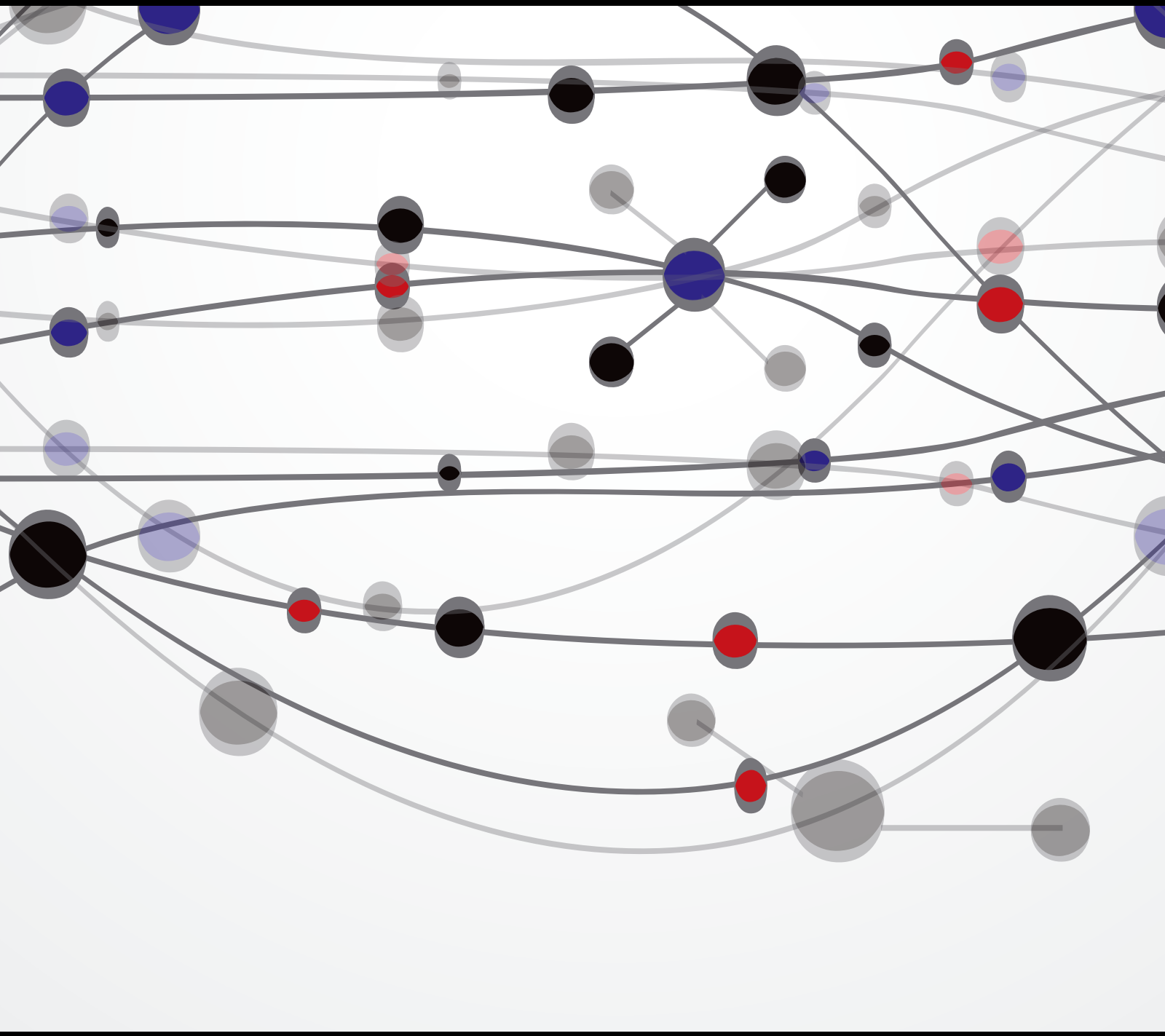
The Scientific World Journal

Optimization and Decision Science

Guest Editors: Farhad Hosseinzadeh Lotfi,
Alireza Amirteimoori, Barnett Parker, Mohammad Khodabakhshi,
Jie Wu, and Mohsen Vaez-Ghasemi



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Editorial

Optimization and Decision Science

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1. Introduction

Operational research (OR) includes a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency. OR involves the construction of mathematical models that attempt to describe the system. Because of the computational and statistical nature of most of these fields, OR also has strong ties to computer science and analytic. Operational researchers faced with a new problem must determine which of these techniques are most appropriate given the nature of the system, the goals for improvement, and constraints on time and computing power. The main focus of this special issue will be on the new research ideas and results for the OR. Considering the wide applications of OR, it seems natural that this journal selected it as the theme of its special issue.

2. Overview

This special issue is composed of ten research papers; the covered subjects include topics such as ranking, congestion, fuzzy numbers and system, supply chain, and optimization problems. Below, a very brief overview of the featured works is given.

M. Rostami-Malkhalifeh et al. in their paper deal with evaluating congestion in free disposal hull (FDH) models. In the DEA literature, there exist many methods dealing with theory and application of congestion in convex technologies. Considering nonconvex technologies, this paper discusses congestion in free disposal hull. The presented method shows the sources of congestion and estimates its amounts and also detects the losses amounts of output due to congestion. In a paper by M. Fallah and A. Mohajeri, green supply chain management is considered. Nowadays, due to global warming and climate changes, it is an important issue. Here, the authors formulate a comprehensive closed-loop model for the logistics planning considering profitability and ecological goals. They noted that the profitability criterion supported the cyclic network with the minimum costs and maximum service level. In the work provided by M. Rostami-Malkhalifeh et al., a new method is presented for solving fuzzy system of linear equations with crisp coefficients matrix and fuzzy or interval right hand side. The authors derived conditions for the existence of a fuzzy or interval solution of linear system. A work by M. Rostami-Malkhalifeh et al. discusses calculating superefficiency and then ranking the units based on it. The authors noted that most of the existing models do not provide the projection of Pareto efficiency. Thus, they introduced a model based on which the projection of Pareto-efficient can be achieved. The presented model is unit invariant and feasible and makes the amount of inefficiency effective in ranking. In a paper by M. Rostami-Malkhalifeh et al., a procedure is provided for ranking DMUs in DEA technique considering ideal and anti-ideal points. They formulated a model which is introduced to compute the performance of units for these two points through using

common set of weights. The important feature of this model is that it can rank all DMUs by solving only three programs and it can rank all the extreme and nonextreme efficient DMUs. In the work by L. G. Moazam and T. Allahviranloo, a new concept of the 2nd power of a fuzzy number is introduced. The authors noted that this method is exponent to production (EP) method that provides an analytical and approximate solution for fully fuzzy quadratic equation (FFQE). The work by M. Rostami-Malkhalifeh et al. studies the inverse DEA technique while the nonradial enhanced Russell model is being used. They also introduced necessary and sufficient conditions for extra inputs or lack outputs determination and also investigated for showing the extra input or lack output. In the work by M. R. Abu Bakar et al., a new method is provided for the directional slack-based measure for the inverse DEA. The authors noted that the presented method elucidates the inverse directional slack-based measure model within a new production possibility set and noted that this approach was investigated in this study with reference to a resource allocation problem. It is possible to simultaneously consider any alterations of certain outputs associated with the efficient decision-making unit. In the paper by S. Kim, it is mentioned that gamification means the use of various elements of game design in nongame contexts including workplace collaboration, marketing, education, military, and medical services. This paper suggested the decision criteria for selection of gamification platform to support a systematic decision-making process for managements. They noted that the criteria are derived from previous works on gamification, introduction of information systems, and analytic hierarchy process.

A. Grilo and J. Santos noted that there exists a shortage in the literature regarding the efficiency evaluation and productivity evolution of the new technology-based firms (NTBFs) in the incubation scope. Thus, they provided a model considering DEA analysis in order to incubate NTBFs for assessment and improvement in the efficiency. Moreover, they also used Malmquist index.

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Research Article

Measuring Efficiency and Productivity Growth of New Technology-Based Firms in Business Incubators: The Portuguese Case Study of Madan Parque

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Business incubators can play a major role in helping to turn a business idea into a technology-based organization that is economically efficient. However, there is a shortage in the literature regarding the efficiency evaluation and productivity evolution of the new technology-based firms (NTBFs) in the incubation scope. This study develops a model based on the data envelopment analysis (DEA) methodology, which allows the incubated NTBFs to evaluate and improve the efficiency of their management. Moreover, the Malmquist index is used to examine productivity change. The index is decomposed into multiple components to give insights into the root sources of productivity change. The proposed model was applied in a case study with 13 NTBFs incubated. From that study, we conclude that inefficient firms invest excessively in research and development (R&D), and, on average, firms have a productivity growth in the period of study.

1. Introduction

The highly competitive environment of new technologies forces firms to seek mechanisms that enable them to achieve sustainable growth. Recent years have seen the emergence of business incubators all over the world. Incubators can play a key role in supporting new technology-based firms (NTBFs) by betting on innovation as a way to help the creation and development of these firms. However, it is still unclear whether the mission of the incubators to encourage the growth of NTBFs has been successful [1]. Since NTBFs are the general typology of incubated firms, it is important to ensure efficient management of the available resources. Costs arising from the lack of efficiency can compromise the survival and growth of incubated firms, regardless of their business area. This situation forces firms to minimize their costs while continuing to provide quality and diversity products. Therefore, performance evaluation and benchmarking could help the NTBFs to become more productive and efficient by avoiding their untimely death. Although decisions about NTBF's strategy are ultimately from the entrepreneurs' responsibility, business incubators may provide efficiency and productivity benchmarking tools in their guidance role for incubated firms.

Data envelopment analysis (DEA) has been highlighted as an assessment tool of technical efficiency in organizations. The DEA allows evaluating the relative efficiency of a combination of units that convert multiple inputs into multiple outputs. For a given set of input and output variables, DEA produces a single comprehensive measure of performance (efficiency score) for each unit [2]. To measure the productivity change over time, this paper suggests a DEA-based Malmquist productivity index methodology [3]. Some studies focus on the importance of efficiency evaluation in the technology sector through the use of DEA, namely, with respect to R&D activities (see [4–8]). However, there is a shortage of research into the subject in the context of incubated firms.

The aim of this paper is to create a model for estimating the technical efficiency and productivity growth of NTBFs

located in a business incubator during the period 2009–2011. The proposed model aims to help decision-making about an NTBF's strategy, and though ultimately this is mainly entrepreneurs' responsibility, business incubators may provide efficiency and productivity benchmarking tools in their guidance role for incubated firms. Managers of NTBFs and business incubators' managers may adopt the proposed model as an internal benchmark management procedure in order to evaluate the relative position of each firm to the efficient frontier, thus enabling a relative orientation in terms of their productivity. Based on these comparisons, both across companies and over a period of time, it will be possible for firms with more specific sources on how to address improvements in terms of management in a macroperspective.

The paper is organized as follows. Section 2 presents the state of art regarding the main characteristics for business incubators and some of their challenges regarding the evaluation of performance. Section 3 describes the concepts and models related to DEA, along with the Malmquist index. The proposed model to evaluate efficiency and productivity growth of NTBFs is presented in Section 4 along with the description, analysis and discussion of the results of the case study of Madan Parque, an incubator in Portugal. In Section 5 conclusions are presented.

2. Incubating New Technology-Based Firms (NTBFs)

The main characteristics of NTBF are [9–11] (a) high percentage of engineers and researchers among employees; (b) fast growth rate and a global market for their products; (c) innovation and advanced technology in the products and services; (d) investment of at least 3% of their revenue in research and development (R&D) activities.

National and regional governments have launched several political programs and policies over the years seeking to develop a nurturing environment for NTBFs. These political measures are often a way to revitalize European regions that have been in economic decline and where there is a belief that NTBF can reverse that downwards trend [12]. Science & Technology Parks (STP) and especially business incubators within STP have had a major role in providing infrastructures for the execution of these national and regional economic policies, with public and private funding. Such efforts have increased considerably due to the recognition of STPs' importance in the development and maturity of NTBFs [13].

Despite NTBFs performance and contribution to the economy, there are factors that may jeopardize their economic potential, for example, the management capacity and the sales ability of their marketing drive, as entrepreneurs often have mainly technological skills and competences. Thus, NTBFs' success often depends on the quality of the management resources made available by business incubators in STP and the sources of capital [14]. NTBF entrepreneurs are more likely to start and grow their companies in SCT business incubators than outside them [15]. Siegel et al. [16] analyzed research productivity in companies located in STP business incubators compared with similar companies located outside STP and their findings demonstrate that NTBFs located in STP are more effective in terms of creation of new products, services, and patents.

According to Monck et al. [14], NTBF performance indicators can be divided into two groups: *inputs* measurement for high-technology activity, such as the number of qualified employees and the R&D effort characterized by investment in R&D as a percentage of overall sales and *output* measurement, such as growth rate and the number of patents and technological innovations. Walter et al. [17] examine academic spin-offs in STP business incubators and find that sales growth rate is a fundamental indicator for measuring the performance of this type of company as it can measure NTBFs management autonomy to explore the market and thereby validate the acceptance of the customers to the R&D results, as does also advocate Oakey [18]. To measure the internal efficiency, the authors propose the ratio of sales per employee.

Wang et al. [7] developed a model to measure the performance of NTBFs through four perspectives: financial, customer, internal processes, and growth perspective. The authors used a combined approach of hierarchical balanced scorecard (HBSC) and nonadditive fuzzy. The aim was therefore to develop a tool for improving performance measures through HBSC in complex environments and high competitiveness. The HBSC serves as a bridge between financial and nonfinancial perspectives and is an integrated system of performance measurement, combining the objectives of the organizations and other traditional functional areas of corporate strategy. For this, the authors use two key performance indicators (results-oriented and company development) in order to measure the strategy's implementation. Their study demonstrates the limitations that may exist in an HBSC survey of performance measures and thus contributes to the improvement of the effectiveness and efficiency of management.

3. Measuring Efficiency and Productivity Growth

3.1. Data Envelopment Analysis. Data envelopment analysis (DEA) measures the relative efficiency of decision-making units (DMUs) with multiple inputs and multiple outputs using a linear programming based model. The technique is nonparametric because it requires no assumption about the weights of the underlying production function [19]. Furthermore, DEA does not require prescribing the functional forms of the relationships between inputs and outputs that are needed in statistical regression approaches, and the variables can be measured in different units [20].

The set of efficient DMUs that form the efficient frontier can be identified. Thus, DEA is also a powerful benchmarking technique since it allows measuring the level of efficiency of nonfrontier units and identifying benchmarks against which such inefficient units can be compared [21].

In the literature, two DEA models are commonly used. The initial basic frontier model, known as the Charnes, Cooper, and Rhodes (CCR) model [22], is built on the assumption of constant returns to the scale (CRS) of activities.

The second model, known as Banker, Charnes, and Cooper (BCC) model [23], is built on the assumption of variable returns to the scale (VRS) of activities. There are two versions for both the CCR and the BCC models to determine the efficient frontier. One is input-oriented and the other output-oriented [24].

The output-oriented score (ϕ) will be greater than or equal to 1, and that $\phi - 1$ is the proportional increase in outputs that could be achieved by the unit under evaluation, with input quantities held constant. It is noted that $1/\phi$ defines a technical efficiency score that varies between 0 and 1 [25]. The existence of input and output slacks shows that additional input reduction or output production is needed in order to make the DMU efficient [2].

According to Banker and Thrall [26], the BCC model allows one to decompose the *technical efficiency* (TE), obtained through the CCR model in *scale efficiency* (SE) and *pure technical efficiency* (PTE). The *score* CCR is denominated TE, and the *score* obtained through the BCC model measures PTE. For a specific DMU, if the technical efficiency *scores* differ in the CCR model and the BCC model, then the DMU presents an inefficiency of scale. Scale efficiency evaluates the capacity of a unit being produced in CCR. If the *scores* of the two models are equal, then the DMU is operating under CCR, that is, in the most efficient scale of production. This scale inefficiency may be computed through the score difference of the technical efficiency of BCC and CCR. A DMU BCC efficiency is always greater than or equal to the efficiency measured in the CCR model [25].

The SE is defined by the ratio of TE to PTE:

$$\mathrm{SE} = \frac{\mathrm{TE}}{\mathrm{PTE}} = \frac{\phi^*_{\mathrm{CCR}}}{\phi^*_{\mathrm{BCC}}}. \tag{1}$$

SE is always lower than 1. Expression (1) is equivalent to TE = SE × PTE. This decomposition describes the sources of inefficiency, which may be caused by an inefficient operation from the DMU (PTE), by disadvantageous conditions under which the DMU is operating (SE), or both [20].

DEA has also been widely applied to different industries, and a number of different DEA models have been developed and improved based on the original DEA model (see [2, 21, 24, 27–31]).

3.2. Malmquist Productivity Index. The Malmquist productivity index (MI) [32], based on DEA models, is one of the prominent indices for measuring the relative productivity change of DMUs over time. From the combination of the inputs and outputs of a DMU in periods t and $t + 1$, it is possible to determine whether the variation in the performance of this DMU is due to technical efficiency change (TEC) of each DMU and/or technological change (TC).

Compared to other indices, the Malmquist productivity index presents some important characteristics and properties. The Malmquist productivity index can be useful in situations in which the objectives of managers differ, are unknown, or are difficult to implement, since it does not require any assumption regarding the cost minimization or profit maximization [33]. Moreover, an assumption associated with application of MI is the existence of a competitive market, which encourages businesses to implement effective strategies [25]. The calculation of the MI requires measurements of two different time periods and two grouped periods. The measures of the two different time periods can be obtained through the DEA CCR model.

Färe et al. [34] specify an output-based MI calculated for year t and $t + 1$ technologies defined by (2). Consider

$$M_0 = \left[\frac{D_0^t\left(x_0^{t+1}, y_0^{t+1}\right)}{D_0^t\left(x_0^t, y_0^t\right)} \cdot \frac{D_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{D_0^{t+1}\left(x_0^t, y_0^t\right)} \right]^{1/2}. \tag{2}$$

The above measure is actually the geometric mean of two Malmquist productivity indices [1]. When $M_0 > 1$, it indicates productivity gain; when $M_0 < 1$, it signifies productivity loss; and $M_0 = 1$ means no change in productivity from t and $t + 1$ [5]. The MI can be decomposed into two components: the first component is the technical efficiency change (TEC); and the second component is the shift in the frontier or the technological change (TC) between periods t and $t + 1$ [3]. Consider

$$M_0 = \frac{D_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{D_0^t\left(x_0^t, y_0^t\right)} \times \left[\frac{D_0^t\left(x_0^t, y_0^t\right)}{D_0^{t+1}\left(x_0^t, y_0^t\right)} \cdot \frac{D_0^t\left(x_0^{t+1}, y_0^{t+1}\right)}{D_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)} \right]^{1/2}, \tag{3}$$

$$M_0 = \mathrm{TEC} \times \mathrm{TC}.$$

The ratio outside the brackets measures the change in relative efficiency; that is, it is also a measure of how close the DMU is to the frontier in period $t + 1$ compared with period t. If TEC = 1, the DMU has the same distance in periods $t + 1$ and t from the respective efficient frontiers. If TEC > 1, the DMU has moved closer to the period $t + 1$ frontier than it was to the period t frontier, and if TEC < 1, the converse occurs. The bracketed term is the index of change in technology between two periods. If TC = 1, it indicates no shift in technology frontier; a value of TC < 1 indicates technological regress; TC > 1 indicates technological progress and is considered to be an evidence of innovation [19, 34].

In relation to the returns to scale assumption, the MI must be calculated in a first step on the basis of CRS, since, if measured according to VRS, the measurement obtained is inaccurate [35]. The TEC and TC indices are obtained under the assumption that the DMU operates according to CRS, that is, assuming that DMU is operating in an optimal scale. So, to deal with more realistic cases with VRS, the TEC calculated under the assumption of CRS technology can be further decomposed into pure technical efficiency change (PTEC) and scale efficiency change (SEC) [36]. SEC quantifies the productivity gain or loss associated with a production unit, evaluating whether movements inside the frontier are in the right direction to attain the CRS point, where changes in outputs result in proportional changes in inputs [37]. While the TEC term is associated with efficiency change

calculated under CRS, the PTEC is the efficiency change calculated under VRS. In this case, MI would comprise three components [34]:

$$M_0 = \text{PTEC} \times \text{SEC} \times \text{TC}. \quad (4)$$

According to Grosskopf [36], a PTEC is defined as

$$\text{PTEC} = \frac{D_{0\text{VRS}}^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{D_{0\text{VRS}}^{t}\left(x_0^{t}, y_0^{t}\right)}. \quad (5)$$

SEC presents the following formulation:

$$\text{SEC} = \frac{D_{0\text{CRS}}^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)/D_{0\text{VRS}}^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{D_{0\text{CRS}}^{t}\left(x_0^{t}, y_0^{t}\right)/D_{0\text{VRS}}^{t}\left(x_0^{t}, y_0^{t}\right)}. \quad (6)$$

While the TEC refers to the changes in technical efficiency calculated under CRS, the PTEC corresponds to changes in technical efficiency with regard to VRS and represents the changes resulting from efficiency improvements in operations and management activities. This decomposition allows us to contemplate situations in which a DMU may be technically effective, since the volume of production uses the least amount of resources, but not operating at the optimal scale production. SEC shows the movements inside the boundary that are in the right direction to reach the CRS point at which the output changes result in proportional changes in inputs [36].

3.3. NTBF and DEA. In recent years, the application of DEA as an assessment tool for NTBFs has grown, especially with regard to the selection of projects for R&D, evaluation of efficiency of R&D processes, or factors affecting the results of R&D. It appears, therefore, that greater emphasis has been given to evaluation activities within the NTBF on the whole, leaving aside the assessment of the efficiency of NTBF.

Lu et al. [6] applied the DEA to study the performance of high-tech industries in R&D. According to these authors, the main factor of success in high-tech companies is to increase the efficiency and performance in R&D. The inputs used in their study were company assets, expenditure on R&D, number of employees, and the number of researchers directly linked to R&D. With regard to outputs, the authors considered number of patents, export volume, return on investment, and sales volume. The results obtained in their study help managers make decisions that make R&D more efficient and innovative.

Chen et al. [8] studied the application of DEA in performance evaluation in R&D companies in the field of computers and peripherals located in STP business incubators. In their study, the CCR and BCC models with three inputs and two outputs were used. The inputs considered were age of the firm, capital expenditure on R&D, and number of employees. With regard to the outputs, annual sales, and the number of patents were used. The authors concluded that the performances differed significantly between the various companies, although the vast majority of firms are technically efficient. Chen et al. [38] assessed the performance of six high-tech industries located in an STP. These authors used four inputs: number of employees, working capital, investment in R&D, and the area occupied. Two outputs were considered: annual sales and number of patents. In addition to studying the efficiency of the six industries each year individually through the CCR and BCC models, the authors used the MI to examine their growth over time.

Studies of performance in R&D NTBFs mostly use the same inputs and outputs, varying depending on data availability, and allow us to highlight the importance of R&D in NTBFs. Despite its drawbacks, the number of patents filed by a company continues to be widely used as a way to measure the level of technology diffusion. However, for many companies, introducing new products and services in the market is the most appropriate output of R&D [39]. For example, Chakrabarti [40] uses the number of patents not only as a measure of output of R&D, but also as the launch of new products and services by businesses. The author states that, for the growth of companies in some industries associated with designing new products, patents have no effect on sales growth.

DEA can help managers to identify NTBF sources of inefficiency, and the best ways to improve performance based on best practices of reference units. DEA does not provide specific information on corrective actions needed to improve business performance, but focuses, rather, on analyzing the reasons why a DMU is inefficient. Thus, managers have the task of evaluating the feasibility of the practical application of the proposed targets for the inputs and outputs [41]. The literature review carried out as part of this study showed that despite the many studies applying DEA to evaluate the efficiency of R&D, the DEA technique to evaluate the efficiency of NTBF in incubators at a more macroscale is not reported.

4. The Case Study of Madan Parque

4.1. Characterization of Madan Parque. To explore the applicability of the DEA and the MI in business incubators in STP, a case study was designed. To collect data, questionnaires were applied to firms incubated in Madan Parque, a business incubator located in Almada, Portugal. Madan Parque was founded in December, 1995, with the Faculty of Science and Technology of the New University of Lisbon (FCT-UNL) as the primary equity partner. The main service of Madan Parque is business incubation. The Parque provides modular office spaces equipped with telephone, electricity, air conditioning, and internet access, as well as access to common spaces, services, and joint activities. By 2012, there were 55 incubated companies that generated 195 jobs. Regarding all companies incubated in Madan Parque, the aggregate turnover was € 6,550,000, with a total investment in R&D of € 450,000, 25 brands registered, 8 national patents, and 3 international patents.

As it was intended to analyze the data available not only from the most recent year, but also from the two previous years, it was decided to restrict the analysis to firms operating in Madan Parque between the years 2009 and 2011, ignoring the companies that started or ended activity in this period. This condition restricted the sample to 21 companies. It

Table 1: Data for each DMU for the year 2009.

DMU	Number of employees	Salary costs (€)	R&D investment (€)	Space costs (€)	Product portfolio	Number of clients	Total sales (€)
1	9	215 000	150 000	450	3	3	350 000
2	3	35 000	30 000	220	5	2	45 000
3	7	65 000	30 000	430	3	1	30 000
4	1	15 000	15 000	220	3	1	20 000
5	19	200 000	75 000	1 200	20	150	310 000
6	5	65 000	35 000	350	4	2	100 000
7	5	75 000	50 000	350	1	20	22 000
8	9	175 000	95 000	540	8	95	275 000
9	9	165 000	75 000	380	3	10	215 000
10	3	65 000	22 500	220	5	14	125 000
11	6	125 000	35 000	220	15	12	210 000
12	2	34 500	10 000	220	4	8	60 000
13	10	185 000	75 000	750	10	35	300 000

Table 2: Data for each DMU for the year 2010.

DMU	Number of employees	Salary costs (€)	R&D investment (€)	Space costs (€)	Product portfolio	Number of clients	Total sales (€)
1	12	250 000	150 000	740	3	3	400 000
2	3	35 000	30 000	220	6	12	125 000
3	7	65 000	30 000	430	3	3	190 000
4	1	15 000	15 000	220	3	2	25 000
5	20	200 000	75 000	1 200	20	150	350 000
6	9	175 000	55 000	350	5	3	255 000
7	5	65 000	25 000	350	1	35	35 000
8	9	180 000	95 000	540	8	150	325 000
9	9	179 000	75 000	380	3	12	225 000
10	3	65 000	22 500	220	5	13	130 000
11	8	175 000	35 000	475	15	10	225 000
12	3	53 000	10 000	220	4	12	72 000
13	12	220 000	100 000	750	15	40	320 000

is important that the DMUs included in the analysis are homogeneous. Thus, the DMUs considered in this work are essentially start-ups/spin-offs incubated from a similar technological baseline. Discarding firms with incomplete data resulted in a final sample of 13 firms.

Tables 1, 2, and 3 report the data collected for the years 2009, 2010, and 2011, respectively.

The data for the year 2011 (Table 3) were analyzed in order to obtain the coefficients of correlation between variables, thus eliminating redundant information. We chose to analyze only the data from the year 2011 because it was the most recent year of the sample. Table 4 shows the matrix of correlations between inputs and outputs.

If two possible inputs present a high correlation, this may indicate that the inclusion of both is useless. The analysis of Table 4 shows that space costs are strongly correlated with the total number of employees, so the space costs variable was excluded from the analysis. The space costs are an indicator of firm size with respect to the occupied office area, so it is natural that the higher the amount of space costs, the greater the number of employees. Similarly, the salary costs have a strong correlation with the number of employees. In this case, we chose to leave out the total number of employees variable, since the pair input/output that has a higher correlation coefficient is the pair salary costs/sales. The correlation coefficients for the outputs do not show high values among themselves. To check if indeed there is any output variable that does not contribute significantly to the efficiency score, scenarios were developed with the two already selected inputs (salary costs and investment in R&D) and three output variables. Table 5 summarizes the sensitivity analysis performed by different combinations of inputs and outputs. We used the *Data Envelopment Analysis Online Software* (DEAOS) and data of Table 5 for the average efficiency scores of the 13 DMUs in each scenario, applying the BCC output-oriented model. Thus, we intend to evaluate

Table 3: Data for each DMU for the year 2011.

DMU	Number of employees	Salary costs (€)	R&D investment (€)	Space costs (€)	Product portfolio	Number of clients	Total sales (€)
1	13	260 000	15 000	740	3	3	450 000
2	3	35 000	30 000	220	12	15	135 000
3	7	65 000	30 000	430	3	4	120 000
4	1	15 000	15 000	220	3	4	30 000
5	24	250 000	75 000	1 200	20	150	350 000
6	15	265 000	55 000	740	5	4	275 000
7	5	55 000	5 000	350	1	135	85 000
8	11	230 000	95 000	540	12	165	295 000
9	9	175 000	75 000	380	3	11	215 000
10	4	83 000	22 500	220	8	12	132 000
11	15	315 000	55 000	475	15	24	280 000
12	3	54 000	10 000	220	8	17	75 000
13	16	275 000	125 000	750	17	55	385 000

Table 4: Correlation coefficients of *inputs* and *outputs*.

	Employees	Salaries	R&D	Space	Sales	Clients	Products
Employees	1,00						
Salaries	0,88	1,00					
R&D	0,65	0,67	1,00				
Space	0,95	0,74	0,54	1,00			
Sales	0,84	0,90	0,61	0,80	1,00		
Clients	0,39	0,20	0,37	0,43	0,21	1,00	
Products	0,61	0,49	0,64	0,51	0,46	0,40	1,00

Table 5: Sensitivity analysis with different combinations of inputs and outputs.

	Inputs	*Outputs*	Efficiency score mean
Scenario 1	Salaries R&D	Sales, clients, products	0,95
Scenario 2	Salaries R&D	Sales, products	0,90
Scenario 3	Salaries R&D	Sales, clients	0,85
Scenario 4	Salaries R&D	Clients, products	0,73

which variables contribute the most to the efficiency and which do not add value to the analysis, adding in the end the fewest possible variables.

We concluded that the output that contributes the least to the average efficiency of DMUs is the number of "clients," so this variable is excluded from the analysis, leaving the model with four variables, two inputs, and two outputs.

After performing the sensitivity analysis, the variables that were considered the most appropriate for the DEA model, as shown in Figure 1, were selected.

The selected variables preferably take low values of input and high output values.

Figure 1: Final *inputs* and *outputs* of the DEA model.

4.2. Analyzing and Discussing Results of DMUs Efficiency Using the DEA-BCC Model. The DEA-BCC model was applied again using the online software DEAOS to obtain the efficiency scores of 2009, 2010, and 2011, which were analyzed separately, as is shown in Table 6. On average, the 13 DMUs have an efficiency increase from year to year, which indicates a better mix of inputs consumed and outputs produced over the three years analyzed. DMUs with efficiency scores higher than 1 are considered inefficient and that the higher the value, the more inefficient the DMU. DMUs 1, 2, 4, 5, and 12 are efficient over the three years since they have an efficiency score equal to 1. DMU 3 shows a large increase in efficiency score from 2009 to 2010 but falls to a lower score in 2011. DMUs 6 and 8 have the same pattern as DMU 3. DMU 7 shows a considerable increase in the efficiency score in 2010 and achieves optimal efficiency in 2011. DMUs 9 and 10, by contrast, have a slight reduction in the efficiency score in 2010 and 2011 compared to that for 2009. DMUs 3, 6, 8,

Table 6: Efficiency scores for each DMU in the years 2009, 2010, and 2011.

DMUs	Efficiency scores (ϕ^*)		
	2009	2010	2011
1	1,00	1,00	1,00
2	1,00	1,00	1,00
3	2,86	1,00	1,47
4	1,00	1,00	1,00
5	1,00	1,00	1,00
6	1,25	1,12	1,56
7	6,25	4,35	1,00
8	1,04	1,00	1,20
9	1,27	1,45	1,54
10	1,00	1,03	1,22
11	1,00	1,00	1,05
12	1,00	1,00	1,00
13	1,00	1,12	1,00
Mean	1,59	1,31	1,16

9, 10, and 11 have efficiency scores higher than 1, and thus are considered ineffective according to the output-oriented model. These DMUs can reduce inefficiency through the proportional increase in their outputs. DMUs 1, 2, 4, 5, 7, 12, and 13 are considered efficient, since they have efficiency scores equal to 1, and the respective slacks are zero. These are therefore the best performing DMUs in the group.

Efficiency scores and the slacks shown in Table 7 allow us to identify the sources of inefficiency of DMUs already identified as inefficient for the year 2011. It is thus possible to identify specific situations of excessive use of resources and/or lack of results. Data illustrate the absence of DMUs with low efficiency, that is, with an efficiency score equal to 1 and nonzero slacks. The inefficient DMUs, that is, those with efficiency score greater than 1, reveal a shortage of results in all outputs. In this case, the sources of inefficiency differ whether DMUs have slacks or not. Inefficient DMUs that have slacks with zero values, in both the inputs and outputs, indicate that the lack of results for each output is contributing equally to inefficiency. Hence, for these DMUs to become efficient, a proportional increase of all outputs should occur, linked to the efficiency score. DMU 10 is the only one that in this situation should increase its outputs by 22% to become efficient. However, if there are nonzero slacks in inefficient DMU outputs, a proportional increase of outputs is insufficient to make the DMU efficient, since there is a lack of results from one (or more) output(s) contributing in particular further to the inefficiency. Thus, in addition to the proportional increase of all outputs, a further increase in outputs with nonzero slack is necessary. It is also possible that there are nonzero input slacks, suggesting that resources are being used in excess and that the DMU should therefore also make a corresponding reduction in the amount of slack in the input to move the DMU to the efficient frontier. DMUs 3, 6, 8, and 9 have considerable slacks in the variable "R&D Investment," while DMU 11 has considerable slack in the variable "salary costs." For example, DMUs 3 and 9 must proportionally increase their outputs, particularly, the "product portfolio," as it has nonzero slack. Simultaneously, these DMUs should reduce their investment in R&D. This might seem contradictory, but it reinforces the idea that DMUs 3 and 9 are using too many of their resources for the results obtained. Thus, investment in R&D applied by DMUs 3 and 9 is not reflected in the results obtained in the same way as with the other DMUs. In this situation, managers should investigate the reasons for this excessive spending on R&D and improve its processes, producing more with fewer resources.

Analyzing the sources of inefficiency discussed above, it is possible to quantify the degree to which a DMU should increase its outputs or decrease its inputs to become efficient. In general, it is possible to set targets for the input and output variables of each DMU, which are in Table 8.

Note that these targets serve only as a diagnosis and that it is the responsibility of corporate managers to set realistic strategies to address the sources of inefficiency.

The classification of some DMUs as efficient or inefficient can be perceived immediately when examining the data. However, the classification of DMUs as efficient is not always intuitive and depends on the combination of its inputs and outputs and their comparison with the other DMUs. Units that could be considered efficient at the outset might not be efficient because there are units with a similar combination of inputs and outputs but can perform better. An example of this is the case of DMU 11, which has quite satisfactory results in both sales volume and product portfolio level, but it is classified as inefficient because it is the unit that has the most salary costs. Analyzing the value of the efficiency score for this unit, we can calculate that besides a 5% increase in its outputs, it would also be necessary to reduce salary costs by approximately € 109,263.

Moreover, the DEA provides information on the sources that cause such inefficiencies, which can be very useful for managers in identifying the factors that are deviating from the DMU's optimal performance. These work output-oriented inefficiencies are related not only to impaired production values, but also to the overuse of a particular input or poor production of an output. In all of the inefficient DMUs, we see the existence of at least one input or output that contributed especially to the inefficiency.

It is possible that, in the particular case under study, some of the observed quantitative targets are unreasonable for companies due to the fact that they do not already possess a high level of maturity, which may limit them in developing new products. Although the goals outlined may be of interest for managers to have a better perception of excessive costs and productive needs, they must not focus on those values only.

Finally, DEA identifies the benchmarks that inefficient units should take as an example to achieve their goals and become efficient. Inefficient units should identify the reasons why they cannot operate efficiently and realize what competing units do best and adapt those practices to their own unit.

Table 7: DEA-BCC model results.

DMU	Efficiency Score (ϕ)	Slacks				Benchmarks
		Salary costs (€)	R&D investment (€)	Product portfolio	Total sales (€)	
1	1,00	0,00	0,00	0,00	0,00	—
2	1,00	0,00	0,00	0,00	0,00	—
3	1,47	0,00	2 000,00	6,38	0,00	1; 2
4	1,00	0,00	0,00	0,00	0,00	—
5	1,00	0,00	0,00	0,00	0,00	—
6	1,56	0,00	2 774,87	0,00	0,00	1; 4; 13
7	1,00	0,00	0,00	0,00	0,00	—
8	1,20	0,00	41 403,27	0,00	0,00	1; 2; 5
9	1,54	0,00	54 333,33	1,78	0,00	1; 2
10	1,22	0,00	0,00	0,00	0,00	1; 10; 5; 12
11	1,05	109 263,91	0,00	0,00	0,00	1; 5; 12
12	1,00	0,00	0,00	0,00	0,00	—
13	1,00	0,00	0,00	0,00	0,00	—

Table 8: *Input* and *output* targets for each DMU.

DMU	Salary costs (€)	R&D investment (€)	Product portfolio	Total sales (€)
1	260 000,00	15 000,00	3,00	450 000,00
2	35 000,00	30 000,00	12,00	135 000,00
3	65 000,00	28 000,00	10,79	177 000,00
4	15 000,00	15 000,00	3,00	30 000,00
5	250 000,00	75 000,00	20,00	350 000,00
6	265 000,00	52 225,13	7,78	42 7732,98
7	55 000,00	5 000,00	1,00	85 000,00
8	230 000,00	20 666,67	14,44	354 955,30
9	175 000,00	75 000,00	6,40	331 000,00
10	83 000,00	22 500,00	9,72	160 409,42
11	205 736,07	55 000,00	15,80	295 119,26
12	54 000,00	10 000,00	8,00	75 000,00
13	275 000,00	125 000,00	17,00	385 000,00

4.3. Analyzing and Discussing of DMUs Efficiency-Malmquist Productivity Index. Initially, we applied the BCC model, in which the efficiency scores for each DMU in the years 2009, 2010, and 2011 were analyzed independently. The results allowed us to identify for each DMU improvement or a step backward in efficiency over the three years. However, it was not possible to identify the factors underpinning the improvement of their performance. The application of MI to the same data set yielded not only the proportion of the productivity gains of each DMU from year to year, but also the identity of the components that were the source of those gains. One of the components analyzed under MI was PTEC, which is calculated in considering VRS. Thus, changes in technical efficiency identified in the MI were in accordance with the assumption of VRS, that is, PTEC, and should be consistent with the changes that occurred in the BCC model.

For example, DMU 6 increases its efficiency score of 1,25 (year 2009) to 1,12 (2010); that is, the increase is 11.25%. Examining the PTEC of the same DMU concerning MI in 2009-2010, we see that there is a change in pure technical efficiency of 11.6%, which is quite close to the value obtained from the BCC model. This consistency prevails in the remaining DMUs. According to Barros and Alves [42], improvements in pure technical efficiency indicate that there was an investment in the company organizational factors, which may include a better balance between inputs and outputs, investments for marketing or quality improvements.

The BCC model indicated that DMUs 1, 2, 4, 5, and 12 are efficient over the three years, since they have an efficiency score that is equal to 1. When analyzing the data component of PTEC in 2009-2010 and 2010-2011, it is confirmed that there are no changes in pure technical efficiency for DMUs mentioned, since PTEC is equal to 1. This comparison confirms that when the aim is to evaluate the performance of a group of DMUs over time, the interpretation of the components resulting from the decomposition of the MI is more intuitive. Moreover, MI has the advantage of identifying

Table 9: Malmquist index and its components for period 2009-2010.

DMU	2009-2010				
	PTEC	SEC	TEC	TC	M_0
1	1,000	0,600	0,600	1,679	1,008
2	1,036	1,123	1,164	1,438	1,674
3	2,818	1,078	3,037	1,155	3,509
4	1,000	1,000	1,000	1,070	1,070
5	1,000	1,043	1,043	1,009	1,053
6	1,118	0,784	0,876	1,265	1,108
7	1,449	1,012	1,466	1,400	2,053
8	1,041	0,714	0,743	1,534	1,140
9	0,875	0,792	0,692	1,445	1,001
10	0,973	0,998	0,970	1,066	1,034
11	1,000	1,000	1,000	0,931	0,931
12	1,000	1,000	1,000	1,007	1,007
13	0,891	0,843	0,751	1,224	0,919
Mean	1,169	0,922	1,103	1,248	1,347

Table 10: Malmquist index and its components for period 2010-2011.

DMU	2010-2011				
	PTEC	SEC	TEC	TC	M_0
1	1,000	1,969	1,969	2,049	4,033
2	1,000	1,000	1,000	1,443	1,443
3	0,678	0,971	0,658	0,996	0,655
4	1,000	0,583	0,583	1,733	1,011
5	1,000	0,676	0,676	1,343	0,908
6	0,719	0,967	0,695	1,298	0,903
7	4,371	0,851	3,719	1,751	6,512
8	0,834	0,946	0,789	1,091	0,861
9	0,939	1,027	0,964	0,999	0,963
10	0,846	0,930	0,786	1,277	1,005
11	0,949	0,532	0,504	1,377	0,695
12	1,000	1,000	1,000	1,573	1,573
13	1,123	0,707	0,793	1,202	0,954
Mean	1,189	0,935	1,088	1,395	1,655

the sources that contribute most to the productivity gains by calculating various indices.

The evolution of the performance of DMUs between the year 2009 (period t) and 2010 (period $t+1$) shows an increase of 34.7% in the mean productivity of the 13 DMUs. The main contribution to this result is the increase of 24.8% in TC greater than the 10.3% increase for the TEC. The innovation was therefore the main source for the recorded mean productivity gains, suggesting that the adoption of new technologies by the DMUs led to considerable improvements. The TEC is the result of multiplying PTEC and SEC. On average, improvements in PTEC, that is, operations and management activities, are the main reason for the improvements in TEC. The average value of PTEC, which measures changes in technical efficiency under VRS, indicates that there was an improvement of 16.9% over the period.

From the five components analyzed in Table 9, only the SEC component has an average value below 1, which suggests a worsening of the scale efficiency. This situation indicates that the DMUs are operating above or below the optimal range, that is, they are having a too high cost for what they produce, or, conversely, they could increase production and reduce costs. Regarding the TEC, we see that DMUs 1, 6, 8, 9, 10, and 13 decreased their technical efficiency between 2009 and 2010. This suggests that these DMUs in 2010 are further away from the efficiency frontier in that period than they were in 2009. Moreover, DMUs 2, 3, 5, and 7 showed an increase in their technical efficiency, with special emphasis on DMU 3. Thus, these DMUs are closer to the frontier in 2010 when compared to the frontier of 2009. DMUs 4, 11, and 12 show the values of TEC that are equal to 1, so that there were no changes in technical efficiency from 2009 to 2010. Analyzing TC data, only DMU 11 has a value lower than 1, that is, technological regression. All other DMUs have TC values greater than 1, and therefore, between 2009 and 2010, there was a positive change in their technological frontier; that is, there was technological progress. This means that, for a given level of input, it is possible to obtain a higher level of output in 2010 than in 2009. This is due to the expansion of the frontier between the two periods.

Relative to MI, data suggest that DMUs 11 and 13 decreased their productivity, since they have values lower than 1 for this index. The remaining DMUs increased their productivity between 2009 and 2010 (MI > 1). It is interesting that the DMUs that show decreases of technical efficiency, with the exception of DMU 13, managed to overcome this situation with very positive changes in their technological frontiers, which contributed strongly to the productivity gains recorded. Regarding PTEC, DMUs 2, 3, 6, 7, and 8 present management improvements that translate into increased productivity. With the exception of DMUs 9, 10, and 13, all the others improved or maintained their PTEC between 2009 and 2010. With respect to SEC, we find that DMUs 2, 3, 5, and 7 increased their scale (size) in this period since they have values higher than 1. DMUs 4, 11, and 12 do not have scale issues and are operating on the frontier of CRS (optimal scale).

The analysis of the period 2010-2011 (Table 10) shows that the average productivity gain of the 13 DMUs was 65.5%, almost double that of the 2009-2010 period. Again, technological progress contributes considerably to productivity improvement, with an increase of 39.5%. This highlights the focus of DMUs in innovation with the introduction of new technologies in their processes. Once again, the SEC is the only component to register a negative average change between 2010 and 2011. However, its value is close to 1, so that, on average, the 13 DMUs are operating very close to their optimum level.

At the individual level, we should highlight DMUs 1 and 7, which have very high productivity gains. In the case of DMU 1, TC and TEC components contribute almost equally to the gains, while the DMU 7 improvements in technical efficiency played an important role in determining its achievements. DMU 3, which had considerably increased its productivity

between 2009 and 2010 and dropped its productivity in 2011 due to the deterioration of technical efficiency. This decrease is probably due to the substantial reduction of its turnover between 2010 and 2011.

Since, for all DMUs except DMU 7, TC > TEC, we confirm that the productivity improvements are strongly due to technological progress. PTEC values show that the relationship between inputs and outputs in DMUs 3, 6, 8, 9, 10, and 11 worsened between 2010 and 2011; that is, these DMUs in 2011 were farther away from the VRS frontier formed by the reference DMU compared to the frontiers of 2010.

The number of DMUs that have productivity gains in 2010-2011 is lower than that in the 2009-2010 period. However, as mentioned above, the average improvements were approximately doubled. As such, it is concluded that although there are fewer DMUs with productivity gains, the improvements for these DMUs are quite positive (DMU 1 and DMU 7). In 2011, there are DMUs that suffered a considerable decrease in productivity compared to 2010, but there are also DMUs with a considerable growth, though these are fewer. Among the DMUs, with a fall in productivity in 2011, are DMUs 3, 8, and 9, which have their sales decline compared to the previous year and, in addition, there is an increase in their costs. The economic situation of Portugal at that time may help to explain this situation, with companies experiencing major difficulties in selling their products and seeking to counter this problem with greater investment in innovation.

The application of MI to collected data allows us to explore changes in productivity. With regard to the values obtained by the MI, we found that during both the period 2009-2010 and 2010-2011, gains in productivity were primarily due to changes in technology. These changes can be interpreted as investments in new technologies, which may include new methodologies, procedures, or techniques in order to improve results. The technological expansion could also mean that companies have improved their productivity by technical experience of its employees, taking advantage of modern facilities and equipment.

5. Conclusion

Business incubators can play a major role in helping to turn a business idea into a technology-based organization that is economically efficient. Performance evaluation and benchmarking could help the NTBFs to become more productive and efficient by avoiding their untimely death. However, there is a shortage in the literature regarding the efficiency evaluation and productivity evolution of the new technology-based firms (NTBFs) in the incubation context. Although decision-making about NTBF's strategy is ultimately the entrepreneurs' responsibility, business incubators may provide efficiency and productivity benchmarking tools in their guidance role for incubated firms. Hence, it is recommended for managers of NTBFs and business incubators' managers to adopt an internal benchmark management procedure in order to evaluate the relative position of each firm to the efficient frontier.

To explore the ability of DEA models to help NTBF in business incubators, a case study of Madan Parque, in Lisbon, Portugal, was conducted. Of the 13 units studied, six were identified as inefficient and probably these results were influenced by the smallness of the data set. The units identified as inefficient should increase all outputs in the proportion indicated in the score of efficiency. It was concluded that four of the six inefficient DMUs have R&D expenditures that are too high and therefore should use their resources more efficiently, since these investments are not having the desired reflection in results. Moreover, the Malmquist productivity index allows us to measure the productivity change over the period 2009 to 2011. The results showed an improvement of 34.7% in productivity between 2009 and 2010 and 65.5% between 2010 and 2011. This productivity growth was mainly due to an expansion in the efficiency frontiers, indicating that companies have invested in new technologies in order to improve their productivity.

The authors are conscious of the data limitations and the need for further work in this area. Future work should include the use of other inputs and outputs and DEA extensions to adapt the model to particular circumstances. In order to confirm the importance of incubation in the NTBFs growth, the application of DEA to technology firms that are not incubated, for further comparison, is suggested.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Estimation of Congestion in Free Disposal Hull Models Using Data Envelopment Analysis

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This paper deals with evaluating congestion in free disposal hull (FDH) models. There are several approaches in data envelopment analysis (DEA) literatures which discuss the theory and application of congestion. However, almost all of these approaches considered convex DEA technologies. So, in the case of nonconvex technologies, including FDH technology, this field is almost nil. This paper makes an attempt to fill in this void. To do so, this study provides a pairwise comparisons-based algorithm to evaluate congestion in FDH model. This algorithm identifies the sources of congestion and estimates its amounts. It is also capable of detecting the losses amounts of output due to congestion. The validity of the proposed model is demonstrated using some numerical and empirical examples.

1. Introduction

Evaluation of decision making units (DMUs) is an important task especially from a managerial point of view. DEA is a nonparametric and mathematical programming based approach for evaluating the performance of a set of homogeneous DMUs using multiple inputs to produce multiple outputs. In performance analysis, in particular in DEA, the concept of congestion plays a seminal role in theory and application. Congestion is a special phenomenon in the production process which is defined in economics where outputs are reduced due to excessive amount of inputs or an increase in one or more outputs results in a reduction in one or more inputs. For an actual example of congestion in a coal mine where a large crowd of the miners are working in a narrow underground, the amount of minerals excavated will be reduced [1].

Heretofore, various approaches have been presented in DEA for the treatment of congestion. The concept of congestion was first introduced in the literatures by Färe and Grosskopf [2] in the context of DEA. Subsequently an operationally implementable form was given by Färe et al. [3] and Cooper et al. [4–6]. Afterwards, Tone and Sahoo [7] developed a new slack-based approach to evaluate the scale elasticity in the presence of congestion with a unified framework. Wei and Yan [8] used DEA output additive models and proposed a necessary and sufficient condition for existence of congestion. Jahanshahloo and Khodabakhshi [9, 10] provided an approach of input congestion based on the relaxed combinations of inputs. Later on, Khodabakhshi [11] provided a one-model approach of input congestion based on input relaxation model. Also Khodabakhshi [12] proposed a method to detect the input congestion in the stochastic DEA. To see more references about this approach, the readers are referred to [13, 14]. Jahanshahloo et al. [15] and Khodabakhshi et al. [16] proposed some methods for computing the congestion in DEA models with production trade-offs and weight restrictions. Sueyoshi and Sekitani [17] proposed a modified approach which is able to measure congestion under the occurrence of multiple solution. There exist some papers which reviewed congestion papers, as that of Khodabakhshi et al. [18].

All of the above-mentioned investigations deal with congestion in convex technologies. In convex models, the targets resulting from efficiency assessment correspond to the points on the continuous efficiency frontiers. This means that DMUs might be compared with unreal DMUs which sometimes is meaningless in real life, for example, when we want to evaluate the efficiency of various car engines.

