general, we consider the problem

$$
y^{(\alpha)} = \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad y(0) = 0. \qquad (31)
$$

We can get the following three solutions:

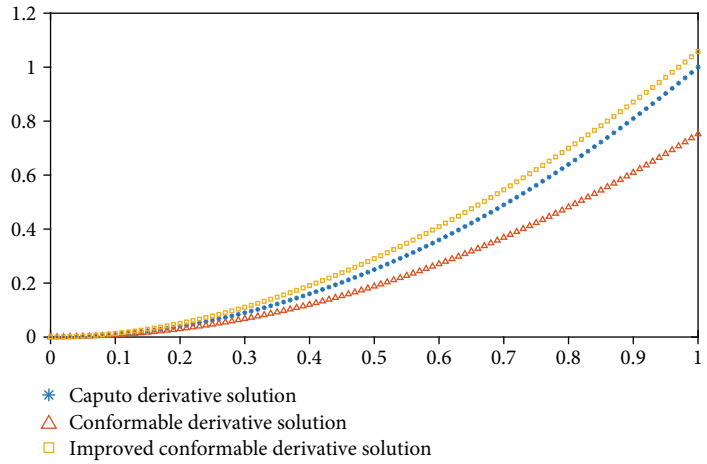

Figure 10: Solution to Example 1.

(i) If $y^{(\alpha)}$ is the Riemann-Liouville or Caputo fractional derivative, the solution is $y_1(x) = x^2$, and

$$\lim_{\alpha\to 0} y_1(x) = \lim_{\alpha\to 0} x^2 = x^2 \tag{32}$$

(ii) If $y^{(\alpha)}$ is the conformable fractional derivative, the problem can be reduced to

$$\begin{cases} x^{1-\alpha}y' = \dfrac{2}{\Gamma(3-\alpha)}x^{2-\alpha}, \\ y(0) = 0. \end{cases} \tag{33}$$

The solution is

$$y_2(x) = \frac{1}{\Gamma(3-\alpha)}x^2. \tag{34}$$

We can see that

$$\lim_{\alpha\to 0} y_2(x) = \lim_{\alpha\to 0} \frac{1}{\Gamma(3-\alpha)} = \frac{1}{2}x^2 \tag{35}$$

(iii) If $y^{(\alpha)}$ is the improved conformable fractional derivative, the problem can be reduced to

$$\begin{cases} (1-\alpha)y + \alpha x^{1-\alpha}y' = \dfrac{2}{\Gamma(3-\alpha)}x^{2-\alpha}, \\ y(0) = 0 \end{cases} \tag{36}$$

Its solution $y_3(x)$ can be obtained through the variation of the constant method.

Since

$$(1-\alpha)y_3 + \alpha x^{1-\alpha}y_3' = \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}, \tag{37}$$

we have

$$y_3 = \alpha y_3 - \alpha x^{1-\alpha}y_3' + \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}. \tag{38}$$

Therefore,

$$\begin{aligned} \lim_{\alpha\to 0} y_3 &= \lim_{\alpha\to 0}\left(\alpha y_3 - \alpha x^{1-\alpha}y_3' + \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}\right) \\ &= \lim_{\alpha\to 0}\frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} = x^2 = \lim_{\alpha\to 0} y_1. \end{aligned} \tag{39}$$

The result for the general α equation shows that the improved conformable derivative has advantages over other fractional derivatives. First, it is a good approximation to the classical Riemann-Liouville or Caputo fractional derivative so it has a similar physical meaning with the Riemann-Liouville and Caputo fractional derivative. Second, it is a local derivative operator so it is easy for numerical computing.

Example 3. Consider the following fractional differential equation:

$$y^{(1.5)} = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad y(0) = 0, y'(0) = 0. \tag{40}$$

If $y^{(1.5)}$ is the Caputo fractional derivative, the solution to this problem is

$$y = x^3. \tag{41}$$

If $y^{(1.5)}$ is the conformable fractional derivative, the problem reduces to

$$x^{0.5}y'' = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad y(0) = 0, y'(0) = 0. \tag{42}$$

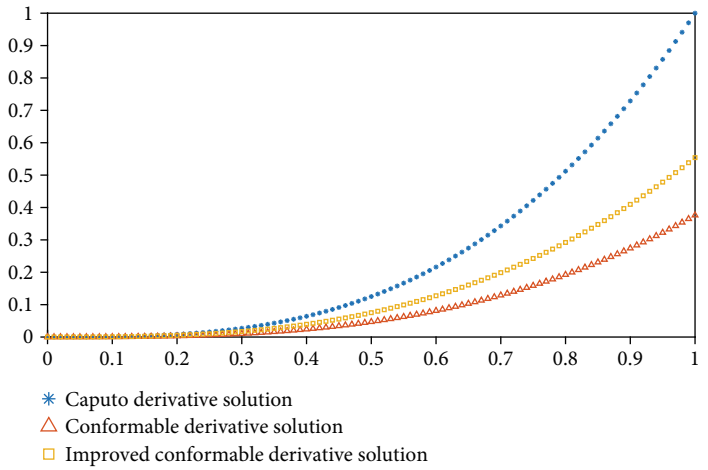

Figure 11: Solution of Example 3.

The solution to this problem is

$$y = \frac{1}{2\Gamma(2.5)}x^3. \tag{43}$$

If $y^{(1.5)'}$ is the improved conformable fractional derivative, the problem reduces to

$$\frac{1}{2}y' + \frac{1}{2}x^{0.5}y'' = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad y(0) = 0, y'(0) = 0. \tag{44}$$

Let $y' = p(x)$, the problem further reduces to

$$\frac{1}{2}p + \frac{1}{2}x^{0.5}p' = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad p(0) = 0. \tag{45}$$

Solve this problem by using the method of constant variation, we get

$$p = \frac{6}{\Gamma(2.5)}\left(x^{3/2} - \frac{3}{2}x + \frac{3}{2}\sqrt{x} - \frac{3}{4} + \frac{3}{4}e^{-2\sqrt{x}}\right), \tag{46}$$

$$y = \frac{6}{\Gamma(2.5)}\left(\frac{2}{5}x^{5/2} - \frac{3}{4}x^2 + x^{3/2} - \frac{3}{4}x - \frac{3}{8}e^{-2\sqrt{x}} - \frac{3}{4}\sqrt{x}e^{-2\sqrt{x}}\right) + \frac{9}{4\Gamma(2.5)}. \tag{47}$$

The solutions to Example 3 can be seen in Figure 11.

4. Conclusion

We propose a kind of improved conformable fractional derivative in this paper. This improved conformable fractional derivative is also local by its definition, and meanwhile, a kind of historical memory parameter is introduced to its definition. The advantage of the improved conformable derivative is that its physical behavior approximates the Riemann-Liouville and Caputo fractional derivative better than the conformable fractional derivative. So this improved conformable fractional derivative has much potential in modelling many physical problems where the Riemann-Liouville and Caputo fractional derivative is usually used.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Positive Solutions for a Weakly Singular Hadamard-Type Fractional Differential Equation with Changing-Sign Nonlinearity

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In this paper, we focus on the existence of positive solutions for a class of weakly singular Hadamard-type fractional mixed periodic boundary value problems with a changing-sign singular perturbation. By using nonlinear analysis methods combining with some numerical techniques, we further discuss the effect of the perturbed term for the existence of solutions of the problem under the positive, negative, and changing-sign cases. The interesting points are that the nonlinearity can be singular at the second and third variables and be changing-sign.

1. Introduction

In this paper, we study the existence of positive solutions for the following Hadamard-type fractional differential equation with mixed periodic boundary conditions:

$$\begin{cases} -\left(\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)\right) - b(t)\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t) = f\left(t, z(t), -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)\right) + \chi(t), \quad 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0,\ \mathscr{D}_{\mathbf{t}}{}^{\beta}z(1) = \mathscr{D}_{\mathbf{t}}{}^{\beta}z(e),\ \mathscr{D}_{\mathbf{t}}{}^{\beta+1}z(1) = \mathscr{D}_{\mathbf{t}}{}^{\beta+1}z(e), \end{cases} \tag{1}$$

where $2 < \beta \le 3$, $\mathscr{D}_{\mathbf{t}}{}^{\beta}$ is the Hadamard fractional derivative of order α, β, $b \in L^p(1, e)$ and $f : [1, e] \times \mathbb{R} \setminus \{0\} \times \mathbb{R}\{0\} \to (0, \infty)$ is a continuous function with singularity at the second and third space variables; $\chi \in L^p([1, e], \mathbb{R}), p \ge 1$ is a variable sign function.

Nonlocal characteristics are the most important property of the fractional differential operator; because of this, the fractional differential equations can describe many viscoelasticities and memory phenomena of natural science. For example, some viscoelastic materials, such as the silicone gel with the property of weak frequency dependency, often involve a complicated strain-stress relationship; if let $\delta(t)$ and $\rho(t)$ be the stress and the strain, respectively, then the stress decays after a shear jump is governed by the following fractional order viscoelasticity Kelvin-Voigt equation [1, 2]:

$$\delta(t) = \omega\tau^{\alpha}\mathscr{D}_{\mathbf{t}}{}^{\alpha}\rho(t) + \omega\tau^{\beta}\mathscr{D}_{\mathbf{t}}{}^{\beta}\rho(t), \tag{2}$$

where $\alpha > \beta > 0$, ω, τ are constants and $\mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\beta}$ are Riemann-Liouville fractional derivatives. In practice, the study of the qualitative properties of solutions for the corresponding fractional models such as existence, uniqueness, multiplicity, and stability is necessary to analyze and control the model under consideration [3–21]. In [3], Zhang et al. considered a singular fractional differential equation with signed measure

$$\begin{cases} (-\mathscr{D}_{\mathbf{t}}{}^{\alpha}x)(t) = f\left(t, x(t), \mathscr{D}_{\mathbf{t}}{}^{\beta}x(t)\right), \quad t \in (0,1), \\ \mathscr{D}_{\mathbf{t}}{}^{\beta}x(0) = 0, \mathscr{D}_{\mathbf{t}}{}^{\beta}x(1) = \int_0^1 \mathscr{D}_{\mathbf{t}}{}^{\beta}x(s)dA(s), \end{cases} \tag{3}$$

where $\mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\beta}$ are the standard Riemann-Liouville derivatives, $\int_0^1 x(s)dA(s)$ is denoted by a Riemann-Stieltjes integral and $0 < \beta \le 1 < \alpha \le 2$, $\alpha - \beta > 1$, A is a function of bounded variation and dA can be a signed measure; the nonlinearity $f(t, x, y)$ may be singular at both $t = 0, 1$ and $x = y = 0$. By using the spectral analysis of the relevant linear operator and Gelfand's formula combining the calculation of a fixed point index of the nonlinear operator, some sufficient conditions for the existence of positive solutions were established. Recently, by using the fixed point index theory, Wang [10] established the existence and multiplicity of positive solutions for a class of singular fractional differential equations with nonlocal boundary value conditions, where the nonlinearity may be singular at some time and space variables.

In the recent years, to improve and develop the fractional calculus, there are several kinds of fractional derivatives and integral operators with different kernels such as Hadamard, Erdelyi-Kober, Caputo-Fabrizio derivatives, Hilfer derivatives, and integrals to be given to enrich the application of the fractional calculus such as the Rubella disease and human liver model [22–25]. In particular, the Hadamard derivative is a nonlocal fractional derivative with singular logarithmic kernel. So the study of Hadamard fractional differential equations is relatively difficult; see [26–32].

In this paper, we are interested in the existence of solutions for the Hadamard-type fractional differential equation (1) which involves a singular perturbed term χ; we will further discuss the effect of the perturbed term for the existence of solutions when χ is positive, negative, and changing-sign. Our main tools rely on nonlinear analysis methods as well as some numerical techniques. Thus, in order to make our work be more self-contained, a brief overview for these methods and techniques should be necessary. In recent work [33–41], some fixed point theorems were employed to study the qualitative properties of solutions for various types of differential equations. For obtaining numerical and analytical results, many authors [42–59] also developed iterative techniques to solve some nonlinear problems with practical applications. In addition, variational methods [60–78] and upper and lower solution methods [2, 25, 79, 80] also offered wonderful tools for dealing with various nonlinear ordinary and partial differential equations arising from natural science fields. These analysis and techniques not only improved and perfected the relative theoretical framework of differential equations but also gave some new understand for the corresponding natural phenomena.

Our work has some new features. Firstly, the equation contains a Hadamard-type fractional derivative which has a singular logarithmic kernel. Secondly, the nonlinearity involves a perturbed term which can be positive, negative, or changing-sign. Thirdly, the nonlinearity is allowed to have weakly singular at space variables, which is a class of interesting natural phenomena. In the end, the effect of the perturbed term for the existence of solutions of the equation is discussed, and the criteria on the existence of positive solutions are established for all cases of the perturbed term including positive, negative, and changing-sign cases. The rest of this paper is organized as follows. In Section 2, we firstly introduce the concept of the Hadamard fractional integral and differential operators and then give the logarithmic kernel and Green function of periodic boundary value problem and their properties. Our main results are summarized in Section 3 which includes three theorems for three different situations. An example is given in Section 4.

2. Basic Definitions and Preliminaries

Before starting our main results, we firstly recall the definition of the Hadamard-type fractional integrals and derivatives; for detail, see [81].

Let $\alpha \in \mathbb{C}$, $\mathbf{R}e(\alpha) > 0$, $n = [\mathbf{R}e(\alpha)]$, and (a, b) be a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$(I_a^{\alpha}x)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}ds, \quad t \in (a, b), \tag{4}$$

and the α left Hadamard fractional derivative is defined by

$$(\mathscr{D}_{\mathbf{t}}{}^{\alpha})x(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{x(s)}{s}ds, \quad t \in (a, b). \tag{5}$$

In what follows, we firstly consider the linear auxiliary problem

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t) = x(t), \quad 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0. \end{cases} \tag{6}$$

It follows from [26] that the problem (6) has a unique solution:

$$z(t) = \int_1^e H(t, s)x(s)\frac{ds}{s}, \tag{7}$$

where

$$H(t, s) = \frac{1}{\Gamma(\beta)}\begin{cases} (\ln t)^{\beta-1}(1-\ln s)^{\beta-2} - (\ln t - \ln s)^{\beta-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\beta-1}(1-\ln s)^{\beta-2}, & 1 \le t \le s \le \mathrm{e}, \end{cases} \tag{8}$$

is Green's function of (6). Now, let $x(t) = -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)$, then the Hadamard-type fractional differential equation (1) reduces to the following second order integrodifferential equation:

$$\begin{cases} x''(t) + b(t)x(t) = f\left(t, \int_1^e H(t, s)x(s)\frac{ds}{s}, x(t)\right) + \chi(t), \quad t \in (1, e), \\ x(1) = x(e), \quad x'(1) = x'(e). \end{cases} \tag{9}$$

In order to obtain the solution of the second order integrodifferential equation (9), let us consider the linear periodic boundary value problem

$$\begin{cases} x'' + b(t)x = 0, \\ x(1) = x(e), \quad x'(1) = x'(e). \end{cases} \tag{10}$$

Clearly, by Fredholm's alternative, the nonhomogeneous equation $x'' + b(t)u = a(t)$ has a unique $(e-1)$-periodic solution:

$$x(t) = \int_1^e G(t,s)a(s)ds, \tag{11}$$

where $G(t,s)$ is the Green function of line equation (10) subject to periodic boundary conditions $x(1) = x(e)$, $x'(1) = x'(e)$.

Thus, it follows from (10) that the equation (9) is equivalent to the following integral equation:

$$x(t) = \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right) + \chi(s)\right]ds. \tag{12}$$

As a result, in order to obtain the solution of equation (1), it is sufficient to find the fixed point of the following operator:

$$Tx(t) = \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right) + \chi(s)\right]ds. \tag{13}$$

Lemma 1 (see [26]). *Green's function H has the following properties:*

(i) $H \in C([1,e]\times[1,e], \mathbb{R}^+)$

(ii) For all $t, s \in [1,e]$, the following inequality holds:

$$\frac{1}{\Gamma(\beta)}(\ln t)^{\beta-1}\ln s(1-\ln s)^{\beta-2} \le H(t,s) \le \frac{1}{\Gamma(\beta)}\ln s(1-\ln s)^{\beta-2} \tag{14}$$

In order to obtain the properties of the Green function G, define the best Sobolev constants as

$$Y(\zeta) = \begin{cases} \dfrac{2\pi}{\zeta(e-1)^{1+(1/\zeta)}}\left(\dfrac{2}{2+\zeta}\right)^{1-(2/\zeta)}\left(\dfrac{\Gamma(1/\zeta)}{\Gamma((1/2)+(1/\zeta))}\right)^2, & \text{if } 1 \le \zeta < \infty, \\ \dfrac{4}{e-1}, & \text{if } \zeta = \infty, \end{cases} \tag{15}$$

where Γ is the gamma function. Let $b \in L^p(1,e)$ and denote $b \succ 0$ means that $b(t) \ge 0$ for all $t \in [1,e]$ and $b(t) > 0$ for t in a subset of positive measure. Define

$$\begin{aligned} p^* &= \frac{p}{p-1}, \quad \text{if } 1 \le p < \infty, \\ p^* &= 1, \quad \text{if } p = \infty. \end{aligned} \tag{16}$$

Then, the following lemma is a direct consequence of Theorem 2.1 and Corollary 2.3 in [82]:

Lemma 2. *Let $1 \le p \le \infty$, assume $b \in L^p(0,T)$ and $b(t) \succ 0$. If $\|b\|_p \le Y(2p^*)$, then $G(t,s) > 0$ for all $(t,s) \in [1,e]\times[1,e]$.*

3. Main Results

In this section, we firstly give our work space $E = C[1,e]$ which equips the maximum norm $\|x\| = \max_{1\le t\le e}|x(t)|$. Let

$$P = \{x \in E : x(t) \ge 0, \quad t \in [1,e]\}, \tag{17}$$

then $(E, \|\cdot\|)$ is a real Banach space and P is a normal cone of E with normal constant 1. For convenience, we denote a set of functions as

$$\Lambda = \left\{b \in L^p(1,e): b \succ 0, \|b\|_p \le Y(2p^*), \quad \text{for } 1 \le p \le \infty\right\}. \tag{18}$$

Now, let us list the hypotheses to be used in the rest the paper.

(F0) $b \in \Lambda$

(F1) There exist a constant $\lambda \in (0,1)$ and two functions $h_1, h_2 \in P$ with $h_1, h_2 \equiv 0$ on any subinterval of $(1,e)$, such that

$$\begin{aligned} &h_1(t)(u+v)^{-\lambda} \le f(t,u,v) \le h_2(t)(u+v)^{-\lambda}, \\ &(u,v) \in (0,\infty)\times(0,\infty), \quad t \in [1,e]. \end{aligned} \tag{19}$$

Suppose $\mu(t)$ is the unique solution of the following equation:

$$\begin{cases} x'' + b(t)x = \chi(t), \quad t \in (1,e),\ x(1) = x(e), \\ x'(1) = x'(e), \end{cases} \tag{20}$$

then from (10), $\mu(t)$ can be written as

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds. \tag{21}$$

Similar to (21), we denote $\kappa_1(t), \kappa_2(t)$ as

$$\begin{aligned} \kappa_1(t) &= \left(1 + \frac{1}{(\beta-1)\Gamma(\beta)}\right)^{-\lambda}\int_1^e G(t,s)h_1(s)ds, \kappa_2(t) \\ &= \int_1^e G(t,s)h_2(s)ds. \end{aligned} \tag{22}$$

Let

$$\mu_* = \inf_{1\le t\le e}\mu(t), \quad \mu^* = \sup_{1\le t\le e}\mu(t),$$
$$\kappa_{1*} = \min_{1\le t\le e}\kappa_1(t), \quad \kappa_1^* = \max_{1\le t\le e}\kappa_1(t), \tag{23}$$
$$\kappa_{2*} = \min_{1\le t\le e}\kappa_2(t), \quad \kappa_2^* = \max_{1\le t\le e}\kappa_2(t).$$

By Lemma 2 and (F0)–(F1), we know that $\kappa_i^*, \kappa_{i*} > 0$, $i = 1, 2$.

Now we shall divide χ into three cases to discuss its influence for the solution of equation (1).

3.1. Positive Case $\chi(t) \ge 0$. In this case, we have the following result:

Theorem 3. *Assume that (F0)–(F1) hold. If $\mu_* \ge 0$, then the Hadamard-type fractional differential equation (1) has at least one positive solution.*

Proof. Let l and L be fixed positive constants; we introduce a closed convex set of cone P:

$$B = \{x \in P : l \le x(t) \le L, \quad t \in [1, e]\}. \tag{24}$$

For any $x \in B$, it follows from Lemma 1 that

$$\frac{l(\ln s)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \le \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} \le \frac{L}{(\beta-1)\Gamma(\beta)}. \tag{25}$$

With the help of (25), for any $s \in [1, e]$, from (19), one gets

$$\begin{aligned}\frac{h_1(s)}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda} &\le f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\\ &\le \frac{h_2(s)}{l^\lambda\left(1+\left((\ln s)^{\beta-1}/(\beta-1)\Gamma(\beta)\right)\right)^\lambda} \le \frac{h_2(s)}{l^\lambda}.\end{aligned} \tag{26}$$

It follows from (26) and (F1) that

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\ge \frac{1}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda}\int_1^e G(t,s)h_1(s)ds \ge \frac{\kappa_{1*}}{L^\lambda},\end{aligned} \tag{27}$$

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\le \frac{1}{l^\lambda}\int_1^e G(t,s)h_2(s)ds + \mu^* \le \frac{\kappa_2^*}{l^\lambda} + \mu^*.\end{aligned} \tag{28}$$

Thus, (27), (28), Lemma 1, (F1), and the Arzela-Ascoli theorem guarantee that the operator $T : B \to P$ is completely continuous.

In the following, we shall choose a suitable $L > 0$ such that T maps the closed convex set B into itself. To do this, we only need to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^\lambda} \ge l, \ \frac{\kappa_2^*}{l^\lambda} + \mu^* \le L. \tag{29}$$

In fact, take $l = 1/L$, it follows from $\kappa_{1*} > 0$ and $0 < \lambda < 1$ that there exists a sufficiently large $L > 1$ such that

$$\kappa_{1*}L^{1-\lambda} \ge 1, \ L^\lambda\kappa_2^* + \mu^* \le L, \tag{30}$$

which imply that (29) holds and $l < L$.

Thus, from Schauder's fixed point theorem, T has a fixed point $x^* \in B$, and hence, the Hadamard-type fractional differential equation (1) has at least one positive solution:

$$z^*(t) = \int_1^e H(t,s)x^*(s)\frac{ds}{s}. \tag{31}$$

3.2. Negative Case $\chi(t) \le 0$. For this case, we have the following existence result:then the Hadamard-type fractional differential equation (1) has at least one positive solution.

Theorem 4. *Assume that (F0)–(F1) hold. If $\mu^* \le 0$, and*

$$\mu_* \ge \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)}\left(1 - \frac{1}{\lambda^2}\right), \tag{32}$$

Proof. Firstly, from (26) and (F1), we have

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\ge \frac{1}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda}\int_1^e G(t,s)h_1(s)ds + \mu_*\\ &\ge \frac{\kappa_{1*}}{L^\lambda} + \mu_*,\end{aligned} \tag{33}$$

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\le \frac{1}{l^\lambda}\int_1^e G(t,s)h_2(s)ds \le \frac{\kappa_2^*}{l^\lambda}.\end{aligned} \tag{34}$$

Thus, by (33) and (34), in order to guarantee $T : B \to B$, it is sufficient to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^\lambda} + \mu_* \ge l, \ \frac{\kappa_2^*}{l^\lambda} \le L. \tag{35}$$

Let us fix $L = \kappa_2^*/l^\lambda$; to ensure (35) holds, we only need to choose some $l > 0$ such that

$$0 < l < \kappa_2^{*1/(1+\lambda)}, \mu_* \geq l - \frac{\kappa_{1*}}{\kappa_2^{*\lambda}} l^{\lambda^2} = g(l). \tag{36}$$

Obviously, the function $g(l)$ in $(0, \infty)$ has a minimum at

$$\tilde{l} = \left(\frac{\lambda^2 \kappa_{1*}}{\kappa_2^{*\lambda}}\right)^{1/\left(1-\lambda^2\right)}. \tag{37}$$

Taking $l = \tilde{l}$, since $\kappa_{1*} \leq \kappa_2^*, 0 < \lambda^2 < 1$, we have

$$0 < l = \tilde{l} = \left(\frac{\lambda^2 \kappa_{1*}}{\kappa_2^{*\lambda}}\right)^{1/\left(1-\lambda^2\right)} < \left(\frac{\kappa_2^*}{\kappa_2^{*\lambda}}\right)^{1/\left(1-\lambda^2\right)} = \kappa_2^{*1/(1+\lambda)}, \tag{38}$$

and from (32), we also have

$$\mu_* \geq \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/\left(1-\lambda^2\right)} \left(1 - \frac{1}{\lambda^2}\right) = g\left(\tilde{l}\right) \geq g(l), \tag{39}$$

which implies that (36) holds if $l = \tilde{l}, L = \kappa_2^*/\tilde{l}^{\lambda}$. Consequently, we have $T : B \to B$.

Thus, from Schauder's fixed point theorem, T has a fixed point $x^* \in B$, and hence, the Hadamard-type fractional differential equation (1) has at least one positive solution:

$$z^*(t) = \int_1^e H(t,s) x^*(s) \frac{ds}{s}. \tag{40}$$

Remark 5. Note that the right side of inequality (32) is negative because of the weak force condition $0 < \lambda < 1$, so if $\mu_* = 0$, the inequality (32) is always valid. Thus, if $\mu_* = 0$, we can omit the assumption (32).

3.3. Changing-Sign Case for $\chi(t)$. In order to establish the existence result under the case where $\chi(t)$ is changing-sign, we need the following lemma.

Lemma 6. *If $0 < \lambda < 1$ and $x \in (0,\infty)$, the equation*

$$x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda} = \lambda^2 \kappa_2^* \kappa_{1*}, \tag{41}$$

has a unique solution $\tilde{l}$ which satisfies

$$0 < \tilde{l} < \varepsilon := \left(\lambda^2 \kappa_2^* \kappa_{1*}\right)^{1/(2+2\lambda)}. \tag{42}$$

Proof. Let

$$\varphi(x) = \lambda^2 \kappa_2^* \kappa_{1*} - x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda}, \tag{43}$$

then

$$\varphi(0) = \lim_{x \to 0} \varphi(x) = \lambda^2 \kappa_2^* \kappa_{1*} > 0. \tag{44}$$

On the other hand, since $0 < \lambda < 1, \kappa_2^* \geq \kappa_{1*}$, one has

$$0 < \varepsilon \leq \kappa_2^{*1/(1+\lambda)}. \tag{45}$$

Therefore,

$$\begin{aligned} \varphi(\varepsilon) &= \varepsilon^{2+2\lambda} - \varepsilon^{1-\lambda^2}\left(\kappa_2^* + \mu^* \varepsilon^{\lambda}\right)^{1+\lambda} \\ &= \varepsilon^{1-\lambda^2}\left[\varepsilon^{(1+\lambda)^2} - \left(\kappa_2^* + \mu^* \varepsilon^{\lambda}\right)^{1+\lambda}\right] \\ &< \varepsilon^{1-\lambda^2}\left[\varepsilon^{(1+\lambda)^2} - \kappa_2^{*1+\lambda}\right] < 0. \end{aligned} \tag{46}$$

Moreover,

$$\begin{aligned} \varphi'(x) = &-\left(1-\lambda^2\right)x^{-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda} - \mu^* \lambda(1+\lambda)x^{1-\lambda^2} \\ &\cdot \left(\kappa_2^* + \mu^* x^{\lambda}\right)^{\lambda} x^{\lambda-1} < 0, \quad x \in (0,\infty). \end{aligned} \tag{47}$$

Thus, it follows from (44) to (47) that equation (41) has a unique positive solution $\tilde{l} \in (0, \varepsilon)$.

Theorem 7. *Assume that (F0)–(F1) hold. Let $\tilde{l}$ be the unique solution of equation (41), if $\mu_* \leq 0$, $\mu^* \geq 0$ and satisfy*

$$\mu_* \geq \tilde{l} - \frac{\kappa_{1*}\tilde{l}^{2\lambda}}{\left(\kappa_2^* + \mu^* \tilde{l}^{\lambda}\right)^{\lambda}}, \tag{48}$$

then the Hadamard-type fractional differential equation (1) has at least one positive solution.

Proof. In this case, following the strategy and notations of (33) and (34), we have

$$\begin{aligned} (Tx)(t) &\geq \frac{1}{L^{\lambda}(1 + (1/(\beta-1)\Gamma(\beta)))^{\lambda}} \int_1^e G(t,s) h_1(s) ds + \mu_* \\ &\geq \frac{\kappa_{1*}}{L^{\lambda}} + \mu_*, (Tx)(t) \leq \frac{1}{l^{\lambda}} \int_1^e G(t,s) h_2(s) ds + \mu^* \\ &\leq \frac{\kappa_2^*}{l^{\lambda}} + \mu^*. \end{aligned} \tag{49}$$

Thus, to ensure $T : B \to B$, it is sufficient to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^{\lambda}} + \mu_* \geq l, \ \frac{\kappa_2^*}{l^{\lambda}} + \mu^* \leq L. \tag{50}$$

To do this, we fix $L = \kappa_2^*/l^{\lambda} + \mu^*$. Clearly, if l satisfies

$$0 < l < L = \frac{\kappa_2^*}{l^{\lambda}} + \mu^*, \tag{51}$$

$$\mu_* \geq l - \frac{\kappa_{1*} l^{2\lambda}}{\left(\kappa_2^* + \mu^* l^\lambda\right)^\lambda}, \tag{52}$$

then the inequalities of (50) hold.

Let

$$\psi(x) = x - \frac{\kappa_{1*} x^{2\lambda}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^\lambda}, \tag{53}$$

notice that

$$\begin{aligned}\psi'(x) &= 1 - \frac{\kappa_{1*}\left[\lambda^2 x^{\lambda^2-1}\left(\kappa_2^* + \lambda^2\mu^* x^\lambda\right)^\lambda - \lambda^* x^{\lambda^2}\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda-1} x^{\lambda-1}\right]}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{2\lambda}} \\ &= 1 - \frac{\kappa_{1*}\lambda^2 x^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^\lambda}\left[1 - \frac{\mu^* x^\lambda}{\kappa_2^* + \mu^* x^\lambda}\right] = 1 - \frac{\kappa_{1*}\kappa_2^*\lambda^2 x^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+1}},\end{aligned} \tag{54}$$

and $\psi'(0) = -\infty, \psi'(\infty) = 1$; consequently, there exists $\tilde{l}$ such that

$$\psi'\left(\tilde{l}\right) = 1 - \frac{\kappa_{1*}\kappa_2^*\lambda^2 \tilde{l}^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* \tilde{l}^\lambda\right)^{\lambda+1}} = 0, \tag{55}$$

which implies that $\tilde{l}$ solves the equation

$$x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^\lambda\right)^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}, \tag{56}$$

i.e., $\tilde{l}$ is the unique solution of equation (41).

On the other hand, since

$$\psi''(x) = \frac{\left(1-\lambda^2\right)\lambda^2\kappa_{1*}\kappa_2^* x^{\lambda^2-2}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+1}} + \frac{(1+\lambda)\lambda^3\mu^*\kappa_{1*}\kappa_2^* x^{\lambda^2+\lambda-2}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+2}} > 0, \tag{57}$$

the function $\psi(x)$ gets the minimum at $\tilde{l}$, i.e.,

$$\psi\left(\tilde{l}\right) = \min_{x\in(0,\infty)} \psi(x). \tag{58}$$

Taking $l = \tilde{l}$, then the assumption (48) implies that (52) holds.

Thus, we only need to prove that the inequality $L > l$ is also satisfied. Notice that $0 < l = \tilde{l} < \varepsilon =: \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)}$ solves (41), we have

$$l^{1-\lambda^2}\left(\kappa_2^* + \mu^* l^\lambda\right)^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}. \tag{59}$$

It follows from $L = \kappa_2^*/l^\lambda + \mu^*$ that

$$l^{1+\lambda}L^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}, \tag{60}$$

and then

$$L = \frac{1}{l}\left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(1+\lambda)}. \tag{61}$$

Thus, it follows from

$$l < \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)}, \tag{62}$$

that

$$L = \frac{1}{l}\left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(1+\lambda)} > \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)} > l. \tag{63}$$

That is, under the assumptions of Theorem 7, (51) and (52) all hold. Thus, according to Schauder's fixed point theorem, the Hadamard-type fractional differential equation (1) has at least one positive solution

$$z^*(t) = \int_1^e H(t,s)x^*(s)\frac{ds}{s}. \tag{64}$$

Remark 8. In this case, the nonlinearity $f(t,u,v)$ of equation (1) may be singular at $u = 0$ and $v = 0$; moreover, $\chi(t)$ can be changing-sign $L^p(1,e)$ function, which is allowed to be singular at some $t \in [1,e]$.

4. Examples

In this section, we give some examples with positive, negative, and changing-sign perturbations to demonstrate the application of our main results.

We consider the following Hadamard-type fractional differential equation with different perturbations:

$$\begin{cases} \mathscr{D}_{\mathrm{t}}^{9/2} z(t) + \frac{1}{4}\mathscr{D}_{\mathrm{t}}^{5/2} z(t) = \frac{39}{2\left(z(t) + |\mathscr{D}_{\mathrm{t}}^{5/2} z(t)|\right)^{1/2}} + \chi(t), \quad 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0,\ \mathscr{D}_{\mathrm{t}}^{5/2} z(1) = \mathscr{D}_{\mathrm{t}}^{5/2} z(e),\ \mathscr{D}_{\mathrm{t}}^{9/2} z(1) = \mathscr{D}_{\mathrm{t}}^{9/2} z(e). \end{cases} \tag{65}$$

Example 9. Consider the case of equation (65) with positive force term $\chi(t) = 1$.

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. Take $\lambda = 1/2$, $h_1(t) = 19$, $h_2(t) = 20$ and

$$f(t,u,v) = \frac{39}{2(u+v)^{1/2}}, (u,v) \in (0,\infty)\times(0,\infty), \quad t \in (1,e), \tag{66}$$

then

$$\frac{19}{(u+v)^{1/2}} \leq f(t,u,v) \leq \frac{20}{(u+v)^{1/2}}, (u,v) \in (0,\infty)\times(0,\infty), \quad t \in (1,e). \tag{67}$$

It is easy to check that the Green function of the equation $x''(t) + (1/4)x(t) = a(t)$ with periodic boundary conditions $x(1) = x(e)$, $x'(1) = x'(e)$ is

$$G(t,s) = \begin{cases} \sin\dfrac{(e-1)-t+s}{2}\sin\dfrac{t-s}{2}, & 1 \le s \le t \le e, \\ \sin\dfrac{(e-1)-s+t}{2}\sin\dfrac{s-t}{2}, & 1 \le t \le s \le e. \end{cases} \tag{68}$$

Thus, $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = \sin\left(\frac{1}{2}e - \frac{1}{2}\right) + \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right)e = 0.1935. \tag{69}$$

By simple computations, one has

$$\mu_* = \mu^* = 0.1935,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{70}$$

Now, take $L = 15.3615$ and let $l = 1/L = 0.0651$. Clearly, L satisfies the inequalities

$$3.0004L^{1/2} \ge 1,\ 3.87L^{1/2} + 0.1935 \le L, \tag{71}$$

thus, according to Theorem 3, the Hadamard-type fractional differential equation (65) has at least one positive solution.

Example 10. We consider the case of Hadamard-type fractional differential equation (65) with negative force term $\chi(t) = -3$.

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. We still take $\lambda = 1/2$, $h_1(t) = 19$, $h_2(t) = 20$, then as Example 9, we have $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = -3\left(\sin\left(\frac{1}{2}e - \frac{1}{2}\right) + \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right)e\right) = -0.5805. \tag{72}$$

By simple computations, one has

$$\mu_* = \mu^* = -0.5805,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{73}$$

So $\mu^* = -0.5805 \le 0$, and

$$\mu_* = -0.5805 \ge \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/\left(1-\lambda^2\right)}\left(1 - \frac{1}{\lambda^2}\right) = -0.8295, \tag{74}$$

thus, according to Theorem 4, the Hadamard-type fractional differential equation (65) has at least one positive solution.

Example 11. Consider the case of Hadamard-type fractional differential equation (65) with changing-sign force term

$$\chi(t) = \begin{cases} -1, & t \in [1,2), \\ \dfrac{10000}{729}, & t \in [2,e]. \end{cases} \tag{75}$$

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. It follows from Example 9 that $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = \begin{cases} -\left[\sin\dfrac{e-1}{2} - \dfrac{1}{2}\cos\dfrac{e-1}{2} - \sin\dfrac{e-2}{2}\cos\left(\dfrac{3}{2} - t\right)\right], & 1 \le t < 2, \\ \dfrac{10000}{729}\left[-\dfrac{1}{2}e\cos\dfrac{e-1}{2} + \cos\dfrac{e-1}{2} + \sin\dfrac{e-1}{2} - \sin\dfrac{1}{2}\cos\left(\dfrac{e+2}{2} - t\right)\right], & 2 \le t \le \mathrm{e}, \end{cases}$$

$$\kappa_1(t) = \left(1 + \frac{1}{(\beta-1)\Gamma(\beta)}\right)^{-\lambda}\int_1^e G(t,s)h_1(s)ds = 15.5059\int_1^e G(t,s)ds, \kappa_2(t) = \int_1^e G(t,s)h_2(s)ds = 20\int_1^e G(t,s)ds. \tag{76}$$

By simple computations, we have

$$\mu_* = -0.0798,\ \mu^* = 1,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{77}$$

Consequently, equation (41) reduces to

$$x^{3/4}\left(3.8700 + x^{1/2}\right)^{3/2} = 2.9029, \tag{78}$$

which has a unique solution $\tilde{l} = 0.2200$. Thus, one has

$$\mu_* = -0.0798 \geq \tilde{l} - \frac{\kappa_{1*}\tilde{l}^{2\lambda}}{\left(\kappa_2^* + \mu^*\tilde{l}^{\lambda}\right)^{\lambda}} = -0.0969, \tag{79}$$

which implies that (48) holds.

According to Theorem 7, the Hadamard-type fractional differential equation (65) has at least one positive solution.

5. Conclusion

The force effect from outside in many complex processes often leads to the system governed equations possessing perturbations. In this work, we establish several criteria on the existence of positive solutions for a Hadamard-type fractional differential equation with singularity in all of the cases where the perturbed term is positive, negative, and changing-sign. The main advantage of our assumption is that it provides an effective bound for the perturbed term χ. This classification is valid and reasonable, and it is also easier to get the solution of the target equation by simple calculation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Research Article

Variational Method to p-Laplacian Fractional Dirichlet Problem with Instantaneous and Noninstantaneous Impulses

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In this paper, a research has been done about the existence of solutions to the Dirichlet boundary value problem for p-Laplacian fractional differential equations which include instantaneous and noninstantaneous impulses. Based on the critical point principle and variational method, we provide the equivalence between the classical and weak solutions of the problem, and the existence results of classical solution for our equations are established. Finally, an example is given to illustrate the major result.

1. Introduction

Fractional calculus involves arbitrary order derivatives and integration, so it plays a very important role in various fields such as physical engineering, medical image processing, mathematics, chemical engineering, and electricity. For this reason, many scholars did research on the theory of fractional differential equations continuously and have made enormous achievements; readers who are interested in these kinds of researches can refer to relevant literature (see [1–6] and the references therein). Of course, in the science field, an impulsive phenomenon has also been spread widely when dealing with practical problems, and it has become a very effective tool for describing changes in sudden discontinuous jumps. In addition, from the perspective of the duration of the change, impulses can be divided into instantaneous impulses and noninstantaneous impulses. The difference between them is that the duration of the instantaneous impulse continuous change is relatively short compared to the duration of the entire process, and the noninstantaneous impulse change is to keep moving from any fixed point and at a certain time interval. In [7], Agarwal et al. provide a more detailed introduction to these two impulsive differential equations. We also noticed that some experts combine the theory of fractional calculus, of noncompactness, with Sadovskii's fixed-point theorem and even more excellent methods have obtained more and better properties for fractional order equations with impulsive conditions, which greatly promotes its development (see, for example, [8–15], and the references therein).

For a long time, many scholars have conducted in-depth research on instantaneous impulsive differential equations. By using fixed-point theorems, critical point theorems, and variational methods, they obtained the existence of solutions (see [16–18]). But in many cases, instantaneous impulses cannot describe the development of certain dynamics. In 2013, Hernández and O'Regan first proposed the concept of noninstantaneous impulsive differential equations (see [19]). Since then, the existence of solutions to noninstantaneous impulsive problems has been gradually expanded by using some methods such as fixed-point theory and analytical semigroup theory, but the variational structure of general noninstantaneous impulsive differential equations has not received widespread attention. Among them, the existence of solutions for second-order differential equations with instantaneous and noninstantaneous impulses was investigated. Tian and Zhang obtained the existence results through the principle of variation (see [20]). In addition, it is worth noting that by using the critical point theory and variational method, Zhao et al. proved the existence and multiplicity of nontrivial solutions to the problem of nonlinear nontransient impulsive differential equations (see [21]). At the same time,

some scholars have obtained the existence results of solutions for noninstantaneous impulsive differential equations by variational methods (see [22–24]).

In [25], Zhang and Liu generalized linear fractional differential equations to nonlinearities on the interval $t \in (s_i, t_i + 1]$ and removed the restriction ${}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(0)) = \text{constant}$. By using a variational method, they considered the following Dirichlet problems with instantaneous and noninstantaneous impulse differential equations:

$$\begin{cases} {}_tD_T^\alpha\left({}_0^CD_t^\alpha\right)u(t) = f_i(t, u(t)), \quad t \in (s_i, t_i + 1], i = 0, 1, 2, \cdots, n, \\ \Delta_t D_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t_i^+), \quad t \in (t_i, s_i], i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(s_i^+), \quad i = 1, 2, \cdots, n, \\ u(0) = u(T) = 0. \end{cases} \tag{1}$$

In [26], Zhou et al. discussed the existence of solutions for fractional differential equations of p-Laplacian with instantaneous and noninstantaneous impulses. The innovation was that when $p = 2$, problem (2) can be regarded as problem (1), and when $\alpha = 1$, it can be simplified to a more general integer order case. Finally, they got the classical solution and prove that the weak solution was equivalent to the classical solution:

$$\begin{cases} {}_tD_T^\alpha\phi_P\left({}_0^CD_t^\alpha u(t)\right) + g(t)|u|^{P-2}u(t) = f_i(t, u(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, n, \\ \Delta_t D_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i^+), \quad t \in (t_i, s_i], i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^+), \quad i = 1, 2, \cdots, n, \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

Motivated by the above-mentioned work, the paper focuses on the existence of solutions for the following p-Laplacian fractional differential equations with instantaneous and noninstantaneous impulses, and if $D_xF_i(t, u(t) - u(t_i + 1)) = f_i(t, u(t))$, the following problem reduces to (2):

$$\begin{cases} {}_tD_T^\alpha\phi_P\left({}_0^CD_t^\alpha u(t)\right) + \lambda(t)|u|^{p-2}u(t) = D_xF_i(t, u(t) - u(t_i + 1)), \\ t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, N, \\ \Delta_t D_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, N, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i^+), \quad t \in (t_i, s_i], i = 1, 2, \cdots, N, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^+), \quad i = 1, 2, \cdots, N, \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $\alpha \in (1/p, 1], p \geq 2$, ${}_tD_T^\alpha$, and ${}_0^CD_T^\alpha$ denote the right Riemann-Liouville fractional derivatives and left Caputo fractional derivatives. $0 = s_0 < t_0 < s_1 < t_1 < \cdots < t_N = T, I_i \in C(R, R)$, and there exists $i \in \{1, 2, \cdots, N\}$ such that $I(u(t_i)) \neq 0$. $f_i \in C((s_i, t_{i+1}] \times R, R), \phi_p : R \longrightarrow R$ is the p-Laplacian function defined as $\phi_p(s) = |s|^{p-2}s(s \neq 0), \phi_p(0) = 0. \lambda \in L^\infty[0, T]$, $\operatorname{essinf}_{t\in(0,T)}\lambda(t) > -(\Gamma(\alpha + 1)/T^\alpha)^p$. The nonlinear functions $D_xF_i(t, x)$ are the derivatives of $F_i(t, x)$, for every

$i = 0, 1, 2, \cdots, N$. To the best of our knowledge, the instantaneous impulses start abruptly at point t_i and the noninstantaneous impulses continue during the intervals $(t_i, s_i]$, and

$$\begin{aligned}
&\Delta_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i) = {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^+) - {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^-),\\
&{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^\pm) = \lim_{t\to t_i^\pm}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t)\right),\\
&{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(s_i^\pm) = \lim_{t\to s_i^\pm}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t)\right).
\end{aligned} \tag{4}$$

The rest of the paper is organized as follows. In Section 2, some basic knowledge and lemmas used in the latter are presented. In Section 3, we first give the equivalent form of solution of problem (3), and secondly, we establish the equivalence of the classical solution and the weak solution by using the critical point principle for the variational structure of problem (3) and finally prove the existence of solution. An illustrative example is given to show the practical usefulness of the analytical results in Section 4.

2. Preliminaries

In this section, we introduce some important definitions, lemmas, and theorems that are important to use later. For the definitions of the left and right fractional integrals and derivatives, we can refer to References [7, 23].

Definition 1 (see [26]). Let $\alpha \in (1/p, 1]$, $p \in [2,\infty)$, the fractional space

$$E_0^{\alpha,p} = \left\{u : [0,T] \longrightarrow R\big|_U, {}_0^C D_t^\alpha u \in L^p([0,T],R), u(0) = u(T)\right\} \tag{5}$$

is defined by the closure of $C_0^\infty([0,T],R)$ with the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T \lambda(t)|u(t)|^p dt + \int_0^T |{}_0^c D_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{6}$$

Definition 2 (see [26]). Let $0 < \alpha \le 1$, the fractional derivative space $E_0^{\alpha,p}$ is a reflexive as well as separable Banach space.

Remark 3 (see [25]). Let $p = 2$, we define $E_0^\alpha = E_0^{\alpha,2}$. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2([0,T],R)$ having an α-order Caputo fractional derivative ${}_0^c D_t^\alpha u \in L^2([0,T],R)$ and $u(0) = u(T)$.

Proposition 4 (see [26]). *Let $0 < \alpha \le 1$ and $u \in AC([a,b])$, then for $t \in [a,b]$,*

$$\begin{aligned}
{}_a^c D_t^\alpha u(t) &= {}_a D_t^\alpha u(t) - \frac{u(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha},\\
{}_t^c D_b^\alpha u(t) &= {}_t D_b^\alpha u(t) - \frac{u(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.
\end{aligned} \tag{7}$$

Proposition 5 (see [26]). *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$, for any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L_p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{1/p}. \tag{8}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \Omega\left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{1/q}, \tag{9}$$

where $\Omega = T^{\alpha-(1/p)}/\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}$.

Proposition 6 (see [26]). *Let $\alpha \in (1/p, 1)$, the space $E_0^{\alpha,p}$ is compactly embedded in $C([0,T],R)$.*

Proposition 7 (see [26]). *Let $\alpha \in (0,1]$, and $u, v \in L^p(a,b)$:*

$$\int_a^b ({}_t D_b^\alpha u(t))v(t)dt = \int_a^b ({}_a D_t^\alpha v(t))u(t)dt. \tag{10}$$

Definition 8. A function

$$\begin{aligned}
u \in \Bigg\{ & u \in AC([0,T]): \int_{s_i}^{t_{i+1}} \left(|u(t)|^p + \left|{}_0^C D_t^\alpha u(t)\right|^p\right)dt\\
& < +\infty, i = 0, 1, 2, \cdots, N \Bigg\}
\end{aligned} \tag{11}$$

is a classical solution of problem (3) if u satisfies the conditions of problem (3) and the boundary condition $u(0) = u(T) = 0$ holds.

Lemma 9. *Let X be a reflexive Banach space and let $\varphi : X \longrightarrow (-\infty, +\infty]$ be weakly lower semicontinuous on X. If φ has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 10. *If $\varphi \longrightarrow (-\infty, +\infty]$ is coercive, then φ has a bounded minimizing sequence.*

Lemma 11 (see [26]). *For $u \in E_0^{\alpha,p}$, we define the norm*

$$\|u\|_\alpha = \left(\int_0^T |u(t)|^p dt + \int_0^T \left(\left|{}_0^C D_t^\alpha u(t)\right|\right)^p dt\right)^{1/p} \tag{12}$$

is equivalent to the norm $\|u\|_{\alpha,p}$.

3. Main Results

In this section, we further discuss the variational structure of problem (3) and define the functional φ to prove that the critical point of φ is the weak solution of problem (3). In addition, we also give the equivalence between weak solution and the classical solution of this problem. Finally, the

existence of classical solution for problem (3) is given by Theorem 16.

For each $i = 0, 1, 2, \cdots, N$, the nonlinear functions F_i satisfy the following assumptions:

(H1). $F_i(t,x)$ is measurable in t for every $x \in R$ and continuously differentiable in x for a.e. $t \in (s_i, t_{i+1}]$, and there exist functions $k_1 \in C(R^+, R^+)$, $k_2 \in L^1((s_i, t_{i+1}), R^+)$ such that

$$\begin{aligned} |F_i(t,x)| &\le k_1(x)k_2(t), \\ |D_x F_i(t,x)| &\le k_1(|x|)k_2(t). \end{aligned} \tag{13}$$

(H2). There exist constants $\delta \in (0,2)$, $a > 0$, and the functions $b_0, b_1 \in L^1((s_i, t_i), R^+)$ such that

$$|F_i(t,x)| \le a|x|^2 + b_0(t)|x|^{\delta} + b_1(t), \quad t \in (s_i, t_{i+1}]. \tag{14}$$

(H3). There exist constants $c_i, d_i > 0$, $\sigma_i \in [0,1)$ $(i = 1, 2, \cdots, N)$ such that $|I_i(u)| \le c_i + d_i|u|^{\sigma_i}$, for any $u \in R$.

Lemma 12. *A function $u \in E_0^{\alpha,p}$, $\forall v \in E_0^{\alpha,p}$, the following form is an equivalent form of the problem (3):*

$$\begin{aligned} &\int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt \\ &\quad = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t_i)dt \\ &\qquad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt - \sum_{i=1}^{N}(I_i u(t_i)v(t_i)). \end{aligned} \tag{15}$$

Proof. For $v \in E_0^{\alpha,p}$, by (10) have

$$\begin{aligned} &\int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt \\ &\quad = \int_0^T {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)v'(t)dt \\ &\quad = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)v'(t)dt \\ &\qquad + \sum_{i=1}^{N}\int_{t_i}^{s_i} {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)v'(t)dt \\ &\quad = {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t_1^-)\right)v(t_1) \\ &\qquad - \int_{s_0}^{t_1}\frac{d}{dt}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\qquad + \sum_{i=1}^{N}\left[{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right]v(t)\Big|_{t_i}^{s_i} \\ &\qquad - \sum_{i=1}^{N}\int_{t_i}^{s_i}\frac{d}{dt}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\qquad + \sum_{i=1}^{N-1}\left[{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right]v(t)\Big|_{s_i}^{t_{i+1}} \\ &\qquad - {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(s_N^+)\right)v(s_N) \\ &\qquad - \sum_{i=0}^{N-1}\int_{s_i}^{t_{i+1}}\frac{d}{dt}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\qquad - \int_{s_N}^{T}\frac{d}{dt}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\quad = \int_0^T\left({}_t D_T^\alpha\phi_p\left({}_0^C D_t^\alpha u(t)\right)v(t)\right)dt \\ &\qquad + \sum_{i=1}^{N}\left[{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(s_i^-)\right) - {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)\right]v(s_i) \\ &\qquad + \sum_{i=1}^{N}\left[{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t_i^-)\right) - {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right)\right]v(t_i). \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &\int_0^T\left({}_t D_T^\alpha\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\quad = \int_0^T\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt \\ &\qquad + \sum_{i=1}^{N}(I_i u(t_i))v(t_i). \end{aligned} \tag{17}$$

As for problem (3), we have

$$\begin{aligned} &\int_0^T\left({}_t D_T^\alpha\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\quad = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\left({}_t D_T^\alpha\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\qquad + \sum_{i=1}^{N}\int_{t_i}^{s_i}\left({}_t D_T^\alpha\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\quad = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t_i)dt \\ &\qquad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\lambda(t)|u(t)|^{p-2}u(t)v(t)dt \\ &\qquad - \sum_{i=1}^{N}\int_{t_i}^{s_i}\frac{d}{dt}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt \\ &\quad = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t_i)dt \\ &\qquad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\lambda(t)|u(t)|^{p-2}u(t)v(t)dt. \end{aligned} \tag{18}$$

Above all, problem (3) has the following equivalent form:

$$\int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t_i)dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt - \sum_{i=1}^{N}(I_i u(t_i))v(t_i). \tag{19}$$

Definition 13. A function $u \in E_0^{\alpha,p}$ is a weak solution of problem (3) if (15) holds for all $v \in E_0^{\alpha,p}$.

Define the function $\varphi : E_0^{\alpha,p} \longrightarrow R$ as

$$\varphi(u) = \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\phi_i(u) + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)ds, \tag{20}$$

where $\phi_i(u) = \int_{s_i}^{t_{i+1}} F_i(t, u(t)-u(t_{i+1}))dt$.

From (9), we have

$$|u(t)-u(t_{i+1})| \le 2\|u\|_\infty \le \frac{T^{\alpha-(1/p)}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}\|u\|_\alpha = 2\Omega\|u\|_\alpha. \tag{21}$$

Under condition (H1), for a.e. $t \in (s_i, t_{i+1}]$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[F_i(t, u(t)-u(t_{i+1})+\varepsilon v(t)) - F_i(t, u(t)-u(t_{i+1})] = D_x F_i(t, u(t)-u(t_{i+1}))v(t). \tag{22}$$

From (21), for all $\gamma \in (0,1)$,

$$|u(t)-u(t_{i+1})+\gamma\varepsilon v(t)| \le 2\Omega\|u\|_\alpha + \|v\|_\infty \le 2\Omega\|u\|_\alpha + \Omega\|v\|_\alpha. \tag{23}$$

According to the mean value theorem and condition (H1), for some $\gamma \in (0,1)$,

$$\left|\frac{1}{\varepsilon}[F_i(t, u(t)-u(t_{i+1})+\varepsilon v(t)) - F_i(t, u(t)-u(t_{i+1}))]\right| = |D_x F_i(t, u(t)-u(t_{i+1})+\gamma\varepsilon v(t))v(t)| \le \Omega\|v\|_\alpha \max_{\omega\in[0,2\Omega\|u\|_\alpha+\Omega\|v\|_\alpha)} k_1(\omega)k_2(t). \tag{24}$$

By the Lebesgue dominated convergence theorem, it is easy to show that ϕ_i has a directional derivative at each point u:

$$\left\langle \phi_i'(u), v\right\rangle = \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t)dt,$$

$$\left|\left\langle \phi_i'(u), v\right\rangle\right| \le \int_{s_i}^{t_{i+1}} |D_x F_i(t, u(t)-u(t_{i+1}))||v(t)|dt \le \Omega\|v\|_\alpha \max_{\omega\in[0,2\Omega\|u\|_\alpha+\Omega\|v\|_\alpha)} k_1(\omega)\int_{s_i}^{t_{i+1}} k_2(t)dt. \tag{25}$$

Thus, $\phi_i \in (E_0^{\alpha,p})^*$, let $u_k \longrightarrow u$ in $E_0^{\alpha,p}$, then $\{u_k\}$ converges uniformly to u for all $t \in [0, T]$, we have

$$\left\|\phi_i'(u_n) - \phi_i'(u)\right\|_\alpha \le \Omega\|v\|_\alpha \int_{s_i}^{t_{i+1}} D_x F_i(t, u_n(t)-u_n(t_{i+1})) - D_x F_i(t, u(t)-u(t_{i+1}))dt. \tag{26}$$

Therefore, ϕ_i' is continuous from $E_0^{\alpha,p}$ into $(E_0^{\alpha,p})^*$, $\varphi \in C^1(E_0^{\alpha,p}, R)$ and

$$\left\langle \varphi'(u), v\right\rangle = \int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^{N}(I_i u(t_i))v(t_i) - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t_i)dt + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt. \tag{27}$$

This yields that the critical points of φ are weak solutions of problem (3).

Lemma 14. *The functional $\varphi : E_0^{\alpha,p} \longrightarrow R$ is a weak lower semicontinuous.*

Proof. Let sequence $\{u_k\}_{k=1}^\infty$ in $E_0^{\alpha,p}$ be weakly convergent to u in $E_0^{\alpha,p}$, then $\|u\| \le \lim_{k\to\infty}\inf\ \|u_k\|$.

From Proposition 6, sequence $\{u_k\}_{k=1}^\infty$ is convergent uniformly to u in $C([0,T], R)$; thus, we have

$$\lim_{k\to\infty}\inf \varphi(u_k) = \left[\lim_{k\to\infty}\inf \frac{1}{q}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i\left(t, u_k(t) - u_k(t_{i+1})\right) + \sum_{i=1}^{N}\int_0^{u_k(t_i)} I_i(s)ds\right] \ge \frac{1}{q}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t, u(t)-u(t_{i+1})) + \sum_{i=1}^{N}\int_0^{u_k(t_i)} I_i(s)ds = \varphi(u). \tag{28}$$

Lemma 15. *$u \in E_0^{\alpha,p}$ is a weak solution of the problem (3), if and only if u is a classical solution of the problem (3).*

Proof. From the definition of classical solutions, without loss of generality, if u is a classical solution of problem (3), then it is a weak solution of (3). Conversely, let $u \in E_0^{\alpha,p}$ be a weak solution of (3); thus, $u(0) = u(T) = 0$ and (15) holds. Now, we prove that u is a classical solution of the problem (3). We take a test function $v \in C_0^\infty(s_i t_{i+1}]$ satisfying $v(t) = 0$, where $t \in [0, s_i) \cup [t_{i+1}, T)$, $i = 0, 1, 2, \cdots, N$. Substituting $v(t)$ into (15) and by Proposition 7, we have

$$\int_{s_i}^{t_{i+1}} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt = \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t_i)dt, \tag{29}$$

$$\int_{s_i}^{t_{i+1}} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \int_{s_i}^{t_{i+1}} D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right)v(t)dt < +\infty, \tag{30}$$

that is to say,

$${}_t D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right) + \lambda(t)|u(t)|^{p-2}u(t) = D_x F_i(t, u(t) - u(t_{i+1})), \quad t \in (s_i, t_{i+1}], i = 0, 1, \cdots, N. \tag{31}$$

Since $u \in E_0^{\alpha,p} \subset C([0, T])$, we have

$$\int_{s_i}^{t_{i+1}} \left(|u(t)|^p + \left|{}_0^C D_t^\alpha u(t)\right|^p\right)dt < +\infty. \tag{32}$$

Because of $D_x F_i(t, u(t) - u(t_{i+1})) \in C((s_i, t_{i+1}] \times R, R)$, by (31), we have

$${}_t D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right) \in AC([s_i, t_{i+1}]). \tag{33}$$

Therefore,

$$\begin{aligned} {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(s_i^+) &= \lim_{t \to s_i^+} \left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t)\right), \\ {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_{i+1}^-) &= \lim_{t \to t_{i+1}^-} \left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t)\right). \end{aligned} \tag{34}$$

In view of (15) and (31), we have

$$\int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^N I_i(u(t_i))v(t_i) + \sum_{i=1}^N \int_{s_i}^{t_{i+1}} \frac{d}{dt}\left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt = 0, \tag{35}$$

that is,

$$\sum_{i=1_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+) - \sum_{i=0_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_{i+1}^-)\right)v(t_{i+1}^-) = \sum_{i=1_t}^N \int_{t_i}^{s_i} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^N (I_i u(t_i))v(t_i). \tag{36}$$

Without loss of generality, we take the test function $v \in C_0^\infty(t_i, s_i]$ such that $v(t) = 0$, where $t \in [0, t_i) \cup [s_i, T)$, $i = 1, 2, \cdots, N$. Substituting $v(t)$ into (30), we obtain ${}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right)$, $t \in (t_i, s_i]$, $i = 1, 2, 3 \cdots N$, is a constant, a.e.

$$\begin{aligned} &{}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right) \\ &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^-)\right) \\ &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right), \quad t \in (t_i, s_i], i = 1, 2, \cdots N. \end{aligned} \tag{37}$$

By using (36) and (37), we have

$$\sum_{i=1_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+) - \sum_{i=0_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_{i+1}^-)\right)v(t_{i+1}^-) = \sum_{i=1_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right)v(s_i) + \sum_{i=1}^N (I_i u(t_i))v(t_i). \tag{38}$$

Hence,

$$\begin{aligned} {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right) &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^-)\right) - I_i u(t_i)v(t_i) \\ &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right)v(s_i) \\ &\quad - \sum_{i=1_t}^N D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+). \end{aligned} \tag{39}$$

Then, combining with (37), we have

$$\begin{aligned} {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_i^+) - {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_i^-) &= I_i u(t_i), \\ {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right) &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^-)\right). \end{aligned} \tag{40}$$

Therefore, u is a classical solution of the problem (3). The proof is completed.

Theorem 16. *Suppose that (H2) and (H3) hold, then problem (3) has at least one classical solution.*

Proof. From Proposition 5, there exist constants A_2, A_3 such that

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1}))dt + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s) \\ &\geq \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_1(t)dt - \sum_{i=1}^{N} c_1\|u\|_\infty \\ &\quad - \sum_{i=1}^{N}\frac{d_i}{\sigma_i+1}\|u\|_\infty - \sum_{i=0}^{N}\left(\int_{s_i}^{t_{i+1}} a|u(t) - u(t_{i+1})|^2 dt\right. \\ &\quad \left. - \int_{s_i}^{t_{i+1}} b_0(t)|u(t) - u(t_{i+1})|^\delta dt\right) \\ &\geq \frac{1}{p}\|u\|^p - a4\Omega^2\|u\|_\alpha^2\sum_{i=0}^{N}\left(t_{i+1-si}\right) - (2\Omega)^\delta\|u\|_\alpha^\delta A_0 \\ &\quad - A_1 - \|u\|A_2 - \|u\|_\alpha^{\sigma_i+1}A_3,\end{aligned} \tag{41}$$

where

$$\begin{aligned}A_0 &= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_0(t)dt, \\ A_1 &= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_1(t)dt.\end{aligned} \tag{42}$$

Since $p > 2$, we get $\lim_{\|u\|_\alpha \to +\infty}\varphi(u) = +\infty$, then φ is coercive. From Lemma 10 and Lemma 14 that φ satisfied all the conditions of Lemma 9, so φ has a minimum on $E_0^{\alpha,p}$ which is a critical point of φ. Since $I_i(u(t_i)) \neq 0$ $(i = 1, 2, \cdots, N)$, then $u \neq$ constant (u is a critical point of φ).

Therefore, problem (3) has at least one classical solution. The proof is completed.

4. Example

Considering the following p-Laplacian fractional differential equations with instantaneous and noninstantaneous impulses,

$$\begin{cases} {}_tD_T^{3/4}\phi_p\left({}_0^CD_T^{3/4}u(t)\right) + |u|^{p-2}u(t) = D_xF_i(t, u(t) - u(t_{i+1})), \\ t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, N, \\ \Delta\left({}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(t_i) = I_i(u(t_i))\right), \quad i = 1, 2, \cdots, N, \\ {}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(t) = {}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^{\alpha}u\right)(t_i^+), \quad t \in (t_i, s_i], i = 0, 1, 2, \cdots, N, \\ {}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^{a}u\right)(s_i^+), \quad i = 1, 2, \cdots, N, \\ u(0) = u(T) = 0, \end{cases} \tag{43}$$

where $\alpha = 3/4$, $T = 2$, $\lambda(t) = 1$, $I_i(u(t_i)) = 1/3 + 1/2\sin u^{2/3}$, $D_xF_i(t, x) = [1/9(u(t) - u(t_{i+1}))^2 - 3(u(t) - u(t_{i+1})) - 2](1 + t)$, we can find $a = 1/3$, $b_0 = 3(1 + t)$, $b_1 = 2(1 + t)$, $\delta = 1$, $c_i = d_i = 1$, $\sigma_i = 2/3 (i = 1, 2, \cdots, N)$. It is easy to check that (H2) and (H3) hold. Then, by Theorem 16, problem (3) has at least one classical solution.

5. Conclusions

In this paper, we use the variational method to discuss the existence of solutions for Dirichlet boundary value problems with instantaneous and noninstantaneous fractional impulsive differential equations, and the existence results of classical solution for our equations are shown. In addition to extending the linear differential operator to a nonlinear differential operator, it is more important to use the p-Laplacian operator; as far as the author knows, there is not much work to study the solution of the generalized p-Laplacian impulsive fractional system using the variational method. Compared with (2), (3) contains the existence results of (2) when $D_xF_i(t, u(t) - u(t_{i+1})) \equiv f_i(t, u(t))$. Without loss of generality, if $p = 2$, $\alpha = 1$ it evolves into an integer order differential equation which has a certain auxiliary role in solving nonlinear similar problems in the future. At the same time, we also noticed the existence of multiple solutions or the existence of nontrivial solutions to the problem, so we will pay special attention in the following work. Overall, our work summarizes and supplements some of the results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

Multiplicity Results for Variable-Order Nonlinear Fractional Magnetic Schrödinger Equation with Variable Growth

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In this paper, we prove the multiplicity of nontrivial solutions for a class of fractional-order elliptic equation with magnetic field. Under appropriate assumptions, firstly, we prove that the system has at least two different solutions by applying the mountain pass theorem and Ekeland's variational principle. Secondly, we prove that these two solutions converge to the two solutions of the limit problem. Finally, we prove the existence of infinitely many solutions for the system and its limit problems, respectively.

1. Introduction

In this paper, we consider the multiplicity of nontrivial solutions of the following concave-convex elliptic equation involving variable-order nonlinear fractional magnetic Schrödinger equation:

$$\begin{cases} (-\Delta)_A^{s(\cdot)} u + V_\lambda(x)u = f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $N \geq 1$; $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ is a continuous function; Ω is a bounded subset in $\mathbb{R}^N$ with $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$; $(-\Delta)_A^{s(\cdot)}$ is the variable-order fractional magnetic Laplace operator; the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$; $\lambda > 0$ is a parameter; magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$; $f, g > 0$ are two bounded nonnegative measurable function; $p, q \in C(\Omega)$; and $u : \mathbb{R}^N \longrightarrow \mathbb{C}$. In [1], the fractional magnetic Laplacian has been defined as

$$(-\Delta)_A^s u(x) = \lim_{r \to 0} \int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)}{|x-y|^{N+2s}} dy, \tag{2}$$

for $x \in \mathbb{R}^N$. In [2], the variable-order fractional magnetic Laplace $(-\Delta)^{s(\cdot)}$ is defined as follows: for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{s(\cdot)}\varphi(x) = 2P.V \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s(x,y)}} dy, \tag{3}$$

along any $\varphi \in C_0^\infty(\Omega)$. Inspired by them, we define the variable-order fractional magnetic Laplacian $(-\Delta)_A^{s(\cdot)}$ as follows: for each $x \in \mathbb{R}^N$,

$$(-\Delta)_A^{s(\cdot)} u(x) = \lim_{r \to 0} \int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)}{|x-y|^{N+2s(x,y)}} dy. \tag{4}$$

Since $s(\cdot)$ is a function, magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$, we see that operator $(-\Delta)_A^{s(\cdot)}$ is variable order fractional magnetic Laplace operator. Especially, when $s(\cdot) \equiv$ constant, $(-\Delta)_A^{s(\cdot)}$ reduce to the usual fractional magnetic Laplace operator. When $s(\cdot) \equiv$ constant, $A = 0$, $(-\Delta)_A^{s(\cdot)}$ reduce to the usual fractional Laplace operator. Very recently, for $s(\cdot) = 1$, $p(x), q(x) \equiv$ constant, and $A = 0$; in [3], under appropriate assumptions, the authors obtained the multiplicity and concentration of the positive solution of the following indefinite semilinear elliptic equations involving concave-convex nonlinearities by the variational method:

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{5}$$

For $s(\cdot) = \alpha, p(x), q(x) \equiv$ constant, and $A = 0$; in [4], the authors obtained the existence, multiplicity, and concentration of nontrivial solutions for the following indefinite fractional elliptic equation by using Nehari manifold decomposition:

$$\begin{cases} (-\Delta)^\alpha u + V_\lambda(x)u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{6}$$

When $A = 0$, $V^-(x) = 0$, and $f(x), g(x) \equiv$ constant, the authors in [2] give some sufficient conditions to ensure the existence of two different weak solutions and use the variational method and the mountain pass theorem to obtain the two weak solutions of problem (12) which converge to two solutions of its limit problems and the existence of infinitely many solutions to its limit problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7}$$

For $s(\cdot) = s, p(x), q(x) \equiv$ constant, in [1], the authors study the existence of solutions for the following equation on the whole space by using the method of Nehari manifold decomposition,and obtain some sufficient conditions for the existence of nontrivial solutions of the following equation:

$$(-\Delta)_A^s u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{8}$$

In recent years, with the continuous deepening of research, the fractional magnetic problem has attracted extensive attention of researchers. More and more researchers have studied the solvability of the fractional magnetic problem (see [5–8]). We know that the fractional magnetic Laplacian operator $(-\Delta)_A^s$ is introduced in literature [9]; $(-\Delta)_A^s$ comes from magnetic Laplacian $(\nabla - iA)^{2s}$. However, as far as we know, up to now, few papers have studied the existence and multiplicity of solutions for the variable-order fractional magnetic Schrödinger equation. Therefore, motivated by the above literature, we are interested in the existence and multiplicity of solutions to problem (1) with variable growth. As far as we know, this is the first time to study the existence and multiplicity of nontrivial solutions for the variable-order nonlinear fractional magnetic Schrödinger equation with variable exponents.

It is worth noting that in this paper, we not only prove that there exist two different nontrivial solutions to problem (1), but we also show that the two nontrivial solutions of problem (1) converge to two solutions of the limit problem for problem (1). The novelty of this paper is that, compared with [1], we write the original fractional magnetic Schrödinger equation without variable growth to a variable-order fractional magnetic Schrödinger equation with variable growth. In addition, compared with [2], we write the variable order fractional Schrödinger equation to a variable order fractional magnetic Schrödinger equation.

Inspired by the above works, we assume that $s : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ and V are continuous functions satisfying the following:

(S_1): $0 < s^- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x, y) \leq s^+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x, y) < 1$.

(S_2): $s(\cdot)$ is symmetric, that is, $s(x, y) = s(y, x)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(V_1): $X = \text{int}\,((V^+)^{-1}(0)) \subset \Omega$ is a nonempty bounded domain and $\tilde{X} = (V^+)^{-1}(0)$.

(V_2): there exists a nonempty open domain $\Omega_0 \subset X$ such that $V^+ \equiv 0$, $V^- \equiv 0$ for all $x \in \bar{\Omega}_0$.

(V_3): V^+ is a continuous function on Ω and $V^- \in L^{N/2}(\Omega)$.

(V_4): there exists a constant $\vartheta_0 > 1$ such that

$$\inf_{u\in H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \left(\left(|u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)|^2 \right) / |x-y|^{N+2s(x,y)} \right) dxdy + \lambda \int_\Omega V^+ u^2 dx}{\int_\Omega V^- u^2 dx} \geq \vartheta_0, \tag{9}$$

or all $\lambda > 0$, where $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ is the Hilbert space related to magnetic field A (see Section 2).

For the variable exponents p, q, we assume that $p, q \in C(\bar{\Omega})$ and satisfy the following assumption:

(H_1): $2 < p(x) < 2N/(N - 2s(x, x))$ for all $x \in \bar{\Omega}$.

(H_2): $1 < q(x) < 2$ for all $x \in \bar{\Omega}$.

In addition, we assume that f, g satisfy the following assumption:

(H_3): $f, g : \mathbb{R}^N \longrightarrow [0, \infty)$ are bounded nonnegative measurable function such that $f > 0$, $g > 0$ on open interval $\Omega_f, \Omega_g \subset \Omega$ and

$$\begin{aligned} \|f\|_\infty = \|f\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{p^-(2-q^+)(\vartheta_0-1)}{\max\left\{C_p^{p^+}, C_p^{p^-}\right\} 2\vartheta_0(p^+-q^+)}, \\ \|g\|_\infty = \|g\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{q^-(p^+-2)(\vartheta_0-1)}{\max\left\{C_q^{q^+}, C_q^{q^-}\right\} 2\vartheta_0(p^+-q^+)} \\ &\quad \cdot \left[\frac{(2-q^+)(\vartheta_0-1)}{2A_1\vartheta_0(p^+-q^+)}\right]^{(2-q^+)/(p^+-2)}. \end{aligned} \tag{10}$$

Based on the hypothesis (S_2), we can give the following definition of weak solutions for problem (1).

Definition 1. We say that $u \in E_\lambda$ is a weak solution of problem (1), if

$$\Re \int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy + \lambda \Re \int_\Omega V^+ u\bar{v}dx - \Re \int_\Omega V^- u\bar{v}dx - \Re \int_\Omega \left(f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u\right)\bar{v}dx = 0, \tag{11}$$

for any $v \in E_\lambda$, where E_λ will be defined in Section 2.

Now, we will describe the first main result as follows.

Theorem 2. *Assume that* $(S_1), (S_2), (V_1) - (V_4)$, *and* $(H_1) - (H_3)$ *hold. Let* $N > 2s^+$. *Then, the problem (1) allows at least two different solutions for all* $\lambda > 0$.

Theorem 3. *Let* u_λ^1 *and* u_λ^2 *be two solutions obtained in Theorem 2. Then,* $u_\lambda^1 \longrightarrow u^1$ *and* $u_\lambda^2 \longrightarrow u^2$ *in* $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ *as* $\lambda \longrightarrow \infty$, *where* $u^1 \neq u^2$ *are two nontrivial solutions of the following problem:*

$$\begin{cases} (-\Delta)_A^{s(\cdot)}u - V^-(x)u = f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \text{ in } \Omega_0, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega_0. \end{cases} \tag{12}$$

Remark 4. In general, if $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ is a continuous function, magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$, then the variable-order fractional magnetic Laplacian can be defined as follows: for each $u \in C_0^\infty(\Omega, \mathbb{C})$,

$$\left\langle (-\Delta)_A^{s(\cdot)}u, v \right\rangle = \Re \int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy, \tag{13}$$

along any $v \in C_0^\infty(\Omega, \mathbb{C})$.

2. Preliminaries and Notations

In this section, we first give the definition of the variable exponential Lebesgue space. Secondly, we define variable-order fractional magnetic Sobolev spaces and prove the compact conditions between them. Finally, we give the variational setting for problem (1) and theorems that will be used later.

In this paper, we use $|\Omega|$ to represent n-dimensional Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^N$. In addition, for each $a \in \mathbb{C}$, we will use $\Re a$ to represent the real part of a and $\bar{a}$ to represent the complex conjugate of a. Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a nonempty set. A measurable function $r : \bar{\Omega} \longrightarrow [1, \infty)$ is called a variable exponent, and we define $r^+ = \operatorname{esssup}_{x\in\Omega} r(x)$, $r^- = \operatorname{essinf}_{x\in\Omega} r(x)$. If r^+ is finite, then the exponent r is said to be bounded. The variable exponent Lebesgue space is

$$L^{r(x)}(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function}; \int_\Omega |u(x)|^{r(x)}dx < \infty \right\} \tag{14}$$

with the Luxemburg norm

$$\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} = \inf \left\{ \mu > 0 : \int_\Omega \left(\frac{|u(x)|}{\mu}\right)^{r(x)} dx \leq 1 \right\}, \tag{15}$$

then $L^{r(x)}(\Omega, \mathbb{C})$ is a Banach space, and when r is bounded, we have the following relations

$$\min\left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^-}, \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^+} \right\} \leq \int_\Omega |u(x)|^{r(x)}dx \leq \max\left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^-}, \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^+} \right\}. \tag{16}$$

For bounded exponent, the dual space $(L^{r(x)}(\Omega, \mathbb{C}))'$ can be identified with $L^{r'(x)}(\Omega, \mathbb{C})$, where the conjugate exponent r' is defined by $r' = r/(r-1)$. If $1 < r^- \leq r^+ < \infty$, then the variable exponent Lebesgue space $L^{r(x)}(\Omega, \mathbb{C})$ is a separable and reflexive. In particular,

$$L^2(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function}; \int_\Omega |u(x)|^2 dx < \infty \right\} \tag{17}$$

with the scalar product

$$\langle u, v \rangle_{L^2(\Omega,\mathbb{C})} = \Re \int_\Omega u\bar{v}dx. \tag{18}$$

By Lemma 3.2.20 of [10] and $\|\cdot\|_{L^{r(x)}(\Omega,\mathbb{C})} = \||\cdot|\|_{L^{r(x)}(\Omega,\mathbb{R})}$, we know that in the variable exponent Lebesgue space, Hölder inequality is still valid. For all $u \in L^{r(x)}(\Omega, \mathbb{C})$, $v \in L^{r'(x)}(\Omega, \mathbb{C})$ with $r(x) \in (1, \infty)$, the following inequality holds:

$$\int_\Omega |u||v|dx \le \left(\frac{1}{r^-} + \frac{1}{\left(r'\right)^-}\right)\|u\|_{L^{r(x)}(\Omega,\mathbb{C})}\|v\|_{L^{r'(x)}(\Omega,\mathbb{C})} \le 2\|u\|_{L^{r(x)}(\Omega,\mathbb{C})}\|v\|_{L^{r'(x)}(\Omega,\mathbb{C})}. \quad (19)$$

Let Ω be a nonempty open subset of $\mathbb{R}^N$, and let $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ be a measurable function, and there exist two constants $0 < s_0 < s_1 < 1$ such that $s_0 < s(x,y) < s_1$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Set

$$H^{s(\cdot)}(\Omega,\mathbb{C}) = \left\{u \in L^2(\Omega,\mathbb{C}): \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{1/2} < \infty\right\}. \quad (20)$$

Equip $H^{s(\cdot)}(\Omega,\mathbb{C})$ with the norm

$$\|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} = \|u\|^2_{L^2(\Omega,\mathbb{C})} + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}dxdy. \quad (21)$$

Especially, if $s(\cdot) \equiv$ constant, then the space $H^{s(\cdot)}(\Omega,\mathbb{C})$ is the usual fractional Sobolev space $H^s(\Omega,\mathbb{C})$.

Lemma 8. *Let Ω be a smooth bounded subset of $\mathbb{R}^N$ and let $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ satisfying (S_1) and $r : \bar{\Omega} \longrightarrow (1,\infty)$ satisfying $1 \le r \le 2N/(N-2s(x,x))$. Then, there exists $\tilde{C}_r = C(N, r^+, s^+, s^-) > 0$ such that $\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} \le \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{C})}$, for any $u \in H^{s(\cdot)}(\Omega,\mathbb{C})$. That is, the embedding $H^{s(\cdot)}(\Omega,\mathbb{C}) \circ L^{r(x)}(\Omega,\mathbb{C})$ is continuous. Moreover, $H^{s(\cdot)}(\Omega,\mathbb{C}) \circ L^{r(x)}(\Omega,\mathbb{C})$ is compact.*

Proof. By Theorem 2.1 of [2], we know that $H^{s(\cdot)}(\Omega,\mathbb{R}) \circ L^{r(x)}(\Omega,\mathbb{R})$ is continuous and compact, there exists $\tilde{C}_r = C(N, r^+, s^+, s^-) > 0$ such that $\|u\|_{L^{r(x)}(\Omega,\mathbb{R})} \le \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{R})}$. then, for any $u \in H^{s(\cdot)}(\Omega,\mathbb{C})$, we have

$$\begin{aligned}\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} &= \||u|\|_{L^{r(x)}(\Omega,\mathbb{R})} \le \tilde{C}_r\||u|\|_{H^{s(\cdot)}(\Omega,\mathbb{R})}\\ &= \tilde{C}_r\left(\||u|\|^2_{L^2(\Omega,\mathbb{R})} + \int_\Omega\int_\Omega \frac{||u(x)|-|u(y)||^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{1/2}\\ &\le \tilde{C}_r\left(\|u\|^2_{L^2(\Omega,\mathbb{C})} + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{1/2}\\ &= \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{C})}.\end{aligned} \quad (22)$$

Hence, the embedding $H^{s(\cdot)}(\Omega,\mathbb{C}) \circ L^{r(x)}(\Omega,\mathbb{C})$ is continuous and compact.

Let $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous function and $A \in L^\infty_{loc}(\mathbb{R}^N,\mathbb{R}^N)$. For a function $u : \mathbb{R}^N \longrightarrow \mathbb{C}$, define

$$[u]^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}dxdy \quad (23)$$

and the corresponding norm denoted by $\|u\|^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} = \|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})} + [u]^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})}$. We consider the space H of measurable functions $u : \mathbb{R}^N \longrightarrow \mathbb{C}$ such that $\|u\|_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} < \infty$; then, $(H, \langle\cdot,\cdot\rangle_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})})$ is a Hilbert space. Define $H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N,\mathbb{C})$ in H; then, $H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space. Especially, if $A = 0$, then the space $H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ is the variable-order fractional Sobolev space $H^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$; if $A = 0$ and $s(\cdot) \equiv$ constant, then the space $H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ is the usual fractional Sobolev space $H^s(\mathbb{R}^N,\mathbb{C})$.

In order to define weak solutions of problem (1), we introduce the functional space

$$H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C}) = \left\{u \in H^{s(\cdot)}_A\left(\mathbb{R}^N,\mathbb{C}\right): u = 0\ a.e.\text{in}\ \mathbb{R}^N \setminus \Omega\right\}, \quad (24)$$

equipping $H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})$ with the scalar product

$$\langle u, v\rangle_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} := \Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy, \quad (25)$$

which induces the following norm $\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} := \langle u,u\rangle^{1/2}_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}$. Hence, $H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})$ generalizes to the variable-order fractional Sobolev space (see [2]) and the magnetic framework the space introduced in [9]. Next, we state and prove some properties of space $H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})$, which will be useful in the sequel.

Lemma 6. *There exists a constant $C_2 > 0$, depending only on N, s_1 and Ω, such that*

$$\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} \le \|u\|_{H^{s(\cdot)}_A\left(\mathbb{R}^N,\mathbb{C}\right)} \le C_2\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, \quad (26)$$

for any $u \in H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})$. Thus, $\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}$ is an equivalent norm of $H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$.

Proof. For any $u \in H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$, by Lemma 3.1 in [9], we have the pointwise diamagnetic inequality

$$\begin{aligned} &\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right| \\ &\quad \geq \big||u(x)| - |u(y)|\big|, \quad \text{for } a.e.x, y \in \mathbb{R}^N, \end{aligned} \tag{27}$$

from which we immediately have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big||u(x)| - |u(y)|\big|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &\geq \int_{\mathbb{R}^N\setminus\Omega}\left(\int_{\Omega} \frac{\big||u(x)| - |u(y)|\big|^2}{|x-y|^{N+2s(x,y)}}\,dy\right)dx \\ &= \int_{\mathbb{R}^N\Omega}\left(\int_{\Omega} \frac{|u(y)|^2}{|x-y|^{N+2s(x,y)}}\,dy\right)dx. \end{aligned} \tag{28}$$

Since Ω is bounded, there exists $r > 1/2$ such that $\Omega \subset B_r$ and $|B_r \setminus \Omega| > 0$; then, we have

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus\Omega}\left(\int_{\Omega} \frac{|u(y)|^2}{|x-y|^{N+2s(x,y)}}\,dy\right)dx \\ &\geq \int_{B_r\Omega}\left(\int_{\Omega} \frac{|u(y)|^2}{|2r|^{N+2s(x,y)}}\,dy\right)dx \\ &\geq \int_{B_r\setminus\Omega}\left(\int_{\Omega} \frac{|u(y)|^2}{|2r|^{N+2s_1}}\,dy\right)dx \\ &= \frac{1}{(2r)^{N+2s_1}}\int_{B_r\setminus\Omega}\left(\int_{\Omega}|u(y)|^2dy\right)dx \\ &= \frac{|B_r\setminus\Omega|}{(2r)^{N+2s_1}}\,\|u\|^2_{L^2(\Omega,\mathbb{C})}. \end{aligned} \tag{29}$$

Thus, we obtain

$$\begin{aligned} \|u\|^2_{L^2(\Omega,\mathbb{C})} &\leq \frac{(2r)^{N+2s_1}}{|B_r\setminus\Omega|}\int_{\mathbb{R}^{2N}} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &= C_1\|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}, \\ \|u\|^2_{H_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})} &= \|u\|^2_{L^2(\Omega,\mathbb{C})} + [u]^2_{H_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})} \\ &\leq C_1\|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} + \|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} \\ &= C_2^2\|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}, \end{aligned} \tag{30}$$

where $C_1 = (2r)^{N+2s_1}/|B_r\setminus\Omega|$ and $C_2^2 = C_1 + 1$.

In addition,

$$\begin{aligned} \|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} &\leq \int_{\Omega}|u(x)|^2dx + \int_{\mathbb{R}^{2N}} \\ &\quad \cdot \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &= \|u\|^2_{H_A^{s(\cdot)}\left(\mathbb{R}^N,\mathbb{C}\right)}. \end{aligned} \tag{31}$$

Combining the above two aspects, we have

$$\|u\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} \leq \|u\|_{H_A^{s(\cdot)}\left(\mathbb{R}^N,\mathbb{C}\right)} \leq C_2\|u\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}, \tag{32}$$

which implies that $\|u\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}$ is the equivalent norm of a norm $\|u\|_{H_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})}$.

Lemma 2.3. *Let Ω be a bounded subset of $\mathbb{R}^N$. Assume that $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ is a continuous function satisfying (S_1) and $r : \bar{\Omega} \longrightarrow [1,\infty)$ is a continuous function satisfying $1 \leq r \leq 2N/(N - 2s(x,x))$. If $u \in H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})$, then*

$$H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C}) \circ H^{s(\cdot)}(\Omega,\mathbb{C}) \tag{33}$$

is continuous and

$$H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C}) \circ L^{r(x)}(\Omega,\mathbb{C}) \tag{34}$$

is compact, that is, there exists $C_r = C(N, r^+, s^+, s^-) > 0$ such that

$$\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} \leq C_r\|u\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}. \tag{35}$$

Proof. For any $u \in H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})$, we have

$$\begin{aligned} \|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} &= \int_{\Omega}|u(x)|^2dx + \int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &\leq \int_{\Omega}|u(x)|^2dx \\ &\quad + 2\int_{\Omega}\int_{\Omega}\frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &\quad + 2\int_{\Omega}\int_{\Omega}\frac{|u(y)|^2\left|e^{i(x-y)\cdot A((x+y)/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}}\,dxdy \\ &= 2\|u\|^2_{H_A^{s(\cdot)}(\Omega,\mathbb{C})} + 2J, \end{aligned} \tag{36}$$

where

$$\begin{aligned} J &= \int_\Omega\int_\Omega \frac{|u(y)|^2\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dxdy \\ &= \int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\ &\quad + \int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\ &= J_1 + J_2. \end{aligned} \tag{37}$$

Since $|e^{it}-1|\le 2$, we have

$$\begin{aligned} J_1 &\le 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s(x,y)}}dx\right)dy \\ &\le 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s_0}}dx\right)dy \\ &= 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|z|>1\}} \frac{1}{|z|^{N+2s_0}}dz\right)dy \\ &\le C_3\int_\Omega |u(y)|^2dy = C_3\|u\|^2_{L^2(\Omega,\mathbb{C})}. \end{aligned} \tag{38}$$

In view of Ω which is bounded, there exists a compact set $K\subset\mathbb{R}^N$ such that $\Omega\subset K$. By Lemma 2.2 of [11], we know that A is locally bounded and $K\subset\mathbb{R}^N$is compact, $\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2\le C_4|x-y|^2$, for $|x-y|\le 1$, $x,y\in K$.

Thus, we obtain

$$\begin{aligned} J_2 &\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\ &\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{C_4}{|x-y|^{N+2s(x,y)-2}}dx\right)dy \\ &\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{C_4}{|x-y|^{N+2s_1-2}}dx\right)dy \\ &\le C_4\int_K |u(y)|^2\left(\int_{K\cap\{|z|\le 1\}} \frac{1}{|z|^{N+2s_1-2}}dz\right)dy \\ &\le C_5\int_K |u(y)|^2dy = C_5\int_{K\setminus\Omega} |u(y)|^2dy + C_5\int_\Omega |u(y)|^2dy \\ &= C_5\int_\Omega |u(y)|^2dy = C_5\|u\|^2_{L^2(\Omega,\mathbb{C})}. \end{aligned} \tag{39}$$

Equations (36)–(39) together with Lemma 6, we have

$$\begin{aligned} \|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} &\le 2\|u\|^2_{H^{s(\cdot)}_A(\Omega,\mathbb{C})} + 2C_3\|u\|^2_{L^2(\Omega,\mathbb{C})} \\ &\quad + 2C_5\|u\|^2_{L^2(\Omega,\mathbb{C})} \le C_6\|u\|^2_{H^{s(\cdot)}_A(\Omega,\mathbb{C})} \\ &\le C_6\|u\|^2_{H^{s(\cdot)}_A\left(\mathbb{R}^N,\mathbb{C}\right)} \le C_7\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, \end{aligned} \tag{40}$$

which implies that the embedding $H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})\hookrightarrow H^{s(\cdot)}(\Omega,\mathbb{C})$ is continuous. In addition, by Lemma 5, we know that $H^{s(\cdot)}(\Omega,\mathbb{C})\hookrightarrow L^{r(x)}(\Omega,\mathbb{C})$ is compact. Therefore, the embedding $H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})\hookrightarrow L^{r(x)}(\Omega,\mathbb{C})$ is compact.

Next, we give the variational setting for problem (1). For $\lambda>0$, we need the following scalar product and norm:

$$\langle u,v\rangle_\lambda := \Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x)-e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\left(\overline{v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)}\right)}{|x-y|^{N+2s(x,y)}}dxdy + \lambda\cdot\Re\int_\Omega V^+u\bar{v}dx,\ \|u\|_\lambda := \langle u,u\rangle^{1/2}_\lambda. \tag{41}$$

Let $E=\{u\in H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})\colon \int_\Omega V^+u^2dx<\infty\}$ be equipped with the norm $\|u\|_E=\|u\|_1$ (that is, $\lambda=1$ in $\|u\|_\lambda$). Obviously, $\|u\|_E\le\|u\|_\lambda$ for $\lambda\ge 1$. Set $E_\lambda=(E,\|\cdot\|_\lambda)$. Moreover, for $r(x)\in(1,2N/(N-2s(x,x)))$, we can get

$$\begin{aligned} \int_\Omega |u(x)|^{r(x)}dx &\le \max\left\{\|u\|^{r^+}_{L^{r(\cdot)}(\Omega,\mathbb{C})}, \|u\|^{r^-}_{L^{r(\cdot)}(\Omega,\mathbb{C})}\right\} \\ &\le \max\left\{C^{r^+}_r\|u\|^{r^+}_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, C^{r^-}_r\|u\|^{r^-}_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}\right\} \\ &\le \max\left\{C^{r^+}_r\|u\|^{r^+}_\lambda, C^{r^-}_r\|u\|^{r^-}_\lambda\right\}. \end{aligned} \tag{42}$$

For simplicity, we let $\|u\|^2_{\lambda,V}:=\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}+\int_\Omega V_\lambda u^2dx$. Therefore, by condition (V_4),

$$\|u\|^2_\lambda \ge \|u\|^2_{\lambda,V} \ge \frac{\vartheta_0-1}{\vartheta_0}\|u\|^2_\lambda, \quad \text{for all } \lambda\ge 0. \tag{43}$$

Associated with problem (1), we consider the energy functional $\Psi_\lambda: E_\lambda\longrightarrow\mathbb{R}$,

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2}\int_\Omega V^-u^2dx \\ &\quad - \int_\Omega\left(\frac{f(x)}{p(x)}|u|^{p(x)} + \frac{g(x)}{q(x)}|u|^{q(x)}\right)dx \\ &= \frac{1}{2}\|u\|^2_{\lambda,V} - \int_\Omega\left(\frac{f(x)}{p(x)}|u|^{p(x)} + \frac{g(x)}{q(x)}|u|^{q(x)}\right)dx. \end{aligned} \tag{44}$$

In fact, one can verify that Ψ_λ is well-defined of class C^1 in E_λ and

$$\Re\int_{\mathbb{R}^{2N}}\frac{\left(u(x)-e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy+\lambda\Re\int_{\Omega}V^{+}u\bar{v}dx$$
$$-\Re\int_{\Omega}V^{-}u\bar{v}dx-\Re\int_{\Omega}\left(f(x)|u|^{p(x)-2}u+g(x)|u|^{q(x)-2}u\right)\bar{v}dx=0, \tag{45}$$

for all $u, v \in E_\lambda$. Therefore, if $u \in E_\lambda$ is a critical point of Ψ_λ, then u is a solution of problem (1).

Now we give the theorems that we need later.

Theorem 8 (see [2, 12]). *Let X be a real infinite dimensional Banach space and $I \in C^1(X)$ a functional satisfying the $(PS)_c$ condition as well as the following three properties:*

(1) $I(0) = 0$, and there exists two constants $\rho, \delta > 0$ such that $I(u) \geq \delta$ for all $u \in X$ with $\|u\| = \rho$

(2) I is even

(3) For all finite dimensional subspaces $Y \subset X$, there exists $R = R(Y) > 0$ such that $I(u) \leq 0$ for all $u \in X \setminus B_R(Y)$, where $B_R(Y) = \{u \in Y : \|u\| \leq R\}$. Then, I poses an unbounded sequence of critical values characterized by a minimax argument

Theorem 9 (see [13]). *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. Suppose I satisfies the (PS) condition, which is even and bounded from below, and $I(0) = 0$. If for any $k \in N$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} I < 0$, where $S_{\rho_k} = \{u \in X : \|u\|_X = \rho_k\}$, then at least one of the following conclusions holds:*

(i) There exists a sequence of critical points $\{u_k\}$ satisfying $I(u_k) < 0$ for all k and $\|u_k\|_X \longrightarrow 0$ as $k \longrightarrow \infty$

(ii) There exists $r > 0$ such that for any $0 < a < r$, there exists a critical point u such that $\|u\|_X = a$ and $I(u) = 0$

It is easy to verify that E_λ is a separable Hilbert space. Let $X_i = span\{e_i\}$, then $E_\lambda = \oplus_{i\geq 1} X_i$. we define

$$Y_k = \oplus_{i=1}^{k} X_i, Z_k = \overline{\oplus_{i=k+1}^{\infty} X_i}. \tag{46}$$

Theorem 10 (see [14], fountain theorem). *Suppose that $I \in C^1(E, \mathbb{R})$ satisfying $I(-u) = I(u)$. Assume that for every $k \in N$, there exist $r_k > \gamma_k > 0$ such that*

(D_1): $a_k = \max\{I(u): u \in Y_k, \|u\| = r_k\} \leq 0$.

(D_2): $b_k = \inf\{I(u): u \in Z_k, \|u\| = \gamma_k\} \longrightarrow \infty$ as $k \longrightarrow \infty$.

(D_3): I satisfies $(PS)_c$ condition for every $c > 0$.

Then, I has an unbounded sequence of critical values which have the form

$$c_k = \inf_{r\in\Gamma_k}\max_{u\in B_k} I(\eta(u)), \tag{47}$$

where $\Gamma_k = \{\eta \in C(B_k, E)$: η is equivariant and $\eta|_{\partial B_k} = id\}$.

Theorem 11 (see [15], dual fountain theorem). *Suppose that $I \in C^1(E, \mathbb{R})$ satisfying $I(-u) = I(u)$. Assume that for every $k \geq k_0$, there exist $r_k > \gamma_k > 0$ such that*

(D_4): $a_k = \inf\{I(u): u \in Z_k, \|u\| = r_k\} \geq 0$.

(D_5): $b_k = \max\{I(u): u \in Y_k, \|u\| = \gamma_k\} < 0$.

(D_6): $d_k = \max\{I(u): u \in Z_k, \|u\| \leq \gamma_k\} \longrightarrow 0$ as $k \longrightarrow \infty$.

(D_7): I satisfies $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0]$.

Then, I has a sequence of negative critical values converging to 0.

3. Proof of Theorem 1

In this part, we first recall that definitions of functional Ψ_λ satisfies the $(PS)_c$ condition and $(PS)_c^*$ condition in E_λ at the level $c \in \mathbb{R}$ and use the usual mountain pass theorem (see [2]) to find a $(PS)_c$ sequence in E_λ. Second, we show that functional Ψ_λ satisfies the $(PS)_c^*$ condition in E_λ at the level $c < c_0$. Finally, we give the proof of problem (1).

Definition 12 (see [2]). Let $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset E$ such that $I(u_n) \longrightarrow c$ and $I'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ admits a strongly convergent subsequence in E.

Definition 13 (see [16]). Let $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the $(PS)_c^*$ condition (with respect to Y_n) if any sequence $\{u_n\} \subset E$ such that $\{u_n\} \in Y_n$, $I(u_n) \longrightarrow c$ and $I'|_{Y_n}(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ admits a strongly convergent subsequence in E.

Remark 14. From Remark 2.1 in [16], we get that the $(PS)_c^*$ condition means the $(PS)_c$ condition.

Theorem 15 (Theorem 3.1, [2]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with $I(0) = 0$. Suppose that*

(i) there exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for each $u \in E$ subject to $\|u\|_E = \rho$

(ii) there exists $e \in E$ with $\|e\|_E > \rho$ such that $I(e) < 0$

Define $\Gamma = \{\gamma \in C^1([0,1], E)$: $\gamma(0) = 1, \gamma(1) = e\}$. Then,

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)) \ge \delta, \tag{48}$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n \subset E$.

In order to obtain our main results by using the mountain pass theorem, we first prove that Ψ_λ satisfies the mountain pass geometry (i) and (ii).

Lemma 16. *Assume that (S_1), (V_1)–(V_4), and (H_1)–(H_3) hold. Then, for each $\lambda > 0$, there exists $\rho > 0$ and $\tau > 0$ such that*

$$\Psi_\lambda(u) > \tau \quad \text{for all } u \in E_\lambda \text{ with } \|u\|_\lambda = \rho. \tag{49}$$

Proof. In view of (42) and the fractional Sobolev inequality, for each $u \in E_\lambda$, one has

$$\begin{aligned}
&\int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
&\quad \le \frac{\|f\|_\infty}{p^-} \int_\Omega |u|^{p(x)} dx + \frac{\|g\|_\infty}{q^-} \int_\Omega |u|^{q(x)} dx \\
&\quad \le \frac{\|f\|_\infty}{p^-} \max \left\{ C_p^{p^+} \|u\|_\lambda^{p^+}, C_p^{p^-} \|u\|_\lambda^{p^-} \right\} \\
&\qquad + \frac{\|g\|_\infty}{q^-} \max \left\{ C_q^{q^+} \|u\|_\lambda^{q^+}, C_q^{q^-} \|u\|_\lambda^{q^-} \right\},
\end{aligned} \tag{50}$$

where C_p, C_q are two constants of embedding from variable-order fractional Sobolev space $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ to $L^{p(x)}(\Omega, \mathbb{C})$ and $L^{q(x)}(\Omega, \mathbb{C})$, respectively. Making use of (43) and (50), we obtain that

$$\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{2} \int_\Omega V^- u^2 dx \\
&\quad - \int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
&= \frac{1}{2} \|u\|_{\lambda,V}^2 - \int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
&\ge \frac{1}{2} \frac{\vartheta_0 - 1}{\vartheta_0} \|u\|_\lambda^2 - \|f\|_\infty \frac{\max \left\{ C_p^{p^+}, C_p^{p^-} \right\}}{p^-} \|u\|_\lambda^{p^+} \\
&\quad - \|g\|_\infty \frac{\max \left\{ C_q^{q^+}, C_q^{q^-} \right\}}{q^-} \|u\|_\lambda^{q^+} \\
&= \frac{\vartheta_0 - 1}{2\vartheta_0} \|u\|_\lambda^2 - A_1 \|u\|_\lambda^{p^+} - A_2 \|u\|_\lambda^{q^+} \\
&= \|u\|_\lambda^{q^+} \left(\frac{\vartheta_0 - 1}{2\vartheta_0} \|u\|_\lambda^{2-q^+} - A_1 \|u\|_\lambda^{p^+ - q^+} - A_2 \right),
\end{aligned} \tag{51}$$

for each $u \in E_\lambda$ with $\|u\|_\lambda \ge 1$.

Set

$$\begin{aligned}
A_1 &= \frac{\|f\|_\infty \max \left\{ C_p^{p^+}, C_p^{p^-} \right\}}{p^-}, \\
A_2 &= \frac{\|g\|_\infty \max \left\{ C_q^{q^+}, C_q^{q^-} \right\}}{q^-}.
\end{aligned} \tag{52}$$

Define $\Phi_1(t)$: $[0, \infty) \longrightarrow \mathbb{R}$ as follows

$$\Phi_1(t) = \Phi_2(t) t^{q^+} \quad \text{for all } t \ge 0, \tag{53}$$

where

$$\Phi_2(t) = \frac{\vartheta_0 - 1}{2\vartheta_0} t^{2-q^+} - A_1 t^{p^+ - q^+} - A_2. \tag{54}$$

As long as

$$A_2 < \left(\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right)^{(2-q^+)/(p^+-2)} \frac{(\vartheta_0 - 1)(p^+ - 2)}{2\vartheta_0 (p^+ - q^+)}, \tag{55}$$

that is,

$$A_1^{2-q^+} A_2^{p^+-2} \le \left(\frac{(2 - q^+)(\vartheta_0 - 1)}{2\vartheta_0 (p^+ - q^+)} \right)^{2-q^+} \left(\frac{(\vartheta_0 - 1)(p^+ - 2)}{2\vartheta_0 (p^+ - q^+)} \right)^{p^+-2}, \tag{56}$$

that is,

$$\begin{aligned}
\|g\|_\infty \le{}& \frac{q^- (p^+ - 2)(\vartheta_0 - 1)}{\max \left\{ C_q^{q^+}, C_q^{q^-} \right\} 2\vartheta_0 (p^+ - q^+)} \\
&\cdot \left[\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right]^{(2-q^+)/(p^+-2)},
\end{aligned} \tag{57}$$

we can easily show that for $t = \tilde{t} = [(2 - q^+)(\vartheta_0 - 1)/2A_1\vartheta_0 (p^+ - q^+)]^{1/(p^+-2)}$, we have

$$\Phi_2(\tilde{t}) = \max_{t \ge 0} \Phi_2(t) > 0. \tag{58}$$

By

$$\|f\|_\infty \le \frac{p^- (2 - q^+)(\vartheta_0 - 1)}{\max \left\{ C_p^{p^+}, C_p^{p^-} \right\} 2\vartheta_0 (p^+ - q^+)}, \tag{59}$$

it is easy to derive that

$$\tilde{t} = \left[\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right]^{1/(p^+-2)} \ge 1. \tag{60}$$

By letting $\rho = \tilde{t} > 0$ and $\tau = \Phi(\tilde{t}) > 0$, we can easily get $\Psi_\lambda(u) > \tau$ for all $u \in E_\lambda$ with $\|u\|_\lambda = \rho$.

Lemma 17. *Suppose that (S_1), $(V_1)-(V_4)$, and $(H_1)-(H_3)$ hold. Then, there exists $e \in E_\lambda$ with $\|e\|_\lambda > \rho$ such that $\Psi_\lambda(e) < 0$ for all $\lambda > 0$, where $\rho > 0$ is given by Lemma 16.*

Proof. Notice that $f : \mathbb{R}^N \longrightarrow [0, \infty)$ is a bounded nonnegative measurable function such that $f > 0$ on open interval $\Omega_f \subset \Omega$; then, we can select $w_0 \in E_\lambda$ such that

$$\|w_0\|_\lambda = 1 \text{ and} \int_\Omega f(x)|w_0(x)|^{p(x)}dx > 0. \tag{61}$$

For all $t \geq 1$, combining (43) with (44), we have

$$\begin{aligned}
\Psi_\lambda(tw_0) &= \frac{1}{2}\|tw_0\|_\lambda^2 - \frac{1}{2}\int_\Omega V^-(tw_0)^2dx \\
&\quad - \int_\Omega \left(\frac{f(x)}{p(x)}|tw_0|^{p(x)} + \frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx \\
&= \frac{1}{2}\|tw_0\|_{\lambda,V}^2 - \int_\Omega \left(\frac{f(x)}{p(x)}|tw_0|^{p(x)} + \frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx \\
&\leq \frac{1}{2}\|tw_0\|_\lambda^2 - \int_\Omega \left(\frac{f(x)}{p(x)}|tw_0|^{p(x)} + \frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx \\
&\leq \frac{t^2}{2}\|w_0\|_\lambda^2 - \frac{t^{p^-}}{p^+}\int_\Omega f(x)|w_0|^{p(x)}dx.
\end{aligned} \tag{62}$$

Since $p^- > 2$, then there exists $t^* \geq 1$ large enough such that $\|t^*w_0\|_\lambda > \rho$ and $\Psi_\lambda(t^*w_0) < 0$. By letting $e = t^*w_0$, we can easily reach the conclusion.

Define

$$\begin{aligned}
c_\lambda &= \inf_{\gamma\in\Gamma_1}\max_{0\leq t\leq 1}\Psi_\lambda(\gamma(t)), \\
c(\Omega_0) &= \inf_{\gamma\in\Gamma_2}\max_{0\leq t\leq 1}\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}(\gamma(t)),
\end{aligned} \tag{63}$$

where $\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}$ is a restriction of Ψ_λ on $H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})$ and

$$\begin{aligned}
\Gamma_1 &= \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e\}, \\
\Gamma_2 &= \left\{\gamma \in C\left([0,1], H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})\right) : \gamma(0) = 0, \gamma(1) = e\right\}.
\end{aligned} \tag{64}$$

Observe that

$$\begin{aligned}
\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}(u) &= \frac{1}{2}\|u\|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}^2 - \frac{1}{2}\int_{\Omega_0} V^-u^2dx \\
&\quad - \int_{\Omega_0}\left(\frac{f(x)}{p(x)}|u|^{p(x)} + \frac{g(x)}{q(x)}|u|^{q(x)}\right)dx,
\end{aligned} \tag{65}$$

for all $u \in H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})$. Obviously, $c(\Omega_0)$ is independent of λ. From the proofs of Lemma 16 and Lemma 17, we can easily derive that $\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}$ satisfies the mountain pass geometry. Since $H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C}) \subset E_\lambda$ for all $\lambda > 0$, we have $0 < \tau \leq c_\lambda \leq c(\Omega_0)$ for all $\lambda > 0$. Evidently, for any $t \in [0,1]$, $te \in \Gamma_2$. Consequently, there exists $c_0 > 0$ such that

$$c(\Omega_0) \leq \max_{0\leq t\leq 1}\Psi_\lambda(te) \leq c_0 < \infty, \tag{66}$$

being $p^- > 2$. Then,

$$0 < \tau \leq c_\lambda \leq c(\Omega_0) < c_0, \tag{67}$$

for all $\lambda > 0$. In view of Lemma 16, Lemma 17, and Theorem 15, it is easy to get that for all $\lambda > 0$, there exists $\{u_n\} \subset E_\lambda$ such that

$$\begin{aligned}
&\Psi_\lambda(u_n) \longrightarrow c_\lambda > 0, \\
&\Psi'_\lambda(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{68}$$

Lemma 18. *Assume that (S_1),(V_1)–(V_4) and (H_1)–(H_3) hold. Then, Ψ_λ satisfies the $(PS)_c^*$ condition in E_λ for all $c < c_0$ and $\lambda > 0$.*

Proof. Assume that $\{u_n\}$ be a $(\mathrm{PS})_c^*$ sequence in E_λ with $c < c_0$; then, $\{u_n\} \in Y_n$, $\Psi_\lambda(u_n) \longrightarrow c_\lambda$ and $\Psi'_\lambda|_{Y_n}(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. It follows from (42) and (43) and the Hölder inequality that

$$\begin{aligned}
&c_\lambda + o(1)\|u_n\|_\lambda \\
&\geq \Psi_\lambda(u_n) - \frac{1}{p^-}\left\langle \Psi'_\lambda(u_n), u_n\right\rangle = \frac{1}{2}\|u_n\|_{\lambda,V}^2 \\
&\quad - \int_\Omega \left(\frac{f(x)}{p(x)}|u_n|^{p(x)} + \frac{g(x)}{q(x)}|u_n|^{q(x)}\right)dx - \frac{1}{p^-}\|u_n\|_{\lambda,V}^2 \\
&\quad + \frac{1}{p^-}\int_\Omega \left(f(x)|u_n|^{p(x)} + g(x)|u_n|^{q(x)}\right)dx \\
&= \left(\frac{1}{2} - \frac{1}{p^-}\right)\|u_n\|_{\lambda,V}^2 - \int_\Omega f(x)\left(\frac{1}{p(x)} - \frac{1}{p^-}\right) \\
&\quad \cdot |u_n|^{p(x)}dx - \int_\Omega g(x)\left(\frac{1}{q(x)} - \frac{1}{p^-}\right)|u_n|^{q(x)}dx \\
&\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\|u_n\|_\lambda^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\int_\Omega g(x)|u_n|^{q(x)}dx \\
&\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\|u_n\|_\lambda^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right) \\
&\quad \cdot \|g\|_\infty \max\left\{C_q^{q^+}\|u_n\|_\lambda^{q^+}, C_q^{q^-}\|u_n\|_\lambda^{q^-}\right\}.
\end{aligned} \tag{69}$$

On the contrary, we suppose that $\{u_n\}$ is not bounded in E_λ. Then, there exists a subsequence still denoted by

$\{u_n\}$ such that $\|u_n\|_\lambda \longrightarrow \infty$ as $n \longrightarrow \infty$. Then, it follows from (69) that

$$\begin{aligned}&\frac{c_\lambda}{\|u_n\|_\lambda^2} + o(1)\frac{1}{\|u_n\|_\lambda}\\ &\quad\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0} - \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\\ &\qquad\cdot \|g\|_\infty \max\left\{C_q^{q^+}\|u_n\|_\lambda^{q^+-2}, C_q^{q^-}\|u_n\|_\lambda^{q^--2}\right\},\end{aligned} \tag{70}$$

which leads to a contradiction. Hence, $\{u_n\}$ is bounded in E_λ for all $\lambda > 0$. Therefore, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and u_0 in E_λ such that

$$\begin{aligned}&u_n \rightharpoonup u_0 \text{ in } E_\lambda,\ u_n \longrightarrow u_0\ a.e. \text{in } \Omega,\\ &|u_n|^{r(x)-2}u_n \rightharpoonup |u_0|^{r(x)-2}u_0 \text{ in } L^{r'(x)}(\Omega, \mathbb{C}),\end{aligned} \tag{71}$$

where $r'(x) = r(x)/r(x) - 1$. The next step is to show that $u_n \longrightarrow u_0$ in E_λ. By Lemma 7, we can get $u_n \longrightarrow u_0$ in $L^{r(x)}(\Omega, \mathbb{C})$. Thus,

$$\lim_{n\to\infty}\int_\Omega |u_n - u_0|^{p(x)}dx = 0, \tag{72}$$

$$\lim_{n\to\infty}\int_\Omega |u_n - u_0|^{q(x)}dx = 0. \tag{73}$$

Making use of Hölder inequality, we can obtain

$$\begin{aligned}&\int_\Omega \left|f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}\right|dx\\ &\quad\leq \|f\|_\infty \int_\Omega \left|\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}\right|dx\\ &\quad= \|f\|_\infty \int_\Omega \left||u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right|\left|\overline{(u_n u_0)}\right|dx\\ &\quad= \|f\|_\infty \int_\Omega \left||u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right||u_n - u_0|dx\\ &\quad\leq \|f\|_\infty \left(\int_\Omega \left||u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right|^{p(x)/(p(x)-1)}dx\right)^{(p(x)-1)/p(x)}\\ &\qquad\cdot \left(\int_\Omega |u_n - u_0|^{p(x)}dx\right)^{1/p(x)}.\end{aligned} \tag{74}$$

Combining (H_3), (71), and (72), we have

$$\lim_{n\to\infty}\int_\Omega f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}dx = 0. \tag{75}$$

Similarly, we have

$$\lim_{n\to\infty}\int_\Omega g(x)\left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx = 0. \tag{76}$$

We notice that $u_n \rightharpoonup u_0$ in E_λ and $\Psi'_\lambda(u_n) \longrightarrow 0$; then, we obtain

$$\lim_{n\to\infty}\left\langle \Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0\right\rangle = 0. \tag{77}$$

Therefore,

$$\begin{aligned}o(1) &= \left\langle \Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0\right\rangle\\ &= \langle u_n, u_n - u_0\rangle_{\lambda,V} - \Re\int_\Omega \Big(f(x)|u_n|^{p(x)-2}u_n\\ &\quad + g(x)|u_n|^{q(x)-2}u_n\Big)\overline{(u_n u_0)}dx - \langle u_0, u_n - u_0\rangle_{\lambda,V}\\ &\quad + \Re\int_\Omega \left(f(x)|u_0|^{p(x)-2}u_0 + g(x)|u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx\\ &= \langle u_n - u_0, u_n - u_0\rangle_{\lambda,V} - \Re\int_\Omega f(x)\Big(|u_n|^{p(x)-2}u_n\\ &\quad - |u_0|^{p(x)-2}u_0\Big)\overline{(u_n u_0)}dx - \Re\int_\Omega g(x)\\ &\quad\cdot \left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx,\end{aligned} \tag{78}$$

which means that

$$\lim_{n\to\infty}\|u_n - u_0\|_{\lambda,V} = 0. \tag{79}$$

It follows from (43) that $\lim_{n\to\infty}\|u_n - u_0\|_\lambda = 0$.

Proof of Theorem 2. In view of Lemma 16, Lemma 17, and Theorem 15, we can easily infer that for all $\lambda > 0$, there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}$ for Ψ_λ on E_λ. It derives from Lemma 18 and $0 < c_\lambda < c(\Omega_0) < c_0$ for all $\lambda > 0$ that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $u_\lambda^1 \in E_\lambda$ such that $u_n \longrightarrow u_\lambda^1$ in E_λ. Furthermore, $\Psi_\lambda(u_n) \longrightarrow c_\lambda \geq \tau$ and u_λ^1 is a solution of problem (1).

The next step is to prove that system (1) has another solution. Set

$$\bar{c}_\lambda = \inf\left\{\Psi_\lambda(u)\colon u \in \overline{B_\rho}\right\}, \tag{80}$$

where $B_\rho = \{u \in E_\lambda : \|u\|_\lambda < \rho\}$ and $\rho > 0$ is given by Lemma 16. Then, $\bar{c}_\lambda < 0$ for all $\lambda > 0$. For this purpose, we first prove there exists $v_0 \in E_\lambda$ such that $\Psi_\lambda(\sigma v_0) < 0$ for all $\sigma > 0$

sufficiently small. Let $v_0 \in H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$ such that $\int_\Omega g(x)|v_0|^{q(x)}dx > 0$. Making use of the hypothesis (H_2) and (43), we obtain that for $\sigma \in (0,1)$ small enough,

$$\begin{aligned}\Psi_\lambda(\sigma v_0) &= \frac{\sigma^2}{2}\|v_0\|^2_\lambda - \frac{1}{2}\int_{\mathbb{R}^N} V^-(\sigma v_0)^2 dx \\ &\quad - \int_\Omega \left(\frac{f(x)}{p(x)}|\sigma v_0|^{p(x)} + \frac{g(x)}{q(x)}|\sigma v_0|^{q(x)}\right)dx \\ &\le \frac{\sigma^2}{2}\|v_0\|^2_{\lambda,V} - \sigma^{p^+}\int_\Omega \frac{f(x)}{p(x)}|v_0|^{p(x)}dx \\ &\quad - \sigma^{q^+}\int_\Omega \frac{g(x)}{q(x)}|v_0|^{q(x)}dx \\ &\le \frac{\sigma^2}{2}\|v_0\|^2_\lambda - \frac{\sigma^{q^+}}{q^+}\int_\Omega g(x)|v_0|^{q(x)}dx < 0.\end{aligned} \tag{81}$$

Consequently, there exists $v_0 \in E_\lambda$ such that $\Psi_\lambda(\sigma v_0) < 0$ for all $\sigma > 0$ sufficiently small.

Applying Lemma 16 and the Ekeland variational principle to $\overline{B}_\rho$, there exists a sequence $\{u_n\}$ such that

$$\begin{aligned}\bar{c}_\lambda \le \Psi_\lambda(u_n) &\le \bar{c}_\lambda + \frac{1}{n}, \\ \Psi_\lambda(v) &\ge \Psi_\lambda(u_n) - \frac{\|u_n - v\|_\lambda}{n},\end{aligned} \tag{82}$$

for all $v \in \overline{B_\rho}$. Now we prove that $\|u_n\|_\lambda < \rho$ for n enough large. On the contrary, we suppose that $\|u_n\|_\lambda = \rho$ for infinitely many n. Without loss of generality, we can suppose that $\|u_n\|_\lambda = \rho$ for $n \in N$. It follows from Lemma 16 that

$$\Psi_\lambda(u_n) \ge \tau > 0. \tag{83}$$

Combining with (82), we obtain $\bar{c}_\lambda \ge \tau > 0$, which is contradictive with $\bar{c}_\lambda < 0$. Next, we prove $\Psi'_\lambda(u_n) \longrightarrow 0$ in E^*_λ as $n \longrightarrow \infty$. Let

$$y_n = u_n + \sigma v, \quad \text{for all } v \in B_1 := \{u \in E_\lambda : \|u\|_\lambda = 1\}, \tag{84}$$

where $\sigma > 0$ small enough such that $2\sigma\rho + \sigma^2 \le \rho^2 - \|u_n\|^2_\lambda$ for fixed n large. Then,

$$\begin{aligned}\|y_n\|^2_\lambda &= \|u_n\|^2_\lambda + 2\sigma\rho\langle u_n, v\rangle_\lambda + \sigma^2 \\ &\le \|u_n\|^2_\lambda + 2\sigma\rho + \sigma^2 = \rho^2,\end{aligned} \tag{85}$$

which implies that $y_n \in \overline{B}_\rho$. Hence, by using (50), we get

$$\Psi_\lambda(y_n) \ge \Psi_\lambda(u_n) - \frac{\|u_n - y_n\|_\lambda}{n}, \tag{86}$$

that is,

$$\frac{\Psi_\lambda(u_n + \sigma v) - \Psi_\lambda(u_n)}{\sigma} \ge -\frac{1}{n}. \tag{87}$$

Set $\sigma \longrightarrow 0^+$; we obtain $\langle \Psi'_\lambda(u_n), v\rangle \ge -1/n$ for each fixed n large. Similarly, choosing $\sigma < 0$ and $|\sigma|$ small enough and repeating the process above, we can easily get that

$$\left\langle \Psi'_\lambda(u_n), v \right\rangle \le \frac{1}{n}, \tag{88}$$

for each fixed n large.

In short, we have

$$\lim_{n\to\infty} \sup_{v\in B_1} \left|\Psi'_\lambda(u_n), v\right| = 0, \tag{89}$$

which immediately concludes that $\Psi'_\lambda(u_n) \longrightarrow 0$ in E^*_λ as $n \longrightarrow \infty$. Therefore, $\{u_n\}$ is a $(PS)_{\bar{c}_\lambda}$ sequence for the functional Ψ_λ. Making use of a similar proof as Lemma 18, there exists u^2_λ such that $u_n \longrightarrow u^2_\lambda$ in E_λ. Therefore, we obtain a nontrivial solution u^2_λ of problem (1) satisfying

$$\begin{aligned}\Psi_\lambda\left(u^2_\lambda\right) &\le \xi < 0, \\ \left\|u^2_\lambda\right\|_\lambda &< \rho.\end{aligned} \tag{90}$$

Hence, it is easy to conclude that

$$\Psi_\lambda\left(u^2_\lambda\right) = \bar{c}_\lambda \le \xi < 0 < \tau < c_\lambda = \Psi_\lambda\left(u^1_\lambda\right), \quad \text{for all } \lambda > 0, \tag{91}$$

which completes the proof.

4. Proof of Theorem 3

In this section, we mainly give the proof of Theorem 3. In addition, inspired by [2, 17], we obtain the method to prove Theorem 3.

Proof of Theorem 3. For each sequence $\{\lambda_n\}$ such that $1 \le \lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$, set $u^{(i)}_n$ to be the critical points of Ψ_λ obtained in Theorem 2, where $i = 1, 2$. Therefore, one has

$$\begin{aligned}&\Psi_{\lambda_n}\left(u^{(2)}_n\right) \le \xi < 0 < \tau < c_{\lambda_n} = \Psi_{\lambda_n}\left(u^{(1)}_n\right) < c_0, \\ &\Psi'_{\lambda_n}\left(u^{(1)}_n\right) = \Psi'_{\lambda_n}\left(u^{(2)}_n\right) = 0\end{aligned} \tag{92}$$

$$\begin{aligned}\Psi_{\lambda_n}\left(u_n^{(i)}\right) &= \frac{1}{2}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 - \frac{1}{2}\int_\Omega V^-\left(u_n^{(i)}\right)^2 dx \\ &\quad - \int_\Omega \frac{f(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \\ &= \frac{1}{2}\left\|u_n^{(i)}\right\|_{\lambda_n,V}^2 - \int_\Omega \frac{f(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx \\ &\quad - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \geq \frac{1}{2}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 \\ &\quad - \int_\Omega \frac{g(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \\ &\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 \\ &\quad + \left(\frac{1}{p^-} - \frac{1}{q^-}\right)\int_\Omega g(x)\left|u_n^{(i)}\right|^{q(x)} dx \\ &\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right) \\ &\quad \cdot \|g\|_\infty \max\left\{C_q^{q^+}\left\|u_n^{(i)}\right\|_{\lambda_n}^{q^+}, C_q^{q^-}\left\|u_n^{(i)}\right\|_{\lambda_n}^{q^-}\right\}.\end{aligned} \tag{93}$$

In view of (92) and (93), it gains

$$\left\|u_n^{(i)}\right\|_{\lambda_n} \leq c_2, \tag{94}$$

where $c_2 > 0$ is independent of λ_n. So we can suppose that $u_n^{(i)} \rightharpoonup u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ and $u_n^{(i)} \longrightarrow u^i$ in $L^{r(x)}(\Omega, \mathbb{C})$. Making use of Fatou's lemma, we can easily obtain

$$\int_\Omega V^+\left|u^{(i)}\right|^2 dx \leq \liminf_{n\to\infty}\int_\Omega V^+\left|u_n^{(i)}\right|^2 dx \leq \liminf_{n\to\infty}\frac{\left\|u_n^{(i)}\right\|_{\lambda_n}^2}{\lambda_n} = 0. \tag{95}$$

Consequently, $u^{(i)} = 0$ a.e. in $\mathbb{R}^N \setminus (V^+)^{-1}(0)$ and $u^{(i)} \in H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})$.

Next, we will prove that $u_n^{(i)} \longrightarrow u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$. Indeed, combining $u_n^{(i)} \longrightarrow u^{(i)}$ in $L^{r(x)}(\Omega, \mathbb{C})$ and (H_3), one has

$$\begin{aligned}&\int_\Omega f(x)\left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx \leq \|f\|_\infty \int_\Omega \left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx, \\ &\int_\Omega g(x)\left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx \leq \|g\|_\infty \int_\Omega \left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx.\end{aligned} \tag{96}$$

Then,

$$\lim_{n\to\infty}\int_\Omega f(x)\left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx = 0, \tag{97}$$

$$\lim_{n\to\infty}\int_\Omega g(x)\left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx = 0. \tag{98}$$

We notice that

$$\lim_{n\to\infty}\left\langle \Psi_{\lambda_n}'\left(u_n^{(i)}\right), u_n^{(i)}\right\rangle = \lim_{n\to\infty}\left\langle \Psi_{\lambda_n}'\left(u_n^{(i)}\right), u^{(i)}\right\rangle = 0. \tag{99}$$

Therefore, we have

$$\begin{aligned}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 &= \int_\Omega V^-\left(u_n^{(i)}\right)^2 dx \\ &\quad + \int_\Omega \left(f(x)\left|u_n^{(i)}\right|^{p(x)} + g(x)\left|u_n^{(i)}\right|^{q(x)}\right) dx + o(1),\end{aligned} \tag{100}$$

$$\begin{aligned}\left\langle u_n^{(i)}, u^{(i)}\right\rangle_{\lambda_n} &= \Re\int_\Omega V^- u_n^{(i)}\overline{u^{(i)}} dx \\ &\quad + \Re\int_\Omega f(x)\left|u_n^{(i)}\right|^{p(x)-2} u_n^{(i)}\overline{u^{(i)}} dx \\ &\quad + \Re\int_\Omega g(x)\left|u_n^{(i)}\right|^{q(x)-2} u_n^{(i)}\overline{u^{(i)}} dx + o(1).\end{aligned} \tag{101}$$

By (97)–(101), we have

$$\lim_{n\to\infty}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 = \lim_{n\to\infty}\left\langle u_n^{(i)}, u^{(i)}\right\rangle_{\lambda_n} = \left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2. \tag{102}$$

On the other hand, the weak lower semicontinuity of norm yields that

$$\begin{aligned}\left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 &\leq \liminf_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 \\ &\leq \limsup_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 \\ &\leq \lim_{n\to\infty}\left\|u_n^{(i)}\right\|_{\lambda_n}^2.\end{aligned} \tag{103}$$

To sum up, we can see that

$$\limsup_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} \leq \left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}. \tag{104}$$

By Proposition 3.32 of [18], we can obtain that $u_n^{(i)} \longrightarrow u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$. We notice that $\lim_{n\to\infty}\langle \Psi_{\lambda_n}'(u_n^{(i)}), v\rangle = 0$, for any $v \in C_0^\infty(\Omega_0, \mathbb{C})$. Hence,

$$\begin{aligned}&\Re\int_{\mathbb{R}^{2N}} \frac{\left(u^{(i)}(x) - e^{i(x-y)\cdot A((x+y)/2)}u^{(i)}(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}} dxdy \\ &- \Re\int_{\Omega_0} V^- u^{(i)}\bar{v} dx - \Re\int_{\Omega_0}\left(f(x)\left|u^{(i)}\right|^{p(x)-2}u^{(i)}\right. \\ &\left. + g(x)\left|u^{(i)}\right|^{q(x)-2}u^{(i)}\right)\bar{v} dx = 0.\end{aligned} \tag{105}$$

Since the density of $C_0^\infty(\Omega_0,\mathbb{C})$ in $H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})$, we can obtain that $u^{(i)}$ is a weak solution of problem (12).

Together with (92), $u^{(i)}=0$ a.e. in $\mathbb{R}^N\setminus(V^+)^{-1}(0)$ and the constants ξ,τ are independent of λ; we have

$$\begin{aligned}
&\frac{1}{2}\left\|u^{(1)}\right\|^2_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}-\frac{1}{2}\int_{\Omega_0}V^-\left(u^{(1)}\right)^2dx\\
&\quad-\int_{\Omega_0}\left(\frac{f(x)}{p(x)}\left|u^{(1)}\right|^{p(x)}+\frac{g(x)}{q(x)}\left|u^{(1)}\right|^{q(x)}\right)dx\geq\tau>0,\\
&\frac{1}{2}\left\|u^{(2)}\right\|^2_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}-\frac{1}{2}\int_{\Omega_0}V^-\left(u^{(2)}\right)^2dx\\
&\quad-\int_{\Omega_0}\left(\frac{f(x)}{p(x)}\left|u^{(2)}\right|^{p(x)}+\frac{g(x)}{q(x)}\left|u^{(2)}\right|^{q(x)}\right)dx\leq\xi<0,
\end{aligned}\tag{106}$$

which implies that $u^{(i)}\neq 0$ and $u^{(1)}\neq u^{(2)}$.

Now we consider the case where $f(x),g(x)\equiv$ constants; that is

$$\begin{cases}(-\Delta)_A^{s(\cdot)}u+V_\lambda(x)u=a|u|^{p(x)-2}u+b|u|^{q(x)-2}u\ \text{in}\ \Omega,\\ u=0\ \text{in}\ \mathbb{R}^N\setminus\Omega,\end{cases}\tag{107}$$

where a,b are two nonnegative constants. Correspondingly, the energy functional $\Psi_\lambda\colon E_\lambda\longrightarrow\mathbb{R}$ is

$$\begin{aligned}
\Psi_\lambda(u)&=\frac{1}{2}\|u\|^2_\lambda-\frac{1}{2}\int_\Omega V^-u^2dx\\
&\quad-\int_\Omega\left(\frac{a}{p(x)}|u|^{p(x)}+\frac{b}{q(x)}|u|^{q(x)}\right)dx\\
&=\frac{1}{2}\|u\|^2_{\lambda,V}-\int_\Omega\left(\frac{a}{p(x)}|u|^{p(x)}+\frac{b}{q(x)}|u|^{q(x)}\right)dx.
\end{aligned}\tag{108}$$

Next, we mainly prove the existence of infinitely many solutions to problem (107) by using four different methods.

Theorem 19. *Assume that (S_1), (S_2), (V_1)–(V_4), and (H_1)–(H_3) hold. Let $N>2s^+$. Then, problem (107) has infinitely many solutions.*

Proof. Method 1: It is easy to verify that functional Ψ_λ is even and satisfies $\Psi_\lambda(0)=0$. Furthermore, Lemma 18 shows that functional Ψ_λ is bounded from below in E_λ and satisfies the (PS) condition. For any $k\in N$ and $\rho_k>0$, let $S_{\rho_k}=\{u\in E_\lambda,\ \|u\|_\lambda=\rho_k\}$; then, for any $u\in S_{\rho_k}$, one has

$$\begin{aligned}
\Psi_\lambda(u)&=\frac{1}{2}\|u\|^2_{\lambda,V}-\int_\Omega\frac{a}{p(x)}|u|^{p(x)}dx-\int_\Omega\frac{b}{q(x)}|u|^{q(x)}dx\\
&\leq\frac{1}{2}\|u\|^2_\lambda-\frac{b}{q^+}\int_\Omega|u|^{q(x)}dx\\
&\leq\frac{1}{2}\|u\|^2_\lambda-\frac{b}{q^+}\min\left\{\|u\|^{q^+}_{L^{q(x)}(\Omega,\mathbb{C})},\|u\|^{q^-}_{L^{q(x)}(\Omega,\mathbb{C})}\right\}.
\end{aligned}\tag{109}$$

We find that there exists $C_{S_{\rho_k}}>0$ such that $\|u\|_{L^{q(x)}(\Omega)}\geq C_{S_{\rho_k}}\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space. Then, by $1<q(x)<2$, it gains

$$\sup_{u\in X^k\cap S_{\rho_k}}\Psi_\lambda(u)\leq\frac{1}{2}\|u\|^2_\lambda-\frac{b}{p^+}\min\left\{C^{q^+}_{S_{\rho_k}},C^{q^-}_{S_{\rho_k}}\right\}\|u\|^{q^+}_\lambda.\tag{110}$$

Letting $\|u\|_\lambda=\rho_k$ small enough, we can obtain $\sup_{u\in X^k\cap S_{\rho_k}}\Psi_\lambda(u)<0=\Psi_\lambda(0)$. Furthermore, we assert that (ii) of Theorem 9 does not work. In fact, (109) means $\Psi_\lambda(u)\neq 0$ since $\Psi_\lambda(tu)<0$ as t small enough with the given $u\in E_\lambda$. Thus, by Theorem 9, we get that problem (1) has a sequence of solutions $\{u_k\}$ with $\|u_k\|_\lambda\longrightarrow 0$ as $k\longrightarrow\infty$. In short, problem (1) has infinitely many solutions for all $\lambda>0$.

Method 2. To start with, we assert that for any finite dimensional subspace X of E_λ, there exists $r_1=r_1(X)$ such that $\Psi_\lambda(u)<0$ for all $u\in E_\lambda\setminus B_{r_1}(X)$, where $B_{r_1}(X)=\{u\in E_\lambda:\|u\|_\lambda<r_1\}$. Indeed, for each $t\geq 1$, we can easily get that

$$\begin{aligned}
\Psi_\lambda(tu)&=\frac{t^2}{2}\|u\|^2_\lambda-\int_\Omega\frac{a}{p(x)}|tu|^{p(x)}dx-\int_\Omega\frac{b}{q(x)}|tu|^{q(x)}dx\\
&\leq\frac{t^2}{2}\|u\|^2_\lambda-\frac{at^{p^-}}{p^+}\int_\Omega|u|^{p(x)}dx\\
&\leq\frac{t^2}{2}\|u\|^2_\lambda-\frac{at^{p^-}}{p^+}\min\left\{\|u\|^{p^+}_{L^{p(x)}(\Omega,\mathbb{C})},\|u\|^{p^-}_{L^{p(x)}(\Omega,\mathbb{C})}\right\}.
\end{aligned}\tag{111}$$

We observe that there exists $C_X>0$ such that $\|u\|_{L^{p(x)}(\Omega)}\geq C_X\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space X. Then, by $p^->2$, it gains

$$\begin{aligned}
\Psi_\lambda(tu)&\leq\frac{t^2}{2}\|u\|^2_\lambda-\frac{at^{p^-}}{p^+}\min\left\{C^{p^+}_X,C^{p^-}_X\right\}\|u\|^{p^+}_\lambda\\
&\longrightarrow-\infty\text{as}\ t\longrightarrow\infty.
\end{aligned}\tag{112}$$

Thus, there exists $r_1>0$ large enough such that $\Psi_\lambda(u)<0$ for all $u\in E_\lambda$, with $\|u\|_\lambda=r_2$ and $r_2\geq r_1$. Consequently, the assertion is valid.

From Lemma18, we know that Ψ_λ satisfies the $(\mathrm{PS})_c$ condition for any $c\in\mathbb{R}$. Obliviously, $\Psi_\lambda(0)=0$ and Ψ_λ is an even functional. In short, it follows from Theorem 8 that there

exists an unbounded sequence of solutions of problem (1) for all $\lambda > 0$.

Lemma 19 (see Lemma 4.1, [10]). *Let $1 < r(x) < 2N/(N - 2s(x,x))$ for all $x \in \bar{\Omega}$. For any $k \in N$, define*

$$\beta_k := \left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})} : u \in Z_k, \|u\|_\lambda = 1 \right\}. \tag{113}$$

Then, $\beta_k \longrightarrow 0$, as $k \longrightarrow \infty$.

Method 3. By Remark 14 and Lemma 18, we know that Ψ_λ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To start with, we will prove (D_1) is satisfied. It follows from (36) and (43) that

$$\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\|u\|^2_{\lambda,V} - \int_\Omega \frac{a}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|u|^{q(x)}dx \\
&\le \frac{1}{2}\|u\|^2_\lambda - \frac{a}{p^+}\int_\Omega |u|^{p(x)}dx \\
&\le \frac{1}{2}\|u\|^2_\lambda - \frac{a}{p^+}\min\left\{ \|u\|^{p^+}_{L^{p(x)}(\Omega,\mathbb{C})}, \|u\|^{p^-}_{L^{p(x)}(\Omega,\mathbb{C})} \right\}.
\end{aligned} \tag{114}$$

We find that there exists $C_{Y_k} > 0$ such that $\|u\|_{L^{p(x)}(\Omega)} \ge C_{Y_k}\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space Y_k. Then,

$$\Psi_\lambda(u) \le \frac{1}{2}\|u\|^2_\lambda - \frac{a}{p^+}\min\left\{ C^{p^+}_{Y_k}, C^{p^-}_{Y_k} \right\}\|u\|^{p^-}_\lambda. \tag{115}$$

Then, by $p^- > 2$, we can easily obtain that (D_1) is satisfied for $\|u\|_\lambda = r_k > 1$ big enough. Next, we will show (D_2) is fulfilled. In view of (43) and Lemma 20, for $u \in E_\lambda$, one has

$$\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2}\int_{\mathbb{R}^N} V^- u^2 dx \\
&\quad - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)} \right) dx \\
&= \frac{1}{2}\|u\|^2_{\lambda,V} - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)} \right) dx \\
&\ge \frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|^2_\lambda - \frac{a}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{b}{q^-}\int_\Omega |u|^{q(x)}dx \\
&\ge \frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|^2_\lambda - \frac{a}{p^-}\beta_k^{p^-}\|u\|^{p^+}_\lambda - \frac{b}{q^-}\beta_k^{q^-}\|u\|^{q^+}_\lambda \\
&= \|u\|^{q^+}_\lambda \left[\frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|^{2-q^+}_\lambda - \frac{a}{p^-}\beta_k^{p^-}\|u\|^{p^+-q^+}_\lambda - \frac{b}{q^-}\beta_k^{q^-} \right] \\
&= \|u\|^{q^+}_\lambda \left[\|u\|^{2-q^+}_\lambda \left(\frac{\vartheta_0 - 1}{2\vartheta_0} - \frac{a}{p^-}\beta_k^{p^-}\|u\|^{p^+-2}_\lambda \right) - \frac{b}{q^-}\beta_k^{q^-} \right] \\
&= \gamma_k^{q^+} \left[\gamma_k^{2-q^+} \left(\frac{\vartheta_0 - 1}{2\vartheta_0} - \frac{a}{p^-}\beta_k^{p^-}\gamma_k^{p^+-2} \right) - \frac{b}{q^-}\beta_k^{q^-} \right].
\end{aligned} \tag{116}$$

Choosing $\gamma_k = ((p^-(\vartheta_0 - 1))/8a\vartheta_0\beta_k^{p^-})^{1/(p^+-2)}$. Combing with Lemma 20, we know that $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $\gamma_k \longrightarrow +\infty$ as $k \longrightarrow \infty$ and

$$\Psi_\lambda(u) \ge \gamma_k^{q^+}\left(\frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\gamma_k^{2-q^+} - \frac{b}{q^-}\beta_k^{q^-} \right) \longrightarrow +\infty, \tag{117}$$

as $k \longrightarrow \infty$. In conclusion, (D_2) is fulfilled. It is easy to check that satisfying $\Psi_\lambda(-u) = \Psi_\lambda(u)$. Thus, by Theorem 10, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

Method 4. First, we will show that (D_4) is fulfilled. By (42) and (43), one has

$$\begin{aligned}
\Psi_\lambda(u) &\ge \frac{1}{2}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|^2_\lambda - \frac{a}{p^-}\max\left\{ C_p^{p^+}\|u\|^{p^+}_\lambda, C_p^{p^-}\|u\|^{p^-}_\lambda \right\} \\
&\quad - \frac{b}{q^-}\max\left\{ \|u\|^{q^+}_{L^{q(x)}(\Omega,\mathbb{C})}, \|u\|^{q^-}_{L^{q(x)}(\Omega,\mathbb{C})} \right\}.
\end{aligned} \tag{118}$$

Since $1 < q^- \le q^+ < 2 < p^- \le p^+ < 2N/(N - 2s(x,x))$, we can choose $r_3 \in (0,1)$ small enough such that for all $u \in E_\lambda$ with $\|u\|_\lambda \le r_3$,

$$\frac{1}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|^2_\lambda \ge \frac{a}{p^-}\max\left\{ C_p^{p^+}\|u\|^{p^+}_\lambda, C_p^{p^-}\|u\|^{p^-}_\lambda \right\} \tag{119}$$

hold. Then,

$$\Psi_\lambda(u) \ge \frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|^2_\lambda - \frac{b}{q^-}\beta_k^{q^-}\|u\|^{q^-}_\lambda. \tag{120}$$

Choosing

$$r_k = \left(\frac{8b\vartheta_0\beta_k^{q^-}}{3q^-(\vartheta_0 - 1)} \right)^{1/(2-q^-)}. \tag{121}$$

Combing with Lemma 20, we know that $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $r_k \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there exists $\bar{k}$ such that $r_k \le r_3$ as $k \ge \bar{k}$. Consequently, for $k \ge \bar{k}$ and $u \in Z_k$ with $\|u\|_\lambda = r_k$, we can obtain that $\Psi_\lambda(u) \ge 0$. Next, we will show (D_5) is fulfilled. For any $u \in Y_k$, $\|u\|_\lambda = \gamma_k$ with $r_k > \gamma_k > 0$, we get

$$\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\|u\|^2_{\lambda,V} - \int_\Omega \frac{a}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|u|^{q(x)}dx \\
&\le \frac{1}{2}\|u\|^2_\lambda - \frac{a}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{b}{q^+}\int_\Omega |u|^{q(x)}dx \\
&\le \frac{1}{2}\|u\|^2_\lambda - \frac{a}{p^+}\min\left\{ \|u\|^{p^+}_{L^{p(x)}(\Omega,\mathbb{C})}, \|u\|^{p^-}_{L^{p(x)}(\Omega,\mathbb{C})} \right\} \\
&\quad - \frac{b}{q^+}\min\left\{ \|u\|^{q^+}_{L^{q(x)}(\Omega,\mathbb{C})}, \|u\|^{q^-}_{L^{q(x)}(\Omega,\mathbb{C})} \right\}.
\end{aligned} \tag{122}$$

We find that there exists $C_{Y_k} > 0$ such that $\|u\|_{L^{q(x)}(\Omega)} \geq C_{Y_k}\|u\|_\lambda$ and $\|u\|_{L^{p(x)}(\Omega)} \geq C_{Y_k}\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space Y_k. Then,

$$\begin{aligned}\Psi_\lambda(u) &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\min\left\{C_{Y_k}^{p^+}\|u\|_\lambda^{p^+}, C_{Y_k}^{p^-}\|u\|_\lambda^{p^-}\right\} \\ &\quad - \frac{b}{q^+}\min\left\{C_{Y_k}^{q^+}\|u\|_\lambda^{q^+}, C_{Y_k}^{q^-}\|u\|_\lambda^{q^-}\right\} \\ &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\min\left\{C_{Y_k}^{p^+}, C_{Y_k}^{p^-}\right\}\|u\|_\lambda^{p^+} \\ &\quad - \frac{b}{q^+}\min\left\{C_{Y_k}^{q^+}, C_{Y_k}^{q^-}\right\}\|u\|_\lambda^{q^+} < 0,\end{aligned} \tag{123}$$

as $\gamma_k > 0$ small enough. Now we check (D_6) is fulfilled. It follows from (D_4) that for $k \geq \bar{k}$ and $u \in Z_k$ with $\|u\|_\lambda \leq r_k$,

$$\begin{aligned}\Psi_\lambda(u) &\geq \frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|_\lambda^2 - \frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^-} \\ &\geq -\frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^-} \geq -\frac{b}{q^-}\beta_k^{q^-} r_k^{q^-},\end{aligned} \tag{124}$$

thanks to $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $r_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, (D_6) is also satisfied. In conclusion, by Theorem 11, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Research Article

Existence and Compactness Results for a System of Fractional Differential Equations

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The existence and uniqueness, boundedness, and continuous dependence of solutions for fractional differential equations with Caputo fractional derivative is proven by Perov's fixed point theorem in vector Banach spaces. We study the existence and compactness of solution sets and the u.s.c. of operator solutions.

1. Introduction

In the past twenty years, the fractional differential equation has aroused great consideration not only in its application in mathematics but also in other applications in physics, engineering, finance, fluid mechanics, viscoelastic mechanics, electroanalytical chemistry, and biological and other sciences [1–7].

In recent decades, the Riemann-Liouville, Caputo, and Hadamard fractional calculus are paid more attention; see the monographs [5, 8–13].

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for initial conditions containing $y(0), y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville type and the Caputo type, see Podlubny [12] and Diethelm [14].

The theory of fractional differential equations and inclusions has been extensively studied and developed by many authors; see [15–21] and the references therein.

Perov in 1964 [22] and Perov and Kibenko [23] extended the classical Banach contraction principle for contractive maps on space endowed with a vector-valued metric. Later, they attempted to generalize the Perov fixed point theorem in several directions which has a number of applications in various fields of nonlinear analysis, semilinear differential equations, and system of ordinary differential equations.

In [24], Dezideriu and Precup studied the following system of semilinear equations

$$\begin{cases} A_1 x = F^1(x, y) \\ A_2 y = F^2(x, y) \end{cases} \tag{1}$$

where $A_1, A_2 : D(A) \subset X \longrightarrow X$ are linear operators and F^1, $F^2 : J \times X \times X \longrightarrow X$ are nonlinear operators.

Precup, in [25], explained the advantage of vector-valued norms and the role of matrices that are convergent to zero in the study of semilinear operator systems.

Many authors studied the existence of solutions for a system of differential equations and impulsive differential

equations by using the vector version fixed point theorem; their results are given in [26–30].

Our goal of this paper is to treat the systems of fractional differential equations. More precisely, we will consider the following problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t, x(t), y(t)), \\ {}^{c}D^{\beta}y(t) = g(t, x(t), y(t)), \quad t \in J, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{2}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Caputo fractional derivatives, α, $\beta \in (0, 1]$, $J = [0, \infty)$, $f, g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions, and $x_0, y_0 \in \mathbb{R}$.

In the case where $\alpha = \beta = 1$, the above system was used to analyze initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [31] and mathematical economics [32] which can be set in the operator from ((2)).

The plan of this paper is as follows: in Section 2, we introduce all the background material used in this paper such as some properties of generalized Banach spaces, fixed point theory, and fractional calculus theory. In Section 3, we state and prove our main results by using Perov's fixed point type theorem in generalized Banach spaces. By the Leray-Schauder fixed point in vector Banach space, we prove the existence and compactness of solution sets of the above problems.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 1 [22]. Let X be a nonempty set. The mapping $d : X \times X \longrightarrow \mathbb{R}_{+}^{m}$ which satisfies all the usual axioms of the metric is called a generalized metric in Perov's sense and (X, d) is called a generalized metric space.

In a generalized metric space in Perov's sense, the concepts of Cauchy sequence, convergent sequence, completeness, and open and closed subsets are similarly defined as those for usual metric space.

If $v, r \in \mathbb{R}^m, v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by $v \leq r$, we mean $v_i \leq r_i$ for each $i \in \{1, \cdots, m\}$, and by $v < r$, we mean $v_i < r_i$ for each $i \in \{1, \cdots, m\}$. Also $|v| := (|v_1|, \cdots, |v_m|)$ and $\max(u, v) := (\max(u_1, v_1), \cdots, \max(u_m, v_m))$. If $c \in \mathbb{R}$, then $v \leq c$ means $v_i \leq c$ for each $i \in \{1, \cdots, m\}$. Denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}, \tag{3}$$

the open ball centered in x_0 with radius r, and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}, \tag{4}$$

the closed ball centered in x_0 with radius r.

Definition 2. A square matrix M of real numbers is said to be convergent to zero if and only if $M^n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 3 [33]. *Let $M \in \mathcal{M}_{m,m}(\mathbb{R}_{+})$. The following statements are equivalent:*

(i) M is a matrix convergent to zero

(ii) The eigenvalues of M are in open disc, i.e., $|\mu| < 1$, for every $\mu \in \mathbb{C}$ with $\det(M - \mu I) = 0$

(iii) The matrix $I - M$ is nonsingular and $(I - M)^{-1} = I + M + \cdots + M^n + \cdots$

(iv) The matrix $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements

(v) $M^n q \longrightarrow 0$ and ${}^{t}qM^n \longrightarrow 0$ as $n \longrightarrow \infty$, for any $q \in \mathbb{R}^m$

Example 4. Some examples of matrix convergent to zero are

$$1) \quad M = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \tag{5}$$

where $a, b \in \mathbb{R}_{+}$ and $a + b < 1$,

$$2) \quad M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \tag{6}$$

where $a, b \in \mathbb{R}_{+}$ and $a + b < 1$, and

$$3) \quad M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \tag{7}$$

where $a, b, c \in \mathbb{R}_{+}$ and $\max\{a, c\} < 1$.

Definition 5 [34]. Let (X, d) be a generalized metric space. An operator $N : X \longrightarrow X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq Md(x, y) \quad \text{for all } x, y \in X. \tag{8}$$

Notice now that the Banach fixed point theorem can be extended to generalized metric spaces in the sense of Perov.

Theorem 6 [22, 28]. *Let (X, d) be a complete generalized metric space and $N : X \longrightarrow X$ be a contractive operator with Lipschitz matrix M. Then, N has a unique fixed point x^*, and for each $x_0 \in X$, we have*

$$d\left(N^k(x_0), x^*\right) \leq M^k(I - M)^{-1}d(x_0, N(x_0)) \quad \text{for all } k \in \mathbb{N}. \tag{9}$$

We recall now the following Leary-Schauder type theorem.

Theorem 7 [28, 35]. *Let X be a generalized Banach space and let $N : X \longrightarrow X$ be a completely continuous operator. Then, either*

(i) the equation $N(x) = x$ has at least one solution, or

(ii) the set $\mathcal{M} = \{x \in X \mid \mu N(x) = x, \mu \in (0, 1)\}$ is unbounded

We will use the following notations. Let (X, d) and (Y, ρ) be two metric spaces and $G : X \longrightarrow \mathcal{P}(Y)$.

$$\begin{aligned}\mathcal{P}(X) &= \{Y \subset X : Y = \varnothing\}, \mathcal{P}_{cl}(X)\\ &= \{Y \in \mathcal{P}(X): Y \text{ is closed}\}, \mathcal{P}_b(X)\\ &= \{Y \in \mathcal{P}(X): Y \text{ bounded}\}.\end{aligned} \tag{10}$$

Definition 8 [28, 36]. A multivalued map $\varphi : X \longrightarrow \mathcal{P}(Y)$ is called upper semicontinuous (u.s.c.) at a point $x_0 \in X$ provided that for every open subset $V \subset Y$ with $V \supseteq \varphi(x_0)$, there exists $U \in \mathcal{V}(x_0)$ such that

$$\forall x \in U, \varphi(x) \subset V. \tag{11}$$

φ is called upper semicontinuous if it is u.s.c. at every point $x \in X$.

The mapping G is said to be completely continuous if it is *u.s.c.*, and for every bounded subset $C \subseteq X$, $G(C)$ is relatively compact, i.e., there exists a relatively compact set $K = K \subset X$ such that

$$G(C) = \cup\{G(x): x \in C\} \subset K. \tag{12}$$

Also, G is compact if $G(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set W containing x, such that $G(W)$ is relatively compact.

Theorem 9 [36]. *Let $G : X \longrightarrow \mathcal{P}_{cp}(Y)$ be a closed locally compact multifunction. Then, G is u.s.c.*

Now, we recall some notations and definitions of fractional calculus theory.

Definition 10 [5]. The Riemann-Liouville fractional integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^a h(t) = \frac{1}{\Gamma(a)} \int_0^t (t - s)^{a-1} h(s) ds, \tag{13}$$

where Γ is the Euler gamma function defined by $\Gamma(a) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$.

Definition 11 [5]. For a function $h \in AC^n(J, \mathbb{R})$, the Caputo fractional-order derivative of order α of h is defined by

$$({}^c D_0^\alpha h)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s) ds, \tag{14}$$

where $n = [\alpha] + 1$.

We recall Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 of [37].

Lemma 12. *Let $v : [0, b) \longrightarrow [0, \infty)$ be a real function; $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b)$(some $b \leq +\infty$); $a(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$; $a(t) \leq M$ (constant); and suppose $v(t)$ is nonnegative and locally integrable on $0 \leq t < b$. Assume $\gamma > 0$ such that*

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t - s)^{1-\gamma}} ds, \tag{15}$$

then

$$v(t) \leq w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t - s)^{n\gamma-1} w(s) ds, \tag{16}$$

for every $t \in [0, b)$.

3. Existence, Uniqueness, and Bounded Solutions

In order to define a solution for problem (2), consider the following functional spaces. Let $J = [0, \infty)$ and $C(J, \mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$.

$$C_b = \{y \in C(J, \mathbb{R}): y \text{ is bounded}\}. \tag{17}$$

C_b is a Banach with norm

$$\|y\| C_b = \sup \{|y(t)| : t \in J\}. \tag{18}$$

We need the following auxiliary result.

Lemma 13 [14]. *Concerning the problem,*

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x), \quad 0 < \alpha < 1, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{19}$$

where the function $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. The function $x : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is the unique solution of the problem (19) if and only if

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s)) ds, \quad t \in \mathbb{R}_+. \tag{20}$$

Definition 14. A function $x, y \in C_b$ is said to be a solution of (2) if and only if

$$\begin{cases} x(t) = x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))ds, \\ y(t) = y_0 + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))ds, \\ \quad t \in J. \end{cases} \tag{21}$$

In this section, we assume the following conditions.

(H1). There exists functions $h_{i,\alpha}, h_{j,\beta} \in L^1(J, \mathbb{R}^+)$, $i = 1, 2$, $j = 3, 4$ such that

$$\begin{aligned} &|f(s, x, y) - f(s, \bar{x}, \bar{y})| \\ &\quad \le h_{i,\alpha}(s)|x - \bar{x}| + h_{2,\alpha}(s)|y - \bar{y}| \quad \text{for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}, \\ &|g(s, x, y) - g(s, \bar{x}, \bar{y})| \\ &\quad \le h_{3,\beta}(s)|x - \bar{x}| + h_{4,\beta}(s)|y - \bar{y}| \quad \text{for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \left\|h_{i,\alpha}\right\|_\infty &= \sup_{t\in J}\int_0^t \theta^\alpha(t-s)h_{i,\alpha}(s)ds < \infty, \quad i = 1, 2, \\ \left\|h_{j,\beta}\right\|_\infty &= \sup_{t\in J}\int_0^t \theta^\beta(t-s)h_{j,\beta}(s)ds < \infty, \quad j = 3, 4, \end{aligned} \tag{23}$$

where for $\gamma = \alpha, \beta$,

$$\theta_\gamma(t) = \begin{cases} t^{\gamma-1}, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{24}$$

(H2). The functions $f_\alpha, g_\beta : J \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} f_\alpha(t) &= \theta_\alpha(t-s)f(s, 0, 0), g_\beta(t) \\ &= \theta_\beta(t-s)g(s, 0, 0), \quad t \in J, \end{aligned} \tag{25}$$

satisfies

$$\begin{aligned} \|f_\alpha\|_\infty &= \sup_{t\in J}\int_0^t \theta_\alpha(t-s)|f(s, 0, 0)|ds < \infty, \\ \left\|g_\beta\right\|_\infty &= \sup_{t\in J}\int_0^t \theta_\beta(t-s)|g(s, 0, 0)|ds < \infty. \end{aligned} \tag{26}$$

Now, we are in a position to prove our existence and uniqueness solution for the problem (2) using the Perov fixed point theorem and show that for each initial condition (x_0, y_0), the solution is bounded.

Theorem 15. *Assume that (H1)-(H2) are satisfied. If the matrix*

$$M = \begin{pmatrix} \dfrac{\left\|h_{1,\alpha}\right\|_\infty}{\Gamma(\alpha)} & \dfrac{\left\|h_{2,\alpha}\right\|_\infty}{\Gamma(\alpha)} \\ \dfrac{\left\|h_{3,\beta}\right\|_\infty}{\Gamma(\beta)} & \dfrac{\left\|h_{4,\beta}\right\|_\infty}{\Gamma(\beta)} \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{R}^+), \tag{27}$$

converges to zero. Then, the problem (2) has a unique bounded solution.

Proof. Transform the problem (2) into a fixed point theorem of the operator $N : C_b \times C_b \longrightarrow C_b \times C_b$ defined by $N(x, y) = (N_1(x, y), N_2(x, y))$, where

$$\begin{aligned} N_1(x, y)(t) &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))ds, \\ N_2(x, y)(t) &= y_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))ds, \\ &\quad t \in [0, \infty). \end{aligned} \tag{28}$$

First, we show that the operator N is well-defined. Let $(x, y) \in C_b \times C_b$ and $t \in [0, \infty)$, then we have

$$\begin{aligned} |N_1(x, y)(t)| &\le |x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x(s), y(s))|ds \\ &\le |x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot |f(s, x(s), y(s)) - f(s, 0, 0)|ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|f(s, 0, 0)|}{(t-s)^{1-\alpha}}ds. \end{aligned} \tag{29}$$

Then,

$$\begin{aligned} \|N_1(x, y)\|_{c_b} &\le |x_0| + \frac{\left\|h_{1,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|x\|c_b \\ &\quad + \frac{\left\|h_{2,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|y\|c_b + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\ \|N_2(x, y)\|_{c_b} &\le |y_0| + \frac{\left\|h_{3,\beta}\right\|_\infty}{\beta}\|x\|c_b \\ &\quad + \frac{\left\|h_{4,\beta}\right\|_\infty}{\beta}\|y\|c_b + \frac{\left\|g_\beta\right\|_\infty}{\Gamma(\beta)}. \end{aligned} \tag{30}$$

Hence, the operator N is well-defined.

Clearly, the fixed points of operator N are solutions of problem (2). Now, we show that N is a contraction. For all $(x, y), (\bar{x}, \bar{y}) \in C_b \times C_b$, we have

$$
\begin{aligned}
&|N_1(x,y)(t) - N_1(\bar{x},\bar{y})(t)| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(t),y(t)) - f(s,\bar{x}(t),\bar{y}(t))|ds \\
&\quad \le \frac{1}{\Gamma(\alpha)}\int_0^t \left[h_{1,\alpha}(s)\left|x(t)-\bar{x}(t)\right| + h_{2,\alpha}(s)|y(t)-\bar{y}(t)|\right]ds.
\end{aligned} \tag{31}
$$

Then,

$$
\begin{aligned}
&\|N_1(x,y) - N_1(\bar{x},\bar{y})\|c_b \\
&\quad \le \frac{\left\|h_{1,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|x-\bar{x}\|c_b + \frac{\left\|h_{2,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|y-\bar{y}\|c_b.
\end{aligned} \tag{32}
$$

Similarly, we have

$$
\begin{aligned}
&\|N_2(x,y) - N_2(\bar{x},\bar{y})\|_{C_b} \\
&\quad \le \frac{\left\|h_{3,\beta}\right\|_\infty}{\Gamma(\beta)}\|x-\bar{x}\|_{C_b} + \frac{\left\|h_{4,\beta}\right\|_\infty}{\Gamma(\beta)}\|y-\bar{y}\|_{C_b}.
\end{aligned} \tag{33}
$$

Therefore,

$$
\|N(x,y) - N(\bar{x},\bar{y})\|_{C_b\times C_b} \le M\begin{pmatrix} \|x-\bar{x}\|_{C_b} \\ \|y-\bar{y}\|_{C_b} \end{pmatrix}. \tag{34}
$$

According to Theorem 6, we deduce that the operator N has unique fixed point which is a solution of problem (2). Now, we will prove that the solution (x,y) of problem (2) is bounded. For all $t \in [0,\infty)$, we have

$$
\begin{aligned}
|x(t)| &\le |x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))|ds \\
&\le |x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t [(h_{1,\alpha}(s)|x(s)| + h_{2,\alpha}(s)|y(s)|]ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |f_\alpha(s,0,0)|ds \\
&\le |x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t [(h_{1,\alpha}(s)|x(s)| + h_{2,\alpha}(s)|y(s)|]ds + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\
|y(t)| &\le |y_0| + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|g(s,x(s),y(s))|ds \\
&\le |y_0| + \frac{1}{\Gamma(\beta)}\int_0^t \left[\left(h_{3,\beta}(s)|x(s)| + h_{4,\beta}(s)|y(s)|\right]ds \right. \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^t \left|g_\beta(s,0,0)\right|ds \\
&\le |y_0| + \frac{1}{\Gamma(\beta)}\int_0^t \left[\left(h_{3,\beta}(s)|x(s)| + h_{4,\alpha}(s)|y(s)|\right]ds + \frac{\left\|g_\beta\right\|_\infty}{\Gamma(\beta)}.
\end{aligned} \tag{35}
$$

Therefore,

$$
\begin{aligned}
\|x\|_{C_b} &\le |x_0| + \frac{\left\|h_{1,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|x\|_{C_b} + \frac{\left\|h_{2,\alpha}\right\|_\infty}{\Gamma(\alpha)}\|y\|_{C_b} + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\
\|y\|_{C_b} &\le |y_0| + \frac{\left\|h_{3,\beta}\right\|_\infty}{\Gamma(\beta)}\|x\|_{C_b} + \frac{\left\|h_{4,\beta}\right\|_\infty}{\Gamma(\beta)}\|y\|_{C_b} + \frac{\left\|g_\beta\right\|_\infty}{\Gamma(\beta)}.
\end{aligned} \tag{36}
$$

Hence,

$$
\begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} \le M\begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} + \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \tag{37}
$$

where

$$
\sigma_1 = |x_0| + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \sigma_2 = |y_0| + \frac{\left\|g_\beta\right\|_\infty}{\Gamma(\beta)}. \tag{38}
$$

Then,

$$
\begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} \le (I-M)^{-1}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{39}
$$

From (H1) and (H2), we deduce that the solution (x,y) is bounded.

For the next result, we prove the continuous dependence of solutions on initial conditions.

Theorem 16. *Assume that (H1) and (H2) hold. If $f(t,0,0) = g(t,0,0) = 0$, $t \in J$ and the matrix M defined in (27) converges to zero.*

For every $(x_0,y_0) \in \mathbb{R}\times\mathbb{R}$, we denoted by $(x(t,x_0), y(t,y_0))$ the solution of problem (2). Then, the map $(x_0,y_0) \longrightarrow (x(\cdot,x_0), y(\cdot,y_0))$ is continuous.

Proof. From Theorem 15, for each initial condition (x_0,y_0), $(\bar{x}_0,\bar{y}_0) \in \mathbb{R}\times\mathbb{R}$, there exists unique solution $(x(\cdot,x_0), y(\cdot,y_0))$, $(\bar{x}(\cdot,\bar{x}_0), \bar{y}(\cdot,\bar{y}_0))$, then we get

$$
\begin{cases}
x(t,x_0) = x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds, \\
y(t,y_0) = y_0 + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} g(s,x(s),y(s))ds, \\
\bar{x}(t,\bar{x}_0) = \bar{x}_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,\bar{x}(s),\bar{y}(s))ds, \\
\bar{y}(t,\bar{y}_0) = \bar{y}_0 + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} g(s,x(s),\bar{y}(s))ds.
\end{cases} \tag{40}
$$

Therefore,

$$|x(t,x_0)-\bar{x}(t,\bar{x}_0)| \le |x_0-\bar{x}_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \left[(h_{1,\alpha}(s)|x(s)-\bar{x}(s)| + h_{2,\alpha}(s)|y(s)-\bar{y}(s)|\right] ds,$$
$$|y(t,y_0)-\bar{y}(t,\bar{y}_0)| \le |y_0-\bar{y}_0| + \frac{1}{\Gamma(\beta)}\int_0^t \left[(h_{3,\beta}(s)|x(s)-\bar{x}(s)| + h_{4,\beta}(s)|y(s)-\bar{y}(s)|\right] ds. \tag{41}$$

Hence,

$$\begin{pmatrix} \|x(\cdot,x_0)-\bar{x}(\cdot,\bar{x}_0)\|C_b \\ \|y(\cdot,y_0)-\bar{y}(\cdot,\bar{y}_0)\|C_b \end{pmatrix} \le (I-M)^{-1}\begin{pmatrix} |x_0-\bar{x}_0| \\ |y_0-\bar{y}_0| \end{pmatrix} \longrightarrow 0, \quad \text{as } (x_0,y_0)\longrightarrow(\bar{x}_0,\bar{y}_0). \tag{42}$$

4. Existence and Compactness of Solution Sets

For the existence and compactness result of problem (2), we consider the following Banach space:

$$\mathrm{BC}_b(J,\mathbb{R}) = \left\{u\in C(J,\mathbb{R}):\ \sup_{t\in J}\frac{|u(t)|}{1+t}<\infty\right\}, \tag{43}$$

with norm

$$\|u\|_B = \sup_{t\in J}\frac{|u(t)|}{1+t}. \tag{44}$$

It is evident that $\mathrm{BC}_b(J,\mathbb{R})$ is a Banach space. The following compactness criterion on unbounded domains is called Corduneanu compactness criterion in which the proof is easy and similar to the classical one in $C_b(\mathbb{R}^+,\mathbb{R})$ (see [38]).

Lemma 17. *Let $H\subset BC_b(\mathbb{R}^+,\mathbb{R})$. Then, H is relatively compact if the following conditions hold:*

(a) H is uniformly bounded in $BC_b(\mathbb{R}^+,\mathbb{R}^n)$

(b) The functions belonging to H are almost equicontinuous on $\mathbb{R}^+$, i.e., for all $I\subset\mathbb{R}_+$ compact interval, for any $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that for every $t_1,t_2\in J$ with $|t_1-t_2|<\delta(\epsilon)$, we have for all $u\in H$

$$\left|\frac{u(t_2)}{1+t_2}-\frac{u(t_1)}{1+t_1}\right|<\epsilon \tag{45}$$

(c) The functions from H are equiconvergent, that is, given $\epsilon>0$, there corresponds $T(\epsilon)>0$ such that

$$\left|\frac{u(t_1)}{1+t_1}-\frac{u(t_2)}{1+t_2}\right|<\epsilon \quad \text{for any } t_1,t_2\ge(\epsilon) \text{ and } u\in H \tag{46}$$

In the sequel of this section, we will consider the following assumption.

(H3). For every $l>0$, the functions $(x,y)\longrightarrow f(t,x,y)$, $g(t,x,y)$ are uniformly continuous on the sets $[-l,l]\times[-l,l]$ uniformly with respect to $t\in\mathbb{R}_+$, i.e., f and g satisfied the following condition: for all $l>0$, $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that for all $(x,y),(\bar{x},\bar{y})\in[-l,l]\times[-l,l]$ and for all $t\in\mathbb{R}_+$ with $|x/(1+t)-\bar{x}/(1+t)|<\delta(\epsilon)$, $|y/(1+t)-\bar{y}/(1+t)|<\delta(\epsilon)$, we have

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|<\epsilon, |g(t,x,y)-g(t,\bar{x},\bar{y})|<\epsilon. \tag{47}$$

(H4). There exist $a_i,b_i,c_i>0$, $i=1,2$ such that

$$|f(t,x,y)|\le a_1\frac{|x|}{1+t}+b_1\frac{|y|}{1+t}+c_1,$$
$$|g(t,x,y)|\le a_2\frac{|x|}{1+t}+b_2\frac{|y|}{1+t}+c_2, \quad \forall x,y\in\mathbb{R}, t\in\mathbb{R}_+. \tag{48}$$

Theorem 18. *Assume that (H3) and (H4) hold. Then, the problem (2) has at least one bounded solution. Moreover, the solution set*

$$S(x_0,y_0)=\{(x,y)\in BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R}):\ (x,y) \text{ is solution of } (2)\}, \tag{49}$$

is compact and the multivalued map $S:(x_0,y_0)\longrightarrow S(x_0,y_0)$ is u.s.c.

Proof. Let $N=(N_1,N_2)$ is defined in the proof of Theorem 15.

Step 1. N is well defined. Let $(x,y)\in \mathrm{BC}_b(J,\mathbb{R})\times\mathrm{BC}_b(J,\mathbb{R})$, then

$$\begin{aligned}\frac{|N_1(x(t),y(t))|}{1+l} &\le \frac{|x_0|}{1+l}+\frac{1}{(1+l)\Gamma(\alpha)}\int_0^t \cdot(t-s)^{\alpha-1}|f(s,x(s),y(s))|ds \\ &\le \frac{|x_0|}{1+l}\frac{1}{(1+l)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1} \cdot\left(a_1\frac{(|x(s)|)}{1+s}+b_1\frac{(|y(s)|)}{1+s}+c_1\right)ds \\ &\le \frac{|x_0|}{1+l}+\frac{c_1t^\alpha}{\Gamma(\alpha+1)(1+t)} + \frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1} \cdot\left(a_1\frac{(|x(s)|)}{1+s}+b_1\frac{(|y(s)|)}{1+s}+\right)ds \\ &\le |x_0|+\frac{c_1}{\Gamma(\alpha+1)}+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t \cdot(t-s)^{\alpha-1}ds(a_1\|x\|_B+b_1\|y\|_B).\end{aligned} \tag{50}$$

Hence,

$$\sup_{t\in J}\frac{|N_1(x(t),y(t))|}{1+t}\le|x_0|+\frac{c_1}{\Gamma(\alpha+1)}+\frac{a_1\|x\|_B+b_1\|y\|_B}{\Gamma(\alpha+1)},$$
$$\sup_{t\in J}\frac{|N_2(x(t),y(t))|}{1+t}\le|y_0|+\frac{c_1}{\Gamma(\beta+1)}+\frac{a_2\|x\|_B+b_2\|y\|_B}{\Gamma(\beta+1)}. \tag{51}$$

Step 2. N is continuous.

Let $(x_n,y_n)\longrightarrow(x,y)$ in $\mathrm{BC}_b(J,\mathbb{R})\times\mathrm{BC}_b(J,\mathbb{R})$. Then, there exists $l>0$ such that for any $t\in\mathbb{R}_+$ and $n\in\mathbb{N}$, we have

$$\frac{x_n(t)}{1+t},\frac{y_n(t)}{1+t},\frac{x(t)}{1+t},\frac{y(t)}{1+t}\in[-l,l]. \tag{52}$$

By (H3), for every $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that for all $x,y,\bar{x},\bar{y}\in[-l,l]$ and for all $t\in\mathbb{R}_+$ with $|x/(1+t)-\bar{x}/(1+t)|<\delta(\epsilon)$, $|y/(1+t)-\bar{y}/(1+t)|<\delta(\epsilon)$ we have

$$\begin{aligned}&|f(t,x,y)-f(t,\bar{x},\bar{y})|\\&\quad<\epsilon\Gamma(\alpha+1),|g(t,x,y)-g(t,\bar{x},\bar{y})|<\epsilon\Gamma(\beta+1).\end{aligned} \tag{53}$$

Since (x_n,y_n) converge to (x,y), then there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$,

$$\left|\frac{x_n(t)}{1+t}-\frac{x(t)}{1+t}\right|<\delta(\epsilon),\left|\frac{y_n(t)}{1+t}-\frac{y(t)}{1+t}\right|<\delta(\epsilon),\quad\forall t\in\mathbb{R}_+. \tag{54}$$

Hence,

$$\begin{aligned}&\left|\frac{N_1(x_n,y_n)(t)}{1+t}-\frac{N_1(x,y)(t)}{1+t}\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t}|f(s,x_n(s),y_n(s))-f(s,x(s),y(s))|ds\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t}ds\Gamma(\alpha+1)\epsilon\le\frac{t^\alpha}{1+t}\epsilon\le\epsilon.\end{aligned} \tag{55}$$

Thus

$$\begin{aligned}&\|N_1(x_n,y_n)-N_1(x,y)\|_B\longrightarrow0,\\&\|N_2(x_n,y_n)-N_2(x,y)\|_B\longrightarrow0,\\&\qquad\text{as } n\longrightarrow\infty.\end{aligned} \tag{56}$$

Step 3. We will show that N maps bounded sets into bounded sets in $BC_b(J,R)\times BC_b(J,R)$.

Let $B_r:=\{(x,y)\in\mathrm{BC}_b(J,\mathbb{R})\times\mathrm{BC}_b(J,\mathbb{R}):\|(x,y)\|_\infty\le r\}$, where $r=(r_1,r_2)$ and if $(x,y)\in B_r$, then we obtain

$$\begin{aligned}\left|\frac{N_1(x(t),y(t))}{1+t}\right|&\le\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t}|f(s,x(s),y(s))|ds\\&\le\frac{1}{\Gamma(\alpha)}[a_1r_1+b_1r_2+c_1].\end{aligned} \tag{57}$$

Similarly, we have

$$\left|\frac{N_2(x(t),y(t))}{1+t}\right|\le\frac{1}{\Gamma(\beta)}[a_2r_1+r_2b_2+c_2]. \tag{58}$$

Hence,

$$(\|N_1(x,y)\|,\|N_2(x,y)\|)\le(l_1,l_2), \tag{59}$$

where

$$l_1=\frac{1}{\Gamma(\alpha)}[a_1r_1+b_1r_2+c_1],l_2=\frac{1}{\Gamma(\beta)}[a_2r_1+r_2b_2+c_2]. \tag{60}$$

Step 4. Now, we prove that N maps bounded sets in $\mathrm{BC}_b(J,\mathbb{R})\times\mathrm{BC}_b(J,\mathbb{R})$ into almost equicontinuous sets of $\mathrm{BC}_b(J,\mathbb{R})\times\mathrm{BC}_b(J,\mathbb{R})$.

Let $(x,y)\in B_r$. Then, for all $t_1<t_2$,

$$\begin{aligned}&\left|\frac{N_1(x(t_2),y(t_2))}{1+t_2}-\frac{N_1(x(t_1),y(t_1))}{1+t_1}\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2}-\frac{(t_1-s)^{\alpha-1}}{1+t_1}\right||f(s,x(s),y(s))|ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2}|f(s,x(s),y(s))|ds\\&\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2}-\frac{(t_1-s)^{\alpha-1}}{1+t_1}\right|(a_1r_1+b_1r_2+c_1)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2}(a_1r_1+b_1r_2+c_1)ds\\&\le\frac{a_1r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\left|\frac{t_2^\alpha}{1+t_2}-\frac{t_1^\alpha}{1+t_1}\right|+\frac{2(t_2-t_1)^\alpha}{1+t_2}\right).\end{aligned} \tag{61}$$

Thus,

$$\left|\frac{N_1(x(t_2),t(t_2))}{1+t_2}-\frac{N_1(x(t_1),t(t_1))}{1+t_1}\right|\longrightarrow0\quad\text{as } t_1\longrightarrow t_2. \tag{62}$$

Similarly, we have

$$\left|\frac{N_2(x(t_2),y(t_2))}{1+t_2}-\frac{N_2(x(t_1),y(t_1))}{1+t_1}\right|\longrightarrow 0 \quad \text{as } t_1\longrightarrow t_2. \tag{63}$$

Step 5. The set $N(\bar{B}(0,r))$ is equiconvergent, i.e., for every $\epsilon>0$, there exists $T(\epsilon)>0$ such that $|N(x(t_2),y(t_2))-N(x(t_1),y(t_1))|<\epsilon$, for every $t_1,t_2>T(\epsilon)$ and each $(x,y)\in\bar{B}(0,r)$.

$$\begin{aligned}&\left|\frac{N_1(x(t_2),y(t_2))}{1+t_2}-\frac{N_1(x(t_1),y(t_1))}{1+t_1}\right|\\&\quad\le\frac{a_1r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\frac{3t_2^{\alpha}}{1+t_2}+\frac{t_1^{\alpha}}{1+t_1}\right),\\&\left|\frac{N_2(x(t_2),y(t_2))}{1+t_2}-\frac{N_2(x(t_1),y(t_1))}{1+t_1}\right|\\&\quad\le\frac{a_2r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\frac{3t_2^{\beta}}{1+t_2}+\frac{t_1^{\beta}}{1+t_1}\right).\end{aligned} \tag{64}$$

It is clear that

$$\frac{3t_2^{\beta}}{1+t_2}\longrightarrow 0,\ \frac{t_1^{\beta}}{1+t_1}\longrightarrow 0,\quad \text{as } t_2,t_1\longrightarrow\infty. \tag{65}$$

Then, for any $\epsilon>0$, there exists $T(\epsilon)>0$ such that for all $t_1,t_2>T(\epsilon)$, we have

$$\begin{aligned}&\left|\frac{N_1(x(t_2),y(t_2))}{1+t_2}-\frac{N_1(x(t_1),y(t_1))}{1+t_1}\right|\\&\quad\le\epsilon,\quad \text{for all } (x,y)\in\bar{B}(0,r),\\&\left|\frac{N_2(x(t_2),y(t_2))}{1+t_2}-\frac{N_2(x(t_1),y(t_1))}{1+t_1}\right|\\&\quad\le\epsilon,\quad \text{for any } (x,y)\in\bar{B}(0,r).\end{aligned} \tag{66}$$

Step 6. The set is defined as follows:

$\mathscr{M}=\{(x,y)\in \mathrm{BC}_b(J,\mathbb{R})\times \mathrm{BC}_b(J,\mathbb{R})\colon (x,y)=\mu N(x,y),\ \mu\in(0,1)\}$ is bounded. Let $(x,y)\in\mathscr{M}$, then

$$\begin{aligned}\frac{|x(t)|}{1+t}&\le\frac{|x_0|}{1+t}+\frac{\mu}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,x(s),y(s))|ds\\&\le\frac{|x_0|}{1+t}+\frac{\mu}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\cdot\left(a_1\frac{|x(s)|}{1+s}+b_1\frac{|y(s)|}{1+s}+c_1\right)ds\\&\le\frac{|x_0|}{1+t}+\frac{c_1t^{\alpha}}{\Gamma(\alpha+1)(1+t)}\\&\quad+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\\&\quad\cdot\left(a_1\frac{|x(s)|}{1+s}+b_1\frac{|y(s)|}{1+s}\right)ds.\end{aligned} \tag{67}$$

Similarly, we get

$$\begin{aligned}&\frac{|y(t)|}{1+t}+\frac{|y_0|}{1+t}+\frac{c_2t^{\beta}}{\Gamma(\beta+1)(1+t)}\\&\quad+\frac{1}{(1+t)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\left(a_2\frac{|x(s)|}{1+s}+b_2\frac{|y(s)|}{1+s}\right)ds.\end{aligned} \tag{68}$$

Thus,

$$\begin{aligned}\frac{|x(t)|}{1+t}+\frac{|y(t)|}{1+t}&\le\frac{X_0}{1+t}+\frac{Ct^{\gamma}}{1+t}+\frac{C_*}{1+t}\int_0^t(t-s)^{\gamma-1}\\&\quad\cdot\left(\frac{|x(s)|}{1+s}+b_2\frac{|y(s)|}{1+s}\right)ds,\end{aligned} \tag{69}$$

where

$$\begin{aligned}X_0&=|x_0|+|y_0|,\\C&=\frac{c_1}{\Gamma(\alpha+1)}+\frac{c_2}{\Gamma(\beta+1)},\\C_*&=\frac{a_1+b_1}{\Gamma(\alpha)}+\frac{a_2+b_2}{\Gamma(\beta)},\\\gamma&=\max(\alpha,\beta).\end{aligned} \tag{70}$$

Hence,

$$\begin{aligned}\frac{|x(t)|}{1+t}+\frac{|y(t)|}{1+t}&\le X_0+C+\frac{C_*}{1+t}\int_0^t(t-s)^{\gamma-1}\\&\quad\cdot\left(\frac{|x(t)|}{1+s}+\frac{|y(s)|}{1+s}\right)ds.\end{aligned} \tag{71}$$

By the Gronwall-Bellman Lemma 12, we have

$$\begin{aligned}\frac{|x(t)|}{1+t}+\frac{|y(t)|}{1+t}&\le X_0+C+\int_0^t\sum_{n=1}^{\infty}\frac{(C_*\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1}ds\\&\le X_0+C+\sum_{n=1}^{\infty}\frac{(t^{\gamma}C_*\Gamma(\gamma))^n}{(1+t)^n\Gamma(n\gamma+1)}(X_0+C)\\&\le X_0+C+\sum_{n=0}^{\infty}\frac{(C_*\Gamma(\gamma))^n}{\Gamma(n\gamma+1)}(X_0+C).\end{aligned} \tag{72}$$

Then, for every $t \in [0, \infty)$, we have

$$\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} \leq (X_0 + C)(1 + E\gamma(C_* \Gamma(\gamma)) := M, \quad (73)$$

where E_γ is the Mittag-Leffler function defined by

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)}. \quad (74)$$

Then,

$$\|x\|_{C_b} \leq M, \|y\|_{C_b} \leq M. \quad (75)$$

According to Theorem 7, problem (2) has at least one solution.

Now, we show that the set

$$S(x_0, y_0) = \{(x, y) \in \mathrm{BC}_b(J, \mathbb{R}) \times \mathrm{BC}_b(J, \mathbb{R}) : (x, y) \text{ is solution of } (2)\}, \quad (76)$$

is compact. It is clear that $S(x_0, y_0) \subset N(S(x_0, y_0))$. From (75), we deduced that $S(x_0, y_0)$ is bounded sets in $BC_b(J, \mathbb{R})$. Since N is compact, then $S(x_0, y_0)$ is compact if and only if $S(x_0, y_0)$ is closed. Let $\{(x_n, y_n)\}_{n \geq 1} \subset S(x_0, y_0)$ be a sequence converge to (x, y). Thus,

$$\begin{aligned} x_n(t) &= x_0 + \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s, x_n(s), y_n(s))ds, \\ y_n(t) &= y_0 + \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} g(s, x_n(s), y_n(s))ds, \\ &t \in J. \end{aligned} \quad (77)$$

Similarly to Step 2, we can prove that

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))ds, \\ y(t) &= y_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))ds, \\ &t \in J. \end{aligned} \quad (78)$$

This implies that $S(x_0, y_0)$ is compact.

The solution operator S is u.s.c.

(a) S has a closed graph:

To see this, first note that the graph of S is the set

$$G_S = \{((a, b), (x, y)) \in (\mathbb{R} \times \mathbb{R}) \times (\mathrm{BC}_b(J, \mathbb{R}) \times \mathrm{BC}_b(J, \mathbb{R})) \mid (x, y) \in S(a, b)\}. \quad (79)$$

Let $((a_q, b_q), (x_q, y_q))_q$ be a sequence in G_S, and let $((a_q, b_q), (x_q, y_q))_q \longrightarrow ((a, b), (x, y))$ as $q \longrightarrow \infty$.

Since $(x_q, y_q) \in S(a_q, b_q)$, then we have

$$\begin{aligned} x_q(t) &= a_q + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x_q(s), y_q(s)\right)ds, \\ y_q(t) &= b_q + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g\left(s, x_q(s), y_q(s)\right)ds, \\ &t \in J. \end{aligned} \quad (80)$$

Let

$$\begin{aligned} Z(t) = (Z_1(t), Z_2(t)) = \Bigg(a + \frac{1}{\Gamma(\alpha)}\int_0^t \\ \cdot (t-s)^{\alpha-1} f(s, x(s), y(s))ds, b \\ + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))ds\Bigg) \quad t \in J. \end{aligned} \quad (81)$$

As in Step 2, we can prove that

$$\left(x_q, y_q\right) \longrightarrow (x, y) \quad \text{as } q \longrightarrow \infty. \quad (82)$$

(b) Using the same method as Steps 3 to 4, we find that for each bounded set $C \subset \mathrm{BC}_b(J, \mathbb{R}) \times \mathrm{BC}_b(J, \mathbb{R})$, we can show that $S(C)$ is compact.

From (a) and (b), we concluded that, S is u.s.c.

5. Applications

In this section, we show the applicability of our main result. We begin by illustrating by Theorem 15.

Example 19. Consider the problem:

$$\begin{cases} {}^cD^{1/2}x = h_1(t)(1 + k_1 \sin|x| + k_2 \cos|x|), \\ {}^cD^{1/2}y = h_2(t)(1 + k_3|x| + k_4|y|)^{1/2}, \quad t \in \mathbb{R}, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \quad (83)$$

where

$$\begin{aligned} h_1(t) &= \frac{\sqrt{y}\int_0^t s e^{-s2} ds}{100(1+t^2)} \text{ and } h_2(s) \\ &= \frac{t\sqrt{t}}{200(1+t^2)}, \quad t \in \mathbb{R}_+, k_i \geq 0, i = 1, 2, 3, 4, \\ f(t, x, y) &= h_1(t)(1 + k_1 \sin|x| + k_2 \cos|y|), g(t, x, y) \\ &= h_2(t)(1 + k_3|x| + k_4|y|). \end{aligned} \quad (84)$$

It is clear that

$$\begin{aligned}\int_0^t \frac{h_1(s)}{(t-s)^{1/2}}ds &= \int_0^t \frac{\sqrt{s}\int_0^s re^{-r^2}dr}{100(1+s^2)(t-s)^{1/2}} \\ &\le \frac{\int_0^\infty e^{-r^2}dr}{100(1+s^2)(t-s)^{1/2}}\int_0^t \frac{s\sqrt{s}}{(1+s^2)(t-s)^{1/2}}ds \\ &= \frac{\int_0^\infty e^{-r^2}dr}{100}\int_0^t \frac{t^2 l\sqrt{l}}{\left(1+(tl)^2\right)(1-l)^{1/2}}dl \\ &\le \frac{\int_0^\infty e^{-r^2}dr}{100}\int_0^t (1-l)^{-(1/2)}dl \\ &\le \frac{\int_0^\infty e^{-r^2}dr}{50} = \frac{\sqrt{\pi}}{100}, \\ \int_0^t \frac{h_2(s)}{(t-s)^{1/2}}ds &= \int_0^t \frac{s\sqrt{s}}{200(1+s^2)(t-s)^{1/2}}ds = \le \frac{1}{100}.\end{aligned} \tag{85}$$

Then,

$$\sup_{t\in\mathbb{R}_+}\int_0^t \frac{h_i(s)}{(t-s)^{1/2}}ds < \infty, \quad i = 1, 2. \tag{86}$$

Observe that every $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}|f(t,x,y) - f(t,\bar{x},\bar{y})| &\le k_1 h_1(t)|x-\bar{x}| + k_2 h_1(t)|y-\bar{y}|, \\ |g(t,x,y) - g(t,\bar{x},\bar{y})| &\le k_3 h_2(t)|x-\bar{x}| + k_4 h_2(t)|y-\bar{y}|.\end{aligned} \tag{87}$$

Hence, the condition (H1) holds.

We see that

$$|f(t,0,0)| = h_1(t), |g(t,0,0)| = h_2(t), \quad t \in \mathbb{R}_+. \tag{88}$$

Hence, the condition (H2) holds. Assume that $k_1k_4 - k_2k_3 \neq 0$. For this, we have

$$\tilde{M} = \begin{pmatrix} \frac{k_1\sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_2\sqrt{\pi}}{\Gamma(1/2)100} \\ \frac{k_3\sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_4\sqrt{\pi}}{\Gamma(1/2)100} \end{pmatrix}. \tag{89}$$

$\det \tilde{M} \neq 0$. If we add that $k_2 = 0$ or k_3 and $0 < k_1 < (\Gamma(1/2)100)/\sqrt{\pi}$, $0 < k_4 < \Gamma(1/2)100 \le 1$. The $\rho(\bar{M}) < 1$. By Theorem 15, it follows that problem (83) has a unique solution.

In this last example, we illustrate the applicability of Theorem 18.

Lemma 20. *Let us consider the function* $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *defined by*

$$f(t,x,y) = a(t)\phi(x,y), \tag{90}$$

where $a : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfy the following assumptions:*

(S1) $a \in C_b(\mathbb{R})$.

(S2) The function φ *is continuous and bounded. Then, the function*

$$f_*(t,x,y) = a(t)\phi\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, (x,y) \in \mathbb{R}\times\mathbb{R}, \tag{91}$$

satisfied the condition (H3).

Proof. Since a is bounded, then there exists $A > 0$ such that

$$|a(t)| < A, \quad \forall t \in \mathbb{R}_+. \tag{92}$$

Let $l > 0$, then for any $(x,y), (\bar{x},\bar{y}) \in [-l,l] \times [-l,l]$, we obtain

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| = |a(t)||\phi(x,y) - \phi(\bar{x},\bar{y})|. \tag{93}$$

Hence,

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le A|\phi(x,y) - \phi(\bar{x},\bar{y})|. \tag{94}$$

It is clear that form (S2) ϕ is uniformly continuous on the $[-l,l] \times [-l,l]$. Thus, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $(x,y), (\bar{x},\bar{y}) \in [-l,l] \times [-l,l]$ with $|x-\bar{x}| < \delta(\epsilon)$, $|y-\bar{y}| < \delta(\epsilon)$ then,

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le A\epsilon. \tag{95}$$

This implies that for every $t \in \mathbb{R}_+$ such that for all (x,y), $(\bar{x},\bar{y}) \in [-l,l] \times [-l,l]$ with $|x/(1+t) - \bar{x}/(1+t)| < \delta(\epsilon)$, $|y/(1+t) - \bar{y}/(1+t)| < \delta(\epsilon)$, we get

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le A\epsilon. \tag{96}$$

Theorem 21. *Let* $a_1, a_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $\phi_1, \phi_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be functions that satisfy the conditions (S1) and (S2).*

Assume

(S3) There exists $d_i > 0, i = 1, 4$ *such that*

$$\begin{aligned}|\phi_1(x,y)| &\le d_1|x| + d_2|y| + d_3, \quad \forall x, y \in \mathbb{R}, t \in \mathbb{R}_+, \\ |\phi_2(x,y)| &\le d_4|x| + d_5|y| + d_6, \quad \forall x, y \in \mathbb{R}, t \in \mathbb{R}_+.\end{aligned} \tag{97}$$

Then, the following problem

$$\begin{cases} {}^cD^\alpha x(t) = f(t,x,(t)), \\ {}^cD^\beta x(t) = g(t,x,(t)), \quad t \in J, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{98}$$

where

$$f(t,x,y)=a_1(t)\phi_1\left(\frac{x}{1+t},\frac{y}{1+t}\right),\quad t\in\mathbb{R}_+,x,y\in\mathbb{R},$$
$$g(t,x,y)=a_2(t)\phi_2\left(\frac{x}{1+t},\frac{y}{1+t}\right),\quad t\in\mathbb{R}_+,x,y\in\mathbb{R}. \tag{99}$$

Proof. From Lemma 20, we deduce that f and g satisfies (H3). By (S1) and (S3), there exist $A_1, A_2 > 0$ such that

$$f(t,x,y)\le A_1\left(\frac{|x|}{1+t},\frac{|y|}{1+t}\right),$$
$$g(t,x,y)\le A_2\left(\frac{x}{1+t},\frac{y}{1+t}\right),\quad t\in\mathbb{R}_+,x,y\in\mathbb{R}. \tag{100}$$

Therefore, all the conditions of Theorem 18 hold. Then, problem (98) has at least one solution in BC_b.

Remark 22. For another application, we can replace the condition (S2) with (S4). The functions φ_1 and φ_2 are uniformly continuous and bounded.

6. Conclusion

In this paper, we investigated a system of fractional differential equations under various assumptions on the right-hand-side nonlinearity and we obtain a number of results regarding the existence and uniqueness of solutions in an appropriate space of continuous functions. In this paper, we have focused on the dependence continuity of a solution, compactness of solution sets, and upper semicontinuity of operator solutions. We hope this paper can provide some contribution to the questions of existence and topological structure for the system of fractional differential equations on unbounded domains.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Article

Existence and Uniqueness of Positive Solutions for Singular Nonlinear Fractional Differential Equation via Mixed Monotone Operator Method

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In this article, we discuss the existence and uniqueness of positive solution for a class of singular fractional differential equations, where the nonlinear term contains fractional derivative and an operator. By applying the fixed point theorem in cone, we get the existence and uniqueness of positive solutions for the fractional differential equation. Moreover, we give an example to demonstrate our main result.

1. Introduction

In this article, we consider the existence and uniqueness of solutions for the following singular fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + p(t)f\left(t, u(t), D_{0+}^{\beta}u(t)\right) + q(t)g(t, u(t), (Hu)(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ \left[D_{0^+}^{\gamma}u(t)\right]_{t=1} = k(u(1)), \end{cases} \tag{1}$$

where $n-1 < \alpha \le n$, $n > 3$, $1 \le \beta \le \gamma \le n-2$, $p, q \in C((0,1), [0,\infty))$, $p(t)$ and $q(t)$ are allowed to be singular at $t = 0$ or $t = 1$, $f : (0,1) \times (0,\infty) \times (0,\infty) \longrightarrow [0,\infty)$ is continuous and $f(t,u,v)$ may be singular at $t = 0, 1$ and $u = v = 0$, $g : (0,1) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is continuous and $g(t, u, v)$ may be singular at $t = 0, 1$, and $k : [0,\infty) \longrightarrow [0,\infty)$ is continuous function.

In recent years, we can see that differential equations have more and more attention in many fields, like electrical networks, quantum physics, and probability. There are a lot of interesting results from fractional differential equation, Schrodinger equation and k-hessian equations. In [1–4], the authors studied the solutions of fractional differential equations, and obtained some interesting results. Such as uniqueness of iterative positive solution, existence of multiple positive solutions or maximum and minimum solutions. In [5–9], the authors studied the solutions of Schrödinger equation. From these paper, there have several results about existence of infinitely solutions, existence and nonexistence of blow-up solutions, existence and asymptotic properties of solutions or existence and nonexistence of entire large solutions. In [10–12], the authors considered existence of blow-up radial solutions of the k-hessian equations or convergence analysis and uniqueness of blow-up solutions of the k-hessian equations. Inspired by these conclusions,we realize that the questions about differential equations are very interesting and varied. So we focus on fractional differential equation. The authors in [13] studied the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t, x(t), x(t)) + g(t, x(t)) = 0, \quad 0 < t < 1, \\ x^{(i)}(0) = 0, \quad i = 0, 1, 2, \cdots, n-2, \\ \left[D_{0+}^{\beta}x(t)\right]_{t=1} = 0, \quad 2 \le \beta \le n-2, \end{cases} \tag{2}$$

where $n-1<\alpha\le n$, $n>3$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, and $f:[0,1]\times[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ is a continuous function.

In [14], by using cone expansion fixed point theorem, Li et al. studied the existence of positive solutions for the following fractional differential equation with Riemann-Stieltjes integral:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+p(t)f(t,x(t))+q(t)g(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0,\\ x(1)=\displaystyle\int_0^1 k(s)x(s)dA(s), \end{cases} \tag{3}$$

where $n-1<\alpha\le n$, $n\ge 2$, $p,q:(0,1)\longrightarrow[0,\infty)$ are continuous, $p,q\in L^1(0,1)$, and allowed to be singular at $t=0$ or $t=1$, $f,g:[0,1]\times(0,\infty)\longrightarrow[0,\infty)$ are continuous and $f(t,x)$, $g(t,x)$ may be singular at $x=0$, $k:(0,1)\longrightarrow[0,\infty)$ is continuous with $k\in L^1(0,1)$, where $\int_0^1 k(s)x(s)dA(s)$ is the Riemann-Stieltjes integral with a signed measure and $A:[0,1]\longrightarrow\mathbb{R}$ is a bounded variation function, and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative of order α.

In [15], Liu et al. investigated the iterative positive solutions for the following singular nonlinear fractional differential equation with integral boundary conditions.

$$\begin{cases} D_{0^+}^{\alpha}x(t)+p(t)f(t,x(t),x(t))+q(t)g(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0,\\ x(1)=\displaystyle\int_0^1 k(s)x(s)dA(s), \end{cases} \tag{4}$$

where $n-1<\alpha\le n$, $n\ge 2$, $p,q:(0,1)\longrightarrow[0,\infty)$ are continuous and allowed to be singular at $t=0$ or $t=1$, $f:(0,1)\times(0,\infty)\times(0,\infty)\longrightarrow[0,\infty)$ is continuous and $f(t,x,y)$ may be singular at $t=0,1$ and $x=y=0$, $g(t,x)$: $(0,1)\times(0,\infty)\longrightarrow[0,\infty)$ is continuous and may be singular at $t=0,1$ and $x=0$, $k:(0,1)\longrightarrow[0,\infty)$ is continuous with $k\in L^1(0,1)$, where $\int_0^1 k(s)x(s)dA(s)$ is the Riemann-Stieltjes integral with a signed measure and $A:[0,1]\longrightarrow\mathbb{R}$ is a bounded variation function, and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative of order α.

In [16], Zhang and Tian considered the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\nu}x(t)+f\left(t,x(t),D_{0+}^{\gamma}x(t)\right)+g(t,x(t))=0, & 0<t<1,\\ x^{(i)}(0)=0, & i=0,1,2,\cdots,n-2,\\ \left[D_{0+}^{\beta}x(t)\right]_{t=1}=k(x(1)), \end{cases} \tag{5}$$

where $n-1<\nu<n$, $n>3$, $1\le\gamma\le\beta\le n-2$, and $f:[0,1]\times[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$, $g:[0,1]\times[0,\infty)\longrightarrow[0,\infty)$, and $k:[0,\infty)\longrightarrow[0,\infty)$ are continuous functions.

Motivated by the abovementioned papers, the purpose of this article is to study the existence and uniqueness of positive solution for FBVP (1). Obviously, the problem in our article is more general. We not only consider the nonlinear term containing the derivative term, but also study the nonlinear term containing an operator. And we do not need the linearity of operator in nonlinear term. It may be a nonlinear operator.

Comparing with the results in [17], boundary value problem (1) has a more general form. First, we discuss the singularity. It means that p, q are allowed to be singular at $t=0,1$, f may be singular at $t=0,1$ and $x=y=0$, and g may be singular at $t=0$. Second, we not only consider the derivative term but also consider the operator term, where the operator can be linear or nonlinear. Especially, when $p(t)=q(t)=1$ and $Hu=0$, we can see that the problem in [16] is a special case of problem (1). Comparing with [13], firstly, when $k(u)=0$, $\beta=0$, $p(t)=q(t)=1$, and $Hu(t)=u(t)$, then our problem (1) reduced compared to the problem in [13]. Third, in [13], the author did not consider the singularity, but in problem (1), we consider the singularity. Compared with [14], first, our method is different from that of [14]. We use the fixed point theorem for the sum of operators rather than the fixed point theorem of cone expansion and compression. Second, $f(t,x,y)$ not only has three variables but also is singular for both time and space variables, and $g(t,x,y)$ also has three variables. Compared with [15], we consider the derivative term and the operator term. Our results extended and improved some existing results, such as [13–16].

The rest of the paper is organized as follows: in Section 2, we give some preliminaries and lemmas that will be used in our main result. In Section 3, we study the existence and uniqueness of a positive solution of the BVP (1). In Section 4, an example is provided to demonstrate our main results.

2. Preliminaries and Lemmas

In this section, we present some notations, definitions and lemmas that will be used in the paper.

Let $(E,\|\cdot\|)$ be a Banach space, and θ is defined as the zero element of E. A nonempty closed convex set $P\subset E$ is a cone if it satisfies (1) $x\in P,\lambda\ge 0\Longrightarrow\lambda x\in P$ and (2) $x\in P$, $-x\in P\Longrightarrow x=\theta$. Putting $P^{\circ}=\{x\in P|x$ is an interior point of $P\}$, a cone P is said to be solid if P° is nonempty. In this paper, suppose that $(E,\|\cdot\|)$ is a Banach space partially ordered by a cone $P\subset E$, that is, $x\le y$ if and only if $y-x\in P$. A cone P called normal if there exists a constant $N>0$ such that for all $x,y\in E,\theta\le x\le y$ implies $\|x\|\le N\|y\|$, and the smallest such N is called the normality constant of P. For all $x,y\in P$, we denote the notation $x\sim y$ if there have constants $\lambda,\mu>0$ such that $\lambda y\le x\le\mu y$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta$, we denote by P_h the set $P_h=\{x\in E|x\sim h\}$. And it is easy to know that $P_h\subset P$.

Definition 1 (see [18]). An operator $A:P\times P\longrightarrow P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in the first component and decreasing in the second component, i.e., $u_i,v_i(i=1,2)\in P,u_1\le u_2,v_1\ge v_2$, implies

$A(u_1, v_1) \le A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2 (see [19]). Let $D = P$ or $D = P^\circ$ and α be a real number with $0 \le \alpha \le 1$. An operator $A : D \longrightarrow D$ is said to be α concave if it satisfies

$$A(tx) \ge t^\alpha Ax, \quad \forall t \in (0,1), x \in D. \tag{6}$$

Definition 3 (see [19]). An operator $B : P \longrightarrow P$ is said to be subhomogeneous if it satisfies

$$B(tx) \ge tBx, \quad \forall t \in (0,1), x \in D. \tag{7}$$

Definition 4 (see [20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$I_{0^+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{8}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 5 (see [20]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$D_{0^+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{9}$$

where $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 6 (see [20]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D_{0^+}^\alpha u(t) = 0 \tag{10}$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R}, i = 1, 2, \cdots, N, \tag{11}$$

as the unique solution, where $N = [\alpha] + 1$.

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

Lemma 7 (see [20]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then,*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{12}$$

for some $C_i \in \mathbb{R} (i = 1, 2, \cdots, N)$, where $N = [\alpha] + 1$.

In the following, we present the Green's function of FBVP (1).

Lemma 8 (see [16]). *Let $h \in C[0,1]$, then the unique solution of the linear problem*

$$\begin{cases} D_{0^+}^\alpha u(t) + h(t) = 0, & 0 < t < 1, n-1 < \alpha \le n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ \left[D_{0^+}^\gamma u(t)\right]_{t=1} = k(u(1)), & 1 \le \gamma \le n-2 \end{cases} \tag{13}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) ds + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} k(u(1)) t^{\alpha-1}, \tag{14}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\gamma-1}, & 0 \le t \le s \le 1. \end{cases} \tag{15}$$

Lemma 9 (see [16]). *The Green's function $G(t,s)$ defined by (15) satisfies the following properties:*

(1) $G : [0,1] \times [0,1] \longrightarrow [0, \infty)$ is continuous

(2) For any $t, s \in [0,1]$, there is

$$\begin{aligned} 0 &\le t^{\alpha-1}(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^\gamma\right] \\ &\le \Gamma(\alpha) G(t,s) \le t^{\alpha-1}(1-s)^{\alpha-\gamma-1} \end{aligned} \tag{16}$$

(3) For any $t, s \in [0,1]$, there is

$$\begin{aligned} 0 &\le t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^{\gamma-\beta}\right] \\ &\le \Gamma(\alpha-\beta) D_{0+}^\beta G(t,s) \le t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1} \end{aligned} \tag{17}$$

In [17], the operator equation $A(x,x) + B(x,x) = x$ was studied, where A, B are mixed monotone operators, and for all $t \in (0,1), \psi(t) \in (t, 1]$, there have $A(tx, t^{-1}y) \ge \psi(t) A(x,y)$, $B(tx, t^{-1}y) \ge tB(x,y)$. Based on the result in [17], we get the following lemma:

Lemma 10. *Let P be a normal cone in E, and let $A, B : P \times P \longrightarrow P$ be two mixed monotone operators and $C : P \longrightarrow P$ is a decreasing operator, suppose that*

(A1) for all $t \in (0,1)$, there have $\psi(t) \in (t, 1]$ such that

$$A\left(tx, t^{-1}y\right) \ge \psi(t) A(x,y), \quad \text{for all } x, y \in P \tag{18}$$

(A2) for all $t \in (0,1)$,

$$B\left(tx, t^{-1}y\right) \ge tB(x,y), \quad \text{for all } x, y \in P \tag{19}$$

(A3) for all $t \in (0,1)$,

$$C\left(t^{-1}y\right) \geq tC(y), \quad \text{for all } y \in P \tag{20}$$

(A4) there exists $h > \theta$ and $h \in P_h$, such that $A(h,h) \in P_h$, $B(h,h) \in P_h$, $C(h) \in P_h$

(A5) there exists a constants $\delta > 0$, such that for all x, $y \in P$,

$$A(x,y) \geq \delta(B(x,y) + C(y)) \tag{21}$$

Then

(1) $A : P_h \times P_h \longrightarrow P_h$, $B : P_h \times P_h \longrightarrow P_h$, $C : P_h \longrightarrow P_h$

(2) There exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that

$$\begin{aligned} rv_0 \leq u_0 < v_0,\ u_0 &\leq A(u_0,v_0) + B(u_0,v_0) + Cv_0 \\ &\leq A(v_0,u_0) + B(v_0,u_0) + Cu_0 \leq v_0 \end{aligned} \tag{22}$$

(3) The operator equation $A(x,x) + B(x,x) + Cx = x$ has a unique solution x^* in P_h

(4) For any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned} x_n &= A(x_{n-1},y_{n-1}) + B(x_{n-1},y_{n-1}) + Cy_{n-1}, \quad n = 1,2,\cdots, \\ y_n &= A(y_{n-1},x_{n-1}) + B(y_{n-1},x_{n-1}) + Cx_{n-1}, \quad n = 1,2,\cdots, \end{aligned} \tag{23}$$

we have $x_n \longrightarrow x^*$ and $y_n \longrightarrow x^*$ as $n \longrightarrow \infty$

Proof. Now we define the operator $T : P \times P \longrightarrow P$ by

$$T(x,y) = A(x,y) + B(x,y) + Cy, \quad x,y \in P. \tag{24}$$

So it is easy to know that T is a mixed monotone operator. Next, we will prove that $T(h,h) \in P_h$ and $T(tx,t^{-1}y) \geq \varphi(t)T(x,y)$, where $\varphi(t) \in (t,1]$, Then according to the results of [21], the conclusions of Lemma 10 holds.

Firstly, we will show that $T(h,h) \in P_h$ for $h > \theta$ and $h \in P_h$. From (18)–(20), for any $t \in (0,1)$, $x,y \in P$, there have

$$A\left(t^{-1}x,ty\right) \leq (\psi(t))^{-1}A(x,y), \tag{25}$$

$$B\left(t^{-1}x,ty\right) \leq t^{-1}B(x,y), \tag{26}$$

$$C(ty) \leq t^{-1}Cy. \tag{27}$$

According to (A4), we may assume that there exist constants $\mu_i > 0, \nu_i > 0\ (i = 1,2,3)$ such that

$$\begin{aligned} \mu_1 h \leq A(h,h) &\leq \nu_1 h, \\ \mu_2 h \leq B(h,h) &\leq \nu_2 h, \\ \mu_3 h \leq Ch &\leq \nu_3 h. \end{aligned} \tag{28}$$

Choose sufficiently small number $a \in (0,1)$ such that $ah \leq x,y \leq (1/a)h$ for any $x,y \in P_h$; then from (18)–(20) and (25)–(28), there have

$$\begin{aligned} A(x,y) &\leq A\left(\frac{1}{a}h, ah\right) \leq \frac{1}{\psi(a)}A(h,h) \leq \frac{1}{\psi(a)}\nu_1 h, \\ A(x,y) &\geq A\left(ah, \frac{1}{a}h\right) \geq \psi(a)A(h,h) \geq \psi(a)\mu_1 h, \end{aligned} \tag{29}$$

and we know $(1/(\psi(a)))\nu_1, \psi(a)\mu_1 > 0$. So $A(x,y) \in P_h$ and $A : P_h \times P_h \longrightarrow P_h$. Similarly, we can easily get that $B(x,y) \in P_h$, $B : P_h \times P_h \longrightarrow P_h$, $C(y) \in P_h$, and $C : P_h \longrightarrow P_h$. Then, we have $T(h,h) = A(h,h) + B(h,h) + Ch \in P_h$.

Second, we will prove that $T(tx,t^{-1}y) \geq \varphi(t)T(x,y)$ is satisfied for any $x,y \in P$, $t \in (0,1)$, $\varphi(t) \in (t,1]$.

For any $x,y \in P$, from (A5), we have

$$A(x,y) + \delta A(x,y) \geq \delta(B(x,y) + Cy) + \delta A(x,y). \tag{30}$$

Then $(1+\delta)A(x,y) \geq \delta(A(x,y) + B(x,y) + Cy) = \delta T(x,y)$. So

$$A(x,y) \geq \frac{T(x,y)}{1+\delta^{-1}}. \tag{31}$$

Thus, there have

$$\begin{aligned} T\left(tx,t^{-1}y\right) - tT(x,y) &= A\left(tx,t^{-1}y\right) + B\left(tx,t^{-1}y\right) + C\left(t^{-1}y\right) \\ &\quad - t(A(x,y) + B(x,y) + Cy) \\ &\geq (\psi(t) - t)A(x,y) + B\left(tx,t^{-1}y\right) \\ &\quad - tB(x,y) + C\left(t^{-1}y\right) - tCy \\ &\geq (\psi(t) - t)A(x,y) \geq \frac{\psi(t) - t}{1+\delta^{-1}}T(x,y). \end{aligned} \tag{32}$$

So

$$T\left(tx,t^{-1}y\right) \geq \left(t + \frac{\psi(t) - t}{1+\delta^{-1}}\right)T(x,y). \tag{33}$$

Now we note $\varphi(t) = t + ((\psi(t) - t)/(1+\delta^{-1}))$; it is easy to know that $\varphi(t) \in (t,1]$, where $t \in (0,1)$. Then, $T(tx,t^{-1}y) \geq \varphi(t)T(x,y)$.

Thus, by the results of Zhai and Zhang in [21], we know what the conclusions of Lemma 10 hold.

3. Main Results

For convenience, in this section, we set

$$E = \left\{x | x \in C[0,1], D_{0+}^{\beta} x \in C[0,1]\right\}. \tag{34}$$

Clearly, E is a Banach space with the norm

$$x = \max\left\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1]}\left|D_{0+}^{\beta}x(t)\right|\right\}. \tag{35}$$

Now, we let the cone P be defined by

$$P = \left\{x \in E | x(t) \geq 0, D_{0+}^{\beta} x(t) \geq 0, \forall t \in [0,1]\right\}. \tag{36}$$

There have $P \subset E$, and P is a normal cone. And, the order relation in E is given by $x \preceq y$ if $x(t) \leq y(t), D_{0+}^{\beta} x(t) \leq D_{0+}^{\beta} y(t)$.

From Lemma 8, we get that the unique solution of the problem (1) satisfies the following equation:

$$\begin{aligned} x(t) = &\int_0^1 G(t,s)p(s)f\left(s, x(s), D_{0+}^{\beta}x(s)\right)ds \\ &+ \int_0^1 G(t,s)q(s)g(s, x(s), (Hx)(s))ds \\ &+ \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(x(1))t^{\alpha-1}, \end{aligned} \tag{37}$$

where $G(t,s)$ is the Green's function given in (15).

Theorem 11. *Assume that the following assumptions hold:*

(H1) $p, q : (0,1) \longrightarrow [0,\infty)$ *are continuous, and* $p(t), q(t)$ *are allowed to be singular at* $t = 0$ *or* $t = 1$. *For all* $t \in (0,1)$, *there is* $p(t) \geq q(t) \geq m$, *where* m *is a constant*

(H2) $f : (0,1) \times (0,\infty) \times (0,\infty) \longrightarrow [0,\infty)$, $g : (0,1) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, $k : (0,1) \longrightarrow [0,\infty)$ *are continuous,* $f(t,u,v)$ *may be singular at* $t = 0, 1$, *and* $u = v = 0$, $g(t,u,v)$, *may be singular at* $t = 0, 1$

(H3) For fixed $t \in (0,1)$, $v \in (0,\infty)$, $f(t,u,v)$ *is increasing in* $u \in (0,\infty)$ *and decreasing in* $v \in (0,\infty)$; *for fixed* $t \in (0,1)$, $v \in (0,\infty)$, $g(t,u,v)$ *is increasing in* $u \in (0,\infty)$ *and decreasing in* $v \in (0,\infty)$; *for fixed* $t \in (0,1)$, $k(v)$ *is decreasing in* $v \in (0,\infty)$ *with* $k(v(1)) \neq 0$

(H4) There exists a constant $\vartheta \in (0,1)$ *such that for all* $\eta, t \in (0,1)$ *and* $u, v \in (0,\infty)$,

$$\begin{aligned} f\left(t, \eta u, \eta^{-1}v\right) &\geq \eta^{\vartheta} f(t,u,v), \\ g\left(t, \eta u, \eta^{-1}v\right) &\geq \eta g(t,u,v), \\ k\left(\eta^{-1}v\right) &\geq \eta k(v) \end{aligned} \tag{38}$$

(H5) $g(s,1,1) \equiv 0$. *There exists a constant* $\delta_1 > 0, \delta_2 > 0$ *such that for all* $t \in (0,1)$ *and* $u, v \in (0,\infty)$, $f(t,u,v) \geq \delta_1 g(t,u,0), f(t,u,v) \geq \delta_2 k(v)$

(H6) $\int_0^1 (1-s)^{\alpha-\gamma-1} p(s) s^{\vartheta(1-\alpha)} f(s,1,1)ds < \infty$ *and* $\int_0^1 (1-s)^{\alpha-1} q(s) s^{1-\alpha} g(s,1,1)ds < \infty$

(H7) $H : C[0,1] \longrightarrow C[0,1]$. $Hu \geq 0$ *and* Hu *is increasing in* $u \in P$. *For* $\lambda \in (0,1)$ *and* $u \in P$, *there is* $H(\lambda u) \geq \lambda Hu$

Then problem (1) has a unique positive solution $u^* \in P_h$, *where* $h(t) = t^{\alpha-1}$. *For any initial value* $u_0, v_0 \in P_h$, *there are two iterative sequences* $\{u_n\}, \{v_n\}$ *for approximating* u^*, v^*, *that is,* $u_n \longrightarrow u^*, v_n \longrightarrow v^*$, *as* $n \longrightarrow \infty$, *where*

$$\begin{aligned} u_n(t) = &\int_0^1 G(t,s)f\left(s, u_{n-1}(s), D_{0+}^{\beta}v_{n-1}(s)\right)ds \\ &+ \int_0^1 G(t,s)g(s, u_{n-1}(s), Hv_{n-1}(s))ds \\ &+ \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v_{n-1}(1))t^{\alpha-1}, \quad n = 1,2,\cdots, \\ v_n(t) = &\int_0^1 G(t,s)f\left(s, v_{n-1}(s), D_{0+}^{\beta}u_{n-1}(s)\right)ds \\ &+ \int_0^1 G(t,s)g(s, v_{n-1}(s), Hu_{n-1}(s))ds \\ &+ \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(u_{n-1}(1))t^{\alpha-1}, \quad n = 1,2,\cdots. \end{aligned} \tag{39}$$

Proof. We define operators A, B, C as follows:

$$A(u,v)(t) = \int_0^1 G(t,s)p(s)f\left(s, u(s), D_{0+}^{\beta}v(s)\right)ds, \quad t \in (0,1), \tag{40}$$

$$B(u,v)(t) = \int_0^1 G(t,s)q(s)g(s, u(s), Hv(s))ds, \quad t \in (0,1), \tag{41}$$

$$Cv(t) = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v(1))t^{\alpha-1}, \quad t \in (0,1). \tag{42}$$

Furthermore, for $u, v \in P$, we obtain that

$$\begin{aligned} D_{0+}^{\beta}A(u,v)(t) = &\int_0^1 D_{0+}^{\beta}G(t,s)p(s)f \\ &\cdot \left(s, u(s), D_{0+}^{\beta}v(s)\right)ds, \quad SSt \in (0,1), \end{aligned} \tag{43}$$

$$\begin{aligned} D_{0+}^{\beta}B(u,v)(t) = &\int_0^1 D_{0+}^{\beta}G(t,s)q(s)g \\ &\cdot (s, u(s), Hv(s))ds, \quad t \in (0,1), \end{aligned} \tag{44}$$

$$D_{0+}^{\beta}Cv(t) = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k(v(1))t^{\alpha-\beta-1}, \quad t \in (0,1). \tag{45}$$

Clearly, u is the solution of problem (1) if and only if $u = A(u,u) + B(u,u) + Cu$. So, if we can prove that the operators A, B, C satisfy all conditions in Lemma 10, then we will obtain the conclusions in Theorem 11.

First, we will prove that A, B are well defined. Evidently, C is well defined, so we omit it.

From (H4), for all $t \in (0,1), \eta \in (0,1), u, v \in (0,\infty)$, we have

$$\begin{aligned} f(t,u,v) &= f\left(t, \eta\eta^{-1}u, \eta^{-1}\eta v\right) \geq \eta^{\vartheta} f\left(t, \eta^{-1}u, \eta v\right), \\ g(t,u,v) &= g\left(t, \eta\eta^{-1}u, \eta^{-1}\eta v\right) \geq \eta g(t, \eta - 1u, \eta v). \end{aligned} \tag{46}$$

Furthermore, we have

$$\begin{aligned} f\left(t, \eta^{-1}u, \eta v\right) &\leq \eta^{-\vartheta} f(t,u,v), \\ g\left(t, \eta^{-1}u, \eta v\right) &\leq \eta^{-1} g(t,u,v). \end{aligned} \tag{47}$$

Taking $u = v = 1$ in (H4) and (47), for $\eta \in (0,1), t \in (0,1)$, we have

$$\begin{aligned} f\left(t, \eta, \eta^{-1}\right) &\geq \eta^{\vartheta} f(t,1,1), \\ f\left(t, \eta^{-1}, \eta\right) &\leq \eta^{-\vartheta} f(t,1,1), \\ g\left(t, \eta, \eta^{-1}\right) &\geq \eta g(t,1,1), \\ g\left(t, \eta^{-1}, \eta\right) &\leq \eta^{-1} g(t,1,1). \end{aligned} \tag{48}$$

For any $u, v \in P_h$, there exists a constant $b > 0$ such that $bh \leq u, v \leq (1/b)h$, where $h(t) = t^{\alpha-1}, t \in (0,1)$. So, from the above inequalities, we obtain

$$\begin{aligned} f(t,u(t),v(t)) &\leq f\left(t, \frac{1}{b}t^{\alpha-1}, bt^{\alpha-1}\right) \leq f\left(t, \frac{1}{b}t^{1-\alpha}, bt^{\alpha-1}\right) \\ &\leq t^{\vartheta(1-\alpha)} f\left(t, \frac{1}{b}, b\right) \leq t^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(t,1,1), \\ f(t,u(t),v(t)) &\geq f\left(t, bt^{\alpha-1}, \frac{1}{b}t^{\alpha-1}\right) \geq f\left(t, bt^{\alpha-1}, \frac{1}{b}t^{1-\alpha}\right) \\ &\geq t^{\vartheta(\alpha-1)} f\left(t, b, \frac{1}{b}\right) \geq t^{\vartheta(\alpha-1)} b^{\vartheta} f(t,1,1), \\ g(t,u(t),v(t)) &\leq g\left(t, \frac{1}{b}t^{\alpha-1}, bt^{\alpha-1}\right) \leq g\left(t, \frac{1}{b}t^{1-\alpha}, bt^{\alpha-1}\right) \\ &\leq t^{(1-\alpha)} g\left(t, \frac{1}{b}, b\right) \leq t^{(1-\alpha)}\left(\frac{1}{b}\right) g(t,1,1), \\ g(t,u(t),v(t)) &\geq g\left(t, bt^{\alpha-1}, \frac{1}{b}t^{\alpha-1}\right) \geq g\left(t, bt^{\alpha-1}, \frac{1}{b}t^{1-\alpha}\right) \\ &\geq t^{(\alpha-1)} g\left(t, b, \frac{1}{b}\right) \geq t^{(\alpha-1)} b g(t,1,1). \end{aligned} \tag{49}$$

Then, combining assumption (H6), we get

$$\begin{aligned} &\int_0^1 G(t,s)p(s) f\left(s, u(s), D_{0+}^{\beta} v(s)\right) ds \\ &\quad \leq \int_0^1 G(t,s)p(s) s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1) ds \\ &\quad \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} p(s) s^{\vartheta(1-\alpha)} \\ &\qquad \cdot \left(\frac{1}{b}\right)^{\vartheta} f(s,1,1) ds < \infty, \\ &\int_0^1 D_{0+}^{\beta} G(t,s)p(s) f\left(s, u(s), D_{0+}^{\beta} v(s)\right) ds \\ &\quad \leq \int_0^1 D_{0+}^{\beta} G(t,s)p(s) s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1) ds \\ &\quad \leq \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\gamma-1} p(s) s^{\vartheta(1-\alpha)} \\ &\qquad \cdot \left(\frac{1}{b}\right)^{\vartheta} f(s,1,1) ds < \infty, \\ &\int_0^1 G(t,s)q(s) g(s, u(s), Hv(s)) ds \\ &\quad \leq \int_0^1 G(t,s)q(s) s^{1-\alpha} \frac{1}{b} g(s,1,1) ds \\ &\quad \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} q(s) s^{1-\alpha} \frac{1}{b} g \\ &\qquad \cdot (s,1,1) ds < \infty, \\ &\int_0^1 D_{0+}^{\beta} G(t,s)q(s) g(s, u(s), Hv(s)) ds \\ &\quad \leq \int_0^1 D_{0+}^{\beta} G(t,s)q(s) s^{1-\alpha} \frac{1}{b} g(s,1,1) ds \\ &\quad \leq \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\gamma-1} q \\ &\qquad \cdot (s) s^{1-\alpha} \frac{1}{b} g(s,1,1) ds < \infty. \end{aligned} \tag{50}$$

Thus, we know that A, B are well defined.

Second, we will prove that A, B, C satisfy conditions (A1)–(A3) of Lemma 10. From (H1)–(H3), for all $u, v \in P, t \in (0,1)$, we have $A(u,v)(t) \geq 0, D_{0+}^{\beta} A(u,v)(t) \geq 0, B(u,v)(t) \geq 0, D_{0+}^{\beta} B(u,v)(t) \geq 0, Cv(t) \geq 0, Cv(t) \geq 0$. Therefore, $A : P \times P \longrightarrow P, B : P \times P \longrightarrow P, C : P \times P \longrightarrow P$. And for all $u, v \in P, t \in (0,1)$, $u \preceq v$ means $u(t) \leq v(t), D_{0+}^{\beta} u(t) \leq D_{0+}^{\beta} v(t)$. Thus, from assumptions (H2) and (H3), it is clear that operators A, B, C satisfy conditions (A1)–(A3) of Lemma 10. And A, B are mixed monotone operators and C is a decreasing operator.

Third, we show that $A(h,h) \in P_h$, $B(h,h) \in P_h$, $C(h) \in P_h$. From (H2) and (H3) and Lemma 9, we have

$$\begin{aligned}
A(h,h) &= \int_0^1 G(t,s)p(s)f\left(s, s^{\alpha-1}, D_{0+}^{\beta}s^{\alpha-1}\right)ds \\
&\le \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}p(s)s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1)ds \\
&= h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}p(s)s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1)ds, \\
A(h,h) &= \int_0^1 G(t,s)p(s)f\left(s, s^{\alpha-1}, D_{0+}^{\beta}s^{\alpha-1}\right)ds \\
&\ge \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}[1-(1-s)^{\gamma}]}{\Gamma(\alpha)} \\
&\quad \cdot p(s)s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1)ds \\
&= h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}[1-(1-s)^{\gamma}]}{\Gamma(\alpha)} \\
&\quad \cdot p(s)s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1)ds, \\
D_{0+}^{\beta}A(h,h) &= \int_0^1 D_{0+}^{\beta}G(t,s)p(s)f\left(s, s^{\alpha-1}, D_{0+}^{\beta}s^{\alpha-1}\right)ds \\
&\le \int_0^1 \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}p(s)s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1)ds \\
&= D_{0+}^{\beta}h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}p(s)s^{\vartheta(1-\alpha)} \\
&\quad \cdot \left(\frac{1}{b}\right)^{\vartheta} f(s,1,1)ds, \\
D_{0+}^{\beta}A(h,h) &= \int_0^1 D_{0+}^{\beta}G(t,s)p(s)f\left(s, s^{\alpha-1}, D_{0+}^{\beta}s^{\alpha-1}\right)ds \\
&\ge \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^{\gamma-\beta}\right]}{\Gamma(\alpha)} \\
&\quad \cdot p(s)s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1)ds \\
&= D_{0+}^{\beta}h(t)\int_0^1 \left(\frac{(1-s)^{\alpha-\gamma-1}\left(1-(1-s)^{\gamma-\beta}\right)}{\Gamma(\alpha)}p(s)\right) \\
&\quad \cdot s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1)ds.
\end{aligned} \tag{51}$$

Set

$$\begin{aligned}
c_1 &= \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^{\gamma-\beta}\right]}{\Gamma(\alpha)}p(s)s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1)ds, c_2 \\
&= \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}p(s)s^{\vartheta(1-\alpha)}\left(\frac{1}{b}\right)^{\vartheta} f(s,1,1)ds.
\end{aligned} \tag{52}$$

Now, in a similar way, for all $t \in (0,1)$, we obtain that

$$\begin{aligned}
B(h,h) &\le h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}q(s)s^{1-\alpha}\frac{1}{b}g(s,1,1)ds, \\
B(h,h) &\ge h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}[1-(1-s)^{\gamma}]}{\Gamma(\alpha)} \\
&\quad \cdot q(s)s^{\alpha-1}bg(s,1,1)ds, \\
D_{0+}^{\beta}B(h,h) &\le D_{0+}^{\beta}h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}q(s)s^{1-\alpha}\frac{1}{b}g(s,1,1)ds, \\
D_{0+}^{\beta}B(h,h) &\ge D_{0+}^{\beta}h(t)\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^{\gamma-\beta}\right]}{\Gamma(\alpha)} \\
&\quad \cdot q(s)s^{\alpha-1}bg(s,1,1)ds.
\end{aligned} \tag{53}$$

Set

$$\begin{aligned}
c_3 &= \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}\left[1-(1-s)^{\gamma-\beta}\right]}{\Gamma(\alpha)}q(s)s^{\alpha-1}bg(s,1,1)ds, \\
c_4 &= \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}q(s)s^{1-\alpha}\frac{1}{b}g(s,1,1)ds.
\end{aligned} \tag{54}$$

Because of $g(t,1,1) \equiv 0$, obviously, $c_4 \ge c_3 > 0$. Then $c_3 h \preceq B(h,h) \preceq c_4 h$. And we have

$$s^{\vartheta(\alpha-1)}b^{\vartheta}f(s,1,1) \ge s^{\alpha-1}bf(s,1,1) \ge s^{\alpha-1}b\delta_1 g(s,1,1). \tag{55}$$

Combining with the condition $p(t) \ge q(t)$, it is clear that $0 < \delta_1 c_3 \ge c_1 \le c_2$. So, there is $c_1 h \preceq A(h,h) \preceq c_2 h$. And from $k(\nu(1)) \ne 0$ and (42) and (45), obviously, we obtain that $Ch \in P_h$.

Finally, we show that operators A, B, C satisfy assumption (A5) of Lemma 10.

According to assumption (H6), for $t \in (0,1)$, we know $h(t) \in (0,1)$. So $0 \ge Hh \ge H1$. From assumption (H5), for all $t \in (0,1)$, $u, v \in P_h$, there is

$$\begin{aligned}
A(u,v) &= \int_0^1 G(t,s)p(s)f\left(s, u(s), D_{0+}^{\beta}v(s)\right)ds \\
&\ge \int_0^1 G(t,s)q(s)\delta_1 g(s,u(s),0)ds \\
&\ge \int_0^1 G(t,s)q(s)\delta_1 g(s,u(s),Hv(s))ds \\
&= \delta_1 B(u,v),
\end{aligned}$$

$$\begin{aligned} D_{0+}^{\beta}A(u,v) &= \int_0^1 D_{0+}^{\beta}G(t,s)p(s)f\left(s,u(s),D_{0+}^{\beta}v(s)\right)ds \\ &\geq \int_0^1 D_{0+}^{\beta}G(t,s)q(s)\delta_1 g(s,u(s),0)ds \\ &\geq \int_0^1 D_{0+}^{\beta}G(t,s)q(s)\delta_1 g(s,u(s),Hv(s))ds \\ &= \delta_1 D_{0+}^{\beta}B(u,v). \end{aligned} \tag{56}$$

Then, $A(u,v) \preceq \delta_1 B(u,v)$.

$$\begin{aligned} A(u,v) &= \int_0^1 G(t,s)p(s)f\left(s,u(s),D_{0+}^{\beta}v(s)\right)ds \\ &\geq \int_0^1 G(t,s)m\delta_2 ds \geq m\delta_2\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha}\right) \\ &\geq mk(v(1))\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right) \\ &= \frac{m}{\Gamma(\alpha-\gamma)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right)Cv, \end{aligned}$$

$$\begin{aligned} D_{0+}^{\beta}A(u,v) &= \int_0^1 D_{0+}^{\beta}G(t,s)p(s)f\left(s,u(s),D_{0+}^{\beta}v(s)\right)ds \\ &\geq \int_0^1 D_{0+}^{\beta}G(t,s)m\delta_2 k(v(1))ds \\ &\geq m\delta_2 k(v(1))\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right) \\ &= \frac{m\delta_2}{\Gamma(\alpha-\beta)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right)D_{0+}^{\beta}Cv. \end{aligned} \tag{57}$$

Then, $A(u,v) \succeq \delta' Cv$, where $\delta' = (m\delta_2/(\Gamma(\alpha-\beta)))((1/(\alpha-\gamma)) - (1/(\alpha-\beta)))$. Now, choose $\delta = \min\{\delta_1, \delta'\}$, then $A(u,v) \succeq \delta(B(u,v) + Cv)$.

So, applying Lemma 10, the proof of Theorem 11 is completed.

Remark 12. In Lemma 10, if we assume that $A, B : P_h \times P_h \longrightarrow P_h$ and $C : P_h \times P_h$, then it is easy to prove the results in Lemma 10 are also satisfied. So, we also can use the changed lemma to prove Theorem 11.

4. Example

Example 13. We consider the following FBVP:

$$\begin{cases} D_{0^+}^{9/2}u(t) + \dfrac{2}{t}\left(t^2+\sqrt{u(t)}\right) + \dfrac{1}{t\sqrt{D_{0+}^{3/2}u(t)+1}} + \dfrac{1}{t}\displaystyle\int_0^t u(t)dt = 0, \quad 0<t<1, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \\ \left[D_{0+}^{5/2}u(t)\right]_{t=1} = \dfrac{1}{\sqrt{u(1)+1}}. \end{cases} \tag{58}$$

Let $p(t) = q(t) = 1/t, f(t,u,v) = t^2 + \sqrt{u} + (1/\sqrt{v+1}), g(t,u,v) = t^2 + \sqrt{u} + v, k(v) = 1/\sqrt{v+1}$. Clearly, p, q, f, g, k satisfied conditions (H1)–(H3). And for $Hu(t) = \int_0^t u(t)dt$, we can see that assumption (H7) is satisfied. For $\eta \in (0,1)$, we have

$$\begin{aligned} f\left(t,\eta u,\eta^{-1}v\right) &= t^2 + \sqrt{\eta u} + \frac{1}{\sqrt{\eta^{-1}v+1}} = t^2\sqrt{\eta u} + \frac{\sqrt{\eta}}{\sqrt{v+\eta}} \\ &\geq \sqrt{\eta}\left(t^2+\sqrt{u}+\frac{1}{\sqrt{v+1}}\right) = \eta^{1/2}f(t,u,v), \\ g\left(t,\eta u,\eta^{-1}v\right) &= t^2 + \sqrt{\eta u} + \eta^{-1}v \geq \eta t^2 + \eta\sqrt{u} + \eta v \\ &= \eta\left(t^2+\sqrt{u}+v\right) = \eta g(t,u,v), \\ k\left(\eta^{-1}v\right) &= \frac{1}{\sqrt{\eta^{-1}v+1}} \geq \eta\frac{1}{\sqrt{v+1}} = \eta k(v). \end{aligned} \tag{59}$$

Then, we know that assumption (H4) of Theorem 11 is satisfied.

Clearly, $g(t,1,1)\equiv 0$. Moreover, if we let $\delta_1 = \delta_2 = 1$, then we have

$$\begin{aligned} f(t,u,v) &= t^2 + \sqrt{u} + \frac{1}{\sqrt{v+1}} \geq t^2 + \sqrt{u} = g(t,u,0), \\ f(t,u,v) &= t^2 + \sqrt{u} + \frac{1}{\sqrt{v+1}} \geq \frac{1}{\sqrt{v+1}} = k(v). \end{aligned} \tag{60}$$

Then, assumption (H5) of Theorem 11 is satisfied.

Finally, we show that assumption (H6) of Theorem 11 is also satisfied. It is easy to see from the following inequalities:

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-\gamma-1}p(s)s^{\vartheta(1-\alpha)}f(s,1,1)ds \\ &\quad = \int_0^1 (1-s)s^{-1}s^{-(7/4)}\left(s^2+1+\frac{\sqrt{2}}{2}\right) < \infty, \end{aligned}$$

$$\int_0^1 (1-s)^{\alpha-1} q(s) s^{1-\alpha} g(s,1,1)\,ds = \int_0^1 (1-s)^{(7/4)} s^{-1} s^{-(7/4)} \left(s^2+1+s\right) ds < \infty. \tag{61}$$

Therefore the assumptions of Theorem 11 are satisfied. Thus, problem (58) possesses the uniqueness solution.

Remark 14. It is clear that $Hu(t) = \int_0^1 u(t)dt$ is linear in this example. If we let operator H be a nonlinear operator, for example, $Hu(t) = u^a(t)$, where $0 < a < 1$, it is obvious that assumption (H7) of Theorem 11 is also satisfied. So using different operators, we also can obtain that BVP (58) has a unique positive solution from the same discussion process. This fact means that our theorem has a wide range of applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

New Modified Conformable Fractional Integral Inequalities of Hermite–Hadamard Type with Applications

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In this study, a few inequalities of Hermite–Hadamard type are constructed via the conformable fractional operators so that the normal version is recovered in its limit for the conformable fractional parameter. Finally, we present some examples to demonstrate the usefulness of conformable fractional inequalities in the context of special means of the positive numbers.

1. Introduction

Recently, fractional differential equations have attracted more and more attention, which can be used to describe some biological, chemical, and physical phenomenon more accurately than the classical differential equations of integer order. Nowadays, these appear naturally in modeling long-term behaviors, particularly in the areas of viscous fluid dynamics, control systems, physics, engineering, and viscoelastic materials [1–3]. The fractional calculus has many interesting applications when it was applied on a new mathematical model in thermoelectricity theory. Also, in modern physical engineering, the fractional system order is used to modify the mathematical models that describe the governing equations of those models to be more economical than the classic uses [4, 5].

There are many possible ways of defining fractional operators: Riemann–Liouville, Caputo, tempered, Marchaud, Atangana-Baleanu, and Hilfer. To find all these definitions, we advice our readers to visit the references [6–10]. Furthermore, all of these operators are considered special cases of a single, unifying, model of fractional operator by Fernandez et al. in [11].

In 2014, Khalil et al. [12] defined a new well-behaved simple fractional derivative, namely, the conformable fractional derivative relying only on the basic limit definition of the classical first derivative. It was first introduced as a conformable fractional derivative, but it lacks some of the desired properties for fractional derivatives [13–16]. This operator and its properties and applications have been intensely studied in other works, of which we mention [13] in particular. In [13], the following conformable derivative was defined:

$$D_\rho(f)(t) = \lim_{\epsilon \to 0} \frac{f\left(t+\epsilon t^{1-\rho}\right) - f(t)}{\epsilon}, \tag{1}$$

where $\rho \in (0, 1)$, $t > 0$, and $f : [0,\infty) \to \mathscr{R}$ is a function. This operator relates to the large and thriving theory of conformable fractional derivatives, and it arises naturally in control theory.

Now, if f is ρ-differentiable in some $(0, \rho), \rho > 0$, $\lim_{t\to 0^+} f^{(\rho)}(t)$ exist, then we define

$$f^{(\rho)}(0) = \lim_{t \to 0^+} f^{(\rho)}(t). \tag{2}$$

Moreover, if f is differentiable, then we have

$$D_\rho(f)(t) = t^{1-\rho} f'(t), \tag{3}$$

where

$$f'(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}. \tag{4}$$

Briefly, we can write $f^{(\rho)}(t)$ for $\mathscr{D}_\rho(f)(t)$ or $(d_\rho/d_\rho t)(f(t))$ to denote the conformable fractional derivatives of f of order ρ at t. In addition, if $f^{(\rho)}(t)$ exists, then we say that f is ρ-differentiable.

Theorem 1 ([13]). *Let $\rho \in (0, 1]$ and f, g be ρ -differentiable at a point $t > 0$. Then,*

(i) $\mathscr{D}_\rho(x_1 f + x_2 g) = x_1 \mathscr{D}_\rho(f) + x_2 \mathscr{D}_\rho(g)$ *for all* $x_1, x_2 \in \mathscr{R}$

(ii) $\mathscr{D}_\rho(fg) = f\mathscr{D}_\rho(g) + g\mathscr{D}_\rho(f)$

(iii) $\mathscr{D}_\rho(f/g) = (g\mathscr{D}_\rho(f) - f\mathscr{D}_\rho(g))/g^2$

(iv) $\mathscr{D}_\rho(\kappa) = 0$ *for any constant* $\kappa \in \mathscr{R}$

(v) $\mathscr{D}_\rho((1/\rho)t^\rho) = 1$

Now, conformable fractional integral and its few basic properties are stated.

Definition 2 ([13]). For any $\rho \in (0, 1]$ and $0 \le x_1 \le x_2$, the function $f : [x_1, x_2] \to \mathscr{R}$ is ρ -fractional integrable on $[x_1, x_2]$, if

$$\int_{x_1}^{x_2} f(t) d_\rho t = \int_{x_1}^{x_2} f(t) t^{\rho-1} dt \tag{5}$$

exists and is finite.

Remark 3.

(a) We indicate each ρ -fractional integrable functions by $L_\rho^1([x_1, x_2])$ on $[x_1, x_2]$

(b) The usual Riemann improper integral is defined by

$$I_{x_1}^\rho(f)(t) = I_{x_1}^1\left(t^{\rho-1} f\right) = \int_{x_1}^{t} x^{\rho-1} f(x) dx, \tag{6}$$

for each $\rho \in (0, 1]$.

Theorem 4 ([13, 16]). *For the differentiable function $f : (x_1, x_2) \to \mathscr{R}$ and $\rho \in (0, 1]$, we have*

$$I_{x_1}^\rho D_\rho^{x_1}(f)(t) = f(t) - f(x_1), \tag{7}$$

for all $t > x_1$.

Theorem 5 ([13]). *Let $\rho \in (n, n+1]$ and $f^{(n)}$ be a continuous function. Then, for the function $f : [x_1, \infty) \to \mathscr{R}$ and for all $t > x_1$, we have*

$$D_\rho^{x_1} I_{x_1}^\rho(f)(t) = f(t). \tag{8}$$

This property is often called the inverse property.

Theorem 6 ([13, 16]). *For the two functions $f, g : [x_1, x_2] \longrightarrow \mathscr{R}$, we have*

$$\int_{x_1}^{x_2} f(x) D_?^{x_1}(g)(x) d_\rho x = fg\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} g(x) D_\rho^{x_1}(f)(x) d_\rho x, \tag{9}$$

such that fg is differentiable.

Theorem 7 ([13, 16]). *Let $\rho \in (0, 1]$ and $f : [x_1, x_2] \longrightarrow \mathscr{R}$ be a continuous function on $[x_1, x_2]$ with $0 \le x_1 \le x_2$; then, we have*

$$\left|\mathscr{J}_{x_1}^\rho(f)(t)\right| \mathscr{J}_{x_1}^\rho |f|(t). \tag{10}$$

Definition 8 ([17]). A function $f : \mathscr{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on the interval $\mathscr{I}$, if the inequality

$$f(\eta x_1 + (1-\eta) x_2) \le \eta f(x_1) + (1-\eta) f(x_2) \tag{11}$$

holds for all $x_1, x_2 \in \mathscr{I}$ and $\eta \in [0, 1]$.

In view of recent results in theory of differential, integral, and fractional differential equations, it is becoming extremely worthless to ignore the existence of integral inequalities which are useful in determining bounds of unknown functions (see, e.g., [17–23]). Also, there are various integral inequalities in the literature and their numbers increase sharply every year. But the common inequality is the Hermite–Hadamard (HH) integral inequality, which was first found in 1893 by Hadamard in [24]. It has the following formula:

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x) dx \le \frac{f(x_1) + f(x_2)}{2}, \tag{12}$$

where $f : E \subseteq \mathscr{R} \longrightarrow \mathscr{R}$ is supposed to be convex on E with $x_1, x_2 \in E$ and $x_1 < x_2$.

The inequality of HH type (12) has been extended and applied in all models of fractional calculus, such as Riemann–Liouville [25–29], ψ-Riemann–Liouville [30], conformable fractional [31, 32], generalized fractional [33–35], time-scale fractional [36, 37], tempered fractional [38], and Atangana-Baleanu [39] models. Since our paper is about conformable fractional integral inequality, here, we

give some inequalities involving conformable fractional integrals that are obtained in [32]:

Lemma 9 ([32], Lemma 1). *Let $h : [x_1, x_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an α -fractional differentiable mapping on (x_1, x_2) with $0 \le x_1 < x_2$. If $D_\alpha(h) \in L_\alpha^1([x_1, x_2])$; then, the following identity for conformable fractional integral holds:*

$$\Phi_\alpha(x_1, x_2) = \sum_{i=1}^{4} \delta_i, \tag{13}$$

where

$$\begin{aligned}
\delta_1 &= \frac{x_2 - x_1}{4} \int_0^1 \left[\left(x_1\zeta + (1-\zeta)\frac{3x_1 + x_2}{4} \right)^{2\alpha-1} \right. \\
&\quad \left. - \left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \left(x_1\zeta + (1-\zeta)\frac{3x_1 + x_2}{4} \right)^{\alpha-1} \right] \\
&\quad \times D_\alpha(h)\left(x_1\zeta + (1-\zeta)\frac{3x_2 + x_2}{4} \right) d_\alpha\zeta, \\
\delta_2 &= \frac{x_2 - x_1}{4} \int_0^1 \left[\left(\frac{3x_1 + x_2}{4}\zeta + (1-\zeta)\frac{x_1 + x_2}{2} \right)^{2\alpha-1} \right. \\
&\quad \left. - \left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \left(\frac{3x_1 + x_2}{4}\zeta + (1-\zeta)\frac{x_1 + x_2}{2} \right)^{\alpha-1} \right] \\
&\quad \times D_\alpha(h)\left(\frac{3x_1 + x_2}{4}\zeta + (1-\zeta)\frac{x_1 + x_2}{2} \right) d_\alpha\zeta, \\
\delta_3 &= \frac{x_2 - x_1}{4} \int_0^1 \left[\left(\frac{x_1 + x_2}{2}\zeta + (1-\zeta)\frac{x_1 + 3x_2}{4} \right)^{2\alpha-1} \right. \\
&\quad \left. - \left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} \left(\frac{x_1 + x_2}{2}\zeta + (1-\zeta)\frac{x_1 + 3x_2}{4} \right)^{\alpha-1} \right] \\
&\quad \times D_\alpha(h)\left(\frac{x_1 + x_2}{2}\zeta + (1-\zeta)\frac{x_1 + 3x_2}{4} \right) d_\alpha\zeta, \\
\delta_4 &= \frac{x_2 - x_1}{4} \int_0^1 \left[\left(\frac{x_1 + 3x_2}{4}\zeta + (1-\zeta)x_2 \right)^{2\alpha-1} \right. \\
&\quad \left. - \left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} \left(\frac{x_1 + 3x_2}{4}\zeta + (1-\zeta)x_2 \right)^{\alpha-1} \right] \\
&\quad \times D_\alpha(h)\left(\frac{x_1 + 3x_2}{4}\zeta + (1-\zeta)x_2 \right) d_\alpha\zeta,
\end{aligned}$$

$$\begin{aligned}
\Phi_\alpha(x_1, x_2) &= \left[\left(\frac{3x_1 + x_2}{4} \right)^{\alpha} - x_1^{\alpha} \right] h(x_1) + \left[x_2^{\alpha} - \left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} \right] h(x_2) \\
&\quad + \left[\left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} - \left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \right] h\left(\frac{x_1 + x_2}{2} \right) \\
&\quad - \alpha \int_{x_1}^{x_2} h(x) d_\alpha x.
\end{aligned} \tag{14}$$

Theorem 10 ([32], Theorem 2). *Let $h : [x_1, x_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an α-fractional differentiable mapping on (x_1, x_2) with $0 \le x_1 < x_2$. If $D_\alpha(h) \in L_\alpha^1([x_1, x_2])$ and $|h'|$ is convex on $[x_1, x_2]$, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}
|\Phi_\alpha(x_1, x_2)| &\le \frac{x_2 - x_1}{4} \left[d_1(\alpha) \mid h'(x_1)| + d_2(\alpha) \mid h'(x_2)| + d_3(\alpha) \right. \\
&\quad \cdot \left| h'\left(\frac{3x_1 + x_2}{4} \right) \right| + d_4(\alpha) \left| h'\left(\frac{x_1 + x_2}{2} \right) \right| \\
&\quad \left. + d_5(\alpha) \left| h'\left(\frac{x_1 + 3x_2}{4} \right) \right| \right],
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
d_1(\alpha) &= \frac{1}{12} \left[4x_1^{\alpha} + \left(\frac{3x_1 + x_2}{4} \right) x_1^{\alpha-1} \right. \\
&\quad \left. + \left(\frac{3x_1 + x_2}{4} \right)^{\alpha-1} x_1 - 5\left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \right], \\
d_3(\alpha) &= \frac{1}{12} \left[x_1^{\alpha} + \left(\frac{3x_1 + x_2}{4} \right) x_1^{\alpha-1} + \left(\frac{3x_1 + x_2}{4} \right)^{\alpha-1} x_1 \right. \\
&\quad + \left(\frac{x_1 + x_2}{2} \right) \left(\frac{3x_1 + x_2}{4} \right)^{\alpha-1} + \left(\frac{x_1 + x_2}{2} \right)^{\alpha-1} \left(\frac{3x_1 + x_2}{4} \right) \\
&\quad \left. + \left(\frac{x_1 + x_2}{2} \right)^{\alpha} - 5\left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \right], \\
d_4(\alpha) &= \frac{1}{12} \left[7\left(\frac{x_1 + x_2}{2} \right)^{\alpha} + \left(\frac{x_1 + x_2}{2} \right) \left(\frac{3x_1 + x_2}{4} \right)^{\alpha-1} \right. \\
&\quad + \left(\frac{x_1 + x_2}{2} \right)^{\alpha-1} \left(\frac{3x_1 + x_2}{4} \right) + \left(\frac{x_1 + x_2}{2} \right) \left(\frac{x_1 + 3x_2}{4} \right)^{\alpha-1} \\
&\quad + \left(\frac{x_1 + x_2}{2} \right)^{\alpha-1} \left(\frac{x_1 + 3x_2}{4} \right) - 5\left(\frac{3x_1 + x_2}{4} \right)^{\alpha} \\
&\quad \left. - 5\left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} \right], \\
d_5(\alpha) &= \frac{1}{12} \left[x_2^{\alpha} + \left(\frac{x_1 + x_2}{2} \right)^{\alpha} + \left(\frac{x_1 + 3x_2}{4} \right) \left(\frac{x_1 + x_2}{2} \right)^{\alpha-1} \right. \\
&\quad + \left(\frac{x_1 + 3x_2}{4} \right)^{\alpha-1} \left(\frac{x_1 + x_2}{2} \right) + \nu\left(\frac{x_1 + 3x_2}{4} \right)^{\alpha-1} \\
&\quad \left. + x_2^{\alpha-1} \left(\frac{x_1 + 3x_2}{4} \right) - 5\left(\frac{x_1 + 3x_2}{4} \right)^{\alpha} \right].
\end{aligned} \tag{16}$$

Theorem 11 ([32], Theorem 3). *Let $h : [x_1, x_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be an α - fractional differentiable mapping on (x_1, x_2) with $0 \le x_1 < x_2$. If $D_\alpha(h) \in L_\alpha^1([x_1, x_2])$ and $|h'|^q$ is convex on $[x_1, x_2]$, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}
|\Phi_\alpha(x_1, x_2)| &\le \frac{x_2 - x_1}{4} \left[(A_1(\alpha))^{1-(1/q)} \left\{ A_2(\alpha) |h'(x_1)|^q + A_3(\alpha) \left| h'\left(\frac{3x_1 + x_2}{4} \right) \right|^q \right\}^{1/q} \right. \\
&\quad + (B_1(\alpha))^{1-(1/q)} \left\{ B_2(\alpha) \left| h'\left(\frac{3x_1 + x_2}{4} \right) \right|^q + B_3(\alpha) \left| h'\left(\frac{x_1 + x_2}{2} \right) \right|^q \right\}^{1/q} \\
&\quad + (C_1(\alpha))^{1-(1/q)} \left\{ C_2(\alpha) \left| h'\left(\frac{x_1 + x_2}{2} \right) \right|^q + C_3(\alpha) \left| h'\left(\frac{x_1 + 3x_2}{4} \right) \right|^q \right\}^{1/q} \\
&\quad \left. + (D_1(\alpha))^{1-(1/q)} \left\{ D_2(\alpha) \left| h'\left(\frac{x_1 + 3x_2}{4} \right) \right|^q + D_3(\alpha) |h'(x_2)|^q \right\}^{1/q} \right],
\end{aligned} \tag{17}$$

where

$$A_1(\alpha)=\frac{4u(3x_1+x_2)^{\alpha}-(4x_1)^{\alpha+1}}{4^{\alpha}(x_2-x_1)(\alpha+1)}-\frac{\alpha(3x_1+x_2)^{\alpha}}{4^{\alpha}},$$

$$A_2(\alpha)=\frac{(3x_1+x_2)^{\alpha+2}-(4x_1)^{\alpha+1}[\alpha(x_2-x_1)+(x_1+x_2)]}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)}-\frac{(3x_1+x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$A_3(\alpha)=\frac{(4x_1)^{\alpha+2}-(3x_1+x_2)^{\alpha+2}(5u-v)-\alpha(3x_1+x_2)^{\alpha}(3u-v)(x_1+x_2)}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)}-\frac{(3x_1+x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$B_1(\alpha)=\frac{(2(x_1+x_2))^{\alpha+2}-(3x_1+x_2)^{\alpha+1}}{4^{\alpha}(x_2-x_1)(\alpha+1)}-\frac{(3x_1+x_2)^{\alpha}}{4^{\alpha}},$$

$$B_2(\alpha)=\frac{(2(x_1+x_2))^{\alpha+2}-(3x_1+x_2)^{\alpha+1}(x_1+3x_2)-\alpha(3x_1+x_2)^{\alpha+1}(x_2-x_1)}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)}-\frac{(3x_1+x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$B_3(\alpha)=\frac{(3x_1+x_2)^{\alpha+2}+4(2(x_1+x_2))^{\alpha+1}+\alpha(2(x_1+x_2))^{\alpha+1}(x_2-x_1)}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)}-\frac{(3x_1+x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$C_1(\alpha)=\frac{4(x_1+3x_2)^{\alpha+1}-(2(x_1+x_2))^{\alpha+2}}{4^{\alpha}(x_2-x_1)(\alpha+1)}-\frac{(x_1+3x_2)^{\alpha}}{4^{\alpha}},$$

$$C_2(\alpha)=\frac{\alpha^2(x_2-x_1)^2(x_1+3x_2)^{\alpha}+16b(x_1+x_2)(x_1+3x_2)^{\alpha}-3\alpha\left(x_1^2+x_2^2\right)(x_1+3x_2)^{\alpha}}{2^{2\alpha+1}(x_2-x_1)^2(\alpha+1)(\alpha+2)}$$
$$-\frac{8b(2(x_1+x_2))^{\alpha+1}+2\alpha(x_2-x_1)(2(x_1+x_2))^{\alpha+1}}{2^{2\alpha+1}(x_2-x_1)^2(\alpha+1)(\alpha+2)}-\frac{(x_1+3x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$C_3(\alpha)=\frac{(2(x_1+x_2))^{\alpha+2}+(x_1+3x_2)^{\alpha+2}+\alpha(x_1+3x_2)^{\alpha+2}-5(x_1+x_2)(x_1+3x_2)^{\alpha+1}}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)},$$

$$D_1(\alpha)=\frac{(4x_2)^{\alpha+1}-(x_1+3x_2)^{\alpha+1}}{4^{\alpha}(\alpha+1)(x_2-x_1)}-\frac{(x_1+3x_2)^{\alpha}}{2^{2\alpha+1}},$$

$$D_2(\alpha)=\frac{(4x_2)^{\alpha+2}+(x_1+3x_2)^{\alpha+1}(x_1-5x_2)+\alpha(x_1+3x_2)^{\alpha+1(x_2-x_1)}}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)},$$

$$D_3(\alpha)=\frac{\alpha(4x_2)^{\alpha+2}(x_2-x_1)-2(4x_2)^{\alpha+1}(x_1+x_2)+(x_1+3x_2)^{\alpha+2}}{4^{\alpha}(x_2-x_1)^2(\alpha+1)(\alpha+2)}. \tag{18}$$

Theorem 12 ([32], Theorem 4). *Let $h:[x_1,x_2]\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be an α -fractional differentiable mapping on (x_1,x_2) with $0\le x_1<x_2$. If $D_\alpha(h)\in L_\alpha^1([x_1,x_2])$ and $|h'|^q$ is concave on $[x_1,x_2]$, then the following inequality for conformable fractional integral holds:*

$$|\Phi_\alpha(x_1,x_2)|\le\frac{x_2-x_1}{4}\left[A_1(\alpha)h'\left(\frac{P_1(\alpha)}{A_1(\alpha)}\right)+B_1(\alpha)h'\left(\frac{P_2(\alpha)}{B_1(\alpha)}\right)+C_1(\alpha)h'\left(\frac{P_3(\alpha)}{C_1(\alpha)}\right)+D_1(\alpha)h'\left(\frac{P_4(\alpha)}{D_1(\alpha)}\right)\right], \tag{19}$$

where $A_1(\alpha), B_1(\alpha), C_1(\alpha)$, and $D_1(\alpha)$ are given in Theorem 11 and

$$
\begin{aligned}
P_1(\alpha) &= \frac{2^{1-\alpha}(3x_1+x_2)^{\alpha+2} - 2^{2\alpha+5} - 2^{2\alpha+1}(7x_1+6x_2)(x_2-x_1)(3x_1+x_2)^{\alpha} - \alpha(7x_1+x_2)(x_2-x_1)(3x_1+x_2)^{\alpha}}{2^{2\alpha+3}(x_2-x_1)(\alpha+2)},\\
P_2(\alpha) &= \frac{2(2(x_1+x_2))^{\alpha+2} - 8(x_1+x_2)^2(3x_1+x_2)^{\alpha} - \alpha(5x_1+3x_2)(x_2-x_1)(3x_1+x_2)^{\alpha}}{2^{2\alpha+3}(x_2-x_1)(\alpha+2)},\\
P_3(\alpha) &= \frac{8(x_1+x_2)^2(x_1+3x_2)^{\alpha} - 4\alpha(x_1+x_2)^{\alpha}(x_1+3x_2)^{\alpha} - \alpha(x_1+3x_2)^{\alpha+2} - 2^{\alpha+3}(x_1+x_2)^{\alpha+2}}{2^{2\alpha+1}(x_2-x_1)(\alpha+2)},\\
P_4(\alpha) &= \frac{2^{2\alpha-1}\alpha(x_1+3x_2) + 4^{\alpha}(x_1+3x_2)^3 + 4(4x_2)^{\alpha+2} - 4^{\alpha+2}x_2^2(x_1+3x_2) - 4(x_1+3x_2)^{\alpha+2} - 2^{2\alpha+3}\alpha x_2^2(x_1+3x_2)}{4^{\alpha+2}(x_2-x_1)(\alpha+2)}.
\end{aligned}
\tag{20}
$$

In view of the above indices, we establish new integral inequalities of HH type for the convex functions via the conformable fractional operators. By doing this, we will try to demonstrate the usefulness of conformable fractional inequalities in the context of special means of the positive numbers.

2. Main Results

The next result is necessary in the rest of the work.

Lemma 13. *Let $f : [x_1, x_2] \subseteq \mathcal{R} \longrightarrow \mathcal{R}$ be an ρ-differentiable function on (x_1, x_2), $0 \le x_1 < x_2$. If $\mathcal{D}_\rho(f) \in L^1_\rho([x_1, x_2])$, then we have*

$$
\begin{aligned}
\Delta_\rho(x_1,x_2) = \frac{x_2-x_1}{4}\Bigg\{ &\int_0^1 \Bigg[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{2\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{\rho-1}\Bigg]\\
&\times \mathcal{D}_\rho(f)\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)d_\rho t\\
&+\int_0^1 \Bigg[\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)^{2\rho} - \left(\frac{x_1+x_2}{2}\right)^{1+\rho}\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)^{\rho-1}\Bigg]\\
&\times \mathcal{D}_\rho(f)\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)d_\rho t\\
&+\int_0^1 \Bigg[\left(\frac{x_1+x_2}{2}t+(1-t)\frac{x_1+3x_2}{4}\right)^{2\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\left(\frac{x_1+x_2}{2}t+(1-t)\frac{x_1+3x_2}{4}\right)^{\rho-1}\Bigg]\\
&\times \mathcal{D}_\rho(f)\left(\frac{x_1+x_2}{2}t+(1-t)\frac{x_1+3x_2}{4}\right)d_\rho t\\
&+\int_0^1 \Bigg[\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)^{2\rho} - x_2^{1+\rho}\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)^{\rho-1}\Bigg]\\
&\times \mathcal{D}_\rho(f)\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)d_\rho t\Bigg\},
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
\Delta_\rho(x_1,x_2) \coloneqq &\left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right]f(x_1)\\
&+\left[{x_2}^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]f(x_2)\\
&+\left[\left(\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]f\left(\frac{x_1+x_2}{2}\right)\\
&-(1+\rho)\Bigg[\mathcal{J}^{\rho}_{x_1}f\left(\frac{3x_1+x_2}{4}\right)\\
&+\mathcal{J}^{\rho}_{(3x_1+x_2)/4}f\left(\frac{x_1+x_2}{2}\right) + \mathcal{J}^{\rho}_{(x_1+x_2)/2}f\left(\frac{x_1+3x_2}{4}\right)\\
&+\mathcal{J}^{\rho}_{(x_1+3x_2)/4}f(x_2)\Bigg].
\end{aligned}
\tag{22}
$$

Proof. Making use of Definition 8, we have

$$
\begin{aligned}
\hbar_1 \coloneqq \frac{x_2-x_1}{4}\int_0^1 &\Bigg[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{2\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\\
&\cdot\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{\rho-1}\Bigg]\\
&\times \mathcal{D}_\rho(f)\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)\mathrm{d}_\rho\\
t = \frac{x_2-x_1}{4}\int_0^1 &\Bigg[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho}\\
&-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\Bigg]f'\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)dt.
\end{aligned}
\tag{23}
$$

On integrating by parts, one can have

$$\begin{aligned}\hbar_1 &:= \left\{\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho}\right\} f \\ &\cdot \left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)\Bigg|_0^1 - \frac{(1+\rho)(x_2-x_1)}{4}\int_0^1 \\ &\cdot \left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{\rho} f\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right) \\ d &= \left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] f(x_1) \\ &- (1+\rho)\int_0^1 \frac{f(x_1 t + (1-t)(3x_1+x_2)/4)}{(x_1 t + (1-t)(3x_1+x_2)/4)^{\rho}}\, dt,\end{aligned} \tag{24}$$

making the change of variable $x := x_1 t + (1-t)(3x_1+x_2)/4$, $t \in [0,1]$, to get

$$\begin{aligned}\hbar_1 &= \left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] f(x_1) - (1+\rho)\int_{x_1}^{(3x_1+x_2)/4} \frac{f(x)}{x^{\rho}} \\ dx &= \left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] f(x_1) - (1+\rho)\,\mathcal{J}^{\rho}_{x_1} f\left(\frac{3x_1+x_2}{4}\right).\end{aligned} \tag{25}$$

Similarly, we get

$$\begin{aligned}\hbar_2 &:= \frac{x_2-x_1}{4}\int_0^1 \left[\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)^{2\rho}\right. \\ &\left. - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)^{\rho-1}\right] \\ &\cdot \mathscr{D}_{\rho}(f)\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right) \\ \mathrm{d}_{\rho}t &= \left[\left(\frac{x_1+x_2}{2}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] f\left(\frac{x_1+x_2}{2}\right) \\ &- (1+\rho)\,\mathcal{J}^{\rho}_{(3x_1+x_2)/4} f\left(\frac{x_1+x_2}{2}\right),\end{aligned}$$

$$\begin{aligned}\hbar_3 &:= \frac{x_2-x_1}{4}\int_0^1 \left[\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)^{2\rho}\right. \\ &\left. - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)^{\rho-1}\right] \\ &\cdot \mathscr{D}_{\rho}(f)\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right) \\ \mathrm{d}_{\rho}t &= \left[\left(\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1+x_2}{2}\right)^{1+\rho}\right] f\left(\frac{x_1+x_2}{2}\right) \\ &- (1+\rho)\,\mathcal{J}^{\rho}_{(x_1+x_2)/2} f\left(\frac{x_1+3x_2}{4}\right),\end{aligned}$$

$$\begin{aligned}\hbar_4 &:= \frac{x_2-x_1}{4}\int_0^1 \left[\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)^{2\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right. \\ &\left. \cdot \left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)^{\rho-1}\right] \mathscr{D}_{\rho}(f)\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right) \\ \mathrm{d}_{\rho}t &= \left[{x_2}^{\rho} - \left(\frac{x_1+3x_2}{4}\right)^{\rho}\right] f(x_2) - (1+\rho)\,\mathcal{J}^{\rho}_{(x_1+3x_2)/4} f(b).\end{aligned} \tag{26}$$

Considering $\hbar_1, \hbar_2, \hbar_3$, and $\hbar_4$, we get (21) and thus, our proof is completed.

Theorem 14. *Let $f : [x_1, x_2] \subseteq \mathscr{R} \longrightarrow \mathscr{R}$ be an ρ-differentiable function on (x_1, x_2), $0 \le x_1 < x_2$. If $\mathscr{D}_{\rho}(f) \in L^1_{\rho}([x_1, x_2])$ and $|f'|$ is a convex function on $[x_1, x_2]$, then we have*

$$\left|\Delta_{\rho}(x_1, x_2)\right| \le \left(\frac{x_2-x_1}{24}\right)\sum_{\ell=1}^{4} D_{\ell}, \tag{27}$$

where

$$\begin{aligned}D_1 &= \left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right]\left(2\left|f'(x_1)\right| + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|\right), \\ D_2 &= \left[\left(\frac{x_1+x_2}{2}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]\left(2\left|f'\left(\frac{3x_1+x_2}{4}\right)\right| + \left|f'\left(\frac{x_1+x_2}{2}\right)\right|\right), \\ D_3 &= \left[\left(\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1+x_2}{2}\right)^{1+\rho}\right]\left(2\left|f'\left(\frac{x_1+x_2}{2}\right)\right| + \left|f'\left(\frac{x_1+3x_2}{4}\right)\right|\right), \\ D_4 &= \left[{x_2}^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]\left(2\left|f'\left(\frac{x_1+3x_2}{4}\right)\right| + \left|f'(x_2)\right|\right).\end{aligned} \tag{28}$$

Proof. Making use of Lemma 13 and the properties of $\mathscr{D}_{\rho}$, we have

$$\begin{aligned}\left|\Delta_{\rho}(x_1, x_2)\right| &\le \frac{x_2-x_1}{4}\int_0^1 \left|\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right| \\ &\times \left|f'\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)\right| dt + \frac{x_2-x_1}{4}\int_0^1 \\ &\cdot \left|\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)^{1+\rho} - \left(\frac{x_1+x_2}{2}\right)^{1+\rho}\right| \\ &\times \left|f'\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)\right| dt + \frac{x_2-x_1}{4}\int_0^1 \\ &\cdot \left|\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right| \\ &\times \left|f'\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)\right| dt + \frac{x_2-x_1}{4}\int_0^1 \\ &\cdot \left|\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right| \\ &\times \left|f'\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)\right| dt \frac{x_2-x_1}{4}\left[\kappa_1+\kappa_2+\kappa_3+\kappa_4\right].\end{aligned} \tag{29}$$

For $\rho \in (0,1]$ and $x > 0$, the function $x^{1+\rho}$ is convex. So, we have

$$\begin{aligned}\kappa_1 &= \int_0^1 \left| \left(x_1 t + (1-t)\frac{3x_1 + x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} \right| \\ &\quad \cdot \left| f'\left(x_1 t + (1-t)\frac{3x_1 + x_2}{4}\right)\right| dt \\ &\le \int_0^1 \left[\left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] t \\ &\quad \cdot \left| f'\left(x_1 t + (1-t)\frac{3x_1 + x_2}{4}\right)\right| dt.\end{aligned} \tag{30}$$

Then, by the convexity of $|f'(t)|$, we can deduce

$$\begin{aligned}\kappa_1 &= \left[\left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] \int_0^1 t \left| f'\left(x_1 t + (1-t)\frac{3x_1 + x_2}{4}\right)\right| dt \\ &\le \left[\left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] \int_0^1 \\ &\quad \cdot \left[t\left(t\left|f'(x_1)\right| + (1-t)\left|f'\left(\frac{3x_1 + x_2}{4}\right)\right|\right)\right] dt \\ &\le \left[\left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right] \left\{\frac{2\left|f'(x_1)\right| + \left|f'((3x_1 + x_2)/4)\right|}{6}\right\}.\end{aligned} \tag{31}$$

Analogously, we can deduce

$$\begin{aligned}\kappa_2 &= \int_0^1 \left| \left(\frac{3x_1 + x_2}{4} t + (1-t)\frac{x_1 + x_2}{2}\right)^{1+\rho} - \left(\frac{x_1 + x_2}{2}\right)^{1+\rho} \right| \\ &\quad \cdot \left| f'\left(\frac{3x_1 + x_2}{4} t + (1-t)\frac{x_1 + x_2}{2}\right)\right| dt \\ &\le \left[\left(\frac{x_1 + x_2}{2}\right)^{1+\rho} - \left(\frac{3x_1 + x_2}{4}\right)^{1+\rho}\right] \\ &\quad \cdot \left\{\frac{2\left|f'((3x_1 + x_2)/4)\right| + \left|f'((x_1 + x_2)/2)\right|}{6}\right\},\end{aligned}$$

$$\begin{aligned}\kappa_3 &= \int_0^1 \left| \left(\frac{x_1 + x_2}{2} t + (1-t)\frac{x_1 + 3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho} \right| \\ &\quad \cdot \left| f'\left(\frac{x_1 + x_2}{2} t + (1-t)\frac{x_1 + 3x_2}{4}\right)\right| dt \\ &\le \left[\left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1 + x_2}{2}\right)^{1+\rho}\right] \\ &\quad \cdot \left\{\frac{2\left|f'(x_1 + x_2/2)\right| + \left|f'(x_1 + 3x_2/4)\right|}{6}\right\},\end{aligned}$$

$$\begin{aligned}\kappa_4 &= \int_0^1 \left| \left(\frac{x_1 + 3x_2}{4} t + (1-t)x_2\right)^{1+\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho} \right| \\ &\quad \cdot \left| f'\left(\frac{x_1 + 3x_2}{4} t + (1-t)x_2\right)\right| dt \\ &\le \left[{x_2}^{1+\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho}\right] \\ &\quad \cdot \left\{\frac{2\left|f'((x_1 + 3x_2)/4)\right| + \left|f'(x_2)\right|}{6}\right\}.\end{aligned} \tag{32}$$

Substituting κ_i's into Δ, we obtain the desired result.

Corollary 15. *Theorem 14 with $\rho = 1$ becomes*

$$|\Delta_1(x_1, x_2)| \le \left(\frac{x_2 - x_1}{24}\right) \sum_{\ell=1}^{4} \bar{D}_\ell, \tag{33}$$

where

$$\begin{aligned}\bar{D}_1 &= \left[\left(\frac{3x_1 + x_2}{4}\right)^2 - {x_1}^2\right]\left(2\left|f'(x_1)\right| + \left|f'\left(\frac{3x_1 + x_2}{4}\right)\right|\right), \\ \bar{D}_2 &= \left[\left(\frac{x_1 + x_2}{2}\right)^2 - \left(\frac{3x_1 + x_2}{4}\right)^2\right]\left(2\left|f'\left(\frac{3x_1 + x_2}{4}\right)\right| + \left|f'\left(\frac{x_1 + x_2}{2}\right)\right|\right), \\ \bar{D}_3 &= \left[\left(\frac{x_1 + 3x_2}{4}\right)^2 - \left(\frac{x_1 + x_2}{2}\right)^2\right]\left(2\left|f'\left(\frac{x_1 + x_2}{2}\right)\right| + \left|f'\left(\frac{x_1 + 3x_2}{4}\right)\right|\right), \\ \bar{D}_4 &= \left[{x_2}^2 - \left(\frac{x_1 + 3x_2}{4}\right)^2\right]\left(2\left|f'\left(\frac{x_1 + 3x_2}{4}\right)\right| + \left|f'(x_2)\right|\right).\end{aligned} \tag{34}$$

Theorem 16. *Let $f : [x_1, x_2] \subseteq \mathscr{R} \longrightarrow \mathscr{R}$ be an ρ-differentiable function on (x_1, x_2) such that $0 \le x_1 < x_2$. If $\mathscr{D}_\rho(f) \in L_\rho^1([x_1, x_2])$, $|f'|^q$ is a convex function on $[x_1, x_2]$, then we have*

$$\begin{aligned}|\Delta_\rho(x_1, x_2)| \le \frac{x_2 - x_1}{4}\Bigg\{ &(A_1(\rho))^{1-(1/q)}\left[\frac{1}{(x_2 - x_1)^2}\left[B_1(\rho)\left|f'(x_1)\right|^q + C_1(\rho)\left|f'\left(\frac{3x_1 + x_2}{4}\right)\right|^\rho\right]\right]^{1/\rho} + (A_2(\rho))^{1-(1/\rho)} \\ &\cdot \left[\frac{1}{(x_2 - x_1)^2}\left[B_2(\rho)\left|f'\left(\frac{3x_1 + x_2}{4}\right)\right|^\rho + C_2(\rho)\left|f'\left(\frac{x_1 + x_2}{2}\right)\right|^\rho\right]\right]^{1/\rho} + (A_3(\rho))^{1-1/\rho} \\ &\cdot \left[\frac{1}{(x_2 - x_1)^2}\left[B_3(\rho)\left|f'\left(\frac{x_1 + x_2}{2}\right)\right|^\rho + C_3(\rho)\left|f'\left(\frac{x_1 + 3x_2}{4}\right)\right|^q\right]\right]^{1/q} + (A_4(\rho))^{1-(1/q)} \\ &\cdot \left[\frac{1}{(x_2 - x_1)^2}\left[B_4(\rho)\left|f'\left(\frac{x_1 + 3x_2}{4}\right)\right|^q + C_4(\rho)\left|f'(x_2)\right|^q\right]\right]^{1/\rho}\Bigg\},\end{aligned} \tag{35}$$

where

$$A_1(\rho)=\frac{2}{2+\rho}\left\{\left[{x_1}^{2+\rho}-\left(\frac{3x_1+x_2}{4}\right)^{2+\rho}\right]-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right\},$$

$$A_2(\rho)=\frac{2}{2+\rho}\left\{\left[\left(\frac{3x_1+x_2}{4}\right)^{2+\rho}-\left(\frac{x_1+x_2}{2}\right)^{2+\rho}\right]-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right\},$$

$$A_3(\rho)=\frac{2}{2+\rho}\left\{\left[\left(\frac{x_1+x_2}{2}\right)^{2+\rho}-\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}\right]-\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right\},$$

$$A_4(\rho)=\frac{2}{2+\rho}\left\{\left[\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}-{x_2}^{2+\rho}\right]-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right\},$$

$$B_1(\rho)=\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{3+\rho}}{3+\rho}-\left(\frac{3x_1+x_2}{4}\right)\cdot\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{2+\rho}}{2+\rho}+\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\cdot\frac{((3x_1+x_2)/4)^2-{x_1}^2}{2}+\left(\frac{3x_1+x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$B_2(\rho)=\frac{((3x_1+x_2)/4)^{3+\rho}-((x_1+x_2)/2)^{3+\rho}}{3+\rho}-\left(\frac{x_1+x_2}{2}\right)\cdot\frac{((3x_1+x_2)/4)^{2+\rho}-((x_1+x_2)/2)^{2+\rho}}{2+\rho}+\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\cdot\frac{((3x_1+x_2)/4)^2-((x_1+x_2)/2)^2}{2}-\left(\frac{x_1+x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$B_3(\rho)=\frac{((x_1+x_2)/2)^{3+\rho}-((x_1+3x_2)/4)^{3+\rho}}{3+\rho}-\left(\frac{x_1+3x_2}{4}\right)\cdot\frac{((x_1+x_2)/2)^{3+\rho}-((x_1+3x_2)/4)^{3+\rho}}{2+\rho}+\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\cdot\frac{((x_1+x_2)/2)-((x_1+3x_2)/4)^2}{2}-\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$B_4(\rho)=\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{3+\rho}}{3+\rho}-x_2\frac{((x_1+3x_2)/4)^{3+\rho}-{x_2}^{3+\rho}}{2+\rho}+\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\frac{((x_1+3x_2)/4)^2-{x_2}^2}{2}-\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$C_1(\rho)=\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{3+\rho}}{3+\rho}-x_1\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{2+\rho}}{2+\rho}+\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\frac{((3x_1+x_2)/4)^2-{x_1}^2}{2}+\left(\frac{3x_1+x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$C_2(\rho)=\frac{((3x_1+x_2)/4)^{3+\rho}-((x_1+x_2)/2)^{3+\rho}}{3+\rho}-\left(\frac{3x_1+x_2}{4}\right)\cdot\frac{((3x_1+x_2)/4)^{2+\rho}-((x_1+x_2)/2)^{2+\rho}}{2+\rho}+\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\cdot\frac{((3x_1+x_2)/4)^2-((x_1+x_2)/2)^2}{2}-\left(\frac{3x_1+x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$C_3(\rho)=\frac{((x_1+x_2)/2)^{3+\rho}-((x_1+3x_2)/4)^{3+\rho}}{3+\rho}-\left(\frac{x_1+x_2}{2}\right)\cdot\frac{((x_1+x_2)/2)^{3+\rho}-((x_1+3x_2)/4)^{3+\rho}}{2+\rho}+\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\cdot\frac{((x_1+x_2)/2)-((x_1+3x_2)/4)^2}{2}-\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4},$$

$$C_4(\rho)=\frac{{x_1}^{2+\rho}-((3x_1+x_2)/4)^{3+\rho}}{3+\rho}-\left(\frac{x_1+3x_2}{4}\right)\cdot\frac{((x_1+3x_2)/4)^{3+\rho}-{x_2}^{3+\rho}}{2+\rho}+\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\cdot\frac{((x_1+3x_2)/4)^2-{x_2}^2}{2}-\left(\frac{x_1+3x_2}{4}\right)^{2+\rho}\frac{x_1-x_2}{4}.\tag{36}$$

Proof. From Lemma 13, we have

$$\left|\Delta_\rho(x_1,x_2)\right|\le\left(\frac{x_2-x_1}{4}\right)\{|I_1|+|I_2|+|I_3|+|I_4|\},\tag{37}$$

where

$$|I_1|=\int_0^1\left[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho}-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]\cdot\left|f'\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)\right|dt,$$

$$|I_2|=\int_0^1\left[\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)^{1+\rho}-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]\cdot\left|f'\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)\right|dt,$$

$$|I_4|=\int_0^1\left[\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)^{1+\rho}-\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]\cdot\left|f'\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)\right|dt,$$

(38)

making use of the power-mean inequality to get

$$|I_1| \le \left(\int_0^1 \left[\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] dt\right)^{1-(1/q)} \times \left(\int_0^1 \left[\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] \cdot \left|f'\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)\right|^q dt\right)^{1/q}. \tag{39}$$

Since $|f'|^q$ is convex on $[x_1, x_2]$, then for any $t \in [0,1]$, we have

$$|I_1| \le \left(\int_0^1 \left[\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] dt\right)^{1-(1/q)} \times \left(\int_0^1 \left[\left(x_1 t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] \cdot \left\{\left|f'(x_1)\right|^q t + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q (1-t)\right\} dt\right)^{(1/q)}. \tag{40}$$

So we have

$$|I_1| \le (A_1(\rho))^{1-1/q}\left\{B_1(\rho)\left|f'(x_1)\right|^q + C_1(\rho)\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q\right\}^{1/q}. \tag{41}$$

Analogously, we can deduce

$$|I_2| \le (A_2(\rho))^{1-(1/q)}\left\{B_2(\rho)\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q + C_2(\rho)\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q\right\}^{(1/q)},$$
$$|I_3| \le (A_3(\rho))^{1-(1/q)}\left\{B_3(\rho)\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q + C_3(\rho)\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q\right\}^{1/q},$$
$$|I_4| \le (A_4(\rho))^{1-1/q}\left\{B_4(\rho)\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q + C_4(\rho)\left|f'(x_2)\right|^q\right\}^{1/q}, \tag{42}$$

where $A_i(\rho)$, $B_i(\rho)$, and $C_i(\rho)$ are as before. Then, by substituting the above inequalities into (37), we get the desired result.

Remark 17. Theorem 16 with $q = 1$ reduces to Theorem 14.

Corollary 18. *Theorem 16 with $\rho = 1$ becomes*

$$|\Delta_1(x_1,x_2)| \le \left(\frac{x_2-x_1}{4}\right)\left\{(A_1(1))^{1-(1/q)}\left[\frac{1}{(x_2-x_1)^2}\left[B_1(1)\left|f'(x_1)\right|^q + C_1(1)\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q\right]\right]^{1/q} + (A_2(1))^{1-(1/q)} \cdot \left[\frac{1}{(x_2-x_1)^2}\left[B_2(1)\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q + C_2(1)\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q\right]\right]^{1/q} + (A_3(1))^{1-(1/q)}\left[\frac{1}{(x_2-x_1)^2}\left[B_3(1)\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q + C_3(1)\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q\right]\right]^{1/q} + (A_4(1))^{1-(1/q)}\left[\frac{1}{(x_2-x_1)^2}\left[B_4(1)\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q + C_4(1)\left|f'(x_2)\right|^q\right]\right]^{1/q}\right\}, \tag{43}$$

where

$$A_1(1) = \frac{2}{3}\left\{\left[{x_1}^3 - \left(\frac{3x_1+x_2}{4}\right)^3\right] - \left(\frac{3x_1+x_2}{4}\right)^2\right\},$$

$$A_2(1) = \frac{2}{3}\left\{\left[\left(\frac{3x_1+x_2}{4}\right)^3 - \left(\frac{x_1+x_2}{2}\right)^3\right] - \left(\frac{3x_1+x_2}{4}\right)^2\right\},$$

$$A_3(1) = \frac{2}{3}\left\{\left[\left(\frac{x_1+x_2}{2}\right)^3 - \left(\frac{x_1+3x_2}{4}\right)^3\right] - \left(\frac{x_1+3x_2}{4}\right)^2\right\},$$

$$A_4(1) = \frac{2}{3}\left\{\left[\left(\frac{x_1+3x_2}{4}\right)^3 - {x_2}^3\right] - \left(\frac{3x_1+x_2}{4}\right)^2\right\},$$

$$B_1(1) = \frac{{x_1}^3 - ((3x_1+x_2)/4)^4}{4} - \left(\frac{3x_1+x_2}{4}\right)\frac{{x_1}^3 - ((3x_1+x_2)/4)^3}{3} + \left(\frac{3x_1+x_2}{4}\right)^2\frac{((3x_1+x_2)/4)^2 - {x_1}^2}{2} + \left(\frac{3x_1+x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$B_2(1) = \frac{((3x_1+x_2)/4)^4 - ((x_1+x_2)/2)^4}{4} - \left(\frac{x_1+x_2}{2}\right) \cdot \frac{((3x_1+x_2)/4)^3 - ((x_1+x_2)/2)^3}{3} + \left(\frac{3x_1+x_2}{4}\right)^2 \cdot \frac{((3x_1+x_2)/4)^2 - ((x_1+x_2)/2)^2}{2} - \left(\frac{3x_1+x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$B_3(1) = \frac{((x_1+x_2)/2)^4 - ((x_1+3x_2)/4)^4}{4} - \left(\frac{x_1+3x_2}{4}\right) \cdot \frac{((x_1+x_2)/2)^4 - ((x_1+3x_2)/4)^4}{3} + \left(\frac{x_1+3x_2}{4}\right)^2 \cdot \frac{((x_1+x_2)/2) - ((x_1+3x_2)/4)^2}{2} - \left(\frac{x_1+3x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$B_4(1) = \frac{{x_1}^3 - ((3x_1+x_2)/4)^4}{4} - x_2\frac{((x_1+3x_2)/4)^4 - {x_2}^4}{3} + \left(\frac{x_1+3x_2}{4}\right)^2\frac{((x_1+3x_2)/4)^2 - {x_2}^2}{2} - \left(\frac{x_1+3x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$C_1(1) = \frac{{x_1}^3 - ((3x_1+x_2)/4)^4}{4} - x_1\frac{{x_1}^3 - ((3x_1+x_2)/4)^3}{3} + \left(\frac{3x_1+x_2}{4}\right)^2\frac{((3x_1+x_2)/4)^2 - {x_1}^2}{2} + \left(\frac{3x_1+x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$C_2(1) = \frac{((3x_1+x_2)/4)^4 - ((x_1+x_2)/2)^4}{4} - \left(\frac{3x_1+x_2}{4}\right) \cdot \frac{((3x_1+x_2)/4)^3 - ((x_1+x_2)/2)^3}{3} + \left(\frac{3x_1+x_2}{4}\right)^2 \cdot \frac{((3x_1+x_2)/4)^2 - ((x_1+x_2)/2)^2}{2} - \left(\frac{3x_1+x_2}{4}\right)^3\frac{x_1-x_2}{4},$$

$$C_3(1) = \frac{((x_1+x_2)/2)^4 - ((x_1+3x_2)/4)^4}{4} - \left(\frac{x_1+x_2}{2}\right) \cdot \frac{((x_1+x_2)/2)^4 - ((x_1+3x_2)/4)^4}{3} + \left(\frac{x_1+3x_2}{4}\right)^2 \cdot \frac{((x_1+x_2)/2) - ((x_1+3x_2)/4)^2}{2} - \left(\frac{x_1+3x_2}{4}\right)^3 \frac{x_1-x_2}{4},$$

$$C_4(1) = \frac{{x_1}^3 - ((3x_1+x_2)/4)^4}{4} - \left(\frac{x_1+3x_2}{4}\right) \cdot \frac{((x_1+3x_2)/4)^4 - {x_2}^4}{3} + \left(\frac{x_1+3x_2}{4}\right)^2 \cdot \frac{((x_1+3x_2)/4)^2 - {x_2}^2}{2} - \left(\frac{x_1+3x_2}{4}\right)^3 \frac{x_1-x_2}{4}. \tag{44}$$

Theorem 19. *Let $f : [x_1, x_2] \subseteq \mathscr{R} \longrightarrow \mathscr{R}$ be an ρ-differentiable function on (x_1, x_2) such that $0 \le x_1 < x_2$. If $\mathscr{D}_\rho(f) \in L_\rho^1([x_1, x_2])$, $|f'|^q$ is a convex function on $[x_1, x_2]$, then for $q > 1$ and $(1/p) + (1/q) = 1$, we have*

$$|\Delta_\rho(x_1,x_2)| \le \left(\frac{x_2-x_1}{4\sqrt[q]{2}}\right)\left\{E_1^{1/\rho}(p,\rho)\left[|f'(x_1)|^q + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q\right]^{1/q} + E_2^{1/p}(p,\rho)\left[\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q + \left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q\right]^{1/q} + E_3^{1/p}(p,\rho)\left[\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q + \left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q\right]^{1/q} + E_4^{1/p}(p,\rho)\left[\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q + |f'(x_2)|^q\right]^{1/q}\right\}, \tag{45}$$

where

$$E_1(p,\rho) = \int_0^1 \left[\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]^p dt,$$

$$E_2(p,\rho) = \int_0^1 \left[\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]^p dt,$$

$$E_3(p,\rho) = \int_0^1 \left[\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]^p dt,$$

$$E_4(p,\rho) = \int_0^1 \left[\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]^p dt. \tag{46}$$

Proof. From Lemma 13, we have

$$\left|\Delta_\rho(x_1,x_2)\right| \le \left(\frac{x_2-x_1}{4}\right)\{|I_1| + |I_2| + |I_3| + |I_4|\}, \tag{47}$$

where

$$|I_1| \le \int_0^1 \left[\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] \cdot \left|f'\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)\right| dt,$$

$$|I_2| \le \int_0^1 \left[\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right] \cdot \left|f'\left(\frac{3x_1+x_2}{4}t + (1-t)\frac{x_1+x_2}{2}\right)\right| dt,$$

$$|I_3| \le \int_0^1 \left[\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right] \cdot \left|f'\left(\frac{x_1+x_2}{2}t + (1-t)\frac{x_1+3x_2}{4}\right)\right| dt,$$

$$|I_4| \le \int_0^1 \left[\left(\frac{x_1+3x_2}{4}t + (1-t)x_2\right)^{1+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right] \cdot \left|f'^{(((x_1+3x_2)x_1+3x_2/4)t+(1-t)x_2)}\right| dt, \tag{48}$$

making use of Hölder's inequality to get

$$|I_1| \le \left(\int_0^1 \left[\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]^p dt\right)^{1/p} \times \left(\int_0^1 \left|f'\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)\right|^q dt\right)^{1/q}. \tag{49}$$

Since $|f'|^q$ is convex on $[x_1, x_2]$, then for any $t \in [0, 1]$, we have

$$|I_1| \le \left(\int_0^1 \left[\left(x_1t + (1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]^p dt\right)^{1/p} \times \left(\int_0^1 \left\{|f'(x_1)|^q t + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q (1-t)\right\} dt\right)^{1/q}. \tag{50}$$

It follows that

$$|I_1| \le E_1^{1/p}(p,\rho)\left(\frac{|f'(x_1)|^q + |f'((3x_1+x_2)/4)|^q}{2}\right)^{1/q}. \tag{51}$$

Analogously, we can deduce

$$|I_2| \le E_2^{1/p}(p,\rho)\left(\frac{\left|f'((3x_1+x_2)/4)\right|^q + \left|f'((x_1+x_2)/2)\right|^q}{2}\right)^{1/q},$$

$$|I_3| \le E_3^{1/p}(p,\rho)\left(\frac{\left|f'^{((x_1+x_2)/2)}\right|^q + \left|f'^{((x_1+3x_2)/4)}\right|^q}{2}\right)^{1/q},$$

$$|I_4| \le E_4^{1/p}(p,\rho)\left(\frac{\left|f'((x_1+3x_2)/4)\right|^q + \left|f'(x_2)\right|^q}{2}\right)^{1/q}. \tag{52}$$

Then, by substituting $|I_i|$ s (for $i = 1, 2, 3, 4$) into (47), we get the desired result.

Corollary 20. *Theorem 19 with $\rho = 1$ becomes*

$$\begin{aligned}|\Delta_1(x_1,x_2)| \le \left(\frac{x_2-x_1}{4\sqrt[q]{2}}\right)\Bigg\{&E_1^{1/p}(p)\left[\left|f'(x_1)\right|^q + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q\right]^{1/q}\\ &+ E_2^{1/p}(p)\left[\left|f'\left(\frac{3x_1+x_2}{4}\right)\right|^q + \left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q\right]^{1/q}\\ &+ E_3^{1/p}(p)\left[\left|f'\left(\frac{x_1+x_2}{2}\right)\right|^q + \left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q\right]^{1/q}\\ &+ E_4^{1/p}(p)\left[\left|f'\left(\frac{x_1+3x_2}{4}\right)\right|^q + \left|f'(x_2)\right|^q\right]^{1/q}\Bigg\},\end{aligned} \tag{53}$$

where

$$\begin{aligned}E_1(p) &= \int_0^1\left[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^2 - \left(\frac{3x_1+x_2}{4}\right)^2\right]^p dt,\\ E_2(p) &= \int_0^1\left[\left(\frac{3x_1+x_2}{4}t+(1-t)\frac{x_1+x_2}{2}\right)^2 - \left(\frac{3x_1+x_2}{4}\right)^2\right]^p dt,\\ E_3(p) &= \int_0^1\left[\left(\frac{x_1+x_2}{2}t+(1-t)\frac{x_1+3x_2}{4}\right)^2 - \left(\frac{x_1+3x_2}{4}\right)^2\right]^p dt,\\ E_4(p) &= \int_0^1\left[\left(\frac{x_1+3x_2}{4}t+(1-t)x_2\right)^2 - \left(\frac{x_1+3x_2}{4}\right)^2\right]^p dt.\end{aligned} \tag{54}$$

Theorem 21. *Let $f : [x_1, x_2] \subseteq \mathscr{R} \longrightarrow \mathscr{R}$ be an ρ-differentiable function on (x_1, x_2) such that $0 \le x_1 < x_2$. If $\mathscr{D}_\rho(f) \in L_\rho^1([x_1, x_2])$ and $|f'|^q$ is a concave function on $[x_1, x_2]$, we have*

$$\begin{aligned}|\Delta_\rho(x_1,x_2)| \le \frac{x_2-x_1}{4}\Bigg\{&A_1(\rho)\left|f'\left(\frac{J_1(\rho)}{A_1(\rho)}\right)\right| + A_2(\rho)\left|f'\left(\frac{J_2(\rho)}{A_2(\rho)}\right)\right|\\ &+ A_3(\rho)\left|f'\left(\frac{J_3(\rho)}{A_3(\rho)}\right)\right| + A_4(\rho)\left|f'\left(\frac{J_4(\rho)}{A_4(\rho)}\right)\right|\Bigg\},\end{aligned} \tag{55}$$

where $A_i(\rho)$, $i = 1, 2, 3, 4$ are given in Theorem 16 and

$$\begin{aligned}J_1(\rho) &= -\frac{2^{2+\rho}}{(3+\rho)(x_2-x_1)}\left[\left(\frac{3x_1+x_2}{4}\right)^{3+\rho} - {x_1}^{3+\rho}\right]\\ &\quad - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\left(\frac{7x_1+x_2}{8}\right),\\ J_2(\rho) &= \frac{2^{2+\rho}}{(3+\rho)(x_2-x_1)}\left[\left(\frac{x_1+x_2}{2}\right)^{3+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{3+\rho}\right]\\ &\quad - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\left(\frac{5x_1+3x_2}{8}\right),\\ J_3(\rho) &= -\frac{2^{2+\rho}}{(3+\rho)(x_2-x_1)}\left[\left(\frac{x_1+3x_2}{4}\right)^{3+\rho} - \left(\frac{x_1+x_2}{2}\right)^{3+\rho}\right]\\ &\quad - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\left(\frac{3x_1+5x_2}{8}\right),\\ J_4(\rho) &= \frac{2^{2+\rho}}{(3+\rho)(x_2-x_1)}\left[{x_2}^{3+\rho} - \left(\frac{x_1+3x_2}{4}\right)^{3+\rho}\right]\\ &\quad - \left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\left(\frac{x_1+7x_2}{8}\right).\end{aligned} \tag{56}$$

Proof. Since $|f'|^q$ is concave on $[x_1, x_2]$, then for any $t \in [0, 1]$, we have

$$\begin{aligned}&\left|f'\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)\right|^q\\ &\quad\ge \left(\left|f'(x_1)\right|t + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|(1-t)\right)^q,\end{aligned} \tag{57}$$

or we can say

$$\left|f'\left(x_1t+(1-t)\tfrac{3x_1+x_2}{4}\right)\right| \ge \left(\left|f'(x_1)\right|t + \left|f'\left(\frac{3x_1+x_2}{4}\right)\right|(1-t)\right). \tag{58}$$

This tells us that the function $|f'|$ is also concave. Then, by making use of I_1 and Jensen's inequality with the inequality (58), we get

$$\begin{aligned}|I_1| &\le \int_0^1\left[\left(x_1t+(1-t)\frac{3x_1+x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]dt \times \left|f'\left(\frac{\int_0^1\left[(x_1t+(1-t)(3x_1+x_2)/4)^{1+\rho} - ((3x_1+x_2)/4)^{1+\rho}\right](x_1t+(1-t)(3x_1+x_2)/4)dt}{\int_0^1\left[(x_1t+(1-t)(3x_1+x_2)/4)^{1+\rho} - ((3x_1+x_2)/4)^{1+\rho}\right]dt}\right)\right|\\ &\le \frac{2}{2+\rho}\left\{\left[{x_1}^{2+\rho} - \left(\frac{3x_1+x_2}{4}\right)^{2+\rho}\right] - \left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right\} \times \left|f'\left(\frac{-2^{2+\rho}/((3+\rho)(x_2-x_1))\left[((3x_1+x_2)/4)^{3+\rho} - {x_1}^{3+\rho}\right] - ((3x_1+x_2)/4)^{1+\rho}((7x_1+x_2)/8)}{2/(2+\rho)\left\{\left[{x_1}^{2+\rho} - ((3x_1+x_2)/4)^{2+\rho}\right] - ((3x_1+x_2)/4)^{1+\rho}\right\}}\right)\right|\\ &= A_1(\rho)\left|f'\left(\frac{J_1(\rho)}{A_1(\rho)}\right)\right|.\end{aligned} \tag{59}$$

By the same way, can deduce $|I_i| \le A_i(\rho)|f'(J_i(\rho)/A_i(\rho))|$, $i = 2, 3, 4$, where $A_i(\rho)$ are as before and

$$J_2(\rho) = \frac{2^{2+\rho}}{(3+\rho)(x_2 - x_1)}\left[\left(\frac{x_1 + x_2}{2}\right)^{3+\rho} - \left(\frac{3x_1 + x_2}{4}\right)^{3+\rho}\right] - \left(\frac{3x_1 + x_2}{4}\right)^{1+\rho}\left(\frac{5x_1 + 3x_2}{8}8\right),$$

$$J_3(\rho) = -\frac{2^{2+\rho}}{(3+\rho)(x_2 - x_1)}\left[\left(\frac{x_1 + 3x_2}{4}\right)^{3+\rho} - \left(\frac{x_1 + x_2}{2}\right)^{3+\rho}\right] - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho}\left(\frac{3x_1 + 5x_2}{8}\right),$$

$$J_4(\rho) = \frac{2^{2+\rho}}{(3+\rho)(x_2 - x_1)}\left[{x_2}^{3+\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{3+\rho}\right] - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho}\left(\frac{x_1 + 7x_2}{8}\right). \tag{60}$$

These rearrange to the desired (55).

Corollary 22. *Theorem 21 with $\rho = 1$ becomes*

$$|\Delta_1(x_1, x_2)| \le \left(\frac{x_2 - x_1}{4}\right)\left\{A_1(1)\,|f'\left(\frac{J_1(1)}{A_1(1)}\right)| + A_2(1)\,|f'\left(\frac{J_2(1)}{A_2(1)}\right)| + A_3(1)\,|f'\left(\frac{J_3(1)}{A_3(1)}\right)| + A_4(1)\,|f'\left(\frac{J_4(1)}{A_4(1)}\right)|\right\}, \tag{61}$$

where $A_i(1)$, $i = 1, 2, 3, 4$ are as in Corollary 18 and

$$J_1(1) = -\frac{2}{(x_2 - x_1)}\left[\left(\frac{3x_1 + x_2}{4}\right)^4 - {x_1}^4\right] - \left(\frac{3x_1 + x_2}{4}\right)^2\left(\frac{7x_1 + x_2}{8}\right),$$

$$J_2(1) = \frac{2}{(x_2 - x_1)}\left[\left(\frac{x_1 + x_2}{2}\right)^4 - \left(\frac{3x_1 + x_2}{4}\right)^4\right] - \left(\frac{3x_1 + x_2}{4}\right)^2\left(\frac{5x_1 + 3x_2}{8}\right),$$

$$J_3(1) = -\frac{2}{(x_2 - x_1)}\left[\left(\frac{x_1 + 3x_2}{4}\right)^4 - \left(\frac{x_1 + x_2}{2}\right)^4\right] - \left(\frac{x_1 + 3x_2}{4}\right)^2\left(\frac{3x_1 + 5x_2}{8}\right),$$

$$J_4(1) = \frac{2}{(x_2 - x_1)}\left[{x_2}^4 - \left(\frac{x_1 + 3x_2}{4}\right)^4\right] - \left(\frac{x_1 + 3x_2}{4}\right)^2\left(\frac{x_1 + 7x_2}{8}\right). \tag{62}$$

3. Applications to Special Means

Consider the two special means of positive real numbers ($x_1, x_2 \in \mathscr{R}^+$):

(i) The arithmetic mean:

$$A(x_1, x_2) = \frac{x_1 + x_2}{2} \tag{63}$$

(ii) The (ρ, r)-th generalized logarithmic mean:

$$L_{(\rho,r)}(x_1, x_2) = \left[\frac{\rho\left(x_2^{r+\rho} - x_1^{r+1}\right)}{(r+\rho)\left(x_2^{\rho} - x_1^{\rho}\right)}\right]^{1/r}, \quad r \in \mathscr{R} \setminus \{-\rho, 0\}, \rho \in (0, 1]. \tag{64}$$

Proposition 23. *For any $0 < x_1 < x_2$, $r > 1$ and $\rho \in (0, 1]$, we have*

$$\begin{aligned}
&\left|\left[\left(\frac{3x_1 + x_2}{4}\right)^{1+\rho} - {x_1}^{1+\rho}\right]x_1^r + \left[{x_2}^{1+\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho}\right]x_2^r \right.\\
&\quad + \left[\left(\frac{x_1 + 3x_2}{4}\right)^{1+\rho} - \left(\frac{3x_1 + x_2}{4}\right)^{1+\rho}\right]A^r(x_1, x_2)\\
&\quad - \left(\frac{1+\rho}{\rho}\right)\left[\left(\left(\frac{3x_1 + x_2}{4}\right)^{\rho} - x_1^{\rho}\right) \times L^r_{(\rho,r)}\left(x_1, \frac{3x_1 + x_2}{4}\right)\right.\\
&\quad + \left(A^{\rho}(x_1, x_2) - \left(\frac{3x_1 + x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{3x_1 + x_2}{4}, A(x_1, x_2)\right)\\
&\quad + \left(\left(\frac{x_1 + 3x_2}{4}\right)^{\rho} - A^{\rho}(x_1, x_2)\right)L^r_{(\rho,r)}\left(A(x_1, x_2), \frac{x_1 + 3x_2}{4}\right)\\
&\quad \left.\left. + \left(x_2^{\rho} - \left(\frac{x_1 + 3x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{x_1 + 3x_2}{4}, x_2\right)\right]\right|\\
&\le r\left(\frac{x_2 - x_1}{24}\right)\sum_{\ell=1}^{4} U_\ell,
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
U_1 &= \left[\left(\frac{3x_1 + x_2}{4}\right)^{\rho+1} - {x_1}^{\rho+1}\right]\left(2x_1^{r-1} + \left(\frac{3x_1 + x_2}{4}\right)^{r-1}\right),\\
U_2 &= \left[A^{\rho+1}(x_1, x_2) - \left(\frac{3x_1 + x_2}{4}\right)^{\rho+1}\right]\left(2\left(\frac{3x_1 + x_2}{4}\right)^{r-1} + A^{r-1}(x_1, x_2)\right),\\
U_3 &= \left[\left(\frac{x_1 + 3x_2}{4}\right)^{\rho+1} - A^{\rho+1}(x_1, x_2)\right]\left(2A^{r-1}(x_1, x_2) + \left(\frac{x_1 + 3x_2}{4}\right)^{r-1}\right),\\
U_4 &= \left[{x_2}^{\rho+1} - \left(\frac{x_1 + 3x_2}{4}\right)^{\rho+1}\right]\left(2\left(\frac{x_1 + 3x_2}{4}\right)^{r-1} + x_2^{r-1}\right).
\end{aligned} \tag{66}$$

Proof. Let $f(x) = x^r$ for $x > 0$ and $r > 1$. Then, by applying Theorem 14 for the convex function $f(x)$, we get the desired result.

Proposition 24. *Let $0 < x_1 < x_2$, $r > 1$ and $\rho \in (0, 1]$; then, for any $q \ge 1$, we have*

$$
\begin{aligned}
&\left|\left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}-{x_1}^{1+\rho}\right]x_1^r+\left[{x_2}^{1+\rho}-\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]x_2^r\right.\\
&+\left[\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]A^r(x_1,x_2)\\
&-\left(\frac{1+\rho}{\rho}\right)\left[\left(\left(\frac{3x_1+x_2}{4}\right)^{\rho}-x_1^{\rho}\right)\times L^r_{(\rho,r)}\left(x_1,\frac{3x_1+x_2}{4}\right)\right.\\
&+\left(A^{\rho}(x_1,x_2)-\left(\frac{3x_1+x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{3x_1+x_2}{4},A(x_1,x_2)\right)\\
&+\left(\left(\frac{x_1+3x_2}{4}\right)^{\rho}-A^{\rho}(x_1,x_2)\right)L^r_{(\rho,r)}\left(A(x_1,x_2),\frac{x_1+3x_2}{4}\right)\\
&\left.\left.+\left(x_2^{\rho}-\left(\frac{x_1+3x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{x_1+3x_2}{4},x_2\right)\right]\right|\\
&\le\frac{r}{4}(x_2-x_1)^{(q-2)/q}\left\{(A_1(\rho))^{1-(1/q)}\left[B_1(\rho)x_1^{(r-1)q}\right.\right.\\
&\left.+C_1(\rho)\left(\frac{3x_1+x_2}{4}\right)^{(r-1)q}\right]^{1/q}+(A_2(\rho))^{1-1/q}\\
&\cdot\left[B_2(\rho)\left(\frac{3x_1+x_2}{4}\right)^{(r-1)q}+C_2(\rho)A^{(r-1)q}(x_1,x_2)\right]^{1/q}\\
&+(A_3(\rho))^{1-(1/q)}\left[B_3(\rho)A^{(r-1)q}(x_1,x_2)+C_3(\rho)\left(\frac{x_1+3x_2}{4}\right)^{(r-1)q}\right]^{1/q}\\
&\left.+(A_4(\rho))^{1-(1/q)}\left[B_4(\rho)\left(\frac{x_1+3x_2}{4}\right)^{(r-1)q}+C_4(\rho)x_2^{(r-1)q}\right]^{(1/q)}\right\},
\end{aligned}
\tag{67}
$$

where $A_i(\rho), B_i(\rho)$, and $C_i(\rho)$ (for $i=1,2,3,4$) are as in Theorem 16.

Proof. Let $f(x)=x^r$ for $x>0$ and $r>1$. Then, by applying Theorem 16 for the convex function $f(x)$, we get the desired inequality.

Proposition 25. *For any $0<x_1<x_2$, $r>1$ and $\rho\in(0,1]$, then for $q>1$ and $(1/p)+(1/q)=1$, we have*

$$
\begin{aligned}
&\left|\left[\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}-{x_1}^{1+\rho}\right]x_1^r+\left[{x_2}^{1+\rho}-\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}\right]x_2^r\right.\\
&+\left[\left(\frac{x_1+3x_2}{4}\right)^{1+\rho}-\left(\frac{3x_1+x_2}{4}\right)^{1+\rho}\right]A^r(x_1,x_2)\\
&-\left(\frac{1+\rho}{\rho}\right)\left[\left(\left(\frac{3x_1+x_2}{4}\right)^{\rho}-x_1^{\rho}\right)\times L^r_{(\rho,r)}\left(x_1,\frac{3x_1+x_2}{4}\right)\right.\\
&+\left(A^{\rho}(x_1,x_2)-\left(\frac{3x_1+x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{3x_1+x_2}{4},A(x_1,x_2)\right)\\
&+\left(\left(\frac{x_1+3x_2}{4}\right)^{\rho}-A^{\rho}(x_1,x_2)\right)L^r_{(\rho,r)}\left(A(x_1,x_2),\frac{x_1+3x_2}{4}\right)\\
&\left.\left.+\left(x_2^{\rho}-\left(\frac{x_1+3x_2}{4}\right)^{\rho}\right)L^r_{(\rho,r)}\left(\frac{x_1+3x_2}{4},x_2\right)\right]\right|\\
&\le r\left(\frac{x_2-x_1}{4\sqrt[q]{2}}\right)\left\{E_1^{1/p}(p,\rho)\left[x_1^{(r-1)q}+\left(\frac{3x_1+x_2}{4}\right)^{(r-1)q}\right]^{1/q}\right.\\
&+E_2^{1/p}(p,\rho)\left[\left(\frac{3x_1+x_2}{4}\right)^{(r-1)q}+A^{(r-1)q}(x_1,x_2)\right]^{1/q}\\
&+E_3^{1/p}(p,\rho)\left[A^{(r-1)q}(x_1,x_2)+\left(\frac{x_1+3x_2}{4}\right)^{(r-1)q}\right]^{1/q}\\
&\left.+E_4^{1/p}(p,\rho)\left[\left(\frac{x_1+3x_2}{4}\right)^{(r-1)q}+x_2^{(r-1)q}\right]^{1/q}\right\},
\end{aligned}
\tag{68}
$$

where $E_i(p)$ for $i=1,2,3,4$, are as in Theorem 19.

Proof. Let $f(x)=x^r$ for $x>0$ and $r>1$. Then, the result follows by applying Theorem 19 for the convex function $f(x)$.

4. Conclusion

In this article, we have considered new integral inequalities for conformable fractional operators through convexity of the function f, its derivative $|\mathscr{D}_\rho(f)|$, and the power of its derivative $|\mathscr{D}_\rho(f)|^q$. Finally, we see that Propositions 23–25 confirm the usefulness of conformable fractional inequalities in the context of special means of the positive numbers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Estimates of Upper Bound for Differentiable Functions Associated with k-Fractional Integrals and Higher Order Strongly s-Convex Functions

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In this paper, we establish two integral identities associated with differentiable functions and the k-Riemann-Liouville fractional integrals. The results are then used to derive the estimates of upper bound for functions whose first or second derivatives absolute values are higher order strongly s-convex functions.

1. Introduction

Fractional calculus also known as noninteger calculus is a branch of mathematical analysis in which we discuss the integrals and derivatives of arbitrary order. The study of fractional calculus has a very long history, which can be traced back to the end of the 17th century; in 1695, L'Hospital wrote to Leibniz to discuss fractional derivative about a function. For hundreds of years, many mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, and Riemann, have carried out in-depth research on this subject (see [1]). Especially, in recent decades, the fractional calculus has found numerous applications in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Due to the backgrounds in practical applications, the fractional calculus has developed rapidly and has become a hot research topic (see [2–4]).

Among several known forms of fractional integrals, the Riemann-Liouville fractional integral has been investigated extensively, which is defined as follows:

Definition 1 ([3]). Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(v) = \frac{1}{\Gamma(\alpha)}\int_a^v (v-u)^{\alpha-1}f(u)\mathrm{d}u, v > a,$$
$$J^{\alpha}_{b^-}f(v) = \frac{1}{\Gamma(\alpha)}\int_v^b (u-v)^{\alpha-1}f(u)\mathrm{d}u, v < b, \tag{1}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}\mathrm{d}u \tag{2}$$

is the gamma function.

In recent years, several researchers have utilized the concepts of fractional calculus to obtain the fractional analogues of classical inequalities. For example, Sarikaya et al. [5] established a generalized Hermite-Hadamard inequality via the Riemann-Liouville fractional integrals; Set [6] gave some

fractional integral inequalities of the Ostrowski type through s-convex functions; Du et al. [7] deduced some variants of the fractional Hermite-Hadamard's inequality using the class of generalized (α,m)-preinvex functions; Noor et al. [8] used the class of the s-Godunova-Levin convex functions to obtain refinements of the fractional Hermite-Hadamard inequalities; Peng et al. [9] obtained the Riemann-Liouville fractional Simpson inequalities through generalized (m,h_1,h_2)-preinvex functions; Wu and Awan [10] used the class of h-convex functions to derive the upper bound estimates of function involving fractional integrals; Wu et al. [11] established some fractional integral inequalities using k-th order differentiable strongly h-preinvex functions; Zhang et al. [12] provided some variations of the fractional Hermite-Hadamard's inequalities. For more results related to this topic, we refer the interested reader to [13–17] and references cited therein.

In [18], Mubeen and Habibullah introduced the k-fractional integral of the Riemann-Liouville type as follows:

Definition 2 ([18]). Let $F \in L[a,b]$. The k-Riemann-Liouville fractional integrals ${}_{a^+}^{\alpha}J F$ and ${}_{b^-}^{\alpha}J F$ of order $\alpha > 0$ with $a \geq 0$, $k > 0$ are defined by

$$
\begin{aligned}
{}_{k}J_{a^+}^{\alpha}F(v) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^v (v-u)^{\alpha/k-1}F(u)\mathrm{d}u,\ v > a,\\
{}_{k}J_{b^-}^{\alpha}F(v) &= \frac{1}{k\Gamma_k(\alpha)}\int_v^b (u-v)^{\alpha/k-1}F(u)\mathrm{d}u,\ v < b,
\end{aligned}
\tag{3}
$$

where

$$
\Gamma_k(\alpha) = \int_0^{\infty} e^{-u^k/k}u^{\alpha-1}\mathrm{d}u \tag{4}
$$

is the k-gamma function.

Note that if $k \longrightarrow 1$, then the k-Riemann-Liouville fractional integrals reduces to the classical Riemann-Liouville fractional integral.

Sarikaya et al. [19, 20] generalized the k-Riemann-Liouville fractional integrals and discussed their properties. Moreover, for the k-gamma function, they showed that $\Gamma_k(\alpha) = k^{\alpha/k-1}\Gamma(\alpha/k)$ and $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$. For k-beta function, it is defined by

$$
B_k(x,y) = \frac{1}{k}\int_0^1 u^{x/k-1}(1-u)^{y/k-1}\mathrm{d}u,\ x>0,\ y>0,\ k>0, \tag{5}
$$

which implies that $B_k(x,y) = (1/k)B((x/k),(y/k))$ and $B_k(x,y) = \Gamma_k(x)\Gamma_k(y)/\Gamma_k(x+y)$.

Motivated by the ideas of [19, 20], in this paper, we first establish two identities for the k -Riemann-Liouville fractional integrals associated with differentiable functions. We then apply the results to derive some estimates of the upper bound for differentiable functions involving k-fractional integrals via higher order strongly s-convex functions.

2. Preliminaries and Lemmas

Let us briefly summarize the concepts on generalized convex functions which are related to the contents of this paper.

As a strengthening property of convexity, Polyak [21] introduced the strongly convex functions, as follows:

Definition 3. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly convex function with modulus $\mu > 0$, if

$$
\begin{aligned}
f((1-t)x+ty) \leq{}& (1-t)f(x)+tf(y)\\
&-\mu t(1-t)|y-x|^2, \forall x,y \in I, t \in [0,1].
\end{aligned}
\tag{6}
$$

Angulo et al. [22] presented an extension of strongly convex functions which is called strongly s-convex functions, i.e.:

Definition 4. Let $I \subset \mathbb{R}$ be an interval, $s \in (0,1]$. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly s -convex function with modulus $\mu > 0$, if

$$
\begin{aligned}
f((1-t)x+ty) \leq{}& (1-t)^s f(x)+t^s f(y)\\
&-\mu t(1-t)|y-x|^2, \forall x,y \in I, t \in [0,1].
\end{aligned}
\tag{7}
$$

Here we provide a further extension of strongly s-convex functions, as follows:

Definition 5. Let $I \subset \mathbb{R}$ be an interval, $s \in (0,1]$, $\mu > 0$. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly s-convex functions of order $\sigma > 0$, if

$$
\begin{aligned}
f((1-t)x+ty) \leq{}& (1-t)^s f(x)+t^s f(y)-\mu(t^{\sigma}(1-t)\\
&+t(1-t)^{\sigma})|y-x|^{\sigma}, \forall x,y \in I, t \in [0,1].
\end{aligned}
\tag{8}
$$

Remark 6. If $\sigma = 2$, then the class of strongly s-convex functions of order $\sigma > 0$ reduces to the class of strongly s-convex functions. For $\sigma = 2$ and $s = 1$, we have the class of classical strongly convex functions.

In the following, we establish two integral identities associated with differentiable functions and the k-Riemann-Liouville fractional integrals. These integral identities play important role in dealing with subsequent results.

Lemma 7. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$. Then for any $\alpha > 0$, $k > 0$, and $x \in (a,b)$, we have*

$$
\begin{aligned}
&\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\\
&-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}{}_{k}J_{x^-}^{\alpha}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}{}_{k}J_{x^+}^{\alpha}f(b)\right]\\
&=\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)f'((1-t)a+tx)dt\\
&\quad-\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)f'((1-t)x+tb)dt.
\end{aligned}
\tag{9}
$$

Proof. Let

$$\begin{aligned} I &= \frac{(x-a)^2}{(b-a)^2}\int_0^1 \left(at^{\alpha/k}+x\right)f'((1-t)a+tx)\mathrm{d}t \\ &\quad - \frac{(b-x)^2}{(b-a)^2}\int_0^1 \left(b(1-t)^{\alpha/k}+x\right)f'((1-t)x+tb)\mathrm{d}t \\ &\triangleq I_1 - I_2. \end{aligned} \tag{10}$$

Integrating by parts gives

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{(b-a)^2}\int_0^1 \left(at^{\alpha/k}+x\right)f'((1-t)a+tx)\mathrm{d}t \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad - \frac{a\alpha(x-a)}{k(b-a)^2}\int_0^1 t^{\alpha/k-1}f((1-t)a+tx)\mathrm{d}t \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad - \frac{a\Gamma_k(\alpha+k)}{(x-a)^{\alpha/k-1}(b-a)^2 k\Gamma_k(\alpha)}\int_a^x (u-a)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad - \frac{a\Gamma_k(\alpha+k)}{(x-a)^{\alpha/k-1}(b-a)^2}\,{}_kJ^{\alpha}_{x-}f(a). \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{(b-x)^2}{(b-a)^2}\int_0^1 \left(b(1-t)^{\alpha/k}+x\right)f'((1-t)x+tb)\mathrm{d}t \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad - \frac{b\alpha(b-x)}{k(b-a)^2}\int_0^1 (1-t)^{\alpha/k-1}f((1-t)x+tb)\mathrm{d}t \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad + \frac{b\Gamma_k(\alpha+k)}{(b-x)^{\alpha/k-1}(b-a)^2 k\Gamma_k(\alpha)}\int_x^b (b-u)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad + \frac{b\Gamma_k(\alpha+k)}{(b-x)^{\alpha/k-1}(b-a)^2}\,{}_kJ^{\alpha}_{x+}f(b). \end{aligned} \tag{12}$$

Using (11) and (12) in (10) leads to (9). This completes the proof of Lemma 7.

Lemma 8. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be twice differentiable function and $f''\in L[a,b]$. Then for any $\alpha>0$, $k>0$ and $x\in(a,b)$, we have*

$$\begin{aligned} &\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right. \\ &\quad \left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right] \\ &\quad + \frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a)\right. \\ &\quad \left.+\frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right] \\ &\quad = \frac{(x-a)}{(b-a)^2}\int_0^1 \left(t^{\alpha/k+1}x+ta\right)f''(tx+(1-t)a)dt \\ &\qquad + \frac{(b-x)}{(b-a)^2}\int_0^1 \left((1-t)^{\alpha/k+1}a+(1-t)x\right)f'' \\ &\qquad \cdot (tb+(1-t)x)dt. \end{aligned} \tag{13}$$

Proof. Let

$$\begin{aligned} I &= \frac{(x-a)}{(b-a)^2}\int_0^1 \left(t^{\alpha/k+1}x+ta\right)f''(tx+(1-t)a)\mathrm{d}t \\ &\quad + \frac{(b-x)}{(b-a)^2}\int_0^1 \left((1-t)t^{\alpha/k+1}a+(1-t)x\right)f'' \\ &\quad \cdot (tb+(1-t)x)\mathrm{d}t \triangleq I_1 + I_2. \end{aligned} \tag{14}$$

Integrating by parts, we obtain

$$\begin{aligned} I_1 &= \frac{(x-a)}{(b-a)^2}\int_0^1 \left(t^{a/k+1}x+ta\right)f''(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2} - \frac{1}{(b-a)^2}\int_0^1 \\ &\quad \cdot \left(\left(\frac{\alpha}{k}+1\right)t^{\alpha/k}x+a\right)f'(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2} + \frac{((\alpha+k)x+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad + \frac{x\alpha(\alpha+k)}{k^2(x-a)(b-a)^2}\int_0^1 t^{\alpha/k-1}f(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2} + \frac{((\alpha+k)x+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad + \frac{x(\alpha+k)\Gamma_k(\alpha+k)}{\Gamma_k(\alpha)k^2(x-a)^{\alpha/k+1}(b-a)^2}\int_a^x (u-a)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(x+a)f'(x)}{(b-a)^2} + \frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad + \frac{x(\alpha+k)\Gamma_k(\alpha+k)}{k(x-a)^{\alpha/k+1}(b-a)^2}\,{}_kJ^{\alpha}_{x-}f(a). \end{aligned} \tag{15}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \frac{(b-x)}{(b-a)^2}\int_0^1 \left((1-t)^{a/k+1}a+(1-t)x\right)f''(tb+(1-t)x)\mathrm{d}t \\
&= -\frac{(x+a)f'(x)}{(b-a)^2} + \frac{1}{(b-a)^2}\int_0^1 \\
&\quad \cdot \left(\left(\frac{\alpha}{k}+1\right)(1-t)^{\alpha/k}a+x\right)f'(tb+(1-t)x)\mathrm{d}t \\
&= -\frac{(x+a)f'(x)}{(b-a)^2} + \frac{((\alpha+k)a+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2} \\
&\quad + \frac{a\alpha(\alpha+k)}{k^2(b-x)(b-a)^2}\int_0^1 (1-t)^{\alpha/k-1}f(tb+(1-t)x)\mathrm{d}t \\
&= -\frac{(x+a)f'(x)}{(b-a)^2} + \frac{((\alpha+k)a+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2} \\
&\quad + \frac{a(\alpha+k)\Gamma_k(\alpha+k)}{\Gamma_k(\alpha)k^2(b-x)^{\alpha/k+1}(b-a)^2}\int_x^b (b-u)^{\alpha/k-1}f(u)\mathrm{d}u \\
&= -\frac{(x+a)f'(x)}{(b-a)^2} + \frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2} \\
&\quad + \frac{a(\alpha+k)\Gamma_k(\alpha+k)}{k(b-x)^{\alpha/k+1}(b-a)^2}\,{}_kJ^{\alpha}_{x+}f(b).
\end{aligned} \tag{16}$$

Combining (14), (15), and (16) leads to (13). Lemma 8 is proved.

3. Main Results

3.1. Estimates of the Upper Bound for Functions Whose First Derivatives Absolute Values Are Higher Order Strongly s-Convex Functions

Theorem 9. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function and $f'\in L[a,b]$. If $|f'|$ is strongly s-convex functions of order $\sigma>0$, then for any $\alpha>0$, $k>0$, $\mu>0$, $s\in(0,1]$, and $x\in(a,b)$, we have*

$$\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right. \\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right| \\
&\quad\le \frac{(x-a)^2}{(b-a)^2}\left(L_1|f'(a)|+L_2|f'(x)|-\mu L_3|x-a|^{\sigma}\right) \\
&\qquad + \frac{(b-x)^2}{(b-a)^2}\left(L_4|f'(x)|+L_5|f'(b)|-\mu L_6|b-x|^{\sigma}\right),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
L_1 &= akB_k(\alpha+k,sk+k)+\frac{x}{s+1}, \\
L_2 &= \frac{ak}{\alpha+ks+k}+\frac{x}{s+1}, \\
L_3 &= \frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)}+akB_k(\alpha+2k,k\sigma+k) \\
&\quad + \frac{x}{(\sigma+1)(\sigma+2)}+xkB_k(2k,k\sigma+k), \\
L_4 &= \frac{bk}{\alpha+ks+k}+\frac{x}{s+1}, \\
L_5 &= bkB_k(ks+k,\alpha+k)+\frac{x}{s+1}, \\
L_6 &= bkB_k(k\sigma+k,\alpha+2k)-bkB_k(k\sigma+2k,\alpha+k) \\
&\quad + bkB_k(2k,\alpha+k\sigma+k)+\frac{x}{(\sigma+1)(\sigma+2)} \\
&\quad + xkB_k(2k,k\sigma+k).
\end{aligned} \tag{18}$$

Proof. By Lemma 7 and the property of absolute value, we have

$$\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right. \\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right| \\
&\quad\le \frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|\mathrm{d}t \\
&\qquad + \frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|\mathrm{d}t.
\end{aligned} \tag{19}$$

Utilizing the fact that $|f'|$ is strongly s-convex functions of the order $\sigma>0$, we obtain

$$\begin{aligned}
&\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|\mathrm{d}t \\
&\quad + \frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|\mathrm{d}t \\
&\quad\le \frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left[(1-t)^s\left|f'(a)\right|+t^s\left|f'(x)\right|\right. \\
&\qquad \left.-\mu\{t^{\sigma}(1-t)+t(1-t)^{\sigma}\}|x-a|^{\sigma}\right]\mathrm{d}t \\
&\qquad + \frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left[(1-t)^s\left|f'(x)\right|\right. \\
&\qquad \left.+t^s\left|f'(b)\right|-\mu\{t^{\sigma}(1-t)+t(1-t)^{\sigma}\}|b-x|^{\sigma}\right]\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(x-a)^2}{(b-a)^2}\left[\left(akB_k(\alpha+k,sk+k)+\frac{x}{s+1}\right)\left|f'(a)\right|\right.\\
&\quad+\left(\frac{ak}{\alpha+ks+k}+\frac{x}{s+1}\right)\left|f'(x)\right|\\
&\quad-\mu\left(\frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)}\right.\\
&\quad+akB_k(\alpha+2k,k\sigma+k)+\frac{x}{(\sigma+1)(\sigma+2)}\\
&\quad\left.\left.+xkB_k(2k,k\sigma+k)\right)|x-a|^{\sigma}\right]+\frac{(b-x)^2}{(b-a)^2}\\
&\quad\cdot\left[\left(\frac{bk}{\alpha+ks+k}+\frac{x}{s+1}\right)\left|f'(x)\right|\right.\\
&\quad+\left(bkB_k(ks+k,\alpha+k)+\frac{x}{s+1}\right)\left|f'(b)\right|\\
&\quad-\mu\left(bkB_k(k\sigma+k,\alpha+2k)-bkB_k(k\sigma+2k,\alpha+k)\right.\\
&\quad+bkB_k(2k,\alpha+k\sigma+k)+\frac{x}{(\sigma+1)(\sigma+2)}\\
&\quad\left.\left.+xkB_k(2k,k\sigma+k)\right)|b-x|^{\sigma}\right],
\end{aligned}
\tag{20}
$$

which implies the desired inequality (17). The proof of Theorem 9 is complete. is complete.

Theorem 10. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function and $f'\in L[a,b]$, and let $p>1$, $q>1$, $1/p+1/q=1$. If $|f'|^q$ is strongly s-convex functions of the order $\sigma>0$, then for any $\alpha>0$,*

$k>0$, $\mu>0$, $s\in(0,1]$, and $x\in(a,b)$, we have

$$
\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\left(\frac{ak}{\alpha+k}+x\right)^{1/p}\left(L_1\left|f'(a)\right|^q\right.\\
&\quad\left.+L_2\left|f'(x)\right|^q-\mu L_3|x-a|^{\sigma}\right)^{1/q}+\frac{(b-x)^2}{(b-a)^2}\\
&\quad\cdot\left(\frac{bk}{\alpha+k}+x\right)^{1/p}\left(L_4\left|f'(x)\right|^q+L_5\left|f'(b)\right|^q\right.\\
&\quad\left.-\mu L_6|b-x|^{\sigma}\right)^{1/q},
\end{aligned}
\tag{21}
$$

where L_1, L_2, L_3, L_4, L_5, and L_6 are given by the same expressions as described in Theorem 9.

Proof. Using Lemma 7, Hölder's inequality, and the fact that $|f'|^q$ is strongly s-convex functions of the order $\sigma>0$, it follows that

$$
\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|\mathrm{d}t\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|\mathrm{d}t\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\left(\int_0^1\left(at^{\alpha/k}+x\right)\mathrm{d}t\right)^{1/p}\\
&\quad\cdot\left(\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|^q\mathrm{d}t\right)^{1/q}\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\left(\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\mathrm{d}t\right)^{1/p}\\
&\quad\cdot\left(\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|^q\mathrm{d}t\right)^{1/q}\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\left(\int_0^1\left(at^{\alpha/k}+x\right)\mathrm{d}t\right)^{1/p}\\
&\quad\times\left(\int_0^1\left(at^{\alpha/k}+x\right)\left[(1-t)^s\left|f'(a)\right|^q+t^s\left|f'(x)\right|^q\right.\right.\\
&\quad\left.\left.-\mu(t^{\sigma}(1-t)+t(1-t)^{\sigma})|x-a|^{\sigma}\right]\mathrm{d}t\right)^{1/q}\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\left(\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\mathrm{d}t\right)^{1/p}\\
&\quad\times\left(\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left[(1-t)^s\left|f'(x)\right|^q\right.\right.\\
&\quad\left.\left.+t^s\left|f'(b)\right|^q-\mu(t^{\sigma}(1-t)+t(1-t)^{\sigma})|b-x|^{\sigma}\right]\mathrm{d}t\right)^{1/q}\\
&\quad=\frac{(x-a)^2}{(b-a)^2}\left(\frac{ak}{\alpha+k}+x\right)^{1/p}\times\left[\left(akB_k(\alpha+k,sk+k)\right.\right.\\
&\quad\left.+\frac{x}{s+1}\right)\left|f'(a)\right|^q+\left(\frac{ak}{\alpha+ks+k}+\frac{x}{s+1}\right)\left|f'(x)\right|^q\\
&\quad-\mu\left(\frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)}+akB_k(\alpha+2k,k\sigma+k)\right.\\
&\quad\left.\left.+\frac{x}{(\sigma+1)(\sigma+2)}+xkB_k(2k,k\sigma+k)\right)|x-a|^{\sigma}\right]^{1/q}\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\left(\frac{bk}{\alpha+k}+x\right)^{1/p}\times\left[\left(\frac{bk}{\alpha+ks+k}+\frac{x}{s+1}\right)\right.\\
&\quad\left.\cdot\left|f'(x)\right|^q+\left(bkB_k(ks+k,\alpha+k)+\frac{x}{s+1}\right)\left|f'(b)\right|^q\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \mu\Bigg(bkB_k(k\sigma + k, \alpha + 2k) - bkB_k(k\sigma + 2k, \alpha + k) \\
&+ bkB_k(2k, \alpha + k\sigma + k) + \frac{x}{(\sigma + 1)(\sigma + 2)} \\
&+ xkB_k(2k, k\sigma + k)\Bigg) |b - x|^{\sigma}\Bigg]^{1/q} = \frac{(x - a)^2}{(b - a)^2} \\
&\cdot \left(\frac{ak}{\alpha + k} + x\right)^{1/p} \left(L_1 \left|f'(a)\right|^q + L_2 \left|f'(x)\right|^q - \mu L_3 |x - a|^{\sigma}\right)^{1/q} \\
&+ \frac{(b - x)^2}{(b - a)^2} \left(\frac{bk}{\alpha + k} + x\right)^{1/p} \left(L_4 \left|f'(x)\right|^q + L_5 \left|f'(b)\right|^q \right. \\
&\left. - \mu L_6 |b - x|^{\sigma}\right)^{1/q}.
\end{aligned} \tag{22}
$$

This completes the proof of Theorem 10.

3.2. Estimates of the Upper Bound for Functions Whose Second Derivatives Absolute Values Are Higher Order Strongly s-Convex Functions

Theorem 11. *Let $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function and $f'' \in L[a, b]$. If $|f''|$ is strongly s -convex functions of the order $\sigma > 0$, then for any $\alpha > 0$, $k > 0$, $\mu > 0$, $s \in (0, 1]$, and $x \in (a, b)$, we have*

$$
\begin{aligned}
&\left| \frac{1}{(b - a)^2} \left[\frac{(x(\alpha + k) + ka) f(x) - kaf(a)}{k(a - x)} \right.\right. \\
&\left. + \frac{(a(\alpha + k) + kx) f(x) - kxf(b)}{k(x - b)} \right] \\
&+ \frac{(\alpha + k)\Gamma_k(\alpha + k)}{k(b - a)^2} \left[\frac{x}{(x - a)^{\alpha/k+1}} {}_k J^{\alpha}_{x-} f(a) \right. \\
&\left.\left. + \frac{a}{(b - x)^{\alpha/k+1}} {}_k J^{\alpha}_{x+} f(b) \right] \right| \\
&\quad \le \frac{(x - a)}{(b - a)^2} \left(M_1 \left|f''(x)\right| + M_2 \left|f''(a)\right| - \mu M_3 |x - a|^{\sigma} \right) \\
&\qquad + \frac{(b - x)}{(b - a)^2} \left(M_4 \left|f''(b)\right| + M_5 \left|f''(x)\right| - \mu M_6 |b - x|^{\sigma} \right),
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
M_1 &= \frac{xk}{\alpha + ks + 2k} + \frac{a}{s + 2}, \\
M_2 &= xkB_k(\alpha + 2k, ks + k) + akB_k(2k, ks + k), \\
M_3 &= \frac{xk^2}{(\alpha + k\sigma + 2k)(\alpha + k\sigma + 3k)} \\
&\quad + xkB_k(\alpha + 3k, k\sigma + k) + \frac{a}{(\sigma + 2)(\sigma + 3)} \\
&\quad + akB_k(3k, k\sigma + k), \\
M_4 &= akB_k(ks + k, \alpha + 2k) + \frac{x}{(s + 1)(s + 2)}, \\
M_5 &= \frac{ka}{\alpha + ks + 2k} + \frac{x}{s + 2}, \\
M_6 &= kaB_k(k\sigma + k, \alpha + 3k) + kaB_k(2k, \alpha + k\sigma + 2k) \\
&\quad + xkB_k(k\sigma + k, 3k) + xkB_k(2k, k\sigma + 2k).
\end{aligned} \tag{24}
$$

Proof. By Lemma 8 and the property of absolute value, we have

$$
\begin{aligned}
&\left| \frac{1}{(b - a)^2} \left[\frac{(x(\alpha + k) + ka) f(x) - kaf(a)}{k(a - x)} \right.\right. \\
&\left. + \frac{(a(\alpha + k) + kx) f(x) - kxf(b)}{k(x - b)} \right] + \frac{(\alpha + k)\Gamma_k(\alpha + k)}{k(b - a)^2} \\
&\cdot \left.\left[\frac{x}{(x - a)^{\alpha/k+1}} {}_k J^{\alpha}_{x-} f(a) + \frac{a}{(b - x)^{\alpha/k+1}} {}_k J^{\alpha}_{x+} f(b) \right] \right| \\
&\quad = \left| \frac{(x - a)}{(b - a)^2} \int_0^1 \left(t^{\alpha/k+1} x + ta\right) f''(tx + (1 - t)a) \mathrm{d}t \right. \\
&\qquad + \frac{(b - x)}{(b - a)^2} \int_0^1 \left((1 - t)^{\alpha/k+1} a + (1 - t)x\right) f'' \\
&\qquad \left. \cdot (tb + (1 - t)x) \mathrm{d}t \right| \le \frac{(x - a)}{(b - a)^2} \int_0^1 \left(t^{\alpha/k+1} x + ta\right) \\
&\qquad \cdot \left|f''(tx + (1 - t)a)\right| \mathrm{d}t + \frac{(b - x)}{(b - a)^2} \int_0^1 \\
&\qquad \cdot \left((1 - t)^{\alpha/k+1} a + (1 - t)x\right) \left|f''(tb + (1 - t)x)\right| \mathrm{d}t.
\end{aligned} \tag{25}
$$

Utilizing the fact that $|f''|$ is strongly s-convex functions of the order $\sigma > 0$, we obtain

$$
\begin{aligned}
&\frac{(x - a)}{(b - a)^2} \int_0^1 \left(t^{\alpha/k+1} x + ta\right) \left|f''(tx + (1 - t)a)\right| \mathrm{d}t \\
&\quad + \frac{(b - x)}{(b - a)^2} \int_0^1 \left((1 - t)^{\alpha/k+1} a + (1 - t)x\right) \left|f''(tb + (1 - t)x)\right| \mathrm{d}t \\
&\quad \le \frac{(x - a)}{(b - a)^2} \int_0^1 \left(t^{\alpha/k+1} x + ta\right) \left[\left(t^s \left|f''(x)\right| + (1 - t)^s \left|f''(a)\right|\right) \right. \\
&\qquad \left. - \mu(t^{\sigma}(1 - t) + t(1 - t)^{\sigma}) |x - a|^{\sigma} \right] \mathrm{d}t \\
&\qquad + \frac{(b - x)}{(b - a)^2} \int_0^1 \left((1 - t)^{\alpha/k+1} a + (1 - t)x\right) \left[\left(t^s \left|f''(b)\right| \right. \right. \\
&\qquad \left. \left. + (1 - t)^s \left|f''(x)\right|\right) - \mu(t^{\sigma}(1 - t) + t(1 - t)^{\sigma}) |b - x|^{\sigma} \right] \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x-a)}{(b-a)^2}\left[\left(\frac{xk}{\alpha+ks+2k}+\frac{a}{s+2}\right)\left|f''(x)\right|\right.\\
&\quad+\left(xkB_k(\alpha+2k,ks+k)+akB_k(2k,ks+k)\right)\left|f''(a)\right|\\
&\quad-\mu\left(\frac{xk^2}{(\alpha+k\sigma+2k)(\alpha+k\sigma+3k)}+xkB_k(\alpha+3k,k\sigma+k)\right.\\
&\quad\left.\left.+\frac{a}{(\sigma+2)(\sigma+3)}+akB_k(3k,k\sigma+k)\right)|x-a|^{\sigma}\right]\\
&\quad+\frac{(b-x)}{(b-a)^2}\left[\left(akB_k(ks+k,\alpha+2k)+\frac{x}{(s+1)(s+2)}\right)\right.\\
&\quad\cdot\left|f''(b)\right|+\left(\frac{ka}{\alpha+ks+2k}+\frac{x}{s+2}\right)\left|f''(x)\right|\\
&\quad-\mu\left(kaB_k(k\sigma+k,\alpha+3k)+kaB_k(2k,\alpha+k\sigma+2k)\right.\\
&\quad\left.\left.+xkB_k(k\sigma+k,3k)+xkB_k(2k,k\sigma+2k)\right)|b-x|^{\sigma}\right],
\end{aligned}
\tag{26}
$$

which implies the desired inequality (23). This completes the proof of Theorem 11 .

Theorem 12. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be twice differentiable function and $f''\in L[a,b]$, and let $p>1$, $q>1$, $1/p+1/q=1$. If $|f''|^q$ is strongly s -convex functions of the order $\sigma>0$, then for any $\alpha>0$, $k>0$, $\mu>0$, $s\in(0,1]$, and $x\in(a,b)$, we have*

$$
\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right.\right.\\
&\quad\left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right]+\frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\\
&\quad\cdot\left.\left[\frac{x}{(x-a)^{\alpha/k+1}}{}_kJ^{\alpha}_{x-}f(a)+\frac{a}{(b-x)^{\alpha/k+1}}{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)}{(b-a)^2}\left(\frac{xk}{\alpha+2k}+\frac{a}{2}\right)^{1/p}\left(M_1\left|f''(x)\right|^q\right.\\
&\quad\left.+M_2\left|f''(a)\right|^q-\mu M_3|x-a|^{\sigma}\right)^{1/q}+\frac{(b-x)}{(b-a)^2}\\
&\quad\cdot\left(\frac{ak}{\alpha+2k}+\frac{x}{2}\right)^{1/p}\left(M_4\left|f''(b)\right|^q+M_5\left|f''(x)\right|^q\right.\\
&\quad\left.-\mu M_6|b-x|^{\sigma}\right)^{1/q},
\end{aligned}
\tag{27}
$$

where M_1, M_2, M_3, M_4, M_5, and M_6 are given by the same expressions as described in Theorem 11.

Proof. Using Lemma 8, Hölder's inequality, and the fact that $|f''|^q$ is strongly s-convex functions of the order $\sigma>0$, we obtain

$$
\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right.\right.\\
&\quad\left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right]+\frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\\
&\quad\cdot\left.\left[\frac{x}{(x-a)^{\alpha/k+1}}{}_kJ^{\alpha}_{x-}f(a)+\frac{a}{(b-x)^{\alpha/k+1}}{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left|f''(tx+(1-t)a)\right|\mathrm{d}t\\
&\qquad+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\\
&\qquad\cdot\left|f''(tb+(1-t)x)\right|\mathrm{d}t\\
&\quad\le\frac{(x-a)}{(b-a)^2}\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\mathrm{d}t\right)^{1/p}\\
&\qquad\times\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left|f''(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{1/q}\\
&\qquad+\frac{(b-x)}{(b-a)^2}\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\mathrm{d}t\right)^{1/p}\\
&\qquad\times\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left|f''(tb+(1-t)x)\right|^q\mathrm{d}t\right)^{1/q}\\
&\quad\le\frac{(x-a)}{(b-a)^2}\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\mathrm{d}t\right)^{1/p}\\
&\qquad\times\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left[\left(t^s\left|f''(x)\right|^q+(1-t)^s\left|f''(a)\right|^q\right)\right.\right.\\
&\qquad\left.\left.-\mu(t^{\sigma}(1-t)+t(1-t)^{\sigma})|x-a|^{\sigma}\right]\mathrm{d}t\right)^{1/q}\\
&\qquad+\frac{(b-x)}{(b-a)^2}\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\mathrm{d}t\right)^{1/p}\\
&\qquad\times\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left[t^s\left|f''(b)\right|^q\right.\right.\\
&\qquad\left.\left.+(1-t)^s\left|f''(x)\right|^q-\mu(t^{\sigma}(1-t)+t(1-t)^{\sigma})\right.\right.\\
&\qquad\left.\left.\cdot|b-x|^{\sigma}\right]\mathrm{d}t\right)^{1/q}=\frac{(x-a)}{(b-a)^2}\left(\frac{xk}{\alpha+2k}+\frac{a}{2}\right)^{1/p}\\
&\qquad\cdot\left(\left(\frac{xk}{\alpha+ks+2k}+\frac{a}{s+2}\right)\left|f''(x)\right|^q\right.\\
&\qquad+\left(xkB_k(\alpha+2k,ks+k)+akB_k(2k,ks+k)\right)\\
&\qquad\cdot\left|f''(a)\right|^q-\mu\left(\frac{xk^2}{(\alpha+k\sigma+2k)(\alpha+k\sigma+3k)}\right.\\
&\qquad+xkB_k(\alpha+3k,k\sigma+k)+\frac{a}{(\sigma+2)(\sigma+3)}\\
&\qquad\left.\left.+akB_k(3k,k\sigma+k)\right)|x-a|^{\sigma}\right)^{1/q}\\
&\qquad+\frac{(b-x)}{(b-a)^2}\left(\frac{ak}{\alpha+2k}+\frac{x}{2}\right)^{1/p}\\
&\qquad\cdot\left(\left(akB_k(ks+k,\alpha+2k)+\frac{x}{(s+1)(s+2)}\right)\right.
\end{aligned}
$$

$$\cdot \left|f''(b)\right|^q + \left(\frac{ka}{\alpha + ks + 2k} + \frac{x}{s+2}\right)\left|f''(x)\right|^q - \mu(kaB_k(k\sigma + k, \alpha + 3k) + kaB_k(2k, \alpha + k\sigma + 2k) + xkB_k(k\sigma + k, 3k) + xkB_k(2k, k\sigma + 2k))|b - x|^\sigma\Bigg)^{1/q}. \tag{28}$$

The proof of Theorem 12 is complete.

4. Conclusion

In this paper, we establish two identities involving differentiable functions and the k-Riemann-Liouville fractional integrals. Utilizing the identities, we obtain the estimates of the upper bound for functions whose first or second derivatives absolute values are higher order strongly s-convex functions. It is worth mentioning that our results contain, as a special case $(k, s, \sigma) = (1,1,2)$, the estimates of the upper bound of functions for the classical Riemann-Liouville fractional integrals and strongly convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

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Research Article

Analysis and Simulation of a Fractional Order Optimal Control Model for HBV

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In this paper, we propose a fractional optimal control model of anti-HBV infection based on saturation incidence and logistic proliferation of uninfected cells for the first time. We derive the basic reproduction number R_0 and the cytotoxic T lymphocyte immune response reproductive number R_1 and give the stable analysis based on R_0 and R_1. We analyse the optimal control condition and give two optimal control strategies about entecavir monotherapy and combination therapy of traditional Chinese medicine and entecavir with different fractional orders by simulation. The simulation shows that combination therapy may be a good choice in anti-HBV infection therapy. We also compare the objective function values of the optimal control strategies with other constant control strategies; the comparison shows that the optimal therapy can get similar or better treatment effect with less drug dose and side effects.

1. Introduction

Hepatitis B virus infection is a major global public health problem. Based on the information published by the World Health Organization on World Hepatitis Day 2019, hepatitis B is the second deadly epidemic, and the number of people infected with the hepatitis virus is 9 times that of HIV [1]. The prevalence of chronic hepatitis B has brought a huge medical burden on society. At present, the most widely used drugs for chronic hepatitis B are nucleot(s)ide analogues (NUCs), such as tenofovir and entecavir, which are used to inhibit viral DNA polymerase and reverse transcriptase activity [2]. The combined therapy of NUCs plus Chinese herbal medicine (CHM) is widely accepted in China, which has been recognized as a prospective alternative approach [3, 4].

In order to better understand the transmission mechanism of various infectious diseases, many mathematical models were set up to enhance our understanding of the dynamics of infectious diseases and chronic viral infections [5–8]. Mathematic models are also used in the study of anti-HBV infection treatment. The initial model of HBV infection was proposed in 1996 by Zeuzem et al. [9]:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \beta xv, \\ \dfrac{dy}{dt} = \beta xv - ay, \\ \dfrac{dv}{dt} = ky - \mu v, \end{cases} \tag{1}$$

where $x(t), y(t)$, and $v(t)$ represent the concentration of uninfected cells, infected cells, and viruses at time t, respectively. Uninfected cells are assumed to be produced at the constant rate λ, die at the rate of dx, and become infected at the rate of βxv. Infected cells are thus produced at the rate of βxv and die at the rate ay. Free virions are generated from infected cells at the rate of ay and decay at the rate of μv.

The infection rate in model (1) is assumed to be linear with respect to the virus. However, the basic reproduction number of model (1) is given by $R_0 = (\beta k/a\mu)\hat{x}$, where $\hat{x}$

represents the number of total liver cells, which implies that an individual with a larger liver will be more difficult to be cured than a person with a smaller one. Paper [10] changed βxv in (1) to $\beta xv/(x+v)$ and gave the following model:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \dfrac{\beta xv}{x+v}, \\ \dfrac{dy}{dt} = \dfrac{\beta xv}{x+v} - ay, \\ \dfrac{dv}{dt} = ky - \mu v. \end{cases} \tag{2}$$

The reproductive number R_0 of (2) is $\beta k/a\mu$ which seems more reasonable because it is no more dependent on the total number of liver cells. On the other hand, when modelling virus infection, the host's immune response should not be ignored; Nowak and Bangham [11] proposed the following immune model:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \beta xv, \\ \dfrac{dy}{dt} = \beta xv - ay - pyz, \\ \dfrac{dv}{dt} = ky - \mu v, \\ \dfrac{dz}{dt} = cyz - bz, \end{cases} \tag{3}$$

where z represents the number of CTL cells, cyz means the activated rate of the immune cells, and pyz indicates the rate at which infected liver cells are eliminated by CTL immune cells. Then, Su et al. changed βxv into $\beta xv/(x+v)$ and gave the model as follows [12]:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \dfrac{\beta xv}{x+v}, \\ \dfrac{dy}{dt} = \dfrac{\beta xv}{x+v} - ay - pyz, \\ \dfrac{dv}{dt} = ky - \mu v, \\ \dfrac{dz}{dt} = cyz - bz. \end{cases} \tag{4}$$

Perelson and Nelson [13] added logistic function in his HIV model, that is $dx/dt = \lambda - dx - px(1 - (x/x_{\max})) - \beta xv$. Based on paper [13], Song and Neumann proposed a HIV viral infection model with logistic function and saturated mass action incidence [14]. Then, Yu et al. gave a HBV model with logistic function and standard mass action incidence [15]:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx + px\left(1 - \dfrac{x}{x_{\max}}\right) - \dfrac{\beta xv}{x+y}, \\ \dfrac{dy}{dt} = \dfrac{\beta xv}{x+y} - ay, \\ \dfrac{dv}{dt} = ky - \mu v. \end{cases} \tag{5}$$

In recent years, more and more fractional order models were used in the biological immune system, because the fractional order model has the memory, while the characteristic of the immune response contains the memory [16, 17]. So, when we set the virus immune model, fractional mathematical models have become an important choice. Many fractional order HIV infection models were set up [18, 19].

So far, some fractional order models about HBV have been set up [20–25]. Papers [20–22] considered the epidemic transmission SEIR model of HBV. In paper [23], a within-host fractional order HBV model was proposed:

$$\begin{cases} D^\alpha x = \lambda - dx - \beta xv + \delta y, \\ D^\alpha y = \beta xv - (a+\delta)y, \\ D^\alpha v = ky - \mu v. \end{cases} \tag{6}$$

In this model, $x(t)$, $y(t)$, and $v(t)$ have the same meaning of (1); infected hepatocytes are cured by noncytolytic processes at a constant rate δ per cell. But the model still used the bilinear mass action incidence βxv. Paper [24] also considered the HBV models based on the bilinear mass action incidence βxv. Paper [25] added δy to the first equation in (4) and changed it to fractional order. The model is given as follows:

$$\begin{cases} D^\alpha x = \lambda - dx - \dfrac{\beta xv}{x+v} + \delta y, \\ D^\alpha y = \dfrac{\beta xv}{x+v} - ay - \delta y - pyz, \\ D^\alpha v = ky - \mu v, \\ D^\alpha z = cyz - bz. \end{cases} \tag{7}$$

On the other hand, in the process of anti-HBV treatment, we hope not only to lower the levels of HBV and the infected hepatocytes during and at the end of therapy but also to minimize the therapeutic side effects and the cost of drugs, so it is very important to use optimal control theory to study the anti-HBV treatment model.

Paper [26] provided a control model, which uses the linear incidence. Here, $u(t)(0 \le u(t) \le 1)$ means the control and $\mu(0 \le \mu \le 1)$ is the efficacy of the antiviral drug:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \beta xv, \\ \dfrac{dy}{dt} = \beta xv - ay, \\ \dfrac{dv}{dt} = [1 - \mu u(t)]y - cv. \end{cases} \tag{8}$$

For the fractional order model, paper [27] gave the analysis of the control model. Here, $x(t)$, $y(t)$, and $v(t)$ have the

same meaning of (1) and u_1 and u_2 represent the drug effect on HBV by interferon (IFN) and lamivudine (LAM):

$$\begin{cases} D^\alpha x = \Pi - \delta x - (1-u_1)\dfrac{\beta x v}{1+ax} + py, \\ D^\alpha y = (1-u_1)\dfrac{\beta x v}{1+ax} - (b+p)y, \\ D^\alpha v = cy - \gamma v - u_2 v. \end{cases} \tag{9}$$

Based on the above discussion, according to the clinical anti-HBV combination therapy of traditional Chinese and western medicine [4], we consider a HBV fractional order model as follows:

$$\begin{cases} D^\alpha x = \lambda - dx - \dfrac{\beta x v}{x+v} + \delta y + qx\left(1 - \dfrac{x}{x_{\max}}\right), \\ D^\alpha y = \dfrac{\beta x v}{x+v} - ay - \delta y - (1+u_3)pyz, \\ D^\alpha v = (1-u_1)ky - (1+u_2)\mu v, \\ D^\alpha z = (1+u_3)cyz - bz. \end{cases} \tag{10}$$

Here, $x, y, v,$ and z have the same meaning as those in model (3). $0 < \alpha < 1$.$u_1(0 \le u_1 \le 1)$ represents the treatment effect of entecavir (ETV), which can block the replication of virus. $u_2(0 \le u_2 \le 1)$ represents the treatment effect of Tiaogan Jianpi Jiedu Granules (TGJPJD), which can accelerate the death of virus. $u_3(0 \le u_3 \le 1)$ represents the treatment effect of Tiaogan-Yipi Granule (TGYP) which can enhance immunity.

The structure of this paper is given as follows. In Section 2, we show some definitions of fractional order and related lemmas. In Section 3, the existence and uniqueness of the positive solution are discussed. In Section 4, we give the stable analysis of our system. The necessary conditions for an optimal control are derived in Section 5. The numerical simulation and the conclusion are given, respectively, in Sections 6 and 7.

2. Basic Concepts and Lemmas

First of all, we give some basic definitions and inferences about fractional order. There are several definitions of fractional derivatives, but we only consider the Caputo derivation used in this paper.

Definition 1. The Caputo fractional derivative of order $\alpha(\alpha > 0)$ of a function $f(f : [0, \infty] \longrightarrow R)$ is given:

$$D^\alpha f = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}}{(t-s)^{\alpha-n+1}} ds. \tag{11}$$

Here, $n = [\alpha] + 1$, and $\Gamma(\cdot)$ is the Euler gamma function. In this paper, we only discuss the situation that $0 < \alpha < 1$. Function (11) will become

$$D^\alpha f = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'}{(t-s)^\alpha} ds. \tag{12}$$

Lemma 2 (see [28] (Laplace transform). *The Laplace transform of formula (12) is*

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-k-1}. \tag{13}$$

Lemma 3 (see [29] (Generalized mean value theorem). *Suppose that $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$, for $0 < \alpha < 1$, we have*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\xi)(x-a)^\alpha. \tag{14}$$

Notice. Suppose that $f(t) \in C[a,b]$ and $D^\alpha f(t) \in C[a,b]$, for $0 < \alpha < 1$, we have the following:

(1) If $D^\alpha f(t) > 0, t_0 \in (a,b)$, then neighbourhood N satisfies $f(t) > f(a), \forall t \in N$

(2) If $D^\alpha f(t) < 0, t_0 \in (a,b)$, then neighbourhood N satisfies $f(t) < f(a), \forall t \in N$

Lemma 4 (see [30]). *For a fractional order system (15) of which $0 < \alpha < 1$ and $x \in R^n$,*

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{15}$$

Assume $f(t,x)$: $R^+ \times R^n \longrightarrow R^n$ satisfies the following conditions:

(1) $f(t, x(t))$ is Lebesgue measurable with respect to t on R^+

(2) $f(t, x(t))$ and $\partial f(t, x(t))/\partial x$ are continuous for all $x \in R^n$

(3) $\|f(t, x(t))\| \le \omega + \lambda\|x\|, \forall t \in R^+, X \in R^n$; here, $\omega, \lambda > 0$ are two positive constants

Then, the initial problem (15) has unique and positive solution on $[t_0, \infty]$.

Lemma 5 (see [31]). *For the system (15), the equilibrium point is locally asymptotically stable, if all eigenvalues λ_i of Jacobian matrix $J = \partial f/\partial x$ evaluated at the equilibrium point satisfies*

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}. \tag{16}$$

Lemma 6. *For the discriminant $D(f)$ of polynomial equation*

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \tag{17}$$

is defined by $D(f) = (-1)^{n(n-1)/2} R(f, f')$. Here, $R(f \cdot f')$ represents the determinant for the corresponding $(2n-1) \otimes (2n-1)$ Sylvester matrix.

Lemma 7. *For the polynomial equation*

$$P(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_n, \tag{18}$$

the conditions which make all the roots of (18) satisfy (16) are displayed as follows:

(1) *For $n = 1$, the condition is $d > 1$*

(2) *For $n = 2$, the conditions are either Routh-Hurwitz conditions or*

$$d_1 < 0, 4d_2 > {d_1}^2, \left|\tan^{-1}\left(\frac{\sqrt{4d_2 - {d_1}^2}}{d_1}\right)\right| > \frac{\alpha\pi}{2} \tag{19}$$

For $n = 3$

(a) *If the discriminant $D(P)$ of (18) is positive, then Routh-Hurwitz conditions are the necessary and sufficient conditions, that is $d_1 > 0, d_3 > 0, d_1d_2 > d_3$*

(b) *If $D(P) < 0, d_1 > 0, d_2 > 0$, and $d_3 > 0$, then (16) holds when $\alpha < 2/3$*

(c) *If $D(P) < 0, d_1 < 0, d_2 < 0$, and $d_3 \geq 0$, then (16) holds when $\alpha > 2/3$*

(d) *If $D(P) < 0, d_1 > 0, d_2 > 0$, and $d_1d_2 = d_3$, then (16) holds for all $\alpha \in [0, 1]$*

(4) *When $n > 3$, suppose $\Delta_1, \Delta_2, \cdots, \Delta_n$ is the Routh-Hurwitz discriminant, that is,*

$$\begin{aligned} \Delta_1 &= a_1 > 0, \\ \Delta_2 &= \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \\ \Delta_3 &= \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0. \end{aligned} \tag{20}$$

For $\alpha \in (0, 1]$, (16) is sufficient but not necessary.

3. The Existence and Uniqueness of Positive Solutions

We first analyse the system (10) without control, that is, $u_1 = u_2 = u_3 = 0$.

Theorem 8. *The solution of system (10) is always nonnegative.*

Proof. For system (10), we can get that

$$\begin{cases} D^\alpha x|_{x=0} = \lambda + \delta y \geq 0, \\ D^\alpha y|_{y=0} = \dfrac{\beta x v}{x + v} \geq 0, \\ D^\alpha v|_{v=0} = ky \geq 0, \\ D^\alpha z|_{z=0} = 0. \end{cases} \tag{21}$$

Based on Lemma 2, we can prove that $x(t), y(t), v(t) \geq 0$ for $t \in [t_0, \infty)$. For $z(t)$, assume that there exists t_1 satisfying that

$$\begin{cases} z(t_1) = 0, \\ z(t) > 0, \quad \forall t \in [t_0, t_1). \end{cases} \tag{22}$$

We can find that $D^\alpha z \geq -bz, t \in [t_0, t_1)$ and $z(t) \geq z(0)E_\alpha[-bt^\alpha], t \in [t_0, t_1)$, which implies $z(t_1) > 0$. This result is contradicted with the assumption, so $z(t) > 0$ for any $t \geq t_0$.

Theorem 9. *There exists an $M > 0$, such that $x(t), y(t), v(t), z(t) < M$, for any $t \in [0, \infty)$.*

Proof. Let

$$\begin{cases} N(t) = x(t) + y(t) + \dfrac{p}{c}z(t), \\ N(0) = N_0 = x_0 + y_0 + \dfrac{p}{c}z_0. \end{cases} \tag{23}$$

Calculate the α-order derivatives on both sides, respectively,

$$\begin{aligned} D^\alpha N(\mathrm{t}) &= \lambda - dx - ay - \frac{pb}{c}z + qx\left(1 - \frac{x}{x_{\max}}\right) \leq \lambda + \frac{qx_{\max}}{4} \\ &\quad - dx - ay - \frac{pb}{c}z \leq h_1 - h_2\left(x + y + \frac{p}{c}z\right). \end{aligned} \tag{24}$$

Here, $h_1 = \lambda + (qx_{\max}/4)$ and $h_2 = \min\{d, a, b\}$. So, we can get

$$\begin{aligned} N(t) &\leq h_1 t^\alpha E_{\alpha,\alpha+1}(-h_2 t^\alpha) + N_0 E_{\alpha,1}(-h_2 t^\alpha) \\ &= \left(-\frac{h_1}{h_2} + N_0\right)E_\alpha(-h_2 t^\alpha) + \frac{h_1}{h_2}. \end{aligned} \tag{25}$$

$E_\alpha(-x)$ is completely monotonic for $x > 0$ [32], $\lim_{t\to\infty} E_\alpha(-\mu t^\alpha) = 0$ and $E_\alpha(0) = 1$. Here, we assume $M_1 = \max\{(h_1/h_2), N_0\}$. So, $N(t) \leq M_1$, $x(t) \leq M_1, y(t) \leq M_1$, and $z(t) \leq M_1$. Similarly, we can have the following inequation:

$$v(t) \leq \left(-\frac{M_1 k}{\mu} + v_0\right)E_\alpha(-\mu t^\alpha) + \frac{M_1 k}{\mu}. \tag{26}$$

Assume that $M_2 = \max\{(M_1k/\mu), v_0\}$, $v(t) \leq M_2$.
Let $M = \max\{M_1, M_2\}$, so $x(t), y(t), v(t), z(t) < M$.

Theorem 10. *System (10) has a unique positive solution on $[0, \infty)$.*

Proof. Let $R_4^+ = \{(x, y, v, z), x, y, v, z\} \geq 0$. Suppose $X(t) = (x(t), y(t), v(t), z(t))$, system (10) can be transformed into the following form:

$$f(t, X) = \begin{pmatrix} \lambda - dx - \dfrac{\beta x v}{x + v} + \delta y + qx\left(1 - \dfrac{x}{x_{\max}}\right) \\ \dfrac{\beta x v}{x + v} - ay - \delta y - pyz \\ ky - \mu v \\ cyz - bz \end{pmatrix}. \tag{27}$$

Obviously, (27) satisfies conditions (1) and (2) of Lemma 4. We only prove that system (33) satisfies condition (3) of Lemma 4. Let

$$\eta = \begin{pmatrix} \lambda + qx\left(1 - \dfrac{x}{x_{\max}}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -d & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -b \end{pmatrix},$$

$$C = \begin{pmatrix} -\beta & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & \delta & 0 & 0 \\ 0 & -\delta & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ v(t) \\ z(t) \end{pmatrix}. \tag{28}$$

So, we can get

$$\begin{aligned} \|f(t, X)\| &= \left\| A_1 X(t) + \frac{v(t)}{x(t) + v(t)} A_2 X(t) \right. \\ &\quad \left. + z(t) A_3 X(t) + y(t) A_4 X(t) + A_5 y(t) + \eta \right\| \\ &\leq \left\| A_1 X(t) + A_2 X(t) + M A_3 X(t) \right. \\ &\quad \left. + \frac{c}{p} M A_4 X(t) + A_5 X(t) + \eta \right\| \\ &\leq \left(\|A_1\| + \|A_2\| + M\|A_3\| \right. \\ &\quad \left. + \frac{c}{p} M\|A_4\| + \|A_5\| \right) X(t) + h_1. \end{aligned} \tag{29}$$

Here, $h_1 = \lambda + (qx_{\max}/4)$, and M is a bound value mentioned in Theorem 9. By Lemma 4, system (10) has a unique solution on $[0, \infty)$.

4. Stable Analysis

In this section, we will discuss the stability of the system (10). The system always has an infection-free equilibrium $E_0 = (x_0, 0, 0, 0)$, where

$$x_0 = \frac{x_{\max}}{2q}\left[q - d + \sqrt{(q - d)^2 + \frac{4q\lambda}{x_{\max}}}\right]. \tag{30}$$

Here, we have the basic reproduction number as

$$R_0 = \frac{\beta k}{\mu(a + \delta)}. \tag{31}$$

When $R_0 > 1$, the system (10) will have an immune-absence equilibrium $E_1 = (x_1, y_1, v_1, 0)$; here, $N_1 = ((q - d)/(R_0 - 1)) - (\mu q/k)$ and

$$\begin{aligned} x_1 &= \frac{1}{R_0 - 1} v_1, \\ y_1 &= \frac{\mu}{k} v_1, \\ v_1 &= \frac{(R_0 - 1)x_{\max}}{2q}\left[-N_1 + \sqrt{N_1^2 + \frac{4q\lambda}{(R_0 - 1)^2 x_{\max}}}\right], \end{aligned} \tag{32}$$

which means that the infected cells and virus coexist but the immune response is not activated yet, that is, $cy_1 < b$. Further, we will have the cytotoxic T lymphocyte immune response reproductive number

$$R_1 = \frac{\mu c}{kb}\frac{(R_0 - 1)x_{\max}}{2q}\left[-N_1 + \sqrt{N_1^2 + \frac{4q\lambda}{(R_0 - 1)^2 x_{\max}}}\right]. \tag{33}$$

When $R_1 > 1$, it means that immune response is activated; the system (10) will have an immune-response equilibrium $E_2 = (x_2, y_2, v_2, z_2)$, where

$$\begin{aligned} x_2 &= m + n - \frac{a_2}{3a_1}, \\ y_2 &= \frac{b}{c}, \\ v_2 &= \frac{kb}{\mu c}, \\ z_2 &= \frac{\lambda - dx_2 - ay_2 + qx_2(1 - (x_2/x_{\max}))}{py_2}. \end{aligned} \tag{34}$$

Here m, n are given as follows:

$$\begin{aligned} &a_1 {x_2}^3 + a_2 {x_2}^2 + a_3 x_2 + a_4 = 0, \\ &a_1 = -\frac{q}{x_{\max}}, \\ &a_2 = q - d + \frac{v_2 q}{x_{\max}}, \\ &a_3 = (q - d - \beta)v_2 + \lambda + \delta y_2, \\ &a_4 = v_2(\lambda + \delta y_2), \\ &uu = \frac{9a_1a_2a_3 - 27a_1^2a_4 - 2a_2^3}{54a_1^3}, \\ &vv = \frac{\sqrt{3\left(4a_1a_3^3 - a_2^2a_3^2 - 18a_1a_2a_3a_4 + 27a_1^2a_4^2 + 4a_4a_2^3\right)}}{18a_1^2}, \\ &m = \sqrt[3]{uu + vv}, \\ &n = \frac{a_2^2 - 3a_1a_3}{9a_1m}. \end{aligned} \tag{35}$$

Theorem 11. *For the model of (10): when $R_0 < 1$, the infection-free equilibrium point E_0 is locally asymptotically stable; when $R_0 > 1$, E_0 is unstable.*

Proof. The Jacobi matrix for equilibrium $E_0 = (x_0, 0, 0, 0)$ is shown as

$$J_{E_0} = \begin{pmatrix} -\sqrt{(q-d)^2 + \frac{4q\lambda}{x_{\max}}} & \delta & -\beta & 0 \\ 0 & -(a+\delta) & \beta & 0 \\ 0 & k & -\mu & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}. \tag{36}$$

So, the characteristic equation for E_0 is

$$\begin{aligned} \left|lE - J_{E_0}\right| &= \left(l + \sqrt{(q-d)^2 + \frac{4q\lambda}{x_{\max}}}\right)(l + b) \\ &\cdot \left[l^2 + (a + \delta + \mu)l - \beta k\left(1 - \frac{1}{R_0}\right)\right] = 0. \end{aligned} \tag{37}$$

Suppose that $B = a + \delta + \mu$, $C = -\beta k(1 - (1/R_0))$, we can have the four characteristic roots: $l_1 = -\sqrt{(q-d)^2 + (4q\lambda/x_{\max})} < 0$, $l_2 = -b < 0$, $l_{3,4} = -(B/2) \pm \sqrt{(B^2 - 4C)/4}$. When $R_0 < 1$, $-(B/2) + \sqrt{(B^2 - 4C)/4} < 0$, from Lemma 5, E_0 is locally asymptotically stable. When $R_0 > 1$, $-(B/2) + \sqrt{(B^2 - 4C)/4} > 0$, from Lemma 5, E_0 is unstable.

Theorem 12. *When $R_0 > 1$ and $R_1 < 1$, if $D(P) > 0$, E_1 is locally asymptotically stable for $0 < \alpha < 1$. If $D(P) < 0$, then E_1 is locally asymptotically stable for $0 < \alpha < 2/3$. If $R_1 > 1$, E_1 is unstable.*

Proof. The characteristic equation for E_1 is given as follows:

$$\left|lE - J_{E_1}\right| = (l - cy_1 + b)\left[l^3 + a_1l^2 + a_2l + a_3\right] = 0. \tag{38}$$

Let

$$\begin{aligned} N_1 &= \frac{q-d}{R_0 - 1} - \frac{\mu q}{k}, \\ N_2 &= N_1^2 + \frac{4q\lambda}{(R_0 - 1)^2 x_{\max}}, \\ a_1 &= a + \delta + \mu + \frac{\beta v_1^2}{(x_1 + v_1)^2} + (R_0 - 1)\left[-N_1 + \sqrt{N_2}\right] > 0, \\ a_2 &= \mu(a + \delta)\left(1 - \frac{1}{R_0}\right) + \frac{\beta v_1^2}{(x_1 + v_1)^2}(\mu + k) \\ &\quad + (a + \delta + \mu)(R_0 - 1)\left[-N_1 + \sqrt{N_2}\right] > 0, \\ a_3 &= \mu(a + \delta)(R_0 - 1)\left[-N_1 + \sqrt{N_2}\right]\left(1 - \frac{1}{R_0}\right) \\ &\quad + \frac{a\beta v_1^2 \mu}{(x_1 + v_1)^2} > 0. \end{aligned} \tag{39}$$

When $R_1 < 1$, the characteristic root $l_1 = cy_1 - b < 0$.

Let $N_3 = (R_0 - 1)[-N_1 + \sqrt{N_2}]$, we have

$$\begin{aligned} a_1 a_2 - a_3 = {} & \frac{\beta k x_1^2}{(x_1 + v_1)^2} + \mu^2 \left(N_3 + \frac{\beta v_1^2}{(x_1 + v_1)^2} \right) \\ & + \left(a + \delta + \frac{\beta v_1^2}{(x_1 + v_1)^2} + N_3 \right) \\ & \cdot \left(\mu \left(N_3 + \frac{\beta v_1^2}{(x_1 + v_1)^2} \right) \right. \\ & + \frac{\beta v_1^2}{(x_1 + v_1)^2} + N_3 (a + \delta) \\ & \left. + \mu (a + \delta) \left(1 - \frac{1}{R_0} \right) \right) > 0. \end{aligned} \tag{40}$$

From Lemma 6, the discriminant $D(P)$ of the polynomial equation $l^3 + a_1 l^2 + a_2 l + a_3$ is as follows:

$$D(P) = 18 a_1 a_2 a_3 - (a_1 a_2)^2 - 4 {a_1}^2 - 4 {a_2}^2 - 27 {a_1}^3. \tag{41}$$

According to Lemma 7, when $R_1 < 1$, if $D(P) > 0$,E_1 is locally asymptotically stable for $0 < \alpha < 1$.If $D(P) < 0$, then E_1 is locally asymptotically stable for $0 < \alpha < 2/3$. If $R_1 > 1$, the equilibrium E_1 is unstable.

Theorem 13. *When $R_0 > 1$ and $R_1 > 1$, E_2 is locally asymptotically stable for $0 < \alpha < 1$.*

Proof. The characteristic equation for E_2 is given as follows:

$$\left| lE - J_{E_2} \right| = l^4 + a_1 l^3 + a_2 l^2 + a_3 l + a_3 = 0. \tag{42}$$

Here, let $(d - q + 2q(x_2/x_m)) = N_4 \geq 0$,

$$\begin{aligned} a_1 &= a + \delta + \mu + N_4 + p z_2 + \frac{\beta v_2^2}{(x_2 + v_2)^2}, \\ a_2 &= N_4 \delta + \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) (\mu + a + p z_2) \\ &\quad + (\mu + b) p z_2 + \beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right), \\ a_3 &= b p z_2 (\mu + N_4) + N_4 \mu p z_2 \\ &\quad + \frac{\beta v_2^2}{(x_2 + v_2)^2} (b p z_2 + a\mu + \mu p z_2) \\ &\quad + N_4 \beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right), \\ a_4 &= \mu b p z_2 \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right). \end{aligned} \tag{43}$$

When $R_0 > 1$, $R_1 > 1$, we have $x_2^2/(x_2 + v_2)^2 < 1/R_0$, which ensures $a_n \geq 0$, $n = 1, 2, 3$. So,

$$\begin{aligned} \Delta_2 = {} & a^2 \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) + (\delta + \mu) \left(\beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right) \right. \\ & \left. + N_4 (\delta + \mu + N_4) \right) + p z_2 \left(N_4^2 + \beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right) \right. \\ & \left. + 2 (N_4 + \mu)(\delta + \mu) + b\delta \right) + p^2 z_2^2 (N_4 + \mu + b) \\ & + \frac{\beta v_2^2}{(x_2 + v_2)^2} \left(\beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right) + (\mu + p z_2) \right. \\ & \left. \cdot (\delta + \mu + p z_2) + N_4 (\delta + 2(\mu + p z_2)) \right) \\ & + a \left(N_4^2 + \beta k \left(\frac{1}{R_0} - \frac{x_2^2}{(x_2 + v_2)^2} \right) + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right. \\ & \cdot \left(\frac{\beta v_2^2}{(x_2 + v_2)^2} + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) + p z_2 \left(\frac{2\beta v_2^2}{(x_2 + v_2)^2} + \mu + c \right) \\ & \left. + 2 N_4 \left(\frac{\beta v_2^2}{(x_2 + v_2)^2} + \delta + \mu + p z_2 \right) \right) \\ & + \left(\frac{\beta v_2^2}{(x_2 + v_2)^2} \right)^2 (\mu + p z_2) \geq 0, \Bigg) \\ \Delta_3 > {} & a^3 \frac{\beta v_2^2}{(x_2 + v_2)^2} \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) \\ & + a^2 \left(\frac{\beta v_2^2 \mu}{(x_2 + v_2)^2} \left(N_4^2 + 2 N_4 \left(\frac{\beta v_2^2}{(x_2 + v_2)^2} + \delta + \mu \right) \right) \right) \\ & + \left(b \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right)^2 + p z_2 \left(\mu \left(\frac{b \beta v_2^2}{(x_2 + v_2)^2} \right. \right. \right. \\ & \left. \left. + \left(N_4 + \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) \left(N_4 + 3 \frac{\beta v_2^2}{(x_2 + v_2)^2} \right) \right) \right. \\ & \left. + \frac{\beta v_2^2 \mu^2}{(x_2 + v_2)^2} \right) \Bigg) + a \left(\frac{\beta v_2^2 \mu}{(x_2 + v_2)^2} \left(N_4 + \frac{\beta v_1^2}{(x_2 + v_2)^2} \right. \right. \\ & \left. \left. + \delta + \mu \right) \left(\frac{\beta v_2^2 \mu}{(x_2 + v_2)^2} + N_4 (\delta + \mu) \right) \right) \geq 0. \end{aligned} \tag{44}$$

According to Lemma 7, Theorem 13 can be proofed.

5. Optimal Control Problem for Combination of Traditional Chinese and Western Medicines

In this section, we analyse the optimal control problem for system (10). Considering the high cost and side effects of long-term use of the highest dose drugs and in order to achieve the treatment effect, we need to analyse the optimal

Table 1: The numerical value of the parameters for system.

Parameters	Value	Reference	Parameters	Value	Reference
λ	$6.67 \times 10^6 \times d$	[35]	p	1×10^{-4}	[35]
d	3.785×10^{-3}	[37]	k	1	[35]
β	1×10^{-4}	[10]	μ	0.35	[35]
δ	3.8×10^{-3}	[23]	c	0.95×10^{-7}	[35]
a	$3.38 \times d$	[35]	b	4.5×10^{-3}	[35]
q	0.3	[15]	x_m	1×10^8	[36]

control model to find optimal treatment strategy. Here, u_1, u_2, and u_3 represent the effects of the drugs ETV, TGJPJD, and TGYP, respectively. Let t_f be the endpoint of treatment, we choose the objective function of the optimal control as follows [33]:

$$J(u) = \frac{1}{2}\left(a_2 y^2 + a_3 v^2\right) + \frac{1}{2}\int_0^{t_f}\left(b_2 y^2 + b_3 v^2 + c_1 {u_1}^2 + c_2 {u_2}^2 + c_3 {u_3}^2 dt\right). \tag{45}$$

Here, $a_2, a_3, b_2, b_3, c_1, c_2, c_3$ represent the corresponding weight of each variable. Let $X(t) = (x(t), y(t), v(t), z(t))^T$, $u(t) = (u_1(t), u_2(t), u_3(t))^T$, $X(0) = (x_0, y_0, v_0, z_0)^T$, then (9) can be rewritten as follows:

$$\begin{cases} D^\alpha X(t) = f(X(t), u(t), t), \\ X(0) = X_0. \end{cases} \tag{46}$$

Then, its Hamilton function of $J(u)$ can be expressed as

$$\begin{aligned} H = {} & p_1\left(\lambda - dx - \frac{\beta x v}{x+v} + \delta y + qx\left(1 - \frac{x}{x_{\max}}\right)\right) \\ & + p_2\left(\frac{\beta x v}{x+v} - ay - \delta y - (1+u_3)pyz\right) \\ & + p_4((1+u_3)cyz - bz) + p_3((1-u_1)ky - (1+u_2)\mu v) \\ & + \left(b_2 y^2 + b_3 v^2 + c_1 {u_1}^2 + c_2 {u_2}^2 + c_3 {u_3}^2\right). \end{aligned} \tag{47}$$

Using Pontryagin's minimum principle, the necessary conditions for the existence of the optimal solution of system (10) are shown as follows:

$$\begin{cases} D^\alpha x = \lambda - dx - \dfrac{\beta x v}{x+v} + \delta y + qx\left(1 - \dfrac{x}{x_{\max}}\right), \\ D^\alpha y = \dfrac{\beta x v}{x+v} - ay - \delta y - (1+u_3)pyz, \\ D^\alpha v = (1-u_1)ky - (1+u_2)\mu v, \\ D^\alpha z = (1+u_3)cyz - bz, \end{cases}$$

$$X(0) = X_0,$$

$$\begin{cases} D^\alpha p_1 = p_1\left(q - d - \dfrac{\beta v^2}{(x+v)^2} - \dfrac{2qx}{x_{\max}}\right) + \dfrac{\beta v^2}{(x+v)^2} p_2, \\ D^\alpha p_2 = \delta p_1 - (a + \delta + (1+u_3)pz)p_2 + (1-u_1)kp_3 + (1+u_3)czp_4, \\ D^\alpha p_3 = -\dfrac{\beta x^2}{(x+v)^2} p_1 + \dfrac{\beta x^2}{(x+v)^2} p_2 - (1+u_2)\mu p_3, \\ D^\alpha p_4 = (1+u_3)pyp_2 - (cy - b)p_4, \end{cases}$$

$$\begin{cases} -kyp_3 + c_1 u_1 = 0, \\ -\mu p_3 v + c_2 u_2 = 0, \\ c_3 u_3 - p_2 pyz + p_4 cyz = 0, \end{cases}$$

$$\begin{aligned} p_1(t_f) &= 0, \\ p_2(t_f) &= a_2 y(t_f), \\ p_3(t_f) &= a_3 v(t_f), \\ p_4(t_f) &= 0. \end{aligned} \tag{48}$$

So, we get the optimal control as

$$\begin{cases} u_1 = \min\left\{\max\left\{\dfrac{kyp_3}{c_1}, 0\right\}, 1\right\}, \\ u_2 = \min\left\{\max\left\{\dfrac{\mu p_3 v}{c_2}, 0\right\}, 1\right\}, \\ u_3 = \min\left\{\max\left\{\dfrac{p_2 pyz - p_4 cyz}{c_3}, 0\right\}, 1\right\}. \end{cases} \tag{49}$$

6. Simulation

In this section, we use the numerical method provided in [34] to simulate our system (10). The parameters are given in Table 1.