FDH models were first formulated by Deprins et al. [19]. The PPS of FDH model is made by deterministic (or observed activities) and free disposability postulates. So the PPS of FDH model is nonconvex. One appealing characteristic of FDH model due to nonconvexity nature of FDH efficiency frontier is that, in FDH model, targets correspond to observed units which is more compatible with real life because, in some circumstances, the observed unit is more comfortable when compared with a real unit rather than with a virtual one. As can be seen from the foregoing, there are several methods for evaluating congestion in convex DEA models, but for FDH models, although there are a few papers which are concerned with the field of estimation returns to scale (RTS), see, for example, [20–24], methods to estimate congestion can be hardly found. Therefore a new scheme is required to deal with congestion in FDH models.

In this paper, we first present definitions of output efficiency for DMUs under a series of DEA output additive models. Then, using these definitions, we develop a necessary and sufficient condition for existence of congestion in FDH model. Afterwards, we provide a polynomial time algorithm based on pairwise comparisons which evaluates congestion for DMUs using certain differences of inputs and outputs. This algorithm simply identifies the sources of congestion and estimates its amounts for congested DMUs.

The rest of the paper unfolds as follows. In the next section, FDH model and some of its properties and definitions will be presented to facilitate later discussions. In Section 3, we present a method with many computational advantages for evaluating congestion in FDH model. The validity of the proposed model is demonstrated using three numerical examples in Section 4. Finally, Section 5 gives the conclusion of this paper.

2. Preliminaries

In this section we first briefly describe some characteristic property of FDH model. Consider n DMUs where each DMU_j $(j = 1, \ldots, n)$ utilizes m inputs x_{ij} $(i = 1, \ldots, m)$ to produce s outputs y_{rj} $(r = 1, \ldots, s)$. Let $x_j = (x_{1j}, \ldots, x_{mj})^T$ and $y_j = (y_{1j}, \ldots, y_{sj})^T$. We will also assume that $x_j \geq 0$, $x_j \neq 0$ and $y_j \geq 0$, $y_j \neq 0$. The production possibility set T is represented as

$$T = \left\{(x, y) \in R_+^{m+s} \mid y \text{ can be produced from } x\right\}. \quad (1)$$

Deprins et al. [19] have deduced the following production possibility set. This set is denoted by T_{FDH}, regarding the assumptions of deterministic and free disposability of the production technology:

$$T_{\mathrm{FDH}} = \left\{(x, y) : \sum_{j=1}^{n} \lambda_j x_j \leq x, \sum_{j=1}^{n} \lambda_j y_j \geq y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0, 1\}, j = 1, \ldots, n\right\}. \quad (2)$$

The additive FDH model to evaluate the efficiency of a special DMU_p $(p \in \{1, \ldots, n\})$ under the T_{FDH} is as follows:

$$\begin{aligned}
\text{Max} \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip} \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \in \{0, 1\}, \ j = 1, \ldots, n, \\
& s_i^+ \geq 0 \quad i = 1, \ldots, m, \\
& s_r^+ \geq 0 \quad r = 1, \ldots, s.
\end{aligned} \quad (3)$$

Definition 1 (FDH efficiency). Consider model (3). If the optimal objective value is zero, then DMU_p is said to be FDH efficient.

It is worth noting that different to CCR and BCC models, the FDH model does not operate with the convexity assumption. Therefore, this model has a discrete nature which causes the efficient target point for an inefficient DMU simply to be assigned as a point among only actually observed DMUs. Thus, the efficiency analysis is done relative to the other given DMUs instead of a hypothetical efficiency frontier. This has the advantage that the achievement goal for an inefficient DMU given by its efficient target point will be more credible than in cases of CCR and BCC models.

Definition 2 (FDH output efficiency). Consider the following model. If $Z_{\mathrm{FDH}} = 0$, then DMU_p is said to be FDH output efficient:

$$\begin{aligned}
Z_{\mathrm{FDH}} = \text{Max} \quad & \sum_{r=1}^{s} s_r^+ \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip} \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \in \{0, 1\}, \ j = 1, \ldots, n, \\
& s_r^+ \geq 0 \quad r = 1, \ldots, s.
\end{aligned} \quad (4)$$

Definition 3 (congestion). Evidence of congestion is present in the performance of any DMU, when a decrease in one or more inputs is associated with increases that are maximally possible in one or more outputs without worsening other inputs or outputs. Conversely, congestion is said to occur when some of the outputs that are maximally possible are

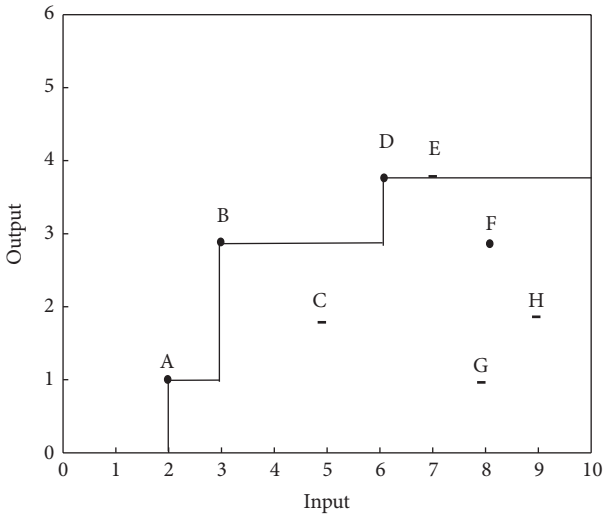

Figure 1: T_{FDH}.

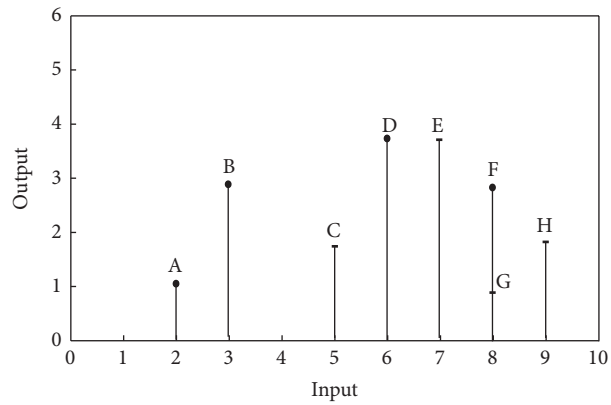

Figure 2: T_{NFDH}.

reduced by increasing one or more inputs without improving any other inputs or outputs [25].

A very restrictive form of the above definition yields the definition of strong congestion as follows.

Definition 4 (strong congestion). If a proportionate reduction in all inputs of a DMU warrants an increase in all maximally possible outputs, then strong congestion occurs [7].

Definition 5 (technical efficiency). Efficiency is achieved by DMU_0 if and only if it is not possible to improve some of its inputs or outputs without worsening some of its other inputs or outputs [25].

Definition 6 (technical inefficiency). Technical inefficiency is said to be present in the performance of DMU_0 when the evidence shows that it is possible to improve some input or output without worsening some other inputs or outputs [25].

3. Congestion in FDH Model

In T_{FDH}, the efficiency surface is a staircase based on those given DMUs that are not dominated by other given DMUs. Figure 1 describes an illustrative example of T_{FDH} which is made by eight DMUs denoted by A, B, ..., H with one input and one output.

It should be noted that evaluating congestion in customary models for convex PPS has been studied on T_{NEW}, which is a PPS without input disposability postulate. Let us denote T_{NEW} corresponding to T_{FDH} as T_{NFDH}, which can be defined as follows:

$$T_{\text{NFDH}} = \left\{ (x, y) \mid x = \sum_{j=1}^{n} \lambda_j x_j, y \le \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0, 1\}, j = 1, \ldots, n \right\}. \tag{5}$$

Figure 2 exhibits T_{NFDH} for the example in Figure 1. As seen from Figure 2, T_{NFDH} has a discrete nature and so the study of congestion on it is complicated. So to overcome this difficulty we introduce a new set as follows:

$$\text{FDH}^{-1} = \left\{ (x, y) \mid x \le \sum_{j=1}^{n} \lambda_j x_j, y \le \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0, 1\}, j = 1, \ldots, n \right\}. \tag{6}$$

Apparently, the set of FDH^{-1} is gained by reversing the sign of input inequalities in T_{FDH}. FDH^{-1} set corresponding to the example in Figure 1 is illustrated in Figure 3.

We use the following model to deal with the congestion phenomenon in FDH model:

$$\begin{aligned} Z_{\text{FDH}^{-1}} = \text{Max} \quad & \sum_{r=1}^{s} s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{ip} \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad r = 1, \ldots, s \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \in \{0, 1\}, \ j = 1, \ldots, n \\ & s_r^+ \ge 0 \quad r = 1, \ldots, s. \end{aligned} \tag{7}$$

We call the above model "FDH^{-1} output additive model."

To see what is involved, we note that the input (like the output) constraints take the form $\sum_{j=1}^{n} \lambda_j x_{ij} \ge x_{ip}$. Hence, in this adaptation of additive models, the objective is to maximize the outputs without reducing any of the inputs.

Definition 7 (FDH^{-1} output efficiency). Consider the model (7). If $Z_{\text{FDH}^{-1}} = 0$, then DMU_p is said to be FDH^{-1} output efficient.

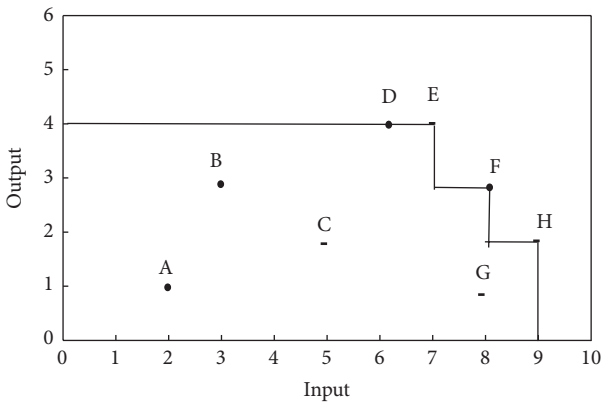

Figure 3: FDH^{-1}.

Lemma 8. *DMU_p is FDH^{-1} output efficient if and only if the following system has no solution:*

$$\sum_{j=1}^{n}\lambda_j x_j \geq x_p,$$
$$\sum_{j=1}^{n}\lambda_j y_j \geq y_p, \qquad \sum_{j=1}^{n}\lambda_j y_j \neq y_p, \tag{8}$$
$$\sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \in \{0,1\}, \ j = 1,\ldots,n.$$

Proof. It is clear using Definition 7. □

Definition 9 (congestion in FDH model). Let $\mathrm{DMU}_p = (x_p, y_p)$ be FDH^{-1} output efficient; if there exists $\mathrm{DMU}_k = (x_k, y_k)$, such that $x_k \leq x_p$, $x_k \neq x_p$ and $y_k \geq y_p$, $y_k \neq y_p$, then DMU_p has evidence of congestion.

Based upon Definition 9, units F, G, and H in Figure 3 have evidence of congestion and unit C is technically inefficient.

Definition 10 (strong congestion in FDH model). Let $\mathrm{DMU}_p = (x_p, y_p)$ be congested in FDH model; if there exists $\mathrm{DMU}_k = (x_k, y_k)$, such that $x_k < x_p$ and $y_k > y_p$, then DMU_p has evidence of strong congestion.

Lemma 11. *Let DMU_p be FDH^{-1} output efficient; then DMU_p has evidence of congestion if and only if the following system has a solution:*

$$\sum_{j=1}^{n}\lambda_j x_j \leq x_p,$$
$$\sum_{j=1}^{n}\lambda_j y_j \geq y_p, \qquad \sum_{j=1}^{n}\lambda_j y_j \neq y_p, \tag{9}$$
$$\sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \in \{0,1\}, \ j = 1,\ldots,n.$$

Proof. Let DMU_p has evidence of congestion, so from Definition 9, there exists $\mathrm{DMU}_k = (x_k, y_k)$, such that $x_k \leq x_p$, $x_k \neq x_p$ and $y_k \geq y_p$, $y_k \neq y_p$. Thus $\lambda_k = 1$ and $\lambda_j = 0$ $(j = 1,\ldots,n, j \neq k)$ is a solution of (9).

Conversely, suppose that $\widetilde{\lambda} = (0,\ldots,0,1,0,\ldots,0)$, whose qth component is one, is a solution of (9). So, we have $x_q \leq x_p$ and $y_q \geq y_p$, $y_q \neq y_p$. Also, certainly $x_q \neq x_p$, since, according to the assumption of lemma, DMU_p is FDH^{-1} output efficient, so $x_q = x_p$ contradicts Lemma 8. Hence, by Definition 9, DMU_p has evidence of congestion. □

Lemma 12. *DMU_p is not FDH output efficient if and only if the following linear system has a solution:*

$$\sum_{j=1}^{n}\lambda_j x_j \leq x_p,$$
$$\sum_{j=1}^{n}\lambda_j y_j \geq y_p, \qquad \sum_{j=1}^{n}\lambda_j y_j \neq y_p, \tag{10}$$
$$\sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \in \{0,1\}, \ j = 1,\ldots,n.$$

Proof. Using definition of FDH output efficiency, the proof is completed. □

We now present the main result of the proposed method.

Theorem 13. *Let DMU_p be FDH^{-1} output efficient; then DMU_p has evidence of congestion if and only if DMU_p is not FDH output efficient.*

Proof. Using Lemmas 11 and 12, the proof is completed. □

Now, using Theorem 13, we can provide the following procedure to evaluate congestion in FDH model.

(1) Solve model (7) corresponding to (x_p, y_p); let (λ^*, s^{+*}) be the optimal solution of it. Let $\widehat{y}_p = y_p + s^{+*}$. It is evident that $(x_p, \widehat{y}_p)$ is FDH^{-1} output efficient.

(2) Solve model (4) for $(x_p, \widehat{y}_p)$.

(3) If $Z_{\mathrm{FDH}} > 0$, then DMU_p is congested.

Remark 14. Models (4) and (7) are mixed-integer programming, but we can simply show that it does not need any mathematical programming problem to solve. Indeed, an enumeration algorithm based on pairwise comparisons, similar to Tulken's enumeration algorithm for the case of radial FDH model [26], can be used.

Now, based upon foregoing procedure and Remark 14, we propose the following algorithm. The proposed algorithm includes two parts. In Part (a), we recognize the existence of congesting in performance of DMU_p and in Part (b), if DMU_p is recognized to be congested in Part (a), the amount

Table 1: Data set and results of Part (a) of the proposed algorithm in Example 1.

DMUs	I	O	D_p	$Z_{\text{FDH}^{-1}}$	s^{+*}	$\widehat{y}_p$	$\widehat{D}_p$	Z_{FDH}	Status
A	2	1	{A, B, C, D, E, F}	3	3	4	{A}	0	No congestion
B	3	3	{B, D, E, F}	1	1	4	{B}	0	No congestion
C	5	2	{C, E, F, H}	2	2	4	{}		No congestion
D	6	4	{D, E}	0	0	4	{D}	0	No congestion
E	7	4	{E}	0	0	4	{D}	0	No congestion
F	8	3	{F}	0	0	3	{B, D, E}	1	Congestion
G	8	1	{F, G, H}	2	2	3	{B, D, E, F}	1	Congestion
H	9	2	{H}	0	0	2	{B, C, D, E, F}	2	Congestion

of congestion for each input as well as the reduction amount of each output due to congestion will be estimated.

Proposed Algorithm

Part (a)

Step 1. Calculate the optimal value of model (7) by the following equation:

$$Z_{\text{FDH}^{-1}} = \sum_{r=1}^{s} \left(y_{rq} - y_{rp}\right) = \max_{j \in D_p} \sum_{r=1}^{s} \left(y_{rj} - y_{rp}\right), \tag{11}$$

where

$$D_p = \left\{ j \in \{1, \ldots, n\} \mid x_j \geq x_p \text{ and } y_j \geq y_p \right\}. \tag{12}$$

Step 2. Let $\widehat{y}_p = y_p + s^{+*}$, where $s^{+*} = y_q - y_p$. Obtain the optimal value of model (4) by

$$Z_{\text{FDH}} = \max_{j \in \widehat{D}_p} \sum_{r=1}^{s} \left(y_{rj} - \widehat{y}_{rp}\right), \tag{13}$$

where

$$\widehat{D}_p = \left\{ j \in \{1, \ldots, n\} \mid x_j \leq x_p \text{ and } y_j \geq \widehat{y}_p \right\}. \tag{14}$$

Step 3. If $Z_{\text{FDH}} > 0$, then DMU_p is congested, so go to Part (b); furthermore, if there exist $j \in \widehat{D}_p$ such that $x_j < x_p$ and $y_j > \widehat{y}_p$, then, based on Definition 4, DMU_p is strongly congested. If $Z_{\text{FDH}} = 0$, then DMU_p is not congested and stop.

Part (b)

Step 4. Define K_p as follows:

$$K_p = \left\{ j \in \widehat{D}_p \mid Z_{\text{FDH}} = \sum_{r=1}^{s} \left(y_{rj} - \widehat{y}_{rp}\right) \right\}. \tag{15}$$

Then calculate

$$\alpha^* = \min_{j \in k_p} \sum_{i=1}^{m} \left(x_{ip} - x_{ij}\right). \tag{16}$$

Table 2: Results of Part (b) of the proposed algorithm in Example 1.

DMUs	k_p	α^*	T_p	s^{c*}	$\widehat{s}^{+*}$
F	{E, D}	1	{E}	1	1
G	{E, D}	1	{E}	1	1
H	{E, D}	2	{E}	2	2

Step 5. Define T_p as follows:

$$T_p = \left\{ j \in K_p \mid \alpha^* = \sum_{i=1}^{m} \left(x_{ip} - x_{ij}\right) \right\}. \tag{17}$$

For $j \in T_p$ define s_i^{c*} as the amount of congestion in ith input of DMU_p and $\widehat{s}_r^{+*}$ as reduction amount of rth output due to congestion as follows:

$$\begin{aligned} s_i^{c*} &= x_{ip} - x_{ij}, \quad i = 1, \ldots, m, \\ \widehat{s}_r^{+*} &= y_{rj} - \widehat{y}_{rp}, \quad r = 1, \ldots, s. \end{aligned} \tag{18}$$

$\alpha^* = \sum_{i=1}^{m} s_i^{c*}$ is the total amount of congestion in all inputs of DMU_p.

Corollary 15. *If $\widehat{D}_p = \emptyset$, congestion has no appearance at DMU_p.*

Proof. In this case, it is obvious that system (10) has no solution. So, by Lemma 12, $(x_p, \widehat{y}_p)$ is FDH output efficient and regarding Theorem 13 congestion has no appearance at $(x_p, \widehat{y}_p)$. Since congestion is a frontier concept, then DMU_p has no congestion □

Remark 16. If $\text{Card}(\{j \in D_p \mid Z_{\text{FDH}^{-1}} = \sum_{r=1}^{s}(y_{rj} - y_{rp})\}) > 1$, then the projection of DMU_p, $(x_p, \widehat{y}_p)$, is not determined uniquely.

Remark 17. $\text{Card}(T_p) \geq 1$. If $\text{Card}(T_p) = 1$, then the amount of congestion can be uniquely determined; otherwise there are alternatives for the amount of congestion for DMU_p.

Using the above algorithm, we can evaluate the congestion for each DMU in FDH model without any mathematical programming problems and with only some certain pairwise differences of inputs and outputs with regard to (11), (16), and (18). Therefore, the following theorem is obviously true.

Table 3: Data set and results of Part (a) of the proposed algorithm in Example 2.

DMUs	(I_1, I_2)	(O_1, O_2)	D_p	$Z_{\text{FDH}^{-1}}$	(s_1^{+*}, s_2^{+*})	$\widehat{y}_p$	$\widehat{D}_p$	Z_{FDH}	Status
A	(1, 1)	(1, 1)	A, B, C, D	2	(1, 1)	(2, 2)	{}		No congestion
B	(2, 2)	(2, 2)	B	0	(0, 0)	(2, 2)	{B}	0	No congestion
C	(2, 3)	(2, 1)	C	0	(0, 0)	(2, 1)	{B, C}	1	Congestion
D	(3, 3)	(1, 1)	D	0	(0, 0)	(1, 1)	{A, B, C, D}	2	Congestion

Table 4: Result of Part (b) of the proposed algorithm in Example 2.

DMUs	K_p	α^*	T_p	(s_1^{c*}, s_2^{c*})	$(\widehat{s}_1^{+*}, \widehat{s}_2^{+*})$
C	{B}	1	{B}	(0, 1)	(0, 1)
D	{B}	2	{B}	(1, 1)	(1, 1)

Theorem 18. *The proposal algorithm is a polynomial time algorithm.*

4. Numerical Examples

In this section, we apply our proposed procedure to measure the congestion effect on two numerical examples and an empirical example.

Example 1. We consider the illustrative example provided in Section 3 which includes eight DMUs, A, B, C, D, E, F, G, and H, with one output and one input each, as shown in Figure 1. The data set of DMUs as well as the results of Part (a) of the proposed algorithm is given in Tables 1 and 2 displays results of Part (b). As shown in Table 1, there is no congestion in DMUs A, B, C, D, and E and congestion has appeared in DMUs F, G, and H. The input congestion amount and reduction amount of output due to congestion for congested units, resulting from the proposed algorithm, have been provided in the two last columns of Table 2, respectively. Obviously, in the case of one input and one output, each congested unit has evidence of strong congestion.

Example 2. We consider an example adopted from Tone and Sahoo ([7], page 756) which has been listed in Table 3 of our study. This example consists of four DMUs, A, B, C, and D, using two inputs and producing two outputs. The results of Part (a) of the proposed algorithm are given in Table 3. From Table 3 we can see that there is no congestion in DMUs A and B. As shown in Table 4, for unit C, the congestion amount of inputs is $(s_1^{c*}, s_2^{c*}) = (0, 1)$ and $(\widehat{s}_1^{+*}, \widehat{s}_2^{+*}) = (0, 1)$ is output losses of C due to congestion. For unit D, the congestion amount of inputs is $(s_1^{c*}, s_2^{c*}) = (1, 1)$ and $(\widehat{s}_1^{+*}, \widehat{s}_2^{+*}) = (1, 1)$ is output losses of D due to congestion. Therefore considering Definition 10, unit D is strongly congested.

Example 3. Table 5 presents the input and output data of the Chinese textile industry during 1981–1997 assembled by Cooper et al. [5]. Each year has been treated as a DMU with two inputs and one output: labor (X_1) measured in units of 1000 persons, capital (X_2) measured in units of one million Ren Min Be (Chinese monetary unit), and output (Y) measured in units of one million Ren Min Be too. Note that the capital and output values have been adjusted to a 1991 base period to eliminate the impact of price variations. The congestion amount regarding FDH technology using the proposed method as well as the congestion amounts using Cooper et al.'s method [6] in BCC technology which are shown by (s_1^{c*}, s_2^{c*}) is provided in Table 6. As can be seen from Table 6, congestion appeared in performance of DMUs 8, 9, 10, and 15 in T_{FDH}. Besides DMUs 2, 8, 9, 10, 12, 13, and 15 have been recognized to be congested in T_{BCC}. Comparison of these two computational results shows that the DMUs which are congested in T_{FDH} are among the ones that are congested in T_{BCC}. That is, the set of congested DMUs in T_{FDH} is a subset of set of congested DMUs in T_{BCC}. It seems reasonable because $T_{\text{FDH}} \subseteq T_{\text{BCC}}$.

Table 5: Data set in Example 3.

DMU = year	Labor (X_1)	Capital (X_2)	Output (Y)
DMU_1 = 1981	389.00	19.86	856.02
DMU_2 = 1982	412.30	21.16	866.85
DMU_3 = 1983	423.50	17.08	956.04
DMU_4 = 1984	417.30	18.10	1082.94
DMU_5 = 1985	570	12.61	1273.20
DMU_6 = 1986	600.50	13.45	1230.72
DMU_7 = 1987	641.10	15.91	1410.66
DMU_8 = 1988	715.30	23.72	1728.16
DMU_9 = 1989	736.00	25.97	2109.57
DMU_{10} = 1990	745.00	18.24	2291.08
DMU_{11} = 1991	756.00	14.40	2533.27
DMU_{12} = 1992	743.00	17.50	2899.16
DMU_{13} = 1993	684.00	25.08	3520.74
DMU_{14} = 1994	691.00	25.45	4949.93
DMU_{15} = 1995	673.00	29.35	4604.00
DMU_{16} = 1996	634.00	23.05	4722.29
DMU_{17} = 1997	596.00	25.02	4760.28

5. Conclusion

In this paper we proposed a method based on pairwise comparison to evaluate congestion in FDH model. The results of the study have been proved with some lemmas and theorems. Our proposed method is able to identify congestion in performance of DMUs and it can determine the amount of excessive inputs for congested DMUs based on the calculation of certain pairwise differences of inputs and outputs. It also

Table 6: Comparison results of congestion treatment in BCC and FDH technologies in Example 3.

DMU = year	(s_1^{c*}, s_2^{c*}) in FDH	(s_1^{c*}, s_2^{c*}) in BCC	Results of the proposed approach in T_{FDH}	Results of Cooper et al.'s approach in T_{BCC} [6]
$\mathrm{DMU}_1 = 1981$	(0, 0)	(0, 0)	No congestion	No congestion
$DMU_2 = 1982$	(0, 0)	(0, 0.72)	Congestion	No congestion
$\mathrm{DMU}_3 = 1983$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_4 = 1984$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_5 = 1985$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_6 = 1986$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_7 = 1987$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_8 = 1988$	(81.30, 0.67)	(65.39, 0)	Congestion	Congestion
$\mathrm{DMU}_9 = 1989$	(45.00, 0.52)	(45.00, 0)	Congestion	Congestion
$\mathrm{DMU}_{10} = 1990$	(2.00, 0.74)	(43.16, 0)	Congestion	Congestion
$\mathrm{DMU}_{11} = 1991$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_{12} = 1992$	(0, 0)	(30.72, 0)	No congestion	Congestion
$\mathrm{DMU}_{13} = 1993$	(0, 0)	(1.79, 0)	No congestion	Congestion
$\mathrm{DMU}_{14} = 1994$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_{15} = 1995$	(7.70, 4.33)	(0, 3.98)	Congestion	Congestion
$\mathrm{DMU}_{16} = 1996$	(0, 0)	(0, 0)	No congestion	No congestion
$\mathrm{DMU}_{17} = 1997$	(0, 0)	(0, 0)	No congestion	No congestion

is capable of detecting the losses amounts of output due to congestion. One of the advantages of this method is that in all stages there is no need to solve any mathematical programming problems and so a polynomial time algorithm to identify congestion in FDH model is provided. Hence, it is superior from a computational point of view. The numerical examples demonstrated the compatibility of the proposed approach and so can be developed in performance analysis and large practical projects.

Because of having low complexity of computation, this method can be developed for imprecise data for further research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Closed-Loop Supply Chain Models with Considering the Environmental Impact

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Global warming and climate changes created by large scale emissions of greenhouse gases are a worldwide concern. Due to this, the issue of green supply chain management has received more attention in the last decade. In this study, a closed-loop logistic concept which serves the purposes of recycling, reuse, and recovery required in a green supply chain is applied to integrate the environmental issues into a traditional logistic system. Here, we formulate a comprehensive closed-loop model for the logistics planning considering profitability and ecological goals. In this way, we can achieve the ecological goal reducing the overall amount of CO_2 emitted from journeys. Moreover, the profitability criterion can be supported in the cyclic network with the minimum costs and maximum service level. We apply three scenarios and develop problem formulations for each scenario corresponding to the specified regulations and investigate the effect of the regulation on the preferred transport mode and the emissions. To validate the models, some numerical experiments are worked out and a comparative analysis is investigated.

1. Introduction

The issue of supply chain management has received increasing attention among the researchers over the last few decades or so. Nowadays, due to the existence of global and competitive market, it is necessary that enterprises work together to enhance their adaptive ability and viability in the market. Hereby, these enterprises achieve common goals such as minimizing the total costs and the delay of deliveries in the whole chain [1–3]. Three main flows exist in the chain: the material flow, the information flow, and the fund flow. Coordination and integration of these flows across enterprises are called a supply chain management (SCM) [4]. The global economic growth from the 20th to the 21st century has led to a rise in consumption of goods. Consequently, large streams of goods all over the world have been founded. In this way, the production and all aspects of logistics such as transportation, warehousing, and inventories have created large environmental problems such as global warming and climate changes [5]. In 1955, an assessment was published by the Intergovernmental Panel on Climate Change (IPCC). This assessment claimed that Earth's surface warming is a result of increase in greenhouse gas concentrations [6, 7]. Greenhouse gases are a collection of gases among which are CO_2 (carbon dioxide), CH_4 (methane), N_2O (nitrous oxide), HFCs (hydrofluorocarbons), PFCs (perfluorocarbons), and SF_6 (sulphur hexafluoride) [8]. Department of the Environment, Transport and the Regions (DETR) estimated that, among these greenhouse gases, CO_2 is present in the atmosphere in significant quantities and accounts for two-thirds of global warming [9]. CO_2 is released from several sources such as transportation, industrial processes, and other commercial sectors. As a result, a greenhouse effect is increased [10]. Integration of SCM concept with the issue of environment protection confirms sharp decline in pollution problem. Research on this approach has received considerable attention recently and led to the creation of new research agenda: green supply chain management (GSCM). So, GSCM is a new paradigm where the supply chain will have a direct relation to the environment. Due to the quality and supply chain revolution in the late 1980s and 1990s, respectively, most enterprises have been motivated to become

environmentally conscious and have been faced with pressure to protect the environment in their supply chains [11, 12]. Nowadays, most research on GSCM has had a tendency to the reverse logistics and closed-loop supply chains such as researches done by Blumberg [13] and Pochampally et al. [14]. In the reverse logistics/closed-loop supply chain systems, a product returns to the manufacturer after use and can be repaired or remanufactured to be delivered again to the end consumers. A top environmental issue for an enterprise is how to reduce the utilization of the materials by reusing and remanufacturing the used products. This brings about the GSCM concept and has led to a problem of the closed-loop supply chain management. The closed-loop logistics are divided into two parts. These two parts are given as follows.

(i) Forward logistics: after manufactory, the distributors are responsible to deliver the final products to the end consumers satisfying their demands.

(ii) Reverse logistics: the flow of used products is processed from the customers back to the dismantlers to do the sorting or disassembling for recovery, reuse, or disposal [13, 14].

With well-managed reverse logistics, the environment protection can be achieved with minimizing of total costs in the whole closed-loop supply chain. Most of the previous studies focused on reverse logistics and only formulated models corresponding to this field. Some researchers presented the closed-loop models, but they did not consider the relation between forward and reverse flows in their proposed models [15–17]. These models often assumed the unlimited capacities for the reverse logistics, which is not a valid assumption for representing the real situations. In real life situations, the DC also plays such role as a collector in a recovery system. So, the capacity of DC is restricted to both distribution and collection. Now, there is an interaction between amounts of the distribution and the collection so that, when the amounts of the collection are larger, the amounts of distribution must decrease under the same capacity. The closed-loop supply chain is characterized by these interactions. With the lack of such kind of relations, the model can be separated into two parts independently and become a supply chain including forward and reverse chains but not a loop. There exist a few studies in which closed-loop models were proposed with realistic assumption. In these studies, researchers shared the same capacity for the reverse logistics and stated the relation between forward and reverse flows [18]. These authors proposed a generalized closed-loop model for the logistics planning. They formulated an integer linear programming model in which the integration between forward and reverse logistics and the decisions for selecting the places such as DCs was considered. Due to NP-hard nature of their model, a genetic algorithm based on spanning tree structure was developed.

Reviewing the literature on closed-loop supply chain, it is concluded that a few studies consider the relations between forward and reverse logistics. In this study, we extend Wang and Hsu's model [18] doing more to protect the environment. First of all, in addition to managing properly reverse logistics to reduce negative impact of greenhouse gases emissions, we suggest another strategy for achieving an expected goal, simultaneously. Here, we focus on a different and important aspect of green supply chains: we focus on transport mode selection as a way to reduce emissions. For this, in addition to minimizing the total cost in the whole closed-loop chain, we consider two types of regulations to reduce carbon emissions coming from freight transport. The first mechanism specifies a cost for carbon emissions and the second one is a constraint on emissions. In this study, we pursue three scenarios and develop problem formulations for each scenario corresponding to these regulations and investigate the effect of the regulation on the preferred transport mode and the emissions. These scenarios are given as follows:

(i) model without emissions (basic),

(ii) model with emissions (emission-constraint problem),

(iii) model with emissions (emission cost-minimization problem).

Here, we use empirical data to estimate the carbon emissions for various transport modes accurately. The transport modes differ with respect to unit transportation cost, lead time, and unit emissions. For the first scenario in which carbon emissions resulting from freight transport are not considered, our model trades off between long lead times and lower transport costs and short lead times and higher transport costs for transport modes. The carbon emissions are taken into account for the rest of the scenarios where a tradeoff exists between lead time, unit transportation cost, and unit emissions for transport mode. For example, air transport has a shorter lead time, higher unit transportation costs, and carbon emissions than water transport. Defining the three scenarios, we analyze the effect of the regulation on the preferred transport mode and the emissions. Second, we focus on structure of closed-loop supply chain. Many procedures are available in this field. One of these procedures is related to traveling salesman problem (TSP) concept in which, having N cities, a salesman should start from home city, visit all customers once, and come back to the home city finding a minimal route. Multiple traveling salesman problem (mTSP) is a well-known problem in which several salesmen should start and return to a single home city somehow all customers are visited exactly once. Now, we suppose that there are multi-DCs in the proposed supply network. Any of them has a number of salesmen. Multiple DCs, multiple traveling salesmen problem (MDMTSP) finds tours for all salesmen such that all customers are visited exactly once and the total cost of the tours is minimized, while salesmen departure from DCs and arrival to the single destination is called the multiple departures single destination multiple TSP [19]. Third, we consider time windows in our proposed closed-loop supply chain. There are four layers of supply network (manufacturers, DCs, customers, and dismantlers). Customers send order lists and wait to deliver them. The purposes are determination of proper locations of manufactories, DCs, and dismantlers among candidates set and a suitable distribution of goods throughout the network minimizing

cost of all tours. Selection of proper manufactories, DCs, and dismantlers to supply customers depends on satisfying time windows on customer's viewpoints. So, embedding the transport mode selection, MDMTSP, and time window concepts in a closed-loop system with respect to the overall amount of CO_2 emitted from journeys, it is noted that our closed-loop network design is more precisely planned with the aim of protecting environment. To our knowledge, this study is the first paper which considers these concepts simultaneously in the closed-loop supply chain. The remainder of our work is organized as follows. The proposed problem is fully explained and justified in Section 2. The methodology based on empirical data to estimate the carbon emissions for different modes of transport is also discussed. Next, the mathematical formulation for three scenarios is developed. In Section 4, the numerical experiments to illustrate the effectiveness of the proposed methodology are given. A comparative analysis is presented in Section 5. Finally, conclusions are drawn.

2. Problem Description

There are essentially four stages along a green logistic network: manufacturers, DCs, customers, and dismantlers. Here, we consider multiple manufactories, DCs, dismantlers, and customers being serviced with one supplier, various transport modes, and one commodity with deterministic demands. The initial problem is making decisions for choosing the proper places of manufactories, DCs, and dismantlers among candidates set while pursuing minimal operations cost, carbon emission, and maximal profits, considering inventory constraints, and satisfying customer demands. Distribution of products from DCs to customers plays critical role. MDMTSP approach can be appropriate for this problem. Any salesman located at DC must depart and visit customers and then go back to the similar or dissimilar DC. In this problem, we suppose that any customer is supplied by only one DC. Meanwhile, the total demands are satisfied. We use the basic conditions for our closed-loop chain and consider them as our assumptions in modeling. These basic conditions are given as follows.

(i) The customers' demands must be satisfied.

(ii) The flow transferred between two inconsecutive stages must be prevented.

(iii) The number of opened facilities and their capacities are limited.

Recycling rate issue is only discussed in the closed-loop logistics literature. This contains the recovery and landfilling rates. In our model, the recovery amount is assumed to be a percentage of the customer demand corresponding to the Van Der Laan et al. [20] assumption based on the dependence of the amounts of returned products on the demand of the products. So, the following assumption is considered by our model.

(iv) The recovery and landfilling rates are given.

The framework of proposed closed-loop chain is illustrated in Figure 1. One of the main advantages of our proposed model is integrating the transport mode selection and closed-loop logistics in the supply network. In this study, we design a closed-loop supply chain with the aim of both minimizing the total cost and reducing the environmental impact in the whole chain by choosing the optimal locations of the facilities, the flows of operation units, and the transportation modes along each capacity-constrained stage when the demand of customers and the recycling rates are given. In relation to the transportation issue, it has a significant impact on air pollution so that the overall amount of CO_2 emitted from it is about 14% of total emissions at the global level [21, 22]. Transportation mode is one of the main choices in transport. There is a variety of transportation modes in our closed-loop chain such as transport by plane, ship, truck, or rail. Costs, transit time, and environmental performance are factors by which each mode is distinguished from other modes. Here, the transport mode is chosen using financial and environmental considerations. Besides, the time window constraints play a key role in selecting the transport mode. Due to the air pollution impacts resulting from freight transportation, this paper pays a special attention to this issue from CO_2 emission's viewpoint. With respect to the emission calculation issue, there are several methodologies to measure carbon emissions accurately: Greenhouse Gas (GHG) Protocol [23], Artemis [24], EcoTransIT [25], NTM [26], and STREAM [27]. Here, we use the NTM method which specifies emissions for four types of transport: air, rail, road, and water. The NTM method has a high level of detail and focuses on Europe. In this section, we describe the calculation method for the total emissions for each type of transport. This method calculates the total emissions for an average-loaded vehicle and allocates part of the emissions to one unit of product. Below, emissions calculated for four types of transport based on NTM method are given.

Air Transport. The emission factor and the distance are the two main elements determining the total emissions coming from the air transport. The emission factor is in two parts: a constant emission factor (CEF) and a variable emission factor (VEF). Estimation of the emission factors from aircraft is based on aircraft type, engine type, and maximum load. With respect to this type of transport, the flight distance (D_a) is considered to calculate the distance between the origin and destination location. The bend of the earth is taken into account when we need to calculate the flight distance. The total emissions for an aircraft are calculated by the following equation:

$$\mathrm{EM}_{\mathrm{total}} = \mathrm{CEF} + \mathrm{VEF} \cdot D_a. \tag{1}$$

Defining (1), the total emissions for an average-loaded vehicle have been calculated. If we want to allocate part of the emissions to one unit of product (e_a), we also have to define the dimensional weight (w_d) which is determined by the density (ρ) multiplied by volume (v) of one unit of product. Corresponding to [28], if a product has a higher density than 167 (kg/m^3), the actual weight is considered to calculate the dimensional weight. In contrast to this, the volume multiplied

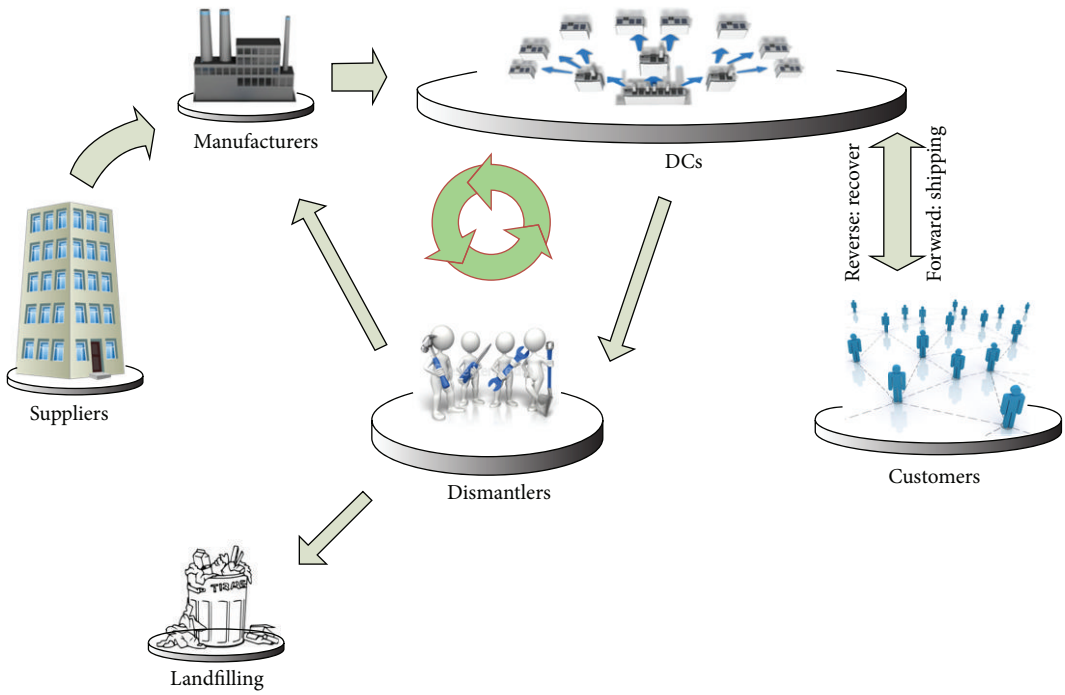


Figure 1: Framework of proposed closed-loop chain.

by 167 (kg/m^3) is substituted for the actual weight when a product has a low density. Then,

$$w_d = \max(w, 167v) = \max(\rho v, 167v) = v \max(\rho, 167). \quad (2)$$

Since the amount of goods carried by a vehicle depends on the weight and the volume of the load, the emissions allocated to one unit of the product (e_a in kg) are calculated as follows:

$$e_a = \mathrm{EM}_{\mathrm{total}} \frac{w_d}{\mathrm{LO}_{\max}\mathrm{LF}}, \quad (3)$$

where $\mathrm{LO}_{\max}$ and LF are the maximum load of an aircraft (in kg) and the average load factor of the aircraft, respectively.

Railway Transport. Here, the emission calculation method for only diesel engine in railway transportation is described [29]. The unit emissions are calculated (e_d) based on the emission factor, the distance, and the weight of the product. The amount of CO_2 emitted when transporting 1 net tonne over 1 km in way is known as the emission factor (EF in kg CO_2/net tonne km). It depends on several factors outlined as follows.

(i) The gross weight of the train (W_{gr} in tonne) includes the weight of the locomotive and the carriages.

(ii) An emission constant (T) determines the fuel consumption for a way.

(iii) A correcting factor for the terrain (ξ_t) is different based on the topography of the way. For example, the factor for *hilly* and *mountainous* terrain is greater than for *flat*. Hence, $\xi_f = 1$ and $\xi_m > \xi_h > 1$, where $t \in \{\text{flat, mountainous, hilly}\}$.

(iv) The load factor (LF) equals the ratio of net and gross weight of the train.

(v) The fuel emissions (FE) denote the emissions per liter of fuel burnt.

The emission factor for the diesel rail transport (EF_d in (kg CO_2/net tonne km)) is defined by the following equation:

$$\mathrm{EF}_d = \frac{\xi_t \cdot T \cdot \mathrm{FE}}{10^6 \sqrt{W_{\mathrm{gr}}} \cdot \mathrm{LF}}. \quad (4)$$

The emissions allocated to one unit of the product (e_d in kg) are a function of the distance (D in km), the weight of the product (w in tonne), and the emission factor. The formula for the unit emissions for the diesel engine in railway transportation is then

$$e_d = \mathrm{EF}_d \cdot D \cdot w. \quad (5)$$

Road Transport. In this section, the fuel consumption, the fuel emissions, and the distance are three main factors to calculate

the total emissions of the vehicle. Below, each factor is given in more detail.

(i) The fuel consumption (FC in L/km) is based on two factors, load factor (LF) and the type of vehicle, and is calculated as follows:

$$\mathrm{FC} = \mathrm{FC}_{\mathrm{empty}} + \left(\mathrm{FC}_{\mathrm{full}} - \mathrm{FC}_{\mathrm{empty}}\right) \cdot \mathrm{LF}, \tag{6}$$

where $\mathrm{FC}_{\mathrm{full}}$ and $\mathrm{FC}_{\mathrm{empty}}$ are the fuel consumption for a full loaded vehicle and the fuel consumption for an unladen vehicle, respectively.

(ii) The fuel emissions (FE) are defined as gram of CO_2 emitted per liter of fuel.

(iii) The distance (D in km) is the distance between the locations.

Combining these factors yields the following equation for the total emissions of the vehicle for road transport ($\mathrm{EM}_{\mathrm{total}}$ in g):

$$\mathrm{EM}_{\mathrm{total}} = \mathrm{FE} \cdot \mathrm{FC} \cdot D. \tag{7}$$

Defining (7), the emissions of the entire vehicle have been calculated. If we want to allocate part of the emissions to one unit of product (e_r), we also have to define the dimensional weight (w_d) of one unit of product, which is defined as

$$w_d = \max(w, 250v) = \max(\rho v, 250v) = v \max(\rho, 250), \tag{8}$$

where 250 is a default density used by transport companies [30]. So, if a product has a density higher than 250 (kg/m^3), the actual weight is considered to calculate the dimensional weight. In contrast to this, the volume multiplied by 250 (kg/m^3) is substituted for the actual weight when a product has a low density. The emissions allocated to one unit of the product (e_r in g) are calculated as follows:

$$e_r = \mathrm{EM}_{\mathrm{total}} \frac{w_d}{\mathrm{LO}_{\max}\mathrm{LF}}, \tag{9}$$

where $\mathrm{LO}_{\max}$ and LF are the maximum load of a vehicle (in kg) and the average load factor of the vehicle, respectively.

Water Transport. Short-sea transport with diesel oil-powered vessels is known as water transport [31]. Here, the total emissions ($\mathrm{EM}_{\mathrm{total}}$ in kg) depend on three factors: the fuel consumption (FC), the fuel emissions (FE), and the distance (D_w). The fuel consumption (FC) (in L per km) is given in [31] for both a given vessel type and an average load factor. The distance D_w (in km) is the distance between two locations over waterways which is larger than the distance over road. The fuel emissions (FE) factor (in kg) is also the amount of CO_2 emitted when 1 L of diesel is burnt. The total emissions ($\mathrm{EM}_{\mathrm{total}}$ in kg) of the vessel are calculated by the following equation:

$$\mathrm{EM}_{\mathrm{total}} = \mathrm{FE} \cdot \mathrm{FC} \cdot D_w. \tag{10}$$

The unit emissions for the vessel in waterway transportation are obtained defining the allocation fraction $\alpha \in (0, 1]$ as follows:

$$\alpha = \frac{\text{unit capacity}}{\text{total capacity}}, \tag{11}$$

where the type of ship plays a critical role in determining the unit of capacity; here, it can be weight for bulk vessels. The formula for the unit emissions (e_w in kg) of the vessel is then

$$e_w = \alpha \cdot \mathrm{EM}_{\mathrm{total}} = \alpha \cdot \mathrm{FE} \cdot \mathrm{FC} \cdot D_w. \tag{12}$$

3. Supply Chain Models with Considering the Environmental Impact

Here, we pursue three scenarios and develop problem formulations for each scenario. A mixed integer linear programming (MILP) optimization model to minimize the total construction cost of this network is presented.

3.1. Basic Model. In order to simplify this problem, we suppose that there is only one product in the concerned closed-loop chain, and the carbon emission resulting from freight transport is not considered. In order to formulate this simplified problem mathematically, the following notations are necessary.

Notations

I: Set of candidate manufactories

J: Set of candidate DCs

K: Set of customers

M: Set of candidate dismantlers

V: Set of transport mode types

V_I: Set of transport mode types at manufactory; $V_I \subset V$

V_J: Set of transport mode types at DC; $V_J \subset V$

V_M: Set of transport mode types at dismantler; $V_M \subset V$.

Parameters

Cm_i: Capacity of manufactory i

Tc_j: Total capacity of DCj (forward and reverse)

Cd_m: Capacity of dismantler m

Pc_j: The percentage of total capacity for reverse logistics in DCj

pr_k: The percentage of recovery of customer k

Pl_m: The percentage of landfilling of dismantler m

dc_k: Demand of customer k

P_cost_i: Unit cost of production in manufactory i

CMD_{v_i}: Unit cost of transportation from manufactory to DC by vehicle v_i per km

CDC_{v_j}: Unit cost of transportation from DC to customer by vehicle v_j per km

CDM_{v_m}: Unit cost of transportation from dismantler to manufactory by vehicle v_m per km

FM_i: Fixed cost for operating manufactory i

FDC_j: Fixed cost for operating DCj

FD_m: Fixed cost for operating dismantler m

Cl: Fixed cost for landfilling per unit

dis_MD_{ij}: Distance between manufactory i and DCj

dis_DC_{jk}: Distance between DCj and customer k

dis_CC_{kl}: Distance between customer k and customer l

dis_DD_{jm}: Distance between DCj and dismantler m

dis_DM_{mi}: Distance between dismantler m and manufactory i

$t\_DC_{jkv_j}$: The time of transportation from DCj to customer k using vehicle v_j

$t\_CC_{klv_j}$: The time of transportation from customer k to customer l using vehicle v_j

a_c_k: The lower bound of expected time for delivering product at customer k

b_c_k: The upper bound of expected time for delivering product at customer k

Rc_{kj}: The recovery cost in DCj from customer k

NVM_{iv_i}: Number of vehicles v_i at manufactory i

NVD_{jv_j}: Number of vehicles v_j at DCj

$NVDi_{mv_m}$: Number of vehicles v_m at dismantler m

CVM_{v_i}: Capacity of vehicle v_i

CVD_{v_j}: Capacity of vehicle v_j

$CVDi_{v_m}$: Capacity of vehicle v_m

$LO_{\max}\_M_{v_i}$: Maximum load for vehicle v_i

$LO_{\max}\_D_{v_j}$: Maximum load for vehicle v_j

$LO_{\max}\_Di_{v_m}$: Maximum load for vehicle v_m

$LF\_M_{v_i}$: Average load factor for vehicle v_i

$LF\_D_{v_j}$: Average load factor for vehicle v_j

$LF\_Di_{v_m}$: Average load factor for vehicle v_m

vol: Volume of product

ρ_{v_i}: Density of product for vehicle v_i

wp: Weight of product

capw: Total capacity of cargo vessel

Q: The maximum number of nodes a salesman may visit

L: The minimum number of nodes a salesman must visit

M: A large number.