According to the clinical trials [4], we choose 108 weeks as a treatment period, that is, the initial time $t_0 = 0$ and the end time $t_f = 7 \times 108$ (days). $u_1, u_2, u_3 \in [0, 1]$, by which 0 means no medicine treatment and 1 means the medicine is fully functional. Due to the fact that the human body cannot fully absorb all of the medicine, we assume a maximum drug effect of 0.98. In this simulation, we assume that there is a

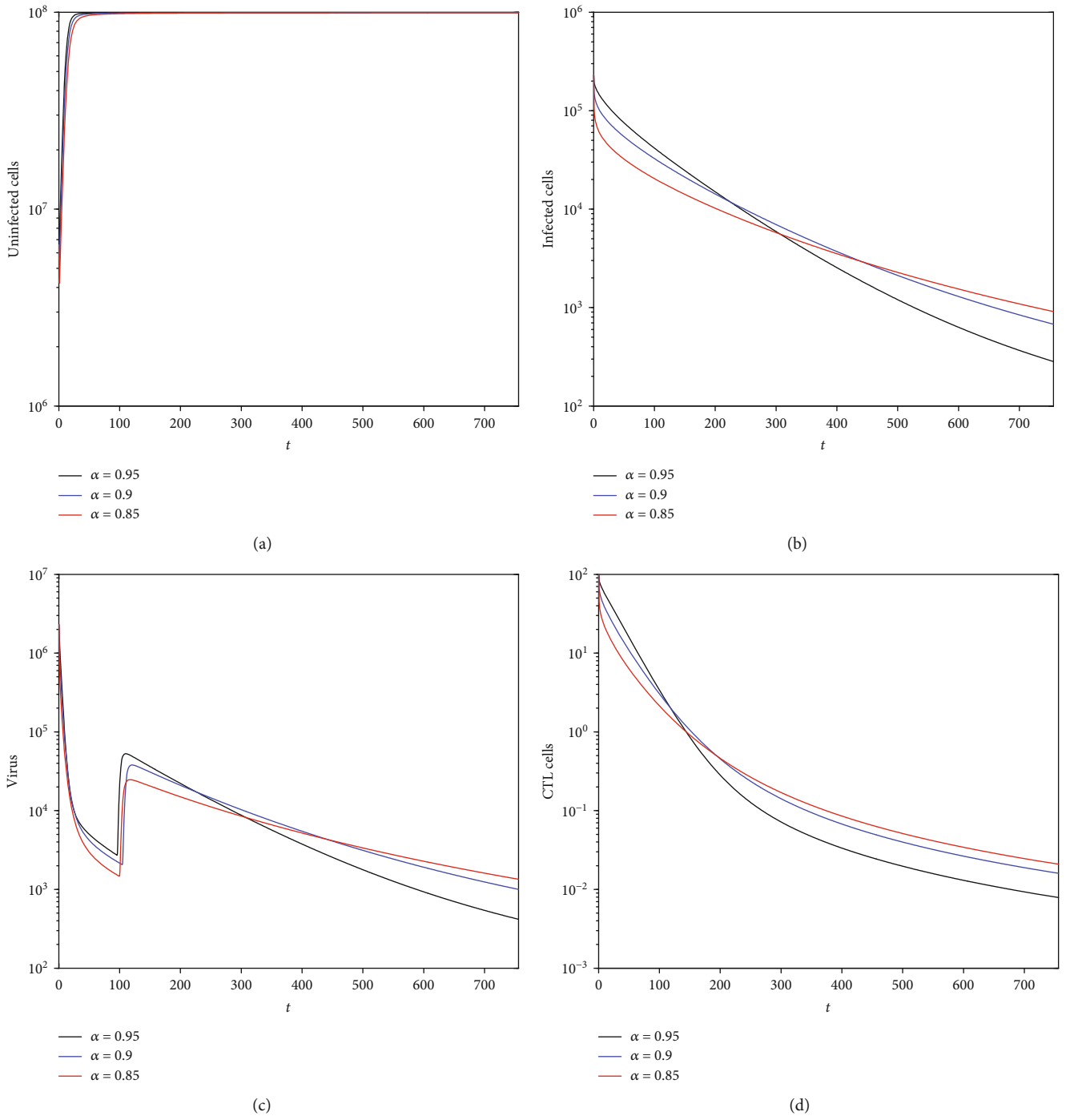

Figure 1: State dynamics of uninfected and infected cells, free virus, and CTL cells with Strategy 1.

positive correlation between the efficacy and the dose in a certain range. As $\alpha \longrightarrow 1$, the influence of memory decreases [37]. Here, we choose $\alpha = 0.95, 0.9$, and 0.85 to see the difference between different fractional orders with the lower memory effect.

In the following part, we will give optimal control strategies according to different treatment protocols.

Strategy 1. ETV monotherapy (i.e.,$u_1 \neq 0, u_2 = u_3 = 0$).

In this case, the objective function (45) is transformed to

$$J(u) = \frac{1}{2}\left(a_2 y^2 + a_3 v^2\right) + \frac{1}{2}\int_0^{t_f}\left(b_2 y^2 + b_3 v^2 + c_1 {u_1}^2\right)dt. \quad (50)$$

The initial condition is $(x_0 = 6.6 \times 10^6, y_0 = 2.3 \times 10^6, v_0 = 2.3 \times 10^7, z_0 = 200)$. We give the simulation with different orders $(\alpha = 0.95, 0.9, 0.85)$; the corresponding dynamic

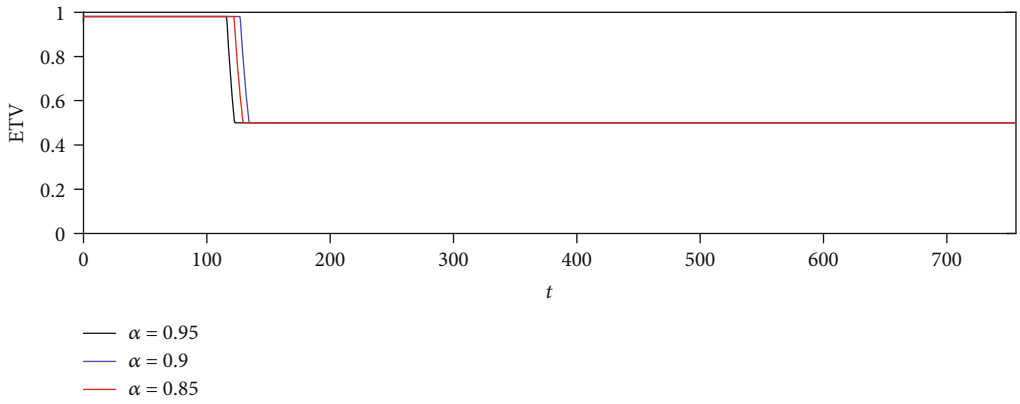

Figure 2: Optimal control Strategy 1 with ETV monotherapy and $\alpha = 0.95, 0.9, 0.85$.

Table 2: The corresponding number of objective function of Strategy 1 with different order and control strategies.

α	Optimal control	Maximum effect	Minimum effect
0.95	3.27×10^8	3.59×10^8	4.76×10^8
0.9	2.06×10^8	2.36×10^8	2.87×10^8
0.85	1.22×10^8	1.55×10^8	1.52×10^8

route simulation is shown in Figure 1, and the optimal control strategy is shown in Figure 2.

From the simulation in Figure 1, we can see that the change rate and the terminal value of infected cells, virus, and CTL are obviously different with different order α even with the same initial condition and the same change trend, which is consistent with the clinical fact that there do exist individual difference, so individual difference may be reflected by different α, and the fractional order model should be a good tool to describe HBV infection.

From the simulation in Figure 2, we find that the maximum dose should only be used at the earlier stage of treatment. And the drug dose can be lowered after a period of high-dose treatment. But the time point of lowering the drug dose is different with different order α. If we use constant control $u_1 = 0.98$ (maximum effect) and $u_1 = 0.5$ (minimum effect), we give the objective function value of the optimal control and constant control with different orders in Table 2. We can see that the objective function values are lower than those of the maximum and minimum constant control, which shows that the optimal therapy can get similar or better treatment effect with less drug dose and side effects.

Strategy 2. Combination of ETV and CHM (i.e.,$u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$).

In this section, we will give the optimal control strategies of combination therapy of ETV and Chinese medicine with different orders as $\alpha = 0.95, 0.9, 0.85$. We choose the same initial condition and the same parameters as those of Strategy 1 and the objective function (45). As mentioned before, ETV can block the replication of virus; we use u_1 to represent ETV treatment effect, and u_2 represents the treatment effect of TGJPJD, which can accelerate the death of virus. u_3 represents the treatment effect of TGYP which can enhance immunity. The simulations are shown in Figures 3 and 4.

From the simulation in Figure 3, we can see that dynamic routes are similar as that in Figure 1. But the terminal values of infected cells and virus are obviously lower than those in Figure 1, which show that the Chinese medicine can enhance the death of infected cells and virus. The simulation can also show that the individual difference may be reflected by different α.

From the simulation in Figure 4, we also find that the maximum dose should only be used at the earlier stage of treatment. The drug dose can be lowered after a period of high-dose treatment with different order α. And the time point of lowering the drug dose is also different with different order α. We also find that the time point of lowering the drug dose is different with different medicine; that is, ETV can be lowered at the earliest time, and then, TGJPJD and TGYP need to be taken at the maximum dose for the longest time which shows the importance of enhancing the immune in anti-HBV therapy. Moreover, for Strategy 1, the time point of lowering the ETV drug dose is about 100 days, but for strategy 2, the time point is only about 60 days.

We also compared the objective function values of constant control $u_1 = u_2 = u_3 = 0.98$ (maximum effect) and $u_1 = u_2 = u_2 = 0.5$ (minimum effect) with the optimal control. The results are shown in Table 3. We can see that the optimal objective function values are always the lowest for different orders. By comparing the optimal objective function values in Table 2 with those in Table 3, we find that combination of ETV and CHM can greatly reduce objective function values, which show that combination of ETV and CHM may be a good choice in anti-HBV infection therapy.

7. Conclusion and Discussion

In this paper, based on the combination therapy of traditional Chinese medicine and Western medicine, we proposed a fractional optimal control model of anti-HBV infection based on saturation incidence and logistic proliferation of

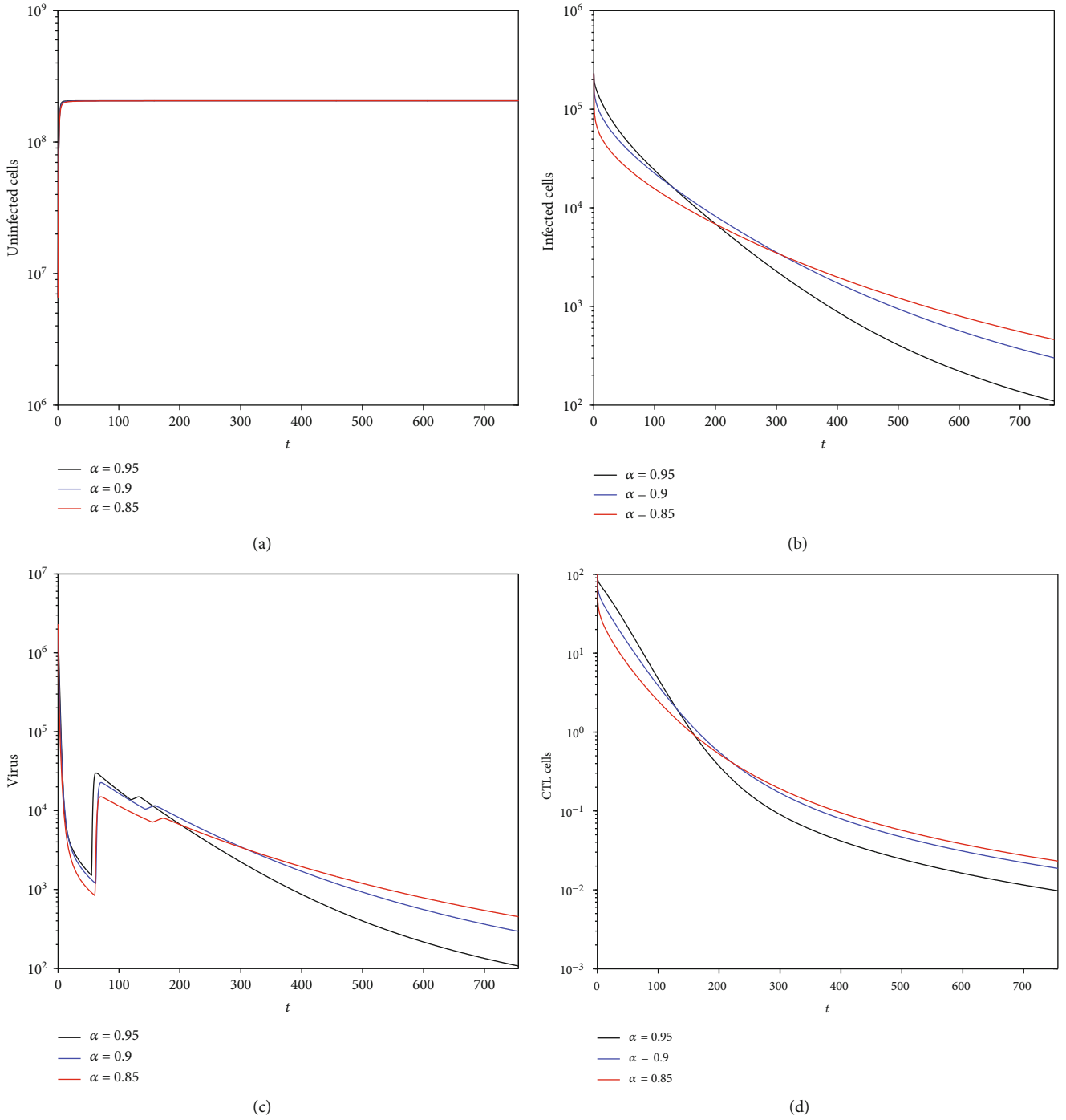

Figure 3: State dynamics of uninfected and infected cells, free virus, and CTL cells with Strategy 2.

uninfected cells for the first time. After giving the stable analysis of our system, we discussed the necessary conditions of the optimal control problem. Two optimal control strategies about ETV monotherapy and combination therapy of ETV and CHM with different fractional orders were given by simulation.

In the simulation, we suppose that there is a positive correlation between drug use and effectiveness. From the simulations, we know that the dynamic routes were different with different orders even with the same parameters, which may show that the individual difference could be reflected by different α.

By comparing the dynamic routes between ETV monotherapy and combination therapy of ETV and CHM, we found that combination therapy can not only obtain better treatment effect but also reduce the taking time and dose of ETV. The simulation shows that combination therapy may be a good choice in anti-HBV infection therapy. We also compared the objective function values of the optimal control strategies with other constant control strategies; the

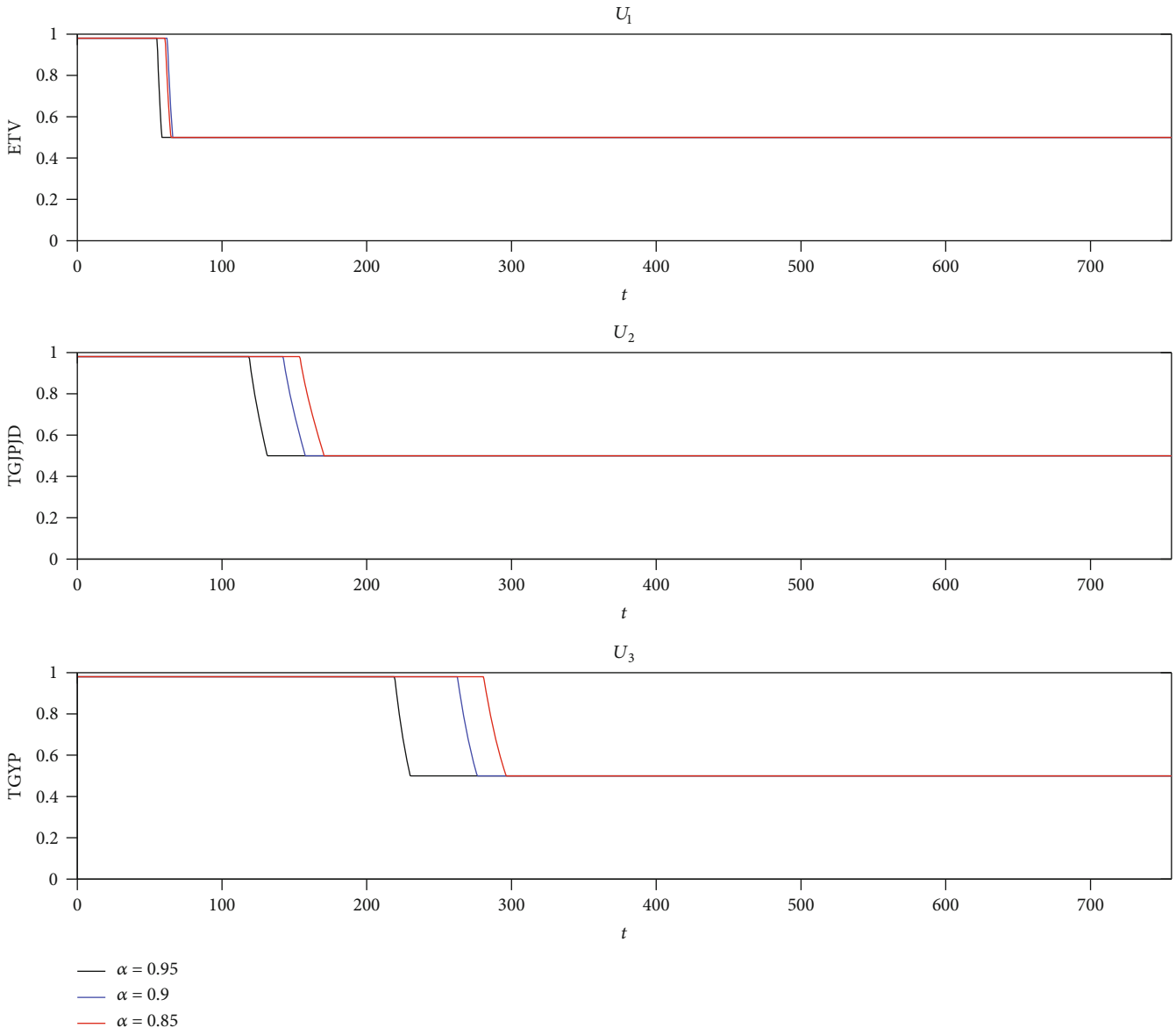

Figure 4: Optimal control strategy with ETV, TGJPJD, and TGYP and $\alpha = 0.95, 0.9, 0.85$.

Table 3: The corresponding number of objective function of Strategy 2 with different order and control strategies.

α	Optimal control	Maximum effect	Minimum effect
0.95	1.81×10^8	2.12×10^8	5.78×10^8
0.9	1.28×10^8	1.59×10^8	3.44×10^8
0.85	1.01×10^8	1.34×10^8	1.75×10^8

comparison showed the optimal therapy can get similar or better treatment effect with less drug dose and side effects.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

On the Exact Solutions of Two (3+1)-Dimensional Nonlinear Differential Equations

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In this article, exact solutions of two (3+1)-dimensional nonlinear differential equations are derived by using the complex method. We change the (3+1)-dimensional B-type Kadomtsev-Petviashvili (BKP) equation and generalized shallow water (gSW) equation into the complex differential equations by applying traveling wave transform and show that meromorphic solutions of these complex differential equations belong to class W, and then, we get exact solutions of these two (3+1)-dimensional equations.

1. Introduction and Main Results

Nonlinear differential equations (NLDEs) play an important role in the research of nonlinear science, which has attracted a lot of attentions of the researchers [1–8]. The investigation of NLDEs is helpful for well understanding of nonlinear physical phenomena [9–16]. Numerous methods have been developed for seeking traveling wave exact solutions to NLDEs, such as sine-Gordon expansion method [17], Kudryashov method [18], modified simple equation method [19], Jacobi elliptic function expansion [20], $\exp(-\psi(z))$-expansion method [21, 22], modified extended tanh method [23, 24], generalized (G'/G) expansion method [25], and improved F-expansion method [26].

In recent years, Yuan et al. [27] introduced an efficient method named complex method to get exact solutions for NLDEs. The complex method is developed by complex analysis and complex differential equations. More details about the complex method can be found in [28–34]. In this work, we will utilize the complex method to achieve exact solutions of the following two (3+1)-dimensional NLDEs.

The (3+1)-dimensional BKP equation [35] is given by

$$u_{xxxy} + \theta\left(u_x u_y\right)_x + \left(u_x + u_y + u_s\right)_t - \left(u_{xx} + u_{ss}\right) = 0, \quad (1)$$

where θ is a constant.

The (3+1)-dimensional gSW equation [36] is given by

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xs} = 0. \quad (2)$$

Class W consists of elliptic function or their degeneration. Substituting traveling wave transform

$$u(x, y, s, t) = U(z), \quad z = n_1 x + n_2 y + n_3 s + n_4 t, \quad (3)$$

into Eq. (1), and then integrating it we get

$$n_1^3 n_2 U''' + \theta n_1^2 n_2 \left(U'\right)^2 + \left(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2\right)U' + r = 0, \quad (4)$$

where r is the integration constant.

Theorem 1. *If $\theta n_1 n_2 \neq 0$, then meromorphic solutions w of Eq. (4) belong to class W and Eq. (4) has the following solutions where $c_i (i = 1, 2, 3, 4)$ are the integral constants.*

(i) The rational function solutions

$$U_r(z) = \frac{6n_1}{\theta}\frac{1}{z - z_0} + \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z - z_0) + c_1, \tag{5}$$

where $r = -(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2, z_0 \in \mathbb{C}$.

(ii) The simply periodic solutions

$$\begin{aligned} U_{1s}(z) = &-\frac{3n_1\mu}{\theta}\coth\frac{\mu(z - z_0)}{2} \\ &- \frac{3n_1\mu}{2\theta}\ln\left(\frac{\coth(\mu/2)(z - z_0) - 1}{\coth(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z - z_0) + c_2, \\ U_{2s}(z) = &-\frac{3n_1\mu}{\theta}\tanh\frac{\mu(z - z_0)}{2} \\ &- \frac{3n_1\mu}{2\theta}\ln\left(\frac{\tanh(\mu/2)(z - z_0) - 1}{\tanh(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z - z_0) + c_3, \end{aligned} \tag{6}$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

(iii) The elliptic function solutions

$$\begin{aligned} U_d(z) = &-\frac{6n_1}{\theta}[\zeta(z) - \zeta(z_0)] - \frac{3n_1}{\theta}\frac{\wp'(z) + G}{\wp(z) - H} \\ &+ \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z - z_0) + c_4, \end{aligned} \tag{7}$$

where $G^2 = 4H^3 - g_2 H - g_3$, $g_2 = 4n_1^2 n_2 r\theta + (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2/12n_1^6 n_2^2$, H and g_3 are arbitrary.

Substituting traveling wave transform

$$u(x, y, s, t) = V(z), z = m_1 x + m_2 y + m_3 s + m_4 t, \tag{8}$$

into Eq. (2), and then integrating it we get

$$m_1^3 m_2 V''' + 3m_1^2 m_2 \left(V'\right)^2 - (m_1 m_3 + m_2 m_4)V' + \lambda = 0, \tag{9}$$

where λ is the integration constant.

Theorem 2. *If $m_1 m_2 \neq 0$, then meromorphic solutions w of Eq. (9) belong to the class W and Eq. (9) has the following solutions where $c_i (i = 1, 2, 3, 4)$ are the integral constants.*

(i) The rational function solutions

$$V_r(z) = \frac{2m_1}{z - z_0} + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_1, \tag{10}$$

where $\lambda = (m_1 m_3 + m_2 m_4)^2/12m_1^2 m_2, z_0 \in \mathbb{C}$.

(ii) The simply periodic solutions

$$\begin{aligned} V_{1s}(z) = &-4m_1\mu\coth\frac{\mu(z - z_0)}{2} \\ &- 4m_1\mu\ln\left(\frac{\coth(\mu/2)(z - z_0) - 1}{\coth(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{2m_1^3 m_2\mu^2 + m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_2, \\ V_{2s}(z) = &-4m_1\mu\tanh\frac{\mu(z - z_0)}{2} \\ &- 4m_1\mu\ln\left(\frac{\tanh(\mu/2)(z - z_0) - 1}{\tanh(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{2m_1^3 m_2\mu^2 + m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_3, \end{aligned} \tag{11}$$

where $\lambda = (m_1 m_3 + m_2 m_4)^2 - m_1^6 m_2^2\mu^4/12m_1^2 m_2$, $z_0 \in \mathbb{C}$..

(iii) The elliptic function solutions

$$\begin{aligned} V_d(z) = &-2m_1[\zeta(z) - \zeta(z_0)] - m_1\frac{\wp'(z) + E}{\wp(z) - F} \\ &+ \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_4, \end{aligned} \tag{12}$$

where $E^2 = 4F^3 - g_2 F - g_3$, $g_2 = (m_1 m_3 + m_2 m_4)^2 - 12\lambda m_1^2 m_2/12m_1^6 m_2^2$, F and g_3 is arbitrary.

2. Preliminaries

Set $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$, $r_i \in \{0, 1, 2, \cdots\}$, $i = 0, 1, \cdots, m$, $r = (r_0, r_1, \cdots, r_m)$, and

$$K_r[U](z) := \prod_{i=0}^{m}\left[U^{(i)}(z)\right]^{r_i}. \tag{13}$$

The degree of $Kr[U]$ is defined as $d(r) := \sum_{i=0}^{m} r_i$. The differential polynomial is given by

$$P\left(U, U', \cdots, U^{(m)}\right) := \sum_{r \in J} a_r K_r[U], \tag{14}$$

where J is a finite index set, then $\deg P(U, U', \cdots, U^{(m)}) := \max_{r\in J}\{d(r)\}$ is the degree of $P(U, U', \cdots, U^{(m)})$.

Considering the following equation:

$$P\left(U, U', \cdots, U^{(m)}\right) = aU^n + d, \tag{15}$$

where $n \in \mathbb{N}$, and $a \neq 0$, d are constants.

Assume that meromorphic solutions w of Eq. (13) have at least one pole and let p, $q \in$ N. Substitute the Laurent series

$$U(z) = \sum_{k=-q}^{\infty} \beta_k z^k, \beta_{-q} \neq 0, q > 0, \tag{16}$$

into Eq. (15) to determine p distinct Laurent principal parts

$$\sum_{k=-q}^{-1} \beta_k z^k, \tag{17}$$

then, Eq. (15) is said to satisfy weak $<p, q>$ condition.

It is know that Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ has double periods and satisfies:

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{18}$$

Weierstrass zeta function $\zeta(z)$ is a meromorphic function which satisfies

$$\wp(z) = -\zeta'(z). \tag{19}$$

These two Weierstrass functions have the following addition formulas:

$$\begin{aligned} \wp(z - z_0) &= -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]^2 - \wp(z_0), \\ \zeta(z - z_0) &= \zeta(z) - \zeta(z_0) + \frac{1}{2}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]. \end{aligned} \tag{20}$$

Eremenko et al. [37] had investigated the following m -order Briot-Bouquet equation (BBEq)

$$P\left(U, U^{(m)}\right) = \sum_{j=0}^{n} P_j(U)\left(U^{(m)}\right)^j = 0, \tag{21}$$

in which $m \in$ N, and $P_j(U)$ are constant coefficient polynomials.

Lemma 1 [38–40]. *Let $m, n, p, s \in N$, $degP(U,U^{(m)}) < n$. If the m-order BBEq*

$$P\left(U, U^{(m)}\right) = aU^n + c \tag{22}$$

satisfies weak $<p, q>$ condition; then, meromorphic solutions w belong to class W. Assume that some values of parameters such solutions w exist; then, other meromorphic solutions should form 1 parameter family $U(z - z_0), z0 \in C$. In addition, each elliptic solution U with a pole at $z = 0$ is.

$$\begin{aligned} U(z) = &\sum_{i=1}^{s-1}\sum_{j=2}^{q} \frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+D_i}{\wp(z)-B_i}\right]^2 - \wp(z)\right) \\ &+ \sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+D_i}{\wp(z)-B_i} + \sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \end{aligned} \tag{23}$$

where β_{-ij} are determined by (16), $D_i^2 = 4B_i^3 - g_2B_i - g_3$ and $\sum_{i=1}^{s}\beta_{-i1} = 0$.

Each rational function solution is

$$R(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z - z_i)^j} + \beta_0, \tag{24}$$

which contains $s(\leq p)$ distinct poles of multiplicity q.

Each simply periodic solution is a rational function $R(\eta)$ of $\eta = e^{\alpha z}(\alpha \in \mathbb{C})$, that is

$$R(\eta) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta - \eta_i)^j} + \beta_0, \tag{25}$$

which contains $s(\leq p)$ distinct poles of multiplicity q.

3. Proofs of Main Results

Proof of Theorem 1. Let $u = U'$, then Eq. (4) becomes

$$n_1^3n_2u'' + \theta n_1^2n_2u^2 + \left(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2\right)u + r = 0. \tag{26}$$

Substituting (16) into Eq. (4), we have $p = 1, q = 2, \beta_{-2} = 6n_1/\theta, \beta_{-1} = 0, \beta_0 = n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2/2\theta n_1^2n_2, \beta_1 = 0, \beta_2 = 4n_1^2n_2r\theta + (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2/40\theta n_1^5n_2^2$, $\beta_3 = 0$, and β_4 is an arbitrary constant. Thus, Eq. (26) is a second-order BBEq as well as satisfies weak $h1, 2i$ condition. Therefore, by Lemma 1, we know that the meromorphic solutions of Eq. (26) belong to class W.

From (23) of Lemma 1, we have the form of elliptic solutions of Eq. (26)

$$u_{d0}(z) = \beta_{-2}\wp(z) + \beta_{10} \tag{27}$$

with pole at $z = 0$.

Put $u_{d0}(z)$ into Eq. (26) to yield

$$u_{d0}(z) = \frac{6n_1}{\theta}\wp(z) + \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2n_2}, \tag{28}$$

where $g_2 = 4n_1^2n_2r\theta + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/12n_1^6n_2^2$ and g_3 is arbitrary.

Therefore, the elliptic solutions of Eq. (26) with arbitrary pole are

$$u_d(z) = \frac{6n_1}{\theta}\wp(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}, \quad (29)$$

where $z_0 \in C$.

Therefore, the solutions of Eq. (4) are

$$\begin{aligned} U_d(z) &= \int u_d(z)dz = \int \left(\frac{6n_1}{\theta}\wp(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}\right)dz \\ &= \frac{6n_1}{\theta}\zeta(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_4 \\ &= -\frac{6n_1}{\theta}[\zeta(z) - \zeta(z_0)] - \frac{3n_1}{\theta}\frac{\wp'(z)+G}{\wp(z)-H} \\ &\quad + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_4, \end{aligned} \quad (30)$$

where $G^2 = 4H^3 - g_2H - g_3$, $g_2 = 4n_1^2n_2r\theta + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/12n_1^6n_2^2$, c_4 is the integral constant, and H and g_3 are arbitrary.

By (24), we infer that the indeterminant rational solutions of Eq.(26) are

$$R_1(z) = \frac{\beta_{11}}{z^2} + \frac{\beta_{12}}{z} + \beta_{20} \quad (31)$$

with pole at $z = 0$.

Substitute $R_1(z)$ into Eq. (26) to yield

$$R_1(z) = \frac{6n_1}{\theta}\frac{1}{z^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}, \quad (32)$$

where $r = -(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

So the rational solutions of Eq. (26) with arbitrary pole are

$$u_r(z) = \frac{6n_1}{\theta}\frac{1}{(z-z_0)^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}. \quad (33)$$

Therefore, the solutions of Eq. (4) are

$$\begin{aligned} U_r(z) &= \int u_r(z)dz = \int \left(\frac{6n_1}{\theta}\frac{1}{(z-z_0)^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}\right)dz \\ &= -\frac{6n_1}{\theta}\frac{1}{z-z_0} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_1, \end{aligned} \quad (34)$$

where c_1 is the integral constant, $r = -(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

Let $\eta = e^{\mu z}$. To obtain simply periodic solutions, we insert $u = R(\eta)$ into Eq. (26) and get

$$\begin{aligned} n_1^3 n_2 \mu^2 \left(\eta R' + \eta^2 R''\right) + \theta n_1^2 n_2 R^2 + (n_4(n_1+n_2+n_3) \\ - n_1^2 - n_3^2)R + r = 0. \end{aligned} \quad (35)$$

Substituting $R_2(z)$ into the Eq.(35), we obtain that

$$\begin{aligned} R_{21}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(\eta-1)^2} + \frac{6n_1}{\theta}\frac{\mu^2}{(\eta-1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2}, \end{aligned} \quad (36)$$

$$\begin{aligned} R_{22}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(\eta+1)^2} - \frac{6n_1}{\theta}\frac{\mu^2}{(\eta+1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2}, \end{aligned} \quad (37)$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

Substituting $\eta = e^{\mu z}$ into Eq. (36) and Eq. (37) yields simply periodic solutions to Eq. (26) with pole at $z = 0$

$$\begin{aligned} u_{1s0}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}-1)^2} + \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}-1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2} \\ &= \frac{6n_1}{\theta}\mu^2\frac{e^{\mu z}}{(e^{\mu z}-1)^2} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2} \\ &= \frac{3n_1}{2\theta}\mu^2 \coth^2\frac{\mu z}{2} \\ &\quad + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \\ u_{2s0}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}+1)^2} - \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}+1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2} \\ &= -\frac{6n_1}{\theta}\mu^2\frac{e^{\mu z}}{(e^{\mu z}+1)^2} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + \left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2}{2\theta n_1^2 n_2} \\ &= \frac{3n_1}{2\theta}\mu^2 \tanh^2\frac{\mu z}{2} \\ &\quad + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \end{aligned} \quad (38)$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

So simply periodic solutions of Eq. (4) with arbitrary pole are

$$u_{1s}(z) = \frac{3n_1}{2\theta}\mu^2 \coth^2\frac{\mu(z-z_0)}{2} + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \tag{39}$$

and

$$u_{2s}(z) = \frac{3n_1}{2\theta}\mu^2 \tanh^2\frac{\mu(z-z_0)}{2} + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \tag{40}$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

Therefore, the solutions of Eq. (4) are

$$\begin{aligned} U_{1s}(z) &= \int u_{1s}(z)dz = \int \left(\frac{3n_1}{2\theta}\mu^2 \coth^2\frac{\mu(z-z_0)}{2} + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}\right)dz \\ &= -\frac{3n_1\mu}{\theta}\coth\frac{\mu(z-z_0)}{2} - \frac{3n_1\mu}{2\theta}\ln\left(\frac{\coth(\mu/2)(z-z_0)-1}{\coth(\mu/2)(z-z_0)+1}\right) + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z-z_0) + c_2, \end{aligned} \tag{41}$$

and

$$\begin{aligned} U_{2s}(z) &= \int u_{2s}(z)dz = \int \left(\frac{3n_1}{2\theta}\mu^2 \tanh^2\frac{\mu(z-z_0)}{2} + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}\right)dz \\ &= -\frac{3n_1\mu}{\theta}\tanh\frac{\mu(z-z_0)}{2} - \frac{3n_1\mu}{2\theta}\ln\left(\frac{\tanh(\mu/2)(z-z_0)-1}{\tanh(\mu/2)(z-z_0)+1}\right) + \frac{\left(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z-z_0) + c_3, \end{aligned} \tag{42}$$

where c_2 and c_3 are the integral constants, $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

Proof of Theorem 2. Let $v = V'$, then Eq. (9) becomes

$$m_1^3 m_2 v'' + 3m_1^2 m_2 v^2 - (m_1 m_3 + m_2 m_4)v + \lambda = 0. \tag{43}$$

Substituting (16) into Eq.(9), we have $p = 1, q = 2, \beta_{-2} = -2m_1, \beta_{-1} = 0, \beta_0 = m_1 m_3 + m_2 m_4/6m_1^2 m_2, \beta_1 = 0, \beta_2 = 12\lambda m_1^2 m_2 - (m_1 m_3 + m_2 m_4)^2/120 m_1^5 m_2, \beta_3 = 0$, and β_4 is an arbitrary constant. Thus, Eq. (43) is a second-order BBEq as well as satisfies weak $h1, 2i$ condition. Therefore, by Lemma 1, we know that the meromorphic solutions of Eq. (43) belong to class W.

From (23) of Lemma 1, we have the form of elliptic solutions of Eq. (43)

$$v_{d0}(z) = \beta_{-2}\wp(z) + \beta_{10} \tag{44}$$

with pole at $z = 0$.

Put $v_{d0}(z)$ into Eq. (43) to yield

$$v_{d0}(z) = -2m_1\wp(z) + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}, \tag{45}$$

where $g_2 = (m_1 m_3 + m_2 m_4)^2 - 12\lambda m_1^2 m_2/12 m_1^6 m_2^2$ and g_3 is arbitrary.

Therefore, the elliptic solutions of Eq. (43) with arbitrary pole are

$$v_d(z) = -2m_1\wp(z - z_0) + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}, \tag{46}$$

where $z_0 \in C$.

Therefore, the solutions of Eq. (9) are

$$\begin{aligned} V_d(z) &= \int v_d(z)dz = \int \left(-2m_1\wp(z-z_0) + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}\right)dz \\ &= -2m_1\zeta(z-z_0) + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z-z_0) + c_4 \\ &= -2m_1[\zeta(z) - \zeta(z_0)] - m_1\frac{\wp'(z)+E}{\wp(z)-F} + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z-z_0) + c_4, \end{aligned} \tag{47}$$

where $E^2 = 4F^3 - g_2 F - g_3$, $g_2 = (m_1 m_3 + m_2 m_4)^2 - 12\lambda m_1^2 m_2/12 m_1^6 m_2^2$, c_4 is the integral constant, and F and g_3 are arbitrary.

By (24), we infer that the indeterminant rational solutions of Eq. (43) are

$$R_1(z) = \frac{\beta_{11}}{z^2} + \frac{\beta_{12}}{z} + \beta_{20} \tag{48}$$

with pole at $z = 0$.

Substitute $R_1(z)$ into Eq. (43) to yield

$$R_1(z) = -\frac{2m_1}{z^2} + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}, \tag{49}$$

where $\lambda = (m_1 m_3 + m_2 m_4)^2/12 m_1^2 m_2$.

So the rational solutions of Eq. (43) with arbitrary pole are

$$v_r(z) = -\frac{2m_1}{(z-z_0)^2} + \frac{m_1m_3 + m_2m_4}{6m_1^2m_2}. \tag{50}$$

Therefore, the solutions of Eq. (9) are

$$\begin{aligned} V_r(z) &= \int v_r(z)dz = \int \left(-\frac{2m_1}{(z-z_0)^2} + \frac{m_1m_3 + m_2m_4}{6m_1^2m_2}\right)dz \\ &= \frac{2m_1}{z-z_0} + \frac{m_1m_3 + m_2m_4}{6m_1^2m_2}(z-z_0) + c_1, \end{aligned} \tag{51}$$

where c_1 is the integral constant, $\lambda = (m_1m_3 + m_2m_4)^2/12 m_1^2m_2$, $z_0 \in \mathbb{C}$.

Let $\eta = e^{\mu z}$. To obtain simply periodic solutions, we insert $v = R(\eta)$ into Eq. (43) and get

$$m_1^3m_2\mu^2\left(\eta R' + \eta^2 R''\right) + 3m_1^2m_2R^2 - (m_1m_3 + m_2m_4)R + \lambda = 0. \tag{52}$$

Substituting $R_2(z)$ into the Eq. (35), we obtain that

$$R_{21}(z) = -\frac{2m_1\mu^2}{(\eta-1)^2} - \frac{2m_1\mu^2}{(\eta-1)} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2}, \tag{53}$$

$$R_{22}(z) = -\frac{2m_1\mu^2}{(\eta+1)^2} + \frac{2m_1\mu^2}{(\eta+1)} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2}, \tag{54}$$

where $\lambda = (m_1m_3 + m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

Substituting $\eta = e^{\mu z}$ into Eq. (53) and Eq. (54) yields simply periodic solutions to Eq. (43) with pole at $z = 0$

$$\begin{aligned} v_{1s0}(z) &= -\frac{2m_1\mu^2}{(e^{\mu z}-1)^2} - \frac{2m_1\mu^2}{(e^{\mu z}-1)} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -\frac{2m_1\mu^2e^{\mu z}}{(e^{\mu z}-1)^2} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -2m_1\mu^2 \coth^2\frac{\mu z}{2} + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}, \\ v_{2s0}(z) &= -\frac{2m_1\mu^2}{(e^{\mu z}+1)^2} + \frac{2m_1\mu^2}{(e^{\mu z}+1)} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= \frac{2m_1\mu^2e^{\mu z}}{(e^{\mu z}+1)^2} + \frac{m_1m_3 + m_2m_4 - m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -2m_1\mu^2 \tanh^2\frac{\mu z}{2} + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}, \end{aligned} \tag{55}$$

where $\lambda = (m_1m_3 + m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

So simply periodic solutions of Eq. (9) with arbitrary pole are

$$v_{1s}(z) = -2m_1\mu^2 \coth^2\frac{\mu(z-z_0)}{2} + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}, \tag{56}$$

and

$$v_{2s}(z) = -2m_1\mu^2 \tanh^2\frac{\mu(z-z_0)}{2} + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}, \tag{57}$$

where $\lambda = (m_1m_3 + m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

Therefore, the solutions of Eq. (9) are

$$\begin{aligned} V_{1s}(z) &= \int v_{1s}(z)dz = \int \left(-2m_1\mu^2 \coth^2\frac{\mu(z-z_0)}{2}\right. \\ &\quad \left. + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}\right)dz \\ &= -4m_1\mu \coth\frac{\mu(z-z_0)}{2} \\ &\quad - 4m_1\mu \ln\left(\frac{\coth(\mu/2)(z-z_0) - 1}{\coth(\mu/2)(z-z_0) + 1}\right) \\ &\quad + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}(z-z_0) + c_2, \end{aligned} \tag{58}$$

and

$$\begin{aligned} V_{2s}(z) &= \int v_{2s}(z)dz = \int \left(-2m_1\mu^2 \tanh^2\frac{\mu(z-z_0)}{2}\right. \\ &\quad \left. + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}\right)dz \\ &= -4m_1\mu \tanh\frac{\mu(z-z_0)}{2} \\ &\quad - 4m_1\mu \ln\left(\frac{\tanh(\mu/2)(z-z_0) - 1}{\tanh(\mu/2)(z-z_0) + 1}\right) \\ &\quad + \frac{2m_1^3m_2\mu^2 + m_1m_3 + m_2m_4}{6m_1^2m_2}(z-z_0) + c_3, \end{aligned} \tag{59}$$

where c_2 and c_3 are the integral constants, $\lambda = (m_1m_3 + m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$, $z_0 \in \mathbb{C}$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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Research Article

Existence of Solutions for Fractional Differential Equations with *p*-Laplacian Operator and Integral Boundary Conditions

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In this paper, we investigate a class of integral boundary value problems of fractional differential equations with a *p*-Laplacian operator. Existence of solutions is obtained by using the fixed point theorem, and an example is given to show the applicability of our main result.

1. Introduction

In this paper, we consider the nonlinear fractional differential equations with a *p*-Laplacian operator and integral boundary conditions

$$\begin{cases} {}^{c}D_{0+}^{\beta}\left(\phi_p({}^{c}D_{0+}^{\alpha}u(t))\right)+f(t,u(t))=0, \quad t\in[0,1], \\ u(1)=\lambda\int_0^1 u(s)ds, \quad u'(1)=0, \\ {}^{c}D_{0+}^{\alpha}u(1)=b{}^{c}D_{0+}^{\alpha}u(\xi), \end{cases} \tag{1}$$

where $1<\alpha\le 2$, $0<\beta\le 1$, $0<\xi$, b, $\lambda<1$, ${}^{c}D_{0+}^{\alpha}$ and ${}^{c}D_{0+}^{\beta}$ are the Caputo fractional derivative.

$\phi_p(s)=|s|^{p-2}s$ is the p-Laplacian operator such that $(1/p)+(1/q)=1$, $p>1$, and $\phi_p^{-1}(s)=\phi_q(s)$, and $f(t,u)$: $[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is a given continuous function.

In recent years, boundary value problems of fractional differential equations have significantly been discussed by some researchers because fractional calculus theory and methods have been widely used in various fields of natural sciences and social sciences. In the field of physical mechanics, fractional calculus not only provides suitable mathematical tools for the study of soft matter but also provides new research ideas and plays an irreplaceable role in the modeling of soft matter [1–3]. Some nonlinear analysis tools such as coincidence degree theory [4, 5], upper and lower solution method [6–8], fixed point theorems [9–11], and variational methods [12–14] have been widely used to discuss existence of solutions for boundary value problems of fractional differential equations.

On the other hand, it is well known that differential equation models with p-Laplacian operators are often used to simulate practical problems such as tides caused by celestial gravity and elastic deformation of beams and rich results of fractional differential equations with a p-Laplacian operator have been obtained [15–18]. In particular, in [15], by using the fixed point theorem, Yan et al. studied the existence of solutions for boundary value problems of fractional differential equations with a p-Laplacian operator:

$$\begin{cases} -D_{0+}^{\alpha}\left(\phi_p\left(D_{0+}^{\beta}u(t)\right)\right)=f(t,u(t)), \\ u(0)=u(1)=u'(0)=u'(1), \\ D_{0+}^{\beta}u(0)=0, \quad D_{0+}^{\beta}u(1)=bD_{0+}^{\beta}u(\eta), \end{cases} \tag{2}$$

where $1<\alpha\le 2$, $3<\beta\le 4$, $0<\eta<1$, $0<b<\eta^{(1-\alpha)/(p-1)}$, $(1/p)+(1/q)=1$, $1<p$, $\phi_p^{-1}(s)=\phi_q(s)$,D_{0+}^{α}, D_{0+}^{β} is the standard Riemann-Liouville derivative, and $f(t,u)$: $(0,1)\times(0,\infty)\longrightarrow[0,\infty)$ is a given countinuous function.

Moreover, during the last decade, the integral boundary value problem of fractional differential equations is also a

hot issue for scholars and some good results have been achieved [19–23]. In [24], by using the method of the upper and lower solutions and Schauder's and Banach's fixed points theorem, Abdo et al. obtained the existence and uniqueness of a positive solution of the fractional differential equations with integral boundary equations:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, 1], \\ u(0) = \lambda \int_0^1 u(s)ds + d, \end{cases} \tag{3}$$

where $0 < \alpha \le 1$, $\lambda \ge 0$, $d > 0$, ${}^{c}D_{0+}^{\alpha}$ is the standard Caputo derivative, and $f : [0, 1] \times [0,\infty) \longrightarrow [0,\infty)$ is a given countinuous function.

In [25], Bai and Qiu discuss the existence of positive solutions for boundary value problems of fractional differential equations:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1], \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{4}$$

where $2 < \alpha \le 3$, $f \in C([0, 1] \times [0,+\infty), [0,+\infty))$,and ${}^{c}D_{0+}^{\alpha}$ is the standard Caputo derivative.

Motivated by the works mentioned above, we concentrate on the solutions for the nonlinear fractional differential equation (1). We obtain the existence result of the fractional differential equations with integral boundary equations by using the Schauder fixed point theorem and other mathematical analysis techniques.

The rest of this paper is organized as follows. In Section 2, we give some notations and lemmas. Section 3 is devoted to study existence of solutions for boundary value problems of fractional differential equations. Finally, we provide an example to illustrate our results.

2. Preliminaries

In the section, we present some definitions and lemmas, which are required for building our theorems.

Definition 1 (see [1]). *The fractional integral of order $\alpha(\alpha > 0)$ of function $f : [0,\infty) \longrightarrow R$ is given by*

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}f(s)ds, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [2]). *The Caputo fractional derivative of order $\alpha(\alpha > 0)$ of function $f : [0,\infty) \longrightarrow R$ is given by*

$${}^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n - \alpha)}\int_0^t (t - s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{6}$$

where $t > 0$, $n = [\alpha] + 1$, $\Gamma(\alpha)$ is the Gamma function.

Lemma 3 (see [26]). *For $\alpha > 0$, the solution of fractional differential equation ${}^{c}D_{0+}^{\alpha}u(t) = 0$ is given by $u(t) = c_0 + c_1t + \cdots + c_{n-1}t^{n-1}$, $c_i \in R$, $i = 0, 1, 2, \cdots, n - 1$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.*

Lemma 4 (see [1]). *For $\alpha > 0$, then (i) $I_{0+}^{\alpha}({}^{c}D_{0+}^{\alpha}u(t)) = u(t) + c_0 + c_1t + \cdots + c_{n-1}t^{n-1}$, $c_i \in R$, $n = [\alpha] + 1$ and (ii) ${}^{c}D_{0+}^{\alpha}I_{0+}^{\alpha}u(t) = u(t)$.*

Lemma 5 (see [1]). *Let X be a Banach space and $\Omega \subset X$ a convex, closed, and bounded set. If $T : \Omega \longrightarrow \Omega$ is a continuous operator such that $T\Omega \subset X$, $T\Omega$ is relatively compact, then T has at least one fixed point in Ω.*

Let $\phi_p({}^{c}D_{0+}^{\alpha}u(t)) = v(t)$, then $v(1) = b^{p-1}v(\xi)$. We now consider the following equations:

$$\begin{cases} -{}^{c}D_{0+}^{\beta}v(t) = y(t), \quad t \in [0, 1], \\ v(1) = b^{p-1}v(\xi). \end{cases} \tag{7}$$

Lemma 6. *Let $y \in C[0, 1]$, then (7) has a unique solution*

$$v(t) = \int_0^1 H(t, \tau)y(\tau)d\tau, \tag{8}$$

where

$$H(t,\tau) = \begin{cases} \dfrac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \dfrac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le \tau \le t \le 1, \quad \xi \le \tau, \\ \dfrac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \dfrac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} - \dfrac{b^{p-1}(\xi-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \le \tau \le t \le 1, \quad \xi \le \tau, \\ \dfrac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \le t \le \tau \le 1, \quad \xi \le \tau, \\ \dfrac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \dfrac{b^{p-1}(\xi-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \le t \le \tau \le 1, \quad \xi \le \tau. \end{cases} \tag{9}$$

Proof. Suppose v satisfies boundary value problem (7), by (i) of Lemma 4, we can obtain

$$v(t) = -I_{0+}^{\beta} y(t) - c_0 = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds - c_0. \quad (10)$$

Using the boundary condition $v(1) = b^{p-1}v(\xi)$, we can obtain

$$c_0 = \frac{b^{p-1}}{\left(1-b^{p-1}\right)\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} y(s)ds - \frac{1}{\left(1-b^{p-1}\right)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)ds. \quad (11)$$

Thus,

$$v(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds - \frac{b^{p-1}}{\left(1-b^{p-1}\right)\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} y(s)ds + \frac{1}{\left(1-b^{p-1}\right)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)ds = \int_0^1 H(t,s)y(s)ds. \quad (12)$$

From the above analysis, the equation

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\left(\phi_p({}^{c}D_{0^+}^{\alpha}u(t))\right) + y(t) = 0, \\ u(1) = \lambda \int_0^1 u(s)ds, \quad u'(1) = 0, \\ {}^{c}D_{0^+}^{\alpha}u(1) = b\,{}^{c}D_{0^+}^{\alpha}u(\xi) \end{cases} \quad (13)$$

is equivalent to

$$\begin{cases} {}^{c}D_{0^+}^{\alpha}u(t) = \phi_q\left(\int_0^1 H(t,\tau)y(\tau)d\tau\right), \quad 0 < t < 1, \\ u'(1) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds. \end{cases} \quad (14)$$

Lemma 7. *Lety* $\in C[0,1]$. *Then (14) has a unique solution:*

$$u(t) = \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \lambda A, \quad (15)$$

where

$$A = \frac{1}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds\right]. \quad (16)$$

Proof. By (i) of Lemma (4), we can obtain $u(t) = (1/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} \phi_q(\int_0^1 H(s,\tau)y(\tau)d\tau)ds - c_1 - c_2 t$. Then, $u'(t) = (1/\Gamma(\alpha-1)) \int_0^t (t-s)^{\alpha-2} \phi_q(\int_0^1 H(s,\tau)y(\tau)d\tau)ds - c_2$, using the boundary condition $u'(1) = 0$, we can obtain

$$c_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds. \quad (17)$$

Another, because

$$u(1) = \lambda \int_0^1 u(s)ds, \quad (18)$$

$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - c_1 - c_2, \quad (19)$$

we have

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \lambda \int_0^1 u(s)ds. \quad (20)$$

Now, we express $\int_0^1 u(s)ds$, let

$$A = \int_0^1 u(t)dt = \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)dsdt - \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)dsdt + \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)dsdt - \int_0^1 \int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)dsdt + \int_0^1 \lambda A dt = \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \int_0^1 \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} \phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \lambda A. \quad (21)$$

We obtain

$$A = \frac{1}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \frac{1}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds\right]. \tag{22}$$

Therefore,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha-1)}\int_0^1 t(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \lambda A. \tag{23}$$

Reverse, if

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds - \frac{1}{\Gamma(\alpha-1)}\int_0^1 t(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)y(\tau)d\tau\right)ds + \lambda A, \tag{24}$$

by (ii) of Lemma (4), we can obtain that $u(t)$ is a solution of (14).

The proof is completed.

Lemma 8. *The functions H is continuous on $[0,1]\times[0,1]$ and has the following properties:*

(1) $H(t,\tau) \le H(\tau,\tau),\quad t,\tau \in [0,1]$

(2) $\int_0^1 H(\tau,\tau)d\tau \le (1/(1-b^{p-1})\Gamma(\beta+1))$

Proof. (1) For any $t,\tau\in[0,1]$, by (9), it is obvious that $H(t,\tau)\le H(\tau,\tau)$. (2) For any $t,\tau\in[0,1]$, by (9), we conclude that

$$H(\tau,\tau) \le \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}. \tag{25}$$

Therefore,

$$\int_0^1 H(\tau,\tau)d\tau \le \frac{1}{(1-b^{p-1})\Gamma(\beta+1)}. \tag{26}$$

This completes the proof.

3. Main Results

In this section, we will show the existence results for boundary value problem (1) by the Schauder fixed point theorem.

Let $I=[0,1]$, $U=\{u(t)\mid u(t)\in C(I)\}$, and definite the norm$\|u\| = \max_{t\in[0,1]}|u(t)|$, $(U,\|\bullet\|)$ is a Banach space.

Theorem 9. *Assume that the following conditions (H_1) and (H_2) are satisfied:*

(H_1) $f(t,u)$: $[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous

(H_2)There exists a constant $k>0$, satisfying $f(t,u)\le L\phi_p(u)$, $t\in[0,1]$, $\|u\|\le k$, where

$$0 < L \le \left[\phi_p\left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)}\right)\left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)}\right)^{-1}\right], \tag{27}$$

then the problem (1) has at least one solution.

Proof. Let $P=\{u(t)\mid \|u(t)\|\le k, t\in[0,1]\}$; thus, $P\subset U$ is convex, bounded, and closed.

Define an operator $T: P\longrightarrow U$ by

$$Tu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds - \frac{1}{\Gamma(\alpha-1)}\int_0^1 t(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds + \frac{\lambda}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds + \frac{1}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right]. \tag{28}$$

For any $u\in P$, then by (H_2), we have

$$f(t,u)\le L\phi_p(u)\le L\phi_p(k). \tag{29}$$

By Lemma (8), we conclude that

$$
\begin{aligned}
|Tu(t)| &\le \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\right.\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad +\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad -\frac{1}{\Gamma(\alpha-1)}\int_0^1 t(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad +\frac{\lambda}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\right.\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad \left.\left.+\frac{1}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)ds\right]\right|\\
&= \left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right.\\
&\quad -\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad +\frac{1}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad -\frac{t}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad +\frac{\lambda}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right.\\
&\quad -\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad \left.\left.+\frac{1}{2\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right]\right|\\
&\le \left|\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right.\\
&\quad -\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad +\frac{1}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad +\frac{\lambda}{1-\lambda}\left[\frac{1}{\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right.\\
&\quad -\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad \left.\left.+\frac{1}{2\Gamma(\alpha)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right]\right|\\
&= \left|\frac{2\alpha^2+2\alpha-\lambda\alpha^2-3\lambda\alpha}{2(1-\lambda)\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right|\\
&\le \left|\frac{(2-\lambda)(\alpha^2+\alpha)}{2(1-\lambda)\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)f(\tau,u(\tau))d\tau\right)\right|\\
&\le \left|\frac{(2-\lambda)(\alpha^2+\alpha)}{2(1-\lambda)\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)L\phi_p(u(\tau))d\tau\right)\right|\\
&\le \left|\frac{(2-\lambda)(\alpha^2+\alpha)}{2(1-\lambda)\Gamma(\alpha+2)}\phi_q\left(\int_0^1 H(\tau,\tau)L\phi_p(k)\right)d\tau\right)\right|\\
&\le \frac{(2-\lambda)(\alpha^2+\alpha)}{2(1-\lambda)\Gamma(\alpha+2)}\left|\phi_q\left(\int_0^1 H(\tau,\tau)d\tau\right)\right|\phi_q(L)k\\
&\le \frac{(2-\lambda)(\alpha^2+\alpha)}{2(1-\lambda)\Gamma(\alpha+2)}\left|\phi_q\left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)}\right)\right|\phi_q(L)k \le k.
\end{aligned}
\tag{30}
$$

Thus, $T(P)\subseteq P$. By (H_2), we have

$$
\begin{aligned}
&\left|\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right|\\
&\le \left|\phi_q\left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)}\right)\right| \le \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)}\right)^{q-1}.
\end{aligned}
\tag{31}
$$

We express

$$
M := \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)}\right)^{q-1}. \tag{32}
$$

For each $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we get

$$
\begin{aligned}
|Tu(t_2)-Tu(t_1)| &= \left|\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right.\right.\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\Big)\\
&\quad -\left(\frac{1}{\Gamma(\alpha-1)}\int_0^{t} t_2(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right.\\
&\quad \left.\left.-\frac{1}{\Gamma(\alpha-1)}\int_0^1 t_1(1-s)^{\alpha-2}\phi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds\right)\right|\\
&\le \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)}\right)^{q-1}\left|\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}-\frac{t_2-t_1}{\Gamma(\alpha)}\right|\\
&\le M\left|\frac{\left(t_2^{\alpha-1}+t_2^{\alpha-2}t_1+t_2^{\alpha-3}t_1^2+\cdots+t_2t_1^{\alpha-2}+t_1^{\alpha-1}\right)}{\Gamma(\alpha+1)}-\frac{1}{\Gamma(\alpha)}\right||t_2-t_1|.
\end{aligned}
\tag{33}
$$

As $t_2 \longrightarrow t_1$, the right-hand side of the previous inequality is independent of u and tends to zero; thus, $T(P)$ is equicontinuous. From the Arzela-Ascoli Theorem, T is compact. Applying Schauder's fixed point theorem, T has at least one fixed point $u \in P$. Therefore, the problem (1) has at least one positive solution u in P.

4. Applications

In this section, we will give an example to illustrate our main results.

Example 1. Consider the following equation:

$$
\begin{cases}
{}^cD_{0^+}^{1/2}\left(\phi_{4/3}\left({}^cD_{0^+}^{3/2}u(t)\right)\right) = \dfrac{\phi_p|u(t)|}{(t+5)^2}, \quad t\in(0,1),\\
u(1) = \dfrac{1}{2}\displaystyle\int_0^1 u(s)ds,\ u'(1)=0, \quad {}^cD_{0^+}^{\alpha}u(1) = \dfrac{1}{2}{}^cD_{0^+}^{\alpha}u\left(\dfrac{1}{2}\right),
\end{cases}
\tag{34}
$$

where $\beta = 1/2$, $\alpha = 3/2$, $p = 4/3$, $q = 4$, $\lambda = \xi = b = 1/2$, and $f(t,u) = \phi_p|u(t)|/(t+5)^2$. Since f is continuous and

$$
|f(t,u)| \le \frac{1}{25}\phi_p|u(t)|, \tag{35}
$$

for $(t, u) \in [0, 1] \times [0,\infty)$, we obtain

$$L = \frac{1}{25}. \tag{36}$$

Then,

$$\left[\phi_p\left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)}\right)\left(\frac{1}{\left(1-b^{p-1}\right)\Gamma(\beta+1)}\right)^{-1}\right] \approx 0.1534. \tag{37}$$

It is obvious that

$$L < \left[\phi_p\left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)}\right)\left(\frac{1}{\left(1-b^{p-1}\right)\Gamma(\beta+1)}\right)^{-1}\right] \approx 0.1534. \tag{38}$$

By Theorem (9), we conclude that the problem (34) has at least one solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors made an equal contribution.

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Research Article

Nonexistence of Global Weak Solutions of a System of Nonlinear Wave Equations with Nonlinear Fractional Damping

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We consider the system of nonlinear wave equations with nonlinear time fractional damping $\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^{\alpha}(t^{\sigma}|u|^q) = |v|^p, t > 0, x \in \mathbb{R}^N, \\ v_{tt} + (-\Delta)^m v + {}^C D_{0,t}^{\beta}(t^{\delta}|v|^r) = |v|^s, t > 0, x \in \mathbb{R}^N, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), x \in \mathbb{R}^N, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), x \in \mathbb{R}^N, \end{cases}$ where $(u, v) = (u(t,x), v(t,x))$, m and N are positive natural numbers, $p, q, r, s > 1$, $\sigma, \delta \geq 0$, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^{\kappa}$, $0 < \kappa < 1$, is the Caputo fractional derivative of order κ. Namely, sufficient criteria are derived so that the system admits no global weak solution. To the best of our knowledge, the considered system was not previously studied in the literature.

1. Introduction

In this paper, we investigate the system of nonlinear wave equations with nonlinear time fractional damping:

$$\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^{\alpha}(t^{\sigma}|u|^q) = |v|^p, & t > 0, x \in \mathbb{R}^N, \\ v_{tt} + (-\Delta)^m v + {}^C D_{0,t}^{\beta}\left(t^{\delta}|v|^r\right) = |v|^s, & t > 0, x \in \mathbb{R}^N, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $(u, v) = (u(t,x), v(t,x))$, m and N are positive natural numbers, $p, q, r, s > 1$, σ and δ are nonnegative numbers that will be specified later, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^{\kappa}$, $0 < \kappa < 1$, is the Caputo fractional derivative (with respect to t) of order κ. Namely, we are interested in obtaining sufficient conditions for which the considered system admits no global weak solution. Our approach is based on the nonlinear capacity method (see, e.g., [1]).

Before we state and prove our result, let us dwell on existence literature. Single wave equations or systems of wave equations have been studied in large; we may mention the books of Lions [2], Reed [3], Georgiev [4], and Strauss [5] and the papers of Aliev et al. [6], Said-Houari [7], Takeda [8], Goergiev and Todorova [9], Todorova and Yordanov [10], Zhang [11], and Kirane and Qafsaoui [12] for equations and systems with classical linear or nonlinear damping and Tatar [13], Kirane and Tatar [14], and Kirane and Laskri [15] for wave equations with fractional damping. In particular, in [13], the following problem was considered:

$$\begin{cases} u_{tt} + {}^C D_{(0,t)}^{\alpha} u - \Delta u = a|u|^{(p-1)}u, & t > 0, x \in \Omega, \\ u(t,x) = 0, t > 0, & x \in \partial\Omega, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $a > 0$, $p > 1$, and $0 < \alpha < 1$. It was shown that, if u is a solution to (2), then there exist $T^* \leq \infty$ and sufficiently large

initial data so that u blows up at T^*. Problem (2) was also considered in [14]. Namely, it was shown that the solution of (2) is unbounded and grows up exponentially in the L^{p+1} -norm for sufficiently large initial data. In [15], the following problem was studied:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\beta/2} D^{\alpha}_{(0,t)} u = |u|^p, & t > 0, x \in \mathbb{R}^N, \\ (u(0,x), u_t(0,x)) = u_0(x), u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{3}$$

where $p > 1$, $1 \le \beta \le 2$, $(-\Delta)^{\beta/2}$ is the fractional Laplacian of order $\beta/2$, $0 < \alpha < 1$, and $D^{\alpha}_{0,t}$ is the Riemann-Liouville fractional derivative of order α. It was shown that for all $p > 1$, if $\liminf_{|x| \to +\infty} u_1(x) = +\infty$,, then (3) does not admit a local weak solution for any $T > 0$.

In the next section, we recall some notions on fractional calculus. In Section 3, we define global weak solutions of system (1) and state our main result. Moreover, as a consequence, we deduce a nonexistence result in the case of a single equation. Finally, the proof of the main result is given in Section 4.

2. Preliminaries

Some preliminaries on fractional calculus (see, e.g., [16–21]) are provided in this section. Given $\varsigma > 0$, the fractional integrals $I^{\varsigma}_0 f$ and $I^{\varsigma}_T f$ of $f \in L^1[0, T]$ are defined by

$$\begin{aligned} (I^{\varsigma}_0 f)(t) &= \frac{1}{\Gamma(\varsigma)} \int_0^t (t - \xi)^{\zeta - 1} f(\xi) d\xi, \quad t \in [0, T] \text{a.e.}, \\ (I^{\varsigma}_T f)(t) &= \frac{1}{\Gamma(\varsigma)} \int_t^T (\xi - t)^{\zeta - 1} f(\xi) d\xi, \quad t \in [0, T] \text{a.e.}, \end{aligned} \tag{4}$$

where Γ denotes the gamma function.

Lemma 1. *Let $\varsigma > 0$, $f \in L^1[0, T]$, and $g \in L^{\infty}[0, T]$. Then,*

$$\int_0^T (I^{\varsigma}_0 f)(t) g(t) dt = \int_0^T f(t) (I^{\varsigma}_T g)(t) dt. \tag{5}$$

Given $0 < \varsigma < 1$, the Caputo fractional derivative ${}^{C}D^{\varsigma}_0 f$ of $f \in AC[0, T]$ is defined by

$$\left({}^{C}D^{\varsigma}_0 f\right)(t) = \left(I^1_0 - \varsigma\right) f'(t), \quad t \in [0, T] a.e. \tag{6}$$

The following lemma will be used later in the proof of our main result.

Lemma 2. *Let $0 < \varsigma < 1$ and $A : [0, T] \to \mathbb{R}$ be the function defined by*

$$A(t) = T^{-\gamma} (T - t)^{\gamma}, \tag{7}$$

where $\gamma > 0$. Then,

$$\left(I^{1-\varsigma}_T A\right)(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(2 - \varsigma + \gamma)} T^{-\gamma} (T - t)^{\gamma + 1 - \varsigma}, \tag{8}$$

$$\left(I^{1-\varsigma}_T A\right)(t) = \frac{-\Gamma(\gamma + 1)}{\Gamma(1 - \varsigma + \gamma)} T^{-\gamma} (T - t)^{\gamma - \varsigma}, \tag{9}$$

for all $t \in (0, T)$.

Proof. For $0 < t < T$, one has

$$\begin{aligned} \left(I^{1-\varsigma}_T A\right)(t) &= \frac{1}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} A(\xi) d\xi \\ &= \frac{T^{-\gamma}}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} (T - \xi)^{\gamma} d\xi \\ &= \frac{T^{-\gamma} (T - t)^{\gamma}}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} \left(1 - \frac{\xi - t}{T - t}\right)^{\gamma} d\xi. \end{aligned} \tag{10}$$

Taking $\rho = (\xi - t)/(T - t)$, one obtains (8). Next, (9) follows immediately from (8).

3. Main Result

We begin with the definition of the intended solutions of system (1). Given $0 < T < \infty$, let $\mathcal{Q}_T = [0, T] \times \mathbb{R}^N$ and Φ_T be the set of functions $\varphi = \varphi(t, x) \in C^{2,2m}_{t,x}(\mathcal{Q}_T)$ satisfying the following conditions:

(a) $\varphi(T, \cdot) = \varphi_t(T, \cdot) \equiv 0$

(b) $\operatorname{supp}_x \varphi \subset \subset \mathbb{R}^N$, i.e., $\operatorname{supp}_x \varphi(t, x) \subset K \subset \mathbb{R}^N$ uniformly in $t \in [0, T]$, where K is a compact

(c) $\left(I^{1-\kappa}_T \varphi\right)_t \in L^{\infty}(\mathcal{Q}_T)$, for all $0 < \kappa < 1$

Definition 3. Let $u_i, v_i \in L^1_{\text{loc}}(\mathbb{R}^N)$, $i = 0, 1$. A pair of functions

$$(u, v) \in L^{\max\{q,s\}}_{\text{loc}}\left([0,\infty) \times \mathbb{R}^N\right) \times L^{\max\{p,r\}}_{\text{loc}}\left([0,\infty) \times \mathbb{R}^N\right) \tag{11}$$

is said to be a global weak solution of system (1), if for all $0 < T < \infty$ and $\varphi \in \Phi_T$,

$$\begin{aligned} &\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx \\ &\quad = \int_{\mathcal{Q}_T} u \varphi_{tt} \, dx \, dt + \int_{\mathcal{Q}_T} u (-\Delta)^m \varphi \, dx \, dt \\ &\qquad - \int_{\mathcal{Q}_T} t^{\sigma} |u|^q \left(I^{1-\alpha}_T \varphi\right)_t dx dt, \end{aligned} \tag{12}$$

$$\int_{\mathcal{Q}_T} |u|^s \varphi \, dx\, dt + \int_{\mathbb{R}^N} \varphi(0,x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0,x) v_0 \, dx = \int_{\mathcal{Q}_T} v \varphi_{tt} \, dx\, dt + \int_{\mathcal{Q}_T} v(-\Delta)^m \varphi \, dx\, dt - \int_{\mathcal{Q}_T} t^\delta |v|^r \left(I_T^{1-\beta} \varphi\right)_t \, dx\, dt. \tag{13}$$

Next, we introduce the parameters

$$\begin{aligned}
\rho_1 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-1}{p}\right) - 2(s+1),\\
\rho_2 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-r}{p}\right) - 2s + \delta - \beta,\\
\rho_3 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-q}{p}\right) + s(\sigma-\alpha) - 2q,\\
\rho_4 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-rq}{p}\right) + s(\sigma-\alpha) + q(\delta-\beta)
\end{aligned} \tag{14}$$

$$\begin{aligned}
\bar{\rho}_1 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-1}{s}\right) - 2(p+1),\\
\bar{\rho}_2 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-q}{s}\right) - 2p + \sigma - \alpha,\\
\bar{\rho}_3 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-r}{s}\right) + p(\delta-\beta) - 2r,\\
\bar{\rho}_4 &= \left(\frac{N}{m} + 1\right)\left(\frac{ps-rq}{s}\right) + p(\delta-\beta) + r(\sigma-\alpha).
\end{aligned} \tag{15}$$

Here, m and N are positive natural numbers, $p, q, r, s > 1$, σ and δ are positive numbers that will be specified later, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^{\kappa}$, $0 < \kappa < 1$.

Theorem 4. *Let $0 < \alpha, \beta < 1$, $\sigma, \delta \geq 0$, $s > q > 1$, and $p > r > 1$. Suppose that $u_i, v_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$, and*

$$\min\left\{\int_{\mathbb{R}^N} u_1(x)\, dx, \int_{\mathbb{R}^N} v_1(x)\, dx\right\} > 0. \tag{16}$$

If

$$\min\left\{\max\{\rho_i : i = 1,2,3,4\}, \max\{\bar{\rho}_i : i = 1,2,3,4\}\right\} < 0, \tag{17}$$

then there exists no global weak solution of system (1).

Consider now the case of a single equation. Namely,

$$\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^{\alpha}\left(t^\sigma |u|^q\right) = |u|^p, & t > 0, x \in \mathbb{R}^N,\\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \end{cases} \tag{18}$$

where $0 < \alpha < 1$, $\sigma \geq 0$, and $p, q > 1$. A global weak solution of (18) can be defined in a similar way as in Definition 3. Taking in Theorem 4 $u = v$, $\alpha = \beta$, $\sigma = \delta$, $q = r$, $p = s$, and $(u_0, u_1) = (v_0, v_1)$, one deduces the following corollary.

Corollary 5. *Let $p > q > 1$ and $u_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$. Suppose that*

$$\int_{\mathbb{R}^N} u_1(x) dx > 0. \tag{19}$$

If

$$\frac{N}{m} < \min\left\{\frac{p+1}{p-1}, \frac{p^2 + (\alpha-\sigma)p + q}{p^2 - q}, \frac{(\alpha-\sigma-1)p^2 + 2pq + q}{p^2 - q}, \cdot \frac{(\alpha-\sigma-1)p + q}{p-q}\right\}, \tag{20}$$

then there exists no global weak solution of problem (18).

Example 6. Consider the equation

$$\begin{cases} u_{tt} + (-\Delta)^4 u + {}^C D_{0,t}^{2/3}\left(|u|^2\right) = |u|^3, & t > 0, x \in \mathbb{R}^2,\\ (u(0,x), u_t(0,x)) = \left(e^{-|x|}, |x|^2 e^{-|x|}\right), & x \in \mathbb{R}^2. \end{cases} \tag{21}$$

Observe that (21) is a special case of (18) with $m = 4$, $N = 2$, $\alpha = 2/3$, $\sigma = 0$, $q = 2$, $p = 3$, $u_0(x) = e^{-|x|}$, and $u_1(x) = |x|^2 e^{-|x|}$. One has $p > q > 1$, $u_0, u_1 \in L^1(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} u_1(x)\, dx = \int_{\mathbb{R}^2} |x|^2 e^{-|x|} dx > 0. \tag{22}$$

On the other hand, one can check easily that

$$\min\left\{\frac{p+1}{p-1}, \frac{p^2 + (\alpha-\sigma)p + q}{p^2 - q}, \frac{(\alpha-\sigma-1)p^2 + 2pq + q}{p^2 - q}, \cdot \frac{(\alpha-\sigma-1)p + q}{p-q}\right\} = 1. \tag{23}$$

Since

$$\frac{N}{m} = \frac{2}{4} < 1, \tag{24}$$

it follows from Corollary 5 where (21) admits no global weak solution.

4. Proof of Theorem 4

Before proving Theorem 4, we need some preliminary results.

Given $0 < T < \infty$, let

$$\varphi(t,x) = A(t)B(x), \quad (t,x) \in \mathcal{Q}_T, \tag{25}$$

where A is defined by (7) with $\gamma \gg 1$ and

$$B(x) = \Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{\mu}, \quad x \in \mathbb{R}^N, \tag{26}$$

where $\mu \gg 1$, $\ell > 0$ is a certain parameter that will be specified later, and $\Xi \in C_0^{\infty}([0,\infty))$ is a decreasing function satisfying

$$\Xi(\rho) = \begin{cases} 1 \text{ if } 0 \le \rho \le 1, \\ 0 \text{ if } \rho \ge 2. \end{cases} \tag{27}$$

Lemma 7. *For all $0 < T < \infty$, the function φ defined by (25) belongs to Φ_T.*

Proof. One can check easily that $\varphi \in C_{t,x}^{2,2m}(\mathcal{Q}_T)$ and satisfies conditions (a) and (b). On the other hand, by (9), for all $0 < \kappa < 1$, one obtains

$$\begin{aligned} \left(I_T^{1-\kappa}\varphi\right)_t(t,x) &= B(x)\left(I_T^{1-\kappa}A\right)_t(t) \\ &= \frac{-\Gamma(\gamma+1)}{\Gamma(1-\kappa+\gamma)}T^{-\gamma}(T-t)^{\gamma-\kappa}B(x) \\ &:= -C(\gamma,\kappa)T^{-\gamma}(T-t)^{\gamma-\kappa}B(x), \quad (t,x) \in \mathcal{Q}_T, \end{aligned} \tag{28}$$

which yields

$$\left|\left(I_T^{1-\kappa}\varphi\right)_t(t,x)\right| \le C(\gamma,\kappa)T^{-\kappa}, \quad (t,x) \in \mathcal{Q}_T. \tag{29}$$

This shows that φ satisfies condition (c).

The following estimate follows from elementary calculations.

Lemma 8. *There exists a constant $C > 0$ such that*

$$\left|\Delta^m\left[\Xi\left(|y|^{2m}\right)\right]^{\mu}\right| \le C\Xi\left(|y|^{2m}\right)^{\mu-2m}, \quad y \in \mathbb{R}^N, \tag{30}$$

for any positive natural number m.

Proof of Theorem 4. Suppose that

$$(u,v) \in L_{\mathrm{loc}}^{\max\{q,s\}}\left([0,\infty)\times\mathbb{R}^N\right) \times L_{\mathrm{loc}}^{\max\{p,r\}}\left([0,\infty)\times\mathbb{R}^N\right) \tag{31}$$

is a global weak solution of system (1). For $T \gg 1$, using (12) and Lemma 7, one obtains

$$\begin{aligned} &\int_{\mathcal{Q}_T} |v|^p\varphi\,dx\,dt + \int_{\mathbb{R}^N}\varphi(0,x)u_1\,dx - \int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx \\ &\quad\le \int_{\mathcal{Q}_T}|u||\varphi_{tt}|dx\,dt + \int_{\mathcal{Q}_T}|u||\Delta^m\varphi|dx\,dt \\ &\qquad + \int_{\mathcal{Q}_T} t^{\sigma}|u|^q\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|dx\,dt, \end{aligned} \tag{32}$$

where φ is defined by (25). Similarly, using (13), one obtains

$$\begin{aligned} &\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt + \int_{\mathbb{R}^N}\varphi(0,x)v_1\,dx - \int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx \\ &\quad\le \int_{\mathcal{Q}_T}|v||\varphi_{tt}|dx\,dt + \int_{\mathcal{Q}_T}|v||\Delta^m\varphi|dx\,dt \\ &\qquad + \int_{\mathcal{Q}_T} t^{\delta}|v|^r\left|\left(I_T^{1-\beta}\varphi\right)_t\right|dx\,dt. \end{aligned} \tag{33}$$

Using Hölder's inequality, the following holds:

$$\begin{aligned} \int_{\mathcal{Q}_T}|u||\varphi_{tt}|dx\,dt &\le \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{1/s} \\ &\quad\cdot\left(\int_{\mathcal{Q}_T}\varphi^{-1/(s-1)}|\varphi_{tt}|^{s/(s-1)}\,dx\,dt\right)^{(s-1)/s}. \end{aligned} \tag{34}$$

Similarly, one has

$$\begin{aligned} \int_{\mathcal{Q}_T}|u||\Delta^m\varphi|dx\,dt &\le \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{1/s} \\ &\quad\cdot\left(\int_{\mathcal{Q}_T}\varphi^{-1/(s-1)}|\Delta^m\varphi|^{s/(s-1)}\,dx\,dt\right)^{(s-1)/s}, \end{aligned} \tag{35}$$

$$\begin{aligned} \int_{\mathcal{Q}_T} t^{\sigma}|u|^q\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|dx\,dt &\le \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{q/s} \\ &\quad\cdot\left(\int_{\mathcal{Q}_T} t^{\sigma s/(s-q)}\varphi^{-q/(s-q)}\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|^{s/(s-q)}dx\,dt\right)^{(s-q)/s}. \end{aligned} \tag{36}$$

It follows from (32), (34), (35), and (36) that

$$\begin{aligned} &\int_{\mathcal{Q}_T}|v|^p\varphi\,dx\,dt + \int_{\mathbb{R}^N}\varphi(0,x)u_1\,dx - \int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx \\ &\quad\le \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{1/s}\left(\int_{\mathcal{Q}_T}\varphi^{-1/(s-1)}|\varphi_{tt}|^{s/(s-1)}\,dx\,dt\right)^{(s-1)/s} \\ &\qquad + \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{1/s} \\ &\qquad\cdot\left(\int_{\mathcal{Q}_T}\varphi^{-1/(s-1)}|\Delta^m\varphi|^{s/(s-1)}\,dx\,dt\right)^{(s-1)/s} \\ &\qquad + \left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{q/s} \\ &\qquad\cdot\left(\int_{\mathcal{Q}_T} t^{\sigma s/(s-q)}\varphi^{-q/(s-q)}\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|^{s/(s-q)}dx\,dt\right)^{(s-q)/s}. \end{aligned} \tag{37}$$

Similarly, using (33) and Hölder's inequality, one obtains

$$\begin{aligned}\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt+\int_{\mathbb{R}^N}\varphi(0,x)v_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx\\ \le\left(\int_{\mathcal{Q}_T}|v|^p\varphi\,dx\,dt\right)^{1/p}\left(\int_{\mathcal{Q}_T}\varphi^{-1/(p-1)}|\varphi_{tt}|^{p/(p-1)}\,dx\,dt\right)^{(p-1)/p}\\ +\left(\int_{\mathcal{Q}_T}|v|^p\varphi\,dx\,dt\right)^{1/p}\left(\int_{\mathcal{Q}_T}\varphi^{-1/(p-1)}|\Delta^m\varphi|^{p/(p-1)}\,dx\,dt\right)^{(p-1)/p}\\ +\left(\int_{\mathcal{Q}_T}|v|^p\varphi\,dx\,dt\right)^{r/p}\left(\int_{\mathcal{Q}_T}t^{\delta p/(p-r)}\varphi^{-r/(p-r)}\left|\left(I_T^{1-\beta}\varphi\right)_t\right|^{p/(p-r)}dx\,dt\right)^{(p-r)/p}.\end{aligned}\tag{38}$$

Now, set

$$\begin{aligned}I&=\left(\int_{\mathcal{Q}_T}|u|^s\varphi\,dx\,dt\right)^{1/s},\\ J&=\left(\int_{\mathcal{Q}_T}|v|^p\varphi\,dx\,dt\right)^{1/p}.\end{aligned}\tag{39}$$

For $\iota>1$, let

$$\begin{aligned}K_\iota&=\left(\int_{\mathcal{Q}_T}\varphi^{-1/(\iota-1)}|\varphi_{tt}|^{\iota/(\iota-1)}\,dx\,dt\right)^{(\iota-1)/\iota},\\ L_\iota&=\left(\int_{\mathcal{Q}_T}\varphi^{-\iota/(\iota-1)}|\Delta^m\varphi|^{\iota/(\iota-1)}\,dx\,dt\right)^{(\iota-1)/\iota}.\end{aligned}\tag{40}$$

For $\zeta\ge0$, $0<\kappa<1$, and $a>b>1$, let

$$M_{\zeta,\kappa,a,b}=\left(\int_{\mathcal{Q}_T}t^{\zeta a/(a-b)}\varphi^{-b/(a-b)}\left|\left(I_T^{1-\kappa}\varphi\right)_t\right|^{a/(a-b)}dx\,dt\right)^{(a-b)/a}.\tag{41}$$

From (37) and (38), we may write

$$\begin{aligned}J^p+\int_{\mathbb{R}^N}\varphi(0,x)u_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx\\ \le I(K_s+L_s)+I^qM_{\sigma,\alpha,s,q},\end{aligned}\tag{42}$$

$$\begin{aligned}I^s+\int_{\mathbb{R}^N}\varphi(0,x)v_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx\\ \le J\left(K_p+L_p\right)+J^rM_{\delta,\beta,p,r}.\end{aligned}\tag{43}$$

On the other hand, from (25), for $i=0,1$, one has

$$\int_{\mathbb{R}^N}\varphi(0,x)u_i\,dx=\int_{\mathbb{R}^N}\Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{\mu}u_i\,dx.\tag{44}$$

Since $u_i\in L^1(\mathbb{R}^N)$, $i=0,1$, by the dominated convergence theorem, one obtains

$$\lim_{T\to\infty}\int_{\mathbb{R}^N}\varphi(0,x)u_i\,dx=\int_{\mathbb{R}^N}u_i\,dx<\infty.\tag{45}$$

Similarly, since $v_i\in L^1(\mathbb{R}^N)$, $i=0,1$, the following holds:

$$\lim_{T\to\infty}\int_{\mathbb{R}^N}\varphi(0,x)v_i\,dx=\int_{\mathbb{R}^N}v_i\,dx<\infty.\tag{46}$$

Again, from (25), one has

$$-\int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx=T^{-1}\int_{\mathbb{R}^N}\varphi(0,x)u_0\,dx,\tag{47}$$

$$-\int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx=T^{-1}\int_{\mathbb{R}^N}\varphi(0,x)v_0\,dx.\tag{48}$$

Using (16), (45), and (47), one obtains

$$\begin{aligned}\lim_{T\to\infty}\int_{\mathbb{R}^N}\varphi(0,x)u_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx\\ =\int_{\mathbb{R}^N}u_1\,dx>0.\end{aligned}\tag{49}$$

Similarly, using (16), (46), and (48), one obtains

$$\lim_{T\to\infty}\int_{\mathbb{R}^N}\varphi(0,x)v_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx=\int_{\mathbb{R}^N}v_1\,dx>0.\tag{50}$$

Consequently, one has

$$\int_{\mathbb{R}^N}\varphi(0,x)u_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)u_0\,dx>0,\quad T\gg1,\tag{51}$$

$$\int_{\mathbb{R}^N}\varphi(0,x)v_1\,dx-\int_{\mathbb{R}^N}\varphi_t(0,x)v_0\,dx>0,\quad T\gg1.\tag{52}$$

Next, it follows from (42) and (51) that

$$J^p\le I(K_s+L_s)+I^qM_{\sigma,\alpha,s,q}.\tag{53}$$

Similarly, (43) and (52) yield

$$I^s\le J\left(K_p+L_p\right)+J^rM_{\delta,\beta,p,r}.\tag{54}$$

Using the inequality

$$(w+z)^\iota\le2^{\iota-1}(w^\iota+z^\iota),\quad w,z\ge0,\iota>1,\tag{55}$$

it follows from (53) that

$$\begin{aligned}J^{ps}&\le2^{s-1}\left[I^s(K_s+L_s)^s+I^{qs}M^s_{\sigma,\alpha,s,q}\right]\\ &\le2^{s-1}\left[2^{s-1}I^s(K_s^s+L_s^s)+I^{qs}M^s_{\sigma,\alpha,s,q}\right],\end{aligned}\tag{56}$$

where upon

$$J^{ps} \le 2^{2s-2}\left[\left(K_s^s + L_s^s\right)I^s + I^{qs}M^s_{\sigma,\alpha,s,q}\right]. \tag{57}$$

Similarly, it follows from (54) that

$$I^{qs} \le 2^{2q-2}\left[\left(K_p^q + L_p^q\right)J^q + J^{rq}M^q_{\delta,\beta,p,r}\right]. \tag{58}$$

Using (54), (57), and (58), the following holds:

$$\begin{aligned} J^{ps} \le C \times \Bigg[& \left(K_p + L_p\right)\left(K_s^s + L_s^s\right)J + \left(K_s^s + L_s^s\right)M_{\delta,\beta,p,r}J^r \\ & + M^s_{\sigma,\alpha,s,q}\left(K_p^q + L_p^q\right)J^q + M^s_{\sigma,\alpha,s,q}M^q_{\delta,\beta,p,r}J^{rq}\Bigg], \end{aligned} \tag{59}$$

where $C > 0$ is a constant that depends only on s and q. Here and below, any positive constant independent of T is denoted by C. Next, using theε-Young inequality with$\varepsilon > 0$which is small enough, one deduces from (59) that

$$\begin{aligned} J^{ps} \le C \times \Bigg[& \left(K_p + L_p\right)^{ps/(ps-1)}\left(K_s^s + L_s^s\right)^{ps/(ps-1)} \\ & + \left(K_s^s + L_s^s\right)^{ps/(ps-r)}M_{\delta,\beta,p,r}^{ps/(ps-r)} \\ & + \left(M^s_{\sigma,\alpha,s,q}\left(K_p^q + L_p^q\right)\right)^{ps/(ps-q)} \\ & + \left(M^s_{\sigma,\alpha,s,q}M^q_{\delta,\beta,p,r}\right)^{ps/(ps-rq)}\Bigg]. \end{aligned} \tag{60}$$

Similarly, one obtains

$$\begin{aligned} I^{ps} \le C \times \Bigg[& \left(K_s + L_s\right)^{ps/(ps-1)}\left(K_p^p + L_p^p\right)^{ps/(ps-1)} \\ & + \left(K_p^p + L_p^p\right)^{ps/(ps-q)}M_{\sigma,\alpha,s,q}^{ps/(ps-q)} \\ & + \left(M^p_{\delta,\beta,p,r}\left(K_s^r + L_s^r\right)\right)^{ps/(ps-r)} \\ & + \left(M^p_{\delta,\beta,p,r}M^r_{\sigma,\alpha,s,q}\right)^{ps/(ps-rq)}\Bigg]. \end{aligned} \tag{61}$$

Further, we shall estimate K_ι and L_ι for $\iota > 1$ and $M_{\zeta,\kappa,a,b}$ for $\zeta > 0$, $0 < \kappa < 1$, and $a > b > 1$. From (25), one has

$$\begin{aligned} K_\iota &= \left(\int_{\mathbb{Q}_T} \varphi^{-1/(\iota-1)}\left|\varphi_{tt}\right|^{\iota/(\iota-1)}\,dx\,dt\right)^{(\iota-1)/\iota} \\ &= \left(\int_0^T A(t)^{-1/(\iota-1)}\left|A_{tt}(t)\right|^{\iota/(\iota-1)}\,dt\right)^{(\iota-1)/\iota} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} B(x)\,dx\right)^{(\iota-1)/\iota} \\ &= C\left(T^{-\gamma}\int_0^T (T-t)^{\gamma-(2\iota/(\iota-1))}\,dt\right)^{(\iota-1)/\iota} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{\mu}dx\right)^{(\iota-1)/\iota} \\ &= CT^{-((i+1)/i)}\left(T^{\ell N}\int_{0<|y|^{2m}<2} \Xi\left(|y|^{2m}\right)dy\right)^{(\iota-1)/\iota}, \end{aligned} \tag{62}$$

which yields

$$K_\iota = CT^{\ell N-1-((\ell N+1)/i)},\ \iota > 1. \tag{63}$$

Similarly, using (25) and Lemma 8, one obtains

$$\begin{aligned} L_\iota &= \left(\int_{\mathbb{Q}_T} \varphi^{-1/(\iota-1)}\left|\Delta^m\varphi\right|^{\iota/(\iota-1)}\,dx\,dt\right)^{(\iota-1)/\iota} \\ &= \left(\int_0^T A(t)\,dt\right)^{(\iota-1)/\iota}\left(\int_{\mathbb{R}^N} B(x)^{-1/(\iota-1)}\left|\Delta^m B(x)\right|^{\iota/(\iota-1)}\,dx\right)^{(\iota-1)/\iota} \\ &= \left(\int_0^T T^{-\gamma}(T-t)^{\gamma}\,dt\right)^{(\iota-1)/\iota} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{-\mu/(\iota-1)}\left|\Delta^m\Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{\mu}\right|^{\iota/(\iota-1)}dx\right)^{(\iota-1)/\iota} \\ &= CT^{(\iota-1)/\iota}\left(T^{\ell N}\int_{\mathbb{R}^N} E\left(|y|^{2m}\right)^{-\mu/(\iota-1)}\left|T^{-2\ell m}\Delta^m E\left(|y|^{2m}\right)^{\mu}\right|^{\iota/(\iota-1)}dy\right)^{(\iota-1)/\iota} \\ &= CT^{(\iota-1)/\iota}T^{((\iota-1)/\iota)(\ell N-(2\ell m\iota/(\iota-1)))}\left(\int_{1<|y|^{2m}<2} E\left(|y|^{2m}\right)^{-\mu/(\iota-1)}\right. \\ &\quad \left.\cdot\left|\Delta^m E\left(|y|^{2m}\right)^{\mu}\right|^{\iota/(\iota-1)}dy\right)^{(\iota-1)/\iota} \\ &\le CT^{\ell N-2\ell m+1-((\ell N+1)/i)}\left(\int_{1<|y|^{2m}<2} E\left(|y|^{2m}\right)^{\mu-(2m\iota/(\iota-1))}dy\right)^{(\iota-1)/\iota}, \end{aligned} \tag{64}$$

so

$$L_\iota \le CT^{\ell N-2\ell m+1-((\ell N+1)/i)},\quad \iota > 1. \tag{65}$$

Next, using (9) and (25), for $\zeta \ge 0$, $0 < \kappa < 1$, and $a > b > 1$, one obtains

$$\begin{aligned} M_{\zeta,\kappa,a,b} &= \left(\int_{\mathbb{Q}_T} t^{\zeta a/(a-b)}\varphi^{-b/(a-b)}\left|\left(I_T^{1-\kappa}\varphi\right)_t\right|^{a/(a-b)}dx\,dt\right)^{(a-b)/a} \\ &= \left(\int_0^T t^{\zeta a/(a-b)}A(t)^{-b/(a-b)}\left|\left(I_T^{1-\kappa}A\right)(t)\right|^{a/(a-b)}dt\right)^{(a-b)/a} \\ &\quad \cdot \left(\int_{\mathbb{R}^N}\Xi\left(\frac{|x|^{2m}}{T^{2\ell m}}\right)^{\mu}dx\right)^{(a-b)/a} \\ &= C\left(T^{-\gamma}\int_0^T t^{\zeta a/(a-b)}(T-t)^{\gamma-(\kappa a/(a-b))}\,dt\right)^{(a-b)/a} \\ &\quad \cdot \left(T^{\ell N}\int_{0<|y|^{2m}<2}\Xi\left(|y|^{2m}\right)^{\mu}dy\right)^{(a-b)/a}, \end{aligned} \tag{66}$$

whereupon

$$M_{\zeta,\kappa,a,b} = CT^{((a-b)/a)(\ell N+1)+\zeta-\kappa}, \quad \zeta \geq 0, 0 < \kappa < 1, a > b > 1. \tag{67}$$

Next, taking $\ell = 1/m$, it follows from (63), (65), and (67) that

$$K_\iota = L_\iota = O\left(T^{((N/m)+1)(((\iota-1)/\iota)-2)}\right), \quad \iota > 1, T \gg 1, \tag{68}$$

$$M_{\zeta,\kappa,a,b} = CT^{((a-b)/a)((N/m)+1)+\zeta-\kappa}, \quad \zeta \geq 0, 0 < \kappa < 1, a > b > 1. \tag{69}$$

Using (60), (68), and (69), the following holds:

$$J^p \leq C\left(T^{\rho_1(ps/(ps-1))} + T^{\rho_2(ps/(ps-r))} + T^{\bar{\rho}_3(ps/(ps-q))} + T^{\rho_4(ps/(ps-rq))}\right)^{1/s}, \tag{70}$$

where ρ_i, $i = 1, 2, 3, 4$, are defined by (14). Similarly, using (61), (68), and (69), one obtains

$$I^s \leq C\left(T^{\bar{\rho}_1(ps/(ps-1))} + T^{\bar{\rho}_2(ps/(ps-q))} + T^{\bar{\rho}_3(ps/(ps-r))} + T^{\bar{\rho}_4(ps/(ps-rq))}\right)^{1/p}, \tag{71}$$

where $\bar{\rho}_i$, $i = 1, 2, 3, 4$, are defined by (15).

Now, from condition (17), we have two cases.

Case 1. If $\max\{\rho_i : i = 1, 2, 3, 4\} < 0$, in this case, taking the infimum limit as $T \to \infty$ in (70) and using Fatou's lemma, the following holds:

$$v(t, x) = 0, (t, x) \in [0,\infty) \times \mathbb{R}^N \text{ a.e.} \tag{72}$$

Hence, by (33), the following holds:

$$\int_{\mathbb{R}^N} \varphi(0, x)v_1\, dx - \int_{\mathbb{R}^N} \varphi_t(0, x)v_0\, dx \leq 0, \quad T \gg 1, \tag{73}$$

which contradicts (52).

Case 2. If $\max\{\bar{\rho}_i : i = 1, 2, 3, 4\} < 0$, as in the previous case, taking the infimum limit as $T \to \infty$ in (71), one obtains

$$u(t, x) = 0, (t, x) \in [0,\infty) \times \mathbb{R}^N \text{a.e.} \tag{74}$$

Hence, by (32), the following holds:

$$\int_{\mathbb{R}^N} \varphi(0, x)u_1\, dx - \int_{\mathbb{R}^N} \varphi_t(0, x)u_0\, dx \leq 0, \quad T \gg 1, \tag{75}$$

which contradicts (51).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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Research Article

A Fixed-Point Theorem for Ordered Contraction-Type Decreasing Operators in Banach Space with Lattice Structure

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In this work, we mainly improve the results in Amini-Harandi and Emami (2010). By introducing a new kind of ordered contraction-type decreasing operator in Banach space, we obtain a unique fixed point by using the iterative algorithm. An example is also presented to illustrate the theorem.

1. Introduction

In this work, we obtain a unique fixed point for a kind of ordered contraction-type decreasing operator in Banach space by using the iterative algorithm. The fixed-point study is mainly focused on two aspects. On the one hand, it is about the research of contraction-type mapping, for example, in [1–8]. On the other hand, it is about the study of monotone operators with concavity and convexity, for example, in [9–15]. There is little research on operators that only satisfy the partial-order constrictions. Applications of operator theory in fractional differential equations can be seen in [16–43].

The following generalization of Banach's contraction principle is due to Geraghty [44].

Lemma 1. *Let (M, d) be a complete metric space and let $f : M \longrightarrow M$ be a map. Suppose there exists $\beta \in \zeta$ such that for each $x, y \in M$,*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \tag{1}$$

where ζ denotes the class of those functions $\beta : [0, \infty) \longrightarrow [0, 1)$ which satisfy the condition $\beta(t_n) \longrightarrow 1 \Rightarrow t_n \longrightarrow 0$. Then f has a unique fixed-point $z \in M$, and $\{f^n(x)\}$ converges to z, for each $x \in M$.

Very recently, Amini-Harandi and Emami [6] proved the following existence theorem which is a version of Lemma 1 in the context of partially ordered complete metric spaces:

Lemma 2. *Let $(M, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in M such that (M, d) is a complete metric space. Let $f : M \longrightarrow M$ be an increasing map such that there exists an element $x_0 \in M$ with $x_0 \preccurlyeq f(x_0)$. Suppose that there exists $\beta \in \zeta$ such that*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \quad x, y \in M, y \leq x. \tag{2}$$

Assume that either f is continuous or M is such that if an increasing sequence $\{x_n\} \longrightarrow x$ in M, then $x_n \preccurlyeq x, \forall n$.

Besides, if for each $x, y \in M$, there exists $z \in M$, which is comparable to x and y.

Then f has a unique fixed point.

We found that in (2), the contraction is concerning a metric. But in fact, the relation of partial order does not play any role in (2). That is to say, in (1), the constriction $y \preccurlyeq x$ is not effective because $d(x, y) = d(y, x)$. A question appears naturally in the authors' minds: "Can the contraction condition be merely about partial order so that $y \preccurlyeq x$ can be

effective?" The authors have been haunted by this question since it was found. Driven by this idea, we introduce a new kind of ordered contraction-type decreasing operator in Banach space with lattice structure and obtain a unique fixed point of the operator. Our results are helpful and meaningful for studies of fixed point. Comparing to [6], our improvements are in three aspects.

First, the contraction is merely about partial order, and the relation of partial-order $y \preccurlyeq x$ does play an important role in the contraction condition. This has never been seen. Second, we consider the situation when the operator is decreasing. Third, we only use the iterative algorithm, and we can start the iterative process with any initial point, i.e., we do not need any assumptions of the existence of upper or lower solutions. An example is also presented to illustrate the theorem.

The outline of this paper is as follows. In the remainder of this section, we will give some preliminaries. In Section 2 of this paper, we present the existence and uniqueness theorem. In Section 3, an example is illustrated.

Definition 3 (see [45]). Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is called a cone if

(i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$

(ii) $x \in P, -x \in P \Rightarrow x = \theta$; θ is the zero element in E

In the case that P is a given cone in a real Banach space $(E, \|.\|)$, a partial order "$\leq$" can be induced on E by $x \leq y \Leftrightarrow y - x \in P$. The cone P is called normal if there exists a constant $N > 0$, such that for all $x, y \in E$, $\theta \leq x \leq y$ implies that $\|x\| \leq N\|y\|$. Details about cones and fixed point of operators can be found in [45, 46].

Definition 4 (see [47, 48]). We call a set $X \subset E$ a lattice under the partial ordering $\leq$, if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for arbitrary $x, y \in X$.

Lemma 5 (see [45]). *A cone P is normal if and only if there exists a norm $\|.\|_1$ in E which is equivalent to $\|.\|$ such that for any $\theta \leq x \leq y$, $\|x\|_1 \leq \|y\|_1$, i.e., $\|.\|_1$ is monotone. The equivalence of $\|.\|$ and $\|.\|_1$ means that there exist $M > m > 0$ such that $m\|.\|_1 \leq \|.\| \leq M\|.\|_1$.*

Lemma 6 (see [45]). *Let P be a normal cone in a real Banach space E. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{n_i}\}$ converging to x^*, then $\{x_n\}$ also converges to x^*. Moreover, if $\{x_n\}$ is an increasing sequence, then $\{x_n\} \leq x^* (n = 1, 2, 3, \cdots)$; if $\{x_n\}$ is a decreasing sequence, then $x * \leq \{x_n\} (n = 1, 2, 3, \cdots)$.*

2. The Main Results

We suppose that E is a partially ordered Banach space. P is a normal cone. The partial-order "$\leq$" on E is induced by the cone P.

Let ζ denote the class of those functionals $\beta : P \longrightarrow [0, 1)$ which satisfy the condition

$$\begin{aligned} &\beta(w_n) \longrightarrow 1 \Rightarrow w_n \\ &\quad \longrightarrow \theta \text{ for any monotonic sequence } \{w_n\} \subset P \end{aligned} \tag{3}$$

Theorem 7 (main theorem). *Suppose that $X \subset E$ is a closed subset, $P \subset X$. X is a lattice. $A : X \longrightarrow X$ is a decreasing operator and satisfies the following ordered contraction condition:*

(H) Suppose that there exists $\beta \in \zeta$ such that

$$Au - Av \leq \beta(v - u)(v - u), \quad \forall u, v \in X, u \leq v. \tag{4}$$

Then A has unique fixed-point $u_* \in X$. Moreover, constructing successively sequence

$$u_n = Au_{n-1} (n = 1, 2, \cdots), \tag{5}$$

for any initial $u_0 \in X$, we have

$$\lim_{n \to \infty} u_n = u_*. \tag{6}$$

Remark 8. Here, we study the decreasing operator while most of the contractions are about increasing operators. And the contraction condition (4) is merely about the partial order, while most of the contractions are about metric.

Remark 9. Two elements x and y in an ordered set $(X, \leq)$ are said to be comparable if either $x \leq y$ or $y \leq x$, and we denote it as $x \leq \geq y$.

Proof. Let $u_0 \in X$, we have $Au_0 \in X$. So we have the following two cases.

Case 1. When u_0 is comparable with Au_0. Firstly, without loss of generality, we suppose that

$$u_0 \leq Au_0. \tag{7}$$

If $u_0 = Au_0$, then the proof is finished. Suppose that $u_0 < Au_0$. Since A is decreasing, we obtain $Au_0 \geq A^2 u_0$, and it is easy to prove that A^2 is increasing. Using the contractive condition (4), we have

$$Au_0 - A^2 u_0 \leq \beta(Au_0 - u_0)(Au_0 - u_0) \leq Au_0 - u_0, \tag{8}$$

hence,

$$u_0 \leq A^2 u_0. \tag{9}$$

From (4), we have

$$\begin{aligned} A^2 v - A^2 u &\leq \beta(Au - Av)(Au - Av) \\ &\leq \beta(Au - Av)\beta(v - u)(v - u) \\ &\leq \beta(v - u)(v - u), \quad \forall u, v \in X, u \leq v. \end{aligned} \tag{10}$$

Let

$$Bu = A^2u, \quad \forall u \in X. \tag{11}$$

From (9) and (10), we have the following two conclusions:

(a) There exists a functional $\beta \in \zeta$ such that for $u, v \in X$ with $u \leq v$

$$Bv - Bu \leq \beta(v-u)(v-u) \tag{12}$$

(b) There exists $u_0 \in X$ such that $u_0 \leq Bu_0$

We assert that the operator B has unique fixed point in X. And the unique fixed point of B is also the unique fixed point of A. In order to be clear, we divide the process of proof into three steps.

Step 1. We will use the method of iteration to construct a fixed point of B. In fact, consider the iterative sequence

$$u_{n+1} = Bu_n, \quad n = 0, 1, 2, \cdots. \tag{13}$$

Since $u_0 \leq Bu_0$ and the operator B is increasing, we have

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots. \tag{14}$$

This means that $\{u_n\}$ is an increasing sequence. It follows from (12) that

$$\theta \leq u_{n+1} - u_n = Bu_n - Bu_{n-1} \leq \beta(u_n - u_{n-1})(u_n - u_{n-1}). \tag{15}$$

Since P is normal, from the equivalence of $\|.\|$ and $\|.\|_1$ in Lemma 5 we have

$$\|u_{n+1} - u_n\|_1 = \|Bu_n - Bu_{n-1}\|_1 \leq \beta(u_n - u_{n-1})\|u_n - u_{n-1}\|_1. \tag{16}$$

Then, $\{\|u_{n+1} - u_n\|_1\}$ is a decreasing sequence and bounded as follows. So

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|_1 = r \geq 0. \tag{17}$$

Assume $r > 0$. Then, from (16), we have

$$\frac{\|u_{n+1} - u_n\|_1}{\|u_n - u_{n-1|}\|_1} \leq \beta(u_n - u_{n-1}). \tag{18}$$

The above inequality yields

$$\lim_{n\to\infty} \beta(u_n - u_{n-1}) = 1. \tag{19}$$

And $\beta \in \zeta$ implies $\lim_{n\to\infty}(u_n - u_{n-1}) = \theta$. Then,

$$\lim_{n\to\infty} \|u_n - u_n - 1\|_1 = 0 \tag{20}$$

and $r = 0$.

Now we show that $\{u_n\}$ is a Cauchy sequence. On the contrary, assume that

$$\lim_{m,n\to\infty} \sup \|u_n - u_m\|_1 > 0. \tag{21}$$

For any fixed natural number $n, m\, n > m$. From (16), by the triangle inequality $\|u_n - u_m\|_1 \leq \|u_n - u_{n+1}\|_1 + \|u_{n+1} - u_{m+1}\|_1 + \|u_{m+1} - u_m\|_1 \leq \|u_n - u_{n+1}\|_1 + \beta(u_n - u_m)\|u_n - u_m\|_1 + \|u_{m+1} - u_m\|_1$.

Hence, we have $\|u_n - u_m\|_1 \leq (1 - \beta(u_n - u_m))^{-1}(\|u_n - u_{n+1}\|_1 + \|u_{m+1} - u_m\|_1)$.

Since $\limsup_{m,n\to\infty}\|u_n - u_m\|_1 > 0$ and $\lim_{n\to\infty}\|u_n - u_{n+1}\|_1 = 0$, then

$$\limsup_{m,n\to\infty}(1 - \beta(u_n - u_m))^{-1} = \infty, \tag{22}$$

from which we obtain

$$\limsup_{m,n\to\infty}\beta(u_n - u_m) = 1. \tag{23}$$

But since $\beta \in \zeta$, we get

$$\limsup_{m,n\to\infty}\|u_n - u_m\|_1 = 0. \tag{24}$$

This contradicts (21) and shows that $\{u_n\}$ is a Cauchy sequence in X.

Since X is closed, we can suppose that there exists a $u_* \in X$ such that

$$u_n \longrightarrow u_*. \tag{25}$$

Since P is normal, (14) together with Lemma 5 implies that

$$u_n \leq u_*. \tag{26}$$

(26), together with (12) and the equivalence of $\|.\|_1$ and $\|.\|$, implies that

$$\|Bu_* - Bu_n\|_1 \leq \beta(u_* - u_n)\|u_* - u_n\|_1. \tag{27}$$

So $\|u_* - Bu_*\|_1 \leq \|u_* - u_{n+1|}\|_1 + \|Bu_* - u_{n+1}\|_1 = \|u_* - u_{n+1|}\|_1 + \|Bu_* - Bu_n\|_1 \leq \|u_* - u_{n+1}\|_1 + \beta(u_* - u_n)\|u_* - u_n\|_1$.

Let $n \longrightarrow \infty$, we obtain $\|u_* - Bu_*\|_1 = 0$. So $u_* = Bu_*$.

This proves that u_* is a fixed point of B in X and

$$u_* = \lim_{n\to\infty} B^n u_0. \tag{28}$$

Step 2. We will obtain the uniqueness of the fixed point of B. On the contrary, if $\bar{u}$ is another fixed point of B, we will get $\bar{u} = u_*$.

In fact, the first case, when $\bar{u}$ is comparable with u_0, without loss of generality, we suppose that $\bar{u} \le u_0$. Since B is increasing,

$$\bar{u} = B^n \bar{u} \le B^n u_0. \tag{29}$$

Moreover, by (12)

$$\begin{aligned} B^n u_0 - \bar{u} = B^n u_0 - B^n \bar{u} &\le \beta\left(B^{n-1}u_0 - B^{n-1}\bar{u}\right)\left(B^{n-1}u_0 - B^{n-1}\bar{u}\right) \\ &= \beta\left(B^{n-1}u_0 - B^{n-1}\bar{u}\right)\left(B^{n-1}u_0 - \bar{u}\right), \end{aligned} \tag{30}$$

and so

$$\left\|B^n u_0 - \bar{u}\right\|_1 \le \beta\left(B^{n-1}u_0 - \bar{u}\right)\left\|\left(B^{n-1}u_0 - \bar{u}\right)\right\|_1. \tag{31}$$

Consequently, the sequence

$$\gamma_n = \left\|B^n u_0 - \bar{u}\right\|_1, \tag{32}$$

is nonnegative and decreasing, and so $\lim_{n\to\infty} \gamma_n = \gamma \ge 0$. Now we show that $\gamma = 0$.

On the contrary, assume that $\gamma > 0$. By passing to subsequences, if necessary, we may assume that $\lim_{n\to\infty} \beta(B^n u_0 - \bar{u}) = \lambda$ exists. From (31), we obtain $\gamma \le \lambda\gamma$, and so $\lambda = 1$. Since $\beta \in \zeta$, then $\lim_{n\to\infty} (B^n u_0 - \bar{u}) = \theta$, and $\gamma = \lim_{n\to\infty} \left\|B^n u_0 - \bar{u}\right\|_1 = 0$.

This contradiction proves $\gamma = 0$.

It can be obtained that

$$\bar{u} = \lim_{n\to\infty} B^n u_0 = u_*. \tag{33}$$

The second case, when $\bar{u}$ cannot compare with u_0. From X which is a lattice, we obtain

$$y_1 = \inf\{\bar{u}, u_0\} \in X, \tag{34}$$

satisfying

$$y_1 \le \bar{u}, \quad y_1 \le u_0, \tag{35}$$

i.e., $\bar{u}$ is comparable with y_1 and u_0 is comparable with y_1. Since B is increasing, we know

$$B^n y_1 \le B^n \bar{u}, B^n y_1 \le B^n u_0, \quad n = 1, 2, \cdots. \tag{36}$$

Moreover, by (12)

$$B^n u_0 - B^n y_1 \le \beta\left(B^{n-1}u_0 - B^{n-1}y_1\right)\left(B^{n-1}u_0 - B^{n-1}y_1\right). \tag{37}$$

So we have

$$\begin{aligned} \left\|B^n u_0 - B^n y_1\right\|_1 &\le \beta\left(B^{n-1}u_0 - B^{n-1}y_1\right)\left\|B^{n-1}u_0 - B^{n-1}y_1\right\|_1 \\ &\le \left\|B^{n-1}u_0 - B^{n-1}y_1\right\|_1. \end{aligned} \tag{38}$$

Similar to the process of (31), (32), and (33),

$$\lim_{n\to\infty} B^n y_1 = \lim_{n\to\infty} B^n u_0 = u_*. \tag{39}$$

From (36), we have

$$\lim_{n\to\infty} B^n y_1 = \lim_{n\to\infty} B^n \bar{u} = \bar{u}. \tag{40}$$

So from (39) and (40), we get

$$\bar{u} = u_*. \tag{41}$$

(33) together with (41) implies that u_* is unique fixed point of B.

Step 3. We will point that the unique fixed point of B is also the unique fixed point of A. Since

$$A^2 u_* = B u_* = u_*. \tag{42}$$

Thus,

$$A^2(Au_*) = A\left(A^2 u_*\right) = Au_*, \tag{43}$$

i.e., $B(Au_*) = Au_*$. From the uniqueness of the fixed point of B, we know

$$Au_* = u_*. \tag{44}$$

So u_* is the unique fixed point of A in X.

Case 2. Another case, when u_0 is not comparable to Au_0. From X which is a lattice, we know there exists $v_0 \in E$ such that $\inf\{Au_0, u_0\} = v_0$. That is, $v_0 \le Au_0$, $v_0 \le u_0$. Since A is a decreasing operator, we have

$$Au_0 \le Av_0. \tag{45}$$

This shows that

$$v_0 \le Av_0. \tag{46}$$

For any initial $v_0 \in X$, constructing successively sequence

$$v_n = Av_{n-1}. \tag{47}$$

From $B = A^2$, we can get

$$v_{2n} = A^{2n} v_0 = B^n v_0, \quad (n = 1, 2, \cdots). \tag{48}$$

From (28), we know

$$\lim_{n\to\infty} v_{2n} = \lim_{n\to\infty} B^n v_0 = u_*. \tag{49}$$

Since $v_{2n+1} = A^{2n+1}v_0 = B^n(Av_0)$ and from the arbitrary of v_0 in (28), we obtain

$$\lim_{n\to\infty} v_{2n+1} = \lim_{n\to\infty} B^n(Av_0) = u_*. \tag{50}$$

(49) and (50) imply that

$$\lim_{n\to\infty} v_n = u_*, \tag{51}$$

holds. Similarly to the proof of Step 2 and Step 3 in case 1, we get that u_* is the unique fixed point of A.

3. An Example

Let $E = R$, equipped with usual normal $\|\cdot\| = |\cdot|$ and usual partial order $\leq$. $X = [0, \infty), P = [0, \infty)$. Then, $P \subset X \subset E$. P is a normal cone in X. $(X, \|\cdot\|, \leq)$ is a partially ordered Banach space. And X is a lattice under the partial order $\leq$ induced by the cone P.

Then, $X = [0, \infty), P = [0, \infty)$ satisfying the assumptions of Theorem 7. Define the mapping $A : X \longrightarrow X$ by $Ax = (m + 1 + mx)/(1 + x), x \in X$ where m is a fixed real number. Then, A is nonincreasing. Define $\beta : P \longrightarrow [0, \infty)$ by $\beta(w) = 1/(1 + w)$, then $\beta \in \zeta$. Now, for all $x, y \in X$ with $x \leq y$, we have

$$\begin{aligned} Ax - Ay &= \frac{m + 1 + mx}{1 + x} - \frac{m + 1 + my}{1 + y} \\ &= \frac{y - x}{1 + x + y + xy} \leq \frac{y - x}{1 + y - x} \\ &= \beta(y - x)(y - x), \end{aligned} \tag{52}$$

so that A and β satisfy the assumption of Theorem 7. Observing that all the other conditions of Theorem 7 are also satisfied, A has a unique fixed-point $x_* > 0$. Moreover, constructing successively sequence

$$x_{n+1} = \frac{m + 1 + mx_n}{1 + x_n} \quad (n = 0, 1, 2, \cdots), \tag{53}$$

for any initial $x_0 \geq 0$, we have

$$\lim_{x\to\infty} x_n = x_*. \tag{54}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fixed Point Theorems for Mizoguchi-Takahashi Type Contraction in Bicomplex-Valued Metric Spaces and Applications

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The main aim of this paper is to study and establish some new fixed point theorems for contractive maps that satisfied Mizoguchi-Takahashi's condition in the setting of bicomplex-valued metric spaces. These new results improve and generalize the Banach contraction principle and some well-known results in the literature. Finally, as applications of our results, we give the existence and uniqueness of the solution of a nonlinear integral equation.

1. Introduction

Fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology. In 2011, Azam et al. [1] introduced the so-called complex-valued metric spaces and proved the existence of fixed points under some contraction conditions. Very recently, Choi et al. [2] introduced the notion of bicomplex-valued metric space which is a generalization of the complex-valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition.

The following new fixed point theorem is one of the main results of this paper. It can be considered a bicomplex-valued metric version of the Banach contraction principle and will generalize and improve some well-known results in the literature. Finally, we strengthen our results by giving an application to find an analytical solution for a nonlinear integral equation. The results thus derived in this study are generally extended and generalized in character. Moreover, the results presented here may be very useful in the study of engineering problems like boundary value problems (see, e. g., [3–7]) in future works.

2. Preliminaries

In this section, we recall some definitions and terminologies, used to prove the main results.

2.1. Bicomplex Numbers. The set of bicomplex numbers denoted by $\mathbb{BC}$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}$. Here, we recall the set of bicomplex numbers $\mathbb{BC}$ (see, for example, [8–10]):

$$\mathbb{BC} = \{w = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3 \, ; x_k \in \mathbb{R}, (k = 0, 1, 2, 3)\}. \tag{1}$$

Since each element w in $\mathbb{BC}$ can be written as $w = x_0 + i_1 x_1 + i_2(x_2 + i_1 x_3)$ or $w = z_1 + i_2 z_2 \, ; (z_1, z_2 \in \mathbb{C})$, we can also express $\mathbb{BC}$ as

$$\mathbb{BC} := \{w = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}\}, \tag{2}$$

where $z_1 = x_0 + i_1 x_1$, $z_2 = x_2 + i_1 x_3$, and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product

of units is commutative and is defined as

$$\begin{aligned} i_1 i_2 &= j, \\ i_1 j &= -i_2, \\ i_2 j &= -i_1. \end{aligned} \tag{3}$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring. Three important subsets of $\mathbb{BC}$ can be specified as

$$\begin{aligned} \mathbb{C}(i_k) &:= \{x + y i_k \,; x \,; y \in \mathbb{R}\}, k = 1, 2, \\ \mathbb{D} &:= \{x + y j \,; x, y \in \mathbb{R}\}. \end{aligned} \tag{4}$$

Each of the set $\mathbb{C}(i_k)$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of the so-called hyperbolic numbers.

2.2. Conjugation and Moduli. Three kinds of conjugation $w^{\dagger_1}, w^{\dagger_2}$, and $w^{\dagger_3}$ can be defined on bicomplex numbers. With the bicomplex number w specified as in (2) and the bar $(^-)$ denoting complex conjugation in $\mathbb{C}$, we define

$$\begin{aligned} w^{\dagger_1} &:= \bar{z}_1 + \bar{z}_2 i_2, \\ w^{\dagger_2} &:= z_1 - z_2 i_2, \\ w^{\dagger_3} &:= \bar{z}_1 - \bar{z}_2 i_2. \end{aligned} \tag{5}$$

It is easy to check that each conjugation has the following properties:

$$\begin{aligned} (u + v)^{\dagger_k} &= u^{\dagger_k} + v^{\dagger_k}, \\ \left(u^{\dagger_k}\right)^{\dagger_k} &= u, \\ (u.v)^{\dagger_k} &= u^{\dagger_k}.v^{\dagger_k}. \end{aligned} \tag{6}$$

Here, $u, v \in \mathbb{BC}$ and $k = 1, 2, 3$.

With each kind of conjugation, one can define a specific bicomplex modulus as

$$\begin{aligned} |w|^2_{i_1} &:= w.w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i_1) \\ |w|^2_{1_2} &:= w.w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\,\mathrm{Re}\,(z_1 \bar{z}_2) i_2 \in \mathbb{C}(i_2) \\ |w|^2_j &:= w.w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\,\mathrm{Im}\,(z_1 \bar{z}_2) j \in D \end{aligned} \tag{7}$$

It can be shown that $|u.v|^2_s = |u|^2_s.|v|^2_s$, where $s = i_1, i_2$ or j.

A norm of a bicomplex number $w = z_1 + i_2 z_2$ denoted by $\|w\|$ is defined by

$$\|w\| = \|z_1 + i_2 z_2\| = \left(|z_1|^2 + |z_2|^2\right)^{1/2}, \tag{8}$$

which, upon choosing $w = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3$, $(x_k \in \mathbb{R}, k = 0, 1, 2, 3)$, gives

$$\|w\| = \left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right)^{1/2}. \tag{9}$$

For any two bicomplex numbers $u, v \in \mathbb{BC}$, one can easily verify that $0 \prec_{i_2} u \prec_{i_2} v \Longrightarrow \|u\| \le \|v\|$, $\|u + v\| \le \|u\| + \|v\|$, $\|\alpha u\| = \alpha \|u\|$, where α is a nonnegative real number. Further, for any two bicomplex numbers, $u, v \in \mathbb{BC}$, $\|uv\| \le \sqrt{2}\|u\|\|v\|$ holds.

Next, we recall some necessary definitions and lemmas.

First, define a partial order $\preceq$ on $\mathbb{BC}$ as follows:

Let $u = u_1 + i_2 u_2 \in \mathbb{BC}$ and $v = v_1 + i_2 v_2 \in \mathbb{BC}$. Define a partial order relation $\preceq_{i_2}$ on $\mathbb{BC}$ as follows:

$u \preceq_{i_2} v$ if and only if $u_1 \preceq_{i_1} v_1$ and $u_2 \preceq_{i_1} v_2$. We find that $u \preceq_{i_2} v$ if any one of the following properties holds:

$$\begin{aligned} &(a_1) \\ &u_1 = v_1 \\ &u_2 = v_2 \\ &(a_2) \\ &u_1 \prec_{i_1} v_1 \\ &u_2 = v_2 \\ &(a_3) \\ &u_1 = v_1 \\ &u_2 \prec_{i_1} v_2 \\ &(a_4) \\ &u_1 \prec_{i_1} v_1 \\ &u_2 \prec_{i_1} v_2 \end{aligned} \tag{10}$$

where $\preceq_{i_1}$ is a partial order relation on $\mathbb{C}$. So that

$$\begin{aligned} &(b_1)\ \mathrm{Re}\,(u_1) = \mathrm{Re}\,(v_1),\ \ \mathrm{Im}\,(u_1) = \mathrm{Im}\,(v_1), \\ &\quad \mathrm{Re}\,(u_2) = \mathrm{Re}\,(v_2),\ \ \mathrm{Im}\,(u_2 = \mathrm{Im}\, v_2) \\ &(b_2)\ \mathrm{Re}\,(u_1) < \mathrm{Re}\,(v_1),\ \ \mathrm{Im}\,(u_1) < \mathrm{Im}\,(v_1), \\ &\quad \mathrm{Re}\,(u_2) = \mathrm{Re}\,(v_2),\ \ \mathrm{Im}\,(u_2) = \mathrm{Im}\,(v_2) \\ &(b_3)\ \mathrm{Re}\,(u_1) = \mathrm{Re}\,(v_1),\ \ \mathrm{Im}\,(u_1) = \mathrm{Im}\,(v_1), \\ &\quad \mathrm{Re}\,(u_2) < \mathrm{Re}\,(v_2),\ \ \mathrm{Im}\,(u_2) < \mathrm{Im}\,(v_2) \\ &(b_4)\ \mathrm{Re}\,(u_1) < \mathrm{Re}\,(v_1),\ \ \mathrm{Im}\,(u_1) < \mathrm{Im}\,(v_1), \\ &\quad \mathrm{Re}\,(u_2) < \mathrm{Re}\,(v_2),\ \ \mathrm{Im}\,(u_2) < \mathrm{Im}\,(v_2) \end{aligned} \tag{11}$$

We write $u \underset{\sim i_2}{\prec} v$ if $u \preceq_{i_2} v$ and $u \neq v$, i.e., one of (b_2), (b_3), and (b_4) is satisfied and we write $u \prec_{i_2} v$ if only (b_4) is satisfied.

Given the definition of the complex metric space introduced in [1], we extend the definition of bicomplex analysis as follows.

Definition 1. Let X be a nonempty set. A functional $d_{\mathbb{BC}} : X \times X \longrightarrow \mathbb{BC}$ is called a bicomplex-valued metric on X if for $x, y, z \in X$ the following conditions are satisfied:

$$\begin{aligned} &(m_1) 0 \preceq_{i_2} d_{\mathbb{BC}}(x, y) \\ &(m_2) d_{\mathbb{BC}}(x, y) = 0 \text{ if and only if } x = y \\ &(m_3) d_{\mathbb{BC}}(x, y) = d_{\mathbb{BC}}(y, x) \\ &(m_4) d_{\mathbb{BC}}(x, y) \preceq_{i_2} d_{\mathbb{BC}}(x, z) + d_{\mathbb{BC}}(z, y) \end{aligned} \tag{12}$$

Then, $(X, d_{\mathbb{BC}})$ is called a bicomplex-valued metric space.

Some known examples of bicomplex-valued metric, which show a bicomplex-valued metric space, are the following.

Example 2. The set of real numbers together with the functional

$$\begin{aligned} d^1_{\mathbb{BC}}(x, y) &= (1 + i_1 + i_2 + i_1 i_2)|x - y|, \\ d^2_{\mathbb{BC}}(x, y) &= i_1 i_2 |x - y|, \end{aligned} \tag{13}$$

for all $x, y \in \mathbb{R}$ where $|\cdot|$ is the usual real modulus. One can easily check that $(X, d^1_{\mathbb{BC}})$, $(X, d^2_{\mathbb{BC}})$ are bicomplex-valued metric spaces.

Definition 3. Let (x_n) be a sequence in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$. The sequence (x_n) is said to converge to $x \in X$ if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x) \prec_{i_2} \varepsilon$ for all $n > N$. It is denoted by $x_n \longrightarrow x$ as $n \longrightarrow +\infty$ or $\lim_{n\to+\infty} x_n = x$.

Definition 4. A sequence (x_n) in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be a Cauchy sequence if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x_m) \prec_{i_2} \varepsilon$ for all $n, m > N$.

Definition 5. A bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be complete if and only if every Cauchy sequence in X converges in X.

Lemma 6 (see [2]). *Let $(X, d_{\mathbb{BC}})$ be a bicomplex-valued metric space and (x_n) be a sequence in X. Then, (x_n) converges to $x \in X$ if and only if $\|d_{\mathbb{BC}}(x_n, x)\| \longrightarrow 0$ as $n \longrightarrow +\infty$.*

Lemma 7 (see [2]). *Let $(X, d_{\mathbb{BC}})$ be a bicomplex-valued metric space and (x_n) be a sequence in X such that $\lim_{n\to+\infty} x_n = x$. Then, for any $a \in X$, $\lim_{n\to+\infty} \|d_{\mathbb{BC}}(x_n, a)\| = \|d_{\mathbb{BC}}(x, a)\|$.*

3. Statement of Results

Definition 8 (see [11, 12]). *A function $\psi : [0,+\infty) \longrightarrow [0, 1)$ is said to be a MT-function if it satisfies Mizoguchi-Takahashi's condition, i.e., $\limsup_{s\to t^+} \psi(s) < 1$ for all $t \in [0,+\infty)$.*

Clearly, if $\psi : [0,+\infty) \longrightarrow [0, 1)$ is a nondecreasing function, then ψ is a MT-function.

Notice that $\psi : [0,+\infty) \longrightarrow [0, 1)$ is a MT-function if and only if for each $t \in [0,+\infty)$ there exist $r_t \in [0, 1)$ and $\varepsilon_t > 0$ such that $\psi(s) \le r_t$ for all $s \in [t, t + \varepsilon_t)$, for more detail, (see [4], Remark 2.5 (iii)).

An example which is not an MT-function is given hereunder. Let $\psi : [0,+\infty) \longrightarrow [0, 1)$ be defined by

$$\psi(t) = \begin{cases} \dfrac{\sin t}{t}, & \text{if } t \in \left(0, \dfrac{\pi}{2}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Since $\limsup_{s\to 0^+} \psi(s) = 1$, ψ is not a MT-function.

Recently, Du [13] first proved the following characterizations of MT-functions which are quite useful for proving our main results.

Theorem 9. *[13] Let $\psi : [0,+\infty) \longrightarrow [0, 1)$ be a function. Then, the following statements are equivalent.*

(a) ψ is a MT-function

(b) For each $t \in [0,+\infty)$, there exist $r^1_t \in [0, 1)$ and $\varepsilon^1_t > 0$ such that $\psi(s) \le r^1_t$ for all $s \in (t, t + \varepsilon^1_t)$

(c) For each $t \in [0,+\infty)$, there exist $r^2_t \in [0, 1)$ and $\varepsilon^2_t > 0$ such that $\psi(s) \le r^2_t$ for all $s \in [t, t + \varepsilon^2_t]$

(d) For each $t \in [0,+\infty)$, there exist $r^3_t \in [0, 1)$ and $\varepsilon^3_t > 0$ such that $\psi(s) \le r^3_t$ for all $s \in (t, t + \varepsilon^3_t]$

(e) For each $t \in [0,+\infty)$, there exist $r^4_t \in [0, 1)$ and $\varepsilon^4_t > 0$ such that $\psi(s) \le r^4_t$ for all $s \in [t, t + \varepsilon^4_t)$

(f) For any nonincreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, +\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \psi(x_n) < 1$

(g) ψ is a function of contractive factor, that is, for any strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, +\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \psi(x_n) < 1$

Theorem 10. *Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex-valued metric space and let $f : X \longrightarrow X$ be a mapping on X. Suppose that there exists a MT-function $\psi : [0,+\infty) \longrightarrow [0, 1)$ such that*

$$d_{\mathbb{BC}}(fx, fy) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x, y)\|) d_{\mathbb{BC}}(x, y), \text{for all } x, y \in X. \tag{15}$$

Then, f has a unique fixed point on X.

Notice that f need not be continuous.

Proof. Let $x_0 \in X$ be given. Define the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_n = f^n x_0 = f x_{n-1}, \text{ for each } n \in \mathbb{N}. \tag{16}$$

For each $n \in \mathbb{N}$, by (15), we have

$$d_{\mathbb{BC}}(x_n, x_{n+1}) = d_{\mathbb{BC}}(fx_{n-1}, fx_n) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) \cdot d_{\mathbb{BC}}(x_{n-1}, x_n), \tag{17}$$

which implies

$$\|d_{\mathbb{BC}}(x_n, x_{n+1})\| \leq \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|)\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|. \tag{18}$$

Let $\delta_n = \|d_{\mathbb{BC}}(x_{n-1}, x_n)\|$ for $n \in \mathbb{N}$. Then, by (18), we have

$$\delta_{n+1} \leq \psi(\delta_n)\delta_n < \delta_n, \forall n \in \mathbb{N}. \tag{19}$$

So, we know that $(\delta_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $[0, +\infty)$. Applying (g) of Theorem 9, we obtain

$$0 \leq \sup \psi(\delta_n) < 1. \tag{20}$$

That is,

$$0 \leq \sup \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) < 1. \tag{21}$$

Let

$$k = \sup \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|). \tag{22}$$

Then, $k \in [0, 1)$. For each $n \in \mathbb{N}$, by (18) again, we have

$$\begin{aligned}\|d_{\mathbb{BC}}(x_n, x_{n+1})\| &\leq \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|)\|d_{\mathbb{BC}}(x_{n-1}, x_n)\| \\ &\leq k\|d_{\mathbb{BC}}(x_{n-1}, x_n)\| \leq k^2\|d_{\mathbb{BC}}(x_{n-2}, x_{n-1})\| \\ &\leq \cdots \leq k^n\|d_{\mathbb{BC}}(x_0, x_1)\|.\end{aligned} \tag{23}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned}\|d_{\mathbb{BC}}(x_n, x_m)\| &\leq \|d_{\mathbb{BC}}(x_n, x_{n+1})\| \\ &\quad + \|d_{\mathbb{BC}}(x_{n+1}, x_{n+2})\| + \cdots + \|d_{\mathbb{BC}}(x_{m-1}, x_m)\| \\ &\leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right)\|d_{\mathbb{BC}}(x_0, x_1)\| < \frac{k^n}{1-k}\|d_{\mathbb{BC}}(x_0, x_1)\|.\end{aligned} \tag{24}$$

Since $k \in [0, 1)$, $\lim_{n\to+\infty}(k^n/1-k)\|d_{\mathbb{BC}}(x_0, x_1)\| = 0$. Hence, by the last inequality, we obtain

$$\|d_{\mathbb{BC}}(x_n, x_m)\| \longrightarrow 0, \text{ as } m, n \longrightarrow +\infty, \tag{25}$$

that is, $d_{\mathbb{BC}}(x_n, x_m) \longrightarrow 0$ as $n \longrightarrow +\infty$. Therefore, (x_n) is a Cauchy sequence in the complete bicomplex-valued metric space. Then, there exists $u \in X$ such that

$$\lim_{n\to+\infty} d_{\mathbb{BC}}(x_n, u) = 0. \tag{26}$$

Next, we prove that $fu = u$. Assume that $fu \neq u$. For each $n \in \mathbb{N}$, by (15), we have

$$d_{\mathbb{BC}}(x_{n+1}, fu) = d_{\mathbb{BC}}(fx_n, fu) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x_n, u)\|)d_{\mathbb{BC}}(x_n, u), \tag{27}$$

which deduces

$$\|d_{\mathbb{BC}}(x_{n+1}, fu)\| \leq \psi(\|d_{\mathbb{BC}}(x_n, u)\|)\|d_{\mathbb{BC}}(x_n, u)\| < \|d_{\mathbb{BC}}(x_n, u)\|. \tag{28}$$

Since $x_n \longrightarrow u$ as $n \longrightarrow +\infty$, from Lemma 6 and Lemma 7 and by taking the limit from both sides of (29), we get

$$\|d_{\mathbb{BC}}(u, fu)\| \leq \|d_{\mathbb{BC}}(u, u)\| = 0. \tag{29}$$

Since $0 \prec d_{\mathbb{BC}}(u, fu)$, then

$$0 < \|d_{\mathbb{BC}}(u, fu)\| \leq 0, \tag{30}$$

which is a contradiction. Therefore, $fu = u$, that is, u is a fixed point of f.

Finally, we want to show the uniqueness of the fixed point of f. Let v be a fixed point of f and $v \neq u$. By (15), we obtain

$$d_{\mathbb{BC}}(fv, fu) = d_{\mathbb{BC}}(v, u) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(v, u)\|)d_{\mathbb{BC}}(v, u), \tag{31}$$

which implies

$$\|d_{\mathbb{BC}}(v, u)\| \leq \psi(\|d_{\mathbb{BC}}(v, u)\|)\|d_{\mathbb{BC}}(v, u)\|, \tag{32}$$

then

$$(1 - \psi(\|d_{\mathbb{BC}}(v, u)\|))\|d_{\mathbb{BC}}(v, u)\| \leq 0. \tag{33}$$

Since $\psi(\|d_{\mathbb{BC}}(v, u)\|) \in [0, 1)$, we have

$$\|d_{\mathbb{BC}}(v, u)\| \leq 0, \tag{34}$$

which deduces

$$\|d_{\mathbb{BC}}(v, u)\| = 0, \tag{35}$$

and hence $d_{\mathbb{BC}}(v, u) = 0$ which is a contradiction. Therefore $v = u$, that is, f has a unique fixed point.

Example 11. Let $X = \mathbb{R}$ and $d_{\mathbb{BC}} : X \times X \longrightarrow \mathbb{BC}$ be defined by $d_{\mathbb{BC}}(x, y) = i_1 i_2 |x - y|$, $(x, y \in X)$, where $|\cdot|$ is the usual real modulus. One can easily check that $(X, d_{\mathbb{BC}})$ is a complete bicomplex-valued metric on X and $\|d_{\mathbb{BC}}(x, y)\| = |x - y|$ for all $x, y \in X$.

Define $f : X \longrightarrow X$ and $\psi : [0,+\infty) \longrightarrow [0,1)$ by $f(x) = 1/12x$ for $x \in X$,

$$\psi(t) = \begin{cases} \frac{5}{6} & \text{if } t = 0 \\ \frac{1}{4} & \text{if } t > 0 \end{cases} \tag{36}$$

Then, ψ is a MT-function.

For all $x, y \in X$ we have

$$d_{\mathbb{BC}}(fx, fy) = i_1 i_2 |fx - fy| = \begin{cases} \frac{1}{2} i_1 i_2 |x - y| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$$\psi(\|d_{\mathbb{BC}}(x,y)\|) d_{\mathbb{BC}}(x,y) = \psi(|x-y|) d_{\mathbb{BC}}(x,y) = \begin{cases} \frac{1}{4} i_1 i_2 |x \neq y| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{37}$$

then, f satisfies (15).

So all the hypotheses of Theorem 10 are fulfilled. It is therefore possible to apply Theorem 10 to get the fact that f has a unique fixed point on X (Here, 0 is the unique fixed point of f).

The following fixed point theorem established in metric space is immediate from Theorem 10.

Theorem 12. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping on X. Suppose that there exists a MT-function $\psi : [0,+\infty) \longrightarrow [0,1)$ such that*

$$d(fx, fy) \leq \psi(d(x,y)) d(x,y), \text{ for all } x, y \in X. \tag{38}$$

Then, f has a unique fixed point on X.

Since any nondecreasing function or any nonincreasing function $\psi : [0,+\infty) \longrightarrow [0,1)$ is an MT-function, by applying Theorem 10, we have the following results.

Corollary 13. *Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex-valued metric space and let $f : X \longrightarrow X$ be a mapping on X. Suppose that there exists a nondecreasing (or nonincreasing) function $\psi : [0,+\infty) \longrightarrow [0,1)$ such that*

$$d_{\mathbb{BC}}(fx, fy) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x,y)\|) d_{\mathbb{BC}}(x,y), \text{for all } x, y \in X, \tag{39}$$

Then, f has a unique fixed point on X.

Corollary 14. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping on X. Suppose that there exists a nondecreasing (nonincreasing) function $\psi : [0,+\infty) \longrightarrow [0,1)$ such that*

$$d(fx, fy) \leq \psi(d(x,y)) d(x,y), \text{ for all } x, y \in X. \tag{40}$$

Then, f has a unique fixed point on X.

Corollary 15 (*bicomplex-valued metric version of Banach contraction principle*). *Let $(X, d_{\mathbb{BC}})$ be a bicomplex-valued complete metric space and let $f : X \longrightarrow X$ be a contraction mapping on X, that is,*

$$d_{\mathbb{BC}}(fx, fy) \preceq_{i_2} k d_{\mathbb{BC}}(x,y), \tag{41}$$

for all $x, y \in X$, where $k \in [0,1)$. Then, f has a unique fixed point on X.

The aim of the next work is to generalize our results by finding a common fixed point for k functions f_i, $i = 1, 2, \cdots, k$ defined on a bicomplex-valued complete metric space.

Theorem 16. *Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex-valued metric space and $f_i : X \longrightarrow X$ be k mappings on X. Suppose that there exists a MT-function $\psi : [0,+\infty) \longrightarrow [0,1)$ such that*

$$d_{\mathbb{BC}}(f_i x, f_{i+1} y) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x,y)\|) d_{\mathbb{BC}}(x,y), \tag{42}$$

for all $x, y \in X$ and $i = 1, 2, \cdots, k$, then, there exists a unique common fixed point $u \in X$ such that $u = f_1 u = f_2 u = \cdots = f_k u$

Proof. For any $x_0 \in X$ Putting $x_1 = f_{i+1} x_0$ and $x_2 = f_i x_1$, then let $x_3 = f_{i+1} x_2$ and $x_4 = f_i x_3$. Inductively, choose a sequence $(x_n)_{n \in \mathbb{N}}$ in X so that $x_{2n+2} = f_i x_{2n+1}$ and $x_{2n+1} = f_{i+1} x_{2n}$ for all $n \geq 0$.

It follows from the property of the function ψ that if n is odd and for $x_{n+1} \neq x_n$, then

$$d_{\mathbb{BC}}(x_{n+1}, x_n) = d_{\mathbb{BC}}(f_i x_n, f_{i+1} x_{n-1}) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x_n, x_{n-1})\|) \cdot d_{\mathbb{BC}}(x_n, x_{n-1}). \tag{43}$$

Similarly, if n is even, we also obtain that

$$d_{\mathbb{BC}}(x_{n+1}, x_n) = d_{\mathbb{BC}}(f_{i+1} x_n, f_i x_{n-1}) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) \cdot d_{\mathbb{BC}}(x_{n-1}, x_n), \tag{44}$$

which implies that, for all $n \in \mathbb{N}$

$$\|d_{\mathbb{BC}}(x_{n+1}, x_n)\| < \|d_{\mathbb{BC}}(x_n, x_{n-1})\| \tag{45}$$

Arguing like in the proof of Theorem 16, we prove that (x_n) is a Cauchy sequence in the complete bicomplex-valued metric space. Then, there exists $u \in X$ such that

$$\lim_{n \to +\infty} d_{\mathbb{BC}}(x_n, u) = 0. \tag{46}$$

Next, we prove that $f_i u = f_{i+1} u = u$, $i = 1, 2, \cdots, k$. Assume that $f_i u \neq u$. For each $n \in \mathbb{N}$, by (42), we have

$$d_{\mathbb{BC}}(x_{2n+1}, f_i u) = d_{\mathbb{BC}}(f_{i+1} x_{2n}, f_i u) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(u, x_{2n})\|) \cdot d_{\mathbb{BC}}(u, x_{2n}), \tag{47}$$

then

$$\|d_{\mathbb{BC}}(x_{2n+1}, f_i u)\| \leq \psi(\|d_{\mathbb{BC}}(u, x_{2n})\|) \cdot \|d_{\mathbb{BC}}(u, x_{2n})\| < \|d_{\mathbb{BC}}(u, x_{2n})\|, \tag{48}$$

then letting $n \longrightarrow +\infty$ and by Lemma 6 and Lemma 7, we have

$$\|d_{\mathbb{BC}}(u, f_i u)\| \leq \|d_{\mathbb{BC}}(u, u)\| = 0. \tag{49}$$

We obtain a contradiction. Hence, $f_i u = u$. As

$$d_{\mathbb{BC}}(u, f_{i+1} u) = d_{\mathbb{BC}}(f_i u, f_{i+1} u) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(u, u)\|) d_{\mathbb{BC}}(u, u), \tag{50}$$

then

$$\|d_{\mathbb{BC}}(u, f_{i+1} u)\| < \|d_{\mathbb{BC}}(u, u)\| = 0. \tag{51}$$

Therefore, $\|d_{\mathbb{BC}}(u, f_{i+1} u)\| = 0$, i.e., $u = f_{i+1} u = f_i u$. If there exists another point $v \in X$ such that $v = f_i v = f_{i+1} v$, then using an argument similar to the above, we get

$$d_{\mathbb{BC}}(u, v) = d_{\mathbb{BC}}(f_i u, f_{i+1} v) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(u, v)\|) d_{\mathbb{BC}}(u, v), \tag{52}$$

i.e.,

$$\|d_{\mathbb{BC}}(u, v)\| < \|d_{\mathbb{BC}}(u, v). \tag{53}$$

We obtain a contradiction, hence $u = v$.

By the same way, if there exists a unique point $w \in X$ such that $w = f_{i+1} w = f_{i+2} w$. We will show that $u = w$. Suppose that $u \neq w$, we have

$$d_{\mathbb{BC}}(u, w) = d_{\mathbb{BC}}(f_i u, f_{i+1} w) \preceq_{i_2} \psi(\|d_{\mathbb{BC}}(u, w)\|) d_{\mathbb{BC}}(u, w), \tag{54}$$

then

$$\|d_{\mathbb{BC}}(u, w)\| < \|d_{\mathbb{BC}}(u, w)\|, \tag{55}$$

which is a contradiction. Then, $u = w$ so $u = f_i u = f_{i+1} u = f_{i+2} u$.

The proof is completed.

4. Application

In this section, we give existence theorems for the Volterra-type integral equation. For this purpose, let $X = C([a, b])$, $\mathbb{R}$ be the space of all continuous realvalued functions on $[a, b]$. Note that X is a complete bicomplex-valued metric space by considering $d_{\mathbb{BC}}(x, y) = i_1 i_2 \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty$, then $\|d_{\mathbb{BC}}(x, y)\| = \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty$.

Consider the Volterra-type integral equation as

$$x(t) = \int_a^b K(t, s, x(s)) ds + g(t), \tag{56}$$

where $x(t)$ is the unknown solution, $t \in [a, b] \subset \mathbb{R}$ and x, $g \in X$.

The kernel K of the integral equation is defined on $[a, b] \times [a, b] \times \mathbb{R}$.

Theorem 17. *Assume that the following conditions are satisfied:*

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous function*

(ii) there exists $\lambda \in [0, 1)$ *such that*

$$|K(t, s, y_1) - K(t, s, y_2)| \leq \frac{\lambda}{b - a} |y_1 - y_2| \tag{57}$$

for all $t, s \in [a, b]$ *and* $y_1, y_2 \in \mathbb{R}$.

Then, the nonlinear Volterra Equation (56) has a unique solution.

Proof. Define a mapping $f : X \longrightarrow X$ as follows:

$$fx(t) = \int_a^b K(t, s, x(s)) ds + g(t). \tag{58}$$

Let $x, y \in X$. By the simple calculation and applying assumption (i_2), for each $t \in [a, b]$, we can write

$$|fx(t) - fy(t)| \leq \int_a^b |K(t, s, x(s)) - K(t, s, y(s))| \cdot |ds \leq \frac{\lambda}{b - a} \int_a^b |x(s) - y(s)| ds. \tag{59}$$

Then,

$$d_{\mathbb{BC}}(fx, fy) = i_1 i_2 \max_{t \in [a,b]} \|fx(t) - fy(t)\|_\infty \preceq \lambda i_1 i_2 \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty = \lambda d_{\mathbb{BC}}(x, y). \tag{60}$$

It follows that f satisfies all the conditions of Theorem 10 where $w(t) = \lambda$ for all $t \in [a, b]$. Hence, f has a unique fixed point. This yields that there exists a unique solution of the nonlinear Volterra integral equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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Research Article

Internal Perturbation Projection Algorithm for the Extended Split Equality Problem and the Extended Split Equality Fixed Point Problem

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In this article, we study the extended split equality problem and extended split equality fixed point problem, which are extensions of the convex feasibility problem. For solving the extended split equality problem, we present two self-adaptive stepsize algorithms with internal perturbation projection and obtain the weak and the strong convergence of the algorithms, respectively. Furthermore, based on the operators being quasinonexpansive, we offer an iterative algorithm to solve the extended split equality fixed point problem. We introduce a way of selecting the stepsize which does not need any prior information about operator norms in the three algorithms. We apply our iterative algorithms to some convex and nonlinear problems. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithms.

1. Introduction

Let H_1, H_2, and H_3 be three real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed, and convex sets. The split feasibility problem (SFP) is formulated

$$\text{to find } x \in C \text{ such that } Ax \in Q, \tag{1}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear operator. The SFP was first introduced by Censor and Elfving [1], which was used in modeling various inverse problems arising from phase retrievals and medical image reconstruction and further studied by many researchers. See, for instance, [2–10].

Moudafi [11, 12] introduced the following split equality feasibility problem (SEFP), which is

$$\text{to find } x \in C, y \in Q \text{ such that } Ax = By, \tag{2}$$

where $A : H_1 \longrightarrow H_3$ and $B : H_2 \longrightarrow H_3$ are two bounded linear operators. Obviously, if $B = I$ and $H_3 = H_2$, then (2) reduces to (1). The split equality feasibility problem (2) allows asymmetric and partial relations between the variables x and y. In order to solve SEFP, many researchers proposed their suggestions, such as [13–17] and references therein. Moudafi [11] introduced the following iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \lambda_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \lambda_k B^*(Ax_{k+1} - By_k)). \end{cases} \tag{3}$$

Under some suitable conditions, he proved that the sequence $\{(x_n, y_n)\}$ weakly converges to the solution of (2) in Hilbert spaces. In addition, Yu and Wang [18] proposed the following iterative algorithm:

$$\begin{cases} x_{k+1} = x_k - \lambda[(x_k - P_C(x_k)) + A^*(Ax_k - By_k)], \\ y_{k+1} = y_k - \lambda[(y_k - P_C(y_k)) - B^*(Ax_{k+1} - By_k)], \end{cases} \tag{4}$$

where $0 < \lambda < (1 + c)^{-1}$ with $c = \max\{\|A\|^2, \|B\|^2\}$. They studied the weak convergence of scheme (4).

Recently, Che et al. [19] proposed the following extended split equality problem (ESEP) which is an extension of the

convex feasibility problem. Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, assume C_i are nonempty closed convex subsets of real Hilbert spaces H_i, respectively. The extended split equality problem is

$$\begin{aligned} &\text{to find } x_1 \in C_1, x_2 \in C_2, \cdots, x_n \in C_n \text{ such that } A_1 x_1 \\ &\quad = A_2 x_2 = \cdots = A_n x_n, \end{aligned} \tag{5}$$

where $A_i : H_i \longrightarrow H$ are linear operators. They presented the following simultaneous iterative algorithm:

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^{n} A_i x_i^k}{n}, \\ x_1^{k+1} = P_{C_1}\left(x_1^k - \lambda_k A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_2^{k+1} = P_{C_2}\left(x_2^k - \lambda_k A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \\ \cdots \\ x_n^{k+1} = P_{C_n}\left(x_n^k - \lambda_k A_n^*\left(A_n x_n^k - \omega_k\right)\right). \end{cases} \tag{6}$$

Under some suitable conditions, they obtained the weak convergence of (6).

In order to avoid using the projection, Moudafi [11] introduced and studied the following problem. Let $S : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ be nonlinear operators such that $\mathrm{Fix}(S) \neq \varnothing$ and $\mathrm{Fix}(T) \neq \varnothing$, where $\mathrm{Fix}(S)$ and $\mathrm{Fix}(T)$ denote the sets of fixed points of S and T, respectively. If $C = \mathrm{Fix}(S)$ and $Q = \mathrm{Fix}(T)$, then SEFP (2) reduces

$$\text{to find } x \in \mathrm{Fix}(S), y \in \mathrm{Fix}(T) \text{ such that } Ax = By, \tag{7}$$

which is called the split equality fixed point problem (SEFPP). Many scholars have studied this issue, such as [20–22].

To solve problem (7), Che and Li [23] proposed the following iterative algorithm:

$$\begin{cases} u_k = x_k - \lambda_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k) S u_k, \\ v_k = y_k + \lambda_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1 - \alpha_k) T v_k. \end{cases} \tag{8}$$

They established the weak convergence of scheme (8) under the conditions that the operators S and T are quasinonexpansive mappings.

Similarly, Che et al. [19] proposed the following extended split equality fixed point problem (ESEFPP), which is

$$\begin{aligned} &\text{to find } x_1 \in \mathrm{Fix}(G_1), x_2 \in \mathrm{Fix}(G_2), \cdots, x_n \\ &\quad \in \mathrm{Fix}(G_n) \text{ such that } A_1 x_1 = A_2 x_2 = \cdots = A_n x_n, \end{aligned} \tag{9}$$

and presented the following simultaneous iterative algorithm:

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^{n} A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k A_1^*\left(A_1 x_1^k - \omega_k\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1 - \alpha_k) G_1\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k A_2^*\left(A_2 x_2^k - \omega_k\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1 - \alpha_k) G_2\left(u_2^k\right), \\ \cdots \\ u_n^k = x_n^k - \lambda_k A_n^*\left(A_n x_n^k - \omega_k\right), \\ x_n^{k+1} = \alpha_k u_n^k + (1 - \alpha_k) G_n\left(u_n^k\right), \end{cases} \tag{10}$$

where G_i are the G-mapping generated by $\{T_i\}_{i=1}^{N}$ which is a finite family of k_i-strictly pseudononspreading. They obtained the weak convergence of (10).

Motivated by the works mentioned above, we continue to study the ESEP (5) and ESEFPP (9) with internal perturbation projection and do not need any prior information about the operator norms. The paper is organized as follows. In Section 2, we introduce some preliminaries to be employed in the subsequent analysis. In Section 3, we present two simultaneous iterative algorithms to solve ESEP (5) and establish the weak and the strong convergence of the proposed algorithms, respectively. We propose a simultaneous iterative algorithm to solve ESEFPP (9) and obtain the weak convergence of the proposed algorithm in Section 4. In Section 5, we apply our iterative algorithms to some convex and nonlinear problems. In the concluding section, several numerical results are shown to confirm the effectiveness of our algorithms.

2. Preliminaries

In this paper, we use $\longrightarrow$ and $\rightharpoonup$ to denote the strong convergence and the weak convergence, respectively. We use $\omega_\omega(x_i^k) = \{x : \exists x_i^{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_i^k\}$. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{11}$$

It is well known that P_C is nonexpansive and firmly nonexpansive. And P_C has the following well-known properties.

Lemma 1. *Let $C \subset H$ be a nonempty, closed, and convex set. The following conclusions hold:*

(1) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0, \forall x \in H, \forall z \in C$

(2) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \forall x, y \in H$

(3) $\langle P_C(x) - x, y - x\rangle \geq \|P_C(x) - x\|^2, \forall x \in H, \forall y \in C$

Definition 2. A mapping $T : C \longrightarrow C$ is said to be

(1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$

(2) quasinonexpansive if $\|Tx - x^*\| \leq \|x - x^*\|, \forall x \in C, x^* \in \mathrm{Fix}(T)$

(3) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C$

(4) firmly quasinonexpansive if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|(I - T)x\|^2, \forall x \in C, x^* \in \mathrm{Fix}(T)$

(5) k-strictly pseudononspreading if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \exists k \in [0, 1), \forall x, y \in C$

Lemma 3 [24]. *In the real Hilbert space H, for $\forall x, y \in H$, the following relations hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(2) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

(3) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \alpha \in [0, 1]$

(4) $\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i \neq j} \alpha_i \alpha_j \|x_i - x_j\|^2$, *for* $\sum_{i=1}^m \alpha_i = 1, \alpha_i \in [0, 1], i = \{1, 2, \cdots, m\}$

Definition 4 [25]. A mapping $T : C \longrightarrow C$ is said to be demiclosed at 0 if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and with $\|x_n - T(x_n)\| \longrightarrow 0$, $T(x) = x$.

Lemma 5 [26]. *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} \leq \Gamma_{k_j+1}$ for all $j \in N$. Define the integer sequence $\{\tau(k)\}$, for $k > k_0$ (such that $\Gamma_{k_0} \leq \Gamma_{k_0+1}$) is as follows:*

$$\tau(k) = \max\{l \in N \mid k_0 \leq l < k, \Gamma_l \leq \Gamma_{l+1}\}. \tag{12}$$

Then, there hold the following properties:

(1) $\tau(k_0) \leq \tau(k_0 + 1) \leq \cdots$ *and* $\tau(k) \longrightarrow \infty$

(2) $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ *and* $\Gamma_k \leq \Gamma_{\tau(k)+1}, \forall k \geq k_0$

Lemma 6 [27]. *Assume that $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \alpha_k)a_k + \alpha_k \delta_k, \forall k \geq 0, \tag{13}$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence in $\Re$ such that

(1) $\sum_{k=1}^\infty \alpha_k = \infty$ *and* $\lim_{k\to\infty} \alpha_k = 0$

(2) $\limsup_{k\to\infty} \delta_k \leq 0$ *or* $\sum_{k=1}^\infty \alpha_k \mid \delta_k \mid < \infty$

Then, $\lim_{k\to\infty} a_k = 0$.

Definition 7 [19]. Let C be a nonempty closed and convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself. For $i = 1, 2, \cdots, N$, let $\pi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. We define a mapping $G : C \longrightarrow C$ as follows:

$$\begin{cases} U_0 = I, \\ U_1 = \alpha_1 T_1^2 U_0 + \beta_1 T_1 U_0 + \gamma_1 U_0 + \delta_1 I, \\ U_2 = \alpha_2 T_2^2 U_1 + \beta_2 T_2 U_1 + \gamma_2 U_1 + \delta_2 I, \\ U_3 = \alpha_3 T_3^2 U_2 + \beta_3 T_3 U_2 + \gamma_3 U_2 + \delta_3 I, \\ \cdots \\ U_{N-1} = \alpha_{N-1} T_{N-1}^2 U_{N-2} + \beta_{N-1} T_{N-1} U_{N-2} + \gamma_{N-1} U_{N-2} + \delta_{N-1} I, \\ G = U_N = \alpha_N T_N^2 U_{N-1} + \beta_N T_N U_{N-1} + \gamma_N U_{N-1} + \delta_N I. \end{cases} \tag{14}$$

Such a mapping G is called the G-mapping generated by $T_1, T_2, \cdots, T_N$ and $\pi_1, \pi_2, \cdots, \pi_N$.

Lemma 8 [19]. *Let C be a nonempty closed and convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of ρ_i-strictly pseudononspreading mappings of C into itself with $\bigcap_{i=1}^N Fix(T_i) \neq \varnothing$ and $\rho = \max_{i=1,2,\cdots,N}\{\rho_i\}$. For $i = 1, 2, \cdots, N$, let $\pi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. Assume that $\rho \leq \beta_i$ for $i = 1, 2, \cdots, N$, and $(\alpha_i + \beta_i)\rho \leq \beta_i\gamma_i$ for $i = 2, 3, \cdots, N$, and $(\alpha_1 + \beta_1)\rho \leq \beta_1(\gamma_1 + \delta_1)$. If G is the G-mapping generated by $T_1, T_2, \cdots, T_N$ and $\pi_1, \pi_2, \cdots, \pi_N$, then $Fix(G) = \bigcap_{i=1}^N Fix(T_i)$ and G is a quasinonexpansive mapping.*

3. Iterative Algorithms for ESEP

In this section, we introduce two simultaneous iterative algorithms with internal perturbation projection to solve ESEP (5) and define the solution set of ESEP (5) as Ω_1.

Algorithm 9. **Initialization:** take $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^{n} A_i x_i^k}{n}, \\ x_1^{k+1} = x_1^k - \lambda_k \left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^* \left(A_1 x_1^k - \omega_k \right) \right), \\ x_2^{k+1} = x_2^k - \lambda_k \left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^* \left(A_2 x_2^k - \omega_k \right) \right), \\ \cdots \\ x_n^{k+1} = x_n^k - \lambda_k \left(\left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^* \left(A_n x_n^k - \omega_k \right) \right), \end{cases} \tag{15}$$

where the stepsize λ_k is chosen in such a way that if $k \in \Omega$, then

$$\lambda_k \in \left(\varepsilon, \rho_k \frac{2\sum_{i=1}^{n} \left\| x_i^k - P_{C_i}\left(x_i^k\right)\right\|^2 + \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2}{\sum_{i=1}^{n} \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2} \right), \tag{16}$$

where the index set $\Omega = \{k : A_i x_i^k - \omega_k \neq 0, \exists i \in \{1, 2, \cdots, n\}\}$, small enough $\varepsilon > 0$ and $0 < \delta_1 < \rho_k < \delta_2 < 1$, set $k := k + 1$ and go to (15). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2, \cdots, n$, the iteration stops.

Remark 10. Note that in (16), the choice of the stepsize $\{\lambda_k\}$ is independent of the norm $\|A_i\|$, for $i = 1, 2, \cdots, n$. Furthermore, we will show from Lemma 11 that $\{\lambda_k\}$ is well defined.

Lemma 11. *Assume the solution set Ω_1 of ESEP (5) is nonempty, then $\{\lambda_k\}$ defined by (16) is well defined.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_1$, then $A_1 z_1 = A_2 z_2 = \cdots = A_n z_n$. Noting that for $i = 1, 2, \cdots, n$, we have

$$\begin{aligned} &\left\langle \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right), x_i^k - z_i \right\rangle \\ &\quad = \left\langle x_i^k - P_{C_i}\left(x_i^k\right), x_i^k - z_i \right\rangle + \left\langle A_i x_i^k - \omega_k, A_i x_i^k - A_i z_i \right\rangle \\ &\quad = \left\langle x_i^k - P_{C_i}\left(x_i^k\right), x_i^k - z_i \right\rangle + \left\langle A_i x_i^k - \omega_k, A_i x_i^k - \omega_k \right\rangle \\ &\qquad + \left\langle A_i x_i^k - \omega_k, \omega_k - A_i z_i \right\rangle. \end{aligned} \tag{17}$$

As a result,

$$\begin{aligned} &\left\langle \left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^* \left(A_1 x_1^k - \omega_k \right), x_1^k - z_1 \right\rangle \\ &\quad = \left\langle x_1^k - P_{C_1}\left(x_1^k\right), x_1^k - z_1 \right\rangle + \left\langle A_1 x_1^k - \omega_k, A_1 x_1^k - \omega_k \right\rangle \\ &\qquad + \left\langle A_1 x_1^k - \omega_k, \omega_k - A_1 z_1 \right\rangle, \\ &\qquad \left\langle \left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^* \left(A_2 x_2^k - \omega_k \right), x_2^k - z_2 \right\rangle \\ &\quad = \left\langle x_2^k - P_{C_2}\left(x_2^k\right), x_2^k - z_2 \right\rangle + \left\langle A_2 x_2^k - \omega_k, A_2 x_2^k - \omega_k \right\rangle \\ &\qquad + \left\langle A_2 x_2^k - \omega_k, \omega_k - A_2 z_2 \right\rangle, \\ &\qquad \left\langle \left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^* \left(A_n x_n^k - \omega_k \right), x_n^k - z_n \right\rangle \\ &\quad = \left\langle x_n^k - P_{C_n}\left(x_n^k\right), x_n^k - z_n \right\rangle + \left\langle A_n x_n^k - \omega_k, A_n x_n^k - \omega_k \right\rangle \\ &\qquad + \left\langle A_n x_n^k - \omega_k, \omega_k - A_n z_n \right\rangle. \end{aligned} \tag{18}$$

Summing the above equalities and applying Lemma 1 (3) as well as the condition $A_1 z_1 = A_2 z_2 = \cdots = A_n z_n$, one has

$$\begin{aligned} &\sum_{i=1}^{n} \left\| x_i^k - P_{C_i}\left(x_i^k\right)\right\|^2 + \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2 \\ &\quad \leq \sum_{i=1}^{n} \left\langle \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right), x_i^k - z_i \right\rangle \\ &\quad \leq \sum_{i=1}^{n} \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\| \left\| x_i^k - z_i \right\|. \end{aligned} \tag{19}$$

Consequently, for $k \in \Omega$, we have $\sum_{i=1}^{n} \|A_i x_i^k - \omega_k\|^2 \neq 0$, then $\sum_{i=1}^{n} \|(x_i^k - P_{C_i}(x_i^k)) + A_i^*(A_i x_i^k - \omega_k)\| \neq 0$, which leads that λ_k is well defined.

Theorem 12. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, assume that C_i are nonempty closed convex subsets of real Hilbert spaces H_i, and $A_i : H_i \longrightarrow H$ are bounded linear operators with their adjoint operators A_i^*. Then, the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 9 converges weakly to a solution of ESEP (5). Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - P_{C_i}(x_i^k)\| \longrightarrow 0$, $\|A_i x_i^k - \omega_k\| \longrightarrow 0$ and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_1$ and

$$z = A_1 z_1 = A_2 z_2 = \cdots = A_n z_n. \tag{20}$$

For $i = 1, 2, \cdots, n$, from (15) and Lemma 3, we have

$$\begin{aligned} &\left\| x_i^{k+1} - z_i \right\|^2 \\ &\quad = \left\| x_i^k - \lambda_k \left(\left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right) - z_i \right\|^2 \\ &\quad = \left\| x_i^k - z_i \right\|^2 + \lambda_k^2 \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2 \\ &\qquad - 2\lambda_k \left\langle x_i^k - z_i, \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle \\ &\quad = \left\| x_i^k - z_i \right\|^2 + \lambda_k^2 \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right)\right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2 \\ &\qquad - 2\lambda_k \left\langle x_i^k - z_i, x_i^k - P_{C_i}\left(x_i^k\right) \right\rangle \\ &\qquad - 2\lambda_k \left\langle x_i^k - z_i, A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle. \end{aligned} \tag{21}$$

According to Lemma 1 and

$$\begin{aligned} &-2\left\langle x_i^k - z_i, A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle \\ &\quad = -2\left\langle A_i x_i^k - A_i z_i, A_i x_i^k - \omega_k \right\rangle \\ &\quad = -\left\| A_i x_i^k - A_i z_i \right\|^2 - \left\| A_i x_i^k - \omega_k \right\|^2 + \|A_i z_i - \omega_k\|^2, \end{aligned} \tag{22}$$

(21) can be written as

$$\begin{aligned}\left\|x_i^{k+1}-z_i\right\|^2 \le{}& \left\|x_i^k-z_i\right\|^2 \\ &+\lambda_k^2\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2 \\ &-2\lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2-\lambda_k\left\|A_ix_i^k-A_iz_i\right\|^2 \\ &-\lambda_k\left\|A_ix_i^k-\omega_k\right\|^2+\lambda_k\|A_iz_i-\omega_k\|^2.\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}\|A_iz_i-\omega_k\|^2 &= \left\|A_iz_i-\frac{\sum_{i=1}^n A_ix_i^k}{n}\right\|^2=\left\|\frac{\sum_{i=1}^n\left(A_iz_i-A_ix_i^k\right)}{n}\right\|^2 \\ &\le \frac{\sum_{i=1}^n\left\|A_iz_i-A_ix_i^k\right\|^2}{n}.\end{aligned} \tag{24}$$

Summing (23) for i from 1 to n, we can obtain

$$\begin{aligned}\sum_{i=1}^n\left\|x_i^{k+1}-z_i\right\|^2 \le{}& \sum_{i=1}^n\left\|x_i^k-z_i\right\|^2+\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)\right. \\ &\left.+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2-2\lambda_k\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2 \\ &-\lambda_k\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\le\sum_{i=1}^n\left\|x_i^k-z_i\right\|^2+\lambda_k\rho_k \\ &\cdot\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right) \\ &-\lambda_k\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right) \\ ={}& \sum_{i=1}^n\left\|x_i^k-z_i\right\|^2-\lambda_k(1-\rho_k) \\ &\cdot\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right).\end{aligned} \tag{25}$$

Let $\Gamma_k(z_1,z_2,\cdots,z_n)=\sum_{i=1}^n\|x_i^k-z_i\|^2$, we have

$$\Gamma_{k+1}(z_1,z_2,\cdots,z_n)\le\Gamma_k(z_1,z_2,\cdots,z_n). \tag{26}$$

Therefore, the sequence $\{\Gamma_k(z_1,z_2,\cdots,z_n)\}$ is nonincreasing and lower bounded by 0. Hence, $\{\Gamma_k(z_1,z_2,\cdots,z_n)\}$ converges to some finite limit, suppose as $l(z_1,z_2,\cdots,z_n)$, and sequence $\{x_i^k\}$ is bounded. Letting $k\longrightarrow\infty$ and taking the limit in the two sides of (25), for $0<\delta_1<\rho_k<\delta_2<1$, we can obtain that

$$\lambda_k(1-\rho_k)\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right)\longrightarrow 0, \tag{27}$$

which implies

$$2\lambda_k\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2\longrightarrow 0,\quad \lambda_k\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\longrightarrow 0. \tag{28}$$

Since $\lambda_k>\varepsilon$, then

$$\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|\longrightarrow 0,\ \left\|A_ix_i^k-\omega_k\right\|\longrightarrow 0. \tag{29}$$

From the definition of λ_k, we have

$$\begin{aligned}&\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2 \\ &\le\lambda_k\rho_k\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right)\longrightarrow 0,\end{aligned} \tag{30}$$

then,

$$\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2\longrightarrow 0, \tag{31}$$

which implies

$$\lambda_k\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0. \tag{32}$$

From (32), for $k\longrightarrow\infty$, we can get

$$\begin{aligned}\left\|x_i^{k+1}-x_i^k\right\| &= \left\|\lambda_k\left(\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right)\right\| \\ &= \lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0.\end{aligned} \tag{33}$$

Consequently, the sequence $\{x_i^k\}$ is asymptotically regular.

Let $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^k,x_2^k,\cdots,x_n^k)$, then for $i=1,2,\cdots,n$, there exists a subsequence of $\{x_i^k\}$ which converges weakly to $\bar{x}_i$. From (29) and the lower semicontinuity of the squared norm, we have

$$\begin{aligned}0\le\left\|\bar{x}_i-P_{C_i}(\bar{x}_i)\right\| &\le\liminf_{j\to\infty}\left\|x_i^{k_j}-P_{C_i}x_i^{k_j}\right\| \\ &=\lim_{k\to\infty}\left\|x_i^k-P_{C_i}x_i^k\right\|=0,\end{aligned} \tag{34}$$

which implies $\bar{x}_i\in C_i$.

Furthermore, it follows from (29) and the lower semicontinuity of the squared norm

$$\|A_i\bar{x}_i - \bar{\omega}\|^2 \le \liminf_{k\to\infty} \left\|A_i x_i^k - \omega_k\right\|^2 = 0, \tag{35}$$

where $\bar{\omega} = (\sum_{i=1}^n A_i\bar{x}_i)/n$, which means that $A_i\bar{x}_i - \bar{\omega} = 0$. That is,

$$\begin{aligned} &A_2\bar{x}_2 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n = (n-1)A_1\bar{x}_1, \\ &A_1\bar{x}_1 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n = (n-1)A_2\bar{x}_2, \\ &\cdots \\ &A_1\bar{x}_1 + A_2\bar{x}_2 + \cdots + A_{n-1}\bar{x}_{n-1} = (n-1)A_n\bar{x}_n. \end{aligned} \tag{36}$$

Obviously,

$$A_1\bar{x}_1 = A_2\bar{x}_2 = \cdots = A_n\bar{x}_n. \tag{37}$$

Hence, $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Omega_1$.

In the sequel, we will show the uniqueness of the weak cluster points of $\{x_i^k\}$. Assume that x_i^* is another weak cluster point of $\{x_i^k\}$. It follows from the definition of Γ_k that

$$\begin{aligned} \Gamma_k(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) &= \sum_{i=1}^n \left\|x_i^k - \bar{x}_i\right\|^2 = \sum_{i=1}^n \left\|x_i^k - x_i^*\right\|^2 \\ &\quad + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 + 2\sum_{i=1}^n \left\langle x_i^k - x_i^*, x_i^* - \bar{x}_i\right\rangle \\ &= \Gamma_k(x_1^*, x_2^*, \cdots, x_n^*) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 \\ &\quad + 2\sum_{i=1}^n \left\langle x_i^k - x_i^*, x_i^* - \bar{x}_i\right\rangle. \end{aligned} \tag{38}$$

Without loss of generality, we can suppose $x_i^k \rightharpoonup x_i^*$, and then

$$l(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) = l(x_1^*, x_2^*, \cdots, x_n^*) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2. \tag{39}$$

Reversing the role of $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ and $(x_1^*, x_2^*, \cdots, x_n^*)$, we can obtain

$$l(x_1^*, x_2^*, \cdots, x_n^*) = l(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2. \tag{40}$$

Adding (39) and (40), we deduce

$$\sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 = 0, \tag{41}$$

which yields that $x_i^* = \bar{x}_i$. Hence, the sequence $\{x_i^k\}$ weakly converges to a solution of ESEP (5), which completes the proof.

In the following, we introduce another simultaneous iterative algorithm with internal perturbation projection to solve ESEP (5) and prove the strong convergence of the algorithm.

Algorithm 13. **Initialization:** let $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k\left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k v_1 + (1-\alpha_k)u_1^k, \\ u_2^k = x_2^k - \lambda_k\left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k v_2 + (1-\alpha_k)u_2^k, \\ \cdots \\ u_n^k = x_n^k - \lambda_k\left(\left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^*\left(A_nx_n^k - \omega_k\right)\right), \\ x_n^{k+1} = \alpha_k v_n + (1-\alpha_k)u_n^k, \end{cases} \tag{42}$$

where $\{\lambda_k\}$ is the same as Algorithm 9, $(v_1, v_2, \cdots, v_n) \in H_1 \times H_2 \times \cdots \times H_n$.

Theorem 14. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let C_i be nonempty closed convex subsets of real Hilbert spaces H_i, respectively. $A_i : H_i \longrightarrow H$ are bounded linear operators with their adjoint operators A_i^*, respectively. If $\alpha_k \in (0, 1)$, $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_{k=1}^\infty \alpha_k = \infty$, then the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 13 converges strongly to a solution of ESEP (5) denoted by $(z_1, z_2, \cdots, z_n) = P_{\Omega_1}(v_1, v_2, \cdots v_n)$. Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - P_{C_i}(x_i^k)\| \longrightarrow 0$, $\|A_ix_i^k - \omega_k\| \longrightarrow 0$ and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) = P_{\Omega_1}(v_1, v_2, \cdots v_n)$, which means $(z_1, z_2, \cdots, z_n) \in \Omega_1$. Let

$$z = A_1z_1 = A_2z_2 = \cdots = A_nz_n. \tag{43}$$

Similar to the proof of the Theorem 12, we have the following inequality

$$\begin{aligned} \sum_{i=1}^n \left\|u_i^k - z_i\right\|^2 &\le \sum_{i=1}^n \left\|x_i^k - z_i\right\|^2 - \lambda_k(1-\rho_k) \\ &\quad \cdot \left(2\sum_{i=1}^n \left\|x_i^k - P_{C_i}\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n \left\|A_ix_i^k - \omega_k\right\|^2\right). \end{aligned} \tag{44}$$

From Lemma 3, we have

$$\begin{aligned}\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2 &= \sum_{i=1}^{n}\left\|\alpha_k v_i+(1-\alpha_k)u_i^k-z_i\right\|^2\\ &\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|u_i^k-z_i\right\|^2\\ &\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\\ &\quad\cdot\left(\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2-\lambda_k(1-\rho_k)\right.\\ &\quad\cdot\left.\left(2\sum_{i=1}^{n}\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^{n}\left\|A_ix_i^k-\omega_k\right\|^2\right)\right)\\ &\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\ &\le \max\left\{\sum_{i=1}^{n}\|v_i-z_i\|^2,\ \sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\right\}.\end{aligned} \tag{45}$$

By induction, we can get

$$\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2\le\max\left\{\sum_{i=1}^{n}\|v_i-z_i\|^2,\ \sum_{i=1}^{n}\left\|x_i^1-z_i\right\|^2\right\}, \tag{46}$$

which implies that sequence $\{\Gamma_k\}$ is bounded by setting $\Gamma_k=\sum_{i=1}^{n}\|x_i^k-z_i\|^2$. In the sequel, we divide the proof into two cases.

Case 1. Assume that there exists integer $k_0>0$ such that $\{\Gamma_k\}$ is decreasing sequence for all $k>k_0$, which implies $\lim_{k\to\infty}\Gamma_k$ exists. Consequently, (45) can be rewritten as

$$\begin{aligned}&\lambda_k(1-\rho_k)\left(2\sum_{i=1}^{n}\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^{n}\left\|A_ix_i^k-\omega_k\right\|^2\right)\\ &\le-\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\ &\quad+\alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2.\end{aligned} \tag{47}$$

Similar to the proof of Theorem 12, taking limit in the two sides of (47) for $k\longrightarrow\infty$, $i=1,2,\cdots n$, we can obtain

$$\begin{aligned}&\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|\longrightarrow 0\,;\ \left\|A_ix_i^k-\omega_k\right\|\longrightarrow 0,\\ &\lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0.\end{aligned} \tag{48}$$

For another, since $\{x_i^k\}$ is bounded, $\{x_i^k\}$ has a convergent subsequence. Without loss of generality, we have $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^k,x_2^k,\cdots,x_n^k)$. From (42), for $k\longrightarrow\infty$, we have

$$\begin{aligned}\left\|u_i^k-x_i^k\right\| &= \left\|\lambda_k\left(\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right)\right\|\\ &= \lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0,\end{aligned} \tag{49}$$

which implies $\{u_i^k\}$ is bounded. Since $\lim_{k\to\infty}\alpha_k=0$, for $k\longrightarrow\infty$, we can infer

$$\begin{aligned}\left\|x_i^{k+1}-u_i^k\right\| &= \left\|\alpha_k v_i+(1-\alpha_k)u_i^k-u_i^k\right\|\\ &= \left\|\alpha_k v_i-\alpha_k u_i^k\right\|=\alpha_k\left\|v_i-u_i^k\right\|\longrightarrow 0.\end{aligned} \tag{50}$$

Combining (49) and (50), we have

$$\left\|x_i^{k+1}-x_i^k\right\|\le\left\|x_i^{k+1}-u_i^k\right\|+\left\|u_i^k-x_i^k\right\|\longrightarrow 0. \tag{51}$$

Similar to the proof of Theorem 12, we can get $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\Omega_1$. Since $\{x_i^k\}$ is asymptotically regular, we can get $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^{k+1},x_2^{k+1},\cdots,x_n^{k+1})$. Depending on the property of the projection, we have

$$\begin{aligned}&\limsup_{k\to\infty}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle\\ &=\lim_{j\to\infty}\left\langle v_i-z_i,x_i^{k_j+1}-z_i\right\rangle\\ &=\left\langle v_i-z_i,\bar{x}_i-z_i\right\rangle\le 0.\end{aligned} \tag{52}$$

$$\begin{aligned}\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2 &= \sum_{i=1}^{n}\left\|\alpha_k v_i+(1-\alpha_k)u_i^k-z_i\right\|^2\\ &\le(1-\alpha_k)^2\sum_{i=1}^{n}\left\|u_i^k-z_i\right\|^2\\ &\quad+2\alpha_k\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle\\ &\le(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\ &\quad+2\alpha_k\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle,\end{aligned} \tag{53}$$

which combining with Lemma 6 and (52), we conclude the sequence $\{(x_1^k,x_2^k,\cdots,x_n^k)\}$ converges strongly to $(z_1,z_2,\cdots,z_n)$.

Case 2. Assume $\{\Gamma_k\}$ is not a decreasing sequence. Then suppose that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$, for $j\ge 0$, such that $\Gamma_{k_j}\le\Gamma_{k_j+1}$. For all $k>k_0$ (such that $\Gamma_{k_0}\le\Gamma_{k_0+1}$), define a integer sequence as follows:

$$\tau(k)=\max\{l\in N\mid k_0\le l<k,\Gamma_l\le\Gamma_{l+1}\}. \tag{54}$$

It is clear that $\tau(k)$ is a nondecreasing sequence satisfying $\lim_{k\to\infty}\tau(k)=\infty$, and $\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}$.

Similar to the proof of Case 1, we can obtain

$$\begin{aligned}\lambda_{\tau(k)}\left(1-\rho_{\tau(k)}\right)&\left(2\sum_{i=1}^{n}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)\right\|^2\right.\\&\left.+\sum_{i=1}^{n}\left\|A_ix_i^{\tau(k)}-\omega_{\tau(k)}\right\|^2\right)\\&\le-\Gamma_{\tau(k)+1}+\left(1-\alpha_{\tau(k)}\right)\Gamma_{\tau(k)}+\alpha_{\tau(k)}\sum_{i=1}^{n}\|v_i-z_i\|^2.\end{aligned}\tag{55}$$

Taking the limit in the two sides of (55) for $k\longrightarrow\infty$, according to Case 1, we have

$$\begin{aligned}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)\right\|&\longrightarrow0,\\\left\|A_ix_i^{\tau(k)}-\omega_{\tau(k)}\right\|&\longrightarrow0,\\\lambda_{\tau(k)}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)+A^*\left(A_ix_i^{\tau(k)}-\omega_{\tau(k)}\right)\right\|&\longrightarrow0,\\\left\|x_i^{\tau(k)+1}-x_i^{\tau(k)}\right\|&\longrightarrow0.\end{aligned}\tag{56}$$

That implies $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^{\tau(k)+1},x_2^{\tau(k)+1},\cdots,x_n^{\tau(k)+1})$. Depending on the property of the projection, we have

$$\limsup_{\tau(k)\to\infty}\left\langle v_i-z_i,x_i^{\tau(k)+1}-z_i\right\rangle\le0.\tag{57}$$

From (53), we can get

$$\Gamma_{\tau(k)+1}\le\left(1-\alpha_{\tau(k)}\right)\Gamma_{\tau(k)}+2\alpha_{\tau(k)}\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{\tau(k)+1}-z_i\right\rangle.\tag{58}$$

Since $\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}$, we have

$$\Gamma_{\tau(k)}\le2\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{\tau(k)+1}-z_i\right\rangle,\tag{59}$$

which implies $\limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)}\le0$, that is,

$$\lim_{\tau(k)\to\infty}\Gamma_{\tau(k)}=0.\tag{60}$$

By (58), we obtain

$$\limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)+1}\le\limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)}\le0,\tag{61}$$

which implies

$$\lim_{\tau(k)\to\infty}\Gamma_{\tau(k)+1}=0.\tag{62}$$

By Lemma 5, we have $\Gamma_k\le\Gamma_{\tau(k)+1}$, which means the sequence $\{(x_1^k,x_i^k,\cdots,x_n^k)\}$ converges strongly to $(z_1,z_2,\cdots,z_n)$. Thus, we complete the proof.

4. Iterative Algorithm for ESEFPP

In this section, we introduce a simultaneous iterative algorithm with internal perturbation projection to solve ESEFPP (9) and define the solution set of ESEFPP (9) as Ω_2.

Algorithm 15. **Initialization:** let $0<a<\alpha_k<b<1$, $(x_1^1,x_2^1,\cdots,x_n^1)\in H_1\times H_2\times\cdots\times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k,x_2^k,\cdots,x_n^k)\in H_1\times H_2\times\cdots\times H_n$, we calculate the next iterate $(x_1^{k+1},x_2^{k+1},\cdots,x_n^{k+1})$ by

$$\begin{cases}\omega_k=\dfrac{\sum_{i=1}^{n}A_ix_i^k}{n},\\u_1^k=x_1^k-\lambda_k\left(\left(x_1^k-G_1\left(x_1^k\right)\right)+A_1^*\left(A_1x_1^k-\omega_k\right)\right),\\x_1^{k+1}=\alpha_ku_1^k+(1-\alpha_k)G_1\left(u_1^k\right),\\u_2^k=x_2^k-\lambda_k\left(\left(x_2^k-G_2\left(x_2^k\right)\right)+A_2^*\left(A_2x_2^k-\omega_k\right)\right),\\x_2^{k+1}=\alpha_ku_2^k+(1-\alpha_k)G_2\left(u_2^k\right),\\\cdots\\u_n^k=x_n^k-\lambda_k\left(\left(x_n^k-G_n\left(x_n^k\right)\right)+A_n^*\left(A_nx_n^k-\omega_k\right)\right),\\x_n^{k+1}=\alpha_ku_n^k+(1-\alpha_k)G_n\left(u_n^k\right),\end{cases}\tag{63}$$

where the stepsize λ_k is chosen in such a way that if $k\in\Omega$, then

$$\lambda_k\in\left(\varepsilon,\rho_k\frac{\sum_{i=1}^{n}\left\|x_i^k-G_i\left(x_i^k\right)\right\|^2+\sum_{i=1}^{n}\left\|A_ix_i^k-\omega_k\right\|^2}{\sum_{i=1}^{n}\left\|\left(x_i^k-G_i\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2}\right),\tag{64}$$

where the index set $\Omega=\{k:A_ix_i^k-\omega_k\ne0,\exists i\in\{1,2,\cdots,n\}\}$, small enough $\varepsilon>0$ and $0<\delta_1<\rho_k<\delta_2<1$, set $k:=k+1$ and go to (63). Otherwise, $A_ix_i^k-\omega_k=0$, for $i=1,2,\cdots,n$, the iteration stops.