Decision Variables

$x\_MD_{ijv_i}$: 1, if a product can be shipped by vehicle v_i from manufactory i to DCj; 0, otherwise

$x\_DC_{jkv_j}$: 1, if a product can be shipped by vehicle v_j from DCj to customer k; 0, otherwise

$x\_DD_{jmv_j}$: 1, if a recovered product can be shipped by vehicle v_j from DCj to dismantler m; 0, otherwise

$x\_CD_{kjv_j}$: 1, if a vehicle v_j returned from customer k to DCj; 0, otherwise

$x\_DM_{miv_m}$: 1, if a reused product can be shipped by vehicle v_m from dismantler m to manufactory i; 0, otherwise

α_i: 1, if production takes place on manufactory i; 0, otherwise

β_j: 1, if DCj is opened; 0, otherwise

γ_m: 1, if dismantler m is opened; 0, otherwise

$z\_CC_{klv_j}$: 1, if a product can be shipped by vehicle v_j from customer k to customer l; 0, otherwise

$y\_MD_{ijv_i}$: Amount shipped by vehicle v_i from manufactory i to DCj

$y\_DC_{jkv_j}$: Amount shipped by vehicle v_j from DCj to customer k

$y\_DD_{jmv_j}$: Amount of recovered product shipped by vehicle v_j from DCj to dismantler m

$y\_CD_{kjv_j}$: Recovered amount shipped by vehicle v_j from customer k to DCj

$y\_DM_{miv_m}$: Reused amount shipped by vehicle v_m from dismantler m to manufactory i

$y\_CC_{klv_j}$: Recovered amount shipped by vehicle v_j from customer k to customer l

PM_i: Quantity produced at manufactory i

u_k: The number of nodes visited by travelers from DC to node k

$congR_k$: Amount of congested product at customer k

$congF_k$: Amount of congested recovered product at customer k

S_k: The arrival time of product at customer k.

Using these definitions, the basic model for the proposed closed-loop chain can be described as follows.

Objective Function

Consider the following:

$$\begin{aligned} f = {} & \sum_{i\in I}\alpha_i\cdot \mathrm{FM}_i + \sum_{j\in J}\beta_j\cdot \mathrm{FDC}_j + \sum_{m\in M}\gamma_m\cdot \mathrm{FD}_m \\ & + \sum_{i\in I}\sum_{j\in J}\sum_{v_i\in V_I} y\_\mathrm{MD}_{ijv_i}\cdot \mathrm{dis\_MD}_{ij}\cdot \mathrm{CMD}_{v_i} \\ & + \sum_{j\in J}\sum_{k\in K}\sum_{v_j\in V_J} y\_\mathrm{DC}_{jkv_j}\cdot \mathrm{dis\_DC}_{jk}\cdot \mathrm{CDC}_{v_j} \\ & + \sum_{k\in K}\sum_{l\in K}\sum_{v_j\in V_J} y\_\mathrm{CC}_{klv_j}\cdot \mathrm{dis\_CC}_{kl}\cdot \mathrm{CDC}_{v_j} \\ & + \sum_{k\in K}\sum_{j\in J}\sum_{v_j\in V_J} y\_\mathrm{CD}_{kjv_j}\cdot \mathrm{dis\_DC}_{jk}\cdot \mathrm{CDC}_{v_j} \\ & + \sum_{j\in J}\sum_{m\in M}\sum_{v_j\in V_J} y\_\mathrm{DD}_{jmv_j}\cdot \mathrm{dis\_DD}_{jm}\cdot \mathrm{CDC}_{v_j} \\ & + \sum_{m\in M}\sum_{i\in I}\sum_{v_m\in V_M} y\_\mathrm{DM}_{miv_m}\cdot \mathrm{dis\_DM}_{mi}\cdot \mathrm{CDM}_{v_m} \\ & + \sum_{i\in I}\mathrm{PM}_i\cdot P\_\mathrm{cost}_i \\ & + \sum_{k\in K}\sum_{j\in J}\sum_{v_j\in V_J} y\_\mathrm{CD}_{kjv_j}\cdot \mathrm{Rc}_{kj} + \mathrm{CL} \\ & \cdot \sum_{m\in M}\left[\mathrm{Pl}_m\cdot \sum_{j\in J}\sum_{v_j\in V_J} y\_\mathrm{DD}_{jmv_j}\right]. \end{aligned} \tag{13}$$

Constraints

Consider the following:

$$\sum_{i\in I}\alpha_i \ge 1, \tag{14}$$

$$\sum_{j\in J}\beta_j \ge 1, \tag{15}$$

$$\mathrm{PM}_i \ge 1 - M\left(1-\alpha_i\right), \quad \forall i\in I, \tag{16}$$

$$\sum_{v_i\in V_I}\sum_{j\in J} x\_\mathrm{MD}_{ijv_i} \ge 1 - M\left(1-\alpha_i\right), \quad \forall i\in I, \tag{17}$$

$$\sum_{v_i\in V_I}\sum_{i\in I} x\_\mathrm{MD}_{ijv_i} \ge 1 - M\left(1-\beta_j\right), \quad \forall j\in J, \tag{18}$$

$$\begin{aligned} & y\_\mathrm{MD}_{ijv_i} \ge 1 - M\left(1 - x\_\mathrm{MD}_{ijv_i}\right), \\ & \forall i\in I, \quad \forall j\in J, \quad \forall v_i\in V_I, \end{aligned} \tag{19}$$

$$\sum_{v_i\in V_I}\sum_{j\in J} y\_\mathrm{MD}_{ijv_i} \le \mathrm{Cm}_i, \quad \forall i\in I, \tag{20}$$

$$\sum_{j\in J} x\_\mathrm{MD}_{ijv_i} \le \mathrm{NVM}_{iv_i}, \quad \forall i\in I,\ \forall v_i\in V_I, \tag{21}$$

$$\sum_{v_i\in V_I} x\_\mathrm{MD}_{ijv_i} \le 1, \quad \forall i\in I,\ \forall j\in J, \tag{22}$$

$$\begin{aligned} & w_p\cdot y\_\mathrm{MD}_{ijv_i} \le \mathrm{CVM}_{v_i}\cdot \mathrm{LF\_}M_{v_i}, \\ & \forall i\in I, \quad \forall j\in J, \quad \forall v_i\in V_I, \end{aligned} \tag{23}$$

$$\sum_{v_j\in V_J}\sum_{k\in K} x\_\mathrm{DC}_{jkv_j} \ge 1 - M\left(1-\beta_j\right), \quad \forall j\in J, \tag{24}$$

$$\sum_{v_j\in V_J}\sum_{k\in K} x\_\mathrm{CD}_{kjv_j} \ge 1 - M\left(1-\beta_j\right), \quad \forall j\in J, \tag{25}$$

$$\sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{DC}_{jkv_j} + \sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{CD}_{kjv_j} \le 1, \quad \forall k\in K, \tag{26}$$

$$\sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{DC}_{jkv_j} + \sum_{v_j\in V_J}\sum_{l\in k} z\_\mathrm{CC}_{lkv_j} = 1, \quad \forall k\in K, \tag{27}$$

$$\sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{CD}_{kjv_j} + \sum_{v_j\in V_J}\sum_{l\in k} z\_\mathrm{CC}_{klv_j} = 1, \quad \forall k\in K, \tag{28}$$

$$\begin{aligned} & \sum_{l\in K} z\_\mathrm{CC}_{klv_j} + \sum_{j\in J} x\_\mathrm{CD}_{kjv_j} = \sum_{e\in K} z\_\mathrm{CC}_{ekv_j} + \sum_{j\in J} x\_\mathrm{DC}_{jkv_j}, \\ & \forall k\in K, \quad \forall v_j\in V_J, \end{aligned} \tag{29}$$

$$\begin{aligned} & u(k) - u(l) + \left(Q\cdot z\_\mathrm{CC}_{klv_j}\right) + \left((Q-2)\cdot z\_\mathrm{CC}_{lkv_j}\right) \\ & \le Q-1, \quad \forall k,l\in K, \forall v_j\in V_J, \end{aligned} \tag{30}$$

$$\begin{aligned} & u(k) + \left((Q-2)\cdot \sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{DC}_{jkv_j}\right) - \sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{CD}_{kjv_j} \\ & \le Q-1, \quad \forall k\in K, \end{aligned} \tag{31}$$

$$\begin{aligned} & \sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{DC}_{jkv_j} + \left((2-L)\cdot \sum_{v_j\in V_J}\sum_{j\in J} x\_\mathrm{CD}_{kjv_j}\right) \ge 2, \\ & \forall k\in K, \end{aligned} \tag{32}$$

$$\mathrm{cong}R_k = \left(\sum_{v_j\in V_J}\sum_{l\in k} z\_\mathrm{CC}_{klv_j}\cdot \mathrm{cong}R_l\right) + \mathrm{dc}_k, \quad \forall k\in K, \tag{33}$$

$$\begin{aligned} & y\_\mathrm{DC}_{jkv_j} \ge 1 - M\left(1 - x\_\mathrm{DC}_{jkv_j}\right), \\ & \forall k\in K, \quad \forall v_j\in V_J, \quad \forall j\in J, \end{aligned} \tag{34}$$

$$y\_\mathrm{DC}_{jkv_j} \ge \mathrm{cong}R_k, \quad \forall k\in K,\ \forall v_j\in V_J,\ \forall j\in J, \tag{35}$$

$$\begin{aligned} & w_p\cdot y\_\mathrm{DC}_{jkv_j} \le \mathrm{CVD}_{v_j}\cdot \mathrm{LF\_}D_{v_j}, \\ & \forall k\in K, \quad \forall v_j\in V_J, \quad \forall j\in J, \end{aligned} \tag{36}$$

$$\sum_{k\in K} x\_\mathrm{DC}_{jkv_j} \le \mathrm{NVD}_{jv_j}, \quad \forall v_j\in V_J,\ \forall j\in J, \tag{37}$$

$$\sum_{v_i \in V_I} \sum_{i \in I} y\_\mathrm{MD}_{ijv_i} = \sum_{v_j \in V_J} \sum_{k \in K} y\_\mathrm{DC}_{jkv_j}, \quad \forall j \in J, \tag{38}$$

$$\mathrm{cong}F_k = \left(\sum_{v_j \in V_J} \sum_{l \in k} z\_\mathrm{CC}_{lkv_j} \cdot \mathrm{cong}F_l \right) + \lceil \mathrm{pr}_k \cdot \mathrm{dc}_k \rceil, \quad \forall k \in K, \tag{39}$$

$$y\_\mathrm{CD}_{kjv_j} \geq 1 - M\left(1 - x\_\mathrm{CD}_{kjv_j}\right), \quad \forall k \in K, \quad \forall v_j \in V_J, \quad \forall j \in J, \tag{40}$$

$$y\_\mathrm{CD}_{kjv_j} \geq \mathrm{cong}F_k, \quad \forall k \in K, \ \forall v_j \in V_J, \ \forall j \in J, \tag{41}$$

$$y\_\mathrm{CC}_{klv_j} \geq \left(\sum_{j \in J} y\_\mathrm{DC}_{jkv_j} + \sum_{h \in K} y\_\mathrm{CC}_{hkv_j} \right) - \lceil (1 - \mathrm{pr}_k) \cdot \mathrm{dc}_k \rceil - M\left(1 - z\_\mathrm{CC}_{klv_j}\right), \quad \forall k, l \in K, \quad \forall v_j \in V_J, \tag{42}$$

$$\sum_{v_j \in V_J} \sum_{j \in J} x\_\mathrm{DD}_{jmv_j} \geq 1 - M(1 - \gamma_m), \quad \forall m \in M, \tag{43}$$

$$\sum_{v_j \in V_J} x\_\mathrm{DD}_{jmv_j} \leq 1, \quad \forall j \in J, \ \forall m \in M, \tag{44}$$

$$\sum_{v_j \in V_J} \sum_{m \in M} x\_\mathrm{DD}_{jmv_j} \geq 1 - M\left(1 - \beta_j\right), \quad \forall j \in J, \tag{45}$$

$$y\_\mathrm{DD}_{jmv_j} \geq 1 - M\left(1 - x\_\mathrm{DD}_{jmv_j}\right), \quad \forall j \in J, \quad \forall m \in M, \quad \forall v_j \in V_J, \tag{46}$$

$$\sum_{v_j \in V_J} \sum_{k \in K} y\_\mathrm{CD}_{kjv_j} = \sum_{v_j \in V_J} \sum_{m \in M} y\_\mathrm{DD}_{jmv_j}, \quad \forall j \in J, \tag{47}$$

$$\sum_{v_j \in V_J} \sum_{k \in K} y\_\mathrm{DC}_{jkv_j} + \sum_{v_j \in V_J} \sum_{m \in M} y\_\mathrm{DD}_{jmv_j} \leq Tc_j \cdot \beta_j, \quad \forall j \in J, \tag{48}$$

$$\sum_{m \in M} x\_\mathrm{DD}_{jmv_j} \leq \mathrm{NVD}_{jv_j}, \quad \forall j \in J, \ \forall v_j \in V_J, \tag{49}$$

$$w_p \cdot y\_\mathrm{DD}_{jmv_j} \leq \mathrm{CVD}_{v_j} \cdot \mathrm{LF}\_D_{v_j}, \quad \forall j \in J, \quad \forall m \in M, \quad \forall v_j \in V_J, \tag{50}$$

$$\sum_{v_j \in V_J} \sum_{m \in M} y\_\mathrm{DD}_{jmv_j} \leq \lfloor \mathrm{Pc}_j \cdot \mathrm{Tc}_j \cdot \beta_j \rfloor, \quad \forall j \in J, \tag{51}$$

$$\sum_{v_m \in V_m} \sum_{i \in I} x\_\mathrm{DM}_{miv_m} \geq 1 - M(1 - \gamma_m), \quad \forall m \in M, \tag{52}$$

$$\sum_{v_m \in V_M} x\_\mathrm{DM}_{miv_m} \leq 1, \quad \forall m \in M, \ \forall i \in I, \tag{53}$$

$$\sum_{v_m \in V_m} \sum_{m \in M} x\_\mathrm{DM}_{miv_m} \geq 1 - M(1 - \alpha_i), \quad \forall i \in I, \tag{54}$$

$$y\_\mathrm{DM}_{miv_m} \geq 1 - M\left(1 - x\_\mathrm{DM}_{miv_m}\right), \quad \forall m \in M, \quad \forall i \in I, \quad \forall v_m \in V_M, \tag{55}$$

$$\sum_{v_m \in V_M} \sum_{m \in M} y\_\mathrm{DM}_{miv_m} + \mathrm{PM}_i = \sum_{v_i \in V_i} \sum_{j \in J} y\_\mathrm{MD}_{ijv_i}, \quad \forall i \in I, \tag{56}$$

$$\sum_{v_j \in V_J} \sum_{j \in J} y\_\mathrm{DD}_{jmv_j} = \left\lfloor \mathrm{Pl}_m \cdot \sum_{v_j \in V_J} \sum_{j \in J} y\_\mathrm{DD}_{jmv_j} \right\rfloor + \sum_{v_m \in V_M} \sum_{i \in I} y\_\mathrm{DM}_{miv_m}, \quad \forall m \in M, \tag{57}$$

$$\sum_{v_m \in V_M} \sum_{i \in I} y\_\mathrm{DM}_{miv_m} + \left\lfloor \mathrm{Pl}_m \cdot \sum_{v_j \in V_J} \sum_{j \in J} y\_\mathrm{DD}_{jmv_j} \right\rfloor \leq \gamma_m \cdot \mathrm{Cd}_m, \quad \forall m \in M, \tag{58}$$

$$\sum_{i \in I} x\_\mathrm{DM}_{miv_m} \leq \mathrm{NVDi}_{mv_m}, \quad \forall m \in M, \ \forall v_m \in V_M, \tag{59}$$

$$w_p \cdot y\_\mathrm{DM}_{miv_m} \leq \mathrm{CVDi}_{v_m} \cdot \mathrm{LF}\_Di_{v_m}, \quad \forall m \in M, \quad \forall i \in I, \quad \forall v_m \in V_M, \tag{60}$$

$$S_k \geq a\_c_k, \quad \forall k \in K, \tag{61}$$

$$S_k \leq b\_c_k, \quad \forall k \in K, \tag{62}$$

$$S_k + t\_\mathrm{CC}_{klv_j} - M\left(1 - z\_\mathrm{CC}_{klv_j}\right) \leq S_l, \quad \forall k, l \in K, \quad \forall v_j \in V_J, \tag{63}$$

$$S_k + t\_\mathrm{CC}_{klv_j} + M\left(1 - z\_\mathrm{CC}_{klv_j}\right) \geq S_l, \quad \forall k, l \in K, \quad \forall v_j \in V_J, \tag{64}$$

$$t\_\mathrm{DC}_{jkv_j} - M\left(1 - x\_\mathrm{DC}_{jkv_j}\right) \leq S_k, \quad \forall k \in K, \quad \forall v_j \in V_J, \quad \forall j \in J, \tag{65}$$

$$t\_\mathrm{DC}_{jkv_j} + M\left(1 - x\_\mathrm{DC}_{jkv_j}\right) \geq S_k, \quad \forall k \in K, \quad \forall v_j \in V_J, \quad \forall j \in J, \tag{66}$$

$$x\_\mathrm{MD}_{ijv_i}, \quad x\_\mathrm{DC}_{jkv_j}, \quad x\_\mathrm{DD}_{jmv_j}, \quad x\_\mathrm{CD}_{kjv_j}, \quad x\_\mathrm{DM}_{miv_m}, \quad z\_\mathrm{CC}_{klv_j}, \quad \alpha_i, \beta_j, \gamma_m \in \{0, 1\}, \quad \forall i \in I, \quad \forall j \in J, \quad \forall k, l \in K, \quad \forall m \in M, \quad \forall v_i \in V_I, \quad \forall v_j \in V_J, \quad \forall v_m \in V_M, \tag{67}$$

$$y\_\mathrm{MD}_{ijv_i}, y\_\mathrm{DC}_{jkv_j}, y\_\mathrm{DD}_{jmv_j},$$
$$y\_\mathrm{CD}_{kjv_j}, y\_\mathrm{DM}_{miv_m}, y\_\mathrm{CC}_{klv_j},$$
$$\mathrm{PM}_i, u_k, \mathrm{cong}R_k, \mathrm{cong}F_k, S_k \geq 0, \quad (68)$$
$$\forall i \in I, \quad \forall j \in J, \quad \forall k, l \in K, \quad \forall m \in M,$$
$$\forall v_i \in V_I, \quad \forall v_j \in V_J, \quad \forall v_m \in V_M.$$

Equation (13) is the objective function which minimizes cost of opening manufactory, distribution center, and dismantler, minimizes the total cost of both forward and backward distances, and minimizes the total cost of operations. Constraints (14) and (15) show that there exist at least one activated manufactory and one DC in the chain, respectively. Constraint (16) ensures that each manufactory can produce an amount of product just after it is selected. Each activated manufactory covers at least one DC, and Constraint (17) represents this goal. On the contrary, each DC receives at least one link from manufactories just after it is selected (Constraint (18)). Constraint (19) represents the amount of flow between manufactory and DC. Constraint (20) represents the limit of the capacity for manufactories in forward logistics. Constraint (21) imposes that the number of traveling vehicles from manufactory would not exceed the existing vehicles. Constraint (22) prevents the route between manufactory and DC from accepting its vehicle more than once. The capacity constraint of each vehicle that traveled from manufactory to DC is shown by Constraint (23). Constraint (24) guarantees that each activated DC covers at least one customer. Each activated DC receives at least one link from customers, and Constraint (25) represents this goal. Constraint (26) represents that a salesman from DC must visit at least two customers. Constraint (27) requires that any customer be supplied by either DC or other customers. In addition, it either comes back to DC or supplies other customers. This concept is represented by Constraint (28). Each customer is supplied and supplies by the same vehicle. This is represented by Constraint (29). Constraints (30), (31), and (32) prevent any subtour in network. Constraint (33) indicates the amount of congested product for supplying other customers by each customer. Constraint (34) represents the amount of flow between DC and customer. Constraint (35) is to satisfy the customer demand. The capacity constraint of each vehicle traveling from DC to customer is shown by Constraint (36). Constraint (37) imposes that the number of traveling vehicles from DC would not exceed the existing vehicles. Constraint (38) satisfies the law of the flow conservation by in-flow equal to out-flow. The amount of congested product for recovering from other customers by each customer is indicated by Constraint (39). Constraints (40)-(41) represent the amount of flow between customer and DC. The amount of flow among customers is represented by Constraint (42). Constraint (43) guarantees that each activated dismantler receives at least one link from DCs. Constraint (44) prevents the route between DC and dismantler from accepting its vehicle more than once. Constraint (45) guarantees that each activated DC covers at least one dismantler. The amount of flow between DC and dismantler is shown by Constraint (46). Constraint (47) satisfies the law of the flow conservation by in-flow equal to out-flow. Constraint (48) indicates that the total forward and backward flows cannot exceed the total capacity of DC. Constraint (49) imposes that the number of traveling vehicles from DC to dismantler would not exceed the existing vehicles. The capacity constraint of each vehicle traveling from DC to dismantler is shown by Constraint (50). Constraint (51) means the reverse limit of the capacity for DCs. Constraint (52) ensures that each activated dismantler covers at least one manufactory. Constraint (53) prevents the route between dismantler and manufactory from accepting its vehicle more than once. Constraint (54) guarantees that each activated manufactory receives at least one link from dismantlers. The amount of flow between dismantler and manufactory is shown by Constraint (55). Constraints (56) and (57) satisfy the law of the flow conservation by in-flow equal to out-flow. Constraint (58) means the reverse limit of the capacity for dismantlers. Constraint (59) imposes that the number of traveling vehicles from dismantler to manufactory would not exceed the existing vehicles. The capacity constraint of each vehicle traveling from dismantler to manufactory is shown by Constraint (60). Constraints (61)–(66) satisfy time windows. Constraint (67) denotes the binary variables, and Constraint (68) restricts all other variables from taking nonnegative values.

3.1.1. Linearization. To improve the performance of the proposed mathematical model we act out the following linearization for the nonlinear equations. As Constraint (33) is nonlinear, we turn it into the following equations:

Equation (33)

$$\longrightarrow \mathrm{cong}R_k \geq M \cdot \left(z\_\mathrm{CC}_{klv_j} - 1\right) + \left(\mathrm{dc}_k + \mathrm{cong}R_l\right), \quad (69)$$
$$\forall v_j \in V_J, \quad \forall l, k \in K,$$

$$\mathrm{cong}R_k \leq (-M) \cdot \left(z\_\mathrm{CC}_{klv_j} - 1\right) + \left(\mathrm{dc}_k + \mathrm{cong}R_l\right),$$
$$\forall v_j \in V_J, \quad \forall l, k \in K, \quad (70)$$

$$\mathrm{cong}R_k \leq \left(\sum_{v_j \in V_J} \sum_{l \in k} z\_\mathrm{CC}_{klv_j}\right) \cdot M + \mathrm{dc}_k, \quad \forall k \in K, \quad (71)$$

$$\mathrm{cong}R_k \geq \left(\sum_{v_j \in V_J} \sum_{l \in k} z\_\mathrm{CC}_{klv_j}\right) \cdot (-M) + \mathrm{dc}_k, \quad (72)$$
$$\forall k \in K.$$

As Constraint (39) is nonlinear, we turn it into the following equations:

Equation (39)

$$\longrightarrow \text{cong}F_k \geq M \cdot \left(z\_\text{CC}_{klv_j} - 1\right) + \left(\left(\text{pr}_k \cdot \text{dc}_k\right) + \text{cong}F_l\right), \quad \forall v_j \in V_J, \quad \forall l, k \in K, \tag{73}$$

$$\text{cong}F_k \leq (-M) \cdot \left(z\_\text{CC}_{klv_j} - 1\right) + \left(\left(\text{pr}_k \cdot \text{dc}_k\right) + \text{cong}F_l\right), \quad \forall v_j \in V_J, \quad \forall l, k \in K, \tag{74}$$

$$\text{cong}F_k \leq \left(\sum_{v_j \in V_J} \sum_{l \in k} z\_\text{CC}_{klv_j}\right) \cdot M + \left(\text{pr}_k \cdot \text{dc}_k\right), \quad \forall k \in K, \tag{75}$$

$$\text{cong}F_k \geq \left(\sum_{v_j \in V_J} \sum_{l \in k} z\_\text{CC}_{klv_j}\right) \cdot (-M) + \left(\text{pr}_k \cdot \text{dc}_k\right), \quad \forall k \in K. \tag{76}$$

3.2. Emissions. Here, we describe how the carbon emissions are incorporated into our model and the methodology to calculate the emissions. In Section 3.2.1 we define the emission cost-minimization problem in which a unit cost for emission is charged. In Section 3.2.2 we define the emission-constraint problem in which we have a hard constraint on the carbon emissions.

3.2.1. Emission Cost-Minimization Model. The objective of the proposed basic model is to minimize the total construction and operations costs while considering structural, product flow, capacity, customers' demands, and time windows constraints. It has ignored the carbon emission as an important factor for green supply chain. Below, we will extend the basic model by adding a cost for carbon emissions. In the Emission Trading Scheme the carbon cost is expressed in €/(metric) tonne emissions. We therefore specify a carbon emission cost CE (CE > 0) per tonne of CO_2 emitted. For any transportation mode, let EM_total_MD_{ij} and $e\_u\_\text{MD}_{ij}$, EM_total_DC_{jK} and $e\_u\_\text{DC}_{jk}$, EM_total_CC_{Kl} and $e\_u\_\text{CC}_{kl}$, EM_total_CD_{Kj} and $e\_u\_\text{CD}_{kj}$, EM_total_DD_{jm} and $e\_u\_\text{DD}_{jm}$, and EM_total_DM_{mi} and $e\_u\_\text{DM}_{mi}$ denote its total carbon emissions and the emissions allocated to one unit of the product for transportation from the ith manufactory to the jth DC, from the jth DC to the kth customer, from the kth customer to the lth customer, from the kth customer to the jth DC, from the jth DC to the mth dismantler, and from the mth dismantler to the ith manufactory, respectively. In this model, the following objectives and constraints are added to the proposed basic closed-loop model to consider the carbon emission issue. In order to formulate this emission cost-minimization model mathematically, the following notations are necessary.

Parameters

CEF: Constant emission factor

VEF: Variable emission factor

$\text{FC\_}D_{v_j}$: The fuel consumption for vehicle v_j

$\text{FE\_}D_{v_j}$: The fuel emissions for diesel fuel for vehicle v_j

FC_M: The fuel consumption for semitrailer stated in manufactory

FE_M: The fuel emissions for diesel fuel for semitrailer stated in manufactory

FC_Di: The fuel consumption for semitrailer stated in dismantler

FE_Di: The fuel emissions for diesel fuel for semitrailer stated in dismantler

T: The fuel consumption factor for diesel train

FER: The fuel emissions for diesel train

W_gr: The gross weight of the train

FEW: The fuel emissions for cargo vessel

FCW: The fuel consumption for cargo vessel

CE: The price of carbon emission (expressed in €/(metric) tonne emissions).

Decision Variables

EM_total_MD_{ij}: Total emissions of the vehicle from manufactory i to DCj

EM_total_DC_{jk}: Total emissions of the vehicle from DCj to customer k

EM_total_CD_{kj}: Total emissions of the vehicle from customer k to DCj

EM_total_CC_{kl}: Total emissions of the vehicle from customer k to customer l

EM_total_DD_{jm}: Total emissions of the vehicle from DCj to dismantler m

EM_total_DM_{mi}: Total emissions of the vehicle from dismantler m to manufactory i

$e\_u\_\text{MD}_{ij}$: Unit emissions of the vehicle from manufactory i to DCj

$e\_u\_\text{DC}_{jk}$: Unit emissions of the vehicle from DCj to customer k

$e\_u\_\text{CD}_{kj}$: Unit emissions of the vehicle from customer k to DCj

$e\_u\_\text{CC}_{kl}$: Unit emissions of the vehicle from customer k to customer l

$e\_u\_\text{DD}_{jm}$: Unit emissions of the vehicle from DCj to dismantler m

$e\_u\_\text{DM}_{mi}$: Unit emissions of the vehicle from dismantler m to manufactory i.

Objective Function

Consider the following:

$$\begin{aligned} f' = & \sum_{i\in I}\sum_{j\in J}\sum_{v_i\in V_I} y\_\mathrm{MD}_{ijv_i}\cdot e\_u\_\mathrm{MD}_{ij}\cdot \mathrm{CE} \\ & + \sum_{j\in J}\sum_{k\in K}\sum_{v_j\in V_J} y\_\mathrm{DC}_{jkv_j}\cdot e\_u\_\mathrm{DC}_{jk}\cdot \mathrm{CE} \\ & + \sum_{k\in K}\sum_{l\in K}\sum_{v_j\in V_J} y\_\mathrm{CC}_{klv_j}\cdot e\_u\_\mathrm{CC}_{kl}\cdot \mathrm{CE} \\ & + \sum_{k\in K}\sum_{j\in J}\sum_{v_j\in V_J} y\_\mathrm{CD}_{kjv_j}\cdot e\_u\_\mathrm{CD}_{kj}\cdot \mathrm{CE} \\ & + \sum_{j\in J}\sum_{m\in M}\sum_{v_j\in V_J} y\_\mathrm{DD}_{jmv_j}\cdot e\_u\_\mathrm{DD}_{jm}\cdot \mathrm{CE} \\ & + \sum_{m\in M}\sum_{i\in I}\sum_{v_m\in V_M} y\_\mathrm{DM}_{miv_m}\cdot e\_u\_\mathrm{DM}_{mi}\cdot \mathrm{CE}. \end{aligned} \tag{77}$$

Constraints

Consider the following:

$$\begin{aligned} \mathrm{EM\_total\_MD}_{ij} \geq & \left(\mathrm{CEF} + \left(\mathrm{VEF}\cdot 0.801\mathrm{dis\_MD}_{ij}\right)\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ija}\right) \\ & - M\left(x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{78}$$

$$\begin{aligned} e\_u\_\mathrm{MD}_{ij} \geq & \left(\frac{\left(v\cdot\rho_a\cdot \mathrm{EM\_total\_MD}_{ij}\right)}{\left(\mathrm{LO}_{\max}\_M_a\cdot \mathrm{LF}\_M_a\right)}\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ija}\right) \\ & - M\left(x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{79}$$

$$\begin{aligned} \mathrm{EM\_total\_MD}_{ij} \geq & \left(\mathrm{FE}\_M\cdot \mathrm{FC}\_M\cdot\left(\mathrm{dis\_MD}_{ij}\right)\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijr}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{80}$$

$$\begin{aligned} e\_u\_\mathrm{MD}_{ij} \geq & \left(\frac{\left(v\cdot\rho_r\cdot \mathrm{EM\_total\_MD}_{ij}\right)}{\left(\mathrm{LO}_{\max}\_M_r\cdot \mathrm{LF}\_M_r\right)}\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijr}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{81}$$

$$\begin{aligned} \mathrm{EM\_total\_MD}_{ij} \geq & \left(10^{-3}\cdot\frac{\left(\xi_f\cdot T\cdot \mathrm{FER}\right)}{10^6\left(\sqrt{W\_\mathrm{gr}}\cdot \mathrm{LF}\_M_t\right)}\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijt}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{82}$$

$$\begin{aligned} e\_u\_\mathrm{MD}_{ij} \geq & \left(\mathrm{EM\_total\_MD}_{ij}\cdot \mathrm{dis\_MD}_{ij}\cdot w_p\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijt}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijw}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{83}$$

$$\begin{aligned} \mathrm{EM\_total\_MD}_{ij} \geq & \left(\mathrm{FCW}\cdot \mathrm{FEW}\cdot 1.2\mathrm{dis\_MD}_{ij}\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijw}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{84}$$

$$\begin{aligned} e\_u\_\mathrm{MD}_{ij} \geq & \left(\frac{\left(w_p\cdot \mathrm{EM\_total\_MD}_{ij}\right)}{\left(\mathrm{cap}_w\cdot 1000\right)}\right) \\ & - M\left(1 - x\_\mathrm{MD}_{ijw}\right) \\ & - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt}\right), \\ & \forall i\in I, \quad \forall j\in J, \end{aligned} \tag{85}$$

$$\begin{aligned} \mathrm{EM\_total\_DC}_{jK} \geq & \left(\mathrm{FE}\_D_{v_j}\cdot \mathrm{FC}\_D_{v_j}\cdot\left(\mathrm{dis\_DC}_{jk}\right)\right) \\ & - M\left(1 - x\_\mathrm{DC}_{jkv_j}\right), \\ & \forall k\in K, \quad \forall j\in J, \quad \forall v_j\in V_J, \end{aligned} \tag{86}$$

$$\begin{aligned} e\_u\_\mathrm{DC}_{jk} \geq & \left(\frac{\left(v\cdot\rho_r\cdot \mathrm{EM\_total\_DC}_{jk}\right)}{\left(\mathrm{LO}_{\max}\_D_{v_j}\cdot \mathrm{LF}\_D_{v_j}\right)}\right) \\ & - M\left(1 - x\_\mathrm{DC}_{jkv_j}\right), \\ & \forall k\in K, \quad \forall j\in J, \quad \forall v_j\in V_J, \end{aligned} \tag{87}$$

$$\begin{aligned}\text{EM\_total\_CC}_{Kl} \geq\ & \left(\text{FE\_}D_{v_j} \cdot \text{FC\_}D_{v_j} \cdot (\text{dis\_CC}_{kl})\right) \\ & - M\left(1 - z\text{\_CC}_{klv_j}\right), \\ & \forall k, l \in K, \quad \forall v_j \in V_J,\end{aligned} \tag{88}$$

$$\begin{aligned}e\text{\_}u\text{\_CC}_{kl} \geq\ & \left(\frac{(v \cdot \rho_r \cdot \text{EM\_total\_CC}_{kl})}{\left(\text{LO}_{\max}\text{\_}D_{v_j} \cdot \text{LF\_}D_{v_j}\right)}\right) \\ & - M\left(1 - z\text{\_CC}_{klv_j}\right), \\ & \forall k, l \in K, \quad \forall v_j \in V_J,\end{aligned} \tag{89}$$

$$\begin{aligned}\text{EM\_total\_CD}_{Kj} \geq\ & \left(\text{FE\_}D_{v_j} \cdot \text{FC\_}D_{v_j} \cdot \left(\text{dis\_DC}_{jk}\right)\right) \\ & - M\left(1 - x\text{\_CD}_{kjv_j}\right), \\ & \forall k \in K, \quad \forall j \in J, \quad \forall v_j \in V_J,\end{aligned} \tag{90}$$

$$\begin{aligned}e\text{\_}u\text{\_CD}_{kj} \geq\ & \left(\frac{\left(v \cdot \rho_r \cdot \text{EM\_total\_CD}_{kj}\right)}{\left(\text{LO}_{\max}\text{\_}D_{v_j} \cdot \text{LF\_}D_{v_j}\right)}\right) \\ & - M\left(1 - x\text{\_CD}_{kjv_j}\right), \\ & \forall k \in K, \quad \forall j \in J, \quad \forall v_j \in V_J,\end{aligned} \tag{91}$$

$$\begin{aligned}\text{EM\_total\_DD}_{jm} \geq\ & \left(\text{FE\_}D_{v_j} \cdot \text{FC\_}D_{v_j} \cdot \left(\text{dis\_DD}_{jm}\right)\right) \\ & - M\left(1 - x\text{\_DD}_{jmv_j}\right), \\ & \forall m \in M, \quad \forall j \in J, \quad \forall v_j \in V_J,\end{aligned} \tag{92}$$

$$\begin{aligned}e\text{\_}u\text{\_DD}_{jm} \geq\ & \left(\frac{\left(v \cdot \rho_r \cdot \text{EM\_total\_DD}_{jm}\right)}{\left(\text{LO}_{\max}\text{\_}D_{v_j} \cdot \text{LF\_}D_{v_j}\right)}\right) \\ & - M\left(1 - x\text{\_DD}_{jmv_j}\right), \\ & \forall m \in M, \quad \forall j \in J, \quad \forall v_j \in V_J,\end{aligned} \tag{93}$$

$$\begin{aligned}\text{EM\_total\_DM}_{mi} \geq\ & (\text{FE\_Di} \cdot \text{FC\_Di} \cdot (\text{dis\_DM}_{mi}) \cdot 1.05) \\ & - M\left(1 - x\text{\_DM}\_mir'\right) \\ & - M\left(x\text{\_DM}\_mit'\right), \\ & \forall m \in M, \quad \forall i \in I,\end{aligned} \tag{94}$$

$$\begin{aligned}e\text{\_}u\text{\_DM}_{mi} \geq\ & \left(\frac{(v \cdot \rho_r \cdot \text{EM\_total\_DM}_{mi})}{(\text{LO}_{\max}\text{\_Di}_{r'} \cdot \text{LF\_Di}_{r'})}\right) \\ & - M\left(1 - x\text{\_DM}_{mir'}\right) \\ & - M\left(x\text{\_DM}_{mir'}\right), \quad \forall m \in M, \ \forall i \in I,\end{aligned} \tag{95}$$

$$\begin{aligned}\text{EM\_total\_DM}_{mi} \geq\ & \left(10^{-3} \cdot \frac{(\xi_h \cdot T \cdot \text{FER})}{10^6\left(\sqrt{W\text{\_gr}} \cdot \text{LF\_Di}_{t'}\right)}\right) \\ & - M\left(1 - x\text{\_DM}_{mit'}\right) \\ & - M\left(x\text{\_DM}_{mir'}\right), \quad \forall m \in M, \ \forall i \in I,\end{aligned} \tag{96}$$

$$\begin{aligned}e\text{\_}u\text{\_DM}_{mi} \geq\ & \left(\text{EM\_total\_DM}_{mi} \cdot \text{dis\_DM}_{mi} \cdot w_p\right) \\ & - M\left(1 - x\text{\_DM}_{mit'}\right) \\ & - M\left(x\text{\_DM}_{mir'}\right), \\ & \forall m \in M, \quad \forall i \in I.\end{aligned} \tag{97}$$

Nonlinear equation (77) is the objective function which minimizes total cost of the carbon emissions allocated to whole units of the product for transportation from manufactory to DC, DC to customer, customer to customer, customer to DC, DC to dismantler, and dismantler to manufactory, respectively. Constraints (78)–(85) show the emissions allocated to one unit of the product for transportation from the ith manufactory to the jth DC, where $x\text{\_MD}_{ija} \cdots x\text{\_MD}_{ijw}$ are the binary variables to link carbon emissions constraints to the related types of transport. Constraints (78)-(79), (80)-(81), (82)-(83), and (84)-(85) measure carbon emissions of the aircraft, vehicle, diesel train, and vessel based on NTM method for air transport, road transport, rail transport, and water transport.

Constraints (86) and (87) show the emissions allocated to one unit of the product for transportation from the jth DC to the kth customer. Constraints (88) and (89) show the emissions allocated to one unit of the product for transportation from the kth customer to the lth customer. Constraints (90) and (91) show the emissions allocated to one unit of the product for transportation from the kth customer to the jth DC. Constraints (92) and (93) show the emissions allocated to one unit of the product for transportation from the jth DC to the mth dismantler. Constraints (86)–(93) measure carbon emissions of the vehicle based on NTM method for road transport. Constraints (94)–(97) show the emissions allocated to one unit of the product for transportation from the mth dismantler to the ith manufactory, where $x\text{\_DM}_{mir'}$ and $x\text{\_DM}_{mit'}$ are the binary variables to link carbon emissions constraints to the related types of transport. Constraints (94)-(95) and (96)-(97) measure carbon emissions of the vehicle and diesel train based on NTM method for road transport and rail transport.

(1) Linearization. To improve the performance of the proposed mathematical model we act out the following linearization for the nonlinear equations. As (77) is nonlinear, we turn it into the following equation and turn the constraints related to the definition of $\text{EM\_total\_MD}_{ij}, \ldots, \text{EM\_total\_DM}_{mi}$ into the following equations:

$$\text{Equation (77)} \longrightarrow \sum_{i \in I} \sum_{j \in J} e\text{\_}u\text{\_MD}_{ij} \cdot \text{CE}$$

$$+\sum_{j\in J}\sum_{k\in K} e\_u\_\mathrm{DC}_{jk}\cdot \mathrm{CE} + \sum_{k\in K}\sum_{l\in K} e\_u\_\mathrm{CC}_{kl}\cdot \mathrm{CE} + \sum_{k\in K}\sum_{j\in J} e\_u\_\mathrm{CD}_{kj}\cdot \mathrm{CE} + \sum_{j\in J}\sum_{m\in M} e\_u\_\mathrm{DD}_{jm}\cdot \mathrm{CE} + \sum_{m\in M}\sum_{i\in I} e\_u\_\mathrm{DM}_{mi}, \tag{98}$$

$$\text{Constraint (78)} \longrightarrow \mathrm{EM\_total\_MD}_{ij} \geq y\_\mathrm{MD}_{ija}\cdot\left(\mathrm{CEF}+\left(\mathrm{VEF}\cdot 0.801\mathrm{dis\_MD}_{ij}\right)\right) - M\left(1-x\_\mathrm{MD}_{ija}\right) - M\left(x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \quad \forall i\in I, \quad \forall j\in J, \tag{99}$$

$$\text{Constraint (80)} \longrightarrow \mathrm{EM\_total\_MD}_{ij} \geq y\_\mathrm{MD}_{ijr}\cdot\left(\mathrm{FE\_}M\cdot \mathrm{FC\_}M\cdot\left(\mathrm{dis\_MD}_{ij}\right)\right) - M\left(1-x\_\mathrm{MD}_{ijr}\right) - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijt} + x\_\mathrm{MD}_{ijw}\right), \quad \forall i\in I, \quad \forall j\in J, \tag{100}$$

$$\text{Constraint (82)} \longrightarrow \mathrm{EM\_total\_MD}_{ij} \geq y\_\mathrm{MD}_{ijt}\cdot\left(10^{-3}\cdot\frac{\left(\xi_f\cdot T\cdot \mathrm{FER}\right)}{10^6\left(\sqrt{W\_\mathrm{gr}}\cdot \mathrm{LF\_}M_t\right)}\right) - M\left(1-x\_\mathrm{MD}_{ijt}\right) - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijw}\right), \quad \forall i\in I, \quad \forall j\in J, \tag{101}$$

$$\text{Constraint (84)} \longrightarrow \mathrm{EM\_total\_MD}_{ij} \geq y\_\mathrm{MD}_{ijw}\cdot\left(\mathrm{FCW}\cdot \mathrm{FEW}\cdot 1.2\mathrm{dis\_MD}_{ij}\right) - M\left(1-x\_\mathrm{MD}_{ijw}\right) - M\left(x\_\mathrm{MD}_{ija} + x\_\mathrm{MD}_{ijr} + x\_\mathrm{MD}_{ijt}\right), \quad \forall i\in I, \quad \forall j\in J, \tag{102}$$

$$\text{Constraint (86)} \longrightarrow \mathrm{EM\_total\_DC}_{jK} \geq y\_\mathrm{DC}_{jkv_j}\cdot\left(\mathrm{FE\_}D_{v_j}\cdot \mathrm{FC\_}D_{v_j}\cdot\left(\mathrm{dis\_DC}_{jk}\right)\right) - M\left(1-x\_\mathrm{DC}_{jkv_j}\right), \quad \forall k\in K,\ \forall j\in J, \quad \forall v_j\in V_J, \tag{103}$$

$$\text{Constraint (88)} \longrightarrow \mathrm{EM\_total\_CC}_{Kl} \geq y\_\mathrm{CC}_{klv_j}\cdot\left(\mathrm{FE\_}D_{v_j}\cdot \mathrm{FC\_}D_{v_j}\cdot\left(\mathrm{dis\_CC}_{kl}\right)\right) - M\left(1-z\_\mathrm{CC}_{klv_j}\right), \quad \forall k,l\in K, \quad \forall v_j\in V_J, \tag{104}$$

$$\text{Constraint (90)} \longrightarrow \mathrm{EM\_total\_CD}_{Kj} \geq y\_\mathrm{CD}_{kjv_j}\cdot\left(\mathrm{FE\_}D_{v_j}\cdot \mathrm{FC\_}D_{v_j}\cdot\left(\mathrm{dis\_DC}_{jk}\right)\right) - M\left(1-x\_\mathrm{CD}_{kjv_j}\right), \quad \forall k\in K,\ \forall j\in J, \quad \forall v_j\in V_J, \tag{105}$$

$$\text{Constraint (92)} \longrightarrow \mathrm{EM\_total\_DD}_{jm} \geq y\_\mathrm{DD}_{jmv_j}\cdot\left(\mathrm{FE\_}D_{v_j}\cdot \mathrm{FC\_}D_{v_j}\cdot\left(\mathrm{dis\_DD}_{jm}\right)\right) - M\left(1-x\_\mathrm{DD}_{jmv_j}\right), \quad \forall m\in M,\ \forall j\in J, \quad \forall v_j\in V_J, \tag{106}$$

$$\text{Constraint (94)} \longrightarrow \mathrm{EM\_total\_DM}_{mi} \geq y\_\mathrm{DM}_{mir'}\cdot\left(\mathrm{FE\_Di}\cdot \mathrm{FC\_Di}\cdot\left(\mathrm{dis\_DM}_{mi}\right)\cdot 1.05\right) - M\left(1-x\_\mathrm{DM}_{mir'}\right) - M\left(x\_\mathrm{DM}_{mit'}\right), \quad \forall m\in M, \quad \forall i\in I, \tag{107}$$

$$\text{Constraint (96)} \longrightarrow \mathrm{EM\_total\_DM}_{mi} \geq y\_\mathrm{DM}_{mit'}\cdot\left(10^{-3}\cdot\frac{\left(\xi_h\cdot T\cdot \mathrm{FER}\right)}{10^6\left(\sqrt{W\_\mathrm{gr}}\cdot \mathrm{LF\_Di}_{t'}\right)}\right)$$

$$- M\left(1 - x\_\mathrm{DM}_{mit'}\right)$$
$$- M\left(x\_\mathrm{DM}_{mir'}\right),$$
$$\forall m \in M, \quad \forall i \in I, \tag{108}$$

3.2.2. Emission-Constraint Model. This problem extends the basic model by constraining the carbon emissions, which is denoted by EM_Average (in Kg). We note that the constraint for carbon emissions is equal to the average of total carbon emissions for the basic model and emission cost-minimization problem. In this model, the objectives are the same as the basic model ones. Constraints (79), (81), (83), (85), (87), (89), (91), (93), (95), (97), (99)–(108) and the new one are added to the proposed basic closed-loop model to define and limit the carbon emissions issue. In order to formulate this emission-constraint model mathematically, the following notation is necessary.

Parameters

EM_Average: The average carbon emissions of the entire system.

The new constraint added to the emission cost-minimization model is as follows.

The New Constraint

Consider the following:

$$\begin{aligned}&\sum_{i\in I}\sum_{j\in J} e\_u\_\mathrm{MD}_{ij} + \sum_{j\in J}\sum_{k\in K} e\_u\_\mathrm{DC}_{jk}\\&\quad+\sum_{k\in K}\sum_{l\in K} e\_u\_\mathrm{CC}_{kl} + \sum_{k\in K}\sum_{j\in J} e\_u\_\mathrm{CD}_{kj}\\&\quad+\sum_{j\in J}\sum_{m\in M} e\_u\_\mathrm{DD}_{jm}\\&\quad+\sum_{m\in M}\sum_{i\in I} e\_u\_\mathrm{DM}_{mi} \le \mathrm{EM\_Average}.\end{aligned} \tag{109}$$

The carbon emissions constraint is shown by Constraint (109).

4. Numerical Experiments

Here, we propose a numerical example to indicate the effectiveness of the proposed mathematical models. Our models are tested in small scale of data. Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 are the given data. The numbers of potential locations for the manufactory, DC, and dismantler are three, four, and two, respectively. Manufactories, DCs, and dismantlers are selected to secure 57 customers having definite demands. While the applied optimization software is not able to provide solutions for 57 customers in a reasonable time, we categorized the customers into 7 more comprehensive zones with aggregated demands. There are four types of transportation mode (air, rail, road, and water) used to transfer product from manufactories to DCs; one type of transportation modes (road) is used to transfer product from DCs to customers and dismantlers, and two types of transportation modes (rail and road) are used to transfer product from dismantlers to manufactories.

For each of the four transport classes used to transfer product from manufactories to DCs, we select a representative vehicle to which we apply the NTM method.

Air Transport. We select a cargo aircraft whose emission factors are most similar to the average values [28]. For the cargo aircraft we select the maximum load ($\mathrm{LO}_{\max}\_M_a$) to be 29029 kg. We note that the distance over road (D_r) between two locations is always more than the air distance (D_a) and we find the following value $D_a = 0.801\, D_r$ on average in Google Maps [32].

Road Transport. We assumed that a semitrailer is used, because it is a common type to use for longer distance. The road type is supposed to be a motorway. We assume a load factor of 70%, which is typical for transport via integrating terminals [30]. The maximum load ($\mathrm{LO}_{\max}\_M_r$) is 40 tonne.

Rail Transport. It is supposed that the rail network is designed for only diesel trains. All constants below are taken from NTM Rail [29]. We assume that the gross weight (W_gr) of the train is 1000 tonne, which is the average value specified by NTM Rail [29]. The entire track from manufactories to DCs is flat and we find the following value $\xi_f = 1$ in NTM Rail [29]. We assume that the rail distance between two locations is equal to the road distance. For a diesel train we take the following parameter values.

Water Transport. We assume that inland waterways are used for transport and that a general cargo vessel is used. For inland waterways NTM assumes a load factor of 50% [31]. The cargo capacity (maximum load) of a general cargo vessel for inland waterways is 1920 tonne. We assume that the distance between two locations over inland waterways is larger than the distance over road. The distance (D_w) is therefore 1.2 times the road distance (D_r). For a general cargo vessel we take the following parameter values.

For one type of transport modes used to transfer product from DCs to customers and dismantlers, we select two representative vehicles to which we apply the NTM method.

Road Transport. We assumed that two lorries are used: 5-tonne lorry and 40-tonne lorry. The road type is supposed to be a motorway. For two lorries we take the following parameter values.

For each of the two transport classes used to transfer product from dismantlers to manufactories, we select a representative vehicle to which we apply the NTM method.

Road Transport. We assumed that a semitrailer is used, because it is a common type to use for longer distance. The road type is supposed to be a hilly terrain. We assume a load factor of 50%. The maximum load ($\mathrm{LO}_{\max}\_\mathrm{Di}_{r'}$) is 40 tonne. To account for hilly terrain we add 5% [30] to the total emissions.

Rail Transport. It is supposed that the rail network is designed for only diesel trains. All constants below are taken from

Table 1: Emission factors for representative vehicle from manufactories to DCs.

Cargo aircraft		Semitrailer (40 tonne)		Diesel train		Cargo vessel	
Load factor (%) (LF_M_a)	80	Load factor (%) (LF_M_r)	70	Load factor (%) (LF_M_t)	50	Load factor (%) (LF_M_w)	50
CEF (kg)	4139.6	FC (L/km) (FC_M)	0.3198	T	122.46	FEW (kg/tonne)	3178
VEF (kg)	15.353	FE (kg/L) (FE_M)	2.621	FER (g/L)	3175	FCW (tonne/km)	0.007

Table 2: Emission factors for representative vehicle from DCs to customers and dismantlers.