Theorem 16. *Let H be a real Hilbert space. For $i=1,2,\cdots,n$, let $A_i:H_i\longrightarrow H$ be bounded linear operators with their adjoint operators A_i^*, respectively. And $G_i:H_i\longrightarrow H_i$ are quasinonexpansive mappings and demiclosed at 0. Then, the sequence $\{(x_1^k,x_2^k,\cdots,x_n^k)\}$ generated by Algorithm 15 weakly converges to a solution of ESEFPP (9). Furthermore, for $i=1,2,\cdots,n$, $\|x_i^k-G_i(x_i^k)\|\longrightarrow0$, $\|A_ix_i^k-\omega_k\|\longrightarrow0$, $\|G_i(u_i^k)-u_i^k\|\longrightarrow0$ and $\|x_i^{k+1}-x_i^k\|\longrightarrow0$ as $k\longrightarrow\infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_2$ and

$$z = A_1 z_1 = A_2 z_2 = \cdots = A_n z_n. \tag{65}$$

For $i = 1, 2, \cdots, n$, from (63) and Lemma 3 (2), we have

$$\begin{aligned}
\left\|u_i^k - z_i\right\|^2 &= \left\|x_i^k - \lambda_k\left(\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\right. \\
&\quad \left.\left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right) - z_i\right\|^2 \\
&= \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 \\
&\quad - 2\lambda_k\left\langle x_i^k - z_i, \left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle \\
&= \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 \\
&\quad - 2\lambda_k\left\langle x_i^k - z_i, x_i^k - G_i\left(x_i^k\right)\right\rangle \\
&\quad - 2\lambda_k\left\langle x_i^k - z_i, A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle.
\end{aligned} \tag{66}$$

It follows from

$$\begin{aligned}
&-2\left\langle x_i^k - z_i, x_i^k - G_i\left(x_i^k\right)\right\rangle \\
&\quad = -2\left\|x_i^k - z_i\right\|^2 - 2\left\langle x_i^k - z_i, z_i - G_i\left(x_i^k\right)\right\rangle \\
&\quad = -2\left\|x_i^k - z_i\right\|^2 + \left\|x_i^k - z_i\right\|^2 + \left\|G_i\left(x_i^k\right) - z_i\right\|^2 \\
&\qquad - \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 \le -\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2,
\end{aligned} \tag{67}$$

$$\begin{aligned}
&-2\left\langle x_i^k - z_i, A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle \\
&\quad = -2\left\langle A_i x_i^k - A_i z_i, A_i x_i^k - \omega_k\right\rangle \\
&\quad = \|A_i z_i - \omega_k\|^2 - \left\|A_i x_i^k - A_i z_i\right\|^2 - \left\|A_i x_i^k - \omega_k\right\|^2,
\end{aligned}$$

then (66) can be rewritten as

$$\begin{aligned}
\left\|u_i^k - z_i\right\|^2 &\le \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 \\
&\quad - \lambda_k\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \lambda_k\|A_i z_i - \omega_k\|^2 \\
&\quad - \lambda_k\left\|A_i x_i^k - A_i z_i\right\|^2 - \lambda_k\left\|A_i x_i^k - \omega_k\right\|^2.
\end{aligned} \tag{68}$$

Since G_i are quasinonexpansive mappings, by (68), we have

$$\begin{aligned}
\left\|x_i^{k+1} - z_i\right\|^2 &= \left\|\alpha_k u_i^k + (1-\alpha_k) G_i\left(u_i^k\right) - z_i\right\|^2 \\
&= \alpha_k\left\|u_i^k - z_i\right\|^2 + (1-\alpha_k)\left\|G_i\left(u_i^k\right) - z_i\right\|^2 \\
&\quad - \alpha_k(1-\alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2 \\
&\le \left\|u_i^k - z_i\right\|^2 - \alpha_k(1-\alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2 \\
&\le \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 - \lambda_k\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 \\
&\quad + \lambda_k\|A_i z_i - \omega_k\|^2 - \lambda_k\left\|A_i x_i^k - A_i z_i\right\|^2 \\
&\quad - \lambda_k\left\|A_i x_i^k - \omega_k\right\|^2 - \alpha_k(1-\alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2.
\end{aligned} \tag{69}$$

Note that

$$\begin{aligned}
\|A_i z_i - \omega_k\|^2 &= \left\|A_i z_i - \frac{\sum_{i=1}^n A_i x_i^k}{n}\right\|^2 = \left\|\frac{\sum_{i=1}^n \left(A_i z_i - A_i x_i^k\right)}{n}\right\|^2 \\
&\le \frac{\sum_{i=1}^n \left\|A_i z_i - A_i x_i^k\right\|^2}{n}.
\end{aligned} \tag{70}$$

Summing (69) for i from 1 to n, we can obtain

$$\begin{aligned}
\sum_{i=1}^n \left\|x_i^{k+1} - z_i\right\|^2 &\le \sum_{i=1}^n \left\|x_i^k - z_i\right\|^2 + \lambda_k^2 \sum_{i=1}^n \left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right. \\
&\quad \left. + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 - \lambda_k \sum_{i=1}^n \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 \\
&\quad - \lambda_k \sum_{i=1}^n \left\|A_i x_i^k - \omega_k\right\|^2 - \alpha_k(1-\alpha_k) \sum_{i=1}^n \\
&\quad \cdot \left\|G_i\left(u_i^k\right) - u_i^k\right\|^2 \le \sum_{i=1}^n \left\|x_i^k - z_i\right\|^2 + \lambda_k \rho_k \\
&\quad \cdot \left(\sum_{i=1}^n \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n \left\|A_i x_i^k - \omega_k\right\|^2\right) \\
&\quad - \lambda_k\left(\sum_{i=1}^n \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n \left\|A_i x_i^k - \omega_k\right\|^2\right) \\
&\quad - \alpha_k(1-\alpha_k) \sum_{i=1}^n \left\|G_i\left(u_i^k\right) - u_i^k\right\|^2 \\
&= \sum_{i=1}^n \|x_i^k - z_i\|^2 - \lambda_k(1-\rho_k) \\
&\quad \cdot \left(\sum_{i=1}^n \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n \left\|A_i x_i^k - \omega_k\right\|^2\right) \\
&\quad - \alpha_k(1-\alpha_k) \sum_{i=1}^n \left\|G_i\left(u_i^k\right) - u_i^k\right\|^2.
\end{aligned} \tag{71}$$

Let $\Gamma_k(z_1, z_2, \cdots, z_n) = \sum_{i=1}^n \left\|x_i^k - z_i\right\|^2$. According to the definition of $\{\lambda_k\}$, we have

$$\Gamma_{k+1}(z_1, z_2, \cdots, z_n) \le \Gamma_k(z_1, z_2, \cdots, z_n). \tag{72}$$

Therefore, the sequence $\{\Gamma_k(z_1, z_2, \cdots, z_n)\}$ is a nonincreasing sequence and lower bounded by 0. As a result $\{\Gamma_k(z_1, z_2, \cdots, z_n)\}$ converges to some finite limit, suppose as $l(z_1, z_2, \cdots, z_n)$, and sequence $\{x_i^k\}$ is bounded. Letting $n \longrightarrow \infty$ and taking limit in the two sides of (71) for $k \longrightarrow \infty$, we obtain that

$$\begin{aligned} \lim_{k\to\infty} \lambda_k \sum_{i=1}^{n} \left\| x_i^k - G_i\left(x_i^k\right) \right\|^2 = 0, \lim_{k\to\infty} \lambda_k \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2 \\ = 0, \lim_{k\to\infty} \left\| G_i\left(u_i^k\right) - u_i^k \right\| = 0. \end{aligned} \tag{73}$$

Since $\lambda_k > \varepsilon$, we have

$$\lim_{k\to\infty} \left\| x_i^k - G_i\left(x_i^k\right) \right\| = 0, \lim_{k\to\infty} \left\| A_i x_i^k - \omega_k \right\| = 0. \tag{74}$$

From the definition of λ_k, we have

$$\begin{aligned} &\lambda_k^2 \left(\sum_{i=1}^{n} \left\| \left(x_i^k - G_i\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2 \right) \\ &\le \lambda_k \rho_k \left(\sum_{i=1}^{n} \left\| x_i^k - G_i\left(x_i^k\right) \right\|^2 + \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2 \right) \longrightarrow 0, \end{aligned} \tag{75}$$

which implies

$$\lambda_k \left\| \left(x_i^k - G_i\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\| \longrightarrow 0. \tag{76}$$

Now, let us prove that $\{x_i^k\}$ is asymptotically regular. Indeed, from

$$\begin{aligned} \left\| u_i^k - x_i^k \right\| &= \left\| x_i^k - \lambda_k \left(\left(x_i^k - G_i\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right) - x_i^k \right\| \\ &= \lambda_k \left\| \left(x_i^k - G_i\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|, \end{aligned} \tag{77}$$

we have

$$\lim_{k\to\infty} \left\| u_i^k - x_i^k \right\| = 0. \tag{78}$$

Consequently,

$$\begin{aligned} \lim_{k\to\infty} \left\| x_i^{k+1} - x_i^k \right\| &= \lim_{k\to\infty} \left\| \alpha_k u_i^k + (1-\alpha_k) G_i\left(u_i^k\right) - x_i^k \right\| \\ &\le \lim_{k\to\infty} \left(\alpha_k \left\| u_i^k - x_i^k \right\| + (1-\alpha_k) \left\| G_i\left(u_i^k\right) - x_i^k \right\| \right) \\ &\le \lim_{k\to\infty} \left(\alpha_k \left\| u_i^k - x_i^k \right\| + (1-\alpha_k) \left\| G_i\left(u_i^k\right) - u_i^k \right\| \right. \\ &\quad \left. + (1-\alpha_k) \left\| u_i^k - x_i^k \right\| \right) = 0, \end{aligned} \tag{79}$$

which yields $\{x_i^k\}$ is asymptotically regular.

Let $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \omega_\omega(x_1^k, x_2^k, \cdots, x_n^k)$, then there exists a subsequence of $\{x_i^k\}$ which weakly converges to $\bar{x}_i$. For $\lim_{k\to\infty} \|u_i^k - x_i^k\| = 0$, we can obtain $\bar{x}_i \in \omega_\omega(u_i^k)$. Due to the demiclosednesses of G_i at 0 and $\lim_{k\to\infty} \|G_i(u_i^k) - u_i^k\| = 0$, it yields $G_i\bar{x}_i = \bar{x}_i$, which implies $\bar{x}_i \in \mathrm{Fix}(G_i)$. Furthermore, it follows from (74) and the lower semicontinuity of the squared norm, we have

$$\|A_i\bar{x}_i - \bar{\omega}\|^2 \le \liminf_{k\to\infty} \left\| A_i x_i^k - \omega_k \right\|^2 = 0, \tag{80}$$

where $\bar{\omega} = (\sum_{i=1}^{n} A_i\bar{x}_i)/n$, which means that $A_i\bar{x}_i - \bar{\omega} = 0$. That is,

$$\begin{aligned} A_2\bar{x}_2 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n &= (n-1)A_1\bar{x}_1, \\ A_1\bar{x}_1 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n &= (n-1)A_2\bar{x}_2, \\ &\cdots \\ A_1\bar{x}_1 + A_2\bar{x}_2 + \cdots + A_{n-1}\bar{x}_{n-1} &= (n-1)A_n\bar{x}_n. \end{aligned} \tag{81}$$

Obviously,

$$\begin{aligned} A_1\bar{x}_1 = A_2\bar{x}_2 = \cdots = A_n\bar{x}_n, \\ (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Omega_2. \end{aligned} \tag{82}$$

Finally, following the proof of Theorem 12, we can prove the uniqueness of the weak cluster points of $\{x_i^k\}$. Thus, we complete the proof.

Similar to the proof of Theorem 16, from Definition 7 and Lemma 8, we have the following corollary.

Corollary 17. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let $A_i : H_i \longrightarrow H$ be bounded linear operators with their adjoint operators A_i^* and C_i be nonempty closed convex subsets of real Hilbert spaces H_i. Let $\{T_j^i\}_{i=1}^N$ be a finite family of ρ_j^i-strictly pseudononspreading mappings of C_i into itself with $\bigcap_{i=1}^N Fix(T_j^i) \neq \varnothing$ and $\rho^i = \max_{j=1,2,\cdots,N}\{\rho_j^i\}$. For $i = 1, 2, \cdots, N$, let $\pi_j^i = (\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i)$, where $\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i \in [0, 1]$ and $\alpha_j^i + \beta_j^i + \gamma_j^i + \delta_j^i = 1$. Assume that $\rho^i \le \beta_j^i$ for $j = 1, 2, \cdots N$, and $(\alpha_j^i + \beta_j^i)\rho^i \le \beta_j^i\gamma_j^i$ for $j = 2, 3, \cdots, N$, and $(\alpha_1^i + \beta_1^i)\rho^i \le \beta_1^i(\gamma_1^i + \delta_1^i)$. If G_i are the G-mapping generated by $T_1^i, T_2^i, \cdots, T_N^i$ and $\pi_1^i, \pi_2^i, \cdots, \pi_N^i$ and demiclosed at 0, then the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 15 weakly converges to a solution of ESEFPP (9). Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - G_i(x_i^k)\| \longrightarrow 0$, $\|A_i x_i^k - \omega_k\| \longrightarrow 0$, $\|G_i(u_i^k) - u_i^k\| \longrightarrow 0$, and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

5. Application

In this section, we apply our iterative algorithms to some convex and nonlinear problems, for example, [11, 28].

5.1. Split Equality Feasibility Problem. Note that ESEP (5) reduces to SEFP (2) as $n = 2$. Then, we can obtain the following algorithm to study SEFP (2) from Algorithm 9.

Algorithm 18. **Initialization:** let $(x_1^1, x_2^1) \in H_1 \times H_2$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k) \in H_1 \times H_2$, we calculate the next iterate (x_1^{k+1}, x_2^{k+1}) by

$$\begin{cases} \omega_k = \dfrac{A_1 x_1^k + A_2 x_2^k}{2}, \\ x_1^{k+1} = x_1^k - \lambda_k \left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_2^{k+1} = x_2^k - \lambda_k \left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \end{cases} \tag{83}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 9. Set $k := k + 1$ and go to (83). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2$, iteration stops.

5.2. Split Equality Fixed Point Problem. Note that ESEFPP (9) reduces to SEFPP (7) as $n = 2$. Then, we can obtain the following algorithm to study SEFPP (7) from Algorithm 15.

Algorithm 19. **Initialization:** let $0 < a < \alpha_k < b < 1$, $(x_1^1, x_2^1) \in H_1 \times H_2$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k) \in H_1 \times H_2$, we calculate the next iterate (x_1^{k+1}, x_2^{k+1}) by

$$\begin{cases} \omega_k = \dfrac{A_1 x_1^k + A_2 x_2^k}{2}, \\ u_1^k = x_1^k - \lambda_k \left(\left(x_1^k - G_1\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1 - \alpha_k) G_1\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k \left(\left(x_2^k - G_2\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1 - \alpha_k) G_2\left(u_2^k\right), \end{cases} \tag{84}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 15. Set $k := k + 1$ and go to (84). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1$, 2, iteration stops.

5.3. Extend Split Equation Problem. Similar to Algorithm 15, we have another algorithm to solve ESEP (5).

Algorithm 20. **Initialization:** let $0 < a < \alpha_k < b < 1$, $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k \left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1 - \alpha_k) P_{C_1}\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k \left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1 - \alpha_k) P_{C_2}\left(u_2^k\right), \\ \cdots \\ u_n^k = x_n^k - \lambda_k \left(\left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^*\left(A_n x_n^k - \omega_k\right)\right), \\ x_n^{k+1} = \alpha_k u_n^k + (1 - \alpha_k) P_{C_n}\left(u_n^k\right), \end{cases} \tag{85}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 9. Set $k := k + 1$ and go to (85). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2, \cdots, n$, iteration stops.

6. Numerical Examples

We are in a position to show some numerical examples to demonstrate the performance and convergence of Algorithms 9 and 13. The whole programs are written in MATLAB 2017b. All the numerical results are carried out on a personal Lenovo computer with Intel(R) Core(TM) i7-7500U CPU 2.70 GHz RAM 4.00 GB. We denote the vector with all elements 1 by e in what follows.

Example 21. Let

$$\begin{aligned} A &= \begin{pmatrix} 0.694828622975817 & 0.950222048838355 & 0.438744359656398 \\ 0.317099480060861 & 0.0344460805029088 & 0.381558457093008 \end{pmatrix}, \\ B &= \begin{pmatrix} 0.765516788149002 & 0.186872604554379 \\ 0.795199901137063 & 0.489764395788231 \end{pmatrix}, \\ C &= \begin{pmatrix} 0.445586200710900 & 0.709364830858073 & 0.276025076998578 & 0.655098003973841 \\ 0.646313010111265 & 0.754686681982361 & 0.679702676853675 & 0.162611735194631 \end{pmatrix}. \end{aligned} \tag{86}$$

Table 1: The numerical results of Example 22.

	P	M	N	Q	Algorithm 9		Algorithm 13		ECCL	
					n	s	n	s	n	s
Case 1	10	20	9	25	191	0.016657	144	0.012685	206	0.027401
	25	35	30	35	475	0.019802	248	0.023921	555	0.044554
	50	30	40	50	1684	0.201384	857	0.103873	2675	0.202256
	10	20	9	25	184	0.019802	113	0.009622	230	0.026396
Case 2	25	35	30	35	367	0.048778	146	0.0017966	433	0.022374
	50	30	40	50	2147	0.264869	758	0.089149	2533	0.179859
	10	20	9	25	230	0.019426	170	0.015182	232	0.011593
Case 3	25	35	30	35	347	0.038227	269	0.028535	433	0.039384
	50	30	40	50	1935	0.235992	1446	0.175387	2087	0.131646
	10	20	9	25	192	0.016825	121	0.009010	213	0.014185
Case 4	25	35	30	35	361	0.047210	167	0.018008	467	0.040100
	50	30	40	50	2065	0.256091	562	0.066533	2307	0.180630

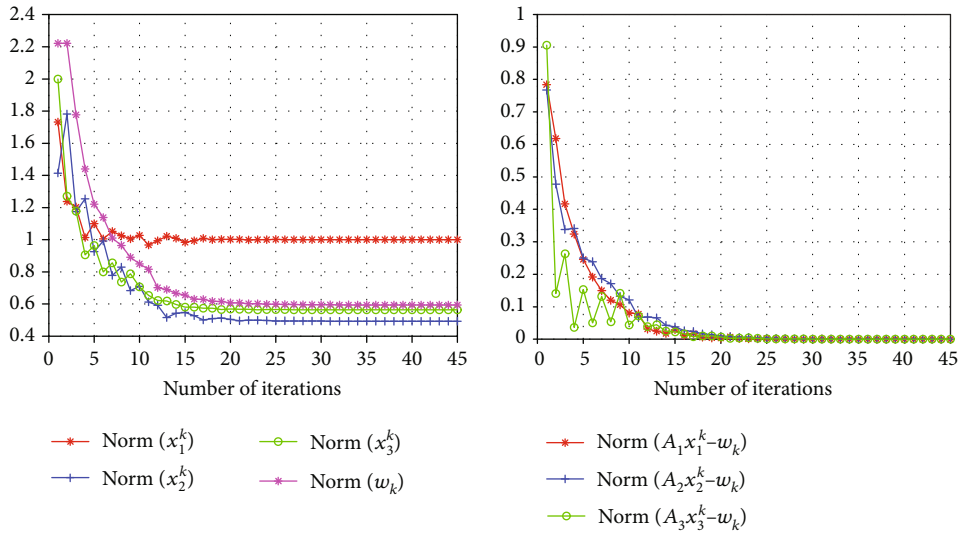

Figure 1: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_ix_i^k - \omega_k\}$ of Algorithm 9 for Example 21.

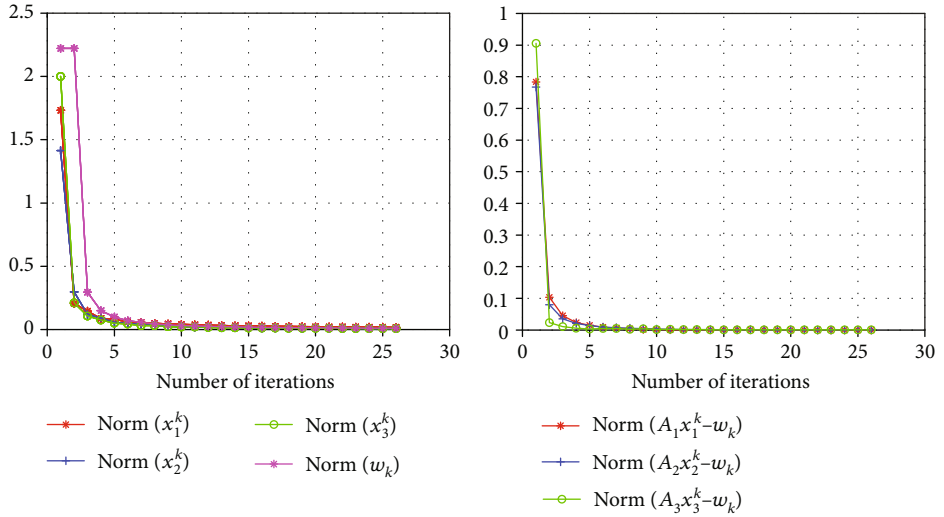

Figure 2: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_ix_i^k - \omega_k\}$ of Algorithm 13 for Example 21.

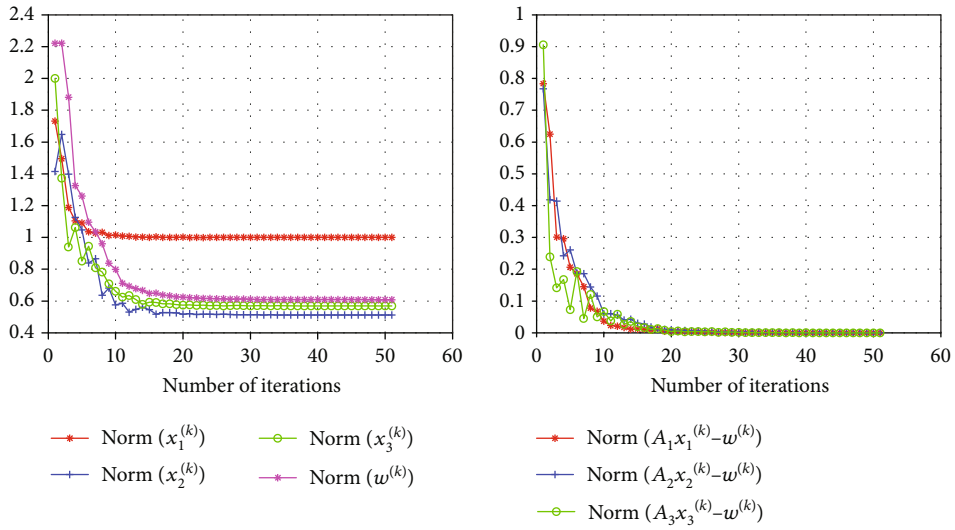

FIGURE 3: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_ix_i^k - \omega_k\}$ of ECCL for Example 21.

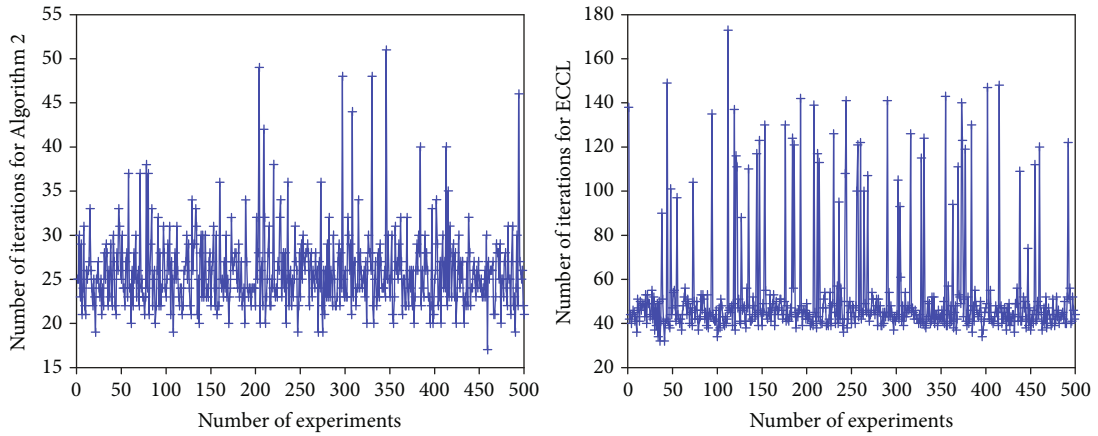

FIGURE 4: The iterative number of Algorithm 13 and ECCL for Case A.

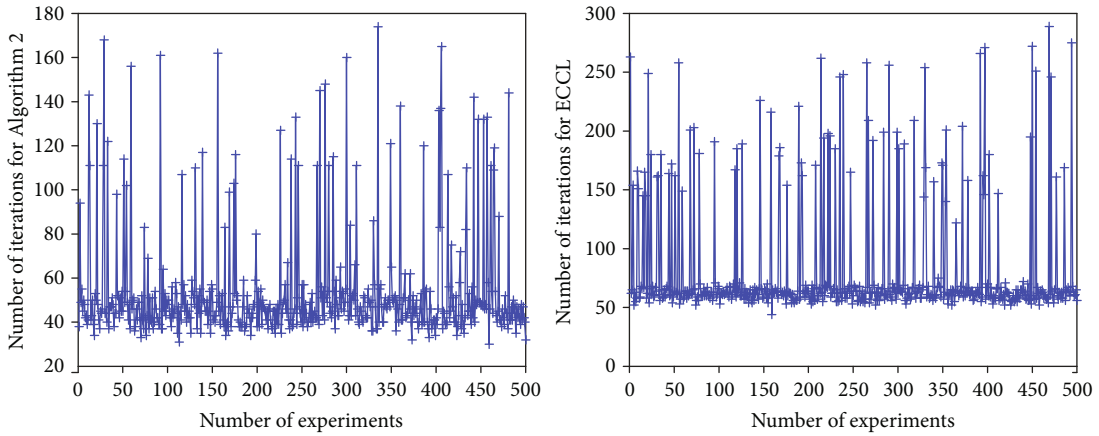

FIGURE 5: The iterative number of Algorithm 13 and ECCL for Case B.

$C_1 = \{x_1 \in R^3 : \|x_1\| \le 1\}$, $C_2 = \{x_2 \in R^2 : -3e \le x_2 \le e\}$, and $C_3 = \{x_3 \in R^4 : -2e \le x_3 \le 2e\}$, to find $(x, y, z) \in C_1 \times C_2 \times C_3$ such that $Ax = By = Cz$.

Example 22. Let $A_1 = (a_{ij})_{P\times M}$, $A_2 = (b_{ij})_{P\times N}$, $A_3 = (c_{ij})_{P\times Q}$, $C_1 = \{x_1 \in R^M : \|x_1\| \le 1\}$, $C_2 = \{x_2 \in R^N : \|x_2\| \le 2\}$, and $C_3 = \{x_3 \in Q : \|x_3\| \le 3\}$, where $a_{ij}, b_{ij}, c_{ij} \in [0, 1]$ are all generated randomly.

In the Example 22, we consider $P = 10, M = 20, N = 9, Q = 25$; $P = 25, M = 35, N = 30, Q = 35$; $P = 50, M = 30, N = 40, Q = 50$, and four initial values.

Case 1. $x = \text{ones}(M, 1)$, $y = \text{ones}(N, 1)$, and $z = 10 * \text{ones}(Q, 1)$.

Case 2. $x = -10 * \text{ones}(M, 1)$, $y = 0 * \text{ones}(N, 1)$, and $z = 10 * \text{ones}(Q, 1)$.

Case 3. $x = 100 * \text{ones}(N, 1)$, $y = \text{ones}(M, 1)$, and $z = -10 * \text{ones}(Q, 1)$.

Case 4. $x = 10 * \text{ones}(N, 1)$, $y = -10 * \text{ones}(M, 1)$, and $z = \text{ones}(Q, 1)$.

We take $\alpha_k = 5/6k$, $\rho_k = 1/2 + 1/10^k$, $(v_1, v_2, \cdots, v_n) = (0, 0, \cdots, 0)$ in Algorithms 9 and 13. In the following tables and figures, we denote Algorithm 5.4 in reference [19] by ECCL. And we set "n" and "s" to express the number of iteration and CPU time in seconds, respectively. We use $\sum_{i=1}^{3} \|A_i x_i^k - \omega_k\| \le 10^{-4}$ as the stop criterion, where $\omega_k = (\sum_{i=1}^{3} A_i x_i^k)/3$. The numerical results can be seen from Table 1 and Figures 1–5.

We take the initial point $x = \text{ones}(3, 1)$, $y = \text{ones}(2, 1)$, and $z = \text{ones}(4, 1)$ in the Example 21. Figures 1–3 express the behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_i x_i^k - \omega_k\}$, for $i = 1, 2, 3$, of Algorithms 9 and 13 and ECCl for Example 21, respectively.

Furthermore, for testing the stationary property of iterative number, we carry 500 experiments for different initial points which are presented randomly, such as

$$\begin{aligned} &\text{Case A } x = \text{rand}\,(3, 1), y = \text{rand}\,(2, 1), z = \text{rand}\,(4, 1), \\ &\text{Case B } x = 1000 * \text{rand}\,(3, 1), y \\ &\qquad = 1000 * \text{rand}\,(2, 1), z = \text{rand}\,(4, 1), \end{aligned} \tag{87}$$

in Example 21 for Algorithm 13 and ECCL, the results can be found in Figures 4 and 5, respectively.

Table 1 shows the numerical results of Example 22 when Algorithms 9 and 13 and ECCL select different initial points and matrix dimensions.

From Figures 1–3 and Table 1, we can see that the iterative numbers of Algorithms 9 and 13 are less than ECCL. From Figures 4 and 5 and Table 1, we can see that the stationary property of the iterative number and CPU time of Algorithm 13 are exceeded by ECCL. And Algorithm 2 is better than Algorithm 13 in these aspects.

Data Availability

All data generated or analysed during this study are included in this manuscript.

Conflicts of Interest

The authors declare that they have no competing interests regarding the present manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Oscillation Results for a Class of Nonlinear Fractional Order Difference Equations with Damping Term

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The paper studies the oscillation of a class of nonlinear fractional order difference equations with damping term of the form $\Delta[\psi(\lambda)z^{\eta}(\lambda)] + p(\lambda)z^{\eta}(\lambda) + q(\lambda)F(\sum_{s=\lambda_0}^{\lambda-1+\mu}(\lambda-s-1)^{(-\mu)}y(s)) = 0$, where $z(\lambda) = a(\lambda) + b(\lambda)\Delta^{\mu}y(\lambda)$, Δ^{μ} stands for the fractional difference operator in Riemann-Liouville settings and of order μ, $0 < \mu \le 1$, and $\eta \ge 1$ is a quotient of odd positive integers and $\lambda \in \mathbb{N}_{\lambda_0+1-\mu}$. New oscillation results are established by the help of certain inequalities, features of fractional operators, and the generalized Riccati technique. We verify the theoretical outcomes by presenting two numerical examples.

1. Introduction and Background

The objective of this paper is to provide oscillation theorems for the equation

$$\Delta[\psi(\lambda)z^{\eta}(\lambda)] + p(\lambda)z^{\eta}(\lambda) + q(\lambda)F \cdot \left(\sum_{s=\lambda_0}^{\lambda-1+\mu}(\lambda-s-1)^{(-\mu)}y(s)\right) = 0, \tag{1}$$

where $\lambda \in \mathbb{N}_{\lambda_0+1-\mu}$, $\eta > 0$ is a quotient of odd positive integers, Δ^{μ} is the fractional difference operator in the sense of Riemann-Liouville (RL) and of order μ, $0 < \mu \le 1$, and

$$z(\lambda) = a(\lambda) + b(\lambda)\Delta^{\mu}y(\lambda). \tag{2}$$

The μ^{th} fractional sum for $\mu > 0$, (see [1]) is defined by

$$\Delta_a^{-\mu}f(\lambda) = \frac{1}{\Gamma(\mu)}\sum_{s=a}^{\lambda-\mu}(\lambda-s-1)^{(\mu-1)}f(s), \quad \lambda \in \mathbb{N}_{a+\mu}, \tag{3}$$

where the fractional sum $\Delta^{-\mu}$ is defined from $\mathbb{N}_a$ to $\mathbb{N}_{a+\mu}$, $f(s)$ is defined for $s \equiv a \bmod(1)$ and $\Delta_a^{-\mu}f(\lambda)$ is defined for $\lambda \equiv (a+\mu) \bmod(1)$. The falling function is

$$\lambda^{(\mu)} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\mu)}. \tag{4}$$

where Γ is the *Gamma function*, given by

$$\Gamma(t) := \int_0^{\infty} s^{t-1}e^{-s}ds, \tag{5}$$

for $t \in \mathbb{R}_+ := (0,\infty)$.

Let μ and N be positive integers such that $N = \lceil\mu\rceil$, namely, $\mu \in (N-1, N)$. Set $\nu = N - \mu$. Then, μ^{th} fractional difference (see [2]) is defined as

$$\Delta^{\mu}f(\lambda) = \Delta^{N-\nu}f(\lambda) = \Delta^{N}\Delta^{-\nu}f(\lambda). \tag{6}$$

The following conditions are assumed to hold throughout this work:

(i) (H_1) $\psi(\lambda) \in C([t_0,\infty))$, $p(\lambda) \in C([t_0,\infty))$, and $q(\lambda) \in C([t_0,\infty))$ are positive sequences with $\psi(\lambda) > p(\lambda)$

(ii) (H_2)

$$\Delta\left[\frac{a(\lambda)}{b(\lambda)}\right] \neq 0 \tag{7}$$

for $\lambda \in [\lambda_0,\infty)$ and

$$\lim_{\lambda\to\infty} \sum_{k=\lambda_0}^{\lambda-1} \left[\frac{a(k)}{b(k)}\right] < \infty \tag{8}$$

(iii) (H_3) $a(\lambda)$ is a nonnegative sequence on $[\lambda_0, \infty)$ for some $\lambda_0 > 0$. There exists a $n_0 > 0$ such that $b(\lambda) \geq n_0$ for $\lambda \in [\lambda_0,\infty)$

(iv) (H_4) F is a monotone decreasing function satisfying $\Psi F(\Psi) > 0$ such that

$$\frac{F(\Psi)}{\Psi^{\eta}} \geq L > 0 \text{ for } \Psi \neq 0 \tag{9}$$

A nontrivial solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

In the literature, Wang et al. [3] extended some oscillation results from integer-order differential equation to the fractional-order differential equation

$$\mathfrak{D}_a^{\alpha} x(t) + q(t) f(x(t)) = 0\,; \quad t \in [a,+\infty), a > 0, \tag{10}$$

where $\mathfrak{D}_a^{\alpha}$ denotes the standard RL differential operator of order α with $0 < \alpha \leq 1$, $q(t)$ is a positive real-valued function, $f(\cdot)$: $[0,\infty) \longrightarrow [0,\infty)$ is a continuous function satisfying

$$\frac{f(u)}{u} I^{\alpha-2} \geq K > 0, \tag{11}$$

and $I^{2-\alpha}$ denotes RL integral operator. Xiang et al. [4] investigated conditions for oscillation of fractional-order differential equation

$$[a(t)w^{\gamma}(t)] - b(\lambda) f\left(\int_t^{\infty} (s-t)^{-\alpha} x(s)\mathrm{d}s\right) = 0\,; \quad t \geq t_0 > 0, \tag{12}$$

where $w(t) = p(t) + q(t)(\mathfrak{D}_{-}^{\alpha} x)(t)$, $\alpha \in (0,1)$ is a fractional-order, $\gamma > 0$ is a quotient of odd positive integers, and $(\mathfrak{D}_{-}^{\alpha} x)$ is the RL right-sided fractional derivative of order α of x defined by

$$(\mathfrak{D}_{-}^{\alpha} x)(t) \coloneqq \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^{\infty} (s-t)^{-\alpha} x(s)\mathrm{d}s, \tag{13}$$

for $t \in \mathbb{R}_+$.

In this paper and motivated by the above work, we intend to carry forward the oscillation results from fractional differential Eq. (12) to the fractional difference Eq. (1). Moreover, we consider Eq. (1) under a damping term.

Fractional difference calculus is evolving as a powerful tool for studying problems in many fields such as biology, mechanics, control systems, ecology, electrical networks, electrochemical of corrosion, chemical physics, optics, and signal processing, economics and so forth (see [5–12]) and the references therein. Basic definitions and properties of fractional difference calculus were presented by Goodrich and Peterson [1]. In addition, there are other research works dealing with fractional difference equations which have helped to build up some of the basic theory in this area (see [2, 13–15]).

Fractional difference equations are applied to model physical processes which vary with time and space and its nonlocal property enables to model systems with memory effect. The study of the qualitative analysis of these equations has gained momentum in recent years and thus numerous publications have been reviled [16–20].

Of late, the investigation of the oscillation of solutions for fractional order difference equations has accelerated with several articles (see [21–29]). In what follows, we state some lemmas and preliminaries that will contribute in proving our main results.

Definition 1 (see [2]). For any $\mu \geq 0$, the falling factorial is known as

$$s^{(\mu)} = \frac{\Gamma(s+1)}{\Gamma(s+1-\mu)}, \tag{14}$$

where Γ is as in (5).

Lemma 2 (see [23]). *Let $y(\lambda)$ be a solution of (1) and*

$$\Psi(\lambda) = \sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda - s - 1)^{(-\mu)} y(s), \tag{15}$$

then

$$\Delta\Psi(\lambda) = \Gamma(1-\mu)\Delta^{\mu} y(\lambda). \tag{16}$$

Lemma 3 (see [30]). *The product and quotient rules of the difference operator Δ are defined to be*

$$\Delta[x(\lambda)y(\lambda)] = \Delta x(\lambda) y(\lambda+1) + x(\lambda)\Delta y(\lambda), \tag{17}$$

and

$$\Delta\left[\frac{x(\lambda)}{y(\lambda)}\right] = \frac{\Delta x(\lambda) y(\lambda) - x(\lambda)\Delta y(\lambda)}{y(\lambda)y(\lambda+1)}, \tag{18}$$

where $\Delta x(\lambda) = x(\lambda+1) - x(\lambda)$.

Lemma 4 (see [31]). *Let $\eta \geq 1$ be a quotient of two positive odd integers. If*

$$\Psi(\lambda+1) > \Psi(\lambda) > 0 (\Psi(\lambda+1) < \Psi(\lambda) < 0), \tag{19}$$

then

$$\Delta\Psi^{\eta}(\lambda) \geq (\Delta\Psi(\lambda))^{\eta} (\Delta\Psi^{\eta}(\lambda) \leq (\Delta\Psi(\lambda))^{\eta}). \tag{20}$$

Lemma 5 (see [32]). *Let $\xi, \zeta \in \mathbb{R}$ with $\zeta > 0$. Then, the inequality*

$$\xi X - \zeta X^2 \leq \frac{\xi^2}{4\zeta}, \tag{21}$$

holds for all $X \in \mathbb{R}$.

2. Main Results

Herein, new oscillation theorems for Eq. (1) are established by using mathematical inequalities, the properties of RL sum and difference operators, and the generalized Riccati technique.

Define the sequence

$$u(\lambda) = \prod_{s=\lambda_0}^{\lambda-1} \frac{\psi(s)}{\psi(s) - p(s)}. \tag{22}$$

Then $u(\lambda) > 0$,

$$u(\lambda+1) = \frac{\psi(\lambda)}{\psi(\lambda) - p(\lambda)} u(\lambda), \tag{23}$$

and hence

$$\Delta u(\lambda) = u(\lambda+1) - u(\lambda) = \frac{p(\lambda)}{\psi(\lambda) - p(\lambda)} u(\lambda). \tag{24}$$

Theorem 6. *Assume that $(H_1) - (H_4)$ hold. If*

$$\lim_{\lambda\to\infty} \sum_{\nu=\lambda_0}^{\lambda-1} \left[\frac{1}{u(\nu)\psi(\nu)}\right]^{1/\eta} = \infty, \tag{25}$$

and

$$\lim_{\lambda\to\infty} \sum_{\nu=\lambda_2}^{\lambda-1} \left[Lu(\nu)q(\nu) - \frac{(\Delta u(\nu))^2 \psi(\nu+1) N^{\eta}}{4u(\nu+1)}\right] = \infty, \tag{26}$$

then Eq. (1) is oscillatory.

Proof. Suppose that $y(\lambda)$ is a nonoscillatory solution of Eq. (1). Without loss of generality, we may assume that $y(\lambda)$ is an eventually positive solution of Eq. (1). Then, there exits $\lambda_1 \in [\lambda_0, \infty)$ such that $y(\lambda) > 0$, $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$, where $\Psi(\lambda)$ is defined in Lemma 2. Considering the assumption (H_4) and using Eq. (1), we get

$$\Delta[\psi(\lambda) z^{\eta}(\lambda)] + p(\lambda) z^{\eta}(\lambda) = -q(\lambda) F(\Psi(\lambda)), \tag{27}$$

or

$$\Delta[\psi(\lambda) z^{\eta}(\lambda)] + p(\lambda) z^{\eta}(\lambda) \leq -Lq(\lambda) \Psi^{\eta}(\lambda), \tag{28}$$

where $z(\lambda)$ is defined in (2). Hence, we proceed from (22) and (17) to

$$\begin{aligned}
\Delta[u(\lambda)\psi(\lambda) z^{\eta}(\lambda)] &= u(\lambda+1)\Delta[\psi(\lambda) z^{\eta}(\lambda)] \\
&\quad + \Delta u(\lambda)\psi(\lambda) z^{\eta}(\lambda) \\
&= u(\lambda+1)\Delta[\psi(\lambda) z^{\eta}(\lambda)] \\
&\quad + u(\lambda) \frac{p(\lambda)}{\psi(\lambda) - p(\lambda)} \psi(\lambda) z^{\eta}(\lambda) \\
&= u(\lambda+1)\Delta[\psi(\lambda) z^{\eta}(\lambda)] \\
&\quad + u(\lambda+1) \frac{\psi(\lambda) - p(\lambda)}{\psi(\lambda)} \\
&\quad \cdot \frac{p(\lambda)}{\psi(\lambda) - p(\lambda)} \psi(\lambda) z^{\eta}(\lambda) \\
&= u(\lambda+1)[\Delta[\psi(\lambda) z^{\eta}(\lambda)] + p(\lambda) z^{\eta}(\lambda)] \\
&\leq -Lu(\lambda+1) q(\lambda) \Psi^{\eta}(\lambda) < 0.
\end{aligned} \tag{29}$$

Then, $u(\lambda)\psi(\lambda) z^{\eta}(\lambda)$ is strictly decreasing on $[\lambda_1, \infty)$ and is eventually of constant sign. Since $u(\lambda) > 0$, $\psi(\lambda) > 0$, and $\eta > 0$ is a quotient of odd positive integers, we observe that $z(\lambda)$ is eventually of constant sign.

First, we show that

$$z(\lambda) > 0 \text{ for } \lambda \in [\lambda_1, \infty). \tag{30}$$

If not, then there exits $\lambda_2 \geq \lambda_1$ such that $z(\lambda_2) < 0$, and we obtain

$$u(\lambda)\psi(\lambda) z^{\eta}(\lambda) < u(\lambda_2)\psi(\lambda_2) z^{\eta}(\lambda_2) = d < 0, \tag{31}$$

which implies

$$z^{\eta}(\lambda) < \frac{d}{u(\lambda)\psi(\lambda)} < 0, \tag{32}$$

for $\lambda \in [\lambda_2, \infty)$ that is

$$z(\lambda) < \left[\frac{d}{u(\lambda)\psi(\lambda)}\right]^{1/\eta} < 0. \tag{33}$$

So we arrive at that $z(\lambda) < 0$ on $[\lambda_2, \infty)$. Hence for $\lambda \in [\lambda_2, \infty)$, we get

$$\frac{a(\lambda)}{b(\lambda)} + \Delta^{\mu} y(\lambda) < \frac{z(\lambda)}{b(\lambda)} < \frac{1}{b(\lambda)} \left[\frac{d}{u(\lambda)\psi(\lambda)}\right]^{1/\eta} < \frac{1}{n_0} \left[\frac{d}{u(\lambda)\psi(\lambda)}\right]^{1/\eta}. \tag{34}$$

From (16), we have

$$\frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^{\alpha} y(\lambda) < \frac{1}{n_0} \left[\frac{d}{u(\lambda)\psi(\lambda)}\right]^{1/\eta}, \tag{35}$$

and summing from λ_2 to $\lambda - 1$, we obtain

$$\sum_{\nu=\lambda_2}^{\lambda-1} \Delta\Psi(\nu) < \Gamma(1-\mu) \sum_{\nu=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(\nu)\psi(\nu)}\right]^{\frac{1}{\eta}} - \frac{a(\nu)}{b(\nu)}\right], \tag{36}$$

and hence we have

$$\Psi(\lambda) < \Psi(\lambda_2) + \Gamma(1-\mu) \sum_{\nu=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(\nu)\psi(\nu)}\right]^{1/\eta} - \frac{a(\nu)}{b(\nu)}\right]. \tag{37}$$

Now, by letting $\lambda \longrightarrow \infty$, we get

$$\lim_{\lambda\to\infty} \Psi(\lambda) < \lim_{\lambda\to\infty} \left\{\Psi(\lambda_2) + \Gamma(1-\mu) \sum_{\nu=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(\nu)\psi(\nu)}\right]^{1/\eta} - \frac{a(\nu)}{b(\nu)}\right]\right\} < \Psi(\lambda_2) + \Gamma(1-\mu) \sum_{\nu=\lambda_2}^{\infty} \left[\frac{1}{n_0} \left[\frac{d}{u(\nu)\psi(\nu)}\right]^{1/\eta} - \frac{a(\nu)}{b(\nu)}\right] = -\infty, \tag{38}$$

which contradicts that $\Psi(\lambda) > 0$, $\lambda \in [\lambda_1, \infty)$. Therefore $z(\lambda) > 0$ for $\lambda \in [\lambda_2, \infty)$.

Now, since

$$\frac{z(\lambda)}{b(\lambda)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^{\mu} y(\lambda) = \frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)}, \tag{39}$$

and we have

$$\frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = \frac{z(\lambda) - a(\lambda)}{b(\lambda)} < \frac{z(\lambda) - a(\lambda)}{n_0} < \frac{z(\lambda)}{n_0}, \tag{40}$$

from (16), we obtain

$$\Delta\Psi(\lambda) < \frac{z(\lambda)}{n_0}. \tag{41}$$

Define the generalized Riccati function

$$w(\lambda) = u(\lambda) \frac{\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)}; \quad \lambda \in [\lambda_1, \infty). \tag{42}$$

It is clear that $w(\lambda) > 0$. Using (17) and (18) for $\lambda \in [\lambda_0, \infty)$, we have

$$\begin{aligned}
\Delta w(\lambda) &= \Delta u(\lambda) \frac{\psi(\lambda+1) z^{\eta}(\lambda+1)}{\Psi^{\eta}(\lambda+1)} + u(\lambda) \Delta \left[\frac{\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)}\right] = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\
&\quad + u(\lambda) \left[\frac{\Delta\{\psi(\lambda) z^{\eta}(\lambda)\} \Psi^{\eta}(\lambda) - \psi(\lambda) z^{\eta}(\lambda) \Delta\Psi^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)}\right] \\
&= \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} + u(\lambda) \frac{\Delta\{\psi(\lambda) z^{\eta}(\lambda)\}}{\Psi^{\eta}(\lambda+1)} \\
&\quad - u(\lambda) \frac{\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} \Delta\Psi^{\eta}(\lambda) = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\
&\quad - u(\lambda) \left[\frac{p(\lambda) z^{\eta}(\lambda) + q(\lambda) f(\Psi(\lambda))}{\Psi^{\eta}(\lambda+1)}\right] \\
&\quad - \frac{u(\lambda)\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} \Delta\Psi^{\eta}(\lambda) = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\
&\quad - \frac{u(\lambda) p(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda+1)} - \frac{u(\lambda) q(\lambda) f(\Psi(\lambda))}{\Psi^{\eta}(\lambda+1)} \\
&\quad - u(\lambda) \frac{\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} \Delta\Psi^{\eta}(\lambda).
\end{aligned} \tag{43}$$

Since

$$\frac{u(\lambda)\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} > 0, \tag{44}$$

and $\Delta\Psi(\lambda) > 0$, then $\Psi(\lambda+1) - \Psi(\lambda) > 0$; it follows from (H_4) that

$$\frac{f(\Psi(\lambda))}{\Psi^{\eta}(\lambda+1)} > \frac{f(\Psi(\lambda+1))}{\Psi^{\eta}(\lambda+1)} \geq L, \tag{45}$$

and $\Delta\Psi^{\eta}(\lambda) > (\Delta\Psi(\lambda))^{\eta}$ by Lemma 4.

Using the fact that $u(\lambda)\psi(\lambda) z^{\eta}(\lambda)$ is strictly decreasing, we arrive at

$$\frac{u(\lambda)\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} \Delta\Psi^{\eta}(\lambda) \geq \frac{u(\lambda)\psi(\lambda) z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)} (\Delta\Psi(\lambda))^{\eta}. \tag{46}$$

Also we have

$$(\Delta\Psi(\lambda))^{\eta} < \left[\frac{z(\lambda)}{n_0}\right]^{\eta}, \tag{47}$$

from (41).

Now substituting (45), (46), and (47) in (43), we get

$$\begin{aligned}\Delta w(\lambda) &< \Delta u(\lambda)\frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda)\\ &\quad - \frac{u(\lambda)\psi(\lambda)z^{\eta}(\lambda)}{\Psi^{\eta}(\lambda)\Psi^{\eta}(\lambda+1)}\frac{z^{\eta}(\lambda)}{n_0^{\eta}}\\ &< \Delta u(\lambda)\frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda)\\ &\quad - \frac{u(\lambda+1)\psi(\lambda+1)z^{\eta}(\lambda+1)}{\Psi^{\eta}(\lambda+1)\Psi^{\eta}(\lambda+1)}\frac{z^{\eta}(\lambda+1)}{n_0^{\eta}}\\ &< \Delta u(\lambda)\frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda)\\ &\quad - \frac{[u(\lambda+1)\psi(\lambda+1)z^{\eta}(\lambda+1)]^2}{u(\lambda+1)\psi(\lambda+1)[\Psi^{\eta}(\lambda+1)]^2 n_0^{\eta}}\\ &< \Delta u(\lambda)\frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda)\\ &\quad - \left[\frac{u(\lambda+1)\psi(\lambda+1)z^{\eta}(\lambda+1)}{\Psi^{\eta}(\lambda+1)}\right]^2\\ &\quad \times \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0^{\eta}} < \Delta u(\lambda)\frac{w(\lambda+1)}{u(\lambda+1)}\\ &\quad - Lu(\lambda)q(\lambda) - \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0^{\eta}}w^2(\lambda+1).\end{aligned} \tag{48}$$

Using (48) and (21) with $X = w(\lambda+1)$, $a = \Delta u(\lambda)/u(\lambda+1)$, and

$$b = \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0^{\eta}}, \tag{49}$$

we obtain

$$\begin{aligned}\Delta w(\lambda) &< -Lu(\lambda)q(\lambda) + \frac{(\Delta u(\lambda))^2\psi(\lambda+1)u(\lambda+1)n_0^{\eta}}{4(u(\lambda+1))^2}\\ &= -Lu(\lambda)q(\lambda) + \frac{(\Delta u(\lambda))^2\psi(\lambda+1)n_0^{\eta}}{4u(\lambda+1)},\end{aligned} \tag{50}$$

and hence

$$\left[Lu(\lambda)q(\lambda) - \frac{(\Delta u(\lambda))^2\psi(\lambda+1)n_0^{\eta}}{4u(\lambda+1)}\right] < -\Delta w(\lambda). \tag{51}$$

Summing up (51) from λ_2 to $\lambda-1$, we have

$$\begin{aligned}\sum_{\nu=\lambda_2}^{\lambda-1}\left[Lu(\nu)q(\nu) - \frac{(\Delta u(\nu))^2\psi(\nu+1)n_0^{\eta}}{4u(\nu+1)}\right] &< -\sum_{\nu=\lambda_2}^{\lambda-1}\Delta w(\nu)\\ &= -[w(\lambda) - w(\lambda_2)] = w(\lambda_2) - w(\lambda),\end{aligned} \tag{52}$$

and hence

$$\sum_{\nu=\lambda_2}^{\lambda-1}\left[Lu(\nu)q(\nu) - \frac{(\Delta u(\nu))^2\psi(\nu+1)n_0^{\eta}}{4u(\nu+1)}\right] < w(\lambda_2). \tag{53}$$

Letting $\lambda \longrightarrow \infty$,

$$\lim_{\lambda\to\infty}\sum_{\nu=\lambda_2}^{\lambda-1}\left[Lu(\nu)q(\nu) - \frac{(\Delta u(\nu))^2\psi(\nu+1)n_0^{\eta}}{4u(\nu+1)}\right] < w(\lambda_2) < \infty, \tag{54}$$

which contradicts with (26).

Theorem 7. *Assume that (25)and (26) hold and there exists a positive sequence $H(t,s)$ such that*

(1) $H(\lambda,\lambda) = 0$ for $\lambda \geq \lambda_0$; $H(\lambda,\nu) > 0$ for $\lambda > \nu \geq \lambda_0$

(2) $\Delta_2 H(\lambda,\nu) = H(\lambda,\nu+1) - H(\lambda,\nu) < 0$ for $\lambda > \nu \geq \lambda_0$

If

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{H(\lambda,\lambda_0)}\sum_{\nu=\lambda_0}^{\lambda-1}&\left[Lu(\nu)q(\nu)H(\lambda,\nu)\right.\\ &\left. - \frac{h_+^2(\lambda,\nu)u(\nu+1)\psi(\nu+1)n_0^{\eta}}{4H(\lambda,\nu)}\right] = \infty,\end{aligned} \tag{55}$$

then Eq. (1) is oscillatory, where $u(\lambda)$ is defined in Theorem 6 and

$$h_+(\lambda,\nu) = \Delta_2 H(\lambda,\nu) + \frac{\Delta u(\nu)H(\lambda,\nu)}{u(\nu+1)}. \tag{56}$$

Proof. Suppose that $y(\lambda)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $y(\lambda)$ is an eventually positive solution of (1). Then, there exists $\lambda_1 \in [\lambda_0,\infty)$ such that $y(\lambda) > 0$, $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1,\infty)$. Proceeding as in proof of Theorem 6 we arrive at (48). Multiplying (48) by $H(\lambda,\nu)$ and summing up from λ_2 to $\lambda-1$, we obtain

$$\begin{aligned}\sum_{\nu=\lambda_2}^{\lambda-1}\Delta w(\nu)H(\lambda,\nu) &< \sum_{\nu=\lambda_2}^{\lambda-1}\frac{\Delta u(\nu)w(\nu+1)H(\lambda,\nu)}{u(\nu+1)}\\ &\quad - \sum_{\nu=\lambda_2}^{\lambda-1}Lu(\nu)q(\nu)H(\lambda,\nu)\\ &\quad - \sum_{\nu=\lambda_2}^{\lambda-1}\frac{H(\lambda,\nu)w^2(\nu+1)}{u(\nu+1)\psi(\nu+1)n_0^{\eta}}.\end{aligned} \tag{57}$$

Using summation by parts formula, we get

$$-\sum_{v=\lambda_2}^{\lambda-1}\Delta w(v)H(\lambda,v)=-H(\lambda,v)w(\lambda)\Big|_{v=\lambda_2}^{v=\lambda-1}+\sum_{v=\lambda_2}^{\lambda-1}w(\lambda+1)\Delta_2H(\lambda,v)$$
$$=H(\lambda,\lambda_2)w(\lambda_2)+\sum_{v=\lambda_2}^{\lambda-1}w(v+1)\Delta_2H(\lambda,v). \tag{58}$$

Hence, from (57)

$$\begin{aligned}\sum_{v=\lambda_2}^{\lambda-1}Lu(v)q(v)H(\lambda,v)&<H(\lambda,\lambda_2)w(\lambda_2)\\&+\sum_{v=\lambda_2}^{\lambda-1}w(v+1)\Delta_2H(\lambda,v)\\&+\sum_{v=\lambda_2}^{\lambda-1}\frac{\Delta u(v)w(v+1)H(\lambda,v)}{u(v+1)}\\&-\sum_{v=\lambda_2}^{\lambda-1}\frac{H(\lambda,v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta}\\&<\sum_{v=\lambda_2}^{\lambda-1}\left[\Delta_2H(\lambda,v)+\frac{\Delta u(v)H(\lambda,v)}{u(v+1)}\right]w(v+1)\\&-\sum_{v=\lambda_2}^{\lambda-1}\frac{H(\lambda,v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta}+H(\lambda,\lambda_2)w(\lambda_2)\\&<H(\lambda,\lambda_2)w(\lambda_2)+\sum_{v=\lambda_2}^{\lambda-1}\left[h_+(\lambda,v)w(v+1)\right.\\&\left.-\frac{H(\lambda,v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta}\right].\end{aligned} \tag{59}$$

Now by using (21) with $X=w(v+1)$, $\xi=h_+(\lambda,v)$ and

$$\zeta=\frac{H(\lambda,v)}{u(v+1)\psi(v+1)n_0^\eta}, \tag{60}$$

we obtain

$$\sum_{v=\lambda_2}^{\lambda-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right] \tag{61}$$
$$<H(\lambda,\lambda_2)w(\lambda_2)<H(\lambda,\lambda_0)w(\lambda_2),$$

for $\lambda>\lambda_2>\lambda_1>\lambda_0$. Then

$$\begin{aligned}&\sum_{v=\lambda_0}^{\lambda-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]\\&=\sum_{v=\lambda_0}^{\lambda_2-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]\\&+\sum_{v=\lambda_2}^{\lambda-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]\\&<\sum_{v=\lambda_0}^{\lambda_2-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]\\&+H(\lambda,\lambda_0)w(\lambda_2)<H(\lambda,\lambda_0)\sum_{v=\lambda_0}^{\lambda_2-1}Lu(v)q(v)\\&+H(\lambda,\lambda_0)w(\lambda_2),\end{aligned} \tag{62}$$

which yields

$$\frac{1}{H(\lambda,\lambda_0)}\sum_{v=\lambda_0}^{\lambda-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]$$
$$<\sum_{v=\lambda_0}^{\lambda_2-1}Lu(v)q(v)+w(\lambda_2). \tag{63}$$

Taking limit as $\lambda\longrightarrow\infty$, we get

$$\lim_{\lambda\to\infty}\frac{1}{H(\lambda,\lambda_0)}\sum_{v=\lambda_0}^{\lambda-1}\left[Lu(v)q(v)H(\lambda,v)-\frac{h_+^2(\lambda,v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda,v)}\right]$$
$$<\sum_{v=\lambda_0}^{\lambda_2-1}Lu(v)q(v)+w(\lambda_2)<\infty, \tag{64}$$

which contradicts with (55).

Next, we consider the condition

$$\lim_{\lambda\to\infty}\sum_{v=\lambda_0}^{\lambda-1}\left[\frac{1}{u(v)\psi(v)}\right]^{1/\eta}<\infty, \tag{65}$$

which implies that (25) does not hold. Under this condition, we have the following result.

Theorem 8. *Assume that $(H_1)-(H_4)$, (26), and (65) hold. If*

$$\lim_{\lambda\to\infty}\sum_{s=\lambda_2}^{\lambda-1}\left[\left[\frac{1}{u(s)\psi(s)}\sum_{v=s}^{\infty}u(v+1)q(v)\right]^{1/\eta}-\frac{a(s)}{b(s)}\right]=\infty, \tag{66}$$

then every solution $y(\lambda)$ is oscillatory or satisfies $\lim_{\lambda\to\infty}\Psi(\lambda)=0$, where $\Psi(\lambda)$ is defined in Lemma 2.

Proof. Assume that $y(\lambda)$ is a nonoscillatory solution of (1). Without loss of generality, assume that $y(\lambda)$ is eventually a positive solution of (1). Proceeding like in the proof of Theorem 6, we get that (28) holds. Then, there are two signs of $z(\lambda)$. When $z(\lambda) > 0$ is eventually positive, we conclude from the proof of Theorem 6 that equation (1) is oscillatory.

Next, assume that $z(\lambda)$ is eventually negative, then there exits $\lambda_2 > \lambda_0$ such that $z(\lambda) < 0$ for $\lambda \geq \lambda_2$. Since $z(\lambda) = a(\lambda) + b(\lambda)\Delta^{\mu} y(\lambda)$, we have

$$a(\lambda) + b(\lambda)\frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = z(\lambda) < 0, \tag{67}$$

and hence

$$\Delta\Psi(\lambda) < -\Gamma(1-\mu)\frac{a(\lambda)}{b(\lambda)}. \tag{68}$$

On the other hand, Since (H_2) holds, we get

$$\lim_{\lambda\to\infty}\frac{a(\lambda)}{b(\lambda)} = 0. \tag{69}$$

Now, taking the limit of the both sides of (68) as λ tends to ∞, we get

$$\lim_{\lambda\to\infty}\Delta\Psi(\lambda) \leq 0. \tag{70}$$

Since $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$, we have

$$\lim_{\lambda\to\infty}\Psi(\lambda) = \beta \geq 0. \tag{71}$$

Claim that $\beta = 0$. If not, then $\Psi(\lambda) \geq \beta$ for $\lambda \in [\lambda_2, \infty)$. Now we have

$$\begin{aligned}\Delta[u(\lambda)\psi(\lambda)z^{\eta}(\lambda)] &\leq -Lu(\lambda+1)q(\lambda)\Psi^{\eta}(\lambda),\\ &\leq -Lu(\lambda+1)q(\lambda)\beta^{\eta},\end{aligned} \tag{72}$$

by (29). Summing up from λ to ∞, we have

$$\begin{aligned}\sum_{\nu=\lambda}^{\infty}\Delta[u(\nu)\psi(\nu)z^{\eta}(\nu)] &= \lim_{\nu\to\infty}u(\nu)\psi(\nu)z^{\eta}(\nu) - u(\lambda)\psi(\lambda)z^{\eta}(\lambda)\\ &\leq -\beta^{\eta}L\sum_{\nu=\lambda}^{\infty}u(\nu+1)q(\nu),\end{aligned} \tag{73}$$

which yields

$$\begin{aligned}u(\lambda)\psi(\lambda)z^{\eta}(\lambda) &\geq \lim_{\nu\to\infty}u(\nu)\psi(\nu)z^{\eta}(\nu) + \beta^{\eta}L\sum_{\nu=\lambda}^{\infty}u(\nu+1)q(\nu)\\ &\geq \beta^{\eta}L\sum_{\nu=\lambda}^{\infty}u(\nu+1)q(\nu),\end{aligned} \tag{74}$$

and hence

$$\frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = z(\lambda) > L^{1/\eta}\beta\left[\frac{1}{u(\lambda)\psi(\lambda)}\sum_{\nu=\lambda}^{\infty}u(\nu+1)q(\nu)\right]^{1/\eta}. \tag{75}$$

The last inequality above implies that

$$\Delta\Psi(\lambda) > \Gamma(1-\mu)\left[L^{1/\eta}\beta\left[\frac{1}{u(\lambda)\psi(\lambda)}\sum_{\nu=\lambda}^{\infty}u(\nu+1)q(\nu)\right]^{1/\eta} - \frac{a(\lambda)}{b(\lambda)}\right]. \tag{76}$$

Summing up from λ_2 to $\lambda - 1$, we get

$$\begin{aligned}\Psi(\lambda) &> \Psi(\lambda_2) + \Gamma(1-\mu)\\ &\cdot\left[L^{1/\eta}\beta\sum_{s=\lambda_2}^{\lambda-1}\left[\left[\frac{1}{u(s)\psi(s)}\sum_{\nu=s}^{\infty}u(\nu+1)q(\nu)\right]^{1/\eta} - \frac{a(s)}{b(s)}\right]\right].\end{aligned} \tag{77}$$

Taking the limit of the both sides of the above inequality as λ tends to ∞, we end up with

$$\lim_{\lambda\to\infty}\Psi(\lambda) = \infty, \tag{78}$$

which contradicts to $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$. Therefore, we obtain $\beta = 0$, that is

$$\lim_{\lambda\to\infty}\Psi(\lambda) = 0. \tag{79}$$

3. Applications

Example 9. Consider the equation

$$\Delta\left[\lambda^2 x^{\eta}(\lambda)\right] + lx^{\eta}(\lambda) + \frac{1}{\lambda^2}\left(\sum_{s=\lambda_0}^{\lambda-1+\mu}(\lambda-s-1)^{-\mu}y(s)\right)^{-3} = 0, \tag{80}$$

where $\mu \in (0, 1)$,

$$x(\lambda) = e^{-\lambda} + \frac{1}{\lambda}\Delta^{\mu}y(\lambda), \quad \lambda \in \mathbb{N}_2 = \{2, 3, 4, \cdots\}, \tag{81}$$

and that $\eta > 0$ is a quotient of odd positive integers in $\mathbb{N}_2$.

Comparing with Eq. (1), we get $\psi(\lambda) = \lambda^2$, $a(\lambda) = e^{-\lambda}$, $b(\lambda) = 1/\lambda$, $p(\lambda) = \lambda$, $F(y) = y^{-3}$, $q(\lambda) = 1/\lambda^2$, $\eta = 3$, $\lambda_0 = 2$, $n_0 = 4$, $L = 1/3$, and that

$$\frac{f(y)}{y^{\eta}} = \frac{1}{y^6} > \varepsilon = L > 0 \text{ for } y \neq 0, \tag{82}$$

where ε is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have

$$\begin{aligned}
u(\lambda) &= \prod_{v=\lambda_0}^{\lambda-1} \frac{\psi(v)}{\psi(v) - p(v)} = \prod_{v=2}^{\lambda-1} \frac{v^2}{v^2 - v} = \lambda - 1;\\
\Delta u(\lambda) &= u(\lambda+1) - u(\lambda) = 1;\\
\Delta\left[\frac{a(\lambda)}{b(\lambda)}\right] &= \Delta\left[\lambda e^{-\lambda}\right] = (\lambda+1)e^{-(\lambda+1)}(1-e) + e^{-\lambda} \neq 0;\\
\lim_{\lambda\to\infty} \sum_{v=\lambda_0}^{\lambda-1} \frac{a(v)}{b(v)} &= \lim_{\lambda\to\infty} \sum_{v=1}^{\lambda-1} v e^{-v} < \infty;\\
\lim_{\lambda\to\infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)}\right]^{1/\eta} &= \lim_{\lambda\to\infty} \sum_{v=2}^{\lambda-1} \left[\frac{1}{v^2(v-1)}\right]^{1/3}\\
&> \sum_{v=2}^{\infty} \frac{1}{v} = \infty;\\
\lim_{\lambda\to\infty} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1) n_0^{\eta}}{4u(v+1)}\right]&\\
= \lim_{\lambda\to\infty} \sum_{v=2}^{\lambda-1} \left[\frac{(v-1)}{3v^2} - \frac{16(v+1)^2}{v}\right] &= \infty.
\end{aligned} \tag{83}$$

Thus, conditions (25) and (26) are satisfied. Therefore, all solutions of (80) are oscillatory by Theorem 6.

Example 10. Consider the equation

$$\Delta\left[\frac{1}{\lambda} z^{\eta}(\lambda)\right] + \frac{1}{\lambda^2} z^{\eta}(\lambda) + \frac{1}{\lambda - 1}\left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda - s - 1)^{-\mu} y(s)\right)^{-1} = 0, \tag{84}$$

where $\mu \in (0,1)$, $z(\lambda) = \lambda^2 + \lambda\Delta^{\mu} y(\lambda)$, $\lambda \in \mathbb{N}_2$, and that $\eta > 0$ is a quotient of odd positive integers in $\mathbb{N}_2$. Comparing with (1), we have $\psi(\lambda) = 1/\lambda$, $a(\lambda) = \lambda^2$, $b(\lambda) = \lambda$, $p(\lambda) = 1/\lambda^2$, $F(y) = y^{-1}$, $q(\lambda) = 1/(\lambda - 1)$, $\eta = 1$, $\lambda_0 = 2$, $n_0 = 2$, $L = 1/2$, and that

$$\frac{f(y)}{y^{\eta}} = \frac{1}{y^2} > \varepsilon = L > 0 \text{ for } y \neq 0, \tag{85}$$

where ε is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have

$$\begin{aligned}
u(\lambda) &= \prod_{v=\lambda_0}^{\lambda-1} \frac{\psi(v)}{\psi(v) - p(v)} = \prod_{v=2}^{\lambda-1} \frac{v}{v-1} = \lambda - 1;\\
\Delta u(\lambda) &= u(\lambda+1) - u(\lambda) = 1;\\
\Delta\left[\frac{a(\lambda)}{b(\lambda)}\right] &= \Delta\left[\frac{\lambda^2}{\lambda}\right] = \Delta[\lambda] \neq 0;\\
\lim_{\lambda\to\infty} \sum_{v=\lambda_0}^{\lambda-1} \frac{a(v)}{b(v)} &= \lim_{\lambda\to\infty} \sum_{v=1}^{\lambda-1} \frac{v^2}{v} = \lim_{\lambda\to\infty} \sum_{v=\lambda_0}^{\lambda-1} v < \infty;\\
\lim_{\lambda\to\infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)}\right]^{1/\eta} &= \lim_{\lambda\to\infty} \sum_{v=2}^{\lambda-1} \left[\frac{v}{(v-1)}\right] = \infty.
\end{aligned} \tag{86}$$

We define the double sequence $H(\lambda, v)$ as follows:

(1) $H(\lambda, v) = (2\lambda - v)^2 > 0$ for $\lambda > v > 2$

(2) $H(\lambda, 2) = (\lambda - 2)^2 > 0$ for $\lambda > s = 2$

(3) $\Delta_2 H(\lambda, v) = H(\lambda, v+1) - H(\lambda, v) = [2\lambda - (v+1)]^2 - (2\lambda - v)^2$

(4) $\Delta_2 H(\lambda, v) = 2v - 4\lambda + 1 < 0$ for $\lambda > v \geq 2$

and

$$\begin{aligned}
h_+(\lambda, v) &= \Delta_2 H(\lambda, v) + \frac{\Delta u(v) H(\lambda, v)}{u(v+1)}\\
&= \frac{\left(3v^2 - 8lv + v + 4\lambda^2\right)}{v}.
\end{aligned} \tag{87}$$

Then

$$\begin{aligned}
\frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v) u(v+1)\psi(v+1) n_0^{\eta}}{4H(\lambda, v)}\right]&\\
= \frac{1}{(\lambda-2)^2} \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{(2\lambda - v)^2}{2} - \frac{\left(3v^2 - 8lv + v + 4\lambda^2\right)^2}{2(v+1)(2\lambda - v)^2}\right],&
\end{aligned} \tag{88}$$

for $\lambda_2 > \lambda_1$, and hence

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v) u(v+1)\psi(v+1) n_0^{\eta}}{4H(\lambda, v)}\right]&\\
= \lim_{\lambda\to\infty} \frac{1}{(\lambda-2)^2} \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{(2\lambda - v)^2}{2} - \frac{\left(3v^2 - 8lv + v + 4\lambda^2\right)^2}{2(v+1)(2\lambda - v)^2}\right] = \infty.&
\end{aligned} \tag{89}$$

Therefore, condition (55) is satisfied. We deduce that all solutions of Eq. (84) are oscillatory by Theorem 7.

4. A Concluding Remark

In this paper, we obtained new oscillation theorems for a class of fractional difference equation. The main outcomes are proved via the means of mathematical inequalities, properties of fractional operators, and generalized Riccati technique. We claim that the concluded results have merit and considered as an extension for the corresponding fractional differential equations. Particular examples that are consistent to the main results are demonstrated at the end of the paper.

The consideration of Eq. (1) with forcing term of the form

$$\Delta[\psi(\lambda)z^{\eta}(\lambda)] + p(\lambda)z^{\eta}(\lambda) + q(\lambda)F \cdot \left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda - s - 1)^{(-\mu)} y(s)\right) = g(\lambda), \tag{90}$$

could be further investigated. We leave this for future consideration.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

A Characterization of Central BMO Space via the Commutator of Fractional Hardy Operator

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This paper is devoted in characterizing the central BMO $(\mathbb{R}^n)$ space via the commutator of the fractional Hardy operator with rough kernel. Precisely, by a more explicit decomposition of the operator and the kernel function, we will show that if the symbol function belongs to the central BMO $(\mathbb{R}^n)$ space, then the commutator are bounded on Lebesgue space. Conversely, the boundedness of the commutator implies that the symbol function belongs to the central BMO $(\mathbb{R}^n)$ space by exploiting the center symmetry of the Hardy operator deeply.

1. Introduction

In this paper, we focus on the need for characterizing the central BMO $(\mathbb{R}^n)$ space via the boundedness of the commutators of the following fractional Hardy operators

$$H_{\Omega,\alpha}f(x) = \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|}\Omega(x-y)f(y)dy,$$
$$H^*_{\Omega,\alpha}f(x) = \int_{|y|\geq|x|}\frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy, \quad (1)$$
$$0<\alpha<n.$$

$H^*_{\Omega,\alpha}$ is the dual operator of $H_{\Omega,\alpha}$. Here, Ω satisfies

$$\Omega(tx)=\Omega(x), \forall t>0, x\in\mathbb{R}^n; \quad (2)$$

$$\int_{\mathbb{S}^{n-1}}\Omega\left(x'\right)d\sigma\left(x'\right)=0; \quad (3)$$

$$\Omega\in L^q\left(\mathbb{S}^{n-1}\right)\forall q>1; \quad (4)$$

and $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ denotes the unit sphere in $\mathbb{R}^n$.

For a function b, the commutators of $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$ can be written as

$$[b,H_{\Omega,\alpha}]f \coloneqq b(H_{\Omega,\alpha}f)-H_{\Omega,\alpha}(bf),$$
$$\left[b,H^*_{\Omega,\alpha}\right]f \coloneqq b\left(H^*_{\Omega,\alpha}f\right)-H^*_{\Omega,\alpha}(bf). \quad (5)$$

In [1], Fu et al. considered the boundedness of $H_{\Omega,\alpha}$ and $[b,H_{\Omega,\alpha}]$ on homogeneous Herz spaces and Lebesgue spaces under the assumption that Ω satisfies (2) and (4). We recall the results from ([1], Proposition 3.1 and Theorem 3.1) as

Suppose that

$$\begin{cases} 1<p_1,p_2<\infty \text{ with } \dfrac{1}{p_1}=\dfrac{1}{p_2}+\dfrac{\alpha}{n}; \\ \dfrac{1}{p_1}+\dfrac{1}{s}+\dfrac{1}{q}=1 \text{ for some } s>p_1'=\dfrac{p_1}{p_1-1}; \\ \Omega \text{ satisfies } (1.1) \text{ and } (1.3); \\ b\in \mathrm{CBMO}^{\max\{p_2,s\}}(\mathbb{R}^n). \end{cases} \quad (6)$$

Then,

$$\begin{cases} H_{\Omega,\alpha} \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \\ [b; H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n). \end{cases} \tag{7}$$

For the boundedness of the classical fractional Hardy operator, see [2]. For a ball $B_r := B(0,r) \subset \mathbb{R}^n$ (i.e., a ball centered at 0 with radius $r > 0$) and $p \geq 1$, $\mathrm{CBMO}^p(\mathbb{R}^n)$ is the central BMO $(\mathbb{R}^n)$ function space, which was introduced by Lu and Yang [3] via the norm

$$\begin{aligned} \|b\|_{\mathrm{CBMO}^p(\mathbb{R}^n)} &= \sup_{r>0} \left(\frac{1}{|B_r|} \int_{B_r} \left| b(x) - b_{B_r} \right|^p dx \right)^{1/p} \quad \text{with } b_{B_r} \\ &= \frac{1}{|B_r|} \int_{B_r} b(x) dx. \end{aligned} \tag{8}$$

It is easy to see that CBMO $(\mathbb{R}^n)$ can be understood as a local version of the classical BMO $(\mathbb{R}^n)$ space at the origin [4] and

$$\begin{aligned} \mathrm{BMO}(\mathbb{R}^n) &\subset \mathrm{CBMO}(\mathbb{R}^n), \\ \mathrm{CBMO}^{p>1}(\mathbb{R}^n) &\subset \mathrm{CBMO}(\mathbb{R}^n). \end{aligned} \tag{9}$$

Hence, the famous John-Nirenberg inequality for BMO $(\mathbb{R}^n)$ space is not true for CBMO $(\mathbb{R}^n)$ space, which reveals that they have quite different properties.

In the study of harmonic analysis, the characterization of function spaces via the boundedness of the commutators plays an important role in the field of PDEs, see, for example, [5–11] and the references therein. However, there are less attention paid for the commutators with rough kernels since the characterization depends heavily on the smoothness of the kernel function Ω. Under the premise that Ω is smooth enough, i.e., $\Omega \in C^\infty(\mathbb{S}^{n-1})$ or $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$, see, for example, [12–15]. It is difficult to weaken the smoothness of Ω, Chen and Ding [16] considered a characterization of BMO $(\mathbb{R}^n)$ space under the condition that Ω satisfies the following Hölder condition of log type

$$\left| \Omega\left(x'\right) - \Omega\left(y'\right) \right| \leq \frac{A}{\left(\log\left(2/\left| x' - y' \right| \right) \right)^\gamma} \quad \text{with } A > 0, \gamma > 1, x', y' \in \mathbb{S}^{n-1}. \tag{10}$$

It is easy to check that (10) is weaker than the Lipschitz condition and stronger than the condition (4). For $q \geq 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$, we call Ω satisfies the L^q–Dini condition if $\int_0^1 ((w_q(\delta))/(\delta)) < \infty$, where $w_q(\delta)$ is defined as

$$\begin{aligned} w_q(\delta) &= \sup_{\|\tau\| \leq \delta} \left(\int_{\mathbb{S}^{n-1}} \left| \Omega\left(\tau x'\right) - \Omega\left(x'\right) \right|^q d\sigma\left(x'\right) \right)^{1/q} \\ &\text{with } \|\tau\| = \sup_{x' \in \mathbb{S}^{n-1}} \left| \tau x' - x' \right|. \end{aligned} \tag{11}$$

As a useful supplement of [1], we give a characterization of the CBMO $(\mathbb{R}^n)$ space via the boundedness of $[b, H_{\Omega,\alpha}]$ and $[b, H^*_{\Omega,\alpha}]$ as follows.

Theorem 1. *Let $1 < p_1, p_2 < \infty$ with $(1/p_1) = (1/p_2) + (\alpha/n)$ and $(1/p_1) + (1/s)(1/q) + 1$ for some $s > {p_1}'$.*

(a) If

$$\begin{cases} \Omega \text{ satisfies } (1.1) \text{ and } (1.3); \\ b \in \mathrm{CBMO}^{\max\{p_2, s\}}(\mathbb{R}^n), \end{cases} \tag{12}$$

then

$$\begin{cases} [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n). \end{cases} \tag{13}$$

(b) if

$$\begin{cases} \Omega \text{ satisfies } (1.1), (1.2), \text{ and } (1.4); \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \text{ then } b \in \mathrm{CBMO}(\mathbb{R}^n). \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n), \end{cases} \tag{14}$$

A part of Theorem 1 has been proven in [1], we will show the rest of Theorem 1 in Section 2. In what follows, we will denote C by a positive constant which may vary from line to line. The symbol $A \lesssim B$ means $A \leq CB$ and $\mathbb{Z}$ for the set of all integers. Last, but not least, $B_r := B(0,r)$, $B_k := B_{2^k}$, $C_k := B_k \setminus B_{k-1}$, and $\chi_k := \chi_{C_k}$ with $k \in \mathbb{Z}$.

2. Proof of Theorem 1.1

We prove Theorem 1 in this section. To do so, we need one lemma about the estimates of the kernel function Ω, which plays a key role in the proof.

Lemma 2. *Let Ω satisfy (2) and (10). Then,*

(a) $|\Omega(x-y)-\Omega(x)| \le C/(\log(|x|/|y|))^{\gamma}$ with γ be given in (10) and $|x| \ge 4|y|$.

(b) If Ω furthermore satisfies the $L^{q\ge 1}$–Dini condition, then, there is a $C>0$ such that for $0<C<1/2$, $r>0$, $x \in \mathbb{R}^n$ with $|x|<Cr$; we have

$$\begin{cases} \left(\int_{r<|y|<2r} |\Omega(y-x)-\Omega(y)|^q dy\right)^{1/q} \le Cr^{n/q} \int_{|x|/2r}^{|x|/r} \dfrac{w_q(\delta)}{\delta} d\delta; \\ \left(\int_{r<|y|<2r} \dfrac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{(n-\alpha)q}} dy\right)^{1/q} \le Cr^{-(n/q')+\alpha} \int_{|x|/2r}^{|x|/r} \dfrac{w_q(\delta)}{\delta} d\delta \quad with\ 0<\alpha<\dfrac{n}{p_1}. \end{cases} \tag{15}$$

Proof. We give the proof by a slight modification from [17]. For $|x| \ge 4|y|$, we first show that

$$\left|(x-y)' - x'\right| \le \frac{3|y|}{|y|}, \tag{16}$$

$$|1/|x| - 1/|x-y|| \le C|y|/|x|^2. \tag{17}$$

Indeed, the first inequality can be obtained immediately from ([18], Lemma 2). Since

$$|1/|x| - 1/|x-y|| \le 4|y|/3|x|^2 \le C|y|/|x|^2, \tag{18}$$

(a) can be shown by (2) and (10) as

$$\begin{aligned} |\Omega(x-y)-\Omega(x)| &\le \frac{C}{\left(\log\left(2/\left|(x-y)'-x'\right|\right)\right)^{\gamma}} \\ &\le \frac{C}{(\log(|x|/|y|))^{\gamma}}. \end{aligned} \tag{19}$$

We are left to show (b). The first estimate can be obtained directly by the L^q–Dini condition as

$$\begin{aligned} &\left(\int_{r<|y|<2r} |\Omega(y-x)-\Omega(y)|^q dy\right)^{1/q} \\ &= \left(\int_r^{2r}\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right)-\Omega\left(y'\right)\right|^q d\delta\left(y'\right) t^{n-1} dt\right)^{1/q} \\ &\le Cr^{n/q} \int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \end{aligned} \tag{20}$$

and the estimate

$$\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right)-\Omega\left(y'\right)\right|^q d\delta\left(y'\right) \le Cw_q^q\left(\frac{|x|}{t}\right). \tag{21}$$

follows from ([19], p.65-77).

Accordingly, the second can be deduced similarly. In fact,

$$\begin{aligned} &\left(\int_{r<|y|<2r} \frac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{(n-\alpha)q}} dy\right)^{1/q} \\ &= Cr^{-(n/q')+\alpha} \left(\int_r^{2r}\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right)\right. \right. \\ &\quad \left.\left. -\Omega\left(y'\right)\right|^q d\delta\left(y'\right) \frac{dt}{t}\right)^{1/q} \\ &\le Cr^{-(n/q')+\alpha} \int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \end{aligned} \tag{22}$$

which is the desired one.

The following is the boundedness of the fractional Hardy operators.

Theorem 3. *Let $1<p_1,p_2<\infty$ with $(1/p_1)=(1/p_2)+(\alpha/n)$ and Ω satisfy (2) and (4). Then, both $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$ are bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$.*

Proof. Since the $(L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))$ boundedness for $H_{\Omega,\alpha}$ is contained in ([1], Proposition 3.1), it is enough to check the boundedness for $H^*_{\Omega,\alpha}$. Namely, the task is now left to show that there exist constants $C>0$ such that for any $f \in L^{p_1}(\mathbb{R}^n)$, one has

$$\|H^*_{\Omega} f\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{23}$$

To do so, we first recall a useful estimates from [1] as

$$\begin{aligned} &|x-y| \le |x| + 2^i \le 2^{k+1} \,\&\, \int_{C_i} |\Omega(x-y)|^q dy \\ &\le \int_0^{2^{k+1}} \int_{\mathbb{S}^{n-1}} \left|\Omega\left(x'\right)\right|^q d\delta\left(x'\right) t^{n-1} dt \le C2^{kn} \end{aligned} \tag{24}$$

for $x \in C_k$, $y \in C_i$, and $i \le k$. Since

$$\begin{aligned}\int_{\mathbb{R}^n} |H_\Omega^* f(x)|^{p_2} dx &\le \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|x|\le|y|\le 2^{kn}} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\quad + \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|>2^{kn}} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= I_1 + I_2. \end{aligned} \tag{25}$$

Applying Hölder's inequality to q, p_1, s for $s > 1$ and (24), we have

$$\begin{aligned} I_1 &\le \sum_{k=-\infty}^{+\infty} 2^{-k(n-\alpha)p_2} \int_{C_k} \left| \sum_{i=-\infty}^{k} \left(\int_{C_i} |\Omega(x-y)|^q dy \right)^{1/q} \right. \\ &\quad \cdot \left(\int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \left(\int_{C_i} dy \right)^{1/s} \Bigg|^{p_2} dx \\ &= C \sum_{k=-\infty}^{+\infty} \left| \sum_{i=-\infty}^{k} 2^{(i-k)n/s} \left(\int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \right|^{p_2} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k} 2^{(i-k)np_2/2s} \|f\|_{L^{p_1}(B_i)}^{p_2} \left(\sum_{i=-\infty}^{k} 2^{(i-k)np_2'/2s} \right)^{p_2/p_2'} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k} 2^{(i-k)np_2/2s} \|f\|_{L^{p_1}(B_i)}^{p_2} \\ &\le C \sum_{i=-\infty}^{+\infty} \|f\|_{L^{p_1}(B_i)}^{p_2} \sum_{k=i}^{+\infty} 2^{(i-k)np_2/2s} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}, \\ I_2 &\le \sum_{k=-\infty}^{+\infty} \int_{B_k} \left| \sum_{i=k}^{\infty} \left(\int_{C_i} |\Omega(x-y)|^q dy \right)^{1/q} \right. \\ &\quad \cdot \left(\int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \left(\int_{C_i} |y|^{-(n-\alpha)s} dy \right)^{1/s} \Bigg|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} \left| \sum_{i=k}^{+\infty} 2^{(k-i)n/p_2} \|f\|_{L^{p_1}(B_i)} \right|^{p_2} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=k}^{+\infty} 2^{(k-i)n/2p_2} \|f\|_{L^{p_1}(B_i)}^{p_2} \\ &\le C \sum_{i=-\infty}^{+\infty} \|f\|_{L^{p_1}(B_i)}^{p^2} \sum_{k=-\infty}^{i} 2^{(k-i)n/2p_2} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}, \end{aligned} \tag{26}$$

which is our desired result.

Now, we can prove Theorem 1. Without loss of generality, we can assume that $\|b\|_{CBMO^{\max\{p_2,s\}}(\mathbb{R}^n)} = 1$ in the proof of (a) since $b \in \mathrm{CBMO}^{\max\{p_2,s\}}(\mathbb{R}^n)$. We see at once that the $(L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))$ boundedness of $[b, H_{\Omega,\alpha}]$ is just ([1], Theorem 3.1). To complete the proof of (a), what is left is to show is that

$$\left\| \left[b, H_{\Omega,\alpha}^*\right] f \right\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{27}$$

It is easy to check that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \left[b, H_{\Omega,\alpha}^*\right] f(x) \right|^{p_2} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|\ge|x|} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{2^{kn}\ge|y|\ge|x|} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right. \\ &\quad \left. + \int_{|y|>2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|x|\le|y|\le 2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\quad + C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|>2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= J_1 + J_2. \end{aligned} \tag{28}$$

Using Hölder's inequality, we have

$$\begin{aligned} J_1 &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \frac{1}{|x|^{n-\alpha}} \int_{|y|\le 2^{kn}} \Omega(x-y)(b(x)-b(y))f(y) dy \right|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} 2^{-k(n-\alpha)p_2} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} \Omega(x-y)(b(x) \right. \\ &\quad \left. - b(y))f(y) dy \right|^{p_2} dx \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}. \end{aligned} \tag{29}$$

For the term J_2, we see at once that

$$\begin{aligned} J_2 &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)\left(b(x)-b_{B_k}\right)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\quad + C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)\left(b(y)-b_{B_k}\right)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= J_{21} + J_{22}. \end{aligned} \tag{30}$$

The Hölder inequality, along with (24), implies

$$\begin{aligned}
J_{21} &\le C\sum_{k=-\infty}^{+\infty}\int_{C_k}\left|b(x)-b_{B_k}\right|^{p_2}\left|\sum_{i=k}^{+\infty}\int_{C_i}\frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy\right|^{p_2}dx\\
&\le C\sum_{k=-\infty}^{+\infty}\int_{B_k}\left|b(x)-b_{B_k}\right|^{p_2}\left|\sum_{i=k}^{+\infty}\|f\|_{L^{p_1}(B_i)}\right.\\
&\quad\left.\cdot\left(\int_{C_i}|\Omega(x-y)|^q dy\right)^{1/q}\left(\int_{C_i}|y|^{-(n-\alpha)s}dy\right)^{1/s}\right|^{p_2}dx\\
&\le C\sum_{k=-\infty}^{+\infty}\sum_{i=k}^{+\infty}2^{(k-i)n/2}\|f\|^{p_2}_{L^{p_1}(B_i)}\left(\sum_{i=k}^{+\infty}2^{(k-i)np_2'/2p_2}\right)^{p_2/p_2'}\\
&\le C\|f\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}\tag{31}$$

The term J_{22} need a further decomposition as follows:

$$\begin{aligned}
J_{22} &\le C\sum_{k=-\infty}^{+\infty}\int_{B_k}\left|\sum_{i=k}^{+\infty}\int_{C_i}\frac{\Omega(x-y)\left(b(y)-b_{B_i}\right)f(y)}{|y|^{n-\alpha}}dy\right|^{p_2}dx\\
&\quad+C\sum_{k=-\infty}^{+\infty}\int_{B_k}\left|\sum_{i=k}^{+\infty}\int_{C_i}\frac{\Omega(x-y)\left(b_{B_i}-b_{B_k}\right)f(y)}{|y|^{n-\alpha}}dy\right|^{p_2}dx\\
&:= J_{221}+J_{222}.
\end{aligned}\tag{32}$$

Applying the Hölder inequality, we deduce

$$\begin{aligned}
J_{221} &\le C\sum_{k=-\infty}^{+\infty}\int_{B_k}\left|\sum_{i=k}^{+\infty}\left(\int_{C_i}\frac{|f(y)|^{p_1}}{|y|^{(n-\alpha)p_1}}dy\right)^{1/p}\right.\\
&\quad\left.\cdot\left(\int_{C_i}|\Omega(x-y)|^q dy\right)^{1/q}\left(\int_{C_i}\left|\left(b(y)-b_{B_i}\right)\right|^s dy\right)^{1/s}\right|^{p_2}dx\\
&\le C\|b\|^p_{\mathrm{CBMO}^s(\mathbb{R}^n)}\sum_{k=-\infty}^{+\infty}\left|\sum_{i=k}^{+\infty}2^{(k-i)n/p}\|f\|_{L^p(B_i)}\right|^p\\
&\le C\sum_{k=-\infty}^{+\infty}\sum_{i=k}^{+\infty}2^{(k-i)n/2}\|f\|^{p_2}_{L^{p_1}(B_i)}\left(\sum_{i=k}^{+\infty}2^{(k-i)np_2'/2p_2}\right)^{p_2/p_2'}\\
&\le C\|f\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}\tag{33}$$

From the fact that

$$\begin{aligned}
\left|b_{B_k}-b_{B_i}\right| &\le \sum_{j=k}^{i-1}\left|b_{B_j}-b_{B_{j+1}}\right|\\
&\le \sum_{j=k}^{i-1}\frac{1}{|B_j|}\int_{B_{j+1}}\left|b(y)-b_{B_{j+1}}\right|dy\\
&\le C(i-k)\|b\|_{\mathrm{CBMO}^{p_2}(B_i)},
\end{aligned}\tag{34}$$

the term J_{222} can be estimated as follows:

$$\begin{aligned}
J_{222} &\le C\|b\|^{p_2}_{CBMO^{p_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{+\infty}\int_{B_k}\\
&\quad\cdot\left|\sum_{i=k}^{+\infty}(i-k)\int_{C_i}\frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy\right|^{p_2}dx\\
&\le C\sum_{k=-\infty}^{+\infty}\int_{B_k}\left|\sum_{i=k}^{+\infty}(i-k)\|f\|_{L^{p_1}(B_i)}\left(\int_{C_i}|\Omega(x-y)|^q dy\right)^{1/q}\right.\\
&\quad\left.\cdot\left(\int_{C_i}|y|^{-(n-\alpha)s}dy\right)^{1/s}\right|^{p_2}dx\\
&\le C\sum_{k=-\infty}^{+\infty}\left|\sum_{i=k}^{+\infty}2^{(k-i)n/p}\|f\|_{L^p(B_i)}\right|^p\\
&\le C\sum_{i=-\infty}^{+\infty}\|f\|^{p_2}_{L^{p_1}(B_i)}\sum_{k=-\infty}^{i}(i-k)2^{(k-i)n/2}\\
&\le C\|f\|^{p_2}_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}\tag{35}$$

which is the desired result and (a) is obtained.

Next, we verify (b) inspired by [18]. Namely, we need to show that there is a constant $C(\Omega,\alpha,p,n)$ such that

$$\Theta := \Theta(b,r) := \frac{1}{|B_r|}\int_{B_r}\left|b(y)-b_{B_r}\right|dy \le C \text{ for any } r\in\mathbb{R}^+.\tag{36}$$

For abbreviation, we assume that $\|[b,H_{\Omega,\alpha}]\|_{L^{p_1}\to L^{p_2}}=1$, $\|[b,H^*_{\Omega,\alpha}]\|_{L^{p_1}\to L^{p_2}}=1$, and $b_{B_r}=0$ since $[b-b_{B_r},H_{\Omega,\alpha}]=[b,H_{\Omega,\alpha}]$. Let $f(y)=\mathrm{sgn}\,(b(y))\chi_{B_r}(y)$. It is easy to check that $(1/|B_r|)\int_{\mathbb{R}^n}f(y)b(y)dy=\Theta$.

Applying (3) and (10), we deduce that for A and γ appearing in (10), there is a constant $C_1<1$ such that $\sigma(D)>0$ for $D=\{x'\in\mathbb{S}^{n-1}:\Omega(x')\ge((2A)/((\log(2/C_1))^\gamma))\}$. For $x'\in D$ and $y'\in\mathbb{S}^{n-1}$ with $|x'-y'|\le C_1$, we obtain from (10) that

$$\begin{aligned}
\Omega\left(y'\right) &= \Omega\left(x'\right)-\left(\Omega\left(x'\right)-\Omega\left(y'\right)\right)\\
&\ge \left|\Omega\left(x'\right)\right|-\left|\Omega\left(x'\right)-\Omega\left(y'\right)\right|\\
&\ge \frac{A}{(\log(2/C_1))^\gamma}.
\end{aligned}\tag{37}$$

Writing $E=\{x\in\mathbb{R}^n:|x|>C_2r \text{ and } \&\, x'\in D\}$ *with* $C_2=3C_1^{-1}+1>4$, we see at once that for $x\in E$, $|[b,H_{\Omega,\alpha}]f(x)|\ge|H_{\Omega,\alpha}(bf)(x)|-|b(x)||H_{\Omega,\alpha}f(x)|:=K_1(x)-K_2(x)$. We conclude from (16) that $|(x-y)'-x'|\le 3|y|/|x|\le C_1$ since $|y|<r$ and $|x|>C_2|y|>4|y|$, and hence, $\Omega((x-y)')\ge A/(\log(2/C_1))^\gamma$, and finally, that

$$K_1(x) \geq \left| \frac{A}{(\log(2/C_1))^{\gamma}} \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} b(y) f(y)\, dy \right| = \frac{A}{(\log(2/C_1))^{\gamma} |x|^{n-\alpha}} \int_{B_r} b(y) f(y)\, dy = \frac{C_3 r^n}{|x|^{n-\alpha}} \Theta. \tag{38}$$

Furthermore,

$$\begin{aligned} K_2(x) &\leq \frac{|b(x)|}{|x|^{n-\alpha}} \int_{B_r} |f(y)| \left| \Omega\left((x-y)'\right) - \Omega\left(x'\right) \right| dy \\ &\leq \frac{C|b(x)|}{|x|^{n-\alpha}} \int_{B_r} \frac{|f(y)|}{(\log(|x|/|y|))^{\gamma}}\, dy \\ &\leq \frac{C|b(x)|}{|x|^{n-\alpha}} \int_{B_r} \frac{|f(y)|}{(\log(|x|/r))^{\gamma}}\, dy \\ &\leq \frac{C_4 r^n |b(x)|}{|x|^{n-\alpha} (\log(|x|/r))^{\gamma}}. \end{aligned} \tag{39}$$

For abbreviation, we write

$$\begin{cases} F = \left\{ x \in E : |b(x)| > \dfrac{C_3 \Theta}{2C_4} \left(\log\left(\dfrac{|x|}{r} \right) \right)^{\gamma}, |x| < \Theta^{p_2/n} r \right\}; \\ F_1 = \{E \setminus F\} \cap \left\{ |x| < \Theta^{p_2/n} r \right\}; \\ F_2 = \{E\} \cap \left\{ C_5 (|F| + (C_2 r)^n)^{1/n} < |x| < \Theta^{p_2/n} r \right\}. \end{cases} \tag{40}$$

This, along with the estimates for K_1 and K_2, one has

$$\begin{aligned} &\|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \|f\|_{L^{p_2'}(\mathbb{R}^n)} \\ &\quad \geq \|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \left\| [b, H_{\Omega,\alpha}] f \right\|_{L^{p_1'}(\mathbb{R}^n)} \\ &\quad \geq \int_{F_1} \left| [b, H_{\Omega,\alpha}] f(x) \right| dx \\ &\quad \geq \int_{F_1} \left| \frac{C_3 r^n \Theta}{|x|^{n-\alpha}} - \frac{C_3 r^n \Theta}{2|x|^{n-\alpha}} \right|^p dx \\ &\quad \geq \left(\frac{C_3 r^n \Theta}{2} \right) \int_{C_3(|F|+(C_2 r)^n)^{1/n}}^{\Theta^{p_2/n} r} \rho^{\alpha-1} d\rho \int_D d\sigma\left(x'\right) \\ &\quad \geq \sigma(D) \frac{(C_3 r^n \Theta)}{\alpha} \left[\Theta^{\alpha p_2/n} r^{\alpha} - C_5^{\alpha} (|F| + (C_2 r)^n)^{\alpha/n} \right]. \end{aligned} \tag{41}$$

Consequently,

$$\begin{aligned} (|F| + (C_2 r)^n)_{\alpha/n} &\geq C_5^{-\alpha} \Theta^{\alpha p_2/n} r^{\alpha} \\ &\quad - \frac{C_5^{-\alpha} \alpha}{C_3 \sigma(D)} \frac{\|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \|f\|_{L^{p_2'}(\mathbb{R}^n)}}{\Theta r^n} \\ &\geq C_6 \Theta^{\alpha p_2/n} r^{\alpha}. \end{aligned} \tag{42}$$

This in turn implies that $C_6 \Theta^{p_2} r^n \leq |F| + (C_2 r)^n$. Thus, (b) is proved if $\Theta \leq (2C_6^{-1} C_2^n)^{1/p_2}$. If $\Theta > (2C_6^{-1} C_2^n)^{1/p_2}$, we see immediately that

$$|F| \geq C_6 \Theta^{p_2} r^n - C_2^n r^n \geq \frac{C_6 \Theta^{p_2} r^n}{2}. \tag{43}$$

According to $F \subset E$, $C_2 > 4$, $|y| > C_2 |x|$, Lemma 2 and (37), we can obtain that for $x \in B_r$ and $y \in F$,

$$|x - y| \approx |y|, \Omega\left((x-y)'\right) \geq \frac{A}{(\log(2/C_1))^{\gamma}}. \tag{44}$$

We continue to choose $f^*(y) = (sgn(b(y)))\chi_F(y)$ for $x \in B_r$ and get

$$\begin{aligned} \left| \left[b, H^*_{\Omega,\alpha} \right] f^*(x) \right| &\geq \int_F \frac{|\Omega(x-y) b(y)|}{|y|^{n-\alpha}}\, dy \\ &\quad - |b(x)| \int_F \frac{|\Omega(x-y) f^*(y)|}{|y|^{n-\alpha}}\, dy \\ &:= L_1(x) + L_2(x). \end{aligned} \tag{45}$$

It is easy to check that

$$\begin{aligned} L_2(x) &\leq C_7 |b(x)| \int_{C_2 r}^{\Theta^{p_2/n} r} \rho^{\alpha-1} d\rho \int_{\mathbb{S}^{n-1}} d\sigma\left(y'\right) \\ &\leq C_8 |b(x)| \left(\Theta^{\alpha p_2/n} r^{\alpha} - C_2^{\alpha} r^{\alpha} \right). \end{aligned} \tag{46}$$

To deal with the term $L_1(x)$, we first obtain from (44) that

$$L_1(x) \geq \frac{A C_3 \Theta}{2C_4 (\log 1/C_1)^{\gamma}} \int_F \frac{(\log(|y|/r))^{\gamma}}{|y|^{n-\alpha}}\, dy. \tag{47}$$

Then, the estimate for $L_1(x)$ consists of two cases.

Case 1. $\gamma \geq n$. Since $4r < C_2 r < |y| < \Theta^{p_2/n} r$ for $y \in F$, we conclude from (47) and $|F| \geq (C_6 \Theta^{p_2} r^n)/2$ that

$$\begin{aligned} L_1(x) &\geq \frac{C_9 \Theta}{r^n} \int_F \left(\frac{\log(|y|/r)}{|y|/r} \right)^n \frac{(\log(|y|/r))^{\gamma-n}}{|y|^{\alpha}}\, dy \\ &\geq \frac{C_9 \Theta^{1+(\alpha p_2/n)}}{r^{n-\alpha}} \left(\frac{\log(p_2/n)}{\Theta^{p_2/n}} \right)^n (\log C_2)^{\gamma-n} |F| \\ &\geq C_{10} \Theta^{1+\alpha p_2/n} (\log \Theta)^n r^{\alpha}. \end{aligned} \tag{48}$$

Case 2. $1 < \gamma < n$. In this case,

$$\begin{aligned} L_1(x) &\geq \frac{C_9 \Theta^{1+(\alpha p_2/n)}}{r^{n-\alpha}} \int_F \frac{(\log(|y|/r))^{\gamma}}{(|y|/r)^n}\, dy \\ &\geq C_{11} \Theta^{1+\alpha p_2/n} (\log \Theta)^{\gamma} r^{\alpha}. \end{aligned} \tag{49}$$

This in turn reveals that

$$\begin{aligned} \left| \left[b, H^*_{\Omega,\alpha} \right] f^*(x) \right| &\geq C_{11} \Theta^{1+\alpha p_2/n} (\log \Theta)^{\gamma} r^{\alpha} \\ &\quad - C_9 |b(x)| r^{\alpha} \Theta^{\alpha p_2/n}, \quad \forall x \in B_r. \end{aligned} \tag{50}$$

Having disposed of the above estimates, we can now obtain

$$\begin{aligned} C_{12}r^{n+\alpha}\Theta^{1+\alpha p_2/n} &\geq \left(\int_{|x|<\Theta^{p_2/n}r} dx\right)^{1/p_1} r^{n/p_2'} \geq r^{n/p_2'}\left\|f^*\right\|_{L^{p_1}(\mathbb{R}^n)} \\ &\geq \left\|\chi_{B_r}\right\|_{L^{p_2'}(\mathbb{R}^n)}\left\|\left[b, H^*_{\Omega,\alpha}\right]f^*\right\|_{L^{p_2}(\mathbb{R}^n)} \\ &\geq \int_{B_r}\left|\left[b, H^*_{\Omega,\alpha}\right]f^*\right|dx \\ &\geq C_{11}\Theta(\log\Theta)^{\gamma}r^{\alpha}\int_{B_r} dx \\ &\quad - C_9\Theta^{\alpha p_2/n}r^{\alpha}\int_{B_r}|b(x)|dx \\ &\geq C_{11}\Theta^{1+\alpha p_2/n}r^{n+\alpha}(\log\Theta)^{\gamma} - C_9\Theta^{1+\alpha p_2/n}r^{n+\alpha}. \end{aligned} \tag{51}$$

Namely, $C_{11}(log\Theta)^{\gamma} - C_9 \leq C_{12}$. This in turn shows that there is a constant $C \coloneqq C(\Omega, \alpha, n, p, \gamma)$ such that $\Theta \leq C$, whence reaching the desired fact.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Averaging Method for Neutral Stochastic Delay Differential Equations Driven by Fractional Brownian Motion

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In this paper, we investigate the stochastic averaging method for neutral stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. By using the linear operator theory and the pathwise approach, we show that the solutions of neutral stochastic delay differential equations converge to the solutions of the corresponding averaged stochastic delay differential equations. At last, an example is provided to illustrate the applications of the proposed results.

1. Introduction

Since Kolmogorov's work began in 1940 [1], the fractional Brownian motion $\{B^H(t), t \geq 0\}$ with Hurst parameter $H \in (0, 1)$ has been studied by some authors [2, 3]. $B^H(t)$ is a zero mean Gaussian stochastic process with a covariance function determined by parameter H. It is self-similar and has the stationary increments and its increment process has long-range dependence when Hurst parameter $1/2 < H < 1$, which makes $B^H(t)$ a suitable candidate to model many complex phenomena in finance and other practical problems. When $H \neq 1/2$, the $B^H(t)$ neither is a semimartingale nor a Markov process. These properties mean that the analysis tools for the classical stochastic differential equation theory no longer work. The most obvious problem is how to define a proper notion of stochastic integral with respect to $B^H(t)$. Three main integration techniques with regard to $B^H(t)$ have been researched, see, for example, [4–8] and the references therein.

In recent years, there has been much interest in a stochastic averaging method, which provides a powerful tool to approximate the original dynamical systems under random fluctuations by a simpler system. The first work, which was introduced by Khasminskii [9], investigated a stochastic averaging method for stochastic differential equations with Gaussian random fluctuations. Since then, the stochastic averaging method has been developed for many types of stochastic differential equations, see, e.g., [10–15].

Some phenomena of biological dynamics, engineering, and financial markets can be better understood when the effect of time delays is considered in the models, see, e.g., [16]. Hence, stochastic delay differential equations driven by fractional Brownian motion are proposed and have recently attracted great attention [17–19]. However, except [20] studied averaging method for stochastic delay differential equations of neutral type driven by G-Brownian motion, the averaging method for neutral stochastic delay differential equations is seldom considered. In this paper, our aim is to study the stochastic averaging method for neutral stochastic delay differential equations driven by fractional Brownian motion.

2. Model Description and Preliminaries

The aim of this section is to introduce the model and some preliminary lemmas.

The following notations are needed in this paper.

Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space. $B^H(t)$ is a m-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ defined on the space

$(\Omega, F, \{F_t\}_{t\geq 0}, P)$. Let $|\bullet|$ denote the Euclidean norm. Let $\mathbb{C} = C((-\infty,0];\mathbb{R}^d)$ denote the family of all continuous functions $\xi : (-\infty,0] \longrightarrow \mathbb{R}^d$ with the norm $\|\xi\|_{\mathbb{C}} = \sup_{-\infty<s\leq 0} |\xi(s)|$. For a stochastic process $\xi = \{\xi(t), t\in(-\infty,T]\}$, $\xi_t \in \mathbb{C}$ defined by $\xi_t(s) = \xi(t+s), s \in (-\infty,0], t \in [0,T]$. Throughout the paper, the symbol N stands for a generic constant, whose value can change from one line to another.

In this paper, we will discuss the following neutral stochastic delay differential equations driven by fractional Brownian motion:

$$d[x(t) + c(t)x(t-\tau(t))] = f(t,x(t),x(t-\tau(t)))dt + g(t,x(t),x(t-\tau(t)))dB^H(t), \tag{1}$$

with initial condition $x_0 = \xi \in C((-\infty,0];\mathbb{R}^d)$, Where $c \in C((-\infty,T],[0,1))$, $0 < T < \infty$. The function τ: $[0,T] \longrightarrow \mathbb{R}^+$ is the time delay. The mappings $f = (f^i)_{d\times 1}, f^i : [0,T]\times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $g = (g^{i,j})_{d\times m}, g^{i,j} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}$ are Borel measurable.

The following operator theory is useful to obtain our main results.

Lemma 1 (see [21]). *Suppose that D is a bounded linear operator on Banach space X, if $\|D\| < 1$, then $I - D$ has a bounded inverse operator $(I-D)^{-1}$, and $\|(I-D)^{-1}\| \leq 1/(1-\|D\|)$.*

Lemma 2 (see [21]). *Assume that D is a bounded linear operator on Banach space X and has an inverse bounded operator, for arbitrary $\Delta D : X \longrightarrow X$, if $\|\Delta D\| < 1/\|D^{-1}\|$, then $S = D + \Delta D$ has a bounded inverse, and $S^{-1} = \sum_{j=0}^{\infty}(-1)^j D^{-1}(D^{-1}\Delta D)^j$.*

Let the Banach space

$$Q = \left\{x : x|_{(-\infty,T]} \in C\left((-\infty,0];\mathbb{R}^d\right)\right\}. \tag{2}$$

$\Phi : Q \longrightarrow Q$ be defined by

$$(\Phi x)(t) = x(t) + c(t)x(t-\tau(t)). \tag{3}$$

Lemma 3. *Let $c^M = \max_{t\in(-\infty,T]} c(t)$. If $0 \leq c(t) < 1$, then Φ has a bounded inverse Φ^{-1} on Q, and for all $x \in Q$*

$$\left(\Phi^{-1}x\right)(t) = \sum_{j\geq 0}\prod_{0\leq i\leq j-1}(-1)^j c(t-i\tau(t))x(t-j\tau(t)), \tag{4}$$

$$\|\Phi^{-1}x\| \leq \|x\|/1 - c^M.$$

Proof. The proof processes are similar to Lemma 2.6 in [22], so we omit it here.

We note that Φ and Φ^{-1} are both linear operators. Then, the system (1) can be changed by the inverse transformation of Φ into the following:

$$du(t) = f\left(t,\left(\Phi^{-1}u\right)(t),\left(\Phi^{-1}u\right)(t-\tau(t))\right)dt + g\left(t,\left(\Phi^{-1}u\right)(t),\left(\Phi^{-1}u\right)(t-\tau(t))\right)dB^H(t), \tag{5}$$

with initial condition $u_0 = (\Phi^{-1}x_0)$. So, we can conclude that $u(t)$ is a solution of system (5) if and only if $(\Phi^{-1}u)(t)$ is a solution of system (1).

System (5) can be written in the integral form:

$$u(t) = u_0 + \int_0^t f\left(s,\left(\Phi^{-1}u\right)(s),\left(\Phi^{-1}u\right)(s-\tau(s))\right)ds + \int_0^t g\left(s,\left(\Phi^{-1}u\right)(s),\left(\Phi^{-1}u\right)(s-\tau(s))\right)dB^H(s), \tag{6}$$

where $\int_0^t g(s,x(s),x,(s-\tau(s)))dB^H(s)$ is a pathwise-defined integral which can be represented by fractional derivatives, see, e.g., Young et al. [4], Zähle [5], and Nualart and Răşcanu [7].

Let $H \in (1/2,1)$, $1-H < \alpha < 1/2$, denoted by $W^{\alpha,\infty}_{(-\infty,T]}$, the space of measurable functions $h : (-\infty,T] \longrightarrow R^d$ with the norm

$$\|h\|_t = \|h\|_{\infty,t} + \int_0^t \|h_{\cdot+t-s} - h_\cdot\|_{\infty,s}(t-s)^{-(\alpha+1)}ds < \infty, \tag{7}$$

where $\|h\|_{\infty,t} = \sup_{-\infty<s\leq t}|h(s)|$.

Let $\alpha \in (0,1/2)$ be fixed. Denoting $W_0^{\alpha,1}$ the space of measurable functions $h : [0,T] \longrightarrow \mathbb{R}^d$ such that

$$\|h\|_{\alpha,1} := \int_0^T \frac{|h(s)|}{s^\alpha}ds + \int_0^T\int_0^s \frac{|h(s)-h(y)|}{(s-y)^{\alpha+1}}dyds < \infty. \tag{8}$$

For $h \in W_0^{\alpha,1}$, defining the left-side fractional derivative $D^\alpha_{0+}h(u)$ by

$$D^\alpha_{0+}h(u) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{h(u)}{u^\alpha} + \alpha\int_0^u \frac{h(u)-h(y)}{(u-y)^{1+\alpha}}dy\right]1_{(0,T)}(u). \tag{9}$$

Also denoted by $W_T^{1-\alpha,\infty}$ is the space of measurable functions $h : [0,T] \longrightarrow \mathbb{R}^m$ such that

$$\|h\|_{1-\alpha,\infty,T} := \sup_{0\leq s<t\leq T}\left(\frac{|h(t)-h(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|h(y)-h(s)|}{(y-s)^{2-\alpha}}dy\right) < \infty. \tag{10}$$

For any $h \in W_T^{1-\alpha,\infty}$, setting $h_{t-}(u) := h(u) - h(t)$ and defining the right-sided fractional derivative $D_{t-}^{1-\alpha} h_{t-}(u)$ by

$$D_{t-}^{1-\alpha} h_{t-}(u) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left[\frac{h(u) - h(t)}{(t-u)^{1-\alpha}} + (1-\alpha) \int_u^t \frac{h(u) - h(y)}{(y-u)^{2-\alpha}} dy \right] 1_{(0,t)}(u). \tag{11}$$

For $z \in W_T^{1-\alpha,\infty}$, we have

$$\Lambda_\alpha(z) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| \left(D_{t-}^{1-\alpha} z_{t-} \right)(s) \right| \le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|z\|_{1-\alpha,\infty,T} < \infty, \tag{12}$$

where $\Gamma(\cdot)$ is the gamma function. Moreover, if $h \in W_0^{\alpha,1}$ and $z \in W_T^{1-\alpha,\infty}$, one can define the integral $\int_0^t h dz$ in the sense of Zähle [5] for all $t \in [0, T]$ and it follows from Nualart and Răşcanu [7] that

$$\left| \int_0^t h dz \right| \le \Lambda_\alpha(z) \|h\|_{\alpha,1}. \tag{13}$$

For our purpose, we adopt the following hypotheses on the coefficients.

(H_1) The function $g(t, x, y)$ is continuously differentiable in variables x and y and there exist some constants $\beta, \delta, 0 < \beta, \delta \le 1$ such that the following properties hold for all $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d, t \in [0, T]$:

$$\begin{aligned} |g(t,x,y) - g(t,\tilde{x},\tilde{y})| &\le N(|x-\tilde{x}| + |y-\tilde{y}|), \left|\partial_{x_i} g(t,x,y) - \partial_{x_i} g(t,\tilde{x},\tilde{y})\right| + \left|\partial_{y_i} g(t,x,y) - \partial_{y_i} g(t,x,\tilde{y})\right| \\ &\le N\left(|x-\tilde{x}|^\delta + |y-\tilde{y}|^\delta\right), |g(t,x,y) - g(s,x,y)| + \left|\partial_{x_i} g(t,x,y) - \partial_{x_i} g(s,x,y)\right| \\ &\quad + \left|\partial_{y_i} g(t,x,y) - \partial_{y_i} g(s,x,y)\right| \le N|t-s|^\beta, \left|\partial_{x_i} g(t,x,y)\right| + \left|\partial_{y_i} g(t,x,y)\right| \\ &\le N, |g(t,x,y)| \le N(1 + |x| + |y|). \end{aligned} \tag{14}$$

(H_2) The function $f(t, x, y)$ is Lipschitz continuous and has linear growth in the variable x and y, uniformly in $t \in [0, T]$, that is, $|f(t,x,y) - f(t,\tilde{x},\tilde{y})| \le N(|x-\tilde{x}| + |y-\tilde{y}|)$, $|f(t,x,y)| \le N(1 + |x| + |y|)$.

Let $\alpha_0 = \min\{1/2, \beta, \delta/(1+\delta)\}$. Under the hypotheses $(H_1) - (H_2)$ with $\beta > 1 - H$ and $\delta > (1/H) - 1$, we can follow the work of Shevchenko [8] to testify that system (5) has a unique solution $u(t) \in W_{(-\infty,T]}^{\alpha,\infty}$, for all $\alpha \in (1 - H, \alpha_0)$. Hence, system (1) has a unique solution $(\Phi^{-1}u)(t) = x(t) \in W_{(-\infty,T]}^{\alpha,\infty}$. Furthermore, we can deduce $g \in W_0^{\alpha,1}$ according to Nualart and Răşcanu [7]. On the other hand, the trajectories of B^H are almost surely locally β'-Hölder-continuous for all $0 < \beta' < H$ see Zähle [5]. Then, B^H belongs to the space $W_T^{1-\alpha,\infty}$. Hence, from the above discussion, we have the following estimate:

$$\left| \int_0^t g(s) dB^H(s) \right| \le \Lambda_\alpha(B^H) \|g\|_{\alpha,1}, \tag{15}$$

where $\Lambda_\alpha(B^H)$ has moments of all orders.

Given that $\epsilon \in [0, \epsilon_0]$ with $\epsilon_0 > 0$ is a fixed number, consider the following standard form of system (5)

$$du_\epsilon(t) = \epsilon f\left(t, \left(\Phi^{-1}u\right)_\epsilon(t), \left(\Phi^{-1}u\right)_\epsilon(t - \tau(t))\right) dt + \epsilon^H g\left(t, \left(\Phi^{-1}u\right)_\epsilon(t), \left(\Phi^{-1}u\right)_\epsilon(t - \tau(t))\right) dB^H(t), \tag{16}$$

with $u_\epsilon(0) = u_0$. Correspondingly, the standard form of system (1) is

$$d[x_\epsilon(t) + c(t)x_\epsilon(t - \tau(t))] = \epsilon f(t, x_\epsilon(t), x_\epsilon(t - \tau(t)))dt + \epsilon^H g(t, x_\epsilon(t), x_\epsilon(t - \tau(t)))dB^H(t), \tag{17}$$

with $x_\epsilon(0) = x_0$.

Let $\bar{f}(x,y)$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\bar{g}(x,y)$: $R^d \times R^d \longrightarrow R^{(d\times m)}$ be measurable functions and satisfy the conditions in $(H_1) - (H_2)$ and the following inequalities:

$$\begin{aligned} &(I_1) \frac{1}{\tilde{T}} \int_0^{\tilde{T}} \left| f(s,x,y) - \tilde{f}(x,y) \right|^2 ds \le \psi_1\left(\tilde{T}\right)\left(1 + |x|^2 + |y|^2\right), \\ &(I_2) \frac{1}{\tilde{T}} \int_0^{\tilde{T}} |g(s,x,y) - \tilde{g}(x,y)|^2 ds \le \psi_2\left(\tilde{T}\right)\left(1 + |x|^2 + |y|^2\right), \end{aligned} \tag{18}$$

where $\tilde{T} \in [0, T], \psi_i(\tilde{T})$ are positive bounded functions with $\sup_{0 \le \tilde{T} \le T} |\psi_i(\tilde{T})| \le N$, $\lim_{\tilde{T} \to \infty} \psi_i(\tilde{T}) = 0, i = 1, 2$.

Let $w_\epsilon(t)$ denote the solution to the following averaged system:

$$dw_\epsilon(t) = \epsilon\bar{f}\left(\left(\Phi^{-1}w\right)_\epsilon(t), \left(\Phi^{-1}w\right)_\epsilon(t-\tau(t))\right)dt + \epsilon^H\bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(t), \left(\Phi^{-1}w\right)_\epsilon(t-\tau(t))\right)dB^H(t), \tag{19}$$

with $w_\epsilon(0) = u_0$. Then, $y_\epsilon(t) \coloneqq (\Phi^{-1}w)_\epsilon(t)$ is the solution to the following averaged system:

$$dy_\epsilon(t) = \epsilon\bar{f}\left(\left(\Phi^{-1}y\right)_\epsilon(t), \left(\Phi^{-1}y\right)_\epsilon(t-\tau(t))\right)dt + \epsilon^H\bar{g}\left(\left(\Phi^{-1}y\right)_\epsilon(t), \left(\Phi^{-1}y\right)_\epsilon(t-\tau(t))\right)dB^H(t), \tag{20}$$

with $y_\epsilon(0) = x_0$.

Now, from the above hypotheses, we are in conditions to demonstrate the relationship between $x_\epsilon(t)$ and $y_\epsilon(t)$.

3. Main Results and Proofs

The main goal of this section is to use the averaging principle to investigate the neutral stochastic delay differential equations driven by fractional Brownian motion. Theorem 6 shows that the solution $y_\epsilon(t)$ of the averaged system (20) converges to solution $x_\epsilon(t)$ of the original system (17) in the sense of mean square.

The following lemmas are crucial for our analysis and are proved in Shevchenko [8] and Nualart and Răşcanu [7].

Lemma 4 (see [8]). *Assume that hypotheses $(H1)-(H2)$ hold. Then, there exists a constant $G_0 > 0$, for $t \in (-\infty, T]$ such that*

$$E\|u_\epsilon\|_T^2 \le G_0. \tag{21}$$

Lemma 5 (see [7]). *Assume that hypotheses $(H_1)-(H_2)$ hold. Then, there exists a constant $G_1 > 0$, for $s, t \in [0, T]$, we have*

$$|u_\epsilon(t) - u_\epsilon(s)| \le G_1\left(1 + 2\left\|\Phi^{-1}\right\|\|u_\epsilon\|_T\right)|t-s|^{1-\alpha}. \tag{22}$$

From Lemma 5, we can prove similarly that $|w_\epsilon(t) - w_\epsilon(s)| \le G_1(1 + 2\|\Phi^{-1}\|\|w_\epsilon\|T)|t-s|^{1-\alpha}$.

Theorem 6. *Assume that hypotheses $(H_1)-(H_2)$ and $(I_1)-(I_2)$ hold. For $\epsilon \in (0, \epsilon_0]$ is a positive small parameter with ϵ_0 a constant, $\theta > 0$, there exist constants $\epsilon_1 \in (0, \epsilon_0]$, $O > 0$ such that for any $\epsilon \in (0, \epsilon_1]$, $t \in [0, O\epsilon^{-\lambda}]$, $0 < \lambda < 1$, we get*

$$\sup_{0\le t\le O\epsilon^{-\lambda}} E|u_\epsilon(t) - w_\epsilon(t)|^2 \le \theta, \tag{23}$$

and then, we obtain

$$\sup_{0\le t\le O\epsilon^{-\lambda}} E|x_\epsilon(t) - y_\epsilon(t)|^2 \le \left\|\Phi^{-1}\right\|^2\theta. \tag{24}$$

Proof. According to systems (16) and (19), on account of the elementary inequality, we obtain

$$\begin{aligned} E|u_\epsilon(t) - w_\epsilon(t)|^2 &\le 2\epsilon^2E\left|\int_0^t\left(f\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right)\right.\right. \\ &\quad \left.\left. - \bar{f}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right)ds\right|^2 \\ &\quad + 2\epsilon^{2H}E\left|\int_0^t\left(g\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right)\right.\right. \\ &\quad \left.\left. - \bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right)dB^H(s)\right|^2 \\ &=: J_1(t) + J_2(t), \end{aligned} \tag{25}$$

where $[0, t] \subset [0, b] \subset [0, T]$. On the one hand, for term $J_1(t)$, we derive

$$\begin{aligned} J_1(t) &= 2\epsilon^2E\left|\int_0^t\left(f\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right)\right.\right. \\ &\quad \left.\left. - \bar{f}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right)ds\right|^2 \\ &\le 2b\epsilon^2E\int_0^t\left|f\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right)\right. \\ &\quad \left. - \bar{f}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right|^2ds. \end{aligned} \tag{26}$$

On the other hand, applying (15) for term $J_2(t)$, it follows that

$$\begin{aligned} J_2(t) &\le 4\epsilon^{2H}\Lambda_\alpha^2\left(B^H\right)b^{1-2\alpha}E\int_0^t\left|g\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right) - \bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right|^2ds \\ &\quad + 4\epsilon^{2H}\Lambda_\alpha^2\left(B^H\right)E\left(\int_0^t\int_0^s\frac{1}{(s-r)^{1+\alpha}}\left|g\left(s, \left(\Phi^{-1}u\right)_\epsilon(s), \left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right) - \bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(s), \left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)\right.\right. \\ &\quad \left.\left. - g\left(r, \left(\Phi^{-1}u\right)_\epsilon(r), \left(\Phi^{-1}u\right)_\epsilon(r-\tau(r))\right) + \bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(r), \left(\Phi^{-1}w\right)_\epsilon(r-\tau(r))\right)\right|drds\right)^2 \coloneqq J_{21}(t) + J_{22}(t). \end{aligned} \tag{27}$$

Furthermore, by (H_1), Lemma 4 and Lemma 5 yields

$$\begin{aligned} J_{22}(t) &\le 4\epsilon^{2H}\Lambda_\alpha^2(B^H)\times E\left(\int_0^t\left(\int_0^s \frac{\left|g\left(s,\left(\Phi^{-1}u\right)_\epsilon(s),\left(\Phi^{-1}u\right)_\epsilon(s-\tau(s))\right)-g\left(r,\left(\Phi^{-1}u\right)_\epsilon(r),\left(\Phi^{-1}u\right)_\epsilon(r-\tau(r))\right)\right|}{(s-r)^{1+\alpha}}dr\right.\right. \\ &\quad \left.\left.+\int_0^s \frac{\left|\bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(s),\left(\Phi^{-1}w\right)_\epsilon(s-\tau(s))\right)-\bar{g}\left(\left(\Phi^{-1}w\right)_\epsilon(r),\left(\Phi^{-1}w\right)_\epsilon(r-\tau(r))\right)\right|}{(s-r)^{1+\alpha}}dr\right)ds\right)^2 \\ &\le N\epsilon^{2H}\Lambda_\alpha^2(B^H)\left[b^{2(1+\beta-\alpha)}+\left\|\Phi^{-1}\right\|^2\left(G_1^2\left(1+4\left\|\Phi^{-1}\right\|^2G_0\right)b^{4-4\alpha}+G_0b^{2-2\alpha}\right)\right]. \end{aligned} \tag{28}$$

Hence, under hypotheses $(H_1)-(H_2)$, $(I_1)-(I_2)$ and Lemma 4, we get

$$\begin{aligned} E|u_\varepsilon(t)-w_\varepsilon(t)|^2 &\le 2b\varepsilon^2E\int_0^t\left|f\left(s,\left(\Phi^{-1}u\right)_\varepsilon(s),\left(\Phi^{-1}u\right)_\varepsilon(s-\tau(s))\right)-\bar{f}\left(\left(\Phi^{-1}w\right)_\varepsilon(s),\left(\Phi^{-1}w\right)_\varepsilon(s-\tau(s))\right)\right|^2ds \\ &\quad+4\varepsilon^{2H}\Lambda_\alpha^2(B^H)b^{1-2\alpha}\times E\int_0^t\left|g\left(s,\left(\Phi^{-1}u\right)_\varepsilon(s),\left(\Phi^{-1}u\right)_\varepsilon(s-\tau(s))\right)-\bar{g}\left(\left(\Phi^{-1}w\right)_\varepsilon(s),\left(\Phi^{-1}w\right)_\varepsilon(s-\tau(s))\right)\right|^2ds \\ &\quad+N\varepsilon^{2H}\Lambda_\alpha^2(B^H)\left[b^{2(1+\beta-\alpha)}+\left\|\Phi^{-1}\right\|^2\left(G_1^2\left(1+4\left\|\Phi^{-1}\right\|^2G_0\right)b^{4-4\alpha}+G_0b^{2-2\alpha}\right)\right] \\ &\le N\left(\varepsilon^2b+\varepsilon^{2H}\Lambda_\alpha^2(B^H)b^{1-2\alpha}\right)\left\|\Phi^{-1}\right\|^2E\int_0^t\sup_{0\le\sigma\le s}|u_\varepsilon(\sigma)-w_\varepsilon(\sigma)|^2ds+N\left(\varepsilon^2b^2+\varepsilon^{2H}\Lambda_\alpha^2(B^H)b^{2-2\alpha}\right)\left(1+2\left\|\Phi^{-1}\right\|^2G_0\right) \\ &\quad+N\varepsilon^{2H}\Lambda_\alpha^2(B^H)\left[b^{2(1+\beta-\alpha)}+\left\|\Phi^{-1}\right\|^2\left(G_1^2\left(1+4\left\|\Phi^{-1}\right\|^2G_0\right)b^{4-4\alpha}+G_0b^{2-2\alpha}\right)\right], \end{aligned} \tag{29}$$

and so

$$\begin{aligned} \sup_{0\le t\le b}E|u_\varepsilon(t)-w_\varepsilon(t)|^2 &\le N\left(\varepsilon^2b+\varepsilon^{2H}\Lambda_\alpha^2(B^H)b^{1-2\alpha}\right)\left\|\Phi^{-1}\right\|^2E\int_0^b\sup_{0\le\sigma\le s}|u_\varepsilon(\sigma)-w_\varepsilon(\sigma)|^2ds \\ &\quad+N\left(\varepsilon^2b^2+\varepsilon^{2H}\Lambda_\alpha^2(B^H)b^{2-2\alpha}\right)\left(1+2\left\|\Phi^{-1}\right\|^2G_0\right) \\ &\quad+N\varepsilon^{2H}\Lambda_\alpha^2(B^H)\left[b^{2(1+\beta-\alpha)}+\left\|\Phi^{-1}\right\|^2\left(G_1^2\left(1+4\left\|\Phi^{-1}\right\|^2G_0\right)b^{4-4\alpha}+G_0b^{2-2\alpha}\right)\right]. \end{aligned} \tag{30}$$

Then, select $\lambda\in(0,1)$, $O>0$, such that for each $t\in[0, O\varepsilon^{-\lambda}]\subset[0,T]$, we obtain

$$\sup_{0\le t\le O\varepsilon^{-\lambda}}E|u_\varepsilon(t)-w_\varepsilon(t)|^2\le K\varepsilon^{1-\lambda}, \tag{31}$$

where

$$\begin{aligned} K &= N\varepsilon^{\lambda-1}\left\{\left[\left(1+2\left\|\Phi^{-1}\right\|^2G_0\right)\left(\varepsilon^{2-2\lambda}O^2+\varepsilon^{2H+2\alpha\lambda-2\lambda}\Lambda_\alpha^2(B^H)O^{2-2\alpha}\right)\right.\right. \\ &\quad+\varepsilon^{2H}\Lambda_\alpha^2(B^H)\left(\varepsilon^{2(\alpha\lambda-\lambda-\beta\lambda)}O^{2(1+\beta-\alpha)}+\left\|\Phi^{-1}\right\|^2\right. \\ &\quad\left.\left.\left.\cdot\left(G_1^2\left(1+4\left\|\Phi^{-1}\right\|^2G_0\right)\varepsilon^{4\alpha\lambda-4\lambda}O^{4-4\alpha}+G_0\varepsilon^{2\alpha\lambda-2\lambda}O^{2-2\alpha}\right)\right)\right]\right\} \\ &\quad\times\exp\left(N\left\|\Phi^{-1}\right\|^2\left(\varepsilon^{2-2\lambda}O^2+\varepsilon^{2H+2\alpha\lambda-2\lambda}\Lambda_\alpha^2(B^H)O^{2-2\alpha}\right)\right). \end{aligned} \tag{32}$$

Thus, for any given number $\theta > 0$, we can choose $\varepsilon_1 \in (0, \varepsilon_0]$ such that for each $\varepsilon \in (0, \varepsilon_1]$ and $t \in [0, O\varepsilon^{-\lambda}] \subset [0, T]$,

$$\sup_{0\le t\le O\varepsilon^{-\lambda}} E|u_\varepsilon(t) - w_\varepsilon(t)|^2 \le \theta, \tag{33}$$

and hence,

$$\begin{aligned}\sup_{0\le t\le O\varepsilon^{-\lambda}} E|x_\varepsilon(t) - y_\varepsilon(t)|^2 &= \sup_{t\in[0,O\varepsilon^{-\lambda}]} E\left|\left(\Phi^{-1}u\right)_\varepsilon(t) - \left(\Phi^{-1}w\right)_\varepsilon(t)\right|^2 \\ &\le \left\|\Phi^{-1}\right\|^2\theta.\end{aligned} \tag{34}$$

The proof is therefore complete.

Finally, as an application, we consider the following example.

Example 1. Consider the neutral stochastic delay differential equations driven by fractional Brownian motion:

$$\begin{aligned}d[x_\varepsilon(t) + 0.8x_\varepsilon(t-\tau)] &= \varepsilon f(t, x_\varepsilon(t), x_\varepsilon(t-\tau))dt \\ &\quad + \varepsilon^H g(t, x_\varepsilon(t), x_\varepsilon(t-\tau))dB^H(t),\end{aligned} \tag{35}$$

where the initial condition $x_\varepsilon(0) = \{\xi(s), -\infty < s \le 0\}$, $E\|\xi\|^2 < \infty$, $\tau > 0$ is a constant, $t \in [0, T]$,

$$\begin{aligned}f(t, x_\varepsilon(t), x_\varepsilon(t-\tau)) &= 2(x_\varepsilon(t) + x_\varepsilon \\ &\quad \cdot (t-\tau))\cos^2 t, g(t, x_\varepsilon(t), x_\varepsilon(t-\tau)) = 2,\end{aligned} \tag{36}$$

$B^H(t)$ is a fractional Brownian motion. Then,

$$\begin{aligned}\bar{f}(x_\varepsilon(t), x_\varepsilon(t-\tau)) &= \frac{1}{\pi}\int_0^\pi 2(x_\varepsilon(t) + x_\varepsilon(t-\tau))\cos^2 t dt \\ &= x_\varepsilon(t) + x_\varepsilon(t-\tau), \bar{g}(x_\varepsilon(t), x_\varepsilon(t-\tau)) = 2.\end{aligned} \tag{37}$$

Let $(\Phi x_\varepsilon)(t) = x_\varepsilon(t) + c(t)x_\varepsilon(t-\tau)$. Thus, $(\Phi^{-1}x_\varepsilon)(t) = \sum_{j\ge0}\prod_{0\le i\le j-1}(-1)^j c(t-i\tau)x_\varepsilon(t-j\tau)$. Define a new averaged stochastic differential delay equation

$$\begin{aligned}dy_\varepsilon(t) &= \varepsilon\bar{f}\left(\left(\Phi^{-1}y\right)_\varepsilon(t), \left(\Phi^{-1}y\right)_\varepsilon(t-\tau)\right)dt \\ &\quad + \varepsilon^H\bar{g}\left(\left(\Phi^{-1}y\right)_\varepsilon(t), \left(\Phi^{-1}y\right)_\varepsilon(t-\tau)\right)dB^H(t),\end{aligned} \tag{38}$$

namely,

$$dy_\varepsilon(t) = \varepsilon\left(\left(\Phi^{-1}y\right)_\varepsilon(t) + \left(\Phi^{-1}y\right)_\varepsilon(t-\tau)\right)dt + 2\varepsilon^H dB^H(t). \tag{39}$$

Obviously, when $t \in [0, \tau]$, the explicit solution of system (39) can be obtained as

$$\begin{aligned}y_\varepsilon(t) = e^{\varepsilon t}\Bigg\{\xi + \varepsilon\int_0^t e^{-\varepsilon s}\Bigg(&\sum_{i\ge1}(-0.8)^i y_\varepsilon(s - i\tau) \\ &+ \sum_{i\ge0}(-0.8)^i y_\varepsilon(s - (i+1)\tau)\Bigg)ds + 2\varepsilon^H\int_0^t e^{-\varepsilon s}dB^H(s)\Bigg\}.\end{aligned} \tag{40}$$

When $t \in [\tau, 2\tau]$, the explicit solution of system (2) is given by

$$\begin{aligned}y_\varepsilon(t) = e^{\varepsilon(t-\tau)}\Bigg\{y_\varepsilon(\tau) + \varepsilon\int_\tau^t e^{-\varepsilon(s-\tau)}\Bigg(&\sum_{i\ge1}(-0.8)^i y_\varepsilon(s - i\tau) \\ &+ \sum_{i\ge0}(-0.8)^i y_\varepsilon(s - (i+1)\tau)\Bigg)ds + 2\varepsilon^H\int_\tau^t e^{-\varepsilon(s-\tau)}dB^H(s)\Bigg\}.\end{aligned} \tag{41}$$

Repeating the same steps, we can deduce solution $y_\varepsilon(t)$ on the entire interval $[0, T]$. It is easy to see that functions $f, g, \bar{f}, \bar{g}$ in systems (35) and (39) satisfy all the conditions in $(H_1) - (H_2)$ and $(I_1) - (I_2)$; therefore, Theorem 6 holds. That is,

$$E|x_\varepsilon(t) - y_\varepsilon(t)|^2 \le \left\|\Phi^{-1}\right\|^2\theta. \tag{42}$$

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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Research Article

The Compact Finite Difference Method of Two-Dimensional Cattaneo Model

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In this paper, we propose and analyze the compact finite difference scheme of the two-dimensional Cattaneo model. The stability and convergence of the scheme are proved by the energy method, the convergence orders are 2 in time and 4 in space. We also use the variables separation method to find the true solution of the problem. On this basis, the validity and accuracy of the scheme are verified by numerical experiments.

1. Introduction

In the diffusion phenomenon described by traditional Fick law and Fourier law, the moment a point passes by, it is disturbed at a great distance, the propagation speed of the disturbance seems to be infinite. However, this property is unphysical. To solve the problem, Cattaneo modified the constitutive equation by introducing a relaxation parameter which plays the role of relaxation time, it should be noted that the relaxation parameter is small and depends on the thermodynamic properties of the material. Thus, he proposed the Cattaneo model [1].

From a mathematical point of view, Cattaneo model transforms the traditional diffusion equation into a hyperbolic equation, the speed of propagation is finite, and improves the property of infinite propagation speed. On the other hand, because of the hyperbolic nature of the Cattaneo model, it may have oscillatory solutions and negative values.

From a physical point of view, the Cattaneo model describes a physical phenomenon which is called heat waves. Although this phenomenon can only be observed under special circumstances, materials, or states [2], it is still gradually accepted by the public. It can be used to describe not only ultrasonic wave propagation in certain diluted gases but also heat pulse propagation in some pure nonmetallic crystals [3]. Straughan et al. [4] studied thermal convection with the Cattaneo-Christov model in horizontal layers of an incompressible Newtonian fluid. Haddad [5] utilized the theory of Cattaneo-Christov to investigate Brinkman's porous media. Cattaneo model is widely used in extended irreversible thermodynamics, cosmological models, and crystalline solids [6–8].

Because the fractional models have the advantage of previous memory, scholars have studied various fractional models [9–13]. With the wide application of the Cattaneo model in physics and theoretical analysis, more and more people pay attention to the generalization and solution of the Cattaneo model. Compte and Metzler [14] generalized the Cattaneo model from three perspectives which called continuous-time random walks, nonlocal transport theory, and delayed flux-force relation. Ferrillo et al. [15] compared the Cattaneo model and fractional Cattaneo model and studied the asymptotic behavior of solutions to the Cattaneo equations. Su et al. [16] studied solutions to the classical Cattaneo equation and compared with the solutions of a phase-

lagging heat transport equation. Čiegis [17] studied the numerical solutions of a class of hyperbolic heat conduction equations, proposed explicit and implicit Euler schemes for such equations, and verified that the convergence order of this scheme was $O(\tau + h^2)$. Huang and Yin [18] proposed a fourth-order compact difference scheme for the 1D Cattaneo model and verified that the convergence order of the scheme was $O(\tau^2 + h^4)$ through numerical experiments. Zhao and Sun [19] presented compact Crank-Nicolson schemes for a class fractional Cattaneo equation in an inhomogeneous medium and used the energy method to verify that the convergence order of the scheme is $O(\tau^{3-\alpha} + h^4)$. Li and Cao [20] proposed an unconditionally stable scheme with convergence order of $O(\tau^{3-\alpha} + h^2)$. Vong et al. [21] proposed a higher-order difference scheme for a class of generalized Cattaneo equations and proved the stability and convergence of the scheme by energy method.

There are many methods to solve partial differential equations, such as the finite element method [22, 23], finite volume element method [24, 25], and finite difference method [26]. Compared with other methods, the finite difference method has the advantages of a low requirement for grid nodes and simple calculation. In 1992, Lele [27] first proposed a general form of compact difference scheme. Compact difference scheme is a kind of finite difference scheme with high precision, which is widely used in solving parabolic equation and hyperbolic equation [28–33].

Previous scholars used different methods to study the integer-order Cattaneo model and the fractional Cattaneo model. The one-dimensional Cattaneo model describes the heat conduction phenomenon of the uniform thin column with side insulation, while the two-dimensional Cattaneo model needs to be studied for the heat conduction phenomenon of the thin plate with upper and lower bottom insulation. In this paper, we construct a compact difference scheme for a two-dimensional Cattaneo model. The time derivative is discretized with a central difference, and the space derivative is discretized with compact operators, so that the convergence order of the scheme reaches $O(\tau^2 + h_x^4 + h_y^4)$. We use the energy method to verify the stability and convergence of the scheme. Finally, we verify the validity and accuracy of the scheme through numerical experiments. It should be noted that the above articles are all about constructing the exact solution of the problem, and then calculating the right term, so as to conduct numerical experiments. In this paper, our example is the definite problem corresponding to the homogeneous equation. The exact solution is obtained through the separation of variables method, and then the numerical experiment is conducted.

The arrangement of the article is as follows. In the second part, we give some lemmas and symbols to derive a compact difference scheme. In the third part, we give a strict proof of the stability and convergence of the scheme by energy method. In the fourth part, we obtain the true solution of the specific problem through the separation of variable method and use numerical experiments to prove the validity of the scheme. In the fifth part, we make a final summary of the article.

2. Construction of the Compact Finite Difference Scheme

In this chapter, we consider the following two-dimensional Cattaneo model:

$$\begin{cases} \varepsilon^2 u_{tt} + u_t = D\left(u_{xx} + u_{yy}\right), & (x, y) \in \Omega, t \in (0, T], \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \\ u_t(x, y, 0) = u_1(x, y), & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y) \in \Gamma, t \in (0, T], \end{cases} \tag{1}$$

where $\Omega = (0, L_x) \times (0, L_y)$ is a rectangular region in R^2, $\Gamma = \partial\Omega$ is the boundary of the rectangular region, ε^2 is a relaxation parameter, D is the diffusion coefficient, and $u_0(x, y)$ and $u_1(x, y)$ are known functions. The above model is considered under homogeneous Dirichlet boundary conditions. The regularity assumption that problem (1) has a solution $u \in C_{x,y,t}^{6,6,4}(\bar{\Omega} \times [0, T])$ will be used in the following analysis.

In order to propose the compact finite difference scheme, we first introduce some basic symbols and preliminary lemmas. Take three positive integers M, N, and K, then let $h_x = L_x/M$, $h_y = L_y/N$, $\tau = T/K$. In this case, the spatial node can be expressed as (x_i, y_j), where $x_i = ih_x$, $i = 0, 1, \cdots, M$; $y_j = jh_y$, $j = 0, 1, \cdots, N$; $t_n = n\tau$, $n = 0, 1, \cdots, K$, let $\bar{\Omega}_h = \{(x_i, y_j) \mid 0 \le i \le M, 0 \le j \le N\}$, $\Omega_\tau = \{u^n \mid 0 \le n \le K\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Gamma_h = \bar{\Omega}_h \cap \Gamma$, $\omega = \{(i, j) \mid (x_i, y_j) \in \Omega_h\}$, $\partial\omega = \{(i, j) \mid (x_i, y_j) \in \Gamma_h\}$, $\bar{\omega} = \omega \cup \partial\omega$. Let $U_h = \{u \mid u = \{u_{ij} \mid (i, j) \in \bar{\omega}\}\}$ be a grid function space defined on $\bar{\Omega}_h$. For any grid function $u \in U_h$, introduce the following notations:

$$\delta_x^2 u_{ij} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2}, \delta_y^2 u_{ij} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2}. \tag{2}$$

The compact operators are defined as follows

$$A_x u_{i,j} = \begin{cases} \left(I + \dfrac{h_x^2}{12}\delta_x^2\right) u_{ij}, & 1 \le i \le M - 1, 0 \le j \le N, \\ u_{i,j}, & i = 0 \text{ or } M, \end{cases}$$

$$A_y u_{i,j} = \begin{cases} \left(I + \dfrac{h_y^2}{12}\delta_y^2\right) u_{ij}, & 1 \le j \le N - 1, 0 \le i \le M, \\ u_{i,j}, & j = 0 \text{ or } N. \end{cases} \tag{3}$$

Let $U_\tau = \{u \mid u = (u^0, u^1, \cdots, u^K)\}$ be grid function space defined on Ω_τ. For any grid function $u \in U_\tau$, introduce the following notations:

$$\delta_t u^n = \frac{u^{n+1} - u^{n-1}}{2\tau}, \delta_{\bar{t}} u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau},$$
$$\delta_{\bar{t}} u^{n-1/2} = \frac{u^n - u^{n-1}}{\tau}, \delta_t^2 u^n = \frac{u^{n-1} - 2u^n + u^{n+1}}{\tau^2}. \tag{4}$$

The following lemmas also play an important role in constructing a scheme.

Lemma 1 [34]. *Suppose $g(x) \in C^2[c, c+h]$, it holds that*

$$g'(c) = \frac{1}{h}[g(c+h) - g(c)] - \frac{h}{2} g''(\xi_7), c < \xi_7 < c+h, \tag{5}$$

where c is a constant.

Lemma 2 [34]. *Suppose $g(x) \in C^3[t_n, t_{n+1}]$, it holds that*

$$g'' = \frac{2}{\tau}\left[\frac{g(t_{n+1}) - g(t_n)}{h} - g'\right] - \frac{\tau}{3} g'''(t_\xi), t_n < t_\xi < t_{n+1}. \tag{6}$$

Lemma 3 [34]. *Suppose $g(x) \in C^4[c-h, c+h]$, it holds that*

$$g''(c) = \frac{g(c+h) - 2g(c) + g(c-h)}{h^2} - \frac{h^2}{12} g^{(4)}(\xi_4), c-h < \xi_4 < c+h, \tag{7}$$

where c is a constant.

Lemma 4 [34]. *Suppose $g(x) \in C^6[c-h, c+h]$, it holds that*

$$\frac{1}{12}\left[g''(c-h) + 10g''(c) + g''(c+h)\right] = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] + \frac{h^4}{240} g^{(6)}(\xi_6), \tag{8}$$

where $\xi_6 \in (c-h, c+h)$, c is a constant.

Define the grid functions as $u_{ij}^n = u(x_i, y_j, t_n)$, $0 \le i \le M$, $0 \le j \le N, 0 \le n \le K$. We discrete at (x_i, y_j, t_n) and refer to the formula [34].

$$\frac{\partial^2 u\left(x_i, y_j, t_n\right)}{\partial x^2} = \frac{1}{2}\left[\frac{\partial^2 u\left(x_i, y_j, t_{n-1}\right)}{\partial x^2} + \frac{\partial^2 u\left(x_i, y_j, t_{n+1}\right)}{\partial x^2}\right] - \frac{\tau^2}{2}\frac{\partial^4 u\left(x_i, y_j, \xi_{in}\right)}{\partial x^2 \partial t^2}, t_{n-1} < \xi_{in} < t_{n+1}. \tag{9}$$

We can obtain

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2}\left(x_i, y_j, t_n\right) + \frac{\partial u}{\partial t}\left(x_i, y_j, t_n\right) = \frac{D}{2}\left[\frac{\partial^2 u}{\partial x^2}\left(x_i, y_j, t_{n-1}\right) + \frac{\partial^2 u}{\partial x^2}\left(x_i, y_j, t_{n+1}\right) + \frac{\partial^2 u}{\partial y^2}\left(x_i, y_j, t_{n-1}\right) + \frac{\partial^2 u}{\partial y^2}\left(x_i, y_j, t_{n+1}\right)\right] + R_{ijn}^{(1)}, (i, j) \in \omega, 1 \le n \le K-1, \tag{10}$$

where

$$R_{ijn}^{(1)} = -\frac{D\tau^2}{2}\left[\frac{\partial^4 u\left(x_i, y_j, \xi_{ijn}\right)}{\partial x^2 \partial t^2} + \frac{\partial^4 u\left(x_i, y_j, \eta_{ijn}\right)}{\partial y^2 \partial t^2}\right], t_{n-1} < \xi_{ijn} < t_{n+1}, t_{n-1} < \eta_{ijn} < t_{n+1}. \tag{11}$$

Utilizing Taylor formula results in

$$\frac{\partial u\left(x_i, y_j, t_n\right)}{\partial t} = \delta_t u_{ij}^n - \frac{\tau^2}{6}\frac{\partial^3 u\left(x_i, y_j, \theta_{ijn}\right)}{\partial t^3}, t_{n-1} < \theta_{ijn} < t_{n+1}. \tag{12}$$

Using the above equation and Lemma 3 result in

$$\varepsilon^2 \delta_t^2 u_{ij}^n + \delta_t u_{ij}^n = \frac{D}{2}\left[\frac{\partial^2 u}{\partial x^2}\left(x_i, y_j, t_{n-1}\right) + \frac{\partial^2 u}{\partial x^2}\left(x_i, y_j, t_{n+1}\right) + \frac{\partial^2 u}{\partial y^2}\left(x_i, y_j, t_{n-1}\right) + \frac{\partial^2 u}{\partial y^2}\left(x_i, y_j, t_{n+1}\right)\right] + R_{ijn}^{(2)}, (i, j) \in \omega, 1 \le n \le K-1, \tag{13}$$

where $R_{ijn}^{(2)} = R_{ijn}^{(1)} + \tau^2(1/6(\partial^3 u(x_i, y_j, \theta_{ijn})/\partial t^3) + (\varepsilon^2/12)(\partial^4 u(x_i, y_j, \rho_{ijn})/\partial t^4))$.

We apply compact operators A_x and A_y to both sides of Eq. (13) and apply Lemma 4, we get

$$\varepsilon^2 A_x A_y \delta_t^2 u_{ij}^n + A_x A_y \delta_t u_{ij}^n = \frac{D}{2}\left(A_y \delta_x^2 u_{ij}^{n-1} + A_y \delta_x^2 u_{ij}^{n+1} + A_x \delta_y^2 u_{ij}^{n-1} + A_x \delta_y^2 u_{ij}^{n+1}\right) + R_{ijn}^{(3)}, (i, j) \in \omega, 1 \le n \le K-1, \tag{14}$$

where

$$R_{ijn}^{(3)} = A_x A_y R_{ijn}^{(2)} + \frac{h_x^4}{240}\left(A_y \frac{\partial^6 u}{\partial x^6}\left(\vartheta_{ijn}, y_j, t_{n-1}\right) + A_y \frac{\partial^6 u}{\partial x^6}\left(\mu_{ijn}, y_j, t_{n+1}\right)\right) + \frac{h_y^4}{240}\left(A_x \frac{\partial^6 u}{\partial y^6}\left(x_i, \zeta_{ijn}, t_{n-1}\right) + A_x \frac{\partial^6 u}{\partial y^6}\left(x_i, \varsigma_{ijn}, t_{n+1}\right)\right). \tag{15}$$

We think about the differential equation (1) at point (x_i, y_j, t_0),

$$\varepsilon^2 \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial t^2} + \frac{\partial u\left(x_i, y_j, t_0\right)}{\partial t} = \frac{D}{2}\left(\frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial x^2} + \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial x^2} + \frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial y^2} + \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial y^2}\right) - \frac{D}{2}\left(\frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial x^2} - \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial x^2} + \frac{\partial^2 u\left(x_i, y_j, t_1\right)}{\partial y^2} - \frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial y^2}\right). \tag{16}$$

By applying Lemma 1 and Lemma 2, we have

$$\frac{\partial^2 u\left(x_i, y_j, t_0\right)}{\partial t^2} = \frac{2}{\tau}\left[\delta_{\bar{t}} u^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right] - \frac{\tau}{3}\frac{\partial^3 u\left(x_i, y_j, \varsigma_1\right)}{\partial t^3}, \varsigma_1 \in (t_0, t_1).$$
$$\frac{\partial u\left(x_i, y_j, t_0\right)}{\partial t} = \delta_{\bar{t}} u^{\frac{1}{2}} - \frac{\tau}{2}\frac{\partial^2 u\left(x_i, y_j, \varsigma_2\right)}{\partial t^2}, \varsigma_2 \in (t_0, t_1). \tag{17}$$

Substituting the above equation into (16) and applying $A_x A_y$ *to both sides, we have*

$$\frac{2\varepsilon^2}{\tau} A_x A_y\left(\delta_{t^-} u_{ij}^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right) + A_x A_y \delta_{t^-} u_{ij}^{\frac{1}{2}} = \frac{D}{2}\left(A_y \delta_x^2 u_{ij}^0 + A_y \delta_x^2 u_{ij}^1 + A_x \delta_y^2 u_{ij}^0 + A_x \delta_y^2 u_{ij}^1\right) + r_{ijn}, \tag{18}$$

where

$$\left|r_{ijn}\right| = O\left(\tau + h_x^4 + h_y^4\right), (i, j) \in \omega. \tag{19}$$

Omitting small terms $R_{ijn}^{(3)}$ *and* r_{ijn}. *The numerical approximate solution is represented by the grid function* U_{ij}^n. *Replacing* u_{ij}^n *with it, the fourth-order compact difference scheme can be obtained*

$$\begin{cases} \varepsilon^2 A_x A_y \delta_t^2 U_{ij}^n + A_x A_y \delta_t U_{ij}^n = \frac{D}{2}\left(A_y \delta_x^2 U_{ij}^{n-1} + A_y \delta_x^2 U_{ij}^{n+1} + A_x \delta_y^2 U_{ij}^{n-1} + A_x \delta_y^2 U_{ij}^{n+1}\right), \\ (i, j) \in \omega, 1 \le n \le K-1, \\ \frac{2\varepsilon^2}{\tau} A_x A_y\left(\delta_{\bar{t}} U_{ij}^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right) + A_x A_y \delta_{\bar{t}} U_{ij}^{\frac{1}{2}} = \frac{D}{2}\left(A_y \delta_x^2 U_{ij}^0 + A_y \delta_x^2 U_{ij}^1 + A_x \delta_y^2 U_{ij}^0 + A_x \delta_y^2 U_{ij}^1\right), \\ (i, j) \in \omega, \\ U_{ij}^0 = u_0\left(x_i, y_j\right), (i, j) \in \omega, \\ U_{ij}^n = 0, (i, j) \in \partial\omega, 0 \le n \le K. \end{cases} \tag{20}$$

From the above analysis, the following theorems can be obtained.

Theorem 5. *The truncation error of compact difference scheme (20) is*

$$|R_{ij}^n| = O\left(\tau^2 + h_x^4 + h_y^4\right), (i,j) \in \omega, 0 \le n \le K. \tag{21}$$

3. Analysis of the Compact Finite Difference Scheme

Let $u, v \in U_h = \{u \mid u \in U_h ; (i,j) \in \partial\omega, u_{ij} = 0\}$, we define the following inner products and the corresponding norms

$$\begin{aligned}
&(u,v) = h_x h_y \sum_{i=1}^{M-1}\sum_{j=1}^{N-1} u_{ij} v_{ij}, \|u\| = \sqrt{(u,u)},\\
&(u,v)_{1,x} = h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N-1} (\delta_x u_{i-1/2,j})\delta_x v_{i-1/2,j}, \|\delta_x u\| = \sqrt{(u,u)_{1,x}},\\
&(u,v)_{1,y} = h_x h_y \sum_{i=1}^{M-1}\sum_{j=1}^{N} (\delta_y u_{i,j-1/2})\delta_y v_{i,j-1/2}, \|\delta_y u\| = \sqrt{(u,u)_{1,y}},\\
&(u,v)_{x,y} = h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N} (\delta_x\delta_y u_{i-1/2,j-1/2})(\delta_x\delta_y v_{i-1/2,j-1/2}), \|\delta_x\delta_y u\| = \sqrt{(u,u)_{x,y}},\\
&(u,v)_A = (A_x A_y u, v) = (u,v) - \frac{h_x^2}{12}(u,v)_{1,x} - \frac{h_y^2}{12}(u,v)_{1,y} + \frac{h_x^2 h_y^2}{144}(u,v)_{x,y},\\
&\|\nabla_h u\|_A^2 = \left(\|\delta_x u\|^2 - \frac{h_y^2}{12}\|\delta_y\delta_x u\|^2\right) + \left(\|\delta_y u\|^2 - \frac{h_x^2}{12}\|\delta_x\delta_y u\|^2\right),\\
&\|u\|_A = \sqrt{(u,u)_A}, \|u\|_\infty = \max_{1\le i\le M-1, 1\le j\le N-1} |u_{ij}|.
\end{aligned} \tag{22}$$

Next, we introduce the Lemma that will be applied to the stability analysis of the scheme.

Lemma 6 [35]. *For any grid function $u \in U_h$, it holds that*

$$\frac{1}{3}\|u\|^2 \le \|u\|_A^2 \le \|u\|^2. \tag{23}$$

Applying the Lemma, we have the following result.

Theorem 7. *Suppose $\{v_{ij}^n \mid 0 \le i \le M, 0 \le j \le N, 0 \le n \le K\}$ is the solution of the following difference scheme*

$$\begin{aligned}
&\varepsilon^2 A_x A_y \delta_t^2 v_{ij}^n + A_x A_y \delta_t v_{ij}^n = \frac{D}{2}\left(A_y\delta_x^2 v_{ij}^{n-1} + A_y\delta_x^2 v_{ij}^{n+1}\right.\\
&\quad\left. + A_x\delta_y^2 v_{ij}^{n-1} + A_x\delta_y^2 v_{ij}^{n+1}\right) + g_{ij}^n, (i,j) \in \omega, 1 \le n \le K-1,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\frac{2\varepsilon^2}{\tau} A_x A_y\left(\delta_{\bar t} v_{ij}^{\frac{1}{2}} - u_1\left(x_i, y_j\right)\right) + A_x A_y \delta_{\bar t} v_{ij}^{\frac{1}{2}}\\
&\quad = \frac{D}{2}\left(A_y\delta_x^2 v_{ij}^0 + A_y\delta_x^2 v_{ij}^1 + A_x\delta_y^2 v_{ij}^0 + A_x\delta_y^2 v_{ij}^1\right)\\
&\quad + g_{ij}^0, (i,j) \in \omega,
\end{aligned} \tag{25}$$

$$v_{ij}^0 = u_0\left(x_i, y_j\right), (i,j) \in \omega, \tag{26}$$

$$v_{ij}^n = 0, (i,j) \in \partial\omega, 0 \le n \le K. \tag{27}$$

It holds that

$$\begin{aligned}
&\varepsilon^2\|\delta_{\bar t} v^{n+1/2}\|_A^2 + \frac{D}{2}\left(\|\nabla_h v^{n+1}\|_A^2 + \|\nabla_h v^n\|_A^2\right) \le D\|\nabla_h v^0\|_A^2\\
&\quad + \frac{3\varepsilon^4}{\tau}\|\varphi\|^2 + \frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2 + \frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2, 0 \le n \le K-1,
\end{aligned} \tag{28}$$

where $\varphi_{ij} = A_x A_y u_1(x_i, y_j)$, g_{ij}^n is a source item.

Proof. Multiplying Eq. (24) by $2\delta_t v^n$, and summing up for i from 1 to $M-1$, we have

$$\begin{aligned}
&2\varepsilon^2\left(A_x A_y \delta_t^2 v^n, \delta_t v^n\right) + 2\left(A_x A_y \delta_t v^n, \delta_t v^n\right)\\
&\quad = D\left(A_y\delta_x^2 v^{n-1} + A_y\delta_x^2 v^{n+1} + A_x\delta_y^2 v^{n-1} + A_x\delta_y^2 v^{n+1}, \delta_t v^n\right)\\
&\quad + 2(g^n, \delta_t v^n), 1 \le n \le K-1.
\end{aligned} \tag{29}$$

Next, we deal with the two terms on the left-hand side of Eq. (29), we have

$$\begin{aligned}
&2\varepsilon^2\left(A_x A_y \delta_t^2 v^n, \delta_t v^n\right)\\
&\quad = 2\varepsilon^2\left[\left(\delta_t^2 v^n, \delta_t v^n\right) - \frac{h_x^2}{12}\left(\delta_t^2 v^n, \delta_t v^n\right)_{1,x} - \frac{h_y^2}{12}\left(\delta_t^2 v^n, \delta_t v^n\right)_{1,y}\right.\\
&\quad\left. + \frac{h_x^2 h_y^2}{144}\left(\delta_t^2 v^n, \delta_t v^n\right)_{x,y}\right] = \frac{\varepsilon^2}{\tau}\left[\left(\|\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_{\bar t} v^{n-1/2}\|^2\right)\right.\\
&\quad - \frac{h_x^2}{12}\left(\|\delta_x\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_x\delta_{\bar t} v^n\right.\\
&\quad\left. - \frac{1}{2}\|^2\right) - \frac{h_y^2}{12}\left(\|\delta_y\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_y\delta_{\bar t} v^{n-1/2}\|^2\right)\\
&\quad + \frac{h_x^2 h_y^2}{144}\left(\|\delta_x\delta_y\delta_{\bar t} v^{n+1/2}\|^2 - \|\delta_x\delta_y\delta_{\bar t} v^{n-1/2}\|^2\right)\Big]\\
&\quad = \frac{\varepsilon^2}{\tau}\left(\|\delta_{\bar t} v^{n+1/2}\|_A^2 - \|\delta_{\bar t} v^{n-1/2}\|_A^2\right), \qquad 1 \le n \le K-1.
\end{aligned} \tag{30}$$

$$2\left(A_x A_y \delta_t v^n, \delta_t v^n\right) = 2\|\delta_t v^n\|_A^2, 1 \le n \le K-1. \tag{31}$$

Then we deal with the first term on the right-hand side of

Eq. (29).

$$\begin{aligned}
&\left(A_y\delta_x^2 v^{n-1} + A_y\delta_x^2 v^{n+1}, \delta_t v^n\right) \\
&= \left[\left(I + \frac{h_y^2}{12}\delta_y^2\right)\delta_x^2 v^{n-1} + \left(I + \frac{h_y^2}{12}\delta_y^2\right)\delta_x^2 v^{n+1}, \delta_t v^n\right] \\
&= \left(\delta_x^2 v^{n-1} + \delta_x^2 v^{n+1}, \delta_t v^n\right) - \frac{h_y^2}{12}\left(\delta_x^2 v^{n-1} + \delta_x^2 v^{n+1}, \delta_t v^n\right)_{1,y} \\
&= -\frac{1}{2\tau}\left(\|\delta_x v^{n+1}\|^2 - \|\delta_x v^{n-1}\|^2\right) \\
&\quad + \frac{h_y^2}{24\tau}\left(\|\delta_x\delta_y v^{n+1}\|^2 - \|\delta_x\delta_y v^{n-1}\|^2\right), 1 \le n \le K-1.
\end{aligned} \tag{32}$$

Analogously

$$\begin{aligned}
&\left(A_x\delta_y^2 v^{n-1} + A_x\delta_y^2 v^{n+1}, \delta_t v^n\right) \\
&= -\frac{1}{2\tau}\left(\|\delta_y v^{n+1}\|^2 - \|\delta_y v^{n-1}\|^2\right) \\
&\quad + \frac{h_x^2}{24\tau}\left(\|\delta_x\delta_y v^{n+1}\|^2 - \|\delta_x\delta_y v^{n-1}\|^2\right), 1 \le n \le K-1.
\end{aligned} \tag{33}$$

Combining (32) and (33), we have

$$\begin{aligned}
&D\left(A_y\delta_x^2 v^{n-1} + A_y\delta_x^2 v^{n+1} + A_x\delta_y^2 v^{n-1} + A_x\delta_y^2 v^{n+1}, \delta_t v^n\right) \\
&= -\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2 - \|\nabla_h v^{n-1}\|_A^2\right), 1 \le n \le K-1.
\end{aligned} \tag{34}$$

Substituting (30), (31), (34) into (29) and applying Lemma 6, Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{\varepsilon^2}{\tau}\left(\|\delta_{\bar{t}} v^{n+1/2}\|_A^2 - \|\delta_{\bar{t}} v^{n-1/2}\|_A^2\right) + 2\|\delta_t v^n\|_A^2 \\
&= -\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2 - \|\nabla_h v^{n-1}\|_A^2\right) + 2(g^n, \delta_t v^n) \\
&\le -\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2 - \|\nabla_h v^{n-1}\|_A^2\right) + \frac{2}{3}\|\delta_t v^n\|^2 + \frac{3}{2}\|g^n\|^2 \\
&\le -\frac{D}{2\tau}\left(\|\nabla_h v^{n+1}\|_A^2 - \|\nabla_h v^{n-1}\|_A^2\right) + 2\|\delta_t v^n\|_A^2 + \frac{3}{2}\|g^n\|^2, 1 \le n \le K-1.
\end{aligned} \tag{35}$$

Sorting Eq. (35) and adding $D/2\|\nabla_h v^n\|_A^2$ to both sides, we have

$$\begin{aligned}
&\varepsilon^2\|\delta_{\bar{t}} v^{n+1/2}\|_A^2 + \frac{D}{2}\left(\|\nabla_h v^{n+1}\|_A^2 + \|\nabla_h v^n\|_A^2\right) \le \varepsilon^2\|\delta_{\bar{t}} v^{n-1/2}\|_A^2 \\
&\quad + \frac{D}{2}\left(\|\nabla_h v^n\|_A^2 + \|\nabla_h v^{n-1}\|_A^2\right) + \frac{3\tau}{2}\|g^n\|^2, 1 \le n \le K-1.
\end{aligned} \tag{36}$$

Let $E^n = \varepsilon^2\|\delta_{\bar{t}} v^{n+1/2}\|_A^2 + (D/2)(\|\nabla_h v^{n+1}\|_A^2 + \|\nabla_h v^n\|_A^2)$, $1 \le n \le K-1$.

Eq. (36) is written as

$$E^n \le E^{n-1} + \frac{3\tau}{2}\|g^n\|^2, 1 \le n \le K-1. \tag{37}$$

Summing over n results in

$$E^n \le E^0 + \frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2, 1 \le n \le K-1. \tag{38}$$

Multiplying Eq. (25) by $\delta_{\bar{t}} v^{1/2}$, summing up for i from 1 to $M-1$, applying Lemma 4 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{2\varepsilon^2}{\tau}\left(A_x A_y\delta_{\bar{t}} v^{\frac{1}{2}}, \delta_{\bar{t}} v^{\frac{1}{2}}\right) - \frac{2\varepsilon^2}{\tau}\left(\varphi, \delta_{\bar{t}} v^{\frac{1}{2}}\right) + \left(A_x A_y\delta_{\bar{t}} v^{\frac{1}{2}}, \delta_{\bar{t}} v^{\frac{1}{2}}\right) \\
&\quad + \frac{D}{2\tau}\left(\|\nabla_h v^1\|_A^2 - \|\nabla_h v^0\|_A^2\right) = \left(g^0, \delta_{\bar{t}} v^{\frac{1}{2}}\right),
\end{aligned} \tag{39}$$

where $\varphi_{ij} = A_x A_y u_1(x_i, y_j)$.

Sorting Eq. (39), we have

$$\begin{aligned}
&\frac{2\varepsilon^2}{\tau}\|\delta_{\bar{t}} v^{\frac{1}{2}}\|_A^2 + \|\delta_{\bar{t}} v^{\frac{1}{2}}\|_A^2 + \frac{D}{2\tau}\left(\|\nabla_h v^1\|_A^2 - \|\nabla_h v^0\|_A^2\right) \\
&= \frac{2\varepsilon^2}{\tau}\left(\varphi, \delta_{\bar{t}} v^{\frac{1}{2}}\right) + \left(g^0, \delta_{\bar{t}} v^{\frac{1}{2}}\right) \le \frac{1}{3}\|\delta_{\bar{t}} v^{\frac{1}{2}}\|^2 + \frac{3\varepsilon^4}{\tau^2}\|\varphi\|^2 \\
&\quad + \frac{\varepsilon^2}{3\tau}\|\delta_{\bar{t}} v^{\frac{1}{2}}\|^2 + \frac{3\tau}{4\varepsilon^2}\|g^0\|^2 \le \|\delta_{\bar{t}} v^{\frac{1}{2}}\|_A^2 + \frac{\varepsilon^2}{\tau}\|\delta_{\bar{t}} v^{\frac{1}{2}}\|_A^2 \\
&\quad + \frac{3\varepsilon^4}{\tau^2}\|\varphi\|^2 + \frac{3\tau}{4\varepsilon^2}\|g^0\|^2.
\end{aligned} \tag{40}$$

Sorting Eq. (40), we have

$$\begin{aligned}
&\varepsilon^2\|\delta_{\bar{t}} v^{\frac{1}{2}}\|_A^2 + \frac{D}{2}\left(\|\nabla_h v^1\|_A^2 + \|\nabla_h v^0\|_A^2\right) \\
&\quad \le D\|\nabla_h v^0\|_A^2 + \frac{3\varepsilon^4}{\tau}\|\varphi\|^2 + \frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2.
\end{aligned} \tag{41}$$

Substituting (41) into (38), we get

$$\begin{aligned}
E^n &\le D\|\nabla_h v^0\|_A^2 + \frac{3\varepsilon^4}{\tau}\|\varphi\|^2 + \frac{3\tau^2}{4\varepsilon^2}\|g^0\|^2 \\
&\quad + \frac{3\tau}{2}\sum_{k=1}^{n}\|g^k\|^2, 1 \le n \le K-1.
\end{aligned} \tag{42}$$

End of the proof.

Remark 8. According to the results, it can be seen that the derivative term of initial value affects stability and convergence, so this term in the Cattaneo model will be set as zero in general. For details, please refer to the literature [15].

Theorem 9. *Suppose* $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \le n \le K\}$ *is the solution of Eq. (1),* $\{U_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \le n \le K\}$ *is the solution of the difference scheme (20). Let*

$$e_{ij}^n = u_{ij}^n - U_{ij}^n, (i,j) \in \bar{\omega}, 0 \le n \le K, \tag{43}$$

it holds that

$$\begin{aligned} &\varepsilon^2 \|\delta_{\bar{t}} e^{n+\frac{1}{2}}\|_A^2 + \frac{D}{2}\left(\|\nabla_h e^{n+1}\|_A^2 + \|\nabla_h e^n\|_A^2\right) \\ &\quad \le \left(\frac{3}{4\varepsilon^2} + \frac{3T}{2}\right) C\left(\tau^2 + h_x^4 + h_y^4\right)^2, 0 \le n \le K-1. \end{aligned} \tag{44}$$

Proof. Subtracting (14), (18), and (20), we can get the error system

$$\begin{aligned} &\varepsilon^2 A_x A_y \delta_t^2 e_{ij}^n + A_x A_y \delta_t e_{ij}^n \\ &\quad = \frac{D}{2}\left(A_y \delta_x^2 e_{ij}^{n-1} + A_y \delta_x^2 e_{ij}^{n+1} + A_x \delta_y^2 e_{ij}^{n-1} + A_x \delta_y^2 e_{ij}^{n+1}\right) \\ &\qquad + R_{ijn}^{(3)}, (i,j) \in \omega, 1 \le n \le K-1, \end{aligned} \tag{45}$$

$$\begin{aligned} &\frac{2\varepsilon^2}{\tau} A_x A_y \delta_{\bar{t}} e_{ij}^{\frac{1}{2}} + A_x A_y \delta_{\bar{t}} e_{ij}^{\frac{1}{2}} \\ &\quad = \frac{D}{2}\left(A_y \delta_x^2 e_{ij}^0 + A_y \delta_x^2 e_{ij}^1 + A_x \delta_y^2 e_{ij}^0 + A_x \delta_y^2 e_{ij}^1\right) \\ &\qquad + r_{ijn}, (i,j) \in \omega, \end{aligned} \tag{46}$$

$$e_{ij}^0 = 0, (i,j) \in \omega, \tag{47}$$

$$e_{ij}^n = 0, (i,j) \in \partial\omega, 0 \le n \le K. \tag{48}$$

Table 1: Error and convergence order of compact difference scheme.

h	τ	*order*	*order*	$errl^\infty$	$errl^2$
1/4	1/20	—	—	7.1854e-03	3.5927e-03
1/8	1/80	3.517	1.759	5.8076e-04	2.9038e-04
1/16	1/320	4.005	2.003	3.6204e-05	1.8102e-05
1/32	1/1280	3.999	2.000	2.2637e-06	1.3185e-06
1/64	1/5120	4.000	2.000	1.4151e-07	7.0753e-08

Using the Theorem 7 and noticing (47) and (48), we have

$$\begin{aligned} &\varepsilon^2 \|\delta_{\bar{t}} e^{n+1/2}\|_A^2 + \frac{D}{2}\left(\|\nabla_h e^{n+1}\|_A^2 + \|\nabla_h e^n\|_A^2\right) \\ &\quad \le D\|\nabla_h e^0\|_A^2 + \frac{3\tau^2}{4\varepsilon^2}\left(r_{ijn}\right)^2 + \frac{3\tau}{2}\sum_{k=1}^{n}\left(R_{ijn}^{(3)k}\right)^2 \\ &\quad \le \frac{3\tau^2}{4\varepsilon^2} C_1^2 \left(\tau + h_x^4 + h_y^4\right)^2 + \frac{3\tau}{2} n C_2^2 \left(\tau^2 + h_x^4 + h_y^4\right)^2 \\ &\quad \le \left(\frac{3}{4\varepsilon^2} + \frac{3T}{2}\right) C\left(\tau^2 + h_x^4 + h_y^4\right)^2. \end{aligned} \tag{49}$$

End of the proof.

Remark 10. Through Theorem 7 and Theorem 9, we prove the stability and convergence of the scheme. Due to the basic relationship between H^1 semi-norm and infinity norm, we can also obtain that the proposed scheme is stable and convergent in L^∞ norm.

4. Numerical Experiments

In this chapter, we verify the numerical accuracy and validity of the proposed scheme.

In the following numerical experiment, for simplicity, the domain is set to the square domain. The domain is $\Omega = (0,1) \times (0,1)$, and h_x is equal to h_y. $\varepsilon^2 = 0.1$, $D = 1$, $u_0(x,y) = \sin(\pi x)\sin(\pi y)$, $u_1(x,y) = 0$. In order to verify the numerical accuracy of the scheme, we need the following exact solution of the above problem. We can obtain the exact solution by using the separation of variables method,

$$u(x,y,t) = e^{-\frac{1}{2\varepsilon^2}t}\left(\cos\sqrt{\frac{2D\pi^2}{\varepsilon^2} - \frac{1}{4\varepsilon^2}}t + \frac{1}{2\varepsilon^2\sqrt{2D\pi^2/\varepsilon^2 - 1/4\varepsilon^2}}\sin\sqrt{\frac{2D\pi^2}{\varepsilon^2} - \frac{1}{4\varepsilon^2}}t\right)\sin(\pi x)\sin(\pi y). \tag{50}$$

Before we give the experimental results, we first give the definitions of l^2-norm of the error and l^∞-norm of the error.

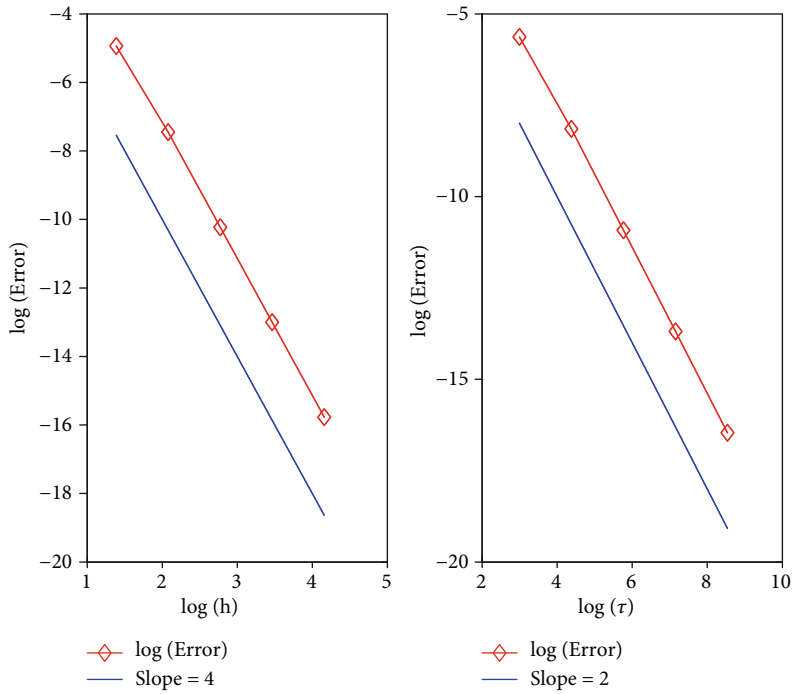

FIGURE 1: Order of spatial and temporal convergence.

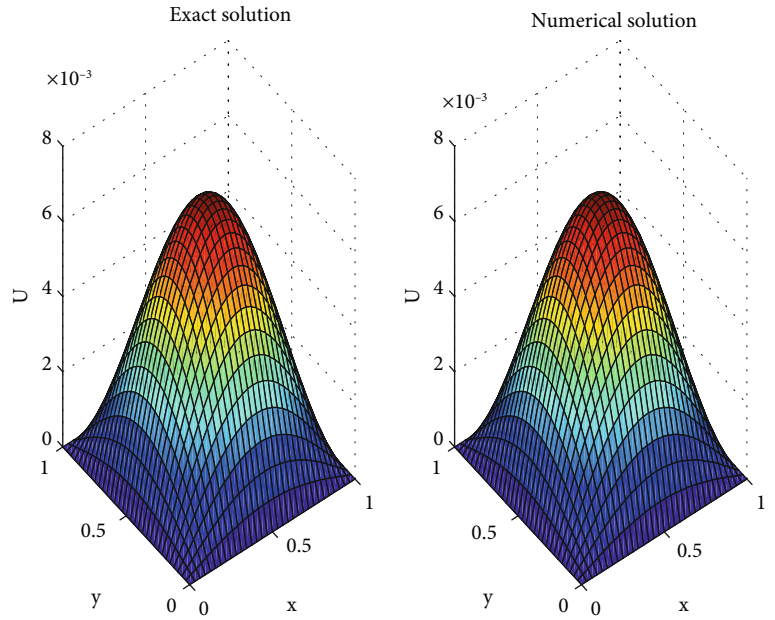

FIGURE 2: The exact solution and the numerical at fixed $D = 1$, $\varepsilon^2 = 0.1$, $T = 1$,$h = 1/32$ and $\tau = 1/1280$.

For all grid points, l^2-norm of the error is defined by

$$Error_{l^2} = \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{N} h_x h_y \mid u_{ij}^K - U_{ij}^K \mid}. \tag{51}$$

For all grid points, l^∞-norm of the error is defined by

$$Error_{l^\infty} = \max_{1\le i\le M, 1\le j\le N} \left| u_{ij}^K - U_{ij}^K \right|, \tag{52}$$

where u_{ij} is the true solution, U_{ij} is the numerical solution, and K is the last moment. Let $N = M$, order of spatial

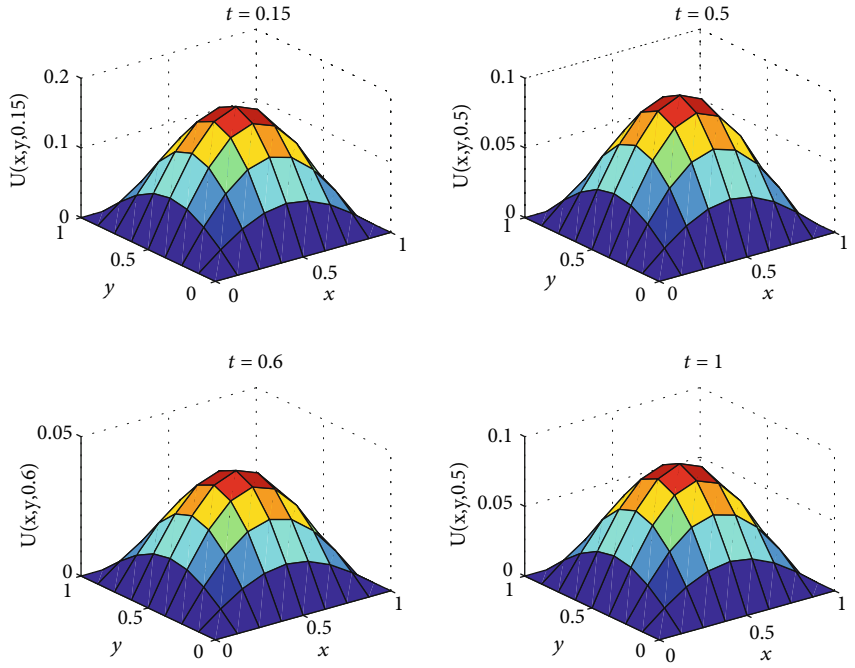

FIGURE 3: Temperature images at different times when $D = 1$, $\varepsilon^2 = 0.1$, $h = 1/16$, and $\tau = 1/80$.

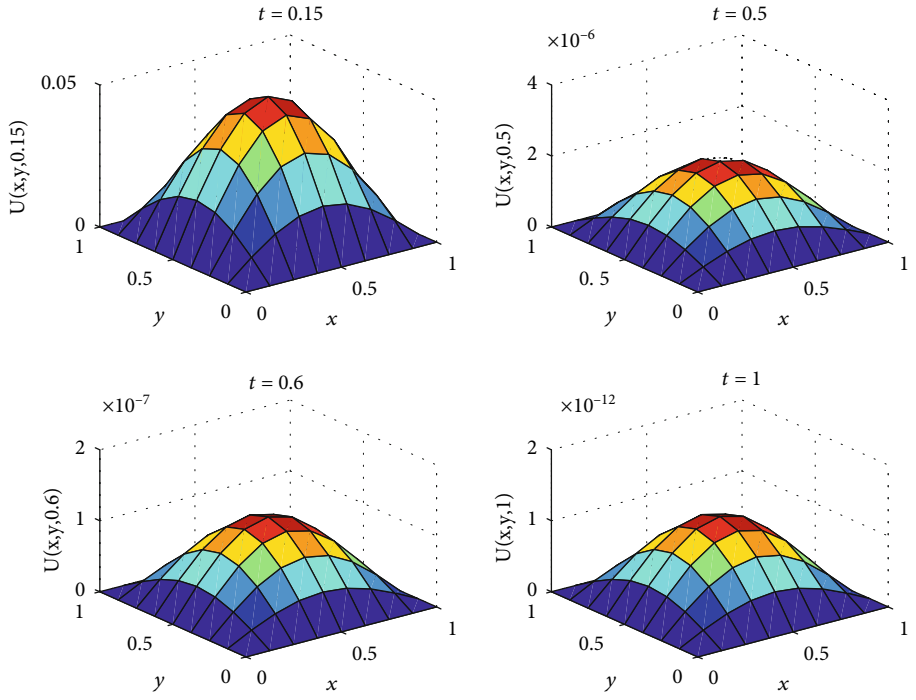

FIGURE 4: Temperature images at different times when $D = 1$, $\varepsilon^2 = 0.01$, $h = 1/16$, and $\tau = 1/80$.

convergence is denoted by P_h and order of temporal convergence is denoted by P_τ, which are defined as follows

$$P_h = \frac{\log\left(Error_1/Error_2\right)}{\log\left(h_1/h_2\right)}, P_\tau = \frac{\log\left(Error_1/Error_2\right)}{\log\left(\tau_1/\tau_2\right)} \tag{53}$$

In Table 1, let $h = h_x = h_y$, and $T = 1$, for h and τ, take the corresponding values, respectively, the compact difference scheme has the following corresponding error, order of temporal convergence and order of spatial convergence. The third column is the order of spatial convergence, and the fourth column is the order of time convergence. We can find that when the spatial step is reduced to one-

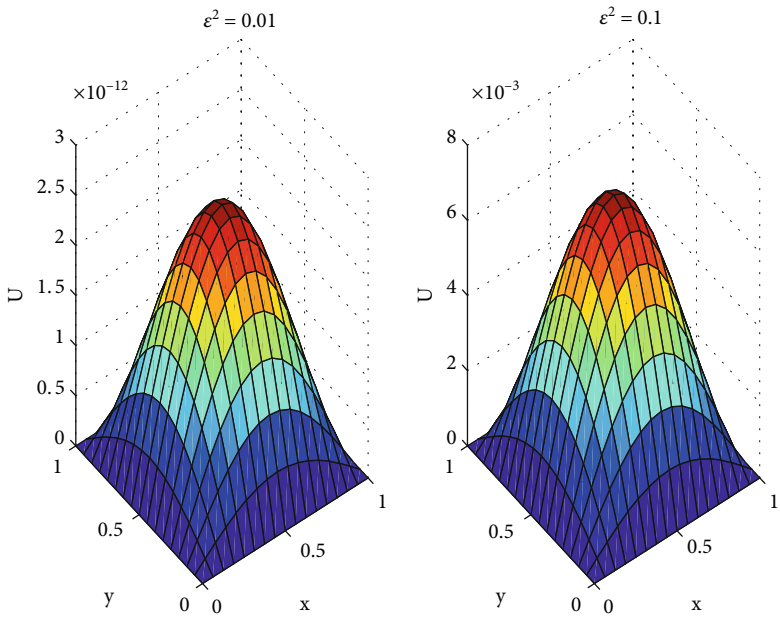

Figure 5: Temperature images with different relaxation parameters when $D = 1$, $h = 1/16$ and $1/320$.

half of the original and the temporal step is reduced to one-quarter of the original, the error is reduced to about one-sixteenth of the original. At the same time, the compact difference scheme keeps the fourth-order spatial rate of convergence and second-order temporal rate of convergency. So we also verify the stability and convergence of the scheme.

In order to verify the applicability of the compact difference scheme more accurately, we give the image of convergence order defined the by l^{∞}-norm. In Figure 1, we can see more clearly that the convergence order of the scheme (20) is $O(\tau^2 + h_x^4 + h_y^4)$, which also verifies that the scheme (20) proposed in this paper is stable and effective. And the relaxation parameter does not affect the convergence rate.

Figure 2 show the true solution and the numerical solution when $h = 1/32$ and $\tau = 1/1280$.

Figure 3 shows the images when t is taken at different moments. It can be seen from the images that as time increases, the temperature also decreases gradually.

Next, the value of ε^2 in the previous example is 0.01, and the temperature image of t at different moments is also obtained. See Figure 4, it has the same property as Figure 3.