	Five-tonne lorry	40-tonne lorry
Load factor (%) (LF_D_{v_j})	75	90
FC_D_{v_j} (L/km)	0.245	0.369
FE_D_{v_j} (kg/L)	2.63	2.63

NTM Rail [29]. We assume that the gross weight (W_gr) of the train is 1000 tonne, which is the average value specified by NTM Rail [29]. The entire track from dismantlers to manufactories is hilly and we find the following value $\xi_h = 1.25$ in NTM Rail [29]. We assume that the rail distance between two locations is equal to the road distance. For a diesel train we take the following parameter values in Table 3.

There are five types of connection links in the proposed closed-loop chain. Possible connection links are as follows:

(i) connection link between the manufactory and DC,

(ii) connection link between the DC and customer,

(iii) connection link among customers,

(iv) connection link between the DC and dismantler,

(v) connection link between the dismantler and manufactory.

Table 4 shows distances related to the defined connection links and transfer times between DCs and customers and among customers using a variety of vehicles. Maximum and minimum waiting time for customers are set to be 500 and 2500 units of time, respectively. Manufactory, distribution center, customer, and dismantler are involved with the respective numbers (capacity, demand, fixed cost, production cost, and rate) as shown in Table 5 and three rates are assumed to be different with respect to each DC, customer, and dismantler, respectively. Table 6 lists the unit cost of transportation. The recovery costs in DC are given in Table 7 and are assumed to be equal for each DC with respect to each customer. Table 8 shows the vehicle properties. The weight and volume of the product are assumed to be 40 (kg) and 0.5 (m^3), respectively. The maximum (Q) and minimum (L) numbers of customers a salesman must visit are supposed to be 4 and 1, respectively. The fixed cost for landfilling is set to be €2 per unit. Since the establishment of the carbon market price has varied between €1 and €30/tonne, we consider the average cost of carbon emission (i.e., €15/tonne).

So far, we present the required data for processing the results. To facilitate the computations in our mixed integer programming (MIP) models, GAMS 22.9 software package is applied. The General Algebraic Modeling System (GAMS) is a high level modeling system for mathematical programming and optimization and is specifically designed for modeling linear, nonlinear, and mixed integer optimization problems. The system is especially useful with large and complex problems [33]. After solving the first two models (basic and emission cost-minimization models) using this software, we have found that the total carbon emissions for the basic and emission cost-minimization problems are 1618.98 (kg) and 1242.89 (kg), respectively. So, the average of these values as a constraint of carbon emissions (EM_Average) for the third model (emission-constraint model) is calculated to be $(1618.98 + 1242.89)/2 = 1430.93 \cong 1431$ (kg). We have reported the results in Table 9 along with the optimal solution obtained for three cases, for comparison purposes. The validity of model is measured for numerical experiment as seen in Figures 2, 3, and 4, schematically. The results are summarized in Table 9. Table 9 presents objective function of three cases. As expected, the basic case has the best objective function value, that is, the lowest cost. The objective function values in the cost-based case and constraint-based case are little higher than the basic case. Product flow rates and amount of CO_2 (kg) emitted from journeys of selective paths are shown in Table 9. There are five types of connection links in the selective path column.

(i) Links connecting between the manufactory and DC are indicated by a-b: [n] format, where a and b are numbers which indicate selective manufactory and DC, respectively. n is a number which indicates a selective path on the figure. [·] is a symbol related to this kind of connection links.

(ii) Links connecting between the DC and customer are indicated by c-d: (n) format, where c and d are numbers which indicate selective DC and customer, and vice versa. (·) is a symbol related to this kind of connection links.

(iii) Links connecting among the customers are indicated by e-f: {n} format, where e and f are numbers which indicate selective customers, respectively. {·} is a symbol related to this kind of connection links.

(iv) Links connecting between the DC and dismantler are indicated by g-h: ⟨n⟩ format, where g and h are numbers which indicate selective DC and dismantlers, respectively. ⟨·⟩ is a symbol related to this kind of connection links.

Table 3: Emission factors for representative vehicle from dismantlers to manufactories.

Semitrailer (40 tonne)		Diesel train	
Load factor (%) ($LF\_Di_{r'}$)	50	Load factor (%) ($LF\_Di_{t'}$)	50
FC (L/km) (FC_Di)	0.293	T	122.46
FE (kg/L) (FE_Di)	2.621	FER (g/L)	3175

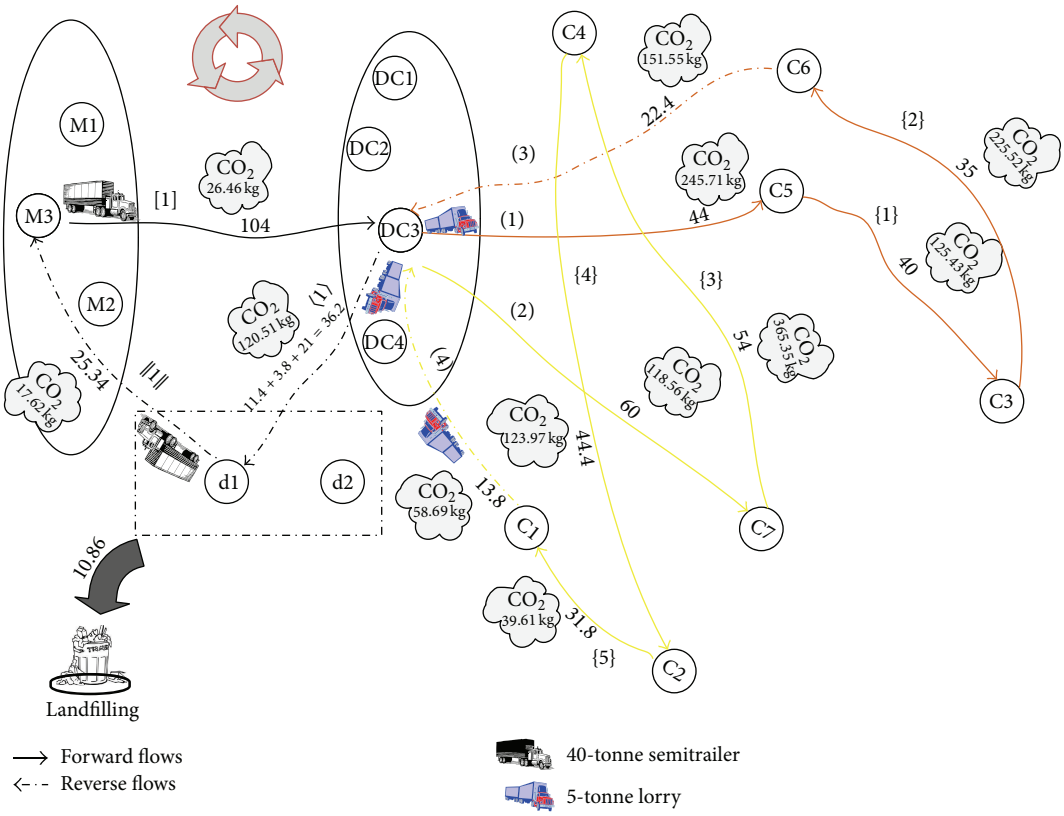


Figure 2: Optimal closed-loop chain of the numerical experiment without considering the carbon emission issue (basic model).

(v) Links connecting between the dismantler and manufactory are indicated by i-j: $\|n\|$ format, where i and j are numbers which indicate selective dismantler and manufactory, respectively. $\|\cdot\|$ is a symbol related to this kind of connection links.

From Table 9, the following is concluded.

(1) For three cases, only one manufactory (number 3) and one DC (number 3) are selected to secure and transport the total sum of customers' demands. Besides, only one dismantler (number 1) is selected to transport the recovered product to the manufactory.

(2) For three cases, one route exits from the manufactory (number 3).

(3) The aggregate value of product flow in exiting paths from a manufactory is equivalent to the total sum of customers' demands. That means that all of customers' demands in the network are met.

(4) The value of product flow in exiting path from a DC is equivalent to the total sum of demands of customers which belongs to the same tour. That means that all of customers' demands in each tour are met.

(5) The value of product flow in exiting path from a customer is equivalent to the total sum of demands of remaining customers which belongs to the same tour plus the amount of recovered product obtained from customer. For example, corresponding to Figure 2, the value of product flow in exiting path from a customer (number 5) is calculated to be $10+14+(0.8\times 20) = 40$ units where customer's (numbers 3, 6, and 5) demands and the percentage of recovery of customer (number 5) are 10, 14, 20, and 80%, respectively.

(6) In the reverse flow, the aggregate value of returned product flow in exiting paths from a DC is equivalent to the total sum of customers' demands of recovered product. For example, corresponding to Figure 2, this value is calculated to be $11.4 + 3.8 + 21 = 36.2$.

Table 4: Distance and transfer times.

DC	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry	Distance	Transfer times using 5-tonne lorry	Transfer times using 40-tonne lorry
											Customers										
		1			2			3			4			5			6			7	
1	85	500	800	149	380	430	75	780	1020	123	270	560	174	660	900	100	984	1329	250	654	780
2	328	300	450	187	280	403	98	857	1220	257	370	460	400	720	950	278	284	629	189	354	870
3	198	1500	1850	447	850	1430	280	780	1470	378	670	1060	260	760	987	315	1084	1629	92	947	1280
4	270	200	705	360	480	830	200	1180	1780	420	770	960	338	1050	1800	173	884	1330	290	554	856
											Customer-customer										
1	0	0	0	58	180	203	128	65	120	117	37	60	68	72	95	150	28	75	249	35	87
2	58	180	203	0	0	0	123	78	147	130	67	106	88	76	98	125	108	162	270	94	128
3	128	65	120	123	78	147	0	0	0	317	77	96	146	105	180	300	88	133	210	55	85
4	117	37	60	130	67	106	317	77	96	0	0	0	190	66	90	250	98	132	315	65	78
5	68	72	95	88	76	98	146	105	180	190	66	90	0	0	0	190	67	97	320	57	110
6	150	28	75	125	108	162	300	88	133	250	98	132	190	67	97	0	0	0	279	85	111
7	249	35	87	270	94	128	210	55	85	315	65	78	320	57	110	279	85	111	0	0	0
DC					Manufactories							Dismantlers						Manufactories-dismantlers			
		1			2			3			1			2			1			2	
1	75	—	—	38	—	—	54	—	—	50	—	—	88	—	—	168	—	—	90	—	—
2	50	—	—	46	—	—	36	—	—	110	—	—	90	—	—	200	—	—	178	—	—
3	24	—	—	25	—	—	68	—	—	155	—	—	182	—	—	138	—	—	119	—	—
4	17	—	—	63	—	—	96	—	—	122	—	—	80	—	—						

Table 5: Capacity, demand, fixed cost, production cost, and rate.

Manufactory			DC			Customer		Dismantler		
Capacity (Cm)	Fixed cost (€) (FM)	Pro. cost (€) (P_cost)	Total capacity (Tc)	Fixed cost (€) (FDC)	Pc (%)	Demand (dc)	pr (%)	Capacity (Cd)	Fixed cost (€) (FD)	Pl (%)
1000000	200000	326	3000	80000	40	20	10	1600	20000	30
1000000	180000	400	5000	50000	20	18	30	2400	25000	38
1000000	150000	300	1500	23000	50	10	50			
			2000	30000	50	12	20			
						20	80			
						14	10			
						10	40			

Table 6: Unit cost (€) of transportation per km.

	DC			
	Cargo aircraft	Semitrailer	Diesel train	Cargo vessel
Manufactory	0.25	0.16	0.2	0.3
	Customer and dismantler			
	Five-tonne lorry		40-tonne lorry	
DC	0.13		0.18	
	Manufactory			
	Semitrailer		Diesel train	
Dismantler	0.15		0.25	

Table 7: The recovery cost (€) in DC from customer.

Rc	1	2	3	4
1	1.5	2.3	1.8	0.8
2	1.5	2.3	1.8	0.8
3	1.5	2.3	1.8	0.8
4	1.5	2.3	1.8	0.8
5	1.5	2.3	1.8	0.8
6	1.5	2.3	1.8	0.8
7	1.5	2.3	1.8	0.8

Table 8: The vehicles' properties.

Vehicle	Number of vehicles (unit) Manufactory (NVM)				Maximum load (kg) (LO_{max}_M)	Capacity (kg) (CVM)	Density of product (ρ)
	1	2	3				
Cargo aircraft	2	1	2		29029	29029	167
Semitrailer	5	3	6		40000	40000	250
Diesel train	1	0	1		1000000	1000000	—
Cargo vessel	3	2	2		1920000	1920000	—
	DC (NVD)				(LO_{max}_D)	(CVD)	
	1	2	3	4			
Five-tonne lorry	5	3	10	4	5000	5000	250
40-tonne lorry	3	6	6	1	40000	40000	250
	Dismantler (NVDi)				(LO_{max}_Di)	(CVDi)	
	1		2				
Semitrailer	4		3		40000	40000	250
Cargo vessel	1		2		1000000	1000000	—

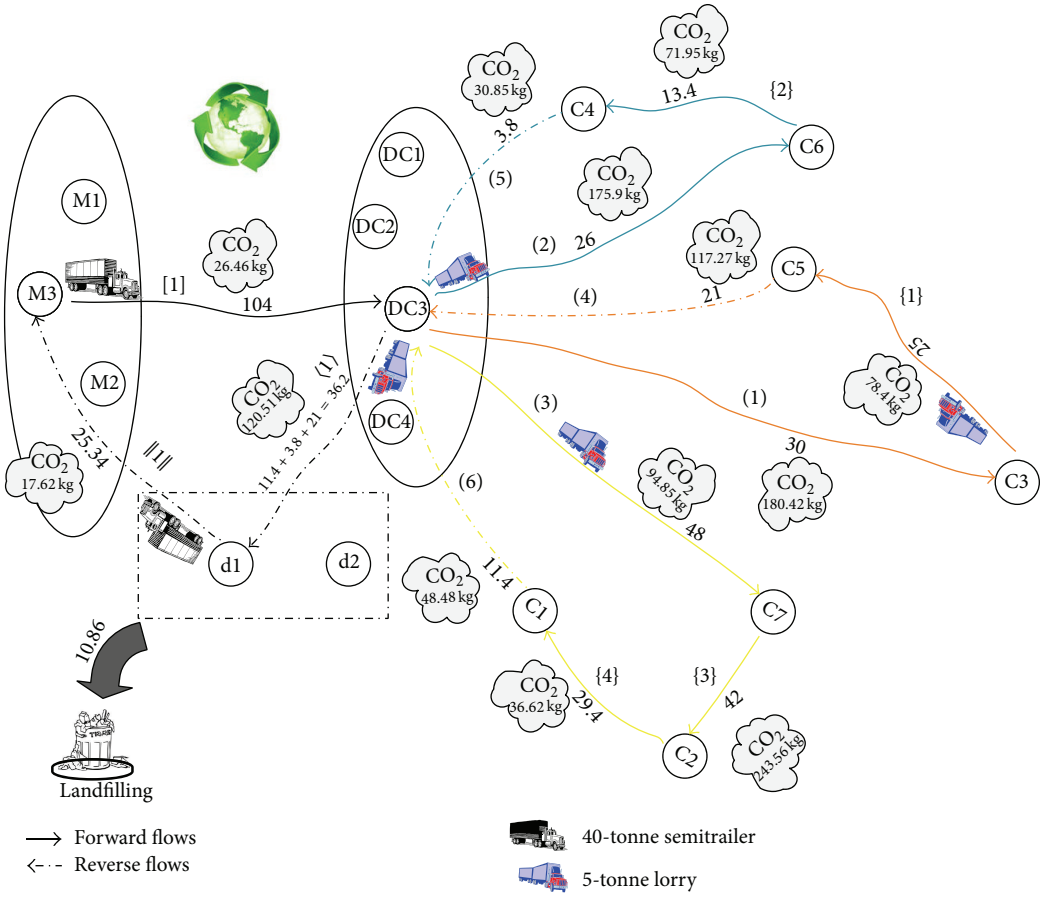


Figure 3: Optimal closed-loop chain of the numerical experiment with considering the carbon emission issue (emission cost-minimization model).

For basic, cost-based, and constraint-based cases, an optimal closed-loop chain is shown in Figures 2, 3, and 4, respectively. In these figures, we consider a particular color for each tour in which a salesman departs from selective DCs and arrives to the customers. So, the selective path given in Table 9 is indicated by different colors. The objective functions for basic, cost-based, and constraint-based cases are 230023.6, 233765.9, and 231159 units in 1886, 1380, and 4610 seconds, respectively. Note that this computation time is needed to be spent for solving a problem with three, four, and two potential locations for the manufactory, DC, and dismantler and seven customers. The suitable paths to deliver product to customers from manufactories and DCs in the forward flows, to deliver recovered product to dismantlers from DCs and customers, and to deliver reused product to manufactories from dismantlers in the reverse flows for basic, emission cost-minimization, and emission-constraint models are shown in Figures 2, 3, and 4. In addition, the selected vehicles for carrying product and the corresponding amount of product are illustrated in them which also include the amount of CO_2 (kg) emitted from journeys and the amount of landfills. The arrival time of product at each customer for three models is reported in Table 10. The traffic light that turned green in all shows that the time window of each customer is satisfied and the arrival time of product is within the allowed range ([500–2500]). The objective values of the three cases (i.e., basic, cost-based, and constraint-based cases) are very close to each other. Therefore to verify the differences, we work out a sensitivity analysis given in Section 5.

5. Comparison of the Three Closed-Loop Supply Chain Models

Here, we investigate the variation of the total costs and CO_2 emissions in different product's weights obtained from all three cases. First, we consider the total cost obtained from basic, cost-based, and constraint-based approaches. We depict the total costs for different product's weights in Figure 5. Since the feasible region of the basic model is larger than or equal to the feasible regions of the cost-based and constraint-based models, the optimal value of the former is no worse than the optimal value of the latter. This implies that the optimal value to the basic model is a lower bound on the optimal value for the problems where we integrated the environmental issues into a traditional logistic system. Thus, the minimum objective value of the solution in the feasible region of the basic model will be less than or equal to the two

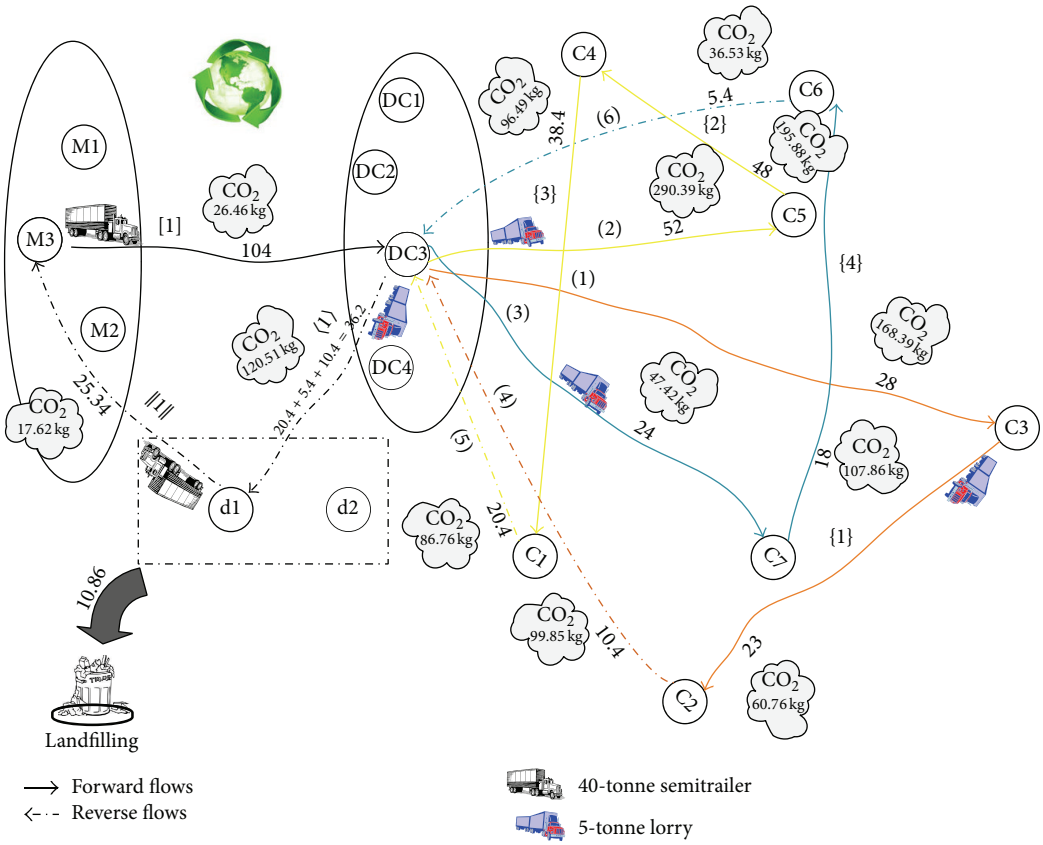
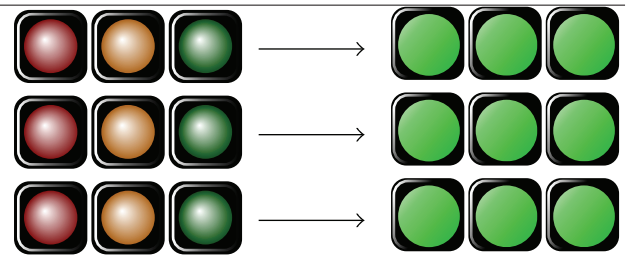


Figure 4: Optimal closed-loop chain of the numerical experiment with considering the carbon emission issue (emission-constraint model).

Table 9: Results for three cases.

Basic model			Emission cost minimization			Emission-constraint model		
Selective path	Amount of product flow	Amount of CO_2 (kg)	Selective path	Amount of product flow	Amount of CO_2 (kg)	Selective path	Amount of product flow	Amount of CO_2 (kg)
3-3: [1]	104	26.46	3-3: [1]	104	26.46	3-3: [1]	104	26.46
3-5: (1)	44	245.71	3-3: (1)	30	180.42	3-3: (1)	28	168.39
3-7: (2)	60	118.56	3-6: (2)	26	175.9	3-5: (2)	52	290.39
5-3: {1}	40	125.43	3-7: (3)	48	94.85	3-7: (3)	24	47.42
3-6: {2}	35	225.52	3-5: {1}	25	78.4	3-2: {1}	23	60.76
7-4: {3}	54	365.35	6-4: {2}	13.4	71.95	5-4: {2}	48	195.88
4-2: {4}	44.4	123.97	7-2: {3}	42	243.56	4-1: {3}	38.4	96.49
2-1: {5}	31.8	39.61	2-1: {4}	29.4	36.62	7-6: {4}	18	107.86
6-3: (3)	22.4	151.55	5-3: (4)	21	117.27	2-3: (4)	10.4	99.85
1-3: (4)	13.8	58.69	4-3: (5)	3.8	30.85	1-3: (5)	20.4	86.76
3-1: ⟨1⟩	36.2	120.51	1-3: (6)	11.4	48.48	6-3: (6)	5.4	36.53
1-3: ‖1‖	25.34	17.62	3-1: ⟨1⟩	36.2	120.51	3-1: ⟨1⟩	36.2	120.51
			1-3: ‖1‖	25.34	17.62	1-3: ‖1‖	25.34	17.62
Landfill	10.86	—	Landfill	10.86	—	Landfill	10.86	—
	Objective			Objective			Objective	
	230023.6			233765.9			231159	

Table 10: The time windows.

Time windows (S_k)	Customer 1	2	3	4	5	6	7	Status
Basic model	1259	1079	865	1012	760	953	947	
Emission cost model	1221	1041	780	1182	885	1084	947	
Emission-constraint model	863	858	780	826	760	1032	947	

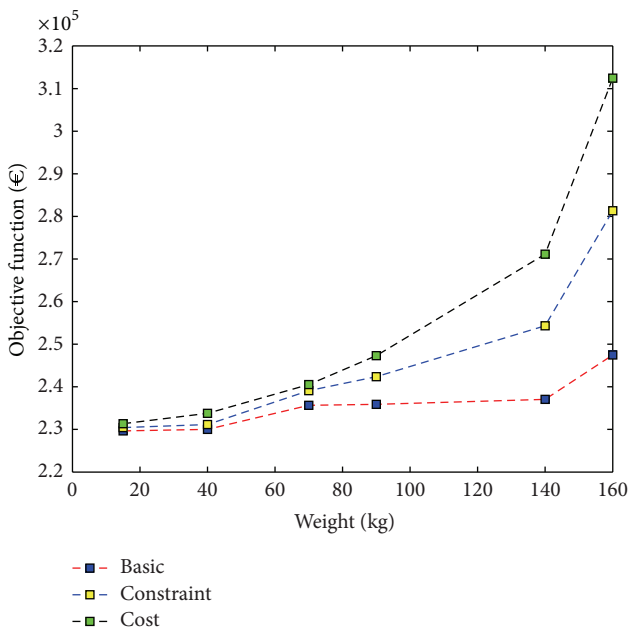


Figure 5: The objective value for the basic, cost-based, and constraint-based models.

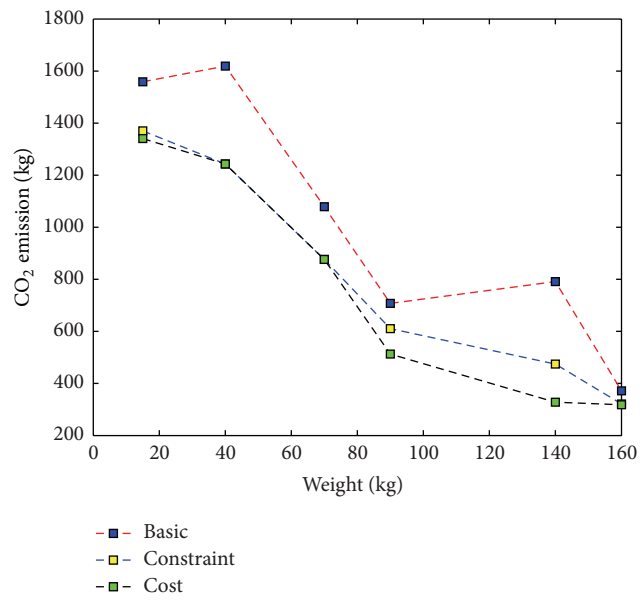


Figure 6: The CO_2 emissions for the basic, cost-based, and constraint-based models.

others. Moreover, increasing the weight of the product leads to high total costs due to the vehicle capacity constraints. Here, we survey the CO_2 emissions obtained as output of the proposed mathematical model.

We see in Figure 6 that the basic model has the highest carbon footprint among the three scenarios due to ignorance of this issue in its assumptions. Thus, the optimal values for emissions of the cost-based and constraint-based models which both consider the carbon footprint are no worse than the value of the basic model. This implies that the emission value to the basic model is an upper bound on the optimal value for the two others. So, the emissions value of the basic model solution will be more than the two others. With the increase of product's weight, the carbon emission is decreasing. This result confirms the finding in [8] that the following ordering for unit emissions is

$$e\_u_{\text{air}} > e\_u_{\text{road}} > e\_u_{\text{rail}} > e\_u_{\text{water}}. \tag{110}$$

Due to the vehicle capacity constraints, the larger vehicles are responsible to carry one commodity from a location to another. For the two cases which consider carbon footprint value in their calculations, the models are more likely to choose larger vehicles such as train and cargo vessel. This is not necessarily because of their capacity to carry heavy loads or time element, since the air transport excels at doing so, but because of the emission element. So, as we can see in Figures 5 and 6, with the increase of product's weight, the use of these vehicles is likely and leads to the increase in costs and the decrease in emission intensity.

Although the basic model is less expensive than other models, it is not a complete model because it does not attempt to consider the carbon emission issue. The upper bound of emission intensity to calculate an emission restriction for constraint-based model is just as it used to be. Then, we will just focus on the other two models. Clearly, with respect to the definition of EM_Average for the third problem, this method is between the other two methods in carbon emissions. These

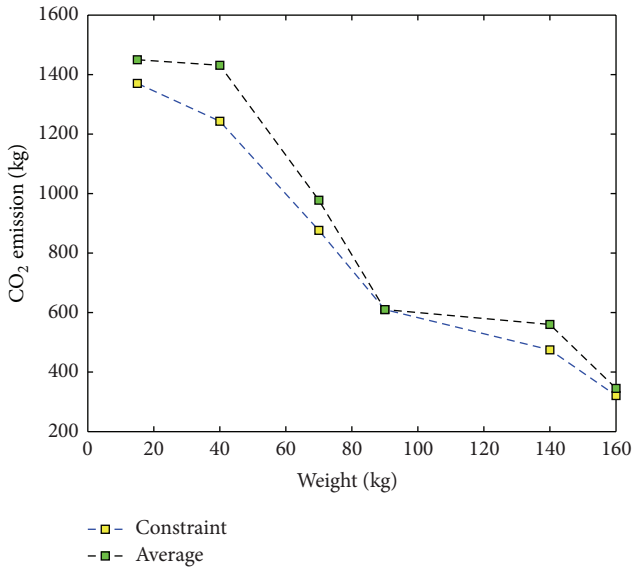


Figure 7: The comparison between the emission decision variable and EM_Average for the third problem.

quantities in constraint-based and cost-based methods are closer than basic method. Because the cost-based approach necessitates a model to adopt environmentally friendly vehicles such as train and cargo vessel, you can expect the total cost estimate to be the highest value. In contrast, the constraint-based approach has more freedom to choose the vehicles that are likely to be cheaper and not greener and safer for the environment. Consequently, the total cost is reduced by 4.6% and the CO_2 emission is increased by 5.9%. Changes in emission intensities obtained from solving the third model over product's weight are compared with the certain amount of EM_Average and shown in Figure 7.

Thus, we find that the performance of constraint-based model is better than cost-based approach considering the total costs and CO_2 emission decision variables.

6. Conclusions

In this work, we presented an extended closed-loop supply chain network to integrate the environmental issues into a traditional logistic system. Our proposed chain contained four layers (manufacturers, DCs, customers, and dismantlers). Finding optimal locations of manufacturers, DCs, and dismantlers and distribution of product satisfying time windows were our purposes that are attained in a mixed integer linear programming approach. In this way, we proposed an approach as multiple DCs multiple traveling salesman problem (MDMTSP) between DCs and customers. In addition to managing properly reverse logistics to reduce negative impact of greenhouse gases emissions, we focused on transport mode selection as a way to reduce emissions. For this, two types of regulations to reduce carbon emissions coming from freight transport were considered. The first mechanism specified a cost for carbon emissions and the second one was a constraint on emissions. Consequently, three models were formulated corresponding to these regulations and the effects of the regulations on the preferred transport mode and the emissions were investigated.

The applicability and effectiveness of our proposed model were tested through numerical example. Also, comparative analysis was investigated on decision variables.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Proposed Method for Solving Fuzzy System of Linear Equations

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This paper proposes a new method for solving fuzzy system of linear equations with crisp coefficients matrix and fuzzy or interval right hand side. Some conditions for the existence of a fuzzy or interval solution of $m \times n$ linear system are derived and also a practical algorithm is introduced in detail. The method is based on linear programming problem. Finally the applicability of the proposed method is illustrated by some numerical examples.

1. Introduction

Systems of simulations linear equations play a major role in various areas such as mathematics, physics, statistics, engineering, and social sciences. Since in many applications at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them [1]. A general model for solving an $m \times n$ fuzzy system of linear equation (FSLE) whose coefficients' matrix is crisp and right hand side column is an arbitrary fuzzy number vector was first proposed by Friedman et al. [1]. Different authors [2–5] have investigated numerical methods for solving such FSLE. Most of mentioned methods in different articles are based on numerical methods such as matrices decomposition and iterative solutions. Previously mentioned papers do not discuss a lot the possibility of solutions. In addition they cannot find alternative solutions. But the proposed method does not include these defects. Allahviranloo et al. [6] have presented that the above-mentioned method is not applicable and does not have solution generally. This paper sets out to investigate the solution of the fuzzy linear system using a linear programming method. Also we are going to explain the necessary and sufficient conditions for existence of the solutions. The idea of this method can join some uses of linear programming to solve the problems of interval data in [7–9]. The structure of this paper is organized as follows.

In Section 2, we provide some basic definitions and results which will be used later.

In Section 3, we prove some theorems which are used for proposed method and present a practical procedure. The introduced method is illustrated by solving some examples in Section 4 and conclusions are drawn in Section 5.

2. Preliminaries

In this section some basic definitions and concepts are brought.

Definition 1 (by [10]). The triangular fuzzy numbers (TFN) are very popular and are denoted by $\tilde{u} = (\alpha, c, \beta)$ and defined by

$$\tilde{u} = \begin{cases} \dfrac{x-\alpha}{c-\alpha}, & \alpha \le x \le c, \\ \dfrac{\beta - x}{\beta - c}, & c \le x \le \beta, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Note. If $\alpha = c$, then TFN is defined:

$$\tilde{u} = \begin{cases} \dfrac{\beta - x}{\beta - \alpha}, & \alpha \le x \le \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

if $\beta = c$, then TFN is defined:

$$\tilde{u} = \begin{cases} \dfrac{x - \alpha}{\beta - \alpha}, & \alpha \le x \le \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

and finally if $\alpha = c = \beta$, then TFN is defined:

$$\tilde{u} = \begin{cases} 1, & x = \beta, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Definition 2 (by [10]). Let $\tilde{u} = (\alpha, c, \beta)$ be a triangular fuzzy number; then one defines

$$\operatorname{supp}(\tilde{u}) = [\alpha, \beta], \qquad \operatorname{core}(\tilde{u}) = c. \tag{5}$$

Lemma 3 (by [10]). *For arbitrary interval* $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$ *the following properties hold:*

(i) $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$,

(ii) $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$,

(iii) *for each* $k \in R$,

$$k[\underline{x}, \overline{x}] = \begin{cases} [k\underline{x}, k\overline{x}], & \text{if } k \ge 0, \\ [k\overline{x}, k\underline{x}], & \text{if } k < 0. \end{cases} \tag{6}$$

Proof. See [10]. □

Definition 4 (by [5]). Let $F(R)$ be a set of all fuzzy numbers on r and $I(R)$ a set of intervals on R. The $m \times n$ linear system of equations is as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m, \end{aligned} \tag{7}$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i \le m$, $1 \le j \le n$, is a crisp $m \times n$ matrix and $y_i \in F(R) \bigcup I(R)$, $1 \le i \le m$, is called a FSLE.

Theorem 5. *If* $\tilde{u}$ *and* $\tilde{v}$ *are triangular fuzzy numbers and if* $t \in R$, *then*

(i) $\tilde{u} = \tilde{v}$ *if and only if* $\operatorname{supp}(\tilde{u}) = \operatorname{supp}(\tilde{v})$ *and* $\operatorname{core}(\tilde{u}) = \operatorname{core}(\tilde{v})$,

(ii)

$$\operatorname{supp}(t\tilde{u} + \tilde{v}) = t \operatorname{supp}(\tilde{u}) + \operatorname{supp}(\tilde{v}), \tag{8}$$

(iii)

$$\operatorname{core}(t\tilde{u} + \tilde{v}) = t \operatorname{core}(\tilde{u}) + \operatorname{core}(\tilde{v}). \tag{9}$$

Proof. See [10]. □

Theorem 6. *Let* A *be* $m \times n$ *matrix and* b *an* m*-vector. The system* $Ax = b$ *with condition* $m \le x \le M$ *has a solution if and only if the optimal solution of the below system is zero:*

$$\begin{aligned} \text{Min} \quad & z = 1x_a \\ \text{s.t.} \quad & Ax + x_a = b, \\ & m \le x \le M, \\ & x_a \ge 0. \end{aligned} \tag{10}$$

Proof. See [11]. □

3. Proposed Method

First of all the following definitions and theorems are introduced.

Definition 7. Let $I = [\underline{x}, \overline{x}]$; it can be written as $I = [-L_I, L_I] + [M_I, M_I]$ such that

$$L_I = \frac{\overline{x} - \underline{x}}{2}, \qquad M_I = \frac{\overline{x} + \underline{x}}{2}. \tag{11}$$

Now one can have the following theorems.

Theorem 8. *Let* $I = [\underline{x}, \overline{x}]$ *and* $J = [\underline{y}, \overline{y}]$ *be intervals and* $k \in R$; *then*

(i) $I = J$ *if and only if* $L_I = L_J$ *and* $M_I = M_J$,

(ii) *if* $I + J = [\underline{x}, \overline{x}] + [\underline{y}, \overline{y}]$, *then* $L_{I+J} = L_I + L_J$ *and* $M_{I+J} = M_I + M_J$,

(iii) *let* $S = kI$ *such that* $k \in \mathbf{R}$; *then* $L_S = |k| L_I$ *and* $M_S = kM_I$.

Proof. Proofs of parts (i) and (ii) are obvious so we prove part (iii). Let $k < 0$ such that

$$S = k[\underline{x}, \overline{x}] = [k\overline{x}, k\underline{x}] = ([-L_S, L_S] + [M_S, M_S]), \tag{12}$$

where

$$\begin{aligned} L_S &= \frac{k\underline{x} - k\overline{x}}{2} = -k\frac{\overline{x} - \underline{x}}{2} = |k| L_I, \\ M_S &= \frac{k\underline{x} + k\overline{x}}{2} = k\frac{\overline{x} + \underline{x}}{2} = kM_I \end{aligned} \tag{13}$$

for $k \ge 0$ the proof is the same. □

Theorem 9. *If* A *is a* $m \times n$ *matrix and* X *is an interval vector, then*

$$AX = [-A^+ L_X, A^+ L_X] + [AM_X, AM_X], \tag{14}$$

where $A_{ij}^+ = |A_{ij}|$ *and* $L_{X_j} = (\overline{X_j} - \underline{X_j})/2$ *and* $M_{X_j} = (\overline{X_j} + \underline{X_j})/2$.

Proof. According to Theorem 8, for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} A^i X &= \sum_{j=1}^{j=n} A_{ij} \left[\underline{X_j}, \overline{X_j}\right] \\ &= \sum_{j=1}^{j=n} A_{ij} \left(\left[-L_{X_j}, L_{X_j}\right] + \left[M_{X_j}, M_{X_j}\right]\right) \\ &= \sum_{j=1}^{j=n} \left|A_{ij}\right| \left[-L_{X_j}, L_{X_j}\right] + A_{ij} \left[M_{X_j}, M_{X_j}\right] \\ &= \sum_{j=1}^{j=n} \left|A_{ij}\right| \left[-L_{X_j}, L_{X_j}\right] + \sum_{j=1}^{j=n} A_{ij} \left[M_{X_j}, M_{X_j}\right] \\ &= \left[-A^{+^i} L_X, A^{+^i} L_X\right] + \left[A^i M_X, A^i M_X\right]. \end{aligned} \tag{15}$$

□

Theorem 10. *If A is a $m \times n$ matrix and X, b are two interval vectors, then the system $AX = b$ has solution(s) if and only if the following systems have solution:*

$$\begin{aligned} A^+ L_X = L_b, \quad L_X \geq 0, \\ AM_X = M_b. \end{aligned} \tag{16}$$

Proof. Let X be an interval solution of $AX = b$; then according to Definition 7 $X = [-L_X, L_X] + [M_X, M_X]$ with $L_X \geq 0$ and according to Theorem 9, we have

$$AX = \left[-A^+ L_X, A^+ L_X\right] + \left[AM_X, AM_X\right] \tag{17}$$

but $AX = b$ and by using part (i) of Theorem 8 this means

$$\begin{aligned} A^+ L_X = L_b, \quad L_X \geq 0, \\ AM_X = M_b; \end{aligned} \tag{18}$$

the converse holds obviously. □

Theorem 11. *The system $A^+ L_X = L_b$ with condition $L_X \geq 0$ has a solution if and only if optimized value of the below linear programing is zero:*

$$\begin{aligned} &\text{Min} \quad z = 1\mathbf{x}_a \\ &\text{s.t.} \quad A^+ L_X + \mathbf{x}_a = L_b, \\ &\qquad Ł_X, \mathbf{x}_a \geq 0. \end{aligned} \tag{19}$$

Proof. This is proved by using Theorem 6. □

Now we are going to apply the same method for solving $A\widetilde{X} = \widetilde{B}$, where $\widetilde{B_i}$ is TFN.

Theorem 12. *If A is a $m \times n$ matrix with crisp coefficients and $\widetilde{B}$ is a TFN vector the same as $\widetilde{B_i} = (\alpha_i, c_i, \beta_i)$, then the system $A\widetilde{X} = \widetilde{B}$ has TFN solution(s), $\widetilde{X}$, the same as $\widetilde{X_i} = (\underline{x_i}, x_i, \overline{x_i})$ if and only if the systems* (20) *have solution:*

$$\begin{aligned} A\left(S_{\widetilde{X}}\right) &= S_{\widetilde{B}}, \\ A\left(C_{\widetilde{X}}\right) &= C_{\widetilde{B}}, \\ C_{\widetilde{X_j}} &\in \operatorname{supp}\left(\widetilde{X_j}\right) \quad j = 1, 2, \ldots, n, \end{aligned} \tag{20}$$

where

$$\begin{aligned} S_{\widetilde{B_i}} &= \operatorname{supp}\left(\widetilde{B_i}\right) = \left[\alpha_i, \beta_i\right], \quad i = 1, \ldots, m, \\ S_{\widetilde{X_j}} &= \operatorname{supp}\left(\widetilde{X_j}\right) = \left[\overline{x_j}, \underline{x_j}\right], \quad j = 1, \ldots, n, \\ C_{\widetilde{B_i}} &= \operatorname{core}\left(\widetilde{B_i}\right) = c_i, \quad i = 1, \ldots, m, \\ C_{\widetilde{X_j}} &= \operatorname{core}\left(\widetilde{X_j}\right) = x_j, \quad j = 1, \ldots, n. \end{aligned} \tag{21}$$

Proof. From part (i) of Theorem 5 $\widetilde{X}$ is a TFN solution of system $A\widetilde{X} = \widetilde{B}$ if and only if

$$\begin{aligned} \operatorname{supp}\left(A^i \widetilde{X}\right) &= \operatorname{supp}\left(\widetilde{B_i}\right), \quad i = 1, 2, \ldots, m, \\ \operatorname{core}\left(A^i \widetilde{X}\right) &= \operatorname{core}\left(\widetilde{B_i}\right), \quad i = 1, 2, \ldots, m. \end{aligned} \tag{22}$$

So according to part (ii) of Theorem 5 for $i = 1, 2, \ldots, m$ we have

$$\begin{aligned} S_{\widetilde{B_i}} &= \operatorname{supp}\left(\widetilde{B_i}\right) \\ &= \operatorname{supp}\left(A^i \widetilde{X}\right) \\ &= \operatorname{supp}\left(\sum_{j=1}^{j=n} A_{ij} \widetilde{X_j}\right) \\ &= \sum_{j=1}^{j=n} A_{ij} \operatorname{supp}\left(\widetilde{X_j}\right) \\ &= A^i S_{\widetilde{X}}. \end{aligned} \tag{23}$$

Also according to part (iii) of Theorem 5 for $i = 1, 2, \ldots, m$ we have

$$\begin{aligned} C_{\widetilde{B_i}} = C\left(\widetilde{B_i}\right) = C\left(A^i \widetilde{X}\right) &= C\left(\sum_{j=1}^{j=n} A_{ij} \widetilde{X_j}\right) \\ &= \sum_{j=1}^{j=n} A_{ij} C\left(\widetilde{X_j}\right) = A^i C_{\widetilde{X}} \end{aligned} \tag{24}$$

but $C_{\widetilde{X_j}} \in \operatorname{supp}(\widetilde{X_j})$, $j = 1, 2, \ldots, n$. □

Theorem 13. *The system $AC_{\widetilde{X}} = C_{\widetilde{B}}$ with condition $C_{\widetilde{X_i}} \in \operatorname{supp}(\widetilde{X_i})$ has a solution if and only if optimal solution of the following linear programing is zero:*

$$\begin{aligned} &\text{Min} \quad 1x_a \\ &\text{s.t.} \quad AC_{\widetilde{X}} + x_a = C_{\widetilde{B}}, \\ &\qquad C_{\widetilde{X_i}} \in \operatorname{supp}\left(\widetilde{X_i}\right), \\ &\qquad x_a \geq 0. \end{aligned} \tag{25}$$

Proof. This is proved by using Theorem 6. □

4. Numerical Example

Here we describe the proposed method completely and step by step by two examples. In the first example, system is

introduced in which matrix A is a definite (crisp) matrix and B is a vector of triangular fuzzy numbers and a solution is then calculated for it. In the second example, an interval system without solution is outlined.

Example 1. Consider the following 4×6 fuzzy system in which $\widetilde{X}$ is a triangular fuzzy vector:

$$\begin{array}{rrrrrrcl} -2\widetilde{X_1} & +4\widetilde{X_2} & -3\widetilde{X_3} & -5\widetilde{X_4} & & -2\widetilde{X_6} & = & (-1,0,1) \\ & +4\widetilde{X_2} & -6\widetilde{X_3} & -2\widetilde{X_4} & +6\widetilde{X_5} & -4\widetilde{X_6} & = & (4,5,8) \\ 5\widetilde{X_1} & & -2\widetilde{X_3} & -3\widetilde{X_4} & +8\widetilde{X_5} & +3\widetilde{X_6} & = & (-5,-3,-1) \\ -9\widetilde{X_1} & +1\widetilde{X_2} & +8\widetilde{X_3} & -8\widetilde{X_4} & +4\widetilde{X_5} & +7\widetilde{X_6} & = & (0,2,5)\,. \end{array} \tag{26}$$

To solve this system, we proceed in two successive stages according to Theorem 12.

Stage 1. Find $\mathrm{Supp}(\widetilde{X})$, where $\mathrm{Supp}(\widetilde{X_j})$ is the interval $[\underline{x_j}, \overline{x_j}]$. Therefore, the following system must be solved:

$$\begin{bmatrix} -2 & 4 & -3 & -5 & 0 & -2 \\ 0 & 4 & -6 & -2 & 6 & -4 \\ 5 & 0 & -2 & -3 & 8 & 3 \\ -9 & 1 & 8 & -8 & 4 & 7 \end{bmatrix} \begin{bmatrix} [\underline{x_1}, \overline{x_1}] \\ [\underline{x_1}, \overline{x_1}] \\ [\underline{x_1}, \overline{x_1}] \\ [\underline{x_1}, \overline{x_1}] \\ [\underline{x_1}, \overline{x_1}] \\ [\underline{x_1}, \overline{x_1}] \end{bmatrix} = \begin{bmatrix} [-1,1] \\ [4,8] \\ [-5,-1] \\ [0,5] \end{bmatrix}. \tag{27}$$

Stage 2. After calculating the intervals $[\underline{x_j}, \overline{x_j}]$ in the first stage, search for $\mathrm{Core}(\widetilde{X_j}) = x_j$ that satisfies $\underline{x_j} \le x_j \le \overline{x_j}$. Therefore, the following system must be solved:

$$\begin{bmatrix} -2 & 4 & -3 & -5 & 0 & -2 \\ 0 & 4 & -6 & -2 & 6 & -4 \\ 5 & 0 & -2 & -3 & 8 & 3 \\ -9 & 1 & 8 & -8 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \tag{28}$$

$$\underline{x_j} \le x_j \le \overline{x_j}, \quad j = 1, 2, \ldots, 6.$$

Here, the system of the first stage (i.e., system (27)) is solved. It is an interval system, so it must be solved in two substages according to Theorem 10. The *first substage* is finding the interval length, that is, L_X. Thus, the following system is solved:

$$\begin{bmatrix} 2 & 4 & 3 & 5 & 0 & 2 \\ 0 & 4 & 6 & 2 & 6 & 4 \\ 5 & 0 & 2 & 3 & 8 & 3 \\ 9 & 1 & 8 & 8 & 4 & 7 \end{bmatrix} \begin{bmatrix} L_{\widetilde{X_1}} \\ L_{\widetilde{X_2}} \\ L_{\widetilde{X_3}} \\ L_{\widetilde{X_4}} \\ L_{\widetilde{X_5}} \\ L_{\widetilde{X_6}} \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 2.5 \end{bmatrix} \tag{29}$$

$$0 \le L_{\widetilde{X_j}} \quad j = 1, 2, \ldots, 6.$$

Here right hand side is $L_{\mathrm{Supp}(\widetilde{B})}$ and $L_{\widetilde{X_j}} = ((\overline{x_j} - \underline{x_j})/2)$. But according to Theorem 11, the presence of solution (29) is equivalent to the following LP problem:

$$\begin{aligned} Z^* = \min \quad & x_{a_1} + x_{a_2} + x_{a_3} + x_{a_4} \\ \text{s.t.} \quad & \begin{bmatrix} 2 & 4 & 3 & 5 & 0 & 2 \\ 0 & 4 & 6 & 2 & 6 & 4 \\ 5 & 0 & 2 & 3 & 8 & 3 \\ 9 & 1 & 8 & 8 & 4 & 7 \end{bmatrix} \begin{bmatrix} L_{\widetilde{X_1}} \\ L_{\widetilde{X_2}} \\ L_{\widetilde{X_3}} \\ L_{\widetilde{X_4}} \\ L_{\widetilde{X_5}} \\ L_{\widetilde{X_6}} \end{bmatrix} \\ & + \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ x_{a_3} \\ x_{a_4} \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 2.5 \end{bmatrix}, \\ & 0 \le L_{\widetilde{X_j}} \quad j = 1, 2, \ldots, 6, \\ & 0 \le x_{\widetilde{a_i}} \quad i = 1, 2, \ldots, 4. \end{aligned} \tag{30}$$

We solve it with the *Simplex* method [11]. Since the optimal value (Z^*) is zero, system (29) has the following solution:

$$L_{\widetilde{X}} = [0.03315,\ 0.06970,\ 0.05475,\ 0.0714,\ 0.16375,\ 0.0668]^t. \tag{31}$$

The *second substage* is to find the center of $[\underline{x_j}, \overline{x_j}]$. So, the solution of the following system is calculated by common methods in linear algebra:

$$\begin{bmatrix} -2 & 4 & -3 & -5 & 0 & -2 \\ 0 & 4 & -6 & -2 & 6 & -4 \\ 5 & 0 & -2 & -3 & 8 & 3 \\ -9 & 1 & 8 & -8 & 4 & 7 \end{bmatrix} \begin{bmatrix} M_{\widetilde{X_1}} \\ M_{\widetilde{X_2}} \\ M_{\widetilde{X_3}} \\ M_{\widetilde{X_4}} \\ M_{\widetilde{X_5}} \\ M_{\widetilde{X_6}} \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -3 \\ 2.5 \end{bmatrix}. \tag{32}$$

Here right hand side is $M_{\mathrm{Supp}(\widetilde{B})}$ and $M_{\widetilde{X_j}} = (\overline{x_j} + \underline{x_j})/2$. Therefore, the solution of the center of $[\underline{x_j}, \overline{x_j}]$ is as follows:

$$M_X = \left[-1.37315,\ 0,\ -0.63725,\ 0.9316,\ 0.67325,\ 0\right]^t. \tag{33}$$

Finally, a solution for $\mathrm{Supp}(\widetilde{X})$ is as follows according to Theorem 10 and solutions (32) and (33):

$$\mathrm{Supp}\left(\widetilde{X}\right) = \begin{bmatrix} S_{\widetilde{X_1}} \\ S_{\widetilde{X_2}} \\ S_{\widetilde{X_3}} \\ S_{\widetilde{X_4}} \\ S_{\widetilde{X_5}} \\ S_{\widetilde{X_6}} \end{bmatrix} = \begin{bmatrix} [\underline{x_1}, \overline{x_1}] \\ [\underline{x_2}, \overline{x_2}] \\ [\underline{x_3}, \overline{x_3}] \\ [\underline{x_4}, \overline{x_4}] \\ [\underline{x_5}, \overline{x_5}] \\ [\underline{x_6}, \overline{x_6}] \end{bmatrix} = \begin{bmatrix} [-1.4063, -1.3400] \\ [-0.0697, 0.0697] \\ [-0.6920, -0.5825] \\ [0.8602, 1.0030] \\ [0.5095, 0.8370] \\ [-0.0668, 0.0668] \end{bmatrix}. \tag{34}$$

Now we come back to the second stage. According to Theorem 13, solving system (28) is equivalent to the presence of a solution for the following system:

$$\begin{aligned} Z^* = \min \quad & x_{a_1} + x_{a_2} + x_{a_3} + x_{a_4} \\ \text{s.t.} \quad & \begin{bmatrix} -2 & 4 & -3 & -5 & 0 & -2 \\ 0 & 4 & -6 & -2 & 6 & -4 \\ 5 & 0 & -2 & -3 & 8 & 3 \\ -9 & 1 & 8 & -8 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \\ & + \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ x_{a_3} \\ x_{a_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \\ & \underline{x_j} \le x_j \le \overline{x_j} \quad j = 1, 2, \ldots, 6, \\ & 0 \le x_{\widetilde{a_i}} \quad i = 1, \ldots, 4. \end{aligned} \tag{35}$$

Again, we solve this problem with the *Simplex* method. Since the optimal value is zero, system (28) has the following solution:

$$\mathrm{Core}\left(\widetilde{X}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -1.3400 \\ -0.0697 \\ -0.6920 \\ 0.8707 \\ 0.5193 \\ 0.0620 \end{bmatrix}. \tag{36}$$

According to Theorem 12 and solutions (34) and (38) a final solution for system (26) is as follows:

$$\widetilde{X} = \begin{bmatrix} (\underline{x_1}, x_1, \overline{x_1}) \\ (\underline{x_2}, x_2, \overline{x_2}) \\ (\underline{x_3}, x_3, \overline{x_3}) \\ (\underline{x_4}, x_4, \overline{x_4}) \\ (\underline{x_5}, x_5, \overline{x_5}) \\ (\underline{x_6}, x_6, \overline{x_6}) \end{bmatrix} = \begin{bmatrix} (-1.4063, -1.3400, -1.3400) \\ (-0.0697, -0.0697, 0.0697) \\ (-0.6920, -0.6920, -0.5825) \\ (0.8602, 0.8707, 1.0030) \\ (0.5095, 0.5193, 0.8370) \\ (-0.0668, 0.0620, 0.0668) \end{bmatrix}. \tag{37}$$

In this example, since different solutions can be obtained in the first or second stage (e.g., in the second substage), other solutions can also be achieved. This is not possible in numerical methods [2–5] or the embedding method [1].