Finally, we use two examples to test the influence of relaxation parameters, as shown in Figure 5. From Figure 3–5, we can see that the larger ε^2 is, the more obvious the fluctuation of temperature distribution is, and the slower the temperature change at the initial time.

In Figure 6, in order to observe the differences between the Cattaneo model and the traditional diffusion equation more conveniently, we give the temperature distribution images of the one-dimensional Cattaneo model and the heat conduction equation at different moments at the midpoint of $x = 0.5$. It can be seen from the Figure 6 that the Cattaneo model has a slower temperature transformation at the initial moment than the heat conduction equation, but at the same

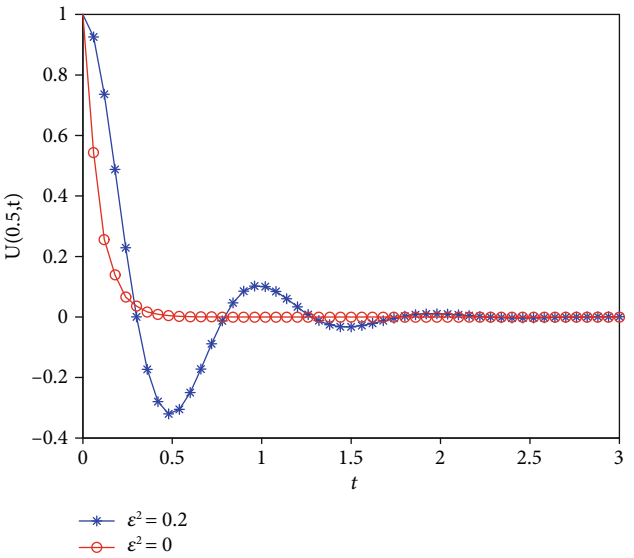

Figure 6: Temperature distribution images at $x = 0.5$.

time, the former has negative temperature and oscillation, which is also affected by the hyperbolic property of Cattaneo model.

5. Conclusion

In this paper, we propose a compact finite difference scheme for a two-dimensional Cattaneo model and prove the stability and convergence of the scheme by energy method. A concrete example is given, and the exact solution is obtained by the method of separating variables, it can be seen from the figures that the fitting degree of the exact solution and numerical solution is higher, it is also verified by numerical

experiments that the convergence order of the scheme is $O(\tau^2 + h_x^4 + h_y^4)$. In conclusion, the proposed scheme is simple, stable, and effective and has higher computational accuracy than some existing methods.

Data Availability

Some or all data, models, or code generated or used during the study are available from the corresponding author by request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Yating Huang carried out the main part of this article. All authors read and approved the final manuscript.

Acknowledgments

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Research Article

Multiple Positive Solutions for a System of Nonlinear Caputo-Type Fractional Differential Equations

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By using fixed-point index theory, we consider the existence of multiple positive solutions for a system of nonlinear Caputo-type fractional differential equations with the Riemann-Stieltjes boundary conditions.

1. Introduction

Fractional order calculus is more widely used than integer order calculus (see [1]). Based on it, many researchers have been focused on the study of the existence of positive solutions for various fractional differential equations with some boundary conditions. We can refer to [2–35] for some recent works about fractional boundary value problems.

In [14], Ma and Cui discussed the following fractional boundary value problem:

$$\begin{cases} {}^{c}D_{0+}^{\theta}p(y) + \mu f(y, p(y)) = 0, \quad y \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(y)dA(y), \end{cases} \tag{1}$$

where $\theta \in (2,3)$, ${}^{c}D_{0+}^{\theta}$ is the Caputo fractional derivative, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\mu > 0$ is a parameter. By using the Guo-Krasnosel'skii fixed-point theorem, they proved that the fractional differential equation has at least a positive solution when μ satisfies some conditions.

A few researchers have studied the existence of solutions for the systems of nonlinear fractional differential equations (see [23–35]). For example, in [24], by monotone iterative technique and cone theory, Zhang et al. have studied the uniqueness of a solution for fractional systems with the Riemann-Stieltjes integral boundary condition. Hao et al. [32] have studied a system of fractional boundary value problems with two parameters; by using the Guo-Krasnosel'skii fixed-point theorem, they obtained the existence of positive solutions for the system in terms of different values of parameters. Meanwhile, a few researchers have been considering the systems for nonlinear integer order boundary value problems, such as in [33], where the researchers considered the existence of multiple positive solutions for a system of nonlinear third-order differential equations. In [34], by Banach's contraction principle, the researchers considered the existence of a unique solution for a system of second order differential equations with coupled integral boundary conditions.

Inspired by [14, 23–35], in this paper, we want to study the following system of fractional differential equations:

$$\begin{cases} -{}^{c}D_{0+}^{\theta_1}x(t) = f_1(t, x(t), y(t)), \quad t \in [0,1], \\ -{}^{c}D_{0+}^{\theta_2}y(t) = f_2(t, x(t), y(t)), \quad t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \int_0^1 x(t)dA_1(t), \\ y(0) = y''(0) = 0, \quad y(1) = \int_0^1 y(t)dA_2(t), \end{cases} \tag{2}$$

where $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous; $\theta_i \in (2,3)$; ${}^cD_{0+}^{\theta_i}$ is the Caputo fractional derivative; A_i is a bounded variation function with positive measure with $B_i = \int_0^1 t dA_i(t) < 1$, $i = 1, 2$.

The main purpose of this paper is that we prove that system (2) has one and multiple positive solutions by the fixed-point index theory.

2. Preliminaries

In this section, we only give the definition of Caputo's fractional derivative. For some of the other definitions and properties of Caputo's fractional derivative, we can refer to [2]. We mainly give the relevant Green functions and the used lemmas.

Definition 1 (see [2, 14]). For a function $x \in C^n[0,+\infty)$, we define Caputo's fractional derivative of order $\theta > 0$ as follows:

$$ {}^cD_{0+}^{\theta} x(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} x^n(s) ds, \quad n-1 < \theta < n, \tag{3} $$

where n is the smallest integer greater than or equal to θ.

From [14], we have the following lemmas.

Lemma 2 (see [14]). *Let $u \in C[0,1]$ and $\theta_1, \theta_2 \in (2,3)$. Then p is the solution of the linear Caputo fractional differential equation*

$$ \begin{cases} {}^cD_{0+}^{\theta i} p(t) + u(t) = 0, & t \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(t) dA_i(t), \end{cases} \tag{4} $$

if and only if p is the solution of the integral equation

$$ p(t) = \int_0^1 K_i(t,s) u(s) ds, \tag{5} $$

where

$$ K_i(t,s) = \frac{1}{\Gamma(\theta_i)} \begin{cases} \dfrac{t}{1-B_i} \left[(1-s)^{\theta_i - 1} - \int_s^1 (t-s)^{\theta_i - 1} dA_i(t) \right] - (t-s)^{\theta_i - 1}, & 0 \le s, t \le 1, \\ \dfrac{t}{1-B_i} \left[(1-s)^{\theta_i - 1} - \int_s^1 (t-s)^{\theta_i - 1} dA_i(t) \right], & 0 \le t \le s \le 1, \end{cases} \tag{6} $$

and $B_i = \int_0^1 t dA_i(t) < 1$, $i = 1, 2$.

Lemma 3 (see [14]). *The above Green's function $K_i(t,s)$ $(i = 1, 2)$ has the following properties:*

(i) $\Gamma(\theta_i) K_i(t,s) \le (1/(1-B_i))(1-s)^{\theta_i - 1}$, *for* $t, s \in [0,1]$

(ii) $\Gamma(\theta_i) K_i(t,s) \ge N_i (1-s)^{\theta_i - 1}$, *for* $t \in [(1/4), (3/4)]$, $s \in [0,1]$

where

$$ N_i = \min \left\{ \frac{1 - \int_0^1 t^{\theta_i - 1} dA_i(t)}{4(1-B_i)}, \min_{t \in [(1/4),(3/4)]} t\left(1 - t^{\theta_i - 2}\right) \right\}, \quad i = 1, 2. \tag{7} $$

Lemma 4 (see [36]). *Let E be a Banach space and $P \subset E$ be a cone. Define $D_r = \{u \in P : \|u\| < r\}$, where $r > 0$. Suppose that $T : \bar{D}_r \longrightarrow P$ is completely continuous, and $Tu \ne u$, for $u \in \partial D_r$.*

(i) If $\|Tu\| \le \|u\|$, $\forall u \in \partial D_r$, then $i(T, D_r, P) = 1$

(ii) If $\|Tu\| \ge \|u\|$, $\forall u \in \partial D_r$, then $i(T, D_r, P) = 0$

3. Main Results

Let $E = C[0,1] \times C[0,1]$ be a Banach space with the norm $\|(u,v)\|_E = \|u\| + \|v\|$ on E, where $\|u\| = \max_{0 \le t \le 1} |u(t)|$.

Define the cone

$$ P = \left\{ (u,v) \in E : u \ge 0, v \ge 0, \min_{(1/4) \le t \le (3/4)} (u(t) + v(t)) \ge K \|(u,v)\|_E \right\}, \tag{8} $$

where

$$ K = \min \{ N_1(1-B_1), N_2(1-B_2) \} < 1. \tag{9} $$

N_1 and N_2 are defined by (7), and the nonlinear operators T_1, T_2, and T are defined by

$$T_1(x,y)(t)=\int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds,\quad t\in[0,1],$$
$$T_2(x,y)(t)=\int_0^1 K_2(t,s)f_2(s,x(s),y(s))ds,\quad t\in[0,1],$$
$$T(x,y)=(T_1(x,y),T_2(x,y)),\quad (x,y)\in E, \tag{10}$$

where $K_i(t,s)(i=1,2)$ is defined by (6).

It is known that fixed points of the operator T in P are positive solutions of the system (2).

Lemma 5. *The operator $T:P\longrightarrow P$ is completely continuous.*

Proof. For $(x,y)\in P$, when $t\in[0,1]$, we have $T_1(x,y)(t)\geq 0$, $T_2(x,y)(t)\geq 0$.

When

$$t\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{11}$$

by Lemma 3, we obtain

$$\begin{aligned}T_1(x,y)(t)&=\int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds\\&\geq\frac{N_1}{\Gamma(\theta_1)}\int_0^1(1-s)^{\theta_1-1}f_1(s,x(s),y(s))ds\\&=\frac{N_1(1-B_1)}{\Gamma(\theta_1)}\int_0^1\frac{(1-s)^{\theta_1-1}}{(1-B_1)}f_1(s,x(s),y(s))ds\\&\geq N_1(1-B_1)\max_{t\in[0,1]}\int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds\\&=N_1(1-B_1)\|T_1(x,y)\|.\end{aligned} \tag{12}$$

Similar to the proof of (12), for

$$(x,y)\in P,\quad t\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{13}$$

we have

$$T_2(x,y)(t)\geq N_2(1-B_2)\|T_2(x,y)\|. \tag{14}$$

By (12) and (14), we get

$$\begin{aligned}&\min_{t\in[(1/4),(3/4)]}(T_1(x,y)(t)+T_2(x,y)(t))\\&\quad\geq N_1(1-B_1)\|T_1(x,y)\|+N_2(1-B_2)\|T_2(x,y)\|\\&\quad\geq K\|T(x,y)\|_E.\end{aligned} \tag{15}$$

By (15), $T(P)\subset P$. Similar to the proof of Lemma 6 in [14], we know that $T_1, T_2: C[0,1]\longrightarrow C[0,1]$ are completely continuous. So $T:P\longrightarrow P$ is completely continuous.

For convenience, some marks and assumptions are given. Some of the marks are as follows:

$$\begin{aligned}g_0&=\lim_{(x,y)\to(0^+,0^+)}\frac{f_1(t,x,y)}{x+y},\\\bar{g}_0&=\lim_{(x,y)\to(0^+,0^+)}\frac{f_2(t,x,y)}{x+y},\\g_\infty&=\lim_{(x,y)\to(+\infty,+\infty)}\frac{f_1(t,x,y)}{x+y},\\\bar{g}_\infty&=\lim_{(x,y)\to(+\infty,+\infty)}\frac{f_2(t,x,y)}{x+y},\\k_1&=\frac{N_1K}{\Gamma(\theta_1)}\int_{(1/4)}^{(3/4)}(1-s)^{\theta_1-1}ds,\\k_3&=\frac{N_2K}{\Gamma(\theta_2)}\int_{(1/4)}^{(3/4)}(1-s)^{\theta_2-1}ds,\end{aligned} \tag{16}$$

where K is defined by (9).

$$\begin{aligned}k_2&=\frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1(1-s)^{\theta_1-1}ds,\\k_4&=\frac{1}{\Gamma(\theta_2)(1-B_2)}\int_0^1(1-s)^{\theta_2-1}ds,\\k&=\min\{k_1,k_3\},\ h=\max\{k_2,k_4\}.\end{aligned} \tag{17}$$

Some of the assumptions are as follows:

S_1. There exist $a>0$ and $b_2>b_1>0$ such that

(i) $f_1(t,x,y)\geq a(x+y), f_2(t,x,y)\geq a(x+y),\ \forall t\in[0,1]$, $0\leq x,y\leq b_1$

(ii) $f_1(t,x,y)<(b_2/2h), f_2(t,x,y)<(b_2/2h),\quad \forall t\in[0,1]$, $0\leq x,y\leq b_2$

where a also satisfies $ak>(1/2)$, k and h are defined by (17).

S_2. There exists $q>0$ such that

$$\begin{aligned}f_1(t,x,y)&>\frac{Kq}{2k},\\f_2(t,x,y)&>\frac{Kq}{2k},\\&\forall t\in[0,1], x+y\in[Kq,q],\end{aligned} \tag{18}$$

where K is defined by (9) and k is defined by (17).

Theorem 6. *Suppose that $g_0=\bar{g}_0=\infty$ and $g_\infty=\bar{g}_\infty=0$ uniformly on $t\in[0,1]$. Then, system (2) has a positive solution.*

Proof. By $g_0 = \bar{g}_0 = \infty$, we know that there exist $r_1 > 0$ and $m_1 > 0$ such that

$$\begin{aligned} &f_1(t,x,y) \geq m_1(x+y),\\ &f_2(t,x,y) \geq m_1(x+y),\\ &\qquad \forall 0 \leq x,y \leq r_1, t \in [0,1], \end{aligned} \tag{19}$$

and $m_1 k_1 > (1/2)$, $m_1 k_3 > (1/2)$.

Set $P_{r_1} = \{(x,y) \in P : \|(x,y)\|_E < r_1\}$. By Lemma 3 and (19), for $(x,y) \in \partial P_{r_1}$, $t \in [(1/4), (3/4)]$, we have

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))ds\\ &\geq \frac{N_1 m_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1 - 1}(x(s)+y(s))ds\\ &\geq m_1 \frac{N_1 K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1-1} ds(\|x\| + \|y\|)\\ &= m_1 k_1 \|(x,y)\|_E > \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{20}$$

Similar to the proof of (20), we have

$$T_2(x,y)(t) > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_1}. \tag{21}$$

So by (20) and (21), we have

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\|\\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_1}. \end{aligned} \tag{22}$$

By Lemma 4 and (22), we have

$$i\left(T, P_{r_1}, P\right) = 0. \tag{23}$$

From $g_\infty = \bar{g}_\infty = 0$, we know that there exist $\tilde{r} > r_1 > 0$ and $m_2 > 0$ such that

$$\begin{aligned} &f_1(t,x,y) \leq m_2(x+y),\\ &f_2(t,x,y) \leq m_2(x+y),\\ &\qquad \forall x+y \geq \tilde{r}, t \in [0,1], \end{aligned} \tag{24}$$

and $m_2 h < (1/4)$. In this part, we divide two cases. One case is that f_1 and f_2 are bounded. Namely, there exists $W > 0$ such that

$$\begin{aligned} &f_1(t,x,y) \leq W,\\ &f_2(t,x,y) \leq W,\\ &\qquad \forall x,y \in [0,+\infty), t \in [0,1]. \end{aligned} \tag{25}$$

Take $r_2 > \max\{2Wh, r_1\}$. Set $P_{r_2} = \{(x,y) \in P : \|(x,y)\|_E < r_2\}$. For $(x,y) \in \partial P_{r_2}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))ds\\ &\leq W \int_0^1 K_1(t,s)ds \leq \frac{W}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} ds\\ &\leq Wh < \frac{r_2}{2} = \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{26}$$

The other case is that f_1 and f_2 are unbounded. By the continuity of f_1, there exists $r_2 > \max\{r_1, \tilde{r}\}$ such that

$$f_1(t,x,y) \leq f_1(t,r_2,r_2), \quad \text{for } 0 < x,y \leq r_2. \tag{27}$$

Hence, set $P_{r_2} = \{(x,y) \in P : \|(x,y)\|_E < r_2\}$. For $(x,y) \in \partial P_{r_2}$ and $t \in [0,1]$ we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))ds\\ &\leq \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s,r_2,r_2)ds\\ &\leq \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} m_2(r_2+r_2)ds\\ &= 2m_2 k_2 r_2 = 2m_2 k_2 \|(x,y)\|_E < 2m_2 h \|(x,y)\|_E\\ &< \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{28}$$

So in either case, there always exists $r_2 > r_1$ such that

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \tag{29}$$

Similarly, we have

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \tag{30}$$

So by (29) and (30), we have

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\|\\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \end{aligned} \tag{31}$$

By Lemma 4 and (31), we have

$$i\left(T, P_{r_2}, P\right) = 1. \tag{32}$$

From (23) and (32), we obtain

$$i\left(T, P_{r_2} \setminus \bar{P}_{r_1}, P\right) = i\left(T, P_{r_2}, P\right) - i\left(T, P_{r_1}, P\right) = 1. \tag{33}$$

So T has a fixed point (x, y) in $P_{r_2} \setminus \bar{P}_{r_1}$. It is obvious that (x, y) is a positive solution of system (2).

Theorem 7. *Suppose that S_1 holds and $g_\infty = \bar{g}_\infty = \infty$. Then, the system (2) has two positive solutions.*

Proof. Set $P_{b_1} = \{(x, y) \in P : \|(x, y)\|_E < b_1\}$. By Lemma 3 and S_1 (i), for $(x, y) \in \partial P_{b_1}$ and $t \in [(1/4), (3/4)]$, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s))\, ds \\ &\geq \frac{N_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s))\, ds \\ &\geq \frac{N_1 a}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} (x(s) + y(s))\, ds \\ &\geq \frac{N_1 a K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} ds (\|x\| + \|y\|) \\ &= a k_1 \|(x, y)\|_E \geq a k \|(x, y)\|_E \geq \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{34}$$

So

$$\|T_1(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \tag{35}$$

Similar to the proof of (35), we have

$$\|T_2(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \tag{36}$$

So by (35) and (36), we have

$$\begin{aligned} \|T(x, y)\|_E &= \|T_1(x, y)\| + \|T_2(x, y)\| \\ &> \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \end{aligned} \tag{37}$$

By Lemma 4 and (37), we have

$$i\left(T, P_{b_1}, P\right) = 0. \tag{38}$$

Set $P_{b_2} = \{(x, y) \in P : \|(x, y)\|_E < b_2\}$. By Lemma 3 and S_1 (ii), for $(x, y) \in \partial P_{b_2}$ and $t \in [0, 1]$, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s))\, ds \\ &\leq \frac{1}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s))\, ds \\ &< \frac{b_2}{2h\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1} ds = \frac{b_2 k_2}{2h} \\ &\leq \frac{b_2}{2} = \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{39}$$

So

$$\|T_1(x, y)\| < \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \tag{40}$$

Similarly,

$$\|T_2(x, y)\| < \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \tag{41}$$

So by (40) and (41), we have

$$\begin{aligned} \|T(x, y)\|_E &= \|T_1(x, y)\| + \|T_2(x, y)\| \\ &< \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \end{aligned} \tag{42}$$

By Lemma 4 and (42), we have

$$i\left(T, P_{b_2}, P\right) = 1. \tag{43}$$

By $g_\infty = \bar{g}_\infty = \infty$, there exist $m_3 > 0$ and $b > b_2$ such that

$$\begin{aligned} &f_1(t, x, y) \geq m_3 (x + y), \\ &f_2(t, x, y) \geq m_3 (x + y), \\ &\quad \forall x + y \geq b, t \in [0, 1], \end{aligned} \tag{44}$$

and $m_3 k > (1/2)$.

Choose $b_3 > \max\{b_2, (b/K)\}$. Set $P_{b_3} = \{(x, y) \in P : \|(x, y)\|_E < b_3\}$. For $(x, y) \in \partial P_{b_3}$

$$\min_{t \in [(1/4), (3/4)]} (x(t) + y(t)) \geq K(\|x\| + \|y\|) = K b_3 > b. \tag{45}$$

Then, by (44) and Lemma 3, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s))\, ds \\ &\geq \frac{N_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s))\, ds \\ &\geq \frac{N_1 m_3}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} (x(s) + y(s))\, ds \\ &\geq \frac{N_1 m_3 K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} ds (\|x\| + \|y\|) \\ &= m_3 k_1 \|(x, y)\|_E \geq m_3 k \|(x, y)\|_E > \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{46}$$

So

$$\|T_1(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_3}. \tag{47}$$

Similarly,

$$\|T_2(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_3}. \tag{48}$$

So by (47) and (48), we have

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{b_3}.\end{aligned} \tag{49}$$

By Lemma 4 and (49), we have

$$i\left(T, P_{b_3}, P\right) = 0. \tag{50}$$

Since $0 < b_1 < b_2 < b_3$, by (38), (43), and (50), we get

$$i\left(T, P_{b_3} \setminus \bar{P}_{b_2}, P\right) = i\left(T, P_{b_3}, P\right) - i\left(T, P_{b_2}, P\right) = -1, \tag{51}$$

$$i\left(T, P_{b_2} \setminus \bar{P}_{b_1}, P\right) = i\left(T, P_{b_2}, P\right) - i\left(T, P_{b_1}, P\right) = 1. \tag{52}$$

From (51), we know that T has a fixed point $(x_1, y_1) \in (P_{b_3} \setminus \bar{P}_{b_2})$. From (52), we know that T has another fixed point$(x_2, y_2) \in (P_{b_2} \setminus P_{b_1})$. So the system (2) has two positive solutions(x_1, y_1) and (x_2, y_2), with $0 < \|(x_2, y_2)\|_E < b_2 < \|(x_1, y_1)\|_E$.

Theorem 8. *Let $g_0 = \bar{g}_0 = 0$ and $g_\infty = \bar{g}_\infty = 0$. In addition, suppose that S_2 holds. Then, system (2) has two positive solutions.*

Proof. By $g_0 = \bar{g}_0 = 0$, we know that there exist $c_1 > 0$ and $h_1 \in (0, q)$ such that

$$\begin{aligned} &f_1(t,x,y) \le c_1(x+y), \\ &f_2(t,x,y) \le c_1(x+y), \\ &\qquad \forall 0 \le x, y \le h_1, t \in [0,1], \end{aligned} \tag{53}$$

and $c_1 h < (1/2)$. Set $P_{h_1} = \{(x,y) \in P : \|(x,y)\|_E < h_1\}$.

By Lemma 3 and (53), for $(x,y) \in \partial P_{h_1}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s, x(s), y(s)) ds \\ &\le \frac{c_1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1 - 1} (x(s) + y(s)) ds \\ &\le \frac{c_1(\|x\| + \|y\|)}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1 - 1} ds \\ &= c_1 k_2 \|(x,y)\|_E < c_1 h \|(x,y)\|_E < \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{54}$$

So

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}. \tag{55}$$

Similarly,

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}. \tag{56}$$

So by (55) and (56), we have

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|\, T_2(x,y)\| \\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}.\end{aligned} \tag{57}$$

By Lemma 4 and (57), we have

$$i\left(T, P_{h_1}, P\right) = 1. \tag{58}$$

From $g_\infty = \bar{g}_\infty = 0$, we know that there exist $c_2 > 0$ and $H > h_1$ such that

$$\begin{aligned} &f_1(t,x,y) \le c_2(x+y), \\ &f_2(t,x,y) \le c_2(x+y), \\ &\qquad \forall x + y \ge H, t \in [0,1], \end{aligned} \tag{59}$$

and $c_2 h < (1/4)$. In this part, we divide two cases. One case is that f_1 and f_2 are bounded. Namely, there exists $W > 0$ such that

$$\begin{aligned} &f_1(t,x,y) \le W, \\ &f_2(t,x,y) \le W, \\ &\qquad \forall x, y \in [0, +\infty), t \in [0,1]. \end{aligned} \tag{60}$$

Take $h_2 > \max\{2Wh, q\}$. Set $P_{h_2} = \{(x,y) \in P : \|(x,y)\|_E < h_2\}$. For $(x,y) \in \partial P_{h_2}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s, x(s), y(s)) ds \le W \int_0^1 K_1(t,s) ds \\ &\le \frac{W}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1 - 1} ds \le Wh < \frac{h_2}{2} \\ &= \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{61}$$

Another case is that f_1 and f_2 are unbounded. By the continuity of f_1, there exists $h_2 > \max\{h_1, q, H\}$ such that

$$f_1(t,x,y) \le f_1(t, h_2, h_2), \quad \text{for } 0 < x, y \le h_2. \tag{62}$$

Hence, set $P_{h_2} = \{(x,y) \in P : \|(x,y)\|_E < h_2\}$. For $(x,y) \in \partial P_{h_2}$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))ds \\ &\le \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s,h_2,h_2)ds \\ &\le \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} c_2(h_2+h_2)ds \\ &= 2c_2k_2h_2 = 2c_2k_2\|(x,y)\|_E < 2c_2h\|(x,y)\|_E \\ &< \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{63}$$

So in either case, there always exists $h_2 > \max\{h_1, q\}$ such that

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}. \tag{64}$$

Similarly, we have

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}. \tag{65}$$

Thus

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}. \end{aligned} \tag{66}$$

So by Lemma 4 and (66), we get

$$i\left(T, P_{h_2}, P\right) = 1. \tag{67}$$

Set $P_q = \{(x,y) \in P : \|(x,y)\|_E < q\}$. For $(x,y) \in \partial P_q$

$$\min_{t\in[(1/4),(3/4)]} (x(t)+y(t)) \ge K(\|x\|+\|y\|) = Kq. \tag{68}$$

Then, by S_2 and Lemma 3, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))ds \\ &\ge \frac{N_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1-1} f_1(s,x(s),y(s))ds \\ &> \frac{N_1 Kq}{2k\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1-1} ds > \frac{q}{2} = \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{69}$$

So

$$\|T_1(x,y)\| > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q. \tag{70}$$

Similarly,

$$\|T_2(x,y)\| > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q. \tag{71}$$

So by (70) and (71), we have

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q. \end{aligned} \tag{72}$$

By Lemma 4 and (72), we have

$$i\left(T, P_q, q\right) = 0. \tag{73}$$

Since $0 < h_1 < q < h_2$, by (58), (67), and (73), we get $i(T, P_q \setminus \bar{P}_{h_1}, P) = i(T, P_q, P) - i(T, P_{h_1}, P) = -1$. So T has a fixed point $(x_1, y_1) \in (P_q \setminus \bar{P}_{h_1})$. $i(T, P_{h_2} \setminus \bar{P}_q, P) = i(T, P_{h_2}, P) - i(T, P_q, P) = 1$. So T has another fixed point $(x_2, y_2) \in P_{h_2} \setminus \bar{P}_q$. Therefore, system (2) has at least two positive solutions.

4. Applications

Example 9. We study the following Caputo-type fractional system:

$$\begin{cases} -{}^cD_{0+}^{(5/2)} x(t) = (2+3t)\sqrt{x(t)+y(t)}, & t \in [0,1], \\ -{}^cD_{0+}^{(5/2)} y(t) = \left(1+t^2+t^4\right)\sqrt[5]{x(t)+y(t)}, & t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \dfrac{1}{2}\displaystyle\int_0^1 x(t)dt, \\ y(0) = y''(0) = 0, \quad y(1) = \dfrac{1}{2}\displaystyle\int_0^1 y(t)dt, \end{cases} \tag{74}$$

where $\theta_1 = \theta_2 = 5/2$, $A_1(t) = A_2(t) = (1/2)t$, $B_1 = B_2 = 1/4$, $f_1(t,x,y) = (2+3t)\sqrt{x+y}$, and $f_2(t,x,y) = (1+t^2+t^4)\sqrt[5]{x+y}$. Obviously, $g_0 = \bar{g}_0 = \infty$ and $g_\infty = \bar{g}_\infty = 0$. From Theorem 6, system (74) has a positive solution.

Example 10. We study the following Caputo-type fractional system:

$$\begin{cases} -{}^cD_{0+}^{5/2} x(t) = f_1(t,x(t),y(t)), & t \in [0,1], \\ -{}^cD_{0+}^{5/2} y(t) = f_2(t,x(t),y(t)), & t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \dfrac{1}{2}\displaystyle\int_0^1 x(t)dt, \\ y(0) = y''(0) = 0, \quad y(1) = \dfrac{1}{2}\displaystyle\int_0^1 y(t)dt, \end{cases} \tag{75}$$

where $\theta_1 = \theta_2 = 5/2$, $A_1(t) = A_2(t) = (1/2)t$, and $B_1 = B_2 = 1/4$.

Take

$$f_1(t,x,y)=\begin{cases}10^4(t+1), & t\in[0,1],0\le x,y\le\dfrac{1}{2},\\ 10^4(t+1)(x+y), & t\in[0,1],\dfrac{1}{2}\le x,y\le 1,\\ 2\times10^4(t+1), & t\in[0,1],1\le x,y\le 10^6,\\ 10^{-8}\left(\dfrac{t+1}{2}\right)(x+y)^2, & t\in[0,1],x,y\ge 10^6,\end{cases}\tag{76}$$

and

$$f_2(t,x,y)=\begin{cases}10^4(t+1), & t\in[0,1],0\le x,y\le\dfrac{1}{2},\\ 10^4(t+1)(x+y), & t\in[0,1],\dfrac{1}{2}\le x,y\le 1,\\ 2\times10^4(t+1), & t\in[0,1],1\le x,y\le 10^6,\\ 10^{-14}\left(\dfrac{t+1}{4}\right)(x+y)^3, & t\in[0,1],x,y\ge 10^6.\end{cases}\tag{77}$$

By simple computation, we get $K=9/256$, $k=(9(9\sqrt{3-1}))/(327680\sqrt{\pi})$, and $h=32/(45\sqrt{\pi})$.

Choose $a=7000$, $b_2=10^6$, and $b_1=1$, then $(b_2/2h)=((45\sqrt{\pi})/64)\cdot 10^6$.

When $0\le x,y\le b_1=1$,

(1) $0\le x,y\le(1/2), f_1(t,x,y)=10^4(t+1)\ge 10^4\ge 7000(x+y); f_2(t,x,y)=10^4(t+1)\ge 10^4\ge 7000(x+y)$

(2) $(1/2)\le x,y\le 1, f_1(t,x,y)=10^4(t+1)(x+y)\ge 10^4(x+y)\ge 7000(x+y);\ f_2(t,x,y)=10^4(t+1)(x+y)\ge 10^4(x+y)\ge 7000(x+y)$

When $0\le x,y\le b_2=10^6$,

(1) $0\le x,\ y\le(1/2),\ f_1(t,x,y)=10^4(t+1)\le 2\times10^4<((45\sqrt{\pi})/64)\cdot 10^6; f_2(t,x,y)=10^4(t+1)\le 2\times10^4<((45\sqrt{\pi})/64)\cdot 10^6$

(2) $(1/2)\le x,\ y\le 1,\ f_1(t,x,y)=10^4(t+1)(x+y)\le 4\times10^4<((45\sqrt{\pi})/64)\cdot 10^6;\ f_2(t,x,y)=10^4(t+1)(x+y)\le 4\times10^4<((45\sqrt{\pi})/64)\cdot 10^6$

(3) $1\le x,\ y\le 10^6, f_1(t,x,y)=2\times10^4(t+1)\le 4\times10^4<((45\sqrt{\pi})/64)\cdot 10^6;\ f_2(t,x,y)=2\times10^4(t+1)\le 4\times10^4<((45\sqrt{\pi})/64)\cdot 10^6$

Obviously, $g_\infty=\bar{g}_\infty=\infty$. So by Theorem 7, system (74) has at least two positive solutions.

Example 11. We study the following Caputo-type fractional system:

$$\begin{cases}-{}^cD_{0+}^{5/2}x(t)=f_1(t,x(t),y(t)), & t\in[0,1],\\ -{}^cD_{0+}^{5/2}y(t)=f_2(t,x(t),y(t)), & t\in[0,1],\\ x(0)=x''(0)=0,\quad x(1)=\dfrac{1}{2}\displaystyle\int_0^1 x(t)dt,\\ y(0)=y''(0)=0,\quad y(1)=\dfrac{1}{2}\displaystyle\int_0^1 y(t)dt,\end{cases}\tag{78}$$

where $\theta_1=\theta_2=(5/2)$, $A_1(t)=A_2(t)=(1/2)t$, and $B_1=B_2=(1/4)$.

Take

$$f_1(t,x,y)=\begin{cases}\dfrac{1}{2}\times10^8(t+1)(x+y)^2, & t\in[0,1],0\le x,y\le 10^{-4},\\ 10^4(t+1)(x+y), & t\in[0,1],10^{-4}\le x,y\le 10^{-2},\\ \dfrac{4(t+1)}{x+y}, & t\in[0,1],x,y\ge 10^{-2},\end{cases}\tag{79}$$

and

$$f_2(t,x,y)=\begin{cases}\dfrac{1}{2}\times10^8(t+1)(x+y)^2, & t\in[0,1],0\le x,y\le 10^{-4},\\ 10^4(t+1)(x+y), & t\in[0,1],10^{-4}\le x,y\le 10^{-2},\\ \dfrac{8\times10^{-2}(t+1)}{(x+y)^2}, & t\in[0,1],x,y\ge 10^{-2}.\end{cases}\tag{80}$$

Take $q=0.001$, then $Kq=(9/25600)$, $(Kq/2k)\approx 0.7778$. When $x+y\in[Kq,q]$, $f_1(t,x,y)=f_2(t,x,y)=10^4(t+1)(x+y)\ge 2\ge(Kq/2k)$. Obviously, $g_0=\bar{g}_0=0$ and $g_\infty=\bar{g}_\infty=0$. So, by Theorem 8, system (78) has at least two positive solutions.

Data Availability

The data set supporting the conclusions is included within this article.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

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Research Article

Inelastic Interaction and Blowup New Solutions of Nonlinear and Dispersive Long Gravity Waves

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In this paper, the fractional Broer–Kaup (BK) system is investigated by studying its novel computational wave solutions. These solutions are constructed by applying two recent analytical schemes (modified Khater method and sech–tanh function expansion method). The BK system simulates the bidirectional propagation of long waves in shallow water. Moreover, it is used to study the interaction between nonlinear and dispersive long gravity waves. A new fractional operator is used to convert the fractional form of the BK system to a nonlinear ordinary differential system with an integer order. Many novel traveling wave solutions are constructed that do not exist earlier. These solutions are considered the icon key in the inelastic interaction of slow ions and atoms, where they were able to explain the physical nature of the nuclear and electronic stopping processes. For more illustration, some attractive sketches are also depicted for the interpretation physically of the achieved solutions.

1. Introduction

Studying the energetic atomic projectiles (atoms or ions) is one of the most exciting recent fields which has impacts on surface physics, plasma, and some related applications [1–3].13;3]. Moreover, it has an essential role which multicharged ions play as part of the solar wind in the space environment [4]. These applications include plasma–wall interaction in controlled thermonuclear fusion devices [5], surface analysis [6], single–particle detection [7], and electrical discharges [8]. The projectile energy deposition is connected with the intensity and nature of ion (atom) surface. Thus, both the nature and intensity of the ion depend on the potential and kinetic energies carried by a projectile toward the surface [9]. The energetic ion that can penetrate a solid surface is able to transfer its kinetic energy to target electrons, leading to atomic excitation, collective electron excitations, or ionization [10].

Partial differential equations (PDEs) have been playing an essential role in the energetic atomic projectiles where many nonlinear evolution equations have been derived to describe the dynamical behaviour of several phenomena in atomic and nuclear physics. Thus, partial differential equations (PDEs) have been playing an important role in the emerging technologies where many nonlinear evolution equations have been derived to describe the dynamical behaviour of some distinct phenomena, for example, nonlinear optics, fluid dynamics, Bose–Einstein condensates, and quantum mechanicss. However, the inadequacy of the PDEs with an integer order has been clarified because of the nonlocal property where this kind of equation does not explain that kind of properties. Therefore, several nature phenomena have been formulated with nonlinear PDEs with fractional order [11–35]. Thus, many fractional operators have been derived such as conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, Caputo–Fabrizio definition, and so on [36–40].

These definitions have been employed to convert the fractional nonlinear partial differential equations to a nonlinear integer–order ordinary differential equation. Then, the computational and numerical schemes can be applied to get various types of solutions for these models and the examples of these schemes.

The $\mathscr{ABR}$ fractional operator is considered one of the most general recent fractional operators that is derived from avoiding the deficiencies and defects of some other fractional operator. This operator is defined as follows.

Definition 1. It is given by [41–44]

$$ {}^{\mathscr{ABR}}D_{a+}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)\, g_\alpha\left(\frac{-\alpha\,(t-\alpha)^{\alpha}}{1-\alpha}\right)dx, \quad (1) $$

where g_α stands for the Mittag–Leffler function, given by [45, 46]

$$ g_\alpha\left(\frac{-\alpha\,(t-\alpha)^{\alpha}}{1-\alpha}\right) = \sum_{n=0}^{\infty}\frac{(-\alpha/1-\alpha)^n (t-x)^{\alpha n}}{\Gamma(\alpha n+1)} \quad (2) $$

and $B(\alpha)$ being a normalization function. Thus,

$$ {}^{\mathscr{ABR}}D_{a+}^{\alpha} f(x) = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{nRL}\mathscr{J}_a^{\alpha n} f(x) \quad (3) $$

This research paper studies the fractional BK system that is given by [47–49].

$$ \begin{cases} \mathscr{D}_t^{\alpha}\mathcal{S} = \dfrac{1}{2}\left(\mathcal{S}^2 + 2\mathscr{W} - \mathcal{S}_x\right)_x, \\ \mathscr{D}_t^{\alpha}\mathscr{W} = \left(\mathcal{S}\mathscr{W} + \dfrac{1}{2}\mathscr{W}_x\right)_x, \end{cases} \quad (4) $$

where $[(0<\alpha<1), \mathcal{S} = \mathcal{S}(x,t), \mathscr{W} = \mathscr{W}(x,t)]$ is unknown functions in time (t) and place (x). System (3) is used to simulate the bidirectional propagation of long waves in shallow water. Moreover, it studies the interaction between dispersive and nonlinear long gravity waves. System (3) is an integrable system and is related to the classical Boussinesq equation that is given by [50]

$$ \mathscr{B}_{tt} - \lambda^2\mathscr{B}_{xx} - \beta\mathscr{B}_{xxxx} - \alpha\left(\mathscr{B}^2\right)_{xx} = 0, \quad (5) $$

where $[\lambda, \alpha, \beta]$ are arbitrary constants. Equation (5) is a model of nonlinear dispersive waves. Employing the next wave transformation (the $\mathscr{ABR}$ fractional operator) $[\mathrm{Y} = (((1-\alpha)(c\,t^{-n\alpha})/B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)) + x), \mathcal{S} = \mathcal{S}(\mathrm{Y}), \mathscr{W} = \mathscr{W}(\mathrm{Y})$, where (c) is an arbitrary constant] to the system ((3)); then, differentiate the results once and substitute the second equation of the system into the first equation leading to convert the system into the next equation

$$ 4\,c^2\,\mathcal{S} - 6\,c\,\mathcal{S}^2 + 2\,\mathcal{S}^3 - \mathcal{S}'' = 0. \quad (6) $$

Applying the homogeneous balance principle to Equation (6), yields $(m=1)$.

The rest of research paper is organized as follows: Section 2 applies the modified Khater method and sech–tanh functions expansion method to the suggested model to get novel solitary wave solutions of it. Section 3 explains the physical interpretation of the shown sketches in our paper. Section 4 shows the novelty of our paper by comparing our results with those obtained in previous research papers. Section 5 explains the conclusion of all the steps of our paper in detail.

2. Application

Here, in this section, the modified Khater method and sech–tanh functions expansion method are applied to the $\mathscr{BK}$ equation to simulate the bi–directional propagation of long waves in sha llow water.

2.1. The Modified Khater Method. Applying the modified Khater method to Equation (6) leads to formulate the general solution of this model in the following formula

$$ \begin{aligned} \mathcal{S}(\mathrm{Y}) &= \sum_{i=1}^{m} a_i\mathscr{K}^{i\mathscr{F}(\mathrm{Y})} + \sum_{i=1}^{m} b_i\mathscr{K}^{-i\mathscr{F}(\mathrm{Y})} + a_0 \\ &= a_1\mathscr{K}^{\mathscr{F}(\mathrm{Y})} + a_0 + b_1\mathscr{K}^{-\mathscr{F}(\mathrm{Y})}, \end{aligned} \quad (7) $$

where $[a_0, a_1, b_1]$ are arbitrary constants to be determined later. Additionally, $\mathscr{F}(\mathrm{Y})$ is the solution function of the following ordinary differential equation

$$ \mathscr{F}'(\mathrm{Y}) = \frac{1}{\ln(\mathscr{K})}\left[\delta\mathscr{K}^{\mathscr{F}(\mathrm{Y})} + \varrho\mathscr{K}^{-\mathscr{F}(\mathrm{Y})} + \chi\right], \quad (8) $$

where $[\delta, \varrho, \chi]$ are arbitrary constants. Substituting Equation (7) along (7) into Equation (6) and collecting all terms with the same power of $[\mathscr{K}^{i\mathscr{F}(\mathrm{Y})}, i = -5,-4,\cdots,4,5]$, give a system of the algebraic equation. Using the Mathematica 12 program for solving this system, yields

Family 2.

$$ \begin{aligned} &\left[a_0 \longrightarrow \frac{1}{2}\left(\sqrt{\chi^2 - 4\delta\varrho} + \chi\right), a_1 \longrightarrow \delta, b_1\right. \\ &\left.\longrightarrow 0, c \longrightarrow \frac{1}{2}\sqrt{\chi^2 - 4\delta\varrho}\right]. \end{aligned} \quad (9) $$

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas

For $[\chi^2 - 4\delta\varrho > 0 \& \delta \neq 0]$,

$$\mathcal{S}_1(x,t) = -\frac{1}{2}\sqrt{\chi^2 - 4\delta\varrho}\left(\tanh\left(\frac{1}{2}x\sqrt{\chi^2 - 4\delta\varrho} - \frac{(\alpha-1)t^{-\alpha}\left(\chi^2 - 4\delta\varrho\right)}{4B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)}\right) - 1\right),$$
$$\mathcal{S}_2(x,t) = -\frac{1}{2}\sqrt{\chi^2 - 4\delta\varrho}\left(\coth\left(\frac{1}{2}x\sqrt{\chi^2 - 4\delta\varrho} - \frac{(\alpha-1)t^{-\alpha}\left(\chi^2 - 4\delta\varrho\right)}{4B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)}\right) - 1\right). \tag{10}$$

For $[\delta\varrho < 0 \& \varrho \neq 0 \& \delta \neq 0 \& \chi = 0]$,

$$\mathcal{S}_3(x,t) = -\sqrt{-\delta\varrho}\left(\tanh\left(\frac{(\alpha-1)\delta\varrho t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)} + x\sqrt{-\delta\varrho}\right) - 1\right),$$
$$\mathcal{S}_4(x,t) = -\sqrt{-\delta\varrho}\left(\coth\left(\frac{(\alpha-1)\delta\varrho t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)} + x\sqrt{-\delta\varrho}\right) - 1\right). \tag{11}$$

For $[\chi = 0 \& \varrho = -\delta]$,

$$\mathcal{S}_5(x,t) = \sqrt{\varrho^2} - \varrho\coth\left(\varrho\left(x - \frac{(\alpha-1)\sqrt{\varrho^2}t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)}\right)\right). \tag{12}$$

For $[\chi = \delta = \kappa \& \varrho = 0]$,

$$\mathcal{S}_6(x,t) = \frac{1}{2}\left(\sqrt{\kappa^2} - \kappa\coth\left(\frac{1}{2}\kappa\left(x - \frac{(\alpha-1)\sqrt{\kappa^2}t^{-\alpha}}{2B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)}\right)\right)\right). \tag{13}$$

For $[\varrho = 0 \& \chi \neq 0 \& \delta \neq 0]$

$$\mathcal{S}_7(x,t) = \frac{1}{2}\left(\chi\left(-\frac{4}{\delta\exp\left(\chi\left(x - (\alpha-1)\sqrt{\chi^2}t^{-\alpha}/2B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)\right)\right) - 2} - 1\right) + \sqrt{\chi^2}\right). \tag{14}$$

Family 3.

$$\left[a_0 \longrightarrow \frac{1}{2}\left(\sqrt{\chi^2 - 4\delta\rho} + \chi\right), a_1 \longrightarrow 0, b_1 \longrightarrow \rho, c \longrightarrow \frac{1}{2}\sqrt{\chi^2 - 4\delta\rho}\right]. \tag{15}$$

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas

For $[\chi^2 - 4\delta\varrho > 0 \& \delta \neq 0]$,

$$\mathcal{S}_8(x,t) = \frac{1}{2}\left(-\frac{4\delta\varrho}{\sqrt{\chi^2 - 4\delta\varrho}\tanh\left(1/2x\sqrt{\chi^2 - 4\delta\varrho} - (\alpha-1)t^{-\alpha}(\chi^2 - 4\delta\varrho)/4B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)\right) + \chi} + \sqrt{\chi^2 - 4\delta\varrho} + \chi\right), \tag{16}$$

$$\mathcal{S}_9(x,t) = \frac{1}{2}\left(-\frac{4\delta\varrho}{\sqrt{\chi^2 - 4\delta\varrho}\coth\left(1/2x\sqrt{\chi^2 - 4\delta\varrho} - (\alpha-1)t^{-\alpha}(\chi^2 - 4\delta\varrho)/4B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^n\Gamma(1-\alpha n)\right) + \chi} + \sqrt{\chi^2 - 4\delta\varrho} + \chi\right). \tag{17}$$

For $[\delta\varrho < 0 \& \varrho \neq 0 \& \delta \neq 0 \& \chi = 0]$,

$$\begin{aligned}\mathcal{S}_{10}(x,t) &= \sqrt{-\delta\varrho}\left(\coth\left(\frac{(\alpha-1)\delta\varrho t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)} + x\sqrt{-\delta\varrho}\right) + 1\right),\\ \mathcal{S}_{11}(x,t) &= \sqrt{-\delta\varrho}\left(\tanh\left(\frac{(\alpha-1)\delta\varrho t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)} + x\sqrt{-\delta\varrho}\right) + 1\right).\end{aligned} \tag{19}$$

For $[\chi = 0 \& \varrho = -\delta]$,

$$\mathcal{S}_{12}(x,t) = \varrho\tanh\left(\rho\left(x - \frac{(\alpha-1)\sqrt{\varrho^2}t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)}\right)\right) + \sqrt{\varrho^2}. \tag{20}$$

For $[\chi = \varrho/2 = \kappa \& \delta = 0]$,

$$\mathcal{S}_{13}(x,t) = \frac{1}{2}\left(\frac{4\kappa}{\exp\left(\kappa\left(x - (\alpha-1)\sqrt{\kappa^2}t^{-\alpha}/2B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)\right)\right) - 2} + \sqrt{\kappa^2} + \kappa\right). \tag{21}$$

For $[\delta = 0 \& \chi \neq 0 \& \varrho \neq 0]$,

$$\mathcal{S}_{14}(x,t) = \frac{1}{2}\left(-\frac{2\chi\varrho}{\varrho - \chi\exp\left(\chi\left(x - (\alpha-1)\sqrt{\chi^2}t^{-\alpha}/2B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)\right)\right)} + \sqrt{\chi^2} + \chi\right). \tag{22}$$

2.2. The Sech–Tanh Function Expansion Method. Applying the sech–tanh function expansion method to Equation (6) leads to formulate the general solution of this model in the following formula:

$$\begin{aligned}\mathcal{S}(\mathrm{Y}) &= \sum_{i=1}^{m}\mathrm{sec\,h}^{i-1}(\mathrm{Y})(a_i\,\mathrm{sech}\,(\mathrm{Y}) + b_i\tanh(\mathrm{Y})) + a_0\\ &= a_1\,\mathrm{sech}\,(\mathrm{Y}) + a_0 + b_1\tanh(\mathrm{Y}),\end{aligned} \tag{23}$$

where $[a_0, a_1, b_1]$ are arbitrary constants to be determined later. Substituting Equation (23) into Equation (6) and collecting all terms with the same power of [sech (Y), tanh (Y)], give a system of the algebraic equation. Using the Mathematica 12 program for solving this system, yields

Family 4.

$$[a_0 \longrightarrow -1, a_1 \longrightarrow 0, b_1 \longrightarrow 1, c \longrightarrow -1] \tag{24}$$

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas:

$$\mathcal{S}_{15}(x,t) = \tanh\left(\frac{(\alpha-1)t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)} + x\right) - 1. \tag{25}$$

Family 5.

$$[a_0 \longrightarrow 1, a_1 \longrightarrow 0, b_1 \longrightarrow 1, c \longrightarrow 1] \tag{26}$$

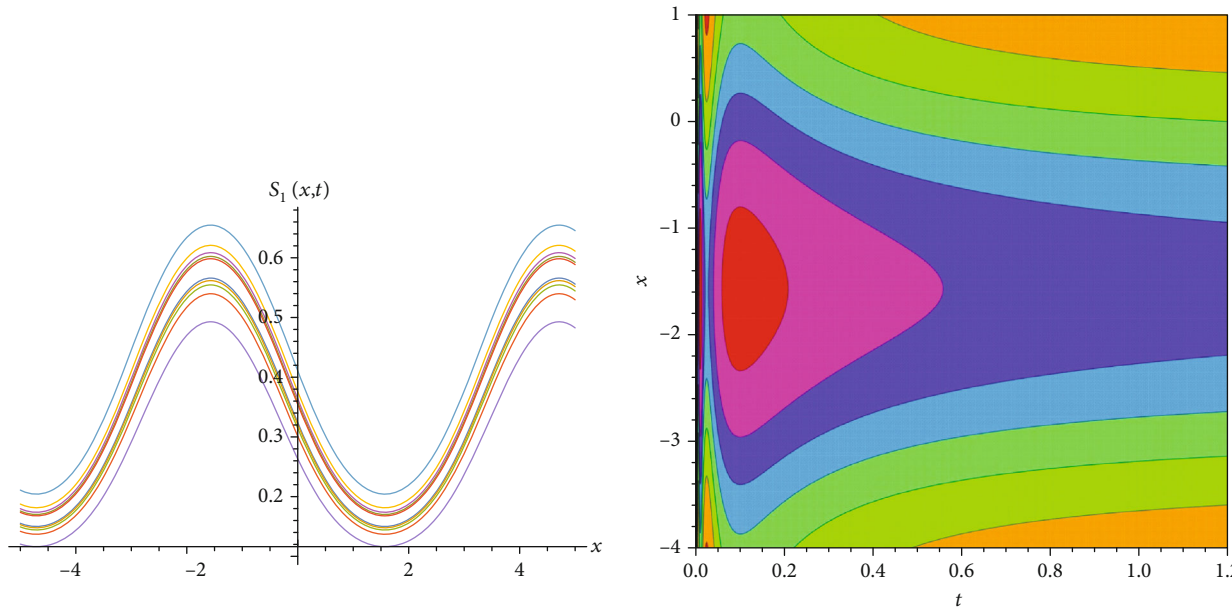

Figure 1: Numerical simulation of Equation (10) in two–dimensional and contour sketches.

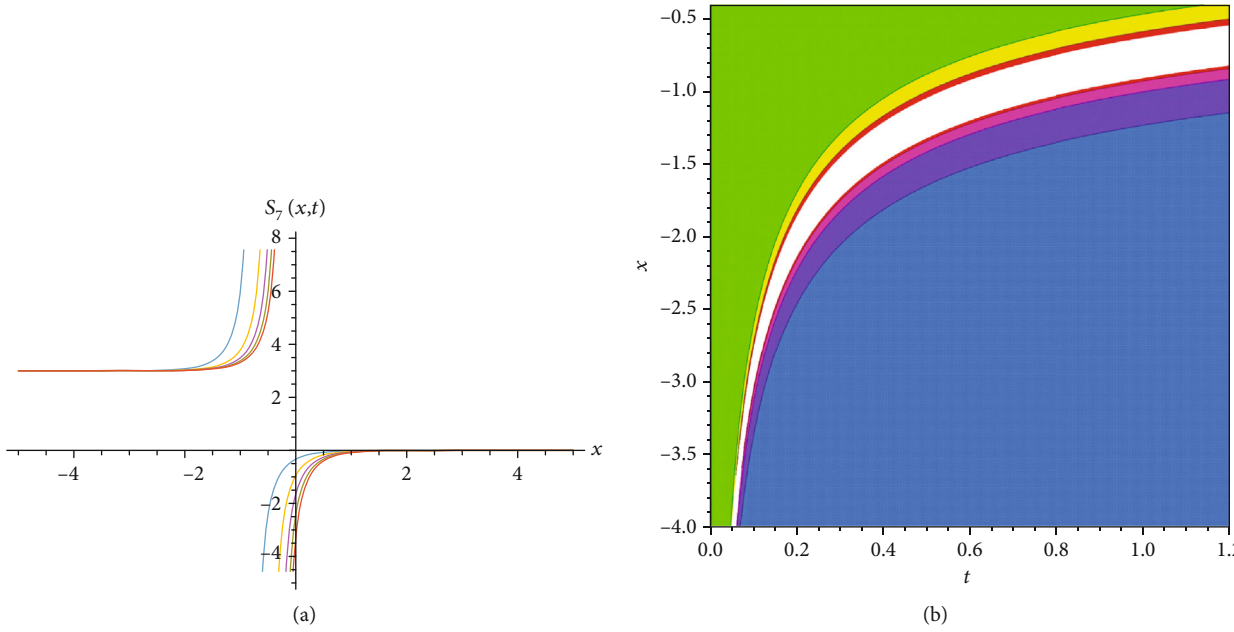

Figure 2: Numerical simulation of Equation (15) in two–dimensional and contour sketches.

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas:

$$\mathcal{S}_{16}(x,t) = \tanh\left(x - \frac{(\alpha-1)t^{-\alpha}}{B(\alpha)\sum_{n=0}^{\infty}(-(\alpha/(1-\alpha)))^{n}\Gamma(1-\alpha n)}\right) + 1. \tag{27}$$

3. Figure Interpretation

This section gives the physical interpretation of the shown figures in our paper. All our obtained solutions are considered traveling wave solutions. In the following lines, we give the physical interpretation of the shown figures:

(1) Figure 1 shows the bell wave solution (8) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2), \delta = 1, \chi = 3, \varrho = 2.]$

(2) Figure 2 shows the singular wave solution (13) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the

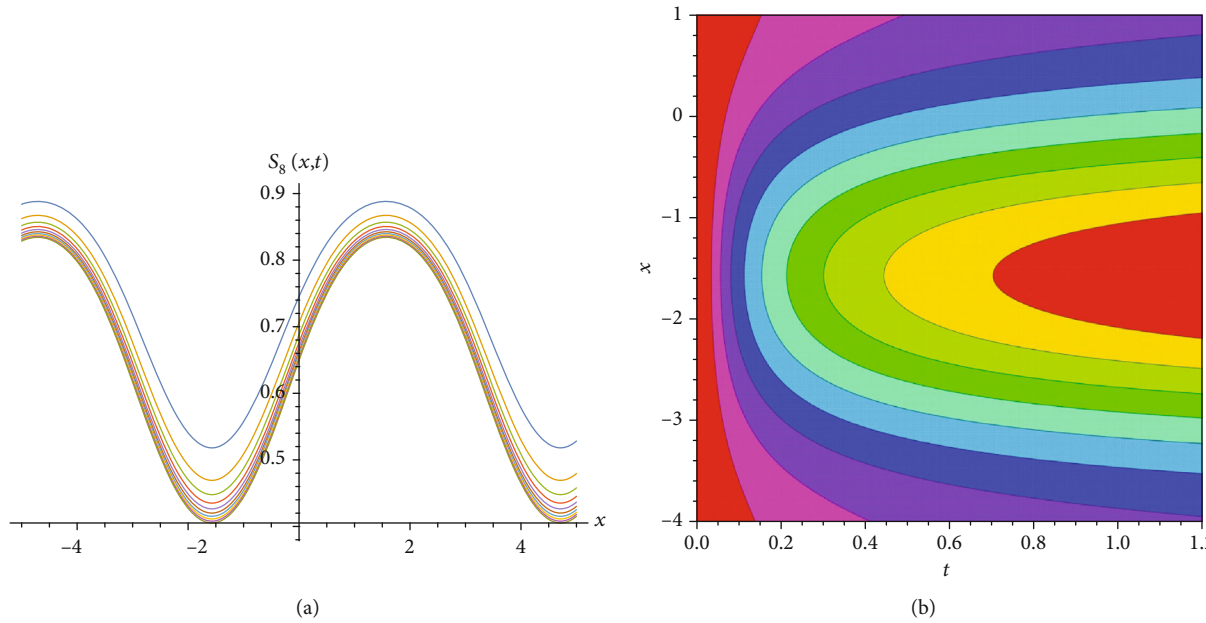

Figure 3: Numerical simulation of Equation (17) in two–dimensional and contour sketches.

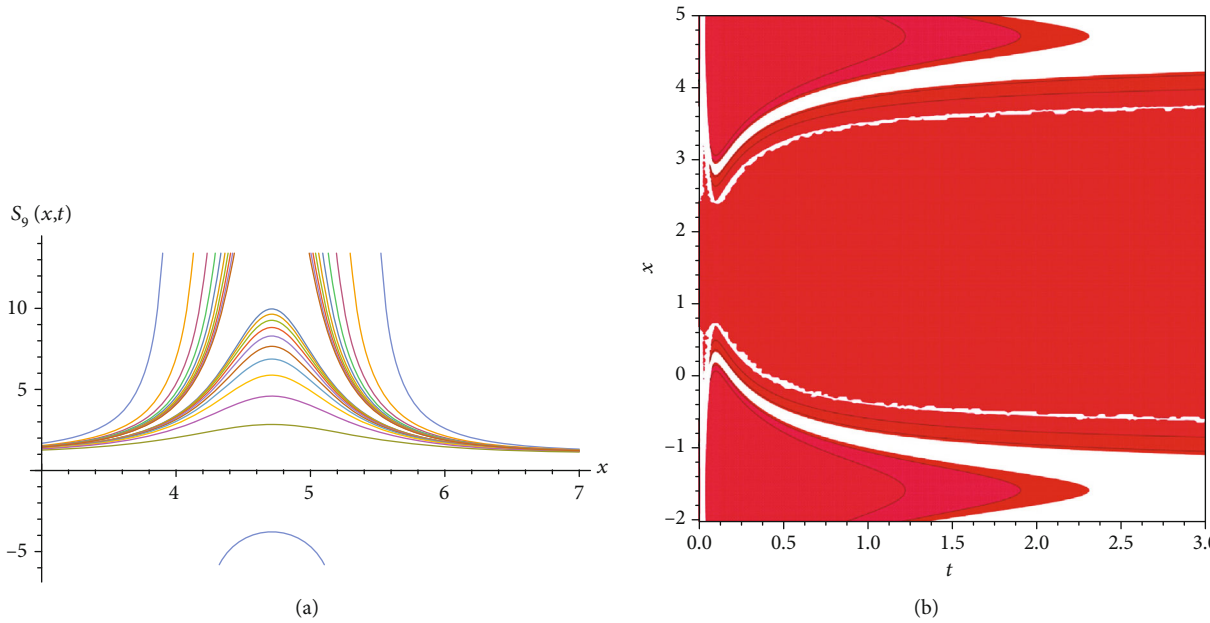

Figure 4: Numerical simulation of Equation (18) in two–dimensional and contour sketches.

contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2), \delta = 1, \chi = 3.]$

(3) Figure 3 shows the W–wave solution (14) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2), \delta = 1, \chi = 5, \varrho = 6.]$

(4) Figure 4 shows the rouge–wave solution (15) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2), \delta = 1, \chi = 5, \varrho = 6.]$

(5) Figure 5 shows the Breath–wave solution (21) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2)]$.

(6) Figure 6 shows the kink–wave solution (22) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the

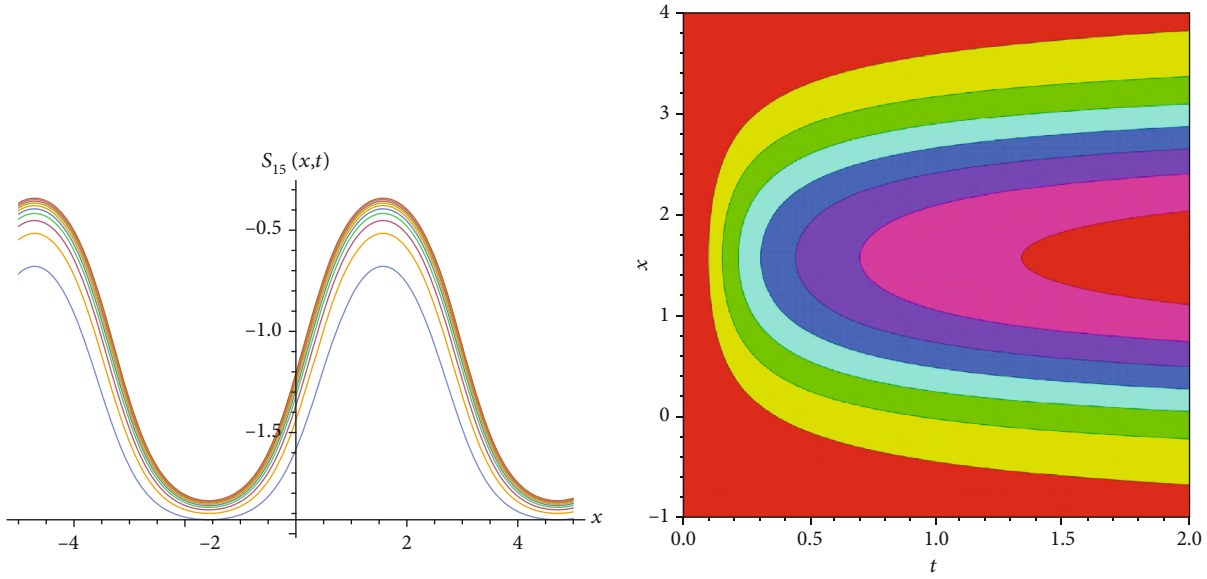

Figure 5: Numerical simulation of Equation (25) in two–dimensional and contour sketches.

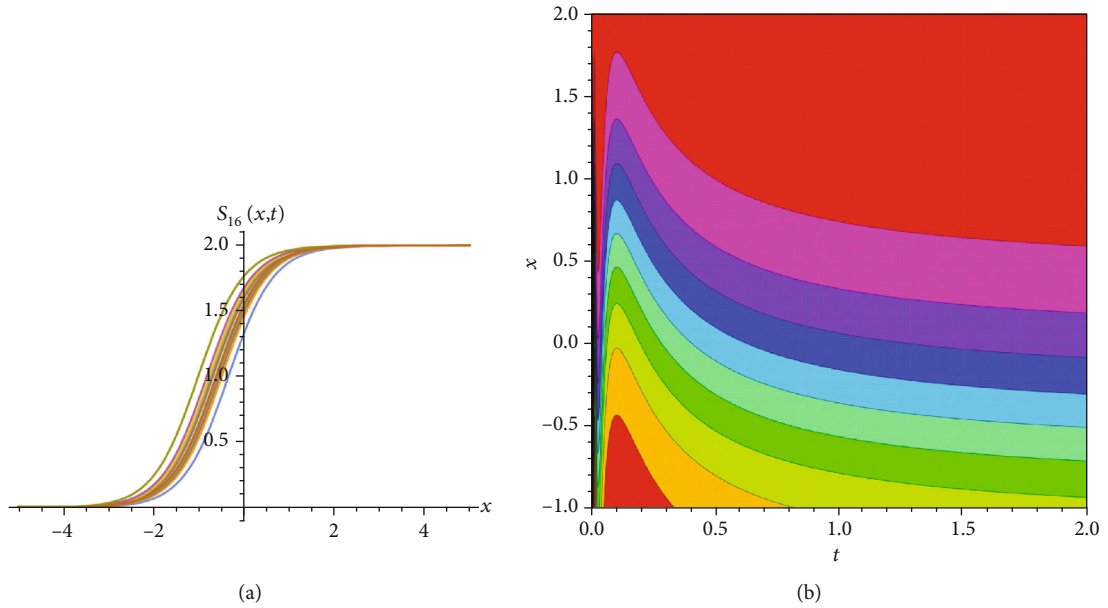

Figure 6: Numerical simulation of Equation (27) in two–dimensional and contour sketches.

contour plot (b) to explain the overhead view of the solution when $[\alpha = (1/2)]$.

4. Results and Discussion

This section shows the novelty of this research paper by explaining the comparison between our obtained solutions and those obtained in previous paper.

4.1. Computational Schemes. This paper investigated the analytical solutions of the fractional BK system by using two recent computational schemes (modified Khater method and sech–tanh function expansion method). These methods are considered recent analytical schemes in this field, and they were not applied to this model yet.

4.2. Obtained Computational Wave Solutions

(i) Equation (10) is equal to Equation (25) when $[\chi^2 - 4\varrho\delta = 4]$

(ii) All our obtained solutions are different from those obtained in [47–49] where the authors of [47–49] applied different methods to solve the BK system with an integer order. On the other hand, we investigated the fractional form of this model

5. Conclusion

In this paper, we investigated new soliton wave solutions of the nonlinear fractional BK system by using two recent analytical schemes. A new fractional operator is used to convert the fractional form of this model to a nonlinear partial differential equation with an integer order. The modified Khater method and sech–tanh function expansion method were applied to this model. Some new soliton wave solutions were obtained, and some of them were explained by plotting them in two–dimensional and contour plots of these solutions. The novelty of our paper was shown by making the comparison between our obtained solutions and that was purchased in previously published articles.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Ethical Approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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Research Article

Improved Fractional Subequation Method and Exact Solutions to Fractional Partial Differential Equations

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In this paper, the improved fractional subequation method is applied to establish the exact solutions for some nonlinear fractional partial differential equations. Solutions to the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the $(3+1)$-space-time fractional Zakharov-Kuznetsov equation are obtained, respectively.

1. Introduction

Fractional differential equations are widely used to describe lots of important phenomena and dynamic processes in physics, engineering, electromagnetics, acoustics, viscoelasticity electrochemistry, material science, stochastic dynamical system, plasma physics, controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics, etc. [1–7]. In order to find the solutions of fractional differential equations, many powerful and efficient methods have been introduced and developed, such as Darboux transformations [8], the hyperbolic function method [9], the variational iteration method [10, 11], the autofinite element method [12, 13], the auxiliary equation method [14], the finite difference method [15, 16], the Adomian decomposition method [17, 18], the homogenous balance method [19], Hirotas bilinear method [20], the homotopy analysis method [21, 22], (G'/G)-expansion method [23, 24], the subequation method [25, 26], the first integral method [27, 28], the improved fractional subequation method [29, 30], the extended Jacobi elliptic function expansion method [31], the generalized Kudryashov method [32], the exponential rational function method [33], the exp-function method [34, 35], the multiple exp-function method [36], the extended simple equation method [37].

In order to deal with nondifferentiable functions, Jumarie [38] has proposed a modification of the Riemann-Liouville definition which appears to provide a framework for a fractional calculus which is quite parallel to the classical calculus.

Jumarie's modified Riemann-Liouville derivative of order α for a function f is defined as follows:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha < 1, \\ \left[f^{(n)}(x) \right]^{(\alpha-n)}, & n \le \alpha < n+1, n \ge 1. \end{cases} \tag{1}$$

Some useful properties of modified Riemann-Liouville derivative are given below:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \gamma > 0, \tag{2}$$

$$D_x^\alpha [f(x)g(x)] = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x), \tag{3}$$

$$D_x^\alpha f[g(x)] = f'_g[g(x)]D_x^\alpha g(x) = D_g^\alpha f[g(x)]\left[g'(x)\right]^\alpha, \tag{4}$$

which holds for nondifferentiable functions. Equations (2), (3), (4) which are important tools for fractional calculus. Based on these merits, the modified Riemann-Liouville derivative was successfully applied to the probability calculus, fractional Laplace problems, and fractional variational calculus.

In this paper, we aim to find new exact solutions of some important partial fractional differential equations under Jumarie's definition by improved fractional subequation method.

In what follows, we introduce the aforementioned fractional partial differential equations. They are the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the (3 + 1)-space-time fractional Zakharov-Kuznetsov equation.

Suppose time $t > 0$, $D_t^\alpha u$ is time modified Riemann-Liouville derivative of order α for a function u, $0 < \alpha \le 1$, the parameters $k_1, k_2, \cdots, k_7$ are any real constants.

The generalized time fractional biological population model is given by

$$D_t^\alpha u + k_1\left(u^2\right)_{xx} + k_2\left(u^2\right)_{yy} + k_3 u_x + k_4 u_y + k_5 u^2 + k_6 u + k_7 = 0, \tag{5}$$

where $u = u(x, y, t)$ is an unknown function.

When $k_1 = -1, k_2 = -1, k_3 = k_4 = k_6 = 0, k_5 = -h \neq 0, k_7 = rh \neq 0$, Equation (5) is the time fractional biological population model [39]:

$$D_t^\alpha u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + h\left(u^2 - r\right), \tag{6}$$

where $u = u(x, y, t)$ denotes the population density and $h(u^2 - r)$ represents the amount of population due to death and birth. Moreover, $h(u^2 - r)$ leads to Verhulst law. Equation (5) has an important role to understand the dynamic process of population changes, and it is also an assistant to achieve precision about it.

The generalized time fractional compound KdV-Burgers equation is given by

$$D_t^\alpha u + k_1 u u_x + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0, \tag{7}$$

where $u = u(x, t)$ is an unknown function.

When $k_1 = k_3 = 0$, Equation (7) becomes the time fractional mKdV equation

$$D_t^\alpha u + k_2 u^2 u_x + k_4 u_{xxx} = 0\,; \tag{8}$$

when $k_1 = k_3 = 0$ Equation (7) becomes the time fractional KdV equation

$$D_t^\alpha u + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0\,; \tag{9}$$

when $k_2 = k_4 = 0$, Equation (7) becomes the time fractional Burgers equation [40]

$$D_t^\alpha u + k_1 u u_x + k_3 u_{xx} = 0\,; \tag{10}$$

when $k_1 = 0$, Equation (7) becomes the time fractional mKdV-Burgers equation

$$D_t^\alpha u + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0\,; \tag{11}$$

when $k_2 = 0$, Equation (7) becomes the time fractional KdV-Burgers equation

$$D_t^\alpha u + k_1 u u_x + k_3 u_{xx} + k_4 u_{xxx} = 0. \tag{12}$$

The space-time fractional regularized long-wave equation is given by [41]:

$$D_t^\alpha u + k_1 D_x^\alpha u + k_2 u D_x^\alpha u + k_3 D_t^\alpha D_x^2 \alpha u = 0, \tag{13}$$

where $u = u(x, t)$ is an unknown function, $D_x^\alpha u$ is the modified Riemann-Liouville derivative of order α for a function u, and $D_x^{2\alpha} u = D_x^\alpha (D_x^\alpha u)$.

The regularized long-wave equation, which describes approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems, was proposed by Benjamin et al. in 1972. The regularized long-wave equation is considered an alternative to the KdV equation, which is modeled to govern a large number of physical phenomena such as shallow waters and plasma waves.

The (3 + 1)-space-time fractional Zakharov-Kuznetsov equation given by [41].

$$D_t^? u + k_1 u D_x^\alpha u + k_2 D_x^{3\alpha} u + k_3 D_x^\alpha D_y^{2\alpha} u + k_4 D_x^\alpha D_z^{2\alpha} u = 0, \tag{14}$$

where $u = u(x, y, z, t)$ is an unknown function; $D_x^\alpha u$, $D_y^\alpha u$, and $D_z^\alpha u$ are the modified Riemann-Liouville derivatives of the function u; $D_x^{2\alpha} u = D_x^\alpha (D_x^\alpha u)$; $D_y^{2\alpha} u = D_y^\alpha (D_y^\alpha u)$; $D_z^{2\alpha} u = D_z^\alpha (D_z^\alpha u)$; and $D_x^{3\alpha} u = D_x^\alpha (D_x^\alpha (D_x^\alpha u))$.

The Zakharov-Kuznetsov equation was first derived for analysing weakly nonlinear ion acoustic waves in heavily magnetized lossless plasma and geophysical flows in two dimensions. The ZK equation is one of the two well-established canonical two-dimensional extensions of the KdV equation. The ZK equation governs the behavior of weakly nonlinear ionacoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field.

Motivated by the above results, in this paper, we use the improved subequation method to find new exact solutions of the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation,

and the (3 + 1)-space-time fractional Zakharov-Kuznetsov equation, respectively.

2. A Brief Description of the Improved Fractional Subequation Method

In this section, basic steps of the improved subequation method [42] are presented.

Consider the following nonlinear fractional differential equation,

$$P\left(u, D_t^\alpha u, D_{x_1}^\alpha u, D_{x_1}^\alpha u^2, \cdots, D_{x_m}^\alpha u, D_{x_1}^\alpha u^2, D_{x_2}^\alpha u^2, \cdots, D_{x_m}^\alpha u^2, \cdots, D_{x_1}^{2\alpha} u, D_{x_m}^{2\alpha} u, \cdots\right) = 0, \tag{15}$$

where $t, x_1, x_2, \cdots, x_m$ are independent variables, $u(t, x_1, x_2, \cdots, x_m)$ is an unknown function, and P is a polynomial of u, $u^2, \cdots$ and their partial fractional derivatives. Also, $D^\alpha(\cdot)$ symbolizes the modified Riemann-Liouville fractional derivative.

Step 1. First of all, using a suitable fractional complex transform,

$$u(t, x_1, x_2, \cdots, x_m) = u(\xi), \xi(t, x_1, x_2, \cdots|, x_m), \tag{16}$$

Equation (15) converts into nonlinear ordinary differential equation given below:

$$Q\left(u, u', u'', u''', \cdots\right) = 0, \tag{17}$$

where $u', u'', u''', \cdots$ denotes the derivations with respect to ξ.

We used to utilize real transformation, but actually, complex transformations are more useful. They make some equations easier to simplify.

Step 2. Suppose that the solution of ordinary differential equation (17) is

$$u = \sum_{i=n}^{-1} a_i \varphi^i + a_0 + \sum_{i=n}^{n} a_i \varphi^i, \tag{18}$$

where constants $a_i (i = -n, \cdots, -1, 1, \cdots, n)$ are going to be determined. Here, n is a positive integer, and it is obtained using the homogeneous balance of the highest order derivative and the nonlinear term seen in Equation (17).

$\varphi = \varphi(\xi)$ is the solution of the Riccati equation

$$\varphi' = \sigma + \varphi^2, \tag{19}$$

where σ is a constant, and the solutions of Equation (19) are obtained by Zhang et al. [34] as follows:

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, & \sigma = 0, \omega = \text{const}, \end{cases} \tag{20}$$

In the previous literatures, $u = a_0 + \sum_{i=1}^{n} a_i \varphi^i$ was considered. In this paper, we assume $u = \sum_{i=-n}^{-1} a_i \varphi^i + a_0 + \sum_{i=1}^{n} a_i \varphi^i$, and we can get a solution that has both hyperbolic tangent function and hyperbolic cotangent function or both tangent function and cotangent function.

Step 3. Putting Equation (18) along with Equation (19) into Equation (17), we obtain a new polynomial in terms of φ. Then, all the coefficients of powers of $\varphi^k (k = 0, 1, 2, \cdots, -1, -2, \cdots)$ are set equal to zero, we get a system of algebraic equations.

Step 4. Finally, the system of algebraic equations is obtained in the previous step for $a_i (-n \le i \le n)$, and σ is solved by the Maple package. By substituting the newly obtained values into Equation (20), we get the exact solutions for the nonlinear fractional differential equation (15).

Applying a suitable fractional complex transform of the improved fractional subequation method and the chain rule, nonlinear fractional differential equations with the modified Riemann-Liouville derivative can be converted into nonlinear ordinary differential equations. Then, using the solutions of a Riccati equation, we can find exact analytical solutions expressed by triangle functions, hyperbolic functions, or power functions.

3. Applications of the Improved Fractional Subequation Method

In this section, the improved fractional subequation method is utilized to solve some nonlinear fractional differential equations introduced in Section 1.

3.1. The Generalized Time Fractional Biological Population Model. The generalized time fractional biological population model is given by

$$D_t^\alpha u + k_1 \left(u^2\right)_{xx} + k_2 \left(u^2\right)_{yy} + k_3 u_x + k_4 u_y + k_5 u^2 + k_6 u + k_7 = 0, \tag{21}$$

where $u = u(x, y, t)$ is an unknown function.

We considered two cases.

Case 1. When $k_1 k_2 > 0$, let

$$u(x, y, t) = u(\xi), \xi = ax + ia\sqrt{\frac{k_1}{k_2}} y + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \tag{22}$$

then Equation (21) is reduced to the ordinary differential equation as follows:

$$\begin{cases} ak_4\sqrt{\dfrac{k_1}{k_2}}u' = 0, \\ (ak_3 + b)u' + k_5u^2 + k_6u + k_7 = 0. \end{cases} \tag{23}$$

When $k_4 \neq 0$, Equation (23) has only constant solutions.

When $k_4 = 0$, Equation (23) has solutions in the form of (18). u' is obtained from the homogeneous balance between the highest order derivative and the nonlinear term u^2. We obtain the solution of Equation (21) as follows:

$$u(\xi) = a - 1\varphi^{-1} + a_0 + a_1\varphi, \tag{24}$$

Substituting Equation (24) together with its necessary derivatives into Equation (21), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} \varphi^{-2} &: -k_3\sigma a_{-1} - b\sigma a_{-1} + k_5 a^2{}_{-1}, \\ \varphi^{-1} &: 2k_5a_{-1}a_0 + a_{-1}k_6, \\ \varphi^0 &: -ak_3a_{-1} - ba_{-1} + ak_3\sigma a_1 + b\sigma a_1 \\ &\quad + k_5a_0^2 + 2k_5a_{-1}a_1 + k_6a_0 + k_7, \\ \varphi^1 &: 2k_5a_0a_1 + k_6a_1, \\ \varphi^2 &: ak_3a_1 + ba_1 + k_5a_1^2. \end{aligned} \tag{25}$$

Let the coefficients be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= a_1, \\ a &= a, \\ b &= -ak_3 - a_1k_5, \\ \sigma &= \frac{\lambda}{4a_1^2k_5^2}. \end{aligned} \tag{26}$$

Set 2

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= \frac{-\lambda}{16k_5^2a_{-1}}, \\ a &= a, \\ b &= \frac{\lambda - 16aa_{-1}k_3k_5}{16k_5a_{-1}}, \\ \sigma &= \frac{16k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{27}$$

Set 3

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= 0, \\ a &= a, \\ b &= \frac{\lambda - 4aa_{-1}k_3k_5}{4k_5a_{-1}}, \\ \sigma &= \frac{4k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{28}$$

where $\lambda = 4k_5k_7 - k_6^2$.

Case 2. When $k_1k_2 < 0$, let

$$u(x, y, t) = u(\xi), \xi = ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \tag{29}$$

then Equation (21) is reduced to the ordinary differential equation as follows:

$$\left(b + k_3a + ak_4\sqrt{-\frac{k_1}{k_2}}\right)u' + k_5u^2 + k_6u + k_7 = 0. \tag{30}$$

The solution of Equation (30) is in the form of (18). $n = 1$ is taken from the homogeneous balance between the highest order derivative u' and the nonlinear term u^2. We obtain the solution of Equation (30) as Equation (24). Substituting Equation (24) together with its necessary derivatives into Equation (30), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} \varphi^{-2} &: \quad -a\sqrt{-\frac{k_1}{k_2}}k_4\sigma a_{-1} - ak_3\sigma a_{-1} - b\sigma a_{-1} + k_5a_{-1}^2, \\ \varphi^{-1} &: \quad 2k_5a_{-1}a_0 + a_{-1}k_6, \\ \varphi^0 &: \quad a\sqrt{-\frac{k_1}{k_2}}k_4\sigma a_1 + ak_3\sigma a_1 + b\sigma a_1 - a\sqrt{-\frac{k_1}{k_2}}k_4a_{-1}, \\ \varphi^1 &: \quad 2k_5a_0a_1 + k_6a_1, \end{aligned}$$

$$\varphi^2: \quad a\sqrt{-\frac{k_1}{k_2}}k_4a_1 + ak_3a_1 + ba_1 + k_5a_1^2. \tag{31}$$

Let the coefficients be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 4

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= a_1, \\ a &= a, \\ b &= -ak_3 - ak_4\sqrt{-\frac{k_1}{k_2}} - a_1k_5, \\ \sigma &= \frac{\lambda}{4a_1^2k_5^2}. \end{aligned} \tag{32}$$

Set 5

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= \frac{-\lambda}{16k_5^2a_{-1}}, \\ a &= a, \\ b &= \frac{\lambda - 16\mu}{16k_5a_{-1}}, \\ \sigma &= \frac{16k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{33}$$

Set 6

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= 0, \\ a &= a, \\ b &= \frac{\lambda - 4\mu}{4k_5a_{-1}}, \\ \sigma &= \frac{4k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{34}$$

where $\mu = aa_{-1}k_3k_5 + aa_{-1}k_4k_5\sqrt{-k_1/k_2}$.

We find that a_{-1}, a_0, a_1, and σ are equal in set 1 and set 4, set 2 and set 5, and set 3 and set 6, respectively. In this study, the solutions of differential equations are symbolized as $u_{(i,j)(x,y,t)}$, $(i, j \in Z^+)$, where i denotes obtained set number and j is the solution number of the Riccati equation, respectively. Thus, using set 1 to set 6, we obtain the solution of Equation (21) as $u_{i,j}(x, y, t)$, $(i = 1, 2, \cdots, 6\,; j = 1, 2, \cdots, 5)$. $u_{i,j}(x, y, t)$ is the following:

When $k_1k_2 > 0$, $\lambda = 4k_5k_7 - k_6^2 < 0$, we have $\sigma < 0$, then

$$u_{1,1}(x,y,t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\tanh\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(ak_3 + a_1k_5)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{1,2}(x,y,t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\coth\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(ak_3 + a_1k_5)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$\begin{aligned} u_{2,1}(x,y,t) = &-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} \\ &- \frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right], \end{aligned}$$

$$\begin{aligned} u_{2,2}(x,y,t) = &-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} \\ &- \frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right], \end{aligned}$$

$$u_{3,1}(x,y,t) = \frac{a_1\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\coth\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(\lambda - 4aa_{-1}k_3k_5)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5},$$

$$u_{3,2}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\tanh\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y-\frac{(\lambda-4aa_{-1}k_3k_5)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}. \tag{35}$$

When $k_1k_2<0$, $\lambda=4k_5k_7-k_6^2<0$, we have $\sigma<0$, then

$$u_{4,1}(x,y,t)=-\frac{k_6}{2k_5}-\frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\tanh\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y-\frac{\left(ak_3+ak_4\sqrt{-k_1/k_2}+a_1k_5\right)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{4,2}(x,y,t)=-\frac{k_6}{2k_5}-\frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\coth\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y-\frac{\left(ak_3+ak_4\sqrt{-k_1/k_2}+a_1k_5\right)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,1}(x,y,t)=-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,2}(x,y,t)=-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{6,1}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\coth\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-4\mu)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5},$$

$$u_{6,2}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\tanh\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-4\mu)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}. \tag{36}$$

When $k_1k_2<0$ and $\lambda=4k_5k_7-k_6^2<0$, we have $\sigma<0$, then

$$u_{1,3}(x,y,t)=-\frac{k_6}{2k_5}+\frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\tan\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(ak_3+a_1k_5)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{1,4}(x,y,t)=-\frac{k_6}{2k_5}+\frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\cot\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(ak_3+a_1k_5)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{2,3}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}+\frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{2,4}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{3,3}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{2|a_1k_5a_{-1}|}\cot\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{\lambda}}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-4aa_{-1}k_3k_5)t^{\alpha}}{4k_5a_{-1}(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}, \tag{37}$$

$$u_{3,4}(x,y,t)=-(a_{-1}\sqrt{\lambda}/2\ |a_1k_5a_{-1}|)\tan\left[(2|a_1k_5a_{-1}|/\sqrt{\lambda})(ax+ia\sqrt{k_1/k_2}y+((\lambda-4aa_{-1}k_3k_5)t^{\alpha}/4k_5a_{-1}(1+\alpha)))\right]-k_6/2k_5,$$

When $k_1k_2 < 0$ and $\lambda = 4k_5k_7 - k_6^2 < 0$, we have $\sigma < 0$, then

$$u_{4,3}(x,y,t) = -\frac{k_6}{2k_5} + \frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\tan\left[\frac{\sqrt{\lambda}}{4|a_1k_5|}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y - \frac{\left(ak_3 + ak_4\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{4,4}(x,y,t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\cot\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y - \frac{\left(ak_3 + ak_4\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,3}(x,y,t) = -\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} + \frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\cdot\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,4}(x,y,t) = -\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} - \frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\cdot\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{6,3}(x,y,t) = \frac{a_{-1}\sqrt{\lambda}}{2|k_5a_{-1}|}\cot\left[\frac{2|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-4\mu)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5},$$

$$u_{6,4}(x,y,t) = \frac{a_{-1}\sqrt{\lambda}}{2|k_5a_{-1}|}\tan\left[\frac{2|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-4\mu)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5}. \tag{38}$$

When $\lambda = 4k_5k_7 - k_6^2 = 0$, we have $\sigma = 0$, then

$$\begin{aligned}
u_{1,5}(x,y,t) &= -\frac{k_6}{2k_5} - \frac{\Gamma(1+\alpha)a_1}{\left[ax + ia\sqrt{(k_1/k_2)}y - \left((ak_3 + a_1k_5)t^\alpha/\Gamma(1+\alpha)\right)\right]^\alpha + \omega},\\
u_{2,5}(x,y,t) &= -\frac{a_{-1}\left[ax + ia\sqrt{(k_1/k_2)}y - (ak_3t^\alpha/\Gamma(1+\alpha))\right]^\alpha + a_{-1}\omega}{\Gamma(1+\alpha)} - \frac{k_6}{2k_5} = u_{3,5}(x,y,t),\\
u_{4,5}(x,y,t) &= -\frac{k_6}{2k_5}\frac{a_1\Gamma(1+\alpha)}{\left[ax + a\sqrt{(-k_1/k_2)}y - \left(\left(ak_3 + ak_3\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha/\Gamma(1+\alpha)\right)\right]^\alpha + \omega},\\
u_{5,5}(x,y,t) &= \frac{-a_{-1}}{\Gamma(1+\alpha)}\left[a^\alpha\left(x + \sqrt{(-k_1/k_2)}y - \frac{\left(k_3 + k_4\sqrt{-k_1/k_2}\right)t^\alpha}{\Gamma(1+\alpha)}\right)^\alpha + \omega\right] - \frac{k_6}{2k_5}.
\end{aligned} \tag{39}$$

Solutions $u_{i,1}, u_{i,2}(i = 1, 2, \cdots, 6)$ describe the soliton. Solitons exist everywhere in the nature; they are special kinds of solitary waves. $u_{2,1}, u_{2,2}, u_{5,1}$, and $u_{5,2}$ describe the multiple soliton solutions. Solutions $u_{i,3}$ and $u_{i,4}$ $(i = 1, 2, \cdots, 6)$ represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions.

Figures 1–6 present the solutions: $u_{1,1}, u_{2,1}, u_{1,3}, u_{2,3}$, $u_{1,5}$, and $u_{5,5}$ of the generalized time fractional biological population model with $-10 < x < 10, 0 < t < 10$. Solution $u_{1,1}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0$, $k_5 = 2, k_6 = 3, k_7 = 1, a_1 = 1, a = 2$, and $y = 0$; solution $u_{2,1}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2$, $k_6 = 3, k_7 = 1, a_{-1} = 1, a = 2$, and $y = 0$; solution $u_{1,3}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2, k_6 = 2, k_7 = 1, a_1 = 1, a = 2$, and $y = 0$; solution $u_{2,3}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = \mathrm{k}_4 = 0, k_5 = 2, k_6 = 3, k_7 =$

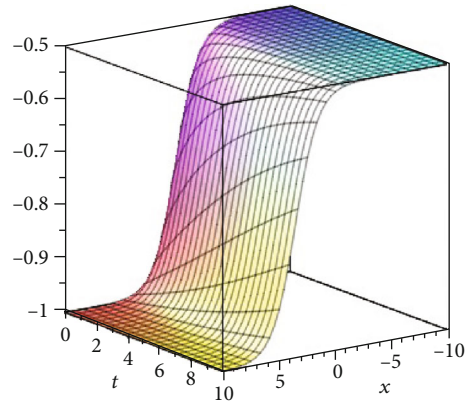

Figure 1: $u_{1,1}(x, 0, t)$ of Equation (21).

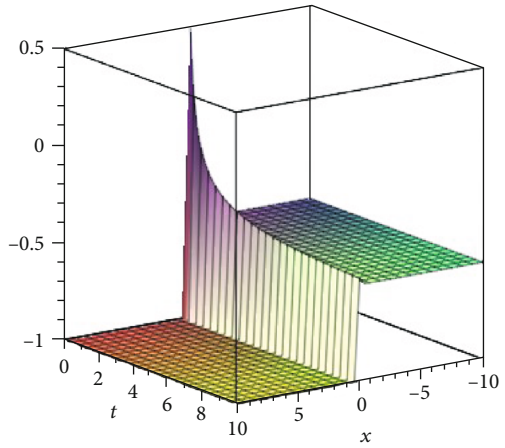

Figure 2: $u_{2,1}(x, 0, t)$ of Equation (21).

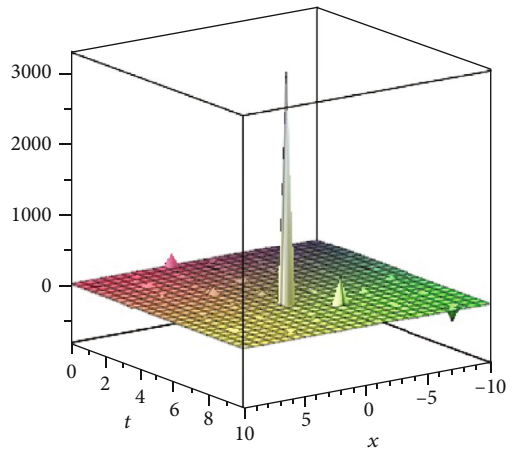

Figure 3: $u_{1,3}(x, 0, t)$ of Equation (21).

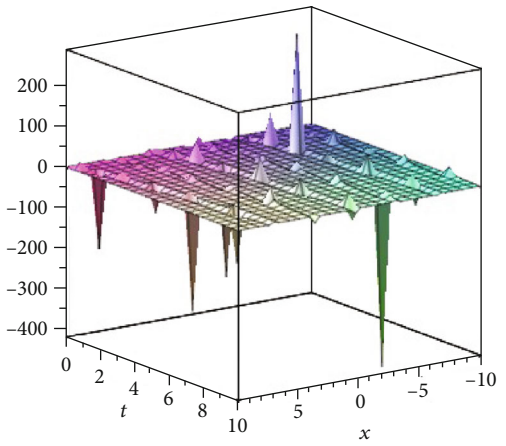

Figure 4: $u_{2,3}(x, 0, t)$ of Equation (21).

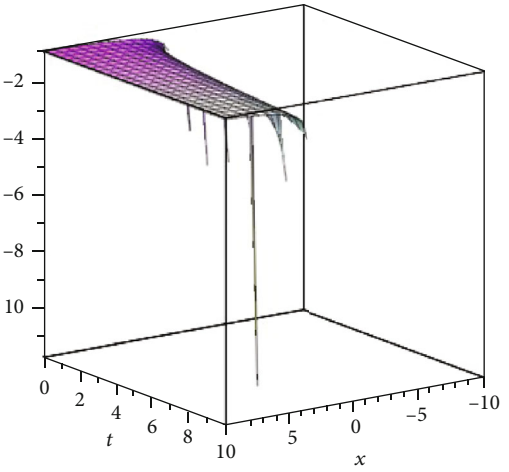

Figure 5: $u_{1,5}(x, 0, t)$ of Equation (21).

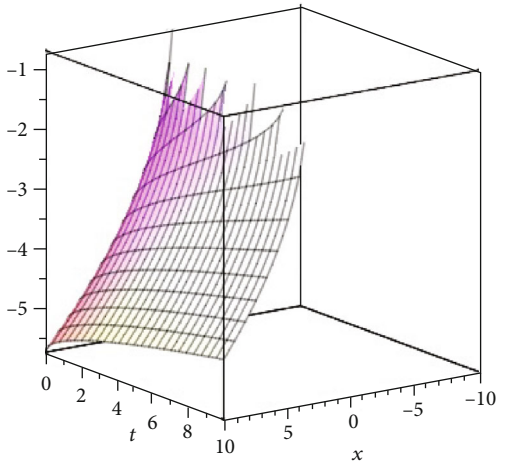

Figure 6: $u_{5,5}(x, 0, t)$ of Equation (21).

1, $a_{-1} = 1$, $a = 2$, and $y = 0$; solution $u_{1,5}$ is presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = 0$, $k_5 = 2$, $k_6 = \sqrt{8}$, $k_7 = 1$, $a = 2$, $a_1 = 1$, $y = 0$, and $\omega = 0$; solution $u_{5,5}$ is presented for values $\alpha = 0.5$, $k_1 = 1$, $k_2 = -1$, $k_3 = 0$, $k_4 = 1$, $k_5 = 2$, $k_6 = \sqrt{8}$, $k_7 = 1$, $a_1 = 1$, $a = 2$, $y = 0$, and $\omega = 0$.

When $k_1 = -1$, $k_2 = -1$, $k_3 = k_4 = k_6 = 0$, $k_5 = -h \neq 0$, and $k_7 = rh \neq 0$, Equation (21) is the time fractional biological population model [39]

$$D_t^{\alpha} u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + h\left(u^2 - r\right). \tag{40}$$

We denote

$$u(x,y,t)=u(\xi),\xi=ax+iay+\frac{bt^{\alpha}}{\Gamma(1+\alpha)},a\neq 0, \tag{41}$$

and find the solution of Equation (40) in the form of $u(\xi)=a_{-1}\varphi^{-1}+a_0+a_1\varphi$. Then, by solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned} a_{-1}&=0,\\ a_0&=0,\\ a_1&\neq 0,\\ \sigma&=-\frac{r}{a_1^2}. \end{aligned} \tag{42}$$

Let $a_1=c/h$, then $\sigma=-rh^2/c^2$, we can obtain set 1 in [42].

Set 2

$$\begin{aligned} a_{-1}&\neq 0,\\ a_0&=0,\\ a_1&=\frac{r}{4a_{-1}},\\ \sigma&=-\frac{4a_{-1}^2}{r}. \end{aligned} \tag{43}$$

Let $a_{-1}=rh/4c$, then $\sigma=-rh^2/4c^2$, we can obtain set 3 in [42].

Set 3

$$\begin{aligned} a_{-1}&\neq 0,\\ a_0&=0,\\ a_1&=0,\\ \sigma&=-\frac{a_{-1}^2}{r}. \end{aligned} \tag{44}$$

Let $a_{-1}=rh/c$, then $\sigma=-rh^2/c^2$, we can obtain set 2 in [42].

Clearly, we get more solutions of the time fractional biological population model (40) than the literature [42].

3.2. The Generalized Compound KdV-Burgers Equation. The generalized compound KdV-Burgers equation is given by

$$D_t^{\alpha}u+k_1uu_x+k_2u^2u_x+k_3u_{xx}+k_4u_{xxx}=0, \tag{45}$$

Let

$$u(x,t)=u(\xi),\xi=ax+\frac{bt^{\alpha}}{\Gamma(1+\alpha)},a\neq 0, \tag{46}$$

then Equation (45) is reduced to ordinary differential equation as

$$bu'+k_1auu'+k_2au^2u'+k_3a^2u''+k_4a^3u'''=0. \tag{47}$$

The solution of Equation (45) is in the form of (18), and here, $n=1$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term u^2u'. We obtain the solution of Equation (45) as follows:

$$u(\xi)=a_{-1}\varphi^{-1}+a_0+a_1\varphi. \tag{48}$$

Substituting Equation (48) together with its necessary derivatives into Equation (47), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} &\varphi^{-4}:-6a^3k_4\sigma^3a_{-1}-ak_2\sigma a_{-1}^3,\\ &\varphi^{-3}:2a^2k_3\sigma^2a_{-1}-2ak_2\sigma a_{-1}^2a_0-ak_1\sigma a_{-1}^2,\\ &\varphi^{-2}:b\sigma a_{-1}-ak_1\sigma a_{-1}a_0-ak_2a_{-1}^3-ak_2\sigma a_{-1}a_0^2\\ &\qquad -ak_2\sigma a_{-1}^2a_1-8a^3k_4\sigma^2a_{-1,}\\ &\varphi^{-1}:-ak_1a_{-1}^2-2ak_2a_{-1}^2a_0+2a^2k_3\sigma a_{-1,} \end{aligned} \tag{49}$$

$$\begin{aligned} &\varphi^{0}:ba_{-1}-b\sigma a_1+ak_1\sigma a_0a_1-ak_1a_{-1}a_0\\ &\qquad +ak_2\sigma a_0^2a_1+ak_2\sigma a_{-1}a_1^2-ak_2a_0^2a_{-1}\\ &\qquad -ak_2a_{-1}^2a_1+2a^3k_4\sigma^2a_1-2a^3k_4\sigma a_{-1},\\ &\varphi^{1}:ak_1\sigma a_1^2+2ak_2\sigma a_0a_1^2+2a^2k_3\sigma a_1,\\ &\varphi^{2}:-ba_1+ak_1a_0a_1+ak_2a_0^2a_1+ak_2a_{-1}a_1^2\\ &\qquad +ak_2\sigma a_1^3+8a^3k_4\sigma a_1,\\ &\varphi^{3}:2ak_2a_0a_1^2+2a^2k_3a_1+ak_1a_1^2,\\ &\varphi^{4}:6a^3k_4a_1+ak_2a_1^3. \end{aligned} \tag{50}$$

Let the coefficients of $\varphi^k(\xi)$ to be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

When $k_2k_4<0$, it denotes $\mu=3ak_1^2k_4+2ak_2k_3^2-12bk_2k_4$ and $\lambda_1=\pm\sqrt{-6k_4/k_2}$, $\lambda_2=-k_1/2\pm((|k_3\mid\sqrt{-6k_2k_4})/6k_4)$, we have

Set 1

$$\begin{aligned} a_{-1}&=-\frac{\lambda_1\mu}{96a^2k_2k_4^2},\\ a_0&=-\frac{\lambda_1k_1+2k_3}{2\lambda_1k_2},\\ a_1&=a\lambda_1,\\ a&=a,\\ b&=b,\\ \sigma&=\frac{\mu}{96a^3k_2k_4^2}. \end{aligned} \tag{51}$$

Set 2

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2}, \\ a_1 &= a\lambda_1, \\ a &= a, \\ b &= b, \\ \sigma &= \frac{\mu}{24a^3 k_2 k_4^2}. \end{aligned} \tag{52}$$

Set 3

$$\begin{aligned} a_{-1} &= -\frac{\mu k_3}{12a^2 k_2 k_4^2 (2\lambda_2 + k_1)}, \\ a_0 &= \frac{\lambda_2}{k_2}, \\ a_1 &= 0, \\ a &= a, \\ b &= b, \\ \sigma &= \frac{\mu}{24a^3 k_2 k_4^2}. \end{aligned} \tag{53}$$

Thus, we obtain the solution of Equation (45) as $u_{i,j}(x,t)$, $(i = 1, 2, 3\,; j = 1, 2, 3, 4, 5)$. Using set 1 to set 3, $u_{i,j}(x,t)$ is as follows:

When $k_2 k_4 < 0$ and $a\mu k_2 < 0$, we have $\sigma < 0$, then

$$\begin{aligned} u_{1,1}(x,t) = {} & \frac{\lambda_1 \mu \mid a \mid}{24a^2 k_4} \sqrt{\frac{-6a}{\mu k_2}} \coth \left[\frac{1}{4 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \tanh \left[\frac{1}{4 \mid ak_4 \mid} \right. \\ & \left. \cdot \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \end{aligned}$$

$$\begin{aligned} u_{1,2}(x,t) = {} & \frac{\lambda_1 \mu \mid a \mid}{24a^2 k_4} \sqrt{\frac{-6a}{\mu k_2}} \tanh \left[\frac{1}{4 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \coth \left[\frac{1}{4 \mid ak_4 \mid} \right. \\ & \left. \cdot \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \end{aligned}$$

$$\begin{aligned} u_{2,1}(\mathrm{x},t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \tanh \\ & \cdot \left[\frac{1}{2 \mid ak_4 \mid} \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \end{aligned}$$

$$u_{2,2}(x,t) = -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \coth,$$

$$u_{3,1}(x,t) = \frac{\mu k_3 |a|}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{-6a}{\mu k_2}} \coth + \frac{\lambda_2}{k_2},$$

$$u_{3,2}(x,t) = \frac{\mu k_3 |a|}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{-6a}{\mu k_2}} \tanh + \frac{\lambda_2}{k_2}. \tag{54}$$

When $k_2 k_4 < 0$ and $a\mu k_2 > 0$, we have $\sigma > 0$, then

$$\begin{aligned} u_{1,3}(x,t) = {} & -\frac{\lambda_1 \mu \mid a \mid}{24a^2 k_4} \sqrt{\frac{6a}{\mu k_2}} \cot \left[\frac{1}{4 \mid ak_4 \mid} \sqrt{\frac{\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & + \frac{a\lambda_1}{4 \mid ak_4 \mid} \sqrt{\frac{\mu}{6ak_2}} \tan \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \end{aligned}$$

$$\begin{aligned} u_{1,4}(x,t) = {} & \frac{\lambda_1 \mu \mid a \mid}{24a^2 k_4} \sqrt{\frac{6a}{\mu k_2}} \tan \left[\frac{1}{4 \mid ak_4 \mid} \sqrt{\frac{\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cot \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \right. \\ & \left. \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \end{aligned}$$

$$\begin{aligned} u_{2,3}(x,t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} + \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \tan \\ & \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \end{aligned}$$

$$\begin{aligned} u_{2,4}(x,t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \coth \\ & \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \end{aligned}$$

$$\begin{aligned} u_{3,3}(x,t) = {} & -\frac{\mu k_3}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{6a^3}{\mu k_2}} \cot \\ & \cdot \left[\frac{1}{2|k_4|} \sqrt{\frac{\mu}{6a^3 k_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] + \frac{\lambda_2}{k_2}, \end{aligned}$$

$$u_{3,4}(x,t) = \frac{\mu k_3 |a|}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{6a}{\mu k_2}} \tan \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] + \frac{\lambda_2}{k_2}. \tag{55}$$

When $k_2 k_4 < 0$ and $\mu = 0$, we have $\sigma = 0$, then

$$u_{1,5}(x,t) = -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} + \frac{a\lambda_1 \Gamma(1+\alpha)}{(ax + (bt^\alpha/\Gamma(1+\alpha)))^\alpha + \omega} = u_{2,5}(x,t). \tag{56}$$

Solutions $u_{i,1}$ and $u_{i,2}$ $(i = 1, 2, 3)$ described the soliton. $u_{1,1}$ and $u_{1,2}$ describe the multiple soliton solutions. Solutions $u_{i,3}$ and $u_{i,4}$ $(i = 1, 2, 3)$ represent the exact periodic traveling wave solutions.

Figures 7–10 present the solutions: $u_{1,1}$, $u_{2,1}$, $u_{1,3}$, and $u_{2,3}$ of the generalized compound KdV-Burgers equation with $-10 < x < 10, 0 < t < 1$. Solutions $u_{1,1}$ and $u_{2,1}$ are presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = -1, a = -4, b = 1$, and $y = 0$; solutions $u_{1,3}$ and $u_{2,3}$ are presented for values $\alpha = 0.5$, $k_1 = k_2 = 1, k_3 = k_4 = -1, a = 4$, b = 1, and $y = 0$.

3.3. The Space-Time Fractional Regularized Long-Wave Equation. Consider the space-time fractional regularized long-wave equation as follows:

$$D_t^\alpha u + k_1 D_x^\alpha u + k_2 u D_x^\alpha u + k_3 D_t^\alpha D_x^{2\alpha} u = 0. \tag{57}$$

Let

$$u(x,t) = u(\xi), \xi = \frac{ax^\alpha}{\Gamma(1+\alpha)} + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \tag{58}$$

then Equation (57) is reduced to the following ordinary differential equation:

$$bu' + k_1 au' + k_2 auu' + k_3 a^2 bu''' = 0. \tag{59}$$

The solution of Equation (57) is in the form of (18), and here, $n = 2$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term uu'. We obtain the solution of Equation (57) as follows:

$$u(\xi) = a_{-2}\varphi^{-2} + a_{-1}\varphi^{-1} + a_0 + a_1\varphi + a_2\varphi^2. \tag{60}$$

Substituting Equation (60) together with its necessary derivatives into Equation (57), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned}
&\varphi^{-5} : -24k_3 a^2 b a_{-2}\sigma^3 - 2k_2 a a_{-2}^2 \sigma,\\
&\varphi^{-4} : -6k_3 a^2 b a_{-1}\sigma^3 - 3k_2 a a_{-1} a_{-2}\sigma,\\
&\varphi^{-3} : -2k_1 a a_{-2}\sigma - 2b a_{-2}\sigma - 2k_2 a a_0 a_{-2}\sigma - k_2 a a_{-1}^2 \sigma - 2k_2 a a_{-2}^2 - 40k_3 a^2 b a_{-2}\sigma^2,\\
&\varphi^{-2} : -k_1 a a_{-1}\sigma - b a_{-1}\sigma - k_2 a a_1 a_{-2}\sigma - 3k_2 a a_{-1} a_{-2} - k_2 a a_0 a_{-1}\sigma - 8k_3 a^2 b a_{-1}\sigma^2,\\
&\varphi^{-1} : -2b a_{-2} - 2k_1 a a_{-2} - k_2 a a_{-1}^2 - 2k_2 a a_0 a_{-2} - 16k_3 a^2 b a_{-2}\sigma,\\
&\varphi^{0} : k_1 a a_1 \sigma - k_1 a a_{-1} + a_1 b\sigma - a_{-1} b - k_2 a a_1 a_{-2} + k_2 a a_{-1} a_2 \sigma + k_2 a a_0 a_1 \sigma - k_2 a a_0 a_{-1} + 2k_3 a^2 b a_1 \sigma^2 - 2k_3 a^2 b a_{-1}\sigma,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\varphi^{1} : 2k_1 a a_2 \sigma + 2b a_2 \sigma + 2k_2 a a_0 a_2 \sigma + k_2 a a_1^2 \sigma + 16k_3 a^2 b a_2 \sigma^2,\\
&\varphi^{2} : k_1 a a_1 + b a_1 + k_2 a a_2 a_{-1} + k_2 a a_1 a_0 + 3k_2 a a_1 a_2 \sigma + 8k_3 a^2 b a_1 \sigma,\\
&\varphi^{3} : 2k_1 a a_2 + 2b a_2 + 2k_2 a a_2^2 \sigma + 2k_2 a a_0 a_2 + k_2 a a_1^2 + 40k_3 a^2 b a_2 \sigma,\\
&\varphi^{4} : 6k_3 a^2 b a_1 + 3k_2 a a_1 a_2,\\
&\varphi^{5} : 24k_3 a^2 b a_2 + 2k_2 a a_2^2.
\end{aligned} \tag{62}$$

Let the coefficients of $\varphi^k(\xi)$ tbe zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned}
a_{-2} &= 0,\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= -\frac{12k_3 ab}{k_2},\\
a &= a,\\
b &= b,\\
\sigma &= -\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}.
\end{aligned} \tag{63}$$

Set 2

$$a_{-2} = -\frac{3(k_1 a + k_2 a_0 a + b)^2}{16k_2 k_3 a^3 b},$$

$$a_{-1} = 0,$$

$$a_0 = a_0,$$

$$
\begin{aligned}
a_1 &= 0,\\
a_2 &= 0,\\
a &= a,\\
b &= b,\\
\sigma &= -\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}.
\end{aligned}
\tag{64}
$$

Set 3

$$
\begin{aligned}
a_{-2} &= -\frac{3(k_1 a + k_2 a_0 a + b)^2}{16k_2 k_3 a^3 b},\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= -\frac{12k_3 ab}{k_2},\\
a &= a,\\
b &= b,\\
\sigma &= -\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}.
\end{aligned}
\tag{65}
$$

Thus, we obtain the solution of Equation (57) as $u_{i,j}(\xi)$, $(i = 1, 2, 3\,; j = 1, 2, 3, 4, 5)$ and $\xi = (ax^{\alpha}/\Gamma(1+\alpha)) + (bt^{\alpha}/\Gamma(1+\alpha))$; $u_{i,j}(\xi)$ is as follows:

When $(k_1 a + k_2 a_0 a + b)/(8k_3 a^2 b) > 0$, we have $\sigma < 0$, then

$$
\begin{aligned}
u_{1,1}(\xi) &= a_0 - \frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a}\tanh^2\\
&\quad\cdot\left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) = u_{2,2}(\xi),\\
u_{2,1}(\xi) &= -\frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a}\coth^2\\
&\quad\cdot\left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) + a_0 = u_{1,2}(\xi),\\
u_{3,1}(\xi) &= -\frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a}\coth^2\left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right)\\
&\quad + a_0 - \frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a}\tanh^2\\
&\quad\cdot\left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) = u_{3,2}(\xi).
\end{aligned}
\tag{66}
$$

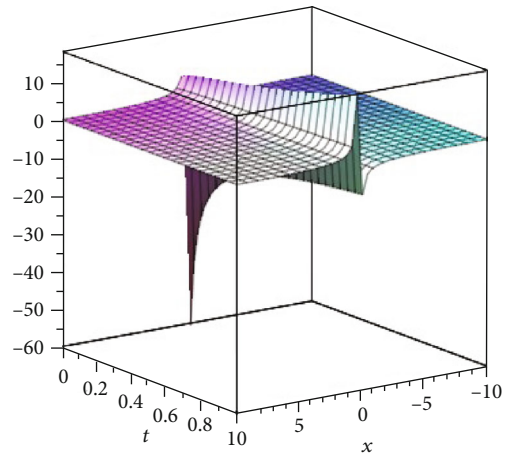

Figure 7: $u_{1,1}(x, t)$ of Equation (45).

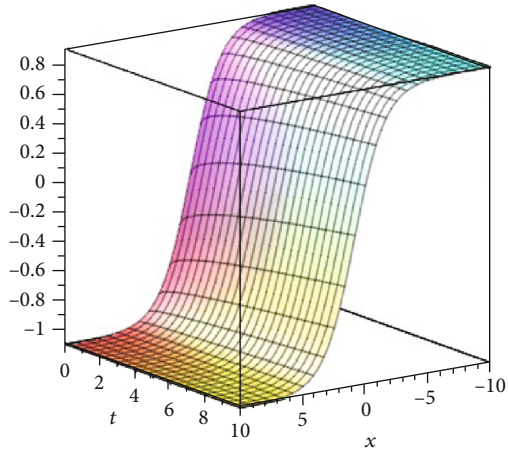

Figure 8: $u_{2,1}(x, t)$ of Equation (45).

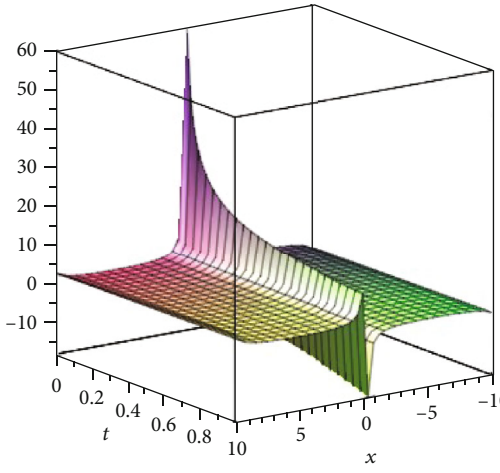

Figure 9: $u_{1,3}(x, t)$ of Equation (45).

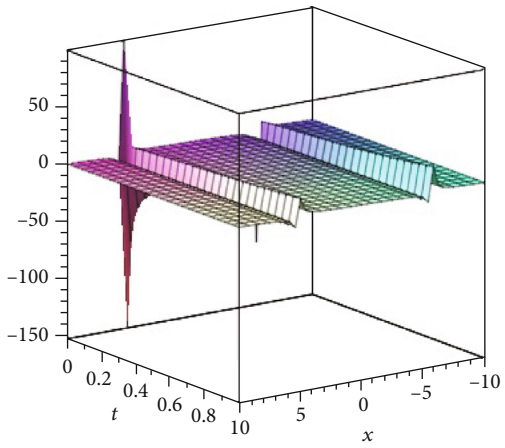

Figure 10: $u_{2,3}(x,t)$ of Equation (45).

When $(k_1a + k_2a_0a + b)/(8k_3a^2b) < 0$, we have $\sigma > 0$, then

$$u_{1,3}(\xi) = a_0 + \frac{3(k_1a + k_2a_0a + b)}{2k_2a} \tan^2 \cdot \left(\sqrt{-\frac{k_1a + k_2a_0a + b}{8k_3a^2b}}\xi\right) = u_{2,4}(\xi),$$

$$u_{2,3}(\xi) = \frac{3(k_1a + k_2a_0a + b)}{2k_2a} \cot^2\left(\sqrt{-\frac{k_1a + k_2a_0a + b}{8k_3a^2b}}\xi\right) + a_0 = u_{1,4}(\xi),$$

$$u_{3,3}(\xi) = \frac{3(k_1a + k_2a_0a + b)}{2k_2a} \cot^2\left(\sqrt{-\frac{k_1a + k_2a_0a + b}{8k_3a^2b}}\xi\right) + a_0 + \frac{3(k_1a + k_2a_0a + b)}{2k_2a} \tan^2 \cdot \left(\sqrt{-\frac{k_1a + k_2a_0a + b}{8k_3a^2b}}\xi\right) = u_{3,4}(\xi). \quad (67)$$

When $(k_1a + k_2a_0a + b)/(8k_3a^2b) = 0, a_{-2} = 0$, we have $\sigma = 0$, then

$$u_{1,5}(\xi) = a_0 - \frac{12k_3ab\Gamma^2(1+\alpha)}{k_2\left(\xi^\alpha + \omega\right)^2} = u_{3,5}(\xi). \quad (68)$$

Solutions $u_{i,1}$ and $u_{i,2}(i = 1, 2, 3)$ describe the multiple soliton. Solutions $u_{i,3}$ and $u_{i,4}(i = 1, 2, 3)$ represent the exact periodic traveling wave solutions.

Figures 11–14 present the solutions: $u_{1,1}, u_{2,1}, u_{2,3}$, and $u_{1,5}$ of the generalized compound KdV-Burgers equation with $0 < x < 1, 0 < t < 2$. Solution $u_{1,1}$ and solution $u_{2,1}$ are presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = -1, a = -4, b = 1, a_0 = 0$, and $y = 0$; $u_{2,3}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = -1, a = 4, b = 1, a_0 = 0$, and $y = 0$; solution $u_{1,5}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = -1, a = 4, b = -4, a_0 = 0, \omega = 0$, and $y = 0$.

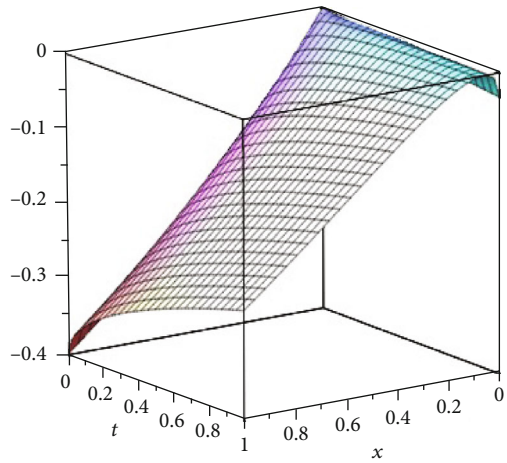

Figure 11: $u_{1,1}(x,t)$ of Equation (57).

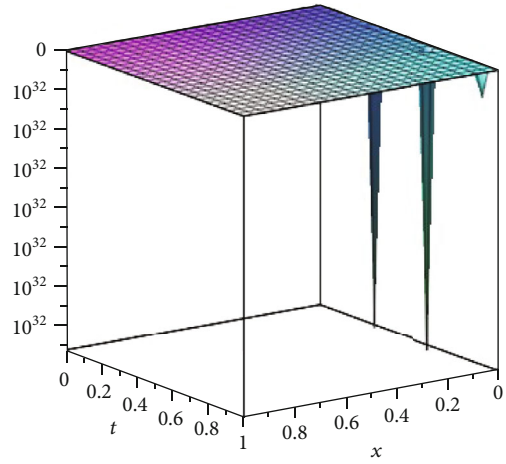

Figure 12: $u_{2,1}(x,t)$ of Equation (57).

3.4. The (3 + 1)-Space-Time Fractional Zakharov-Kuznetsov Equation. The (3 + 1)-space-time fractional Zakharov-Kuznetsov Equation is given by

$$D_t^\alpha u + k_1 u D_x^\alpha u + k_2 D_x^{3\alpha} u + k_3 D_x^\alpha D_y^{2\alpha} u + k_4 D_x^\alpha D_z^{2\alpha} u = 0, \quad (69)$$

Let

$$u(x,t) = u(\xi), \xi = \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{bx^\alpha}{\Gamma(1+\alpha)} + \frac{cy^\alpha}{\Gamma(1+\alpha)} + \frac{dz^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \quad (70)$$

then Equation (69) is reduced to the ordinary differential equation as

$$au' + k_1 buu' + k_2 b^3 u''' + k_3 bc^2 u''' + k_4 bd^2 u''' = 0. \quad (71)$$

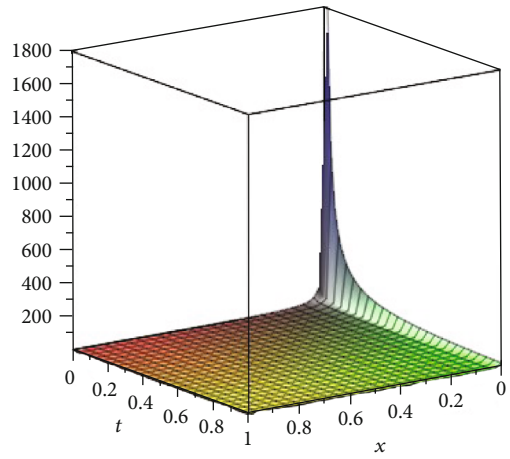

Figure 13: $u_{2,3}(x,t)$ of Equation (57).

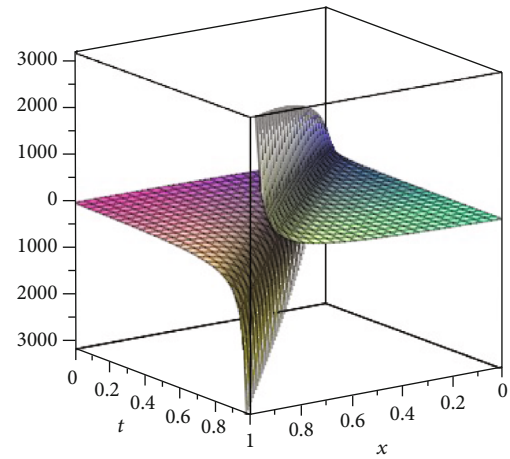

Figure 14: $u_{1,5}(x,t)$ of Equation (57).

The solution of Equation (71) is the form (18), and here, $n=2$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term uu'. We obtain the solution of Equation (71) as

$$u(\xi) = a_{-2}\varphi^{-2} + a_{-1}\varphi^{-1} + a_0 + a_1\varphi + a_2\varphi^2. \tag{72}$$

Substituting Equation (72) together with its necessary derivatives into Equation (71), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned}
\varphi^{-5} &: -2k_1 b a_{-2}^2\sigma + 24\lambda a_{-2}\sigma^3,\\
\varphi^{-4} &: -3k_1 b a_{-1}a_{-2}\sigma + 6\lambda a_{-1}\sigma^3,\\
\varphi^{-3} &: -2aa_{-2}\sigma - 2k_1 b a_0 a_{-2}\sigma - k_1 b a_{-1}^2\sigma\\
&\quad - 2k_1 b a_{-2}^2 + 40\lambda a_{-2}\sigma^2,\\
\varphi^{-2} &: -aa_{-1}\sigma - 3k_1 b a_{-1}a_{-2} - k_1 b a_0 a_{-1}\sigma\\
&\quad - k_1 b a_1 a_{-2}\sigma + 8\lambda a_{-1}\sigma^2,\\
\varphi^{-1} &: -2aa_{-2} - k_1 b a_{-1}^2 - 2k_1 b a_0 a_{-2} + 16\lambda a_{-2}\sigma,\\
\varphi^{0} &: aa_1\sigma - aa_{-1} - k_1 b a_{-2}a_1 + k_1 b a_{-1}a_2\sigma\\
&\quad + k_1 b a_0 a_1\sigma - k_1 b a_0 a_{-1} - 2\lambda a_1\sigma^2 + 2\lambda a_{-1}\sigma,\\
\varphi^{1} &: 2aa_2\sigma + 2k_1 b a_0 a_2\sigma + k_1 b a_1^2\sigma - 16\lambda a_2\sigma^2,\\
\varphi^{2} &: aa_1 + k_1 b a_{-1}a_2 + k_1 b a_0 a_1 + 3k_1 b a_1 a_2\sigma - 8\lambda a_1\sigma,\\
\varphi^{3} &: 2aa_2 + 2k_1 b a_2^2\sigma + 2k_1 b a_0 a_2 + k_1 b a_1^2 - 40\lambda a_2\sigma,\\
\varphi^{4} &: 3k_1 b a_1 a_2 - 6\lambda a_1,\\
\varphi^{5} &: 2k_1 b a_2^2 - 24\lambda a_2.
\end{aligned} \tag{73}$$

where $\lambda = -k_2 b^3 - k_3 b c^2 - k_4 b d^2$.

Let the coefficients of $\varphi^k(\xi)$ be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned}
a_{-2} &= 0,\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= \frac{12\lambda}{k_1},\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a + k_1 a_0 b}{8b\lambda}.
\end{aligned} \tag{74}$$

Set 2

$$\begin{aligned}
a_{-2} &= \frac{3(a + k_1 a_0 b)^2}{16k_1 b^2\lambda},\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= 0,\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a + k_1 a_0 b}{8b\lambda}.
\end{aligned} \tag{75}$$

Set 3

$$
\begin{aligned}
a_{-2} &= \frac{3(a+k_1a_0b)^2}{16k_1b^2\lambda},\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= \frac{12\lambda}{k_1},\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a+k_1a_0b}{8b\lambda}.
\end{aligned}
\tag{76}
$$

Thus, we obtain the solution of Equation (59) as $u_{i,j}(\xi)$, $(i = 1, 2, 3\,; j = 1, 2, 3, 4, 5)$ and $\xi = (at^\alpha/\Gamma(1+\alpha)) + (bx^\alpha/\Gamma(1+\alpha)) + (cy^\alpha/\Gamma(1+\alpha)) + (dz^\alpha/\Gamma(1+\alpha))$; $u_{i,j}(\xi)$ is tas follows:

When $(a + k_1a_0b)/(8b\lambda) < 0$, we have $\sigma < 0$, then

$$
\begin{aligned}
u_{1,1}(\xi) &= a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,1}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right) + a_0,\\
u_{3,1}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{1,2}(\xi) &= a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,2}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right) + a_0,\\
u_{3,2}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right).
\end{aligned}
\tag{77}
$$

When $(a + k_1a_0b)/(8b\lambda) > 0$, we have $\sigma > 0$, then

$$
\begin{aligned}
u_{1,3}(\xi) &= a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\tanh^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,3}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right) + a_0,\\
u_{3,3}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{1,4}(\xi) &= a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,4}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right) + a_0,\\
u_{3,4}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right).
\end{aligned}
\tag{78}
$$

When $(a + k_1a_0b)/(8b\lambda) = 0$, we have $\sigma = 0$, then

$$
u_{1,5}(\xi) = a_0 + \frac{12\lambda\Gamma^2(1+\alpha)}{k_1\left(\xi^\alpha+\omega\right)^2} = u_{3,5}(\xi). \tag{79}
$$

Solutions $u_{i,1}$ and $u_{i,2}(i = 1, 2, 3)$ describe the multiple soliton. Solutions $u_{i,3}$ and $u_{i,4}$ $(i = 1, 2, 3)$ represent the exact periodic traveling wave solutions.

Figures 15–18 present the solutions: $u_{1,1}, u_{1,3}, u_{2,3}$, and $u_{1,5}$ of the generalized compound KdV-Burgers equation with $0 < x < 1, 0 < t < 2$. Solution $u_{1,1}$ is presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = -1$, $a = -4$, $b = 1$, $c = 1$, $d = 1$, $a_0 = 0$, and $y = z = 0$; solution $u_{1,3}$ and $u_{2,3}$ are presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = -1$, $a = 4$, $b = 1$, $c = 1$, $d = 1$, $a_0 = 0$, and $y = z = 0$; solution $u_{1,5}$ is presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = -1$, $a = -4$, $b = 1$, $c = 1$, $d = 1$, $a_0 = 4$, $\omega = 0$, and $y = z = 0$.

4. Results and Discussion

The generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the $(3+1)$-space-time fractional Zakharov-Kuznetsov equation have gained the focus of many studies due to their frequent appearance in various applications.

The improved fractional subequation method has several advantages according to other traditional methods. Applying a suitable fractional complex transform $u(t, x_1, x_2, \cdots, x_m) = u(\xi)$, $\xi = \xi(t, x_1, x_2, \cdots, x_m)$ and chain rule, the nonlinear fractional differential equations with the modified Riemann-Liouville derivative are converted into the nonlinear ordinary differential equations. This is a significant impact because

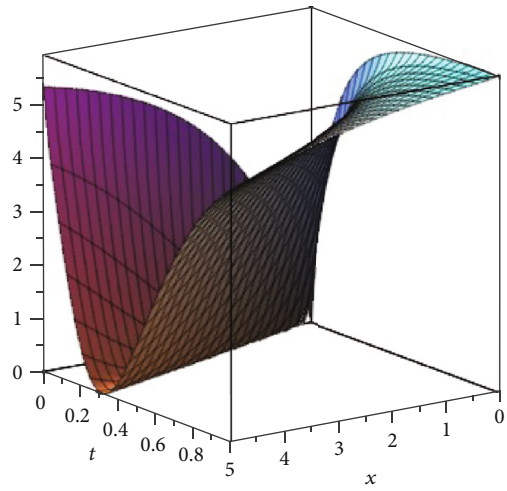

Figure 15: $u_{1,1}(x, 0, 0, t)$ of Equation (69).

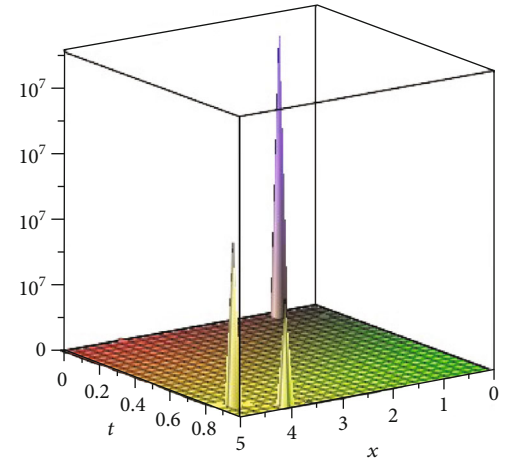

Figure 16: $u_{1,3}(x, 0, 0, t)$ of Equation (69).

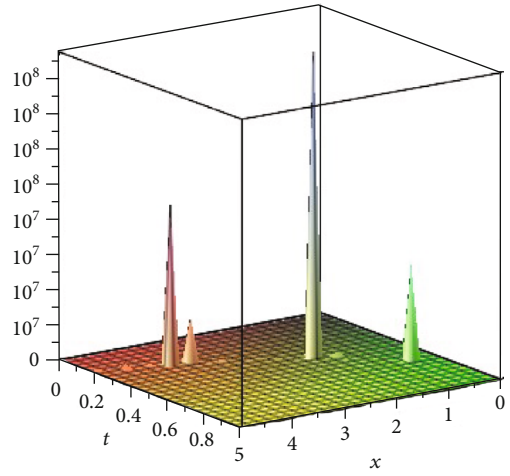

Figure 17: $u_{2,3}(x, 0, 0, t)$ of Equation (69).

neither Caputo definition nor Riemann-Liouville definition satisfies the chain rule. With the help of the Riccati equation, the method has been employed for finding the exact analytical solutions of these equations.

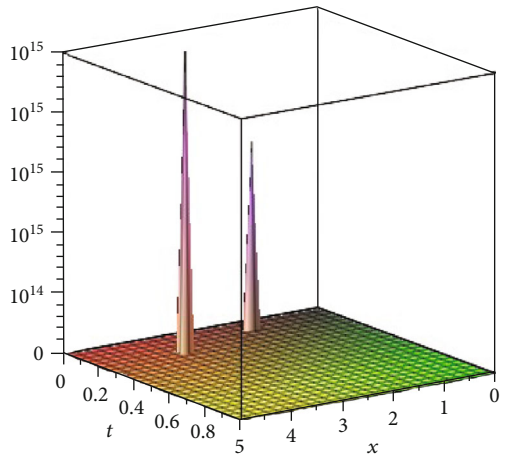

Figure 18: $u_{1,5}(x, 0, 0, t)$ of Equation (69).

These obtained solutions are traveling solutions. Furthermore, solutions $u_{i,1}(\xi)$ and $u_{i,2}(\xi)$ describe the solitons which are everywhere in nature. $u_{i,3}(\xi)$ and $u_{i,4}(\xi)$ represent the exact periodic traveling wave solutions.

The improved fractional subequation method is reliable and effective for finding more solutions for some space-time fractional nonlinear differential equations. We can substitute $\varphi' = r + p\varphi + q\varphi^2$ [39] for $\varphi' = \sigma + \varphi^2$, then, we will obtain more solutions.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Existence of Solutions for Fractional Boundary Value Problems with a Quadratic Growth of Fractional Derivative

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In this paper, we deal with two fractional boundary value problems which have linear growth and quadratic growth about the fractional derivative in the nonlinearity term. By using variational methods coupled with the iterative methods, we obtain the existence results of solutions. To the best of the authors' knowledge, there are no results on the solutions to the fractional boundary problem which have quadratic growth about the fractional derivative in the nonlinearity term.

1. Introduction

It has been seen that fractional differential equations have better effects in many realistic applications than the classical ones. Qualitative theory and its applications in physics, engineering, economics, biology, and ecology are extensively discussed and demonstrated in [1–4] and the references therein. Some recent contributions to the theory of fractional differential equation can be seen in [5–10].

Some classical tools such as fixed point theorems [5], the method of upper and lower solutions, and monotone iterative technique [11, 12] have been widely used to study the fractional differential equation. Recently, the study of fractional differential equations has attracted much attention by using variational methods, for example, [7, 8, 13–19]. We also mention that in the recent works [20, 21], the authors have developed a general approach concerting the existence of solutions.

Fractional differential equations containing left and right fractional differential operators have received attention from scientists due to their applications in physical phenomena exhibiting anomalous diffusion. In [7], appropriate fractional derivative spaces were defined and existence and uniqueness results for a fractional boundary value problem were proven using the Lax-Milgram theorem.

Jiao and Zhou [8] showed the variational structure of a fractional boundary value problem under an appropriate functional space; they used the least action principle and the Mountain Pass theorem to obtain the existence of at least one solution. Sun and Zhang [22] obtained the existence result for a fractional boundary value problem by using the Mountain Pass method and an iterative technique. In [23], the authors discussed the existence of a fractional boundary value problem with linear growth about the fractional derivative in a nonlinearity term. Compared with some integral-order partial differential equations such as [6, 24–31], the fractional derivatives have hereditary and nonlocal properties so that they are much more suitable for describing long-memory processes than the classical integer-order derivatives.

Motivated by the above papers, in this paper, we first investigated the existence of solutions for the following fractional boundary value problems:

$$\begin{cases} {}_tD^{\alpha}_{T\,0}D^{\alpha}_{t}u(t) + {}_0D^{\alpha}_{t\,t}D^{\alpha}_{T}u(t) = f(t, u(t), {}_0D^{\alpha}_{t}u(t)), \text{a.e.t} \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right fractional Riemann-Liouville derivatives of order $1/2 < \alpha < 1$, respectively, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

Note that problem (1) is not variational due to the fractional derivative contained in nonlinearity, so we cannot find a functional such that its critical point is the weak solution corresponding to (1). In order to overcome this difficulty, we consider the following fractional boundary value problem which is independent on the fractional derivative of the solution

$$\begin{cases} {}_tD_{T0}^\alpha D_t^\alpha u(t) + {}_0D_t^\alpha {}_tD_T^\alpha u(t) = f(t, u(t), {}_0D_t^\alpha w(t)), \text{ a.e.t} \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where w is an element of fractional Sobolev space E^α. First, by using variational methods, we obtain the existence of solutions for (2). Then, under the assumption that f is linear growth about the fractional derivative and based on iterative methods, we show there exists a solution for (1). Our conditions are weaker than that in [23].

We also discuss the following fractional boundary value problem:

$$\begin{cases} {}_tD_{T0}^\alpha D_t^\alpha u(t) + {}_0D_t^\alpha {}_tD_T^\alpha u(t) = a(t)g(u(t)) + \lambda|{}_0D_t^\alpha u(t)|^2, \text{ a.e.t} \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $a \in L^1(0, T; \mathbb{R}^+)$, λ is a parameter, and $g \in C(\mathbb{R}; \mathbb{R})$. Compared with (1), the nonlinearity of (3) is quadratic growth about a fractional derivative. By using variational methods and an iterative technique, we obtain that there exists solutions for (3) when λ and a satisfy suitable conditions. To the best of the authors' knowledge, there are no results on the solutions to the fractional boundary problem which have quadratic growth about the fractional derivative in the nonlinearity term .

The paper is organized as follows. In Section 2, we will list some important properties of the basic functional space. We show the existence results for (1) and (3) in Section 3 and Section 4, respectively.

2. Preliminary

Let us briefly recall the property of a fractional derivative which will be used to construct the variational functional.

Lemma 1 [1]. *For $0 < \beta \le 1$, if $f \in L^p([a, b]\mathbb{R})$ with $p(1-\beta) > 1$, ${}_0J_t^{1-\beta} f(t)$ is absolutely continuous, and $g : ([a, b], \mathbb{R})$ is absolutely continuous with $g(b) = 0$, then*

$$\int_a^b \left[{}_aD_t^\beta f(t)\right] g(t)dt = \int_a^b \left[{}_tD_b^\beta g(t)\right] f(t)dt. \tag{4}$$

Now, we recall some properties of the basic function space which have been studied in [32].

Throughout this paper, let $1/2 < \alpha < 1$.

The fractional derivative space E^α is defined by the completion of $C_0^\infty(0, T)$ with respect to the norm $\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |{}_0D_t^\alpha u(t)|^2 dt\right)^{1/2}$, where ${}_0D_t^\alpha$ is the α-order left Riemann-Liouville fractional derivative. Then, E^α is a reflexive and separable Hilbert space. And the Riemann-Liouville fractional derivative exists for the elements in E^α [22].

Lemma 2 [32]. *For all $u \in E^\alpha$, we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^2}, \tag{5}$$

$$\|u\|_\infty \le \frac{T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|{}_0D_t^\alpha u\|_{L^2}. \tag{6}$$

According to (5), one can consider E^α with respect to the equivalent norm

$$\|u\|_\alpha = \|{}_0D_t^\alpha u\|_{L^2}. \tag{7}$$

Lemma 3 [32]. *If the sequence $\{u_k\}$ converges weakly to u in E^α, i.e., $u_k \rightharpoonup u$, then $u_k \longrightarrow u$ in $C([0, T], \mathbb{R}^N)$, i.e., $\|u - u_k\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$.*

By the proof of Proposition 4.1 in [8], we have the following property.

Lemma 4. *For any $u \in E^\alpha$,*

$$|\cos(\pi\alpha)|^2 \|u\|_\alpha^2 \le \|{}_tD_T^\alpha u\|_{L^2}^2 \le \frac{1}{|\cos(\pi\alpha)|^2} \|u\|_\alpha^2. \tag{8}$$

3. Existence Result for (1)

We assume that $f(t, x, y)$ satisfies the following conditions:

(H_0) $f : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable in $t \in [0, T]$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}$ and continuous in $(x, y) \in \mathbb{R} \times \mathbb{R}$ for a.e.t $\in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$, such that

$$|f(t, x, y)| \le a(|x|)b(t) + y^2, \tag{9}$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}$.

(H_1) There are constants $\mu > 2$ and $R > 0$ such that, for $|x| \ge R$, $y \in \mathbb{R}$,

$$0 < \mu F(t, x, y) \le f(t, x, y)x, \tag{10}$$

where $F(t, x, y) = \int_0^x f(t, s, y)ds$.

(H_2) $\lim_{x \to 0}(f(t, x, y)/x) = 0$ uniformly for $t \in [0, T]$ and $y \in \mathbb{R}$.

In order to derive a weak solution of (1), we suppose that u is a solution of (1), and multiplying (1) by an arbitrary $v \in C_0^\infty(0,T)$ and by Lemma 1, we have

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)dt = 0. \tag{11}$$

Since (11) is well defined for $u, v \in E^a$, the weak solution of (1) may be defined as follows.

Definition 5. A weak solution of (1) is a function $u \in E^\alpha$ such that

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)dt = 0, \tag{12}$$

for every $v \in E^\alpha$.

Definition 6. A function u is called a solution of (1), if ${}_0D_t^\alpha u(t)$ and ${}_tD_T^\alpha u(t)$ exist, $({}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_T^\alpha u(t))$ is derivable for almost every $t \in [0,T]$, and u satisfies (1).

For a given $w \in E^\alpha$, we consider the functional $\varphi_w : E^\alpha \longrightarrow \mathbb{R}$, defined by

$$\varphi_w(u) = \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt - \int_0^T F(t,u(t),{}_0D_t^\alpha w(t))dt. \tag{13}$$

In view of assumption (H_0), we know that φ is continuously differentiable and

$$\varphi_w'(u)v = \int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt - \int_0^T f(t,u(t),{}_0D_t^\alpha w(t))v(t)dt, \tag{14}$$

for $u, v \in E^\alpha$. Hence, a critical point of φ_w gives us a weak solution of (2).

Lemma 7. *If u is a weak solution of (1), then u is also a solution of (1).*

Proof. Let u be a weak solution of (1), then $u \in E^\alpha$, so ${}_0D_t{}^\alpha u(t)$ and ${}_tD_T{}^\alpha u(t)$ exist and $u(0) = u(T) = 0$. For every $v \in C_0^\infty$, we have

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)dt = 0. \tag{15}$$

Since $v \in C_0^\infty$, we have ${}_0D_t^\alpha v(t) = {}_0J_t^\alpha v'(t)$ and ${}_tD_T^\alpha v(t) = {}_{-t}J_T^\alpha v'(t)$, so

$$\int_0^T \left({}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_t^\alpha u(t) - \int_0^t f(s,u(s),{}_0D_s^\alpha u(s))ds \right) v'(t)dt = 0. \tag{16}$$

Then, there exists a constant C such that

$${}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_T^\alpha u(t) - \int_0^t f(s,u(s),{}_0D_s^\alpha u(s))ds = C, \tag{17}$$

thus,

$${}_tD_{T0}^\alpha D_t^\alpha u(t) + {}_0D_{t\,t}^\alpha D_T^\alpha u(t) = f(t,u(t),{}_0D_t^\alpha u(t)),\ t \in [0,T]. \tag{18}$$

Lemma 8. *Suppose (H_0) and (H_1) hold, then φ_w satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset E^\alpha$, $\{\varphi_w(u_n)\}$ is bounded, and $\varphi_w'(u_n) \longrightarrow 0$; we first show that $\{u_n\}$ is bounded.

It follows from (H_1) that

$$\begin{aligned}
\mu\varphi_w(u_n) - \varphi_w'(u_n)u_n &= \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad + \int_{\{t\in[0,T]\,|\,|u_n(t)|\geq R\}} f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))dt \\
&\quad + \int_{\{t\in[0,T]\,|\,|u_n(t)|< R\}} f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))dt \\
&\geq \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt \\
&\quad + \int_{\{t\in[0,T]\,|\,|u_n(t)|< R\}} (f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))),
\end{aligned} \tag{19}$$

which implies that $\{u_n\}$ is bounded.

From the reflexivity of E^α, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_n\}$, $u_n \rightharpoonup u$ then $\|u_n - u\|_\infty \longrightarrow 0$. Next, we will prove that $\{u_n\}$ strongly converges to u. By (H_0), we know that

$$\int_0^T (f(t, u_n(t), {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))(u_n(t) - u(t))dt \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{20}$$

Note that

$$\begin{aligned}\left(\varphi_w'(u_n) - \varphi_w'(u)\right)(u_n - u) &= \int_0^T |{}_0D_t^\alpha(u_n(t) - u(t))|^2 \\ &+ |{}_tD_T^a(u_n(t) - u(t))|^2 dt - \int_0^T (f(t, u_n(t), {}_0D_t^a w(t)) \\ &- f(t, u(t), {}_0D_t^a w(t))(u_n(t) - u(t))dt \geq \left(1 + |\cos \pi\alpha|^2\right)\|u_n - u\|_\alpha^2 \\ &- \int_0^T (f(t, u_n(t), {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))(u_n(t) - u(t))dt.\end{aligned} \tag{21}$$

By $\varphi'(u_n) \longrightarrow 0$ and $u_n \rightharpoonup u$, we obtain that

$$\left(\varphi'(u_n) - \varphi'(u)\right)(u_n - u) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{22}$$

Thus, $\|u_n - u\|_\alpha \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, φ_w satisfies the (PS) condition.

Lemma 9. *Let $w \in E^\alpha$ and suppose (H_0), (H_1), and (H_2) hold, then (2) has at least one nontrivial solution.*

Proof. The proof relies on the Mountain Pass theorem [33, 34]. It is clear that $\varphi_w \in C^1(E^\alpha, R)$, $\varphi_w(0) = 0$, and φ_w satisfies the (PS) condition from Lemma 8. By (H_2), for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$F(t, x, y) \leq \varepsilon|x|^2, \ |x| < \delta, \ t \in [0, T], \ y \in \mathbb{R}. \tag{23}$$

Let $\rho = \delta\Gamma(\alpha)(2\alpha - 1)^{1/2}/2T^{\alpha-(1/2)}$ and choose $\|u\|_\alpha = \rho$; by the above inequality and (6), we have

$$\begin{aligned}\varphi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt \\ &- \int_0^T F(t, u(t), {}_0D_t^\alpha w(t))dt \geq \frac{1 + |\cos \pi\alpha|^2}{2}\|u\|_\alpha^2 \\ &- \varepsilon T\|u\|_\infty^2 \geq \left(\frac{1 + |\cos \pi\alpha|^2}{2} - \frac{\varepsilon T^{2\alpha}}{\Gamma(\alpha)^2(2\alpha - 1)}\right)\|u\|_\alpha^2.\end{aligned} \tag{24}$$

It suffices to choose $\varepsilon = (1 + |\cos \pi\alpha|^2)\Gamma(\alpha)^2(2\alpha - 1)/4T^{2\alpha}$ and $\sigma = ((1 + |\cos \pi\alpha|^2)/4)\rho$ to get $\varphi_w(u) \geq \sigma$. Thus, there exist positive numbers ρ and σ which are independent of w such that $\varphi_w(u) \geq \sigma$ for $u \in E^\alpha$ satisfies $\|u\|_\alpha = \rho$.

It follows from (H_1) that there exist $c_1, c_2 > 0$ such that

$$F(t, x, y) \geq c_1|x|^\mu - c_2, \quad \forall t \in [0, T], \ x, y \in \mathbb{R}. \tag{25}$$

Choosing $\tilde{u} \in E^\alpha$ satisfies $\|\tilde{u}\|_\alpha = 1$, and we obtain

$$\begin{aligned}\varphi_w(r\tilde{u}) &= \frac{r^2}{2}\int_0^T |{}_0D_t^\alpha \tilde{u}(t)|^2 + |{}_tD_T^\alpha \tilde{u}(t)|^2 dt \\ &- \int_0^T F(t, r\tilde{u}(t), {}_0D_t^\alpha w(t))dt \leq \frac{r^2}{2}\left(1 + \frac{1}{|\cos \pi\alpha|^2}\right) \\ &- c_1 r^\mu \int_0^T |\tilde{u}|^\mu dt + c_2 T,\end{aligned} \tag{26}$$

which implies that $\varphi(r\tilde{u}) \longrightarrow -\infty$ as $r \longrightarrow \infty$. Hence, we obtain that there exists a $\beta > 0$ independent of u_1 and w such that $\varphi_w(u) \leq 0$ for all $\|u\|_\alpha > \beta$.

The above discussions combined with the Mountain Pass theorem show that (2) has at least one nontrivial solution u_w which can be characterized as

$$\varphi_w'(u_w) = 0, \varphi_w(u_w) = \inf_{g \in \Gamma} \max_{u \in g([0,1])} \varphi_w(u), \tag{27}$$

where $\Gamma = \{g \in C([0, 1], E^\alpha) | g(0) = 0, g(1) = \beta\tilde{u}\}$.

In order to obtain the existence of solutions for (1), we need the following Lipschitz condition.

(H_3) There exist $L_1, L_2 > 0$ such that

$$\begin{aligned}&|f(t, x_1, y) - f(t, x_2, y)| \leq L_1|x_1 - x_2|, \\ &\quad \forall t \in [0, T], \ x_1, x_2 \in [0, r_1], \ y \in \mathbb{R}, \\ &|f(t, x, y_1) - f(t, x, y_2)| \leq L_2|y_1 - y_2|, \\ &\quad \forall t \in [0, T], \ x \in [0, r_1], \ y_1, y_2 \in \mathbb{R},\end{aligned} \tag{28}$$

where $r_1 = c_2 T^{\alpha-(1/2)}/\Gamma(\alpha)(2\alpha - 1)^{1/2}$; C_2 will be determined later.

Theorem 10. *Let $\gamma = L_2 T^\alpha \Gamma(\alpha + 1)/((1 + |\cos \pi\alpha|^2)\Gamma(\alpha + 1)^2 - L_1 T^{2\alpha})$, suppose (H_0)–(H_3) hold, $2R/(1 + |\cos \pi\alpha|^2) < 1$, and $0 < \gamma < 1$, then problem (1) has a nontrivial solution.*

Proof. For $n = 1, 2, \cdots$ and $u_1 = 0$, we construct a sequence $\{u_n\}$, where u_n is the solution of the following problem:

$$\begin{cases} {}_tD_{T0}^\alpha D_t^\alpha u_n(t) + {}_0D_{t\,t}^\alpha D_T^\alpha u_n(t) = F(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)), \text{a.e.t} \in [0, T], \\ u_n(0) = u_n(T) = 0. \end{cases} \tag{29}$$

Now, we assume $\|u_{n-1}\|_\alpha \leq C_2$; by the mathematical induction, we will prove that $\|u_n\|_\alpha \leq C_2$. It follows from u_n satisfying (29) that

$$\varphi_{u_{n-1}}(u_n) = \inf_{g\in\Gamma}\max_{u\in g([0,1])}\varphi_{u_{n-1}}((u) \le \max_{r\in[0,\beta]}\varphi_{u_{n-1}}((r\tilde{u}) \le \max_{r\in[0,\beta]}\left(\frac{r^2}{2}\left(1+\frac{1}{|\cos\pi\alpha|^2}\right)-c_1r^{\mu}\int_0^T|\tilde{u}|^{\mu}dt+c_2T\right). \tag{30}$$

Let

$$h(r) = \frac{r^2}{2}\left(1+\frac{1}{|\cos\pi\alpha|^2}\right)-c_1r^{\mu}\int_0^T|\tilde{u}|^{\mu}dt+c_2T, \tag{31}$$

then $h(r)$ can achieve its maximum at $\tilde{r} = ((|\cos\pi\alpha|^2+1)/\cos\pi\alpha|^2c_1\mu\int_0^T|u_1|^{\mu}dt)^{1/(\mu-2)}$, so $\varphi_{u_{n-1}}(u_n) \le h(\tilde{r})$. By (H_1) and (H_0), we have

$$\begin{aligned}
h(\tilde{r}) \ge \varphi_{u_{n-1}}(u_n) &= \frac{1}{2}\int_0^T|{}_0D_t^{\alpha}u_n(t)|^2+|{}_tD_T^{\alpha}u_n(t)|^2dt\\
&\quad-\int_0^T F(t,u_n(t),{}_0D_t^{\alpha}u_{n-1}(t))dt\\
&\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_{\alpha}^2-\int_{\{t\in[0,T]\|u_n(t)|\le R\}}F(t,u_n(t),{}_0D_t^{\alpha}u_{n-1}(t))dt\\
&\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_{\alpha}^2\int_{\{t\in[0,T]\|u_n(t)|\le R\}}\int_0^{u_n(t)}f(t,s,{}_0D_t^{\alpha}u_{n-1}(t))dsdt\\
&\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_{\alpha}^2-\int_{\{t\in[0,T]\|u_n(t)|\le R\}}\int_0^R a(|s|)b(t)dsdt\\
&\quad-R\int_0^t|{}_0D_t^{\alpha}u_{n-1}(t)|^2dt \ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_{\alpha}^2\\
&\quad-\int_0^T b(t)dt\int_0^R a(|s|)ds-R\|u_{n-1}\|_{\alpha}^2.
\end{aligned} \tag{32}$$

Then,

$$\begin{aligned}
\|u_n\|_{\alpha}^2 &\le \frac{2R}{1+|\cos\pi\alpha|^2}\|u_{n-1}\|_{\alpha}^2+\frac{2\left(h(\tilde{r})+\int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{1+|\cos\pi\alpha|^2}\\
&\le \left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^{n-1}\|u_1\|_{\alpha}^2\\
&\quad+\frac{2\left(h(\tilde{r})+\int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{1+|\cos\pi\alpha|^2}\sum_{k=0}^{n-2}\left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^k\\
&= \left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^{n-1}\|u_1\|_{\alpha}^2\\
&\quad+\frac{2\left(1+|\cos\pi\alpha|^2\right)\left(h(\tilde{r})+\int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{\left(1+|\cos\pi\alpha|^2\right)\left(1+|\cos\pi\alpha|^2-2R\right)}.
\end{aligned} \tag{33}$$

Thus, we can choose $C_2 = 2(1+|\cos\pi\alpha|^2)(h(\tilde{r})+\int_0^T b(t)dt\int_0^R a(|s|)ds)/(1+|\cos\pi\alpha|^2)(1+|\cos\pi\alpha|^2-2R)$.

Since $(\varphi'(u_{n+1})-\varphi'(u_n))(u_{n+1}-u_n) = 0$ and $\|u_n\|\alpha \le C_2$, then $\|u_n\|\infty \le r_1$, where r_1 is given by (H_3). From (H_3), we have

$$\begin{aligned}
&\int_0^T|{}_0D_t^{\alpha}(u_{n+1}(t)-u_n(t))|^2+|{}_tD_T^{\alpha}(u_{n+1}(t)-u_n(t))|^2dt\\
&= \int_0^T(f(t,u_{n+1}(t),{}_0D_t^{\alpha}u_n(t))\\
&\quad-f(t,u_n(t),{}_0D_t^{\alpha}u_{n-1}(t)))(u_{n+1}(t)-u_n(t))dt\\
&= \int_0^T(f(t,u_{n+1}(t){}_0D_t^{\alpha}u_n(t))-f(t,u_n(t),{}_0D_t^{\alpha}u_n(t)))(u_{n+1}(t)\\
&\quad-u_n(t))dt+\int_0^T(f(t,u_n(t),{}_0D_t^{\alpha}u_n(t))\\
&\quad-f(t,u_n(t),{}_0D_t^{\alpha}u_{n-1}(t)))(u_{n+1}(t)-u_n(t))dt\\
&\le L_1\|u_{n+1}-u_n\|_{L^2}^2+L_2\int_0^T|{}_0D_t^{\alpha}u_n(t)-{}_0D_t^{\alpha}u_{n-1}(t)|(u_{n+1}(t)-u_n(t))dt\\
&\le L_1\|u_{n+1}-u_n\|_{L^2}^2+L_2\|u_n-u_{n-1}\|_{\alpha}\|u_{n+1}-u_n\|_{L^2}.
\end{aligned} \tag{34}$$

Combining the above estimates with (8) and (5), we obtain

$$\begin{aligned}
&\left(1+|\cos\pi\alpha|^2-\frac{L_1T^{2\alpha}}{\Gamma(\alpha+1)^2}\right)\|u_{n+1}-u_n\|_{\alpha}\\
&\le \frac{L_2T^{\alpha}}{\Gamma(\alpha+1)}\|u_n-u_{n-1}\|_{\alpha},
\end{aligned} \tag{35}$$

that is $\|u_{n+1}-u_n\|_{\alpha} \le \gamma\|u_n-u_{n-1}\|_{\alpha}$. Since $0<\gamma<1$, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in E^{α}. Thus, there is a u such that $\{u_n\}$ converges strongly to u in E^{α}.

Now, we show that $u\equiv 0$. By the proof of Lemma 9, we know $\varphi(u_n) \ge \sigma$.

If $\varphi(u_n) \longrightarrow \varphi(u)$, then $\varphi(u) > \sigma$, and since $\varphi(0) = 0$, we obtain $u\equiv 0$. In order to show $\varphi(u_n) \longrightarrow \varphi(u)$, we only need to show $\int_0^T F(t,u_n,{}_0D_t^{\alpha}u_{n-1}(t))dt \longrightarrow \int_0^T F(t,u,{}_0D_t^{\alpha}u(t))dt$. In fact,

$$\begin{aligned}
&\int_0^T F(t,u_n,{}_0D_t^{\alpha}u_{n-1}(t))dt-\int_0^T F(t,u,{}_0D_t^{\alpha}u(t))dt\\
&= \int_0^T\int_0^{u_n}f(t,x,{}_0D_t^{\alpha}u_{n-1}(t))dt-\int_0^T\int_0^{u}f(t,x,{}_0D_t^{\alpha}u(t))dt\\
&= \int_0^T\int_u^{u_n}f(t,x,{}_0D_t^{\alpha}u_{n-1}(t))-f(t,x,{}_0D_t^{\alpha}u(t))dt\\
&\le L_2\int_0^T|{}_0D_t^{\alpha}u_{n-1}(t)-{}_0D_t^{\alpha}u(t))\|u_n-u|dt\\
&\le L_2\left\|{}_0D_t^{\alpha}u_{n-1}(t)-{}_0D_t^{\alpha}u(t))\right\|L^2\left\|u_n-u\right\|L^2\\
&\le \frac{L_2T^{\alpha}}{\Gamma(\alpha+1)}\left\|u_n-u\right\|_{\alpha}^2 \longrightarrow 0.
\end{aligned} \tag{36}$$

Next, we show that for any $v \in E^{\alpha}$,

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt = \int_0^T f(t, u(t), {}_0D_t^\alpha u(t))v(t)dt. \tag{37}$$

It remains only to show

$$\int_0^T f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t))v(t)dt \longrightarrow \int_0^T f(t, u(t), {}_0D_t^\alpha u(t))\,v(t)dt. \tag{38}$$

Note that

$$\begin{aligned}
&\int_0^T (f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u(t), {}_0D_t^\alpha u(t)))v(t)dt \\
&= \int_0^T (f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - (f(t, u(t), {}_0D_t^\alpha u_{n-1}(t))v(t)dt \\
&\quad + \int_0^T (f(t, u(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u(t), {}_0D_t^\alpha u(t)))v(t)dt \\
&\le L_1\|u_n - u\|_{L^2}\|v\|_{L^2} + L_2\|u_{n-1} - u\|_\alpha\|v\|_{L^2} \\
&\le C\left(\|u_n - u\|_\alpha + \|u_{n-1} - u\|_\alpha\right)\|v\|_\alpha,
\end{aligned} \tag{39}$$

where C is a constant. Thus, we obtain a nontrivial solution of problem (1).

4. Existence Result for (3)

In Section 3, condition (H_3) implies that the nonlinearity is linear growth about a fractional derivative of solutions; this section will consider the fractional boundary value problem (3) in which the nonlinearity is quadratic growth about the fractional derivative of solutions.

Assume that g satisfies the following conditions:

(H_1') There are constants $\mu > 2$ and $R > 0$ such that, for $|x| \ge R$, $0 < \mu G(x) \le g(x)x$, where $G(x) = \int_0^x g(s)ds$.

(H_2') $\lim_{x\to 0}(g(x)/x) = 0$.

Similar to Section 3, since (3) is not variational, given $w \in E^\alpha$, we consider the following problem which is independent on the fractional derivative of the solution:

$$\begin{cases} {}_tD_T^\alpha\,{}_0D_t^\alpha u(t) + {}_0D_t^\alpha\,{}_tD_T^\alpha u(t) = a(t)g(u(t)) + \lambda|{}_0D_t^\alpha w(t)|^2, \text{ a.e.t} \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{40}$$

Then, the corresponding functional $\phi : E^\alpha \longrightarrow E^\alpha$ is given by

$$\begin{aligned}
\phi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt - \int_0^T a(t)G(u(t))dt \\
&\quad - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 u(t)dt, \\
\phi_w'(u)v &= \int_0^T {}_0D_t^\alpha(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)dt \\
&\quad - \int_0^T a(t)g(u(t))v(t)dt - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 v(t)dt,
\end{aligned} \tag{41}$$

where $v \in E^\alpha$.

(H_1') implies that there are constants $c_1, c_2 > 0$ such that

$$G(x) \ge c_1|x|^\mu - c_2. \tag{42}$$

By (H_2'), there is a $\delta > 0$ such that $|x| \le \delta$ implies

$$g(x) \le \frac{(\Gamma(\alpha))^2(2\alpha - 1)\left(1 + |\cos\ \pi\alpha|^2\right)}{4\int_0^T a(t)dt\,T^{2a-1}}|x|^2. \tag{43}$$

For convenience of our statement, let us give some notations and denote

$$\begin{aligned}
a &= \frac{(\mu - 2)\left(1 + |\cos\pi\alpha|^2\right)}{2\mu}, \\
b &= \frac{\lambda(\mu - 1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha - 1)^{1/2}\mu}, \\
d &= \frac{(\mu - 1)\Gamma(\alpha)(2\alpha - 1)^{1/2}\delta\left(1 + |\cos\pi\alpha|^2\right)}{32T^{\alpha-(1/2)}\mu},
\end{aligned} \tag{44}$$

where μ is given in (H_1') and δ is given in (43). Assuming $a^2 > 4bd$, let

$$\varepsilon_1 = \frac{a - \sqrt{a^2 - 4bd}}{2b},\ \varepsilon_2 = \frac{a + \sqrt{a^2 - 4bd}}{2b}. \tag{45}$$

For a fixed function $v \in E^\alpha$ with $\|v\| = 1$, we denote

$$\begin{aligned}
m(t) &= \left(1 + \frac{1}{|\cos\pi\alpha|^2}\right)r^2 - c_1 r^\mu\int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{r\Gamma(\alpha)(2\alpha - 1)^{1/2}\delta\left(1 + |\cos\pi\alpha|^2\right)}{4T^{\alpha-(1/2)}} + c_2\int_0^T a(t),
\end{aligned} \tag{46}$$

$$M = \max_{r\in(0,\infty)} m(t),\ M_1 = M + \int_0^T a(t)dt\ \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right). \tag{47}$$

For $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, denote

$$R_1 = R_1(\varepsilon) = \frac{M_1 \varepsilon}{a\varepsilon - b\varepsilon^2 - d},\ R_2 = R_2(\varepsilon) = \frac{R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}. \tag{48}$$

We also need the following Lipschitz condition:

(H_3')There exists $L_{R2} > 0$ such that $|g(x_1) - g(x_2)| \le L_R |x_1 - x_2|$ for $0 < x_1, x_2 < R_2$. Then we have the following result.

Theorem 11. *Suppose (H_1')–(H_3') hold, $a^2 > 4bd$; if there exists $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, such that*

$$\lambda R_1^2 \le \frac{\Gamma(\alpha)^2(2\alpha-1)\delta\left(1+|\cos\pi\alpha|^2\right)}{8T^{2\alpha-1}}, \tag{49}$$

$$\frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} < 1 + |\cos\pi\alpha|^2 - \frac{T^{2\alpha-1}L_{R_2}\int_0^T a(t)dt}{(\Gamma(\alpha))^2(2\alpha-1)}. \tag{50}$$

Then, (3) has a nontrivial weak solution.

Proof. We first verify that for a given $w \in E^\alpha$ with $\|w\|_\alpha \le R_1$, (40) has at least a nontrivial weak solution.

In order to use the Mountain Pass theorem, we first show that ϕ satisfies the (PS) condition. Let $\{u_n\} \subset E^\alpha$, $\{\phi_w(u_n)\}$ is bounded, and $\phi_w'(u_n) \longrightarrow 0$, we show that $\{u_n\}$ is bounded.

In fact,

$$\begin{aligned}
\mu\phi_\omega(u_n) - \phi_\omega'(u_n)u_n &= \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad + \int_0^T a(t)(g(u_n)u_n - \mu G(u_n))dt + \lambda(1-\mu)\int_0^T |{}_0D_t^\alpha \omega(t)|^2 u_n(t)dt \\
&\ge \left(\frac{\mu}{2} - 1\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 + \lambda(1-\mu)\|\omega\|_\alpha^2\|u_n\|_\infty \\
&\quad + \int_{\{t\in[0,T]||u_n(t)|\le R\}} a(t)(u_n(t)g(u_n(t)) - \mu G(u_n(t)))dt \\
&\ge \left(\frac{\mu}{2} - 1\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 + \frac{\lambda R_1^2(1-\mu)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n\|_\alpha \\
&\quad + \int_{\{t\in[0,T]||u_n(t)|\le R\}} a(t)(u_n(t)g(u_n(t)) - \mu G(u_n(t)))dt,
\end{aligned} \tag{51}$$

which implies that $\{u_n\}$ is bounded, and similar to the last part of Lemma 8, we get that $\phi_w(u)$ satisfies the (PS) condition.

Let $\rho = \Gamma(\alpha)(2\alpha-1)^{1/2}\delta/T^{\alpha-(1/2)}$ and choose $u \in E^\alpha$ with $\|u\| = \rho$, then $\|u\|_\infty \le \delta$. From (43), (6), and (49), we have

$$\begin{aligned}
\phi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt - \int_0^T a(t)G(u(t))dt \\
&\quad - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 u(t)dt \ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u\|_\alpha^2 \\
&\quad - \frac{\Gamma(\alpha)^2(2\alpha-1)\left(1+|\cos\pi\alpha|^2\right)}{4T^{2\alpha-1}}\|u\|_\infty^2 - \lambda\|w\|_\alpha^2\|u\|_\infty \\
&\ge \frac{1}{4}\left(1+|\cos\pi\alpha|^2\right)\|u\|_\alpha^2 - \frac{\lambda T^{\alpha-(1/2)}R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u\|_\alpha \\
&= \left(\frac{1}{4}\left(1+|\cos\pi\alpha|^2\right)\rho - \frac{\lambda T^{\alpha-(1/2)}R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\right)\rho \\
&\ge \frac{1}{8}\left(1+|\cos\pi\alpha|^2\right)\rho^2.
\end{aligned} \tag{52}$$

Then, we obtain that there exists $\beta > 0$ such that for $\|u\|_\alpha = \rho$, $\phi_w(u) \ge \beta$ uniformly for $w \in E^\alpha$ with $\|w\|_\alpha \le R1$.

Let $v \in E^\alpha$ with $\|v\| = 1$, then

$$\begin{aligned}
\phi_w(rv) &\le \frac{r^2}{2}\left(1 + \frac{1}{|\cos\pi\alpha|^2}\right) - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \lambda r\int_0^T |{}_0D_t^\alpha w(t)|^2 v dt + c_2\int_0^T a(t)dt \\
&\le \frac{r^2}{2}\left(1 + \frac{1}{|\cos\pi\alpha|^2}\right) - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{r\lambda R_1^2 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|v\|_\alpha \\
&\quad + c_2\int_0^T a(t)dt \longrightarrow -\infty \text{ as } r \longrightarrow \infty,
\end{aligned} \tag{53}$$

which implies there exists $r_0 > \rho$ such that $\phi_w(r_0 v) < 0$. Then, from the Mountain Pass theorem, we get that ϕ_w has a nontrivial weak solution u_w which can be characterized as

$$\phi_w'(u_w) = 0,\ \phi_w(u_w) = \inf_{\eta\in\Gamma}\max_{u\in\eta([0,1])}\phi_w(u), \tag{54}$$

where $\Gamma = \{\eta \in C([0,1], E^\alpha) | \eta(0) = 0, \eta(1) = r_0 v\}$.

Let $u_1 = 0$, we can obtain that ϕ_{u1} has a nontrivial critical point u_2. For $n = 1, 2, \cdots$, we construct a sequence $\{u_n\}$, where u_n is the critical point of $\phi_{u_{n-1}}$. Now, we assume that $\|u_{n-1}\|_\alpha \le R_1$; by the mathematical induction, we will prove that $\|u_n\|_\alpha \le R_1$. In fact, by (54), we have

$$\begin{aligned}\phi_{u_{n-1}}(u_n) &\le \max_{r\in(0,r_0)}\phi_{u_{n-1}}(rv) \le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right)r^2 \\ &\quad -\int_0^T a(t)(c_1 r^\mu|v|^\mu - c_2)dt + \lambda r\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 v(t)dt \\ &\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right)r^2 - c_1 r^\mu\int_0^T a(t)|v|^\mu dt \\ &\quad + \frac{\lambda r T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_{n-1}\|_\alpha^2 + c_2\int_0^T a(t)dt \\ &\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right)r^2 - c_1 r^\mu\int_0^T a(t)|v|^\mu dt \\ &\quad + \frac{\lambda r T^{\alpha-(1/2)}R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}} + c_2\int_0^T a(t)dt \\ &\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right)r^2 - c_1 r^\mu\int_0^T a(t)|v|^\mu dt \\ &\quad + \frac{r\Gamma(\alpha)(2\alpha-1)^{1/2}\delta\left(1+|\cos\pi\alpha|^2\right)}{4T^{\alpha-(1/2)}} \\ &\quad + c_2\int_0^T a(t)dt \le M,\end{aligned} \tag{55}$$

where M is given in (47).

Hence,

$$\begin{aligned}M \ge \phi_{u_{n-1}}(u_n) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\ &\quad -\int_0^T a(t)G(u_n(t))dt - \lambda\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)dt \\ &\ge \frac{1}{2}\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\ &\quad -\frac{1}{\mu}\int_0^T a(t)g(u_n(t))u_n(t)dt - \lambda\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)dt \\ &\quad -\int_0^T a(t)dt\ \max_{|x|\le R}\left(G(x)-\frac{1}{\mu}g(x)\right) \\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\ &\quad +\frac{1}{\mu}\phi'_{u_{n-1}}(u_n)u_n - \lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)dt \\ &\quad -\int_0^T a(t)dt\ \max_{|x|\le R}\left(G(x)-\frac{1}{\mu}g(x)\right) \\ &\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \\ &\quad -\lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)dt \\ &\quad -\int_0^T a(t)dt\ \max_{|x|\le R}\left(G(x)-\frac{1}{\mu}g(x)\right).\end{aligned} \tag{56}$$

So,

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \le M \\ &\quad +\lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)dt \\ &\quad +\int_0^T a(t)dt\ \max_{|x|\le R}\left(G(x)-\frac{1}{\mu}g(x)\right) \le M_1 \\ &\quad +\left(1-\frac{1}{\mu}\right)\frac{\lambda T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_{n-1}\|_\alpha^2\|u_n\|_\alpha \le M_1 \\ &\quad +\frac{\lambda(\mu-1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\left(\varepsilon\|u_n\|_\alpha^2 + \frac{R_1^2}{4\varepsilon}\|u_{n-1}\|_\alpha^2\right).\end{aligned} \tag{57}$$

By (49), we obtain

$$\begin{aligned}&\left(\frac{(\mu-2)\left(1+|\cos\pi\alpha|^2\right)}{2\mu} - \frac{\lambda\varepsilon(\mu-1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\right)\|u_n\|_\alpha^2 \\ &\quad \le \frac{\lambda R_1^2(\mu-1)T^{\alpha-(1/2)}}{4\varepsilon\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\|u_{n-1}\|_\alpha^2 + M_1 \\ &\quad \le \frac{(\mu-1)\Gamma(\alpha)(2\alpha-1)^{1/2}\delta\left(1+|\cos\pi\alpha|^2\right)}{32T^{\alpha-(1/2)}\mu\varepsilon}\|u_{n-1}\|_\alpha^2 + M_1.\end{aligned} \tag{58}$$

That is

$$(a-b\varepsilon)\|u_n\|_\alpha^2 \le \frac{d}{\varepsilon}\|u_{n-1}\|_\alpha^2 + M_1. \tag{59}$$

When $\varepsilon\in(\varepsilon_1,\varepsilon_2)$, where ε_1 and ε_2 are given in (45), we have

$$\frac{d}{\varepsilon} < a-b\varepsilon. \tag{60}$$

Then,

$$\begin{aligned}\|u_n\|_\alpha^2 &\le \frac{d/\varepsilon}{a-b\varepsilon}\|u_{n-1}\|_\alpha^2 + \frac{M_1}{a-b\varepsilon} \\ &\le \left(\frac{d/\varepsilon}{a-b\varepsilon}\right)^{n-1}\|u_1\|_\alpha^2 + \frac{M_1}{a-b\varepsilon}\sum_{k=0}^{n-2}\left(\frac{d/\varepsilon}{a-b\varepsilon}\right)^k \\ &\le \|u_1\|_\alpha^2 + \frac{M_1\varepsilon}{a\varepsilon-b\varepsilon^2-d}.\end{aligned} \tag{61}$$

Therefore, the above argument implies that $\|u_n\|_\alpha \le R_1$.

Finally, we show that $\{u_n\}$ is convergent to a nontrivial solution of (3).

Since $\|u_n\|_\alpha \le R_1$, we have $\|u_n\|_\infty \le R_2$, then by (H_3'), we obtain

$$\begin{aligned}
&\left(1+|\cos\pi\alpha|^2\right)\|u_{n+1}-u_n\|_\alpha^2 \le \int_0^T |{}_0D_t^\alpha(u_{n+1}-u_n(t))|^2 \\
&\quad + |{}_tD_T^\alpha(u_{n+1}-u_n(t))|^2 dt \\
&= \left(\phi'_{u_n}(u_{n+1})-\phi'_{u_{n-1}}(u_n)\right)(u_{n+1}-u_n(t)) \\
&\quad + \int_0^T a(t)(f(u_n)-f(u_{n+1}))(u_{n+1}-u_n)dt \\
&\quad + \lambda\int_0^T \left(|{}_0D_t^\alpha u_n(t)|^2 - |{}_0D_t^\alpha u_{n-1}|^2\right)(u_{n+1}-u_n(t))dt \\
&\le L_{R_2}\|u_{n+1}-u_n\|_\infty^2 \int_0^T a(t)dt \\
&\quad + \lambda\int_0^T {}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha(u_n-u_{n-1})dt\,\|u_{n+1}-u_n\|_\infty \\
&\quad + \lambda\int_0^T {}_0D_t^\alpha u_{n-1}(t)\,{}_0D_t^\alpha(u_n-u_{n-1})dt\,\|u_{n+1}-u_n\|_\infty \\
&\le L_{R_2}\|u_{n+1}-u_n\|_\infty^2 \int_0^T a(t)dt + \lambda\|u_n\|_\alpha\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\infty \\
&\quad + \lambda\|u_{n-1}\|_\alpha\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\infty \\
&\le \frac{T^{2\alpha-1}L_{R_2}\int_0^T a(t)dt}{(\Gamma(\alpha))^2(2\alpha-1)}\|u_{n+1}-u_n\|_\alpha^2 \\
&\quad + \frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha.
\end{aligned} \tag{62}$$

So

$$\begin{aligned}
&\left(1+|\cos\pi\alpha|^2 - \frac{T^{2\alpha-1}L_{R_2}\int_0^T a(t)dt}{(\Gamma(\alpha))^2(2\alpha-1)}\right)\|u_{n+1}-u_n\|_\alpha \\
&\quad \le \frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n-u_{n-1}\|_\alpha.
\end{aligned} \tag{63}$$

From (50), we know that $\{u_n\}$ is a Cauchy sequence in E^α and u is a weak solution of (3). Since $\phi_{u_{n-1}}(u_n) > \beta > 0$ for $n = 1, 2, \cdots$ and β does not depend on n, we obtain that u is a nontrivial weak solution of (3).

Corollary 12. *Suppose (H_1')-(H_3') hold; if the right-hand side of (50) is greater than 0, then there exists a constant $\lambda_0 > 0$, such that (1) has a solution when $\lambda \in (0, \lambda_0)$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Multiplicity of Quasilinear Schrödinger Equation

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In this paper, we study the quasilinear Schrödinger equation involving concave and convex nonlinearities. When the pair of parameters belongs to a certain subset of $\mathbb{R}^2$, we establish the existence of a nontrivial mountain pass-type solution and infinitely many negative energy solutions by using some new techniques and dual fountain theorem. Recent results from the literature are improved and extended.

1. Introduction and Main Results

In this paper, we are concerned with the following quasilinear Schrödinger equations:

$$-\Delta u + V(x)u - \Delta\left(|u|^{2\alpha}\right)|u|^{2\alpha-2}u = \lambda g(x,u) + \mu h(x,u),\ x \in \mathbb{R}^N, \tag{1}$$

where $N \geq 3$, $3/4 < \alpha \leq 1$, λ and μ are the parameters, $g, h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and the potential $V(x)$ satisfies the following assumption:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 \coloneqq \inf_{x\in\mathbb{R}^N} V(x)$, and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

In the last two decades, Equation (1) has been studied extensively due to its strong physical background. In particular, under the suitable parameters, Equation (1) can describe the self-channeling of a high-power ultra-short laser in matter, for detail [1–3]. In addition, Equation (1) also appears the superfluid film in plasma physics [4] and some fluid mechanics [5].

For the case where $\alpha = 1$, $\lambda = 1$, and $\mu = 1$, solutions of (1) are standing wave solutions of the following quasilinear Schrödinger equation:

$$\begin{aligned} i\Psi_t + \Delta\Psi - W(x)\Psi + k\Delta\left(h\left(|\Psi|^2\right)\right)h'\left(|\Psi|^2\right)\Psi \\ + g(x,\Psi) + h(x,\Psi) = 0, \quad x \in \mathbb{R}^N. \end{aligned} \tag{2}$$

Many researchers focus on the nonlinear quasilinear Schrödinger Equations (1) and (2). Adachi and Watanabe [6] discussed the uniqueness of the ground state solutions for the following quasilinear Schrödinger equations:

$$-\Delta u + \lambda u - k\Delta\left(|u|^{\alpha}\right)|u|^{\alpha-2}u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{3}$$

via a dual approach. The Nehari method was adopted to establish the existence results of ground state solutions by Liu et al. in [7]. The Lagrange multiplier method was used in [8] to prove the existence of the ground state solutions. Recently, Liu et al. in [9] applied a perturbation method to get the existence of positive solutions and a sequence of high energy solutions. In [10], Zhang et al. established the existence of infinite solutions for a modified nonlinear Schrödinger equation by using a symmetric mountain pass theorem as well as dual approach. Recently, Zhang et al. [11] offered a new iterative technique to get the existence and nonexistence of blow-up radial solutions for a Schrödinger equation involving a nonlinear operator.

The common point of the above works is that the nonlinear terms are all superlinear; to our best knowledge, there are few results on the quasilinear Schrödinger equation involving concave and convex nonlinearities. Furthermore, as mentioned above, there are few results on the existence of negative energy solutions. The reason is that the combination of concave and convex will make it difficult to verify the mountain pass theorem and fountain theorem. Inspired by the above-

mentioned works, this paper is aimed at considering quasilinear Schrödinger Equation (1) involving concave and convex nonlinearities and proving the existence of mountain pass-type of solution and a sequence of infinitely many solutions with negative energy.

Throughout this paper, we assume that g and h satisfy the following assumptions:

(g_1) There exist constants $1 < r_1 < r_2 < \cdots < r_m < 2$ and functions $h_i(x) \in L^{2/2-r_i}(\mathbb{R}^N, \mathbb{R}^+)$, $(i = 1, 2, \cdots, m)$, such that

$$|g(x,u)| \le \sum_{i=1}^{m} h_i(x)|u|^{r_i-1}. \tag{4}$$

(g_2) There exist $\sigma \in (1,2)$, $c_1 > 0$ such that $G(x,u) \ge c_1|u|^\sigma$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(g_3) $g(x,u)$ is odd in u.

(h_1) $3/4 < \alpha \le 1$, $4\alpha < p < 2\alpha \cdot 2^*$, $|h(x,u)| \le C(|u| + |u|^{p-1})$ holds for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(h_2) $h \in C(\mathbb{R}^N, \mathbb{R})$ and $h(x,u) = o(|u|)$, $|u| \longrightarrow 0$ uniformly on $\mathbb{R}^N$.

(h_3) There exists $\theta > 4$ such that

$$0 < H(x,u) := \int_0^u h(x,s)ds \le \frac{u}{\theta}h(x,u), \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{5}$$

(h_4) There exists $R > 0$ such that

$$\inf_{x \in \mathbb{R}^N, |u| > R} H(x,u) > 0. \tag{6}$$

(h_5) $h(x,u)$ is odd in u.

Theorem 1. *Assume (V), (g_1), and (h_1)–(h_4) hold. Then, there exist two positive constants $\bar{\mu}, \bar{\lambda}$ such that for $(\lambda, \mu) \in (0, +\infty) \times (0, \bar{\mu}] \cup [-\bar{\lambda}, \bar{\lambda}] \times (0, +\infty)$, (1) has at least a nontrivial mountain pass-type solution.*

Theorem 2. *Assume (V), (g_1)–(g_3) and (h_1)–(h_5) hold. Then, for $\lambda > 0, \mu > 0$, problem (1) has a sequence of solutions with negative energy.*

Remark 3. (V), (g_1), and (h_1)–(h_3) ensure the compactness of energy functional. Inspired by [12], we adopt a new approach to verify the mountain structure; on the other hand, the Tayler expansion technique plays an important role in verifying a dual fountain theorem. As mentioned above, our results may be regarded as a generalization and improvement of many existing results.

The main tools of this paper are variational methods combined with analysis techniques and suitable estimations; these methods and techniques are important for dealing with our equation and establishing the relative energy estimation. Thus, in order to make readers follow our work more easily, here, we briefly recall these helpful analysis techniques such as variational methods, iterative technique, fixed point theorems, and finite element methods. In [13–15], variational methods and some critical point theorems were employed to study the existence of solution for various elliptic equations. The iterative technique [16, 17] was also developed to establish the existence criterion of solutions as well as the corresponding convergence analysis for Hessian-type elliptic equations. In addition, fixed point theorems [18–21] and finite element methods [21, 22] also provide important theoretical and numerical tools for solving various nonlinear equations.

The rest of this paper is organized as follows: in Section 2, the variational framework and some lemmas are presented. Section 3 is devoted to the Proofs of Theorem 10 and Theorem 11.

2. Preliminaries and Functional Setting

In this paper, we make use of the following notations:

$$\begin{aligned} H^1(\mathbb{R}^N) &:= \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}, \\ \|u\|_{H^1}^2 &:= \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right), \\ D^{1,2}(\mathbb{R}^N) &:= \left\{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}, \\ \|u\|_{D^{1,2}}^2 &:= \int_{\mathbb{R}^N} |\nabla u|^2. \end{aligned} \tag{7}$$

Set $E := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty\}$; E is a Hilbert space with inner product $(u,v)_E = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)$. By $|\cdot|_p$, we denote the usual L^p norm. B_R denotes the open ball centered at the origin and radius $R > 0$. Throughout this paper, C and C_i are used in various places to denote positive constants, which are not essential to the problem.

It follows from ([23], Lemma 2.1) that the embedding $E \hookrightarrow L^r(\mathbb{R}^N)(2 \le r < 2^*)$ is compact, and there is $\eta_r > 0$ such that $|u|_r \le \eta_r \|u\|$ for all $u \in E$. The energy functional $J : E \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4\alpha}\int_{\mathbb{R}^N} \left|\nabla\left(|u^{2\alpha}|\right)\right|^2 \\ &\quad - \int_{\mathbb{R}^N} (\lambda G(x,u) + \mu H(x,u)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) + \alpha\int_{\mathbb{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 \\ &\quad - \int_{\mathbb{R}^N} (\lambda G(x,u) + \mu H(x,u)). \end{aligned} \tag{8}$$

According to [6], we can make the change of variable by $v = f^{-1}(u)$, where f is defined by

$$f' = \frac{1}{\sqrt{1 + 2\alpha|f(t)|^{2(2\alpha-1)}}}, \quad f(-t) = -f(t) \text{ on } t \in [0, +\infty). \tag{9}$$

After the change of variables, we obtain the following functional:

$$I(v) := J(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)f^2(v)\right) - \int_{\mathbb{R}^N}\left(\lambda G(x, f(v)) + \mu H(x, f(v))\right), \tag{10}$$

which is well defined in E under the assumptions (V), (h_1), (h_2), and (g_1). Moreover, the critical points of the functional I correspond to the weak solutions of the following equation:

$$-\Delta v = \frac{1}{\sqrt{1 + 2\alpha|f(v)|^{2(2\alpha-1)}}}\left[g(x, f(v)) + h(x, f(v))V(x)f(v)\right] \text{ in } \mathbb{R}^N. \tag{11}$$

It is shown in [24] that if $v \in E$ is a critical point of the function I, then $u = f(v) \in E$ and u is a solution of (1). The function $f(t)$ enjoys some properties given in [25]. The Mountain Pass lemma in [26] allows us to find Ceramitype sequence.

Lemma 4. *Assume (V), (h_1), (h_2), and (g_1) hold. Then, $I \in C^1(E, \mathbb{R})$ and*

$$\left\langle I'(v), w\right\rangle = \int_{\mathbb{R}^N}\left(\nabla v \nabla w + V(x)f(v)f'(v)w\right) - \int_{\mathbb{R}^N}\left(\lambda g(x, f(v))f'(v)w + \mu h(x, f(v))f'(v)w\right), \tag{12}$$

for all $w \in E$ and $\psi' : E \longrightarrow E^*$ is compact, where $\psi(v) = \int_{\mathbb{R}^N} G(x, f(v))$.

Proof. The proof is standard; we omit it here.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \oplus_{i=1}^{\infty} X_i$ and $\dim X_i < +\infty$ for each $i \in \mathbb{N}$. Further, we set

$$Y_k = \bigoplus_{i=1}^{k} X_i, \qquad Z_k = \bigoplus_{i=k}^{\infty} X_i. \tag{13}$$

Lemma 5 ([27]). *Let $\Phi \in C^1(X, \mathbb{R})$ satisfies $\Phi(-u) = \Phi(u)$. Assume that, for every $k > k_0$, there exists $\rho_k > \gamma_k > 0$ such that*

(A_1) $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi(u) \geq 0$.

(A_2) $b_k := \max_{u \in Y_k, \|u\| = \gamma_k} \Phi(u) < 0$.

(A_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \longrightarrow 0$ as $k \longrightarrow +\infty$.

(A_4) Φ satisfies the $(C)_c$ condition for every $c \in [d_{k_0}, 0)$.

Then Φ has a sequence of negative critical values converging to 0.

Remark 6. If $1 < p < 2^*$, then we have

$$\beta_k := \sup_{u \in Z_k, \|u\| = 1} |u|_p \longrightarrow 0, \; k \longrightarrow \infty. \tag{14}$$

3. Proof of Main Results

Lemma 7. *If $\|v_n\|^2 \longrightarrow +\infty$, then $\|v_n\|_0^2 := \int_{\mathbb{R}^N}(|\nabla v|^2 + V(x)f^2(v)) \longrightarrow +\infty$, as $n \longrightarrow \infty$.*

Proof. If not, there exists a positive constant $C > 0$ such that $\|v_n\|^2 \longrightarrow +\infty$ and $\|v_n\|_0^2 \leq C$, as $n \longrightarrow \infty$. Then, $(\|v_n\|_0^2)/(\|v_n\|^2) \longrightarrow 0$. Set $w_n = v_n/\|v_n\|$ and $h_n = (f^2(v_n))/(\|v_n\|^2)$, one has $\int_{\mathbb{R}^N}(|\nabla w_n|^2 + V(x)h_n(x)) \longrightarrow 0$. Hence,

$$\int_{\mathbb{R}^N}|\nabla w_n|^2 \longrightarrow 0, \int_{\mathbb{R}^N} V(x)h_n(x) \longrightarrow 0, \int_{\mathbb{R}^N} V(x)w_n^2(x) \longrightarrow 1. \tag{15}$$

It follows from [25]; for each $\varepsilon > 0$, there exists C_2 independent of n such that $\text{meas}(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^{\mathbb{N}} : |v_n(x)| \geq C_2\}$. One has

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus\Omega_n} V(x)w_n^2 \leq C_3\int_{\mathbb{R}^{\mathbb{N}}\setminus\Omega_n} V(x)\frac{f^2(v_n)}{\|v_n\|^2} \leq C_3\int_{\mathbb{R}^{\mathbb{N}}} V(x)h_n(x) \longrightarrow 0. \tag{16}$$

On the other hand, there exists $\varepsilon > 0$ such that whenever $\Omega' \subset R^N$ and $\text{meas}(\Omega') < \varepsilon$, $\int_{\Omega'} V(x)w_n^2 \leq 1/2$. Thus,

$$\int_{\mathbb{R}^{\mathbb{N}}} V(x)w_n^2 = \int_{\mathbb{R}^{\mathbb{N}}\setminus\Omega_n} V(x)w_n^2 + \int_{\Omega_n} V(x)w_n^2 \leq o(1) + \frac{1}{2}, \tag{17}$$

which contradicts with (15).

Lemma 8. *Suppose (V), (g_1), and (h_1)–(h_4) hold. Then, there exists two positive constants $\bar{\mu}, \bar{\lambda}$, such that (i) for $(\lambda, \mu) \in (0, +\infty) \times (0, \bar{\mu}] \cup [-\bar{\lambda}, \bar{\lambda}] \times (0, +\infty)$; there exist $\rho > 0$, $\eta > 0$ such that $\inf\{I(v) : v \in E, \|v\| = \rho\} > \eta$; (ii) there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof.

(i) For $r > 0$, $v \in S_r = \{v \in E : \|v\| = r\}$, when r is large enough, by simple computation, one has

$$I(v) \geq \|v\|_0^2\left(\frac{1}{4} - |\lambda|C_4\|v\|^{r_m-2} - \mu\frac{C_5}{p}\|v\|^{p-2}\right). \tag{18}$$

Let $m(s) = |\lambda|C_4 s^{r_m-2} + \mu(C_5/p)s^{p-2}$, $m(s)$ is bounded below; thus, $m(s)$ admits a minimizer $s_0 = ((|\lambda|C_4 p(2 - r_m))/(\mu C_5(p-2)))^{1/(p-r_m)}$, then

$$\inf_{s\in(0,+\infty)} m(s) = m(s_0) = \mu^{(2-r_m)/(p-r_m)} \cdot |\lambda|^{(p-2)/(p-r_m)} \cdot \left[C_4 \left(\frac{C_4 p(2-r_m)}{C_5(p-2)} \right)^{(r_m-2)/(p-r_m)} + \frac{C_5}{p} \left(\frac{C_4 p(2-r_m)}{C_5(p-2)} \right)^{(p-2)/(p-r_m)} \right]. \tag{19}$$

Let $K := C_4(C_4 p(2-r_m)/C_5(p-2))^{(r_m-2)/(p-r_m)} + (C_5/p)(C_4 p(2-r_m)/C_5(p-2))^{(p-2)/(p-r_m)}$. For all $\mu > 0$ and

$$|\lambda| \le \left(\frac{1}{4\mu^{(2-r_m)/(p-r_m)} K} \right)^{(p-r_m)/(p-2)} := \bar{\lambda} \tag{20}$$

or for all $\lambda > 0$ and

$$\mu \le \left(\frac{1}{4\lambda^{(p-2)/(p-r_m)} K} \right)^{(p-r_m)/(2-r_m)} := \bar{\mu}, \tag{21}$$

one has $m(s_0) < 1/4$. For $\|v\| = s_0$, there exists $s' > 0$ such that $\|v\|_0 \in (s', s_0]$. Thus, there exists $\rho > 0$ such that $\inf_{v\in S_\rho} I(v) > 0 = I(0)$

(ii) The conclusion is easy to check, and we omit it here.

Lemma 9. *Suppose (V), (g_1), and (h_1)–(h_3) hold. Then I satisfies the $(C)_c$ condition*

Proof.

(1) Let $\{v_n\}$ be a $(C)_c$ sequence of I, that is, $I(v_n) \longrightarrow c$, $(1 + \|v_n\|)I'(v_n) \longrightarrow 0$. By Lemma 7, after some direct computation, one gets that the sequence $\{v_n\}$ is bounded

(2) Passing to a subsequence if necessary, assume that $v_n \rightharpoonup v$ in E. First, through a proof by contradiction with the help of $\alpha \in (3/4, 1]$, there exists $C_4 > 0$ such that

$$\int_{\mathbb{R}^N} \left[|\nabla(v_n - v)|^2 + V(x)\Big(f(v_n)f'(v_n) - f(v)f'(v)\Big)(v_n - v) \right] \ge C_4 \|v_n - v\|^2. \tag{22}$$

On the other hand, by Lemma 4, ψ' is compact, $\|\psi'(v_n) - \psi'(v)\| \longrightarrow 0$, and

$$\left\langle \psi'(v_n) - \psi'(v), v_n - v \right\rangle \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{23}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \Big(h(x, f(v_n))f'(v_n) - h(x, f(v))f'(v) \Big)(v_n - v) = 0. \tag{24}$$

Therefore, by (22)–, one has

$$o(1) = \left\langle I'(v_n) - I'(v), v_n - v \right\rangle \ge C_6 \|v_n - v\|^2 + o(1), \tag{25}$$

which implies $\|v_n - v\| \longrightarrow 0$ as $n \longrightarrow \infty$

Proof of Theorem 10. As a consequence of Lemmas 8 and 9 and Mountain Pass theorem, we get the desired result.

Proof of Theorem 11. Since E is a reflexive and separable Banach space, there exist $\{e_j\} \subset E$ and $\{e_j^*\} \subset E^*$ such that $E = \overline{\text{span}}\{e_j : j = 1, 2, \cdots\}$, $E^* = \overline{span}\{e_j^* : j = 1, 2, \cdots\}$ in which

$$\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \tag{26}$$

Set $E_j = \text{span}\{e_j\}$, $Y_k = \oplus_{i=1}^{k} E_i$, $Z_k = \oplus_{i=k}^{\overline{\infty}} E_i$. We will use Dual Fountain theorem Lemma 5 to prove Theorem 2. Set

$$Q(v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) f^2(v) \right), \tag{27}$$

Q is a C^2-functional on E. Now, by the Taylor formula and some simple computation, we get

$$Q(v) = Q(0) + \left\langle Q'(0), v \right\rangle + \frac{1}{2} \left\langle Q''(0)v, v \right\rangle + o\left(\|v\|^2\right) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)v^2 \right) dx + o\left(\|v\|^2\right), \text{ as } \|v\| \longrightarrow 0. \tag{28}$$

On the other hand, on Z_k, for $\|v\|$ small enough, $\int_{\mathbb{R}^N} |G(x, f(v))| \le C_6 |v|_2^{r_1}$. Thus, by Remark 6 and simple computation, there exists $C > 0$ such that

$$I(v) \ge \|v\|^2 \left(\frac{C}{8} - \lambda C_6 \beta_k^{r_1} \|v\|^{r_1 - 2} \right), \tag{29}$$

which implies (A_1) holds. By (g_2), for $\|v\|$ which is small enough, we have

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) - \lambda \int_{\mathbb{R}^N} G(x, f(v)) - \mu \int_{\mathbb{R}^N} H(x, f(v)) \le \frac{1}{2} \|v\|^2 - C|v|_\sigma^\sigma + \mu C_2 |v|_2^2. \tag{30}$$

Since Y_k is a finite dimensional space, (A_2) is satisfied for every $\gamma_k > 0$ which small enough, when $\lambda > 0$, $\mu > 0$. From (29), for $k > k_0$, $v \in Z_k$, $\|v\| \le \rho_k$,

$$I(v) \ge -\lambda C_6 \beta_k^{r_1} \|v\|^{r_1} \ge -\lambda C_6 \beta_k^{r_1} \rho_k^{r_1}, \tag{31}$$

since $\beta_k \longrightarrow 0$ and $\rho_k \longrightarrow 0$ as $k \longrightarrow \infty$; condition (A_3) is also checked. (A_4) follows from Lemma 9. Thus, by using Lemma 5, we get the desired results and complete the proof.

Data Availability

The authors declare that there is no data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Relaxed Self-Adaptive Projection Algorithm for Solving the Multiple-Sets Split Equality Problem

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In this article, we introduce a relaxed self-adaptive projection algorithm for solving the multiple-sets split equality problem. Firstly, we transfer the original problem to the constrained multiple-sets split equality problem and a fixed point equation system is established. Then, we show the equivalence of the constrained multiple-sets split equality problem and the fixed point equation system. Secondly, we present a relaxed self-adaptive projection algorithm for the fixed point equation system. The advantage of the self-adaptive step size is that it could be obtained directly from the iterative procedure. Furthermore, we prove the convergence of the proposed algorithm. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithm.

1. Introduction

Let H_1, H_2, and H_3 be real Hilbert spaces. For $i = 1, 2, \cdots, t$, $j = 1, 2, \cdots, r$, C_i and Q_j are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and assume that $A : H_1 \longrightarrow H_3, B : H_2 \longrightarrow H_3$ are two bounded linear operators. The multiple-sets split equality problem (MSSEP) is to find x and y satisfying the property

$$x \in C = \bigcap_{i=1}^{t} C_i,\ y \in Q = \bigcap_{j=1}^{r} Q_j \text{ such that } Ax = By. \quad (1)$$

When $B = I$, MSSEP (1) reduces to the multiple-sets split feasibility problem

$$\text{find a point } x \in C = \bigcap_{i=1}^{t} C_i, Ax \in Q = \bigcap_{j=1}^{r} Q_j, \quad (2)$$

which is applied to intensity-modulated radiation therapy [1–11], signal processing [12–21], and image reconstruction [22–38]. Censor et al. [39] proposed the proximity function $p(x)$ to measure the distance of a point to all sets

$$p(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \left\| x - P_{C_i}(x) \right\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \left\| Ax - P_{Q_j}(Ax) \right\|^2, \quad (3)$$

where $l_i > 0$ for all i, and $\lambda_j > 0$ for all j with $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. To solve (2), they considered the following constrained MSSEP:

$$\text{find a point } x \in \Omega \text{ such that } x \text{ solves (2)}, \quad (4)$$

and then presented the projection method

$$x_{k+1} = P_{\Omega}(x_k - s\nabla p(x)), \quad (5)$$

where $s > 0$ and Ω is an auxiliary simple nonempty closed convex set with $\Omega \cap S \neq \varnothing$, and S denotes the solution set of (2). The convergence of the projection method was obtained under some mild conditions.

When $t = r = 1$, MSSEP (1) reduces to the split equality problem which was introduced by Moudafi [40] as follows:

$$\text{find two points } x \in C, y \in Q \text{ such that } Ax = By, \tag{6}$$

which is applied to the game theory [41] and optimal control and approximation theory [42]. The following alternating CQ algorithm (ACQ) was introduced by Moudafi [40] as follows:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{7}$$

where $\gamma_k, \beta_k \in (\varepsilon, \min\{(1/\lambda_A), (1/\lambda_B)\} - \varepsilon)$ for small enough $\varepsilon > 0$, A^* and B^* denote the adjoint of A and B, respectively. λ_A and λ_B are the spectral radiuses of A^*A and B^*B, respectively. Since the computation of P_C and P_Q onto a closed convex subset might be hard to be implemented, Fukushima [43] suggested a way to compute the projection onto a level set of a convex function by considering a sequence of projections onto half-spaces containing the original level set. Then, Moudafi [44] introduced the following relaxed alternating CQ algorithm (RACQ):

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \beta_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{8}$$

where C_k and Q_k are two sequences of closed convex sets.

Recently, Dang et al. [45] gave the following relaxed two-point projection method to solve MSSEP (1):

$$\begin{cases} x_{k+1} = P_{\Omega_1}\left(x_k - \gamma\left(\sum_{i=1}^{t} \alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right)\right), \\ y_{k+1} = P_{\Omega_2}\left(y_k - \gamma\left(\sum_{j=1}^{r} \beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_{k+1} - By_k)\right)\right), \end{cases} \tag{9}$$

where $\gamma \in (0, \min\{\{1/21/2, 1/(4\|A\|^2)(1/4\|A\|^2), 1/(4\|B\|^2)\})$, $C_{i,k}$, $i = 1, 2, \cdots, r$ and $Q_{j,k}$, $j = 1, 2, \cdots, t$ are two sequences of closed convex sets corresponding to C_i and Q_j, respectively. $\Omega_1 \subset H_1$ and $\Omega_2 \subset H_2$ are auxiliary simple sets. $\alpha_i > 0$ for all i, and $\beta_j > 0$ for all j with $\sum_{i=1}^{t}\alpha_i + \sum_{j=1}^{r}\beta_j = 1$. Under some mild conditions, the weak convergence of the algorithm (9) was obtained.

Noting that the determination of the stepsize γ of algorithm (9) depends on the operator (matrix) norms $\|A\|$ and $\|B\|$. This implies that if we implement the relaxed two-point projection method (9), one first need to calculate operator norms of A and B, which is in general not an easy work in practice. To overcome this weakness, Lopez et al. [46] and Zhao and Yang [47] introduced self-adaptive methods of which the advantage of the methods is that the stepsizes do not need prior knowledge of the operator norms. Motivated by them, we introduce a relaxed self-adaptive projection algorithm for solving the multiple-sets split equality problem. First, we transfer the origin problem to the constrained multiple-sets split equality problem and establish the fixed point equation system. We show the equivalence of the constrained multiple-sets split equality problem and the fixed point equation system. Second, based on the fixed point equation system, we present a relaxed self-adaptive projection algorithm for solving the constrained multiple-sets split equality problem, and the convergence of the proposed algorithm is obtained. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithm.

The remainder of this paper is organized as follows. Section 2 shows some preliminaries and notations used for subsequent analysis. In Section 3, we transfer the origin problem to the constrained multiple-sets split equality problem and establish the fixed point equation system and propose a relaxed self-adaptive projection algorithm for solving the constrained multiple-sets split equality problem. The convergence of the proposed algorithm is obtained. In Section 4, several numerical results are shown to confirm the effectiveness of our algorithm.

2. Preliminaries

Throughout this paper, we use $\longrightarrow$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We write $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to indicate the weak ω-limit set of $\{x_k\}$. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \forall y \in C. \tag{10}$$

It is well known that P_C is nonexpansive and firmly nonexpansive. Moreover, P_C has the following well-known properties (see for example [48]).

Lemma 1. *Let $C \subset H$ be nonempty, closed and convex. Then for all $x, y \in H$ and $z \in C$,*

(i) $\langle x - P_C x, z - P_C x\rangle \le 0$

(ii) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$;

(iii) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2$.

Definition 2. Let $f : H \longrightarrow R$ be convex. The subdifferential of f at x is defined as

$$\partial f(x) = \{\xi \in H \mid f(y) \ge f(x) + \langle \xi, y - x\rangle, \forall y \in H\}. \tag{11}$$

An element of $\partial f(x)$ is said to be a subgradient.

Lemma 3. *Suppose $f : H \longrightarrow R$ is a convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded set of H.*

3. Algorithm and Its Convergence

In this section, we focus on a relaxed self-adaptive projection algorithm and obtain the convergence of the proposed algorithm. Following the idea of Censor et al. [39], we give two additional closed convex sets $\Omega_1 \subset H_1$ and $\Omega_2 \subset H_2$ and consider the constrained multiple-sets split equality problem

$$\text{find } x \in \Omega_1,\ y \in \Omega_2 \text{ such that } (x, y) \text{ solves } (1), \tag{12}$$

where the sets C_i and Q_j can be expressed by

$$\begin{aligned} C_i &= \{x \in H_1 \mid c_i(x) \le 0\}, \\ Q_j &= \left\{y \in H_2 \mid q_j(y_k) \le 0\right\}, \end{aligned} \tag{13}$$

$c_i : H_1 \longrightarrow R$ and $q_j : H_2 \longrightarrow R$ are convex functions for all $i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, r$, and Γ denotes the solution set of (32). Define

$$C_{i,k} = \left\{x \in H_1 \mid c_i(x_k) + \left\langle \xi_{i,k}, x - x_k \right\rangle \le 0\right\}, \tag{14}$$

where $\xi_{i,k} \in \partial c_i(x_k)$ and

$$Q_{j,k} = \left\{y \in H_2 \mid q_j(y_k) + \left\langle \eta_{j,k}, y - y_k \right\rangle \le 0\right\}, \tag{15}$$

where $\eta_{j,k} \in \partial q_j(y_k)$. It is easily seen that $C_i \subset C_{i,k}$ and $Q_j \subset Q_{j,k}$ for all k. Notice that $C_{i,k}$ and $Q_{j,k}$ are half-spaces and thus the corresponding projections have closed-form expressions. Hence, we focus on the following multiple-sets split equality problem (CMSSEP):

$$\begin{aligned} &\text{find } x \in \Omega_1,\ y \in \Omega_2 \text{ to solve } x \\ &\quad \in C = \bigcap_{i=1}^{t} C_{i,k}, y \in Q = \bigcap_{j=1}^{r} Q_{j,k} \text{such that } Ax = By. \end{aligned} \tag{16}$$

Now, we define the proximity function $p_k(x, y)$:

$$\begin{aligned} p_k(x, y) = \frac{1}{2}\sum_{i=1}^{t} \alpha_i \left\| x - P_{C_{i,k}}(x) \right\|^2 + \frac{1}{2}\sum_{j=1}^{r} \beta_j \left\| y - P_{Q_{j,k}}(y) \right\|^2 \\ + \frac{1}{2}\|Ax - By\|^2, \end{aligned} \tag{17}$$

where $\alpha_i > 0$ for all i, and $\beta_j > 0$ for all j with $\sum_{i=1}^{t} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$.

Using the proximity function $p_k(x, y)$, we can obtain the following technical lemmas.

Lemma 4. *Assume that (16) is consistent (i.e.,(16) has a solution) and denotes its solution set by Γ. If $(x\, y) \in \Gamma$, then it solves the fixed point equation system*

$$\begin{cases} x = P_{\Omega_1}\left(x - \lambda\left(\sum_{i=1}^{t} \alpha_i\left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right)\right), \\ y = P_{\Omega_2}\left(y - \beta\left(\sum_{j=1}^{r} \beta_j\left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right)\right). \end{cases} \tag{18}$$

Proof. To solve the problem (16), we consider the minimization problem

$$\min\{p_k(x, y) \mid x \in \Omega_1, y \in \Omega_2\}. \tag{19}$$

(19) leads to the following unconstrained optimization problem:

$$\min_{x \in \Omega_1, y \in \Omega_2} \left\{\delta_{\Omega_1}(x) + \delta_{\Omega_2}(y) + p_k(x, y)\right\}, \tag{20}$$

where δ_{Ω_i} is a indicator function of Ω_i for $i = 1, 2$ defined by

$$\delta_{\Omega_i}(x) = \begin{cases} 0, & x \in \Omega_i, \\ +\infty, & \text{otherwise.} \end{cases} \tag{21}$$

Note that $\partial\delta_{\Omega_1}(x) = N_{\Omega_1}(x)$ and $\partial\delta_{\Omega_2}(y) = N_{\Omega_2}(y)$, where N_{Ω_1} and N_{Ω_2} are the normal cone of the convex sets Ω_1 and Ω_1, respectively. From the optimality conditions of (20), it yields

$$\begin{cases} 0 \in \sum_{i=1}^{t} \alpha_i\left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By) + \partial\delta_{\Omega_1}(x), \\ 0 \in \sum_{j=1}^{r} \beta_j\left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By) + \partial\delta_{\Omega_2}(y), \end{cases} \tag{22}$$

which means that, for $\lambda > 0, \beta > 0$,

$$\begin{aligned} x - \lambda\left(\sum_{i=1}^{t} \alpha_i\left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right) &= x + \lambda\partial\delta_{\Omega_1}(x), \\ y - \beta\left(\sum_{j=1}^{r} \beta_j\left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right) &= y + \beta\partial\delta_{\Omega_2}(y), \end{aligned} \tag{23}$$

that is,

$$\begin{aligned} x &= \left(I + \lambda N_{\Omega_1}\right)^{-1}\left(x - \lambda\left(\sum_{i=1}^{t} \alpha_i\left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right)\right), \\ y &= \left(I + \beta N_{\Omega_2}\right)^{-1}\left(y - \beta\left(\sum_{j=1}^{r} \beta_j\left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right)\right). \end{aligned} \tag{24}$$

Since $(I+\lambda N_{\Omega_1})^{-1}=P_{\Omega_1}$ and $(I+\beta N_{\Omega_2})^{-1}=P_{\Omega_2}$, we obtain

$$\begin{cases} x=P_{\Omega_1}\left(x-\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x-P_{C_{i,k}}(x)\right)+A^T(Ax-By)\right)\right), \\ y=P_{\Omega_2}\left(y-\beta\left(\sum_{j=1}^{r}\beta_j\left(y-P_{Q_{j,k}}(y)\right)-B^T(Ax-By)\right)\right). \end{cases} \tag{25}$$

Thus, the desired result can be obtained.

The following lemma reveals that ESEP (16) is equivalent to the fixed point equation system (18).

Lemma 5. *Assume that the problem (16) is consistent.* $(x^*,y^*)\in\Gamma$ *solves ESEP (2) if and only if* (x^*,y^*) *solves the fixed point equation system (18).*

Proof. From Lemma 4, we reveal that (x^*,y^*) can solve (16); it also can solve (18). Next, we will prove that (x^*,y^*) can solve (18), it also can solve (16). Obviously, one has $x^*\in\Omega_1$, and $y^*\in\Omega_2$. It follows from the proposition of projection that

$$\begin{cases} \left\langle x^*-\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x^*-P_{C_{i,k}}(x^*)\right)+A^T(Ax^*-By^*)\right)-x^*,u-x^*\right\rangle\le 0,u\in\Gamma, \\ \left\langle y^*-\beta\left(\sum_{j=1}^{r}\beta_j\left(y^*-P_{Q_{j,k}}(y^*)\right)-B^T(Ax^*-By^*)\right)-y^*,v-y^*\right\rangle\le 0,v\in\Gamma. \end{cases} \tag{26}$$

which means

$$\begin{cases} \left\langle -\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x^*-P_{C_{i,k}}(x^*)\right)+A^T(Ax^*-By^*)\right),u-x^*\right\rangle\le 0,u\in\Gamma, \\ \left\langle -\beta\left(\sum_{j=1}^{r}\beta_j\left(y^*-P_{Q_{j,k}}(y^*)\right)-B^T(Ax^*-By^*)\right),v-y^*\right\rangle\le 0,v\in\Gamma. \end{cases} \tag{27}$$

Hence, from Lemma 3, we add two inequalities to obtain

$$\begin{aligned} &\sum_{i=1}^{t}\alpha_i\left\|x^*-P_{C_{i,k}}(x^*)\right\|^2+\sum_{j=1}^{r}\beta_j\left\|y^*-P_{Q_{j,k}}(y^*)\right\|^2 \\ &\quad+\left\langle Ax^*-By^*,Bv-Au+Ax^*-By^*\right\rangle\le 0. \end{aligned} \tag{28}$$

Furthermore, from $Au=Bv$, we deduce

$$\begin{aligned} &\|x^*-P_{C_{i,k}}(x^*)\|=0,\text{ for } i=1,2,\cdots,t, \\ &\left\|y^*-P_{Q_{j,k}}(y^*)\right\|^2=0,\text{ for } j=1,2,\cdots,r, \\ &\|Ax^*-By^*\|^2=0. \end{aligned} \tag{29}$$

Thus, (x^*,y^*) solves ESEP (16). This completes the proof.

Based on (18), we can introduce a relaxed self-adaptive projection algorithm to solve (16), with $\sigma_k\in(0,1)$.

Alggorithm 6. Let $x_0\in H_1,y_0\in H_2$ be arbitrary. We calculate the $(k+1)$th iterate via the following formula

$$\begin{cases} u_k=P_{\Omega_1}\left(x_k-\lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\right)\right), \\ x_{k+1}=\gamma_k x_k+(1-\gamma_k)u_k, \\ v_k=P_{\Omega_2}\left(y_k-\lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\right)\right), \\ y_{k+1}=\gamma_k y_k+(1-\gamma_k)v_k, \end{cases} \tag{30}$$

where the stepsize λ_k is chosen by

$$\lambda_k=2\sigma_k\frac{\sum_{i=1}^{t}\alpha_i\|x_k-P_{C_{i,k}}(x_k)\|^2+\sum_{j=1}^{r}\beta_j\|y_k-P_{Q_{j,k}}(y_k)\|^2+\|Ax_k-By_k\|^2}{\left\|\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\right\|^2+\left\|\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\right\|^2}=4\sigma_k\frac{p_k(x_k,y_k)}{\|\nabla p_k(x_k,y_k)\|^2}, \tag{31}$$

with $\sigma_k\in(1,0)$.

Next, we will focus on the convergence analysis of Algorithm 6.

Theorem 7. *Assume* $\lim_{k\to\infty}\gamma_k=0$, $\sum_{k=1}^{\infty}\gamma_k=\infty$ *and* $\sigma_k\in[M_1,M_2]\subset(0,1)$, *then the sequence* (x_k,y_k) *generated by Algorithm 6 converges to a solution of (1).*

Proof. Taking $(x^*,y^*)\in\Gamma$, one has

$$Ax^*=By^*. \tag{32}$$

From (30) and the fact that the projection is nonexpansive, we have

$$\begin{aligned}\|u_k - x^*\|^2 &= \left\| P_{\Omega_i}\left(x_k - \lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right)\right) - x^*\right\|^2 \\ &\le \left\| x_k - \lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right) - x^*\right\|^2 \\ &= \left\| x_k - x^*\|^2 + (\lambda_k)^2\right\| \sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right\|^2 \\ &\quad - 2\lambda_k\left\langle \sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k), x_k - x^*\right\rangle.\end{aligned} \tag{33}$$

Since

$$\begin{aligned}&-2\lambda_k\left\langle \sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k), x_k - x^*\right\rangle \\ &\quad = -2\lambda_k\left\langle \sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right), x_k - x^*\right\rangle \\ &\qquad - 2\lambda_k\left\langle A^T(Ax_k - By_k), x_k - x^*\right\rangle \\ &\quad = -2\lambda_k\sum_{i=1}^{t}\alpha_i\left\langle x_k - P_{C_{i,k}}(x_k), x_k - x^*\right\rangle \\ &\qquad - 2\lambda_k\langle Ax_k - By_k, Ax_k - Ax^*\rangle \\ &\quad \le -2\lambda_k\sum_{i=1}^{t}\alpha_i\|x_k - P_{C_{i,k}}(x_k)\|^2 - \lambda_k\|Ax_k - By_k\|^2 \\ &\qquad - \lambda_k\|Ax_k - Ax^*\|^2 + \lambda_k\|By_k - Ax^*\|^2,\end{aligned} \tag{34}$$

together with (33), we deduce

$$\begin{aligned}\|u_k - x^*\|^2 &\le \|x_k - \lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right) - x^*\|^2 \\ &= \|x_k - x^*\|^2 + (\lambda_k)^2\|\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\|^2 \\ &\quad - 2\lambda_k\sum_{i=1}^{t}\alpha_i\|x_k - P_{C_{i,k}}(x_k)\|^2 - \lambda_k\|Ax_k - By_k\|^2 \\ &\quad - \lambda_k\|Ax_k - Ax^*\|^2 + \lambda_k\|By_k - Ax^*\|^2.\end{aligned} \tag{35}$$

Similarly, we have

$$\begin{aligned}\|v_k - y^*\|^2 &= \|P_{\Omega_2}\left(y_k - \lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k)\right)\right) - y^*\|^2 \\ &\le \|y_k - \lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k)\right) - y^*\|^2 \\ &\le \|y_k - y^*\|^2 + (\lambda_k)^2\|\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k))\|^2 \\ &\quad - 2\lambda_k\sum_{i=1}^{t}\beta_j\|y_k - P_{Q_{j,k}}(y_k)\|^2 - \lambda_k\|By_k - By^*\|^2 \\ &\quad - \lambda_k\|Ax_k - By_k\|^2 + \lambda_k\|Ax_k - By^*\|^2.\end{aligned} \tag{36}$$

From (35) and (36), it follows

$$\begin{aligned}\|u_k - x^*\|^2 + \|v_k - y^*\|^2 &\le \|x_k - x^*\|^2 + \|y_k - y^*\|^2 \\ &\quad - \lambda_k\left(2\left(\sum_{i=1}^{t}\alpha_i\|x_k - P_{C_{i,k}}(x_k)\|^2 + \sum_{i=1}^{t}\beta_j\|y_k\right.\right. \\ &\quad \left. - P_{Q_{j,k}}(y_k)\|^2 + \|Ax_k - By_k\|^2\right) \\ &\quad - \lambda_k\left(\|\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\|^2\right. \\ &\quad \left.\left. + \|\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k)\|^2\right)\right),\end{aligned} \tag{37}$$

which together with (31) means

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \le \|x_k - x^*\|^2 + \|y_k - y^*\|^2. \tag{38}$$

Furthermore, it follows from (31) and (38) that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 &= \|\gamma_k x_k + (1-\gamma_k)u_k \\ &\quad - x^*\|^2 + \|\gamma_k y_k + (1-\gamma_k)v_k - y^*\|^2 \\ &\le \gamma_k\|x_k - x^*\|^2 + (1-\gamma_k)\|u_k - x^*\|^2 \\ &\quad + \gamma_k\|y_k - y^*\|^2 + (1-\lambda_k)\|v_k - y^*\|^2 \\ &\le \gamma_k\left(\|x_k - x^*\|^2 + \|y_k - y^*\|^2\right) \\ &\quad + (1-\gamma_k)\left(\|u_k - x^*\|^2 + \|v_k - y^*\|^2\right) \\ &\le \|x_k - x^*\|^2 + \|y_k - y^*\|^2,\end{aligned} \tag{39}$$

By induction, one has

$$\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \le \|x_0 - x^*\|^2 + \|y_0 - y^*\|^2. \tag{40}$$

Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Following (31), (36), and (39), we have

$$\begin{aligned}\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 &\le \gamma_k\left(\|x_k - x^*\|^2 + \|y_k - y^*\|^2\right) \\ &\quad + (1-\gamma_k)\left(\|u_k - x^*\|^2 + \|v_k - y^*\|^2\right) \le \|x_k - x^*\|^2 \\ &\quad + \|y_k - y^*\|^2 - (1-\gamma_k)\lambda_k\left(2\left(\sum_{i=1}^{t}\alpha_i\|x_k - P_{C_{i,k}}(x_k)\|^2\right.\right. \\ &\quad \left. + \sum_{i=1}^{t}\beta_j\|y_k - P_{Q_{j,k}}(y_k)\|^2 + \|Ax_k - By_k\|^2\right) \\ &\quad - \lambda_k\left(\|\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\|^2\right. \\ &\quad \left.\left. + \|\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k)\|^2\right)\right).\end{aligned} \tag{41}$$

Without loss of generality, we can assume that there is $\sigma > 0$ such that $4(1-\gamma_k)\sigma_k(1-\sigma_k) > \sigma$ for all k. Setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, together with (41), we have the following inequality

$$\sigma \frac{(p_k(x_k,y_k))^2}{\|\nabla p_k(x_k,y_k)\|^2} + s_{k+1} - s_k \le 0. \tag{42}$$

Since s_k is eventually decreasing, we obtain s_k as convergent. From (42), we have $\lim_{k\to\infty} p_k(x_k,y_k) = 0$. Furthermore,

$$\lim_{k\to\infty} \|x_k - P_{C_{i,k}}(x_k)\|^2 = 0, for i = 1,2,\cdots,t, \tag{43}$$

$$\lim_{k\to\infty} \|y_k - P_{Q_{j,k}}(y_k)\|^2 = 0, for j = 1,2,\cdots,r, \tag{44}$$

$$\lim_{k\to\infty} \|Ax_k - By_k\|^2 = 0. \tag{45}$$

Furthermore,

$$\begin{aligned}\|x_{k+1} - x_k\| &= \|\gamma_k x_k + (1-\gamma_k)u_k - x_k\| = (1-\gamma_k)\|u_k - x_k\| \\ &\le (1-\gamma_k)\lambda_k\left(\sum_{i=1}^{t}\alpha_i\|x_k - P_{C_{i,k}}(x_k)\| + \|A^T(Ax_k - By_k)\|\right),\end{aligned} \tag{46}$$

which with (41), (45), and the assumption on γ_k means

$$\lim_{k\to\infty} \|x_{k+1} - x_k\|^2 = 0. \tag{47}$$

Note that

$$\|x_{k+1} - u_k\| = \|\gamma_k x_k + (1-\gamma_k)u_k - u_k\| = \gamma_k\|x_k - u_k\|, \tag{48}$$

we have

$$\lim_{k\to\infty} \|x_{k+1} - u_k\|^2 = 0. \tag{49}$$

(47) and (49) imply

$$\lim_{k\to\infty} \|x_k - u_k\|^2 = 0. \tag{50}$$

Similarly, we have

$$\begin{aligned}\lim_{k\to\infty} \|y_{k+1} - y_k\|^2 &= 0, \\ \lim_{k\to\infty} \|y_{k+1} - v_k\|^2 &= 0, \\ \lim_{k\to\infty} \|y_k - v_k\|^2 &= 0.\end{aligned} \tag{51}$$

Thus, $\{x_k\}$ and $\{y_k\}$ are asymptotically regular. Notice that

$$\begin{aligned}&\left\|\sum_{i=1}^{t}\alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k))\right\|^2 \\ &\quad + \left\|\sum_{j=1}^{r}\beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k))\right\|^2 \\ &\le 2\left(\sum_{i=1}^{t}\alpha_i\|\left(x_k, -, P_{C_{i,k}},(x_k)\right\|^2 + \|A\|^2\|Ax_k - By_k\|^2\right. \\ &\quad \left. + \sum_{j=1}^{r}\beta_j\|y_k - P_{Q_{j,k}}(y_k)\|^2 + \|B\|^2\|Ax_k - By_k\|^2\right) \\ &\le 2\max\left\{1,\|A\|^2 + \|B\|^2\right\}\left(\sum_{i=1}^{t}\alpha_i\|\left(x_k, -, P_{C_{i,k}},(x_k)\right\|^2\right. \\ &\quad \left. + \|Ax_k - By_k\|^2 + \sum_{j=1}^{r}\beta_j\|y_k - P_{Q_{j,k}}(y_k)\|^2\right),\end{aligned} \tag{52}$$

which implies that

$$\lambda_k \ge \sigma_k \frac{1}{\max\left\{1,\|A\|^2 + \|B\|^2\right\}}. \tag{53}$$

Moreover, it follows from (22) that

$$\begin{aligned}\left\|\frac{x_{k+1} - x_k}{\lambda_k}\right\| &= (1-\gamma_k)\frac{1}{\lambda_k}\|u_k - x_k\| \le (1-\gamma_k) \\ &\quad \cdot \left(\sum_{i=1}^{t}\alpha_i\|x_k - \mathrm{P}_{C_{i,k}}(x_k)\| + \|A^T(Ax_k - By_k)\|\right),\end{aligned} \tag{54}$$

which with (43), (45), and the assumption on γ_k yields

$$\lim_{k\to\infty} \left\|\frac{x_{k+1} - x_k}{\lambda_k}\right\| = 0. \tag{55}$$

Similarly, one has

$$\lim_{k\to\infty} \left\|\frac{y_{k+1} - y_k}{\lambda_k}\right\| = 0. \tag{56}$$

Let $\bar{x}$ and $\bar{y}$ be, respectively, weak cluster points of the sequences $\{x_k\}$ and $\{y_k\}$, then there exist two subsequences of $\{x_k\}$ and $\{y_k\}$ (again labeled $\{x_k\}$ and $\{y_k\}$ which converge weakly to $\bar{x}$ and $\bar{y}$). Next, we will show that $(\bar{x},\bar{y}) \in \Gamma$. It follows from (30) that

$$\frac{x_{k+1}-x_k}{\lambda_k(1-\gamma_k)}-\lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\right)\in N_{\Omega_1}\left(\frac{x_{k+1}-\gamma_k x_k}{1-\gamma_k}\right),$$
$$\frac{y_{k+1}-y_k}{\lambda_k(1-\gamma_k)}-\lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\right)\in N_{\Omega_2}\left(\frac{y_{k+1}-\gamma_k y_k}{1-\gamma_k}\right). \tag{57}$$

From the graphs of the maximal monotone operators, N_C and N_Q are weakly-strongly closed, and by passing to the limit in the last inclusions, we obtain that $\bar{x}\in\Omega_1$ and $\bar{y}\in\Omega_2$.

On the other hand, from Lemma 1 and the definition of $C_{i,k}$, one has

$$c_i(x_k)\le\left\langle\xi^{i,k},x_k-P_{C_{i,k}}(x_k)\right\rangle\le\|\xi^{i,k}\|\|x_k-P_{C_{i,k}}(x_k)\|\le M_1\|x_k-P_{C_{i,k}}(x_k)\|, \tag{58}$$

where M satisfies $\|\xi^{i,k}\|\le M_1$ for all k. The lower semicontinuity of function $c_i(x)$ and (41) assert that

$$c_i(\bar{x})\le\liminf_{k\to\infty}c_i(x_k)\le 0. \tag{59}$$

Thus, $\bar{x}\in C_i$ for $i=1,2,\cdots,t$. Likewise, we can obtain

$$q_j(x_k)\le\left\langle\eta^{j,k},y_k-P_{Q_{j,k}}(y_k)\right\rangle\le\|\eta^{j,k}\|\|y_k-P_{Q_{j,k}}(y_k)\|\le M_2\|y_k-P_{Q_{j,k}}(y_k)\|, \tag{60}$$

where M_2 satisfies $\|\eta^{j,k}\|\le M_2$ for all k. The lower semicontinuity of function $q_j(x)$ and (42) lead to

$$q_i(\bar{y})\le\liminf_{k\to\infty}q_i(y_k)\le 0. \tag{61}$$

Thus, $\bar{y}\in Q_j$ for $j=1,2,\cdots,r$. Moreover, the weak convergence of Ax_k-By_k to $A\bar{x}-B\bar{y}$ and the lower semicontinuity of the squared norm imply

$$\|A\bar{x}-B\bar{y}\|\le\liminf_{k\to\infty}\|Ax_k-By_k\|=0, \tag{62}$$

hence, $(\bar{x},\bar{y})\in\Gamma$. This completes the proof.

4. Numerical Examples

We are in a position to show numerical examples to demonstrate the performance and convergence of Algorithm 6. The whole programs are written in MATLAB 7.0. All the numerical results are carried out on a personal Lenovo computer with Intel®Core™ i7-7500 U CPU 2.70 GHz and RAM 4.00 GB. We denote the vector with all elements 1 by e in what follows.

Example 8. Let

$$A=\begin{pmatrix}3&-1&2\\2&1&0\\3&0&3\end{pmatrix},B=\begin{pmatrix}4&-4&2&1\\3&-1&4&3\\5&1&0&4\end{pmatrix} \tag{63}$$

$C_1=\{x\in R^3\mid x_1+5x_2^2+4x_3\le 0\}$, $C_2=\{x\in R^3\mid 3x_1+10x_3\le 0\}$, $Q_1=\{y\in R^4\mid 2y_1-3y_2-2y_3+4y_4\le 0\}$, and $Q_2=\{y\in R^4\mid 2y_1^2-y_2+4y_3-3y_4\le 0\}$. Find $x\in C=C_1\cap C_2$, $y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

Example 9. Let

$$A=\begin{pmatrix}0.2620&0.0268&0.2589\\0.5697&0.5004&0.0458\\0.3595&0.8270&0.2464\end{pmatrix},$$
$$B=\begin{pmatrix}0.6607&0.0130&0.0335&0.9213\\0.3294&0.7180&0.4060&0.9840\\0.6594&0.3911&0.7163&0.9834\end{pmatrix} \tag{64}$$

$C_1=\{x\in R^3\mid x_1^4+x_2^2-2x_3^2-1\le 0\}$, $C_2=\{x\in R^3\mid 2x_1^2+x_2^3-3x_3^2-1\le 0\}$, $Q_1=\{y\in R^4\mid 2y_1^3-y_2^2+2y_3^3+6y_4-2\le 0\}$, and $Q_2=\{y\in R^4\mid 2y_1^2+3y_3^2+2y_4^2-2\le 0\}$. Find $x\in C=C_1\cap C_2$, $y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

Example 10. Let $A=(a_{ij})_{J\times N}$ and $B=(b_{ij})_{J\times M}$. $C_1=\{x_1\in R^N\mid\|x_1\|\le 2\}$, $C_2=\{x_2\in R^N\mid -e\le x_2\le 3e\}$. $Q_1=\{y_1\in R^M\mid -2e\le y_1\le 6e\}$, and $Q_2=\{y_2\in R^M\mid\|y_2\|\le 4\}$, where $\{a_{ij}\},\{b_{ij}\}\in(0,1)$ are all generated randomly; J, N and M are positive integers. Find $x\in C=C_1\cap C_2$, $y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

In this example, we consider $J=10$, $N=10$, and $M=20$; $J=20$, $N=30$, and $M=40$; and $J=40$, $N=50$, and $M=60$ and three initial values:

(i) Case 1 $x=\text{ones}(N,1)$, $y=\text{ones}(M,1)$;

(ii) Case 2 $x=10*\text{ones}(N,1)$, $y=10*\text{ones}(M,1)$;

(iii) Case 3 $x=-10*\text{ones}(N,1)$, $y=-10*\text{ones}(M,1)$.

We take $\Omega_1=C_{1,n}$, $\Omega_2=Q_{1,n}$ when the algorithm iterates to step n, $\gamma_k=1/20k$, $\sigma_k=(1/4)+(1/2k)$, $\alpha_1=\alpha_2=\beta_1=\beta_2=1/4$ in Algorithm 6. In the following tables and figures, we denote Algorithm 6 and the algorithm in reference [45] by QSPA and RTPPM, respectively. And we set "n", "s" and "x^*", " and "y^*" to express the number of iteration, CPU time in seconds, and the final solution, respectively. Init. denote the initial points, and $p_k(x,y)\le\varepsilon=10^{-4}$ is used as the stop

Table 1: The numerical results of Example 8.

Init.	QSPR	RTPPM
$x_1 = (0, 0, 0)^T$	$n = 14, s = 0.000671$	$n = 7684, s = 0.198442$
$y_1 = (1, 0, -1, 1)^T$	$x^* = (0.0196, -0.0222, -0.0055)^T$	$x^* = (0.0342, -0.0734, -0.0153)^T$
	$y^* = (0.0345, 0.0119, 0.0071, -0.0357)^T$	$y^* = (0.0509, 0.0012, -0.0023, -0.0495)^T$
$x_1 = (0, 1, 1)^T$	$n = 15, s = 0.000636$	$n = 49, s = 0.001585$
$y_1 = (0, 0, 0, 0)^T$	$x^* = (-0.0750, 0.0232, 0.0181)^T$	$x^* = (-0.2420, -0.0354, 0.0589)^T$
	$y^* = (-0.0279, 0.0205, 0.0014, -0.0109)^T$	$y^* = (-0.0721, 0.0426, -0.0254, -0.0566)^T$
$x_1 = (1, 1, 1)^T$	$n = 258, s = 0.008365$	$n = 34587, s = 0.816461$
$y_1 = (1, 1, -1, -1)^T$	$x^* = (-0.1554, -0.0125, -0.0255)^T$	$x^* = (0.0836, -0.1013, -0.0566)^T$
	$y^* = (-0.0951, 0.0345, 0.0180, -0.0257)^T$	$y^* = (0.0496, -0.0180, 0.0025, -0.0371)^T$

Table 2: The numerical results of Example 9.

Init.	QSPR	RTPPM
$x_1 = (1, 1, 1)^T$	$n = 16, s = 0.001268$	$n = 478, s = 0.029797$
$y_1 = (1, 1, 1, 1)^T$	$x^* = (0.1710, 0.1525, 0.1824)^T$	$x^* = (1.2294, 0.7761, 0.9712)^T$
	$y^* = (0.1110, 0.1218, 0.1283, 0.0100)^T$	$y^* = (0.6171, 0.7022, 0.6533, 0.1692)^T$
$x_1 = 10(1, 1, 1)^T$	$n = 37, s = 0.001877$	$n = 1687, s = 0.079442$
$y_1 = 10(1, 1, 1, 1)^T$	$x^* = (0.2973, 0.3380, 0.7617)^T$	$x^* = (1.5257, 0.4196, 1.5156)^T$
	$y^* = (0.6560, 0.3316, 0.2638, -0.1878)^T$	$y^* = (0.9622, 0.8416, 0.2119, 0.1513)^T$
$x_1 = 100(1, 1, 1)^T$	$n = 63, s = 0.003101$	$n = 2651, s = 0.110352$
$y_1 = 100(1, 1, 1, 1)^T$	$x^* = (0.5363, -0.1663, 2.0146)^T$	$x^* = (1.5276, 0.3916, 1.5165)^T$
	$y^* = (0.9380, 0.0704, -0.1862, 0.0354)^T$	$y^* = (0.9713, 0.8464, 0.1785, 0.1454)^T$

Table 3: The numerical results of Example 10.

				QSPR with λ_n		QSPR with $0.5\,\lambda_n$		RTPPM with λ_n	
	J	N	M	n	s	n	s	n	s
Case 1	10	10	20	30	0.003122	48	0.004302	808	0.050933
	20	30	40	37	0.005304	94	0.011647	1994	0.258627
	40	50	60	91	0.013088	188	0.028659	4014	1.286561
Case 2	10	10	20	56	0.011269	125	0.031055	3236	0.201379
	20	30	40	107	0.038957	171	0.039266	1762	0.233102
	40	50	60	295	0.041663	351	0.100628	5084	1.673274
Case 3	10	10	20	67	0.008644	98	0.016412	812	0.051558
	20	30	40	118	0.015576	178	0.028107	1953	0.270206
	40	50	60	233	0.058625	302	0.108850	4176	1.360695

criterion. The numerical results can be seen from Tables 1–3 and Figures 1–4. For Figures 3 and 4, take $J = 20, N = 30$, and $M = 40$ in Example 10.

From Tables 1–3, we can see that the iterative number and CPU time of Algorithm 6 is less algorithm RTPPM. Figures 1–4 indicate that Algorithm 6 is more stable than RTPPM.

Furthermore, for testing the stationary property of iterative number, we carry 500 experiments for the initial point which are presented randomly, such as

$$x_1 = \text{rand}\,(3, 1), y_1 = \text{rand}\,(4, 1), \tag{65}$$

in Example 9, the results can be found in Figure 1.

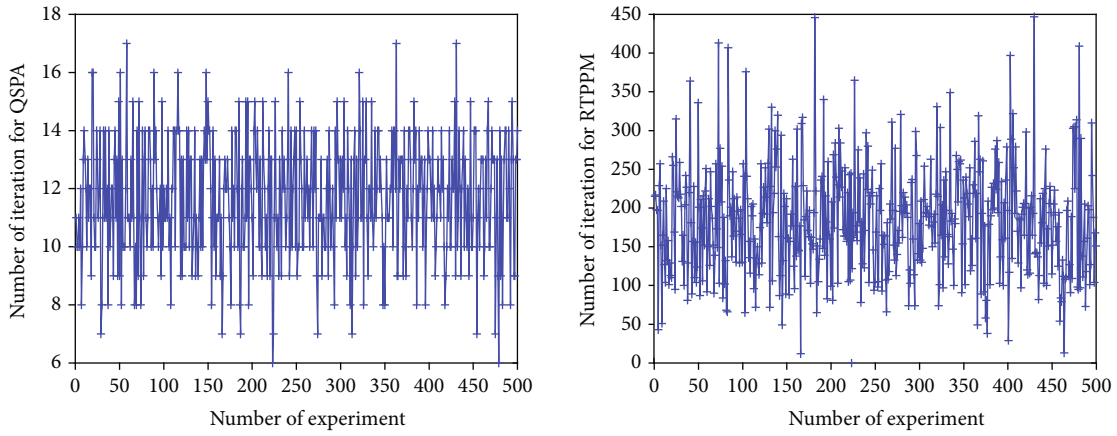

Figure 1: The iteration number of QSPA and RTPPM.

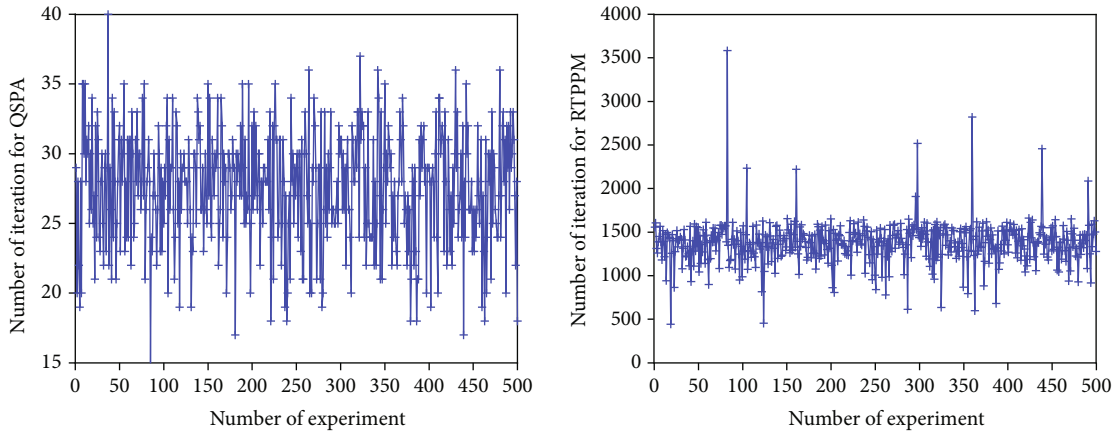

Figure 2: The iteration number of QSPA and RTPPM.

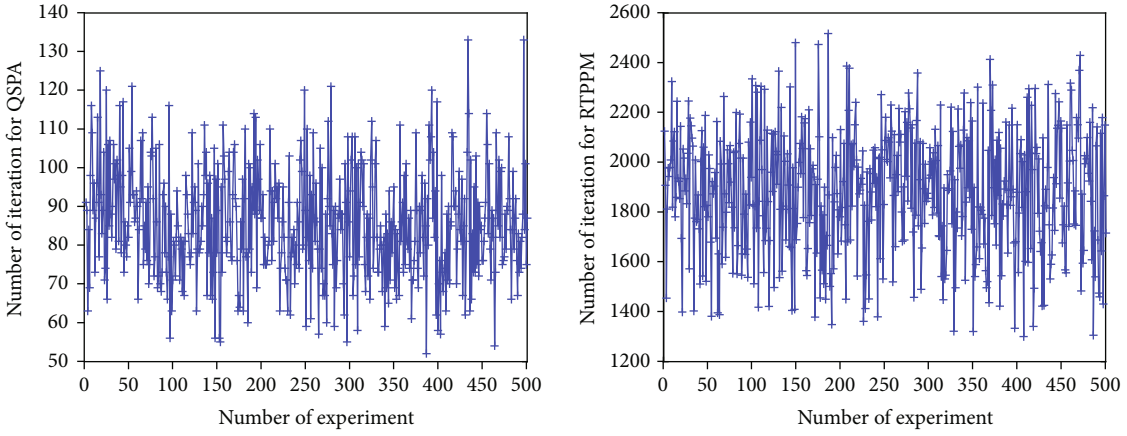

Figure 3: The iteration number of QSPA and RTPPM.

On the other initial point, such as

$$x_1 = \text{rand}\,(3,1) * 10, y_1 = \text{rand}\,(4,1) * 10, \tag{66}$$

in Example 9, the results can be found in Figure 2.

Similarly, we carry 500 experiments for the initial point which are presented randomly, such as

$$x_1 = \text{rand}\,(N,1), y_1 = \text{rand}\,(M,1), \tag{67}$$

in Example 10, the results can be found in Figure 3.

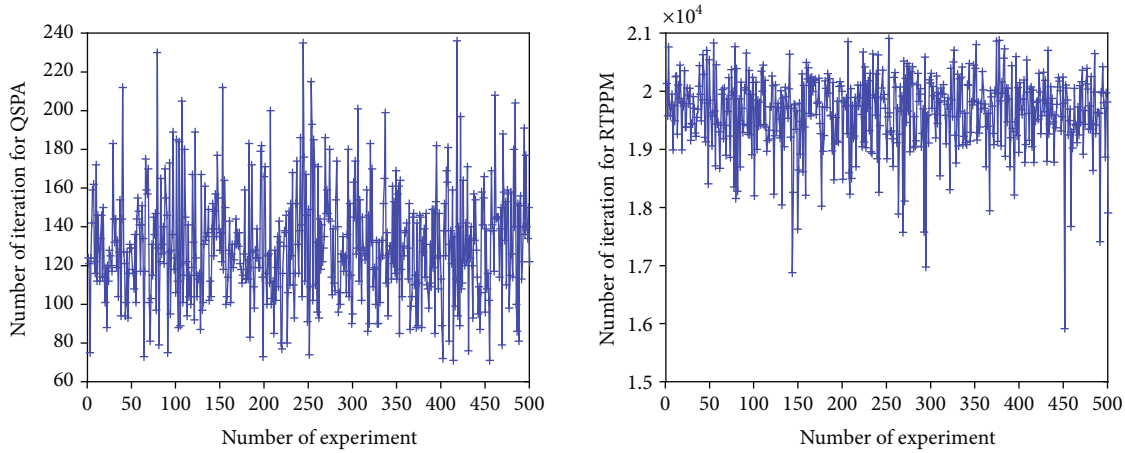

Figure 4: The iteration number of QSPA and RTPPM.

On the other initial point, such as

$$x_1 = \text{rand}\,(N, 1) * 10, y_1 = \text{rand}\,(M, 1) * 10, \quad (68)$$

in Example 10, the results can be found in Figure 4.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

Each author equally contributed to this paper and read and approved the final manuscript.

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Research Article

The Existence of Solution of Diffusion Equation with the General Conformable Derivative

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In this paper, the general conformable fractional derivative is used in the classical diffusion equations, and the corresponding maximum principle is obtained. By the maximum principle, this paper proves the uniqueness of the solution and the continuous dependence on source function and initial-boundary conditions of the solution. Furthermore, by employing the variable separation method, this paper obtains some existence results and the asymptotic behavior of the classical solution.

1. Introduction

It has been several decades since the fractional calculus has attracted extensive attention in both pure and applied fields [1–8]. Fractional calculus has many definitions, such as the Riemann-Liouville type, Caputo type, Hadamard type, Erdélyi-Kober type, and Riesz type. Most of the above fractional derivatives are defined by their corresponding fractional integrals. Compared with some integral-order partial differential equations such as [9–19], fractional derivatives have hereditary and nonlocal property so that they are much more suitable for describing long-memory processes than the classical integer-order derivatives.

However, the traditional fractional calculus have complex expressions which causes many difficulties in applying them in engineering calculation, physical application, numerical modeling, etc. In addition, the traditional fractional calculus loses some basic but important properties, such as product rule and chain rule. Khalil et al. [20] proposed a new local fractional derivative, called conformable derivative, and proved the product rule and the fractional mean value theorem. Abdeljawad supplemented the Taylor power series representation, the fractional chain rule, the Gronwall inequality, and the fractional Laplace transform in [21]. Zhao et al. [22, 23] further extended the definition to the general conformable fractional derivative (GCFD). Zhang et al. defined the conformable variable order derivative in [24]. Owing to its well-behaved properties and the close connection with the classical derivatives, conformable derivative generated great research interest [25–29].

In [30, 31], the author considered a generalized time-fractional diffusion equation by replacing the first-order time derivative to the Caputo fractional derivative. Recently, some literatures about the fractional diffusion equations with various definitions of fractional calculus occur. Al-Refai et al. [32, 33] considered a nonlinear fractional diffusion equation with the Riemann-Liouville time-fractional derivative and obtained their corresponding maximum principle. Borikhanov et al. [34] considered a nonlinear time-fractional diffusion equation with the Atangana-Baleanu derivative. Chen et al. [35] used the finite element method to approximate the solutions to a time-fractional advection-diffusion equations with the Caputo variable order derivative. Zhang et al. [36] obtained the sharp blow-up and global existence of solutions, a time-fractional diffusion system with the Riemann-Liouville derivative. In this paper, we consider the same equation as in [30] with the GCFD on domain $D \times (0, T)$, $D \subset R^n$, i.e.,

$$\left(D_t^{\alpha,\psi}u\right)(t) = -A(\mathrm{u}) + F(x,t), 0 < \alpha \le 1, \tag{1}$$

with the nonhomogeneous initial-boundary conditions

$$\begin{cases} u|_{t=0} = u_0(x), x \in \bar{D}, \\ u|_{\partial D} = v(x,t), (x,t) \in \partial D \times [0,T], \end{cases} \tag{2}$$

where $F(x,t)$, $u_0(x)$, and $v(x,t)$ are continuous functions, $D^{\alpha,\psi}$ is a GCFD of order α, ∂D is the boundary of D, and $-A$ is a linear elliptic operator

$$-A(u) \coloneqq p(x)\Delta u + (\nabla p, \nabla u) - q(x)u, x \in \bar{D}, \tag{3}$$

where Δ is the Laplace operator, ∇ is the gradient operator, $p \in C^1(\bar{D})$, $q \in C(\bar{D})$, $p(x) > 0$, $q(x) \ge 0$, and the domain of definition of operator A is

$$M_A = \{f, f \in C^2(D) \cap C^1(\bar{D}), \\ f \text{ satisfies the boundary condition and } A(f) \in L^2(D)\}. \tag{4}$$

The GCFD is defined in the following sense (see [22]):

$$(D^{\alpha,\psi}f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\psi(t,\alpha)) - f(t)}{\varepsilon}, t > 0, \tag{5}$$

and the GCFD at 0 is defined as $(D^{\alpha,\psi}f)(0) = \lim_{t \to 0+} (D^{\alpha,\psi}f)(t)$, where $\psi(t,\alpha)$ is a continuous real function depending on t and the fractional order α and satisfying the below conditions

$$\psi(t,\alpha) \ne 0, \\ \psi(t,1) = 1, \tag{6}$$

and the relationship between the function $\psi(t,\alpha)$ and the order α should be one-to-one. By the definition (5), we know that the GCFD is an extension of the classical derivative ($\alpha = 1$ or $\psi(t,\alpha) = 1$) and the conformable derivative ($\psi(t,\alpha) = t^{1-\alpha}$) defined in [20, 21]. Compared with the definition of conformable derivative in [20, 21] and the Definition 2.5 in [24], the GCFD in relation (5) becomes the conformable variable order derivative defined in [24] if $\psi(t,\alpha) = t^{1-\alpha}$ and the order α is a time-dependent function $\alpha(t)$, that is

$$\left(D^{\alpha(t)}f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha(t)}\right) - f(t)}{\varepsilon}, t > 0. \tag{7}$$

If the limit (5) exists, it is called that f is α-differentiable. Furthermore, if f is differentiable, then by direct calculation of definition, we can obtain that f is α-differentiable and

$$(D^{\alpha,\psi}f)(t) = f'(t)\psi(t,\alpha). \tag{8}$$

In this paper, we introduce the fractional Taylor power series expansion (see Lemma 5) and prove the theorem of term-by-term integration and differentiation (see Lemma 7 and 8) with the general conformable fractional calculus. Then by using the above and some other properties of the general conformable factional calculus (see Section 2), we obtain the maximum principle (see Theorems 9 and 10) for the classical diffusion Equation (1) with the GCFD and get some existence (see Theorems 17 and 18) and uniqueness (see Theorem 12) results of the classical solution of (1). Finally, we get the asymptotic behavior of the classical solution (see Theorem 19). The problems (1) and (2) have a solution implies that u is α-differentiable and $(D_t^{\alpha,\psi}u)(t)$ is continuous on $(0,T)$. We define that $u(x,t)$ is called a classical solution of problems (1) and (2), if $u(x,t) \in C(\bar{D} \times [0,T]) \cap C_x^2(D) \cap C_t^1(0,T)$ and satisfies Equation (1) and the initial-boundary condition (2).

2. Preliminaries

In this section, we will recall some properties of the GCFD and its fractional integral calculus.

Definition 1 (see [22]). The integral of a function $f : (0,t) \longrightarrow R$ *of order* α *is defined by*

$$(I^{\alpha,\psi}f)(t) = \int_0^t \frac{f(\tau)}{\psi(\tau,\alpha)} d\tau, t > 0, \tag{9}$$

where the integral is the Riemann integral.

Remark 2. To further study the properties of general conformable fractional calculus, we assume that $\psi(t,\alpha) > 0$ *and* $1/\psi(\cdot,\cdot)$: $(0,T] \times (0,1] \longrightarrow (0,+\infty)$ *is Riemann integrable and this assumption is valid in the following sections.*

Lemma 3 (see [20–22]). *If* f, g *is* α*-differentiable at* t*, then*

(1) $D^{\alpha,\psi}(c) = 0$*, where* c *is any real constant,*

(2) $(Linearity) D^{\alpha,\psi}(af(t) + bg(t)) = aD^{\alpha,\psi}f(t) + bD^{\alpha,\psi}g(t), a, b \in R,$

(3) $(Product\ rule) D^{\alpha,\psi}(f(t)g(t)) = f(t)D^{\alpha,\psi}g(t) + g(t)D^{\alpha,\psi}f(t),$
$D^{\alpha,\psi}(f(t)/g(t)) = g(t)D^{\alpha,\psi}f(t) - f(t)D^{\alpha,\psi}g(t)/g^2(t),$

(4) $(Chain\ rule)$ *If* f *is* $\alpha -$ *differentiable at* g *and* $g \ne 0$*, then*

$$D^{\alpha,\psi}(f(g(t))) = (D^{\alpha,\psi}f)(g(t)) \cdot D^{\alpha,\psi}(g(t)) \cdot \frac{1}{\psi(g(t),\alpha)} \tag{10}$$

If f *is differentiable at* g*, then* $D^{\alpha,\psi}(f(g(t))) = f'(g(t))D^{\alpha,\psi}(g(t))$.

Remark 4. Especially, by the definition (5), the relation (8) and Lemma 3 (4), we have

$$D^{\alpha,\psi}\left(\int_0 \frac{1}{\psi(\tau,\alpha)}d\tau\right)=1, D^{\alpha,\psi}\left(e^{\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau}\right)=e^{\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau}, \tag{11}$$

which coincides with the classical derivatives $(\psi(t,\alpha)=1)$ and the conformable fractional derivatives $(\psi(t,\alpha)=t^{1-\alpha})$ defined in [20, 21].

Lemma 5 (Fractional Taylor power series expansions). *Assume that f is an infinite α-differentiable function at a neighbourhood of t_0. Then f has the fractional power series expansion at point t_0:*

$$f(t)=\sum_{k=0}^{+\infty}\frac{(D^{\alpha,\psi}f)^{(k)}(t_0)}{k!}\left(\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k, t_0<t<t_0+R, R>0, \tag{12}$$

where $(D^{\alpha,\psi}f)^{(k)}(t_0)$ means the application of the GCFD k times.

Proof. The proof is similar to that of Theorem 17 in [21]. And this result coincides with the classical derivatives and the conformable derivatives. Especially, the exponential function $e^{\int_{t_0}^t (1/\psi(\tau,\alpha))d\tau}$ has the factional Taylor power series expansion at point t_0:

$$e^{\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau}=\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k. \tag{13}$$

Lemma 6 (see [22]). *Let f be a continuous differentiable function on $(0,T)$, then*

$$I^{\alpha,\psi}(D^{\alpha,\psi}f(t))=f(t)-f(0), D^{\alpha,\psi}(I^{\alpha,\psi}f(t))=f(t). \tag{14}$$

Lemma 7. *Let a sequence of functions $f_i(t)$, $i=1,2,\cdots$ satisfying the following conditions:*

(1) for any given $\alpha(>0)$, there exists fractional integrals $(I^{\alpha,\psi}f_i)(t)$, $i=1,2,\cdots$

(2) both the series $\sum_{i=1}^{+\infty}f_i(t)$ and $\sum_{i=1}^{+\infty}(I^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t\in[0,T]$ then the function $\sum_{i=1}^{+\infty}f_i(t)$ is α-integrable on $(0,T)$ and

$$I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)\right)=\sum_{i=1}^{+\infty}(I^{\alpha,\psi}f_i)(t). \tag{15}$$

Proof. Due to the uniformly convergence of $\sum_{i=1}^{+\infty}f_i(t)$ and $\sum_{i=1}^{+\infty}(I^{\alpha,\psi}f_i)(t)$, then for any $\varepsilon>0$, there exists a positive integer N such that for any $n>N$ and $t\in(0,T)$,

$$\left|\sum_{i=1}^{+\infty}f_i(t)-\sum_{i=1}^{n}f_i(t)\right|<\frac{\varepsilon}{2\int_0^T(1/\psi(\tau,\alpha))d\tau}, \left|\sum_{i=n}^{+\infty}(I^{\alpha,\psi}f_i)(t)\right|<\frac{\varepsilon}{2}. \tag{16}$$

Therefore, by (16) and the linearity of the operator $I^{\alpha,\psi}$, we obtain

$$\begin{aligned}&\left|I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)\right)-\sum_{i=1}^{+\infty}(I^{\alpha,\psi}f_i)(t)\right|\\&\le\left|I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)-\sum_{i=1}^{n}f_i(t)\right)\right|+\left|\sum_{i=n+1}^{+\infty}(I^{\alpha,\psi}f_i)(t)\right|\\&\le\left|\int_0^t\frac{\sum_{i=1}^{+\infty}f_i(\tau)-\sum_{i=1}^{n}f_i(\tau)}{\psi(\tau,\alpha)}d\tau\right|+\left|\sum_{i=n+1}^{+\infty}(I^{\alpha,\psi}f_i)(t)\right|<\varepsilon.\end{aligned} \tag{17}$$

i.e., the function $\sum_{i=1}^{+\infty}f_i(t)$ is α-integrable and $I^{\alpha,\psi}(\sum_{i=1}^{+\infty}f_i(t))=\sum_{i=1}^{+\infty}(I^{\alpha,\psi}f_i)(t)$, then the proof is completed.

Lemma 8. *Let a continuous differentiable sequence of functions $f_i(t)$, $i=1,2,\cdots$ satisfying the following conditions:*

(1) for any given $\alpha(>0)$, there exists fractional integrals $(D^{\alpha,\psi}f_i)(t)$, $i=1,2,\cdots$

(2) both the series $\sum_{i=1}^{+\infty}f_i(t)$ and $\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t\in[0,T]$. Moreover, $\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t)\in L(0,T)$, then the function $\sum_{i=1}^{+\infty}f_i(t)$ is α-differentiable on $(0,T)$ and

$$D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)\right)=\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t). \tag{18}$$

Proof. Since $\sum_{i=1}^{+\infty}f_i(t)$ and $\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t\in[0,T]$, then by Lemma 6 and Lemma 7.

$$\begin{aligned}I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t)\right)&=\sum_{i=1}^{+\infty}(I^{\alpha,\psi}((D^{\alpha,\psi}f_i)(t)))\\&=\sum_{i=1}^{+\infty}f_i(t)-\sum_{i=1}^{+\infty}f_i(0).\end{aligned} \tag{19}$$

The left side of the above equality is α-differentiable; then by Lemma 3(1), $\sum_{i=1}^{+\infty}f_i(t)$ is α-differentiable and

$$\begin{aligned}D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)\right)&=D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty}f_i(t)-\sum_{i=1}^{+\infty}f_i(0)\right)\\&=\sum_{i=1}^{+\infty}(D^{\alpha,\psi}f_i)(t).\end{aligned} \tag{20}$$

3. The Uniqueness of Solution

Theorem 9. *Let u be a classical solution of Equation (1) in the domain $D \times (0, T)$ and $F(x, t) \le 0$, then*

$$u(x,t) \le \max\left\{0, \max_{(x,t)\in S} u(x,t)\right\}, (x,t) \in \bar{D} \times [0,T], \quad (21)$$

where $S := (\bar{D} \times \{0\}) \cup (\partial D \times [0, T])$.

Proof. The proof follows by Lemma 3 (4) and setting the same auxiliary function $w(x, t) := u(x, t) + (\varepsilon/2) \cdot (T - t/T)$ as [30]. Due to u is a classical solution of the problems (1) and (2), then u is differentiable with respect to t. Therefore, we have $(D_t^{\alpha,\psi} u)(t_0) = \psi(t_0, \alpha) u_t(x_0, t_0) = 0$, where (x_0, t_0) is the maximum point of u over the domain $\bar{D} \times [0, T]$.

Theorem 10. *Let u be a classical solution of Equation (1) in the domain $D \times (0, T)$ and $F(x, t) \ge 0$, $(x, t) \in D \times (0, T)$, then*

$$u(x,t) \ge \min_{(x,t)\in S} u(x,t), (x,t) \in \bar{D} \times [0,T]. \quad (22)$$

Theorem 11. *Let u be a classical solution of problems (1) and (2) and $F \in C(\bar{D} \times [0, T])$ with the norm $\|F\|_{C(\bar{D}\times[0,T])} = \sup_{\bar{D}\times[0,T]} |F(x,t)| := M_1$.*

Denote by $M_2 = \|u_0\|_{C(\bar{D})} = \sup_{\bar{D}} |u_0(x)|$, $M_3 = \|v\|_{C(\partial D\times[0,T])} = \sup_{\partial D\times[0,T]} |v(x,t)|$, then the following estimate of the solution norm holds true:

$$\|u\|_{C(\bar{D}\times[0,T])} \le 2CM_1 + \max\{M_2, M_3\}, \quad (23)$$

where $C := \int_0^T (1/\psi(\tau, \alpha)) d\tau$.

Proof. Define the auxiliary function g:

$$g(x,t) = u(x,t) - \int_0^t \frac{M_1}{\psi(\tau,\alpha)} d\tau, (x,t) \in \bar{D} \times [0,T]. \quad (24)$$

By the continuity of ψ and Remark 4, then g is a classical solution of the following problem:

$$\begin{cases} \left(D_t^{\alpha,\psi} g\right)(t) = -A(g) + F_1(x,t), (x,t) \in D \times (0,T), \\ g|_{t=0} = u_0, x \in \bar{D}, \\ g|_{\partial D} = v(x,t) - \int_0^t \dfrac{M_1}{\psi(\tau,\alpha)} d\tau := v_1(x,t), (x,t) \in \partial D \times [0,T], \end{cases} \quad (25)$$

where $F_1(x, t) = F(x, t) - M_1 - q(x) \int_0^t (M_1/\psi(\tau, \alpha)) d\tau \le 0$. According to the conditions, we have

$$\|v_1\|_{C(\partial D\times[0,T])} \le M_3 + CM_1. \quad (26)$$

Then by Theorem 9, the classical solution g satisfies the estimate,

$$g(x,t) \le \max\{M_2, M_3 + CM_1\}, (x,t) \in \bar{D} \times [0,T], \quad (27)$$

which means that

$$u(x,t) \le 2CM_1 + \max\{M_2, M_3\}. \quad (28)$$

Similarly, consider of the auxiliary function g_1 and using the Theorem 10

$$g_1(x,t) = u(x,t) + \int_0^t \frac{M_1}{\psi(\tau,\alpha)} d\tau, (x,t) \in \bar{D} \times [0,T], \quad (29)$$

then we have

$$g_1(x,t) \ge \max\{-M_2, -M_3 - CM_1\}, (x,t) \in \bar{D} \times [0,T], \quad (30)$$

i.e., $u(x, t) \ge -2CM_1 - \max\{M_2, M_3\}$. Combined with (28), we obtain

$$\|u\|_{C(\bar{D}\times[0,+\infty))} \le 2CM_1 + \max\{M_2, M_3\}. \quad (31)$$

Theorem 12. *The problems (1) and (2) have at most one classical solution and the solution continuously depends on the data given in the problem in the sense that if*

$$\begin{aligned} \|F_1 - F_2\|_{C(\bar{D}\times[0,T])} \le \varepsilon_1, \left\|u_0^{(1)} - u_0^{(2)}\right\|_{C(\bar{D})} \\ \le \varepsilon_2, \|v_1 - v_2\|_{C(\partial D\times[0,T])} \le \varepsilon_3, \end{aligned} \quad (32)$$

then for the corresponding classical solution u_1 and u_2, the estimate holds:

$$\|u_1 - u_2\|_{C(\bar{D}\times[0,T])} \le 2C\varepsilon_1 + \max\{\varepsilon_2, \varepsilon_3\}. \quad (33)$$

Proof. Let u_1, u_2 be the classical solution of the following problem, respectively,

$$\begin{cases} \left(D_t^{\alpha,\psi} u\right)(t) = -A(u) + F_i(x,t), (x,t) \in D \times (0,T), \\ u|_{t=0} = u_0^{(i)}(t), x \in \bar{D}, \\ u|_{\partial D} = v_i(x,t), (x,t) \in \partial D \times [0,T], \end{cases} \quad (34)$$

where $i = 1, 2$. Then $u = u_1 - u_2$ is the classical solution of the corresponding problem

$$\begin{cases} \left(D_t^{\alpha,\psi}u\right)(t) = -A(u) + F_1(x,t) - F_2(x,t), (x,t) \in D \times (0,T), \\ u|_{t=0} = u_0^{(1)}(t) - u_0^{(2)}(t), x \in \bar{D}, \\ u|_{\partial D} = v_1(x,t) - v_2(x,t), (x,t) \in \partial D \times [0,T]. \end{cases} \tag{35}$$

By (23), the estimate (33) holds. The uniqueness of the classical solution of (1) and (2) is a direct consequence of Theorem 9, Theorem 10 and the estimate (33).

4. The Existence of Solution

In this section, we will consider some existence results of solution of problems (1) and (2) with homogenous boundary conditions

$$\begin{cases} u|_{t=0} = u_0(x), x \in \bar{D}, \\ u|_{\partial D} = 0, (x,t) \in \partial D \times [0,T]. \end{cases} \tag{36}$$

Firstly, we consider the homogenous equation, i.e., $F(x,t) \equiv 0$. In the following, we seek the solution of Equation (1) by using the variable separation method in the form

$$u(x,t) = X(x)T(t), (x,t) \in D \times (0,T), \tag{37}$$

which satisfies the boundary condition. Substituting expression (37) in Equation (1) and separating the variables, we obtain the following sense:

$$\frac{(D^{\alpha,\psi}T)(t)}{T(t)} = -\frac{A(X(x))}{X(x)} = -\lambda, \tag{38}$$

where λ is a constant independent on t and x. Therefore, addition to the boundary condition, we have

$$(D^{\alpha,\psi}T)(t) + \lambda T(t) = 0, \tag{39}$$

and the eigenvalues problem for the operator A

$$\begin{cases} A(X(x)) = \lambda X(x), (x,t) \in D \times (0,T), \\ X|_{\partial D} = 0, (x,t) \in \partial D \times [0,T]. \end{cases} \tag{40}$$

The properties of A is well known in [37] that A is a positive and self-adjoint linear operator. Moreover, (40) has a counted number of positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_k \longrightarrow +\infty (k \longrightarrow +\infty)$ with finite multiplicity, and any function $f \in M_A$ can be represented through its Fourier series:

$$f(x) = \sum_{i=1}^{+\infty} (f, X_i) X_i(x), \tag{41}$$

where $X_i \in M_A$ is the corresponding eigenfunction of the eigenvalue λ_i. In fact, in the following, we choose $\{X_i\}_{i=1}^{+\infty}$ to be real and orthonormal.

For Equation (39) and its inhomogenous equation

$$(D^{\alpha,\psi}T)(t) + \lambda T(t) = f(t), \tag{42}$$

we consider the following first-order linear differential equation for $\lambda = \lambda_i$:

$$T_i'(t) + \frac{\lambda_i}{\psi(t,\alpha)} T_i(t) = \frac{f(t)}{\psi(t,\alpha)}. \tag{43}$$

It is well known that the solution of Equation (43) is given by

$$T_i(t) = e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \left[c_i + \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{f(\tau)}{\psi(\tau,\alpha)} d\tau \right], \tag{44}$$

where c_i are some constants. Obviously, $T_i(t) \in C[0,T] \cap C^1(0,T)$. Moreover, $T_i(t)$ in (44) satisfies the following equation:

$$\psi(\tau,\alpha) T_i'(t) + \lambda_i T_i(t) = f(t). \tag{45}$$

Hence, according to the relation (8) we have

$$(D^{\alpha,\psi}T_i)(t) + \lambda_i T_i(t) = \psi(\tau,\alpha) T_i'(t) + \lambda_i T_i(t) = f(t), \tag{46}$$

which means the function in (44) is the solution of Equation (42). Correspondingly, the solution of Equation (39) for $\lambda = \lambda_i$ is

$$T_i(t) = c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau}. \tag{47}$$

Naturally, each function

$$T_i(t) X_i(x) = c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} X_i(x), \tag{48}$$

and its finite sum

$$u_k(x,t) = \sum_{i=1}^{k} T_i(t) X_i(x) = \sum_{i=1}^{k} c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} X_i(x), \tag{49}$$

satisfy Equation (1) and the boundary condition.

In addition to the initial condition, we define the formal solution of problems (1) and (36) as the following sense:

$$u(x,t) = \sum_{i=1}^{+\infty} T_i(t) X_i(x) = \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} X_i(x). \tag{50}$$

Next, we consider the case of inhomogenous equation. Similarly to the case of homogenous condition, the formal solution of inhomogenous equation also has the expression (50) with appropriate $T_i(t)$. For any i, we multiply Equation (1) by X_i, integrating over D. Then by Lemma 8 and using the

self-adjoint property of the operator A, combined by the orthogonality of $\{X_i\}_{i=1}^{+\infty}$, we obtain

$$\begin{aligned}(D^{\alpha,\psi}T_i)(t) &= \int_D \left(D_t^{\alpha,\psi}u\right)(t)\cdot X_i(x)dx \\ &= \int_D D_t^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} T_i(t)X_i(x)\right)\cdot X_i(x)dx \\ &= \int_D \left(\sum_{i=1}^{+\infty} D_t^{\alpha,\psi}(T_i(t)X_i(x))\right)\cdot X_i(x)dx \\ &= -(Au, X_i) + (F, X_i) = -\lambda_i(u, X_i) + (F, X_i) \\ &= -\lambda_i T_i + (F, X_i).\end{aligned} \tag{51}$$

That is

$$(D^{\alpha,\psi}T_i)(t) + \lambda_i T_i(t) = (F(x,t), X_i(x)). \tag{52}$$

According to the relation (44), the solution of Equation (52) is

$$T_i(t) = e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\left[c_i + \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau\right], \tag{53}$$

where c_i are some constants. Consequently, combined with the initial condition, we define the formal solution of problems (1) and (36) as the following sense:

$$\begin{aligned}u(x,t) = \sum_{i=1}^{+\infty} T_i(t)X_i(x) &= \sum_{i=1}^{+\infty}(u_0, X_i)e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}X_i(x) \\ &+ \sum_{i=1}^{+\infty} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau), X_i(x))X_i(x)}{\psi(\tau,\alpha)}d\tau.\end{aligned} \tag{54}$$

Remark 13. For Equation (52), we try applying the operator $I^{\alpha,\psi}$ to its both sides, then by Definition 1 and Lemma 6, we have

$$T_i(t) = T_i(0) - \lambda_i\int_0^t \frac{T_i(\tau)}{\psi(\tau,\alpha)}d\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau. \tag{55}$$

Next, we apply the method of successive approximations to solve this integral equation by setting

$$T_i^{(0)}(t) = T_i(0), \tag{56}$$

$$\begin{aligned}T_i^{(m)}(t) = T_i^{(0)}(t) &- \lambda_i\int_0^t \frac{T_i^{(m-1)}(\tau)}{\psi(\tau,\alpha)}d\tau \\ &+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau, m\in N^+.\end{aligned} \tag{57}$$

Taking (56) into account, then

$$T_i^{(1)}(t) = T_i(0) - \lambda_i T_i(0)\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau. \tag{58}$$

Similarly, using (56)–(58), we have

$$\begin{aligned}T_i^{(2)}(t) &= T_i(0) - \lambda_i T_i(0)\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau + \lambda_i^2 T_i(0)\int_0^t\int_0^s \frac{1}{\psi(\tau,\alpha)} \\ &\cdot \frac{1}{\psi(s,\alpha)}dsd\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[1 - \lambda_i\int_\tau^t \frac{1}{\psi(s,\alpha)}ds\right]d\tau \\ &= \sum_{k=0}^{2}\frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\ &+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^{1}\frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.\end{aligned} \tag{59}$$

Continuing this process, we obtain the following expression for $T_i^{(m)}(t)$:

$$\begin{aligned}T_i^{(m)}(t) &= \sum_{k=0}^{m}\frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\ &+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^{m-1}\frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.\end{aligned} \tag{60}$$

Taking the limit as $m \longrightarrow +\infty$, we obtain

$$\begin{aligned}T_i(t) &= \sum_{k=0}^{+\infty}\frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\ &+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^{+\infty}\frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.\end{aligned} \tag{61}$$

By Lemma 5, the expression (61) is the fractional Taylor power series expansion of (53).

Definition 14. A formal solution of the problems (1) and ((36)) *is called the Fourier series in the* form (50) $(F \equiv 0)$ *and* (54) $(F \neq 0)$.

Definition 15. Let the sequences of $u_{0k} \in C(\bar{D})$ *and* $v_k \in C(\partial D \times [0,T])$, $F_k \in C(\bar{D}\times[0,T])$, $k = 1, 2, \cdots$ *such that for* $k \longrightarrow +\infty$, $u_{0k} \longrightarrow u_0$, $v_k \longrightarrow v$, $F_k \Longrightarrow F$ *in* $L^2(D)$ *and* $t \in [0,T]$. *And for any* $k = 1, 2, \cdots$, *there exists a classical solution* u_k *of the following problem*

$$\begin{cases}\left(D_t^{\alpha,\psi}u_k\right)(t) = -A(u_k) + F_k(x,t), \\ u_k|_{t=0} = u_{0k}(x), x \in \bar{D}, \\ u_k|_{\partial D} = v_k(x,t), (x,t) \in \partial D \times [0,T].\end{cases} \tag{62}$$

Suppose there exists a function $u \in C(\bar{D} \times [0,T])$ such that $u_k \longrightarrow u$ as $k \longrightarrow +\infty$, then the function u is called a generalized solution of the problems (1) and (2).

Remark 16. By Theorem 12, the convergence of u_{0k}, v_k, F_k *and the completeness of* $C(\bar{D} \times [0,T], \|\cdot\|_{C(\bar{D}\times[0,T])})$, *we know that there always exists a function* $u \in C(\bar{D} \times [0,T])$ *which is the limit of the sequence* u_k. *In fact, under the assumption* u_0, $F \in M_A$, *the formal solution is also the generalized solution of problems* (1) and (36). *That is the following theorem.*

Theorem 17. *Suppose that* $u_0, F \in M_A$, *then the formal solution (54) is the generalized solution of the problems (1) and (36).*

Proof. For every $k = 1, 2, \cdots$,

$$u_k(x,t) = \sum_{i=1}^{k} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} X_i(x) + \sum_{i=1}^{k} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) X_i(x)}{\psi(\tau,\alpha)} d\tau, \tag{63}$$

is the classical solution of the problems (1) and (36) with the initial condition

$$u_{0k}(x) = \sum_{i=1}^{k} (u_0, X_i) X_i(x). \tag{64}$$

Note that $u_0 \in M_A$, $u_{0k} \longrightarrow u_0$ as $k \longrightarrow +\infty$ in $\bar{D} \times [0,T]$ and $e^{-\int_0^t (\lambda_i/\psi(\tau,\alpha)) d\tau} \le 1$; therefore, $\sum_{i=1}^{k} (u_0, X_i) e^{-\int_0^t (\lambda_i/\psi(\tau,\alpha)) d\tau} X_i(x)$ uniformly converges on $\bar{D} \times [0,T]$.

In view of $F \in M_A$, then $F(x,t) = \sum_{i=1}^{+\infty} (F(x,t), X_i) X_i(x)$ uniformly converges on $\bar{D} \times [0,T]$. Therefore,

$$\begin{aligned} &\sum_{i=1}^{k} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \left| \frac{(F(x,\tau), X_i(x)) X_i(x)}{\psi(\tau,\alpha)} \right| d\tau \\ &\le \int_0^t \left| \frac{1}{\psi(\tau,\alpha)} \cdot \left(\sum_{i=1}^{k} (F(x,\tau), X_i) X_i(x) \right) \right| d\tau \\ &\longrightarrow \int_0^t \left| \frac{1}{\psi(\tau,\alpha)} \cdot \left(\sum_{i=1}^{+\infty} (F(x,\tau), X_i) X_i(x) \right) \right| d\tau \, (ask \longrightarrow +\infty). \end{aligned} \tag{65}$$

Consequently, (54) is the generalized solution of the problems (1) and (36).

Theorem 18. *Suppose that* $u_0, F \in M_A$, $A(u_0), A^2(u_0) \in M_A$ *and for any* $t \in (0,T)$, $A(F), A^2(F) \in M_A$, *then the formal solution (54) is the classical solution of the problems (1) and (36).*

Proof. By Theorem 17, the formal solution (54) is the generalized solution; it remains to prove that (54) is at least twice differentiable with respect to the spatial variable x and differentiable with respect to the time variable t.

Differentiating term-by-term (54) with respect to x, we construct a series

$$\sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \nabla X_i(x) + \sum_{i=1}^{+\infty} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) \nabla X_i(x)}{\psi(\tau,\alpha)} d\tau. \tag{66}$$

Now we shall prove that the above series (66) is uniformly convergent. Since $u_0 \in M_A$, $F \in M_A$, then $A(u_0)$, $A(F) \in L^2(D)$. Moreover, $A(u_0), A(F) \in M_A$, so $A(u_0)$ and $A(F)$ can be expanded into uniformly convergent series

$$A(u_0) = \sum_{i=1}^{+\infty} (A(u_0), X_i) X_i, A(F) = \sum_{i=1}^{+\infty} (A(F), X_i) X_i, \tag{67}$$

for any $t \in (0,T)$. Then by the Parseval equality,

$$\begin{aligned} \|A(u_0)\|_2^2 &= \sum_{i=1}^{+\infty} |(A(u_0), X_i)|^2 = \sum_{i=1}^{+\infty} |(u_0, A(X_i))|^2 \\ &= \sum_{i=1}^{+\infty} \lambda_i^2 |(u_0, X_i)|^2 < +\infty, \end{aligned} \tag{68}$$

$$\begin{aligned} \|A(F)\|_2^2 &= \sum_{i=1}^{+\infty} |(A(F), X_i)|^2 = \sum_{i=1}^{+\infty} |(F, A(X_i))|^2 \\ &= \sum_{i=1}^{+\infty} \lambda_i^2 |(F, X_i)|^2 < +\infty. \end{aligned} \tag{69}$$

Considering that the series $\sum_{i=1}^{+\infty} (|\nabla X_i|^2 / \lambda_i^2)$ uniformly convergent (see [38]), then applying the Cauchy inequality we have

$$\begin{aligned} \sum_{i=1}^{+\infty} |(F(x,t), X_i(x)) \nabla X_i(x)| &\le \sum_{i=1}^{+\infty} |\lambda_i (F(x,t), X_i(x))| \cdot \left| \frac{\nabla X_i(x)}{\lambda_i} \right| \\ &\le \left(\sum_{i=1}^{+\infty} \lambda_i^2 |(F(x,t), X_i(x))|^2 \right)^{1/2} \\ &\quad \cdot \left(\sum_{i=1}^{+\infty} \left(\frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right) \right)^{1/2} \\ &\le \|A(F)\|_2 \left(\sum_{i=1}^{+\infty} \left(\frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right) \right)^{1/2}, \end{aligned} \tag{70}$$

which implies that the series $\sum_{i=1}^{+\infty} |(F(x,t), X_i(x)) \nabla X_i(x)|$ is uniformly convergent on D for any $t \in (0,T)$. In view that $\{X_i\}_{i=1}^{+\infty}$ is a complete orthonormal system in $L^2(D)$, using

the above facts, the estimates (68)–(70) and applying the Cauchy inequality, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\left|(u_0,X_i)e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\nabla X_i(x)\right| \\
&\quad+\sum_{i=1}^{+\infty}\left|e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\int_0^t e^{\int_0^\tau\frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau),X_i(x))\nabla X_i(x)}{\psi(\tau,\alpha)}d\tau\right| \\
&\le\sum_{i=1}^{+\infty}|(u_0,X_i)\nabla X_i(x)| \\
&\quad+\sum_{i=1}^{+\infty}\left|\int_0^t\frac{(F(x,\tau),X_i(x))\nabla X_i(x)}{\psi(\tau,\alpha)}d\tau\right| \\
&\le\sum_{i=1}^{+\infty}|\lambda_i(u_0,X_i)|\cdot\left|\frac{\nabla X_i(x)}{\lambda_i}\right| \\
&\quad+\int_0^t\frac{1}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}|\lambda_i(F(x,\tau),X_i(x))|\cdot\left|\frac{\nabla X_i(x)}{\lambda_i}\right|\right)d\tau \\
&\le\left(\sum_{i=1}^{+\infty}\lambda_i^2(u_0,X_i)^2\right)^{1/2}\cdot\left(\sum_{i=1}^{+\infty}\frac{|\nabla X_i(x)|^2}{\lambda_i^2}\right)^{1/2} \\
&\quad+\int_0^t\frac{1}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}\lambda_i^2(F,X_i)^2\right)^{1/2} \\
&\quad\cdot\left(\sum_{i=1}^{+\infty}\frac{|\nabla X_i(x)|^2}{\lambda_i^2}\right)^{1/2}d\tau \\
&\le\left(\|A(u_0)\|_2+C\|A(F)\|_2\right)\left(\sum_{i=1}^{+\infty}\frac{|\nabla X_i(x)|^2}{\lambda_i^2}\right)^{1/2},
\end{aligned}\tag{71}$$

where $C=\int_0^T(1/\psi(\tau,\alpha))d\tau$. Consequently, the series (66) uniformly converges to ∇u and the generalized solution (54) belongs to $C_x^1(D)$.

Applying the similarly method, we can show the generalized solution (54) belongs to $C_x^2(D)$ as well and the relation

$$\begin{aligned}
\Delta u=&\sum_{i=1}^{+\infty}(u_0,X_i)e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\Delta X_i(x) \\
&+\sum_{i=1}^{+\infty}e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\int_0^t e^{\int_0^\tau\frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau),X_i(x))\Delta X_i(x)}{\psi(\tau,\alpha)}d\tau
\end{aligned}\tag{72}$$

holds true. In fact, since $A(u_0)\in M_A$, $A(F)\in M_A$, then $A^2(u_0)\in L^2(D)$, $A^2(F)\in L^2(D)$. Moreover, due to $A^2(u_0)$, $A^2(F)\in M_A$, so $A^2(u_0)$ and $A^2(F)$ can be expanded into uniformly convergent series for any $t\in(0,T)$. Note that the series $\sum_{i=1}^{+\infty}(|\Delta X_i|^2/\lambda_i^3)$ is uniformly convergent (see [38]), then by the Cauchy inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{+\infty}|(F(x,t),X_i(x))\Delta X_i(x)| \\
&\le\sum_{i=1}^{+\infty}\left|\lambda_i^{\frac{3}{2}}(F(x,t),X_i(x))\right|\cdot\left|\frac{\Delta X_i(x)}{\lambda_i^{3/2}}\right| \\
&\le l_1\left(\sum_{i=1}^{+\infty}\lambda_i^4|(F(x,t),X_i(x))|^2\right)^{1/2} \\
&\quad\cdot\left(\sum_{i=1}^{+\infty}\left(\frac{|\Delta X_i(x)|^2}{\lambda_i^3}\right)\right)^{1/2} \\
&\le l_1\|A^2(F)\|_2\left(\sum_{i=1}^{+\infty}\left(\frac{|\Delta X_i(x)|^2}{\lambda_i^3}\right)\right)^{1/2}.
\end{aligned}\tag{73}$$

Due to $\lim_{i\to+\infty}\lambda_i=+\infty$, there exists a constant l such that $\lambda_i^3<l\lambda_i^4$, $i=1,2,\cdots$, where l_1 is a constant depending on l. Then (73) implies that the series $\sum_{i=1}^{+\infty}|(F(x,t),X_i(x))\Delta X_i(x)|$ is uniformly convergent on D for any $t\in(0,T)$. Using the above facts and applying the Parseval equality and Cauchy inequality, we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty}\left|(u_0,X_i)e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\Delta X_i(x)\right| \\
&\quad+\sum_{i=1}^{+\infty}\left|e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\int_0^t e^{\int_0^\tau\frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau),X_i(x))\Delta X_i(x)}{\psi(\tau,\alpha)}d\tau\right| \\
&\le\sum_{i=1}^{+\infty}|(u_0,X_i)\Delta X_i(x)|+\sum_{i=1}^{+\infty}\left|\int_0^t\frac{(F(x,\tau),X_i(x))\Delta X_i(x)}{\psi(\tau,\alpha)}d\tau\right| \\
&\le\sum_{i=1}^{+\infty}\left|\lambda_i^{\frac{3}{2}}(u_0,X_i)\right|\cdot\left|\frac{\Delta X_i(x)}{\lambda_i^{3/2}}\right| \\
&\quad+\int_0^t\frac{1}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}\left|\lambda_i^{\frac{3}{2}}(F(x,\tau),X_i(x))\right|\cdot\left|\frac{\Delta X_i(x)}{\lambda_i^{3/2}}\right|\right)d\tau \\
&\le l_1\left(\sum_{i=1}^{+\infty}\lambda_i^4(u_0,X_i)^2\right)^{1/2}\cdot\left(\sum_{i=1}^{+\infty}\frac{|\Delta X_i(x)|^2}{\lambda_i^3}\right)^{1/2} \\
&\quad+\int_0^t\frac{l_1}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}\lambda_i^4(F,X_i)^2\right)^{1/2} \\
&\quad\cdot\left(\sum_{i=1}^{+\infty}\frac{|\Delta X_i(x)|^2}{\lambda_i^3}\right)^{1/2}d\tau \\
&\le\left(l_1\|A^2(u_0)\|_2+Cl_1\|A^2(F)\|_2\right)\left(\sum_{i=1}^{+\infty}\frac{|\Delta X_i(x)|^2}{\lambda_i^3}\right)^{1/2}.
\end{aligned}\tag{74}$$

Consequently, the relation (72) holds true and the generalized solution (54) belongs to $C_x^2(D)$. It remains to prove that (54) belongs to $C_t^1(0,T)$. We take a derivative term by term with respect to t (54) and construct the series

$$
\begin{aligned}
&-\sum_{i=1}^{+\infty}\frac{\lambda_i(u_0,X_i)}{\psi(t,\alpha)}e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}X_i(x)\\
&\quad-\sum_{i=1}^{+\infty}\frac{\lambda_i e^{-\int_0^t(\lambda_i/\psi(\tau,\alpha))d\tau}}{\psi(t,\alpha)}\int_0^t e^{\int_0^\tau\frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau),X_i(x))X_i(x)}{\psi(\tau,\alpha)}d\tau\\
&\quad+\sum_{i=1}^{+\infty}\frac{(F(x,\tau),X_i(x))X_i(x)}{\psi(t,\alpha)}.
\end{aligned}
\tag{75}
$$

Now let $t\in[\varepsilon,T]$, $0<\varepsilon<T$. Note that $1/\psi(t,\alpha)$ is the Riemann integrable on $[\varepsilon,T]$; there exists $M>0$ such that $|1/\psi(t,\alpha)|\le M$. Next we shall to prove that the above series (75) is uniformly convergent on any interval $[\varepsilon,T]$, $0<\varepsilon<T$. By the uniformly convergence of $\sum_{i=1}^{+\infty}(|X_i|^2/\lambda_i)$ (see [38]), applying the Parseval equality and the Cauchy inequality, we get

$$
\begin{aligned}
&\sum_{i=1}^{+\infty}\left|\frac{\lambda_i(u_0,X_i)}{\psi(t,\alpha)}e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}X_i(x)\right|+\sum_{i=1}^{+\infty}\left|\frac{(F(x,\tau),X_i(x))X_i(x)}{\psi(t,\alpha)}\right|\\
&\quad+\sum_{i=1}^{+\infty}\left|\frac{\lambda_i e^{-\int_0^t(\lambda_i/\psi(\tau,\alpha))d\tau}}{\psi(t,\alpha)}\int_0^t e^{\int_0^\tau\frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau),X_i(x))X_i(x)}{\psi(\tau,\alpha)}d\tau\right|\\
&\le\sum_{i=1}^{+\infty}M|\lambda_i(u_0,X_i)X_i(x)|+\sum_{i=1}^{+\infty}M|(F(x,\tau),X_i(x))X_i(x)|\\
&\quad+\sum_{i=1}^{+\infty}\int_0^t\frac{M\lambda_i\,|\,(F(x,\tau),X_i(x))X_i(x)\,|}{\psi(\tau,\alpha)}d\tau\\
&\le\sum_{i=1}^{+\infty}M\left|\lambda_i^{3/2}(u_0,X_i)\right|\cdot\left|\frac{X_i(x)}{\lambda_i^{1/2}}\right|+\sum_{i=1}^{+\infty}M\left|\lambda_i^{1/2}(F,X_i)\right|\cdot\left|\frac{X_i(x)}{\lambda_i^{1/2}}\right|\\
&\quad+\int_0^t\frac{M}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}\left|\lambda_i^{3/2}(F(x,\tau),X_i(x))\right|\cdot\left|\frac{X_i(x)}{\lambda_i^{1/2}}\right|\right)d\tau\\
&\le M_1\left(\sum_{i=1}^{+\infty}\lambda_i^4(u_0,X_i)^2\right)^{1/2}\cdot\left(\sum_{i=1}^{+\infty}\frac{|X_i(x)|^2}{\lambda_i}\right)^{1/2}\\
&\quad+M_2\left(\sum_{i=1}^{+\infty}\lambda_i^2(F,X_i)^2\right)^{1/2}\cdot\left(\sum_{i=1}^{+\infty}\frac{|X_i(x)|^2}{\lambda_i}\right)^{1/2}\\
&\quad+\int_0^t\frac{M_1}{\psi(\tau,\alpha)}\left(\sum_{i=1}^{+\infty}\lambda_i^4(F(x,\tau),X_i)^2\right)^{1/2}\\
&\quad\cdot\left(\sum_{i=1}^{+\infty}\frac{|X_i(x)|^2}{\lambda_i}\right)^{1/2}d\tau\\
&\le\left(M_1\left\|A^2(u_0)\right\|_2+M_2\|A(F)\|_2+CM_1\left\|A^2(F)\right\|_2\right)\\
&\quad\cdot\left(\sum_{i=1}^{+\infty}\frac{|X_i(x)|^2}{\lambda_i}\right)^{1/2}.
\end{aligned}
\tag{76}
$$

Once again, due to $\lim_{i\to+\infty}\lambda_i=+\infty$, there exists a constant l such that $\lambda_i^3<l\lambda_i^4$, $\lambda_i<l\lambda_i^2 i=1,2,\cdots$, where M_1, M_2 are some constants depending on l and M. Then, the series (75) uniformly convergent on any interval $[\varepsilon,T]$, $0<\varepsilon<T$. Therefore, the generalized solution (54) belongs to $C_t^1(0,T)$ and the proof is complete.

Theorem 19. *The classical solution (50) of (1) and (36) with $F(x,t)\equiv 0$ decay to zero with the rate $e^{-\int_0^t(\lambda_1/\psi(s,\alpha))ds}$.*

Proof. In fact, since $u_0\in M_A$, then $|\sum_{i=1}^{+\infty}(u_0,X_i)X_i(x)|=|u_0|<+\infty$. Then for any $t>0$ and $x\in\bar{D}$, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{+\infty}(u_0,X_i)e^{-\int_0^t\frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}X_i(x)\right|\\
&\quad\le e^{-\int_0^t\frac{\lambda_1}{\psi(s,\alpha)}ds}\left|\sum_{i=1}^{+\infty}(u_0,X_i)e^{-\int_0^t\frac{\lambda_i-\lambda_1}{\psi(\tau,\alpha)}d\tau}X_i(x)\right|\\
&\quad\le|u_0|e^{-\int_0^t\frac{\lambda_1}{\psi(s,\alpha)}ds}\longrightarrow 0(ast\longrightarrow+\infty).
\end{aligned}
\tag{77}
$$

5. Conclusions

The GCFD is an extension of the classical derivatives and the conformable derivatives. In this paper, we introduce its fractional Taylor power series expansions and prove the theorem of term-by-term integration and differentiation. Utilizing its properties, we obtain the uniqueness and existence of the classical solution of the classical diffusion equations with GCFD by using variable separation method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Research Article

Eigenvalue Criteria for Existence of Positive Solutions to Fractional Boundary Value Problem

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The existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary value problem (BVP) ${}^{C}D_{0+}^{\alpha}y(x)+f(x,y(x))=0,\ 0<x<1,\ y(0)=y'(1)=y''(0)=0$ is established, where $2<\alpha\le 3$, ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, and $f:[0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is a continuous function. The conclusion relies on the fixed-point index theory and the Leray-Schauder degree theory. The growth conditions of the nonlinearity with respect to the first eigenvalue of the related linear operator is given to guarantee the existence and multiplicity.

1. Introduction

In this paper, we concentrate on the existence and multiplicity of positive solutions for the following problem:

$$ {}^{C}D_{0+}^{\alpha}y(x)+f(x,y(x))=0,\quad 0<x<1, \tag{1}$$

$$ y(0)=y'(1)=y''(0)=0, \tag{2}$$

where $2<\alpha\le 3$, ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, and $f:[0,1]\times[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous function.

In the past twenty years, the fractional differential equation has aroused great consideration [1–21] not only in its application in mathematics but also in other applications in science and engineering, for example, fluid mechanics, viscoelastic mechanics, electroanalytical chemistry, and biological engineering. Bai and Qiu [22, 23] have investigated the existence and multiplicity of positive solutions of (1) and (2) by using the nonlinear alternative of the Leray-Schauder type and Krasnoselskii's fixed-point theorem in a cone, but they did not consider its eigenvalue criteria.

The rest of the paper is organized as follows. In Section 2, we recall some concepts relative to fractional calculus and give some lemmas with respect to the corresponding Green function. In Section 3, with the use of the fixed-point theory, some existence and multiplicity results of positive solutions are obtained. At last, two examples are given.

2. Background Materials

For the convenience of the reader, we give some definitions and lemmas.

Definition 1 (see [23]). The Caputo's fractional derivative of order $\alpha>0$ of a continuous function $y:[0,+\infty)\longrightarrow\mathbb{R}$ is given by

$$ {}^{C}D_{0+}^{\alpha}y(x)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt, \tag{3}$$

where $n-1\le\alpha<n$ and $n\in\mathbb{N}$, provided that the right-hand side is pointwise defined on $[0,+\infty)$.

Lemma 2 (see [15]). *Given $h\in C[0,1]$, the unique solution of*

$$ \begin{aligned} &{}^{C}D_{0+}^{\alpha}y(x)+h(x)=0,\quad 0<x<1,\\ &y(0)=y'(1)=y''(0)=0, \end{aligned} \tag{4}$$

is given by

$$y(x) = \int_0^1 G(x,t)h(t)dt, \tag{5}$$

where

$$G(x,t) = \begin{cases} \dfrac{(\alpha-1)x(1-t)^{\alpha-2} - (x-t)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } 0 \le t \le x \le 1, \\ \dfrac{x(1-t)^{\alpha-2}}{\Gamma(\alpha-1)}, & \text{for } 0 \le x \le t \le 1, \end{cases} \tag{6}$$

Lemma 3 (see [23]). *The Green function $G(x,t)$ defined by (6) satisfies the following properties:*

(i) $G(x,t) > 0$, *for all* $x, t \in (0,1)$

(ii) $\min_{(1/4)\le x\le(3/4)} G(x,t) \ge 1/4 \max_{0\le x\le 1} G(x,t) = 1/4 G(1,t)$, *for* $t \in (0,1)$

Lemma 4 (see [24]). *Let K be a cone in a Banach space X, and Ω be a bounded open set in K. Suppose that $T : \bar{\Omega} \longrightarrow K$ is a completely continuous operator. If there exists $y_0 \in K \setminus \{\theta\}$ such that*

$$y - Ty \ne \mu y_0, \quad \text{for all } y \in \partial\Omega, \mu \ge 0, \tag{7}$$

then the fixed-point index $i(T, \Omega, K) = 0$.

Lemma 5 (see [24]). *Let K be a cone in a Banach space X. Suppose that $T : K \longrightarrow K$ is a completely continuous operator. If there exists a bounded open set Ω such that each solution of*

$$y = \sigma Ty, \quad y \in K, \sigma \in [0,1], \tag{8}$$

satisfies $y \in \Omega$, then the fixed-point index $i(T, \Omega, K) = 1$.

Lemma 6 (see [25]). *Suppose that $A : C[0,1] \longrightarrow C[0,1]$ is a completely continuous linear operator and $A(K) \subset K$. If there exist $\psi \in C[0,1] \setminus (-K)$ and a constant $c > 0$ such that $cA\psi \ge \psi$, then the spectral radius $r(A) \ne 0$ and A has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(A))^{-1}$.*

3. Existence and Multiplicity

Let $C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\le x\le 1}|u(x)|$ and the ordering $u \le v$ if $u(x) \le v(x)$ for all $x \in [0,1]$. Define

$$K = \left\{ u \in C[0,1] \mid u(x) \ge 0, \min_{(1/4)\le x\le(3/4)} u(x) \ge \frac{1}{4}\|u\| \right\}. \tag{9}$$

Given $f \in C([0,1] \times [0,\infty), [0,\infty))$. Let $T, A : K \longrightarrow C[0,1]$ be the operators defined by

$$(Tu)(x) := \int_0^1 G(x,t) f(t, u(t))dt, \quad u \in K, x \in [0,1], \tag{10}$$

and

$$(Au)(x) := \int_0^1 G(x,t)u(t)dt, \quad u \in K, x \in [0,1]. \tag{11}$$

It is well known that $T, A : K \longrightarrow K$ are all completely continuous [23].

Denote

$$M = \left(\int_0^1 \max_{0\le x\le 1} G(x,t)dt \right)^{-1}, \qquad N = \left(\int_{(1/4)}^{(3/4)} G\left(\frac{1}{2}, t\right) dt \right)^{-1}, \tag{12}$$

where M, N are positive constants.

Lemma 7. *Suppose A is defined by (11), then the spectral radius $r(A) > 0$ and A has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(A))^{-1}$.*

Proof. The operator $A : C[0,1] \longrightarrow C[0,1]$ is a completely continuous linear operator and $A(K) \subseteq K$ (see [23]). Choose $\psi \in C[0,1]$ and $[x_1, x_2] \subset (0,1)$ such that $\psi(x) \ge 0$ for $x \in [0,1]$; $\psi(x) > 0$ for $x \in (x_1, x_2)$. By the use of Lemma 2, for $x \in [0,1]$, there holds

$$(A\psi)(x) = \int_0^1 G(x,t)\psi(t)dt \ge \int_{x_1}^{x_2} G(x,t)\psi(t)dt > 0. \tag{13}$$

So, we can choose $c \in \mathbb{R}$ so large that

$$c(A\psi)(x) \ge \psi(x), \quad \text{for } x \in [0,1]. \tag{14}$$

By Lemma 6, we complete the proof.

Theorem 8. *Suppose the following conditions hold:*

$$(I_0) \liminf_{y\to 0^+} \frac{f(x,y)}{y} > \lambda_1, \qquad (S_\infty) \limsup_{y\to+\infty} \frac{f(x,y)}{y} < \lambda_1, \tag{15}$$

where λ_1 is the first eigenvalue of the operator A defined by (11). Then, BVP (1) and (2) have at least one positive solution.

Proof. By condition (I_0), there exists $r_1 > 0$ small enough such that

$$f(x,y) \ge \lambda_1 y, \quad \text{for all } 0 \le x \le 1, 0 \le y \le r_1. \tag{16}$$

Let φ^* be the positive eigenfunction of A corresponding to λ_1, thus $\varphi^* = \lambda_1 A\varphi^*$.

For every $\varphi \in \bar{K}_{r_1}$, for $x \in [0,1]$

$$\begin{aligned}(T\varphi)(x) &= \int_0^1 G(x,t) f(t,\varphi(t))\,dt \\ &\geq \lambda_1 \int_0^1 G(x,t)\varphi(t)\,dt = \lambda_1 (A\varphi)(x).\end{aligned} \tag{17}$$

Suppose without loss of generality that T has no fixed point on ∂K_{r_1} (otherwise, the proof is completed). We claim that

$$\varphi - T\varphi \neq \mu\varphi^*, \quad \text{for all } \varphi \in \partial K_{r_1}, \mu > 0. \tag{18}$$

In fact, if there exist $\varphi_1 \in \partial K_{r_1}$ and $\mu_0 > 0$ such that $\varphi_1 - T\varphi_1 = \mu_0\varphi^*$, then

$$\varphi_1 = T\varphi_1 + \mu_0\varphi^* \geq \mu_0\varphi^*. \tag{19}$$

Let

$$\mu^* = \sup\{\mu \mid \varphi_1 \geq \mu\varphi^*\}. \tag{20}$$

It is easy to see that $+\infty > \mu^* \geq \mu_0 > 0$ and $\varphi_1 \geq \mu^*\varphi^*$. Taking into account that A is a linear positive operator, we have

$$\lambda_1 A\varphi_1 \geq \mu^*\lambda_1 A\varphi^* = \mu^*\varphi^*. \tag{21}$$

Therefore, by (17),

$$\varphi_1 = T\varphi_1 + \mu_0\varphi^* \geq \lambda_1 A\varphi_1 + \mu_0\varphi^* \geq (\mu^* + \mu_0)\varphi^*, \tag{22}$$

which contradicts the definition of μ^*. Hence (18) holds and we have from Lemma 3 that

$$i\left(T, K_{r_1}, K\right) = 0. \tag{23}$$

On the other hand, by (S_∞), there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$f(x,y) \leq \sigma\lambda_1 y, \quad \text{for all } y \geq r_2, x \in [0,1]. \tag{24}$$

Define $A_1 : C[0,1] \longrightarrow C[0,1]$ as $A_1\varphi = \sigma\lambda_1 A\varphi, \varphi \in C[0,1]$. Then A_1 is a bounded linear operator and $A_1(K) \subset K$. Denote

$$B = \left(\max_{0\leq x,t\leq 1} G(x,t)\right) \sup_{y\in\bar{B}_{r_2}} \int_0^1 f(t,y(t))\,dt. \tag{25}$$

It is clear that $B < +\infty$. Let

$$W = \{\varphi \in K \mid \varphi = \mu T\varphi, 0 \leq \mu \leq 1\}. \tag{26}$$

In the following, we firstly prove that the set W is bounded.

For any $\varphi \in W$, set $\bar{\varphi}(x) = \min\{\varphi(x), r_2\}$ for $x \in [0,1]$ and denote $E(\varphi) = \{x \in [0,1] \mid \varphi(x) > r_2\}$, then for $x \in [0,1]$,

$$\begin{aligned}\varphi(x) = \mu(T\varphi)(x) &\leq (T\varphi)(x) = \int_0^1 G(x,t) f(t,\varphi(t))\,dt \\ &= \int_{E(\varphi)} G(x,t) f(t,\varphi(t))\,dt + \int_{[0,1]\setminus E(\varphi)} G(x,t) f(t\bar{\varphi}(t))\,dt \\ &\leq \int_0^1 G(x,t)\sigma\lambda_1\varphi(t)\,dt + \int_0^1 G(x,t) f(t,\bar{\varphi}(t))\,dt \\ &\leq (A_1\varphi)(x) + B.\end{aligned} \tag{27}$$

Thus $((I - A_1)\varphi)(x) \leq B, x \in [0,1]$. Since λ_1 is the first eigenvalue of A and $0 < \sigma < 1$, the first eigenvalue of A_1, $(r(A_1))^{-1} > 1$. Therefore, the inverse operator $(I - A_1)^{-1}$ exists and

$$(I - A_1)^{-1} = I + A_1 + A_1^2 + \cdots + A_1^n + \cdots. \tag{28}$$

It follows from $A_1(K) \subset K$ that $(I - A_1)^{-1}(K) \subset K$. So we have $\varphi(x) \leq (I - A_1)^{-1}B, x \in [0,1]$ and the set W is bounded.

Choose $r_3 > \max\{r_2, \|(I - A_1)^{-1}B\|\}$. Then by Lemma 4, we have

$$i\left(T, K_{r_3}, K\right) = 1. \tag{29}$$

By (23) and (29), one has

$$i\left(\frac{T, K_{r_3}}{\bar{K}_{r_1}, K}\right) = i\left(T, K_{r_3}, K\right) - i\left(T, K_{r_1}, K\right) = 1. \tag{30}$$

Then, T has at least one fixed point on $K_{r_3} \setminus \bar{K}_{r_1}$. This means that problem (1) and (2) have at least one positive solution. The proof is complete.

Theorem 9. *Suppose the following conditions are met:*

$$\begin{aligned}&(I_\infty) \liminf_{y\to+\infty} \frac{f(x,y)}{y} < \lambda_1, \\ &(S_0) \limsup_{y\to 0^+} \frac{f(x,y)}{y} < \lambda_1,\end{aligned} \tag{31}$$

where λ_1 is the first eigenvalue of the operator A defined by (11). Then BVP (1) and (2) have at least one positive solution.The proof is similar to Theorem 8.

Theorem 10. *Suppose there exist two numbers $b > a > 0$ such that the following conditions are met:*

$$\begin{aligned}&(C_1)\, f(x,y) \leq Ma, \quad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq a, \\ &(C_2)\, f(x,y) \geq Nb, \quad \text{for } \frac{1}{4} \leq x \leq \frac{3}{4} \text{ and } \frac{1}{4}b \leq y \leq b.\end{aligned} \tag{32}$$

Then, BVP (1) and (2) have at least one positive solution.

Proof. If C_1 and C_2 hold, similar to Lemma 3 [6], we have

$$\begin{aligned} i(A, K_a, K) &= 1, \\ i(A, K_b, K) &= 0. \end{aligned} \tag{33}$$

Consequently, the additivity of the fixed-point index implies

$$i\left(\frac{A, K_b}{\bar{K}_a, K}\right) = i(A, K_b, K) - i(A, K_a, K) = -1. \tag{34}$$

Consequently, A has a fixed point $y(x)$ in $K_b \setminus \bar{K}_a$.

Theorem 11. *The problem in (1) and (2) has at least two positive solutions if conditions (I_0), (I_∞), and C_1 hold, where λ_1 is the first eigenvalue of the operator A defined by (11).*

Proof. Because (I_0) and (I_∞) hold, there exist $0 < r < a < R$ such that

$$\begin{aligned} i(A, K_r, K) &= 0, \\ i(A, K_R, K) &= 0. \end{aligned} \tag{35}$$

On the other hand, C_1 implies $i(A, K_a, K) = 1$. So we have

$$\begin{aligned} i\left(\frac{A, K_a}{\bar{K}_r, K}\right) &= i(A, K_a, K) - i(A, K_r, K) = 1, \\ i\left(\frac{A, K_R}{\bar{K}_a, K}\right) &= i(A, K_R, K) - i(A, K_a, K) = -1, \end{aligned} \tag{36}$$

therefore, A has two fixed points, $y_1 \in K_a \setminus \bar{K}_r$, $y_2 \in K_R \setminus \bar{K}_a$.

Theorem 12. *The problem in (1) and (2) has at least two positive solutions if conditions (S_0), (S_∞), and C_2 hold, where λ_1 is the first eigenvalue of the operator A defined by (10).*

Proof. Because (S_0) and (S_∞) hold, there exist $0 < r < b < R$ such that

$$\begin{aligned} i(A, K_r, K) &= 1, \\ i(A, K_R, K) &= 1. \end{aligned} \tag{37}$$

On the other hand C_2 implies $i(A, K_b, K) = 0$. So we have

$$\begin{aligned} i\left(\frac{A, K_b}{\bar{K}_r, K}\right) &= i(A, K_b, K) - i(A, K_r, K) = -1, \\ i\left(\frac{A, K_R}{\bar{K}_b, K}\right) &= i(A, K_R, K) - i(A, K_b, K) = 1. \end{aligned} \tag{38}$$

Therefore, A has two fixed points, $y_1 \in K_b \setminus \bar{K}_r$, $y_2 \in K_R \setminus \bar{K}_b$.

4. Example

To illustrate the main points, we give two examples.

Example 13. Let

$$f(x, y) = y^{(1/2)} + \frac{2 + \sin x}{4}. \tag{39}$$

Consider the BVP

$${}^{C}D^{\alpha}_{0+} y(x) + [y(x)]^{(1/2)} + \frac{2 + \sin x}{4} = 0, \quad 0 < x < 1, \tag{40}$$

$$y(0) = y'(1) = y''(0) = 0. \tag{41}$$

It is not difficult to see that

$$\frac{f(x, y)}{y} = y^{-(1/2)} + \frac{2 + \sin x}{4y}. \tag{42}$$

Then

$$\begin{aligned} \liminf_{y \to 0^+} \frac{f(x, y)}{y} &= +\infty > \lambda_1, \\ \limsup_{y \to +\infty} \frac{f(x, y)}{y} &= 0 < \lambda_1, \end{aligned} \tag{43}$$

where λ_1 is the first eigenvalue of the operator A defined by (11). By Theorem 11, BVP (40) and (41) have at least one positive solution.

Example 14. Let

$$f(x, y) = y^3 + y^2 \sin x. \tag{44}$$

Consider the BVP

$${}^{C}D^{\alpha}_{0+} y(x) + [y(x)]^3 + [y(x)]^2 \sin x = 0, \quad 0 < x < 1, \tag{45}$$

$$y(0) = y'(1) = y''(0) = 0. \tag{46}$$

It is not difficult to see that

$$\frac{f(x, y)}{y} = y^2 + y \sin x. \tag{47}$$

Then

$$\begin{aligned} \liminf_{y \to +\infty} \frac{f(x, y)}{y} &= +\infty > \lambda_1, \\ \limsup_{y \to +0^+} \frac{f(x, y)}{y} &= 0 < \lambda_1, \end{aligned} \tag{48}$$

where λ_1 is the first eigenvalue of the operator A defined by (11). By Theorem 12, BVP (45) and (46) have at least one positive solution.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Results of Positive Solutions for the Fractional Differential System on an Infinite Interval

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The chief topic of this paper is to investigate the fractional differential system on an infinite interval. By introducing an appropriate compactness criterion in a special function space and applying the Schauder fixed-point theorem and the Banach contraction mapping principle, we established the results for the existence and uniqueness of positive solutions. An example is then given to show the utilization of the main results.

1. Introduction

In this paper, we investigate the following fractional differential system on an infinite interval:

$$\begin{cases} D_{0^+}^{\alpha_1}u(t) + f_1(t,u(t),v(t)) = 0, \\ D_{0^+}^{\alpha_2}v(t) + f_2(t,u(t),v(t)) = 0, & 0 < t < +\infty, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0^+}^{\alpha_1-1}u(+\infty) = \mu_1 \int_0^{\infty} a_1(t)u(t)\mathrm{d}A_1(t), \\ v(0) = v'(0) = v''(0) = \cdots = v^{(m-2)}(0) = 0, & D_{0^+}^{\alpha_2-1}v(+\infty) = \mu_2 \int_0^{\infty} a_2(t)v(t)\mathrm{d}A_2(t), \end{cases} \tag{1}$$

where $1 \le n-1 < \alpha_1 \le n$, $1 \le m-1 < \alpha_2 \le m$, and $n, m \ge 2$, $D_{0^+}^{\alpha_i}$ is the Riemann–Liouville derivative operator. $\mu_i > 0$ is a constant, $\int_0^{\infty} a_1(t)u(t)\mathrm{d}A_1(t)$ and $\int_0^{\infty} a_2(t)v(t)\mathrm{d}A_2(t)$ denote the Riemann–Stieltjes integral, and A_i is a function of bounded positive variation. $a_i \in L[0, +\infty)$, $\mu_i \int_0^{\infty} a_i(t)t^{\alpha_i-1}\mathrm{d}A_i(t) < \Gamma(\alpha_i)$, $\int_0^{\infty} a_i(t)\mathrm{d}A_i(t) < +\infty$, and f_i: $[0, +\infty) \times [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ is continuous, $i = 1, 2$.

Numerous models in physics, chemistry, biology, medicine, and other fields have promoted the research of differential equations, for instance, evaluation of water quality on receiving water [1], and the advection-dispersion equation can be formulated as shown for the case of one-dimensional flow:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D_L\frac{\partial^2 C}{\partial t^2} - f(C), \tag{2}$$

where C is the concentration of a generic pollutant, t is the time, x is the longitudinal displacement, u is the velocity, D_L is the diffusion coefficient, and $f(C)$ is a generic term for reactions involving the pollutant C. Westerlund [2] established a one-dimensional model to describe the transmission of the electromagnetic wave:

$$\mu\varepsilon\frac{\partial^2 E(x,t)}{\partial x^2}+\mu\varepsilon\zeta D_t^{\nu}E(x,t)+\frac{\partial^2 E(x,t)}{\partial t^2}=0, \tag{3}$$

where μ, ε, and ζ are constants and $D_t^{\nu}E(x,t)=(\partial^{\nu}E(x,t))/\partial t^{\nu}$ is a fractional derivative. In the process of establishing the model, k-Hessian equations [3], Sobolev equations [4], and Schrödinger elliptic equations [5, 6], there are also huge applications.

Under the proper initial or boundary conditions, to study the positive solution of the above models is very necessary; especially, for the boundary value problems on the infinite interval, many authors put their interest in it [7–16]. Liang and Zhang [17] applied the fixed-point theorem to obtain the existence of positive solutions for the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+a(t)f(u(t))=0, \quad 0<t<+\infty, \\ u(0)=u'(0)=0, \qquad \lim\limits_{t\longrightarrow+\infty}D_{0^+}^{\alpha-1}u(t)=\sum\limits_{i=1}^{m-2}\beta_i u(\xi_i), \end{cases} \tag{4}$$

where $2<\alpha\le 3$, $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<+\infty$, $\beta_i>0$, $0<\sum_{i=1}^{m-2}\beta_i u(\xi_i)<\Gamma(\alpha)$, and $f:[0,+\infty)\longrightarrow[0,+\infty)$is continuous.

For all we know, there are few studies on fractional differential systems of infinite intervals, although it is necessary to do so. In this paper, we aimed at getting the existence and uniqueness of positive solutions for system (1) on infinite interval. Compared with the existing literature, the innovations of this paper are as follows. Firstly, the paper which we discuss is the system rather than a single equation. Secondly, we study the system with integral boundary value conditions on infinite intervals, which are more general than those of two-point, three-point, and multipoint boundary value condition. At last, we use two different techniques: the Schauder fixed-point theorem and the Banach contraction mapping principle, for system (1), not only to obtain the existence of positive solutions but also the uniqueness of positive solutions.

2. Preliminaries and Lemmas

Definition 1 (see [18, 19]). Let $\alpha>0$ and u be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $[0,+\infty)$. Then, for $t>0$, we call

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\mathrm{d}s \tag{5}$$

the Riemann–Liouville fractional integral of u of order α.

Definition 2 (see [18, 19]). The Riemann–Liouville fractional derivative of order $\alpha>0$, $n-1\le\alpha<n$, $n\in\mathbb{N}$, is defined as

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t(t-s)^{n-\alpha-1}u(s)\mathrm{d}s, \tag{6}$$

where $\mathbb{N}$ denotes the natural number set and the function $u(t)$ is n times continuously differentiable on $[0,+\infty)$.

Lemma 1 (see [18, 19]). *Let $\alpha>0$, and if the fractional derivative $D_{0^+}^{\alpha-1}u(t)$ and $D_{0^+}^{\alpha}u(t)$ are continuous on $[0,+\infty)$, then,*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}, \tag{7}$$

where $c_1, c_2, \ldots, c_n\in(-\infty,+\infty)$, n is the smallest integer greater than or equal to α.

Lemma 2. *Let $y_i\in C(0,+\infty)\cap L[0,+\infty)$; then, the fractional system*

$$\begin{cases} D_{0^+}^{\alpha_1}u(t)+y_1(t)=0, \\ D_{0^+}^{\alpha_1}v(t)+y_2(t)=0, \\ 0<t<+\infty, \\ u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0, \\ D_{0^+}^{\alpha_1-1}u(+\infty)=\mu_1\int_0^{\infty}a_1(t)u(t)\mathrm{d}A_1(t), \\ v(0)=v'(0)=v''(0)=\cdots=v^{(m-2)}(0)=0, \\ D_{0^+}^{\alpha_2-1}v(+\infty)=\mu_2\int_0^{\infty}a_2(t)v(t)\mathrm{d}A_2(t), \end{cases} \tag{8}$$

has a unique integral representation

$$\begin{cases} u(t)=\int_0^{\infty}G_1(t,s)y_1(s)\mathrm{d}s, \\ v(t)=\int_0^{\infty}G_2(t,s)y_2(s)\mathrm{d}s, \end{cases} \tag{9}$$

where

$$G_i(t,s)=G_{i1}=(t,s)+G_{i2}(t,s), \quad i=1,2,$$

$$G_{i1}(t,s)=\frac{1}{\Gamma(\alpha_i)}\begin{cases} t^{\alpha_i-1}-(t-s)^{\alpha_i-1}, & 0\le s\le t\le+\infty, \\ t^{\alpha_i-1}, & 0\le t\le s\le+\infty, \end{cases} \tag{10}$$

$$G_{i2}(t,s)=\frac{\mu_i t^{\alpha_i-1}}{\Gamma(\alpha_i)-\mu_i\int_0^{\infty}a_i(t)t^{\alpha_i-1}\mathrm{d}A_i(t)}\int_0^{\infty}a_i(t)G_{i1}(t,s)\mathrm{d}A_i(t).$$

Proof. By Lemma 1, the equations in system (8) can be transformed into the equivalent integral equations

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha_1} y_1(t) + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}, c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n, \\ v(t) &= -I_{0+}^{\alpha_2} y_2(t) + \overline{c}_1 t^{\alpha_2-1} + \overline{c}_2 t^{\alpha_2-2} + \cdots + c_m t^{\alpha_2-m}, \overline{c}_j \in (-\infty, +\infty), \quad \overline{j} = 1, 2, \ldots, m, \end{aligned} \tag{11}$$

that is,

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s)\mathrm{d}s + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}, c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n, \\ v(t) &= -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s)\mathrm{d}s + \overline{c}_1 t^{\alpha_2-1} + \overline{c}_2 t^{\alpha_2-2} + \cdots + \overline{c}_m t^{\alpha_2-m}, \overline{c}_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, m. \end{aligned} \tag{12}$$

Since

$$\begin{aligned} u(0) = u'(0) = \cdots = u^{(n-2)} = 0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \end{aligned} \tag{13}$$

we have

$$\begin{aligned} c_2 = c_3 = \cdots = c_n = 0, \\ \overline{c}_2 = \overline{c}_3 = \cdots = \overline{c}_m = 0. \end{aligned} \tag{14}$$

So,

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s)\mathrm{d}s + c_1 t^{\alpha_1-1}, \\ v(t) &= -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s)\mathrm{d}s + \overline{c}_1 t^{\alpha_2-1}. \end{aligned} \tag{15}$$

We also have

$$\begin{aligned} D_{0^+}^{\alpha_1-1} u(t) &= c_1 \Gamma(\alpha_1) - \int_0^t y_1(s)\mathrm{d}s, \\ D_{0^+}^{\alpha_2-1} v(t) &= \overline{c}_1 \Gamma(\alpha_2) - \int_0^t y_2(s)\mathrm{d}s. \end{aligned} \tag{16}$$

For

$$\begin{aligned} D_{0^+}^{\alpha_1-1} u(+\infty) &= \mu_1 \int_0^\infty a_1(s) u(s)\mathrm{d}A_1(s), \\ D_{0^+}^{\alpha_2-1} v(+\infty) &= \mu_2 \int_0^\infty a_2(s) v(s)\mathrm{d}A_2(s), \end{aligned} \tag{17}$$

we obtain

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha_1)} \left(\mu_1 \int_0^\infty a_1(s) u(s)\mathrm{d}A_1(s) + \int_0^{+\infty} y_1(s)\mathrm{d}s \right), \\ \overline{c}_1 &= \frac{1}{\Gamma(\alpha_2)} \left(\mu_2 \int_0^\infty a_2(s) v(s)\mathrm{d}A_2(s) + \int_0^{+\infty} y_2(s)\mathrm{d}s \right). \end{aligned} \tag{18}$$

Combining (15) and (18), we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s)\mathrm{d}s + c_1 t^{\alpha_1-1} \\ &= \int_0^\infty G_{11}(t,s) y_1(s)\mathrm{d}s + \frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)} \int_0^\infty a_1(t) u(t)\mathrm{d}A_1(t), \end{aligned} \tag{19}$$

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s)\mathrm{d}s + \overline{c}_1 t^{\alpha_2-1} \\ &= \int_0^\infty G_{21}(t,s) y_2(s)\mathrm{d}s + \frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)} \int_0^\infty a_2(t) v(t)\mathrm{d}A_2(t). \end{aligned} \tag{20}$$

(19) and (20) are multiplied by $a_1(t)$ and $a_2(t)$, respectively, and then solved the integral from 0 to $+\infty$ with respect to $A_1(t)$ and $A_2(t)$; then, we have

$$\begin{aligned} \int_0^\infty a_1(t) u(t)\mathrm{d}A_1(t) &= \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1) - \mu_1 \int_0^\infty a_1(t)\mathrm{d}A_1(t)} \int_0^\infty a_1(t) \int_0^\infty G_{11}(t,s) y_1(s)\mathrm{d}s\,\mathrm{d}A_1(t), \\ \int_0^\infty a_2(t) v(t)\mathrm{d}A_2(t) &= \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2) - \mu_2 \int_0^\infty a_2(t)\mathrm{d}A_2(t)} \int_0^\infty a_2(t) \int_0^\infty G_{21}(t,s) y_2(s)\mathrm{d}s\,\mathrm{d}A_2(t). \end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}
u(t) &= \int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s + \frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_0^\infty a_1(t)u(t)\mathrm{d}A_1(t)\\
&= \int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s + \frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)-\mu_1\int_0^\infty a_1(t)\mathrm{d}A_1(t)}\\
&\quad\cdot\int_0^\infty a_1(t)\int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s\,\mathrm{d}A_1(t)\\
&= \int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s + \int_0^\infty G_{12}(t,s)y_1(s)\mathrm{d}s,\\
v(t) &= \int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s + \frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_0^\infty a_2(t)v(t)\mathrm{d}A_2(t)\\
&= \int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s + \frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)-\mu_2\int_0^\infty a_2(t)\mathrm{d}A_2(t)}\\
&\quad\cdot\int_0^\infty a_2(t)\int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s\,\mathrm{d}A_2(t)\\
&= \int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s + \int_0^\infty G_{22}(t,s)y_2(s)\mathrm{d}s.
\end{aligned} \tag{22}$$

So, (9) holds. The proof is completed.

Lemma 3. *The Green function in Lemma 2 has the following properties:*

(1) $G_{i1}(t,s)$ *is continuous and* $G_{i1}(t,s)\geq 0$, $(t,s)\in[0,+\infty)\times[0,+\infty)$.

(2) $(G_{i1}(t,s)/1+t^{\alpha_i-1})\leq(1/\Gamma(\alpha_i))$, $(G_i(t,s)/1+t^{\alpha_i-1})\leq\bar{\omega}$, $(t,s)\in[0,+\infty)\times[0,+\infty)$,

where

$$\bar{\omega}=\max\left\{\begin{array}{l}\dfrac{1}{\Gamma(\alpha_1)}+\dfrac{\Gamma(\alpha_1)+\mu_1\int_0^\infty a_1(t)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)\left(\Gamma(\alpha_1)-\mu_1\int_0^\infty a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)\right)},\\[2ex] \dfrac{1}{\Gamma(\alpha_2)}+\dfrac{\Gamma(\alpha_2)+\mu_2\int_0^\infty a_2(t)\mathrm{d}A_2(t)}{\Gamma(\alpha_2)\left(\Gamma(\alpha_2)-\mu_2\int_0^\infty a_2(t)t^{\alpha_2-1}\mathrm{d}A_2(t)\right)}\end{array}\right\}. \tag{23}$$

The space $X=E_1\times E_2$ *will be used in the study of system (1), where*

$$\begin{aligned}
E_1&=\left\{u\in C[0,+\infty):\ \sup_{t\in[0,+\infty)}\frac{u(t)}{1+t^{\alpha_1-1}}<+\infty\right\},\\
E_2&=\left\{v\in C[0,+\infty):\ \sup_{t\in[0,+\infty)}\frac{v(t)}{1+t^{\alpha_2-1}}<+\infty\right\}.
\end{aligned} \tag{24}$$

Then, $(E_1,\|\cdot\|)$ *and* $(E_2,\|\cdot\|)$ *are the Banach space with the norm*

$$\begin{aligned}
\|u\|&=\sup_{t\in[0,+\infty)}\frac{u(t)}{1+t^{\alpha_1-1}},\\
\|v\|&=\sup_{t\in[0,+\infty)}\frac{v(t)}{1+t^{\alpha_2-1}}.
\end{aligned} \tag{25}$$

Clearly, $(X,\|\cdot\|)$ *is a Banach space with the norm* $\|u,v\|=\|u\|+\|v\|$. *Define nonlinear integral operators* $T_i: X\longrightarrow E_i$ *and* $T: X\longrightarrow X$ *by*

$$T_i(u,v)(t)=\int_0^\infty G_i(t,s)f_i(s,u(s),v(s))\mathrm{d}s,\quad i=1,2, \tag{26}$$

$$T(u,v)=\left(T_1(u,v),T_2(u,v)\right). \tag{27}$$

Thus, the existence of solutions to system (1) is equivalent to the existence of solutions in X for operator equation $T(u,v)=(u,v)$ *defined by (27).*

Lemma 4 (see [20, 21]). *Let E be defined as (24) and M be any bounded subset of E. Then, M is relatively compact in E if* $\{(x(t)/1+t^{\alpha-1}):x\in M\}$ *is equicontinuous on any finite subinterval of J, and for any given* $\varepsilon>0$, *there exists* $N>0$ *such that* $|(x(t_1)/1+t_1^{\alpha-1})-(x(t_2)/1+t_2^{\alpha-1})|<\varepsilon$ *uniformly with respect to all* $x\in M$, *and* t_1, $t_2>N$.

3. Main Results

We list the conditions to be used later: $(\mathbf{H}_1)$ there exist nonnegative functions $p_i(t)$, $g_i(t)$, $h_i(t)\in L^1[0,+\infty)$ and $t^{\alpha_1-1}g_i(t)$, $t^{\alpha_2-1}h_i(t)\in L^1[0,+\infty)$ such that

$$\begin{aligned}
|f_i(t,u,v)|&\leq p_i(t)+g_i(t)|u|+h_i(t)|v|,\ (t,u,v)\in[0,+\infty)\\
&\quad\times[0,+\infty)\times[0,+\infty).
\end{aligned} \tag{28}$$

$(\mathbf{H}_2)$ $|f_i(t,0,0)|\in L^1[0,+\infty)$, there exist nonnegative functions $g_i(t)$, $h_i(t)\in L^1[0,+\infty)$ and $t^{\alpha_1-1}g_i(t)$, $t^{\alpha_2-1}h_i(t)\in L^1[0,+\infty)$ such that

$$\begin{aligned}
|f_i(t,u_1,v_1)-f_i(t,u_2,v_2)|&\leq g_i(t)|u_1-u_2|+h_i(t)|v_1-v_2|,\\
(t,u_1,v_1),(t,u_1,v_1)&\in[0,+\infty)\times[0,+\infty)\times[0,+\infty).
\end{aligned} \tag{29}$$

Remark 1. If $(\mathbf{H}_1)$ holds, then

$$\int_0^\infty|f_i(t,u(t),v(t))|\mathrm{d}t\leq p_i^*+(g_i^*+h_i^*)\|(u,v)\|,\quad (u,v)\in X, \tag{30}$$

where

$$\begin{aligned}
p_i^*&=\int_0^\infty p_i(t)\mathrm{d}t,\\
g_i^*&=\int_0^\infty\left(1+t^{\alpha_1-1}\right)g_i(t)\mathrm{d}t,\\
h_i^*&=\int_0^\infty\left(1+t^{\alpha_2-1}\right)h_i(t)\mathrm{d}t.
\end{aligned} \tag{31}$$

In fact, by $(\mathbf{H}_1)$, for any $(u, v) \in X$, we have

$$\begin{aligned}
&\int_0^\infty \left| f_i(t, u(t), v(t)) \right| \mathrm{d}t \\
&\leq \int_0^\infty \left(p_i(t) + g_i(t)|u(t)| + h_i(t)|v(t)| \right) \mathrm{d}t \\
&= \int_0^\infty \left(p_i(t) + \left(1 + t^{\alpha_1 - 1}\right) g_i(t) \frac{|u(t)|}{1 + t^{\alpha_1 - 1}} + \left(1 + t^{\alpha_2 - 1}\right) h_i(t) \frac{|v(t)|}{1 + t^{\alpha_2 - 1}} \right) \mathrm{d}t \\
&\leq \int_0^\infty \left(p_i(t) + \left(1 + t^{\alpha_1 - 1}\right) g_i(t) \|u\| + \left(1 + t^{\alpha_2 - 1}\right) h_i(t) \|v\| \right) \mathrm{d}t \\
&= \int_0^\infty p_i(t) \mathrm{d}t + \int_0^\infty \left(1 + t^{\alpha_1 - 1}\right) g_i(t) \mathrm{d}t \|u\| + \int_0^\infty \left(1 + t^{\alpha_2 - 1}\right) h_i(t) \mathrm{d}t \|v\| \\
&= p_i^* + g_i^* \|u\| + h_i^* \|v\| \leq p_i^* + \left(g_i^* + h_i^*\right) \|(u, v)\|.
\end{aligned} \tag{32}$$

Theorem 1. *Assume that $(\boldsymbol{H}_1)$ holds; then, $T: X \longrightarrow X$ is a completely continuous operator.*

Proof. First, we show that $T: X \longrightarrow X$ is continuous. Suppose $\{(u_n, v_n)\} \subset X$, $(u, v) \in X$ with $\|(u_n, v_n) - (u, v)\| \longrightarrow 0$ $(n \longrightarrow +\infty)$, and there exists a constant $r > 0$ such that $\|(u_n, v_n)\| \leq r$ and $\|(u, v)\| \leq r$. By $(\mathbf{H}_1)$ and (30), we have

$$\begin{aligned}
&\left| \int_0^\infty G_1(t, s) f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s - \int_0^\infty G_1(t, s) f_1\left(s, u(s), v(s)\right) \mathrm{d}s \right| \\
&\leq \int_0^\infty G_1(t, s) f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s + \int_0^\infty G_1(t, s) f_1\left(s, u(s), v(s)\right) \mathrm{d}s \\
&\leq \varpi \int_0^\infty f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s + \varpi \int_0^\infty f_1\left(s, u(s), v(s)\right) \mathrm{d}s \\
&\leq 2\varpi \left(p_1^* + g_1^* \|u\| + h_1^* \|v\|\right) \leq 2\varpi \left(p_1^* + \left(g_1^* + h_1^*\right) \|(u, v)\|\right) < +\infty.
\end{aligned} \tag{33}$$

From $(\mathbf{H}_1)$ and (33), for any $\varepsilon > 0$, there exists sufficiently large M_0 such that

$$\begin{aligned}
&\int_{M_0}^\infty G_1(t, s) f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s + \int_{M_0}^\infty G_1(t, s) f_1 \\
&\cdot \left(s, u(s), v(s)\right) \mathrm{d}s < \frac{\varepsilon}{2}.
\end{aligned} \tag{34}$$

On the contrary, by the continuity of $f_1(t, u, v)$ on $[0, M_0] \times [0, (1 + M_0^{\alpha_1 - 1})r] \times [0, (1 + M_0^{\alpha_1 - 1})r]$, there exists $N > 0$ such that when $n > N$ and $t \in [0, M_0]$, we have

$$\left| f_1\left(s, u_n(s), v_n(s)\right) - f_1\left(s, u(s), v(s)\right) \right| < \frac{\varepsilon}{2\varpi M_0}. \tag{35}$$

Hence, for any $t \in [0, +\infty)$ and $n > N$, we obtain

$$\begin{aligned}
&\left| \frac{T_1(u_n, v_n)(t)}{1 + t^{\alpha_1 - 1}} - \frac{T_1(u, v)(t)}{1 + t^{\alpha_1 - 1}} \right| \\
&= \left| \int_0^\infty \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s - \int_0^\infty \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} f_1\left(s, u(s), v(s)\right) \mathrm{d}s \right| \\
&\leq \int_0^{M_0} \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} \left| f_1\left(s, u_n(s), v_n(s)\right) - f_1\left(s, u(s), v(s)\right) \right| \mathrm{d}s \\
&\quad + \int_{M_0}^\infty \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} f_1\left(s, u_n(s), v_n(s)\right) \mathrm{d}s + \int_{M_0}^\infty \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} f_1\left(s, u(s), v(s)\right) \mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^{M_0} \frac{G_1(t,s)}{1+t^{\alpha_1-1}} \left| f_1(s,u_n(s),v_n(s)) - f_1(s,u(s),v(s)) \right| \mathrm{d}s \\
&\quad + \int_{M_0}^{\infty} G_1(t,s) f_1(s,u_n(s),v_n(s)) \mathrm{d}s + \int_{M_0}^{\infty} G_1(t,s) f_1(s,u(s),v(s)) \mathrm{d}s \\
&< \varpi \int_0^{M_0} \left| f_1(s,u_n(s),v_n(s)) \mathrm{d}s - f_1(s,u(s),v(s)) \right| \mathrm{d}s + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{36}$$

Thus, we know that $\|T_1(u_n,v_n) - T_1(u,v)\| \longrightarrow 0$ $(n \longrightarrow +\infty)$. By the similar proof as (33)–(36), we know $\|T_2(u_n,v_n) - T_2(u,v)\| \longrightarrow 0$ $(n \longrightarrow +\infty)$. So, $T: X \longrightarrow X$ is continuous.

Next, we show that $T: X \longrightarrow X$ is relatively compact. Let Ω be a bounded subset of X; then, there exists constant $M > 0$ such that $\|(u,v)\| \le M$, $(u,v) \in \Omega$. For any $(u,v) \in \Omega$, $t \in [0, +\infty)$, and by (32), we obtain

$$\begin{aligned}
\left| \frac{T_1(u_n,v_n)(t)}{1+t^{\alpha_1-1}} \right| &= \left| \int_0^{\infty} \frac{G_1(t,s)}{1+t^{\alpha_1-1}} f_1(s,u_n(s),v_n(s)) \mathrm{d}s \right| \\
&\le \varpi \int_0^{\infty} \left| f_1(s,u_n(s),v_n(s)) \right| \mathrm{d}s \\
&\le \varpi \left(p_1^* + (g_1^* + h_1^*) \|(u,v)\| \right) \\
&\le \varpi \left(p_1^* + (g_1^* + h_1^*) M \right) < +\infty.
\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}
\left| \frac{T_2(u_n,v_n)(t)}{1+t^{\alpha_2-1}} \right| &\le \varpi \left(p_1^* + (g_1^* + h_1^*) \|(u,v)\| \right) \\
&\le \varpi \left(p_1^* + (g_1^* + h_1^*) M \right) < +\infty.
\end{aligned} \tag{38}$$

Consequently, $T(\Omega)$ is uniformly bounded.

Given $I \subset [0, +\infty)$ be any compact interval. For any t_1, $t_2 \in I$, $t_1 < t_2$ and $(u,v) \in \Omega$, we deduce

$$\begin{aligned}
&\left| \frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}} - \frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}} \right| \\
&= \left| \int_0^{\infty} \frac{G_1(t_2,s)}{1+t_2^{\alpha_1-1}} f_1(s,u(s),v(s)) \mathrm{d}s - \int_0^{\infty} \frac{G_1(t_1,s)}{1+t_1^{\alpha_1-1}} f_1(s,u(s),v(s)) \mathrm{d}s \right| \\
&\le \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) \mathrm{d}s \\
&\quad + \int_0^{\infty} \left| \frac{G_{12}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{12}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) \mathrm{d}s \\
&\le \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) \mathrm{d}s \\
&\quad + \left| \frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}} - \frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}} \right| \int_0^{\infty} \frac{\mu_1 \int_0^{\infty} a_1(t) G_{11}(t,s) \mathrm{d}A_1(t)}{\Gamma(\alpha_1) - \mu_1 \int_0^{\infty} a_1(t) t^{\alpha_1-1} \mathrm{d}A_1(t)} f_1(s,u(s),v(s)) \mathrm{d}s \\
&\le \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) \mathrm{d}s \\
&\quad + \int_0^{\infty} \left| \frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) \mathrm{d}s \\
&\quad + \left| \frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}} - \frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}} \right| \int_0^{\infty} \frac{\mu_1 \int_0^{\infty} a_1(t) G_{11}(t,s) \mathrm{d}A_1(t)}{\Gamma(\alpha_1) - \mu_1 \int_0^{\infty} a_1(t) t^{\alpha_1-1} \mathrm{d}A_1(t)} f_1(s,u(s),v(s)) \mathrm{d}s.
\end{aligned} \tag{39}$$

Since

$$\begin{aligned}
&\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\int_0^{t_1}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\int_{t_2}^{+\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s\right|\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|+\left|(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1\\
&\quad\cdot(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|+\left|(t_2-s)^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1(s,u\\
&\quad\cdot(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^{+\infty}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s\\
&\longrightarrow 0,\qquad\qquad\text{as } t_1\longrightarrow t_2.
\end{aligned}\tag{40}$$

In the same way, we can know

$$\int_0^{\infty}\left|\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\longrightarrow 0,\quad\text{as } t_1\longrightarrow t_2.\tag{41}$$

So,

$$\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right|\longrightarrow 0,\quad\text{as } t_1\longrightarrow t_2.\tag{42}$$

Similar to (39)–(41), we have

$$\left|\frac{T_2(u,v)(t_2)}{1+t_2^{\alpha_2-1}}-\frac{T_2(u,v)(t_1)}{1+t_1^{\alpha_2-1}}\right|\longrightarrow 0,\quad\text{as } t_1\longrightarrow t_2.\tag{43}$$

Therefore, $T(\Omega)$ is equicontinuous.

By $(\mathbf{H}_1)$ and (30), for any $\varepsilon>0$, there exists $\kappa>0$ such that

$$\int_{\kappa}^{\infty}f(s,u_n(s),v_n(s))\mathrm{d}s_1<\varepsilon.\tag{44}$$

Due to $\lim_{t\longrightarrow+\infty}(t^{\alpha_1-1}/1+t^{\alpha_1-1})=1$, there exists sufficiently large $N_1>0$ such that, for any t_1, $t_2>N_1$, we have

$$\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|<\varepsilon.\tag{45}$$

Also because of $\lim_{t\longrightarrow+\infty}((t-\kappa)^{\alpha_1-1}/1+t^{\alpha_1-1})=1$, there exists sufficiently large $N_2>\kappa$ such that, for any t_1, $t_2>N_2$ and $0\le s\le\kappa$, we have

$$\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|<\varepsilon.\tag{46}$$

Choose $N>\max\{N_1,N_2\}$; for any t_1, $t_2>N$, we get

$$\begin{aligned}
&\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right|\\
&\le\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\int_0^{\infty}\frac{\mu_1\int_0^{\infty}a_1(t)G_{11}(t,s)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)-\mu_1\int_0^{\infty}a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)}f_1(s,u(s),v(s))\mathrm{d}s.
\end{aligned}\tag{47}$$

In (47),

$$
\begin{aligned}
&\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\int_0^{t_1}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\int_{t_2}^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}\right|\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{t_1}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+1\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\frac{\max\limits_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+1\right)f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\frac{\max\limits_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)\mathrm{d}s\\
&\quad+\frac{9}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}f_1(s,u(s),v(s))\mathrm{d}s\\
&\le\frac{\max\limits_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}2\kappa\varepsilon+\frac{9}{\Gamma(\alpha_1)}\varepsilon.
\end{aligned}
\tag{48}
$$

By (47) and (48), we have

$$\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right|$$

$$\leq\frac{\max_{s\in[0,\kappa],(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}2\kappa\varepsilon+\frac{9}{\Gamma(\alpha_1)}\varepsilon$$

$$+\varepsilon\int_0^\infty\frac{\mu_1\int_0^\infty a_1(t)G_{11}(t,s)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)-\mu_1\int_0^\infty a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)}f_1(s,u(s),v(s))\mathrm{d}s. \tag{49}$$

So, $T_1(u,v)$ is equicontinuous on $+\infty$, proof similar to (49), and we know $T_2(u,v)$ is equicontinuous on $+\infty$; thus, $T(u,v)$ is equicontinuous on $+\infty$. It follows from Lemma 4 that $T:X\longrightarrow X$ is relatively compact. Therefore, $T:X\longrightarrow X$ is completely continuous. The proof is completed. □

Theorem 2. *Assume that (H_1) holds; then, system (1) has at least one positive solution if $\varpi(g_1^*+g_2^*+h_1^*+h_2^*)<1$.*

Proof. Let

$$r\geq\frac{\varpi(p_1^*+p_2^*)}{1-\varpi(g_1^*+g_2^*+h_1^*+h_2^*)},$$
$$K=\{(u,v)\in X,\|(u,v)\|\leq r\}. \tag{50}$$

Now, we illustrate $T(K)\subset K$ for any $(u,v)\in K$ and $t\in[0,+\infty)$; by Lemma 3 and Remark 1, we have

$$\left|\frac{T_1(u,v)(t)}{1+t^{\alpha_1-1}}\right|=\left|\int_0^\infty\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s\right|$$
$$\leq\varpi\int_0^\infty\left|f_1(s,u(s),v(s))\right|\mathrm{d}s$$
$$\leq\varpi(p_1^*+(g_1^*+h_1^*)\|(u,v)\|)$$
$$\leq\varpi(p_1^*+(g_1^*+h_1^*)r)\leq r. \tag{51}$$

Similarly, we have

$$\left|\frac{T_2(u,v)(t)}{1+t^{\alpha_2-1}}\right|\leq\varpi(p_2^*+(g_2^*+h_2^*)r)\leq r. \tag{52}$$

By (51) and (52),

$$\|T(u,v)\|=\|T_1(u,v)\|+\|T_2(u,v)\|\leq\varpi[(p_1^*+(g_1^*+h_1^*)r)+(p_2^*+(g_2^*+h_2^*)r)]\leq r. \tag{53}$$

Therefore, $T(K)\subset K$. By Theorem 1, we know that $T:K\longrightarrow K$ is completely continuous. So, by the Schauder fixed-point theorem, system (1) has at least one positive solution. The proof is completed. □

Theorem 3. *Assume that (H_2) holds; then, system (1) has a unique positive solution if $\varpi(g_1^*+g_2^*+h_1^*+h_2^*)<1$.*

Proof. From ($\mathbf{H}_2$), we know

$$\left|f_i(t,u,v)\right|\leq g_i(t)|u|+h_i(t)|v|+\left|f_i(t,0,0)\right|. \tag{54}$$

So, for any $(u,v)\in X$, we have

$$\begin{aligned}
&\int_0^\infty\left|f_i(t,u(t),v(t))\right|\mathrm{d}t\\
&\leq\int_0^\infty\left(\left(g_i(t)|u(t)|+h_i(t)|v(t)|+\left|f_i(t,0,0)\right|\right)\right)\mathrm{d}t\\
&=\int_0^\infty\left(\left(1+t^{\alpha_1-1}\right)g_i(t)\frac{|u(t)|}{1+t^{\alpha_1-1}}+\left(1+t^{\alpha_2-1}\right)h_i(t)\frac{|v(t)|}{1+t^{\alpha_2-1}}+\left|f_i(t,0,0)\right|\right)\mathrm{d}t\\
&\leq\int_0^\infty\left(\left(1+t^{\alpha_1-1}\right)g_i(t)\|u(t)\|+\left(1+t^{\alpha_2-1}\right)h_i(t)\|v(t)\|+\left|f_i(t,0,0)\right|\right)\mathrm{d}t\\
&=\int_0^\infty\left(1+t^{\alpha_1-1}\right)g_i(t)\mathrm{d}t\|u(t)\|+\int_0^\infty\left(1+t^{\alpha_2-1}\right)h_i(t)\mathrm{d}t\|v(t)\|+\int_0^\infty\left|f_i(t,0,0)\right|\mathrm{d}t\\
&=g_i^*\|u\|+h_i^*\|v\|+\int_0^\infty\left|f_i(t,0,0)\right|\mathrm{d}t\leq(g_i^*+h_i^*)\|(u,v)\|+\int_0^\infty\left|f_i(t,0,0)\right|\mathrm{d}t.
\end{aligned} \tag{55}$$

For any $(u_1, v_1), (u_2, v_2) \in X$ and $t \in [0, +\infty)$, by Lemma 3, we have

$$\begin{aligned}
&\left|\frac{T_1(u_1, v_1)(t) - T_1(u_2, v_2)(t)}{1 + t^{\alpha_1 - 1}}\right| \\
&= \left|\int_0^\infty \frac{G_1(t, s)}{1 + t^{\alpha_1 - 1}} \left|f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s))\right| \mathrm{d}s\right| \\
&\le \varpi \int_0^\infty \left(g_1(s)\left|(u_1(s) - u_2(s))\right| + h_1(s)\left|(v_1(s) - v_2(s))\right|\right) \mathrm{d}s \\
&\le \varpi \int_0^\infty \left(\left(1 + s^{\alpha_1 - 1}\right) g_1(s) \frac{\left|(u_1(s) - u_2(s))\right|}{1 + s^{\alpha_1 - 1}} + \left(1 + s^{\alpha_1 - 1}\right) h_1\right. \\
&\quad \left.(s) \frac{\left|(v_1(s) - v_2(s))\right|}{1 + s^{\alpha_1 - 1}}\right) \mathrm{d}s \\
&\le \varpi\left(g_1^* \|u_1 - u_2\| + h_1^* \|v_1 - v_2\|\right).
\end{aligned} \tag{56}$$

By the similar proof, we have

$$\left|\frac{T_2(u_1, v_1)(t) - T_2(u_2, v_2)(t)}{1 + t^{\alpha_2 - 1}}\right| \le \varpi\left(g_2^* \|u_1 - u_2\| + h_2^* \|v_1 - v_2\|\right). \tag{57}$$

Now, inequalities (56) and (57) can show that

$$\|T(u, v)\| \le \varpi\left(g_1^* + g_2^* + h_1^* + h_2^*\right)\left(\|u_1 - u_2\| + \|v_1 - v_2\|\right). \tag{58}$$

Thus, by the Banach contraction mapping theorem that T has a unique fixed point in X, system (1) has a unique positive solution. The proof is completed. □

4. An Example

Consider the following fractional differential system:

$$\begin{cases}
D_{0+}^{(5/2)} u(t) + f_1(t, u(t), v(t)) = 0, \\
D_{0+}^{(7/2)} v(t) + f_2(t, u(t), v(t)) = 0, \quad 0 < t < +\infty, \\
u(0) = u'(0) = 0, \\
D_{0+}^{(3/2)} u(+\infty) = \dfrac{1}{5} \displaystyle\int_0^\infty e^{-t} u(t) \mathrm{d}t, \\
v(0) = v'(0) = v''(0) = 0, \\
D_{0+}^{(5/2)} v(+\infty) = \dfrac{1}{4} \displaystyle\int_0^\infty e^{-t} v(t) \mathrm{d}t,
\end{cases} \tag{59}$$

where $\alpha_1 = (5/2)$, $\alpha_2 = (7/2)$, $\mu_1 = (1/5)$, $\mu_2 = (1/4)$, $A_1(t) = A_2(t) = t$, $a_1(t) = e^{-t} = a_2(t) = e^{-t}$. Then, we have

$$\begin{aligned}
&\int_0^\infty a_1(t) \mathrm{d}A_1(t) = \int_0^\infty a_2(t) \mathrm{d}A_2(t) = \int_0^\infty e^{-t} \mathrm{d}t = 1 < +\infty, \\
&\int_0^\infty a_1(t) t^{\alpha_1 - 1} \mathrm{d}A_1(t) = \int_0^\infty e^{-t} t^{(3/2)} \mathrm{d}t = 2.3562 < +\infty, \\
&\int_0^\infty a_2(t) t^{\alpha_2 - 1} \mathrm{d}A_2(t) = \int_0^\infty e^{-t} t^{(5/2)} \mathrm{d}t = 3.3233 < +\infty, \\
&\varpi = 2.0929.
\end{aligned} \tag{60}$$

Take

$$\begin{aligned}
f_1(t, u(t), v(t)) &= \frac{1}{1 + e^t} + \frac{u(t)|\cos t|}{\left(1 + t^{(3/2)}\right) e^{7t}} + \frac{v(t)|\sin t|}{7\left(1 + t^{(5/2)}\right) e^{3t}}, \\
f_2(t, u(t), v(t)) &= \frac{1}{2(1 + e^t)} + \frac{u(t)}{\left(1 + t^{(3/2)}\right) e^{6t}} + \frac{v(t)}{3\left(1 + t^{(5/2)}\right) e^{4t}}.
\end{aligned} \tag{61}$$

Let

$$\begin{aligned}
g_1(t) &= \frac{1}{\left(1 + t^{(3/2)}\right) e^{7t}}, \\
g_2(t) &= \frac{1}{\left(1 + t^{(3/2)}\right) e^{6t}}, \\
h_1(t) &= \frac{1}{7\left(1 + t^{(5/2)}\right) e^{3t}}, \\
h_2(t) &= \frac{1}{3\left(1 + t^{(5/2)}\right) e^{4t}}.
\end{aligned} \tag{62}$$

Through calculation, we get

$$\begin{aligned}
g_1^* &= \int_0^\infty \left(1 + t^{\alpha_1 - 1}\right) g_1(t) \mathrm{d}t = \int_0^\infty e^{7t} \mathrm{d}t = \frac{1}{7}, \\
g_2^* &= \int_0^\infty \left(1 + t^{\alpha_1 - 1}\right) g_2(t) \mathrm{d}t = \int_0^\infty e^{6t} \mathrm{d}t = \frac{1}{6}, \\
h_1^* &= \int_0^\infty \left(1 + t^{\alpha_2 - 1}\right) h_1(t) \mathrm{d}t = \frac{1}{7} \int_0^\infty e^{3t} \mathrm{d}t = \frac{1}{21}, \\
h_2^* &= \int_0^\infty \left(1 + t^{\alpha_2 - 1}\right) h_2(t) \mathrm{d}t = \frac{1}{3} \int_0^\infty e^{4t} \mathrm{d}t = \frac{1}{12}, \\
&\varpi\left(g_1^* + g_2^* + h_1^* + h_2^*\right) = 2.0929\left(\frac{1}{7} + \frac{1}{6} + \frac{1}{21} + \frac{1}{12}\right) \\
&= 0.9219 < 1.
\end{aligned} \tag{63}$$

Then, by Theorem 2, system (59) has at least one positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Analytical Solutions to the Caudrey-Dodd-Gibbon-Sawada-Kotera Equation via Symbol Calculation Approach

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In this paper, we derive analytical solutions of the Caudrey-Dodd-Gibbon-Sawada-Kotera (CDGSK) equation via symbol calculation approach. Applying the $\exp(-\varphi(z))$-expansion method, we achieve the trigonometric, exponential, hyperbolic, and rational function solutions for the CDGSK equation. By choosing the appropriate parameters, we give some computer simulation to the analytical solutions of the CDGSK equation.

1. Introduction

Nonlinear fractional and integer order differential equations are widely utilized in fluid dynamics, solid state physics, plasma physics, biology, nonlinear optics, chemistry, etc. It has aroused the strong interest of researchers to these differential equations [1–18]. The study to exact solutions of various NLDEs is extremely important in modern mathematics with ramifications to some areas of physics, mathematics, and other sciences. There are many systematic methods to seek exact solutions of nonlinear differential equations, for example, the Hirota bilinear method [19], Tanh method [20], Lie symmetry [21], modified Kudryashov method [22], Exp-function method [23], sine-Gordon expansion method [24], complex method [25–29], and $\exp(-\varphi(z))$-expansion method [30–32].

Sawada and Kotera [33] proposed one of basic models in soliton theory as follows:

$$u_{xxxxx} + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_t = 0, \quad (1)$$

which is also introduced by Caudrey et al. independently [34]; so in literature, Equation (1) is called the Caudrey-Dodd-Gibbon-Sawada-Kotera equation. Many years have passed by, lots of research results for the CDGSK equation have been developed. As to this equation, the finite dimensional reduction was investigated by Enolski et al. [35], and N-soliton solutions were discovered by Parker [36] via the dressing method. Darboux transformation [37] and Bäcklund transformation in bilinear forms [38] were applied to study the CDGSK equation. Riemann theta function solutions of the CDGSK equation were also established [39]. In this paper, we employ the $\exp(-\varphi(z))$-expansion method [30–32] to obtain the exact solutions of the CDGSK equation.

2. The $\exp(-\varphi(z))$-Expansion Method

In this section, we give the main steps of the mentioned method.

Step 1. Inserting traveling wave transform

$$u(x,t) = u(z), \quad z = \lambda x + \omega t \quad (2)$$

into a nonlinear PDE

$$P\left(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots\right) = 0 \quad (3)$$

yields

$$F\left(u, u', u'', \cdots\right) = 0, \tag{4}$$

in which F is the polynomial of u along with its derivatives.

Step 2. Assume that Equation (4) has analytical solutions as follows:

$$u(z) = \sum_{\tau=0}^{m} B_\tau (\exp(-\varphi(z)))^\tau, \tag{5}$$

where $B_\tau (0 \le \tau \le m)$ are constants which will be ascertained subsequently, such that $B_\tau \neq 0$ and $\varphi = \varphi(z)$ satisfy

$$\varphi'(z) = \gamma + \exp(-\varphi(z)) + \vartheta \exp(\varphi(z)). \tag{6}$$

The solutions of Equation (6) are given in the following:
When $\gamma^2 - 4\vartheta > 0, \vartheta \neq 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\vartheta)}\tanh\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) - \gamma}{2\vartheta}\right), \tag{7}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\vartheta)}\coth\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) - \gamma}{2\vartheta}\right). \tag{8}$$

When $\gamma^2 - 4\vartheta < 0, \vartheta \neq 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\vartheta - \gamma^2)}\tan\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma}{2\vartheta}\right),$$

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\vartheta - \gamma^2)}\cot\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma}{2\vartheta}\right). \tag{9}$$

When $\gamma^2 - 4\vartheta > 0, \gamma \neq 0, \vartheta = 0$,

$$\varphi(z) = -\ln\left(\frac{\gamma}{\exp(\gamma(z+c)) - 1}\right). \tag{10}$$

When $\gamma^2 - 4\vartheta = 0, \gamma \neq 0, \vartheta \neq 0$,

$$\varphi(z) = \ln\left(-\frac{2(\gamma(z+c)+2)}{\gamma^2(z+c)}\right). \tag{11}$$

When $\gamma^2 - 4\vartheta = 0, \gamma = 0, \vartheta = 0$,

$$\varphi(z) = \ln(z+c). \tag{12}$$

In Equations (7)–(12), $B_\tau \neq 0, \gamma, \vartheta, c$ are constants. Taking the homogeneous balance between the nonlinear terms and highest order derivatives of Equation (4) yields the positive integer m.

Step 3. Insert Equation (5) into Equation (4), and collect the function $\exp(-\varphi(z))$ to yield the polynomial to $\exp(-\varphi(z))$. Let all coefficients with the same power of $\exp(-\varphi(z))$ be zero to obtain a system of algebraic equations. In solving these equations, we achieve the values of $B_\tau \neq 0, \gamma, \vartheta$ and substitute them into Equation (5) as well as Equations (7)–(12) to accomplish the determination for analytical solutions of the original PDE.

3. Exact Solutions of the CDGSK Equation

Inserting

$$u(x,t) = u(z), \quad z = \lambda x + \omega t \tag{13}$$

into Equation (1) and then integrating it, we obtain

$$\lambda^5 u'''' + 15\lambda^3 u u'' + 15\lambda u^3 + \omega u + \delta = 0, \tag{14}$$

where δ is the integration constant.

Taking the homogeneous balance between u^3 and u'''' of Equation (14) yields

$$u(z) = B_0 + B_1 \exp(-\varphi(z)) + B_2(\exp(-\varphi(z)))^2, \tag{15}$$

where $B_2 \neq 0$ and B_1 and B_0 are constants.

Substituting u'''', uu'', u^3, u into Equation (14) and letting the coefficients about $\exp(-\varphi(z))$ be zero, we obtain

$$\begin{aligned} e^{0(-\varphi(z))} :\ & \lambda^5 B_1 \gamma^3 \vartheta + 14\lambda^5 B_2 \gamma^2 \vartheta^2 + 8\lambda^5 B_1 \gamma \vartheta^2 \\ & + 16\lambda^5 B_2 \vartheta^3 + 15\lambda^3 B_0 B_1 \vartheta\gamma + 30\lambda^3 B_0 B_2 \vartheta^2 \\ & + 15\lambda B_0^3 + \omega B_0 + \delta = 0, \end{aligned}$$

$$\begin{aligned} e^{1(-\varphi(z))} :\ & B_1\omega + B_1\gamma^4\lambda^5 + 30 B_2 \gamma^3 \lambda^5 \vartheta + 22 B_1 \gamma^2 \lambda^5 \vartheta \\ & + 120 B_2 \gamma \lambda^5 \vartheta^2 + 30\lambda^3 B_0 B_1 \vartheta + 15 B_0 B_1 \gamma^2 \lambda^3 \\ & + 90 B_0 B_2 \gamma \lambda^3 \vartheta + 15 B_1{}^2 \gamma \lambda^3 \vartheta + 30 B_1 B_2 \lambda^3 \vartheta^2 \\ & + 16 B_1 \lambda^5 \vartheta^2 + 45 B_0{}^2 B_1 \lambda = 0, \end{aligned}$$

$$\begin{aligned} e^{2(-\varphi(z))} :\ & 16 B_2 \gamma^4 \lambda^5 + 15\lambda^5 B_1 \gamma^3 + 232 B_2 \gamma^2 \lambda^5 \vartheta \\ & + 60 B_1 \gamma \lambda^5 \vartheta + 136 B_2 \lambda^5 \vartheta^2 + 15 B_1{}^2 \gamma^2 \lambda^3 \\ & + 60 B_0 B_2 \gamma^2 \lambda^3 + 105 B_1 B_2 \gamma \lambda^3 \vartheta + 30 B_2{}^2 \lambda^3 \vartheta^2 \\ & + 45 B_0 B_1 \gamma \lambda^3 + 120 B_0 B_2 \lambda^3 \vartheta + 30 B_1{}^2 \lambda^3 \vartheta \\ & + 45 B_0{}^2 B_2 \lambda + 45 B_0 B_1{}^2 \lambda + B_2 \omega = 0, \end{aligned}$$

$$\begin{aligned} e^{3(-\varphi(z))} :\ & 130 B_2 \gamma^3 \lambda^5 + 50 B_1 \gamma^2 \lambda^5 + 440 B_2 \gamma \lambda^5 \vartheta \\ & + 75 B_1 B_2 \gamma^2 \lambda^3 + 40 B_1 \lambda^5 \vartheta + 15 B_1^3 \lambda \\ & + 150 B_0 B_2 \gamma \lambda^3 + 45 B_1^2 \gamma \lambda^3 + 150 B_1 B_2 \lambda^3 \vartheta \\ & + 30 B_0 B_1 \lambda^3 + 90 B_0 B_1 B_2 \lambda + 90 B_2^2 \gamma \lambda^3 \vartheta = 0, \end{aligned}$$

$$
\begin{aligned}
&e^{4(-\varphi(z))}: 330\,B_2\,\gamma^2\lambda^5 + 60\,B_1\,\gamma\,\lambda^5 + 60\,B_2^2\gamma^2\lambda^3 \\
&\quad + 240\,B_2\,\lambda^5\vartheta + 195\,B_1\,B_2\,\gamma\,\lambda^3 + 30\,B_1^2\lambda^3 \\
&\quad + 120\,B_2^2\lambda^3\vartheta + 90\,B_0\,B_2\,\lambda^3 + 45\,B_0\,B_2^2\lambda \\
&\quad + 45\,B_1^2B_2\,\lambda = 0, \\
&e^{5(-\varphi(z))}: 336\,B_2\,\gamma\,\lambda^5 + 24\,B_1\,\lambda^5 + 150\,B_2^2\gamma\,\lambda^3 \\
&\quad + 120\,B_1\,B_2\,\lambda^3 + 45\,B_1\,B_2^2\lambda = 0, \\
&e^{6(-\varphi(z))}: 120\,B_2\,\lambda^5 + 90\,B_2^2\lambda^3 + 15\,B_2^3\lambda = 0.
\end{aligned}
\tag{16}
$$

We solve the above algebraic equations and then obtain two different cases:

Case 1.

$$
\begin{aligned}
&B_2 = -2\lambda^2, \\
&B_1 = -2\gamma\lambda^2, \\
&B_0 = \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda}, \\
&\delta = \frac{\lambda^7\left(\gamma^2 - 4\,\vartheta\right)^3}{9} \\
&\quad - \frac{\sqrt{5\lambda\left(\lambda^5(\gamma^2 - 4\,\vartheta)^2 - 4\,\omega\right)}\left(2\,\lambda^5\left(\gamma^2 - 4\,\vartheta\right)^2 + \omega\right)}{45\lambda},
\end{aligned}
\tag{17}
$$

where ϑ and γ are arbitrary constants.

Substituting Equation (17) into Equation (15) yields

$$
\begin{aligned}
u(z) = {} & \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda} \\
& - 2\gamma\lambda^2\,\exp\left(-\varphi(z)\right) - 2\lambda^2(\exp\left(-\varphi(z)\right))^2.
\end{aligned}
\tag{18}
$$

Applying Equations (7)–(12) into Equation (18), respectively, yields analytical solutions of the CDGSK equation as follows:

Case 1.1. When $\gamma^2 - 4\vartheta > 0, \vartheta \neq 0$,

$$
\begin{aligned}
u_{11}(z) = {} & \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda} \\
& + \frac{4\lambda^2\gamma\vartheta}{\sqrt{(\gamma^2 - 4\vartheta)}\,\tanh\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) + \gamma} \\
& - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(\gamma^2 - 4\vartheta)}\,\tanh\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) + \gamma\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
u_{12}(z) = {} & \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda} \\
& + \frac{4\lambda^2\gamma\vartheta}{\sqrt{(\gamma^2 - 4\vartheta)}\,\coth\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) + \gamma} \\
& - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(\gamma^2 - 4\vartheta)}\,\coth\left(\left(\sqrt{\gamma^2 - 4\vartheta}/2\right)(z+c)\right) + \gamma\right)^2}.
\end{aligned}
\tag{19}
$$

Case 1.2. When $\gamma^2 - 4\vartheta < 0, \vartheta \neq 0$,

$$
\begin{aligned}
u_{13}(z) = {} & \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda} \\
& - \frac{4\lambda^2\gamma\vartheta}{\sqrt{(4\vartheta - \gamma^2)}\,\tan\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma} \\
& - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(4\vartheta - \gamma^2)}\,\tan\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
u_{14}(z) = {} & \frac{-5\,\lambda^3\left(\gamma^2 + 8\,\vartheta\right) + \sqrt{5\,\lambda^6(\gamma^2 - 4\,\vartheta)^2 - 20\,\lambda\,\omega}}{30\lambda} \\
& - \frac{4\lambda^2\gamma\vartheta}{\sqrt{(4\vartheta - \gamma^2)}\,\cot\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma} \\
& - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(4\vartheta - \gamma^2)}\,\cot\left(\left(\sqrt{4\vartheta - \gamma^2}/2\right)(z+c)\right) - \gamma\right)^2}.
\end{aligned}
\tag{20}
$$

Case 1.3. When $\gamma^2 - 4\vartheta > 0, \gamma \neq 0, \vartheta = 0$,

$$
\begin{aligned}
u_{15}(z) = {} & \frac{-5\lambda^3\gamma^2 + \sqrt{5\lambda\left(\lambda^5\gamma^4 - 4\omega\right)}}{30\lambda} - \frac{2\lambda^2\gamma^2}{\exp\left(\gamma(z+c)\right) - 1} \\
& - \frac{2\lambda^2\gamma^2}{\left(\exp\left(\gamma(z+c)\right) - 1\right)^2}.
\end{aligned}
\tag{21}
$$

Case 1.4. When $\gamma^2 - 4\vartheta = 0, \gamma \neq 0, \vartheta \neq 0$,

$$
\begin{aligned}
u_{16}(z) = {} & \frac{-30\lambda^3\vartheta + \sqrt{-5\lambda\omega}}{15\lambda} + \frac{\lambda^2\gamma^3(z+c)}{\gamma(z+c) + 2} \\
& - \frac{\lambda^2\gamma^4(z+c)^2}{2\left(\gamma(z+c) + 2\right)^2}.
\end{aligned}
\tag{22}
$$

Case 1.5. When $\gamma^2 - 4\vartheta = 0, \gamma = 0, \vartheta = 0$,

$$
u_{17}(z) = \frac{\sqrt{-5\lambda\omega}}{15\lambda} - \frac{2\lambda^2}{(z+c)^2}.
\tag{23}
$$

Case 2.

$$B_2 = -2\lambda^2,$$
$$B_1 = -2\gamma\lambda^2,$$
$$B_0 = -\frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda},$$
$$\delta = \frac{\lambda^7\left(\gamma^2-4\vartheta\right)^3}{9} + \frac{\sqrt{5\lambda\left(\lambda^5(\gamma^2-4\vartheta)^2-4\omega\right)}\left(2\lambda^5\left(\gamma^2-4\vartheta\right)^2+\omega\right)}{45\lambda}, \tag{24}$$

where ϑ and γ are arbitrary constants.

Substituting Equation (24) into (15) yields

$$u(z) = -\frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda} - 2\lambda^2\gamma\exp\left(-\varphi(z)\right) - 2\lambda^2\left(\exp\left(-\varphi(z)\right)\right)^2. \tag{25}$$

Applying Equations (7)–(12) into Equation (25), respectively, yields analytical solutions of the CDGSK equation as follows:

Case 2.1. When $\gamma^2-4\vartheta>0$, $\vartheta\neq 0$,

$$u_{21}(z) = -\frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda} + \frac{4\lambda^2\gamma\vartheta}{\sqrt{(\gamma^2-4\vartheta)}\tanh\left(\left(\sqrt{\gamma^2-4\vartheta}/2\right)(z+c)\right)+\gamma} - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(\gamma^2-4\vartheta)}\tanh\left(\left(\sqrt{\gamma^2-4\vartheta}/2\right)(z+c)\right)+\gamma\right)^2},$$
$$u_{22}(z) = -\frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda} + \frac{4\lambda^2\gamma\vartheta}{\sqrt{(\gamma^2-4\vartheta)}\coth\left(\left(\sqrt{\gamma^2-4\vartheta}/2\right)(z+c)\right)+\gamma} - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(\gamma^2-4\vartheta)}\coth\left(\left(\sqrt{\gamma^2-4\vartheta}/2\right)(z+c)\right)+\gamma\right)^2}. \tag{26}$$

Case 2.2. When $\gamma^2-4\vartheta<0$, $\vartheta\neq 0$,

$$u_{23}(z) - \frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda} - \frac{4\lambda^2\gamma\vartheta}{\sqrt{(4\vartheta-\gamma^2)}\tan\left(\left(\sqrt{4\vartheta-\gamma^2}/2\right)(z+c)\right)-\gamma} - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(4\vartheta-\gamma^2)}\tan\left(\left(\sqrt{4\vartheta-\gamma^2}/2\right)(z+c)\right)-\gamma\right)^2},$$
$$u_{24}(z) = -\frac{5\lambda^3\left(\gamma^2+8\vartheta\right)+\sqrt{5\lambda^6(\gamma^2-4\vartheta)^2-20\lambda\omega}}{30\lambda} - \frac{4\lambda^2\gamma\vartheta}{\sqrt{(4\vartheta-\gamma^2)}\cot\left(\left(\sqrt{4\vartheta-\gamma^2}/2\right)(z+c)\right)-\gamma} - \frac{8\lambda^2\vartheta^2}{\left(\sqrt{(4\vartheta-\gamma^2)}\cot\left(\left(\sqrt{4\vartheta-\gamma^2}/2\right)(z+c)\right)-\gamma\right)^2}. \tag{27}$$

Case 2.3. When $\gamma^2-4\vartheta>0$, $\gamma\neq 0$, $\vartheta=0$,

$$u_{25}(z) = -\frac{5\lambda^3\gamma^2+\sqrt{5\lambda\left(\lambda^5\gamma^4-4\omega\right)}}{30\lambda} - \frac{2\lambda^2\gamma^2}{\exp\left(\gamma(z+c)\right)-1} - \frac{2\lambda^2\gamma^2}{\left(\exp\left(\gamma(z+c)\right)-1\right)^2}. \tag{28}$$

Case 2.4. When $\gamma^2-4\vartheta=0$, $\gamma\neq 0$, $\vartheta\neq 0$,

$$u_{26}(z) = -\frac{30\lambda^3\vartheta+\sqrt{-5\lambda\omega}}{15\lambda} + \frac{\lambda^2\gamma^3(z+c)}{\gamma(z+c)+2} - \frac{\lambda^2\gamma^4(z+c)^2}{2\left(\gamma(z+c)+2\right)^2}. \tag{29}$$

Case 2.5. When $\gamma^2-4\vartheta=0$, $\gamma=0$, $\vartheta=0$,

$$u_{27}(z) = -\frac{\sqrt{-5\lambda\omega}}{15\lambda} - \frac{2\lambda^2}{(z+c)^2}. \tag{30}$$

Figures 1–5 show the properties of the solutions.

4. Conclusions

In this paper, abundant analytical solutions of the CDGSK equation are obtained via symbol calculation approach. The properties of the solutions are shown by some graphs. It shows that the $\exp(-\varphi(z))$-expansion method is an effective method to seek analytical solutions for nonlinear differential equations.

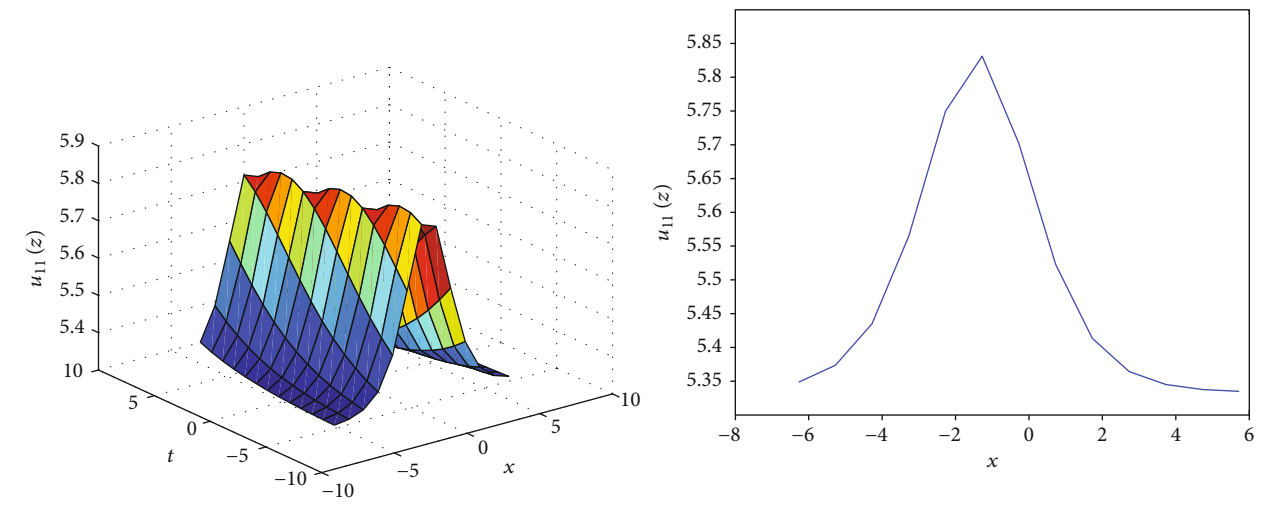

FIGURE 1: The 3D and 2D graphics of $u_{11}(z)$ by choosing the values $\gamma = 5$, $\vartheta = 6$, $\lambda = 1$, $\omega = 1/4$, $c = 1$, and $t = 0$ for the 2D graphic.

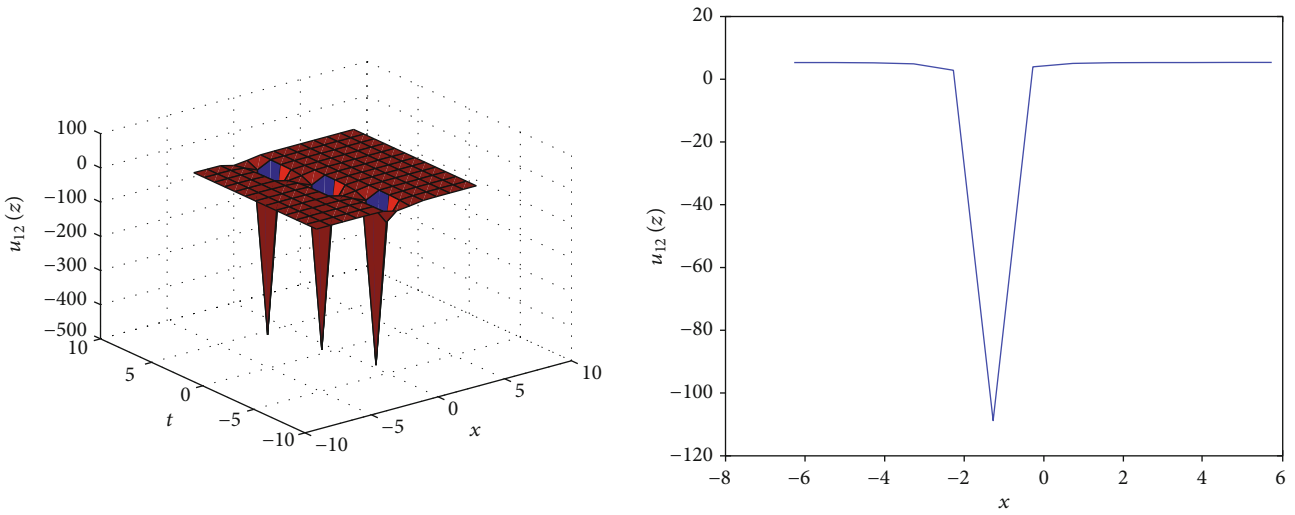

FIGURE 2: The 3D and 2D graphics of $u_{12}(z)$ by choosing the values $\gamma = 5$, $\vartheta = 6$, $\lambda = 1$, $\omega = 1/4$, $c = 1$, and $t = 0$ for the 2D graphic.

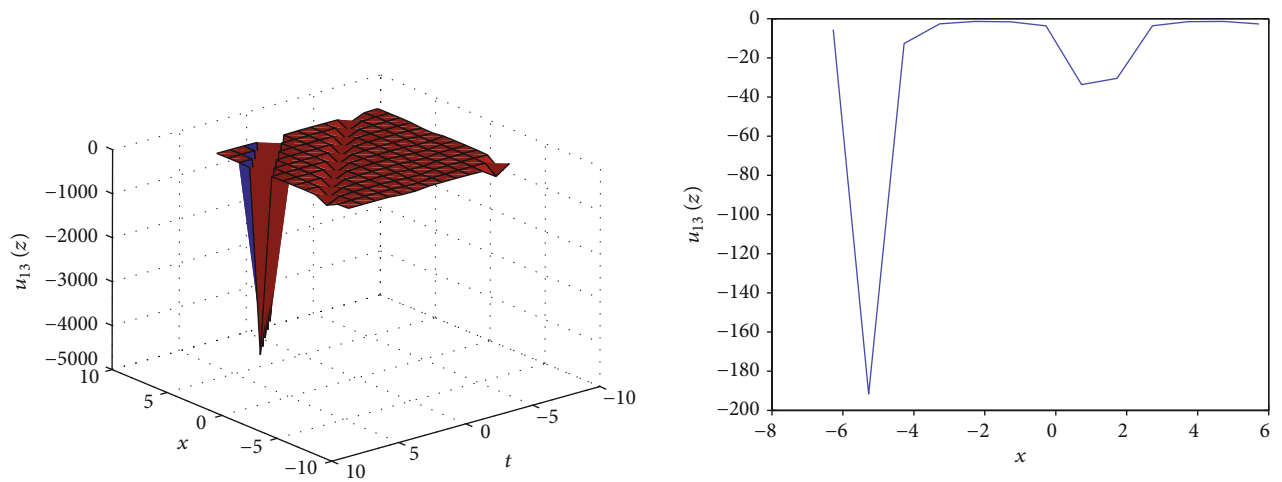

FIGURE 3: The 3D and 2D graphics of $u_{13}(z)$ by choosing the values $\gamma = 4$, $\vartheta = 5$, $\lambda = 1$, $\omega = 1$, $c = 1$, and $t = 0$ for the 2D graphic.

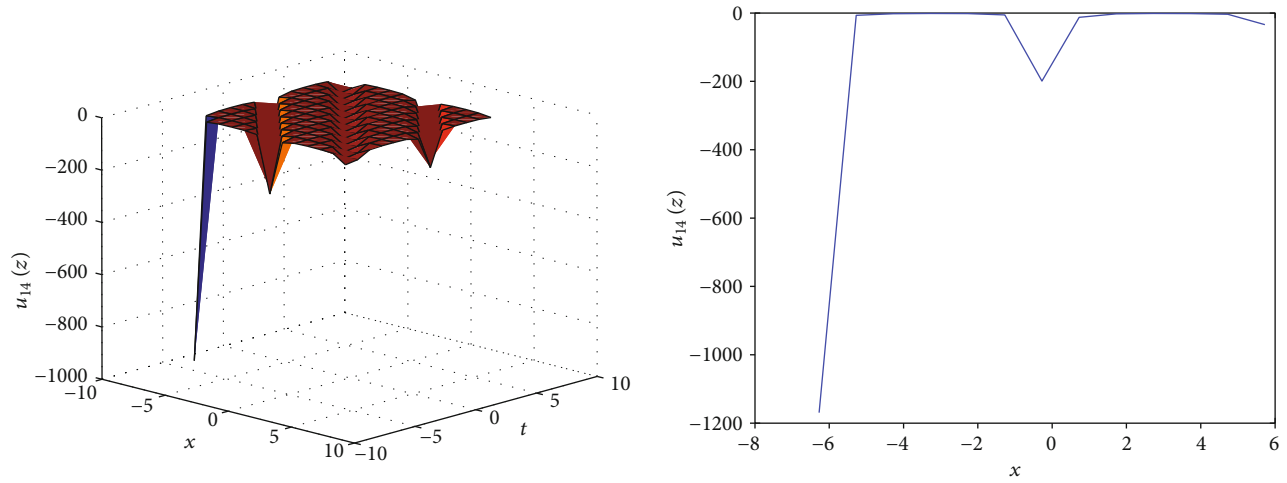

FIGURE 4: The 3D and 2D graphics of $u_{14}(z)$ by choosing the values $\gamma = 4$, $\vartheta = 5$, $\lambda = 1$, $\omega = 1$, $c = 1$, and $t = 0$ for the 2D graphic.