Example 2. Consider the 4×6 interval system:

$$\begin{array}{rrrrrrcl} 6x_1 & -6x_2 & -2x_3 & 8x_4 & -2x_5 & & = & [-2, 0] \\ -5x_1 & -4x_2 & +6x_3 & -4x_4 & +x_5 & 5x_6 & = & [3, 6] \\ 8x_1 & +2x_2 & +x_3 & +5x_4 & -8x_5 & 8x_6 & = & [2, 3] \\ -3x_1 & & & +5x_4 & -8x_5 & -7x_6 & = & [0, 5]. \end{array} \tag{38}$$

Here system (39) is solved. It is an interval system, so it is solved in two stages according to Theorem 10. The first stage is to find the interval length, that is, L_X. Thus, the following system is solved:

$$A^+ L_X = L_b, \tag{39}$$

such that

$$\mathbf{A}^+ = \begin{bmatrix} 6 & 6 & 2 & 8 & 2 & 0 \\ 5 & 4 & 6 & 4 & 1 & 5 \\ 8 & 2 & 1 & 5 & 8 & 8 \\ 3 & 0 & 0 & 5 & 8 & 4 \end{bmatrix}, \qquad \mathbf{L_b} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}. \tag{40}$$

However, according to Theorem 11, the presence of solution (39) is equivalent to the following LP problem:

$$\begin{aligned} Z^* = \mathrm{Min} \quad & 1\mathbf{x}_a \\ \text{s.t.} \quad & A^+ L_X + \mathbf{x}_a = L_b \\ & Ł_X, \mathbf{x}_a \ge \mathbf{0}. \end{aligned} \tag{41}$$

We solve it with the *Simplex* method. Since the optimal value is not zero, in fact $Z^* = 2.794$, system (39) has no solution. As a result, system (38) has no solution.

5. Conclusion

In this paper, we presented a method which is novel for transformed fuzzy system of linear equations. The proposed method is applicable rather than other existing methods. Because the base of this method is linear programming, it can explicitly express the presence or the absence of a solution. In addition, if a solution exists, it expresses the possibility of other solutions. This is not possible in numerical methods based on the embedding method. With a slight change, this method can be used for systems whose right hand side is trapezoidal fuzzy numbers; in previous methods, they did not have this ability. Finding the optimal solutions of fuzzy LP problems is one important application of solving fuzzy linear systems. With this new method, these problems can easily be solved. The presented method can introduce fundamental change in operation research with interval data. It also amends some application of linear programming with fuzzy data as some method in [12–15]. In the numerical solution of fuzzy differential equations, this method provides an explicit method so it is extremely efficient. For example, in three-diagonal matrices, parallel processing techniques can be applied on this method if the number of equations is too high. This method can analytically present the algebraic structure of fuzzy polyhedrons in the same way the representation theorem provides the algebraic structure for crisp polyhedrons. The previous methods are lacking this capability.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Calculating Super Efficiency of DMUs for Ranking Units in Data Envelopment Analysis Based on SBM Model

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There are a number of methods for ranking decision making units (DMUs), among which calculating super efficiency and then ranking the units based on the obtained amount of super efficiency are both valid and efficient. Since most of the proposed models do not provide the projection of Pareto efficiency, a model is developed and presented through this paper based on which in the projection of Pareto-efficient is obtained, in addition to calculating the amount of super efficiency. Moreover, the model is unit invariant, and is always feasible and makes the amount of inefficiency effective in ranking.

1. Introduction

Since in the current world a wide variety of companies and organizations of different areas work in common ground under the supervision of a common manager (like central bank in a country or other banks and respective branches), in order to improve the performance of units, a manager in addition to evaluation should rank them and present an efficient pattern corresponding to inefficient units. Data envelopment analysis (DEA) is a technique for calculating the amount of efficiency in DMUs which have multiple inputs and outputs. According to the obtained efficiency through this technique, these units can be ranked and distinction can be made between efficient units and inefficient ones [1–4].

So far, a number of studies have been conducted for ranking DMUs and according to them various models have been proposed. Banker et al. in [5] using additive model evaluated DMUs and ranked them. Sexton's cross-efficiency method is another method of ranking presented in 1986 [6]. In this method, first, all DMUs are assessed by multiplier model and then optimal weights corresponding to each unit are considered for other units and the amount of objective function is measured. Afterwards, for each unit, the obtained amounts are combined through averaging and, by considering the achieved number, DMUs are ranked. The disadvantages of this method can be attributed to the presence of multiple optimal solutions as well as unreliability of averaging in unit rank. It is worth mentioning that many researchers such as [7–13] have proposed different models for improving sexton method. Andersen and Petersen in 1993 proposed a method in which they could rank extreme efficient units by eliminating unit under evaluation of production possibility set (PPS) and forming a new PPS [14]. In 1998, Mehrabian et al. using weight constraints on input and output weights in A.P. model solved some of its problems such as instability; however, others including the ranking of nonextreme efficient units, lack of presenting Pareto-efficient projection, and infeasibility in some cases still remained [15]. Li et al. (1999) modified Mehrabian et al.'s model and simultaneously by enhancing outputs and reducing inputs to the same extent resolved the mentioned infeasibility [16]. Sueyoshi (1999) adding weight constraints to CCR multiplier model developed an approach named benchmarking method. Their model like A.P. suffered infeasibility in some cases. Sueyoshi introduced AIN parameter for the purpose of ranking extreme efficient units [6]. Common weight is another method developed in 2000 by Hosseinzadeh Lotfi et al. for ranking units. That model through which units were evaluated and ranked was multi-objective; however, after specific transformation, a nonlinear programming model resulted [17]. In 2004, gradient line

method was introduced by Jahanshahloo et al. for ranking extreme efficient units [18]. This method was always feasible despite the fact that it does not provide any suggestion for ranking the nonextreme efficient units. Jahanshahloo et al. presented other methods such as Mont Carlo, norms (L_1 and L_∞), Chebyshev norm, and concept of advantage, all of which rank the units in a way [4, 19–27]. In addition, there exist some other ranking methods not much developed and extended in the literature [4, 7, 18, 20, 21, 28–33].

Among the above-mentioned methods, A.P. is the one which has been mostly used despite its disadvantages like lack of finding Pareto-efficient projection, infeasible cases, lack of ranking nonextreme efficient units, and finally lack of stability corresponding to data transformation. A number of researchers have proposed various models and attempted to modify it and eliminate its problems [14, 16, 27, 34–45].

In this paper, besides ranking DMUs, a projection of Pareto-efficient is presented. Also, the amount of inefficiency has been involved in the ranking. The model is always feasible and stable corresponding to data transformation. In this paper, first SBM model and then Tone super efficiency are presented. Afterwards, the proposed model along with some theorems is introduced. Then, using 2 different numerical examples, the proposed model is compared with Tone model and the results are revealed.

2. Preliminaries

Suppose that there is a set of n DMUs $\{\mathrm{DMU}_j : j = 1, 2, \ldots, n\}$, producing s outputs $y_{rj} (r = 1, 2, \ldots, s)$ by consuming m inputs. Assume that all the vector of $x_{ij} (i = 1, \ldots, m)$ inputs and outputs are not negative and are opposed to zero. The production possibility set (PPS) spanned by all DMUs is defined as follows:

$$P = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_s) \mid x_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1, \ldots, m, \; y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, r = 1, \ldots, s, \lambda_j \geq 0 \right\}. \tag{1}$$

Based upon the constant return to scale (CRS), the SBM model is described as follows. It assumed that $x_{ij} > 0$ ($i = 1, \ldots, m$, $j = 1, 2, \ldots, n$) and $y_{rj} > 0$ ($r = 1, 2, \ldots, s$, $j = 1, 2, \ldots, n$):

$$\begin{aligned} \text{Min:} \quad & \rho = \frac{1 - (1/m) \sum_{i=1}^{m} \left(z_i^- / x_{ik} \right)}{1 + (1/s) \sum_{r=1}^{s} \left(z_r^+ / y_{rk} \right)} \\ \text{s.t.:} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + z_i^- = x_{ik}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - z_r^+ = y_{rk}, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad z_r \geq 0, \quad z_i \geq 0. \end{aligned} \tag{2}$$

Definition 1. DMU_k in model (2) is defined as an efficient unit if and only if $\rho^* = 1$. In other words, DMU_k is SBM-efficient, whenever $z_i^{*-} = 0$, $z_i^{*+} = 0$.

In order to define super efficiency model corresponding to model (2), to obtain new PPS without considering DMU_k, P_c' is as the following:

$$P_c' = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_s) \mid x \geq \sum_{j=1, j \neq k}^{n} \lambda_j x_j, \; y \leq \sum_{j=1, j \neq k}^{n} \lambda_j y_j, \lambda_j \geq 0, j \neq k \right\}. \tag{3}$$

Tone introduced model (4) for calculating super efficiency of DMUs based on production possibility set P_c':

$$\begin{aligned} \text{Min:} \quad & \delta = \frac{(1/m) \sum_{i=1}^{m} \left(\overline{x}_i / x_{ik} \right)}{(1/s) \sum_{r=1}^{s} \left(\overline{y}_r / y_{rk} \right)} \\ \text{s.t.:} \quad & \sum_{j=1, j \neq k}^{n} \lambda_j x_{ij} \leq \overline{x}_i, \quad i = 1, \ldots, m, \\ & \sum_{j=1, j \neq k}^{n} \lambda_j y_{rj} \geq \overline{y}_r, \quad r = 1, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n, \quad j \neq k, \\ & \overline{x}_i \geq x_{ik}, \quad i = 1, \ldots, m, \\ & \overline{y}_r \geq 0, \quad \overline{y}_r \leq y_{rk}, \quad r = 1, \ldots, s. \end{aligned} \tag{4}$$

If nonefficient units were evaluated through the above model, the efficient score for all of them would be equal to one. In other words, nonefficient units could not be distinguished. Therefore, both efficient and nonefficient units should be considered discretely and it firstly requires solving SBM model for all DMUs and then distinguishing efficient units from nonefficient ones and, finally, super efficiency score is obtained.

This paper through the coming sections seeks to present a model based on which super efficiency score for nonefficient units is measured and projection of Pareto-efficient for all units is obtained.

3. Proposed Model

In this section, a model is introduced which its notion is based on the minimum distance from nonradial view. In this model, using the fact that input vector (output vector) corresponding to each unit is not exactly equal to zero, the maximum amount of each component is obtained from input and output vectors. Then, considering these amounts and also PPS of P_c', all units

are solved by model (5). In this model, the point from P_c' frontier to the unit under evaluation is obtained as follows:

$$\begin{aligned}
\text{Min:}\quad & \frac{1+(1/m)\sum_{i=1}^{m}\left(t_i x_i^{\max}/x_{ik}\right)}{1-(1/s)\sum_{r=1}^{s}\left(t_r y_r^{\max}/y_{rk}\right)} \\
\text{s.t.:}\quad & \sum_{j=1,j\neq k}^{n}\lambda_j x_{ij}-t_i^{-}x_i^{\max}\leq x_{ik},\quad \forall i, \\
& \sum_{j=1,j\neq k}^{n}\lambda_j y_{rj}+t_r^{+}y_r^{\max}\geq\ y_{rk},\quad \forall r, \\
& \lambda_j\geq 0,\quad t_i^{-}\geq 0,\quad t_r^{+}\geq 0.
\end{aligned}\tag{5}$$

The point $(x_{ik}+t_i^{-*}x_i^{\max}, y_{rk}-t_r^{+*}y_r^{\max})$ is the point of P_c'. Regarding the position of units in P_c' and the obtained projection of the unit under evaluation by model (5), two cases are probable to happen. The first case is like DMU_f of Figure 1 in which the obtained projection is placed on the strong frontier while the second case is similar to DMU_e in Figure 2 in which its projection is on the weak frontier.

Since projection point may lie on weak frontier, for the purpose of finding Pareto-efficient projection point for all units under evaluation, model (6) should be solved. In this model, $(t_i^{-*}x_i^{\max}, t_r^{+*}y_r^{\max})$ is the optimal solution of model (5):

$$\begin{aligned}
\text{Min:}\quad & \frac{1-(1/m)\sum_{i=1}^{m}\left(s_i^{-}/x_{ik}\right)}{1+(1/s)\sum_{r=1}^{s}\left(s_r^{+}/y_{rk}\right)} \\
\text{s.t.:}\quad & \sum_{j=1,j\neq k}^{n}\lambda_j x_{ij}-t_i^{-*}x_i^{\max}+s_i^{-}=x_{ik},\quad \forall i, \\
& \sum_{j=1,j\neq k}^{n}\lambda_j y_{rj}+t_r^{+*}y_r^{\max}-s_r^{+}=\ y_{rk},\quad \forall r, \\
& \lambda_j\geq 0,\quad s_r\geq 0,\quad s_i\geq 0.
\end{aligned}\tag{6}$$

Through this, first by adding input saving $(t_i^{-*}x_i^{\max})$ and subtracting output surpluses $(t_r^{+*}y_r^{\max})$ to and from the unit under evaluation, it moves to a point of P_c' which is a frontier point. Moreover, since it may lie on the weak frontier, it is projected on Pareto-efficient point by using model (6).

In this way, if DMU is nonefficient, $t_i^{-*}=t_r^{+*}=0$. Consequently, for its projection model (6) which is in fact the same as SBM model is used. Super efficiency score in this method for DMU_k is defined as follows:

$$\varphi^{*}=\begin{cases}\dfrac{1+(1/m)\sum_{i=1}^{m}\left(\left(t_i^{-*}x_i^{\max}-s_i^{*-}\right)/x_{ik}\right)}{1-(1/s)\sum_{r=1}^{s}\left(\left(t_r^{+*}y_r^{\max}-s_r^{*+}\right)/y_{rk}\right)} \\ \qquad \text{if } \delta^{*}=\dfrac{1+(1/m)\sum_{i=1}^{m}\left(t_i^{-*}x_i^{\max}/x_{ik}\right)}{1-(1/s)\sum_{r=1}^{s}\left(t_r^{+*}y_r^{\max}/y_{rk}\right)}>1 \\ \dfrac{1-(1/m)\sum_{i=1}^{m}\left(s_i^{*-}/x_{ik}\right)}{1+(1/s)\sum_{r=1}^{s}\left(s_r^{*+}/y_{rk}\right)} \\ \qquad \text{Otherwise.}\end{cases}\tag{$*$}$$

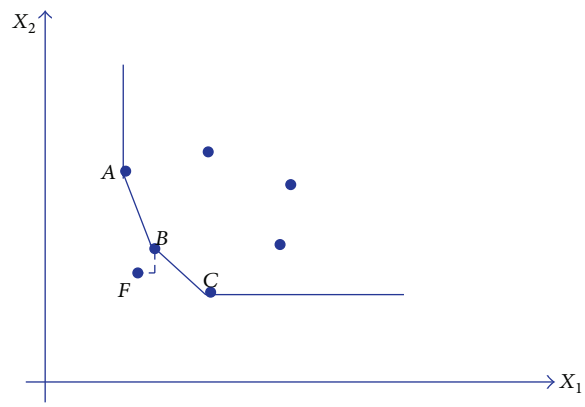

Figure 1

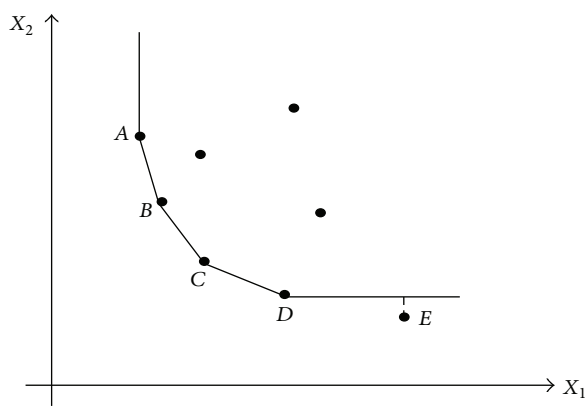

Figure 2

As it is observed in the above definition, if the first projection of the unit under evaluation like DMU_e lies on the weak frontier, the second projection which is Pareto-efficient is considered for this unit. Furthermore, the amount of slack variables (s_i^{-}, s_r^{+}) is included in the definition.

Theorem 2. *Model* (4) *and model* (5) *are equivalent.*

Proof. As can be seen in model (2), $\overline{x}_i\geq x_{ik}$, $\overline{y}_r\leq y_{rk}$. Substitute $\overline{x}_i=x_{ik}+t_i^{-}x_i^{\max}$ and $\overline{y}_r=y_{rk}+t_r^{+}y_r^{\max}$ in model (4) and rewrite the following:

$$\begin{aligned}
\text{Min:}\quad & \delta=\frac{(1/m)\sum_{i=1}^{m}\left(\left(x_{ik}+t_i^{-}x_i^{\max}\right)/x_{ik}\right)}{(1/s)\sum_{r=1}^{s}\left(\left(y_{rk}-t_r^{+}y_r^{\max}\right)/y_{rk}\right)} \\
\text{s.t.:}\quad & \sum_{j=1,j\neq k}^{n}\lambda_j x_{ij}\leq x_{ik}+t_i^{-}x_i^{\max},\quad i=1,\ldots,m, \\
& \sum_{j=1,j\neq k}^{n}\lambda_j y_{rj}\geq y_{rk}+t_r^{+}y_r^{\max},\quad r=1,\ldots,s, \\
& \lambda_j\geq 0,\quad j=1,\ldots,n,\quad j\neq k, \\
& t_i^{-}\geq 0,\quad i=1,\ldots,m, \\
& t_r^{+}\geq 0,\quad r=1,\ldots,s.
\end{aligned}\tag{7}$$

The solution space of both model (6) and model (7) is equal. After rearrangement, simplifying the objective function of model (7), we will have the following:

$$\begin{aligned}
\text{Min:}\quad \delta &= \frac{(1/m)\sum_{i=1}^{m}\left(\left(x_{ik}+t_i^{-}x_i^{\max}\right)/x_{ik}\right)}{(1/s)\sum_{r=1}^{s}\left(\left(y_{rk}-t_r^{+}y_r^{\max}\right)/y_{rk}\right)} \\
&= \frac{1+(1/m)\sum_{i=1}^{m}\left(t_i^{-}x_i^{\max}/x_{ik}\right)}{1-(1/s)\sum_{r=1}^{s}\left(t_r^{+}y_r^{\max}/y_{rk}\right)} \\
\text{s.t.:}\quad & \sum_{j=1,j\neq k}^{n}\lambda_j x_{ij}-t_i^{-}x_i^{\max}\le x_{ik},\quad i=1,\dots,m, \\
& \sum_{j=1,j\neq k}^{n}\lambda_j y_{rj}+t_r^{+}y_r^{\max}\ge y_{rk},\quad r=1,\dots,s, \\
& \lambda_j\ge 0,\quad j=1,\dots,n,\quad j\neq k, \\
& t_i^{-}\ge 0,\quad i=1,\dots,m, \\
& t_r^{+}\ge 0,\quad r=1,\dots,s.
\end{aligned} \tag{8}$$

It can be seen that model (8) is the same as model (5). □

Theorem 3. *If $DMU_k \notin P_c'$, $\varphi^* \le \delta^*$.*

Proof. Since $\mathrm{DMU}_k \notin P_c'$, there exists an i in model (5), in which $t_i^{-*} > 0$ or there exists an r in which $t_r^{+*} > 0$. Thus $(1+(1/m)\sum_{i=1}^{m}(t_i^{-*}x_i^{\max}/x_{ik}))/(1-(1/s)\sum_{r=1}^{s}(t_r^{+*}y_r^{\max}/y_{rk})) > 1$. Depending on the position of projection point of DMU_k on strong frontier or weak frontier, the amounts of s_i^{*-}, s_r^{*+} are greater than or equal to zero in the definition of φ^*. Therefore, $\varphi^* = (1+(1/m)\sum_{i=1}^{m}((t_i^{-*}x_i^{\max}-s_i^{*-})/x_{ik}))/(1-(1/s)\sum_{r=1}^{s}((t_r^{+*}y_r^{\max}-s_r^{*+})/y_{rk})) \le (1+(1/m)\sum_{i=1}^{m}(t_i^{-*}x_i^{\max}/x_{ik}))/(1-(1/s)\sum_{r=1}^{s}(t_r^{+*}y_r^{\max}/y_{rk})) = \delta^*$. □

Theorem 4. *If DMU_k belongs to P_c', $\varphi^* = \rho^*$.*

Proof. Since $\mathrm{DMU}_k \in P_c'$, using model (5), $t_i^{-*} = t_r^{+*} = 0$. Then, substituting them in model (6), model (2) is obtained and also $s_i^{*-} = z_i^{*-}$, $s_r^{*+} = z_r^{*+}$. Therefore, by definition of super efficiency, $(*)$: $\varphi^* = (1-(1/m)\sum_{i=1}^{m}(s_i^{*-}/x_{ik}))/(1+(1/s)\sum_{r=1}^{s}(s_r^{*+}/y_{rk})) = (1-(1/m)\sum_{i=1}^{m}(z_i^{*-}/x_{ik}))/(1+(1/s)\sum_{r=1}^{s}(z_r^{*+}/y_{rk})) = \rho^*$. □

Theorem 5. *If DMU_k is not efficient in model (2), that is, not SBM-efficient, the input excesses (s_i^-) and output shortfalls (s_i^+) identified by model (6) are the same as the those identified by model (2); in other words, $s_i^{*-} = z_i^{*-}$, $s_r^{*+} = z_r^{*+}$.*

Proof. If DMU_k is evaluated by model (4) and is not efficient, then $\mathrm{DMU}_k \in P_c'$ and also $t_i^{-*} = t_r^{+*} = 0$. As a result, model (6) is degenerated to model (2) and then $s_i^{*-} = z_i^{*-}$ and $s_r^{*+} = z_r^{*+}$. □

Theorem 6. *According to the obtained amount of super efficiency φ^*, three cases are identified as follows:*

(a) *If $\varphi^* > 1$, $\varphi^* \le \delta^*$;*

(b) *If $\varphi^* = 1$, $\varphi^* = \delta^* = \rho^*$;*

(c) *If $\varphi^* < 1$, $\varphi^* = \rho^*$.*

Proof. (a) Since $\varphi^* > 1$, DMU_k does not belong to P_c'. Thus, according to Theorem 3, $\varphi^* \le \delta^*$.

(b) If $\varphi^* = 1$, DMU_k is placed on P_c' frontier. This means that in model (5) the amount of $t_i^{-*}, t_r^{+*} = 0$. Therefore, $\delta^* = (1+(1/m)\sum_{i=1}^{m}(t_i^{-*}x_i^{\max}/x_{ik}))/(1-(1/s)\sum_{r=1}^{s}(t_r^{+*}y_r^{\max}/y_{rk})) = 1$. Moreover, $\varphi^* = (1-(1/m)\sum_{i=1}^{m}(s_i^{*-}/x_{ik}))/(1+(1/s)\sum_{r=1}^{s}(s_r^{*+}/y_{rk})) = 1$. Thus, considering the definition of ρ in model (2) and Theorem 3, $\varphi^* = \delta^* = \rho^*$.

(c) In the third case, in which $\varphi^* < 1$, based on the mentioned definition of super efficiency, $\varphi^* = (1-(1/m)\sum_{i=1}^{m}(s_i^{*-}/x_{ik}))/(1+(1/s)\sum_{r=1}^{s}(s_r^{*+}/y_{rk}))$. Besides, according to model (5), $t_i^{-*} = t_r^{+*} = 0$. Consequently, $\delta^* = (1+(1/m)\sum_{i=1}^{m}(t_i^{-*}x_i^{\max}/x_{ik}))/(1-(1/s)\sum_{r=1}^{s}(t_r^{+*}y_r^{\max}/y_{rk})) = 1$. Using model (5), it is shown that $\varphi^* = \rho^*$. □

Theorem 7. *Model (5) is unit invariant.*

Proof. If in model (5) either all inputs or outputs are divided or multiplied by a number, the model is unit invariant and the optimal solution does not change because

$$\begin{aligned}
\text{Min:}\quad \delta &= \frac{1+(1/m)\sum_{i=1}^{m}\left(t_i^{-}\left(x_i^{\max}/K\right)/\left(x_{ik}/K\right)\right)}{1-(1/s)\sum_{r=1}^{s}\left(t_r^{+}\left(y_r^{\max}/K'\right)/\left(y_{rk}/K'\right)\right)} \\
\text{s.t.:}\quad & \sum_{j=1,j\neq k}^{n}\lambda_j\frac{x_{ij}}{K}-t_i^{-}\frac{x_i^{\max}}{K}\le\frac{x_{ik}}{K},\quad \forall i \\
& \sum_{j=1,j\neq k}^{n}\lambda_j\frac{y_{rj}}{K'}+t_r^{+}\frac{y_r^{\max}}{K'}\ge\frac{y_{rk}}{K'},\quad \forall r, \\
& \lambda_j\ge 0,\quad t_i^{-}\ge 0,\quad t_r^{+}\ge 0.
\end{aligned} \tag{9}$$

After simplifying K of objective function and constraints, model (9) degenerates to model (5). □

Theorem 8. *The identified projection from model (6) is Pareto-efficient.*

Proof. 2 cases are considered for DMU_k.

Case 1. DMU_k belongs to spanned production possibility set by $\mathrm{DMU}_i (i \neq k)$. In this case, model (6) degenerates to SBM model and the projection is Pareto-efficient.

Case 2. If DMU_k does not belong to production possibility set, then $(t_i^{-*}x_i^{\max}, t_r^{+*}y_r^{\max})$ is placed on frontier of PPS spanned by all DMU_s excluding DMU_k. It is claimed that the point $(x_{ik}+t_i^{-*}x_i^{\max}-s_i^{*-}, y_{rk}-t_r^{+*}y_r^{\max}+s_r^{*+})$ is Pareto-efficient.

Proof by contradiction: if the above-mentioned point is not Pareto-efficient, then there exists a point like $(x_{ik}+t_i^{-*}x_i^{\max}-\bar{s}_i^-, y_{rk}-t_r^{+*}y_r^{\max}+\bar{s}_r^+)$ which dominates $(x_{ik}+t_i^{-*}x_i^{\max}-s_i^{*-}, y_{rk}-t_r^{+*}y_r^{\max}+s_r^{*+})$. This contradicts the optimal property of s_r^{*+}, s_i^{*-}. □

Theorem 9. *Model* (5) *is always feasible.*

Proof. If $\mathrm{DMU}_k \in P_c'$, according to definition of P_c',

$$\sum_{j=1,j\neq k}^{n} \lambda_j x_{ij} \leq x_{ik},$$
$$\sum_{j=1,j\neq k}^{n} \lambda_j y_{rj} \geq y_{rk}, \tag{10}$$
$$\exists\, \overline{\lambda}, \quad \overline{\lambda} \geq 0.$$

Choosing $(\overline{\lambda}, t_i^- = 0, t_r^+ = 0)$, a feasible solution of model (5) can be obtained. Otherwise, if $\mathrm{DMU}_k \notin P_c'$, the corresponding solution for that should be found. For this purpose, at first, we should suppose that $\overline{\lambda} = 1$. The input constraints of model (5) are $\sum_{j=1,j\neq k}^{n} x_{ij} - x_{ik} \leq t_i^t x_i^{\max}$. Since $x_i^{\max} > 0$, $y_r^{\max} > 0$, $\forall i, r$, $\overline{t_i^-}$ is defined as follows:

$$\overline{t_i^-} = \begin{cases} \dfrac{\sum_{j=1,j\neq k}^{n} x_{ij} - x_{ik}}{x_i^{\max}} & \displaystyle\sum_{j=1,j\neq k}^{n} x_{ij} > x_{ik} \\ 0 & \text{o.w.} \end{cases} \tag{11}$$

Moreover, corresponding to output constraints $\overline{\lambda} = 1$, $y_r^{\max} > 0, \forall r$, $\overline{t_r^+}$ is defined as follows:

$$\overline{t_r^+} = \begin{cases} \dfrac{y_{rk} - \sum_{j=1,j\neq k}^{n} y_{rj}}{y_r^{\max}} & \displaystyle y_{rk} > \sum_{j=1,j\neq k}^{n} y_{rj} \\ 0 & \text{o.w.} \end{cases} \tag{12}$$

Considering $(\overline{\lambda}, \overline{t_i^-}, \overline{t_r^+})$, a feasible solution for model (5) is achieved.

Model (6) is a fractional programming model that can be modified to a linear programming as follows. That occurs by substituting variable t which itself is defined by $1/t = 1 + (1/s)\sum_{i=1}^{s}(s_r^+/y_{rk})$

$$\begin{aligned} \text{Min:} \quad & t - \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\widehat{s}_i^-}{x_{ik}}\right) \\ \text{s.t.:} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s}\left(\frac{\widehat{s}_r^+}{y_{rk}}\right), \\ & \sum_{j=1,j\neq k}^{n} \widehat{\lambda}_j x_{ij} - t t_i^* x_i^{\max} + \widehat{s}_i^- = t x_{ik}, \quad \forall i, \\ & \sum_{j=1,j\neq k}^{n} \widehat{\lambda}_j y_{rj} + t t_r^* y_r^{\max} - \widehat{s}_r^+ = t y_{rk}, \quad \forall r, \\ & \widehat{\lambda}_j \geq 0, \quad \widehat{s}_r \geq 0, \quad \widehat{s}_i \geq 0. \end{aligned} \tag{13}$$

In model (13), it should be considered that $t > 0$. The optimal solution for model (13) is $(\widehat{\lambda}_j^*, \widehat{s}_i^{*-}, \widehat{s}_r^{*+}, t^*)$ by which the optimal solution of model (6) is obtained as follows:

$$\lambda_j^* = \frac{\widehat{\lambda}_j^*}{t^*}, \qquad s_i^{*-} = \frac{\widehat{s}_i^{*-}}{t^*}, \qquad s_r^{*+} = \frac{\widehat{s}_r^{*+}}{t^*}. \tag{14}$$

□

4. Numerical Examples

Two examples of Tone [26, 46] were revisited and ranked through evaluation of units by both our and Tone's methods. In the first example, it is supposed that the 5 decision making units have 2 inputs and 2 outputs. The related data and results are listed in Table 1. The first column represents units. In columns 2 and 3, information about inputs is presented; however, that of outputs is listed in columns 4 and 5. The sixth column (ρ^*) shows the amount of optimal solution for model (2). Column seven (δ^*) represents optimal solution for each unit in model (4). The eighth column indicates the amounts of super efficiency which is obtained through the proposed method. For calculating φ^*, first, the units are evaluated by model (5) and then by substituting point $(x_{ik} + t_i^{-*} x_i^{\max}, y_{rk} - t_r^{+*} y_r^{\max})$ in model (6) the optimal solution is obtained and considering the optimal amounts of (t_i^{-*}, t_r^{+*}), (s_i^{-*}, s_r^{+*}) and substituting them in the proposed definition of super efficiency $(*)$, the amount of φ^* is obtained. Finally, in column 9 the unit ranks are presented based on φ^*.

As it is noticed in Table 2, despite the fact that the amount of φ^* for unit 5 is bigger than that of unit 3, by calculating super efficiency through the proposed method, it is observed that the rank of unit 3 is better than that of unit 5.

In the second example, 6 decision making units with 4 inputs and 2 outputs are considered. In this example, the units are evaluated by both proposed method and Tone's method and then ranked. Table 2 shows the data of those units. As it is noticed, $\rho^* = 1$. This means that all units are SBM-efficient and are located on the frontier. DMU_5 in both methods has the first rank.

As it is seen, ranks of units in proposed method (φ^*) were 4, 2, 5, 3, 1, and 6, respectively, while those of Tone's got 6, 2, 4, 3, 1, and 5, respectively.

5. Conclusion

In data envelopment analysis, a wide variety of models have been presented by using which decision making units can be evaluated and ranked, though most of them do not have properties such as feasibility in all cases, being unit invariant, ranking nonextreme efficient units, and finding strong Pareto-efficient projection for all units.

In this study, in order to calculate super efficiency of units and rank them, Anderson-Peterson's idea was utilized in two stages. In the first stage, the unit under evaluation was projected on production possibility set spanned by the rest of the DMUs and, in the second stage, the first projection point

Table 1

DMU	Inp. 1	Inp. 2	Out. 1	Out. 2	ρ^*	δ^*	φ^*	Rank
1	4	3	2	3	0.7980	1	0.7980	3
2	6	3	2	3	0.5682	1	0.5682	5
3	8	1	6	2	1	1.3333	1.3333	1
4	8	1	6	1	0.6667	1	0.6667	4
5	2	4	1	4	1	1.4545	0.9919	2

Table 2

DMU	Inp. 1	Inp. 2	Inp. 3	Inp. 4	Out. 1	Out. 2	ρ^*	δ^*	φ^*	Rank
1	80	600	54	8	90	5	1	1.0116	0.7798	4
2	65	200	97	1	58	1	1	1.4146	0.9164	2
3	83	400	72	4	60	7	1	1.0781	0.7677	5
4	40	1000	75	7	80	10	1	1.1563	0.7927	3
5	52	600	20	3	72	8	1	1.5858	1.8107	1
6	94	700	36	5	96	6	1	1.0198	0.7590	6

is transferred to a Pareto-efficient point. Then, by applying slack variables in the definition of super efficiency units were ranked. The introduced model in this paper has several advantages including having feasibility, obtaining a Pareto-efficient projection, and being unit invariant. The other advantage of the model is that it involves the amount of inefficiency in the amount of super efficiency and consequently affects ranking units in the case that the first projection point is placed on weak frontier. However, the problem of ranking nonextreme efficient units still remains. The proposed model is similar to SBM and as it was observed in previous sections, it is equivalent to Tone's model, though the obtained results are different due to the new definition of super efficiency.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Full Ranking for Decision Making Units Using Ideal and Anti-Ideal Points in DEA

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We propose a procedure for ranking decision making units in data envelopment analysis, based on ideal and anti-ideal points in the production possibility set. Moreover, a model has been introduced to compute the performance of a decision making unit for these two points through using common set of weights. One of the best privileges of this method is that we can make ranking for all decision making units by solving only three programs, and also solving these programs is not related to numbers of decision making units. One of the other advantages of this procedure is to rank all the extreme and nonextreme efficient decision making units. In other words, the suggested ranking method tends to seek a set of common weights for all units to make them fully ranked. Finally, it was applied for different sets holding real data, and then it can be compared with other procedures.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric method to define the relative efficiency of a group of decision making units (DMUs) that use multiple inputs to produce multiple outputs. Methodology of DEA pioneered by Farrell [1] and later developed by Charnes et al. [2]. DEA computes the relative efficiencies of all DMUs by finding a set of the best weights for every DMU by maximizing its efficiency. On the other hand, flexibility in picking different inputs and outputs weights leads to coming up with DMUs as relative efficient which causes ranking disorders among DMUs. To reduce the flexibility in selecting inputs and outputs weights, researchers have already tried to modify the traditional DEA model and remove the weak points as well. Here are listed some of these offered methods. The assurance region model was firstly presented by Thompson et al. [3] and the common weights model by Roll et al. [4]. Roll and Golany [5] offered an alternative method in which they normalized all inputs and outputs at the beginning, in a way that the magnitude of parameters would not influence the model and then through imposing restrictions on the weights of the model, they could achieve common weights. Mavi et al. [6] presented a common set of weights using ideal point method. Hosseinzadeh Lotfi et al. [7] and Jahanshahloo et al. [8] proposed two different models in DEA by using common weights. They suggested that instead of solving n linear programming models, we can reach the efficiency of DMUs through solving only one nonlinear programming model. Through utilizing multiple objective programming (MOP) and common set of weights (CSW), Hosseinzadeh Lotfi et al. [7] introduced a model to compute the efficiency of DMUs.

When DEA models are applied to calculate the performance of DMUs, usually several DMUs yield with the same efficiencies, that are all equal to one. Therefore, it is necessary to suggest a model to differentiate between these units. Otherwise, we are not able to rank them accordingly. Numerous models have been proposed to reduce the number of efficient units so far: Andersen and Petersen (AP) [9] and Mehrabian, Alirezaee, and Jahanshahloo (MAJ) [10] can be considered as two of the most popular of these methods; however, sometimes they fail in ranking. So we intend to compare the proposed procedure with the two aforementioned methods by some examples in this paper. Additionally, some papers based on cross-efficiency have been prepared such as Sexton et al. [11], Wu et al. [12], Jahanshahloo et al. [13, 14], and Wang et al. [15].

One of the most important and practical procedures in ranking is benchmarking methods, which are suggested by Torgesen et al. [16], Sueyoshi [17], Lu and Lo [18], and Chen and Deng [19]. Wang et al. [20] presented two nonlinear programming models for full ranking which have high complexity for computations. You can see [21] for further study on ranking methods. To overcome the problems in the complete ranking of units, we propose a mixed integer programming which is capable of ranking every (extreme and nonextreme) efficient DMU, although sometimes other methods fail. The main purpose in this paper is introducing a model to evaluate the DEA efficiency of DMUs. We tend to suggest ideal and anti-ideal points in the model; then through using CSW and MOP a comprehensive evaluation of DMUs can be proposed. In addition, we prove that our model is feasible. The rest of this paper is organized as follows. Section 2 briefly introduces the approach of finding a CSW by MOP concepts. In Section 3, a procedure would be proposed to rank DMUs. Section 4 compares the proposed method with the other models using three numerical examples. The paper is concluded in the final section.

2. Common Set of Weights Model

Assume that there is a set of n DMUs. Each DMU_j ($j = 1,\ldots,n$) consumes the amounts $X_j = \{x_{ij}\}$ of m different of inputs ($i = 1,\ldots,m$) and produces the amounts $Y_j = \{y_{rj}\}$ of r outputs ($r = 1,\ldots,s$). Charnes et al. [2] presented following well-known CCR model which measures the relative efficiencies of DMUs:

$$\begin{aligned} \max \quad & \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ \text{s.t.} \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\ldots,n, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m, \end{aligned} \tag{1}$$

where DMU_o represents the DMU under evaluation; u_r ($r = 1,\ldots,s$) and v_i ($i = 1,\ldots,m$) are the weights assigned to the outputs and inputs and ε presents a non-Archimedean infinitesimal. If there is a set of positive weights that makes $\theta_o^* = 1$, then DMU_o is called relative efficient and otherwise it is called relative inefficient. The linear programming equivalent of model (1) is

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\ldots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{2}$$

This problem has a dual which is given by

$$\begin{aligned} \min \quad & \theta_o - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta_o x_{io}, \quad i = 1,\ldots,m, \\ & \sum \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1,\ldots,s, \\ & \lambda_j, s_i^-, s_r^+ \ge 0, \quad j = 1,\ldots,n, \\ & i = 1,\ldots,m, \quad r = 1,\ldots,s. \end{aligned} \tag{3}$$

The constraint space of (3) defines the production possibility set (PPS) T_c. That is,

$$T_c = \left\{(x,y) \mid x \ge \sum_{j=1}^{n} \lambda_j x_j,\ y \le \sum_{j=1}^{n} \lambda_j y_j,\ \lambda_j \ge 0,\ j = 1,\ldots,n\right\}. \tag{4}$$

It should be noted that DMU_o is extreme efficient if and only if the model (3) has a unique optimal solution as follows:

$$\begin{aligned} & \theta_o^* = 1, \\ & \lambda_o^* = 1, \\ & \lambda_j^* = 0, \quad j = 1,\ldots,n,\ j \ne o, \\ & s^{-*} = 0, \qquad s^{+*} = 0. \end{aligned} \tag{5}$$

Extra flexibility to choose weights mostly brings several DMUs with relative efficient DMUs. However, to remove this problem, many attempts have been explored further restricting weights in DEA. One of the most important ones is the common weights method in DEA, which at first initiated by Cook et al. [22]. The other method was proposed by Roll et al. [4] in DEA, where all DMUs can be evaluated by only one common weight. While it is almost tough, it can suggest more precise ranking; therefore each introduced efficient DMU of this method would be efficient DMU in primary DEA models. Hosseinzadeh Lotfi et al. [7] suggested a model to compute the efficiency of DMUs, in which they were only solved by one nonlinear programming model instead of n linear programming models. The following multiobjective fractional programming (MOFP) can be used to maximize the efficiency score of all DMUs together [7]:

$$\begin{aligned} \max \quad & \left\{\frac{\sum_{r=1}^{s} u_r y_{r1}}{\sum_{i=1}^{m} v_i x_{i1}}, \frac{\sum_{r=1}^{s} u_r y_{r2}}{\sum_{i=1}^{m} v_i x_{i2}}, \ldots, \frac{\sum_{r=1}^{s} u_r y_{rn}}{\sum_{i=1}^{m} v_i x_{in}}\right\} \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\ldots,n, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m. \end{aligned} \tag{6}$$

Several methods have been proposed to solve the aforementioned MOFP problem. One of them is goal programming (GP). Based on the GP method, model (6) can be transformed to the following model for attaining a set of common weights [23]:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left(n_j + p_j\right) \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} + n_j - p_j = A_j, \quad j = 1, \ldots, n, \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, \ldots, n, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad n_j \ge 0, \quad p_j \ge 0, \\ & r = 1, \ldots, s, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{aligned} \tag{7}$$

Here A_j is the goal of the jth objective function and n_j, p_j represent the positive deviation and negative deviation of the jth goal, respectively. On the other hand, while in the conventional DEA models, every individual DMU tends to maximize its efficiency, so the amounts of A_j $(j = 1, \ldots, n)$ in model (7) would be one.

3. The Proposed Ranking Method

On the first step we are going to introduce ideal and anti-ideal points.

Definition 1. An ideal point is a point that can consume the least inputs to produce the most outputs.

Definition 2. An anti-ideal point is a point that uses the most inputs only to generate the least outputs.

Due to mentioned definitions we can show the inputs and outputs of ideal point with $x_i^{\min}$ $(i = 1, \ldots, m)$ and $y_r^{\max}$ $(r = 1, \ldots, s)$, respectively. Also, we denote by $x_i^{\max}$ $(i = 1, \ldots, m)$ and $y_r^{\min}$ $(r = 1, \ldots, s)$ the inputs and outputs of anti-ideal point, respectively. These are determined as follows:

$$\begin{aligned} x_i^{\min} &= \min_j \left\{x_{ij}\right\}, \qquad x_i^{\max} = \max_j \left\{x_{ij}\right\}, \\ & \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n, \\ y_r^{\min} &= \min_j \left\{y_{rj}\right\}, \qquad y_r^{\max} = \max_j \left\{y_{rj}\right\}, \\ & \qquad r = 1, \ldots, s, \quad j = 1, \ldots, n. \end{aligned} \tag{8}$$

According to efficiency concept, the efficiency of ideal point can be defined as

$$\theta_I = \frac{\sum_{r=1}^{s} u_r y_r^{\max}}{\sum_{i=1}^{m} v_i x_i^{\min}}, \tag{9}$$

where u_r, v_i are the weights assigned to the rth output and the ith input, respectively. Suppose that θ_I^* is the ideal point efficiency, which results from the following LP model:

$$\begin{aligned} \max \quad & \theta_I = \sum_{r=1}^{s} u_r y_r^{\max} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_i^{\min} = 1, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad r = 1 \ldots, s, \quad i = 1, \ldots, m. \end{aligned} \tag{10}$$

As such, the efficiency score of anti-ideal point can be specified as

$$\theta_A = \frac{\sum_{r=1}^{s} u_r y_r^{\min}}{\sum_{i=1}^{m} v_i x_i^{\max}}. \tag{11}$$

If we consider θ_A^* as the efficiency of the anti-ideal point, then it can be solved by the model below:

$$\begin{aligned} \max \quad & \theta_A = \sum_{r=1}^{s} u_r y_r^{\min} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_i^{\max} = 1, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad r = 1 \ldots, s, \quad i = 1, \ldots, m. \end{aligned} \tag{12}$$

Here we assume that θ_I^* is a goal for all DMUs, in such a way every single DMU tends to get its efficiency approach to it and consequently each DMU inclines toward getting its efficiency farther from θ_A^*. Then in accordance with this idea and goal programming, we obtain the following model:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \left(n_j - p_j\right) \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + n_j}{\sum_{i=1}^{m} v_i x_{ij}} = \theta_I^*, \quad j = 1, \ldots, n, \\ & \frac{\sum_{r=1}^{s} u_r y_{rj} - p_j}{\sum_{i=1}^{m} v_i x_{ij}} = \theta_A^*, \quad j = 1, \ldots, n, \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, \ldots, n, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad n_j \ge 0, \quad p_j \ge 0, \\ & r = 1, \ldots, s, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \end{aligned} \tag{13}$$

where n_j and p_j are deviation variables for the DMU_j. Although model (13) is able to reduce the efficient units in

DEA, it is still possible to evaluate more than one DMU as an efficient unit in DEA; therefore it cannot suggest a comprehensive ranking for n DMUs. To overcome this problem, the following model is offered:

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n}\left(n_j - p_j\right) \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + n_j}{\sum_{i=1}^{m} v_i x_{ij}} = \theta_I^*, \quad j = 1,\ldots,n, \\
& \frac{\sum_{r=1}^{s} u_r y_{rj} - p_j}{\sum_{i=1}^{m} v_i x_{ij}} = \theta_A^*, \quad j = 1,\ldots,n, \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\ldots,n, \\
& \frac{\sum_{r=1}^{s} u_r y_{rj} + t_j}{\sum_{i=1}^{m} v_i x_{ij}} = 1, \quad j = 1,\ldots,n, \\
& \varepsilon d_j \le t_j \le M d_j, \quad j = 1,\ldots,n, \\
& \sum_{j=1}^{n} d_j = n - 1, \quad d_j \in \{0,1\}, \quad j = 1,\ldots,n, \\
& u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad n_j \ge 0, \quad p_j \ge 0, \\
& r = 1,\ldots,s, \quad i = 1,\ldots,m, \quad j = 1,\ldots,n,
\end{aligned} \tag{14}$$

where M is a giant number. In model (14) each DMU by minimizing its efficiency from ideal point efficiency and maximizing from anti-ideal point efficiency tends to attain a set of inputs and outputs weights.

Definition 3. If (u^*, v^*) is an optimal solution of model (2), then

$$H = \left\{(x, y) \mid u^{t*} y - v^{t*} x = 0\right\} \tag{15}$$

is a supporting hyperplane of the PPS.

Theorem 4. *Model* (14) *is feasible and bounded.*

Proof. For evaluating DMUs, we can use model (2). We know that there exists at least one extreme efficient unit by using model (2). Without loss of generality, suppose that the DMU_1 is extreme efficient. On the other hand, there exist an infinite number of supporting hyperplanes passing through any extreme efficient DMU [24]. We will show that there exists a supporting hyperplane that only DMU_1 lies on its intersection and T_c. Since DMU_1 is extreme efficient, hence model (3) has a unique optimal solution

$$\begin{aligned}
& \theta_1^* = 1, \\
& \lambda_1^* = 1, \qquad \lambda_j^* = 0, \quad j = 2,\ldots,n, \\
& s^{-*} = 0, \qquad s^{+*} = 0.
\end{aligned} \tag{16}$$

After including slack variables t_j $(j = 1,\ldots,n)$ in the first n constraints of model (2), according to strong complementary slackness conditions (SCSC) [25], there exist a pair of an optimal solution (u^*, v^*, t^*) of model (2) and an optimal solution $(\theta^*, \lambda^*, s^{-*}, s^{+*})$ of model (3), such that

$$\begin{aligned}
& \lambda_j^* t_j^* = 0, \quad \lambda_j^* + t_j^* > 0, \quad j = 1,\ldots,n, \\
& u_r^* s_r^{+*} = 0, \quad u_r^* + s_r^{+*} > 0, \quad r = 1,\ldots,s, \\
& v_i^* s_i^{-*} = 0, \quad v_i^* + s_i^{-*} > 0, \quad i = 1,\ldots,m,
\end{aligned} \tag{17}$$

where $t_j^* = -\sum_{r=1}^{s} u_r^* y_{rj} + \sum_{i=1}^{m} v_i^* x_{ij}$. Therefore, since (16) is the unique optimal solution model (3); then by (17) we have

$$t_1^* = 0, \qquad t_j^* > 0, \quad j = 2,\ldots,n. \tag{18}$$

Equation (18) implies that

$$\sum_{r=1}^{s} u_r^* y_{r1} - \sum_{i=1}^{m} v_i^* x_{i1} = 0, \qquad \sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} < 0, \quad j = 2,\ldots,n. \tag{19}$$

That is,

$$\frac{\sum_{r=1}^{s} u_r^* y_{r1}}{\sum_{i=1}^{m} v_i^* x_{i1}} = 1, \qquad \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}} < 1, \quad j = 2,\ldots,n. \tag{20}$$

Hence, only DMU_1 lies on the supporting hyperplane $H = \{(x, y) \mid u^{*t} y - v^{*t} x = 0\}$. It is evident that (21) is a feasible solution for model (14):

$$\begin{aligned}
& u_r = u_r^*, \quad v_i = v_i^*, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m, \\
& d_1 = 0, \qquad d_j = 1, \quad j = 2,\ldots,n, \\
& t_1 = 0, \qquad t_j > 0, \quad j = 2,\ldots,n, \\
& n_j = \theta_I^* \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj}, \quad j = 1,\ldots,n, \\
& p_j = \sum_{r=1}^{s} u_r y_{rj} - \theta_A^* \sum_{i=1}^{m} v_i x_{ij}, \quad j = 1,\ldots,n.
\end{aligned} \tag{21}$$

On the other hand, since $n_j \ge 0$ and $0 \le \theta_A^* \le 1$, obviously model (14) is bounded. This completes the proof. □

Let u_r^* $(r = 1,\ldots,s)$ and v_i^* $(i = 1,\ldots,m)$ be the optimal weights in model (14). Then

$$\theta_j^* = \frac{\sum_{r=1}^{s} u_r^* y_{rj}}{\sum_{i=1}^{m} v_i^* x_{ij}}, \quad j = 1,\ldots,n, \tag{22}$$

is referred to as the efficiency of DMU_j. Model (14) introduces a complete ranking for n DMUs and this will be tested and illustrated in the next section through some examples.

Table 1: Data and CCR efficiency for Example 1.

FMS	Inputs		Outputs				CCR efficiency
	X_1	X_2	Y_1	Y_2	Y_3	Y_4	
1	17.02	5	42	45.3	14.2	30.1	1
2	16.46	4.5	39	40.1	13	29.8	1
3	11.76	6	26	39.6	13.8	24.5	0.9682
4	10.52	4	22	36	11.3	25	1
5	9.50	3.8	21	34.2	12	20.4	1
6	4.79	5.4	10	20.1	5	16.5	1
7	6.21	6.2	14	26.5	7	19.7	1
8	11.12	6	25	35.9	9	24.7	0.9614
9	3.67	8	4	17.4	0.1	18.1	1
10	8.93	7	16	34.3	6.5	20.6	0.9536
11	17.74	7.1	43	45.6	14	31.1	0.9831
12	14.85	6.2	27	38.7	13.8	25.4	0.8012

Table 2: The results of Example 1.

FMS	Kao and Hung models [27]			Wang et al. models [20]		Proposed method
	Common model (1)	Common model (2)	Common model (3)	Model (5)	Model (6)	
1	1.0000 (1)	0.9654 (4)	0.9111 (6)	1.0101 (5)	1.0051 (5)	0.8680 (7)
2	0.9766 (5)	0.9616 (6)	0.9026 (7)	0.9867 (7)	0.9818 (7)	0.8370 (9)
3	0.9488 (9)	0.9132 (9)	0.9021 (8)	1.0185 (4)	1.0134 (4)	0.9256 (5)
4	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0702 (2)	1.0649 (2)	0.9783 (3)
5	1.0000 (1)	0.9641 (5)	0.9663 (4)	1.0893 (1)	1.0893 (1)	1.0000 (1)
6	0.9624 (6)	0.9866 (3)	0.9872 (3)	1.0058 (6)	1.0008 (6)	0.9481 (4)
7	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0481 (3)	1.0429 (3)	0.9853 (2)
8	0.9614 (7)	0.9423 (7)	0.9203 (5)	0.9752 (8)	0.9704 (8)	0.8832 (6)
9	0.7528 (12)	0.8462 (10)	0.8760 (9)	0.7190 (12)	0.7155 (12)	0.7462 (12)
10	0.8334 (10)	0.8041 (11)	0.8295 (11)	0.8521 (10)	0.8478 (10)	0.8374 (8)
11	0.9507 (8)	0.9160 (8)	0.8591 (10)	0.9528 (9)	0.9481 (9)	0.8173 (10)
12	0.7943 (11)	0.7750 (12)	0.7602 (12)	0.8501 (11)	0.8460 (11)	0.7618 (11)

4. Numerical Examples

In this section, we provide three numerical examples that all involve a significant number of DEA efficient units. Then we compare them to other methods to show the potential usage of the proposed ranking methodology in the complete ranking of DMUs.

Example 1. Consider the 12 flexible manufacturing systems (FMSs) given in Table 1 with two inputs and four outputs. These data are taken from Shang and Sueyoshi [26]. We solve proposed model for all FMSs and compare results with the Wang et al. [20] and Kao and Hung models [27]. Seeing the results in Table 1, the researcher can realize seven FMSs as CCR efficient FMSs. Therefore, it is impossible to have complete ranking for all FMSs. In order to solve this problem, Kao and Hung [27] suggested to use three common weights of DEA models. The gained results are shown in columns 2, 3, and 4 in Table 2. It is easily understood that in the first model of Kao and Hung [27], four FMSs and other models just generate two FMSs as efficient FMSs. Moreover, these three models illustrate three different ranking, which is a demerit of the proposed models by Kao and Hung [27].

However, the main deficiency in Kao and Hung [27] is that the proposed models related to p as a parameter which is calculated in Example 1 as $P = 1$, $P = 2$, and $P = \infty$. Wang et al. [20] suggested two models utilizing common weights which to some extent eliminate deficiencies of the Kao and Hung models. As it can be seen in Table 2, Wang et al. [20] evaluate FMS_5 as the efficient FMS. The resulted efficiencies and rankings by Wang models are inserted in the fifth and sixth columns of Table 2. Although the two proposed models by Wang et al. [20] have the same ranking, this is not right always. On the other hand, the introduced models are nonlinear ones. Although their method is an interesting approach as a theoretical idea, it could not be efficient from computational point of view. The last column in Table 2 illuminates the results of efficiency and ranking obtained by the proposed model of the paper. It is clear from Table 2 that the proposed model with ideal and anti-ideal points considers FMS_5 as the efficient FMS, and all the other

Table 3: Inputs and Outputs and ranking by AP, MAJ, and new proposed ranking models.

DMU	Input 1	Input 2	Output 1	Output 2	CCR efficiency	θ^*_{AP}	θ^*_{MAJ}	θ^*_{New}
A	2	3	4	5	0.9254 (4)	0.9286 (4)	0.9524 (4)	0.8667 (3)
B	3	4	5	6	0.7602 (5)	0.7619 (5)	0.7619 (5)	0.7173 (5)
C	2	2	4	4	0.9900 (3)	1.0000 (3)	1.0000 (3)	0.8571 (4)
D	2	3	5	4	1.0000 (1)	1.2500 (2)	1.1667 (2)	0.9333 (2)
E	3	4	4	5	0.6174 (7)	0.6100 (7)	0.6190 (7)	0.5909 (7)
F	2	2	4	6	1.0000 (1)	1.5000 (1)	1.2857 (1)	1.0000 (1)
G	3	4	5	4	0.7093 (6)	0.7143 (6)	0.7143 (6)	0.6364 (6)

Table 4: Result of correlation analysis.

	Proposed method	AP method	MAJ method
Proposed method	1	0.945	0.982
AP method	/	1	0.986
MAJ method	/	/	1

Correlation is significant at the 0.01 level (two-tailed).

11 FMSs as nonefficient. As you may notice from Table 2, FMS_9 has the worst performance by Wang et al. and suggested models.

Example 2. Consider the example investigated by Jahanshahloo et al. [28] where all DMUs have two inputs and two outputs. The data are reproduced in Table 3, together with the CCR efficiencies of seven DMUs. The example is solved by two methods: AP and MAJ which are the most popular methods. In accordance with Table 3, the results of ranking by proposed method are almost the same with AP and MAJ ranking methods. On the other hand, we can achieve correlation analysis of the efficiencies gained by imposing these three models. Table 4 states results of analysis. This research has been significant at the level of 0.01. As you can see in Table 4, there is a high correlation between the proposed model and AP and MAJ models. It is worth notifying that if AP and MAJ models are infeasible or unable to present a complete ranking, these three models lack this correlation and ranking declines to its lowest level of correlation analysis. So, please notice the next example.

Example 3. Consider the problem of measuring the performances of five DMUs, where each DMU has two inputs and one output. The data set is shown in Table 5. Here all outputs have been normalized to one for convenience. The Farrell frontier for these DMUs is shown in Figure 1. As can be seen in Figure 1, DMU_A, DMU_D, and DMU_E are extreme efficient DMUs and DMU_B and DMU_C are nonextreme efficient DMUs. The results of this example by using our proposed method, θ^*_{AP} and θ^*_{MAJ}, are documented in Table 5. As you can see in Table 5, AP and MAJ models could not rank DMU_B and DMU_C; however the proposed model here is able to rank both extreme and nonextreme efficient DMUs. It shows the first privilege of the new ranking model over AP and MAJ models. According to the

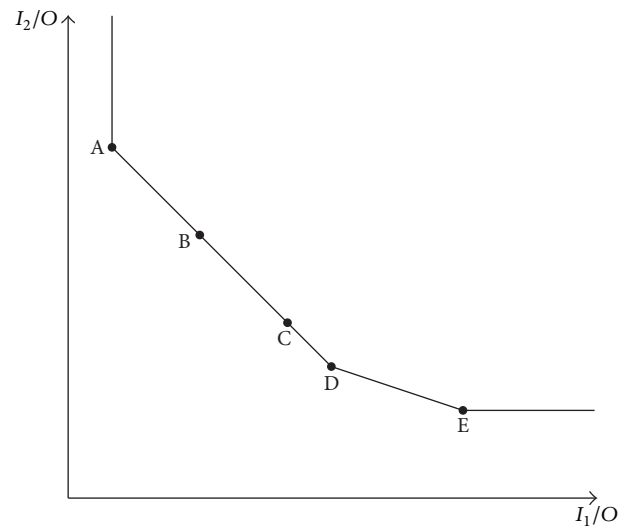

Figure 1: Farrell frontier for five DMUs.

suggested model, performance of the five DMUs is ranked as follows:

$$DMU_A \succ DMU_B \succ DMU_C \succ DMU_D \succ DMU_E, \quad (23)$$

where "$\succ$" denotes superior to. The second advantage of the proposed method is that to obtain a complete ranking for all DMUs; the researcher just needs to solve three programs, though the other models lack this merit. The third privilege is its feasibility; however, in some cases the AP and MAJ models are infeasible. Utilizing correlation analysis at the 0.01 level, then correlation between the suggested method and AP, MAJ is recorded as 0.478 and 0.185, respectively. This poor correlation is for this fact that AP and MAJ are not able to present a comprehensive ranking.

5. Conclusion

In the current paper, we have developed a new mixed integer programming based on ideal and anti-ideal points. In this procedure, firstly we must compute ideal and anti-ideal points to rank all DMUs. Then their efficiency scores could be obtained. Through using the proposed model, all DMUs can be ranked, whereas most of ranking methods cannot do it. One of the prominent features of this model compared

Table 5: Data for Example 3 and their ranking and efficiencies by AP, MAJ, and new proposed ranking models.

DMU	Input 1	Input 2	Output 1	θ^*_{AP}	θ^*_{MAJ}	θ^*_{New}
A	1	8	1	3.0000 (1)	1.2222 (2)	1.0000 (1)
B	3	6	1	1.0000 (4)	1.0000 (4)	0.9048 (2)
C	5	4	1	1.0000 (4)	1.0000 (4)	0.8261 (3)
D	6	3	1	1.0833 (3)	1.0400 (3)	0.7917 (4)
E	9	2	1	1.5000 (2)	1.1250 (1)	0.6129 (5)

to the others is that it is always feasible. On the other hand, traditional DEA models cannot define a DMU with the best performance. However, it can be easily conducted by the proposed model here. The other advantage of this new model is that we are able to rank all the extreme and nonextreme efficient units by solving only three programs. Three numerical examples have been tested and examined by applying the suggested ranking method. The proposed model complies with crisp data. It can be examined further in the future researches in accordance with interval or fuzzy data.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Solution of Fully Fuzzy Quadratic Equation Based on Optimization Theory

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Firstly in this paper we introduce a new concept of the 2nd power of a fuzzy number. It is exponent to production (EP) method that provides an analytical and approximate solution for fully fuzzy quadratic equation (FFQE) : $F(\widetilde{X}) = \widetilde{D}$, where $F(\widetilde{X}) = \widetilde{A}\widetilde{X}^2 + \widetilde{B}\widetilde{X} + \widetilde{C}$. To use the mentioned EP method, at first the 1-cut solution of FFQE as a real root is obtained and then unknown manipulated unsymmetrical spreads are allocated to the core point. To this purpose we find λ and μ as optimum values which construct the best spreads. Finally to illustrate easy application and rich behavior of EP method, several examples are given.

1. Introduction

The problem of finding the roots of equations like the quadratic equation has many applications in applied sciences like finance [1, 2], economy [3–6], and mechanics [7]. Sevastjanov and Dymova [8] proposed a new method for solving interval and fuzzy linear equations. In [9–11] Buckly discussed solving fuzzy equations. Abbasbandy and Otadi in [12] obtained the real valued roots of fuzzy polynomials using fuzzy neural networks. In [13–16] the authors have introduced numerical and neural net solutions to solve fuzzy equations. In the current paper we propose a new method to solve $\widetilde{A}\widetilde{X}^2 + \widetilde{B}\widetilde{X} + \widetilde{C} = \widetilde{D}$ that the complication of arithmetics does not depend on $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{X}$ being positive or negative. This method solves some problems that have no analytical solution and this is an advantage of EP method because, as we know, numerical methods need initial guess to continue and the new method provides this neediness.

The rest of the paper is set out as follows. In the second section some related basic definitions of fuzzy mathematics for the analysis are recalled. In Section 3, a new method, called EP, for solving fully fuzzy quadratic equation is presented. In Section 4, this method is used for an analytical approximate solution of FFQE. In Section 5, the conclusions are drawn.

2. Basic Concepts

The basic definitions are given as follows.

Definition 1 (see [17–20]). A fuzzy number is a function $\widetilde{u} : \mathbb{R} \to [0, 1]$ which satisfies the following:

(1) $\widetilde{u}$ is upper semicontinuous on $\mathbb{R}$;

(2) $\widetilde{u}$ is normal; that is, $\exists x_0 \in \mathbb{R}$ with $\widetilde{u}(x_0) = 1$;

(3) $\widetilde{u}$ is convex fuzzy set;

(4) $\overline{\{x \in \mathbb{R} \mid \widetilde{u}(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Note: in this paper we consider fuzzy numbers which have a unique $x_0 \in \mathbb{R}$ with $\widetilde{u}(x_0) = 1$ [18].

The set of all these fuzzy numbers is denoted by $\mathscr{F}$. Obviously, $\mathbb{R} \subseteq \mathscr{F}$. For $0 < r \leq 1$, we define r-cut of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = \{x \in \mathbb{R} : \widetilde{u}(x) \geq r\}$ and $[\widetilde{u}]_0 = \overline{\{x \in \mathbb{R} : \widetilde{u}(x) \geq 0\}}$. In [8] from (4)–(10) it follows that $[\widetilde{u}]_r$ is a bounded closed interval for each $r \in [0, 1]$. We denote the r-cut of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$.

Definition 2 (see [18]). A fuzzy number $\widetilde{u}$ is positive (negative) if $\widetilde{u}(x) = 0$ for all $x < 0$ $(x > 0)$.

Definition 3 (see [21, 22]). A fuzzy number $\widetilde{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded nondecreasing left continuous function in $[0, 1]$;

(2) $\overline{u}(r)$ is a bounded nonincreasing left continuous function in $[0, 1]$;

(3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Definition 4 (see [21]). For arbitrary $\widetilde{u} = (\underline{u}(r), \overline{u}(r))$ and $\widetilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$, and scalar k, we define addition, subtraction, and scalar product by k and multiplication is, respectively, as follows.

Addition: $\underline{u+v}(r) = \underline{u}(r) + \underline{v}(r)$; $\overline{u+v}(r) = \overline{u}(r) + \overline{v}(r)$.

Subtraction: $\underline{u-v}(r) = \underline{u}(r) - \overline{v}(r)$; $\overline{u-v}(r) = \overline{u}(r) - \underline{v}(r)$.

Scalar product:

$$k\widetilde{u} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases} \tag{1}$$

Multiplication:

$$\begin{aligned} \underline{uv}(r) &= \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\} \\ \overline{uv}(r) &= \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}. \end{aligned} \tag{2}$$

For two important cases multiplication of two fuzzy numbers is defined by the following terms.

If $\widetilde{u} \geq 0$ and $\widetilde{v} \geq 0$, then $\underline{uv}(r) = \underline{u}(r)\underline{v}(r)$ and $\overline{uv}(r) = \overline{u}(r)\overline{v}(r)$.

If $\widetilde{u} \leq 0$ and $\widetilde{v} \leq 0$, then $\underline{uv}(r) = \overline{u}(r)\overline{v}(r)$ and $\overline{uv}(r) = \underline{u}(r)\underline{v}(r)$.

If $\widetilde{u} \geq 0$ and $\widetilde{v} \leq 0$, then $\underline{uv}(r) = \overline{u}(r)\underline{v}(r)$ and $\overline{uv}(r) = \underline{u}(r)\overline{v}(r)$.

If $\widetilde{u} \leq 0$ and $\widetilde{v} \geq 0$, then $\underline{uv}(r) = \underline{u}(r)\overline{v}(r)$ and $\overline{uv}(r) = \underline{u}(r)\overline{v}(r)$.

Arithmetics of r-cuts is similar to arithmetics of the parametric form recalled previously [23, 24].

Definition 5 (see [21]). Two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are said to be equal, if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$, for each $r \in [0, 1]$.

A crisp number α in parametric form is $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. A triangular fuzzy number is popular and represented by $\widetilde{u} = (m, \alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$, which has the parametric form as follows:

$$\underline{u}(r) = m - \alpha + r\alpha, \qquad \overline{u}(r) = m + \beta - r\beta. \tag{3}$$

Definition 6 (see [25]). Let $D : \mathcal{F} \times \mathcal{F} \to \mathbb{R} \cup \{0\}$ and let $D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ be the Hausdorff distance between fuzzy numbers, where $[\widetilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$ and $[\widetilde{v}]_r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well known:

(1) $D(\widetilde{u} \bigoplus \widetilde{w}, \widetilde{v} \bigoplus \widetilde{w}) = D(\widetilde{u}, \widetilde{v})$, for all $\widetilde{u}, \widetilde{v}, \widetilde{w} \in \mathcal{F}$;

(2) $D(k \bigodot \widetilde{u}, k \bigodot \widetilde{v}) = |k| D(\widetilde{u}, \widetilde{v})$, for all $k \in \mathbb{R}$ and $\widetilde{u}, \widetilde{v} \in \mathcal{F}$;

(3) $D(\widetilde{u} \bigoplus \widetilde{v}, \widetilde{w} \bigoplus \widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e})$, for all $\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in \mathcal{F}$.

Therefore $(\mathcal{F}, D)$ is a complete metric space.

3. Exponent to Production Method

Let $\widetilde{A} = (\underline{a}(r), \overline{a}(r))$, $\widetilde{B} = (\underline{b}(r), \overline{b}(r))$, $\widetilde{C} = (\underline{c}(r), \overline{c}(r))$, $\widetilde{D} = (\underline{d}(r), \overline{d}(r))$, $\widetilde{X} = (\underline{x}(r), \overline{x}(r))$ and

$$F(\widetilde{X}) = \widetilde{D}, \tag{4}$$

$$F(\widetilde{X}) = (\underline{F}(\underline{x}, \overline{x}, r), \overline{F}(\underline{x}, \overline{x}, r)), \tag{5}$$

where $F(\widetilde{X}) = \widetilde{A}\widetilde{X}^2 + \widetilde{B}\widetilde{X} + \widetilde{C}$. In this method, we convert the 2nd exponent of a fuzzy number to a product of two fuzzy numbers in parametric form. By this conversion we obtain an analytical and approximate solution for a FFQE. In EP method, at first, we find α, as a real root of crisp 1-cut equation and then we get $\widetilde{X} = (\alpha - f_1(\alpha_1(r), \alpha_2(r)), \alpha + f_2(\alpha_1(r), \alpha_2(r)))$ as solution of (4) and apply the approximation $\widetilde{X}^2 \cong (\alpha - \alpha_1(r), \alpha + \alpha_1(r))(\alpha - \alpha_2(r), \alpha + \alpha_2(r))$, in which $\alpha_1(r) \geqslant 0$, $\alpha_2(r) \geqslant 0$, $\alpha_1'(r) < 0$, and $\alpha_2'(r) < 0$. To find $f_i(\alpha_1(r), \alpha_2(r))$, for $i = 1, 2$, we consider two cases:

(1) FFQE has analytical solution;

(2) FFQE does not have analytical solution.

3.1. Case (1). In this case we have $\underline{x}(0) \leqslant \underline{x}(1) = \overline{x}(1) \leqslant \overline{x}(0)$. Therefore we construct conditions that provide a good approximation. These conditions are as follows:

(1) $f_i(\alpha_1(1), \alpha_2(1)) = 0$, for $i = 1, 2$;

(2) $\alpha - f_1(\alpha_1(0), \alpha_2(0)) = \underline{x}(0)$ and $\alpha + f_2(\alpha_1(0), \alpha_2(0)) = \overline{x}(0)$.

To find $f_i(\alpha_1(r), \alpha_2(r))$, $i = 1, 2$, at first we substitute $\widetilde{X} = (\alpha - f_1(\alpha_1(r), \alpha_2(r)), \alpha + f_2(\alpha_1(r), \alpha_2(r)))$ and $\widetilde{X}^2 \cong (\alpha - \alpha_1(r), \alpha + \alpha_1(r))(\alpha - \alpha_2(r), \alpha + \alpha_2(r))$, in parametric form of (4), and then we obtain

$$\begin{aligned} &(\underline{a}(r), \overline{a}(r))(\alpha - \alpha_1(r), \alpha + \alpha_1(r))(\alpha - \alpha_2(r), \alpha + \alpha_2(r)) \\ &\quad + (\underline{b}(r), \overline{b}(r)) \\ &\quad \times (\alpha - f_1(\alpha_1(r), \alpha_2(r)), \alpha + f_2(\alpha_1(r), \alpha_2(r))) \\ &\quad + (\underline{c}(r), \overline{c}(r)) = (\underline{d}(r), \overline{d}(r)). \end{aligned} \tag{6}$$

Set $r = 1$. Using condition (1) and $\underline{a}(1) = \overline{a}(1)$, $\underline{b}(1) = \overline{b}(1)$, $\underline{c}(1) = \overline{c}(1)$, and $\underline{d}(1) = \overline{d}(1)$, we obtain

$$\alpha(\alpha_1(1) + \alpha_2(1)) = 0, \tag{7}$$

for each $\alpha \in \mathbb{R}$; that means

$$\alpha_1(1) + \alpha_2(1) = 0. \tag{8}$$

By this conclusion we decide to get

$$\begin{aligned} f_1(\alpha_1(r), \alpha_2(r)) &= \lambda(\alpha_1(r) + \alpha_2(r)), \\ f_2(\alpha_1(r), \alpha_2(r)) &= \mu(\alpha_1(r) + \alpha_2(r)), \\ \lambda, \mu > 0, &\quad \lambda, \mu \in \mathbb{R}. \end{aligned} \tag{9}$$

Using condition (2) we have

$$\begin{aligned} \alpha - \lambda(\alpha_1(0) + \alpha_2(0)) &= \underline{x}(0), \\ \alpha + \mu(\alpha_1(0) + \alpha_2(0)) &= \overline{x}(0). \end{aligned} \tag{10}$$

Up to now we have two equations and three unknown λ, μ, and $\chi(0) = \alpha_1(0) + \alpha_2(0)$. The third equation comes from the equality below, for $r = 0$,

$$\begin{aligned} &(\underline{a}(r), \overline{a}(r))(\alpha - \alpha_1(r), \alpha + \alpha_1(r))(\alpha - \alpha_2(r), \alpha + \alpha_2(r)) \\ &\quad + (\underline{b}(r), \overline{b}(r))(\alpha - \lambda(\alpha_1 + \alpha_2)(r), \alpha + \mu(\alpha_1 + \alpha_2)(r)) \\ &\quad + (\underline{c}(r), \overline{c}(r)) = (\underline{d}(r), \overline{d}(r)). \end{aligned} \tag{11}$$

Now we can find λ, μ, and $\chi(0)$. To construct solution of the new method, in the above parametric form, let $\chi(r) = \alpha_1(r) + \alpha_2(r)$ and $\nu(r) = \alpha_1(r) \times \alpha_2(r)$; then, by solving a 2×2 system via $\chi(r)$ and $\nu(r)$, we obtain $\chi(r)$ and we set

$$\widetilde{X} = (\alpha - \lambda\chi(r), \alpha + \mu\chi(r)) \tag{12}$$

as a solution of (4). Notice that always we have real spreads, which means $\alpha_1(r) + \alpha_2(r) \in \mathbb{R}$, because $\alpha_1(r)$ and $\alpha_2(r)$ are the roots of $Z^2 - \chi(r)Z + \nu(r) = 0$.

Using the proposed method we obtain the following set of expressions for $\chi(r)$.

Case (1). $\widetilde{X} > 0$:

(1) if $\widetilde{A} > 0$ and $\widetilde{B} < 0$, then

$$\chi(r) = \frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UU^-(\lambda, r) + \overline{a}(r)LL^-(\mu, r)}; \tag{13}$$

(2) if $\widetilde{A} > 0$ and $\widetilde{B} > 0$, then

$$\chi(r) = \frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UU^+(\mu, r) + \overline{a}(r)LL^+(\lambda, r)}; \tag{14}$$

(3) if $\widetilde{A} < 0$ and $\widetilde{B} > 0$, then

$$\chi(r) = -\frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UU^-(\mu, r) + \overline{a}(r)LL^-(\lambda, r)}; \tag{15}$$

(4) if $\widetilde{A} < 0$ and $\widetilde{B} < 0$, then

$$\chi(r) = -\frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UU^+(\lambda, r) + \overline{a}(r)LL^+(\mu, r)}. \tag{16}$$

Case (2). $\widetilde{X} < 0$:

(1) if $\widetilde{A} > 0$ and $\widetilde{B} < 0$, then

$$\chi(r) = -\frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UL^+(\lambda, r) + \overline{a}(r)LU^+(\mu, r)}; \tag{17}$$

(2) if $\widetilde{A} > 0$ and $\widetilde{B} > 0$, then

$$\chi(r) = -\frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UL^-(\mu, r) + \overline{a}(r)LU^-(\lambda, r)}; \tag{18}$$

(3) if $\widetilde{A} < 0$ and $\widetilde{B} > 0$, then

$$\chi(r) = \frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UL^+(\mu, r) + \overline{a}(r)LU^+(\lambda, r)}; \tag{19}$$

(4) if $\widetilde{A} < 0$ and $\widetilde{B} < 0$, then

$$\chi(r) = \frac{\overline{a}(r)\underline{G}(\alpha, \alpha, r) - \underline{a}(r)\overline{G}(\alpha, \alpha, r)}{\underline{a}(r)UL^-(\lambda, r) + \overline{a}(r)LU^-(\mu, r)}; \tag{20}$$

where

$$\begin{aligned} \underline{G}(\underline{x}, \overline{x}, r) &= \underline{F}(\underline{x}, \overline{x}, r) - \underline{d}(r), \\ \overline{G}(\underline{x}, \overline{x}, r) &= \overline{F}(\underline{x}, \overline{x}, r) - \overline{d}(r), \\ UU^\dagger(\xi, r) &= \alpha\overline{a}(r) \dagger \xi\overline{b}(r), \\ LL^\dagger(\xi, r) &= \alpha\underline{a}(r) \dagger \xi\underline{b}(r), \\ UL^\dagger(\xi, r) &= \alpha\overline{a}(r) \dagger \xi\underline{b}(r), \\ LU^\dagger(\xi, r) &= \alpha\underline{a}(r) \dagger \xi\overline{b}(r), \end{aligned} \tag{21}$$

in which $\dagger \in \{+, -\}$ and $\xi \in \{\lambda, \mu\}$.

3.2. Case (2). In this case we do not have analytical solution, and we do not have $\underline{x}(0)$ and $\overline{x}(0)$ in which $\underline{x}(0) \leqslant \underline{x}(1) = \overline{x}(1) \leqslant \overline{x}(0)$; therefore we propose that $\lambda = \mu = 0.5$ because of Lemma 7.

Lemma 7. *If α is real core point of $F(\widetilde{X}) = \widetilde{D}$ and we do not have analytical solution, then $\lambda = \mu = 0.5$ is the best choice in EP method.*

Proof. Without loss of generality suppose that $\widetilde{X} > 0$. Let

$$\begin{aligned} \widetilde{Y}\widetilde{Z} &= (\alpha - \alpha_1(r), \alpha + \alpha_1(r))(\alpha - \alpha_2(r), \alpha + \alpha_2(r)), \\ \widetilde{X} &= (\alpha - \lambda(\alpha_1(r) + \alpha_2(r)), \alpha + \mu(\alpha_1(r) + \alpha_2(r))). \end{aligned} \tag{22}$$

In EP method we use the approximation $\widetilde{X}^2 \cong \widetilde{Y}\widetilde{Z}$.

To find the optimum parameters λ and μ, we must solve the minimization problem as follows:

$$\begin{aligned} &\text{Min} \quad D\left(\widetilde{X}^2, \widetilde{Y}\widetilde{Z}\right) \\ &\text{st.} \quad \lambda, \mu \in \mathbb{R}, \\ &\quad \lambda > 0, \quad \mu > 0, \end{aligned} \tag{23}$$

where $D(\widetilde{X}^2, \widetilde{Y}\widetilde{Z}) = \sup_{r\in[0,1]} \max\{|\alpha(2\lambda - 1)\chi(r) + \nu(r) - \lambda^2\chi^2(r)|, |\alpha(1 - 2\mu)\chi(r) + \nu(r) - \mu^2\chi^2(r)|\}$.

To delimitate maximum error for any $\alpha \in \mathbb{R}$, we choose $\lambda = \mu = 0.5$. This completes the proof. □

Lemma 8. *Necessary condition for existence of EP solution with $\lambda = \mu = 0.5$ is*

$$\overline{a}(0)\,\underline{G}(\alpha,\alpha,0) - \underline{a}(0)\,\overline{G}(\alpha,\alpha,0) \geq 0, \quad \text{if } \widetilde{A} > 0, \tag{24}$$

$$\overline{a}(0)\,\underline{G}(\alpha,\alpha,0) - \underline{a}(0)\,\overline{G}(\alpha,\alpha,0) \leq 0, \quad \text{if } \widetilde{A} < 0. \tag{25}$$

Proof. Since $\chi(r)$ is sum of $\alpha_1(r)$ and $\alpha_2(r)$ and we want $\alpha_1(r)$ and $\alpha_2(r)$ to be nonnegative, then necessary condition for existence of EP solution with $\lambda = \mu = 0.5$ is $\chi(r) \geq 0$ and since $[\alpha - 0.5\chi(r), \alpha + 0.5\chi(r)] \subseteq [\alpha - 0.5\chi(0), \alpha + 0.5\chi(0)]$, for all $0 \leq r \leq 1$, it is sufficient to have $\chi(0) \geq 0$. Considering denominators of (13)–(20) we find that (24) and (25) hold if $\chi(0) \geq 0$, and this completes the proof. □

Lemma 9. *The EP method with $\lambda = \mu = 0.5$ does not have solution,*

$$\text{if } \overline{a}(0)\,\underline{G}(\alpha,\alpha,0) - \underline{a}(0)\,\overline{G}(\alpha,\alpha,0) < 0, \quad \text{when } \widetilde{A} > 0, \tag{26}$$

$$\text{if } \overline{a}(0)\,\underline{G}(\alpha,\alpha,0) - \underline{a}(0)\,\overline{G}(\alpha,\alpha,0) > 0, \quad \text{when } \widetilde{A} < 0. \tag{27}$$

Proof. This lemma is conclusion of Lemma 8. □

Lemma 10. *Suppose $\lambda = \mu = 0.5$ in EP method and $\chi(0) \geq 0$, then sufficient condition for existence of solution is $\nu(0) \geq 0$.*

Proof. We know $\alpha_1(r)$ and $\alpha_2(r)$ are nonnegative if $\chi(r) \geq 0$ and $\nu(r) \geq 0$.

Because of construction of EP method for $\lambda = \mu = 0.5$ and Lemma 8, it is obvious that we must have $\nu(0) \geq 0$. □

Lemma 11. *Suppose $\chi(0) \geq 0$ and $\lambda = \mu = 0.5$ in EP method, then we have solution with $\widetilde{X} > 0$ as follows:*

(1) *if $\widetilde{A} > 0$ and $\widetilde{B} < 0$, then*

$$\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \geq \left[UU^-(0.5,0) + LL^-(0.5,0)\right]\chi(0); \tag{28}$$

(2) *if $\widetilde{A} > 0$ and $\widetilde{B} > 0$, then*

$$\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \geq \left[UU^+(0.5,0) + LL^+(0.5,0)\right]\chi(0); \tag{29}$$

(3) *if $\widetilde{A} < 0$ and $\widetilde{B} > 0$, then*

$$\begin{aligned} &\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \\ &\quad \geq -\left[UU^-(0.5,0) + LL^-(0.5,0)\right]\chi(0); \end{aligned} \tag{30}$$

(4) *if $\widetilde{A} < 0$ and $\widetilde{B} < 0$, then*

$$\begin{aligned} &\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \\ &\quad \geq -\left[UU^+(0.5,0) + LL^+(0.5,0)\right]\chi(0), \end{aligned} \tag{31}$$

with $\widetilde{X} < 0$;

(1) *if $\widetilde{A} > 0$ and $\widetilde{B} < 0$, then*

$$\begin{aligned} &\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \\ &\quad \geq -\left[LU^+(0.5,0) + UL^+(0.5,0)\right]\chi(0); \end{aligned} \tag{32}$$

(2) *if $\widetilde{A} > 0$ and $\widetilde{B} > 0$, then*

$$\begin{aligned} &\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \\ &\quad \geq -\left[LU^-(0.5,0) + UL^-(0.5,0)\right]\chi(0); \end{aligned} \tag{33}$$

(3) *if $\widetilde{A} < 0$ and $\widetilde{B} > 0$, then*

$$\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \geq \left[LU^+(0.5,0) + UL^+(0.5,0)\right]\chi(0); \tag{34}$$

(4) *if $\widetilde{A} < 0$ and $\widetilde{B} < 0$, then*

$$\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \geq \left[LU^-(0.5,0) + UL^-(0.5,0)\right]\chi(0). \tag{35}$$

Proof. We consider only one case to discuss. We consider $\widetilde{A} > 0$, $\widetilde{B} < 0$, and $\widetilde{X} > 0$ and solve (11) via $\chi(r)$ and $\nu(r)$ in $r = 0$; we obtain

$$\begin{aligned} \nu(0) = &\Big(\underline{G}(\alpha,\alpha,0) - \overline{G}(\alpha,\alpha,0) \\ &- \left(UU^-(0.5,0) + LL^-(0.5,0)\right)\chi(0)\Big) \\ &\times \left[\overline{a}(0) - \underline{a}(0)\right]^{-1}. \end{aligned} \tag{36}$$

This is obvious that $\nu(0) \geq 0$, if the numerator is nonnegative and this completes the proof. □

4. Numerical Examples

In the next examples we use round numbers with approximation less than 10^{-4}.

Example 1. Let $\widetilde{A} = (4/1/1)$, $\widetilde{B} = (2/1/1)$, $\widetilde{C} = (1/1/1)$, and $\widetilde{D} = (3/2/2)$ [9].

We will look for a solution where $\widetilde{X} \geq 0$.

Equation $F(\widetilde{X}) = \widetilde{D}$ becomes in parametric form as follows:

$$(3+r, 5-r)\left(\underline{x}^2(r), \overline{x}^2(r)\right) + (1+r, 3-r)\left(\underline{x}(r), \overline{x}(r)\right) + (r, 2-r) = (1+2r, 5-2r). \tag{37}$$

The real roots of 1-cut and 0-cut equations are $\alpha = \underline{x}(1) = \overline{x}(1) = 0.5$, $\underline{x}(0) = 0.4343$, and $\overline{x}(0) = 0.5307$.

Therefore we have analytical solution and EP solution by (24) and (29).

By (10) and (14), we find $\lambda = 0.7067$, $\mu = 0.3296$, and

$$\chi(r) = \frac{2-2r}{-r^2+2r+15+\mu\left(9-r^2\right)-\lambda\left(r^2-4r-5\right)}. \tag{38}$$

In this example, with $\widetilde{X} = (0.5-\lambda\chi(r), 0.5+\mu\chi(r))$, Hausdorff metric is $D(F(\widetilde{X}), \widetilde{D}) = 0.0228$.

Example 2. Letting $\widetilde{A} = (4/2/2)$, $\widetilde{B} = (2/2/2)$, and $\widetilde{D} = (1/0.5/0.5)$, we have

$$(2+2r, 6-2r)\left(\underline{x}^2(r), \overline{x}^2(r)\right) + (2r, 4-2r)\left(\underline{x}(r), \overline{x}(r)\right) = (0.5+0.5r, 1.5-0.5r). \tag{39}$$

The real roots of 1-cut and 0-cut equations are $\alpha = \underline{x}(1) = \overline{x}(1) = (-1+\sqrt{5})/4 = 0.3090$, $\underline{x}(0) = 0.5$, and $\overline{x}(0) = 0.2676$.

Therefore this example does not have analytical solution. We look for EP solution. By (26) we find that this example does not have EP solution with $\lambda = \mu = 0.5$ too.

Now we consider an example with $\widetilde{A} < 0$, $\widetilde{B} > 0$, and $\widetilde{X} > 0$.

Example 3. Letting $\widetilde{A} = (-2/1/1)$, $\widetilde{B} = (3/1/2)$, and $\widetilde{D} = (1/1/2)$, we have

$$(-3+r, -1-r)\left(\underline{x}^2(r), \overline{x}^2(r)\right) + (2+r, 5-2r)\left(\underline{x}(r), \overline{x}(r)\right) = (r, 3-2r). \tag{40}$$

The real roots of 1-cut and 0-cut equations are $\alpha = \underline{x}(1) = \overline{x}(1) = 0.5, 1$, $\underline{x}(0) = 0.7838, 1.2527$, and $\overline{x}(0) = 0.7229, 0.914$.

Therefore, this example does not have analytical solution. We look for EP solution. By (25), (30), and (15), we find that this example has EP solution with $\lambda = \mu = 0.5$ as follows: $\widetilde{X} = (0.5-0.5\chi(r), 0.5+0.5\chi(r))$ that

$$\chi(r) = \frac{-r^2+6r-5}{-r^2+4r-23}, \tag{41}$$

and 0-cut of EP solution is $\underline{x}(0) = 0.3913$ and $\overline{x}(0) = 0.6087$. In this example by using Hausdorff metric we have $D(F(\widetilde{X}), \widetilde{D}) = 0.3290$.

Notice that the numerical methods needed $\underline{x}(0)$ and $\overline{x}(0)$ to obtain initial guess and often these values achieve analytical solution, but in Example 3 these values achieve EP method because in this example we do not have analytical solution.

5. Conclusion

In this paper we introduced a new method to solve a fully fuzzy quadratic equation. To this purpose we found the optimum spreads to decrease maximum error. One of the advantages of this method is that complications do not depend on the sign of the coefficients and variable. It is possible that these equations do not have any analytical solution, but the proposed method gives us an approximate analytical solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Using Enhanced Russell Model to Solve Inverse Data Envelopment Analysis Problems

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This paper studies the inverse data envelopment analysis using the nonradial enhanced Russell model. Necessary and sufficient conditions for inputs/outputs determination are introduced based on Pareto solutions of multiple-objective linear programming. In addition, an approach is investigated to identify extra input/lack output in each of input/output components (maximum/minimum reduction/increase amounts in each a of input/output components). In addition, the following question is addressed: if among a group of DMUs, it is required to increase inputs and outputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMUs, how much should the inputs and outputs of the DMU increase? This question is discussed as inverse data envelopment analysis problems, and a technique is suggested to answer this question. Necessary and sufficient conditions are established by employing Pareto solutions of multiple-objective linear programming as well.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric method based on linear programming for computing relative efficiencies of a decision making unit (DMU) by comparing it with other DMUs such that they produce the same multiple outputs by consuming the same multiple inputs. This method was first introduced by Charnes et al. (CCR model) [1] and extended by Banker et al. (BCC model) [2]. In the two recent decades, a wide range of research in operations research field has been allocated to this technique; see, for example, [3–5].

Relationships between DEA and MOLP can be applied as instruments in strategic planning and management control. These two types of models have much in common. However, DEA is to assess past performances as part of the management control function while MOLP is to plan future performances. These relationships have been used and developed by many scholars; see, for example, [6–12].

Inverse DEA is one of the most noticeable subjects both practically and theoretically. This concept first was discussed by Zhang and Cui [13]. Some of the questions are introduced by Wei et al. [14] in inverse DEA filed. They considered inverse DEA to answer this question: "if among a group of DMUs, we increase certain inputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMUs, how much should the outputs of the DMU increase?" In order to answer this question, Wei et al. [14] and Yan et al. [15] offered a linear programming problem to estimate the outputs when the unit under assessment is weakly efficient and a MOLP model when the unit under assessment is inefficient.

Moreover, in the following, Hadi-Vencheh et al. [16, 17] attempted to answer this question: "if among a group of DMUs, we increase certain outputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMUs, how much should the inputs of the DMU increase?" In their studies, the proposed models by Wei et al. [14] have been developed. After introducing inverse DEA, some of researchers studied it theoretically and practically [15–21].

In all investigations done on inverse DEA, researchers considered radial models such as CCR [1], BCC [2], ST [22], and FG [23] models. As it is known, the nonradial models have some different properties compared with the radial models. Therefore, in some cases, answering the questions presented in the literature inverse DEA with the

nonradial models can provide more appropriate information. Consequently, for more suitable analysis, one of the nonradial models, for instance, additive models or slack-based models, can be considered. In this research, to solve some of the above problems, the nonradial Enhanced Russell Model (ERM) [24] is considered. In addition, a new problem in inverse DEA field is introduced: "if among a group of DMUs for a particular unit, the decision maker is required to increase inputs and outputs, in which, the DMU maintains its current efficiency level with respect to other DMUs, how much should the inputs and outputs of the DMU increase?" Pareto solutions of MOLP are used to solve this problem.

The paper is organized as follows. In Section 2, DEA models are reviewed and extended and the problem is stated in DEA terms. It is shown that how the inverse DEA problem (increment of the outputs) can be converted to and solved by a multiobjective programming problem, when the DMU is ERM efficient. If there exists a lack in each of the output components, its amount is specified. In Section 3, the question proposed by Wei et al. [14] is answered, when the DMU is ERM efficient. Likewise, if there exists an extra in each of the input components, its amount is specified. The new problem is solved in Section 4. For a special decision making unit providing that the ERM-efficiency score remains unchanged, necessary and sufficient conditions are introduced based on Pareto solutions of multiple-objective linear programming problems to find the minimum and maximum increase of input and output vectors, respectively. In Section 5, two examples are used to illustrate our calculation method. Finally, Section 6 demonstrates some conclusions and suggestions for future research.

2. Estimate Inputs

Let us consider a set of $\mathrm{DMU}_s\{\mathrm{DMU}_j : j = 1,\dots,n\}$, in which DMU_j produces multiple outputs y_{rj} $(r = 1,\dots,s)$, by utilizing multiple inputs x_{ij} $(i = 1,\dots,m)$. Let input and output for DMU_j be denoted by $X_j = (x_{1j}, x_{2j},\dots,x_{mj})^t$ and $Y_j = (y_{1j}, y_{2j},\dots,y_{sj})^t$, respectively. Also, suppose $X_j \gneqq 0$ and $Y_j \gneqq 0$. The nonradial enhanced Russell model (ERM) [24] is considered for measuring the relative efficiency of the unit under assessment DMU_o, $o \in \{1, 2,\dots,n\}$, as follows:

$$\begin{aligned}
\rho_o^* = \min \quad & \frac{(1/m)\sum_{i=1}^m \theta_i}{(1/s)\sum_{r=1}^s \varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \le \theta_i x_{io}, \quad i = 1,\dots,m \\
& \sum_{j=1}^n \lambda_j y_{rj} \ge \varphi_r y_{ro}, \quad r = 1,\dots,s \\
& \theta_i \le 1, \qquad \varphi_r \ge 1, \quad i = 1,\dots,m,\ r = 1,\dots,s, \\
& \lambda \in \Omega,
\end{aligned} \tag{1}$$

where

$$\Omega = \left\{ \lambda \mid \lambda = (\lambda_1,\dots,\lambda_n), \delta_1 \left(\sum_{j=1}^n \lambda_j + \delta_2 (-1)^{\delta_3} \nu \right) = \delta_1, \right.$$
$$\left. \nu \ge 0,\ \lambda_j \ge 0,\ j = 1,\dots,n \right\}. \tag{2}$$

Here δ_1, δ_2, and δ_3 are parameters with 0-1 values. It is obvious that if $\delta_1 = 0$, then model (1) is under constant returns to scale (CRC), if $\delta_1 = 1$ and $\delta_2 = 0$, then model (1) is under variable returns to scale (VRS), if $\delta_1 = \delta_2 = 1$ and $\delta_3 = 0$, then model (1) is under a nonincreasing returns to scale (NIRS), and if $\delta_1 = \delta_2 = \delta_3 = 1$, then model (1) is under a nondecreasing returns to scale (NDRS) assumption of the production technology.

Remark 1. Although in this study under discussion results satisfy CRC, VRS, NIRS, and NDRS assumption of the production technology, but the CRC model is considered only for simplicity.

Definition 2 (ERM-efficiency, see [24]). The optimal value ρ_o^* of the model (1) is called the ERM-efficiency score of DMU_o. DMU_o is ERM efficient, if and only if $\rho_o^* = 1$ (this condition is equivalent to $\theta_i^* = 1$ and $\varphi_r^* = 1$ for each $i = 1,\dots,m$, $r = 1,\dots,s$ in any optimal solution).

In [13] Zhang and Cui introduced inverse DEA. Since then this problem has allocated to itself some of researches in DEA field; see, for example [5, 14, 16–19, 21]. Based upon investigated results by Hadi-Vencheh et al. [16], this question is considered: suppose that the DMU_o is ERM efficient, if the ERM-efficiency score of DMU_o remains unchanged, but the outputs increase, how much should the inputs of the DMU_o increase?

To answer the above question, until the end of this section, presume that the outputs of DMU_o are increased from Y_o to $\beta_o = Y_o + \Delta Y_o$, where $\Delta Y_o \gneqq 0$. The objective of the problem is to estimate the input vector on the condition that DMU_o is still ERM efficient. In fact,

$$\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*,\dots,\alpha_{mo}^*)^t = X_o + \Delta X_o, \quad \Delta X_o \gneqq 0. \tag{3}$$

Assume DMU_{n+1} represents DMU_o after modification of the inputs and outputs. The following model is considered to calculate the ERM efficiency of DMU_{n+1}:

$$\begin{aligned}
\rho_o^{+*} = \min \quad & \frac{(1/m)\sum_{i=1}^m \theta_i}{(1/s)\sum_{r=1}^s \varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + \alpha_{io}^* \lambda_{n+1} \le \theta_i \alpha_{io}^*, \quad i = 1,\dots,m \\
& \sum_{j=1}^n \lambda_j y_{rj} + \beta_{ro} \lambda_{n+1} \ge \varphi_r \beta_{ro}, \quad r = 1,\dots,s
\end{aligned}$$

$$\theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s$$
$$\lambda_j \ge 0, \quad j = 1, \ldots, n+1. \tag{4}$$

Along with [16], the following definition is considered.

Definition 3. If the optimal values of models (1) and (4) are equal, it is said that the ERM efficiency remains unchanged; that is, $\text{eff}(\alpha_o^*, \beta_o) = \text{eff}(X_o, Y_o)$.

To answer the above question, based on the results of [16], the following MOLP model is considered:

$$\begin{aligned} \min \quad & (\alpha_{1o}, \ldots, \alpha_{mo}) \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_i^* \alpha_{io}, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi_r^* \beta_{ro}, \quad r = 1, \ldots, s \\ & \alpha_{io} \ge x_{io}, \quad i = 1, \ldots, m \\ & \lambda_j \ge 0, \quad j = 1, \ldots, n, \end{aligned} \tag{5}$$

where $(\theta^* = (\theta_1^*, \ldots, \theta_m^*), \varphi^* = (\varphi_1^*, \ldots, \varphi_s^*))$ is an optimal solution to problem (1).

Definition 4 (see [16]). Let (λ^*, α_o^*) be a feasible solution to problem (5). If there is no feasible solution (λ, α_o) of (5) such that $\alpha_{io} \le \alpha_{io}^*$ for all $i = 1, \ldots, m$ and $\alpha_{io} < \alpha_{io}^*$ for at least one i, then it is said the (λ^*, α_o^*) is a Pareto (strongly efficient) solution to problem (5).

Definition 5 (see [14]). Let (λ^*, α_o^*) be a feasible solution to problem (5). If there is no feasible solution (λ, α_o) of (5) such that $\alpha_{io} < \alpha_{io}^*$ for all $i = 1, \ldots, m$, then it is said the (λ^*, α_o^*) is a weakly Pareto (efficient) solution to problem (5).