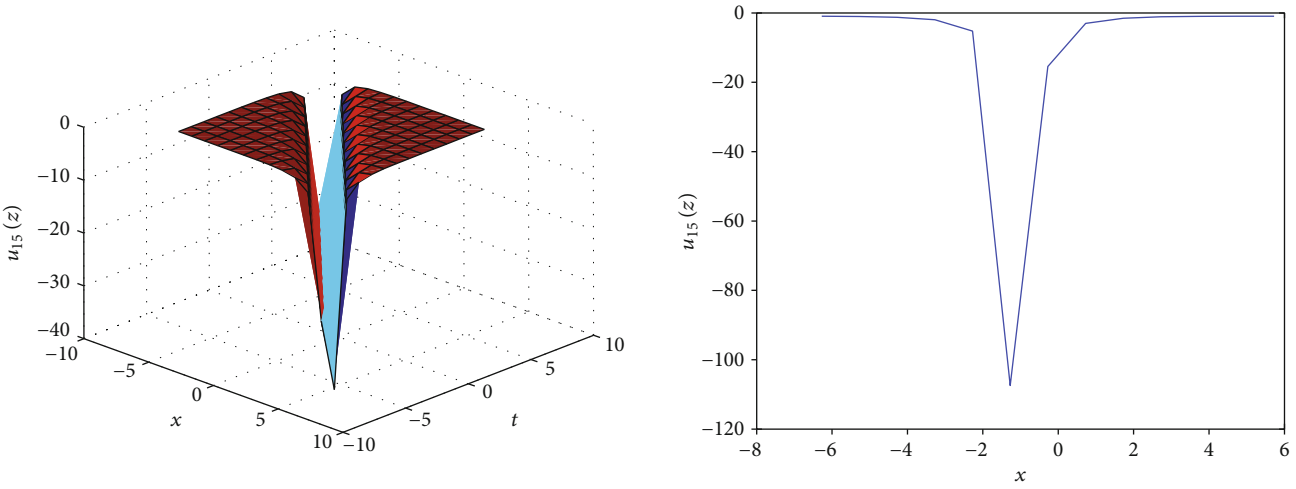

FIGURE 5: The 3D and 2D graphics of $u_{15}(z)$ by choosing the values $\gamma = 2$, $\vartheta = 0$, $\lambda = 1$, $\omega = 1$, $c = 1$, and $t = 0$ for the 2D graphic.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Research Article

On the Reducibility of Quasiperiodic Linear Hamiltonian Systems and Its Applications in Schrödinger Equation

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In this paper, we consider the reducibility of the quasiperiodic linear Hamiltonian system $\dot{x} = (A + \varepsilon Q(t))$, where A is a constant matrix with possible multiple eigenvalues, $Q(t)$ is analytic quasiperiodic with respect to t, and ε is a small parameter. Under some nonresonant conditions, it is proved that, for most sufficiently small ε, the Hamiltonian system can be reduced to a constant coefficient Hamiltonian system by means of a quasiperiodic symplectic change of variables with the same basic frequencies as $Q(t)$. Applications to the Schrödinger equation are also given.

1. Introduction

In this paper, we are concerned with the reducibility of the quasiperiodic linear Hamiltonian system

$$\dot{x} = (A + \varepsilon Q(t))x, x \in R^n, \tag{1}$$

where A is a constant matrix with possible multiple eigenvalues, $Q(t)$ is analytic quasiperiodic with respect to t, and ε is a small parameter.

Firstly, let us recall the definition of the reducibility for quasiperiodic linear systems. Let $A(t)$ be an $n \times n$ quasiperiodic matrix, the differential equation

$$\dot{x} = A(t)x, \quad x \in R^n \tag{2}$$

is called reducible, if there exists a nonsingular quasiperiodic change of variables

$$x = \varphi(t)y, \tag{3}$$

where $\varphi(t)$ and $\varphi^{-1}(t)$ are quasiperiodic and bounded, which changes (2) into

$$\dot{y} = By, \quad y \in R^n, \tag{4}$$

where B is a constant matrix.

The well-known Floquet theorem states that every periodic differential equation (2) can be reduced to a constant coefficient differential equation (4) by means of a periodic change of variables with the same period as $A(t)$. However, this is not true for the quasiperiodic linear system; one can see [1] for more details. In 1981, Johnson and Sell [2] proved that the quasiperiodic linear system (2) is reducible if the quasiperiodic coefficient matrix $A(t)$ satisfies the "full spectrum" condition.

A typical example of quasiperiodic linear systems comes from the (continuous time) quasiperiodic Schrödinger operators, which are defined on $L^2(R)$ as

$$(Ly)(t) = -y''(t) + q(\theta + \omega t)y(t), \tag{5}$$

where $q : T^n \longrightarrow R$ is called the potential and $\theta \in T^n$ is called the phase. It is well known that the spectrum of L does not depend on the phase when ω is rationally independent, but it is closely related to the dynamics of the Schrödinger equations

$$(Ly)(t) = -y''(t) + q(\theta + \omega t)y(t)\, Ey(t) \tag{6}$$

or equivalently the dynamics of the linear systems

$$\dot{x} = V_{E,q}(\theta)x, \quad \dot{\theta} = \omega, \tag{7}$$

where

$$V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(t) - E & 0 \end{pmatrix} \in sl(2, \mathbb{R}). \tag{8}$$

Dinaburg and Sinai [3] proved that linear systems (7) are reducible for most $E > E^*(q, \alpha, \tau)$, which are sufficiently large, if ω is fixed and satisfies the Diophantine condition

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad k \in \mathbb{Z}^r \setminus \{0\}, \tag{9}$$

where α, τ are positive constants. The result was generalized by Rüssmann [4] for ω satisfying the Bruno condition.

Eliasson [5] proved a full measure reducibility result for quasiperiodic linear Schrödinger equations. More precisely, Eliasson proved that (7) is reducible for almost all $E > E*(q, \omega)$ in Lebesgue measure sense, where ω is a fixed Diophantine vector.

In the case that $n = 2$, a stronger reducibility result, called a nonperturbative reducibility, is available. The nonperturbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constant α. Hou and You [6] proved, besides other results, the nonperturbative reducibility for (7).

The reducibility of quasiperiodic linear systems with coefficients in $gl(m, R)$ was considered by Jorba and Simó [7]. Suppose that A is a constant matrix with different eigenvalues, they proved that if the eigenvalues of A and the frequencies of Q satisfy some nonresonant conditions, then there exists sufficiently small $\varepsilon_0 > 0$ and a nonempty Cantor set $E \subset (0, \varepsilon_0)$, such that for any $|\varepsilon| \in E$, system (1) is reducible. Moreover, the relative measure of the set $(0, \varepsilon_0) \setminus E$ in $(0, \varepsilon_0)$ is exponentially small in ε_0. Junxiang [8] obtained the similar result for the multiple eigenvalue case. Later, many authors [7–10] paid attention to the reducibility of the quasiperiodic linear system (1), which is close to a constant coefficient linear system.

In 1996, Jorba and Simó [10] extended the conclusion of the linear system to the nonlinear system

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x + \varepsilon g(t) + h(x, t), \quad x \in R^n. \tag{10}$$

Suppose that A has n different nonzero eigenvalues, they proved that under some nonresonant conditions and nondegeneracy conditions, there exists a nonempty Cantor set $E \subset (0, \varepsilon_0)$, such that for all $|\varepsilon| \in E$, system (10) is reducible. Later, Wang and Xu [11] further investigated the nonlinear quasiperiodic system

$$\dot{x} = Ax + f(t, x, \varepsilon), \quad x \in R^2, \tag{11}$$

where A is a real 2×2 constant matrix, and $f(t, 0, \varepsilon) = O(\varepsilon)$, $\partial_x f(t, 0, \varepsilon) = O(\varepsilon)$ as $\varepsilon \longrightarrow 0$. They proved without any nondegeneracy condition, one of two results holds: (1) system (11) is reducible to $\dot{y} = By + O(y)$ for all $\varepsilon \in (0, \varepsilon_0)$; (2) there exists a nonempty Cantor set $E \subset (0, \varepsilon_0)$, such that system (11) is reducible to $\dot{y} = By + O(y^2)$ for all $|\varepsilon| \in E$.

In [12], Her and You considered one-parameter family of quasiperiodic linear system

$$\dot{x} = (A(\lambda) + g(\omega_1 t, \cdots, \omega_l t, \lambda))x, \tag{12}$$

where $A \in C^\omega(\Lambda, gl(m, \mathbb{C}))(C^\omega(\Lambda, \mathrm{gl}(m, \mathbb{C}))$ be the set of $m \times m$ matrices $A(\lambda)$ depending analytically on parameter λ in a closed interval ($\Lambda \subset \mathbb{R}$), and g is analytic and small. They proved that under some nonresonant conditions and nondegeneracy conditions, there exists an open and dense set $\mathscr{A}$ in $C^\omega(\Lambda, gl(m, \mathbb{C}))$, such that for each $A \in \mathscr{A}$, system (12) is reducible for almost all $\lambda \in \Lambda$.

Instead of a total reduction to a constant coefficient linear system, Jorba et al. [13] investigated the effective reducibility of the following quasiperiodic system:

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x, \quad |\varepsilon| \leq \varepsilon_0, \tag{13}$$

where A is a constant matrix with different eigenvalues. They proved that under nonresonant conditions, by a quasiperiodic transformation, system (13) is reducible to a quasiperiodic system

$$\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t, \varepsilon))y, \quad |\varepsilon| \leq \varepsilon_* \leq \varepsilon_0, \tag{14}$$

where R^* is exponentially small in ε. Li and Xu [14] obtained the similar result for Hamiltonian systems. Later, Xue and Zhao [15] extended the result to the case of multiple eigenvalues.

In this paper, we will study the reducibility of quasiperiodic linear *Hamiltonian* system (1), where matrix A may have multiple eigenvalues. To this end, the following assumptions are made.

Assumption 1. (nonresonant condition). Let all eigenvalues of matrix A be $\lambda_1, \cdots, \lambda_n$, $Q(t)$ be an analytic quasiperiodic function on $D_\rho = \{\theta \in \mathbb{C}^r : |\mathrm{Im}\, \theta_j| \leq \rho, \boxtimes j = 1, 2, \cdots, r\}$ with the frequencies $\omega = (\omega_1, \cdots, \omega_r)$. Suppose that $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\omega = (\omega_1, \cdots, \omega_r)$ satisfy the nonresonant conditions

$$\left|\langle k, \omega \rangle \sqrt{-1} - \lambda_i + \lambda_j\right| \geq \frac{\alpha}{|k|^\tau} \tag{15}$$

for all $k \in \mathbb{Z}^r \setminus \{0\}$, $0 \leq i, j \leq n$, where $\alpha > 0$ is a small constant and $\tau > r - 1$.

Assumption 2. (nondegeneracy condition). Assume that $A + \varepsilon\bar{Q}$ has $n$different eigenvalues $\mu_1, \cdots, \mu_n$ with $|\mu_i| \geq 2\delta\varepsilon$, $|\mu_i - \mu_j| \geq 2\delta\varepsilon$, $i \neq j$, $0 \leq i, j \leq n$, where δ is a positive constant independently of ε. Here, we denote the average of $Q(t)$ by $\bar{Q}$, that is,

$$\bar{Q} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Q(t)dt. \tag{16}$$

We are in a position to state the main result.

Theorem 3. *Suppose that Hamiltonian system (1) satisfies Assumptions 1 and 2. Then there exist some sufficiently small $\varepsilon_0 > 0$ and a nonempty Cantor subset $E_{\varepsilon 0} \subset (0, \varepsilon_0)$ with positive Lebesgue measure, such that for $\varepsilon \in E_{\varepsilon 0}$, Hamiltonian system (1) is reducible, i.e., there is an analytic quasiperiodic symplectic transformation $x = \psi(t)y$, where $\psi(t)$ has same frequencies as $Q(t)$, which changes (1) into the Hamiltonian system $\dot{y} = By$, where B is a constant matrix. Moreover, if ε_0 is small enough, the relative measure of $E_{\varepsilon 0}$ in $(0, \varepsilon_0)$ is close to 1.*

Now, we give some remarks on this result. Firstly, here we deal with the Hamiltonian system and have to find the *symplectic* transformation, which is different from that in [7, 8]. Secondly, we consider the reducibility, other than the effective reducibility in [13, 14]. The last but not the least, we can allow matrix A to have multiple eigenvalues. Of course, if the eigenvalues of A are different, the nondegeneracy condition holds naturally.

As an example, we apply Theorem 3 to the following Schrödinger equation:

$$\ddot{x} + \varepsilon a(t)x = 0, \tag{17}$$

where $a(t)$ is analytic quasiperiodic with the frequencies $\omega = (\omega_1, \cdots, \omega_r)$. Denote the average of $a(t)$ by $\bar{a}$. If $\bar{a} > 0$ and the frequencies ω of $a(t)$ satisfy the Diophantine condition

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{|k|^{\tau}}, \quad k \in \mathbb{Z}^r \setminus \{0\}, \tag{18}$$

where $\alpha > 0$ is a small constant and $\tau > r - 1$, then there exists some sufficiently small $\varepsilon_0 > 0$, equation (17) is reducible and the equilibrium of (17) is stable in the sense of Lyapunov for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$. Moreover, all solutions of equation (17) are quasiperiodic with the frequencies $\Omega = (\omega_1, \cdots, \omega_r, \sqrt{b})$ for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$, where $b = \bar{a}\varepsilon + O(\varepsilon^2)$ as $\varepsilon \longrightarrow 0$. Here, we remark that if we rewrite equation (17) into Hamiltonian system (1), we find that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{19}$$

which has multiple eigenvalues $\lambda_1 = \lambda_2 = 0$. One can see Section 4 for more details about this example.

There are plenty of works about the stability of all kinds of equations, one can refer to [16–23] for a detailed description. In particular, for quasiperiodic equations, in order to determine the type of stability of the equilibria of quasiperiodic Hamiltonian systems, the authors need to assume that the corresponding linearized system is reducible, and some conditions were added to the system after the reducibility. However, as far as we know, the case that the conditions are added to the original system has not been considered in the literature up to now, which we will study in the future.

The paper is organized as follows. In Section 2, we list some basic definitions and results that will be useful in the proof of the main result. In Section 3, we will prove Theorem 3. Equation (17) will be analyzed in Section 4.

2. Some Preliminaries

We first give the definition of quasiperiodic functions.

Definition 4. A function f is said to be a quasiperiodic function with a vector of basic frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$, if $f(t) = F(\theta_1, \theta_2, \cdots, \theta_r)$, where F is 2π periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, 2, \cdots, r$. Moreover, if $F(\theta)$ $(\theta = (\theta_1, \theta_2, \cdots, \theta_r))$ is analytic on $D_\rho = \{\theta \in \mathbb{C}^r : |\mathrm{Im}\ \theta_j| \leq \rho,\ j = 1, 2, \cdots, r\}$, we say that $f(t)$ is analytic quasiperiodic on D_ρ.

It is well known that an analytic quasiperiodic function $f(t)$ can be expanded as Fourier series

$$f(t) = \sum_{k\in\mathbb{Z}^r} f_k e^{\langle k,\omega\rangle\sqrt{-1}t}, \tag{20}$$

with Fourier coefficients defined by

$$f_k = \frac{1}{(2\pi)^r} \int_{T^r} F(\theta) e^{-\langle k,\theta\rangle\sqrt{-1}} d\theta. \tag{21}$$

Denote by $\|f\|_\rho$ the norm

$$\|f\|_\rho = \sum_{k\in\mathbb{Z}^r} |f_k| e^{|k|\rho}. \tag{22}$$

Definition 5. An $n \times n$ matrix $Q(t) = (q_{ij}(t))_{1\leq i,j\leq n}$ is said to be analytic quasiperiodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$, if all $q_{ij}(t)(i, j = 1, 2, \cdots, n)$ are analytic quasiperiodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$.

Define the norm of Q by

$$\|Q\|_\rho = \max_{1\leq i\leq n} \sum_{j=1}^{n} \left\|q_{ij}\right\|_\rho. \tag{23}$$

.

It is easy to see that

$$\|Q_1Q_2\|_\rho \le \|Q_1\|_\rho \|Q_2\|_\rho. \tag{24}$$

If Q is a constant matrix, write $\|Q\| = \|Q\|_\rho$ for simplicity. Denote the average of $Q(t)$ by $\bar{Q} = (\bar{q}_{ij})_{1\le i,j\le n}$, where

$$\bar{q}_{ij} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_{ij}(t)dt. \tag{25}$$

See [24] for the existence of the limit.

Also, we need two lemmas which are provided in this section for the proof of Theorem 3 that were proved in [10].

Lemma 6. *Let $h : B_\sigma(0) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^2 function that satisfies $h(0) = 0$, $D_x h(0) = 0$, $\|D_{xx}h(x)\| \le K$, $x \in B_\sigma(0)$. Then, $\|h(x)\| \le K/2\|x\|^2$, $\|D_x h(x)\| \le K\|x\|$.*

Lemma 7. *Suppose that B_0 is an $n \times n$ matrix with different nonzero eigenvalues satisfying $|\mu_i^0| > \gamma$, $|\mu_i^0 - \mu_j^0| > \gamma$, $i \ne j$, $1 \le i, j \le n$, and S_0 is a regular matrix such that $S_0^{-1}B_0S_0 = \operatorname{diag}(\mu_1^0, \cdots, \mu_n^0)$. Set $\beta_0 = \max\{\|S_0\|, \|S_0^{-1}\|\}$, and choose b such that*

$$0 < b < \frac{\gamma}{(3n-1)\beta_0^2}. \tag{26}$$

If B_1 verifies $\|B_1 - B_0\| \le b$, then the following conclusions hold:

(1) B_1 has n different nonzero eigenvalues $\mu_1^1, \cdots, \mu_n^1$

(2) There exists a regular matrix S_1 such that $S_1^{-1}B_1S_1 = \operatorname{diag}(\mu_1^1, \cdots, \mu_n^1)$, which satisfies $\|S\|, \|S_1^{-1}\| \le \beta_1$, where $\beta_1 = 2\beta_0$.

The next lemma is used to perform a step of the inductive procedure in the proof of Theorem 3.

Lemma 8. *Consider the differential equation*

$$\dot{P}(t) = \Lambda P(t) - P(t)\Lambda + R(t), \tag{27}$$

where Λ is a constant Hamiltonian matrix with n different eigenvalues $\nu_1, \cdots, \nu_n$, R is an analytic quasiperiodic Hamiltonian matrix on D_ρ with frequencies ω satisfying $\bar{R} = 0$.

If

$$\left|\langle k, \omega\rangle\sqrt{-1} - \nu_i + \nu_j\right| \ge \frac{\alpha}{|k|^{3\tau}} \tag{28}$$

for all $0 \ne k \in \mathbb{Z}^r$, and $|\nu_i| \ge \delta\varepsilon$, $|\nu_i - \nu_j| \ge \delta\varepsilon$, for $i \ne j$, $0 \le i, j \le n$, where δ is a positive constant independent of ε, then equation (27) has a unique analytic quasiperiodic Hamiltonian solution $P(t)$ with $\bar{P} = 0$, where $P(t)$ has frequencies ω and satisfies

$$\|P\|_{\rho-s} \le \frac{c}{\alpha s^\nu}\|R\|_\rho \tag{29}$$

with $\nu = 3\tau + r$ and $0 < s < \rho$, where the constant c depends only on τ and r.

Proof. Choosing S such that $S^{-1}\Lambda S = D = \operatorname{diag}(\nu_1, \cdots, \nu_n)$, making the change of variable $P(t) = SX(t)S^{-1}$ and defining $Y(t) = S^{-1}R(t)S$, equation (27) becomes

$$\dot{X}(t) = DX(t) - X(t)D + Y(t), \quad \bar{Y} = 0. \tag{30}$$

Expanding X and Y into Fourier series yields that

$$\begin{aligned} X(t) &= \sum_{k\in\mathbb{Z}^r} X_k e^{\langle k,\omega\rangle\sqrt{-1}t}, \\ Y(t) &= \sum_{k\in\mathbb{Z}^r} Y_k e^{\langle k,\omega\rangle\sqrt{-1}t}, \end{aligned} \tag{31}$$

where $X_k = (x_{ij}^k)_{1\le i,j\le n}$ and $Y_k = (y_{ij}^k)_{1\le i,j\le n}$.

By comparing the coefficients of (30), we obtain that

$$x_{ij}^k = \frac{y_{ij}^k}{\langle k, \omega\rangle\sqrt{-1} - \nu_i + \nu_j}, \quad 1 \le i, j \le n, k \ne 0, \tag{32}$$

$$x_0^{ij} = 0, \quad 1 \le i, j \le n. \tag{33}$$

Since R is analytic on D_ρ, Y is also analytic on D_ρ. Therefore, we have

$$\|Y_k\| \le \|Y\|_\rho e^{-|k|\rho}. \tag{34}$$

Hence,

$$\|X\|_{\rho-s} = \sum_{k\in\mathbb{Z}^r} \|X_k\| e^{|k|(\rho-s)} \le \sum_{0\ne k\in\mathbb{Z}^r} \frac{|k|^{3\tau}e^{-s|k|}}{\alpha}\|Y\|_\rho \le \frac{c}{\alpha s^\nu}\|Y\|_\rho, \tag{35}$$

where $\nu = 3\tau + r$ and $0 < s < \rho$. Here and hereafter, we always use the same symbol c to denote different constants in estimates. Hence,

$$\|P\|_{\rho-s} \le c\|X\|_{\rho-s} \le \frac{c}{\alpha s^\nu}\|Y\|_\rho \le \frac{c}{\alpha s^\nu}\|R\|_\rho. \tag{36}$$

Now, we prove that P is Hamiltonian. Since Λ and R are Hamiltonian, then $\Lambda = J\Lambda_J$ and $R = JR_J$, where Λ_J and R_J are symmetric. Let $P_J = J^{-1}P$, if P_J is symmetric, then P is Hamiltonian. Below, we prove that P_J is symmetric.

Substituting $P = JP_I$ into equation (27) yields

$$\dot{P}_J = \Lambda_J J_P^J - P_J J \Lambda_J + R_J, \tag{37}$$

and transposing equation (37), we get

$$\dot{P}_J^T = \Lambda_J J P_J^T - P_J^T J \Lambda_J + R_J. \tag{38}$$

It is easy to see that JP_J and JP_J^T are solutions of (27); moreover, $\overline{JP_J} = \overline{JP_J^T} = 0$. Since the solution of (27) with $\bar{P} = 0$ is unique, we have $JP_J = JP_J^T$, which implies that P is Hamiltonian. Up to now, we have finished the proof of this lemma.

3. Proof of Theorem 3

From the assumptions of Theorem 3, it follows that $A + \varepsilon\bar{Q}$ is a Hamiltonian matrix with n different eigenvalues $\mu_1, \cdots, \mu_n$, and $|\mu_i| \geq 2\delta\varepsilon$, $|\mu_i - \mu_j| \geq 2\delta\varepsilon$, $i \neq j$, $0 \leq i, j \leq n$, where δ is a positive constant independent of ε. We rewrite Hamiltonian system (1) into

$$\dot{x} = \left[A + \varepsilon\bar{Q} + \varepsilon\left(Q(t) - \bar{Q}\right)\right]x := \left(A_1 + \varepsilon\tilde{Q}(t)\right)x, \tag{39}$$

where $A_1 = A + \varepsilon\bar{Q}$, $\tilde{Q}(t) = Q(t) - Q$, and $\tilde{Q}(t) = 0$.

Introduce the change of variables $x = e^{\varepsilon P(t)}x_1$, where $P(t)$ will be determined later, under this symplectic transformation, Hamiltonian system (39) is changed into the new Hamiltonian system

$$\dot{x}_1 = e^{-\varepsilon P(t)}\left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P}\right)e^{\varepsilon P(t)}x_1. \tag{40}$$

Expand $e^{\varepsilon P}$ and $e^{-\varepsilon P}$ into

$$\begin{aligned} e^{\varepsilon P} &= I + \varepsilon P + B, \\ e^{-\varepsilon P} &= I - \varepsilon P + \tilde{B}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} B &= \frac{(\varepsilon P)^2}{2!} + \frac{(\varepsilon P)^3}{3!} + \cdots, \\ \tilde{B} &= \frac{(\varepsilon P)^2}{2!} - \frac{(\varepsilon P)^3}{3!} + \cdots. \end{aligned} \tag{42}$$

Then, system (40) can be rewritten

$$\begin{aligned} \dot{x}_1 &= \left(I - \varepsilon P + \tilde{B}\right)\left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P}\right)(I + \varepsilon P + B)x_1 \\ &= \left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P} + \varepsilon A_1 P - \varepsilon P A_1 + \varepsilon^2 Q_1\right)x_1, \end{aligned} \tag{43}$$

where

$$\begin{aligned} Q_1 = &-P\left(\tilde{Q} - \dot{P}\right) + \left(\tilde{Q} - \dot{P}\right)P - P\left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P}\right)P + (I - \varepsilon P) \\ &\cdot \left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P}\right)\frac{B}{\varepsilon^2} + e^{\varepsilon P}\left(A_1 + \varepsilon\tilde{Q} - \varepsilon\dot{P}\right)\frac{\tilde{B}}{\varepsilon^2}. \end{aligned} \tag{44}$$

We would like to have

$$\tilde{Q} - \dot{P} + A_1 P - P A_1 = 0, \tag{45}$$

which is equivalent to

$$\dot{P} = A_1 P - P A_1 + \tilde{Q}. \tag{46}$$

By Assumption 2 of Theorem 3, it is easy to see that the inequalities

$$|\mu_i| \geq \delta\varepsilon, \left|\mu_i - \mu_j\right| \geq \delta\varepsilon, \quad i \neq j, 0 \leq i, j \leq n \tag{47}$$

hold. Moreover, if the equalities

$$\left|\langle k, \omega\rangle\sqrt{-1} - \mu_i + \mu_j\right| \geq \frac{\alpha_0}{|k|^{3\tau}}, \quad 0 \neq k \in \mathbb{Z}^r, \tag{48}$$

also hold, where $\alpha_0 = \alpha/2$, thus, by Lemma 8, (46) is solvable for P on a smaller domain, that is, there is a unique quasiperiodic Hamiltonian matrix $P(t)$ with frequencies ω on $D_{\rho-s}$, which satisfies $\bar{P} = 0$ and

$$\|P\|_{\rho-s} \leq \frac{c}{\alpha_0 s^\nu}\left\|\tilde{Q}\right\|_\rho \leq \frac{c}{\alpha_0 s^\nu}\|Q\|_\rho, \tag{49}$$

where $s = (1/2)\rho$.Therefore, by (46), Hamiltonian system (43) becomes

$$\dot{x}_1 = \left(A_1 + \varepsilon^2 Q_1\right)x_1, \tag{50}$$

where

$$\begin{aligned} Q_1 = &P(A_1 P - P A_1) + (P A_1 - A_1 P)P - P(A_1 - \varepsilon(P A_1 - A_1 P)) \\ &\cdot P + (I - \varepsilon P)(A_1 - \varepsilon(P A_1 - A_1 P))\frac{B}{\varepsilon^2} \\ &+ \frac{\tilde{B}}{\varepsilon^2}(A_1 - \varepsilon(P A_1 - A_1 P))e^{\varepsilon P}. \end{aligned} \tag{51}$$

From Lemma 6, it follows that

$$\begin{aligned} \|B\|_{\rho-s} &\leq c\|\varepsilon P\|_{\rho-s}^2, \\ \left\|\tilde{B}\right\|_{\rho-s} &\leq c\|\varepsilon P\|_{\rho-s}^2. \end{aligned} \tag{52}$$

Therefore, if ε is sufficiently small, we have

$$\|Q_1\|_{\rho-s} \le c\|P\|^2_{\rho-s} \le \frac{c}{\alpha_0^2 s^{2\nu}}\|Q\|^2_\rho. \tag{53}$$

Now we consider the iteration step. In the m^{th} step, we consider the Hamiltonian system

$$\dot{x}_m = \left(A_m + \varepsilon^{2m}Q_m(t)\right)x_m,\ m \ge 1, \tag{54}$$

where A_m has n different eigenvalues $\lambda_1^m, \cdots, \lambda_n^m$ with

$$\begin{aligned} |\lambda_i^m| &\ge \delta\varepsilon, \\ \left|\lambda_i^m - \lambda_j^m\right| &\ge \delta\varepsilon, \\ i \ne j,\ 1 \le i, j \le n. \end{aligned} \tag{55}$$

Here, we define $\lambda_i^1 = \mu_i$, $i = 1, \cdots, n$.

Let $A_{m+1} = A_m + \varepsilon^{2m}\bar{Q}_m$, then system (54) becomes

$$\dot{x}_m = \left(A_{m+1} + \varepsilon^{2m}\tilde{Q}_m\right)x_m, \quad m \ge 1, \tag{56}$$

where $\tilde{Q}_m = Q_m(t) - \bar{Q}_m$. We need to solve

$$\dot{P}_m = A_{m+1}P_m - P_mA_{m+1} + \tilde{Q}_m. \tag{57}$$

If

$$\left|\langle k, \omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}\right| \ge \frac{\alpha_m}{|k|^{3\tau}}, \quad 0 \ne k \in \mathbb{Z}^r, \tag{58}$$

and A_{m+1} has n different eigenvalues $\lambda_1^{m+1}, \cdots, \lambda_n^{m+1}$ with

$$\begin{aligned} \left|\lambda_i^{m+1}\right| &\ge \delta\varepsilon, \\ \left|\lambda_i^{m+1} - \lambda_j^{m+1}\right| &\ge \delta\varepsilon, \\ i \ne j,\ 1 \le i, j \le n, \end{aligned} \tag{59}$$

by Lemma 8, there is a unique quasiperiodic Hamiltonian matrix $P_m(t)$ with frequencies ω on $D_{\rho_m - s_m}$, which satisfies

$$\|P_m\|_{\rho_m - s_m} \le \frac{c}{\alpha_m s_m^\nu}\|Q_m\|_{\rho_m}. \tag{60}$$

Thus, under the symplectic change of variables $x_m = e^{\varepsilon^{2m}P_m(t)}x_{m+1}$, system (56) is changed into

$$\dot{x}_{m+1} = \left(A_{m+1} + \varepsilon^{2^{m+1}}Q_{m+1}\right)x_{m+1}, \tag{61}$$

where

$$\begin{aligned} Q_{m+1}(t) &= P_m(A_{m+1}P_m - P_mA_{m+1}) \\ &\quad + (P_mA_{m+1} - A_{m+1}P_m)P_m \\ &\quad - P_m\left(A_{m+1} + \varepsilon^{2m}(P_mA_{m+1} - A_{m+1}P_m)\right)P_m \\ &\quad + \left(I - \varepsilon^{2m}P_m\right)\left(A_{m+1} + \varepsilon^{2m}(P_mA_{m+1} - A_{m+1}P_m)\right)\frac{B_m}{\varepsilon^{2m+1}} \\ &\quad + \frac{\tilde{B}_m}{\varepsilon^{2m+1}}\left(A_{m+1} + \varepsilon^{2m}(P_mA_{m+1} - A_{m+1}P_m)\right)\varepsilon^{2m}P_m, e^{-\varepsilon^{2^m}P_m} \\ &= I + \varepsilon^{2^m}P_m + B_m, e^{-\varepsilon^{2^m}P_m} = I - \varepsilon^{2^m}P_m + \tilde{B}_m, \\ B_m &= \frac{\left(\varepsilon^{2^m}P_m\right)^2}{2!} + \frac{\left(\varepsilon^{2^m}P_m\right)^3}{3!} + \cdots, \\ \tilde{B}m &= \frac{\left(\varepsilon^{2^m}P_m\right)^2}{2!} + \frac{\left(\varepsilon^{2^m}P_m\right)^3}{3!} + \cdots. \end{aligned} \tag{62}$$

From Lemma 6, it follows that

$$\begin{aligned} \|B_m\|_{\rho_m - s_m} &\le c\left\|\varepsilon^{2^m}P_m\right\|^2_{\rho_m - s_m}, \\ \left\|\tilde{B}_m\right\|_{\rho_m - s_m} &\le c\left\|\varepsilon^{2^m}P_m\right\|^2_{\rho_m - s_m}. \end{aligned} \tag{63}$$

Therefore, if $|\varepsilon|$ is sufficiently small, by (60) we have

$$\|Q_{m+1}\|_{\rho_m - s_m} \le \frac{c}{\alpha_m^2 s_m^{2\nu}}\|Q_m\|^2_{\rho_m}. \tag{64}$$

Now we prove that the iteration is convergent as $m \longrightarrow \infty$. When $m = 1$, we choose

$$\begin{aligned} \alpha_1 &= \frac{1}{4}\alpha, \\ \rho_1 &= \frac{1}{2}\rho, \\ s_1 &= \frac{1}{8}\rho, \\ F_1 &= \frac{\left\|\varepsilon^2 Q_1\right\|_{\rho 1}}{\alpha_1^2 s_1^{2\nu}}. \end{aligned} \tag{65}$$

At the m^{th} step, we define

$$\begin{aligned} \alpha_m &= \frac{\alpha}{(m+1)^2}, \\ s_m &= \frac{\rho}{2^{m+2}}, \\ \rho_m &= \rho_1 - (s_1 + s_2 + \cdots + s_{m-1}) \\ F_m &= \frac{\varepsilon^{2^m}\|Q_m\|_{\rho_m}}{\alpha_m^2 s_m^{2\nu}}. \end{aligned} \tag{66}$$

By (64), we have

$$F_{m+1} \le c\frac{\varepsilon^{2^{m+1}}\|Q_m\|_{\rho_m}^2}{\left(\alpha_m^2 s_m^{2\nu}\right)^2} = cF_m^2, \tag{67}$$

where the constant c depends only on α, ρ. Hence, it follows that

$$cF_{m+1} \le (cF_m)^2 \le (cF_1)2^m. \tag{68}$$

If $cF_1 < 1$, then $cF_m \longrightarrow 0$ as $m \longrightarrow \infty$. From (60), it follows that

$$\left\|\varepsilon^{2^m} P_m\right\|_{\rho_m - s_m} < cF_m. \tag{69}$$

Thus, if $cF_1 < 1/2$, then

$$\left\|e^{\pm\varepsilon^{2^m}} P_m\right\|_{\rho_m} \le 2. \tag{70}$$

Since

$$\|A_{m+1} - A_m\| = \left\|\varepsilon^{2^m}\bar{Q}_m\right\| \le \left\|\varepsilon^{2^m} Q_m\right\|_{\rho_m} < cF_m, \tag{71}$$

if $cF_1 \le \delta\varepsilon/(3n-1)\beta_m^2$, it follows from (71) that

$$\|A_{m+1} - A_m\| \le \frac{\delta\varepsilon}{(3n-1)\beta_m^2}, \quad \text{for any } m \ge 1, \tag{72}$$

where $\beta_m = \max\{\|S_m\|, \|S_m^{-1}\|\}$ and S_m is the regular matrix in Lemma 7 such that

$$S_m^{-1} A_m S_m = \operatorname{diag}(\lambda_1^m, \cdots, \lambda_n^m). \tag{73}$$

Thus, it follows from Lemma 7 that A_{m+1} has n different eigenvalues $\lambda_1^{m+1}, \cdots, \lambda_n^{m+1}$.

Moreover,

$$\begin{aligned} \left|\lambda_i^{m+1} - \lambda_j^{m+1}\right| &\ge \delta\varepsilon, \quad i \ne j, \boxtimes 1 \le i, \boxtimes j \le n, \\ \left|\lambda_i^{m+1}\right| &\ge \delta\varepsilon, \quad i = 1, \cdots, n. \end{aligned} \tag{74}$$

In fact,

$$\begin{aligned} \left|\lambda_i^{m+1} - \lambda_j^{m+1}\right| &\ge \left|\lambda_i^1 - \lambda_j^1\right| - \sum_{l=1}^{m}\left(\left|\lambda_i^{l+1} - \lambda_i^l\right| + \left|\lambda_j^{l+1} - \lambda_j^l\right|\right) \\ &\ge \left|\lambda_i^1 - \lambda_j^1\right| - 2\sum_{l=1}^{m}\|A_{l+1} - A_l\| \\ &\ge \left|\lambda_i^1 - \lambda_j^1\right| - 2\sum_{l=1}^{m} cF_l \\ &\ge 2\delta\varepsilon - 2\sum_{l=1}^{m} cF_l. \end{aligned} \tag{75}$$

Moreover, we have

$$\sum_{l=1}^{m} cF_l \le \sum_{l=1}^{\infty} cF_l \le \sum_{m=0}^{\infty}(cF_1)^{2^m} \le \sum_{m=1}^{\infty}(cF_1)^m = \frac{cF_1}{1-cF_1} < 2cF_1. \tag{76}$$

Thus, if $cF_1 \le \min\{1/2, (1/4)\delta\varepsilon, \delta\varepsilon/(3n-1)\beta_m^2\}$, that is, $0 < \varepsilon \le \min\{1, c/\|Q\|_\rho, c/\|Q\|_\rho^2\}$, then by (68), we have

$$\left|\lambda_i^{m+1} - \lambda_j^{m+1}\right| \ge 2\delta\varepsilon - 4\varepsilon cF_1 \ge \delta\varepsilon, \quad i \ne j,\ 1 \le n. \tag{77}$$

In the same way as above, we have

$$\left|\lambda_i^{m+1}\right| \ge \delta\varepsilon, \quad i = 1, \cdots, n. \tag{78}$$

Let $D_* = \cap_{m+1}^{\infty} D_{\rho_m} = D_{\rho/4}$. By (69), the composition of all the changes $e^{\varepsilon^{2^m} P_m}$ converges to ψ as $m \longrightarrow \infty$. Obviously,

$$\left\|\varepsilon^{2m} Q_m\right\|_{D_*} \le cF_m \longrightarrow 0, \quad m \longrightarrow \infty. \tag{79}$$

Furthermore, it follows from (71) that A_m is convergent as $m \longrightarrow \infty$. Define $B = \lim_{m\to\infty} A_m$. Then, under the symplectic change of variables $x = \psi(t)y$, $m \longrightarrow \infty$ Hamiltonian system (1) is changed into $\dot{y} = By$.

Now we prove that, for most sufficiently small ε, such symplectic transformation exists. From the above iteration, we need to prove that the nonresonant conditions

$$\left|\langle k, \omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}\right| \ge \frac{\alpha_m}{|k|^{3\tau}}, \tag{80}$$

for all $0 \ne k \in \mathbb{Z}^r$, $1 \le i, j \le n$, $m = 0, 1, 2, \cdots$, hold for most sufficiently small ε.

Let $f(\varepsilon) = \langle k, \omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}$, $i \ne j$,

$$O_{ijm}^k = \left\{\varepsilon \in (0, \varepsilon_0) : |f(\varepsilon)| < \frac{\alpha_m}{|k|^{3\tau}}\right\}, \tag{81}$$

where we choose

$$\varepsilon_0 = \min\left\{1, \frac{c}{\|Q\|_\rho}, \frac{c}{\|Q\|_\rho^2}\right\}, \tag{82}$$

such that, for $\varepsilon \in (0, \varepsilon_0)$, the above iteration is convergent, and

$$\left|\frac{df}{d\varepsilon}\right| = \left|\frac{d}{d\varepsilon}\left(\lambda_i^{m+1} - \lambda_j^{m+1}\right)\right| \ge \delta. \tag{83}$$

For $\varepsilon \in (0, \varepsilon_0)$, by (71), we have

$$\begin{aligned} \|A_{m+1} - A_1\| &\le \|A_{m+1} - A_m\| + \cdots + \|A_2 - A_1\| \\ &\le cF_m + \cdots + cF_2 \le 2cF_1 \le \frac{1}{2}\delta\varepsilon. \end{aligned} \tag{84}$$

Hence,

$$\begin{aligned} |f(\varepsilon)| &\ge \left|\langle k, \omega\rangle\sqrt{-1} - \lambda_i + \lambda_j\right| - \left|\lambda_i^1 - \lambda_i\right| \\ &\quad - \left|\lambda_j^1 - \lambda_j\right| - \left|\lambda_i^1 - \lambda_i^{m+1}\right| - \left|\lambda_j^1 - \lambda_j^{m+1}\right| \\ &\ge \frac{\alpha}{|k|^\tau} - 2q\varepsilon - 2\|A_{m+1} - A_1\| \ge \frac{\alpha}{|k|^\tau} - 2q\varepsilon - \delta\varepsilon \\ &\ge \frac{\alpha}{|k|^\tau} - 3M\varepsilon_0, \end{aligned} \tag{85}$$

where $M = \max\{q, \delta\}$.

If $1/|k|^\tau > 6M\varepsilon_0/\alpha$, then

$$|f(\varepsilon)| \ge \frac{\alpha}{2|k|^\tau} > \frac{\alpha_m}{|k|^{3\tau}}, \tag{86}$$

and $O_{ijm}^k = \varnothing$.

Suppose that $1/|k|^\tau < 6M\varepsilon_0/\alpha$. By (83), it follows that

$$\operatorname{meas}\left(O_{ijm}^k\right) < \frac{\alpha_m}{|k|^{3\tau}\delta}. \tag{87}$$

Thus,

$$\begin{aligned} \operatorname{meas}\left(\bigcup_{i\ne j0\ne k\in\mathbb{Z}^r} O_{ijm}^K\right) &\le \frac{n^2\alpha_m}{\delta} \sum_{|k|^{-\tau}<6M\varepsilon_0/\alpha} \frac{1}{|k|^{3\tau}} \\ &\le \frac{n^2\alpha_m}{\delta}\cdot\frac{36M^2\varepsilon_0^2}{\alpha^2}\sum_{k\in\mathbb{Z}^r}\frac{1}{|k|^\tau} \le \frac{c\varepsilon_0^2}{m^2}. \end{aligned} \tag{88}$$

Let

$$E_m = \left\{\varepsilon \in (0, \varepsilon_0) : \left|\langle k, \omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}\right| > \frac{\alpha_m}{|k|^{3\tau}}, \quad 0 \ne k \in \mathbb{Z}^r,\ i \ne j\right\}. \tag{89}$$

Then

$$(0, \varepsilon_0) - E_m = \bigcup_{i\ne j0\ne k\in\mathbb{Z}^r} O_{ijm}^k. \tag{90}$$

Thus

$$\operatorname{meas}((0, \varepsilon_0) - E_m) \le \frac{c\varepsilon_0^2}{m^2}. \tag{91}$$

Let $E_{\varepsilon_0} = \cap_{m=1}^{\infty} E_m$, then

$$\begin{aligned} &\operatorname{meas}\left((0, \varepsilon_0) - E_{\varepsilon_0}\right) \le c\varepsilon_0^2, \\ &\lim_{\varepsilon_0\to 0}\frac{\operatorname{meas}\left((0, \varepsilon_0) - E_{\varepsilon_0}\right)}{\varepsilon_0} = 0. \end{aligned} \tag{92}$$

Therefore, E_{ε_0} is a nonempty subset of $(0, \varepsilon_0)$. Thus, for $\varepsilon \in E_{\varepsilon_0}$, Hamiltonian system (1) is reducible, i.e., there exists a symplectic transformation $x = \psi(t)y$, which changes Hamiltonian system (1) into the Hamiltonian system $\dot{y} = By$. Thus, Theorem 3 is proved completely.

4. The Applications

As an example, we apply Theorem 3 to the following Schrödinger equation

$$\ddot{x} + \varepsilon a(t)x = 0, \tag{93}$$

where $a(t)$ is an analytic quasiperiodic function on D_ρ with frequencies $\omega = (\omega_1, \cdots, \omega_r)$. Denote the average of $a(t)$ by $\bar{a}$, and suppose $\bar{a} > 0$.

Let $\dot{x} = y$, then equation (93) can be rewritten in the equivalent form

$$\dot{x} = y, \quad \dot{y} = -\varepsilon a(t)x. \tag{94}$$

To apply Theorem 3, we express (94) in the form

$$\dot{z} = (A + \varepsilon Q(t))z, \tag{95}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ -a(t) & 0 \end{pmatrix}. \tag{96}$$

It is easy to see that A has multiple eigenvalues $\lambda_1 = \lambda_2 = 0$; moreover, $A + \varepsilon\bar{Q}$ has two different eigenvalues $\mu_1 = i\sqrt{\bar{a}\varepsilon}$, $\mu_2 = -i\sqrt{\bar{a}\varepsilon}$, where $\bar{Q}$ stands for the average of the matrix $Q(t)$ and $i = \sqrt{-1}$. It is clear that

$$|\mu_i| = \sqrt{\bar{a}}\sqrt{\varepsilon} \ge 2\delta\varepsilon, \quad i = 1, 2, \tag{97}$$

$$|\mu_1 - \mu_2| = 2\sqrt{\bar{a}}\sqrt{\varepsilon} \ge 2\delta\varepsilon, \tag{98}$$

where we choose $\delta = (1/2)\sqrt{\bar{a}} > 0$, which is a constant independent of ε. Therefore, Theorem 3 can be applied. It follows from Theorem 3 that the following result holds.

Theorem 9. *Assume that $a(t)$ is an analytic quasiperiodic function on D_ρ with frequencies $\omega = (\omega_1, \cdots, \omega_r)$, and $\bar{a} > 0$. If the frequencies ω of $a(t)$ satisfy the Diophantine condition*

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad k \in \mathbb{Z}^r \setminus \{0\}, \tag{99}$$

where $\alpha > 0$ is a small constant and $\tau > r - 1$.

Then, there exist some sufficiently small $\varepsilon_0 > 0$ and a nonempty Cantor subset $E_{\varepsilon_0} \subset (0, \varepsilon_0)$ with positive Lebesgue measure, such that for $\varepsilon \in E_{\varepsilon_0}$, system (95) is reducible. Moreover, if ε_0 is small enough, the relative measure of E_{ε_0} in $(0, \varepsilon_0)$ is close to 1.

Remark 10. From Theorem 9, it follows that equation (93) can be changed into a constant coefficient differential equation for most sufficiently small $\varepsilon > 0$.

Now we want to study the Lyapunov stability of the equilibrium of equation (93), using the results obtained in Section 3. If $a(t)$ is periodic in time (T is the period), one famous stability criterion was given by Magnus and Winkler [25] for Hill's equation

$$\ddot{x} + a(t)x = 0. \tag{100}$$

That is, (100) is stable if

$$a(t) > 0, \int_0^T a(t)dt \leq \frac{4}{T}, \tag{101}$$

which can be shown using a Poincaré inequality. Such a stability criterion had been generalized and improved by Zhang and Li in [26], which now is the so-called L^p criterion. Recently, Zhang in [27] had extended such a criterion to the linear planar Hamiltonian system

$$\begin{aligned} \dot{x} &= m(t)y, \\ \dot{y} &= -n(t)x, \end{aligned} \tag{102}$$

where $m(t)$, $n(t)$ are continuous and T-periodic functions.

However, for quasiperiodic equation (93), the results above cannot be applied directly. Now, we obtain a result about the stability of the equilibrium of equation (93).

Theorem 11. *Under the conditions of Theorem 9, the equilibrium of equation (93) is stable in the sense of Lyapunov for most sufficiently small $\varepsilon > 0$.*

Proof. Theorem 9 tells us that, for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$, there exists an analytic quasiperiodic symplectic transformation $z = \psi(t)z_\infty$, where $\psi(t)$ has the same frequencies as $Q(t)$, which changes (95) into the Hamiltonian system

$$\dot{z}_\infty = Bz_\infty, \tag{103}$$

where B is a constant matrix. Moreover, from the proof of Theorem 3 in Section 3, it follows that B has two different eigenvaluesλ_1^∞, λ_2^∞, satisfying

$$\begin{aligned} |\lambda_i^\infty| &\geq \delta\varepsilon (i = 1, 2), \\ |\lambda_1^\infty - \lambda_2^\infty| &\geq \delta\varepsilon. \end{aligned} \tag{104}$$

Furthermore, by the proof of Theorem 3, we have

$$\left\| B - \left(A + \varepsilon\bar{Q}\right) \right\| \leq cF_1 = O\left(\varepsilon^2\right). \tag{105}$$

Therefore, the two different eigenvalues of B are pure imaginary and can be written in the form

$$\lambda_i^\infty = \pm i\sqrt{b}, \quad i = 1, 2, \tag{106}$$

where b can be written in the following form

$$b = \bar{a}\varepsilon + O\left(\varepsilon^2\right), \tag{107}$$

which depends on $\bar{a}$ and ε only.

Thus, there exists a singular symplectic matrix S such that

$$S^{-1}BS = \begin{pmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{pmatrix}. \tag{108}$$

Let $z_\infty = S\tilde{z}_\infty$, under this symplectic transformation, system (103) is changed into

$$\dot{\tilde{z}}_\infty = S^{-1}BS\tilde{z}_\infty = \begin{pmatrix} i\sqrt{b} & 0 \\ 0 & -i\sqrt{b} \end{pmatrix} \tilde{z}_\infty. \tag{109}$$

Hence, by an analytic quasiperiodic symplectic transformation, equation (93) is changed into

$$\ddot{x}_\infty + bx_\infty = 0. \tag{110}$$

It is easy to see that equation (110) is elliptic. Therefore, the equilibrium of equation (93) is stable in the sense of Lyapunov for most sufficiently small $\varepsilon > 0$.

For the existence of quasiperiodic solution of equation (93), we have the following result.

Theorem 12. *Under the conditions of Theorem 9, all solutions of equation (93) are quasiperiodic with frequencies $\Omega = (\omega_1, \cdots, \omega_r, \sqrt{b})$ for most sufficiently small $\varepsilon > 0$, where b is given by (107).*

Proof. By Theorem 9, we know that, for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$, there exists an analytic quasiperiodic symplectic transformation which has the same frequencies as $a(t)$; by this transformation, equation (93) is changed into (110). On the other hand, it is easy to see that all solutions of equation (110) are periodic, and the frequency of these solutions is $\sqrt{b}$.

Thus, we only need to prove that, for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$, the following nonresonant condition

$$\left|k_1\omega_1+\cdots+k_r\omega_{r+1}\sqrt{b}\right| \geq \frac{\alpha_0}{|k|^{5\tau+4}} \tag{111}$$

holds for all $k = (k_1,\cdots,k_{r+1}) \in \mathbb{Z}^{r+1} \setminus \{0\}$, where α_0 is defined in Section 3, that is, $\alpha_0 = (1/2)\alpha$, and $\omega = (\omega_1,\cdots,\omega_r)$ are the frequencies of $a(t)$.

If $k_{r+1} = 0$, then from the Diophantine condition (99), it follows that (111) holds.

Suppose that $k_{r+1} \neq 0$. Let $g(\varepsilon) = k_1\omega_1+\cdots+k_r\omega_r + k_{r+1}\sqrt{b}$, and

$$O_k = \left\{\varepsilon \in (0, \varepsilon_0): |g(\varepsilon)| < \frac{\alpha_0}{|k|^{5\tau+4}}\right\}. \tag{112}$$

It follows from the nondegeneracy condition that

$$\left|\frac{dg}{d\varepsilon}\right| = \left|\frac{d}{d\varepsilon}\left(k_{r+1}\sqrt{b}\right)\right| \geq |k_{r+1}|\delta. \tag{113}$$

By (107), we have

$$\sqrt{b} \leq 4\delta\sqrt{\varepsilon}. \tag{114}$$

From the Diophantine condition (99), it follows that

$$\begin{aligned} |g(\varepsilon)| &\geq \frac{\alpha}{(|k_1|+\cdots+|k_r|)} - |k_{r+1}|\sqrt{b} \geq \frac{\alpha}{|k_r|^{\tau}} - |k_{r+1}|\sqrt{b} \\ &\geq \frac{\alpha}{|k|^{\tau}} - |k_{r+1}|4\delta\sqrt{\varepsilon} \geq \frac{\alpha}{|k|^{\tau}} - 4\delta|k|\sqrt{\varepsilon_0}. \end{aligned} \tag{115}$$

If $1/|k|^{\tau+1} \geq 8\delta\sqrt{\varepsilon_0}/\alpha$, then

$$|g(\varepsilon)| \geq \frac{\alpha}{2|k|^{\tau}} \geq \frac{\alpha_0}{|k|^{5\tau+4}}, \tag{116}$$

and $O_k = \varnothing$.

Suppose that $1/|k|^{\tau+1} < 8\delta\sqrt{\varepsilon_0}/\alpha$, it follows from (113) that

$$\operatorname{meas}(O_k) < \frac{\alpha_0}{|k|^{5\tau+4}|k_{r+1}|\delta}. \tag{117}$$

Thus,

$$\begin{aligned} \operatorname{meas}\left(\bigcup_{0\neq k\in\mathbb{Z}^{r+1}} O_k\right) &\leq \frac{\alpha_0}{\delta} \sum_{1/|k|^{\tau+1}<8\delta\sqrt{\varepsilon_0}/\alpha} \frac{1}{|k|^{5\tau+4}|k_{r+1}|} \\ &\leq \frac{\alpha_0}{\delta}\frac{(8\delta)^4\varepsilon_0^2}{\alpha^4} \sum_{k\in\mathbb{Z}^{r+1}} \frac{1}{|k|^{\tau}|k_{r+1}|} \\ &\leq c\varepsilon_0^2 \sum_{0\neq k_{r+1}\in\mathbb{Z}} \frac{1}{|k_{r+1}|^{\tau+1}} \leq c\varepsilon_0^2. \end{aligned} \tag{118}$$

Then

$$\lim_{\varepsilon_0\to 0} \frac{\operatorname{meas}(\cup_{0\neq k\in\mathbb{Z}^{r+1}} O_k)}{\varepsilon_0} = 0. \tag{119}$$

Therefore, (111) holds for most sufficiently small $\varepsilon \in (0, \varepsilon_0)$.

Thus, all solutions of equation (93) are quasiperiodic with frequencies $\Omega = (\omega_1,\cdots,\omega_r, \sqrt{b})$ for most sufficiently small $\varepsilon > 0$.

Data Availability

There is no additional data in the manuscript, because the main result is theoretical proof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Rapid Convergence of Approximate Solutions for Fractional Differential Equations

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In this paper, we develop a generalized quasilinearization technique for a class of Caputo's fractional differential equations when the forcing function is the sum of hyperconvex and hyperconcave functions of order m ($m \geq 0$), and we obtain the convergence of the sequences of approximate solutions by establishing the convergence of order k ($k \geq 2$).

1. Introduction

Fractional differential equations have received attention from some researchers because they have extensive application in mechanics, biochemistry, electrical engineering, medicine, and many other fields (see [1–6]). For more information about the basic theory of fractional differential equations, we can refer to the monographs [7–9] and references cited therein. It is well known [10] that the monotone iterative technique offers an approach for obtaining approximate solutions to a wide variety of nonlinear differential equations. Recently, there are some results on the monotone sequences of approximate solutions converging uniformly to a solution of fractional differential equations by employing monotone iterative technique and generalized monotone iterative method coupled with the method of upper and lower solutions, which can be found in [11–21].

In view of applications, it is very significant to study the rate of convergence of solutions. The quasilinearization method [22] is one of the effective methods to obtain a sequence of approximate solutions with quadratic convergence, and it is extremely useful in scientific computations due to its accelerated rate of convergence as in [23, 24]. A few results of quadratic convergence for fractional differential equations were also obtained by applying quasilinearization, such as the initial value problem of Caputo's fractional differential equations [13, 25, 26], fractional differential equations via initial time different lower and upper solutions [27], and the system of fractional differential equations [28]. However, to the best of our knowledge, there are few results of rapid convergence of fractional differential equations. Recently, Wang and others obtained the results on rapid convergence of solutions for various differential equations [29–33]. Inspired and motivated by [34, 35], in the present paper, we will discuss the rapid convergence of approximate solutions of fractional differential equations when the forcing function is the sum of hyperconvex and hyperconcave functions with coupled lower and upper solutions, and construct sequences of approximate solutions that converge rapidly to the extremal solutions of (1) by using an improved quasilinearization method (rate of convergence $k \geq 2$).

2. Preliminaries

Consider the initial value problem of Caputo's fractional differential equations (IVP):

$$^{c}D^{q}x = f(t,x) + g(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $f, g : J \times R \longrightarrow R$ are continuous functions, $J = [t_0, T]$ and $0 < q < 1$.

A function $x(t)$ is called a solution of IVP (1) if it satisfies (1).

Firstly, we give the following definitions and lemmas.

Definition 1. The locally Hölder continuous functions α_0, $\beta_0 : J \longrightarrow R$ are coupled lower and upper solutions of type I of IVP (1) if the following inequalities hold:

$$\begin{cases} {}^cD^q\alpha_0 \le f(t,\alpha_0) + g(t,\beta_0), & \alpha_0(t_0) \le x_0, \\ {}^cD^q\beta_0 \ge f(t,\beta_0) + g(t,\alpha_0), & \beta_0(t_0) \ge x_0. \end{cases} \tag{2}$$

Definition 2. The locally Hölder continuous functions α_0, $\beta_0 : J \longrightarrow R$ are coupled lower and upper solutions of type II of IVP (1) if the following inequalities hold:

$$\begin{cases} {}^cD^q\alpha_0 \le f(t,\beta_0) + g(t,\alpha_0), & \alpha_0(t_0) \le x_0, \\ {}^cD^q\beta_0 \ge f(t,\alpha_0) + g(t,\beta_0), & \beta_0(t_0) \ge x_0. \end{cases} \tag{3}$$

Let $f^{(k)}(t,x) = \partial^k f(t,x)/\partial x^k$ denote the kth partial of f with respect to $x, k \in \mathbf{N}$. $||x|| = \sup_{t\in J}|x(t)|$.

Definition 3. A function f is called m-hyperconvex, $m \in \mathbf{N}$, if $f^{(m+1)}(t,x) \ge 0$; f is called m-hyperconcave if the inequality is reversed.

The linear IVP of Caputo's fractional differential equation is given by

$${}^cD^q x = \lambda x + \theta(t), \quad x(t_0) = x_0, \tag{4}$$

where $\theta : J \longrightarrow R$ is a locally Hölder continuous function, $t \in J, \lambda \in R$. The unique solution of (4) can be expressed in the following form [7]:

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} \left[(t-s)^{q-1} E_{q,q}(\lambda(t-s)^q)\theta(s)\right] ds, \tag{5}$$

where

$$\begin{aligned} E_q(t) &= \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+1)}, \\ E_{q,q}(t) &= \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+q)}, \end{aligned} \tag{6}$$

are Mittag-Leffler's functions, and Γ denotes the *Gamma* function.

Remark 4. We note that (5) remains valid if x and $\theta(t)$ are functions mapping from $J \longrightarrow R^n$, $x_0 \in R^n$, and λ is an $n \times n$ matrix.

We need the following lemmas to prove our main results, which proofs can be found in literature [26].

Lemma 5. *Assume that $x : J \longrightarrow R$ is a locally Hölder continuous function such that for $t_1 \in (t_0, T]$, $x(t_1) = 0$ and $x(t) < 0$ for $t \in [t_0, t_1)$. Then ${}^cD^q x(t_1) \ge 0$.*

Lemma 6. *Assume that $\{x_n(t)\}$ is a family of continuous functions on J. For each $n > 0$ and the Caputo's fractional differential equations, we have the following:*

$${}^cD^q x_n(t) = f(t, x_n(t)), \quad x_n(t_0) = x_0, \tag{7}$$

the functions $f(t, x_n(t))$ satisfy $|f(t, x_n(t))| \le M$ for $t \in J$. Then, the family $\{x_n(t)\}$ is equicontinuous on J.

In our further investigations, we need the following comparison results.

Lemma 7. *Assume that one of the following conditions holds:*

H_1 α_0, β_0 are coupled lower and upper solutions of type I of (1) and

$$\begin{aligned} f(t,x_1) - f(t,x_2) &\le L(x_1 - x_2), \\ g(t,x_1) - g(t,x_2) &\ge -L(x_1 - x_2), \end{aligned} \tag{8}$$

where $x_1 \ge x_2$, $L \ge 0$ is a constant

H_2 α_0, β_0 are coupled lower and upper solutions of type II of (1) and

$$\begin{aligned} f(t,x_1) - f(t,x_2) &\ge -L(x_1 - x_2), \\ g(t,x_1) - g(t,x_2) &\le L(x_1 - x_2), \end{aligned} \tag{9}$$

where $x_1 \ge x_2$, $L \ge 0$ is a constant

Then, $\alpha_0(t_0) \le \beta_0(t_0)$ implies $\alpha_0(t) \le \beta_0(t)$ on J.

Proof. Firstly, we prove that the conclusion is valid when H_1 holds. To do this, let $\tilde{\beta}_0(t) = \beta_0(t) + \varepsilon E_q(3L(t-t_0)^q)$ and $\tilde{\alpha}_0(t) = \alpha_0(t) - \varepsilon E_q(3L(t-t_0)^q)$ for any small $\varepsilon > 0$ so that $\tilde{\beta}_0(t) > \beta_0(t)$, $\tilde{\alpha}_0(t) < \alpha_0(t)$, and $\tilde{\beta}_0(t_0) > \beta_0(t_0) \ge \alpha_0(t_0) > \tilde{\alpha}_0(t_0)$. Then, in view of H_1, we have

$$\begin{aligned} {}^cD^q\tilde{\alpha}_0 &\le f(t,\alpha_0) + g(t,\beta_0) - 3L\varepsilon E_q(3L(t-t_0)q) \\ &< f(t,\tilde{\alpha}_0) + g\left(t,\tilde{\beta}_0\right). \end{aligned} \tag{10}$$

Similarly, we have ${}^cD^q\tilde{\beta}_0 > f(t,\tilde{\beta}_0) + g(t,\tilde{\alpha}_0)$.

We next show that $\tilde{\alpha}_0(t) < \tilde{\beta}_0(t)$ on J, which proves the conclusion as $\varepsilon \longrightarrow 0$. Letting $Q(t) = \tilde{\alpha}_0(t) - \tilde{\beta}_0(t)$, suppose that $Q(t) < 0$ is not true on J. Then, there exists a $t_1 \in (t_0, T]$ such that $Q(t_1) = 0$ and $Q(t) < 0$ for $t \in [t_0, t_1)$. In view of Lemma 5, ${}^cD^qQ(t_1) \ge 0$, that is, ${}^cD^q\tilde{\alpha}_0(t_1) \ge {}^cD^q\tilde{\beta}_0(t_1)$. Thus, we arrive at the following contradiction:

$$\begin{aligned} f(t_1,\tilde{\alpha}_0(t_1)) + g\left(t_1,\tilde{\beta}_0(t_1)\right) &> {}^cD^q\tilde{\alpha}_0(t_1) \ge {}^cD^q\tilde{\beta}_0(t_1) \\ &> f\left(t_1,\tilde{\beta}_0(t_1)\right) + g(t_1,\tilde{\alpha}_0(t_1)). \end{aligned} \tag{11}$$

Similar to the proof process above, we can obtain the result of Lemma 7 when H_2 holds.

Lemma 8. *Assume that $f, g \in C[\Omega, R]$, where $\Omega = \{(t,x)\colon \alpha_0(t) \le x \le \beta_0(t),\ t \in J\}$, and one of the following conditions holds:*

H_3 α_0, β_0 are coupled lower and upper solutions of type I of (1) such that $\alpha_0(t) \le \beta_0(t)$ on J, and $g(t,x)$ is monotone nonincreasing in x for $t \in J$

H_4 α_0, β_0 are coupled lower and upper solutions of type II of (1) such that $\alpha_0(t) \le \beta_0(t)$ on J, and $f(t,x)$ is monotone nonincreasing in x for $t \in J$

Then, there exists a solution $x(t)$ of (1) satisfying $\alpha_0(t) \le x(t) \le \beta_0(t)$ on J.

Proof. Suppose that H_3 holds. Consider the mapping $P : J \times R \longrightarrow R$ defined by

$$P(t,x) = \max\{\alpha_0(t), \min[x(t), \beta_0(t)]\}. \tag{12}$$

Then

$$^cD^q x = f(t, P(t,x)) + g(t, P(t,x)), \tag{13}$$

and it has a solution $x(t)$ on J with $x(t_0) = x_0$.

Firstly, we prove that $\alpha_0(t) \le x(t)$ on J. Letting $\bar{\alpha}_0(t) = \alpha_0(t) - \varepsilon E_q(L(t-t_0)^q)$, for any small $\varepsilon > 0$ such that $\bar{\alpha}_0(t) < \alpha_0(t)$ on J. We can prove that $\bar{\alpha}_0(t) < x(t)$ on J, which shows that $\alpha_0(t) \le x(t)$ as $\varepsilon \longrightarrow 0$. Setting $m(t) = \bar{\alpha}_0(t) - x(t)$, suppose $m(t) < 0$ is not true on J, then there exists a $t_2 \in (t_0, T]$ such that $m(t_2) = 0$ and $m(t) < 0$ for $t \in [t_0, t_2)$. From Lemma 5, it follows that $^cD^q m(t_2) \ge 0$, that is, $^cD^q\bar{\alpha}_0(t_2) \ge {}^cD^q x(t_2)$. Therefore,

$$^cD^q x(t_2) \le {}^cD^q\bar{\alpha}_0(t_2) < {}^cD^q\alpha_0(t_2) \le f(t_2, \alpha_0(t_2)) + g(t_2, \beta_0(t_2)). \tag{14}$$

On the other hand, we have

$$\begin{aligned} ^cD^q x(t_2) &= f(t_2, P(t_2,x)) + g(t_2, P(t_2,x)) \\ &\ge f(t_2, \alpha_0(t_2)) + g(t_2, \beta_0(t_2)), \end{aligned} \tag{15}$$

and it contradicts with (14). This contradiction proves the claim.

Similarly, letting $\bar{\beta}_0(t) = \beta_0(t) + \varepsilon E_q(L(t-t_0)^q)$, we can find that $x(t) \le \beta_0(t)$.

It is easy to construct the proofs of the results relative to H_4. We omit the details.

To obtain the results of this paper, we need to consider two-dimensional Caputo's fractional differential systems:

$$^cD^q X = H(t,X), \quad X(t_0) = X_0, \tag{16}$$

where $H \in C[J \times R^2, R^2]$ and $X : J \longrightarrow R^2$ is a locally Hölder continuous function.

Lemma 9. *Assume that $v, w : J \longrightarrow R^2$ are locally Hölder continuous functions satisfying the following:*

$$\begin{aligned} &^cD^q v \le H(t,v), \\ &v(t_0) \le X_0, \\ &^cD^q w \ge H(t,w), \\ &w(t_0) \ge X_0, \end{aligned} \tag{17}$$

and whenever $X \ge Y$,

$$H_i(t,x_1,x_2) - H_i(t,y_1,y_2) \le L[(x_1-y_1) + (x_2-y_2)], \tag{18}$$

where $L \ge 0$ is a constant, $i = 1, 2$. Then $v(t_0) \le w(t_0)$ implies $v(t) \le w(t)$ on J.

Proof. Let $\bar{w}(t) = w(t) + \varepsilon E_q(2L(t-t_0)^q)$ and $\bar{v}(t) = v(t) - \varepsilon E_q(2L(t-t_0)^q)$ for any small $\varepsilon > 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ so that $\bar{w}(t) > w(t)$, $\bar{v}(t) < v(t)$, and $\bar{w}(t_0) > w(t_0) \ge v(t_0) > \bar{v}(t_0)$. Consequently, we obtain, for each i

$$\begin{aligned} ^cD^q\bar{v}_i &= {}^cD^q v_i - 2L\varepsilon_i E_q(2L(t-t_0)^q) \\ &\le H_i(t, \bar{v}_1, \bar{v}_2) - L\varepsilon_i E_q(3L(t-t_0)^q) \\ &< H_i(t, \bar{v}_1, \bar{v}_2). \end{aligned} \tag{19}$$

Similarly, we have

$$^cD^q\bar{w}_i = {}^cD^q w_i + 2L\varepsilon_i E_q(2L(t-t_0)^q) > H_i(t, \bar{w}_1, \bar{w}_2). \tag{20}$$

We next prove that $\bar{v}(t) < \bar{w}(t)$ on J, which shows the required conclusion as $\varepsilon \longrightarrow 0$. Suppose that $\bar{v}(t) < \bar{w}(t)$ is not true on J, then there exists an index j, and a $t_3 \in (t_0, T]$ such that $\bar{v}_j(t_3) = \bar{w}_j(t_3)$ and $\bar{v}_j(t) < \bar{w}_j(t)$ for $t \in [t_0, t_3)$. Set $m(t) = \bar{v}_j(t) - \bar{w}_j(t)$, it then follows from Lemma 5 that $^cD^q m(t_3) \ge 0$, that is $^cD^q\bar{v}_j(t_3) \ge {}^cD^q\bar{w}_j(t_3)$. Furthermore,

$$H_j(t_3, \bar{v}_1(t_3), \bar{v}_2(t_3)) > {}^cD^q\bar{v}_j(t_3) > H_j(t_3, \bar{w}_1(t_3), \bar{w}_2(t_3)), \tag{21}$$

which leads to a contradiction. This completes the proof.

3. Main Results

In this section, we consider that $f(t,x)$ and $g(t,x)$ are hyperconvex and hyperconcave in x of order $m-1$, respectively. We first give some inequalities depending on whether m is even or odd [34]:

(i) $m = 2k$

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \tag{22}$$

$$f(t,\eta) \leq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}, \tag{23}$$

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \tag{24}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}. \tag{25}$$

(ii) $m = 2k+1$

$$f(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \geq \xi, \tag{26}$$

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \leq \xi, \tag{27}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \geq \xi, \tag{28}$$

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \leq \xi, \tag{29}$$

Based on the above inequalities, we have the following result which is relative to the coupled lower and upper solutions of type I in Definition 1 when m is even.

Theorem 10. *Consider the following assumptions:*

A_1 α_0, β_0 are coupled lower and upper solutions of type I of (1) with $\alpha_0 \leq \beta_0$ on J

A_2 $f, g \in C^{2k}[\Omega, R]$ such that $f(t,x)$ and $g(t,x)$ are the hyperconvex and hyperconcave in x of the order $2k-1$, respectively

A_3 $g_x(t,x)$ satisfies

$$g_x(t,x) \leq \min\left[g^{(2k)}(t,x)\right] \frac{(\beta_0-\alpha_0)^{2k-1}}{(2k-2)!} \leq 0 \text{ on } \Omega \tag{30}$$

Then, there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converging uniformly to the solution $x(t)$ of (1) on J and the convergence is of the order $2k$.

Proof. It follows from assumption A_2 that the inequalities (22), (23), (24), and (25) hold. Consider the following fractional differential equations:

$$\begin{aligned} {}^cD^q v(t) &= F(t,\alpha,\beta\,;v,w) \\ &\triangleq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\alpha)(v-\alpha)^i}{i!} \\ &\quad + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\beta)(w-\beta)^i}{i!} \\ &\quad + \frac{g^{(2k-1)}(t,\alpha)(w-\beta)^{2k-1}}{(2k-1)!}, \quad v(t_0) = x_0, \end{aligned} \tag{31}$$

$$\begin{aligned} {}^cD^q w(t) &= G(t,\alpha,\beta\,;w,v) \\ &\triangleq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\beta)(w\text{-}\beta)^i}{i!} \\ &\quad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\alpha)(v-\alpha)^i}{i!} \\ &\quad + \frac{f^{(2k-1)}(t,\alpha)(w-\beta)^{2k-1}}{(2k-1)!}, \quad w(t_0) = x_0. \end{aligned} \tag{32}$$

Firstly, applying (31) and (32) and taking $\alpha = \alpha_0, \beta = \beta_0$, we obtain

$${}^cD^q v(t) = F(t,\alpha_0,\beta_0\,;v,w), \quad v(t_0) = x_0, \tag{33}$$

$${}^cD^q w(t) = G(t,\alpha_0,\beta_0\,;w,v), \quad w(t_0) = x_0. \tag{34}$$

Condition A_1 and inequalities (22), (23), (24), and (25) imply

$$\begin{aligned} &{}^cD^q \alpha_0 \leq f(t,\alpha_0) + g(t,\beta_0) = F(t,\alpha_0,\beta_0\,;\alpha_0,\beta_0), \quad \alpha_0(t_0) \leq x_0, \\ &{}^cD^q \beta_0 \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!} \\ &\qquad + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} \\ &\qquad + \frac{g^{(2k-1)}(t,\alpha_0)(\alpha_0-\beta_0)^{2k-1}}{(2k-1)!} \\ &\qquad = F(t,\alpha_0,\beta_0\,;\beta_0,\alpha_0), \quad \beta_0(t_0) \geq x_0, \\ &{}^cD^q \alpha_0 \leq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} \\ &\qquad + \frac{f^{(2k-1)}(t,\alpha_0)(\alpha_0-\beta_0)^{2k-1}}{(2k-1)!} \\ &\qquad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\alpha_0)(\alpha_0-\beta_0)^i}{i!} \\ &\qquad = G(t,\alpha_0,\beta_0\,;\alpha_0,\beta_0), \quad \alpha_0(t_0) \leq x_0, \\ &{}^cD^q \beta_0 \geq f(t,\beta_0) + g(t,\alpha_0) = G(t,\alpha_0,\beta_0\,;\beta_0,\alpha_0), \quad \beta_0(t_0) \geq x_0. \end{aligned} \tag{35}$$

Employing assumption A_3 and the Taylor series expansion with the Lagrange remainder, we get

$$F_w(t,\alpha_0,\beta_0;v,w)=g_w(t,w)-\frac{g^{(2k)}(t,\xi_2)(w-\beta_0)^{2k-2}(\xi_1-\alpha_0)}{(2k-2)!}\le 0,$$
$$G_v(t,\alpha_0,\beta_0;w,v)=g_v(t,v)-\frac{g^{(2k)}(t,\eta_1)(v-\alpha_0)^{2k-1}}{(2k-1)!}\le 0, \tag{36}$$

where $\alpha_0\le\xi_2\le\xi_1\le\beta_0$ and $\alpha_0\le\eta_1\le\beta_0$. Therefore, $F(t,\alpha_0,\beta_0;v,w)$ and $G(t,\alpha_0,\beta_0;w,v)$ are nonincreasing in w and v, respectively. By Lemma 8, there exist solutions $\alpha_1(t)$ and $\beta_1(t)$ of (33), (34) on J such that $\alpha_0(t)\le\alpha_1(t)\le\beta_0(t)$ and $\alpha_0(t)\le\beta_1(t)\le\beta_0(t)$. Furthermore, in view of the inequalities (22), (23), (24), and (25) and condition A_2, we have

$$\begin{aligned}{}^cD^q\alpha_1&=F(t,\alpha_0,\beta_0;\alpha_1,\beta_1)\le f(t,\alpha_1)+g(t,\beta_1),\\ &\qquad\alpha_1(t_0)=x_0,\\ {}^cD^q\beta_1&=G(t,\alpha_0,\beta_0;\beta_1,\alpha_1)\ge f(t,\beta_1)+g(t,\alpha_1),\\ &\qquad\beta_1(t_0)=x_0,\end{aligned} \tag{37}$$

and in view of H_1, we have $\alpha_1(t)\le\beta_1(t)$ for $t\in J$. Hence, $\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)$.

By induction, for all n, we can obtain that

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t), \tag{38}$$

where α_n and β_n are solutions of

$$\begin{aligned}{}^cD^qv(t)&=F(t,\alpha_{n-1},\beta_{n-1};v,w),\qquad v(t_0)=x_0,\\ {}^cD^qw(t)&=G(t,\alpha_{n-1},\beta_{n-1};w,v),\qquad w(t_0)=x_0,\end{aligned} \tag{39}$$

and

$${}^cD^q\alpha_n=F(t,\alpha_{n-1},\beta_{n-1};\alpha_n,\beta_n)\le f(t,\alpha_n)+g(t,\beta_n),\quad \alpha_n(t_0)=x_0, \tag{40}$$

$${}^cD^q\beta_n=G(t,\alpha_{n-1},\beta_{n-1};\beta_n,\alpha_n)\ge f(t,\beta_n)+g(t,\alpha_n),\quad \beta_n(t_0)=x_0. \tag{41}$$

According to (40) and (41), α_n and β_n are coupled lower and upper solutions of type I of (1). We have $g_x\le 0$ on Ω from assumption A_3. It then follows from Lemma 8 that $x(t)$ is a solution of (1) on J satisfying $\alpha_n(t)\le x(t)\le\beta_n(t)$. Hence, we have

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le x\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t). \tag{42}$$

By (42), the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded on J. From Lemma 6, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous on J. Consequently, by employing the Ascoli-Arzela Theorem, sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly convergent on J.

Finally, we prove that the convergence of $\{\alpha_n\}$ and $\{\beta_n\}$ is of the order $2k$.

Set $W_n(t)=x(t)-\alpha_n(t)\ge 0$ and $V_n(t)=\beta_n(t)-x(t)\ge 0$ for $t\in J$ with $W_n(t_0)=V_n(t_0)=0$. Using the known conditions and the mean value theorem, we obtain

$$\begin{aligned}{}^cD^qW_{n+1}(t)&=f(t,x)+g(t,x)\\ &\quad-\left[\sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!}\right.\\ &\quad+\sum_{i=0}^{2k-2}\frac{g^{(i)}(t,\beta_n)(\beta_{n+1}-\beta_n)^i}{i!}\\ &\quad\left.+\frac{g^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\right]\\ &=f(t,x)+g(t,x)\\ &\quad-\left[f(t,\alpha_{n+1})-\frac{f^{(2k)}(t,\delta_1)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\right.\\ &\quad+g(t,\beta_{n+1})-\frac{g^{(2k-1)}(t,\delta_2)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\\ &\quad\left.+\frac{g^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\right]\\ &=f_x(t,\delta_3)(x-\alpha_{n+1})-g_x(t,\delta_4)(\beta_{n+1}-x)\\ &\quad+\frac{f^{(2k)}(t,\delta_1)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)(\beta_n-\beta_{n+1})^{2k-1}(\delta_2-\alpha_n)}{(2k-1)!}\\ &\le f_x(t,\delta_3)W_{n+1}(t)-g_x(t,\delta_4)V_{n+1}(t)\\ &\quad+\frac{f^{(2k)}(t,\delta_1)(x-\alpha_n)^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)(\beta_n-x)^{2k-1}[(\beta_n-x)+(x-\alpha_n)]}{(2k-1)!}\\ &=f_x(t,\delta_3)W_{n+1}-g_x(t,\delta_4)V_{n+1}\\ &\quad+\frac{f^{(2k)}(t,\delta_1)W_n^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)V_n^{2k-1}(V_n+W_n)}{(2k-1)!}\\ &\le k_1W_{n+1}+k_2V_{n+1}+k_3W_n^{2k}+k_4V_n^{2k-1}(V_n+W_n),\end{aligned} \tag{43}$$

where $\alpha_n\le\delta_1\le\alpha_{n+1}\le\delta_3\le x\le\delta_4\le\beta_{n+1}\le\delta_2\le\beta_n$, $\alpha_n\le\delta_5\le\delta_2$, $0\le f_x(t,x)\le k_1$, and $0\le -g_x(t,x)\le k_2$ and it follows from assumption A_2 that $f^{(2k)}$ and $g^{(2k)}$ are bounded on Ω, that is, $0\le f^{(2k)}(t,x)/(2k)!\le k_3$, $0\le -g^{(2k)}(t,x)/(2k-1)!\le k_4$.

Similarly, we have

$$
\begin{aligned}
{}^{c}D^{q}V_{n+1}(t) &= \sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\beta_n)(\beta_{n+1}-\beta_n)^{i}}{i!} \\
&\quad + \frac{f^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&\quad + \sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^{i}}{i!} \\
&\quad - f(t,x)-g(t,x) \\
&= f(t,\beta_{n+1}) - \frac{f^{(2k-1)}(t,\sigma_1)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&\quad + \frac{f^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&\quad + g(t,\alpha_{n+1}) - \frac{g^{(2k)}(t,\sigma_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} \\
&\quad - f(t,x)-g(t,x) = f_x(t,\sigma_3)(\beta_{n+1}-x) \\
&\quad - g_x(t,\sigma_4)(x-\alpha_{n+1}) \\
&\quad + \frac{f^{(2k)}(t,\sigma_5)(\beta_n-\beta_{n+1})^{2k-1}(\sigma_1-\alpha_n)}{(2k-1)!} \\
&\quad - \frac{g^{(2k)}(t,\sigma_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} \\
&\le f_x(t,\sigma_3)V_{n+1} - g_x(t,\sigma_4)W_{n+1} \\
&\quad - \frac{g^{(2k)}(t,\sigma_2)(x-\alpha_n)^{2k}}{(2k)!} \\
&\quad + \frac{f^{(2k)}(t,\sigma_5)(\beta_n-x)^{2k-1}[(\beta_n-x)+(x-\alpha_n)]}{(2k-1)!} \\
&= f_x(t,\sigma_3)V_{n+1} - g_x(t,\sigma_4)W_{n+1} \\
&\quad + \frac{f^{(2k)}(t,\sigma_5)V_n^{2k-1}(V_n+W_n)}{(2k-1)!} \\
&\quad - \frac{g^{(2k)}(t,\sigma_2)W_n^{2k}}{(2k)!} \\
&\le k_1V_{n+1} + k_2W_{n+1} + k_5V_n^{2k-1}(V_n+W_n) \\
&\quad + k_6W_n^{2k},
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
&\alpha_n \le \sigma_2 \le \alpha_{n+1} \le \sigma_4 \le x \le \sigma_3 \le \beta_{n+1} \le \sigma_1 \le \beta_n, \\
&\alpha_n \le \sigma_5 \le \sigma_1, \\
&0 \le \frac{f^{(2k)}(t,x)}{(2k-1)} \le k_5, \\
&0 \le -\frac{g^{(2k)}(t,x)}{(2k)!} \le k_6.
\end{aligned}
\tag{45}
$$

For the following inequalities

$$
\begin{aligned}
{}^{c}D^{q}W_{n+1}(t) &\le k_1W_{n+1} + k_2V_{n+1} \\
&\quad + \left\| k_3W_n^{2k} + k_4V_n^{2k-1}(V_n+W_n) \right\|, \quad W_{n+1}(t_0)=0, \\
{}^{c}D^{q}V_{n+1}(t) &\le k_1V_{n+1} + k_2W_{n+1} \\
&\quad + \left\| k_5V_n^{2k-1}(V_n+W_n) + k_6W_n^{2k} \right\|, \quad V_{n+1}(t_0)=0,
\end{aligned}
\tag{46}
$$

we can get

$$
{}^{c}D^{q}\Phi \le \lambda\Phi + \Psi, \quad \Phi(t_0)=\Psi_0,
\tag{47}
$$

where

$$
\begin{aligned}
\Phi &= \begin{pmatrix} W_{n+1} \\ V_{n+1} \end{pmatrix}, \\
\lambda &= \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}, \\
\Psi &= \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \\
\Psi_0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
\tag{48}
$$

in which

$$
\begin{aligned}
\Psi_1 &= \left\| k_3W_n^{2k} + k_4V_n^{2k-1}(V_n+W_n) \right\|, \\
\Psi_2 &= \left\| k_5V_n^{2k-1}(V_n+W_n) + k_6W_n^{2k} \right\|.
\end{aligned}
\tag{49}
$$

Formulas (4) and (5) and Lemma 9 imply

$$
\Phi(t) \le \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^{q})ds\Psi \le M\Psi,
\tag{50}
$$

where

$$
\begin{aligned}
M &= \begin{pmatrix} \dfrac{(T-t_0)^q}{q}E_{q,q}(k_1(T-t_0)^q) & \dfrac{(T-t_0)^q}{q}E_{q,q}(k_2(T-t_0)^q) \\ \dfrac{(T-t_0)^q}{q}E_{q,q}(k_2(T-t_0)^q) & \dfrac{(T-t_0)^q}{q}E_{q,q}(k_1(T-t_0)^q) \end{pmatrix} \\
&= \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}.
\end{aligned}
\tag{51}
$$

Hence, we have

$$\begin{aligned} W_{n+1}(t) &\le M_1\left\|k_3W_n^{2k} + k_4V_n^{2k-1}(V_n + W_n)\right\| \\ &\quad + M_2\left\|k_5V_n^{2k-1}(V_n + W_n) + k_6W_n^{2k}\right\| \\ &\le M_1\left(k_3\|W_n\|^{2k} + k_4\|V_n\|^{2k-1}(\|V_n\| + \|W_n\|)\right) \\ &\quad + M_2\left(k_5\|V_n\|^{2k-1}(\|V_n\| + \|W_n\|) + k_6\|W_n\|^{2k}\right) \\ &= c_1\|W_n\|^{2k} + c_2\|V_n\|^{2k-1}(\|V_n\| + \|W_n\|), \end{aligned} \tag{52}$$

where $c_1 = M_1k_3 + M_2k_6$, $c_2 = M_1k_4 + M_2k_5$. Similarly, we get

$$V_{n+1}(t) \le c_3\|W_n\|^{2k} + c_4\|V_n\|^{2k-1}(\|V_n\| + \|W_n\|). \tag{53}$$

Employing the Binomial Theorem

$$\begin{aligned} (a+b)^{2k} &= C_{2k}^0a^{2k} + C_{2k}^1a^{2k-1}b + \cdots + C_{2k}^ra^{2k-r}b^r + \cdots + C_{2k}^{2k}b^{2k} \\ &\ge a^{2k} + b^{2k} + a^{2k-1}b, \end{aligned} \tag{54}$$

and the inequalities (52) and (53), we obtain

$$\begin{aligned} W_{n+1} + V_{n+1}(t) &\le (c_1 + c_3)\|W_n\|^{2k} \\ &\quad + (c_2 + c_4)\|V_n\|^{2k-1}(\|V_n\| + \|W_n\|) \\ &= c_5\|W_n\|^{2k} + c_6\|V_n\|^{2k} + c_6\|V_n\|^{2k-1}\|W_n\| \\ &\le c_7(\|W_n\| + \|V_n\|)^{2k}, \end{aligned} \tag{55}$$

that is,

$$\|x(t) - \alpha_{n+1}(t)\| + \|\beta_{n+1}(t) - x(t)\| \le c_7[\|x(t) - \alpha_n(t)\| + \|\beta_{n+1}(t) - x(t)\|]^{2k}, \tag{56}$$

where $c_1 + c_3 = c_5$, $c_2 + c_4 = c_6$, $c_7 = \max\{c_5, c_6\}$.

The following theorem is relative to the coupled lower and upper solutions of type II in Definition 1 when m is odd.

Theorem 11. *Consider the following assumptions:*

D_1 α_0, β_0 are coupled lower and upper solutions of type II of (1) with $\alpha_0 \le \beta_0$ on J.

D_2 $f, g \in C^{2k+1}[\Omega, R]$ such that $f(t,x)$ and $g(t,x)$ are hyperconvex and hyperconcave in x of order $2k$, respectively.

D_3 $f_x(t,x)$ satisfies

$$f_x(t,x) \le -\max\left[f^{(2k+1)}(t,x)\right]\frac{(\beta_0 - \alpha_0)^{2k}}{(2k-1)!} \le 0 \text{ on } \Omega. \tag{57}$$

Then, there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converging uniformly to the solution $x(t)$ of (1) on J and the convergence is of the order $2k$.

Proof. It follows from assumption D_2 that the inequalities (26), (27), (28), and (29) hold. In order to construct the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, consider the following fractional differential equations:

$$\begin{aligned} {}^cD^q\alpha_n(t) &= F(t, \alpha_{n-1}, \beta_{n-1}; \beta_n, \alpha_n) \\ &\triangleq \sum_{i=0}^{2k-1}\frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\ &\quad + \frac{f^{(2k)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k}}{(2k)!} \\ &\quad + \sum_{i=0}^{2k-1}\frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &\quad + \frac{g^{(2k)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k}}{(2k)!}, \quad \alpha_n(t_0) = x_0, \\ {}^cD^q\beta_n(t) &= G(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n, \beta_n) \\ &\triangleq \sum_{i=0}^{2k-1}\frac{f^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &\quad + \frac{f^{(2k)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k}}{(2k)!} \\ &\quad + \sum_{i=0}^{2k-1}\frac{g^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\ &\quad + \frac{g^{(2k)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k}}{(2k)!}, \quad \beta_n(t_0) = x_0. \end{aligned} \tag{58}$$

Similar to the proof of Theorem 10, the conclusion can be obtained. We omit the details.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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Research Article

Exact Combined Solutions for the (2 + 1)-Dimensional Dispersive Long Water-Wave Equations

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The homogeneous balance of undetermined coefficient (HBUC) method is presented to obtain not only the linear, bilinear, or homogeneous forms but also the exact traveling wave solutions of nonlinear partial differential equations. Linear equation is obtained by applying the proposed method to the (2 + 1)-dimensional dispersive long water-wave equations. Accordingly, the multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of the (2 + 1)-dimensional dispersive long water-wave equations are obtained directly. The HBUC method, which can be used to handle some nonlinear partial differential equations, is a standard, computable, and powerful method.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used to describe a variety of phenomena not only in physics [1, 2], thermodynamics [3], fluid dynamics [4, 5], and practical engineering [6, 7] but also in several other fields [8]. How to obtain the traveling wave solutions for NLPDEs is very important in the nonlinear phenomena [1, 9, 10]. In recent decades, there are many excellent methods, such as the (G'/G)-expansion method [11, 12], the homotopy perturbation method [13, 14], the Riccati-Bernoulli sub-ODE method [15, 16], the three-wave method [17, 18], the inverse scattering method [19, 20], the first integral method [21, 22], Hirota's bilinear method [23, 24], the homogeneous balance method [25, 26], the iteration method [27, 28], the tanh-sech method [29, 30], and the extended homoclinic test method [31, 32], which are applied to obtain the exact traveling wave solutions of some NLPDEs.

The above traditional methods can be used to handle some well-known NLPDEs. However, there is no unified approach, which can be dealt with all NLPDEs. To obtain the traveling wave solutions of NLPDEs, Hirota's bilinear method, the three-wave method, and the (G'/G)-expansion method are employed to investigate the traveling wave solutions of many NLPDEs. Unfortunately, some exact solutions are omitted by using Hirota's bilinear method, the three-wave method, and the (G'/G)-expansion method if the NLPDEs can be linearized. To solve this problem, the HBUC method is proposed to derive the linear forms of NLPDEs.

In this paper, the (2 + 1)-dimensional dispersive long water-wave equations (DLWEs) [33, 34] are investigated as follows:

$$u_{yt} + u_{xxy} - 2v_{xx} - \left(u^2\right)_{xy} = 0, \tag{1}$$

$$v_t - v_{xx} - 2(uv)_x = 0, \tag{2}$$

where $u = u(x, y, t)$ represents the surface velocity of water along the x-direction and $v = v(x, y, t)$ gives the surface velocity of water along the y-direction.

The DLWEs can also be derived from the well-known Kadomtsev-Petviashvili equation using the symmetry constraint. The DLWEs were used to model nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth. The DLWEs also appear in many scientific applications such as nonlinear fiber optics, plasma physics, fluid dynamics, and coastal engineering. Moreover, the solutions of the DLWEs are very

helpful for coastal and engineers to apply the nonlinear water model to coastal and harbor design [33].

The DLWEs were investigated where different approaches were exploited. Wu et al. reported that the DLWEs exist of many nonpropagating hydrodynamical solitons both in theory and in experiment, and the DLWEs have no Painleve property, though the system is Lax or inverse scattering transformation integrable [35]. Paquin and Winternitz investigated the DLWEs by the Lie group method [36]. The extended mapping approach [37], the extended projective approach [38], and the tanh-sech method [33] are among many other methods that were used to handle the DLWEs. Much effort has been focused on the existence of propagating solitons [36], multiple soliton solutions, and rational solutions [33].

In this paper, the linear equation of the DLWEs is derived by the HBUC method. Then, the multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of the DLWEs are obtained directly.

The remainder of this paper is organized as follows: the HBUC method is presented in Section 2. In Section 3, the HBUC method is used to obtain N-multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of Equations (1) and (2) directly. In Section 4, some important conclusions are given.

2. Description of the HBUC Method

In this section, the following general NLPDE in two variables is considered:

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{3}$$

where P is a polynomial function of its arguments; the subscripts x and t denote the partial derivatives of u, respectively. The HBUC method consists of three steps as follows:

Step 1. Assume that the Equation (3) has a solution of the following form:

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0, m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{4}$$

where $u = u(x,t)$, $w = w(x,t)$, and $(\ln w)_{i,j} = (\partial^{i+j}(\ln w(x,t)))/\partial x^i \partial t^j$, and m, n (balance numbers) and a_{ij} $(i = 0, 1, \cdots, m;\ j = 0, 1, \cdots, n)$ (balance coefficients) $(a_{mn} \neq 0)$ are constants to be determined later.

The balance numbers can be determined by balancing the highest nonlinear terms and the highest order partial derivative terms. A set of algebraic equations for the balance coefficients is obtained by substituting Equation (4) into Equation (3) and balancing the terms with $(w_x/w)^i (w_t/w)^j$.

Step 2. If the NLPDEs can be linearized, the linear equation can be obtained by solving the set of algebraic equations and simplifying Equation (3) directly or after integrating some time (generally, integrating times equal to the orders of the lowest partial derivative of Equation (3)) with respect to x and t.

Step 3. Based on Step 1 and Step 2, by using traveling wave transformations

$$w(x,t) = w(\xi), \tag{5}$$

and

$$\xi = x - Vt, \tag{6}$$

Equation (3) can be reduced to a linear partial differential equation

$$w_t + \alpha_1 w_{xx} + \alpha_2 w_x + \alpha_3 w, \tag{7}$$

where α_1, α_2, and α_3 are constants. Then, solving the linear partial differential equation (7) yields the exact combined solutions of Equation (3). Next, Equations (1) and (2) are chosen to obtain the combined solutions by applying the HBUC method.

3. Application to the (2 + 1)-Dimensional DLWEs

Assume that the solutions of Equations (1) and (2) are of the forms

$$u = a_{mnl}(\ln w)_{m,n,l} + \sum_{\substack{i,j,k=0 \\ i+j+k\neq 0, m+n+l}}^{i=m,j=n,k=l} a_{ijk}(\ln w)_{i,j,k} + a_{000}, \tag{8}$$

$$v = b_{pqr}(\ln w)_{p,q,r} + \sum_{\substack{i,j,k=0 \\ i+j+k\neq 0, p+q+r}}^{i=p,j=q,k=r} b_{ijk}(\ln w)_{i,j,k} + b_{000}, \tag{9}$$

where $u = u(x,y,t)$, $w = w(x,y,t)$, and $(\ln w)_{i,j,k} = (\partial^{i+j+k}(\ln w(x,y,t)))/\partial x^i \partial y^j \partial t^k$, and m, n, l, p, q, r (balance numbers) and a_{ijk} $(i = 0, 1, \cdots, m;\ j = 0, 1, \cdots, n;\ k = 0, 1, \cdots, l)$ $(a_{mnl} \neq 0)$ and b_{ijk} $(i = 0, 1, \cdots, p;\ j = 0, 1, \cdots, q;\ k = 0, 1, \cdots, r)$ $(b_{pqr} \neq 0)$ (balance coefficients) are constants to be determined later.

Balancing u_{xxy}, v_{xx}, and $(u^2)_{xy}$ in Equation (1) and v_{xx} and $(uv)_x$ in Equation (2), it is required that

$$\begin{aligned} m+2 &= 2m+1 = p+2,\\ n+1 &= 2n+1 = q,\\ l &= 2l = r,\\ p+2 &= m+p+1,\\ q &= q+n,\\ l+r &= l. \end{aligned} \tag{10}$$

Solving the above algebraic equations, we get $m=1$, $n=0, l=0, p=1, q=1, r=0$. Then, Equations (8) and (9) can be written as

$$\begin{aligned} u &= a_1(\ln w)_x + a_0,\\ v &= b_3(\ln w)_{xy} + b_2(\ln w)_x + b_1(\ln w)_y + b_0, \end{aligned} \tag{11}$$

where $a_i (i=0,1)$ and $b_i (i=0,1,2,3)$ are constants to be determined later.

Substituting Equation (11) into Equations (1) and (2) and equating the coefficients of $w_x^3 w_y / w^4$ on the left-hand side of Equations (1) and (2) to zero yield a set of algebraic equations for a_1 and b_3 as follows:

$$\begin{aligned} -6\left(a_1^2 + a_1 - 2b_3\right) &= 0,\\ -6b_3(a_1 - 1) &= 0. \end{aligned} \tag{12}$$

Solving the above algebraic equations and noticing $a_1 b_3 \neq 0$, we get $a_1 = b_3 = 1$. Substituting a_1 and b_3 back into Equation (11), we get

$$\begin{aligned} u &= (\ln w)_x + a_0,\\ v &= (\ln w)_{xy} + b_2(\ln w)_x + b_1(\ln w)_y + b_0. \end{aligned} \tag{13}$$

Substituting Equation (13) back into Equations (1) and (2), Equation (1) minus Equation (2) is

$$\frac{A_1}{w} + \frac{A_2}{w^2} + \frac{A_3}{w^3} = 0, \tag{14}$$

where

$$\begin{aligned} A_1 &= b_2(2a_0 w_{xx} - w_{xxx} - w_{xt}) + b_1\left(2a_0 w_{xy} - w_{xxy} - w_{yt}\right) + 2b_0 w_{xx},\\ A_3 &= -6b_2 w_x^3 - 6b_1 w_x^2 w_y,\\ A_2 &= b_2\left(w_x w_t + 7w_{xx} w_x - 2a_0 w_x^2\right) + b_1\left(3w_{xx} w_y + w_y w_t + 4w_{xy} w_x - 2a_0 w_x w_y\right) - 2b_0 w_x^2. \end{aligned} \tag{15}$$

Obviously, setting $b_i = 0\,(i=0,1,2)$, we find that Equation (1) coincides with Equation (2). According to the above analysis, suppose that the solutions of Equations (1) and (2) are of the forms

$$\begin{aligned} u(x,y,t) &= (\ln w)_x + a_0,\\ v(x,y,t) &= (\ln w)_{xy}, \end{aligned} \tag{16}$$

where a_0 is an arbitrary constant and $w = w(x,y,t)$ is a function of x, y, t that will be determined later.

Substituting Equation (16) into Equations (1) and (2) yields a single NLPDE

$$K_0 + K_1 + K_2 + K_3 = 0, \tag{17}$$

where

$$\begin{aligned} K_0 &= a_0\left(\frac{-2w_{xxy}}{w} + \frac{2w_{xx}w_y + 4w_x w_{xy}}{w^2} - \frac{4w_x^2 w_y}{w^3}\right),\\ K_1 &= \frac{w_{xyt} - w_{xxxy}}{w},\\ K_2 &= \frac{w_x w_{xxy} - w_{xy}w_t - w_{xt}w_y + w_{xx}w_{xy} + w_{xxx}w_y - w_x w_{yt}}{w^2},\\ K_3 &= \frac{2w_x w_y w_t - 2w_x w_{xx} w_y}{w^3}. \end{aligned} \tag{18}$$

Simplifying Equation (17) and integrating with respect to x once, we get

$$\frac{\partial}{\partial x}\left(\frac{w_{yt}w - w_y w_t}{w^2} - \frac{w_{xxy}w - w_{xx}w_y}{w^2} - \frac{2a_0\left(w_{xy}w - w_x w_y\right)}{w^2}\right) = 0. \tag{19}$$

Equation (19) is identical to

$$\begin{aligned} &\left(w_{yt}w - w_y w_t\right) - \left(w_{xxy}w - w_{xx}w_y\right)\\ &\quad - 2a_0\left(w_{xy}w - w_x w_y\right) - p(y,t)w^2 = 0. \end{aligned} \tag{20}$$

where $p(y,t)$ is an arbitrary function of y, t.

Particularly, taking $p(y,t) = 0$ in Equation (20), the bilinear equation of Equations (1) and (2) is obtained as follows:

$$\left(w_{yt}w - w_y w_t\right) - \left(w_{xxy}w - w_{xx}w_y\right) - 2a_0\left(w_{xy}w - w_x w_y\right) = 0. \tag{21}$$

Equation (21) can be written concisely in terms of D-operator as

$$D_y(w_t - w_{xx} - 2a_0 w_x)\cdot w = 0, \tag{22}$$

where

$$D_x^m D_t^n a \cdot b = \left(\partial_x - \partial_{x'}\right)^m \left(\partial_t - \partial_{t'}\right)^n a(x,t) b\left(x', t'\right)\Big|_{x'=x, t'=t}. \tag{23}$$

By using the property of D-operator, Equation (22) is identical to

$$w_t - w_{xx} - 2a_0 w_x - q(x,t)w = 0, \tag{24}$$

where $q(x,t)$ is an arbitrary function of x, t.

Particularly, taking $q(x,t) = \alpha = \text{constant}$ in Equation (24), we get a linear equation

$$w_t - w_{xx} - 2a_0 w_x - \alpha w = 0. \tag{25}$$

Remark 1. We note that Equation (25) does not depend on the variable y, instead it depends only on the variables x, t.

Using the transformation

$$w(x,y,t) = w(\xi) = w(x + l(y) - ct), \tag{26}$$

Equation (25) is reduced to

$$w'' + (2a_0 + c)w' + \alpha w = 0, \tag{27}$$

where $l(y)$ is an arbitrary function of y and the prime denotes the derivation with respect to ξ.

There are three types of traveling wave solutions of Equation (27) as follows.

When $\Delta = (2a_0 + c)^2 - 4\alpha > 0$,

$$w = C_1 e^{\left(\left(-(2a_0+c)+\sqrt{\Delta}\right)/2\right)\xi} + C_2 e^{\left(\left(-(2a_0+c)-\sqrt{\Delta}\right)/2\right)\xi}, \tag{28}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution of Equation (25) as follows:

$$w_1 = \sum_{i=1}^{n_1} \left(C_{1,1i} e^{\left(\left(-(2a_0+c_{1i})+\sqrt{(2a_0+c_{1i})^2-4\alpha}\right)/2\right)\xi_{1i}} + C_{2,1i} e^{\left(\left(-(2a_0+c_{1i})-\sqrt{(2a_0+c_{1i})^2-4\alpha}\right)/2\right)\xi_{1i}} \right), \tag{29}$$

where $\Delta_{1i} = (2a_0 + c_{1i})^2 - 4\alpha > 0$ and $\xi_{1i} = x + l_{1i}(y) - c_{1i}t$; $C_{1,1i}, C_{2,1i}, a_0, \alpha$, and c_{1i} are arbitrary constants; and $l_{1i}(y)$ $(i = 1, 2, \cdots, n_1)$ are arbitrary functions of y.

When $\Delta = (2a_0 + c)^2 - 4\alpha < 0$,

$$w = e^{(-(2a_0+c)/2)\xi} \left(C_1 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) \right), \tag{30}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution as follows:

$$w_2 = \sum_{i=1}^{n_2} e^{(-(2a_0+c_{2i})/2)\xi_{2i}} \left(C_{1,2i} \cos\left(\frac{\sqrt{-\Delta_{2i}}}{2}\xi_{2i}\right) + C_{2,2i} \sin\left(\frac{\sqrt{-\Delta_{2i}}}{2}\xi_{2i}\right) \right), \tag{31}$$

where $\Delta_{2i} = (2a_0 + c_{1i})^2 - 4\alpha < 0$ and $\xi_{2i} = x + l_{2i}(y) - c_{2i}t$; $C_{1,2i}, C_{2,2i}, a_0, \alpha$, and c_{2i} are arbitrary constants; and $l_{2i}(y)$ $(i = 1, 2, \cdots, n_2)$ are arbitrary functions of y.

When $\Delta = (2a_0 + c)^2 - 4\alpha = 0$,

$$w = (C_1 + C_2\xi)e^{(-(2a_0+c)/2)\xi}, \tag{32}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution as follows:

$$w_3 = \sum_{i=1}^{2} (C_{1,3i} + C_{2,3i}\xi_{3i}) e^{(-(2a_0+c_{3i})/2)\xi_{3i}}, \tag{33}$$

where $\Delta_{3i} = (2a_0 + c_{3i})^2 - 4\alpha = 0$ and $\xi_{3i} = x + l_{3i}(y) - c_{3i}t$; $C_{1,3i}, C_{2,3i}, a_0, \alpha$, and c_{3i} are arbitrary constants; and $l_{3i}(y)$ $(i = 1, 2)$ are arbitrary functions of y.

Generally, we can get combined solutions of Equation (25) as follows:

$$w = w_1 + w_2 + w_3, \tag{34}$$

where w_1, w_2, and w_3 are given by Equations (29), (31), and (33), respectively.

Accordingly, we can get combined solutions of Equations (1) and (2)

$$\begin{aligned} w &= w_1 + w_2 + w_3, \\ u(x,y,t) &= (\ln w)_x + a_0, \\ v(x,y,t) &= (\ln w)_{xy} = u_y. \end{aligned} \tag{35}$$

Choosing the appropriate parameters in Equation (35) can obtain all solutions of DLWEs in Ref. [15]. For example, setting $C_{1,2i} = C_{2,2i} = 0$ $(i = 1, 2, \cdots, n_2)$, $C_{1,3i} = C_{2,3i} = 0$ $(i = 1, 2)$, $n_1 = N$, $C_{1,1i} = 1/N$, $C_{2,1i} = 1$, $\alpha = a_0 = 0$, and $l_{1i}(y) = l_{1i}y$, we get the N-kink solutions and the N-soliton solutions

$$u(x,y,t)=\frac{-\sum_{i=1}^{N}c_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}}{1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}},$$
$$v(x,y,t)=-\frac{\left(\sum_{i=1}^{N}c_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)\left(\sum_{i=1}^{N}c_{1i}l_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)}{\left(1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)^{2}}-\frac{\sum_{i=1}^{N}c_{1i}l_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}}{1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}},\tag{36}$$

where c_{1i} and $l_{1i}\,(i=1,\cdots,N)$ are arbitrary constants.

Setting $C_{1,1i}=C_{2,1i}=0\,(i=1,2,\cdots,n_1)$, $C_{1,3i}=C_{2,3i}=0\,(i=1,2)$, $n_2=1$, $\xi_{20}=-\arctan(C_{2,21}/C_{1,21})$, and $l_{2i}(y)=l_{2i}y$, we get the singular solution (periodic solutions)

$$u(x,y,t)=-\frac{\sqrt{4\alpha-(2a_0+c_2)^2}}{2}\tan\cdot\left(\frac{\sqrt{4\alpha-(2a_0+c_2)^2}}{2}(x+l_2y-c_2t)+\xi_{20}\right)-\frac{c_2}{2},$$
$$v(x,y,t)=u_y.\tag{37}$$

where $C_{1,21}, C_{2,21}, l_2, \alpha, a_0$, and c_2 are arbitrary constants.

Setting $C_{1,1i}=C_{2,1i}=0\,(i=1,2,\cdots,n_1)$, $C_{1,2i}=C_{2,2i}=0\,(i=1,2,\cdots,n_2)$, $C_{1,31}=C_1$, $C_{1,32}=C_2$, and $C_{1,32}=C_{2,32}=0$, we get the rational solutions

$$u(x,y,t)=\frac{C_2}{C_1+C_2(x+l_3y-c_3t)}-\frac{c_3}{2},$$
$$v(x,y,t)=u_y,\tag{38}$$

where $C_1, C_2, l_3, \alpha, a_0$, and $c_3((2a_0+c_3)^2-4\alpha=0)$ are arbitrary constants.

Remark 2. We can deal with Equation (25) by using some assumptions. For example, suppose that $w=-t\beta(y)+W(x)$, $\alpha=0$, and $a_0\neq 0$, we get

$$w=-t\beta(y)-\frac{C_1(y)e^{-2a_0x}}{2a_0}-\frac{\beta(y)x}{2a_0}+C_2(y),$$
$$u(x,y,t)=\frac{\beta(y)\left(2a_0^2t+a_0x+1\right)-a_0C_1(y)e^{-2a_0x}-2C_2(y)a_0^2}{\beta(y)(2a_0t+x)+C_1(y)e^{-2a_0x}-2a_0C_2(y)},$$
$$v(x,y,t)=u_y,\tag{39}$$

where $C_1(y)$, $C_2(y)$, and $\beta(y)$ are arbitrary functions of y, and $a_0\neq 0$ is an arbitrary constant.

Suppose that $w=-\beta(y)t+W(x)$ and $\alpha=a_0=0$, we get

$$w=-t\beta(y)-\frac{\beta(y)x^2}{2}+xC_1(y)+C_2(y),$$
$$u(x,y,t)=\frac{2x\beta(y)-2C_1(y)}{\beta(y)(x^2+2t)-2xC_1(y)-2C_2(y)},$$
$$v(x,y,t)=u_y,\tag{40}$$

where $C_1(y)$, $C_2(y)$, and $\beta(y)$ are arbitrary functions of y.

Similarly, when $\alpha=a_0=0$, Equation (25) is reduced to $w_t-w_{xx}=0$. We can get the exact solution

$$w=C_1(y)-\frac{C_2(y)}{\sqrt{t}}e^{-\left(x^2/4t\right)},$$
$$u(x,y,t)=-\frac{xC_2(y)e^{-\left(x^2/4t\right)}}{2t\left(\sqrt{t}C_1(y)+C_2(y)e^{-\left(x^2/4t\right)}\right)},$$
$$v(x,y,t)=u_y,\tag{41}$$

where $C_1(y)$ and $C_2(y)$ are arbitrary functions of y.

Similarly, we can assume that $w=\sum_{i=1}^{n}p_i(x)q_i(t)$; then, a new solution of Equation (25) can be obtained. Being similar to the above process, we omit it.

4. Conclusions

The (2 + 1)-dimensional dispersive long water-wave equations can be linearized by the HBUC method. Then, the N-multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of Equations (1) and (2) can be obtained. Many well-known NLPDEs, such as the Whitham-Broer-Kaup equations, the Broer-Kaup equations, and the variant Boussinesq equations, can be handled by the HBUC method. The performance of the HBUC method is found to be simple and efficient. The HBUC method is also a standard, computable, and powerful method, which allows us to solve complicated and tedious algebraic calculations by the availability of computer systems like Maple.

Data Availability

The authors confirm that the data supporting the findings of this article are available within the article and are available on request from the corresponding author.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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Research Article

Bilinear Equation of the Nonlinear Partial Differential Equation and Its Application

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The homogeneous balance of undetermined coefficient method is firstly proposed to derive a more general bilinear equation of the nonlinear partial differential equation (NLPDE). By applying perturbation method, subsidiary ordinary differential equation (sub-ODE) method, and compatible condition to bilinear equation, more exact solutions of NLPDE are obtained. The KdV equation, Burgers equation, Boussinesq equation, and Sawada-Kotera equation are chosen to illustrate the validity of our method. We find that the underlying relation among the (G'/G)-expansion method, Hirota's method, and HB method is a bilinear equation. The proposed method is also a standard and computable method, which can be generalized to deal with other types of NLPDE.

1. Introduction

The nonlinear partial differential equation (NLPDE) is known to describe a wide variety of phenomena not only in physics but also in biology, chemistry, and several other fields [1–3]. The investigation of the exact solutions for NLPDE plays an important role on the study of nonlinear physical phenomena [4–9]. In recent years, many powerful methods are used to obtain the exact solutions of NLPDE, for example, the inverse scattering method [7], Bäcklund and Darboux transformation method [8], homotopy perturbation method [9], first integral method [10–12], variational iteration method [13, 14], sub-ODE method [15–17], Jacobi elliptic function method [18], tanh-sech method [19], (G'/G)-expansion method [20, 21], Hirota's method [22–24], and homogeneous balance (HB) method [25–27].

As the direct methods, the (G'/G)-expansion method, Hirota's method, and HB method are very effective for constructing the exact solutions of NLPDE. Exact traveling wave solutions, N-soliton solutions, and solitary wave solutions of some NLPDE are obtained by using the above three methods [20–30]. Fan improved the HB method to investigate the BT, Lax pairs, symmetries, and exact solutions for some NLPDE [31, 32]. He also showed that there are many links among the HB method, Weiss-Tabor-Carnevale method, and Clarkson-Kruskal method.

However, there is no unified direct method which can be used to deal with all types of NLPDE. And also, no literature is available to illustrate the underlying relations among the three direct methods.

In the present paper, by improving some key steps in the HB method [26], we propose a new method, HB of undetermined coefficient method, which can be used to derive the bilinear equation of NLPDE. Based on the bilinear equation, by applying the perturbation method, sub-ODE method, and compatible condition, more exact solutions of NLPDE are obtained. We illustrate the real meaning of balance numbers. We show the underlying relations among the (G'/G)-expansion method, Hirota's method, and HB method.

This paper is organized as follows: the HB of undetermined coefficient method is described in Section 2. In Sections 3 and 4, the KdV equation and Burgers equation are chosen as examples to illustrate the method, respectively. In Section 5, the bilinear equations of Boussinesq equation and Sawada-Kotera equation are derived, respectively. Some brief conclusions are given in Section 6.

2. Description of the HB of Undetermined Coefficient Method

Let us consider a general NLPDE, say, in two variables

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{1}$$

where P is a polynomial function of its arguments and the subscripts denote the partial derivatives. The HB of undetermined coefficient method consists of three steps.

Step 1. Suppose that the solution of Equation (1) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0, m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{2}$$

where $u = u(x,t)$, $w = w(x,t)$, $(\ln w)_{i,j} = \partial^{i+j}(\ln w(x,t))/\partial x^i \partial t^j$, m, n (balance numbers), and $a_{ij}(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n)$ (balance coefficients) are constants to be determined later. By balancing the highest nonlinear terms and the highest order partial derivative terms, balance numbers are obtained. Substituting Equation (2) into Equation (1) and balancing the terms with $(w_x/w)^i(w_t/w)^j$ yield a set of algebraic equations for $a_{ij}(i = 1, \cdots, m, j = 1, \cdots, n)$.

Step 2. Solving the set of algebraic equations and simplifying Equation (1), we can get the bilinear equation of Equation (1) directly or after integrating some times (Generally, integrating times equal to the orders of the lowest partial derivative of Equation (1).) with respect to x, t.

Step 3. Generally, in order to obtain the exact solutions of Equation (1), there are three methods to deal with the bilinear equation of Equation (1):

(i) Applying the perturbation method to the bilinear equation of Equation (1), N-soliton solution of Equation (1) can be obtained.

(ii) By using traveling wave transformations

$$w(x,t) = w(\xi), \quad \xi = x - Vt, \tag{3}$$

the bilinear equation of Equation (1) satisfies the following ODE:

$$w'' + \lambda w' + \mu w = 0, \tag{4}$$

where the prime denotes the derivation with respect to ξ and λ, μ, and V are constants to be determined later.

Substituting Equations (3) and (4) into Equation (1), it is converted into the following equation:

$$l_1 w^2 + l_2 w w' + l_3 w'^2 = 0, \tag{5}$$

where l_1, l_2, and l_3 are polynomial functions of V, λ, and μ.

Setting $l_1 = l_2 = l_3 = 0$ yields a set of algebraic equations for V, λ, and μ. Solving the set of algebraic equations and using the solutions of Equation (4), w can be determined. Substituting w into Equation (2), the exact traveling wave solutions of Equation (1) are obtained.

(iii) By applying the compatible condition $(w_{xt} = w_{tx})$ to the bilinear equation of Equation (1), it is reduced to an ODE. Solving the ODE, more exact solutions of Equation (1) can be obtained.

Next, we choose the KdV equation and Burgers equation to illustrate our method.

3. Application to the KdV Equation

Let us consider the celebrated KdV equation in the form

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{6}$$

where δ is a constant. Suppose that the solution of Equation (6) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0, m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{7}$$

where m, n, and $a_{ij}\,(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n)$ are constants to be determined later.

Balancing u_{xxx} and uu_x in Equation (6), it is required that $m + 3 = 2m + 1$ and $n = 2n$. Then, Equation (7) can be written as

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}. \tag{8}$$

From Equation (8), one can calculate the following derivatives:

$$u = a_{20}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right) + a_{10}\frac{w_x}{w} + a_{00},$$

$$u_x = a_{20}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right),$$

$$u_t = a_{20}\left(\frac{w_{xxt}}{w} - \frac{w_{xx}w_t + 2w_x w_{xt}}{w^2} + 2\frac{w_x^2 w_t}{w^3}\right) + a_{10}\left(\frac{w_{xt}}{w} - \frac{w_x w_t}{w^2}\right),$$

$$u_{xxx} = a_{20}\left(\frac{w_{xxxxx}}{w} - \frac{5w_{xxxx}w_x + 10w_{xxx}w_{xx}}{w^2} + \frac{20w_{xxx}w_x^2 + 30w_{xx}^2w_x}{w^3} - \frac{60w_{xx}w_x^3}{w^4} + \frac{24w_x^5}{w^5}\right) + a_{10}\left(\frac{w_{xxxx}}{w} - \frac{4w_{xxx}w_x + 3w_{xx}^2}{w^2} + \frac{12w_{xx}w_x^2}{w^3} - \frac{6w_x^4}{w^4}\right). \tag{9}$$

Equating the coefficients of $(w_x/w)^5$ and $(w_x/w)^4$ on the left-hand side of Equation (6) to zero yields a set of algebraic equations for a_{20} and a_{10} as follows:

$$\begin{aligned} -2a_{20}^2 + 24\delta a_{20} &= 0,. \\ 3a_{20}a_{10} - 6\delta a_{10} &= 0. \end{aligned} \tag{10}$$

Solving the above algebraic equations, we get $a_{20} = 12\delta$ and $a_{10} = 0$. Substituting a_{20} and a_{10} back into Equation (8), we get

$$u = 12\delta(\ln w)_{xx} + a_{00}. \tag{11}$$

where a_{00} is an arbitrary constant. Substituting Equation (11) into Equation (6), we get

$$12\delta(K_1 + K_2 + K_3) = 0, \tag{12}$$

where

$$\begin{aligned} K_1 &= \frac{w_{xxt}}{w} - \frac{2w_xw_{xt} + w_{xx}w_t}{w^2} + \frac{2w_x^2w_t}{w^3}, \\ K_2 &= a_{00}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right), \\ K_3 &= \delta\left(\frac{w_{xxxxx}}{w} + \frac{2w_{xxx}w_{xx} - 5w_{xxxx}w_x}{w^2} + \frac{16w_{xxx}w_x^2 - 6w_xw_{xx}^2}{w^3}\right). \end{aligned} \tag{13}$$

Simplifying Equation (12) and integrating once with respect to x, we get

$$\frac{\partial}{\partial x}\left(\frac{(w_{xt}w - w_xw_t) + \delta\left(w_{xxxx}w - 4w_xw_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right)}{w^2}\right) = 0. \tag{14}$$

Equation (14) is identical to

$$(w_{xt}w - w_xw_t) + \delta\left(w_{xxxx}w - 4w_xw_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right) - C(t)w^2 = 0, \tag{15}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, taking $C(t)$ as zero in Equation (15), we get the bilinear equation of Equation (6) as follows:

$$(w_{xt}w - w_xw_t) + \delta\left(w_{xxxx}w - 4w_xw_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right) = 0. \tag{16}$$

Equation (16) can be written concisely in terms of D-operators as

$$\left(D_xD_t + \delta D_x^4 + a_{00}D_x^2\right)w \cdot w = 0, \tag{17}$$

where

$$D_x^mD_t^n a \cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b\left(x', t'\right)\Big|_{x'=x,t'=t}. \tag{18}$$

Remark 1. Applying Hirota's method [22–24], the bilinear equation of Equation (6) can be written as

$$\left(D_xD_t + \delta D_x^4\right)w \cdot w = 0. \tag{19}$$

Equation (19) is obtained by setting $a_{00} = 0$ in Equation (17). Obviously, Equation (19) is a special case of Equation (17).

(i) Now, we apply the perturbation method to Equation (17) to derive N-soliton solution of Equation (6). Suppose that w can be expanded as follows:

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \varepsilon^N w_N + \cdots, \tag{20}$$

where ε is a parameter and $w_i = w_i(x,t)$ $(i = 1, 2, \cdots)$.

Substituting Equation (20) into Equation (17) and arranging it at each order of ε, we get

$$\begin{aligned} \varepsilon &: D_x\left(D_t + a_{00}D_x + \delta D_x^3\right)(w_1 \cdot 1 + 1 \cdot w_1) = 0, \\ \varepsilon^2 &: D_x\left(D_t + a_{00}D_x + \delta D_x^3\right)(w_2 \cdot 1 + w_1 \cdot w_1 + 1 \cdot w_2) = 0, \\ \varepsilon^3 &: D_x\left(D_t + a_{00}D_x + \delta D_x^3\right)(w_3 \cdot 1 + w_2 \cdot w_1 + w_1 \cdot w_2 + 1 \cdot w_3) = 0, \\ &\cdots \end{aligned} \tag{21}$$

The order-ε equation can be rewritten as a linear differential equation for w_1 as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+a_{00}\frac{\partial}{\partial x}+\delta\frac{\partial^3}{\partial x^3}\right)w_1=0. \tag{22}$$

Solving Equation (22), we get

$$w_1=e^{P_1x-\left(a_{00}P_1+\delta P_1^3\right)t}, \tag{23}$$

where P_1 is an arbitrary constant.

The coefficient of ε^2 can be rearranged as follows:

$$\begin{aligned}2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+a_{00}\frac{\partial}{\partial x}+\delta\frac{\partial^3}{\partial x^3}\right)w_2\\ =-D_x\left(D_t+a_{00}D_x+\delta D_x^3\right)w_1\cdot w_1.\end{aligned} \tag{24}$$

Substituting Equation (23) into Equation (24), the right-hand side of Equation (24) equals zero. Therefore, we can choose

$$w_2=0. \tag{25}$$

Substituting Equations (23) and (25) into Equation (20), we get

$$w=1+e^{P_1x-\left(a_{00}P_1+\delta P_1^3\right)t+\xi_1^0}, \tag{26}$$

where P_1, a_{00}, and ξ_1^0 are arbitrary constants.

Substituting Equation (26) into Equation (11), 1-soliton solution of Equation (6) can be obtained. If we choose $w_1=e^{P_1x-(a_{00}P_1+\delta P_1^3)t}+e^{P_2x-(a_{00}P_2+\delta P_2^3)t}$ in Equation (22), similar to the process of obtaining 1-soliton solution, we can get 2-soliton solution of Equation (6) as follows:

$$w=1+e^{\eta_1}+e^{\eta_2}+\frac{\left(P_1-P_2\right)^2}{\left(P_1-P_2\right)^2}e^{\eta_1+\eta_2}, \tag{27}$$

where $\eta_1=P_1x-(a_{00}P_1+\delta P_1^3)t+\xi_1^0$, $\eta_2=P_2x-(a_{00}P_2+\delta P_2^3)t+\xi_2^0$, P_i,ξ_i^0 $(i=1,2)$, and a_{00} are arbitrary constants.

Substituting Equation (27) into Equation (11), 2-soliton solution of Equation (6) can be obtained. Similarly, we can get N-soliton solution of Equation (6).

Remark 2. Obviously, setting $a_{00}=0$ in Equations (23) and (27), 1-soliton and 2-soliton solutions of Equation (6) are identical to Hirota's results [22–24].

Remark 3. By using the properties of D-operators [22–24], a Bäcklund transformation of Equation (17) can be obtained as follows:

$$\begin{aligned}\left(D_t+(a_{00}+\alpha)D_x+\delta D_x^3\right)w^*\cdot w=0,\\ \left(D_x^2-\beta D_x-\alpha\right)w^*\cdot w=0,\end{aligned} \tag{28}$$

where w^* and w satisfy Equation (17) and α, β, and a_{00} are arbitrary constants.

(ii) Now, we discuss Equation (16) by using the sub-ODE method.

Using transformations $w(x,t)=w(\xi),\xi=x-Vt$, Equation (16) is reduced to

$$(a_{00}-V)\left(w''w-w'^2\right)+\delta\left(w''''w-4w'w'''+3w''^2\right)=0, \tag{29}$$

where the prime denotes the derivation with respect to ξ and V is a constant to be determined later.

Noticing the bilinear property of Equation (16), suppose that w satisfies the following ODE:

$$w''+\lambda w'+\mu w=0, \tag{30}$$

where λ and μ are parameters.

Substituting Equation (30) into Equation (16), we get

$$l_1w^2+l_2ww'+l_3w'^2=0, \tag{31}$$

where

$$\begin{aligned}l_1&=\mu\left(V-a_{00}+\delta\left(4\mu-\lambda^2\right)\right),\\ l_2&=\lambda\left(V-a_{00}+\delta\left(4\mu-\lambda^2\right)\right),\\ l_3&=V-a_{00}+\delta\left(4\mu-\lambda^2\right).\end{aligned} \tag{32}$$

Setting $l_1=l_2=l_3=0$ yields a set of algebraic equations for V, λ, and μ. Solving this set of algebraic equations, we get

$$V=a_{00}+\delta\left(\lambda^2-4\mu\right), \tag{33}$$

where λ, μ, and a_{00} are arbitrary constants.

Substituting Equation (30) into Equation (11), we get

$$u=12\delta\left(\frac{w_x}{w}+\frac{\lambda}{2}\right)^2+3\delta\left(\lambda^2-4\mu\right)+a_{00}. \tag{34}$$

Substituting the general solutions of Equation (30) into Equation (34), we get three types of traveling wave solutions of Equation (6) as follows.

When $\lambda^2-4\mu>0$,

$$\begin{aligned}u_1(x,t)=u_1(\xi)=-3\delta A\left(\frac{C_1e^{(\sqrt{A}/2)\xi}-C_2e^{-(\sqrt{A}/2)\xi}}{C_1e^{(\sqrt{A}/2)\xi}-C_2e^{-(\sqrt{A}/2)\xi}}\right)^2\\ +3\delta A+a_{00},\end{aligned} \tag{35}$$

where

$$A = \lambda^2 - 4\mu,$$
$$V = a_{00} + A\delta, \tag{36}$$
$$\xi = x - (a_{00} + A\delta)t,$$

and λ, μ, C_1, C_2, and a_{00} are arbitrary constants.

Taking $C_1 = C_3 + C_4/2$ and $C_2 = C_3 - C_4/2$, Equation (35) can be rewritten as

$$u_2(x,t) = u_2(\xi) = -3\delta A\left(\frac{C_3 \sinh\left(\sqrt{A}/2\right)\xi + C_4 \cosh\left(\sqrt{A}/2\right)\xi}{C_3 \cosh\left(\sqrt{A}/2\right)\xi + C_4 \sinh\left(\sqrt{A}/2\right)\xi}\right)^2 + 3\delta A + a_{00}, \tag{37}$$

where C_3, C_4, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36).

In particular, if $|C_4/C_3| < 1$, then Equation (37) is reduced to

$$u_3(x,t) = u_3(\xi) = 3\delta A \operatorname{sech}^2\left(\frac{\sqrt{A}}{2}\xi + \xi_0\right) + a_{00}, \tag{38}$$

where C_3, C_4, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36), $\xi_0 = \operatorname{arctanh}(C_4/C_3)$.

When $\lambda^2 - 4\mu < 0$,

$$u_4(x,t) = u_4(\xi) = 3\delta A\left(\frac{-C_1 \sin\left(\sqrt{-A}/2\right)\xi + C_2 \cos\left(\sqrt{-A}/2\right)\xi}{C_1 \cos\left(\sqrt{-A}/2\right)\xi + C_2 \sin\left(\sqrt{-A}/2\right)\xi}\right)^2 + 3\delta A + a_{00}, \tag{39}$$

where C_1, C_2, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36).

Obviously, Equation (39) can be written as

$$u_5(x,t) = u_5(\xi) = 3\delta A \sec^2\left(\frac{\sqrt{-A}}{2}\xi + \xi_0\right) + a_{00}, \tag{40}$$

where C_1, C_2, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36), $\xi_0 = -\arctan(C_2/C_1)$.

When $\lambda^2 - 4\mu = 0$,

$$u_6(x,t) = u_6(\xi) = -12\delta\left(\frac{C_2}{C_1 + C_2\xi}\right)^2 + a_{00}, \tag{41}$$

where $V = a_{00}$, $\xi = x - a_{00}t$, C_1, C_2, and a_{00} are arbitrary constants.

(iii) Now, we discuss Equation (16) from the compatible condition. Equation (16) can be written as

$$(\delta w_{xxxx} + w_{xt} + a_{00}w_{xx})w + \left(\delta\left(3w_{xx}^2 - 4w_{xxx}w_x\right) - a_{00}w_x^2 - w_x w_t\right) = 0. \tag{42}$$

Notice $w_x \neq 0$; otherwise, we can only get a trivial solution. Setting the second term of Equation (42) to zero and solving w_t yield

$$w_t = \frac{\delta\left(3w_{xx}^2 - 4w_{xxx}w_x\right) - a_{00}w_x^2}{w_x}. \tag{43}$$

Substituting Equation (43) into Equation (42), we get

$$w_{xxxx} + \frac{w_{xx}^3 - 2w_x w_{xx} w_{xxx}}{w_x^2} = 0. \tag{44}$$

Integrating Equation (44) once with respect to x, we get

$$w_{xxx} - \frac{w_{xx}^2}{w_x} = b(t), \tag{45}$$

where $b(t)$ is an arbitrary function of t.

Using transformations $Y = w_{xx}^2$ and $X = w_x$, Equation (45) is reduced to

$$\frac{dY}{dX} = \frac{2Y}{X} + 2b(t). \tag{46}$$

Solving the above equation, we get

$$Y = c(t)X^2 - 2b(t)X, \tag{47}$$

namely,

$$w_{xx} = \sqrt{c(t)w_x^2 - 2b(t)w_x}, \tag{48}$$

where $b(t)$ and $c(t)$ are arbitrary functions of t.

Case 1. When $b(t) = c(t) = 0$, from Equation (48), we get

$$w = c_1(t)x + c_2(t), \tag{49}$$

where $c_1(t)$ and $c_2(t)$ are arbitrary functions of t.

Substituting the above equation into Equation (48), we get

$$x\frac{dc_1(t)}{dt} + \frac{dc_2(t)}{dt} = -a_{00}c_1(t). \tag{50}$$

Setting the coefficients of $x^i (i = 0, 1)$ to zero in the above equation, we get

$$\frac{dc_1(t)}{dt} = 0,$$
$$\frac{dc_2(t)}{dt} = -a_{00}c_1(t). \tag{51}$$

Solving the above equations, we get

$$c_1(t) = C_1,$$
$$c_2(t) = C_2 - a_{00}C_1 t, \tag{52}$$

where $C_i\,(i = 1, 2)$ are arbitrary constants. Then, we get

$$w = C_1(x - a_{00}t) + C_2. \tag{53}$$

Substituting Equation (53) into Equation (11), we get an exact solution of Equation (6) as follows:

$$u_7(x,t) = a_{00} - \frac{12\delta C_1^2}{(C_1(x - a_{00}t) + C_2)^2}, \tag{54}$$

where $C_i\,(i = 1, 2)$ and a_{00} are arbitrary constants.

Case 2. When $b(t) = 0$ and $c(t) > 0$, similar to Case 1, we get

$$w = C_3 + C_2 e^{C_1\left(x - \left(a_{00} + \delta C_1^{\,2}\right)t\right)}, \tag{55}$$

and an exact solution of Equation (6) as follows:

$$u_8(x,t) = \frac{12\delta C_1^2 C_2 C_3 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)}}{\left(C_3 + C_2 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)}\right)^2} + a_{00}, \tag{56}$$

where $C_i\,(i = 1, 2, 3)$ and a_{00} are arbitrary constants.

Case 3. When $b(t) \neq 0$ and $c(t) = 0$ similar to Case 1, we get

$$w = C_1(x - a_{00}t + C_2)^3 + 12\delta C_1 t + C_3, \tag{57}$$

and an exact solution of Equation (6) as follows:

$$u_9(x,t) = a_{00} - \frac{36\delta C_1(x - a_{00}t + C_2)\left(C_1(x - a_{00}t + C_2)^3 - 24\delta C_1 t - 2C_3\right)}{\left(C_1(x - a_{00}t + C_2)^3 - 12\delta C_1 t + C_3\right)^2}, \tag{58}$$

where $C_i\,(i = 1, 2, 3)$ and a_{00} are arbitrary constants.

Case 4. When $b(t) \neq 0$ and $c(t) > 0$, similar to Case 1, we get

$$w = C_4 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5, \tag{59}$$

and an exact solution of Equation (6) as follows:

$$u_{10}(x,t) = 12\delta(A_1 - B_1) + a_{00}, \tag{60}$$

where

$$A_1 = \frac{C_1^2 C_4 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_1^2 C_3 e^{-C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)}}{C_4 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5},$$
$$B_1 = \frac{\left(C_1 C_4 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} - C_1 C_3 e^{-C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_2\right)^2}{\left(C_4 e^{C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x - \left(a_{00} + \delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5\right)^2}, \tag{61}$$

where $C_2^2 + 4C_1^2 C_3 C_4 = 0$, $C_i\,(i = 1, 2, 3, 4, 5)$, and a_{00} are arbitrary constants.

Case 5. When $b(t) \neq 0$ and $c(t) < 0$, similar to Case 1, we get

$$w = -\frac{C_1 C_2 x + C_1 \cos\left(C_2\left(x + \delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1 C_2^3 t - a_{00}C_1 C_2 t - C_4 C_2^3}{C_2^3}, \tag{62}$$

and an exact solution of Equation (6) as follows:

$$u_{11}(x,t) = 12\delta(A_2 - B_2) + a_{00}, \tag{63}$$

where

$$\begin{aligned} A_2 &= \frac{-C_1C_2^2\cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right)}{C_1C_2x + C_1\cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1C_2^3 t - C_1C_2a_{00}t - C_2^3C_4}, \\ B_2 &= \frac{C_1^2C_2^2\left(-1+\sin\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right)\right)^2}{\left(C_1C_2x + C_1\cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1C_2^3 t - a_{00}C_1C_2t - C_2^3C_4\right)^2}, \end{aligned} \tag{64}$$

where $C_i\,(i=1,2,3,4)$ and a_{00} are arbitrary constants.

So far, based on the bilinear equation which is derived by using the HB of undetermined coefficient method, many exact solutions of the KdV are obtained by applying the perturbation method, sub-ODE method, and compatible condition. Our results can compare with the (G'/G)-expansion method, Hirota's method, and HB method [20–30].

4. Application to the Burgers Equation

Let us consider the Burgers equation in the form

$$u_t + uu_x + \delta u_{xx} = 0, \tag{65}$$

where δ is a constant.

Suppose that the solution of Equation (65) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0,m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{66}$$

where m, n, and $a_{ij}\,(i=0,1,\cdots,m, j=0,1,\cdots,n)$ are constants to be determined later.

Balancing u_{xx} and uu_x in Equation (65), it is required that $m+2 = 2m+1$ and $n = 2n$. Then, Equation (66) can be written as

$$u = a_{10}(\ln w)_x + a_{00}. \tag{67}$$

From Equation (67), one can calculate the following derivatives:

$$\begin{aligned} u &= a_{10}\frac{w_x}{w} + a_{00}, \\ u_x &= a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right), \\ u_t &= a_{10}\left(\frac{w_{xt}}{w} - \frac{w_xw_t}{w^2}\right), \\ u_{xx} &= a_{10}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right). \end{aligned} \tag{68}$$

Equating the coefficients of $(w_x/w)^3$ on the left-hand side of Equation (65) to zero yields an algebraic equation for a_{10} as follows:

$$-a_{10}^2 + 2a_{10}\delta = 0. \tag{69}$$

Solving the above algebraic equation, we get $a_{10} = 2\delta$. Substituting a_{10} back into Equation (67), we get

$$u = 2\delta(\ln w)_x + a_{00}, \tag{70}$$

where a_{00} is an arbitrary constant.

Substituting Equation (70) into Equation (65), we get

$$2\delta(K_1 + K_2 + K_3) = 0, \tag{71}$$

where

$$\begin{aligned} K_1 &= \frac{w_{xt}w - w_xw_t}{w^2}, \\ K_2 &= a_{00}\left(\frac{w_{xx}w - w_x^2}{w^2}\right), \\ K_3 &= \delta\left(\frac{ww_{xxx} - w_xw_{xx}}{w^2}\right). \end{aligned} \tag{72}$$

Simplifying Equation (71), we get

$$(w_{xt}w - w_xw_t) + a_{00}\left(w_{xx}w - w_x^2\right) + \delta(ww_{xxx} - w_xw_{xx}) = 0. \tag{73}$$

Equation (73) can be written concisely in terms of D-operators as

$$D_x(w_t + \delta w_{xx} + a_{00}w_x)\cdot w = 0. \tag{74}$$

Equation (74) is identical to

$$w_t + \delta w_{xx} + a_{00}w_x - C(t)w = 0, \tag{75}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, taking $C(t)$ as constant C in Equation (75), we get

$$w_t + \delta w_{xx} + a_{00} w_x - Cw = 0. \tag{76}$$

Remark 4. Applying Hirota's method [22–24], the bilinear equation of Equation (74) can be written as

$$D_x(w_t + \delta w_{xx}) \cdot w = 0. \tag{77}$$

Equation (77) is obtained by setting $a_{00} = 0$ in Equation (74). Obviously, Equation (77) is a special case of Equation (74).

Remark 5. Equations (70) and (76) are general Cole-Hopf transformations. In fact, setting $a_{00} = C = 0$ in Equations (70) and (76), we get the famous Cole-Hopf transformations

$$u = 2\delta(\ln w)_x, w_t + \delta w_{xx} = 0. \tag{78}$$

(i) Now, we apply the perturbation method to Equation (74) to derive N-soliton solution of Equation (65). Suppose that w can be expanded as follows:

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \varepsilon^N w_N + \cdots, \tag{79}$$

where ε is a parameter and $w_i = w_i(x, t)$ $(i = 1, 2, \cdots)$.

Substituting Equation (79) into Equation (74) and arranging it at each order of ε, we get

$$\begin{aligned}
&\varepsilon : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_1 \cdot 1 + 1 \cdot w_1) + \delta D_x w_{1xx} \cdot 1 = 0,\\
&\varepsilon^2 : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_2 \cdot 1 + w_1 \cdot w_1 + 1 \cdot w_2)\\
&\quad + \delta D_x(w_{1xx} \cdot w_1 + w_{2xx} \cdot 1) = 0,\\
&\varepsilon^3 : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_3 \cdot 1 + w_2 \cdot w_1 + w_1 \cdot w_2 + 1 \cdot w_3)\\
&\quad + \delta D_x(w_{1xx} \cdot w_2 + w_{2xx} \cdot w_1 + w_{3xx} \cdot 1) = 0,\\
&\cdots
\end{aligned} \tag{80}$$

The order-ε equation can be rewritten as a linear differential equation for w_1 as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + a_{00}\frac{\partial}{\partial x} + \delta\frac{\partial^2}{\partial x^2}\right)w_1 = 0. \tag{81}$$

Solving Equation (81), we get

$$w_1 = e^{P_1 x - \left(a_{00}P_1 + \delta P_1^2\right)t}, \tag{82}$$

where P_1 is an arbitrary constant.

The coefficient of ε^2 can be rearranged as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + a_{00}\frac{\partial}{\partial x} + \delta\frac{\partial^2}{\partial x^2}\right)w_2\\
&\quad = -\frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)w_1 \cdot w_1 - \delta D_x w_{1xx} \cdot w_1.
\end{aligned} \tag{83}$$

Substituting Equation (82) into Equation (83), the right-hand side of Equation (83) equals zero. Therefore, we can choose

$$w_2 = 0. \tag{84}$$

Substituting Equations (82) and (84) into Equation (79), we get

$$w = 1 + e^{P_1 x - \left(a_{00}P_1 + \delta P_1^2\right)t + \xi_1^0}, \tag{85}$$

where P_1, a_{00}, and $\xi_1{}^0$ are arbitrary constants.

Substituting Equation (85) into Equation (70), 1-soliton solution of Equation (65) can be obtained.

If we choose $w_1 = e^{P_1 x - (a_{00}P_1 + \delta P_1^2)t} + e^{P_2 x - (a_{00}P_2 + \delta P_2^2)t}$ in Equation (81), Equation (74) has no 2-soliton solution. But there exists a solution as follows:

$$w = 1 + \sum_{i=1}^{n} e^{\eta_1}, \tag{86}$$

where $\eta_i = P_i x - (a_{00}P_i + \delta P_i^2)t + \xi_1^0$, P_i, ξ_1^0 $(i = 1, 2, \cdots, n)$, and a_{00} are arbitrary constants.

(ii) Now, we discuss Equation (73) by using the sub-ODE method.

Using transformations $w(x, t) = w(\xi)$, $\xi = x - Vt$, Equation (73) is reduced to

$$(a_{00} - V)\left(w''w - w'^2\right) + \delta\left(ww''' - w'w''\right) = 0, \tag{87}$$

where the prime denotes the derivation with respect to ξ and V is a constant to be determined later.

Noticing the bilinear property of Equation (87), suppose that w satisfies the following ODE:

$$w'' + \lambda w' + \mu w = 0, \tag{88}$$

where λ and μ are parameters.

Substituting Equation (88) into Equation (87), we get

$$l_1 w^2 + l_2 w w' + l_3 w'^2 = 0, \tag{89}$$

where

$$\begin{aligned} l_1 &= \mu(\delta\lambda + V - a_{00}), \\ l_2 &= \lambda(\delta\lambda + V - a_{00}), \\ l_3 &= \delta\lambda + V - a_{00}. \end{aligned} \tag{90}$$

Setting $l_1 = l_2 = l_3 = 0$ yields a set of algebraic equations for V, λ, and μ. Solving the set of algebraic equations, we get

$$V = a_{00} - \delta\lambda, \tag{91}$$

where λ, μ, and a_{00} are arbitrary constants.

Substituting the general solutions of Equation (88) into Equation (70), we get three types of traveling wave solutions of Equation (65) as follows.

When $\lambda^2 - 4\mu > 0$,

$$u_1(x,t) = u_1(\xi) = \delta\sqrt{A}\left(\frac{C_1 e^{(\sqrt{A}/2)\xi} - C_2 e^{-(\sqrt{A}/2)\xi}}{C_1 e^{(\sqrt{A}/2)\xi} + C_2 e^{-(\sqrt{A}/2)\xi}}\right) + a_{00} - \delta\lambda, \tag{92}$$

where

$$\begin{aligned} A &= \lambda^2 - 4\mu, \\ V &= a_{00} - \delta\lambda, \\ \xi &= x - (a_{00} - \delta\lambda)t, \end{aligned} \tag{93}$$

where C_1, C_2, λ, μ, and a_{00} are arbitrary constants.

Taking $C_1 = (C_3 + C_4)/2$ and $C_2 = (C_3 - C_4)/2$, Equation (93) can be written as

$$\begin{aligned} u_2(x,t) = u_2(\xi) &= \delta\sqrt{A} \\ &\cdot \left(\frac{C_3 \sinh\left(\sqrt{A}/2\right)\xi + C_4 \cosh\left(\sqrt{A}/2\right)\xi}{C_3 \cosh\left(\sqrt{A}/2\right)\xi + C_4 \sinh\left(\sqrt{A}/2\right)\xi}\right) \\ &+ a_{00} - \delta\lambda, \end{aligned} \tag{94}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94).

In particular, if $|C_4/C_3| < 1$, then Equation (94) is reduced to

$$u_3(x,t) = u_3(\xi) = \delta\sqrt{A}\tanh\left(\frac{\sqrt{A}}{2}\xi + \xi_0\right) + a_{00} - \delta\lambda, \tag{95}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94), $\xi_0 = \operatorname{arctanh}(C_4/C_3)$.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} u_4(x,t) = u_4(\xi) &= \delta\sqrt{-A} \\ &\cdot \left(\frac{-C_3 \sin\left(\sqrt{-A}/2\right)\xi + C_4 \cos\left(\sqrt{-A}/2\right)\xi}{C_3 \cos\left(\sqrt{-A}/2\right)\xi + C_4 \sin\left(\sqrt{-A}/2\right)\xi}\right) \\ &+ a_{00} - \delta\lambda, \end{aligned} \tag{96}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94).

Obviously, Equation (96) can be written as

$$u_5(x,t) = u_5(\xi) = -\delta\sqrt{-A}\tan\left(\frac{\sqrt{-A}}{2}\xi - \xi_0\right) + a_{00} - \delta\lambda, \tag{97}$$

where λ, μ, and a_{00} are arbitrary constants and A and ξ are given by Equation (94), $\xi_0 = -\arctan(C_4/C_3)$.

When $\lambda^2 - 4\mu = 0$,

$$u_6(x,t) = u_6(\xi) = \frac{2\delta C^2}{C_1 + C_2\xi} + a_{00} - \delta\lambda, \tag{98}$$

where $\xi = x - (a_{00} - \delta\lambda)t$, C_1, C_2, λ, μ, and a_{00} are arbitrary constants.

(iii) Now, we discuss Equation (73) from the compatible condition.

Using the compatible condition, we can get nothing but Equation (76). Using transformations $w(x,t) = w(\xi)$, $\xi = x - Vt$, Equation (76) is reduced to

$$\delta w'' + (a_{00} - V)w' - Cw = 0. \tag{99}$$

Substituting the general solutions of Equation (99) into Equation (70), we get three types of traveling wave solutions of Equation (65) as follows.

When $(a_{00} - V_1)^2 + 4C\delta > 0$,

$$u_7(x,t) = u_7(\xi) = \sqrt{\Delta}\left(\frac{C_1 e^{(\sqrt{\Delta}/2\delta)\xi} - C_2 e^{-(\sqrt{\Delta}/2\delta)\xi}}{C_1 e^{(\sqrt{\Delta}/2\delta)\xi} + C_2 e^{-(\sqrt{\Delta}/2\delta)\xi}}\right) + V_1, \tag{100}$$

where

$$\begin{aligned} \Delta &= (a_{00} - V_1)^2 + 4C\delta, \\ \xi &= x - V_1 t, \end{aligned} \tag{101}$$

where V_1, C, C_1, C_2, and a_{00} are arbitrary constants.

Taking $C_1 = (C_3 + C_4)/2$ and $C_2 = (C_3 + C_4)/2$, Equation (101) can be written as

$$u_8(x,t) = u_8(\xi) = \sqrt{\Delta} \cdot \left(\frac{C_3 \sinh\left(\sqrt{\Delta}/2\delta\right)\xi + C_4 \cosh\left(\sqrt{\Delta}/2\delta\right)\xi}{C_3 \cosh\left(\sqrt{\Delta}/2\delta\right)\xi + C_4 \sinh\left(\sqrt{\Delta}/2\delta\right)\xi} \right) + V_1, \tag{102}$$

where V_1, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

In particular, if $C_4/C_3 < 1$, then Equation (102) is reduced to

$$u_9(x,t) = u_9(\xi) = \sqrt{\Delta} \tanh\left(\frac{\sqrt{\Delta}}{2\delta}\xi + \xi_0\right) + V_1, \tag{103}$$

where $\xi_0 = \operatorname{arctanh}(C_4/C_3)$, V_1, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

When $(a_{00} - V_2)^2 + 4C\delta < 0$,

$$u_{10}(x,t) = u_{10}(\xi) = \sqrt{-\Delta} \cdot \left(\frac{-C_3 \sin\left(\sqrt{-\Delta}/2\delta\right)\xi + C_4 \cos\left(\sqrt{-\Delta}/2\delta\right)\xi}{C_3 \cos\left(\sqrt{-\Delta}/2\delta\right)\xi + C_4 \sin\left(\sqrt{-\Delta}/2\delta\right)\xi} \right) + V_2, \tag{104}$$

where V_2, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

Obviously, Equation (104) can be written as

$$u_{11}(x,t) = u_{11}(\xi) = -\sqrt{-\Delta} \tan\left(\sqrt{\frac{\sqrt{-\Delta}}{2\delta}}\xi - \xi_0\right) + V_2, \tag{105}$$

where $\xi_0 = -\arctan(C_4/C_3)$, V_2, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

When $(a_{00} - V_3)^2 + 4C\delta = 0$,

$$u_{12}(x,t) = u_{12}(\xi) = 2\delta \frac{C_2}{C_1 + C_2\xi} + V_3, \tag{106}$$

where $\xi = x - V_3 t$, V_3, C_1, C_2, and a_{00} are arbitrary constants.

Moreover, note that Equation (76) is linear, so we can get the solution of Equation (65) as follows:

$$w = \sum_{i=1}^{n_1} w_{1i} + \sum_{i=1}^{n_2} w_{2i} + \sum_{i=1}^{n_3} w_{3i} u_{13} = 2\delta(\ln w) + a_{00}, \tag{107}$$

where

$$\begin{aligned} w_{1i} &= C_{1,1i} e^{\left(\left(-(a_{00}-V_{1i})+\sqrt{\Delta_{1i}}\right)/2\right)\xi_{1i}} + C_{2,1i} e^{\left(\left(-(a_{00}-V_{1i})-\sqrt{\Delta_{1i}}\right)/2\right)\xi_{1i}}, \\ w_{2i} &= \left(C_{1,2i} \cos\left(\frac{\sqrt{-\Delta_{2i}}\xi_{2i}}{2}\right) + C_{2,2i} \sin\left(\frac{\sqrt{-\Delta_{2i}}\xi_{2i}}{2}\right) \right) e^{\left((-(a_{00}-V_{2i})\xi_{2i})/2\right)}, \\ w_{3i} &= (C_{1,3i} + C_{2,3i}\xi_{3i}) e^{\left((-(a_{00}-V_{3i})\xi_{3i})/2\right)}, \end{aligned} \tag{108}$$

where $\xi_{1i} = x - V_{1i}t$, $\Delta_{1i} = (a_{00} - V_{1i})^2 + 4C\delta > 0$ $(i = 1, \cdots, n_1)$; $\xi_{2i} = x - V_{2i}t$, $\Delta_{2i} = (a_{00} - V_{2i})^2 + 4C\delta < 0$ $(i = 1, \cdots, n_2)$; $\xi_{3i} = x - V_{3i}t$, $\Delta_{3i} = (a_{00} - V_{3i})^2 + 4C\delta = 0$ $(i = 1, \cdots, n_3)$; $C_{1,1i}, C_{2,1i}$ $(i = 1, \cdots, n_1)$; $C_{1,2i}, C_{2,2i}$ $(i = 1, \cdots, n_2)$; and $C_{1,3i}, C_{2,3i}$ $(i = 1, \cdots, n_3)$ are arbitrary constants, and n_1, n_2, and n_3 are arbitrary but finite integers.

Remark 6. We can deal with Equation (76) by using some assumptions. For example, when we suppose that $w = -\alpha t + W(x)$ and $C = 0$, we get

$$\begin{aligned} w &= -\alpha t + \frac{\alpha x}{a_{00}} + C_1 + C_2 e^{-a_{00}x/\delta}, \\ u_{14}(x,t) &= \frac{\alpha\left(2\delta - t a_{00}^2 + a_{00}x\right) + a_{00}^2\left(C_1 - C_2 e^{-a_{00}x/\delta}\right)}{\alpha(x - a_{00}t) + a_{00}\left(C_1 + C_2 e^{-a_{00}x/\delta}\right)}, \end{aligned} \tag{109}$$

where C_1, C_2, α, and a_{00} are arbitrary constants.

When we suppose that $w = -\alpha t + W(x)$ and $C = a_{00} = 0$, we get

$$\begin{aligned} w &= -\alpha t + \frac{\alpha x^2}{2\delta} + C_1 x + C_2, \\ u_{15}(x,t) &= \frac{4\delta(\alpha x + \delta C_1)}{2\delta(C_1 x + C_2 - \alpha t) + \alpha x^2}, \end{aligned} \tag{110}$$

where C_1, C_2, and α are arbitrary constants.

Similarly, we can assume that $w = \sum_{i=1}^{n} p_i(x) q_i(t)$; then, a new solution of Equation (65) can be obtained. Being similar to above process, we omit it.

So far, applying the HB of undetermined coefficient method to the Burgers equation, we get the bilinear equation of Burgers equation. Moreover, we reduce the Burgers to a linear equation. Based on them, many exact solutions of the Burgers equation are obtained by applying the perturbation method, sub-ODE method, and compatible condition. Our results can compare with the (G'/G)-expansion method, Hirota's method, and HB method [20–30].

5. Bilinear Equation of the Boussinesq Equation and Sawada-Kotera Equation

In this section, we derive the bilinear equations of the Boussinesq equation and Sawada-Kotera equation by using the HB of undetermined coefficient method. Being similar to Section 4, we omit the process of solving exact solutions.

Example 1. The generalized Boussinesq equation reads

$$u_{tt} + 2\alpha u_{xt} + \left(\alpha^2 + \beta\right)u_{xx} + \gamma u u_{xx} + \delta u_{xxxx} = 0, \tag{111}$$

where α, β, γ, and δ are known constants.

In order to balance u_{xxxx} and uu_x in Equation (111), it is required that $m + 4 = 2m + 2$ and $n = 2n$. Then, we suppose that the solution of Equation (111) is of the form

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}, \tag{112}$$

where a_{i0} $(i = 0, 1, 2)$ are constants to be determined later.

Substituting Equation (112) into Equation (111) and equating the coefficients of $(w_x/w)^6$ and $(w_x/w)^5$ on the left-hand side of Equation (111) to zero yield a set of algebraic equations for a_{20} and a_{10}. Solving the algebraic equations, we get $a_{20} = 6\delta/\gamma$ and $a_{10} = 0$. Substituting a_{20} and a_{10} back into Equation (112), we get

$$u = \frac{6\delta}{\gamma}(\ln w)_{xx} + a_{00}. \tag{113}$$

where a_{00} is an arbitrary constant.

Substituting Equation (113) into Equation (111), we get

$$\frac{6\delta}{\gamma}(K_1 + K_2 + K_3) = 0, \tag{114}$$

where

$$\begin{aligned}
K_1 = {} & \beta\left(\frac{w_{xxxx}}{w} - \frac{3w_{xx}^2 + 4w_x w_{xxx}}{w^2} + \frac{12w_{xx}w_x^2}{w^3} - \frac{6w_x^4}{w^4}\right) \\
& + \alpha^2\left(\frac{12w_{xx}w_x^2}{w^3} - \frac{3w_{xx}^2 + 4w_x w_{xxx}}{w^2} + \frac{w_{xxxx}}{w} - \frac{6w_x^4}{w^4}\right) \\
& + \alpha\left(\frac{2w_{xxxt}}{w} - \frac{6w_{xx}w_{xt} + 6w_{xxt}w_x + 2w_t w_{xxx}}{w^2}\right. \\
& \left. + \frac{12w_x w_t w_{xx} + 12w_x^2 w_{xt}}{w^3} - \frac{12w_t w_x^3}{w^4}\right) + \frac{w_{xxtt}}{w} \\
& - \frac{2w_x w_{xtt} + w_{xx}w_{tt} + 2w_t w_{xxt} + 2w_{xt}^2}{w^2} \\
& + \frac{2w_x^2 w_{tt} + 2w_{xx}w_t^2 + 8w_x w_t w_{xt}}{w^3} - \frac{6w_x^2 w_t^2}{w^4},
\end{aligned}$$

$$\begin{aligned}
K_2 = {} & a_{00}\left(\frac{2\gamma w_{xxxx}}{w} - \frac{8\gamma w_x w_{xxx} + 6\gamma w_{xx}^2}{w^2}\right. \\
& \left. + \frac{24\gamma w_{xx}w_x^2}{w^3} - \frac{12\gamma w_x^4}{w^4}\right),
\end{aligned}$$

$$\begin{aligned}
K_3 = {} & \delta\left(\frac{w_{xxxxxx}}{w} + \frac{2w_{xxx}^2 - 6w_x w_{xxxxx} - 3w_{xx}w_{xxxx}}{w^2}\right. \\
& \left. + \frac{18w_x^2 w_{xxxx} - 6w_{xx}^3}{w^3} + \frac{18w_x^2 w_{xx}^2 - 24w_x^3 w_{xxx}}{w^4}\right).
\end{aligned} \tag{115}$$

Simplifying Equation (114) and integrating twice with respect to x, we get

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(\left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(\frac{ww_{xx} - w_x^2}{w^2}\right) + 2\alpha\left(\frac{ww_{xt} - w_x w_t}{w^2}\right)\right. \\
\left. + \left(\frac{ww_{tt} - w_t^2}{w^2}\right) + \delta\left(\frac{ww_{xxxx} - 4w_x w_{xxx} 3w_{xx}^2}{w^2}\right)\right) = 0.
\end{aligned} \tag{116}$$

Equation (116) is identical to

$$\begin{aligned}
& \left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(ww_{xx} - w_x^2\right) + 2\alpha(ww_{xt} - w_x w_t) \\
& + \left(ww_{tt} - w_t^2\right) + \delta\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) \\
& - (C_1(t)x + C_2(t))w^2 = 0,
\end{aligned} \tag{117}$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of t and a_{00} is an arbitrary constant.

In particular, letting $C_1(t) = C_2(t) = 0$ in Equation (117), we get the bilinear equation of Equation (111) as follows:

$$\begin{aligned}
& \left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(ww_{xx} - w_x^2\right) + 2\alpha(ww_{xt} - w_x w_t) \\
& + \left(ww_{tt} - w_t^2\right) + \delta\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) = 0.
\end{aligned} \tag{118}$$

Equation (118) can be written concisely in terms of D-operators as

$$\left(2\alpha D_x D_t + \delta D_x^4 + \left(\alpha^2 + \beta + 2\gamma a_{00}\right)D_x^2 + D_t^2\right)w \cdot w = 0, \tag{119}$$

where a_{00} is an arbitrary constant.

Example 2. The Sawada-Kotera equation reads

$$u_t + 15\left(u^3 + uu_{xx}\right)_x + u_{xxxxx} = 0. \tag{120}$$

In order to balance u_{xxxxx} and u^3 in Equation (120), it is required that $m + 5 = 3m + 1$ and $n = 3n$. Then, we suppose that the solution of Equation (120) is of the form

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}, \tag{121}$$

where a_{i0} $(i = 0,1,2)$ are constants to be determined later.

Substituting Equation (121) into Equation (120) and equating the coefficients of $(w_x/w)^7$ and $(w_x/w)^6$ on the left-hand side of Equation (120) to zero yield a set of algebraic equations for a_{20} and a_{10}. Solving the algebraic

equations, we get $a_{20} = 2$ and $a_{10} = 0$. Substituting a_{20} and a_{10} back into Equation (121), we get

$$u = 2(\ln w)_{xx} + a_{00}, \tag{122}$$

where a_{00} is an arbitrary constant.

Substituting Equation (122) into Equation (120), we get

$$2(K_1 + K_2 + K_3) = 0, \tag{123}$$

where

$$\begin{aligned}
K_1 &= 45a_{00}^2\left(\frac{w_{xxx}}{w} - \frac{3w_xw_{xx}}{w^2} + \frac{2w_x^3}{w^3}\right),\\
K_2 &= 15a_{00}\left(\frac{w_{xxxxx}}{w} + \frac{2w_{xx}w_{xxx} - 5w_xw_{xxxx}}{w^2}\right.\\
&\quad \left. + \frac{8w_{xxx}w_x^2 - 6w_xw_{xx}^2}{w^3}\right),\\
K_3 &= \left(\frac{w_{xxt}}{\mathrm{w}} - \frac{2w_xw_{xt} + w_{xx}w_t}{w^2} + \frac{2w_x^2w_t}{w^3}\right)\\
&\quad + \left(\frac{w_{xxxxxxx}}{w} + \frac{9w_{xx}w_{xxxxx} - 5w_{xxx}w_{xxxx} - 7w_xw_{xxxxxx}}{w^2}\right.\\
&\quad \left. + \frac{20w_xw_{xxx}^2 - 30w_xw_{xx}w_{xxxx} + 12w_x^2w_{xxxxx}}{w^3}\right).
\end{aligned} \tag{124}$$

Simplifying Equation (123) and integrating once with respect to x, we get

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(45a_{00}^2\left(\frac{ww_{xx} - w_x^2}{w^2}\right)\right.\\
&\quad + 15a_{00}\left(\frac{ww_{xxxx} - 4w_xw_{xxx} + 3w_{xx}^2}{w^2}\right) + \frac{ww_{xt} - w_xw_t}{w^2}\\
&\quad \left. + \frac{ww_{xxxxxx} - 6w_xw_{xxxxx} + 15w_{xx}w_{xxxx} - 10w_{xxx}^2}{w^2}\right) = 0.
\end{aligned} \tag{125}$$

Equation (125) is identical to

$$\begin{aligned}
&45a_{00}^2\left(ww_{xx} - w_x^2\right) + 15a_{00}\left(ww_{xxxx} - 4w_xw_{xxx} + 3w_{xx}^2\right)\\
&\quad + \left(ww_{xt} - w_xw_t\right) + \left(ww_{xxxxxx} - 6w_xw_{xxxxx}\right.\\
&\quad \left. + 15w_{xx}w_{xxxx} - 10w_{xxx}^2\right) - C(t)w^2 = 0,
\end{aligned} \tag{126}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, letting $C(t) = 0$ in Equation (126), we get the bilinear equation of Equation (120) as follows:

$$\begin{aligned}
&45a_{00}^2\left(ww_{xx} - w_x^2\right) + 15a_{00}\left(ww_{xxxx} - 4w_xw_{xxx} + 3w_{xx}^2\right)\\
&\quad + \left(ww_{xt} - w_xw_t\right) + \left(ww_{xxxxxx} - 6w_xw_{xxxxx}\right.\\
&\quad \left. + 15w_{xx}w_{xxxx} - 10w_{xxx}^2\right) = 0.
\end{aligned} \tag{127}$$

Equation (127) can be written concisely in terms of D-operators as

$$\left(45a_{00}^2D_x^2 + 15a_{00}D_x^4 + D_xD_t + D_x^6\right)w \cdot w = 0, \tag{128}$$

where a_{00} is an arbitrary constant.

6. Conclusions

The HB of undetermined coefficient method is successfully used to establish the bilinear equation of NLPDE. By applying the perturbation method, sub-ODE method, and compatible condition to the bilinear equation, more exact solutions of NLPDE are obtained. We illustrate the real meaning of balance numbers. We show the underlying relations among the (G'/G)-expansion method, Hirota's method, and HB method. Many well-known NLPDE can be handled by the HB of undetermined coefficient method. The performance of our method is found to be simple and efficient. The availability of computer systems like Maple facilitates the tedious algebraic calculations. Our method is also a standard and computable method, which allows us to solve complicated and tedious algebraic calculations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no competing interests.

Authors' Contributions

Xiao-Feng Yang analyzed and interpreted the data and wrote the manuscript; Yi Wei designed and optimized the algorithm and program. All authors read the manuscript.

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Research Article

A Method for Solving Reliability of Route Travel Time

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In order to obtain an effective method for solving route travel time reliability, this paper proposes an effective new method to calculate travel time reliability using an independent link travel time function. Based on several months of historical data, the results show that the Edgeworth expansion can better reflect travel time distribution law. In addition, travel time reliability can be calculated more conveniently by combining an approximate discretization algorithm and an empirical distribution function.

1. Introduction

Urban development has made traffic congestion increasingly severe, resulting in a high demand for a reliable transportation system. However, there are many stochastic factors that affect the systems, such as traffic congestion and road maintenance. These factors highlight the importance of a reliable transportation network. At present, the research on traffic network reliability mainly includes connectivity, travel time, and road network capacity. Travel time reliability is particularly important: it allows travelers to measure travel efficiency and perceive travel quality. The value of time is increasing and people are demanding shorter travel times. To achieve this, more attention must be paid to the volatility and reliability of travel time. The study of travel time reliability is of great significance for evaluating road network operations and improving travel quality.

Asakura and Kashiwadani proposed the concept of travel time reliability in 1991 [1]. Subsequently, scholars in different countries continue to develop this concept using time index, time delay index, and other indicators. In general, it is acceptable to use the probability measure function to calculate travel time reliability. Emam and Al-Deek estimated travel time reliability using data comprising the number and speed of vehicles provided by a coil detector on a highway in Orlando, Florida. They then assessed the goodness of fit using Weibull, exponential, lognormal, and Gaussian distributions. The results showed that the lognormal distribution was optimal for estimating reliability [2]. Wang et al. noted that travel behavior was impacted by travel time, travel time reliability, and monetary cost. Using these factors, she established the biobjective user equilibrium model of travel time reliability [3]. Bhouri and Aron studied factors that impacted travel reliability, such as hypersensitive reactions (e.g., tachycardia) on highways in France [4]. Hojatia et al. identified nonrecurring congestion using sudden time and location. The tobit model, which is based on extra buffer time, studies the impact of traffic accidents on travel time [5]. Hou and Tan established models for highway networks based on the Monte Carlo simulation to calculate the postearthquake connectivity reliability [6]. Among others, Tao and Zhang considered travel time reliability for multimodal transportation networks composed of commuters and cars [7]. Pan et al. addressed the dynamic networks to determine and control the risk of stochastic travel times. Then, they used optimization theory to develop algorithms to solve the problem [8].

The literature review revealed that most of the current research on travel time reliability determines the distribution function for travel time and establishes the reliability model. The function uses road and travel time statistics to determine the travel time distribution law. Travel time reliability is mainly used in analytical and simulation methods. Using historical data from a highway in Shanghai, this paper explores a new analytical method to solve for travel time reliability.

2. The Concept of Travel Time Reliability

Travel time reliability is defined as the probability of completing a trip during traffic within a specified travel time. The basic equation is

$$R(t) = P(T \le t_0). \tag{1}$$

In the equation above, $R(t)$ is the travel time reliability within designated sections, routes, or OD (Origin and Destination) pairs in a certain period of time; T is the travel time within them; and t_0 is the specified travel time threshold. If the probability density function for the travel time of sections, routes, and OD pairs is $f(t)$, and the travel time distribution function is $F(t_0)$, then (1)

$$R(t) = P(t \le t_0) = \int_0^{t_0} f(t)dt = F(t_0). \tag{2}$$

3. Solving for Route Travel Time Reliability

Assume that a route L consists of two sections (a_1 and a_2) in series, the travel times of sections a_1 and a_2 are X_1 and X_2, respectively, and the joint density function of X_1 and X_2 is $f(x_1, x_2)$. The travel time of route L is Y. The probability distribution function of $Y = x_1 + x_2$ should be obtained first. The travel time reliability is then calculated using (2).

By definition, the distribution function of travel time $Y = x_1 + x_2$ is

$$F_Y(Y \le y) = P(X_1 + X_2 \le y) = \iint_D f(x_1, x_2)dx_1dx_2, \tag{3}$$

when X_1 and X_2 are independent, assuming that the edge density of $f(x_1, x_2)$, with respect to X_1 and X_2, is $f_{x_1}(x_1)$ and $f_{x_2}(x_2)$. Then, the probability density function of route travel time ($Y = x_1 + x_2$) becomes

$$f_Y(y) = \int_{-\infty}^{\infty} f_{x_1}(y - x_2) f_{x_2}(x_2)dx_2 \tag{4}$$

or

$$f_Y(y) = \int_{-\infty}^{\infty} f_{x_1}(x_1) f_{x_2}(y - x_1)dx_1. \tag{5}$$

The distribution function is

$$F_Y(y) = \int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f_{x_1}(x_1) f_{x_2}(y - x_1)dx_1dz \right] \tag{6}$$

or

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{x_1}(y - x_2) f_{x_2}(x_2)dx_2dz. \tag{7}$$

3.1. Discrete Approximation Using an Empirical Distribution Function. If the travel time distribution function, $F(t)$, and the probability density function, $f(t)$, cannot be expressed in a formula, they can be approximated using empirical distribution. We can use the empirical distribution of travel time to calculate reliability. The travel time of continuous variables is treated as discrete variables.

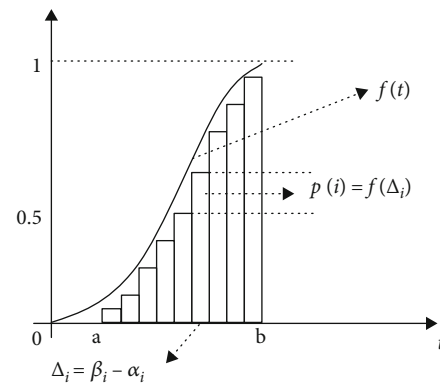

Figure 1: Discretization of empirical distribution of time travel.

As shown in Figure 1, if the time interval of study is $[a, b]$, $t \in [a, b]$, the time interval $[a, b]$ is divided into two subareas: $[a, b] = \cup_1^m [\alpha_i, \beta_i]$ and $\Delta_i = [\alpha_i, \beta_i]$. The probability of travel time in $[\alpha_i, \beta_i]$ is recorded as $p(i)$, then $p(i) = p(\alpha_i \le t \le \beta_i) = F(\beta_i) - F(\alpha_i)$. The travel time reliability of continuous variables can be approximated by a series of discrete values:

$$R(t) = P(t \le t_0) = \int_0^{t_0} f(t)dt \approx \sum_{i=1}^{m} p(i). \tag{8}$$

The travel time probability distribution of the two sections can be discretized using (8). Equation $P(x_1 = i, x_2 = j)$ calculates the probability of travel times $X_1$and X_2 being less than i and X_2 are independent,

$$P(x_1 = i, x_2 = j) = P(x_1 = i) \times P(x_2 = j). \tag{9}$$

Then, the probability of route travel time, r, is

$$P(Y = r) = \sum_{i=1}^{r} P(x_1 = i, x_2 = r - i). \tag{10}$$

The reliability of route travel time not exceeding t is

$$\begin{aligned} R(Y \le t) = P(Y \le t) = \sum_{r=1}^{t} \sum_{i=0}^{r} P(x_1 = i, x_2 = r - i) \\ \cdot \sum_{r=1}^{m} \sum_{i=0}^{r} P(x_1 = i) P(x_2 = r - i). \end{aligned} \tag{11}$$

3.2. Edgeworth Asymptotic Expansion Series Approximation Using Density Function. Since the urban traffic system is complex, the law of travel time distribution is becoming less and less obvious. If the distribution of travel time cannot be expressed by known functions, we can use series expansion to approximate the distribution function and calculate travel time reliability. The Edgeworth expansion series

approximates the characteristic equation of the distribution function using $F(t)$. The density function, $f(t)$, is another known distribution extension of the characteristic equation. The normal distribution is commonly selected and then passed through the inverse transformation of Fourier's equation. This obtains characteristic equation $F(t)$. To understand the main process, refer to references [9–11].

The Edgeworth expansion for probability density function was used to calculate travel time reliability. If the random variable travel time is t, the mean value is μ, and the variance is σ^2, and if $y = (t-u)/\sigma$, then the probability density function is

$$p(y) = \varphi(y)\left(1 + \frac{\kappa_3}{3!}H_3(y) + \frac{\kappa_4}{4!}H_4(y) + \frac{10\kappa_3^2}{6!}H_6(y) + \frac{\kappa_5}{5!}H_5(y) + \frac{35\kappa_3\kappa_4}{7!}H_7(y)+\cdots\right). \tag{12}$$

In this equation, $\phi(y)$ is the standard normal distribution density function. The Hermite orthogonal polynomial, $H_i(y)$, $H_0(y) = 1, H_1(y) = y, H_2(y) = y^2 - 1, H_3(y) = y^3 - 3y$, $H_4(y) = y^4 - 6y^2 + 3\cdots$;

$$\begin{aligned} H_n(y) &= (-1)^n e^{-y^2/2}\frac{d^n}{dy^n}e^{y^2/2} \\ &= y^n\sum_{k=1}^{n/2}(-1)^k(2k-1)!!C_n^{2k}y^{n-2k}, \\ &= \mu_1, \end{aligned} \tag{13}$$

$$\begin{aligned} \kappa_2 &= \mu_2 - \mu_1^2, \\ \kappa_3 &= \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3, \end{aligned} \tag{14}$$

$$\kappa_4 = \mu_4 - 3\mu_2^2 - 4\mu_1\mu_3 + 12\mu_1^2\mu_2 - 6\mu_1^4, \cdots. \tag{15}$$

u_k is the k-order moment estimator that can be calculated using actual observations.

Taking the four items ahead, (12) can be simplified to

$$p(y) = \varphi(y)\left[1 + \frac{s}{3!}\left(y^3 - 3y\right)\right]. \tag{16}$$

In this equation, $\phi(y)$ is the standard normal distribution probability density function and s is the deviation of travel time. The corresponding distribution function is

$$F(y) = \left[\varphi(y) - \frac{s}{3!}\phi(y)\left(y^2 - 1\right)\right] \tag{17}$$

In practice, the convergence of the Edgeworth expansion of probability density function is often neglected. The numerical experiments show that it can fulfill accuracy requirements. Specific details can be found in references [10, 11].

In (12), $\kappa_i$and μ_i are constants. Equation (12) can be simplified to

$$p(y) = \varphi(y)\left(\sum_{i=0}^{\infty}a_i y^i\right). \tag{18}$$

After the travel time probability density of the road section is obtained using equation (13), the route travel time reliability can be calculated using discrete approximation (Section 3.1) if the section travel time are independent. If the asymptotic expression of the travel time density function of the two sections is

$$\begin{aligned} p(x_1) &= \varphi(x_1)\left(\sum_{i=0}^{\infty}a_i {x_1}^i\right), \\ p(x_2) &= \varphi(x_2)\left(\sum_{i=0}^{\infty}b_i {x_2}^i\right). \end{aligned} \tag{19}$$

Then, the probability density function of $y = x_1 + x_2$ is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty}p_{x_1}(y-x_2)p_{x_2}(x_2)dx_2 \\ &= \int_{-\infty}^{\infty}\varphi(y-x_2)\left(\sum_{i=0}^{\infty}a_i(y-x_2)^i\right)\varphi(x_2)\left(\sum_{i=0}^{\infty}b_i {x_2}^i\right)dx_2, \end{aligned} \tag{20}$$

$$F(Y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty}\varphi(z-x_2)\left(\sum_{i=0}^{\infty}a_i(z-x_2)^i\right)\varphi(x_2)\left(\sum_{i=0}^{\infty}b_i {x_2}^i\right)dx_2dz. \tag{21}$$

Equation (21) can be used to calculate the reliability of path travel time.

4. Empirical Research

Existing literature has revealed the diversity and uncertainty of travel time

distribution. The current distribution function is still unable to describe many cases. Empirical research was conducted using the travel time distribution of the Shanghai Outer Ring Expressway at typical peak hour (8:00–9:30 and 17:00–18:30) from May 1st to July 31st, 2015. As shown in Figure 2, the Edgeworth series curve approximation (blue line) for the density function of the on-ramps from Wuzhong Road to Caobao Road is compared with their respective histograms. During rush hour, the saturation flow is close to or already at maximum, and the travel time fluctuation range is large; compared to normal distribution, there is a certain degree of deviation marked by s_1 and s_2. The peak values are at $s_1 = 1.07$ and $s_2 = 1.93$ for morning travel times. The evening rush hour curve deviates to the left.

Compared with normal distribution, the Edgeworth expansion better reflects the distribution law of travel time. By calculating the absolute error between expected probability of the normal distribution, the expected probability of Edgeworth expansion, and the actual frequency of travel

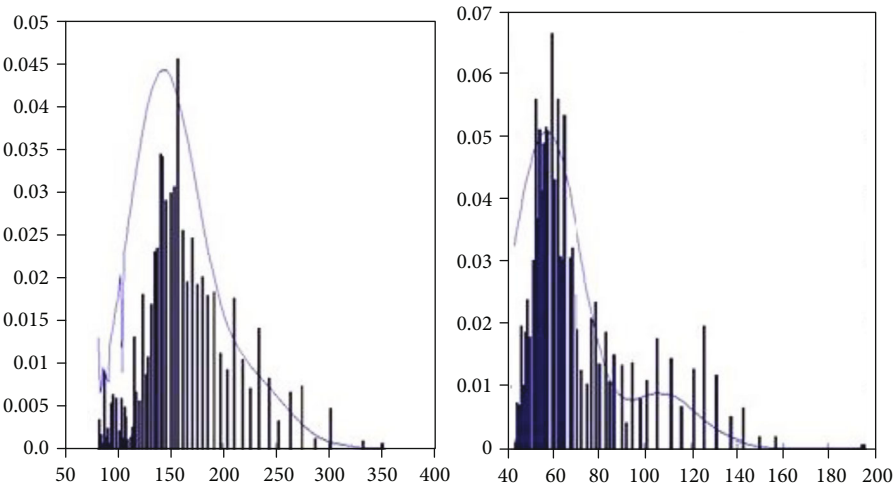

Figure 2: Comparing the Edgeworth series travel time probability density function and histograms.

Table 1: Error analysis of travel time estimation in early peak section.

Category	Normal distribution	Edgeworth expansion distribution
Maximum absolute error	0.0709	0.0646
Average absolute error	0.0169	0.0138
Sum of squares for error	0.0138	0.0099

Table 2: Error analysis of travel time estimation in late peak section.

Category	Normal distribution	Edgeworth expansion distribution
Maximum absolute error	0.2615	0.184
Average absolute error	0.0475	0.0313
Sum of squares of error	0.0906	0.0438

time, to compare the accuracy of travel time reliability, the differences are described using the maximum absolute error, the mean absolute error, and the sum of squares for error. The results are shown in Tables 1 and 2. It can be seen that the maximum absolute error value of travel time estimated by the Edgeworth expansion is less than the normal distribution, and the error between the estimated value of the Edgeworth expansion and the actual frequency is small. The accuracy of travel time distribution fitted by the Edgeworth expansion is higher than that of normal distribution, and the asymptotic distribution of travel time based on the Edgeworth expansion.

Using the distribution of section travel time, (21) can be used to calculate the reliability of travel time.

5. Conclusion

Travel time reliability is of great significance when analyzing the reliability of urban traffic networks. A new method for calculating travel time reliability was designed under the condition that the route travel times were independent. Based on several months of historical data, the results show that the Edgeworth expansion series can better reflect the distribution law of travel time. In addition, travel time reliability can be calculated more conveniently by combining an approximate discretization algorithm and an empirical distribution function. However, this method requires statistical traffic data to determine the empirical distribution function. The accuracy of the obtained results is closely related to the division of time interval. The accuracy and expanded forms of the Edgeworth series approximation for distribution functions are closely related. However, the quantitative relationship between its error and expansion needs to be studied further.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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Research Article

New Sequential Fractional Differential Equations with Mixed-Type Boundary Conditions

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In this paper, we introduce new sequential fractional differential equations with mixed-type boundary conditions $\begin{cases} ({}^{C}D^{q}+k{}^{C}D^{q-1})u(t)=f(t,u(t),{}^{C}D^{q-1}u(t)), t\in(0,1), \\ \alpha_1 u(0)+\beta_1 u(1)+\gamma_1 I^{r}u(\eta)=\varepsilon_1, \eta\in(0,1), \\ \alpha_2 u'(0)+\beta_2 u'(1)+\gamma_2 I^{r}u'(\eta)=\varepsilon_2, \end{cases}$ where $q\in(1,2]$ is a real number, $k,r>0,\alpha_i,\beta_i,\gamma_i,\varepsilon_i\in\mathbb{R}, i=1,2,{}^{C}D^{q}$ is the Caputo fractional derivative, and the boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases. Our approach to treat the above problem is based upon standard tools of fixed point theory and some new inequalities of norm form. Some existence results are obtained and well illustrated through the aid of examples.

1. Introduction

In this paper, we focus on sequential fractional differential equations with mixed-type boundary conditions.

$$\begin{cases} \left({}^{C}D^{q}+k{}^{C}D^{q-1}\right)u(t)=f\left(t,u(t),{}^{C}D^{q-1}u(t)\right), & t\in(0,1), \\ \alpha_1 u(0)+\beta_1 u(1)+\gamma_1 I^{r}u(\eta)=\varepsilon_1, & \eta\in(0,1), \\ \alpha_2 u'(0)+\beta_2 u'(1)+\gamma_2 I^{r}u'(\eta)=\varepsilon_2, \end{cases} \tag{1}$$

where $q\in(1,2]$ is a real number and $k,r>0,\alpha_i,\beta_i,\gamma_i,\varepsilon_i\in\mathbb{R}$, $i=1,2,{}^{C}D^{q}$ is the Caputo fractional derivative of order q. The nonlinearity term f contains the unknown function and its lower order fractional derivatives. The new boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases which can be regarded as the linear combination of the values of the unknown function and its first derivatives at the end points of interval, and the Riemann-Liouville fractional integral value of the unknown function and its first derivatives at an interior point of interval.

Fractional differential equations have attracted significant attention for their wide application in many fields of engineering and applied sciences (see [1–10]). Sequential fractional differential equations as an importance branch have also received wide attention; for instance, see [11–16]. Motivated by the HIV infection model and its application background in [12], the existence and uniqueness of solutions for the following sequential fractional differential system are obtained by means of Leray-Schauder's alternative and Banach's contraction principle

$$\begin{cases} \left({}^{C}D^{p}+\lambda_1{}^{C}D^{p-1}\right)u(t)=f_1(t,u(t),v(t)), & t\in(0,1), \\ \left({}^{C}D^{p}+\lambda_2{}^{C}D^{p-1}\right)v(t)=f_2(t,u(t),v(t)), & t\in(0,1), \\ u(0)=u'(0)=0, u(1)=av(\xi), \\ v(0)=v'(0)=0, v(1)=bu(\eta), \end{cases} \tag{2}$$

where $\lambda_i > 0$ $(i = 1, 2)$ is a parameter; $2 < p, q \leq 3$, ${}^{C}D^{p}$, and ${}^{C}D^{q}$ are the Caputo fractional derivatives; and the nonlinearity terms $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the given continuous function.

Antiperiodic boundary conditions arise in the mathematical problems of certain physical phenomena and processes. Recently, many scholars paid attention to solvability for fractional differential equations involving antiperiodic boundary conditions (see [17–21]). For example, in [21], the authors considered the nonlinear antiperiodic boundary value problems

$$\begin{cases} \left({}^{C}D^{q} + k{}^{C}D^{q-1}\right)u(t) = f(t, u(t)), q \in (2, 3], \quad t \in (0, T), \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \alpha_2 u'(0) + \gamma_2 u'(T) = b, \alpha_3 u''(0) + \gamma_3 u''(T) = c, \end{cases} \tag{3}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivatives of order q, α_i, γ_i $(i = 1,2,3)$, $a, b, c \in \mathbb{R}$, $k > 0$, f is a continuous function.

Integral boundary conditions are believed to be more reasonable than the local boundary conditions, which can describe modeling of blood flow, cellular systems, population dynamics, heat transmission, etc. There are a number of results about fractional differential equations and partial differential equations with integral boundary condition; we refer the reader to see [17, 20, 22–43] and the references cited therein. In [20], the authors discussed the following fractional differential equation with integral boundary conditions given by

$$\begin{cases} \left({}^{C}D^{\alpha+\beta} + k{}^{C}D^{\beta}\right)x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, x(1) = \mu\int_0^1 x(s)ds, {}^{C}D^{\alpha}x(0) + {}^{C}D^{\alpha}x(1) = 0, \end{cases} \tag{4}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives; $0 < \alpha < 1$, $1 < \beta \leq 2$, $k > 0$, and $\mu > 0$ are real numbers; and f is a given continuous function.

Observing the results of the above literature, an interesting and important question is whether antiperiodic and integral boundary conditions can be unified in a system. If we have unified the conditions, how can we obtain the existence of the solutions? Through a literature search, the sequential fractional differential equation (1) has not been given up to now.

Now in this paper, we shall discuss the problem (1) by using the standard tools of fixed point theory and some new inequalities of norm from.

2. Preliminary and Lemmas

In this paper, we provide some necessary definitions and lemmas of the Caputo fractional calculus; for more information, see the books [1–3].

Definition 1. The Caputo derivative of fractional order $\alpha > 0$ for a n-times continuously differentiable function $f : [0, \infty) \longrightarrow \mathbb{R}$ is defined as

$${}^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(x)\mathrm{d}t, \quad n-1 < \alpha < n, \tag{5}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α.

Definition 2. The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \quad \alpha > 0, \tag{6}$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$.