Theorem 6. *Suppose that the DMU_o is ERM efficient and $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem* (1) *($\theta_i^* = 1, \varphi_r^* = 1$ for all i, r and $\rho_o^* = 1$). Let $(\widehat{\lambda}^*, \widehat{\alpha}_o^*)$ be a Pareto solution to problem* (5) *such that $\widehat{\alpha}_o^* > X_o$, and $(\lambda^{+*}, \theta^{+*}, \varphi^{+*})$ is an optimal solution to problem* (4) *with the optimal value of ρ_o^{+*}. In addition, suppose that the optimal value and an optimal solution of the problem*

$$\begin{aligned} K_o^* = \max \quad & \frac{1}{s} \sum_{r=1}^{s} k_{ro} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \left(\lambda_j^{+*} + \lambda_{n+1}^{+*} \widehat{\lambda}_j^* \right) y_{rj} \ge k_{ro} \varphi_r^* \beta_{ro} = k_{ro} \beta_{ro}, \\ & \qquad r = 1, \ldots, s \\ & k_{ro} \ge 1, \quad r = 1, \ldots, s \end{aligned} \tag{6}$$

are K_o^* and $k_o^* = (k_{1o}^*, \ldots, k_{so}^*)$, respectively. If the inputs of the DMU_o are increased to $\widehat{\alpha}_o^*$, then

(i) *if $K_o^* = 1$ (it is clear $k_{ro}^* = 1$, for each r), then $\text{eff}(\widehat{\alpha}_o^*, \beta_o) = \text{eff}(X_o, Y_o)$,*

(ii) *if $K_o^* > 1$, then $\text{eff}(\widehat{\alpha}_o^*, k_o^* \star \beta_o) = \text{eff}(X_o, Y_o)$.*

Remark 7. $k_o^* \star \beta_o = (k_{1o}^* \beta_{1o}, \ldots, k_{so}^* \beta_{so})$ and $(k_o^* \star \beta_o) - \beta_o = (k_{1o}^* \beta_{1o} - \beta_{1o}, \ldots, k_{so}^* \beta_{so} - \beta_{so})$. Note that the $k_{ro}^* \beta_{ro} - \beta_{ro}$ indicates the lack-output amount in rth output component of the DMU_o. In other words, for the decision maker to preserve the ERM-efficiency score of the DMU_o while the inputs increase from X_o to $\widehat{\alpha}_o^*$, they are required to increase the outputs from Y_o to $k_o^* \star \beta_o$.

Proof. Because $(\widehat{\lambda}^*, \widehat{\alpha}_o^*)$ is a feasible solution of (5), the following inequalities are held:

$$\sum_{j=1}^{n} \widehat{\lambda}_j^* x_{ij} \le \theta_i^* \widehat{\alpha}_{io}^* = \widehat{\alpha}_{io}^*, \quad i = 1, \ldots, m \tag{7}$$

$$\sum_{j=1}^{n} \widehat{\lambda}_j^* y_{rj} \ge \varphi_r^* \beta_{ro} = \beta_{ro}, \quad r = 1, \ldots, s \tag{8}$$

$$\widehat{\alpha}_{io}^* \ge x_{io}, \quad i = 1, \ldots, m \tag{9}$$

$$\widehat{\lambda}_j^* \ge 0, \quad j = 1, \ldots, n. \tag{10}$$

With regard to inequalities (7), (8), and (10) it is obvious that $(\overline{\lambda}, \theta^*, \varphi^*)$ is a feasible solution to problem (4), where $\overline{\lambda} = (\widehat{\lambda}^*, 0) \ge 0$, $\theta^* = (\theta_1^*, \ldots, \theta_m^*) \in R_+^m$, and $\varphi^* = (\varphi_1^*, \ldots, \varphi_s^*) \in R_+^s$, therefore, $\rho_o^{+*} \le \rho_o^*$.

Using inequalities (7) and (8) in problem (4), the following results are obtained:

$$\begin{aligned} \theta_i^{+*} \widehat{\alpha}_{io}^* &\ge \sum_{j=1}^{n} \lambda_j^{+*} x_{ij} + \lambda_{n+1}^{+*} \widehat{\alpha}_{io}^* \\ &= \sum_{j=1}^{n} \lambda_j^{+*} x_{ij} + \lambda_{n+1}^{+*} \left(\sum_{j=1}^{n} \widehat{\lambda}_j^* x_{ij} \right), \end{aligned} \tag{11}$$

$$\theta_i^{+*} \widehat{\alpha}_{io}^* \ge \sum_{j=1}^{n} \left(\lambda_j^{+*} + \lambda_{n+1}^{+*} \widehat{\lambda}_j^* \right) x_{ij}, \quad i = 1, \ldots, m$$

$$\begin{aligned} \varphi_r^{+*} \beta_{ro} &\le \sum_{j=1}^{n} \lambda_j^{+*} y_{rj} + \lambda_{n+1}^{+*} \beta_{ro} \\ &\le \sum_{j=1}^{n} \lambda_j^{+*} y_{rj} + \lambda_{n+1}^{+*} \left(\sum_{j=1}^{n} \widehat{\lambda}_j^* y_{rj} \right), \end{aligned} \tag{12}$$

$$\varphi_r^{+*} \beta_{ro} \le \sum_{j=1}^{n} \left(\lambda_j^{+*} + \lambda_{n+1}^{+*} \widehat{\lambda}_j^* \right) y_{rj}, \quad r = 1, \ldots, s.$$

Set $\widetilde{\lambda}_j = \lambda_j^{+*} + \lambda_{n+1}^{+*} \widehat{\lambda}_j^*$ for each $j = 1, \ldots, n$, and it is obviously seen that $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n) \ge 0$.

Now by contradiction assume that $\rho_o^{+*} < \rho_o^*$.

Since $\rho_o^{+*} < \rho_o^*$, then $(1/m)\sum_{i=1}^m \theta_i^{+*} < 1$ or $(1/s)\sum_{r=1}^s \varphi_r^{+*} > 1$, so that there are two cases.

(a) Let $(1/m)\sum_{i=1}^m \theta_i^{+*} < 1$. There exists at least one t, $1 \le t \le m$, such that $\theta_t^{+*} < 1$. With regard to inequalities (11), the following inequality is obtained:

$$\sum_{j=1}^n \tilde{\lambda}_j x_{tj} \le \theta_t^{+*} \hat{\alpha}_{to}^* < \hat{\alpha}_{to}^*. \tag{13}$$

On the other hand, since, for each $i = 1, \ldots, m$, $\hat{\alpha}_{io}^* > x_{io}$, therefore if

$$\eta_t = \min\left\{\hat{\alpha}_{to}^* - \sum_{j=1}^n \tilde{\lambda}_j x_{tj}, \hat{\alpha}_{to}^* - x_{to}\right\}, \tag{14}$$

then $\eta_t > 0$. Now define

$$\tilde{\alpha}_{io} = \begin{cases} \hat{\alpha}_{io}^*, & \text{if } i \ne t, \\ \hat{\alpha}_{io}^* - \eta_t, & \text{if } i = t. \end{cases} \tag{15}$$

Taking (14) into consideration, the following inequalities are obtained:

$$\eta_t \le \hat{\alpha}_{to}^* - \sum_{j=1}^n \tilde{\lambda}_j x_{tj} \Longrightarrow \sum_{j=1}^n \tilde{\lambda}_j x_{tj} \le \hat{\alpha}_{to}^* - \eta_t = \theta_t^* \tilde{\alpha}_{to}, \tag{16}$$

$$\eta_t \le \hat{\alpha}_{to}^* - x_{to} \Longrightarrow x_{to} \le \hat{\alpha}_{to}^* - \eta_t, \Longrightarrow x_{to} \le \tilde{\alpha}_{to}, \tag{17}$$

which implies that $\tilde{\alpha}_o \ge X_o$.

Based on $\tilde{\lambda} \ge 0$ and (11), (12), and (16) and $\tilde{\alpha}_o \ge X_o$, $(\tilde{\lambda}, \tilde{\alpha}_o)$ is a feasible solution to problem (5), where $\tilde{\alpha}_{io} \le \hat{\alpha}_{io}^*$ for all $i = 1, \ldots, m$, and $\tilde{\alpha}_{io} < \hat{\alpha}_{io}^*$ for some $i = 1, \ldots, m$. This contradicts the assumption that $(\hat{\lambda}^*, \hat{\alpha}_o^*)$ is a Pareto solution to problem (5).

(b) Let $(1/s)\sum_{r=1}^s \varphi_r^{+*} > 1$. There exists at least one t, $1 \le t \le s$, such that $\varphi_t^{+*} > 1$. By (12), the following strict inequality is obtained:

$$\sum_{j=1}^n \tilde{\lambda}_j y_{tj} \ge \varphi_t^{+*} \beta_{to} > \beta_{to} \tag{18}$$

and, therefore, there exists a $\mu_t > 1$ that satisfies

$$\sum_{j=1}^n \tilde{\lambda}_j y_{tj} \ge \mu_t \beta_{to} = (\mu_t k_{to}^*) \beta_{to}. \tag{19}$$

Now define

$$\tilde{k}_{ro} = \begin{cases} k_{ro}^* = 1, & \text{if } r \ne t, \\ k_{ro}^* \mu_t = \mu_t, & \text{if } r = t. \end{cases} \tag{20}$$

It is obvious that $\tilde{k}_o = (\tilde{k}_{1o}, \ldots, \tilde{k}_{so})$ is a feasible solution to problem (6), such that

$$K_o^* = \frac{1}{s}\sum_{r=1}^s k_{ro} < \frac{1}{s}\sum_{r=1}^s \tilde{k}_{ro} = \tilde{K}_o. \tag{21}$$

This contradicts the assumption expressing that the optimal value of problem (6) is K_o^*. Consequently, in each case $\rho_o^{+*} \not< \rho_o^*$, and since $\rho_o^{+*} \leqslant \rho_o^*$, therefore, $\rho_o^{+*} = \rho_o^*$.

(ii) with replacing the notation β_{ro} by $k_{ro}^*\beta_{ro}$ for each $r = 1, \ldots, s$, in problem (6), clearly, optimal value of problem (6) is 1. Therefore, according to part (i), then $\text{eff}(\hat{\alpha}_o^*, k_o^* \star \beta_o) = \text{eff}(X_o, Y_o)$. □

Although Theorem 6 satisfies for all units that be ERM-efficient, but the following theorem is the converse version of Theorem 6 that satisfies for all DMU_s.

Theorem 8. *Suppose that the* $(\lambda^*, \theta^*, \varphi^*)$ *is an optimal solution to problem* (1) *with the optimal value of* ρ_o^*. *Let* $(\bar{\lambda}, \bar{\alpha}_o)$ *be a feasible solution to problem* (5). *If* $\text{eff}(\bar{\alpha}_o, \beta_o) = \text{eff}(X_o, Y_o)$, *then* $(\bar{\lambda}, \bar{\alpha}_o)$ *must be a Pareto solution to problem* (5).

Proof. If $(\bar{\lambda}, \bar{\alpha}_o)$ is not a Pareto solution to problem (5), then there would exist another feasible solution of (5), $(\tilde{\lambda}, \tilde{\alpha}_o)$ in which, $\tilde{\alpha}_{io} \le \bar{\alpha}_{io}$ for $i = 1, \ldots, m$ and $\tilde{\alpha}_{io} < \bar{\alpha}_{io}$ for at least one $i \in \{1, \ldots, m\}$. Let $I = \{i \mid \tilde{\alpha}_{io} < \bar{\alpha}_{io}\}$. Then

$$\sum_{j=1}^n \tilde{\lambda}_j x_{ij} \le \theta_i^* \tilde{\alpha}_{io} < \theta_i^* \bar{\alpha}_{io}, \quad i \in I, \tag{22}$$

$$\sum_{j=1}^n \tilde{\lambda}_j x_{ij} \le \theta_i^* \tilde{\alpha}_{io} \le \theta_i^* \bar{\alpha}_{io}, \quad i \notin I, \tag{23}$$

$$\sum_{j=1}^n \tilde{\lambda}_j y_{rj} \ge \varphi_r^* \beta_{ro}, \quad r = 1, \ldots, s, \tag{24}$$

$$\tilde{\lambda}_j \ge 0, \quad j = 1, \ldots, n. \tag{25}$$

By inequality (22) there exists $\mu_i < 1$ for all $i \in I$, such that

$$\sum_{j=1}^n \tilde{\lambda}_j x_{ij} \le (\mu_i \theta_i^*) \bar{\alpha}_{io}, \quad i \in I. \tag{26}$$

For each $i = 1, \ldots, m$ and $r = 1, \ldots, s$, define

$$\bar{\theta}_i = \begin{cases} \mu_i \theta_i^* & \text{if } i \in I, \\ \theta_i^*, & \text{if } i \notin I, \end{cases} \qquad \bar{\varphi}_r = \varphi_r^*, \quad r = 1, \ldots, s. \tag{27}$$

Based on inequalities (23)–(26), $(\tilde{\lambda}, \bar{\theta}, \bar{\varphi})$ is a feasible solution to problem (1) (only restrictions on the right have replaced the X_o and Y_o by $\bar{\alpha}_o$ and β_o in (1), resp.), where $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_m)$ and $\bar{\varphi} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_s)$. Therefore

$$\bar{\rho}_o = \frac{(1/m)\sum_{i=1}^m \bar{\theta}_i}{(1/s)\sum_{r=1}^s \bar{\varphi}_r} < \frac{(1/m)\sum_{i=1}^m \theta_i^*}{(1/s)\sum_{r=1}^s \varphi_r^*} = \rho_o^*, \tag{28}$$

which is against the assumption expressing that $\text{eff}(\bar{\alpha}_o, \beta) = \rho_o^*$. □

Along with Theorem 2 in Section 5 in [14] the following theorem is considered.

Theorem 9. *Suppose that the DMU_o is ERM efficient and the inputs and the outputs for DMU_o are going to increase from X_o and Y_o to $\alpha_o = X_o + \Delta X_o$ and $\beta_o = Y_o + \Delta Y_o$, respectively, where there are $\Delta X_o \gneqq 0$, $\Delta Y_o \gneqq 0$. It is worth mentioning that it is required that the (α_o, β_o) belongs to the production possibility set. Consider the following problem:*

$$\begin{aligned}
\widehat{\rho}_o = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\theta_i}{(1/s)\sum_{r=1}^{s}\varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i \alpha_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r \beta_{ro}, \quad r = 1,\ldots,s \\
& \theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n.
\end{aligned} \tag{29}$$

If the inputs and outputs of DMU_o are increased from X_o to $\widehat{X}_o = (\widehat{\theta}_1 x_{1o},\ldots,\widehat{\theta}_m x_{mo})$ and $\widehat{Y}_o = (\widehat{\varphi}_1 y_{1o},\ldots,\widehat{\varphi}_s y_{so})$, respectively, and $(\widehat{\lambda},\widehat{\theta},\widehat{\varphi})$ is an optimal solution to problem (29) with the optimal value of $\widehat{\rho}$, then $eff(X_o, Y_o) = eff(\widehat{X}_o, \widehat{Y}_o)$.

Proof. When the inputs and outputs of the DMU_o convert to $\widehat{X}_o$ and $\widehat{Y}_o$, respectively, the ERM-efficiency score of the DMU equals the optimal value of the problem below

$$\begin{aligned}
\widetilde{\rho}_o = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\theta_i}{(1/s)\sum_{r=1}^{s}\varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + \lambda_{n+1}\widehat{x}_{io} \le \theta_i \widehat{x}_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} + \lambda_{n+1}\widehat{y}_{ro} \ge \varphi_r \widehat{y}_{ro}, \quad r = 1,\ldots,s \\
& \theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n+1.
\end{aligned} \tag{30}$$

Suppose that $(\widetilde{\lambda},\widetilde{\theta},\widetilde{\varphi})$ is an optimal solution to problem (30) with the optimal value of $\widetilde{\rho}_o$. To prove the theorem, $\widetilde{\rho}_o = 1$ should be shown. By contradiction assume that $\widetilde{\rho}_o < 1$.

Problem (30) is nonlinear, but it can be converted to the following linear programming problem by employing the $t = 1/(1/s)\sum_{r=1}^{s}\varphi_r > 0$:

$$\begin{aligned}
\overline{\widetilde{\rho}}_o = \min \quad & \frac{1}{m}\sum_{i=1}^{m}\overline{\theta}_i \\
\text{s.t.} \quad & \sum_{j=1}^{n}\overline{\lambda}_j x_{ij} + \overline{\lambda}_{n+1}\widehat{x}_{io} \le \overline{\theta}_i \widehat{x}_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\overline{\lambda}_j y_{rj} + \overline{\lambda}_{n+1}\widehat{y}_{ro} \ge \overline{\varphi}_r \widehat{y}_{ro}, \quad r = 1,\ldots,s \\
& \sum_{r=1}^{s}\overline{\varphi}_r = s, \quad t \ge \epsilon, \\
& \overline{\theta}_i \le t, \quad \overline{\varphi}_r \ge t, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\
& \overline{\lambda}_j \ge 0, \quad j = 1,\ldots,n+1,
\end{aligned} \tag{31}$$

where $(\overline{\theta},\overline{\varphi}) = (t\theta, t\varphi)$, $\overline{\lambda}_j = t\lambda_j$ for $j = 1,\ldots,n$, and ϵ is non-Archimedean infinitesimal. Note that $t \ge \epsilon$ is a redundant constraint in problem (31) and hence can be omitted. The dual problem (31) is as follows:

$$\begin{aligned}
\max \quad & sz \\
\text{s.t.} \quad & U^t Y_j - V^t X_j \le 0, \quad j = 1,\ldots,n \qquad (32) \\
& U^t \widehat{Y}_o - V^t \widehat{X}_o \le 0, \\
& v_i \widehat{x}_{io} - \tau_i = \frac{1}{m}, \quad i = 1,\ldots,m \qquad (33) \\
& -u_r \widehat{y}_{ro} + w_r + z = 0, \quad r = 1,\ldots,s \\
& e_m \tau - e_s W = 0, \qquad (34) \\
& V \ge 0, \quad U \ge 0, \quad \tau \ge 0, \quad z \in R,
\end{aligned}$$

where $e_m = (1,\ldots,1) \in R^m$ and $e_s = (1,\ldots,1) \in R^s$. Assume $(\overline{V},\overline{U},\overline{\tau},\overline{W},\overline{z})$ is an optimal solution to the above problem. Based on the duality Theorem, the following relation is held:

$$s\overline{z} = \widetilde{\rho}_o < 1. \tag{35}$$

Also, constraint sets (33) are added together and result in

$$-\overline{U}^t \widehat{Y}_o + e_s \overline{W} + s\overline{z} + \overline{V}^t \widehat{X}_o - e_m \overline{\tau} = 1; \tag{36}$$

taking (35) into consideration, the following inequality is obtained:

$$s\overline{z} = \widetilde{\rho}_o < 1 = -\overline{U}^t \widehat{Y}_o + e_s \overline{W} + s\overline{z} + \overline{V}^t \widehat{X}_o - e_m \overline{\tau}, \tag{37}$$

and, therefore

$$\overline{U}^t \widehat{Y}_o - \overline{V}^t X_o < 0. \tag{38}$$

Since the variable $\overline{\lambda}_{n+1}$ is corresponding to the constraint (32) in the above problem that is unbinding at each optimal solution, therefore, according to the complementary slackness theorem, $\overline{\lambda}_{n+1}$ must be 0, and so $\lambda_{n+1} = 0$.

Consequently, $\widetilde{\rho}_o$ is also the optimal value of the following problem:

$$\begin{aligned}
\widetilde{\rho}_o = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\theta_i}{(1/s)\sum_{r=1}^{s}\varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i \widehat{x}_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r \widehat{y}_{ro}, \quad r = 1,\ldots,s \\
& \theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n.
\end{aligned} \tag{39}$$

Now, consider the following problem:

$$\begin{aligned}
\overline{\rho}_o = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\widehat{\theta}_i\theta_i}{(1/s)\sum_{r=1}^{s}\widehat{\varphi}_r\varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \widehat{\theta}_i\theta_i x_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge \widehat{\varphi}_r\varphi_r y_{ro}, \quad r = 1,\ldots,s \\
& \widehat{\theta}_i\theta_i \le 1, \quad \widehat{\varphi}_r\varphi_r \ge 1, \quad i = 1,\ldots,m, \\
& \qquad\qquad\qquad\qquad\qquad\qquad r = 1,\ldots,s \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n.
\end{aligned} \tag{40}$$

Clearly, $(\widetilde{\lambda},\widetilde{\theta},\widetilde{\varphi})$ is a feasible solution to problem (40), and so

$$\overline{\rho}_o \le \frac{(1/m)\sum_{i=1}^{m}\widehat{\theta}_i\widetilde{\theta}_i}{(1/s)\sum_{r=1}^{s}\widehat{\varphi}_r\widetilde{\varphi}_r}. \tag{41}$$

On the other hand, since $\widetilde{\rho}_o < 1$, there exists at least one t, $1 \le t \le m$ or $1 \le t \le s$, such that $\widetilde{\theta}_t < 1$, or $\widetilde{\varphi}_t > 1$. Therefore, in each case

$$\overline{\rho}_o \le \frac{(1/m)\sum_{i=1}^{m}\widehat{\theta}_i\widetilde{\theta}_i}{(1/s)\sum_{r=1}^{s}\widehat{\varphi}_r\widetilde{\varphi}_r} < \frac{(1/m)\sum_{i=1}^{m}\widehat{\theta}_i}{(1/s)\sum_{r=1}^{s}\widehat{\varphi}_r} = \widehat{\rho}_o. \tag{42}$$

Obviously, problem (40) is just problem (29) (replacing the notation $\widehat{\theta}_i\theta_i$ by θ_i and $\widehat{\varphi}_r\varphi_r$ by φ_r). Therefore, in each case, the optimal value of problem (29) would be $\overline{\rho}_o < \widehat{\rho}_o$, which contradicts the fact that the maximum value of (29) is $\widehat{\rho}_o$. □

3. Estimate Outputs

In this section, the problem provided by Wei et al. [14] is considered. Based on the results of their study, this question is addressed: suppose that the DMU_o is ERM efficient, if the ERM-efficiency score of DMU_o remains unchanged while the inputs increase, and how much should the outputs of the DMU_o increase?

To solve the above problem, to the end of this section, presume that the inputs of DMU_o are increased from X_o to $\alpha_o = X_o + \Delta X_o$, where $\Delta X_o \gneqq 0$. The aim of the problem is to estimate the output vector provided that the DMU_o is still ERM efficient. In fact,

$$\beta_o^* = \left(\beta_{1o}^*, \beta_{2o}^*, \ldots, \beta_{so}^*\right)^t = Y_o + \Delta Y_o, \quad \Delta Y_o \gneqq 0. \tag{43}$$

Assume DMU_{n+1} represents DMU_o after modification of the inputs and outputs. Along the line of [14], the following model is considered to calculate the ERM efficiency of DMU_{n+1}:

$$\begin{aligned}
\rho_o^{+*} = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\theta_i}{(1/s)\sum_{r=1}^{s}\varphi_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + \alpha_{io}\lambda_{n+1} \le \theta_i\alpha_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} + \beta_{ro}^*\lambda_{n+1} \ge \varphi_r\beta_{ro}^*, \quad r = 1,\ldots,s \\
& \theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n+1.
\end{aligned} \tag{44}$$

Based on Definition 3 it is said that the ERM efficiency remains unchanged, if and only if $\mathrm{eff}(\alpha_o, \beta_o^*) = \mathrm{eff}(X_o, Y_o)$. To answer the above question, along the line of [14], the following MOLP model is considered:

$$\begin{aligned}
\max \quad & (\beta_{1o},\ldots,\beta_{so}) \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i^*\alpha_{io}, \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r^*\beta_{ro}, \quad r = 1,\ldots,s \\
& \alpha_{io} \ge x_{io}, \quad i = 1,\ldots,m \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n,
\end{aligned} \tag{45}$$

where (θ^*, φ^*) is an optimal solution to problem (1).

Theorem 10. *Suppose that the DMU_o is ERM efficient and $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem* (1). *Let $(\widehat{\lambda}^*, \widehat{\beta}_o^*)$ be a Pareto solution to problem* (45) *such that $\widehat{\beta}_o^* > Y_o$, and $(\lambda^{+*}, \theta^{+*}, \varphi^{+*})$ is an optimal solution to problem* (4) *with*

the optimal value of ρ_o^{+*}. Also, suppose that the optimal value and an optimal solution of the following problem:

$$\begin{aligned} K_o^* = \max \quad & \frac{1}{m}\sum_{i=1}^{m} k_{io} \\ \text{s.t.} \quad & \sum_{j=1}^{n}\left(\lambda_j^{+*} + \lambda_{n+1}^{+*}\widehat{\lambda}_j^*\right)x_{ij} \le k_{io}\theta_i^*\alpha_{io} = k_{io}\alpha_{io}, \\ & \qquad i = 1,\ldots,m \\ & k_{io} \le 1, \quad i = 1,\ldots,m \end{aligned} \tag{46}$$

are K_o^* and $k_o^* = (k_{1o}^*,\ldots,k_{mo}^*)$. If the outputs of DMU$_o$ are increased to $\widehat{\beta}_o^*$, then

(i) *if $K_o^* = 1$, then $eff(\alpha_o, \widehat{\beta}_o^*) = eff(X_o, Y_o)$,*

(ii) *if $K_o^* < 1$, then $eff(k_o^* \star \alpha_o, \widehat{\beta}_o^*) = eff(X_o, Y_o)$.*

Remark 11. $k_o^* \star \alpha_o = (k_{1o}^*\alpha_{1o},\ldots,k_{mo}^*\alpha_{mo})$ and $\alpha_o - (k_o^* \star \alpha_o) = (\alpha_{1o} - k_{1o}^*\alpha_{1o},\ldots,\alpha_{mo} - k_{mo}^*\alpha_{mo})$. Note that the $\alpha_{io} - k_{io}^*\alpha_{io}$ indicates extra-input amount in ith input component of the DMU. In other words, for the decision maker to preserve the ERM-efficiency score of the DMU$_o$ while the outputs increase from Y_o to $\widehat{\beta}_o^*$, they are required to increase the inputs from X_o to $k_o^* \star \alpha_o$.

Proof. The proof is similar to the proof of Theorem 6. □

Theorem 12. *Assume $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem* (1) *with the optimal value of ρ_o^*. Let $(\overline{\lambda}, \overline{\beta}_o)$ be a feasible solution to problem* (45). *If $eff(\alpha_o, \overline{\beta}_o) = eff(X_o, Y_o)$, then $(\overline{\lambda}, \overline{\beta}_o)$ must be a Pareto solution to problem* (45).

Proof. The proof is similar to the proof of Theorem 8. □

4. Estimate the Minimum Increase of Inputs and the Maximum Increase of Outputs

In order to present suitable patterns to the decision maker to increase inputs and outputs for an ERM-efficient DMU, under preserving the ERM-efficiency index, this new question in field of inverse DEA is addressed: if DMU$_o$ is ERM efficient, while the inputs and outputs are required to be increased, how much should the inputs and outputs of the DMU$_o$ be increased? In other words, under preserving the ERM-efficiency score, how much should the minimum and maximum of input and output vectors of the DMU$_o$ increase, respectively?

By answering the question, the decision maker may be able to take better decisions in order to extend decision making units. That is to say that the decision maker can take necessary actions by choosing a suitable strategy for spreading an ERM-efficient DMU.

The aim of addressing this question is estimating the minimum increase of inputs and the maximum increase of outputs provided that the DMU$_o$ is still ERM efficient. In fact,

$$\begin{aligned} \alpha_o^* &= \left(\alpha_{1o}^*, \alpha_{2o}^*, \ldots, \alpha_{mo}^*\right)^t = X_o + \Delta X_o, \quad \Delta X_o \gneqq 0, \\ \beta_o^* &= \left(\beta_{1o}^*, \beta_{2o}^*, \ldots, \beta_{so}^*\right)^t = Y_o + \Delta Y_o, \quad \Delta Y_o \gneqq 0. \end{aligned} \tag{47}$$

Remark 13. Note that it is required to make the rate of increase inputs and outputs of the unit under assessment bounded; otherwise, it is possible that at least one of the components of α_o^* or β_o^* is unbounded.

Suppose that DMU$_{n+1}$ represents DMU$_o$ after modification of the inputs and outputs. Based on results of [14, 16], the following model is considered to estimate the ERM efficiency of DMU$_{n+1}$:

$$\begin{aligned} \rho_o^{+*} = \min \quad & \frac{(1/m)\sum_{i=1}^{m}\theta_i}{(1/s)\sum_{r=1}^{s}\varphi_r} \\ \text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + \alpha_{io}^*\lambda_{n+1} \le \theta_i\alpha_{io}^*, \quad i = 1,\ldots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj} + \beta_{ro}^*\lambda_{n+1} \ge \varphi_r\beta_{ro}^*, \quad r = 1,\ldots,s \\ & \theta_i \le 1, \quad \varphi_r \ge 1, \quad i = 1,\ldots,m, \quad r = 1,\ldots,s \\ & \lambda_j \ge 0, \quad j = 1,\ldots,n+1. \end{aligned} \tag{48}$$

Definition 14. If the optimal values of problems (1) and (48) are equal, it is said that the ERM efficiency remains unchanged; that is, $\text{eff}(\alpha_o^*, \beta_o^*) = \text{eff}(X_o, Y_o)$.

To solve this new question, that is, to estimate the minimum increase of inputs and the maximum increase of outputs, along the line of [14, 16], the following MOLP problem is considered:

$$\begin{aligned} \min \quad & (\alpha_{1o},\ldots,\alpha_{mo}) \\ \max \quad & (\beta_{1o},\ldots,\beta_{so}) \\ \text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i^*\alpha_{io}, \quad i = 1,\ldots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r^*\beta_{ro}, \quad r = 1,\ldots,s \\ & \alpha_{io} \ge x_{io}, \quad i = 1,\ldots,m \\ & \beta_{ro} \ge y_{ro}, \quad r = 1,\ldots,s \\ & \beta_o \in \Gamma, \quad \alpha_o \in \Lambda, \\ & \lambda_j \ge 0, \quad j = 1,\ldots,n, \end{aligned} \tag{49}$$

where θ^* and φ^* are an optimal solution to problem (1), Γ and Λ are bounded sets, and show the maximum rate of increase

in inputs and outputs of the DMU_o, respectively, such that by the decision maker are considered.

Theorem 15. *Suppose that the DMU_o is ERM efficient and $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem* (1). *Let $(\widehat{\lambda}^*, \widehat{\alpha}_o^*, \widehat{\beta}_o^*)$ be a Pareto solution to problem* (49) *such that $\widehat{\alpha}_o^* > X_o, \widehat{\beta}_o^* \gneqq Y_o$. If the inputs and outputs of DMU_o are increased to $\widehat{\alpha}_o^*$ and $\widehat{\beta}_o^*$, respectively, then eff$(\widehat{\alpha}_o^*, \widehat{\beta}_o^*) =$ eff(X_o, Y_o).*

Proof. Assume $(\lambda^{+*}, \theta^{+*}, \varphi^{+*})$ is an optimal solution to problem (48) with the optimal value of ρ^{+*}. The proof is similar to the proof of Theorem 6, when replacing the notation β_o by $\widehat{\beta}_o^*$. The only difference is in case (b), that is, when $(1/s)\sum_{r=1}^{s} \varphi_r^{+*} > 1$. In this case, there exists at least one p, $1 \le p \le s$, such that $\varphi_p^{+*} > 1$. Taking (12) into consideration, the following inequality is obtained:

$$\sum_{j=1}^{n} \widetilde{\lambda}_j y_{pj} \ge \varphi_p^{+*} \widehat{\beta}_{po}^* > \widehat{\beta}_{po}^*, \tag{50}$$

and, thus, there exists a $\eta_p > 0$ that satisfies

$$\sum_{j=1}^{n} \widetilde{\lambda}_j y_{pj} \ge \left(\widehat{\beta}_{po}^* + \eta_p\right). \tag{51}$$

Now, define

$$\widetilde{\beta}_{ro} = \begin{cases} \beta_{ro}^*, & \text{if } r \ne p, \\ \eta_r + \widehat{\beta}_{ro}^*, & \text{if } r = p. \end{cases} \tag{52}$$

Based on $\widetilde{\lambda} \ge 0$, inequality (9), (11), and (51), and $\widetilde{\beta}_o \gneqq \widehat{\beta}_o^* \gneqq Y_o$, $(\widetilde{\lambda}, \widehat{\alpha}_o^*, \widetilde{\beta}_o)$ is a feasible solution to problem (49), where $\widetilde{\beta}_{ro} \ge \widehat{\beta}_{ro}^*$ for all $r = 1, \ldots, s$, and $\widetilde{\beta}_{ro} > \beta_{ro}^*$ for some $r = 1, \ldots, s$. This contradicts the assumption expressing that the $(\widehat{\lambda}^*, \widehat{\alpha}_o^*, \widehat{\beta}_o^*)$ is a Pareto solution to problem (49). Therefore, in any two cases, because $\rho_o^{+*} \nless \rho_o^*$ and $\rho_o^{+*} \leqslant \rho_o^*$, therefore $\rho_o^{+*} = \rho_o^*$. □

Theorem 16. *Suppose that the $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem* (1) *with the optimal value of ρ_o^*. Let $(\overline{\lambda}, \overline{\alpha}_o, \overline{\beta}_o)$ be a feasible solution to the problem* (49). *If eff$(\overline{\alpha}_o, \overline{\beta}_o) =$ eff(X_o, Y_o), then $(\overline{\lambda}, \overline{\alpha}_o, \overline{\beta}_o)$ must be a Pareto solution to problem* (49).

Proof. If $(\overline{\lambda}, \overline{\alpha}_o, \overline{\beta}_o)$ is not a Pareto solution to problem (49), then there would exist another feasible solution of problem (5), $(\widetilde{\lambda}, \widetilde{\alpha}_o, \widetilde{\beta})$, such that $\widetilde{\alpha}_{io} \le \overline{\alpha}_{io}$ for all i and $\widetilde{\beta}_{ro} \ge \overline{\beta}_{ro}$ for all r; furthermore, there exists at least one i in which $\widetilde{\alpha}_{io} < \overline{\alpha}_{io}$ or at least one r such that $\widetilde{\beta}_{ro} > \overline{\beta}_{ro}$. Let $I = \{i \mid \widetilde{\alpha}_{io} < \overline{\alpha}_{io}\}$ and $O = \{r \mid \widetilde{\beta}_{ro} > \overline{\beta}_{ro}\}$. Therefore

$$\sum_{j=1}^{n} \widetilde{\lambda}_j x_{ij} \le \theta_i^* \widetilde{\alpha}_{io} < \theta_i^* \overline{\alpha}_{io} \quad i \in I \tag{53}$$

$$\sum_{j=1}^{n} \widetilde{\lambda}_j x_{ij} \le \theta_i^* \widetilde{\alpha}_{io} \le \theta_i^* \overline{\alpha}_{io} \quad i \notin I \tag{54}$$

Table 1: The data and ERM efficiency in Example 1.

DMUs	x_1	x_2	y_1	y_2	θ_1^*	θ_2^*	φ_1^*	φ_2^*	ρ^*
A	4.00	2.00	4.00	4.00	1.00	1.00	4.00	2.00	0.33
B	2.00	1.00	8.00	4.00	1.00	1.00	1.00	1.00	1.00
C	1.00	2.00	5.00	5.00	1.00	1.00	1.00	1.00	1.00

$$\sum_{j=1}^{n} \widetilde{\lambda}_j y_{rj} \ge \varphi_r^* \widetilde{\beta}_{ro} > \varphi_r^* \overline{\beta}_{ro} \quad r \in O \tag{55}$$

$$\sum_{j=1}^{n} \widetilde{\lambda}_j y_{rj} \ge \varphi_r^* \widetilde{\beta}_{ro} \ge \varphi_r^* \overline{\beta}_{ro} \quad r \notin O \tag{56}$$

$$\widetilde{\lambda}_j \ge 0, \quad j = 1, \ldots, n. \tag{57}$$

Based on inequalities (53) and (56) there exists $\mu_i < 1$ for $i \in I$ and $\eta_r > 1$ for $r \in O$, such that

$$\sum_{j=1}^{n} \widetilde{\lambda}_j x_{ij} \le \left(\mu_i \theta_i^*\right) \overline{\alpha}_{io}, \quad i \in I \tag{58}$$

$$\sum_{j=1}^{n} \widetilde{\lambda}_j y_{rj} \ge \left(\eta_r \varphi_r^*\right) \overline{\beta}_{ro}, \quad r \in O. \tag{59}$$

For each $i = 1, \ldots, n$ and $r = 1, \ldots, s$, define

$$\overline{\theta}_i = \begin{cases} \mu_i \theta_i^*, & \text{if } i \in I, \\ \theta_i^*, & \text{if } i \notin I, \end{cases} \qquad \overline{\varphi}_r = \begin{cases} \eta_r \varphi_r^*, & \text{if } r \in O, \\ \varphi_r^*, & \text{if } r \notin O. \end{cases} \tag{60}$$

By (54) and (56)–(59), $(\widetilde{\lambda}, \overline{\theta}, \overline{\varphi})$ is a feasible solution of problem (1) (only restrictions on the right have replaced the X_o and Y_o by $\overline{\alpha}_o$ and $\overline{\beta}_o$ in problem (1)) where $\overline{\theta} = (\overline{\theta}_1, \ldots, \overline{\theta}_m)$ and $\overline{\varphi} = (\overline{\varphi}_1, \ldots, \overline{\varphi}_s)$. Therefore

$$\overline{\rho}_o = \frac{(1/m)\sum_{i=1}^{m} \overline{\theta}_i}{(1/s)\sum_{r=1}^{s} \overline{\varphi}_r} < \frac{(1/m)\sum_{i=1}^{m} \theta_i^*}{(1/s)\sum_{r=1}^{s} \varphi_r^*} = \rho_o^* \tag{61}$$

which is against the assumption that is eff$(\overline{\alpha}_o, \overline{\beta}_o) =$ eff$(X_o, Y_o) = \rho_o^*$. □

5. Illustrative Examples

Example 1. Consider a problem of three DMU with two inputs x_1, x_2 and two outputs y_1, y_2. The data of inputs, outputs, and ERM-efficiency score are shown in Table 1.

As can be seen, DMU_C is an ERM-efficient DMU. If the decision maker is interested in increasing the output vector from $Y_C = (5, 5)$ to $\beta_C = (6, 6)$, then, to solve the model (5) by employing the weight-sum method [25], a Pareto solution for this MOLP is generated as $\alpha_C^* = (1.20, 2.40)$. Based on the model (6), $K_C^* = 1$, eff$(X_C, Y_C) =$ eff$(\alpha_C^*, \beta_C) = 1$, which means that the ERM-efficiency remains unchanged. Hence, due to Theorem 6, when the outputs of DMU_A increase to

Table 2: The data in Example 2.

DMUs	x	y	θ^*	φ^*	ERM efficiency
A	1.00	1.00	0.50	1.00	0.50
B	2.00	4.00	1.00	1.00	1.00
C	5.00	5.00	0.67	1.33	0.50
D	6.00	7.00	0.79	1.35	0.58

$\beta_C = (6, 6)$ if the decision maker would like to preserve the efficiency score of this DMU, the inputs should increase to $\alpha_C^* = (1.20, 2.40)$.

Also, if the decision maker is interested in increasing the output vector of $Y_C = (5, 5)$ to $\beta_C = (5.5, 6)$, then to solve the model (5) by employing the weight-sum method [25], must the input vector of the $X_C = (1, 2)$ increase to $\alpha_C^* = (1.20, 2.40)$. Based on model (6), since $K_C^* = 11.5/5 > 1$ and $k_C^* = (6/5, 1)$, we have $\text{eff}(X_C, Y_C) \neq \text{eff}(\alpha_C^*, \beta_C) = 0.96$; therefore, according to part (ii) of Theorem 6, we have $\text{eff}(X_C, Y_C) = \text{eff}(1.20, 2.40, 6/5 * 5.5, 1 * 6) = 1$. This indicates that new DMU in output component the first to amount "0.5" of the lack produce. In other words, using model (4), we have $\text{eff}(X_C, Y_C) = \text{eff}(1.20, 2.40, 6/5 * 5.5, 1 * 6) = 1$.

Example 2. Consider a problem of four DMU with one input x and one output y. The data of input, output, and ERM-efficiency scores are shown in Table 2.

As can be seen, DMU_B is an ERM-efficient DMU. Assume the decision maker identified rate of increase input and output for this DMU, respectively, as $2 \leq x \leq 7$, $4 \leq y \leq 10$. In order to propose patterns to the decision maker to increase input and output for this DMU, under preserving the ERM-efficiency index, based on model (49) the following MOLP model is considered:

$$\begin{aligned}
\min \quad & \alpha_B \\
\max \quad & \beta_B \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 + 5\lambda_3 + 6\lambda_4 \leq \theta_B^* \alpha_B \\
& \lambda_1 + 4\lambda_2 + 5\lambda_3 + 7\lambda_4 \geq \varphi_B^* \beta_B \\
& \alpha_B \geq 2, \qquad \beta_B \geq 4 \\
& \alpha_B \leq 7, \qquad \beta_B \leq 10 \\
& \lambda_j \geq 0, \quad j = 1, \ldots, 4.
\end{aligned} \tag{62}$$

Using the weight-sum method [25], a Pareto solution is generated for this MOLP as $(\alpha_B^*, \beta_B^*) = (5, 10)$. Therefore, according to Theorem 15, when the inputs and outputs of DMU_B increase to 5 and 10, respectively, then the ERM-efficiency score of this DMU is 1.

6. Conclusion

In the present paper, a typical inverse optimization problem on the nonradial enhanced Russell model has been studied: two main questions on inverse DEA have been discussed and the models have been proposed to estimate the input (output) levels of a given DMU when some or all its output (input) levels were increased under the constant ERM-efficiency score. To determine sufficient and necessary conditions of estimated inputs (outputs), Pareto solutions of the MOLP were used. Moreover, in finding inputs (outputs), if there exists lack output (extra input) in each of a output (input) components, the amount of the lack (extra) is specified by auxiliary models. For an ERM-efficient DMU, necessary and sufficient conditions were introduced to find the minimum and maximum increase of input and output levels, respectively, provided that the ERM-efficiency score remains unchanged. Therefore, the patterns can be presented to the decision maker in order to increase inputs and outputs (extending decision making units) for an ERM-efficient DMU such that the ERM-efficiency remains unchanged. In other words, inverse DEA can be used in theoretical and practical purposes such as strategic planning, management control, resource allocation, and ranking. The sufficient conditions were established only for the ERM-efficient DMU, but the given necessary conditions were for each ERM-efficient or ERM-inefficient DMU. Finding sufficient conditions for ERM-inefficient DMU_s can be a suitable research field. In addition, using nonradial models under intertemporal dependence, solving the problems introduced by Wei et al. [14], Hadi-Vencheh et al. [16], and the new problem discussed in this paper can be a useful research topic.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Directional Slack-Based Measure for the Inverse Data Envelopment Analysis

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A novel technique has been introduced in this research which lends its basis to the Directional Slack-Based Measure for the inverse Data Envelopment Analysis. In practice, the current research endeavors to elucidate the inverse directional slack-based measure model within a new production possibility set. On one occasion, there is a modification imposed on the output (input) quantities of an efficient decision making unit. In detail, the efficient decision making unit in this method was omitted from the present production possibility set but substituted by the considered efficient decision making unit while its input and output quantities were subsequently modified. The efficiency score of the entire DMUs will be retained in this approach. Also, there would be an improvement in the efficiency score. The proposed approach was investigated in this study with reference to a resource allocation problem. It is possible to simultaneously consider any upsurges (declines) of certain outputs associated with the efficient decision making unit. The significance of the represented model is accentuated by presenting numerical examples.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric linear programming-based technique for measuring and evaluating the relative performances of a set of decision making units (DMUs). Each of the DMUs uses multiple inputs to generate multiple outputs and they are assumed to operate under similar conditions. By considering the information about available data on the input/output values of the DMUs and some axiomatic foundations, a production possibility set (PPS) is defined. The basic ideas behind the DEA date back to Farrell [1], but the recent series of discussions started with study conducted by Charnes et al. [2], in assessment of an educational center in the USA and extended by Banker et al. [3]. Many researchers have proposed extensions to the DEA technique (see Emrouznejad et al. [4] and Gattoufi et al. [5]).

If it is intended to measure the efficiency, the DEA models can be generally categorized into two main groups, namely, the radial and nonradial models. Farrell [1] and Debreu [6] were pioneers to conduct the first systematic surveys on the radial models. Seemingly, the radial-measures approach entails various striking characteristics. Yet, it suffers from various problems in use notwithstanding the intensity of this approach (see Avkiran et al. [7]). Moreover, Koopmans [8] and Russell [9] conducted some of the earliest efforts on the nonradial models. For the time being, various undertakings were exerted on the basis of the performance approximation with the aim of elucidating the nonradial measure for the technical efficiency (see Charnes et al. [10] and Cooper et al. [11–13]), each of which are concomitant with relevant benefits and drawbacks (see Avkiran et al. [7]). In order to acquire a nonradial measure termed the slacks-based measure (SBM) wherein there is a possibility of diminishing the input and output slacks, a newfangled and proper synthetic method was employed by Tone [14] in a controlled study. The Russell measure model was presented in the proposed model of Tone [14] by Färe and Lovell [15] which was additionally reexamined by Pastor et al. [16] introducing it as

the Enhanced Russell Measure model (ERM). In recent times, the Directional Slack-Based Measures (DSBM) of efficiency was introduced by Jahanshahloo et al. [17] which worked under the Generalized Returns to Scale (GRS). On the basis of the directional distance function, the GRS is reported to encompass numerous remarkable characteristics.

The point of fact is that the DMUs performances are appraised by the DEA; in this context, the production relationship can be examined devoid of any functional specification between them. We should assume a production technology wherein m inputs are required for engendering the s outputs. In turn, we need to represent the inputs and outputs by x and y. We can afterward define the production possibility set (PPS) T; in this fashion: $T = \{(x, y) : y$ can be engendered by $x\}$. T contains the entire possible input/output combinations. Besides, the boundary points of T are titled as the efficient frontier, the frontier line, or the production frontier [1]. It needs to be pinpointed that those DMUs which are the properties of this frontier will be considered efficient whereas the others would be regarded as inefficient. The values of the observations' inputs and outputs in the dataset will influentially affect the efficient frontier representation in the DEA. Any relative variations in the input and output amounts will result in the variations in the efficient frontier structure as well as the DMUs' relative efficiency values. At this point, it is necessary to know how not to deteriorate the relative efficiency value of a deliberated DMU in case the internal technical structure of the deliberated DMU marginally fluctuates in a short run.

Some researchers have severely debated the inverse DEA models through the last two decades. Because of the fact that the constraint parameters include the input and output values of the DMUs for the DEA models, the inverse DEA models can be categorized into two kinds. This classification depends on the kind of parameters which are varying and the ones which have to be changed in order for the optimal objective value to be kept unaffected. A resource allocation problem is underscored to be the first category of the inverse DEA models which is known as an inverse DEA problem associated with specifying the utmost probable inputs for certain outputs with the intention of keeping the current efficiency value of a deliberated DMU unvaried in comparison with the other DMUs. Contrariwise, the other inverse DEA model is said to be an investment analysis problem which is an inverse DEA problem associated with identifying the best possible outputs for certain inputs in order to keep the current efficiency value of a certain DMU unchanged as compared with the remainder of DMUs. Ordinarily, there should not be a drastic change in a DMU's internal technical structure in a short run (Yan et al. [18]). The inverse DEA was initially proposed by Wei et al. [19] in recent times with the aim of addressing this question: if given inputs (outputs) of a certain unit are elevated amid a set of DMUs, how much rise should be given to the outputs (inputs) of that same unit while the DMU maintains its efficiency level in comparison with the other DMUs?

Alternatively, an inverse measure was defined by Cherchye and van Puyenbroeck [20] on the basis of the efficient reference mix properties. A method was also introduced by Pendharkar [21] in which the DEA was being utilized to resolve the inverse classification problem. Such a problem indeed implicates discovering the way predictor attributes of a case are changed so that such a case would be categorized into a diverse and more appropriate category. This study revealed that the DEA might be exploited for the inverse classification problem assuming the classes' conditional monotonicity and convexity. The inversed DEA problem along with the developed solution method proposed by Yan et al. [18] was extended by Jahanshahloo et al. [22] to the case of specifying the outputs of a certain DMU once the entire or a few of the inputs had a rise as well as the time when the efficiency value of such DMU, as compared with the other DMUs, was required to be enhanced by its current efficiency value identified percentage. Through another study executed by Jahanshahloo et al. [23], the inversed DEA models were found to be utilized for calculating the inputs for a DMU once the entire or a few of the outputs as well as the DMU's efficiency value were either elevated or retained. They likewise acknowledged the extra inputs once the outputs could be calculated by making use of the models recommended by Yan et al. [18] and Jahanshahloo et al. [22].

In line with this, a modified inverse DEA model was introduced by Jahanshahloo et al. [24] intended for the sensitivity analysis of both efficient and inefficient DMUs' efficiency categorizations. In this model, the preference cones were used to represent the essential policies over inputs, outputs, and DMUs. Furthermore, a broader case of the aforementioned question was addressed by Hadi-Vencheh and Foroughi [25]. Their study demonstrated that, in the paper of Wei et al. [19], only the upsurge of inputs (outputs) had been taken in to account while a unit might contend with the upsurge of some of inputs (outputs) and the declines of the other inputs (outputs) at the same time. Besides, a method was introduced by Alinezhad et al. [26] which exploited an interactive MOLP aimed at resolving the inverse DEA problems. Nonetheless, it needs to be accentuated that the majority of studies elaborating on the inverse DEA problems have merely concentrated on the considered DMU's efficiency score irrespective of considering the effects of the input (output) changes on the other DMUs' efficiency scores. In recent times, the inverse BCC model was considered by Lertworasirikul et al. [27] for a resource allocation problem in which it was possible to instantaneously deliberate some outputs' upsurges and the other outputs' decreases in the given DMU.

The current research is an attempt to introduce a new inverse DEA model while lending its basis to the efficiency DSBM [17]. So far, this has been discarded by the preceding studies although there have been various methods on the term inverse DEA model. With the intention of enhancing the efficiency scores of some DMUs when there are changes in the output (input) amounts of an efficient DMU, the current study will elucidate the inverse DEA model for the DSBM model. The inverse DSBM model is meant for determining the utmost possible input (output) quantities of the efficient DMU once its outputs (input) are changed. In detail, the deliberated efficient DMU was eliminated from the current PPS, being substituted by the same efficient DMU following

the modification of its inputs and outputs. It is asserted here that the introduced inverse DSBM model seemingly is a kind of resource allocation problem in which it is possible to concurrently consider some inputs' rises (reductions). In practice, the mentioned inverse DEA model is advantageous for the decision makers enabling them to realize they way of allocating the limited resources to an efficient DMU the moment its inputs and outputs are modified consistent with the anticipated trivial upsurges of its outputs (inputs). In this way, the modifications would result in a boost in some DMUs' efficiency scores.

The current paper is organized as follows. Section 2 succinctly introduces the directional distance function and further elaborates on the DSBM model detailing its properties. Section 3 deals with our proposed model for the invers DEA model, exploring it meticulously to determine the best possible inputs for the intended efficient DMU with the purpose of enhancing the efficiency scores of some of the DMUs. Moreover, Section 4 presents a solution approach to the inverse proposed approach while Section 5 provides the illustrative examples for the invers DEA model. At the end, Section 6 concludes the research as well as presenting some further research.

2. Directional Slack-Based Measure

Directional distance function is concisely presented in this section along with describing the DSBM model. Furthermore, its properties as recommended by Jahanshahloo et al. [17] have been also underlined.

Suppose there are n DMU_j and that the DMU_j ($j = 1,\ldots,n$) are under assessment such that they convert m inputs x_{ij} to s outputs y_{kj} where $x_{ij} \geq 0$, $x_{ij} \neq 0$, $y_{kj} \geq 0$, and $y_{kj} \neq 0$ ($i = 1,\ldots,m$, $k = 1,\ldots,s$). Indeed, let the oth DMU consumes inputs $x_{io} = (x_{1o}, x_{2o},\ldots,x_{mo})$ to generate outputs $y_{ko} = (y_{1o}, y_{2o},\ldots,y_{so})$ which $x_{io} \geq 0$, $x_{io} \neq 0$, $y_{ko} \geq 0$, and $y_{ko} \neq 0$ ($i = 1,\ldots,m$, $k = 1,\ldots,s$) and also the PPS is assigned by T.

Chambers et al. [28, 29] were pioneers in examining the structural behavior of the directional distance function by which a value is specified for the technical efficiency. The directional distance function is potent of simultaneously heightening the outputs as well as declining the inputs; in other words, the given vector (x, y) as the input-output vector is reflected onto the PPS production frontier. This can be achieved through utilizing a predetermined direction vector $(-g^-, g^+) \in (-(R_+^m), R_+^s)$ shown in this fashion:

$$\vec{D}_T = (x, y; -g^-, g^+) = \mathrm{Max}\left\{\theta : (x - \theta g^-, y + \theta g^+) \in T\right\}. \tag{1}$$

We can indicate the PPS presented consistent with the standard assumptions by the observed DMUs with general returns to scale (GRS) assumption of technology (Cooper et al. [30]) as

$$T_G = \left\{(x, y) : x \geq \sum_{j=1}^{n} x_{ij}\lambda_j,\ y \leq \sum_{j=1}^{n} y_{kj}\lambda_j,\ U \leq \sum_{j=1}^{n}\lambda_j \leq L,\ \lambda_j \geq 0,\ i = 1,\ldots,m,\ k = 1,\ldots,s,\ j = 1,\ldots,n\right\} \tag{2}$$

in which two nonnegative scalar parameters exist which include L ($0 \leq L \leq 1$) and U (≥ 1) for $\sum_{j=1}^{n}\lambda_j$.

Taking into account $L = 1$ and $U = 1$, the T_G can be altered to T_V which stands as the PPS pertinent variable returns to scale (VRS) assumption of technology (Banker et al. [3]). By bearing in mind $L = 1$, $U = \infty$, and $L = 0$, $U = 1$, T_G converts into T_{ID} and T_{ND} that are referred to as PPS, respectively, pertinent to the increasing returns to scale (IRS) and decreasing returns to scale (DRS) assumption of technology. The reason why the IRS and DRS are recognized as the "variable returns to scale" is that the engendered outputs have a rise more or a smaller amount, corresponding to the rises in the inputs [31]. Besides, if we assume that $L = 0$ and $U = \infty$, the T_G changes into T_C which denotes the PPS with respect to the constant returns to scale (CRS) supposition of technology (Charnes et al. [2]).

It needs to be pinpointed that the radial models are unable to explain the nonradial excesses and shortfalls because of the fact that such models remove the nonzero input and output slacks. Relevant to this, the DSBM model of efficiency was introduced by Jahanshahloo et al. [17] in regard to T_G as shown in

$$\text{Model (1)}\quad \rho^* = \mathrm{Min}\ \frac{1 - (1/m)\sum_{i=1}^{m}\theta_i}{1 + (1/s)\sum_{k=1}^{s}\varphi_k}, \tag{3}$$

$$\text{s.t.}\quad \sum_{j=1}^{n}\lambda_j x_{ij} \leq x_{io} - \theta_i g_i^-,\quad i = 1,\ldots,m, \tag{4}$$

$$\sum_{j=1}^{n}\lambda_j y_{kj} \geq y_{ko} + g_k^+\varphi_k,\quad k = 1,\ldots,s, \tag{5}$$

$$L \leq \sum_{j=1}^{n}\lambda_j \leq U, \tag{6}$$

$$\lambda_j \geq 0,\quad j = 1,\ldots,n, \tag{7}$$

$$\theta_i \geq 0,\quad i = 1,\ldots,m, \tag{8}$$

$$\varphi_k \geq 0,\quad k = 1,\ldots,s. \tag{9}$$

Now, by selecting the direction vector $g = (-g^-, g^+)$ as $g_i^- = x_{io}$ ($i = 1,\ldots,m$), $g_k^+ = y_{ko}$ ($k = 1,\ldots,s$), we consider

the DSBM model by the following model which is a special case of the SBM [14] and the ERM [16] with respect to T_G:

$$\begin{aligned}
\text{Model (2)}\quad \rho^* = \text{Min}\ & \frac{1-(1/m)\sum_{i=1}^{m}\theta_i}{1+(1/s)\sum_{k=1}^{s}\varphi_k} \\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io}\left(1-\theta_i\right), \\
& \qquad i=1,\ldots,m, \\
& \sum_{j=1}^{n}\lambda_j y_{kj} \ge y_{ko}\left(1+\varphi_k\right), \quad k=1,\ldots,s, \\
& L \le \sum_{j=1}^{n}\lambda_j \le U, \\
& \lambda_j \ge 0, \quad j=1,\ldots,n, \\
& \theta_i \ge 0, \quad i=1,\ldots,m, \\
& \varphi_k \ge 0, \quad k=1,\ldots,s,
\end{aligned} \tag{10}$$

while θ_i stands for the contraction rate related to input i and φ_k demonstrates the extension rate of output k for the oth DMU. In this way, it can be reflected onto the production frontier of T_G. Likewise, the objective function of the DSBM model is responsible for increasing θ_i and φ_k ($i = 1,\ldots,m$, $k = 1,\ldots,s$). The subsequent variables λ_j ($j = 1,\ldots,n$) will be utilized for a structural connection that is established between the DMUs within the input/output space; that is to say that $\lambda \in R^n$ is the intensity vector.

Definition 1. A DMU_j ($j = 1,\ldots,n$) will be efficient if and only if $\rho^* = 1$.

In the optimum solutions, the mentioned situation will be equal to ${\theta_i}^* = 0$, ($i = 1,\ldots,m$) and ${\varphi_k}^* = 0$, ($k = 1,\ldots,s$).

Proposition 2. *Consider* $0 \le \rho^* \le 1$.

Proof. Primarily, because $x_{io}(1-\theta_i) \ge 0$, $\rho^* \le 1$. Alternatively, it is quite known that $\rho^* \ge 0$. Therefore, $0 \le \rho^* \le 1$. □

According to Proposition 2, ρ^* is taken to mean as an efficiency score; in other words, there is no input inefficiency (excess) and no output inefficiency (shortage) in any optimal solution within the entire inputs and outputs.

Proposition 3. *The entire input and output constraints in Model (2) are binding in each optimum solution* (5).

Proof. The proof for this proposition resembles that of Pastor et al. [16]. □

It is necessary to highlight that the DSBM model entails several favorable confidants which can be inferred as follows.

(a) The mentioned measure is invariant regarding the measurement unit of every input and output item.

(b) The measure is observed to be monotone intensely while it is capable of diminishing in each θ_i and φ_k.

(c) The Charnes-Cooper conversions can be exploited for easily transforming this model into a linear programming problem [32].

(d) It can be said that this measure seems to be complete due to the fact that it is in contradiction with the oriented measures; in practice, the entire inefficiencies which are concomitant with the nonzero slacks which might be recognized by the model are referred to as nonoriented measures.

3. The Inverse DSBM Model

In the current study, Model (2) was adjusted through the method mentioned below for our target. Indeed, we replaced the objective function by its numerator while we discarded the denominator. Consequently, the input-oriented Model (2) can be presented in the following way:

$$\begin{aligned}
\text{Model (3)}\quad \psi^* = \text{Min}\ & 1-\frac{1}{m}\sum_{i=1}^{m}\theta_i, \\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io}\left(1-\theta_i\right), \quad i=1,\ldots,m, \\
& \sum_{j=1}^{n}\lambda_j y_{kj} \ge y_{ko}, \quad k=1,\ldots,s, \\
& L \le \sum_{j=1}^{n}\lambda_j \le U, \\
& \lambda_j \ge 0, \quad j=1,\ldots,n, \\
& \theta_i \ge 0, \quad i=1,\ldots,m.
\end{aligned} \tag{11}$$

Seemingly, θ_i ($i = 1,\ldots,m$) is the contraction rate in input i for the oth DMU that might be reflected onto the production frontier of T_G. Likewise, ψ^* signifies the DMU_j's efficiency score.

It is to be stated that the linear combination of the observed DMUs failed to cover the efficient frontier related to Model (3); henceforth, the efficient frontier will be moved by every alteration applied to the inputs or outputs of the given efficient DMUs which can put on the efficient frontier. In addition, there might be alterations in the DMUs' efficiency scores in the PPS. This part was started out aimed at recommending an inverse model through making use of Model (3) in order to attain the efficiency scores of every DMU in comparison with the other DMUs on the basis of the acquired efficient frontier. Correspondingly, there might be some improvement in the efficiency scores associated with few of the DMUs. Having regarded the acquired efficient frontier, the PPS would integrate the entire observed DMUs with the objective of removing the given efficient DMU from the present PPS while being substituted by the same efficient DMU following the change in the quantities of its input and output.

We are now enabled to examine Model (3) for the inverse DEA models by being consistent with whatever

stated formerly as well as considering the DMUs, as well as their consistent input and output vectors. In addition, DMU_e can indicate the given efficient DMU with the input and output quantities in progress. Also, the $\mathrm{DMU}_{e'}$ can be deliberated as an efficient DMU once there is an alteration in its inputs and outputs. At last, the advanced inverse DEA model is introduced for a resource allocation problem while it endeavors to tackle the following problem.

The assumption is that a group of DMUs in progress exists. Then $\theta_i^{j^*}$ $(i = 1, \ldots, m, j = 1, \ldots, n)$ indicates the decline rate in ith input of DMU_j from Model (3). Moreover, the output quantities of DMU_e have experienced an improvement from Y_e to $Y_e + \Delta Y_e \geq 0$ $(\Delta Y_e \neq 0)$. At this point, the minimum $X_e + \Delta X_e \geq 0$ $(\Delta X_e \neq 0)$ should be obtained somehow the efficiency score of DMU_e with new input and output amounts $(X_e + \Delta X_e, Y_e + \Delta Y_e)$ remains stable whereas the other DMUs fail to worsen.

The point to be accentuated is that, prior to the modifications of the DMU_e inputs and outputs, the present PPS was integrated with n DMU_j $(j = 1, \ldots, n)$. On the other hand, the new PPS was also integrated with n DMU_j $(j = 1, \ldots, e-1, e', e+1, \ldots, n)$ following the modifications of the DMU_e inputs and outputs. Indeed, DMU_e was omitted from the present PPS while $\mathrm{DMU}_{e'}$ was substituted.

In order to conduct the analyses, initially the efficiency score of DMU_j was acquired prior to the modifications of the DMU_e inputs and outputs. This was fulfilled by utilizing Model (3) in which $\theta_i^{j^*}$ $(i = 1, \ldots, m, j = 1, \ldots, n)$ indicated the decline rate in the ith input of DMU_j prior to the modifications of the DMU_e output amounts. In addition, ψ_j^* signified the DMU_j efficiency score.

What follows is inverse Model (3) recommended for a resource allocation problem.

Step 1. Let us assume that the DMU_e outputs are transformed from Y_e to $Y_e + \Delta Y_e \geq 0$ $(\Delta Y_e \neq 0)$. Thus, the minimum $X_e + \Delta X_e \geq 0$, $(\Delta X_e \neq 0)$ was obtained through the succeeding model:

$$\text{Model (4)} \quad \text{Min } X_e + \Delta X_e, \tag{12}$$

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j x_{ij} \leq (x_{ie} + \Delta x_{ie}) \left(1 - \theta_i^{e^*}\right), \quad i = 1, \ldots, m, \tag{13}$$

$$\sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j y_{kj} \geq (y_{re} + \Delta y_{re}), \quad k = 1, \ldots, s, \tag{14}$$

$$L \leq \sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j \leq U, \tag{15}$$

$$\lambda_j \geq 0, \quad j \neq e, \ j = 1, \ldots, n. \tag{16}$$

Step 2. ΔX_e was used from Model (4) while the efficiency score could be gained by the subsequent model for the other DMU_p $(p = 1, \ldots, n, p \neq e)$:

$$\begin{aligned}
\text{Model (5)} \quad \gamma_p{}^* = \text{Min } & 1 - \frac{1}{m} \sum_{i=1}^{m} \theta_i^p, \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j x_{ij} + \lambda_{e'} (x_{ie} + \Delta x_{ie}) \\
& \leq x_{ip} \left(1 - \theta_i^p\right), \quad i = 1, \ldots, m, \\
& \sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j y_{kj} + \lambda_{e'} (y_{re} + \Delta y_{re}) \geq y_{kp}, \\
& \qquad k = 1, \ldots, s, \\
& L \leq \sum_{\substack{j=1 \\ j \neq e}}^{n} \lambda_j + \lambda_{e'} \leq U, \\
& \lambda_j \geq 0, \quad j \neq e, \ j = 1, \ldots, n, \\
& \lambda_{e'} \geq 0.
\end{aligned} \tag{17}$$

The θ_i^p $(i = 1, \ldots, m)$ signifies the decline rate in the ith input of DMU_p $(p = 1, \ldots, n, p \neq e)$ following the implementation of modifications in the output values of DMU_e. Besides, γ_p^* demonstrated the efficiency score of DMU_p $(p = 1, \ldots, n, p \neq e)$.

It needs to be affirmed that Models (4) and (5) were as multiobjective nonlinear programming (MONLP) forms.

4. The Solution Related to the MONLP

The ΔX_e value was required to be determined in this section as the aim was to solve the inverse DSBM model for the resource allocation problem such that the efficiency score did not decline for the whole DMUs. For fulfilling this aim, Models (4) and (5) were needed to be solved. Nevertheless, it is admitted that solving these models is not such an easy task because they stand in the form of MONLP. Subsequently, a linear programming model was assumed in Proposition 6, yielding an optimum solution for the inverse model.

Proposition 4. *The assumption that the efficiency score of DMU_e in connection with other DMU_j in a set of homogeneous DMU_j $(j = 1, \ldots, n)$ is ψ_e^* and $\theta_i^{e^*}$ $(i = 1, \ldots, m)$ indicates the decline rate in the ith input of DMU_e. By applying the modifications to the output values of DMU_e, $\Delta Y_e \neq 0$, the minimum ΔX_e of the $\mathrm{DMU}_{e'}$, which fails to worsen the efficiency score of the entire DMU_j $(j = 1, \ldots, n)$, can be attained through solving Model (4).*

Proof. $\theta_i^{e^*} = 0$ $(i = 1, \ldots, m)$ because DMU_e is being regarded as an efficient DMU. Consequently, after applying

the modifications to the output values of DMU_e, $\Delta Y_e \neq 0$, the minimum ΔX_e of the $\mathrm{DMU}_{e'}$ will be acquired through solving Model (4) while it fails to worsen the efficiency score of $\mathrm{DMU}_{e'}$. Furthermore, the optimal value of Model (5) cannot be less than that of Model (3). Because Model (3) contains feasible regions which are in turn subspaces of the feasible regions belonging to Model (5), $\theta_i^p \leq \theta_i^{p^*}$ and $1-\theta_i^p \geq 1-\theta_i^{p^*}$ $(i = 1, \ldots, m)$. Consequently, for other DMU_p $(p = 1, \ldots, n,\ p \neq e)$, the efficiency scores could be acquired by Model (5) and we would then have $1 - (1/m)\sum_{i=1}^m \theta_i^p \leq 1 - (1/m)\sum_{i=1}^m \theta_i^{p^*}$; then, $\gamma_p{}^* \leq \psi_p^*$. □

It is denoted by Proposition 5 that once we execute Steps 1 and 2, the efficiency scores related to the entire DMU_j $(j = 1, \ldots, n)$ would not get worse as compared with the efficiency scores related to Model (3). To sum up, by executing the two mentioned steps, it is observed that the efficiency scores associated with all DMU_j $(j = 1, \ldots, n)$ would either remain unchanged or experience a rise.

Proposition 5. *If one assumes that the DMU_e efficiency score in connection with other DMU_j in a set of homogeneous DMU_j $(j = 1, \ldots, n)$ is ψ_e^* and $\theta_i^{e^*}$ $(i = 1, \ldots, m)$ demonstrates the decline rate in the ith input of DMU_e, besides, we need to assume that the DMU_e outputs are changed from Y_e to $Y_e + \Delta Y_e \geq 0$, $(\Delta Y_e \neq 0)$, then, there would be one optimal solution as a minimum to Model (4) if and only if $Y_e + \Delta Y_e \in P_{out}$, where*

$$P_{out} = \left\{ y;\ \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j y_{kj} \geq y\ (k = 1, \ldots, s),\ L \leq \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j \leq U, \lambda_j \geq 0\ (j = 1, \ldots, n, j \neq e) \right\}. \tag{18}$$

Proof. If $Y_e + \Delta Y_e \in P_{\mathrm{out}}$, the constraints will be therefore met in Model (4):

$$\sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j y_{kj} \geq (y_{re} + \Delta y_{re}), \quad k = 1, \ldots, s.$$

$$L \leq \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j \leq U, \tag{19}$$

$$\lambda_j \geq 0, \quad j \neq e,\ j = 1, \ldots, n.$$

Additionally, having estimated the proper bottommost quantity $X_e + \Delta X_e \geq 0$ $(\Delta X_e \neq 0)$, the succeeding constraints would be met:

$$\sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j x_{ij} \leq (x_{ie} + \Delta x_{ie})\left(1 - \theta_i^{e^*}\right), \quad i = 1, \ldots, m. \tag{20}$$

For that reason, there would be one optimal solution to Model (4) as a minimum which will be attained by allowing the objective associated with Model (3). Nonetheless, on the condition that there is one optimal solution as a minimum to Model (4), $Y_e + \Delta Y_e \in P_{\mathrm{out}}$ which is consistent with the constraints (14) in Model (4). □

Proposition 6. *Having considered the ensuing linear programming model in which $\theta_i^{e^*}$ stands for the decline rate in the ith input of DMU_e in Model (3) and $W^T \in R^m$, every optimal solution related to this model would be then a Pareto solution for Model (4):*

$$\begin{aligned}
\textit{Model}\,(6)\ \ \text{Min}\quad & W^T (X_e + \Delta X_e) \\
s.t.\quad & \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j x_{ij} \leq (x_{ie} + \Delta x_{ie})\left(1 - \theta_i^{e^*}\right), \\
& \qquad i = 1, \ldots, m, \\
& \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j y_{kj} \geq (y_{re} + \Delta y_{re}), \quad k = 1, \ldots, s, \\
& L \leq \sum_{\substack{j=1\\ j\neq e}}^{n} \lambda_j \leq U, \\
& \lambda_j \geq 0, \quad j \neq e,\ j = 1, \ldots, n.
\end{aligned} \tag{21}$$

Proof. If we consider $(\lambda^*, \Delta X_e{}^*)$ as an optimal solution for Model (6), we need to assume by contrast where it cannot be a Pareto solution to Model (4). Moreover, we have $(\lambda, \Delta X_e)$ to the Model (4) so that $\Delta X_e \leq \Delta X_e{}^*$ without generality loss while as a minimum a severe inequity would be met in any of the components. Consequently, there would exist $W^T(X_e + \Delta X_e) \leq W^T(X_e + \Delta X_e{}^*)$, $W^T > 0$. Hereafter, it is noticeable that $(\lambda, \Delta X_e)$ can be taken as the solution for Model (4). Indeed, this should be introduced as an inconsistency and the relevant proof would be subsequently established due to the fact that the sets of constraint resembled in Model (4) and (6). □

5. Illustrative Examples

Our proposed technique is exhibited here by two simple numerical examples along with one applied example. The models introduced earlier will be utilized for the case of $L = 1$ and $U = 1$ in this section because it seems to be more plenary as compared with the other cases. Also, the GAMS recognized as an operative software package was employed for solving the mentioned models.

Example 1. We primarily assume that there are 8 DMUs that make use of one input for producing one output. Then, it would be initially necessary to solve Model (3) if it is intended to attain the DMUs efficiency scores entirely prior to any

Table 1: The Data and Results Obtained from Model (3) with One-Input and One-Output.

DMU	Existing		Efficiency
	x	y	ψ^*
A	4	8	1.00
B	7	11	1.00
C	6	8	0.67
D	3	3	1.00
E	8	9	0.56
F	5	10	1.00
G	5	5	0.68
H	3.6	6	1.00

modifications in the inputs and outputs of the given efficient DMU. Table 1 represents all the data and results obtained for Model (3) indicating that technically DMUs A, B, D, F, and H appear to be efficient whereas DMUs C, E, and G seem to be inefficient.

At this point, we need to take into account efficient DMU B and to consider that $W^T = (1, 1)$ for the input weights as well as the DMU B's output is changed from 11 to 10.80. As predicted, we realized that there were no feasible solutions for Model (6) for DMU B once we solved it. In other words, it can be declared that the new output for DMU B did not appear in P_{out}. Once more, we need to take into account DMU B while assuming a reduction in its output from 11 to 9.80. Subsequently, an optimal solution will be gained as $\Delta x = -2.10$. It means that the new output appears in P_{out} once Model (6) is solved for it. At this point, the DMUs' efficiency scores can be estimated by Model (5) through utilizing Δx from Model (6) for the other DMUs. Henceforth, it is claimed that that they do not undergo any declines as they stay unaffected. In other words, they do not get worse.

At this instant, another efficient DMU is taken into account while allowing for DMU A. Then, it is necessary to suppose that there would be a rise in its output from 8 to 9. Afterward, one optimal solution will be obtained as $\Delta x = 0.65$ once Model (6) is solved for it. In other words, the new output will appear in P_{out}. All the DMUs' efficiency scores can be estimated using Model (5) by making use of Δx from Model (6) for the other DMUs. Seemingly, it is discerned that the efficiency scores related to DMU C and E underwent a boost from 0.67 and 0.56 to 0.72 and 0.58, respectively. Also, the efficiency scores related to the other DMUs would stay unaffected.

Example 2. To better understand the proposed method, we also exploited a two-input one-output five-DMU example. Primarily, Model (3) was solved with the aim of estimating the DMUs' efficiency scores prior to modifications applied to the inputs and outputs of the given efficient DMU. Table 2 illustrates all the relevant data and results attained for Model (3). Having a glimpse over the efficiency analysis of DMUs reveals that although technically DMUs 3 and 4 seem to be inefficient, DMUs 1, 2, and 5 prove to be efficient.

Table 2: The Data and Results Obtained from Model (3) with Two-Input and Two-Output.

DMU	Existing				Efficiency
	x_1	x_2	y_1	y_2	ψ^*
1	10	20	70	6	1.00
2	15	15	100	3	1.00
3	20	30	80	5	0.45
4	25	15	100	2	0.86
5	12	9	90	8	1.00

In case we suppose the efficient DMU 2 and that $W^T = (1, 1)$ as well as assuming that the DMU 2 outputs were changed from (100, 3) to (93, 6). Then the optimal objective quantities would be equal to 26.70 while the optimal solution would be $\Delta x_1 = 0.90$, $\Delta x_2 = -4.20$ if we solve Model (6) for it. This implies that the new outputs appear in P_{out}. At this point, we utilized the optimal solution quantities from Model (6) but the DMUs' efficiency scores were all required to be estimated by Model (5) for the other DMUs. Additionally, it was observed that the efficiency scores associated with DMU 4 experienced an enhancement from 0.86 to 1. Furthermore, there was no change in the other DMUs' efficiency scores. It can be pinpointed that they did not get worse. It can be asserted hereby that no feasible solution will exist for Model (6) on the condition that the DMU 2 outputs are converted from (100, 3) to (95, 6), for example.

Example 3. We aim at examining the methods more applicably to expose their competencies; we are then urged to presume an empirical example on the die press division of a motorcycle-part company [27]. In detail, the motorcycle parts are formed by the die press division using the die press machines. In the die press division, there exist a set of 25 die press machines having 80-ton press. In the current research, the die press machines have been assumed as the DMUs. To be more precise, Table 3 indicates that the mentioned company used 9 variables from the dataset having 6 inputs and 4 outputs characterized as x_1, x_2, x_3, x_4, x_5, and x_6 for the inputs as well as y_1, y_2, and y_3 for the outputs. It needs to be emphasized that the 7th input was eliminated from the dataset. Table 3 demonstrates the data and results obtained for Model (3) on the efficiency analysis, indicating that only 7 technically inefficient DMUs exist including DMU 4, DMU 9, DMU 10, DMU 15, DMU 20, DMU 23, and DMU 24 whereas the other DMUs seem to be entirely efficient technically.

Now, we take into account efficient DMU 3, assuming that the output vector related to DMU 3 has been converted from (906, 1.29, 96) to (861, 1.32, 99) as well as assuming that $W^T = (1, 1, 1, 1, 1, 1)$ for the input weights. Then it was revealed that no feasible solutions existed to this problem after having solved Model (3) for DMU 3. The results here resembled that of proposition 4 showing that the DMU 3 new output values were not in P_{out}. Consequently, the new input vector in this case was not detected to retain or enhance the efficiency sores related to some of the DMUs if the DMU 3 outputs are converted to (861, 1.32, 99). Again, we take into

Table 3: The Data and Results Obtained from Model (3) for the Die Press Separation of the Motorcycle-Part Company.

DMU	Existing									Efficiency
	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	ψ^*
1	18	8	93	140	5	110	861	1.27	98	1.00
2	18	7	87	120	5	80	947	1.27	96	1.00
3	18	7	83	110	5	70	906	1.29	96	1.00
4	24	14	163	260	15	210	672	1.27	96	0.47
5	21	5	62	120	5	90	974	1.27	98	1.00
6	21	7	79	150	20	110	867	1.28	100	1.00
7	18	9	102	210	20	150	806	1.27	98	1.00
8	18	8	95	110	20	80	869	1.29	97	1.00
9	24	14	136	320	15	240	696	1.28	97	0.47
10	24	9	98	220	20	170	817	1.28	100	0.80
11	24	8	94	110	15	120	913	1.23	100	1.00
12	24	6	82	170	15	120	989	1.27	99	1.00
13	18	8	97	130	15	170	823	1.29	95	1.00
14	18	7	89	90	15	130	862	1.32	97	1.00
15	21	7	79	180	15	140	888	1.31	98	0.78
16	21	7	87	110	10	130	847	1.34	97	1.00
17	21	9	113	140	10	80	793	1.27	98	1.00
18	21	6	74	130	10	130	957	1.29	99	1.00
19	21	7	79	110	10	110	842	1.32	99	1.00
20	21	8	95	200	10	170	839	1.28	97	0.66
21	18	7	74	170	15	110	887	1.26	98	1.00
22	21	7	76	90	5	120	898	1.31	98	1.00
23	24	16	187	290	15	280	614	1.29	98	0.45
24	24	13	175	210	15	260	735	1.28	95	0.48
25	18	8	91	130	20	120	885	1.31	96	1.00

account DMU 3 while it is assumed that the DMU 3 output vector is transformed from (906, 1.29, 96) to (861, 1.30, 99). Seemingly, the new output amounts appear in P_{out}. Having solved Model (3) for DMU 3, it is observed that the optimal solution would be $\Delta x_1 = 3.28$, $\Delta x_2 = -0.09$, $\Delta x_3 = -4.19$, $\Delta x_4 = 0.94$, $\Delta x_5 = 5$, $\Delta x_6 = 39.06$ while the optimal objective amount would be 337. Consequently, it is observed that the new input vector would be (21.28, 6.91, 78.81, 110.94, 10, 109.06). Moreover, Model (5) is employed for estimating the efficiency score of the entire DMUs utilizing the new input and output vectors for DMU 3 for the other DMUs. Here, it is discerned that the efficiency scores associated with DMU 20 and 24 underwent a boost from 0.66 and 0.48 to 0.67 and 0.49, respectively, while there were no alterations in the efficiency scores of the other DMUs.

At this instant, another efficient DMU is to be considered while regarding DMU 22. In case we assume that the output vector of DMU 22 experiences a variation from (898, 1.31, 98) to (947, 1.30, 100) as well as assuming $W^T = (1, 1, 1, 1, 1, 1)$ for the input weights, no optimal solution was accordingly detected for DMU 22 after solving Model (3). To further elucidate, it needs to be asserted that this was the case since (947, 1.30, 100) did not appear in P_{out}. Again, we need to take into account DMU 22 while there is a variation in the output vector of DMU 5 from (898, 1.31, 98) to (947, 1.29, 99). This time, an optimal solution to Model (3) for DMU 22 is found and the reason behind this is that the new outputs for DMU 22 stay in P_{out}. In further detail, the optimal solution would be $\Delta x_1 = 0.09$, $\Delta x_2 = -0.89$, $\Delta x_3 = -1.36$, $\Delta x_4 = 37.50$, $\Delta x_5 = 5$, $\Delta x_6 = 6.90$ while the optimal objective value would be 366.24. It can be then observed that the new input vector would be (21.09, 6.11, 74.64, 127.50, 10, 126.90). At last, it needs to be asserted that all the DMUs efficiency scores can be estimated by Model (5) while utilizing the new input and output vectors for DMU 22 for the remainder of the DMUs. Moreover, some improvement is discerned in the efficiency scores of DMU 15 and 23, changing from 0.78 and 0.45 to 0.81 and 0.46, respectively. Contrariwise, no change was observed in the efficiency scores of the rest of the DMUs.

6. Concluding Remarks and Further Research

If it is intended to specify the optimum potential values for the inputs (outputs) of any output values (inputs) in a deliberated DMU, the traditional inverse DEA model will be commonly utilized so that the relative efficiency value of the deliberated DMU would be steady in comparison with the other DMUs. In the current paper, a new invers DEA model was introduced which revolves around directional slack-based measures of efficiency with general returns to scale assumption of technology. In fact, the efficiency analyses of the entire DMUs are considered by the proposed inverse model which is unlike the inverse DEA models proposed by other surveys. In further detail, the inverse proposed model can be utilized with the aim of determining the optimum potential input values for certain outputs values of an efficient DMU. In this way, it is emphasized that the efficiency score associated with every DMU does not worsen in a new production possibility set. To be precise, the efficiency index associated with every DMU will be retained while having a possibility of improvement, as well. The considered efficient DMU was eliminated in this method from the present PPS while being substituted by the same efficient DMU following the alterations implemented to its inputs and outputs. Furthermore, the proposed inverse model for the resource allocation problem was examined in this paper in which it was possible to consider the upsurges (decreases) of certain outputs concurrently. Another point to underscore is that the proposed approach is advantaged by its simplicity while it can be simply extended in diverse directions. For example, it is possible to employ the proposed models for discretionary and nondiscretionary data or the fuzzy data.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Decision Support Model for Introduction of Gamification Solution Using AHP

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Gamification means the use of various elements of game design in nongame contexts including workplace collaboration, marketing, education, military, and medical services. Gamification is effective for both improving workplace productivity and motivating employees. However, introduction of gamification is not easy because the planning and implementation processes of gamification are very complicated and it needs interdisciplinary knowledge such as information systems, organization behavior, and human psychology. Providing a systematic decision making method for gamification process is the purpose of this paper. This paper suggests the decision criteria for selection of gamification platform to support a systematic decision making process for managements. The criteria are derived from previous works on gamification, introduction of information systems, and analytic hierarchy process. The weights of decision criteria are calculated through a survey by the professionals on game, information systems, and business administration. The analytic hierarchy process is used to derive the weights. The decision criteria and weights provided in this paper could support the managements to make a systematic decision for selection of gamification platform.

1. Introduction

Motivation of organizational members is one of the most important things and is not an easy problem to solve in corporate and education environments [1]. The study on gamified class showed that the gaming approach is both more effective in improving students' knowledge and more motivational than the nongaming approach [2]. The work of [3] showed that gamification is effective in work collaboration. However, it is a dawn of research on gamification and hard to find the detailed methodologies or works which show how the gamified environments could be designed and developed.

To improve the organizational outputs of the gamified environments and to motivate the organizational members with the gamified environments effectively, the gamified environments should be designed and developed according to the interdisciplinary approaches including psychology, computer science, pedagogy, management science, economics, esthetics, demography, statistics, and industrial engineering because the basis of the gamified environments is a game and those studies are closely related with a game. Managements might easily face with the problem when they select and introduce the gamification platform because the gamification platform has convergent characteristics of interdisciplinary areas described above. Providing a systematic decision making method for gamification process is the purpose of this paper. This paper suggests the decision criteria and weights for the selection of gamification platform to support a systematic decision making process for managements.

The following parts of this paper are organized in three parts. Firstly, previous works are reviewed including the recent approach of gamification, methodologies for selection and introduction of information systems, and analytic hierarchy process (AHP). Secondly, the decision criteria, weights, and a case study are provided. Finally, the implication of this study and further research issues are summarized in the Conclusion section.

2. Related Work

Previous works on the following topics are summarized in this section. Firstly, the definition and trend of gamification research and previous works on gamification supporting

platforms are summarized. Secondly, the methodologies for selection and introduction of information systems are reviewed because the gamification supporting platform is a kind of the information systems. Thirdly, the concept and previous works on AHP are provided.

2.1. Gamification and Supporting Platforms. Gamification is defined as the use of various elements which could be used in game design in nongame contexts including workplace collaboration, marketing, education, military, and medical services [4]. The recent works on gamification are summarized in Table 1.

The work of [5] defined the gamification platform as "It comes complete with reward features for points, levels, badges, virtual goods, Facebook credits, and coupons. There are installable widgets for notifications, progress, avatars, profiles, leaderboards, social sharing. There are published APIs for deep integration, back-end admin consoles for set up, and full reporting and metrics." There are many kinds of platforms for gamification. The list of some leading platforms includes Gamify, Badgeville, Bunchball, Big Door Media, CrowdTwist, Cynergy, SpectrumDNA, Reputely, iActionable, Scvngr, Manumatix, and Leapfrog Builders.

2.2. Methodologies for Selection and Introduction of Information Systems. METHOD/1 supports the introduction of enterprise information systems and breaks down each phase of introduction process into smaller steps named segments and tasks. A series of manuals of METHOD/1 provides these steps in detail [6, 7].

ASAP is SAP's rapid implementation supporting tool designed to streamline and standardize the implementation of SAP products. ASAP aims to optimize time, quality, and efficient use of resources. ASAP supports the entire team which includes internal team members from the customer company and external consultants such as project manager, business process consultants, and the technical staffs.

The work of [8] proposed integrated methodology framework which is composed of patterns, scenarios, road map, components, and repository. The components offer detailed functional tools needed in the implementation path, which includes the support system for solution introduction and evaluation.

The work of [9] provides methodology which consists of process and criteria to support selection activities of the information security systems. It presents the rating approach for prioritizing security systems and the hierarchical structure of process and criteria.

The work of [10] summarized quality attributes of software products. They found that there are different schools/opinions/traditions concerning the properties of critical systems and the best methods to develop them are performance (from the tradition of hard real-time systems and capacity planning), dependability (from the tradition of ultrareliable, fault-tolerant systems), security (from the traditions of the government, banking, and academic communities), and safety (from the tradition of hazard analysis and system safety engineering).

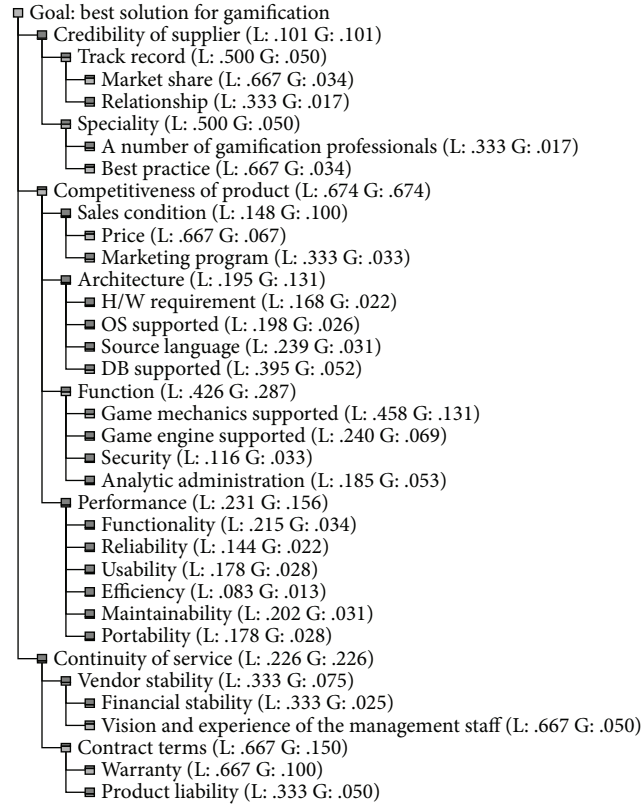

Figure 1: Weights of decision criteria for gamification platform.

ISO/IEC 9126 provides an international standard for the evaluation of software quality. ISO/IEC 9126 aims to solve the problems of human biases that could cause a negative impact on the selection and introduction of software. The human biases include unclear goal of the project, changing priorities after the kickoff of a project. To solve these problems, ISO/IEC 9126 suggests common goals of software selection and introduction projects which are as follows [11].

(i) Functionality: a set of attributes which provide a set of functions and their specified properties. These attributes provide suitability, accuracy, interoperability, security, and functionality compliance.

(ii) Reliability: a set of attributes which guarantee the performance level under stated conditions for a stated period of time. These attributes provide maturity, fault tolerance, recoverability, and reliability compliance.

(iii) Usability: a set of attributes which make easy to use, and the individual assessment of use by a set of users. These attributes provide understandability, learnability, operability, attractiveness, and usability compliance.

(iv) Efficiency: a set of attributes which guarantee the effective balance between inputs and outputs of the system. The inputs mean the amount of resources used for the system. The outputs mean the performance level of the system. The attributes provide time behavior, resource utilization, and efficiency compliance.

Table 1: Previous works on gamification.

Previous works	Key characteristics
[12]	It classifies the gamification approaches into the following: focusing on the technological aspects of computer games, focusing on the behavior evoked by computer games, and focusing on the design of computer games. It provides some case studies which show the process and benefits of gamification
[13]	It describes the characteristics of generation Y and the key elements of games that deserve a place in the enterprise including performance, achievement, and social interaction
[14]	It shows the interaction matrix of basic human desires and game mechanics including points, levels, challenges, virtual goods, leaderboards, and gifting. It provides the recent cases of gamification such as the frequent flyer programs, Foursquare, and Nike Plus
[15]	It describes the concept, benefits, key elements and mechanics of game design, and various cases of gamification
[3]	It shows the patterns of user activity in an enterprise social network service after the removal of game elements. It proved that the removal of game elements reduced overall participation within the SNS
[16]	It provides an experiment in computer science class which provides more frequent commits using a social software application. This study shows that the game elements are effective to motivate engineering students
[17]	It provides examples of social games which show behavioral economic biases related to the loss aversion tendency which is one of the key factors of behavioral economics and prospect theory
[19]	It provides the classification of engineering students based on the Bartle's game player types using the online survey which consists of 24 questionnaires

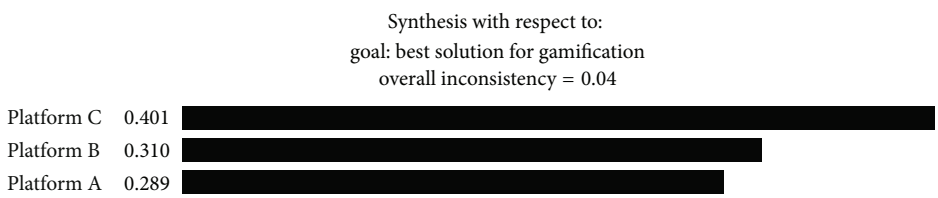

Figure 2: Synthesis result of the selection problem (exported from Expert Choice).

(v) Maintainability: a set of attributes which mean easiness for specified modifications of the system. These attributes provide analyzability, changeability, stability, testability, and maintainability compliance.

(vi) Portability: a set of attributes which support the system to be transferred from one environment to another. These attributes provide adaptability, installability, coexistence, replaceability, and portability compliance.

2.3. AHP. The AHP is a structured technique which supports a complex situation of decision making. It was proposed by Saaty in the 1970s based on mathematics and has been widely studied and used since then. It can be used in group decision making situation and has been used in various fields such as education, industry, and government.

Using the AHP, the decision problem is decomposed and structured into a hierarchy of easily understandable subproblems. One of the most important things is that each subproblem should guarantee independency. Each subproblem might be tangible or intangible aspect of the decision problem. After the building of the structured hierarchy, the decision makers judge pairwise comparison for every element of subproblems. The pairwise comparison is the process which judges the relative impact or importance of each element. In the pairwise comparison process, the decision makers can use concrete data about the elements or use their intuitive and professional judgments about the elements. Using of the human judgments is the essence of the AHP. Table 2 summarizes previous works on AHP.

Table 2: Previous works on AHOP.

Previous works	Key characteristics
[18, 20]	It describes the definition, calculation procedures, and application areas of AHP method
[21]	It provides a decision making model for selection of automobile using AHP
[22]	It provides the decision criteria and decision model, which are based on AHP, for introduction of expert system which could be used in education environments
[23]	It shows a decision model which supports an introduction of multimedia authoring tool for multiple decision makers

3. Decision Supporting Model Using AHP

This section provides the decision criteria and weights for the selection of gamification platform. The decision criteria are

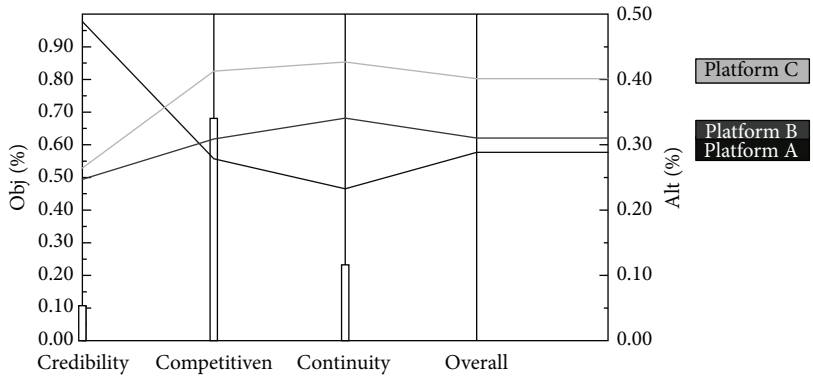

Figure 3: Sensitivity analysis (exported from Expert Choice).

derived from previous works on gamification and information systems.

The weights of decision criteria are calculated through a survey by the professionals using AHP. Also, a case study is provided to show a functionality and practical value of the decision criteria and weights.

3.1. Decision Criteria for Gamification Platform. This paper takes [8, 9] to suggest the first and second level of criteria. [3, 10–17] are used to derive third level of criteria. The decision criteria for the selection of gamification platform are shown in Table 3.

3.2. Weights of Decision Criteria for Gamification Platform. Judgments were elicited from the eight professionals on game, information systems, and business administration.

Expert Choice was used to rate the priorities among criteria. For example, the competitiveness of product was the most important criteria in level 2. After inputting the criteria and their importance into Expert Choice, the priorities from each set of judgments were found and recorded as shown in Figure 1.

The decision model classifies the goal, decision criteria, and variables into four major levels. The highest level of the hierarchy is the overall goal, to select the best gamification platform. Level 2, level 3, and level 4 represent the criteria in selecting the gamification platform. The overall consistency of the input judgments at all levels is within the acceptable ratio of 0.1, as recommended by Saaty et al. [18].

Table 3: Decision criteria for gamification platform.

1st level criteria	2nd level criteria	3rd level criteria
Credibility of supplier	Track record	Market share Relationship
	Speciality	A number of gamification professionals Best practice
Competitiveness of product	Sales condition	Price Marketing program
	Architecture	Hardware requirement OS supported Source language DB supported
	Function	Game mechanics supported Game engine supported Security Analytic administration
	Performance	Functionality Reliability Usability Efficiency Maintainability Portability
Continuity of service	Vendor stability	Financial stability Vision and experience of the management staff
	Contract terms	Warranty Product liability

4. A Case Study

4.1. Background. In this case study, AHP and the proposed selection model for gamification platform were applied to a particular project in which X Company located in South Korea wanted to select gamification platform. There was no relationship in corporate governance structure between gamification platform vendors and X Company, so vendors and products were treated as independent. Three gamification platforms were prepared for decision alternatives. In this paper, the alternatives are called platform A, platform B, and platform C.

4.2. Comparative Judgments on Three Gamification Platforms. Five staffs participated to compare each product using Expert Choice software. Table 4 shows the normalized priority weights of the gamification platforms.

The overall priority of the gamification platform alternatives is calculated by multiplying its global priority with the corresponding weight along the hierarchy. Synthesizing all the elements using Expert Choice, the results are shown in

Table 4: Normalized priority weights of three gamification platforms.

Decision criteria and weights			Platforms' priority weights		
Level 1	Level 2	Level 3	Platform A	Platform B	Platform C
Credibility of supplier (L: 0.101 G: 0.101)	Track record (L: 0.500 G: 0.050)	Market share (L: 0.667 G: 0.034)	0.019	0.01	0.006
		Relationship (L: 0.333 G: 0.017)	0.002	0.003	0.009
	Speciality (L: 0.500 G: 0.050)	A number of gamification professionals (L: 0.333 G: 0.017)	0.009	0.004	0.003
		Best practice (L: 0.667 G: 0.034)	0.019	0.007	0.008
Competitiveness of product (L: 0.674 G: 0.674)	Sales condition (L: 0.148 G: 0.100)	Price (L: 0.667 G: 0.067)	0.011	0.02	0.037
		Marketing program (L: 0.333 G: 0.033)	0.003	0.006	0.019
	Architecture (L: 0.195 G: 0.131)	H/W requirement (L: 0.168 G: 0.022)	0.006	0.012	0.012
		OS supported (L: 0.198 G: 0.026)	0.014	0.006	0.005
		Source language (L: 0.239 G: 0.031)	0.004	0.009	0.017
		DB supported (L: 0.395 G: 0.052)	0.029	0.012	0.008
	Function (L: 0.426 G: 0.287)	Game mechanics supported (L: 0.458 G: 0.131)	0.016	0.073	0.042
		Game engine supported (L: 0.240 G: 0.069)	0.007	0.011	0.038
		Security (L: 0.116 G: 0.033)	0.002	0.006	0.019
	Performance (L: 0.231 G: 0.156)	Analytic administration (L: 0.185 G: 0.053)	0.026	0.011	0.029
		Functionality (L: 0.215 G: 0.034)	0.019	0.009	0.009
		Reliability (L: 0.144 G: 0.022)	0.012	0.004	0.002
		Usability (L: 0.178 G: 0.028)	0.015	0.008	0.008
		Efficiency (L: 0.083 G: 0.013)	0.002	0.004	0.007
		Maintainability (L: 0.202 G: 0.031)	0.003	0.006	0.017
		Portability (L: 0.178 G: 0.028)	0.015	0.006	0.003
Continuity of service (L: 0.226 G: 0.226)	Vendor stability (L: 0.333 G: 0.075)	Financial stability (L: 0.333 G: 0.025)	0.014	0.007	0.007
		Vision and experience of the management staff (L: 0.667 G: 0.050)	0.024	0.028	0.011
	Contract terms (L: 0.667 G: 0.150)	Warranty (L: 0.667 G: 0.100)	0.008	0.031	0.056
		Product liability (L: 0.333 G: 0.050)	0.008	0.015	0.028

Figure 2. It shows that gamification platform C scored the highest in the result, followed by platform B and platform A.

4.3. Sensitivity Analysis. Sensitivity analysis attempts to check the impact of change in the input data or parameters of the proposed gamification platform. Relatively small changes in the hierarchy or judgment may lead to a different outcome. Using Expert Choice, the sensitivity of the outcome can be tested. Figure 3 shows a sensitivity analysis of the alternative priorities with respect to changes in the relative weights of the criteria.

4.4. Validation. The goal of the validation was to ensure that the results derived from this model were reasonable. That is to say, to examine if the gamification expert's knowhow could be substituted with the proposed decision criteria and weighted priorities is the goal of this validation.

Three gamification experts, apart from five staffs who participated in comparative judgments on three gamification platforms, helped to validate the proposed model. The judgments of three gamification experts with their own gamification knowhow and the result derived from this model as described in Section 4.2 were compared to assess the functionality of this model. Three gamification experts were not informed of which platform had been selected using this model. Background information on X Company and whitepapers on three gamification platforms were provided to gamification experts. After reviewing this information, three gamification experts chose the best platform and

the worst platform for X Company. They chose platform C as the best and platform A as the worst. The gamification experts' choice based on their knowhow was matched with the result described in Section 4.2.

5. Conclusion

This paper provides the decision criteria and weights for the selection of gamification platform. The decision criteria are derived from previous works on gamification and information systems. The weights of decision criteria are calculated through a survey by the professionals using AHP. Also, a case study that X Company used the decision criteria and weights for the selection of gamification platform in is provided to show a functionality and practical value of this paper.

The implications of this paper are summarized as follows.

(i) As described at the introduction part of this paper, it is only a short time since gamification approaches began, so the decision criteria and weights provided in this paper could support the selection of gamification platform.

(ii) The decision criteria on gamification platform would support the managements to understand what they should consider for successful gamification.

Limitation and further research issues are summarized as follows.

(i) It lacks providing sufficient pool of survey respondents for pairwise comparison, so a number of survey respondents should be increased to improve the reliability of the weights of decision criteria for gamification platform.

(ii) A case study which validates the functionality of the decision criteria for gamification platform is provided. However, the validation is not suffonsified because it only provides a single case.

(iii) Decision criteria for gamification platform should be enriched and revised through the in-depth and interdisciplinary reviews on human behavior, theory of organizational structure, theory of organizational behavior, content theory, process theory, game design, information systems, and so on.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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