Lemma 3. *The Caputo fractional differential equation* ${}^{C}D^{\alpha}u(t) = 0$ *has the general solution for* $\alpha > 0$

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \tag{7}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, *and* n *is given as in Definition 1.*

Lemma 4. *Let* $h(t) \in C[0, 1]$ *and* $u \in C^2[0, 1]$. *Then, the following sequential fractional differential equations*

$$\begin{cases} \left({}^{C}D^{q} + k{}^{C}D^{q-1}\right)u(t) = h(t), \quad 0 < t < 1, 1 < q \leq 2, \\ \alpha_1 u(0) + \beta_1 u(1) + \gamma_1 I^{r}u(\eta) = \epsilon_1, \quad r > 0, 0 < \eta < 1, \\ \alpha_2 u'(0) + \beta_2 u'(1) + \gamma_2 I^{r}u'(\eta) = \epsilon_2, \end{cases} \tag{8}$$

have a unique solution

$$u(t) = \frac{\Delta_1 - \Delta_2 e^{-kt}}{\Delta_2 \Delta_3}\Phi(h(1), h(\eta)) + \frac{1}{\Delta_2}\psi(h(1), h(\eta)) + \int_0^t e^{-k(t-s)} I^{q-1} h(s)ds, \tag{9}$$

where

$$\Delta_1 = \alpha_1 + \beta_1 e^{-k} + \gamma_1 \int_0^\eta \frac{(\eta - s)^{r-1}}{\Gamma(r)} e^{-ks} ds,$$

$$\Delta_2 = \alpha_1 + \beta_1 + \frac{\gamma_1 \eta^r}{\Gamma(r+1)},$$

$$\begin{aligned}\Delta_3 &= -k\alpha_2 - ke^{-k}\beta_2 - k\gamma_2 \int_0^\eta \frac{(\eta - s)^{r-1}}{\Gamma(r)} e^{-ks} ds, \Phi(h(1), h(\eta)) \\ &= -\beta_2 I^{q-1} h(1) - k\beta_2 \int_0^1 e^{-k(1-s)} I^{q-1} h(s)ds \\ &\quad + \gamma_2 \left[\int_0^\eta \frac{(\eta - s)^{r-1}}{\Gamma(r)} I^{q-1} h(s)ds - k \int_0^\eta \frac{(\eta - s)^{r-1}}{\Gamma(r)} \right. \\ &\quad \left. \cdot \left(\int_0^s e^{-k(s-m)} I^{q-1} h(m)dm \right) ds \right] - \epsilon_2, \Psi(h(1), h(\eta)) \\ &= -\gamma_1 \int_0^\eta \frac{(\eta - s)^{r-1}}{\Gamma(r)} \left(\int_0^s e^{-k(s-m)} I^{q-1} h(m)dm \right) ds \\ &\quad - \beta_1 \int_0^1 e^{-k(1-s)} I^{q-1} h(s)ds + \epsilon 1.\end{aligned} \tag{10}$$

Proof. Using Lemma 3, the general solution of the fractional differential equations $({}^C D^q + k {}^C D^{q-1})u(t) = h(t)$ can be written as

$$u(t) = A_0 e^{-kt} + A_1 + \int_0^t e^{-k(t-s)} \left(I^{q-1} h(s) \right) ds, \tag{11}$$

where $A_0, A_1 \in \mathbb{R}$. Differentiating (11) with respect to t, we get

$$u'(t) = -kA_0 e^{-kt} + I^{q-1} h(t) - k \int_0^t e^{-k(t-s)} I^{q-1} h(s)ds. \tag{12}$$

Applying the boundary condition (8) in (11) and (12), we obtain

$$\Delta_1 A_0 + \Delta_2 A_1 = \Psi(h(1), h(\eta)), \quad \Delta_3 A_0 + \Phi(h(1), h(\eta)) = 0. \tag{13}$$

A simultaneous solution of equation (13) leads to

$$A_0 = -\frac{1}{\Delta_3}\Phi(h(1), h(\eta)), \quad A_1 = \frac{\Delta_1}{\Delta_2 \Delta_3}\Phi(h(1), h(\eta)) + \frac{1}{\Delta_2}\Psi(h(1), h(\eta)). \tag{14}$$

Substituting A_0 and A_1 to (11), we obtain the desired solution in (9). The converse of the lemma follows by direct computation. The proof is completed.

Remark 5. Caputo fractional differentiating (11) with respect to t, we obtain

$$\begin{aligned}{}^C D^{q-1} u(t) &= \frac{kA_0}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} e^{-ks} ds + \frac{1}{\Gamma(2-q)} \\ &\quad \cdot \int_0^t (t-s)^{1-q} I^{q-1} h(s)ds - \frac{k}{\Gamma(2-q)} \\ &\quad \cdot \int_0^t (t-s)^{1-q} \left(\int_0^s e^{-k(s-m)} I^{q-1} h(m)dm \right) ds,\end{aligned} \tag{15}$$

where A_0 is defined as (14).

Set $C[0,1]$ is all the continuous functions on $[0,1]$, $C_{q-1}[0,1] = \{u \in C[0,1]: {}^C D^{q-1} u \in C[0,1]\}$. Let $E = (C_{q-1}[0,1], \|\cdot\|_{q-1})$ denotes the Banach space endowed with the norm defined by $\|u\|_{q-1} = \|u\| + \|{}^C D^{q-1} u\| = \sup_{0 \le t \le 1} |u(t)| + \sup_{0 \le t \le 1} |{}^C D^{q-1} u(t)|$. For the convenience of the proofs in the next main results, we first give the bounds for integrals arising from the sequel, which are very important for us to establish the existence of solutions for problem (1).

Lemma 6. *Suppose that $h \in C([0,1], \mathbb{R})$. Then, we have*

(i) $|\Phi(h(1), h(\eta))| \le [(|\beta_2|(2 - e^{-k})/\Gamma(q)) + (|\gamma_2|\eta^{q+r}(2 - e^{-k\eta})/\Gamma(q)\Gamma(r+1))]\|h\| + |\epsilon_2|$

(ii) $|\Psi(h(1), h(\eta))| \le [(|\beta_1|(1 - e^{-k})/k\Gamma(q)) + (|\gamma_1|\eta^{q+r}(1 - e^{-k\eta})/k\Gamma(q)\Gamma(r+1))]\|h\| + |\epsilon_1|$

Proof.

(i) Obviously, we have

$$\begin{aligned}\left| I^{q-1} h(s) \right| &\le \frac{1}{\Gamma(q\text{-}1)} \int_0^s (s - \omega)^{q-2} \mathrm{d}\omega \|h\| \\ &= \frac{s^q}{\Gamma(q)}\|h\|, \quad \left| I^{q-1} h(1) \right| \le \frac{1}{\Gamma(q)}\|h\|, \\ \left| \int_0^s e^{-k(s-m)} I^{q-1} h(m)dm \right| &\le \frac{s^q}{\Gamma(q)} \int_0^s e^{-k(s-m)} dm \|h\| \\ &= \frac{s^q}{\Gamma(q)} \frac{1 - e^{-ks}}{k}\|h\|, \\ \left| \int_0^1 e^{-k(1-s)} I^{q-1} h(s)ds \right| &\le \frac{1}{\Gamma(q)} \int_0^1 e^{-k(1-s)} ds \|h\| \\ &= \frac{1 - e^{-k}}{k\Gamma(q)}\|h\|, \\ \left| \int_0^\eta \frac{(\eta - s)^{r\text{-}1}}{\Gamma(r)} I^{q-1} h(s)ds \right| &\le \frac{\eta^q}{\Gamma(q)} \int_0^\eta \frac{(\eta - s)}{\Gamma(r)} ds \|h\| \\ &\le \frac{\eta^{q+r}}{\Gamma(q)\Gamma(r+1)}\|h\|.\end{aligned} \tag{16}$$

Furthermore,

$$\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\right)I^{q-1}h(m)dm\right| \le \frac{\eta^{q+r}\left(1-e^{-k\eta}\right)}{k\Gamma(q)\Gamma(r+1)}\|h\|. \tag{17}$$

Hence,

$$\begin{aligned}
&|\Phi(h(1),h(\eta))|\\
&\le \left|\beta_2 I^{q-1}h(1)\right| + k\left|\beta_2\int_0^1 e^{-k(1-s)}I^{q-1}h(s)ds\right|\\
&\quad + |\gamma_2|\left[\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}I^{q-1}h(s)ds\right|\right.\\
&\quad \left. + k\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right)ds\right|\right]\\
&\quad + |\epsilon_2| \le \left\{\frac{|\beta_2|}{\Gamma(q)} + \frac{|\beta_2|\left(1-e^{-k}\right)}{\Gamma(q)}\right.\\
&\quad \left. + \frac{|\gamma_2|\eta^{q+r}}{\Gamma(q)\Gamma(r+1)} + \frac{|\gamma_2|\eta^{q+r}\left(1-e^{-k\eta}\right)}{\Gamma(q)\Gamma(r+1)}\right\}\|h\| + |\epsilon_2|.
\end{aligned} \tag{18}$$

When the proof of (ii) is similar to (i), we omit it.

In view of Lemma 4 and Remark 5, replacing $h(t)$ by $f(t,u(t),{}^C D^{q-1}u(t))$ in (9) and (15), we transform the solution of problem (1) into the fixed point of operator equation $u = \mathbb{F}u$, where operator $\mathbb{F}: E \longrightarrow E$ is defined as

$$\begin{aligned}
(\mathbb{F}u)(t) = {}& \frac{\Delta_1 - \Delta_2 e^{-kt}}{\Delta_2\Delta_3}\Phi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right.\\
&\left. f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right)\\
&+ \frac{1}{\Delta_2}\Psi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right.\\
&\left. f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right)\\
&+ \int_0^t e^{-k(t-s)}I^{q-1}f\left(s,u(s){}^C D^{q-1}u(s)\right)ds,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\left({}^C D^{q-1}\mathbb{F}u\right)(t) = {}& -\frac{k}{\Gamma(2-q)}\Phi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right.\\
&\left. f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right)\int_0^t (t-s)^{1-q}e^{-ks}ds\\
&+ \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}I^{q-1}f\left(s,u(s),\right.\\
&\left.{}^C D^{q-1}u(s)\right)ds - \frac{k}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}\\
&\cdot\left(\int_0^s e^{-k(s-m)}I^{q-1}f\left(m,u(m),\right.\right.\\
&\left.\left.{}^C D^{q-1}u(m)\right)dm\right)ds.
\end{aligned} \tag{20}$$

Lemma 7 (see [44]). *Let $T: E \to E$ be completely continuous (i.e., a map restricted to any bounded set in E is compact). Let*

$$\mathbb{F} = \{x \in E : x = \lambda T(x), \text{for some } 0 < \lambda < 1\}. \tag{21}$$

Then, either set $\mathbb{F}$ is unbounded or $\mathbb{F}$ has at least one fixed point.

Lemma 8 (see [45]). *Let E be a Banach space, $D \subset E$ be closed and $F: D \longrightarrow D$ a strict contraction, i.e., $|Fx - Fy| \le k|x-y|$ for some $k \in (0,1)$ and all $x, y \in D$. Then, F has a unique fixed point.*

3. Main Results

Before starting and introducing the main results, we list our assumption for f:

(i) H0: $f: [0,1]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous

(ii) H1: there exist positive constants $a_{11}, a_{12}, a_{13} \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{11}|u|^{\sigma_1} + a_{12}|v|^{\sigma_2} + a_{13}, \quad \forall t \in [0,1], u, v \in \mathbb{R}, 0 < \sigma_1, \sigma_2 < 1 \tag{22}$$

(iii) H2: there exist positive constants $a_{21}, a_{22}, a_{23} \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{21}|u|^{\tau_1} + a_{22}|v|^{\tau_2} + a_{23}, \quad \forall t \in [0,1], u, v \in \mathbb{R}, 0 < \tau_1, \tau_2 > 1 \tag{23}$$

(iv) H3: there exist positive constants $a_{31}, a_{32}, a_{33} \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{31}|u| + a_{32}|v| + a_{33}, \quad \forall t \in [0,1], u, v \in \mathbb{R} \tag{24}$$

(v) H4: there exist positive constants $a_{41}, a_{42}, a_{43} \in \mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{41}|u|^{\rho_1} + a_{42}(t)|v|^{\rho_2} + a_{43}, \quad \forall t \in [0,1], u, v \in \mathbb{R}, 0 < \rho_1 < 1 < \rho_2 \tag{25}$$

(vi) H5: there exist positive constants $a_{51}, a_{52} \in \mathbb{R}^+$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le a_{51}|u-\bar{u}| + a_{52}|v-\bar{v}|, \quad \forall t \in [0,1], u, v, \bar{u}, \bar{v} \in \mathbb{R} \tag{26}$$

For convenience, we introduce the following symbols:

$$L_1 = \frac{|\beta_2|\left(2-e^{-k}\right)}{\Gamma(q)} + \frac{|\gamma_2|\eta^{q+r}\left(2-e^{-k\eta}\right)}{\Gamma(q)\Gamma(r+1)},$$
$$L_2 = \frac{|\beta_1|\left(1-e^{-k}\right)}{k\Gamma(q)} + \frac{|\gamma_1|\eta^{q+r}\left(1-e^{-k\eta}\right)}{k\Gamma(q)\Gamma(r+1)},$$
$$L_3 = \frac{(|\Delta_1|+|\Delta_2|)L_1 + |\Delta_3|L_2}{|\Delta_2\Delta_3|} + \frac{1}{k\Gamma(q)} + \frac{k\Gamma(q)L_1+2}{\Gamma(3-q)\Gamma(q)},$$
$$L_4 = \frac{(|\Delta_1|+|\Delta_2|)|\varepsilon_2| + |\Delta_3\varepsilon_1|}{|\Delta_2\Delta_3|} + \frac{k|\varepsilon_2|}{\Gamma(3-q)}. \tag{27}$$

Theorem 9. *Suppose that (H0) and (H1) hold. Then, problem (1) has at least one solution.*

Proof. We first define a ball in E as $B_R = \{u | u \in E, \|u\|_{q-1} \le R\}$, where

$$R \ge \max\left\{(3L_3a_{11})^{1/(1-\sigma_1)}, (3L_3a2)^{1/(1-\sigma_1)}, 3(3L_3a_{13}+L_4)\right\}. \tag{28}$$

Then, we show that $T : B_R \longrightarrow B_R$. For $\forall u \in B_R$, using Lemma 6 and the condition (H1), we have

$$\begin{aligned}
|(\mathbb{F})(t)|s \le{}& \frac{|\Delta_1|+|\Delta_2|e^{-kt}}{|\Delta_2||\Delta_3|}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),\right.\right.\\
&\left.\left.f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right| + \frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1,u(1),\right.\right.\right.\\
&\left.\left.\left.{}^{C}D^{q-1}u(1)\right),f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right|\\
&+\int_0^t e^{-k(t-s)}I^{q-1}\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right|ds\\
\le{}& \frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left[L_1\left(a_{11}R^{\sigma_1}+a_{12}R^{\sigma_2}+a_{13}\right)+|\varepsilon_2|\right]\\
&+\frac{1}{|\Delta_2|}\left[L_2\left(a_{11}R^{\sigma_1}+a_{12}R^{\sigma_2}+a_{13}\right)+|\varepsilon_1|\right]\\
&+\frac{1}{k\Gamma(q)}\left[a_{11}R^{\sigma_1}+a_{12}R^{\sigma_2}+a_{13}\right]\\
\le{}& \left[\frac{(|\Delta_1|+|\Delta_2|)L_1+|\Delta_3|L_2}{|\Delta_2\Delta_3|}+\frac{1}{k\Gamma(q)}\right]\\
&\cdot\left(a_{11}R^{\sigma_1}+a_{12}R^{\sigma_2}+a_{13}\right)\\
&+\frac{(|\Delta_1|+|\Delta_2|)|\epsilon_2|+|\Delta_3\epsilon_1|}{|\Delta_2\Delta_3|},
\end{aligned} \tag{29}$$

$$\begin{aligned}
\left|\left({}^{C}D^{q-1}\mathbb{F}u\right)(t)\right| \le{}& \frac{k}{\Gamma(3-q)}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),\right.\right.\\
&\left.\left.f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right|\\
&+\frac{1}{\Gamma(3-q)\Gamma(q)}\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right|\\
&+\frac{1}{\Gamma(3-q)\Gamma(q)}\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right|\\
\le{}& \frac{kL_1\Gamma(q)+2}{\Gamma(3-q)\Gamma(q)}\left(a_{11}R^{\sigma_1}+a_{12}R^{\sigma_2}+a_{13}\right)\\
&+\frac{k|\varepsilon_2|}{\Gamma(3-q)}.
\end{aligned} \tag{30}$$

From (29) and (30), we obtain

$$\|\mathbb{F}u\|_{q-1} \le L_3a_{11}R^{\sigma 1} + L_3a_{12}R^{\sigma 2} + L_4 \le \frac{1}{3}R + \frac{1}{3}R = R. \tag{31}$$

This means $\mathbb{F} : B_R \longrightarrow B_R$.

From the formula (19), it is easy to know that operators $\mathbb{F}(u)(t), {}^{C}D^{q-1}\mathbb{F}(u)(t)$ are continuous on $[0,1]$. Now, we show that operator $\mathbb{F}$ is equicontinuous. Set

$$M = \max_{t\in[0,1]}\left\{\left|f\left(t,u(t),{}^{C}D^{q-1}u(t)\right)\right|\right\}, \quad \forall u \in B_R. \tag{32}$$

Let $t_1, t_2 \in [0,1](t_1 < t_2)$, we have

$$\begin{aligned}
&\left|(\mathbb{F}u)(t_2)-(\mathbb{F}u)(t_1)\right|\\
&\le \frac{\left|e^{-kt_2}-e^{-kt_1}\right|}{|\Delta_3|}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),\right.\right.\\
&\quad\left.\left.f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right| + \int_0^{t_1}\left|e^{-k(t_2-s)}-e^{-k(t_1-s)}\right|\\
&\quad\cdot\left|I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right|ds\\
&\quad+\int_{t_1}^{t_2}\left|e^{-k(t_2-s)}\right|\left|I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right|ds\\
&\le \frac{L_1M+|\varepsilon_2|}{|\Delta_3|}\left(e^{-kt_1}-e^{-kt_2}\right)\\
&\quad+\frac{M}{\Gamma(q)}\int_0^{t_1}\left|e^{-k(t_2-s)}-e^{-k(t_1-s)}\right|ds+\frac{M}{\Gamma(q)}\int_{t_1}^{t_2}e^{-k(t_2-s)}ds\\
&\le \frac{L_1M+|\varepsilon_2|}{|\Delta_3|}\left(e^{-kt_1}-e^{-kt_2}\right)\\
&\quad+\frac{M}{\Gamma(q)}\left[2\left(1-e^{-k(t_2-t_1)}\right)+\left(e^{-kt_1}-e^{-kt_2}\right)\right],
\end{aligned} \tag{33}$$

$$
\begin{aligned}
&\left|\left({}^{C}D^{q-1}\mathbb{F}\right)(u)(t_2)-\left({}^{C}D^{q-1}\mathbb{F}\right)(u)(t_1)\right| \\
&\quad\le \frac{k}{\Gamma(2-q)}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),\right.\right. \\
&\quad\left.\left. f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right|\times\left|\int_0^{t_2}(t_2-s)^{1-q}e^{-ks}ds\right. \\
&\quad\left.-\int_0^{t_1}(t_1-s)^{1-q}e^{-ks}ds\right|+\frac{1}{\Gamma(2-q)} \\
&\quad\cdot\left|\int_0^{t_2}(t_2-s)^{1-q}I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)ds\right. \\
&\quad\left.-\int_0^{t_1}(t_1-s)^{1-q}I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)ds\right| \\
&\quad+\frac{k}{\Gamma(2-q)}\left|\int_0^{t_2}(t_2-s)^{1-q}\left(\int_0^{s}e^{-k(s-m)}I^{q-1}f\left(m,u(m),\right.\right.\right. \\
&\quad\left.\left.\left.{}^{C}D^{q-1}u(m)\right)dm\right)ds-\int_0^{t_1}(t_1-s)^{1-q}\left(\int_0^{s}e^{-k(s-m)}I^{q-1}f\right.\right. \\
&\quad\left.\left.\cdot\left(m,u(m),{}^{C}D^{\alpha-1}u(m)\right)dm\right)ds\right| \\
&\quad\le\frac{kL_1M+k|\varepsilon_2|}{\Gamma(2-q)}\left\{\left|\int_0^{t_1}\left[(t_2-s)^{1-q}-(t_1-s)^{1-q}\right]e^{-ks}ds\right|\right. \\
&\quad\left.+\left|\int_{t_1}^{t_2}(t_2-s)^{1-q}e^{-ks}ds\right|\right\}+\frac{2M}{\Gamma(q)\Gamma(2-q)} \\
&\quad\cdot\left\{\left|\int_0^{t_1}\left[(t_2-s)^{1-q}-(t_1-s)^{1-q}\right]ds\right|+\left|\int_{t_1}^{t_2}(t_2-s)^{1-q}ds\right|\right\}.
\end{aligned}
\tag{34}
$$

In (33) and (34), letting $t_1\longrightarrow t_2$, then,

$$
\begin{aligned}
&\left|(\mathbb{F}u)(t_2)-(\mathbb{F}u)(t_1)\right|\longrightarrow 0,\left|\left({}^{C}D^{q-1}\mathbb{F}u\right)(t_2)-\left({}^{C}D^{q-1}\mathbb{F}u\right)(t_1)\right| \\
&\qquad\longrightarrow 0,
\end{aligned}
\tag{35}
$$

That is, as $t_1\longrightarrow t_2$,

$$
\left\|(\mathbb{F}u)(t_2)-(\mathbb{F}u)(t_2)\right\|_{q-1}\longrightarrow 0. \tag{36}
$$

Therefore, $\mathbb{F}(B_R)$ is an equicontinuous set. Furthermore, it is uniformly bounded because of $\mathbb{F}(B_R)\subset B_R$. Applying the Arzelà-Ascoli theorem, we can infer that $\mathbb{F}$ is a completely continuous operator.

Consider $V=\{u\in B_R\mid u=\mu\mathbb{F}u,0<\mu<1\}$ and show that V is bounded. For $u\in V$, we know $\|u\|_{q-1}<\|(\mathbb{F}u)\|_{q-1}\le R$. By Lemma 7, problem (1) has at least one solution in BR.

Theorem 10. *Suppose that (H0) and (H2) hold. If* $3(L_3a_{23}+L_4)\le\min\{(3L_3a_{21})^{1/(1-\tau_1)},(3L_3a_{22})^{1/(1-\tau_2)}\}$, *then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 9. We just need to make sure that R satisfies $3(L_3a_{23}+L_4)\le R\le\min\{(3L_3a_{21})^{1/(1-\tau_1)},(3L_3a_{22})^{1/(1-\tau_2)}\}$ in B_R.

Theorem 11. *Suppose that (H0) and (H3) hold. If* $L_3(a_{31}+a_{32})<1$, *then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 9, we omit it.

Theorem 12. *Suppose that (H0) and (H4) hold. If* $\max\{(3L_3a_{41})^{1/(1-\rho_1)},3(L_3a_{43}+L_4)\}\le(3L_3a_{42})^{1/(1-\rho_2)}$, *then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 10, we omit it.

Theorem 13. *Suppose that (H0) and (H5) hold. If* $L_3(a_{51}+a_{52})<1$, *then problem (1) has a unique solution.*

Proof. Define $\sup_{t\in[0,1]}|f(t,0,0)|=M<\infty$, such that $r\ge(L_3M+L_4)/(1-L_3(a_{51}+a_{52}))$.

First, we show that $T(B_r)\subset B_r$, where $B_r=\{u|u\in E:\|u\|_{\alpha-1}\le r\}$. For $\forall u\in B_r$, by direct calculation, we have

$$
\begin{aligned}
|(\mathbb{F}u)(t)|&\le\frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)-f(1,0,0)\right.\right. \\
&\quad\left.\left.+f(1,0,0),f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)-f(\eta,0,0)\right.\right. \\
&\quad\left.\left.+f(\eta,0,0)0\right|+\frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)\right.\right.\right. \\
&\quad\left.\left.\left.-f(1,0,0)+f(1,0,0),f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right.\right. \\
&\quad\left.\left.-f(\eta,0,0)+f(\eta,0,0)\right)\right|+\int_0^{t}e^{-k(t-s)}I^{q-1} \\
&\quad\cdot\left[\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)-f(s,0,0)\right|+|f(s,0,0)|\right]ds \\
&\le\frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left[L_1\left(a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right)+|\varepsilon_2|\right] \\
&\quad+\frac{1}{|\Delta_2|}\left[L_2\left(a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right)+|\varepsilon_1|\right] \\
&\quad+\frac{1}{k\Gamma(q)}\left|\left[a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right]\right. \\
&\le\left[\frac{(|\Delta_1|+|\Delta_2|)L_1+|\Delta_3|L_2}{|\Delta_2\Delta_3|}+\frac{1}{k\Gamma(q)}\right]\left[(a_{51}+a_{52})\|u\|_{q-1}\right. \\
&\quad\left.+M\right]+\frac{(|\Delta_1|+|\Delta_2|)|\varepsilon_2|+|\Delta_3\varepsilon_1|}{|\Delta_2\Delta_3|},
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\left|\left({}^{C}D^{q-1}\mathbb{F}u\right)(t)\right|&\le\frac{k}{\Gamma(3-q)}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)\right.\right. \\
&\quad\left.\left.-f(1,0,0)+f(1,0,0),f\left(\eta,u(\eta),\right.\right.\right. \\
&\quad\left.\left.\left.{}^{C}D^{q-1}u(\eta)\right)-f(\eta,0,0)+f(\eta,0,0)\right)\right| \\
&\quad+\frac{2}{\Gamma(3-q)\Gamma(q)}\left[\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right.\right. \\
&\quad\left.\left.-f(s,0,0)\right|+\left|f(s,0,0)\right|\right]
\end{aligned}
$$

$$\leq \frac{k}{\Gamma(3-q)}\left[L_1\left(a_{51}\|u\| + a_{52}\left\|{}^{C}D^{q-1}u\right\| + M\right) + |\varepsilon_2|\right] + \frac{2}{\Gamma(3-q)\Gamma(q)}\left[a_{51}\|u\| + a_{52}\left\|{}^{C}D^{q-1}u\right\| + M\right] \leq \frac{k\Gamma(q)L_1 + 2}{\Gamma(3-q)\Gamma(q)} \cdot \left[(a_{51}+a_{52})\|u\|_{q-1} + M\right] + \frac{k|\varepsilon_2|}{\Gamma(3-q)}. \tag{38}$$

Combining (37) and (38), we obtain

$$\|(\mathbb{F}u)\|_{\alpha-1} \leq L_3(a_{51}+a_{52})\|u\|_{q-1} + L_3M + L_4 \leq r. \tag{39}$$

Now, for any $u, v \in B_r$, we have

$$\begin{aligned}
&|(\mathbb{F}u)(t) - (\mathbb{F}v)(t)| \\
&\leq \frac{|\Delta_1| + |\Delta_2|}{|\Delta_2\Delta_3|}\left|\Phi\left(f\left(1, u(1), {}^{C}D^{q-1}u(1)\right), f\left(\eta, u(\eta), {}^{C}D^{\alpha-1}u(\eta)\right)\right) - \Phi\left(f\left(1, v(1), {}^{C}D^{\alpha-1}v(1)\right), f\left(\eta, v(\eta), {}^{C}D^{\alpha-1}v(\eta)\right)\right)\right| + \frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1, u(1), {}^{C}D^{\alpha-1}u(1)\right), f\left(\eta, u(\eta), {}^{C}D^{\alpha-1}u(\eta)\right)\right) - \Psi\left(f\left(1, v(1), {}^{C}D^{\alpha-1}v(1)\right), f\left(\eta, v(\eta), {}^{C}D^{\alpha-1}v(\eta)\right)\right)\right| \\
&\quad + \int_0^t e^{-k(t-s)}\left(I^{q-1}\left|f\left(s, u(s), {}^{C}D^{\alpha-1}u(s)\right) - f\left(s, v(s), {}^{C}D^{\alpha-1}v(s)\right)\right|ds\right) \\
&\leq \left[\frac{(|\Delta_1| + |\Delta_2|)L_1 + |\Delta_3|L_2}{|\Delta_2\Delta_3|} + \frac{1}{k\Gamma(q)}\right] \cdot \left[a_{51}\|u - v\| + a_{52}\left\|{}^{C}D^{\alpha-1}u - {}^{C}D^{\alpha-1}v\right\|\right] \\
&\leq \left[\frac{(|\Delta_1| + |\Delta_2|)L_1 + |\Delta_3|L_2}{|\Delta_2\Delta_3|} + \frac{1}{k\Gamma(q)}\right] \cdot (a_{51}+a_{52})\|u - v\|_{\alpha-1}.
\end{aligned} \tag{40}$$

Similar to (40), one has

$$\left|\left({}^{C}D^{\alpha-1}\mathbb{F}u\right)(t) - \left({}^{C}D^{\alpha-1}\mathbb{F}v\right)(t)\right| \leq \frac{k\Gamma(q)L_1 + 2}{\Gamma(3-q)\Gamma(q)}(a_{51}+a_{52})\|u - v\|_{\alpha-1}. \tag{41}$$

Therefore,

$$\|(\mathbb{F}u) - (\mathbb{F}v)\|_{\alpha-1} \leq L_3(a_{51}+a_{52})\|u - v\|_{\alpha-1}. \tag{42}$$

Since $L_3(a_{51} + a_{52}) < 1$, $\mathbb{F}$ is a contraction operator. Using Lemma 8, operator $\mathbb{F}$ has a unique fixed point which is the unique solution of problem (1).

In order to illustrate our main results, we consider the following sequential fractional differential equations:

$$\begin{cases} \left({}^{C}D^{1.5} + {}^{C}D^{0.5}\right)u(t) = f\left(t, u(t), {}^{C}D^{0.5}u(t)\right), & t \in (0,1), \\ 0.1u(0) + 0.2u(1) - I^2u(0.5) = 0.01, \\ 0.2u'(0) + 0.1u'(1) + 2I^2u'(0.5) = 0.02, \end{cases} \tag{43}$$

where $q = 1.5$, $k = 1$, $r = 2$, $\eta = 0.5$, $\alpha_1 = 0.1$, $\beta_1 = 0.2$, $\gamma_1 = -1$, $\varepsilon_1 = 0.01$, $\alpha_2 = 0.2$, $\beta_2 = 0.1$, $\gamma_1 = 2$, and $\varepsilon_2 = 0.02$.

Example 1. Let $f\left(t, u(t), {}^{C}D^{0.5}u(t)\right) = 0.01u(t)^{0.2} + 0.02\left({}^{C}D^{0.5}u(t)\right)^{0.1} + 0.15(1-t)^2$. So we have

$$\left|f\left(t, u(t), {}^{C}D^{0.5}u(t)\right)\right| \leq 0.01|u(t)|^{0.2} + 0.02\left|{}^{C}D^{0.5}u(t)\right|^{0.1} + 0.15, \tag{44}$$

where $a_{11} = 0.01$, $a_{12} = 0.02$, $a_{13} = 0.15$, $\sigma_1 = 0.2$, and $\sigma_2 = 0.1$. Theorem 9 implies that problem (43) has at least one solution.

Example 2. Let $f\left(t, u(t), {}^{C}D^{0.5}u(t)\right) = 0.01u(t)^{1.2} + 0.02\left({}^{C}D^{0.5}u(t)\right)^{1.1} + 0.01(1-t)^2$. By direct calculation, we obtain that

$$\begin{aligned}
&\Delta_1 = 0.093438, \Delta_2 = 0.246808, \Delta_3 \\
&\quad = -0.407603, L_1 = 0.323142, L_2 \\
&\quad = 0.161275, L_3 = 5.785852, L_4 = 0.130729.
\end{aligned} \tag{45}$$

In the meantime, we have

$$\left|f\left(t, u(t), {}^{C}D^{0.5}u(t)\right)\right| \leq 0.01|u(t)|^{1.2} + 0.02\left|{}^{C}D^{0.5}u(t)\right|^{1.1} + 0.01, \tag{46}$$

where $a_{21} = 0.01$, $a_{22} = 0.22$, $a_{23} = 0.01$, $\tau_1 = 1.2$, and $\tau_2 = 1.1$. So we obtain $3(L_3a_{23} + L_4) = 0.565763 < \min\left\{(3L_3a_{21})^{1/(1-\tau_1)}, (3L_3a_{22})^{1/(1-\tau_2)}\right\} = 1.111599$. Theorem 10 implies that problem (43) has at least one solution.

Example 3. Let $f\left(t, u(t), {}^{C}D^{0.5}u(t)\right) = 0.01u(t) + 0.02{}^{C}D^{0.5}u(t) + 0.01(1-t)^2$. So we have

$$\left|f\left(t, u(t), {}^{C}D^{0.5}u(t)\right) - f\left(t, v(t), {}^{C}D^{0.5}v(t)\right)\right| \leq 0.01|u(t) - v(t)| + 0.02\left|{}^{C}D^{0.5}u(t) - {}^{C}D^{0.5}v(t)\right|, \tag{47}$$

where $a_{51} = 0.01$ and $a_{52} = 0.02$. Combining with the calculation result of Example 2, we can obtain that $L_3(a_{51} + a_{52}) = 0.173576 < 1$. Hence, Theorem 13 implies that problem (43) has a unique solution.

4. Conclusions

We have obtained some existence results for new sequential fractional differential equations by using some nonlinear growth conditions

$$|f(t,u(v))| \le b_{11}|u|^{\theta_1} + b_{12}|v|^{\theta_1} + b_{13}, \quad \theta_1, \theta_2 \in (0, +\infty), \tag{48}$$

which is different from the existing linear condition. Obviously, these results are easy to verify and apply (see Example 1 and Example 3).

On the other hand, we note that our results contain some special types of results by fixing the parameters in the given problem (1). For instance, let $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = 0$, $\alpha_2 = 1$, $\beta_2 = 1$, and $\gamma_2 = 0$, then the results of this paper are the following sequential fractional differential equations with the boundary value conditions of the form $u(0) + u(1) = \epsilon_1$ and $u'(0) + u'(1) = \epsilon_2$. Further, letting $\alpha_1 = 0$, $\beta_1 = 1$, $\gamma_1 = -1$, $\epsilon_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$, $\gamma_2 = -1$, and $\epsilon_2 = 0$, we obtain the results for the boundary conditions $u(1) = I^r u(\eta)$ and $u'(1) = I^r u'(\eta)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Positive Solutions for Elliptic Problems with the Nonlinearity Containing Singularity and Hardy-Sobolev Exponents

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In this paper, we study multiplicity of positive solutions for a class of semilinear elliptic equations with the nonlinearity containing singularity and Hardy-Sobolev exponents. Using variational methods, we establish the existence and multiplicity of positive solutions for the problem.

1. Introduction and Main Results

Consider the following semilinear elliptic equations with Dirichlet boundary value conditions:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s} u + \lambda u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, $0 < s < 2$, $2^*(s) = 2(N-s)/N-2$ is the Hardy-Sobolev critical exponent, $2^* = 2^*(0) = 2N/(N-2)$ is the Sobolev critical exponent, $\mu < \bar{\mu} = {}^{\Delta} (N-2)^2/4$, and $\gamma \in (0,1)$.

The energy functional associated with problem (1) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\, dx, \tag{2}$$

for any $u \in H_0^1(\Omega)$. In general, a function u is called a weak solution of problem (1) if $u \in H_0^1(\Omega)$ and $u(x) > 0$ for all $x \in \Omega$; it holds

$$\int_\Omega \left(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} - \frac{|u|^{2^*(s)-1}}{|x|^s} v - \lambda |u|^{-\gamma} v\right) dx = 0, \quad \forall v \in H_0^1(\Omega). \tag{3}$$

The paper by Crandall et al. [1] is the starting point on semilinear problem with singular nonlinearity. There is a large literature on singular nonlinearities (see [2–14] and references therein). In particular, the following Dirichlet problem

$$\begin{cases} -\Delta u = u^\beta + \mu u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{4}$$

has been shown in [2], in which the authors proved that problem (4) possesses at least one solution for $\mu > 0$ small enough, and there exists no solution when μ is large. Chabrowski in [15] considered the Neumann boundary problems with singular superlinear nonlinearities by approximation and variational methods. When the superlinear term is subcritical, he obtained two solutions, a mountain-pass solution and a local minimizer solution. And, in the critical case, he obtained a local minimizer solution and proved that there is no moutain-pass solution.

In recent years, people have paid much attention to the existence of solutions for problems with the Sobolev critical

exponent (the case that $s = 0$) (see [16–21] and the references therein); some authors also considered the singular problems with the Hardy-Sobolev critical exponent (the case that $s \neq 0$) (see [22–27] and the references therein).

Up to our knowledge, the literature does not contain any result on the existence of positive solutions to the problem (1) with the nonlinearity containing singularity and Hardy-Sobolev exponents. Motivated by reasons above, the aim of this paper is to show the existence of positive solutions of problem (1). We study problem (1) and obtain at least two solutions via the Nehari method. It is well-known that the singular term leads to the nondifferentiability of the functional I on $H_0^1(\Omega)$, so I does not belong to $C^1(H_0^1(\Omega), \mathbb{R})$. In order to get the existence of multiple positive solutions of problem (1), we use the Nehari method and differentiate the two solutions by their different Nehari-type sets.

The main result can be described as follows.

Theorem 1. *Let $0 < s < 2$, $\mu < \bar{\mu} = {}^{\Delta} (N-2)^2/4$ and $\gamma \in (0, 1)$. Then, there exists $\lambda^* > 0$ small enough, such that problem (1) has at least two positive solutions for any $\lambda \in (0, \lambda^*)$.*

The paper is organized as follows: in Section 2, we give some preliminaries; in Section 3, we prove Theorem 1. This idea is essentially introduced in 20]. Throughout this paper, we make use of the following notations:

(i) The norm in $H_0^1(\Omega)$ is denoted by

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \tag{5}$$

By Hardy inequality [28], we easily derive that the norm is equivalent to the usual norm:

$$\|u\|_0^2 = \int_\Omega |\nabla u|^2 dx \tag{6}$$

(ii) $D^{1,2}(\mathbb{R}^N)(N \geq 3)$ denotes the space of the functions $u \in L^{2^*}(\mathbb{R}^N)$ such that $|\nabla u| \in L^2(\mathbb{R}^N)$, endowed with scalar product and norm, respectively, given by

$$\begin{aligned} \langle u, v \rangle_{\mathbb{R}^N} &= \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} \right) dx, \\ \|u\|_{\mathbb{R}^N}^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \end{aligned} \tag{7}$$

that coincides with the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the L^2-norm of the gradient. By Hardy inequality [28], we easily derive that the norm is equivalent to the usual norm:

$$\|u\|_{0,\mathbb{R}^N}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx \tag{8}$$

in $D^{1,2}(\mathbb{R}^N)$.

(iii) The norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$

(iv) $C, C_0, C_1, C_2, \cdots$ denote positive constants

2. Preliminaries

In this section, we will study the unperturbed problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s} u, & x \in \mathbb{R}^N, \\ u \in D_r^{1,2}(\mathbb{R}^N), u > 0, & x \in \mathbb{R}^N. \end{cases} \tag{9}$$

It is well-known that the nontrivial solutions of problem (9) are equivalent to the nonzero critical points of the energy functional

$$\begin{aligned} I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \\ - \frac{1}{2^*(s)} \int_{\mathbb{R}^N} \frac{|u^+|^{2^*(s)}}{|x|^s} dx, \quad u \in D_r^{1,2}(\mathbb{R}^N). \end{aligned} \tag{10}$$

Obviously, the energy functional $I_0(u)$ is well-defined and is of C^2 with derivatives given by

$$\begin{aligned} \left\langle I_0'(u), v \right\rangle = \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} \right) dx \\ - \int_{\mathbb{R}^N} \frac{|u^+|^{2^*(s)-1}}{|x|^s} v \, dx, \quad u, v \in D_r^{1,2}(\mathbb{R}^N), \end{aligned} \tag{11}$$

$$\begin{aligned} \left\langle I_0''(u)v, w \right\rangle = \int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla w - \mu \frac{vw}{|x|^2} \right) dx \\ - \int_{\mathbb{R}^N} \frac{(2^*(s)-1)|u^+|^{2^*(s)-2}}{|x|^s} vw \, dx, \\ u, v, w \in D_r^{1,2}(\mathbb{R}^N). \end{aligned} \tag{12}$$

For all $\varepsilon > 0$, it is well-known that the function

$$\begin{aligned} z_\varepsilon(x) = \left(\frac{2\varepsilon^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}} (N-s)(\bar{\mu}-\mu)}{\sqrt{\bar{\mu}}} \right)^{\sqrt{\bar{\mu}}/2-s} / \\ \cdot \left(|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} \left(\varepsilon^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}} + |x|^{(2-s)\sqrt{\bar{\mu}-\mu}/\sqrt{\bar{\mu}}} \right. \right. \end{aligned} \tag{13}$$

solves equation (9) and satisfies

$$\int_{\mathbb{R}^N} \left(|\nabla z_\varepsilon|^2 - \mu \frac{z_\varepsilon^2}{|x|^2} \right) dx = \int_{\mathbb{R}^N} \frac{|z_\varepsilon|^{2^*(s)}}{|x|^s} dx. \tag{14}$$

Moreover $z_\varepsilon(x)$ is the extremal function of the minimization problem

$$S := \inf\left\{\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\left(u^2/|x|^2\right)\right)dx}{\left(\int_{\mathbb{R}^N}\left(|u|^{2^*(s)}/|x|^s\right)dx\right)^{2/2^*(s)}} : u\in D^{1,2}\left(\mathbb{R}^N\right),\quad u\neq 0\right\},$$
$$S^{N-s/2-s}=\int_{\mathbb{R}^N}\left(|\nabla z_\varepsilon|^2-\mu\left(z_\varepsilon^2/|x|^2\right)\right)dx=\int_{\mathbb{R}^N}|z_\varepsilon|^{2^*(s)}/|x|^s dx. \tag{15}$$

In view of [27, 29], we have the following exact local behavior of the solutions of (1).

Lemma 2. *Let $0<s<2$, $\mu<\bar{\mu}=^{\Delta}(N-2)^2/4$. If $u\in H_0^1(\Omega)$ is a positive solution of (1), then there exists $r>0$ small enough and some positive constants C_1 and C_2 such that*

$$C_1|x|^{-\left(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}\right)}\le|u(x)|\le C_2|x|^{-\left(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}\right)},\quad x\in B(0,r)\setminus\{0\}. \tag{16}$$

Define $\zeta\in C_0^1(\Omega)$ such that $|\zeta(x)|\le 1$, $|\nabla\zeta(x)|\le c$ for any $x\in\Omega$, $\zeta(x)\ge 0$ for all $\in\Omega$, $0\in\Omega$, and

$$\zeta(x)=\begin{cases}1, & \text{if}\quad |x|\le\delta,\\ 0, & \text{if}\quad |x|\ge 2\delta.\end{cases} \tag{17}$$

Denote $v_\varepsilon(x)=\zeta(x)z_\varepsilon(x)$. Then, using an argument similar to [30], we have the following lemma.

Lemma 3. *Let $u\in H_0^1(\Omega)$ be a weak solution of problem (1). Then, for $\varepsilon>0$ small enough*

$$\begin{aligned}
&\int_\Omega\frac{|u|^{2^*(s)-1}}{|x|^s}v_\varepsilon dx=O\left(\varepsilon^{N-2/4-2s}\right),\\
&\int_\Omega u\frac{|v_\varepsilon|^{2^*(s)-1}}{|x|^s}dx=O\left(\varepsilon^{N-2/4-2s}\right),\\
&\int_\Omega\left(|\nabla v_\varepsilon|^2-\mu\frac{v_\varepsilon^2}{|x|^2}\right)dx=S^{N-s/2-s}+O\left(\varepsilon^{N-s/2-s}\right)+O\left(\varepsilon^{N-2/2-s}\right),\\
&\int_\Omega\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}dx\ge S^{N-s/2-s}-O\left(\varepsilon^{N-2/2-s}\right),\\
&\int_\Omega|v_\varepsilon|^p dx=O\left(\varepsilon^{p(N-2)/4-2s}\right).
\end{aligned} \tag{18}$$

Next, we define some Nehari-type sets, which are relevant in getting multiple positive solutions. Denote

$$\begin{aligned}
\mathcal{N}_\lambda&=\left\{u\in H_0^1(\Omega):\ \|u\|^2=\int_\Omega\left(\frac{|u|^{2^*(s)}}{|x|^s}+\lambda\,|u|^{1-\gamma}\right)dx,\quad u\neq 0\right\},\\
\mathcal{N}_\lambda^+&=\left\{u\in\mathcal{N}_\lambda:(1+\gamma)\|u\|^2>(2^*(s)-1+\gamma)\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx\right\},\\
\mathcal{N}_\lambda^0&=\left\{u\in\mathcal{N}_\lambda:(1+\gamma)\|u\|^2=(2^*(s)-1+\gamma)\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx\right\},\\
\mathcal{N}_\lambda^-&=\left\{u\in\mathcal{N}_\lambda:(1+\gamma)\|u\|^2<(2^*(s)-1+\gamma)\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx\right\}.
\end{aligned} \tag{19}$$

Define the minimization problems

$$d_+(\lambda):=\inf_{u\in\mathcal{N}_\lambda^+}I_\lambda(u). \tag{20}$$

Since

$$I_\lambda(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx-\frac{1}{2^*(s)}\int_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx-\frac{\lambda}{1-\gamma}\int_\Omega|u|^{1-\gamma}dx, \tag{21}$$

it is easy to see that $d_+(\lambda)<0$ for $\lambda>0$ and $d_+(\lambda)\longrightarrow 0$ as $\lambda\longrightarrow 0$. Take $\lambda_3>0$ such that $d_+(\lambda)+N^{-1}S^{N-s/2-s}>0$ for any $\lambda\in(0,\lambda_3)$. Denote

$$\lambda_4=\frac{2^*(s)-2}{2^*(s)+\gamma-1}S^{(1+\gamma)(N-s/4-2s)+1-\gamma/2}\cdot\left(\frac{1+\gamma}{2^*(s)-\gamma-1}\right)^{(1+\gamma)(N-2/4-2s)}|\Omega|^{(1-2^*(s)-\gamma)/2^*(s)}, \tag{22}$$

and set $\lambda^*=\min\{\lambda_3,\lambda_4\}$.

Lemma 4. *If $\lambda\in(0,\lambda^*)$, then $\mathcal{N}_\lambda^0=\varnothing$. Moreover, for any $u\neq 0$, there exists a unique $t^+=t^+(u)>0$ such that $t^+(u)u\in\mathcal{N}_\lambda^-$ and*

$$\begin{aligned}
t^+>T_m&:=\left(\frac{\|u\|^2}{(2^*(s)-1)\int_\Omega\left(|u|^{2^*(s)}/|x|^s\right)dx}\right)^{1/(2^*(s)-2)},\\
I_\lambda(t^+u)&=\max_{t\ge T_m}I_\lambda(tu),
\end{aligned} \tag{23}$$

and there exists a unique $t^-=t^-(u)>0$ such that $t^-(u)u\in\mathcal{N}_\lambda^+$ and

$$t^-<T_m\quad\text{and}\quad I_\lambda(t^-u)=\inf_{0\le t\le T_m}I_\lambda(tu). \tag{24}$$

Proof. The proof is similar to [30]. We omit the details.

3. Proof of Theorem 1

In this section, we will prove Theorem 1. The proof of Theorem 1 is based on solving the minimization problem (18) and the minimization problem

$$d_-(\lambda) := \inf_{u\in\mathcal{N}^-} I_\lambda(u). \tag{25}$$

We divide the proof into two steps. In the first step, we prove that if there is $w \in \mathcal{N}_\lambda^+$ such that $d_+(\lambda) = I_\lambda(w)$ and there is $v \in \mathcal{N}_\lambda^-$ such that $d_-(\lambda) = I_\lambda(v)$, then w and v are two positive weak solutions of (1). In the second step, we prove that the minima $d_+(\lambda)$ in (18) and $d_-(\lambda)$ in (23) are achieved, respectively.

Step 1. Let $w \in \mathcal{N}_\lambda^+$ be such that $d_+(\lambda) = I_\lambda(w)$ and $v \in \mathcal{N}_\lambda^-$ such that $d_-(\lambda) = I_\lambda(v)$.

Lemma 5. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geq 0$, we have the following:*

There is η_0 such that $I_\lambda(w+\eta\varphi) \geq I_\lambda(w)$ for each $0 \leq \eta \leq \eta_0$

(ii) $t_\eta^- \longrightarrow 1$ as $\eta \longrightarrow 0^+$, where for each $\eta \geq 0$, t_η^- is the unique positive number satisfying $t_\eta^-(v+\eta\varphi) \in \mathcal{N}_\lambda^-$

Proof. The proof follows exactly the scheme in the proof of Lemma 3 in [31].

Lemma 6. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geq 0$, we have $w^{-\gamma}\varphi$ and $v^{-\gamma}\varphi \in L^1(\Omega)$. Moreover,*

$$\begin{aligned}&\int_\Omega \left(\nabla w\nabla\varphi - \mu\frac{w\varphi}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\varphi - \lambda|w|^{-\gamma}\varphi\right)dx \geq 0,\\ &\int_\Omega \left(\nabla v\nabla\varphi - \mu\frac{v\varphi}{|x|^2} - \frac{|v|^{2^*(s)-1}}{|x|^s}\varphi - \lambda|v|^{-\gamma}\varphi\right)dx \geq 0.\end{aligned} \tag{26}$$

In particular, $w, v > 0$ for all $x \in \Omega \setminus \{0\}$.

Proof. We only prove (24) since the proof of (25) is similar. Let $\varphi \geq 0$ and $\varepsilon > 0$. By (i) of Lemma 5 and simple computations, we have that

$$\begin{aligned}&\frac{\lambda}{1-\gamma}\int_\Omega \frac{(w+\varepsilon\varphi)^{1-\gamma} - w^{1-\gamma}}{\varepsilon}\mathrm{d}x\\ &\quad\leq \frac{1}{2\varepsilon}\left(\|w+\varepsilon\varphi\|^2 - \|w\|^2\right)\\ &\qquad - \frac{1}{2^*(s)\varepsilon}\int_\Omega\left(\frac{|w+\varepsilon\varphi|^{2^*(s)}}{|x|^s} - \frac{|w|^{2^*(s)}}{|x|^s}\right)\mathrm{d}x.\end{aligned} \tag{27}$$

Since the right hand side of the inequality has a finite limit value as $\varepsilon \longrightarrow 0$, by direct calculations, we conclude $\varepsilon^{-1}((w+\varepsilon\varphi)^{1-\gamma} - w^{1-\gamma})$ increases monotonically as $\varepsilon \longrightarrow 0$ and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\left((w+\varepsilon\varphi)^{1-\gamma} - w^{1-\gamma}\right)}{\varepsilon}\\ &\quad= \begin{cases} 0, & \text{if} \quad \varphi(x) = 0,\\ (1-\gamma)w^{-\gamma}\varphi, & \text{if} \quad \varphi(x) > 0 \quad \text{and} \quad w(x) > 0,\\ \infty, & \text{if} \quad \varphi(x) > 0 \quad \text{and} \quad w(x) = 0.\end{cases}\end{aligned} \tag{28}$$

The Fatou lemma yields $w^{-\gamma}\varphi \in L^1(\Omega)$ and we get (24).

Since $w, v \geq 0$ and $w, v \not\equiv 0$, by the strong maximum principle, it follows that

$$w, v > 0, \quad \forall x \in \Omega \setminus \{0\}. \tag{29}$$

Lemma 7. *We have that w and v are positive weak solutions of (1).*

Proof. For any $\varphi \in H_0^1(\Omega)$ and $\eta > 0$, we define $\psi = (w+\eta\varphi)$ and $\psi^+ = \max\{\psi, 0\}$. Then, $\psi^+ \in H_0^1(\Omega)$. Since $w \in \mathcal{N}_\lambda$, we obtain from (24) that

$$\begin{aligned}0 &\leq \int_\Omega\left(\nabla w\nabla\psi^+ - \mu\frac{w\psi^+}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\psi^+ - \lambda\mid w\mid^{-\gamma}\psi^+\right)\mathrm{d}x\\ &= \int_{\{x\in\Omega, w+\eta\varphi>0\}}\left(\nabla w\nabla\psi^+ - \mu\frac{w\psi^+}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\psi^+ - \lambda|w|^{-\gamma}\psi^+\right)\mathrm{d}x\\ &= \int_\Omega\left(\nabla w\nabla\psi - \mu\frac{w\psi}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\psi - \lambda|w|^{-\gamma}\psi\right)\mathrm{d}x\\ &\quad - \int_{\{x\in\Omega, w+\eta\varphi\leq 0\}}\left(\nabla w\nabla\psi - \mu\frac{w\psi}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\psi - \lambda|w|^{-\gamma}\psi\right)\mathrm{d}x\\ &\leq \eta\int_\Omega\left(\nabla w\nabla\varphi - \mu\frac{w\varphi}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\varphi - \lambda|w|^{-\gamma}\varphi\right)\mathrm{d}x\\ &\quad - \eta\int_{\{x\in\Omega, w+\eta\varphi\leq 0\}}\left(\nabla w\nabla\varphi - \mu\frac{u\varphi}{|x|^2}\right)\mathrm{d}x.\end{aligned} \tag{30}$$

Dividing by η and letting $\eta \longrightarrow 0$, by $w \geq 0, \forall x \in \Omega$, we have the measure of $\{x \in \Omega, w+\eta\varphi \leq 0\}$ which tends to 0 as $\eta \longrightarrow 0$, and we get that

$$\int_{\{x\in\Omega, w+\eta\varphi\leq 0\}} \nabla w\nabla\varphi\mathrm{d}x \longrightarrow 0. \tag{31}$$

Therefore,

$$\int_\Omega\left(\nabla w\nabla\varphi - \mu\frac{w\varphi}{|x|^2} - \frac{|w|^{2^*(s)-1}}{|x|^s}\varphi - \lambda|w|^{-\gamma}\varphi\right)\mathrm{d}x \geq 0. \tag{32}$$

Since φ is arbitrary, we get that w is a solution of (1). Similarly, we can prove that v is also a solution of (1).

Step 2. The minima $d_+(\lambda)$ and $d_-(\lambda)$ are achieved. The proof is exactly the same as [32]. We omit the details here.

We point out that v_ε and the exact local behavior of w (see Lemma 2.) play essential roles. From Lemma 2., we have

$$C_1|x|^{-\left(\sqrt{\mu}-\sqrt{\mu-\mu}\right)} \le |w(x)| \le C_2|x|^{-\left(\sqrt{\mu}-\sqrt{\mu-\mu}\right)}, \quad x \in B(0,r)\setminus\{0\}. \tag{33}$$

So there is $m > 0$ such that $w(x) \ge m$ for $x \in \text{supp } w \setminus \{0\}$.

Lemma 8. *Under the assumptions of Theorem 1,*

$$d_-(\lambda) < I_\lambda(w) + \frac{2-s}{2(N-s)}S^{N-s/2-s}. \tag{34}$$

Proof. First, using an argument similar to the proofs in [31], we have $\eta_* > 0$ such that $w + \eta_* v_\varepsilon \in \mathcal{N}_\lambda^-$. It remains to be proven that

$$\sup\{I_\lambda(w+\eta v_\varepsilon): \eta > 0\} < I_\lambda(w) + \frac{2-s}{2(N-s)}S^{N-s/2-s}. \tag{35}$$

Since w is a solution, we obtain by direct computation that

$$\begin{aligned} I_\lambda(w+\eta v_\varepsilon) - I_\lambda(w) &= \frac{\eta^2}{2}\|v_\varepsilon\|^2 + \eta\int_\Omega\left(\nabla w\nabla v_\varepsilon - \mu\frac{wv_\varepsilon}{|x|^2}\right)dx \\ &\quad - \int_\Omega\left(\frac{|w+\eta v_\varepsilon|^{2^*(s)} - w^{2^*(s)}}{2^*(s)|x|^s}\right)dx \\ &\quad - \lambda\int_\Omega\left(\frac{(w+\eta v_\varepsilon)^{1-\gamma}}{1-\gamma} - \frac{w^{1-\gamma}}{1-\gamma}\right)dx \\ &= \frac{\eta^2}{2}\|v_\varepsilon\|^2 - \int_\Omega\left(\frac{|w+\eta v_\varepsilon|^{2^*(s)} - w^{2^*(s)}}{2^*(s)|x|^s}\right. \\ &\quad \left. - \frac{|w|^{2^*(s)-1}\eta v_\varepsilon}{|x|^s}\right)dx - \lambda\int_\Omega\left(\frac{(w+\eta v_\varepsilon)^{1-\gamma}}{1-\gamma}\right. \\ &\quad \left. - \frac{w^{1-\gamma}}{1-\gamma} - w^{-\gamma}\eta v_\varepsilon\right)dx. \end{aligned} \tag{36}$$

Note that the following inequality (see [31]) holds: there is $\alpha > 0$ and $0 < \delta < N - s/N - 2$ such that $\lambda((r+t)^{1-\gamma}/1-\gamma - r^{1-\gamma}/1-\gamma - r^{-\gamma}t) \ge -\alpha t^\delta$ for each r≥m and t≥0.

By direct calculations, we can get that

$$\begin{aligned} I_\lambda(w+\eta v_\varepsilon) - I_\lambda(w) &\le \frac{\eta^2}{2}\|v_\varepsilon\|^2 - \frac{\eta^{2^*(s)}}{2^*(s)}\int_\Omega\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}dx \\ &\quad - \eta^{2^*(s)-1}\int_\Omega w\frac{|v_\varepsilon|^{2^*(s)-1}}{|x|^s}dx \\ &\quad + \alpha\eta^\delta\int_\Omega|v_\varepsilon|^\delta dx. \end{aligned} \tag{37}$$

So $I_\lambda(w+\eta v_\varepsilon) - I_\lambda(w) \longrightarrow 0$ as $\eta \longrightarrow 0$ and $I_\lambda(w+\eta v_\varepsilon) - I_\lambda(w) \longrightarrow -\infty$ as $\eta \longrightarrow +\infty$. Hence, we only consider the right-hand side of the above inequality in the case of $\eta \in [\eta_0, \eta_1]$ for some $0 < \eta_0 < \eta_1 < +\infty$. Hence, we obtain from Lemma 3 that

$$\begin{aligned} \sup_{\eta>0} I_\lambda(w+\eta v_\varepsilon) - I_\lambda(w) &\le \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{22^*(s)/2^*(s)-2} \\ &\quad - \left(\int_\Omega\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}dx\right)^{-2/2^*(s)-2} \\ &\quad - O\left(\varepsilon^{N-2/4-2s}\right) + O\left(\varepsilon^{\delta(N-2/4-2s)}\right) \\ &= \frac{2-s}{2(N-s)}S^{N-s/2-s} - O\left(\varepsilon^{N-s/4-2s}\right) \\ &\quad - O\left(\varepsilon^{N-2/4-2s}\right) + O\left(\varepsilon^{\delta(N-2/4-2s)}\right) \\ &< \frac{2-s}{2(N-s)}S^{N-s/2-s}, \\ &\quad \text{for } \varepsilon > 0 \text{ sufficiently small.} \end{aligned} \tag{38}$$

The proof is complete.

Lemma 9. *The minimum $d_-(\lambda)$ in (23) is achieved by $v \in \mathcal{N}_\lambda^-$ with $I_\lambda(v) = d_-(\lambda)$.*

Proof. Let $\{v_n\}$ be any sequence in $\mathcal{N}_\lambda^-$ such that $I_\lambda(v_n) \longrightarrow d_-(\lambda)$. It is easy to see that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Then, there exists a $v \in H_0^1(\Omega)$ and a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$ such that

$$v_n \rightharpoonup v, \quad \text{in} \quad H_0^1(\Omega). \tag{39}$$

Set $u_n = v_n - v$ and assume that

$$\begin{aligned} \|u_n\|^2 &\longrightarrow a^2, \\ \int_\Omega\frac{|u_n|^{2^*(s)}}{|x|^s}dx &\longrightarrow b^{2^*(s)}. \end{aligned} \tag{40}$$

Since $v_n \in \mathcal{N}_\lambda^-$, according to Bre'zis-Lieb's Lemma (see Lemma 1.32 in MW [33]) and the Sobolev embedding theorem, one gets

$$a^2 + \|v\|^2 = \lambda\int_\Omega|v|^{1-\gamma}\,dx + b^{2^*(s)} + \int_\Omega\frac{|v|^{2^*(s)}}{|x|^s}\,dx. \tag{41}$$

We claim that $v \ne 0$. Indeed, if $v = 0$, then $a^2 = b^{2^*} \ge S^{N-s/2-s} \ne 0$ (since for any $u \in \mathcal{N}_\lambda^-$, $\|u\|$ is bounded away from zero) and this means that

$$\begin{aligned} d_-(\lambda) &= \lim_{n\longrightarrow\infty} I_\lambda(v_n) = I_\lambda(0) + \frac{1}{2}a^2 - \frac{1}{2^*(s)}b^{2^*(s)} \\ &\ge \frac{2-s}{2(N-s)}S^{N-s/2-s}, \end{aligned} \tag{42}$$

which contradicts the previous lemma.

From the assumption $\lambda \in (0, \lambda^*)$, by Lemma 4, we have $0 < t^+ < T_m < t^-$ such that $t^+ v \in \mathcal{N}_\lambda^+$ and $t^- v \in \mathcal{N}_\lambda^-$. For $t > 0$, we define

$$\begin{aligned} \delta(t) &= \frac{1}{2} a^2 t^2 - \frac{1}{2^*(s)} b^{2^*(s)} t^{2^*(s)}, \\ g(t) &= I_\lambda(tv) + \delta(t). \end{aligned} \tag{43}$$

Now, we consider the cases

(i) $t^- < 1$

(ii) $t^- \geq 1$ and $b > 0$

(iii) $t^- \geq 1$ and $b = 0$

Case (i). From $t^- < 1$,

$$g'(t^-) = \langle I(t^- v), v \rangle + \delta'(t^-) = a^2 t^- - b^{2^*(s)} (t^-)^{2^*(s)-1}. \tag{44}$$

Define $h(t) \doteq a^2 t - b^{2^*(s)} (t)^{2^*(s)-1}$, then $h(t)$ is increasing as $t < T_m$;; $h(t)$ is decreasing as $t > T_m$, so $h(t)$ has maximum value $h(T_m)$. As $t^- > T_m$, we have $g'(t^-) \geq h(T_m) > 0$. $g'(1) = 0$ and $g'(t^-) > 0$, so we can see that g is increasing on $[t^-, 1]$. Then, we have

$$\begin{aligned} d_-(\lambda) = g(1) > g(t^-) \geq I_\lambda(t^- v) = I_\lambda(0) + \frac{(t^-)^2}{2} \left(a^2 - b^{2^*(s)} \right) \\ > I_\lambda(t^- v) \geq d_-(\lambda), \end{aligned} \tag{45}$$

which is a contradiction.

Case (ii). We set $t_0 = \left(a^2 / b^{2^*(s)}\right)^{N-2/2(2-s)}$. We know that δ attains the unique maximum at t_0 and $\delta(t_0) \geq (2 - s/2(N - s)) S^{N-s/2-s}$. Moreover, $\delta'(t) > 0$ for $0 < t < t_0$ and $\delta'(t) < 0$ for $t > t_0$.

From the assumption on $\lambda \in (0, \lambda^*)$, we know $g(1) \geq g(t_0)$. If $t_0 \leq 1$, we have

$$\begin{aligned} d_-(\lambda) = g(1) \geq g(t_0) = I_\lambda(t_0 v) + \delta(t_0) \\ \geq I_\lambda(t_0 v) + \frac{2 - s}{2(N - s)} S^{N-s/2-s}, \end{aligned} \tag{46}$$

which contradicts the previous lemma. Thus, we have $t_0 > 1$. By virtue of $g'(t) \leq 0$ for $t \geq 1$, we obtain $(\partial/\partial t) I_\lambda(tv) \leq -\delta'(t) \leq 0$ for $1 \leq t \leq t_0$ and

$$\begin{aligned} d_-(\lambda) = g(1) = I_\lambda(v) + \frac{1}{2} a^2 - \frac{1}{2^*(s)} b^{2^*(s)} \\ \geq I_\lambda(v) + \frac{2 - s}{2(N - s)} S^{N-s/2-s}, \end{aligned} \tag{47}$$

which also contradicts the previous lemma.

Case (iii). If $a \neq 0$, then we obtain from the fact that $v_n \in \mathcal{N}_\lambda^-$ by some computations that $(\partial/\partial t) I_\lambda(tv)|_{t=1} < 0$ and $(\partial^2/\partial^2 t) I_\lambda(tv)|_{t=1} < 0$, from the definition of $\mathcal{N}_\lambda^-$, which contradicts $t^- \geq 1$. So

$$\begin{aligned} &a = 0, \\ &v_n \longrightarrow v \quad \text{in} \quad H_0^1(\Omega). \end{aligned} \tag{48}$$

Hence, we have $v \in \mathcal{N}_\lambda^-$ with $I_\lambda(v) = d_-(\lambda)$.
The proof is complete.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

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Research Article

Uniqueness of Iterative Positive Solution to Nonlinear Fractional Differential Equations with Negatively Perturbed Term

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We prove that there are unique positive solutions for a new kind of fractional differential equation with a negatively perturbed term boundary value problem. Our methods rely on an iterative algorithm which requires constructing an iterative scheme to approximate the solution. This allows us to calculate the estimation of the convergence rate and the approximation error.

1. Introduction

In this work, we present the uniqueness of a positive solution for the following boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{p}x(t) + p(t)f(t, x(t), y(t)) + q(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases} \tag{1}$$

where D_{0+}^{p} is the standard fractional derivative of order p satisfying $2 < p \le 3$, and $f(t, x, y)$ may be singular at $y = 0$, $t = 0, 1$.

Fractional calculus differential equations are an important branch of differential equations. In recent years, it has attracted the interest of many researchers and has become a hot-button issue [1–14]. Compared with the integer order, it has a wider range of applications as it can be used to describespecific problems more precisely, such as the problem in complex analysis, polymer rheology, physical chemistry, electrical networks, and many other branches of science, For specific applications, see [15, 16, 20–28]. In [3], the authors study the following BVP:

$$\begin{cases} -D_{0+}^{p}x(t) = p(t)f(t, x(t)) - q(t), & 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 0. \end{cases} \tag{2}$$

They obtained the existence of multiple positive solutions by means of the Gou-Krasnoselskii fixed point theorem. In [4], the authors also study the same BVP. By constructing a special u_0-positive operator and using its properties, they obtained a unique solution for BVP (2).

BVP (1) is more general than the problem in papers [3] and [4] in four aspects. First, the nonlinear term has two space variables and can be singular with respect to the second space variable. Second, the method we used is different from papers [3] and [4]. Through constructing an iterative process which can be from any initial value, only by the iterative algorithm, we can prove that it converges uniformly to the unique positive solution. Third, we calculate the estimation of the convergence rate and the approximation error. Finally, we compare with it [3] and [4]. We lower the conditional constraint on the nonlinear term since we do not need the monotone of the nonlinear term. So the result of this paper is most general, not only did it weaken the restrictions but it also strengthened the conclusions in [3] and [4]. Also, we can show that the main result in [4] is a corollary of our work.

The rest of our presentation is as follows. In Section 2, we recall some definitions and lemmas. In Section 3, we establish the result of the uniqueness of the positive solutions to BVP (1).

Finally, in Section 4, an illustrative example is also presented.

2. Preliminaries

We first list some definitions and lemmas which will be used later.

Definition 1 (see [5]). Let $p > 0$, and the Riemann-Liouvill standard fractional integral and the Riemann-Liouvill standard fractional derivative of order $p > 0$ of a function $f : (0,\infty) \longrightarrow R$ are given by

$$I_{0+}^{p} f(x) = \frac{1}{\Gamma(p)} \int_0^x \frac{f(t)}{(x-t)^{1-p}}\, dt,$$
$$D_{0+}^{p} f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}}\, ds, \tag{3}$$

where $n = [p] + 1$ and $[p]$ denotes the integer part of the real number p and provides that the right side integral is pointwise defined in $[0, \infty)$.

Lemma 2 (see [3]). *Suppose that $2 < p \le 3$ and $h \in C(0,1)^T L^1[0,1]$, then the boundary value problem*

$$\begin{cases} D_{0+}^{p} x(t) + h(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases} \tag{4}$$

is given by

$$x(t) = \int_0^1 G(t,s) h(s)\, ds, \tag{5}$$

where

$$G(t,s) = \frac{1}{\Gamma(p)} \begin{cases} t^{p-1}(1-s)^{p-1} - (t-s)p - 1, & 0 \le s \le t \le 1, \\ t^{p-1}(1-s)^{p-1}, & 0 \le t \le s \le 1. \end{cases} \tag{6}$$

Lemma 3 (see [3]). *The function $G(t,s)$ obtained in Lemma 2 is continuous on $[0,1] \times [0,1]$ and satisfies the following properties:*

(1) for all $t, s \in [0,1]$, $G(t,s) \ge 0$

(2) for all $t, s \in [0,1]$

$$\frac{t^{p-1}(1-t)(1-s)^{p-1}s}{\Gamma(p)} \le G(t,s) \le \frac{(p-1)t^{p-1}(1-t)}{\Gamma(p)}. \tag{7}$$

In the rest of this paper, we assume that the following conditions hold:

H_1. $f(t,x,y) \in C[(0,1) \times [0,+\infty) \times (0,+\infty) ; (0,+\infty)]$ is increasing with respect to x, decreasing with respect to y, and for any $(t,x,y) \in (0,1) \times [0,+\infty) \times (0,+\infty)$, $\lambda \in (0,1)$, there exists a constant $k \in (0,1)$ such that

$$f(t, \lambda x, \lambda^{-1} y) \ge \lambda^k f(t,x,y). \tag{8}$$

H_2. $p, q \in C[(0,1) ; (0,+\infty)]$ are Lebesgue integrable, and p does not vanish identically on any subinterval of $(0,1)$.

H_3. The third condition is as follows:

$$0 < \int_0^1 \left[p(s) f\left(s, (1-s)s^{p-1}, (1-s)s^{p-1}\right) + q(s)\right] ds < \infty. \tag{9}$$

The basic space used in this paper is $E = C[0,1]$. Define a set P in E as follows:

$$P = \{x \in E | \text{there exists a positive constant } l_x \in (0,1), \text{ such that } l_x(1-t)t^{p-1} \le x(t) \le (l_x)^{-1}(1-t)t^{p-1},\ t \in (0,1)\}. \tag{10}$$

Denote $h(t) = (1-t)t^{p-1}$. Evidently, $(h(t), h(t)) \in P \times P$, i.e., $P \times P$ is not empty.

Let the operator $T::E \times E \longrightarrow E$ be defined by

$$(T(x,y))(t) = \int_0^1 G(t,s)[p(s) f(s, x(s), y(s)) + q(s)] ds. \tag{11}$$

3. Main Results

In this section, we will prove the existence of the positive solution to BVP (1) and calculate the approximation error and the convergence rate.

Theorem 4. *Assume that H_1, H_2, and H_3 hold. Then, BVP (1) has a unique positive solution $x^*(t)$ satisfying*

$$l(1-t)t^{p-1} \le x^*(t) \le l^{-1}(1-t)t^{p-1}, \quad t \in [0,1], \tag{12}$$

where l is a constant that belongs to $(0,1)$.

Proof. The solution of BVP (1) coincides with the fixed point of operator T. So our goal is to show that operator T has a unique fixed point in P. We divide our proof in four steps.

Step 1. We verify that operator T is well defined. From H_1, H_2, and H_3, for any $(x,y) \in P \times P$ and $t \in (0,1)$, we know that

$$\begin{aligned} (T(x,y))(t) &= \int_0^1 G(t,s)[p(s) f(s, x(s), y(s)) + q(s)] \\ &\quad \cdot ds \le \frac{(p-1)t^{p-1}(1-t)}{\Gamma(p)} \int_0^1 \\ &\quad \cdot \left[p(s) f\left(s, (l_x)^{-1}(1-s)s^{p-1}, l_x(1-s)s^{p-1}\right)\right. \\ &\quad \left. + q(s)\right] ds \le \frac{(l_x)^{-k}(p-1)t^{p-1}(1-t)}{\Gamma(p)} \int_0^1 \\ &\quad \cdot \left[p(s) f\left(s, (1-s)s^{p-1}, (1-s)s^{p-1}\right)\right. \\ &\quad \left. + q(s)\right] ds < +\infty. \end{aligned} \tag{13}$$

So, operator T is well defined.

Step 2. We verify the properties of T.

$$\begin{aligned}(T(x,y))(t) &= \int_0^1 G(t,s)[p(s)f(s,x(s),y(s)) + q(s)] \\ &\cdot ds \geq \frac{\left(t^{(p-1)}(1-t)\right)}{\Gamma(p)} \int_0^1 (1-s)^{(p-1)} \\ &\cdot s\Big[p(s)f\left(s, l_x(1-s)s^{(p-1)}, (l_x)^{-1}(1-s)s^{(p-1)}\right) \\ &+ q(s)]ds \geq \frac{(l_x)^k t^{(p-1)}(1-t)}{\Gamma(p)} \int_0^1 (1-s)^{(p-1)} \\ &\cdot s\Big[p(s)f\left(s, (1-s)s^{(p-1)}, (1-s)s^{(p-1)}\right) + q(s)\Big]ds.\end{aligned} \tag{14}$$

Let

$$\begin{aligned}0 < l_{Tx} < \min \Bigg\{ &1, \frac{(l_x)^k}{\Gamma(p)} \int_0^1 (1-s)^{p-1} s[p(s) \\ &\cdot f\left(s, (1-s)s^{p-1}, (1-s)s^{p-1}\right) + q(s)]ds \Bigg\},\end{aligned} \tag{15}$$

and

$$\begin{aligned}(l_{Tx})^{-1} > \max \Bigg\{ &1, \frac{(l_x)^{-k}(p-1)}{\Gamma(p)} \int_0^1 [p(s) \\ &\cdot f\left(s, (1-s)s^{p-1}, (1-s)s^{p-1}\right) + q(s)]ds \Bigg\}.\end{aligned} \tag{16}$$

Consequently, we can prove that there exists constants $0 < l_{Tx} < 1$ such that

$$l_{Tx}(1-t)t^{p-1} \leq (T(x,y))(t) \leq (l_{Tx})^{-1}(1-t)t^{p-1}, \quad t \in (0,1). \tag{17}$$

Therefore, the operator $T : P \times P \longrightarrow P$ is well defined. It follows from H1 that T is nondecreasing with respect to x and nonincreasing with respect to y.

Step 3. We will establish the existence of a positive solution to BVP (1). Since $(h,h) \in P \times P$, from the above steps, we have $T(h,h) \in P$. According to the definition of P, there exists a constant $0 < l_{Th} < 1$ such that

$$l_{Th}(1-t)t^{p-1} \leq T(h,h)(t) \leq (l_{Th})^{-1}(1-t)t^{p-1}. \tag{18}$$

Taking

$$\begin{aligned}x_0 &= \rho h(t), \\ y_0 &= \rho^{-1} h(t),\end{aligned} \tag{19}$$

$$\begin{aligned}x_n &= T(x_{n-1}, y_{n-1}), \\ y_n &= T(y_{n-1}, x_{n-1}), \\ n &= 1, 2, \cdots,\end{aligned} \tag{20}$$

where ρ is a fixed number satisfying

$$0 < \rho \leq l_{Th}^{1/1-k}. \tag{21}$$

In fact, $0 < \rho < 1$. From (19), we get $x_0, y_0 \in P$ and $x_0 \leq y_0$. Moreover,

$$\begin{aligned}x_1 = (T(x_0,y_0))(t) &= \int_0^1 G(t,s)\left[p(s)f\left(s, \rho h(s), \rho^{-1}h(s)\right)\right. \\ &\left. + q(s)\right]ds \geq \rho^k \int_0^1 G(t,s)[p(s)f(s,h(s),h(s)) + q(s)] \\ &\cdot ds = \rho^k T(h,h)(t) \geq \rho^k l_{Th} h(t) \geq \rho^k \rho^{1-k} h(t) = x_0,\end{aligned} \tag{22}$$

and

$$\begin{aligned}y_1 = (T(y_0,x_0))(t) &= \int_0^1 G(t,s)\left[p(s)f\left(s, \rho^{-1}h(s), \rho h(s)\right)\right. \\ &\left. + q(s)\right]ds \leq \rho^{-k} \int_0^1 G(t,s)[p(s)f(s,h(s),h(s)) + q(s)] \\ &\cdot ds = \rho^{-k} T(h,h)(t) \leq \rho^{-k} (l_{Th})^{-1} h(t) \leq \rho^{-k} \rho^{k-1} h(t) = y_0.\end{aligned} \tag{23}$$

Combining $x_0 \leq y_0$ with T is nondecreasing with respect to x and nonincreasing with respect to y, so we have

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0. \tag{24}$$

On the other hand, for any nature number n, denote ρ^2 by c and we have

$$\begin{aligned}x_n = T(x_{n-1}, y_{n-1}) &= T_n(y_0, x_0) \\ &= T_n\left(\rho h(t), \rho^{-1}h(t)\right) = T_n\left(\rho^2 \rho^{-1} h(t), \rho^{-2} \rho h(t)\right) \\ &\geq \left(\rho^2\right)^{k_n} T_n\left(\rho^{-1}h(t), \rho h(t)\right) = c_{k_n} y_n.\end{aligned} \tag{25}$$

Therefore, for any nature number n and n^*, we obtain

$$0 \leq x_{n+n^*} - x_n \leq y_n - x_n \leq \left(1 - c^{k_n}\right) y_n \leq \left(1 - c^{k_n}\right) \rho^{-1} h(t). \tag{26}$$

So, there exists $x^* \in P$ such that

$$x_n(t) \longrightarrow x^*(t), \tag{27}$$

uniformly on $(0,1)$. By the same argument, we can also prove that

$$y_n(t) \longrightarrow x^*(t), \tag{28}$$

uniformly on $(0,1)$. Since T is continuous, we can take the limits in $x_n = T(x_n, y_n)$ and we get $x^* = T(x^*, x^*)$. Therefore, x^* is a positive solution of BVP (1). Owing to $x^* \in P$, for any $t \in (0,1)$, there exists a constant $l \in (0,1)$ such that

$$l(1-t)t^{p-1} \le x^*(t) \le l^{-1}(1-t)t^{p-1}, \tag{29}$$

holds.

Step 4. We further show its uniqueness. Let $y^*(t)$ be another positive solution of BVP (1), then for any $t \in (0,1)$, there exists a constant $r \in (0,1)$ such that

$$r(1-t)t^{p-1} \le y^*(t) \le r^{-1}(1-t)t^{p-1}. \tag{30}$$

Taking ρ defined in (21) as being small enough such that $\rho < r$. So

$$x_0(t) \le y^*(t) \le y_0(t), \quad t \in (0,1). \tag{31}$$

In view of $T(y^*, y^*) = y^*$, by means of the nondecreasing T, we obtain

$$x_n(t) \le y^*(t) \le y_n(t), \quad t \in (0,1). \tag{32}$$

Taking limits to the above inequality, we get $x^* = y^*$. Therefore, the solutions of BVP (1) are unique which completes the proof of Theorem 4.

Now, we are in a position to construct the successive sequence which converges to the unique solution.

Theorem 5. *Suppose conditions H1, H2, and H3 are satisfied. Then, for any initial value $u_0 \in P$, the successive sequence*

$$u_n(t) = \int_0^1 G(t,s)[p(s)f(s, u_{n-1}(s), u_{n-1}(s)) + q(s)]ds, \tag{33}$$

uniformly converges to the unique positive solution $x^*(t)$ where the error estimation is the same order infinitesimal of $(1 - c^{k_n})$, where $c \in (0,1)$ and determined by u_0.

Proof. According to Theorem 4, notice that the positive solution x^* is unique; for any $u_0 \in P$, there exists a constant $l \in (0,1)$, such that

$$l(1-t)t^{p-1} \le u_0(t) \le l^{-1}(1-t)t^{p-1}. \tag{34}$$

We shall adopt the similar argument as in the proof of Theorem 4; take $\rho < l$ to be a fixed number $0 < \rho \le l_{Th}^{1/1-k}$, thus

$$x_0(t) \le u_0(t) \le y_0(t), \quad t \in (0,1). \tag{35}$$

Let

$$u_n = \int_0^1 G(t,s)[p(s)f(s, u_{n-1}(s), u_{n-1}(s)) + q(s)]ds, \quad n = 1, 2, \cdots, \tag{36}$$

then

$$x_n(t) \le u_n(t) \le y_n(t), \quad t \in (0,1). \tag{37}$$

Taking limits to inequality (37), by the same method of the proof to (26), we can show that $\{u_n(t)\}$ uniformly converges to the unique positive solution x^* of BVP (1), and

$$\max\{|u_n(t) - x^*(t)|\} = o\left(1 - c^{k_n}\right), \tag{38}$$

which means the error estimation is the same order infinitesimal of $(1 - c^{k_n})$, where $c = \rho^2$ and determined by u_0.

The proof is completed.

4. Example

Let us illustrate the main results with an example.

Example. Take $p = (5/2)$, $p(t) = (1-t)^2$, $q(t) = t^{2/3}$, and $f(t, x, y) = a(t)x^{1/4} + b(t)y^{-1/3}$.

We consider the following BVP:

$$\begin{cases} D_{0+}^{5/2}x(t) + (1-t)^2\left[a(t)x^{1/4} + b(t)y^{-1/3}\right] + t^{2/3} = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 0. \end{cases} \tag{39}$$

For any $\lambda \in (0,1)$, take $k = 1/2$, we can prove that

$$f\left(t, \lambda x, \lambda^{-1} y\right) \ge \lambda^k f(t, x, y). \tag{40}$$

Combined with the expression of f, p, and q, it is easy to see that H1 and H2 are holding. In addition,

$$0 < \int_0^1 \left[(1-t)^2 f\left(t, (1-t)t^{3/2}, (1-t)t^{3/2}\right) + t^{3/2}\right]dt < \infty. \tag{41}$$

So all of the assumptions in Theorem 4 are satisfied. Then, BVP (39) has a unique positive solution x^*. For any initial value $x_0 \in P$, we can construct the successive iterative sequence $\{xn(t)\}$ as follows

$$x_n(t) = \int_0^1 G(t,s)[p(s)f(s, x_{n-1}(s), x_{n-1}(s)) + q(s)]ds, \quad n = 1, 2, \cdots, \tag{42}$$

which uniformly converges to the unique positive solution x^* on $(0,1)$. The error estimation is the same order infinitesimal of $(1 - c^{k_n})$, i.e.,

$$\max\{|x_n(t) - x^*(t)|\} = o\left(1 - c^{(1/4)^n}\right), \tag{43}$$

where $c \in (0,1)$ is a constant and determined by the initial value x_0. In addition, for any $t \in (0,1)$, there exists a constant$l \in (0,1)$that satisfies

$$l(1-t)t^{p-1} \le x^*(t) \le l^{-1}(1-t)t^{p-1}. \tag{44}$$

5. Conclusions

We obtain the uniqueness of the positive solutions for a new kind of fractional differential equation with a negatively perturbed term. Through constructing an iterative process u_n, which can be from any initial value $u_0 \in P$, only by the iterative algorithm can we prove that it converges uniformly to the unique positive solution where the error estimation is the same order infinitesimal of $(1 - c^{k_n})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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Research Article

The Splitting Crank–Nicolson Scheme with Intrinsic Parallelism for Solving Parabolic Equations

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In this paper, a splitting Crank–Nicolson (SC-N) scheme with intrinsic parallelism is proposed for parabolic equations. The new algorithm splits the Crank–Nicolson scheme into two domain decomposition methods, each one is applied to compute the values at $(n+1)$th time level by use of known numerical solutions at n-th time level, respectively. Then, the average of the above two values is chosen to be the numerical solutions at $(n+1)$th time level. The new algorithm obtains accuracy of the Crank–Nicolson scheme while maintaining parallelism and unconditional stability. This algorithm can be extended to solve two-dimensional parabolic equations by alternating direction implicit (ADI) technique. Numerical experiments illustrate the accuracy and efficiency of the new algorithm.

1. Introduction

The parallel numerical schemes for parabolic equations have been studied rapidly. In 1985, Evans [1] proposed the Alternating Group Explicit (AGE) scheme for the diffusion equation by Saul'yev asymmetric schemes [2]. The Alternating Segment Crank–Nicolson (ASC-N) scheme was designed in [3]. Afterwards, the alternating segment algorithms (AGE scheme and ASC-N scheme) above became very effective methods for parabolic equations, such as convection-diffusion equation [4], dispersive equation [5–8], and fourth-order parabolic equation [9, 10]. Meanwhile, domain decomposition methods (DDMs) for the partial differential equations have been developed. Since the advantages of the low computation and communication costs at each time step, the nonoverlapping DDMs have been studied extensively [11–20]. The concept called "intrinsic parallelism" was presented in [21–23]. In 1999, the alternating difference schemes were presented, and the unconditional stability analysis was given in [24]. The unconditionally stable domain decomposition method was obtained by the alternating technique in [25–30]. Recently, the numerical methods for parabolic equations have attracted great attention of scholars [31–42]. Meanwhile, the finite volume element methods for elliptic problems are studied by Bi et al. [43, 44], and the finite volume element method for second-order hyperbolic equations is proposed by Chen et al. [45]. Obviously, the Crank–Nicolson scheme is one of the most classical methods for PDEs [46, 47]. But, there is litter work on parallel algorithms that satisfy the accuracy and stability of the Crank–Nicolson scheme.

On this basis, a new parallel algorithm called splitting Crank–Nicolson scheme for parabolic equations will be presented in this paper. The idea of the new algorithm is to divide the classical Crank–Nicolson scheme into two parts, which are DDMs, each of which is used in computations at $(n+1)$th time level utilizing the numerical solution at time level n. Then, the average of the above two values is chosen to be the numerical solutions at $(n+1)$th time level. It can be described as follows:

(1) Split the Crank–Nicolson scheme into DDM I and DDM II by Saul'yev asymmetric schemes.

(2) DDM I is applied to compute the values at $(n+1)$th time level noted as V^{n+1} by use of known value U^n at

n-th time level. DDM II is also used to compute the values at $(n+1)$th time level noted as W^{n+1} by use of U^n at n-th time level.

(3) The value $U^{n+1} = (V^{n+1} + W^{n+1})/2$ are set as numerical solutions at $(n+1)$th time level.

The advantage of the SC-N scheme is spitting the C-N scheme into two parallel algorithms, which can be computed by parallel computers. Then, the average value obtained is restored to C-N scheme approximately. This paper is organized as follows: in Section 2, we introduce a SC-N scheme for parabolic equations. For simplicity of presentation, we focus on a model problem, namely, one-dimensional parabolic equations. The new algorithm and detailed presentations are given. Then, we extend the new algorithm to solve two-dimensional parabolic equations by ADI technique. Finally, numerical experiments illustrated the accuracy of SC-N scheme is approximate to the C-N scheme and the new algorithm is efficient.

2. Algorithm Presentation

Considering the model problem of one-dimensional parabolic equations,

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in (0,l),\ t \in (0,T], \tag{1}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad x \in [0,l], \\ &u(0,t) = g_0(t),\ u(l,t) = g_l(t), \quad t \in (0,T], \end{aligned} \tag{2}$$

where $a > 0$ is a constant.

Let h and τ be the spatial and temporal step sizes, respectively. Denote $x_j = jh$, $j = 0, 1, \ldots, m$ and $t_n = n\tau$, $n = 0, 1, \ldots, N$. Let u_j^n be the approximate solution at (x_j, t_n). $u(x, t)$ represents the exact solution of (1).

The well-known Crank–Nicolson scheme can be written as

$$\begin{aligned} &-\frac{ar}{2}u_{j-1}^{n+1} + (1+ar)u_j^{n+1} - \frac{ar}{2}u_{j+1}^{n+1} \\ &\quad = \frac{ar}{2}u_{j-1}^{n} + (1-ar)u_j^{n} + \frac{ar}{2}u_{j+1}^{n} + \tau f_j^n, \end{aligned} \tag{3}$$

where $r = \tau/h^2$.

Let $\mu = ar$, and (3) can be written as the matrix form

$$AU^{n+1} = BU^n + F^n, \tag{4}$$

where $U^n = (u_1^n, u_2^n, \ldots, u_{m-1}^n)^T$ and $F^n = ((\mu/2)(u_0^n + u_0^{n+1}) + \tau f_1^n, \tau f_2^n, \ldots, \tau f_{m-2}^n, (\mu/2)(u_m^n + u_m^{n+1}) + \tau f_{m-1}^n)^T$.

The matrices A and B are as follows:

$$A = \begin{bmatrix} 1+\mu & -\frac{\mu}{2} & & & \\ -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{\mu}{2} & 1+\mu & -\frac{\mu}{2} \\ & & & -\frac{\mu}{2} & 1+\mu \end{bmatrix}, \tag{5}$$

$$B = \begin{bmatrix} 1-\mu & \frac{\mu}{2} & & & \\ \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\mu}{2} & 1-\mu & \frac{\mu}{2} \\ & & & \frac{\mu}{2} & 1-\mu \end{bmatrix}.$$

Splitting Crank–Nicolson scheme (4), we obtain

$$(A_1 + A_2)U^{n+1} = (B_1 + B_2)U^n + 2F^n, \tag{6}$$

where A_1 and A_2 are block diagonal matrices, respectively.

$$\begin{aligned} A &= \left(\frac{A_1 + A_2}{2}\right), \\ B &= \left(\frac{B_1 + B_2}{2}\right). \end{aligned} \tag{7}$$

V^{n+1} and W^{n+1} can be defined as follows:

$$A_1 V^{n+1} \triangleq B_1 U^n + F^n, \tag{8}$$

$$A_2 W^{n+1} \triangleq B_2 U^n + F^n, \tag{9}$$

where V^{n+1} and W^{n+1} approximate to U^{n+1}, respectively. Then,

$$U^{n+1} \approx \frac{V^{n+1} + W^{n+1}}{2}. \tag{10}$$

Scheme (8) is DDM I, and scheme (9) is called DDM II. The two DDMs aforementioned are suitable for parallel computing.

In order to construct two domain decomposition methods, DDM I and DDM II, at $(n+1)$th time level, we need to consider four forms of Saul'yev asymmetric difference schemes corresponding to (3) as follows:

$$-\frac{ar}{2}u_{j-1}^{n+1}+\left(1+\frac{3ar}{2}\right)u_j^{n+1}-aru_{j+1}^{n+1}=\frac{ar}{2}u_{j-1}^{n}+\left(1-\frac{ar}{2}\right)u_j^{n}+\tau f_j^{n}, \tag{11}$$

$$-aru_{j-1}^{n+1}+\left(1+\frac{3ar}{2}\right)u_j^{n+1}-\frac{ar}{2}u_{j+1}^{n+1}=\left(1-\frac{ar}{2}\right)u_j^{n}+\frac{ar}{2}u_{j+1}^{n}+\tau f_j^{n}, \tag{12}$$

$$\left(1+\frac{ar}{2}\right)u_j^{n+1}-\frac{ar}{2}u_{j+1}^{n+1}=aru_{j-1}^{n}+\left(1-\frac{3ar}{2}\right)u_j^{n}+\frac{ar}{2}u_{j+1}^{n}+\tau f_j^{n}, \tag{13}$$

$$-\frac{ar}{2}u_{j-1}^{n+1}+\left(1+\frac{ar}{2}\right)u_j^{n+1}=\frac{ar}{2}u_{j-1}^{n}+\left(1-\frac{3ar}{2}\right)u_j^{n}+aru_{j+1}^{n}+\tau f_j^{n}. \tag{14}$$

The flow chart of schemes (11)–(14) are displayed in Figure 1.

Assume $m-1=6K$, where K is a positive integer.

2.1. DDM I. For the values u_1^{n+1}, u_2^{n+1}, and u_3^{n+1}, we use the formulas as follows:

$$\begin{cases} -\dfrac{ar}{2}u_0^{n+1}+(1+ar)u_1^{n+1}-\dfrac{ar}{2}u_2^{n+1}=\dfrac{ar}{2}u_0^{n}+(1-ar)u_1^{n}+\dfrac{ar}{2}u_2^{n}+\tau f_1^{n}, \\ -\dfrac{ar}{2}u_1^{n+1}+(1+ar)u_2^{n+1}-\dfrac{ar}{2}u_3^{n+1}=\dfrac{ar}{2}u_1^{n}+(1-ar)u_2^{n}+\dfrac{ar}{2}u_3^{n}+\tau f_2^{n}, \\ -\dfrac{ar}{2}u_2^{n+1}+\left(1+\dfrac{ar}{2}\right)u_3^{n+1}=\dfrac{ar}{2}u_2^{n}+\left(1-\dfrac{3ar}{2}\right)u_3^{n}+aru_4^{n}+\tau f_3^{n}. \end{cases} \tag{15}$$

Find the values $[u_{6k-2}^{n+1}, u_{6k-1}^{n+1}, u_{6k}^{n+1}, u_{6k+1}^{n+1}, u_{6k+2}^{n+1}, u_{6k+3}^{n+1}]$ by using the following formulas ($k=1, 2, \ldots, K-1$):

$$\begin{cases} \left(1+\dfrac{ar}{2}\right)u_{6k-2}^{n+1}-\dfrac{ar}{2}u_{6k-1}^{n+1}=aru_{6k-3}^{n}+\left(1-\dfrac{3ar}{2}\right)u_{6k-2}^{n}+\dfrac{ar}{2}u_{6k-1}^{n}+\tau f_{6k-2}^{n}, \\ -\dfrac{ar}{2}u_{6k-2}^{n+1}+(1+ar)u_{6k-1}^{n+1}-\dfrac{ar}{2}u_{6k}^{n+1}=\dfrac{ar}{2}u_{6k-2}^{n}+(1-ar)u_{6k-1}^{n}+\dfrac{ar}{2}u_{6k}^{n}+\tau f_{6k-1}^{n}, \\ -\dfrac{ar}{2}u_{6k-1}^{n+1}+\left(1+\dfrac{3ar}{2}\right)u_{6k}^{n+1}-aru_{6k+1}^{n+1}=\dfrac{ar}{2}u_{6k-1}^{n}+\left(1-\dfrac{ar}{2}\right)u_{6k}^{n}+\tau f_{6k}^{n}, \\ -aru_{6k}^{n+1}+\left(1+\dfrac{3ar}{2}\right)u_{6k+1}^{n+1}-\dfrac{ar}{2}u_{6k+2}^{n+1}=\left(1-\dfrac{ar}{2}\right)u_{6k+1}^{n}+\dfrac{ar}{2}u_{6k+2}^{n}+\tau f_{6k+1}^{n}, \\ -\dfrac{ar}{2}u_{6k+1}^{n+1}+(1+ar)u_{6k+2}^{n+1}-\dfrac{ar}{2}u_{6k+3}^{n+1}=\dfrac{ar}{2}u_{6k+1}^{n}+(1-ar)u_{6k+2}^{n}+\dfrac{ar}{2}u_{6k+3}^{n}+\tau f_{6k+2}^{n}, \\ -\dfrac{ar}{2}u_{6k+2}^{n+1}+\left(1+\dfrac{ar}{2}\right)u_{6k+3}^{n+1}=\dfrac{ar}{2}u_{6k+2}^{n}+\left(1-\dfrac{3ar}{2}\right)u_{6k+3}^{n}+aru_{6k+4}^{n}+\tau f_{6k+3}^{n}. \end{cases} \tag{16}$$

Obviously, each subdomain contains 6 nodes which can be computed by (16) independently.

For the values $u_{m-3}^{n+1}, u_{m-2}^{n+1}, u_{m-1}^{n+1}$ by using the formulas as follows:

$$\begin{cases} \left(1+\dfrac{ar}{2}\right)u_{m-3}^{n+1}-\dfrac{ar}{2}u_{m-2}^{n+1}=aru_{m-4}^{n}+\left(1-\dfrac{3ar}{2}\right)u_{m-3}^{n}+\dfrac{ar}{2}u_{m-2}^{n}+\tau f_{m-3}^{n}, \\ -\dfrac{ar}{2}u_{m-3}^{n+1}+(1+ar)u_{m-2}^{n+1}-\dfrac{ar}{2}u_{m-1}^{n+1}=\dfrac{ar}{2}u_{m-3}^{n}+(1-ar)u_{m-2}^{n}+\dfrac{ar}{2}u_{m-1}^{n}+\tau f_{m-2}^{n}, \\ -\dfrac{ar}{2}u_{m-2}^{n+1}+(1+ar)u_{m-1}^{n+1}-\dfrac{ar}{2}u_{m}^{n+1}=\dfrac{ar}{2}u_{m-2}^{n}+(1-ar)u_{m-1}^{n}+\dfrac{ar}{2}u_{m}^{n}+\tau f_{m-1}^{n}. \end{cases} \tag{17}$$

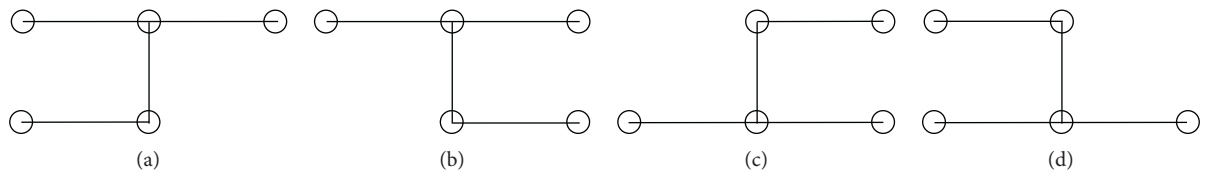

Figure 1: The diagram of Saul'yev asymmetric difference schemes. (a) Scheme (11). (b) Scheme (12). (c) Scheme (13). (d) Scheme (14).

DDM I can be written as the matrix form

$$A_1 U^{n+1} = B_1 U^n + F^n, \tag{18}$$

where $U^n = (u_1^n, u_2^n, \cdots, u_{m-1}^n)^T$, $A_1 = (I + \mu G_1)$, and $B_1 = (I - \mu G_2)$.

The matrices G_1 and G_2 are block diagonal matrices which are as follows:

$$G_1 = \begin{bmatrix} P_1 & & & \\ & Q & & \\ & & \ddots & \\ & & & P_2 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} Q_1 & & & \\ & Q & & \\ & & \ddots & \\ & & & Q_2 \end{bmatrix}, \tag{19}$$

where

$$P_1 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & -1 & \frac{3}{2} & -\frac{1}{2} & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1 & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & -1 & \frac{3}{2} & -\frac{1}{2} & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & -1 & \frac{3}{2} & -\frac{1}{2} & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & 1 \end{bmatrix}. \tag{20}$$

Each block matrix system (i.e., each subdomain) can be solved independently. It is evident that DDM I (18) has intrinsic parallelism.

2.2. DDM II. For the values $u_1^{n+1}, u_2^{n+1}, \ldots, u_6^{n+1}$ by using the following formulas:

$$\begin{cases} -\frac{ar}{2}u_0^{n+1} + (1+ar)u_1^{n+1} - \frac{ar}{2}u_2^{n+1} = \frac{ar}{2}u_0^n + (1-ar)u_1^n + \frac{ar}{2}u_2^n + \tau f_1^n, \\ -\frac{ar}{2}u_1^{n+1} + (1+ar)u_2^{n+1} - \frac{ar}{2}u_3^{n+1} = \frac{ar}{2}u_1^n + (1-ar)u_2^n + \frac{ar}{2}u_3^n + \tau f_2^n, \\ -\frac{ar}{2}u_2^{n+1} + \left(1+\frac{3ar}{2}\right)u_3^{n+1} - aru_4^{n+1} = \frac{ar}{2}u_2^n + \left(1-\frac{ar}{2}\right)u_3^n + \tau f_3^n, \\ -aru_3^{n+1} + \left(1+\frac{3ar}{2}\right)u_4^{n+1} - \frac{ar}{2}u_5^{n+1} = \left(1-\frac{ar}{2}\right)u_4^n + \frac{ar}{2}u_5^n + \tau f_4^n, \\ -\frac{ar}{2}u_4^{n+1} + (1+ar)u_5^{n+1} - \frac{ar}{2}u_6^{n+1} = \frac{ar}{2}u_4^n + (1-ar)u_5^n + \frac{ar}{2}u_6^n + \tau f_5^n, \\ -\frac{ar}{2}u_5^{n+1} + \left(1+\frac{ar}{2}\right)u_6^{n+1} = \frac{ar}{2}u_5^n + \left(1-\frac{3ar}{2}\right)u_6^n + aru_7^n + \tau f_6^n. \end{cases} \tag{21}$$

Find the values $[u_{6k+1}^{n+1}, u_{6k+2}^{n+1}, u_{6k+3}^{n+1}, u_{6k+4}^{n+1}, u_{6k+5}^{n+1}, u_{6k+6}^{n+1}]$ by using the formulas as follows ($k = 1, 2, \ldots, K-2$):

$$\begin{cases} \left(1+\frac{ar}{2}\right)u_{6k+1}^{n+1} - \frac{ar}{2}u_{6k+2}^{n+1} = aru_{6k}^n + \left(1-\frac{3ar}{2}\right)u_{6k+1}^n + \frac{ar}{2}u_{6k+2}^n + \tau f_{6k+1}^n, \\ -\frac{ar}{2}u_{6k+1}^{n+1} + (1+ar)u_{6k+2}^{n+1} - \frac{ar}{2}u_{6k+3}^{n+1} = \frac{ar}{2}u_{6k+1}^n + (1-ar)u_{6k+2}^n + \frac{ar}{2}u_{6k+3}^n + \tau f_{6k+2}^n, \\ -\frac{ar}{2}u_{6k+2}^{n+1} + \left(1+\frac{3ar}{2}\right)u_{6k+3}^{n+1} - aru_{6k+4}^{n+1} = \frac{ar}{2}u_{6k+2}^n + \left(1-\frac{ar}{2}\right)u_{6k+3}^n + \tau f_{6k+3}^n, \\ -aru_{6k+3}^{n+1} + \left(1+\frac{3ar}{2}\right)u_{6k+4}^{n+1} - \frac{ar}{2}u_{6k+5}^{n+1} = \left(1-\frac{ar}{2}\right)u_{6k+4}^n + \frac{ar}{2}u_{6k+5}^n + \tau f_{6k+4}^n, \\ -\frac{ar}{2}u_{6k+4}^{n+1} + (1+ar)u_{6k+5}^{n+1} - \frac{ar}{2}u_{6k+6}^{n+1} = \frac{ar}{2}u_{6k+4}^n + (1-ar)u_{6k+5}^n + \frac{ar}{2}u_{6k+6}^n + \tau f_{6k+5}^n, \\ -\frac{ar}{2}u_{6k+5}^{n+1} + \left(1+\frac{ar}{2}\right)u_{6k+6}^{n+1} = \frac{ar}{2}u_{6k+5}^n + \left(1-\frac{3ar}{2}\right)u_{6k+6}^n + aru_{6k+7}^n + \tau f_{6k+6}^n. \end{cases} \tag{22}$$

Obviously, each subdomain contains 6 nodes which can be computed by (22) independently.

Find the values $u_{m-6}^{n+1}, u_{m-5}^{n+1}, \ldots, u_{m-1}^{n+1}$ by using the formulas as follows:

$$\begin{cases} \left(1+\frac{ar}{2}\right)u_{m-6}^{n+1}-\frac{ar}{2}u_{m-5}^{n+1}=aru_{m-7}^{n}+\left(1-\frac{3ar}{2}\right)u_{m-6}^{n}+\frac{ar}{2}u_{m-5}^{n}+\tau f_{m-6}^{n}, \\ -\frac{ar}{2}u_{m-6}^{n+1}+(1+ar)u_{m-5}^{n+1}-\frac{ar}{2}u_{m-4}^{n+1}=\frac{ar}{2}u_{m-6}^{n}+(1-ar)u_{m-5}^{n}+\frac{ar}{2}u_{m-4}^{n}+\tau f_{m-5}^{n}, \\ -\frac{ar}{2}u_{m-5}^{n+1}+\left(1+\frac{3ar}{2}\right)u_{m-4}^{n+1}-aru_{m-3}^{n+1}=\frac{ar}{2}u_{m-5}^{n}+\left(1-\frac{ar}{2}\right)u_{m-4}^{n}+\tau f_{m-4}^{n}, \\ -aru_{m-4}^{n+1}+\left(1+\frac{3ar}{2}\right)u_{m-3}^{n+1}-\frac{ar}{2}u_{m-2}^{n+1}=\left(1-\frac{ar}{2}\right)u_{m-3}^{n}+\frac{ar}{2}u_{m-2}^{n}+\tau f_{m-3}^{n}, \\ -\frac{ar}{2}u_{m-3}^{n+1}+(1+ar)u_{m-2}^{n+1}-\frac{ar}{2}u_{m-1}^{n+1}=\frac{ar}{2}u_{m-3}^{n}+(1-ar)u_{m-2}^{n}+\frac{ar}{2}u_{m-1}^{n}+\tau f_{m-2}^{n}, \\ -\frac{ar}{2}u_{m-2}^{n+1}+(1+ar)u_{m-1}^{n+1}-\frac{ar}{2}u_{m}^{n+1}=\frac{ar}{2}u_{m-2}^{n}+(1-ar)u_{m-1}^{n}+\frac{ar}{2}u_{m}^{n}+\tau f_{m-1}^{n}. \end{cases} \tag{23}$$

DDM II can be written as the matrix form

$$A_2U^{n+1}=B_2U^n+F^n, \tag{24}$$

where $A_2=(I+\mu G_2)$ and $B_2=(I-\mu G_1)$.

The matrices G_1 and G_2 are block diagonal matrices which are as follows:

$$G_1=\begin{bmatrix} P_1 & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & P_2 \end{bmatrix}, \qquad G_2=\begin{bmatrix} Q_1 & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & Q_2 \end{bmatrix}. \tag{25}$$

Each block matrix system (i.e., each subdomain) can be solved independently. It is evident that DDM II (24) has intrinsic parallelism.

Schemes (15)–(17) and schemes (21)–(23) construct two domain decomposition methods (18) and (24), respectively. The corresponding algorithm can be described in Algorithm 1. The matrix form of Algorithm 1 can be written as follows:

$$\begin{cases} A_1V^{n+1}=B_1U^n+F^n, \\ A_2W^{n+1}=B_2U^n+F^n, \\ U^{n+1}=\frac{1}{2}\left(V^{n+1}+W^{n+1}\right), \\ n=0,1,2,\ldots, \end{cases} \tag{26}$$

where $U^n=(u_1^n,u_2^n,\ldots,u_{m-1}^n)^T$, $V^n=(v_1^n,v_2^n,\ldots,v_{m-1}^n)^T$, and $W^n=(w_1^n,w_2^n,\ldots,w_{m-1}^n)^T$.

Remark 1. Obviously, C-N scheme (4) is equal to $\{((A_1+A_2)/2)U^{n+1}=((B_1+B_2)/2)U^n+F^n\}$.

Remark 2. Because the SC-N scheme is derived from the C-N scheme, it also has the properties of the C-N scheme, i.e., the SC-N scheme is unconditionally stable and maintains the second-order numerical accuracy $O(\tau^2+h^2)$.

3. Extension to Two-Dimensional Parabolic Equations

In this section, we will extend Algorithm 1 to solve two-dimensional parabolic equations:

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right)=f(x,y,t),\ (x,y)\in\Omega,\quad t\in(0,T], \tag{27}$$

$$u(x,y,0)=u_0(x,y),\ (x,y)\in\Omega, \tag{28}$$

$$u(x,y,t)=0,\ (x,y)\in\partial\Omega,\quad t\in(0,T], \tag{29}$$

where domain $\Omega\in(0,L_x)\times(0,L_y)$ and $a>0$ and $b>0$ are diffusion coefficients.

Let $u_{i,j}^n$ be the approximate solution at (x_i,y_j,t_n), and $u(x,y,t)$ represents the exact solution of (27). With the same time and space discretization of Algorithm 1, we obtain its extended algorithm by alternating direction implicit (ADI) technique [48] for equations (27)–(29).

3.1. x-Direction. Let $r_1=a\tau/(2h^2)$ and $r_2=b\tau/(2h^2)$, and the matrix form of the SC-N scheme in x-direction can be written as follows:

Require: Initialization $U^0(x_j) \longleftarrow u_0(x_j)$.
  for $n = 0, 1, \cdots, N$ **do**
    for $j = 0, 1, \cdots, m$ **do**
      Solve the values V_j^{n+1} by using DDM I (18).
      Solve the values W_j^{n+1} by using DDM II (24).
      The average of two values will be calculated, i.e. $U_j^{n+1} = (1/2)(V_j^{n+1} + W_j^{n+1})$.
    end for
  end for
Ensure: Output $U^N(x_j)$.

Algorithm 1: The SC-N scheme for one-dimensional parabolic equations.

$$\begin{cases} (I + r_1 G_1) V_1^{n+(1/2)} = (I - r_1 G_2) U^n + b_1^n, \\ (I + r_1 G_2) W_1^{n+(1/2)} = (I - r_1 G_1) U^n + b_1^n, \\ U^{n+(1/2)} = \dfrac{1}{2}\left(V_1^{n+(1/2)} + W_1^{n+(1/2)}\right), \end{cases} \tag{30}$$

where

$$b_1^n = \begin{bmatrix} \left[\left(r_2 u_{0,j}^n + r_1 u_{0,j}^{n+(1/2)}\right)/2\right)\right] + r_2\left(u_{1,j-1}^n - 2u_{1,j}^n + u_{1,j+1}^n\right) + \left(\tau f_{1,j}^n/2\right) \\ r_2\left(u_{2,j-1}^n - 2u_{2,j}^n + u_{2,j+1}^n\right) + \left(\tau f_{2,j}^n/2\right) \\ \vdots \\ r_2\left(u_{m-2,j-1}^n - 2u_{m-2,j}^n + u_{m-2,j+1}^n\right) + \left(\tau f_{m-2,j}^n/2\right) \\ \left[\left(r_2 u_{m,j}^n + r_1 u_{m,j}^{n+(1/2)}\right)/2\right)\right] + r_2\left(u_{m-1,j-1}^n - 2u_{m-1,j}^n + u_{m-1,j+1}^n\right) + \left(\tau f_{m-1,j}^n/2\right) \end{bmatrix}, \tag{31}$$

$$\begin{cases} U^{n+(1/2)} = \left(u_{1,j}^{n+(1/2)}, u_{2,j}^{n+(1/2)}, \ldots, u_{m-1,j}^{n+(1/2)}\right)^T, \\ V_1^{n+(1/2)} = \left(v_{1,j}^{n+(1/2)}, v_{2,j}^{n+(1/2)}, \ldots, v_{m-1,j}^{n+(1/2)}\right)^T, \\ W_1^{n+(1/2)} = \left(w_{1,j}^{n+(1/2)}, w_{2,j}^{n+(1/2)}, \ldots, w_{m-1,j}^{n+(1/2)}\right)^T, \\ j = 1, 2, \ldots, m-1. \end{cases}$$

3.2. y-Direction. The matrix form of the SC-N scheme in y-direction can be written as follows:

$$\begin{cases} (I + r_2 G_1) V_2^{n+1} = (I - r_2 G_2) U^{n+(1/2)} + b_2^{n+(1/2)}, \\ (I + r_2 G_2) W_2^{n+1} = (I - r_2 G_1) U^{n+(1/2)} + b_2^{n+(1/2)}, \\ U^{n+1} = \dfrac{1}{2}\left(V_2^{n+1} + W_2^{n+1}\right), \end{cases} \tag{32}$$

where

$$b_2^{n+(1/2)} = \begin{bmatrix} \left[\left(r_2 u_{i,0}^{n+(1/2)} + r_1 u_{i,0}^{n+1}\right)/2\right] + r_1\left(u_{i-1,1}^{n+1} - 2u_{i,1}^{n+1} + u_{i+1,1}^{n+1}\right) + \left(\tau f_{i,1}^{n+(1/2)}/2\right) \\ r_1\left(u_{i-1,2}^{n+1} - 2u_{i,2}^{n+1} + u_{i+1,2}^{n+1}\right) + \left(\tau f_{i,2}^{n+(1/2)}/2\right) \\ \vdots \\ r_1\left(u_{i-1,m-2}^{n+1} - 2u_{i,m-2}^{n+1} + u_{i+1,m-2}^{n+1}\right) + \left(\tau f_{i,m-2}^{n+(1/2)}/2\right) \\ \left[\left(r_2 u_{i,m}^{n+(1/2)} + r_1 u_{i,m}^{n+1}\right)/2\right] + r_1\left(u_{i-1,m-1}^{n+1} - 2u_{i,m-1}^{n+1} + u_{i+1,m-1}^{n+1}\right) + \left(\tau f_{i,m-1}^{n+(1/2)}/2\right) \end{bmatrix}, \tag{33}$$

$$\begin{cases} U^{n+1} = \left(u_{i,1}^{n+1}, u_{i,2}^{n+1}, \ldots, u_{i,m-1}^{n+1}\right)^T, \\ V_2^{n+1} = \left(v_{i,1}^{n+1}, v_{i,2}^{n+1}, \ldots, v_{i,m-1}^{n+1}\right)^T, \\ W_2^{n+1} = \left(w_{i,1}^{n+1}, w_{i,2}^{n+1}, \ldots, w_{i,m-1}^{n+1}\right)^T, \\ i = 1, 2, \ldots, m-1. \end{cases}$$

The corresponding algorithm can be described in Algorithm 2.

Similar to Algorithm 1, it is obvious that Algorithm 2 has unconditional stability and parallelism.

Remark 3. In fact, $\{(((I + r_1G_1) + (I + r_1G_2))/2)U^{n+(1/2)} = (((I - r_1G_1) + (I - r_1G_2))/2)U^n + b_1^n\}$ is the C-N scheme in x-direction, and $\{(((I + r_2G_1) + (I + r_2G_2))/2)U^{n+1} = (((I - r_2G_1) + (I - r_2G_2))/2)U^{n+(1/2)} + b_2^{n+(1/2)}\}$ is the C-N scheme in y-direction.

Remark 4. In Algorithm 2, the domain is divided into many subdomains by using two DDMs. In each time interval, we first solve the values along x-direction by (30) at the half-time step and then solve the values along y-direction by (32) at the next half-time step. Schemes (30) and (32) lead to block diagonal algebraic systems that can be solved independently. So, Algorithm 2 not only is suitable for parallel computation but also maintains the accuracy. Based on the advantage of ADI technique, Algorithm 2 reduces computational complexities. Though it is developed for two-dimensional problems, Algorithm 2 can be easily extended to solve high-dimensional parabolic equations.

4. Numerical Experiments

To illustrate the efficiency of the SC-N scheme for parabolic equations, we will compare the accuracy of the new algorithm with the existing method.

Example 1

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = \left(1 + \pi^2\right)e^t \sin(\pi x), \quad x \in (0,1), t \in (0,1], \\ u(x,0) = \sin(\pi x), \quad x \in [0,1], \\ u(0,t) = u(1,t) = 0, \quad t \in (0,1]. \end{cases} \tag{34}$$

The exact solution of Example 1 is

$$u(x,t) = e^t \sin(\pi x). \tag{35}$$

Firstly, we examine the convergence rate of Algorithm 1. We divide the mesh point into many segments, such as $K = 3$, $K = 4$, $K = 5$, and $K = 6$. We calculate errors $L^\infty = \|u(x_j, t_n) - u_j^n\|_\infty$ taking $\tau = 0.001$, and the following rate of convergence in the space is

$$\text{rate} \approx \frac{\log\left(L_{h1}^\infty / L_{h2}^\infty\right)}{\log(h1/h2)}. \tag{36}$$

Clearly, the errors appear to be of order $O(h^2)$ in Table 1.

Next, we present the error results of the SC-N scheme in terms of the absolute errors and the relative errors, where the absolute error (A. E.) is defined by

$$e_j^n = \left|u_j^n - u\left(x_j, t_n\right)\right|, \tag{37}$$

and the relative error (R. E.) is defined by

$$E_j^n = \frac{e_j^n}{\left|u\left(x_j, t_n\right)\right|} \times 100\%. \tag{38}$$

Require: Initialization $U^0(x_i, y_j) \longleftarrow u_0(x_i, y_j)$.
  for $n = 0, 1, \cdots, N$ **do**
    for $i = 0, 1, \cdots, m$ **do**
      for $j = 0, 1, \cdots, m$ **do**
        Solve the values $U_{i,j}^{n+(1/2)}$ by using the scheme (30).
        Solve the values $U_{i,j}^{n+1}$ again by using the scheme (32).
      end for
    end for
  end for
Ensure: Output $U^N(x_i, y_j)$.

Algorithm 2: The SC-N scheme for two-dimensional parabolic equations.

Table 1: Convergence rate of Algorithm 1 for h at $t = 0.4$.

h	L^∞ error	Rate
1/19 ($K = 3$)	$4.998746e-4$	—
1/25 ($K = 4$)	$3.085232e-4$	2.0167
1/31 ($K = 5$)	$2.002844e-4$	1.9955
1/37 ($K = 6$)	$1.412289e-4$	1.9903

Table 2: The absolute errors and relative errors of numerical solutions to Example 1 for $h = 1/19$ (i.e., $K = 3$).

x_j	Algorithm 1 ($t = 0.2$)		Algorithm 1 ($t = 0.4$)		Algorithm 1 ($t = 0.8$)	
	A.E.	R. E.	A. E.	R. E.	A. E.	R. E.
0.11	$0.3731e-3$	$1.7559e-2$	$0.5021e-3$	$1.9312e-2$	$0.7577e-3$	$1.9534e-2$
0.21	$0.6184e-3$	$1.5420e-2$	$0.8428e-3$	$1.7176e-2$	$1.2739e-3$	$1.7399e-2$
0.31	$0.8750e-3$	$1.5999e-2$	$1.1881e-3$	$1.7754e-2$	$1.7950e-3$	$1.7976e-2$
0.42	$0.9753e-3$	$1.5409e-2$	$1.3294e-3$	$1.7165e-2$	$2.0093e-3$	$1.7388e-2$
0.52	$1.0066e-3$	$1.5469e-2$	$1.3715e-3$	$1.7225e-2$	$2.0729e-3$	$1.7447e-2$
0.63	$0.9135e-3$	$1.5280e-2$	$1.2463e-3$	$1.7037e-2$	$1.8839e-3$	$1.7259e-2$
0.73	$0.7582e-3$	$1.5778e-2$	$1.0310e-3$	$1.7534e-2$	$1.5579e-3$	$1.7757e-2$
0.84	$0.5469e-3$	$1.7563e-2$	$0.7361e-3$	$1.9316e-2$	$1.1109e-3$	$1.9538e-2$
0.95	$0.1891e-3$	$1.7556e-2$	$0.2545e-3$	$1.9309e-2$	$0.3841e-3$	$1.9532e-2$

Tables 2 and 3 display the absolute errors and the relative errors obtained by presented Algorithm 1 for $h = 1/19$ (i.e., $K = 3$) and $h = 1/25$ (i.e., $K = 4$) at $t = 0.2$, $t = 0.4$, and $t = 0.8$ when taking $r = 1.5$ ($r = \tau/h^2$). From Tables 2 and 3, it is obvious that our algorithm has high accuracy.

We compare Algorithm 1 with the ASC-N scheme in [3] and C-N scheme (3) by the maximum errors for $h = 1/19$ (i.e., $K = 3$) and $h = 1/25$ (i.e., $K = 4$) at different times $t = 0.2$, $t = 0.4$, $t = 0.6$, and $t = 0.8$. With the increasing of computation time, the errors of the ASC-N scheme in [3] increase more than those of Algorithm 1 for different r ($r = \tau/h^2$) in Tables 4 and 5. Moreover, the results show that Algorithm 1 can achieve the same accuracy as the classic C-N scheme while maintaining parallelism. We consider an example for $h = 1/121$ (i.e., $K = 20$) with large grid ratio $r = 15$ ($r = \tau/h^2$). Table 6 shows that Algorithm 1 has a better accuracy than two others, and it is indicated that Algorithm 1 is stable.

Example 2

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x, y, t), \ (x, y) \in \Omega, \quad t \in (0, 1], \\ u(x, y, 0) = \sin(\pi x)\sin(\pi y), \quad (x, y) \in \Omega, \\ u(0, y, t) = u(1, y, t) = 0, \quad y \in [0, 1], t \in (0, 1], \\ u(x, 0, t) = u(x, 1, t) = 0, \quad x \in [0, 1], t \in (0, 1], \end{cases} \tag{39}$$

where the domain is $\Omega = (0, 1) \times (0, 1)$, and the right side function is

$$f(x, y, t) = \left(1 + 2\pi^2\right)e^t \sin(\pi x)\sin(\pi y). \tag{40}$$

The corresponding exact solution is

Table 3: The absolute errors and relative errors of numerical solutions to Example 1 for $h = 1/25$ (i.e., $K = 4$).

x_j	Algorithm 1 ($t = 0.2$)		Algorithm 1 ($t = 0.4$)		Algorithm 1 ($t = 0.8$)	
	A.E.	R. E.	A. E.	R. E.	A. E.	R. E.
0.04	$0.5579e-4$	$9.0292e-3$	$0.7499e-4$	$9.9277e-3$	$1.1317e-4$	$1.0041e-2$
0.12	$1.6391e-4$	$9.0313e-3$	$2.2031e-4$	$9.9298e-3$	$3.3247e-4$	$1.0043e-2$
0.24	$2.7890e-4$	$8.2703e-3$	$3.7803e-4$	$9.1694e-3$	$5.7102e-4$	$0.9283e-2$
0.36	$3.5674e-4$	$8.0054e-3$	$4.8512e-4$	$8.9048e-3$	$7.3305e-4$	$0.9018e-2$
0.48	$3.8737e-4$	$7.8818e-3$	$5.2760e-4$	$8.7813e-3$	$7.9739e-4$	$0.8895e-2$
0.60	$3.6642e-4$	$7.8242e-3$	$4.9945e-4$	$8.7237e-3$	$7.5491e-4$	$0.8837e-2$
0.72	$2.9685e-4$	$7.8239e-3$	$4.0462e-4$	$8.7235e-3$	$6.1158e-4$	$0.8837e-2$
0.84	$1.8851e-4$	$7.9457e-3$	$2.5655e-4$	$8.8452e-3$	$3.8769e-4$	$0.8959e-2$
0.96	$0.5579e-4$	$9.0292e-3$	$0.7499e-4$	$9.9277e-3$	$1.1317e-4$	$1.0041e-2$

Table 4: Comparisons by the maximum errors to Example 1 for $h = 1/19$ (i.e., $K = 3$).

Methods	r	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
Algorithm 1	0.36	$3.5539e-4$	$4.8387e-4$	$5.9795e-4$	$7.3131e-4$
ASC-N scheme [3]		$4.7468e-4$	$7.1743e-4$	$1.0705e-3$	$1.5969e-3$
C-N scheme		$3.5468e-4$	$4.7723e-4$	$5.7998e-4$	$7.0232e-4$
Algorithm 1	0.72	$5.7866e-4$	$7.8813e-4$	$9.7397e-4$	$1.1912e-3$
ASC-N scheme [3]		$7.3439e-4$	$1.1100e-3$	$1.6563e-3$	$2.4709e-3$
C-N scheme		$5.7499e-4$	$7.8239e-4$	$9.6534e-4$	$1.0928e-3$
Algorithm 1	1.45	$1.0066e-3$	$1.3715e-3$	$1.6949e-3$	$2.0729e-3$
ASC-N scheme [3]		$1.0999e-3$	$1.6591e-3$	$2.4754e-3$	$3.6929e-3$
C-N scheme		$9.9823e-4$	$1.3421e-3$	$1.6034e-3$	$1.8807e-3$
Algorithm 1	1.80	$1.2153e-3$	$1.6560e-3$	$2.0465e-3$	$2.5029e-3$
ASC-N scheme [3]		$1.2843e-3$	$1.9356e-3$	$2.8880e-3$	$4.3084e-3$
C-N scheme		$1.2031e-3$	$1.6423e-3$	$1.9034e-3$	$2.2222e-3$

Table 5: Comparisons by the maximum errors to Example 1 for $h = 1/25$ (i.e., $K = 4$).

Methods	r	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
Algorithm 1	0.63	$2.2660e-4$	$3.0852e-4$	$3.8124e-4$	$4.6627e-4$
ASC-N scheme [3]		$2.8997e-4$	$4.3826e-4$	$6.5392e-4$	$9.7553e-4$
C-N scheme		$2.2632e-4$	$3.0324e-4$	$3.7834e-4$	$4.5342e-4$
Algorithm 1	0.94	$3.0760e-4$	$4.1963e-4$	$5.1826e-4$	$6.3353e-4$
ASC-N scheme [3]		$4.5949e-4$	$6.9190e-4$	$8.3264e-4$	$1.5416e-3$
C-N scheme		$3.0642e-4$	$4.1823e-4$	$5.1654e-4$	$6.1278e-4$
Algorithm 1	1.35	$3.8737e-4$	$5.2761e-4$	$6.5199e-4$	$7.9740e-4$
ASC-N scheme [3]		$4.2810e-4$	$6.4710e-4$	$9.6552e-4$	$1.4404e-3$
C-N scheme		$3.8535e-4$	$5.2252e-4$	$6.4721e-4$	$7.7627e-4$
Algorithm 1	1.76	$4.6541e-4$	$6.3397e-4$	$7.8343e-4$	$9.5815e-4$
ASC-N scheme [3]		$4.6674e-4$	$7.0553e-4$	$1.0527e-3$	$1.5704e-3$
C-N scheme		$4.5121e-4$	$6.2937e-4$	$7.6320e-4$	$9.3729e-4$

Table 6: Maximum error comparison for $r = 15$ and $h = 1/121$ (i.e., $K = 20$).

	Algorithm 1	ASC-N method [3]	AGE method [1]
$t = 0.2$	$1.3994e-4$	$1.4759e-4$	$1.2717e-3$
$t = 0.4$	$3.2238e-4$	$2.8034e-4$	$3.2242e-3$
$t = 0.6$	$5.0639e-4$	$6.5004e-4$	$6.2125e-3$
$t = 0.8$	$6.9237e-4$	$8.3824e-4$	$9.2284e-3$

$$u(x, y, t) = e^t \sin(\pi x)\sin(\pi y). \tag{41}$$

Let $h = 1/19$ (i.e., $K = 3$), $h = 1/25$ (i.e., $K = 4$), $h = 1/31$ (i.e., $K = 5$), and $h = 1/37$ (i.e., $K = 6$), and the maximum errors of Algorithm 2 for $\tau = 0.0004$ at different times are displayed in Table 7. It is obvious that the accuracy of Algorithm 2 is satisfied. Table 8shows the comparison of the CPU calculation time among Algorithm 2, the classical C-N scheme, and the ASC-N scheme. Algorithm 2 not only has high accuracy but also has good parallel efficiency.

Table 7: The maximum errors of Algorithm 2 to Example 2 for $\tau = 0.0004$.

	$h = 1/19$	$h = 1/25$	$h = 1/31$	$h = 1/37$
$t = 0.2$	$6.0275e-4$	$1.5451e-4$	$8.0677e-5$	$4.9096e-5$
$t = 0.4$	$4.4976e-4$	$1.9272e-4$	$1.0052e-4$	$6.1127e-5$
$t = 0.5$	$5.1593e-4$	$2.1305e-4$	$1.1113e-4$	$6.7574e-5$
$t = 0.6$	$5.7284e-4$	$2.3546e-4$	$1.2282e-4$	$7.4684e-5$
$t = 0.8$	$7.0014e-4$	$2.8760e-4$	$1.5001e-4$	$9.1219e-5$

Table 8: Comparison of three schemes' calculation time for $r = 1.2$ at $t = 0.5$.

h	Algorithm 2	C-N scheme	ASC-N scheme
1/37 ($K = 6$)	6.4121s	20.2625s	16.7089s
1/49 ($K = 8$)	21.4226s	49.9863s	41.0513s
1/61 ($K = 10$)	53.2623s	175.2632s	146.0665s
1/91 ($K = 15$)	341.5628s	1167.2047s	966.9273s

5. Conclusion

We have proposed and analyzed the SC-N scheme for parabolic equations. This algorithm consists of two DDMs, which split the Crank–Nicolson scheme; each one is used to solve the values at the same time level. Then, the average of two values is calculated. The SC-N scheme maintains the properties of the C-N scheme, i.e., unconditional stability and second-order numerical accuracy. Then, we extend the new algorithm to two-dimensional parabolic equations by ADI technique, which means that high-dimensional parabolic equations can be solved by the proposed algorithm in this paper. Numerical experiments illustrate the good performance of the new algorithm.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Monotone Iterative Method for Fractional Differential Equations with Integral Boundary Conditions

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In this paper, the existence of extremal solutions for fractional differential equations with integral boundary conditions is obtained by using the monotone iteration technique and the method of upper and lower solutions. The main equations studied are as follows: $\begin{cases} -D_{0+}^{\alpha}u(t) = f(t,u(t)), \ t \in [0,1], \\ u(0) = 0, \ u(1) = \int_0^1 u(t)\mathrm{d}A(t), \end{cases}$ where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order $\alpha \in (1,2)$ and $A(t)$ is a positive measure function. Moreover, an example is given to illustrate the main results.

1. Introduction

The purpose of this paper is to study the following differential equations with integral boundary conditions:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f(t,u(t)), \quad t \in [0,1], \\ u(0) = 0, \\ u(1) = \int_0^1 u(t)\mathrm{d}A(t), \end{cases} \tag{1}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative and $1 < \alpha < 2$ and $A(t)$ is a positive measure function, and it satisfies $\int_0^1 G(t)\mathrm{d}A(t) < G(1)$.

Fractional differential equations have been widely used in physics, chemistry, aerodynamics, electrodynamics of complex media, and rheology of polymers [1–7]. As a result, various nonlinear functional analysis methods have been used to study the existence of solutions for differential equations [8–38]. However, it is difficult to obtain the exact solutions of fractional differential equations, so the monotone iteration method and the upper and lower solutions are generally used to obtain the approximate solutions of fractional differential problems. This method is also applicable to both initial value problems and boundary value problems for integer-order differential equations and differential systems [39–43]. In recent years, many scholars have used this method to study various fractional differential equation problems [44–55]. For example, in [47, 48, 52], the authors paid attention to the Riemann–Liouville fractional differential equations of order $\alpha \in (1,2)$, and in [44, 45, 49, 55], the authors considered the Riemann–Liouville fractional differential equation of order $\alpha \in (0,1)$, while in [46], the authors considered the boundary value problem with Riemann–Liouville fractional order $\alpha \in (2,3)$. Of course, some papers also use the monotone iterative method to deal with nonlinear Caputo fractional differential equations [56].

Based on the upper and lower solutions, this paper presents a method to prove the existence of solutions of Riemann–Liouville fractional differential equation (1). By using the monotone iteration technique coupled with the upper and lower solution method, a new comparison principle is established and the existence of the extremal solution of integral boundary value problems (1) is proved.

This paper is mainly divided into the following two parts: Section 2 mainly introduces the preparation of this article, and then in Section 3, the monotone sequence of solutions is constructed, and the main result of integral boundary value problems (1) is given.

2. Preliminaries

In this section, we will briefly introduce some of the necessary definitions and results that will be used in the main results.

Definition 1 (see [2, 5]). The fractional integral of order $\alpha > 0$ of a function $u: (0,\infty) \longrightarrow R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\mathrm{d}s, \tag{2}$$

provided that the right-hand side is point-wise defined on $(0,+\infty)$.

Definition 2 (see [2, 5]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $u: (0,+\infty) \longrightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\mathrm{d}s, \tag{3}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0,+\infty)$.

Let $E = \{u \in C[0,1]: D_{0+}^{\alpha} u \in C[0,1]\}$ be endowed with the norm $\|u\| = \max\{\|u\|_\infty, \|D_{0+}^{\alpha} u\|_\infty\}$ in which $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$, and then $(E, \|\cdot\|)$ is a Banach space.

Definition 3. We say that $u \in E$ is an upper solution of (1) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha} u(t) \geq f(t, u(t)), & t \in [0,1], \\ u(0) = 0, \quad u(1) \geq \displaystyle\int_0^1 u(t)\mathrm{d}A(t). \end{cases} \tag{4}$$

Definition 4. We say that $v \in E$ is a lower solution of (1) if it satisfies

$$\begin{cases} -D_{0+}^{\alpha} v(t) \leq f(t, v(t)), & t \in [0,1], \\ v(0) = 0, \quad v(1) \leq \displaystyle\int_0^1 v(t)\mathrm{d}A(t). \end{cases} \tag{5}$$

Denote

$$g(t) = \frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma((k+1)\alpha-2)}. \tag{6}$$

It is easy to check that (see [57, 58])

$$\begin{aligned} &g(0) = \frac{\alpha-2}{\Gamma(\alpha-1)} < 0, \\ &g'(t) > 0, \quad t \in [0,+\infty), \\ &\lim_{t\longrightarrow+\infty} g(t) = +\infty. \end{aligned} \tag{7}$$

Therefore, there exists a unique number $b^* > 0$ such that

$$g(b^*) = 0. \tag{8}$$

Set $G(t) = t^{\alpha-1} E_{\alpha,\alpha}(bt^{\alpha})$, where $E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty} (x^k/(\Gamma((k+1)\alpha)))$ is the Mittag-Leffler function (see [2, 5]).

In this article, we list the following assumption for convenience:

$(H1)$: the parameter b satisfies $b \in (0, b^*]$.

$(H2)$: $A(t)$ is a positive measure function and $0 < G(1) - \int_0^1 G(t)\mathrm{d}A(t)$.

$(H3)$: assume that $w_0, v_0 \in E$ are the upper and lower solutions of problem (1), respectively, and $v_0(t) \leq w_0(t)$, $t \in [0,1]$.

$(H4)$: $f \in C([0,1]\times\mathbb{R},\mathbb{R})$ and $f(t,w) - f(t,v) \geq -b(w-v)$ for $v_0(t) \leq v \leq w \leq w_0(t), t \in [0,1]$.

Next, we will consider the following auxiliary linear boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + bu(t) = y(t), & t \in [0,1], \\ u(0) = 0, \\ u(1) = \displaystyle\int_0^1 u(t)\mathrm{d}A(t) + c, & c \in \mathbb{R}. \end{cases} \tag{9}$$

Lemma 1. *Suppose that (H1) and (H2) hold and $y \in C[0,1]$. Then, fractional boundary value problem (9) has the following unique solution:*

$$u(t) = \int_0^1 K(t,s) y(s)\mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}, \tag{10}$$

where

$$\begin{aligned} &K(t,s) = K_0(t,s) + G(t)h(s), \\ &h(s) = \frac{\int_0^1 K_0(t,s)\mathrm{d}A(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}, \\ &K_0(t,s) = \frac{1}{G(1)} \begin{cases} G(t)G(1-s), & 0 \leq t \leq s \leq 1, \\ G(t)G(1-s) - G(t-s)G(1), & 0 \leq s \leq t \leq 1. \end{cases} \end{aligned} \tag{11}$$

Proof. The main idea of Lemma 1 comes from [57]. By [2], we first find the solution of the fractional differential equations

$$-D_{0+}^{\alpha} u(t) + bu(t) = y(t), \quad t \in [0,1], \tag{12}$$

with two-point boundary condition

$$\begin{aligned} &u(0) = 0, \\ &u(1) = a \in \mathbb{R}, \end{aligned} \tag{13}$$

which can be expressed by

$$u(t) = -\int_0^t G(t-s) y(s)\mathrm{d}s + c_1 G(t) + c_2 G'(t). \tag{14}$$

Since $u(0) = 0$, we have $c_2 = 0$. Then, by the condition $u(1) = a$, we calculated that

$$c_1 = \frac{\int_0^1 G(1-s)y(s)\mathrm{d}s + a}{G(1)}. \tag{15}$$

Therefore, the solution of (12) and (13) is

$$\begin{aligned} u(t) &= -\int_0^t G(t-s)y(s)\mathrm{d}s + \frac{\int_0^1 G(1-s)y(s)\mathrm{d}s + a}{G(1)}G(t) \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{u(1)}{G(1)}G(t). \end{aligned} \tag{16}$$

Next, we consider problem (9). Integrating equality (16) with respect to $A(t)$, we have

$$\begin{aligned} \int_0^1 u(t)\mathrm{d}A(t) &= \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) \\ &\quad + \frac{u(1)\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}. \end{aligned} \tag{17}$$

Making use of the condition $u(1) = \int_0^1 u(t)\mathrm{d}A(t) + c$ in the above equality yields

$$u(1) - c = \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + \frac{u(1)\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}, \tag{18}$$

and then we get

$$u(1)\left(1 - \frac{\int_0^1 G(t)\mathrm{d}A(t)}{G(1)}\right) = \int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + c. \tag{19}$$

Obviously,

$$u(1) = \frac{G(1)\int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + cG(1)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \tag{20}$$

Therefore,

$$\begin{aligned} u(t) &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{G(t)\left(G(1)\int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + cG(1)\right)}{G(1)\left(G(1) - \int_0^1 G(t)\mathrm{d}A(t)\right)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{G(t)\left(\int_0^1 \int_0^1 K_0(t,s)y(s)\mathrm{d}s\,\mathrm{d}A(t) + c\right)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{\int_0^1 G(t)\left(\int_0^1 K_0(t,s)\mathrm{d}A(t)\right)y(s)\mathrm{d}s}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \\ &= \int_0^1 K_0(t,s)y(s)\mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \end{aligned} \tag{21}$$

The proof is completed. □

Lemma 2. *Suppose that (H1) and (H2) hold, and $u \in E$ satisfies*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) \geq 0, \\ u(0) = 0, \quad u(1) \geq \int_0^1 u(t)\mathrm{d}A(t). \end{cases} \tag{22}$$

Then, for $t \in [0,1]$, $u(t) \geq 0$.

Proof. Let $y(t) = -D_{0+}^{\alpha}u(t) + bu(t)$ and $c = u(1) - \int_0^1 u(t)\mathrm{d}A(t)$. From (12), we have $y(t) \geq 0$, $c \geq 0$, and

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = y(t), \\ u(0) = 0, \quad u(1) = \int_0^1 u(t)\mathrm{d}A(t) + c. \end{cases} \tag{23}$$

By Lemma 1, we obtain that problem (23) has unique solution u, which can be expressed as follows:

$$u(t) = \int_0^1 K(t,s)y(s)\mathrm{d}s + \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)}. \tag{24}$$

From [57], it follows that $K_0(t,s) \geq 0$ for $t, s \in [0,1]$. This together with $(H1)$ and $(H2)$ yields

$$K(t,s) = K_0(t,s) + G(t)\frac{\int_0^1 K_0(t,s)\mathrm{d}A(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \geq 0, \quad t,s \in [0,1]. \tag{25}$$

Hence, we conclude that

$$u(t) \geq \frac{cG(t)}{G(1) - \int_0^1 G(t)\mathrm{d}A(t)} \geq 0, \quad t \in [0,1], \tag{26}$$

which completes the proof. □

3. Main Results

For $v_0, w_0 \in E$ with $v_0(t) \leq w_0(t)$ for $t \in [0,1]$, we denote an ordered interval:

$$[v_0, w_0] = \{u \in E: v_0(t) \leq u(t) \leq w_0(t), \quad t \in [0,1]\}. \tag{27}$$

Theorem 1. *Suppose (H1)–(H4) hold, and then there exist monotone iterative sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ such that $v_n \longrightarrow v^*, w_n \longrightarrow w^*$ as $n \longrightarrow \infty$ uniformly in $[v_0, w_0]$, and v^*, w^* are a minimal and a maximal solution of (1) in $[v_0, w_0]$, respectively.*

Proof. For $v_{n-1}, w_{n-1} \in E$, $n \geq 1$, we define two sequences $\{v_n\}, \{w_n\} \subset E$ satisfying the following fractional differential equation:

$$\begin{cases} -D_{0+}^{\alpha} v_n(t) + b v_n(t) = f(t, v_{n-1}(t)) + b v_{n-1}(t), & t \in [0,1], \\ v_n(0) = 0, v_n(1) = \int_0^1 v_n(t)\mathrm{d}A(t), \end{cases} \tag{28}$$

$$\begin{cases} -D_{0+}^{\alpha} w_n(t) + b w_n(t) = f(t, w_{n-1}(t)) + b w_{n-1}(t), \\ \quad t \in [0,1], \\ w_n(0) = 0, \quad w_n(1) = \int_0^1 w_n(t)\mathrm{d}A(t). \end{cases} \tag{29}$$

By consideration of Lemma 1, for any $n \geq 1$, problems (28) and (29) have a unique solution $v_{n+1}(t), w_{n+1}(t)$, respectively, which are well defined.

Firstly, we need to show that, for any $t \in [0,1]$, $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Let $x(t) = v_1(t) - v_0(t)$, and the definition of $v_1(t)$ together with $(H3)$ yields

$$\begin{cases} -D_{0+}^{\alpha} x(t) + b x(t) = -D_{0+}^{\alpha}(v_1(t) - v_0(t)) + b(v_1(t) - v_0(t)) = -D_{0+}^{\alpha} v_1(t) + b v_1(t) - [-D_{0+}^{\alpha} v_0(t) + b v_0(t)] \\ \geq f(t, v_0(t)) + b v_0(t) - f(t, v_0(t)) - b v_0(t) = 0, \\ x(0) = v_1(0) - v_0(0) = 0, \\ x(1) = v_1(1) - v_0(1) \geq \int_0^1 v_1(t)\mathrm{d}A(t) - \int_0^1 v_0(t)\mathrm{d}A(t) = \int_0^1 x(t)\mathrm{d}A(t). \end{cases} \tag{30}$$

In the light of Lemma 2, we have $x(t) \geq 0, t \in [0,1]$, namely, $v_1(t) \geq v_0(t)$. Similarly, it can be shown that $w_0(t) \geq w_1(t)$, $t \in [0,1]$.

Secondly, we make $h(t) = w_1(t) - v_1(t)$. From $(H4)$, we get

$$\begin{aligned} -D_{0+}^{\alpha} h(t) &= f(t, w_0(t)) - b[w_1(t) - w_0(t)] - f(t, v_0(t)) + b[v_1(t) - v_0(t)] \\ &\geq -b[w_0(t) - v_0(t)] - b[w_1(t) - w_0(t)] + b[v_1(t) - v_0(t)] \\ &= -bh(t). \end{aligned} \tag{31}$$

Also, $h(0) = 0$ and $h(1) = \int_0^1 h(t)\mathrm{d}A(t)$. Thus, Lemma 2 implies that, for any $t \in [0,1]$, $w_1(t) \geq v_1(t)$.

Thirdly, we prove that w_1, v_1 are upper and lower solutions of problem (1), respectively. Note that

$$\begin{aligned} -D_{0+}^{\alpha} v_1(t) &= f(t, v_0(t)) + b v_0(t) - b v_1(t) - f(t, v_1(t)) + f(t, v_1(t)) \\ &= f(t, v_0(t)) - f(t, v_1(t)) + f(t, v_1(t)) - b[v_1(t) - v_0(t)] \\ &\leq b[v_1(t) - v_0(t)] + f(t, v_1(t)) - b[v_1(t) - v_0(t)] \\ &= f(t, v_1(t)). \end{aligned} \tag{32}$$

And by assumption $(H4)$, $v_1(0) = 0$ and $v_1(1) = \int_0^1 v_1(t)\mathrm{d}A(t)$. This shows that v_1 is a lower solution of problem (1). Similarly, we can infer that w_1 is an upper solution of (1).

Using mathematical induction, it is easy to verify that

$$\begin{aligned} v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq w_n(t) \leq \cdots \\ \leq w_1(t) \leq w_0(t). \end{aligned} \tag{33}$$

Clearly, it is easy to conclude that $\{v_n\}$ and $\{w_n\}$ are uniformly bounded in $C[0,1]$. Moreover, by Lemma 1, problems (28) and (29) are equivalent to the following integral equation:

$$\begin{aligned} v_n(t) &= \int_0^1 K(t,s)\left[f\left(s, v_{n-1}(s) + bv_{n-1}(s)\right]\mathrm{d}s,\right. \\ w_n(t) &= \int_0^1 K(t,s)\left[f\left(s, w_{n-1}(s) + bw_{n-1}(s)\right]\mathrm{d}s,\right. \end{aligned} \tag{34}$$

respectively. Therefore, the continuity of the functions $K(t,s)$ allows us to conclude that $\{v_n\}$ and $\{w_n\}$ are equicontinuous in $C[0,1]$. Using (28) and (29) again, we know that $\{D_{0+}^{\alpha} v_n\}$ and $\{D_{0+}^{\alpha} w_n\}$ are uniformly bounded and equicontinuous in $C[0,1]$. So, $\{v_n\}$ and $\{w_n\}$ are uniformly bounded and equicontinuous in E. Using the standard arguments, we have $\{v_n\}$ and $\{w_n\}$ converging, say, to v^* and w^*, uniformly on $[0,1]$, respectively. That is,

$$\begin{aligned} &\lim_{n\longrightarrow\infty} v_n(t) = v^*(t), \\ &\lim_{n\longrightarrow\infty} w_n(t) = w^*(t), \\ &t \in [0,1]. \end{aligned} \tag{35}$$

Furthermore, $v^*(t)$ and $w^*(t)$ are the solutions of problem (1), and $v_0 \le v^* \le w^* \le w_0$ on $[0,1]$.

Next, we need to prove that v^* and w^* are extremal solutions of (1) in $[v_0, w_0]$. Let $u \in [v_0, w_0]$ be any solution of problem (1). We assume that $v_m(t) \le u(t) \le w_m(t), t \in [0,1]$ for some m. Take $p(t) = u(t) - v_{m+1}(t)$, $q(t) = w_{m+1}(t) - u(t)$. Then, by assumption $(H4)$, we obtain

$$\begin{cases} -D_{0+}^{\alpha} p(t) \ge -bp(t), \quad t \in [0,1], \\ p(0) = 0, \quad p(1) = \int_0^1 p(t)\mathrm{d}A(t), \end{cases}$$
$$\begin{cases} -D_{0+}^{\alpha} q(t) \ge -bq(t), \quad t \in [0,1], \\ q(0) = 0, \quad q(1) = \int_0^1 q(t)\mathrm{d}A(t). \end{cases} \tag{36}$$

By Lemma 2, we have

$$v_{m+1}(t) \le u(t) \le w_{m+1}(t), \quad t \in [0,1]. \tag{37}$$

Applying mathematical induction, one has $v_n(t) \le u(t) \le w_n(t)$ on $[0,1]$ for any n. Taking the limit, we conclude $v^*(t) \le u(t) \le w^*(t), t \in [0,1]$. The proof is complete. □

Example 1. Consider the following problem:

$$\begin{cases} -D_{0+}^{4/3} u(t) = -\dfrac{u^3(t)}{10} + \dfrac{\sin t}{10}, \quad t \in [0,1], \\ u(0) = 0, \\ u(1) = \int_0^1 u(t)\mathrm{d}t, \end{cases} \tag{38}$$

where $\alpha = 4/3$, $A(t) = t$, and $f(t,u) = -(u^3/10) + (\sin t/10)$.

Taking $v_0(t) = -t^{1/3}$ and $w_0(t) = t^{1/3}$, we have

$$\begin{cases} -D_{0+}^{4/3} v_0(t) = 0 \le \dfrac{t}{10} + \dfrac{\sin t}{10} = f\left(t, v_0(t)\right), \quad t \in [0,1], \\ v_0(0) = 0, v_0(1) = -1 < -\dfrac{3}{4} = \int_0^1 v_0(t)\mathrm{d}t, \end{cases}$$
$$\begin{cases} -D_{0+}^{4/3} w_0(t) = 0 \ge -\dfrac{t}{10} + \dfrac{\sin t}{10} = f\left(t, w_0(t)\right), \quad t \in [0,1], \\ w_0(0) = 0,\ w_0(1) = 1 > \dfrac{3}{4} = \int_0^1 w_0(t)\mathrm{d}t, \end{cases} \tag{39}$$

which shows that $v_0(t)$ and $w_0(t)$ are a lower and an upper solution of (38), respectively, and $v_0(t) \le w_0(t)$. So, $(H3)$ holds.

Using the strictly monotone increasing property of gamma function $\Gamma(\cdot)$ on $[2, +\infty)$, we have

$$\begin{aligned} g(t) &= -\frac{2}{3\Gamma(1/3)} + \sum_{k=1}^{\infty} \frac{t^k}{\Gamma((4k/3) - (2/3))} = -\frac{2}{3\Gamma(1/3)} \\ &\quad + \frac{t}{\Gamma(2/3)} + \sum_{k=2}^{\infty} \frac{t^k}{\Gamma(((4(k-2))/3) + 2)} \\ &\le -\frac{2}{3\Gamma(1/3)} + \frac{t}{\Gamma(2/3)} + \sum_{k=2}^{\infty} \frac{t^k}{\Gamma(k)} = -\frac{2}{3\Gamma(1/3)} \\ &\quad + \frac{t}{\Gamma(2/3)} + t\left(e^t - 1\right), \quad t \in [0, +\infty). \end{aligned} \tag{40}$$

By MATLAB, we obtain $g(3/10) < -0.0568$. Therefore, $b^* > b := (3/10)$. In addition, we have

$$f(t,w) - f(t,v) = -\left[\frac{w^3}{10} - \frac{v^3}{10}\right] \ge -\frac{3}{10}(w - v), \tag{41}$$

for $v_0(t) \le v \le w \le w_0(t), t \in [0,1]$. Hence, $(H1)$ and $(H4)$ hold.

Note that $G(t) = \sum_{k=0}^{+\infty} (3/10)^k \left(t^{(4k+1)/3}/(\Gamma((4k+4)/3))\right)$. Then,

$$\begin{aligned} \int_0^1 G(t)\mathrm{d}t &= \sum_{k=0}^{+\infty} \left(\frac{3}{10}\right)^k \frac{\int_0^1 t^{((4k+1)/3)}\mathrm{d}t}{\Gamma((4k+4)/3)} \\ &= \sum_{k=0}^{+\infty} \left(\frac{3}{10}\right)^k \frac{1}{((4k+4)/3)\Gamma(((4k+4)/3))} \\ &< \sum_{k=0}^{+\infty} \left(\frac{3}{10}\right)^k \frac{1}{\Gamma((4k+4)/3)} = G(1). \end{aligned} \tag{42}$$

It shows that $(H2)$ holds. Thus, Theorem 1 ensures that problem (38) has extremal solutions in $[v_0, w_0]$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence and Nonexistence of Positive Solutions for High-Order Fractional Differential Equation Boundary Value Problems at Resonance

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We investigate positive solution for high-order fractional differential equations with resonant boundary value conditions. By using the spectral theory and the fixed point index theory, both the nonexistence and existence results of the resonant boundary value problems are given.

1. Introduction

We study the high-order fractional boundary value problem (FBVP) at resonance:

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t)), \quad n-1<\alpha<n, 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, x(1)=\sum_{i=1}^{m}\eta_i x(\xi_i), \end{cases} \tag{1}$$

where $n>3$, $0<\xi_1<\cdots<\xi_m<1$, and $\eta_i>0$ $(i=1,\ldots,m)$ satisfying $\sum_{i=1}^{m}\eta_i\xi_i^{\alpha-1}=1$ and D_{0+}^{α} denotes the Riemann–Liouville derivative. It is clear that $\lambda=0$ and $ct^{\alpha-1}, c\in\mathbb{R}$, solve the linear problem:

$$\begin{cases} -D_{0+}^{\alpha}x(t) = \lambda x(t), \quad 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, x(1)=\sum_{i=1}^{m}\eta_i x(\xi_i), \end{cases} \tag{2}$$

which implies that FBVP (1) happens to be at resonance.

By virtue of the widespread applications, various differential equations have been studied by many researchers (see [1–13] and the references therein). Fractional-order models can describe many processes more accurately than integer-order models, and a great deal of papers focusing on FBVPs appeared in recent years (see [14–23]). The nonlocal FBVPs have especially drawn much attention (see [24–31]). For instance, in [14], the authors investigated the Dirichlet-type FBVP:

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t)), \quad 1<\alpha\le 2, 0<t<1, \\ x(0)=x(1)=0. \end{cases} \tag{3}$$

By using the contraction mapping principle, Cui et al. established the uniqueness results of solution to FBVP (3).

Zhang and Zhong [24, 25] considered the following FBVP:

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t)), & n-1<\alpha<n, 0<t<1, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, & D_{0+}^{\beta}x(1)=\int_0^{\eta}a(t)D_{0+}^{\gamma}x(t)\mathrm{d}t. \end{cases} \tag{4}$$

In [24], the authors established multiplicity results of positive solutions to (4) by using fixed point theorems. In [25], Zhang and Zhong considered uniqueness of solution for FBVP (4) by using the contraction mapping principle.

While there are many papers focusing on nonlocal problems, papers on resonant boundary value problems (RBVPs) are relatively scarce ([32–37]). It should be noted that nothing but positive solution is meaningful in some practical problems. The main tool used to consider positive solution for RBVPs is the Leggett–Williams theorem for coincidence (see [35, 37]).

Motivated by the above work, we focus on deducing some necessary conditions and sufficient conditions of the existence of positive solutions to the RBVP (1). This article has some new features compared with some existing papers. Firstly, the main tools used in this article are the spectral theory and fixed point index theory which is different from [35, 37]. Secondly, some necessary conditions to RBVP (1) are established by virtue of spectral theory. Finally, we deduce some sufficient conditions concerning the behavior of the nonlinearity f at 0 and at ∞.

2. Basic Definitions and Preliminaries

Let

$$g(z) = \sum_{i=0}^{+\infty} \frac{[(i+1)\alpha + 1 - n][(i+1)\alpha - n]z^i}{\Gamma((i+1)\alpha)}. \tag{5}$$

Clearly, $g(z)$ has a unique positive root written as σ, that is,

$$g(\sigma) = 0. \tag{6}$$

For convenience, we use the notations:

$$f_0 = \liminf_{z\longrightarrow 0+} \min_{t\in[0,1]} \frac{f(t,z)}{z},$$

$$f_\infty = \liminf_{z\longrightarrow +\infty} \min_{t\in[0,1]} \frac{f(t,z)}{z},$$

$$f^0 = \limsup_{z\longrightarrow 0+} \max_{t\in[0,1]} \frac{f(t,z)}{z},$$

$$f^\infty = \limsup_{z\longrightarrow +\infty} \max_{t\in[0,1]} \frac{f(t,z)}{z},$$

$$h(t) = t^{\alpha-1}E_{\alpha,\alpha}(\sigma t^\alpha),$$

$$h_1(t) = t^{\alpha+1-n}E_{\alpha,\alpha}(\sigma t^\alpha),$$

$$G(t,s) = \frac{1}{h(1)} \begin{cases} h(t)h(1-s), & 0 \le t \le s \le 1, \\ h(t)h(1-s) - h(1-s)h(1), & 0 \le s \le t \le 1, \end{cases}$$

$$K(t,s) = G(t,s) + \frac{\sum_{i=1}^m \eta_i G(\xi_i, s)h(t)}{h(1) - \sum_{i=1}^m \eta_i h(\xi_i)}, \tag{7}$$

where

$$E_{\alpha,\alpha}(z) = \sum_{i=0}^{+\infty} \frac{z^i}{\Gamma((i+1)\alpha)}. \tag{8}$$

We assume that the following assumption holds throughout this paper:

(H_0) f is continuous on $[0, 1] \times [0, +\infty)$

We also list here some hypotheses to be used later:

(H_1) $f(t, ct^{\alpha-1}) \not\equiv 0$, $\forall c > 0$

(H_2) $f(t, z) \ge -\sigma z$, $\forall\ (t, z) \in [0, 1] \times [0, +\infty)$

(H_3) $f^\infty < 0 < f_0$

(H_4) $f^0 < 0 < f_\infty$

Lemma 1. *Let $\phi \in C(0, 1) \cap L^1[0, 1]$. Then, the FBVP*

$$\begin{cases} -D_{0+}^\alpha x(t) + \sigma x(t) = \phi(t), \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, x(1) = \sum_{i=1}^m \eta_i x(\xi_i), \end{cases} \tag{9}$$

has a unique solution:

$$x(t) = \int_0^1 K(t,s)\phi(s)\mathrm{d}s. \tag{10}$$

Proof. First, it follows from $\eta_i > 0$, $0 < \xi_i < 1$, and $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} = 1$ that

$$h(1) - \sum_{i=1}^m \eta_i h(\xi_i) = \sum_{j=1}^{+\infty} \frac{\sigma^j \left[1 - \sum_{i=1}^{m-2} \eta_i \xi_i^{(j+1)\alpha-1}\right]}{\Gamma((j+1)\alpha)} > 0. \tag{11}$$

By [38], we can write the solution of (9) as

$$x(t) = -\int_0^t h(t-s)\phi(s)\mathrm{d}s + c_1 h(t) + c_2 h'(t) + \cdots + c_n h^{(n-1)}(t). \tag{12}$$

The conditions

$$x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0 \tag{13}$$

yields that

$$c_2 = c_3 = \cdots = c_n = 0. \tag{14}$$

Therefore, we have

$$\begin{aligned} x(1) &= c_1 h(1) - \int_0^1 h(1-s)\phi(s)\mathrm{d}s, \\ x(\xi_i) &= c_1 h(\xi_i) - \int_0^{\xi_i} h(\xi_i - s)\phi(s)\mathrm{d}s. \end{aligned} \tag{15}$$

It follows from

$$x(1) = \sum_{i=1}^m \eta_i x(\xi_i) \tag{16}$$

that

$$c_1 = \frac{\int_0^1 h(1-s)\phi(s)\mathrm{d}s - \sum_{i=1}^m \eta_i \int_0^{\xi_i} h(\xi_i - s)\phi(s)\mathrm{d}s}{h(1) - \sum_{i=1}^m \eta_i h(\xi_i)}. \quad (17)$$

Then, the solution of (9) is

$$\begin{aligned} x(t) &= -\int_0^t h(t-s)\phi(s)\mathrm{d}s + \frac{\int_0^1 h(1-s)\phi(s)\mathrm{d}s - \sum_{i=1}^m \eta_i \int_0^{\xi_i} h(\xi_i - s)\phi(s)\mathrm{d}s}{h(1) - \sum_{i=1}^m \eta_i h(\xi_i)} h(t) \\ &= \frac{\int_0^1 h(t)h(1-s)\phi(s)\mathrm{d}s - \int_0^t h(1)h(t-s)\phi(s)\mathrm{d}s}{h(1)} - \frac{\int_0^1 h(1-s)\phi(s)\mathrm{d}s}{h(1)} h(t) + \frac{\int_0^1 h(1-s)\phi(s)\mathrm{d}s}{h(1) - \sum_{i=1}^m \eta_i h(\xi_i)} h(t) \\ &\quad - \frac{\sum_{i=1}^m \eta_i \int_0^{\xi_i} h(\xi_i - s)\phi(s)\mathrm{d}s}{h(1) - \sum_{i=1}^m \eta_i h(\xi_i)} h(t) \\ &= \int_0^1 G(t,s)\phi(s)\mathrm{d}s + \frac{\sum_{i=1}^m \eta_i \left[\int_0^1 h(\xi_i)h(1-s)\phi(s)\mathrm{d}s - \int_0^{\xi_i} h(1)h(\xi_i - s)\phi(s)\mathrm{d}s\right]}{h(1)\left[h(1) - \sum_{i=1}^m \eta_i h(\xi_i)\right]} \\ &= \int_0^1 K(t,s)\phi(s)\mathrm{d}s. \end{aligned} \quad (18)$$

□

Lemma 2. *G(t, s) satisfies the following:*

(1) $G(1-s, 1-t) = G(t, s)$, $\forall s, t \in [0, 1]$.

(2) $G(t, s) \geq \Lambda_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}$, $\forall s, t \in [0, 1]$.

(3) $G(t, s) \leq \Lambda_2 s(1-s)^{\alpha-1}$, $\forall s, t \in [0, 1]$, where

$$\Lambda_1 = \frac{1}{h(1)[\Gamma(\alpha)]^2}, \qquad \Lambda_2 = \frac{[h'(1)]^2}{h(1)s_0}, \quad (19)$$

in which, $s_0 \in (0, 1)$ satisfies $s_0 = (1-s_0)^{\alpha-2}$.

Proof. Since (1) is trivially true, we only need to verify (2) and (3). When $1 \geq t > s \geq 0$, it is easy to get

$$h_1''(t) = t^{\alpha-1-n} g(\sigma t^\alpha) \leq g(\sigma) = 0. \quad (20)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial s}\left[h_1(t)h_1(1-s) - h_1(t-s)h_1(1)\right] &= h_1'(t-s)h_1(1) \\ &\quad - h_1(t)h_1'(1-s) \geq 0, \end{aligned} \quad (21)$$

which implies that

$$h_1(t)h_1(1-s) \geq h_1(t-s)h_1(1). \quad (22)$$

Then, we have

$$\begin{aligned} h(1)G(t,s) &= t^{n-2}(1-s)^{n-2}h_1(t)h_1(1-s) \\ &\quad - (t-s)^{n-2}h_1(t-s)h_1(1) \\ &\geq h_1(t)h_1(1-s)\left[t^{n-2}(1-s)^{n-2} - (t-s)^{n-2}\right] \\ &\geq h_1(t)h_1(1-s)t^{n-3}(1-s)^{n-3}[t(1-s) - (t-s)] \\ &\geq \frac{t^{\alpha+1-n}(1-s)^{\alpha+1-n}}{[\Gamma(\alpha)]^2} t^{n-3}(1-s)^{n-3}s(1-t) \\ &= \frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{[\Gamma(\alpha)]^2} \\ &\geq \frac{s(1-t)t^{\alpha-1}(1-s)^{\alpha-1}}{[\Gamma(\alpha)]^2}. \end{aligned} \quad (23)$$

When $1 \geq s \geq t \geq 0$, we can get

$$G(t,s) = \frac{h(t)h(1-s)}{h(1)} \geq \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{h(1)[\Gamma(\alpha)]^2} \geq \frac{s(1-t)t^{\alpha-1}(1-s)^{\alpha-1}}{h(1)[\Gamma(\alpha)]^2}. \quad (24)$$

It follows from (23) and (24) that (2) holds.

In the sequel, we prove that (3) holds.

When $1 \geq s \geq t \geq 0$, we can get

$$G(t,s) = \frac{h(t)h(1-s)}{h(1)} \leq h(1)t^{\alpha-1}(1-s)^{\alpha-1} \leq h(1)s(1-s)^{\alpha-1}. \quad (25)$$

When $1 \geq t > s \geq 0$, we have

$$G(t,s) \le \frac{h(t)h(1-s)}{h(1)} \le h(1)t^{\alpha-1}(1-s)^{\alpha-1} \le h(1)(1-s)^{\alpha-1}. \tag{26}$$

Notice that $h'(t)$ is nondecreasing on $[0, 1]$, from which we can get

$$\frac{\partial}{\partial s}G(t,s) \le \frac{h'(1-s)[h(1)-h(t)]}{h(1)}. \tag{27}$$

Therefore,

$$\begin{aligned} G(t,s) &= \int_0^s \frac{\partial}{\partial \tau}G(t,\tau)\mathrm{d}\tau \\ &\le \int_0^s \frac{h'((-\tau)[h(1)-h(t)]}{h(1)}\mathrm{d}\tau \\ &= \frac{[h(1)-h(1-s)][h(1)-h(t)]}{h(1)} \\ &\le \frac{[h'(1)]^2 s(1-s)}{h(1)}. \end{aligned} \tag{28}$$

Let s_0 be the unique root of the equation $s = (1-s)^{\alpha-2}$ on $[0, 1]$. Then,

$$\min\left\{s, (1-s)^{\alpha-2}\right\} \le \frac{s(1-s)^{\alpha-2}}{s_0}, \quad s \in [0,1]. \tag{29}$$

By direct calculation, we get

$$h(1) < h'(1). \tag{30}$$

Therefore, (26)–(30) yield that

$$\begin{aligned} G(t,s) &\le \min\left\{h(1)(1-s)^{\alpha-1}, \frac{[h'(1)]^2 s(1-s)}{h(1)}\right\} \\ &\le \min\left\{(1-s)^{\alpha-2}, s\right\} \times \frac{[h'(1)]^2(1-s)}{h(1)} \\ &\le \frac{[h'(1)]^2 s(1-s)^{\alpha-1}}{h(1)s_0}. \end{aligned} \tag{31}$$

Then, it follows from (25) and (31) that (3) holds. □

Corollary 1. *It follows from (1) and (3) that*

$$G(t,s) \le \Lambda_2(1-t)t^{\alpha-1}, \quad \forall s,t \in [0,1]. \tag{32}$$

Lemma 3. *Green's function K(t, s) satisfies the following:*

(1) $K(t, s) \ge \Lambda_3 s(1-s)^{\alpha-1}t^{\alpha-1}$, $\forall s, t \in [0, 1]$.

(2) $K(t, s) \le \Lambda_4 s(1-s)^{\alpha-1}$, $\forall s, t \in [0, 1]$.

(3) $K(t, s) \le \Lambda_4 t^{\alpha-1}$, $\forall s, t \in [0, 1]$, where

$$\begin{aligned} \Lambda_3 &= \frac{\Lambda_1\left(1-\sum_{i=1}^m \eta_i \xi_i^\alpha\right)}{\Gamma(\alpha)\left[h(1)-\sum_{i=1}^m \eta_i h(\xi_i)\right]}, \\ \Lambda_4 &= \Lambda_2\left[1 + \frac{h(1)\sum_{i=1}^m \eta_i}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)}\right]. \end{aligned} \tag{33}$$

Proof. First, it follows from (2) of Lemma 2 that

$$\begin{aligned} K(t,s) &\ge \frac{\sum_{i=1}^m \eta_i G(\xi_i,s)h(t)}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)} \\ &\ge \frac{\Lambda_1 s(1-s)^{\alpha-1}\sum_{i=1}^m \eta_i(1-\xi_i)\xi_i^{\alpha-1}}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)} \times h(t) \\ &\ge \frac{\Lambda_1\left(1-\sum_{i=1}^m \eta_i \xi_i^\alpha\right)}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)} \times s(1-s)^{\alpha-1} \times \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &= \Lambda_3 s(1-s)^{\alpha-1}t^{\alpha-1}, \end{aligned} \tag{34}$$

that is, (1) holds.

By (3) of Lemma 2, we have

$$\begin{aligned} K(t,s) &= G(t,s) + \frac{\sum_{i=1}^m \eta_i G(\xi_i,s)h(t)}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)} \\ &\le \Lambda_2 s(1-s)^{\alpha-1}\left[1 + \frac{\sum_{i=1}^m \eta_i h(1)}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)}\right] \\ &= \Lambda_4 s(1-s)^{\alpha-1}. \end{aligned} \tag{35}$$

So, (2) holds.

It follows from Corollary 1 that

$$G(t,s) \le \Lambda_2(1-t)t^{\alpha-1} \le \Lambda_2 t^{\alpha-1}. \tag{36}$$

Therefore,

$$\begin{aligned} K(t,s) &\le G(t,s) + \frac{\sum_{i=1}^m \eta_i G(\xi_i,s)h(1)t^{\alpha-1}}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)} \\ &\le \Lambda_2 t^{\alpha-1}\left[1 + \frac{\sum_{i=1}^m \eta_i \xi_i^{\alpha-1}h(1)}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)}\right] \\ &\le \Lambda_2 t^{\alpha-1}\left[1 + \frac{h(1)\sum_{i=1}^m \eta_i}{h(1)-\sum_{i=1}^m \eta_i h(\xi_i)}\right] \\ &= \Lambda_4 t^{\alpha-1}. \end{aligned} \tag{37}$$

So, (3) holds. □

Let $E = C[0, 1]$ with $\|x\| = \max_{0\le t\le 1}|x(t)|$, θ is the zero element, and $B_r = \{x \in E: \|x\| < r\}$. Define two cones P and Q by

$$P = \{x \in E: x(t) \ge 0, 0 < t < 1\},$$

$$Q = \left\{x \in E: \exists l_x > 0, \text{ s.t., } l_x t^{\alpha-1} \ge x(t) \ge \frac{\Lambda_3\|x\|}{\Lambda_4}t^{\alpha-1}, 0 < t < 1\right\}. \tag{38}$$

Clearly, RBVP (1) is equivalent to the auxiliary FBVP:

$$\begin{cases} -D_{0+}^{\alpha}x(t)+\sigma x(t)=f(t,x(t))+\sigma x(t), \quad 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, x(1)=\sum_{i=1}^{m}\eta_i x(\xi_i). \end{cases} \tag{39}$$

Denote

$$\begin{aligned} Ax(t) &= \int_0^1 K(t,s)[f(s,x(s))+\sigma x(s)]\mathrm{d}s, \\ Lx(t) &= \int_0^1 K(t,s)x(s)\mathrm{d}s. \end{aligned} \tag{40}$$

Then, the fixed point x of A is a solution of RBVP (1). It is clear that $L\colon P \longrightarrow Q$ is continuous and the first eigenvalue of L is $\lambda_1=\sigma$. Moreover, $\psi(t)=t^{\alpha-1}$ is a eigenfunction corresponding to λ_1, that is,

$$\psi = \sigma L\psi. \tag{41}$$

Set

$$L_n x(t) = \int_{1/n}^1 K(t,s)x(s)\mathrm{d}s, \quad n=1,2,3,\ldots. \tag{42}$$

By virtue of [13], one has the sequence $\{r(L_n)\}$ that is nondecreasing and

$$\lim_{n\longrightarrow+\infty} r(L_n) = r(L) = \sigma^{-1}. \tag{43}$$

Lemma 4 (see [39]). *Let E be a Banach space, $Q\subset E$ be a cone, $\Omega\subset E$ be a bounded open set, and $A\colon \overline{\Omega}\cap Q \longrightarrow Q$ be a completely continuous operator. Assume that $\exists x_0\in Q\backslash\{\theta\}$ such that*

$$x - Ax \neq \lambda x_0, \quad \forall \lambda\geq 0, x\in\partial\Omega\cap Q. \tag{44}$$

Then, $i(A, \Omega\cap Q, Q)=0$.

Lemma 5 (see [39]). *Let E be a Banach space, $Q\subset E$ be a cone, $\Omega\subset E$ be a bounded open set, $\theta\in\Omega$, and $A\colon \overline{\Omega}\cap Q\longrightarrow Q$ be a completely continuous operator. Assume that*

$$Ax \neq \lambda x, \quad \forall\lambda\geq 1,\ x\in\partial\Omega\cap Q, \tag{45}$$

and then, $i(A, \Omega\cap Q, Q)=1$.

3. Main Results

Theorem 1. *Assume that $f\geq 0$ and (H_1) holds. RBVP (1) cannot have any positive solution.*

Proof. We will use the method of contradiction to prove this theorem. Suppose that RBVP (1) has a positive solution x, then

$$x = Ax = \int_0^1 K(t,s)[f(s,x(s))+\sigma x(s)]\mathrm{d}s. \tag{46}$$

First, we will show that

$$\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s > 0. \tag{47}$$

If otherwise, we have

$$\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s = 0, \tag{48}$$

which implies $f(t, x(t))\equiv 0$ on $[0, 1]$. Then, (1) yields that x satisfies

$$\begin{cases} D_{0+}^{\alpha}x(t)=0, \quad 0<t<1, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, x(1)=\sum_{i=1}^{m}\eta_i x(\xi_i). \end{cases} \tag{49}$$

It is clear that the solution of the above equation is $ct^{\alpha-1}$. Then, $f(t, ct^{\alpha-1})\equiv 0$, which contradicts with (H_1). So, (47) holds.

By Lemma 3, we can get that $\exists\,\varrho_1>\varrho_2>0$, s.t.

$$\varrho_1 t^{\alpha-1} \geq x(t) \geq \varrho_2 t^{\alpha-1}, \tag{50}$$

that is,

$$\varrho_1\psi \geq x \geq \varrho_2\psi. \tag{51}$$

Therefore,

$$\begin{aligned} x &= \int_0^1 K(t,s)f(s,x(s))\mathrm{d}s + \sigma Lx \\ &\geq \Lambda_3 t^{\alpha-1}\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s + \sigma Lx \\ &= \left[\sigma\Lambda_3\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s\right]L\psi + \sigma Lx \\ &\geq \left[\sigma\Lambda_3\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s\right]L\frac{x}{\varrho_1} + \sigma Lx \\ &= \left[\frac{\sigma\Lambda_3\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s}{\varrho_1} + \sigma\right]Lx. \end{aligned} \tag{52}$$

By induction, we have

$$x \geq \left[\frac{\sigma\Lambda_3\int_0^1 s(1-s)^{\alpha-1}f(s,x(s))\mathrm{d}s}{\varrho_1} + \sigma\right]^n L^n x, \quad n=1,2,3,\ldots, \tag{53}$$

which implies

$$x \geq \left[\frac{\sigma\Lambda_3 \int_0^1 s(1-s)^{\alpha-1} f(s,x(s))\mathrm{d}s}{\varrho_1} + \sigma\right] L^n \varrho_2 \psi = \left[\frac{\Lambda_3 \int_0^1 s(1-s)^{\alpha-1} f(s,x(s))\mathrm{d}s}{\varrho_1} + 1\right] \varrho_2 \psi. \tag{54}$$

This is impossible since

$$\left[\frac{\Lambda_3 \int_0^1 s(1-s)^{\alpha-1} f(s,x(s))\mathrm{d}s}{\varrho_1} + 1\right]^n \longrightarrow +\infty, \quad n \longrightarrow \infty. \tag{55}$$

□

Theorem 2. *Assume that $f \leq 0$ and (H_1) holds. RBVP (1) cannot have any positive solution.*

Proof. Suppose that RBVP (1) has a positive solution x, then we have

$$\begin{cases} D_{0+}^{\alpha} x(t) = -f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, x(1) = \sum_{i=1}^{m} \eta_i x(\xi_i). \end{cases} \tag{56}$$

It follows from

$$\begin{aligned} &D_{0+}^{\alpha} x(t) = -f(t, x(t)), \\ &x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \end{aligned} \tag{57}$$

such that

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))\mathrm{d}s + c_1 t^{\alpha-1}. \tag{58}$$

Noticing $f \leq 0$, we can get

$$t^{\alpha-1} \int_0^t (1-s)^{\alpha-1} f(s,x(s))\mathrm{d}s \leq \int_0^t (t-s)^{\alpha-1} f(s,x(s))\mathrm{d}s \leq 0. \tag{59}$$

This and (58) yield

$$\lim_{t \longrightarrow 0+} \frac{x(t)}{t^{\alpha-1}} = c_1, \tag{60}$$

which implies $c_1 \geq 0$.

Next, we will prove that $c_1 > 0$. If otherwise, it follows from (58) that

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))\mathrm{d}s. \tag{61}$$

By considering $x(1) = \sum_{i=1}^{m} \eta_i x(\xi_i)$ and $\sum_{i=1}^{m} \eta_i \xi_i^{\alpha-1} = 1$, we have

$$\sum_{i=1}^{m} \eta_i \int_0^1 (\xi_i - \xi_i s)^{\alpha-1} f(s,x(s))\mathrm{d}s = \sum_{i=1}^{m} \eta_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(s,x(s))\mathrm{d}s, \tag{62}$$

which implies $f(t, x(t)) \equiv 0$. Thus, (61) yields $x = \theta$, which contradicts with the hypothesis.

From (58), there exists $c_2 > c_1$ such that

$$c_1 t^{\alpha-1} \leq x(t) \leq c_2 t^{\alpha-1}. \tag{63}$$

Then,

$$\begin{aligned} x &= \int_0^1 K(t,s) f(s,x(s))\mathrm{d}s + \sigma L x \\ &\leq \Lambda_4 t^{\alpha-1} \int_0^1 f(s,x(s))\mathrm{d}s + \sigma L x \\ &= \left[\sigma\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s\right] L\psi + \sigma L x \\ &\leq \left[\sigma\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s\right] L\frac{x}{c_1} + \sigma L x \\ &= \left[\frac{\sigma\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s}{c_1} + \sigma\right] L x. \end{aligned} \tag{64}$$

By induction, we have

$$x \leq \left[\frac{\sigma\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s}{c_1} + \sigma\right]^n L^n x, \quad n = 1, 2, 3, \ldots, \tag{65}$$

which implies

$$x \leq \left[\frac{\sigma\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s}{c_1} + \sigma\right]^n L^n c_2 \psi = \left[\frac{\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s}{c_1} + 1\right]^n c_2 \psi. \tag{66}$$

It is easy to get that $f(t, x(t)) \not\equiv 0$ (If otherwise, (58) yields $x(t) = c_1 t^{\alpha-1}$. Then, $D_{0+}^{\alpha} x(t) = 0$, which implies $f(t, c_1 t^{\alpha-1}) \equiv 0$, which contradicts with (H_1)). Therefore, we have

$$0 < \frac{\Lambda_4 \int_0^1 f(s,x(s))\mathrm{d}s}{c_1} + 1 < 1. \tag{67}$$

Let $n \longrightarrow \infty$, and (66) yields $x = \theta$, which contradicts with the hypothesis. □

Corollary 2. *Assume that (H_1) holds. The necessary condition for RBVP (1) to have positive solutions is that f admits changing sign.*

Theorem 3. *Assume that (H_2) and (H_3) hold. RBVP (1) has a positive solution.*

Proof. First, by virtue of the Arzela–Ascoli theorem and Lemma 3, we have that $A: Q \longrightarrow Q$ is completely continuous. In the sequel, we will prove that A has a positive fixed point.

By $f_0 > 0$, there exists $r_1 > 0$, s.t.

$$f(t,z) \geq 0, \quad \forall (t,z) \in [0,1] \times [0,r_1]. \tag{68}$$

Suppose the operator A has no fixed point on $\partial B_{r_1} \cap Q$. Next, we will show

$$x - Ax \neq \lambda\psi, \quad \forall \lambda > 0, x \in \partial B_{r_1} \cap Q. \tag{69}$$

If otherwise, there exist $\lambda_0 > 0$ and $x_1 \in \partial B_{r_1} \cap Q$, s.t.

$$x_1 - Ax_1 = \lambda_0\psi. \tag{70}$$

Thus,

$$x_1 \geq \lambda_0\psi. \tag{71}$$

Denote

$$\lambda^* = \sup\{\mu: x_1 \geq \lambda\psi\}. \tag{72}$$

Then, $\lambda^* \geq \lambda_0$ and $x_1 \geq \lambda^*\psi$. Noticing $Ax_1 \geq \sigma L x_1$, we have

$$x_1 = Ax_1 + \lambda_0\psi \geq \sigma L x_1 + \lambda_0\psi \geq \lambda^*\sigma L\psi + \lambda_0\psi = (\lambda^* + \lambda_0)\psi. \tag{73}$$

This contradicts with the notation of λ^*. Thus, (69) holds. Then, it follows from Lemma 4 that

$$i(A, B_{r_1} \cap Q, Q) = 0. \tag{74}$$

By $f^\infty < 0$, there exist $\varepsilon \in (0, 1)$ and $r_2 > r_1$, s.t.

$$f(t,z) \leq -\sigma\varepsilon z, \quad \forall (t,z) \in [0,1] \times [r_2, +\infty). \tag{75}$$

Denote

$$S = \{x \in Q \mid x = \lambda Ax, \quad 0 \leq \lambda \leq 1\}. \tag{76}$$

Next, we will show S is bounded.

For any $x \in S$, let $\overline{x}(t) = \min\{x(t), r_2\}$. Then, we have

$$f(t, x(t)) \leq -\sigma\varepsilon x(t) + f(t, \overline{x}(t)) + \sigma\overline{x}(t), \quad t \in [0,1]. \tag{77}$$

Clearly, we have

$$A\overline{x} = \int_0^1 K(t,s)[f(s, \overline{x}(s)) + \sigma\overline{x}(s)]\mathrm{d}s \leq \max_{[0,1]\times[0,r_2]} \Lambda_4 \times \{f(t,z) + \sigma z\}. \tag{78}$$

Set $T_1 = \sigma(1-\varepsilon)L$. Then,

$$x = \lambda Ax \leq Ax \leq \sigma(1-\varepsilon)Lx(t) + A\overline{x} \leq T_1 x + M. \tag{79}$$

Here,

$$M = \max_{[0,1]\times[0,r_2]} \Lambda_4 \times \{f(t,z) + \sigma z\}. \tag{80}$$

Therefore,

$$(I - T_1)x \leq M. \tag{81}$$

Clearly, we have $r(T_1) = 1 - \varepsilon$. So, the inverse operator of $I - T_1$ exists. Thus, (81) yields

$$\|x\| \leq M\left\|(I - T_1)^{-1}\right\|, \tag{82}$$

which implies S is bounded.

Let $r_3 = r_2 + M\|(I - T_1)^{-1}\|$. Then, it follows from Lemma 5 that

$$i(A, B_{r_3} \cap Q, Q) = 1. \tag{83}$$

By (74) and (83), we have

$$i(A, (B_{r_3} \setminus \overline{B}_{r_1}) \cap Q, Q) = 1. \tag{84}$$

This means A has a fixed point on $(B_{r_3} \setminus \overline{B}_{r_1}) \cap Q$. So, RBVP (1) has a positive solution. □

Theorem 4. *Assume that (H_2) and (H_4) hold. RBVP (1) has a positive solution.*

Proof. It follows from $f^0 < 0$ that $\exists R_1 > 0$, s.t.

$$f(t,z) \leq 0, \quad \forall (t,z) \in [0,1] \times [0,R_1]. \tag{85}$$

Set $T_2 = \sigma L$. Clearly, we have $r(T_2) = 1$.

Suppose the operator A has no fixed point on $\partial B_{R_1} \cap Q$. Next, we will show

$$Ax \neq \lambda x, \quad \forall \lambda > 1, \ x \in \partial B_{R_1} \cap Q. \tag{86}$$

If otherwise, there exist $\lambda_0 > 1$ and $x_1 \in \partial B_{R_1} \cap Q$, s.t. $Ax_1 = \lambda_0 x_1$. Noticing that

$$Ax_1 = \int_0^1 K(t,s)[f(s, x_1(s)) + \sigma x_1(s)]\mathrm{d}s \leq T_2 x_1, \tag{87}$$

we have

$$\lambda_0 x_1 \leq T_2 x_1. \tag{88}$$

By induction, we have

$$\lambda_0^n x_1 \leq T_2^n x_1, \quad n = 1, 2, \ldots, \tag{89}$$

which implies

$$r(T_2) = \lim_{i \longrightarrow +\infty} \sqrt[i]{\left\|T_2^i\right\|} \geq \lambda_0. \tag{90}$$

This contradicts with $r(T_2) = 1$. So, (86) holds. By Lemma 5, we get

$$i(A, B_{R_1} \cap Q, Q) = 1. \tag{91}$$

From $f_\infty > 0$, we know that there exist $\delta > 0$ and $R_2 > R_1$, s.t.

$$f(t,z) \geq \delta z, \quad \forall (t,z) \in (0,1] \times [R_2, +\infty). \tag{92}$$

Since $\{r(L_n)\}$ is nondecreasing and converges to σ^{-1}, one can choose m satisfies

$$r(L_m) > \frac{1}{\sigma + \delta}. \tag{93}$$

It follows from Lemma 3 and the Krein–Rutman theorem that there exists ψ_m satisfies

$$L_m \psi_m = r(L_m)\psi_m. \tag{94}$$

Let

$$R_3 = \frac{\Lambda_4 R_2 m^{\alpha-1}}{\Lambda_3} + R_2. \tag{95}$$

For any $x \in \partial B_{R_3} \cap Q$, we have

$$x(t) \geq \frac{\Lambda_3 \|x\|}{\Lambda_4} t^{\alpha-1} > R_2, \quad \forall t \in \left[\frac{1}{m}, 1\right]. \tag{96}$$

Then,

$$\begin{aligned} Ax(t) &\geq \int_{1/m}^{1} K(t,s)[f(s,x(s)) + \sigma x(s)]\mathrm{d}s \\ &\geq \int_{1/m}^{1} K(t,s)[\delta x(s) + \sigma x(s)]\mathrm{d}s \\ &= (\delta + \sigma)(L_m x)(t). \end{aligned} \tag{97}$$

Suppose the operator A has no fixed point on $\partial B_{R_3} \cap Q$. In the following, we will show

$$x - Ax \neq \lambda \psi_m, \quad \forall x \in \partial B_{R_3} \cap Q, \lambda > 0. \tag{98}$$

If otherwise, then there exist $\lambda_0 > 0$ and $x_1 \in \partial B_{R_3} \cap Q$ such that

$$x_1 - Ax_1 = \lambda_0 \psi_m. \tag{99}$$

Denote

$$\lambda^* = \sup\{\mu : x_1 \geq \lambda \psi_m\}. \tag{100}$$

Then, $\lambda^* \geq \lambda_0$ and $x_1 \geq \lambda^* \psi_m$. Noticing (97), we have

$$\begin{aligned} x_1 &= Ax_1 + \lambda_0 \psi_m \\ &\geq (\delta + \sigma) L_m x_1 + \lambda_0 \psi_m \\ &\geq (\delta + \sigma) L_m \lambda^* \psi_m + \lambda_0 \psi_m \\ &= (\delta + \sigma)\lambda^* r(L_m)\psi_m + \lambda_0 \psi_m \\ &\geq (\lambda^* + \lambda_0)\psi_m, \end{aligned} \tag{101}$$

which contradicts with the notation of λ^*. Hence, (98) holds. By virtue of Lemma 4, one has

$$i\left(A, B_{R_3} \cap Q, Q\right) = 0. \tag{102}$$

From (91) and (102), one has

$$i\left(A^*, \left(B_R \backslash \overline{B}_{r_3}\right) \cap Q, Q\right) = -1. \tag{103}$$

Therefore, RBVP (1) has a positive solution on $(B_{R_3} \backslash \overline{B}_{R_1}) \cap Q$. □

4. Examples

Example 1. Let us consider the existence of positive solutions for the RBVP:

$$\begin{cases} -D_{0+}^{7/2} x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = 0, x(1) = 19x\left(\frac{1}{9}\right) + 7x\left(\frac{4}{9}\right), \end{cases} \tag{104}$$

with

$$f(t,z) = z^2 \cos t. \tag{105}$$

It is clear that (H_0) and (H_1) hold. Then, Theorem 1 yields that RBVP (104) has no positive solution.

Example 2. Consider the following RBVP:

$$\begin{cases} -D_{0+}^{5/2} x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = 0, x(1) = 2x\left(\frac{1}{9}\right) + \frac{8}{3}x\left(\frac{1}{4}\right) + 2x\left(\frac{4}{9}\right), \end{cases} \tag{106}$$

with

$$f(t,z) = z^2 \cos t - \frac{z}{4}. \tag{107}$$

By direct calculation, we have $f^0 = -(1/4)$, $f_\infty = +\infty$. Thus, (H_4) holds. Moreover,

$$\begin{aligned} g\left(\frac{1}{4}\right) &= -\frac{1}{3\sqrt{\pi}} + \sum_{k=1}^{+\infty} \frac{4^{-k}}{((5k/2) + (3/2))\Gamma((5k/2) - (1/2))} \\ &\leq -\frac{1}{3\sqrt{\pi}} + \frac{1}{3}\sum_{k=1}^{+\infty} \frac{4^{-k}}{\Gamma(2k)} \\ &= -\frac{1}{3\sqrt{\pi}} + \frac{e^{1/2} - e^{-(1/2)}}{12} \\ &\approx -0.1012136, \end{aligned} \tag{108}$$

which implies $\sigma > 1/4$. Therefore, (H_2) holds. It follows from Theorem 4 that RBVP (106) has a positive solution.

5. Conclusions

In this article, we focus on the existence results of positive solutions to high-order FBVPs at resonance. The main tool used in this article is different from most of articles dealing with positive solution for resonant problems. First, we rewrite the RBVPs as equivalent nonresonant FBVPs. By means of the spectral theory, we obtain the necessary conditions for RBVP (1). By means of the fixed point index theory, we establish some sufficient conditions for RBVP (1) concerning the behavior of the nonlinearity f at 0 and at ∞.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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